3

The Poisson–Hermite Semigroup

In this chapter, we consider the Poisson–Hermite semigroup, which is the semigroup subordinated to the Ornstein–Uhlenbeck semigroup. This is analogous to the classical case in which the Poisson semigroup is obtained by subordination of the heat semigroup (for more details see the Appendix). Then, we study the characterization of the $\frac{\partial^2}{\partial t^2} + L$-harmonic functions, the generalized Poisson–Hermite semigroups, and the conjugated Poisson–Hermite semigroup which, as in the classical case, is closely related to the notion of singular integrals.

3.1 Definition and Basic Properties

We define the *Poisson–Hermite semigroup* as the semigroup subordinated to the Ornstein–Uhlenbeck semigroup using *Bochner's subordination formula*,[1]

$$e^{-\lambda} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\lambda^2/4u} du, \tag{3.1}$$

(see E. Stein [252]). Thus, making the change of variables $r = e^{-t^2/4u}$, we have

$$\begin{aligned} P_t f(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{(t^2/4u)} f(x) du \\ &= \frac{1}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{\exp\left(\frac{-|y-e^{-t^2/4u}x|^2}{1-e^{-t^2/2u}}\right)}{(1-e^{-t^2/2u})^{d/2}} du f(y) dy \\ &= \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 t \frac{\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r} f(y) dy. \end{aligned} \tag{3.2}$$

[1] In [133], C. Herz considered more general subordination relations between semigroups.

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4_3

Then,

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy, \tag{3.3}$$

with what we will call the *Poisson–Hermite kernel*,

$$p(t,x,y) = \frac{1}{\pi^{(d+1)/2}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{\exp\left(\frac{-|y-e^{-t^2/4u}x|^2}{1-e^{-t^2/2u}}\right)}{(1-e^{-t^2/2u})^{d/2}} du \tag{3.4}$$

$$= \frac{1}{4\pi^{(d+1)/2}} \int_0^1 t \frac{\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r}, \tag{3.5}$$

using the change of variables $r = e^{-t^2/4u}$. Moreover, making the change of variables $v = 1 - r^2$, we get

$$p(t,x,y) = \frac{1}{8\pi^{(d+1)/2}} \int_0^1 \frac{e^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})^{3/2}} \frac{e^{-\frac{|y-\sqrt{1-v}x|^2}{v}}}{v^{d/2}} \frac{dv}{1-v}. \tag{3.6}$$

The subordination of the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$ can be expressed alternatively in the following way. Let $\mu_t^{(1/2)}$ be the Borel measure on $[0,\infty)$ whose Laplace transform satisfies

$$\int_0^\infty e^{-\lambda s} \mu_t^{(1/2)}(ds) = e^{-\sqrt{\lambda}t}.$$

It is easy to check that the family of measures $\{\mu_t^{(1/2)}\}_{t\geq 0}$ forms a convolution semigroup [81]. Moreover, using Bochner's subordination formula (3.1) (with $\lambda = t\sqrt{\alpha}$ and the change of variables $s = \frac{t^2}{4u}$), it yields the following explicit expression of the measure $\mu_t^{(1/2)}$:

$$\mu_t^{(1/2)}(du) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4u} u^{-3/2} du. \tag{3.7}$$

Then, P_t can be defined by

$$P_t f(x) = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(ds). \tag{3.8}$$

The Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$ is a strongly continuous, symmetric, conservative semigroup of positive contractions in $L^p(\gamma_d)$, $1 \leq p < \infty$, with infinitesimal generator $(-L)^{1/2}$. More precisely,

Theorem 3.1. *The family of operators $\{P_t\}_{t\geq 0}$ satisfies the following properties:*

i) Semigroup property:

$$P_{t_1+t_2} = P_{t_1} \circ P_{t_2},\ t_1, t_2 \geq 0.$$

ii) Positivity and conservative property:

$$P_t f \geq 0, \quad \textit{for} \quad f \geq 0,\, t \geq 0,$$

and

$$P_t 1 = 1.$$

iii) Contractivity property:

$$||P_t f||_{p,\gamma} \leq ||f||_{p,\gamma}, \quad t \geq 0,\, 1 \leq p \leq \infty.$$

iv) Strong $L^p(\gamma_d)$-continuity property: The mapping $t \to P_t f$ is continuous from $[0,\infty)$ to $L^p(\gamma_d)$, for $1 \leq p < \infty$ and $f \in L^p(\gamma_d)$.

v) Symmetry property: P_t is a self-adjoint operator in $L^2(\gamma_d)$, i.e.,

$$\int_{\mathbb{R}^d} P_t f(x) g(x) \gamma_d(dx) = \int_{\mathbb{R}^d} f(x) P_t g(x) \gamma_d(dx),\, t \geq 0. \tag{3.9}$$

vi) Infinitesimal generator: $(-L)^{1/2}$ is the infinitesimal generator of $\{P_t : t \geq 0\}$, that is to say,

$$\lim_{t \to 0} \frac{P_t f - f}{t} = (-L)^{1/2} f. \tag{3.10}$$

Proof. These results can be obtained immediately from Theorem 2.5 using Bochner's subordination formula (3.1). □

As the Poisson–Hermite semigroup is subordinated to the Ornstein–Uhlenbeck semigroup and, therefore, $(-L)^{1/2}$ is its infinitesimal generator, we conclude that P_t can be defined in the spectral sense as $e^{-t(-L)^{1/2}}$. Therefore,

$$P_t \mathbf{h}_\nu = e^{-t\sqrt{|\alpha|}} \mathbf{h}_\nu. \tag{3.11}$$

Proposition 3.2. *(B. Muckenhoupt)*

i) If f has a Hermite expansion $f = \sum_{k=0}^{\infty} \mathbf{J}_k f$, then for all $t \geq 0$, $P_t f$ has a Hermite expansion

$$P_t f = \sum_{k=0}^{\infty} e^{-t\sqrt{k}} \mathbf{J}_k f. \tag{3.12}$$

ii) If $f \in L^2(\gamma_d)$ then $\sum_{k=0}^{\infty} e^{-t\sqrt{k}} \mathbf{J}_k f(x)$ converges absolutely to $P_t f(x)$ almost everywhere (a.e.) x.

Proof.

i) By arguments analogous to those given in Proposition 2.3, and using Bochner's subordination formula (3.1), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} P_t f(x)\mathbf{h}_\nu(x)\gamma_d(dx) &= \int_{\mathbb{R}^d}\left(\int_0^1 T(t,r)T_{(-\log r)}f(x)dr\right)\mathbf{h}_\nu(x)\gamma_d(dx)\\ &= \int_0^1\int_{\mathbb{R}^d} T_{(-\log r)}f(x)\mathbf{h}_\nu(x)\gamma_d(dx)T(t,r)dr\\ &= \langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\int_0^1 r^\nu T(t,r)dr = e^{-t\sqrt{\nu}}\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}.\end{aligned}$$

ii) As the sequence $\{\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu(x)\}_\nu$ is bounded for each x, by the Weierstrass M-test, the series $\sum_{k=0}^\infty e^{-t\sqrt{k}}\mathbf{J}_k f(x)$ converges absolutely for each x. Given that $L^2(\gamma_d)\subset L^1(\gamma_d)$, then according to *i*), $P_t f(x)$ has an expansion $P_t f(x) = \sum_{k=0}^\infty e^{-t\sqrt{k}}\mathbf{J}_k f(x)$; this must be the limit a.e. □

B. Muckenhoupt obtained this result for $d=1$ (see [193]). It was extended to higher dimensions by C. P. Calderón [44] .

To study higher-order Gaussian Littlewood–Paley g functions and the Gaussian Besov–Lipschitz and Triebel–Lizorkin spaces and Riesz transform, we need some results for the k-th derivatives of the Poisson–Hermite semigroup $\frac{\partial^k P_t f(x)}{\partial t^k}$. Let us consider explicitly their expressions. First, recall that $p(t,x,y)$ can be written as

$$p(t,x,y) = \frac{1}{2\pi^{(d+1)/2}}\int_0^1 t\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}.$$

Therefore, using Rodrigues' formula (1.28),

$$\begin{aligned}\frac{\partial p(t,x,y)}{\partial t} &= \frac{1}{2\pi^{(d+1)/2}}\int_0^1\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\left(1-\frac{t^2}{2(-\log r)}\right)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}.\\ &= \frac{1}{2^2\pi^{(d+1)/2}}\int_0^1\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\left(2-4\frac{t^2}{4(-\log r)}\right)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}.\\ &= -\frac{1}{2^2\pi^{(d+1)/2}}\int_0^1\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}H_2\left(\frac{t}{2\sqrt{-\log r}}\right)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}.\end{aligned}\tag{3.13}$$

where H_2 is the Hermite polynomial of order 2.

Moreover, by induction, again using Rodrigues' formula (1.28) and the three-term recurrence relation of the Hermite polynomials (10.19), it can be proved that, for $k>1$

$$\frac{\partial^k p(t,x,y)}{\partial t^k} = C_d\int_0^1\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{H_{k+1}\left(\frac{t}{2(-\log r)^{1/2}}\right)}{(-\log r)^{\frac{k-1}{2}}}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r},$$

where H_{k+1} is the Hermite polynomial of order $k+1$.

On the other hand, for $j = 1,\cdots,d$,

$$\begin{aligned}\frac{\partial p(t,x,y)}{\partial x_j} &= \frac{1}{\pi^{(d+1)/2}}\int_0^1 t\,\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\,\frac{(y_j - rx_j)}{(1-r^2)^{(d+1)/2}}\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)dr\\ &= \int_0^1 t\,\frac{\exp\left(t^2/4\log r\right)}{(-\log r)}\,\omega(r)\,\frac{(y_j - rx_j)}{(1-r^2)^{(d+3)/2}}\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)dr,\end{aligned} \tag{3.14}$$

where $\omega(r) = C_d(\frac{1-r^2}{-\log r})^{1/2}$ is a Lipschitz function on $[0,1]$, and

$$\begin{aligned}\frac{\partial^{|\beta|}p(t,x,y)}{\partial x_1^{\beta_1}\cdots x_d^{\beta_d}} &= \frac{(-1)^{|\beta|}}{2\pi^{(d+1)/2}}\int_0^1 t\,\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\,r^{|\beta|}\mathbf{H}_\beta\Big(\frac{y-rx}{(1-r^2)^{1/2}}\Big)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\,\frac{dr}{r}\\ &= \int_0^1 t\,\frac{\exp\left(t^2/4\log r\right)}{(-\log r)}\,\omega(r)r^{|\beta|}\mathbf{H}_\beta\Big(\frac{y-rx}{(1-r^2)^{1/2}}\Big)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{(d+1)/2}}\,\frac{dr}{r}.\end{aligned} \tag{3.15}$$

Now, we will also need the following technical result about the L^1-norm of the derivatives of the kernel $p(t,x,y)$.

Lemma 3.3. *If $p(t,x,y)$ is the Poisson–Hermite kernel, then*

$$\int_{\mathbb{R}^d}\left|\frac{\partial p(t,x,y)}{\partial t}\right|dy \le \frac{C}{t}, \tag{3.16}$$

where C is a constant independent of x and t. Moreover, for any positive integer k, we have

$$\int_{\mathbb{R}^d}\left|\frac{\partial^k p(t,x,y)}{\partial t^k}\right|dy \le \frac{C}{t^k}. \tag{3.17}$$

Proof. Let us first prove (3.16). Using Tonelli's theorem, using the fact that

$$\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}dy = 1,$$

we have

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\frac{\partial p(t,x,y)}{\partial t}\right|dy &\le \frac{1}{2\pi^{(d+1)/2}}\int_{\mathbb{R}^d}\int_0^1\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\left|1+\frac{t^2}{2\log r}\right|\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\,\frac{dr}{r}\,dy\\ &= \frac{1}{2\pi^{(d+1)/2}}\int_0^1\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\left|1+\frac{t^2}{2\log r}\right|\int_{\mathbb{R}^d}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\,dy\,\frac{dr}{r}\\ &= \frac{1}{2\pi^{1/2}}\int_0^1\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\left|1+\frac{t^2}{2\log r}\right|\frac{dr}{r}.\end{aligned}$$

Thus, what we need to prove is

$$\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \left|1+\frac{t^2}{2\log r}\right| \frac{dr}{r} \le \frac{C}{t}. \tag{3.18}$$

Making the change of variables $s=-\log r$, we get

$$\begin{aligned}\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \left|1+\frac{t^2}{2\log r}\right| \frac{dr}{r} &= \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\left|1-\frac{t^2}{2s}\right| ds \\ &\le \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} ds + \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\frac{t^2}{2s} ds\end{aligned}$$

Now, making the change of variables $v=\frac{t^2}{4s}$, $ds=-\frac{t^2}{4v^2}dv$, we get

$$\begin{aligned}\int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} ds &= \int_0^\infty e^{-v}\left(\frac{t^2}{4v}\right)^{-3/2}\frac{t^2}{4v^2}dv = \int_0^\infty e^{-v}\frac{(4v)^{3/2}}{t^3}\frac{t^2}{4v^2}dv \\ &= \frac{C}{t}\int_0^\infty e^{-v}v^{-1/2}dv = \frac{C\Gamma(1/2)}{t} = \frac{C'}{t}\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\frac{t^2}{4s} ds &= 2\int_0^\infty e^{-v}\left(\frac{t^2}{4v}\right)^{-3/2} v\frac{t^2}{4v^2}dv = 2\int_0^\infty e^{-v}\frac{(4v)^{3/2}}{t^3}v\frac{t^2}{4v^2}dv \\ &= \frac{C}{t}\int_0^\infty e^{-v}v^{1/2}dv = \frac{C\Gamma(3/2)}{t} = \frac{C'}{t}.\end{aligned}$$

For the proof of the general case (3.17), we use induction. As the case $k=1$ is already proved, let us assume that (3.17) holds for certain k and prove that it also holds for $k+1$. According to the semigroup property, and taking $u=t+s$, we have

$$\begin{aligned}\frac{\partial^{k+1}p(u,x,y)}{\partial u^{k+1}} &= \frac{\partial}{\partial s}\frac{\partial^k}{\partial t^k}p(t+s,x,y) = \frac{\partial}{\partial s}\frac{\partial^k}{\partial t^k}\left[\int_{\mathbb{R}^d} p(s,x,v)p(t,v,y)dv\right] \\ &= \int_{\mathbb{R}^d}\frac{\partial p(s,x,v)}{\partial s}\frac{\partial^k p(t,v,y)}{\partial t^k}dv.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\frac{\partial^{k+1}p(u,x,y)}{\partial u^{k+1}}\right| dy &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\frac{\partial p(s,x,v)}{\partial s}\right|\left|\frac{\partial^k p(t,v,y)}{\partial t^{ks}}\right| dvdy \\ &\le \int_{\mathbb{R}^d}\left|\frac{\partial p(s,x,v)}{\partial s}\right|\int_{\mathbb{R}^d}\left|\frac{\partial^k p(t,v,y)}{\partial t^k}\right| dydv \le \frac{C}{s}\frac{C}{t^k}.\end{aligned}$$

Finally, taking $s=t=u/2$, the case $k+1$ is proved. □

Using the representation of the Poisson–Hermite semigroup (3.8) using the one-sided stable measure

$$\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}}\frac{e^{-t^2/4s}}{s^{3/2}}ds = g(t,s)ds,$$

we can rephrase the result of Lemma 3.3 in terms of $\mu_t^{(1/2)}$ as follows. First, for any $k \in \mathbb{N}$, the notation $\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)$ denotes

$$\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds) := \frac{\partial^k g(t,s)}{\partial t^k}ds.$$

Then, by induction, it can be seen that

$$\frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(ds) = \left(\sum_{\substack{i\in\mathbb{Z}, j\in\mathbb{N},\\ 0\le j\le k, 2j-i=k}} a_{i,j}\frac{t^i}{s^j}\right)\mu_t^{(1/2)}(ds) \tag{3.19}$$

where $\{a_{i,j}\}$ is a (finite) set of constants.

Moreover, using the change of variables $u = \frac{t^2}{4s}$, it is easy to see that given $k \in \mathbb{N}$ and $t > 0$

$$\int_0^{+\infty} \frac{1}{s^k}\mu_t^{\frac{1}{2}}(ds) = \frac{C_k}{t^{2k}}, \tag{3.20}$$

and then, if $k \in \mathbb{N}$ and $t > 0$

$$\int_0^{+\infty} \left|\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}\right|(ds) \le \frac{C_k}{t^k}. \tag{3.21}$$

Additionally a pointwise estimate of the k-th derivative of the Poisson–Hermite semigroup is needed in what follows.

Lemma 3.4.

$$\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right| \le C_k T^* f(x) t^{-k}, \tag{3.22}$$

where $T^* f$ is the maximal Ornstein–Uhlenbeck function.

Proof. Using (3.21) and the dominated convergence theorem, we have

$$\begin{aligned}\left|\frac{\partial^k P_t f(x)}{\partial t^k}\right| &= \left|\int_0^{+\infty} T_s f(x)\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)\right| \le \int_0^{+\infty} |T_s f(x)|\left|\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)\right| \\ &\le \int_0^{+\infty} T^* f(x)\left|\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)\right| \le C_k T^* f(x) t^{-k}. \qquad \square\end{aligned}$$

Now, we need an estimate of the $L^p(\gamma_d)$-norms of the derivatives of the Poisson–Hermite semigroup.

Lemma 3.5. *Suppose $f \in L^p(\gamma_d)$, then for any integer k, the function $\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma}$ is a non-increasing function of t, for $0 < t < +\infty$. Moreover,*

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} \leq C\|f\|_{p,\gamma}\, t^{-k},\ t > 0. \tag{3.23}$$

Proof. Let us consider first the case $k = 0$. Let us fix $t_1, t_2 > 0$, then by using the semigroup property, we get

$$u(x, t_1 + t_2) = P_{t_1+t_2} f(x) = P_{t_1}(P_{t_2} f(x)) = P_{t_1}(u(x, t_2))$$

Therefore, by definition of P_t, Jensen's inequality and the invariance of γ_d

$$\begin{aligned}
\int_{\mathbb{R}^d} |u(x,t_1+t_2)|^p \gamma_d(dx) &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} p(t_1,x,y) u(y,t_2)\, dy \right|^p \gamma_d(dx) \\
&\leq \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} p(t_1,x,y) |u(y,t_2)|^p dy \Big) \gamma_d(dx) \\
&= \int_{\mathbb{R}^d} P_{t_1}(|u(x,t_2)|^p) \gamma_d(dx) = \int_{\mathbb{R}^d} |u(x,t_2)|^p \gamma_d(dx).
\end{aligned}$$

Thus,

$$\|P_{t_1+t_2} f\|_{p,\gamma} \leq \|P_{t_2} f\|_{p,\gamma}.$$

Now, we prove the general case $k > 0$. Differentiating the identity

$$u(x, t_1 + t_2) = P_{t_1}(u(x, t_2))$$

k-times with respect to t_2, we get

$$\frac{\partial^k u(x, t_1 + t_2)}{\partial (t_1 + t_2)^k} = P_{t_1}\Big(\frac{\partial^k u(x, t_2)}{\partial t_2^k}\Big)$$

and then we use an analogous argument to the one above.

To prove (3.23), we again use the representation of the Poisson–Hermite semigroup with a one-sided stable measure (3.8), and differentiating it k-times with respect to t, we get

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds).$$

Thus, using Minkowski's integral inequality, the contractive property of the Ornstein–Uhlenbeck semigroup and inequality (3.21), we get for $t > 0$

$$\begin{aligned}
\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} &\leq \int_0^{+\infty} \left\| T_s f \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) \right\| = \int_0^{+\infty} \|T_s f\|_{p,\gamma} \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) \right| \\
&\leq \|f\|_{p,\gamma} \int_0^{+\infty} \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) \right| \leq \frac{C_k}{t^k} \|f\|_{p,\gamma}. \qquad \square
\end{aligned}$$

Definition 3.6. *The maximal function of the Poisson–Hermite semigroup or Poisson–Hermite maximal function $\{P_t\}_{t\geq 0}$ is defined as*

$$P^* f(x) = \sup_{t>0} |P_t f(x)|. \tag{3.24}$$

In Theorem 4.28 of Chapter 4, we study the boundedness properties of P^*, proving that it is bounded in $L^p(\gamma_d)$ for $1 < p \leq \infty$, and it is of weak type $(1,1)$ with respect to the measure γ_d. Moreover, from the boundedness property of P^*, it follows that

$$P_0 f(x) = \lim_{t\to 0^+} P_t f(x) = f(x) \quad a.e.\ x \in \mathbb{R}^d, \tag{3.25}$$

and

$$P_\infty f(x) := \lim_{t\to\infty} P_t f(x) = \int_{\mathbb{R}^d} f(y)\, \gamma_d(dy) \quad a.e.\ x \in \mathbb{R}^d, \tag{3.26}$$

for all $f \in L^p(\gamma_d)$, $1 \leq p \leq \infty$; see Theorem 4.46,. Observe that this says that the Poisson–Hermite semigroup does not decay at infinity, i.e., it is not true that $P_t \to 0$ as $t \to \infty$, unless $\int_{\mathbb{R}^d} f(y)\, \gamma_d(dy) = 0$. In this case, one can obtain a precise estimate of the decay, as it is proved in the following result.

Lemma 3.7. *The Poisson–Hermite semigroup $\{P_t\}_{t>0}$ has exponential decay on $\mathscr{C}_0^\perp = \bigoplus_{k=1}^\infty \mathscr{C}_k$. More precisely, if $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$,*

$$|P_t f(x)| \leq C_{d,f}(d + |x|)e^{-t}. \tag{3.27}$$

Proof. As $\{P_t\}_{t>0}$ is a strongly continuous semigroup, we have

$$\lim_{t\to 0^+} P_t f(x) = f(x) \tag{3.28}$$

and according to the hypothesis, because we are assuming that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$,

$$\lim_{t\to\infty} P_t f(x) = 0. \tag{3.29}$$

Let us prove that

$$\left|\frac{\partial}{\partial t} P_t f(x)\right| \leq C_{d,f}(d + |x|)e^{-t}.$$

As

$$\frac{\partial T_t f}{\partial t}(x) = L(T_t f)(x),$$

differentiating in (2.36), we have

$$\nabla_x (T_t f)(x) = \left(e^{-t} T_t \left(\frac{\partial f}{\partial x_1} \right)(x), \ldots, e^{-t} T_t \left(\frac{\partial f}{\partial x_d} \right)(x) \right)$$

and

$$\triangle_x (T_t f)(x) = \sum_{j=1}^d e^{-2t} T_t \left(\frac{\partial^2 f}{\partial x_j^2} \right)(x).$$

Therefore, taking $f \in C_b^2(\mathbb{R}^d)$ and using (3.2), we have that

$$\frac{\partial P_t f}{\partial t}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{2u} L(T_{t^2/4u} f)(x)du.$$

Carrying on the computations, as in [122], we get

$$\begin{aligned}\left|\frac{\partial P_t f}{\partial t}(x)\right| &\leq C_d \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} \left[\sum_{j=1}^d \frac{e^{-t^2/2u}}{2} + |x_j| e^{-t^2/4u}\right] f(u)du \\ &\leq C_{d,f}(d+|x|)e^{-t}.\end{aligned}$$

Then,

$$|P_t f(x)| \leq \int_t^\infty \left|\frac{\partial}{\partial s} P_s f(x)\right| ds \leq C_{d,f}(d+|x|)e^{-t}. \qquad \square$$

On the other hand, because the Poisson–Hermite semigroup is the subordinated semigroup of the Ornstein–Uhlenbeck semigroup, it is easy to see that it is also hypercontractive.

Additionally, we have the following result.

Proposition 3.8. *If $f \in L^p(\gamma_d)$, $u(x,t) = P_t f(x)$ is a $C^\infty(\mathbb{R}_+^{d+1})$ solution of the elliptic equation,*[2]

$$\frac{\partial^2 u}{\partial t^2}(x,t) + Lu = 0, \quad x \in \mathbb{R}^d, t > 0, \tag{3.30}$$

with boundary condition $u(x,0) = f(x), x \in \mathbb{R}^d$.

Proof. By the general theory of semigroups, given that $(-L)^{1/2}$ is the infinitesimal generator of $\{P_t\}_{t\geq 0}$, we have

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial}{\partial t}\left[\frac{\partial P_t f}{\partial t}(x)\right] = \frac{\partial}{\partial t}[(-L)^{1/2} P_t f(x)] \\ &= (-L)^{1/2}\left[\frac{\partial P_t f}{\partial t}(x)\right] = (-L)^{1/2}[(-L)^{1/2} P_t f(x)] = -Lu(x,t).\end{aligned}$$

Alternatively, if we assume first that $f \in L^2(\gamma_d)$, because the sequence $\{\langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu(x)\}_{\nu \geq 0}$ is bounded for each x, we know that

$$P_t f(x) = \sum_{k=0}^\infty e^{-t\sqrt{k}} \mathbf{J}_k f(x) = \sum_{k=0}^\infty e^{-t\sqrt{k}} \sum_{|\nu|=k} f_H(\nu) \mathbf{h}_\nu(x)$$

converges absolutely for each x; therefore, we can differentiate term by term. Now, because the Hermite polynomials are eigenfunctions of L, we have

[2] Sometimes called the wave equation (see for instance [59]).

$$\frac{\partial^2 P_t f}{\partial t^2}(x)+LP_t f(x)=\sum_{k=0}^{\infty}\frac{e^{-t\sqrt{k}}}{(2^k k!)^{1/2}}\sum_{|\nu|=k} f_H(\nu)[k\mathbf{H}_\nu(x)-k\mathbf{H}_\nu(x)]=0.$$

Differentiation under the integral sign is justified by showing that the derivatives of the kernel are bounded in y for each (t,x) in a neighborhood of (t_0,x_0), and this is easy to check by estimating the derivatives of $T(t,r)M_{(-\log r)}(x,y)$ and integrating with respect to r. The boundary condition holds by (3.25). □

Therefore, $u(x,t)=P_t f(x)$ satisfies:

$$2\frac{\partial^2 u}{\partial t^2}(x,t)+\Delta_x u(x,t)-2\langle x,\bigtriangledown_x u(x,t)\rangle=0, \tag{3.31}$$

and we will say that u is $\frac{\partial^2}{\partial t^2}+L$*-harmonic*. Moreover, $u(x,t)=P_t f(x)$, which can also be called the *Poisson–Hermite integral*, can be thought of as the $\frac{\partial^2}{\partial t^2}+L$-harmonic extension of f in $\mathbb{R}^d$ to the upper half-plane $\mathbb{R}^{(n+1)}_+$.

In [106], G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, and B. Viviani find optimal integrability conditions to guarantee the existence of solutions of (3.31).

3.2 Characterization of $\frac{\partial^2}{\partial t^2}+L$-Harmonic Functions

In the classical case, it is well known that Δ-harmonic functions on the disc $\mathbb{D}$, and in the case of the semiplane $\mathbb{R}^{d+1}_+$, are characterized by being the Poisson integral of $L^p(\mathbb{R}^d)$-functions, $1<p\leq\infty$, see for instance, [252, Chapter VII, §1] and [299, Vol I Chapter VII, 7].

In his famous paper [199], B. Muckenhoupt and E. Stein defined the notion of Poisson integral for the case of the ultraspherical expansions and then they gave the corresponding characterization of functions that are Poisson integrals of L^p-functions in that case.

In Gaussian harmonic analysis, the analogous problem is the characterization of $\frac{\partial^2}{\partial t^2}+L$-harmonic functions on the half-plane $\mathbb{R}^{d+1}_+$ that are Poisson–Hermite integrals of functions $\mathbb{R}^d$. This was studied by L. Forzani and W. Urbina in [94]. Let us start with the bounded case. The proof of this result essentially follows, with the necessary variations, the classical proof that can be found in Stein's book [252].

Theorem 3.9. *Given a function u defined in $\mathbb{R}^{d+1}_+$, u is $\frac{\partial^2}{\partial t^2}+L$-harmonic and bounded if and only if u is the Poisson–Hermite integral of a function in $L^\infty(\gamma_d)$.*

Proof. It is enough to prove the sufficient condition, because the necessary condition is immediate, as the Poisson–Hermite integral of a bounded function is $\frac{\partial^2}{\partial t^2}+L$-harmonic and bounded. Now, assume that u is a $\frac{\partial^2}{\partial t^2}+L$-harmonic function such that

$|u| \leq M$ in $\mathbb{R}^{d+1}_+$. For each $k \in \mathbb{N}$ set $f_k(x) = u(x, 1/k)$ and let $u_k(x,t)$ be the Poisson–Hermite integral of f_k. Let us consider

$$\Delta_k(x,t) = u(x,t+1/k) - u_k(x,t).$$

It is enough to prove that $\Delta_k \equiv 0$ because, assuming that, we have

$$u(x,t+1/k) = u_k(x,t) = \int_{\mathbb{R}^d} p(t,x,y) f_k(y)\, \gamma_d(dy)$$

and hence, by the boundedness condition

$$||f_k||_{L^\infty(\gamma)} = ||u(\cdot, 1/k)||_{L^\infty(\gamma)} \leq M < \infty.$$

Thus, $\{f_k\}$ is a bounded sequence in $L^\infty(\gamma_d) = (L^1(\gamma_d))^*$, and then, according to the Bourbaki–Alaoglu theorem, there is an $f \in L^\infty(\gamma_d)$ and a subsequence $\{f_{k'}\}$ such that $f_{k'} \to f$ in the weak* topology, that is,

$$\int_{\mathbb{R}^d} f_{k'}(y)\phi(y)\gamma_d(dy) \longrightarrow \int_{\mathbb{R}^d} f(y)\phi(y)\gamma_d(dy),$$

for all $\phi \in L^1(\gamma_d)$.

For a fixed $(x,t) \in \mathbb{R}^{d+1}_+$, choosing $\phi(\cdot) = p(t,x,\cdot)$, in the limit, we have that

$$u(x,t) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, \gamma_d(dy).$$

Then, to prove that $\Delta_k \equiv 0$; define, for $\varepsilon > 0$, the auxiliary function

$$U(x,t) = \Delta_k(x,t) + 2M\varepsilon t + \varepsilon h(x,t),$$

where $h(x,t) = e^{-2t}(\frac{2}{n}|x|^2 - 1) + 1$ is strictly positive, radial in x, and $\frac{\partial^2}{\partial t^2} + L$-harmonic function.

$U(x,t)$ is clearly $\frac{\partial^2}{\partial t^2} + L$-harmonic on $\mathbb{R}^{d+1}_+$ and continuous on $\overline{\mathbb{R}^{d+1}_+}$. We restrict our attention to the bounded domain $\Sigma = \{(x,t) : 0 < t < 1/\varepsilon, |x| < R\}$, where R is sufficiently large, to be chosen later. Then, on its boundary,

$$\begin{aligned}\partial\Sigma &= \{(x,0) : |x| < R\} \cup \{(x, 1/\varepsilon) : |x| < R\} \cup \{(x,t) : 0 < t < 1/\varepsilon, |x| = R\} \\ &= \partial\Sigma_1 \cup \partial\Sigma_2 \cup \partial\Sigma_3,\end{aligned}$$

we have the following two conditions:

- On $\partial\Sigma_1$, $\Delta_k(x,0) = 0$ and
$$U(x,0) = \varepsilon h(x,0) \geq 0.$$
- On $\partial\Sigma_2$,
$$U(x,1/\varepsilon) = \Delta_k(x,1/\varepsilon) + 2M + \varepsilon h(x,1/\varepsilon) \geq 0$$
since $|\Delta_k(x,t)| \leq 2M$.

- Finally, on $\partial\Sigma_3$, because $\Delta_k(x,t)$ is bounded and $h(x,t)$ is radially increasing in x, $U(x,t)$ is positive for R big enough (note that R depends on ε).

Then, by using the maximum principle,[3] we get that $U(x,t)\geq 0$ in the region Σ and this implies that for all $(x,t)\in\Sigma$

$$\Delta_k \geq -\varepsilon(2Mt+h(x,t)).$$

By a similar argument, considering $-\Delta_k$ instead of Δ_k, we get that for all $(x,t)\in\Sigma$

$$\Delta_k \leq \varepsilon(2Mt+h(x,t)).$$

Now, consider an arbitrary point $(x,t)\in\mathbb{R}^{d+1}_+$. For any ε small enough $(x,t)\in\Sigma$; thus, we can get both inequalities for $\Delta_k(x,t)$ and, therefore, $\Delta_k(x,t)=0$. □

The characterization result, mentioned above, is the following theorem (see [94]).

Theorem 3.10. *Given a function u defined in $\mathbb{R}^{d+1}_+$, u is $\frac{\partial^2}{\partial t^2}+L$-harmonic and uniformly $L^p(\gamma_d)$-bounded, $1\leq p<\infty$, that is to say*

$$sup_{t>0}||u(\cdot,t)||_{p,\gamma}\leq M, \tag{3.32}$$

if and only if u is the Poisson–Hermite integral of a function in $L^p(\gamma_d)$, if $p>1$. In the case p = 1, u is the Poisson–Hermite integral of a measure μ as above.

In the classical case, the analogous result of Theorem 3.10 is simply a corollary of the corresponding result of Theorem 3.9, but that is not the case here. The proof of Theorem 3.10 is a combination of the classical proof and specific estimates for the Gaussian measure. One of the necessary ingredients is the following result, which first appeared in [87].

Theorem 3.11. *Let us consider the operators*

$$L_1 u = \frac{\partial^2 u}{\partial t^2}+Lu, \text{ and } L_2 u = L_1 u - 2u. \tag{3.33}$$

If u satisfies $L_1 u=0$ or $L_2 u=0$, then:

i) Mean value inequality. There exists a constant C, dependent only on dimension, such that

$$|u(x,t)|\leq \frac{C}{|B((x,t),r)|}\int_{B((x,t),r)}|u(y,s)|dy\,ds, \tag{3.34}$$

for $r\leq t$, and $t\leq m(x)$, where, as before, $m(x)=1\wedge\frac{1}{|x|}$ is the admissibility function. Thus, the mean value inequality is valid for radii that are small enough.

ii) If $u\geq 0$ in $B((x,t),2r)$, then

$$u(z,l)\approx \frac{1}{|B((x,t),r)|}\int_{B((x,t),r)}u(y,s)dy\,ds, \tag{3.35}$$

for any $(z,l)\in B((x,t),r)$, with $r\leq t$ and $t\leq m(x)$.

[3] The weak maximum principle on bounded domains can be applied here as L is a uniformly elliptic differential operator with continuous coefficients.

iii) Harnack's inequality. There exists a constant $C > 0$ such that if $u \geq 0$ in $B((x,t),2r)$

$$\sup_{B((x,t),r)} u \leq C \inf_{B((x,t),r)} u \tag{3.36}$$

if $r \leq t$ and $t \leq m(x)$.

Proof. For each $(x_0,t_0) \in \mathbb{R}^{d+1}_+$, $x_0 \neq 0, |x_0| > 1$, set $B = B\Big((x_0,t_0), \frac{1}{|x_0|}\Big)$. Let us define on B the transformation

$$x = x_0 + \frac{1}{|x_0|}x',$$

$$t = t_0 + \frac{1}{|x_0|}t'.$$

Then $(x,y) \in B$ if and only if $(x',y') \in B((0,0),1)$. Define the function

$$U(x',t') = u\left(x_0 + \frac{1}{|x_0|}x', t_0 + \frac{1}{|x_0|}t'\right).$$

The function U satisfies the equation

$$\Delta_{x',t'}U - 2\frac{1}{|x_0|}\left(x_0 + \frac{1}{|x_0|}x'\right)\nabla_{x'}U = 0$$

and because $(x',t') \in B((0,0),1)$, then $\frac{1}{|x_0|}\left(x_0 + \frac{1}{|x_0|}x'\right)$ is bounded by a constant. Given that the (classical) mean value inequality is still true for differential operators with bounded first-order coefficients (see D. Gilbarg, N. S. Trudinger [113], page 244), we have

$$U(0,0) \leq \frac{1}{s^{d+1}}\int_{B((0,0),s)} U(x',t')dx'dt'$$

for all $s \leq 1$.

Now, according to the definition of U, the latter inequality can be rewritten as

$$\begin{aligned} u(x_0,t_0) &\leq \frac{1}{s^{d+1}}\int_{B((0,0),s)} u\left(x_0 + \frac{1}{|x_0|}x', t_0 + \frac{1}{|x_0|}t'\right)dx'dt' \\ &= \frac{|x_0|^{d+1}}{s^{d+1}}\int_{B((x_0,y_0),\frac{s}{|x_0|})} u(x,t)dxdt. \end{aligned}$$

Hence, to obtain the inequality, if $t_0 < \frac{1}{|x_0|}$, take $s = |x_0|t_0$ and if $t_0 > \frac{1}{|x_0|}$, $s = 1$.

To prove (3.35) and (3.36) we use, as before, the results we know for classical positive solutions (see D. Gilbarg, N. S. Trudinger [113, pages 244–250]). □

We are now ready to prove Theorem 3.10.

Proof. The necessary condition is immediate because the Poisson–Hermite integral of a $L^p(\gamma_d)$ function is $\frac{\partial^2}{\partial t^2}+L$-harmonic and $L^p(\gamma_d)$-bounded. We then just have to prove the sufficient condition.

For each $(x,t)\in\mathbb{R}^{d+1}_+$, consider an admissible ball $B((x,t),r)$ radius $r\le t$, and $t\le m(x)$, because, as we already know the values of Gaussian density $e^{-|y|^2}$ are equivalents for points (y,s) on that ball, it is clear that

$$B((x,t),r)\subset\Big\{(y,s): t-r<s<t+r\Big\},$$

and $|B((x,t),r)|=Cr^{d+1}$; therefore, using these facts, the mean value inequality (3.34) and Hölder's inequality, we get, for $1\le p<\infty$,

$$\begin{aligned}|u(x,t)|^p &\le \frac{C}{|B((x,t),r)|}\int_{B((x,t),r)}|u(y,s)|^p dy\,ds\\ &\le \frac{Ce^{|x|^2}}{r^{d+1}}\int_{t-r}^{t+r}\Big(\int_{\mathbb{R}^d}|u(y,s)|^p\gamma_d(dy)\Big)ds.\end{aligned}$$

Thus, according to the $L^p(\gamma_d)$-boundedness

$$|u(x,t)|\le Cr^{-d/p}e^{|x|^2/p},$$

with $r\le t$ and $t\le m(x)$.

As before, consider, for each $k\in\mathbb{N}$, $f_k(x)=u(x,1/k)$, $u_k(x,t)$ its Poisson–Hermite integral and

$$\Delta_k(x,t)=u(x,t+1/k)-u_k(x,t).$$

According to the weak compactness argument, it is again enough to prove that

$$\Delta_k\equiv 0.$$

Observe that, according to the previous inequality,

$$\begin{aligned}|u(x,t+1/k)| &\le C\left(\left(t+\frac{1}{k}\right)1\wedge\frac{1}{|x|}\right)^{-d/p}e^{|x|^2/p}\\ &\le C(k\vee 1\vee|x|)^{d/p}e^{|x|^2/p}.\end{aligned}$$

Now, consider the auxiliary function

$$U(x,t)=\Delta_k+2C\varepsilon(k^2+|x|^2)^d e^{|x|^2/p}t+\varepsilon h(x,t),$$

where h is as in the proof of Theorem 3.9. Then, $U(x,t)$ is clearly $\frac{\partial^2}{\partial t^2}+L$-subharmonic on $\mathbb{R}^{d+1}_+$ and continuous on $\overline{\mathbb{R}^{d+1}_+}$. Thus, according to an analogous argument to that of the proof of Theorem 3.9, to apply the maximum principle on the bounded domain

$$\Sigma = \left\{(x,t) : 0 < t < 1/\varepsilon, |x| < R\right\}$$

we get that $U(x,t) \geq 0$ in the region Σ; thus, this implies for all $(x,t) \in \Sigma$

$$\Delta_k \geq -\varepsilon \left(2C(k^2 + |x|^2)^d e^{|x|^2/p} + h(x,t)\right).$$

Analogously, considering $-\Delta_k$ instead of Δ_k, we get that for all $(x,t) \in \Sigma$

$$\Delta_k \leq \varepsilon \left(2C(k^2 + |x|^2)^d e^{|x|^2/p} + h(x,t)\right).$$

Now, consider an arbitrary point $(x,t) \in \mathbb{R}^{d+1}_+$. For any $\varepsilon > 0$ small enough $(x,t) \in \Sigma$; thus, we can get both inequalities for $\Delta_k(x,t)$, i.e., $\Delta_k(x,t) = 0$. Therefore, for $p > 1$, there exist $f \in L^p(\gamma_d)$ and a subsequence $\{f_{k'}\}$ such that $f_{k'} \to f$ in the weak* topology. Thus, $u(x,t)$ is the Poisson–Hermite integral of that f.

For $p = 1$ there exists a measure μ, such that $e^{-|y|^2}\mu(dy)$ is a finite measure, and a subsequence $\{f_{k'}\}$ such that $f_{k'} \to \mu$ in the weak* topology; therefore $u(x,t)$ is the Poisson–Hermite integral of μ. □

3.3 Generalized Poisson–Hermite Semigroups

The Poisson–Hermite semigroup can be generalized to a family of semigroups obtained from the Ornstein–Uhlenbeck semigroup, by using the *generalized subordination formula*. Let $\mu_t^{(\alpha)}$ be the Borel measure on $[0,\infty)$ such that its Laplace transform satisfies

$$\int_0^\infty e^{-\lambda s}\mu_t^{(\alpha)}(ds) = e^{-\lambda^\alpha t}, 0 < \alpha < 1. \tag{3.37}$$

The measures $\mu_t^{(\alpha)}$ are probability measures, which are known as *one-sided stable measures* in $[0,\infty)$ *of order* α; moreover, for each α fixed, $\{\mu_t^{(\alpha)}\}_{t\geq 0}$ is a convolution semigroup (see [81]).

Definition 3.12. *The generalized Poisson–Hermite semigroup of order α, $\{P_t^\alpha\}_{t\geq 0}$ is defined as*

$$P_t^\alpha f(x) = \int_0^\infty T_s f(x)\mu_t^{(\alpha)}(ds). \tag{3.38}$$

The proof that $\{P_t^\alpha\}_{t\geq 0}$ is a strongly continuous, symmetric, conservative semigroup of positive contractions on $L^p(\gamma_d)$, $1 \leq p < \infty$, with infinitesimal generator $(-L)^\alpha$ can be obtained by adapting the proof for the case $\alpha = 1/2$. Hence, formally

$$P_t^\alpha = e^{-(-L)^\alpha t},$$

which means that for any ν multi-index,

$$P_t^{\alpha} \mathbf{h}_{\nu} = e^{-t|\nu|^{\alpha}} \mathbf{h}_{\nu}, \tag{3.39}$$

and, therefore, if $f = \sum_{k=0}^{\infty} \mathbf{J}_k f$,

$$P_t^{\alpha} f = \sum_{k=0}^{\infty} e^{-tk^{\alpha}} \mathbf{J}_k f.$$

Again, these semigroups turn out to be hypercontractive; therefore,

Lemma 3.13. *If* $1 < p < \infty$

$$||P_t^{\alpha}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma} \leq e^{-tn^{\alpha}} ||f||_{p,\gamma}. \tag{3.40}$$

Proof. From Lemma 2.18, we have

$$||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma} \leq e^{-tn} ||f||_{p,\gamma}.$$

Then, using (3.37) and Minkowski's integral inequality, we get

$$\begin{aligned} ||P_t^{\alpha}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma} &\leq \left|\left| \int_0^{\infty} T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f \, \mu_t^{\alpha}(ds) \right|\right|_{p,\gamma} \\ &\leq \int_0^{\infty} ||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma} \, \mu_t^{\alpha}(ds) \\ &\leq \int_0^{\infty} e^{-nt} ||f||_{p,\gamma} \, \mu_t^{\alpha}(ds) \leq Ce^{-n^{\alpha}t} ||f||_{p,\gamma}. \end{aligned}$$ □

Now, if instead of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t \geq 0}$, in formula (2.73), we use the generalized Poisson–Hermite semigroups, $\{P_t^{\alpha}\}_{t \geq 0}$, we get *generalized potential operators*

$$U_{n,\alpha} f = \int_0^{\infty} P_t^{\alpha}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f \, ; dt, \tag{3.41}$$

and obtain similar $L^p(\gamma_d)$ estimates, as in (2.74), using Lemma 3.13 and Minkowski's integral inequality,

$$\|U_{n,\alpha} f\|_{p,\gamma} \leq \int_0^{\infty} \|P_t^{\alpha}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f\|_{p,\gamma} \, dt \leq C \frac{1}{n^{\alpha}} \|f\|_{p,\gamma}. \tag{3.42}$$

In particular, if $f \in \mathscr{C}_k$ with $k \geq n$,

$$U_{n,\alpha} f = \int_0^{\infty} P_t^{\alpha} f dt = \frac{1}{n^{\alpha}} f.$$

These results will be key in the proof of Meyer's multiplier theorem (see Theorem 6.2).

3.4 Conjugate Poisson–Hermite Semigroup

The investigation of conjugacy for discrete and continuous non-trigonometric orthogonal expansions was initiated and extensively studied in the seminal article by B. Muckenhoupt and E. M. Stein [199]. B. Muckenhoupt introduced in [194] the conjugate Hermite expansions for dimension $d = 1$. According to (3.31), we know that given $f \in L^1(\gamma_1)$, if $u(x,t) = P_t f(x)$, then $u(x,t)$ satisfies

$$2\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) - 2x\frac{\partial u}{\partial x}(x,t) = 0, \tag{3.43}$$

or equivalently,

$$2\frac{\partial^2 u}{\partial t^2}(x,t) + e^{x^2}\frac{\partial}{\partial x}\left(e^{-x^2}\frac{\partial u}{\partial x}(x,t)\right) = 0.$$

B. Muckenhoupt introduced the Gaussian *conjugate function* v of u by considering the *Gaussian Cauchy–Riemann* equations,

$$\begin{aligned} \frac{\partial u}{\partial x}(x,t) &= -\frac{\partial v}{\partial t}(x,t) \\ \frac{\partial u}{\partial t}(x,t) &= e^{x^2}\frac{\partial}{\partial x}(e^{-x^2}v(x,t)). \end{aligned} \tag{3.44}$$

Then, the function $v(x,t)$ must be defined as

$$v(x,t) = \int_{-\infty}^{\infty} Q(t,x,y) f(y)\, dy, \; t > 0, \tag{3.45}$$

where

$$Q(t,x,y) = \frac{\sqrt{2}}{\pi}\int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \exp\left(\frac{t^2}{4\log r}\right) \frac{y - rx}{(1-r^2)^2} \exp\left(\frac{-r^2x^2 + 2rxy - r^2y^2}{1-r^2}\right) dr. \tag{3.46}$$

Observe that (3.46) can be obtained from (3.4), for $d = 1$, differentiating with respect to x, integrating with respect to t, using the fact that Q must tend to 0 as $t \to \infty$ and multiplying by -1, i.e.,

$$Q(t,x,y) = -\int_t^{\infty} \frac{\partial p(s,x,y)}{\partial x} ds. \tag{3.47}$$

By construction v satisfies the first Cauchy–Riemann equation. Additionally, it is easy to see that v satisfies,

$$2\frac{\partial^2 v}{\partial t^2}(x,t) + \frac{\partial^2 v}{\partial x^2}(x,t) - 2x\frac{\partial v}{\partial x}(x,t) = -2v(x,t), \tag{3.48}$$

which is equivalent to

$$2\frac{\partial^2 v}{\partial t^2}(x,t) + \frac{\partial}{\partial x}\left[e^{x^2}\frac{\partial(e^{-x^2}v(x,t))}{\partial x}\right] = 0.$$

Now, because u satisfies (3.43), i.e., it is a $\frac{\partial^2}{\partial t^2} + L$-harmonic, but v does not, then it seems that probably this is not the best notion of conjugacy. More on the problem of notions of conjugacy for orthogonal polynomials can be found at [39].

Definition 3.14. *The conjugate Poisson–Hermite integral of f, is defined as*

$$P_t^c f(x) = v(x,t).$$

Therefore,

$$P_t^c f(x) = -\int_t^\infty \frac{\partial P_s f}{\partial x}(x)ds. \tag{3.49}$$

In [194], B. Muckenhoupt proved that $P_t^c f$ is bounded on $L^p(\gamma_1)$, $1 < p < \infty$ and as we see later in Chapter 9, if $t \to 0$, $P_t^c f$ tends to the *Gaussian Hilbert transform* $\mathscr{H} f$, in L^p-norm and a.e.

In his doctoral dissertation, R. Scotto [244] extended Muckenhoupt's notion of conjugacy in higher dimensions, $d > 1$, considering the Gaussian Cauchy–Riemann equations in $\mathbb{R}^d$:

$$\begin{aligned}
\frac{\partial u}{\partial x_j}(x,t) &= -\frac{\partial v_j}{\partial t}(x,t),\ j = 1,\dots,d\\
\frac{\partial v_i}{\partial x_j}(x,t) &= \frac{\partial v_j}{\partial x_i}(x,t),\ i,j = 1,\dots,d\\
\frac{\partial u}{\partial t}(x,t) &= \frac{1}{2}\sum_{j=1}^d e^{|x|^2}\frac{\partial}{\partial x_j}(e^{-|x|^2}v_j(x,t)).
\end{aligned} \tag{3.50}$$

From these relations, R. Scotto defined a system of conjugates,

$$(u(x,t), v_1(x,t), v_2(x,t), \dots, v_d(x,t)).$$

Again, following Muckenhoupt's argument, the functions $v_i(x,t)$ verify that the first equation of (3.50); thus,

Definition 3.15. *The i-th conjugate Poisson kernel of f, is defined as*

$$P_{i,t}^c f = v_i(x,t),\ i = 1,\dots,d.$$

Therefore,

$$P_{i,t}^c f = \int_{\mathbb{R}^d} Q_i(t,x,y) f(y)\,dy,\ t > 0, \tag{3.51}$$

where

$$Q_j(t,x,y) = -\int_t^\infty \frac{\partial p}{\partial x_j}(s,x,y)ds$$
$$= \frac{1}{\pi^{(d+1)/2}} \int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \exp\left(\frac{t^2}{4\log r}\right) \frac{y_j - rx_j}{(1-r^2)^{(d+3)/2}}$$
$$\times \quad \exp\left(\frac{-r^2(|x|^2+|y|^2)+2r\langle x,y\rangle}{1-r^2}\right) dr,$$

Thus,

$$P^c_{i,t} f(x) = -\int_t^\infty \frac{\partial P_s f}{\partial x_i}(x)ds, \tag{3.52}$$

for any $i = 1,\ldots,d$.

Thus, again following Muckenhoupt [194], we have the following result:

Proposition 3.16.

i) $Q_i(t,x,y)$ *is a bounded function in y, for any* $i = 1,\ldots,d$.
ii) If $f \in L^1(\gamma_d)$, *then for any* $i = 1,\ldots,d$ $P^c_{i,t} f$ *exists for any* $t > 0$ *and they verify an analogous equation as (3.48),*

$$\frac{\partial^2 v}{\partial t^2}(x,t) + Lv(x,t) = -v(x,t), \tag{3.53}$$

and the Gaussian Cauchy–Riemann equations (3.50).
iii) If f has a Hermite expansion $f = \sum_{k=0}^\infty \sum_{|\nu|=k} \hat{f}_H(\nu)\mathbf{h}_\nu$, *then, for any* $t \geq 0$, $P^c_{i,t} f$ *has a Hermite expansion*

$$P^c_{i,t} f = -\sum_{k=1}^\infty \sum_{|\nu|=k} \hat{f}_H(\nu) e^{-t\sqrt{|\nu|}} \sqrt{\frac{2}{|\nu|}} \nu_i \mathbf{h}_{\nu-\mathbf{e}_i}, \tag{3.54}$$

where $\mathbf{e_i}$ *is the unitary vector with zeros in all j-coordinates* $j \neq i$ *and one in the i-th coordinate. These series are called* conjugate Poisson series.
iv) If $f \in L^2(\gamma_d)$ *and* $t > 0$, *the series (3.54) converges a.e.*

Proof.

i) Let $i = 1,\ldots,d$ fixed. Considering the cases $0 \leq r < 1/2$ and $1/2 \leq t < 1$, and replacing $-\log r$ by a multiple of $1-r$, in the second case it can be proved that

$$\left(\frac{1-r^2}{-\log r}\right)^{1/2} \frac{\exp\left(\frac{t^2}{4\log r}\right)}{(1-r^2)^{(d+3)/2}} < C\Big(1 + \frac{1}{x^{d+4}}\Big).$$

Then, we get that $Q_i(t,x,y)$ is bounded.

ii) If $f \in L^1(\gamma_d)$ then $P^c_{i,t} f$ is well defined by*i*). The differentiation under the integral sign can be done as it can be proved that all the kernels are properly bounded (for more details see [244]).

$Q_i(t,x,y)$ satisfies the first equation of (3.50) by construction; therefore $P^c_{i,t} f(x)$ will verify it too. According to an analogous argument to that done for v verifying (3.48), we have that $P^c_{i,t} f(x)$ satisfies (3.53), because, as $u(x,t) = P_t f(x)$ satisfies (3.30), it follows that $\frac{\partial P^c_{i,t} f}{\partial x_i}(x)$ verifies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial P^c_{i,t} f}{\partial x_i}\right)(x) + L\frac{\partial P^c_{i,t} f}{\partial x_i}(x) = \frac{\partial P^c_{i,t} f}{\partial x_i}(x).$$

The second equation of (3.50) is satisfied immediately, as

$$\frac{\partial P^c_{i,t} f}{\partial x_j}(x) = -\int_t^\infty \frac{\partial^2 P_s f}{\partial x_i \partial x_j}(x)ds = \frac{\partial P^c_{j,t} f}{\partial x_i}(x).$$

Finally, the last equation of (3.50) is satisfied, because

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^d e^{|x|^2}\frac{\partial}{\partial x_i}(e^{-|x|^2}P^c_{i,t} f(x)) &= \sum_{i=1}^d \left[-x_i P^c_{i,t} f(x) + \frac{1}{2}\frac{\partial P^c_{i,t} f}{\partial x_i}(x)\right] \\
&= -\int_t^\infty \sum_{i=1}^d \left[-x_i \frac{\partial P_s f}{\partial x_i}(x) + \frac{1}{2}\frac{\partial^2 P_s f}{\partial x_i^2}(x)\right] ds \\
&= -\int_t^\infty L P_s f(x) ds = -\int_t^\infty \frac{\partial^2 P_s f}{\partial s^2}(x) ds \\
&= \frac{\partial P_t f}{\partial t}(x).
\end{aligned}$$

iii) Following an analogous argument as in Proposition 2.3, we can prove that

$$Q_i(t,x,y) = -\sum_{k=1}^\infty \sum_{|\nu|=k} \hat{f}_H(\nu) e^{-t\sqrt{|\nu|}} \sqrt{\frac{2}{|\nu|}}\, \nu_i \mathbf{h}_{\nu - \mathbf{e}_i}.$$

and

$$\int_{\mathbb{R}^d} Q_i(t,x,y)\mathbf{h}_{\nu-\mathbf{e}_i}(x)\gamma_d(dx) = e^{-t\sqrt{|\nu|}}\mathbf{h}_\nu(y),$$

and from there, using Fubini's theorem, we can prove that $P^c_{i,t} f$ has the expansion (3.54).

iv) It can be proved by an analogous argument to that in the proof of Proposition 2.3. □

3.5 Notes and Further Results

1. Following B. Muckenhoupt [193], the Poisson–Hermite kernel can also be written as
$$p(t,x,y) = \int_0^1 U(t,r)M_{(-\log r)}(x,y)dr,$$
where $M_t(x,y)$ is Mehler's kernel, and
$$U(t,r) = \frac{1}{2\pi^{1/2}} \frac{t\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{1}{r}.$$
P_t can also be written as
$$P_t f(x) = \int_0^1 U(t,r)T_{(-\log r)}f(x)dr. \tag{3.55}$$
Observe that the definition of the Poisson–Hermite semigroup given here, for $d=1$, differs from that in [193] by a constant, because in that case
$$T(t,r) = \frac{1}{(2\pi)^{1/2}} \frac{t\exp(t^2/2\log r)}{(-\log r)^{3/2}} \frac{1}{r},$$
which implies, essentially, similar relations, but with different constants.

2. Similar to the case of the Ornstein–Uhlenbeck semigroup, for the Jacobi semigroup and the Laguerre semigroup, using Bochner's subordination formula (3.1), we can define the *Jacobi–Poisson semigroup* $\{P_t^{\alpha,\beta}\}_{t\geq 0}$and the *Laguerre–Poisson semigroup* $\{P_t^{\alpha}\}_{t\geq 0}$, in addition to their conjugate semigroups (see for instance [213] and [209]). In an expository and very interesting paper [276], J. L. Torrea considers the semigroup theory as a tool for developing harmonic analysis for general differential second operators, based on the seminal papers of B. Muckenhoupt and E. Stein [199, 193] and [194].

3. Associated with the family of translated semigroups $\{T_t^{(\kappa)}\}_{t\geq 0}$, defined in (2.78), we have their *subordinated semigroups* $\{P_t^{(\kappa)}\}_{t\geq 0}$, defined by using the Bochner subordination formula; these are referred to as the *translated Poisson–Hermite semigroups*. Therefore,
$$P_t^{(\kappa)}\mathbf{h}_\nu = e^{-t\sqrt{|\nu|+\kappa}}\mathbf{h}_\nu. \tag{3.56}$$
Moreover, $P_t^{(\kappa)} f \leq P_t f$ for any $t \geq 0$ and $f \geq 0$. These translated semigroups are important in Chapter 5 and in Chapter 9.