2

The Ornstein–Uhlenbeck Operator and the Ornstein–Uhlenbeck Semigroup

In this chapter we are going to define and study the Ornstein–Uhlenbeck operator and the Ornstein–Uhlenbeck semigroup. They are analogous, in the Gaussian harmonic analysis, to the Laplacian and the heat semigroup in the classical case. Then, we study an important property of the Ornstein–Uhlenbeck semigroup, the hypercontractivity property, and some of its applications.

2.1 The Ornstein–Uhlenbeck Operator

In the classical case, we consider the Laplacian differential operator $\Delta_x = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^d$ and the eigenvalue problem

$$\Delta_x u = \lambda u \tag{2.1}$$

with boundary condition

$$u(x) = O(1), \quad \text{as } |x| \to \infty.$$

Then, the set of eigenvalues of this problem consists of all non-positive real numbers, and given $\lambda < 0$ the eigenfunctions corresponding to λ are

$$e^{i\langle \cdot, y\rangle}, \quad |y|^2 = -\lambda. \tag{2.2}$$

The *Ornstein–Uhlenbeck operator* in $\mathbb{R}^d$ is a second-order differential operator defined as

$$L = \frac{1}{2}\Delta_x - \langle x, \nabla_x\rangle = \sum_{i=1}^{d}\left[\frac{1}{2}\frac{\partial^2}{\partial x_i^2} - x_i\frac{\partial}{\partial x_i}\right], \tag{2.3}$$

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where $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d})$ is the gradient, and Δ_x is the Laplace operator defined in the space of test functions $C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$. The operator L has a self-adjoint extension to $L^2(\gamma_d)$, that is also denoted as L, that is,

$$\int_{\mathbb{R}^d} Lf(x)g(x)\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\gamma_d(dx); \tag{2.4}$$

thus, L is the natural *"symmetric" Laplacian* in this context.

The Ornstein–Uhlenbeck operator L can also be written as

$$L = \sum_{i=1}^{d} L_i, \tag{2.5}$$

where $L_i = \frac{1}{2}\partial_i^2 - x_i\partial_i$, $i = 1, \cdots d$, is the one-dimensional Ornstein–Uhlenbeck operator in the i-th variable. Hence, for $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$ and $\nu = (\nu_, \cdots, \nu_d)$ a multi-index,

$$\begin{aligned}
L\mathbf{H}_\nu(x) &= \sum_{i=1}^{d}\Big[\frac{1}{2}\frac{\partial^2\mathbf{H}_\nu}{\partial x_i}(x) - x_i\frac{\partial \mathbf{H}_\nu(x)}{\partial x_i}(x)\Big] \\
&= \sum_{i=1}^{d}\Big[\frac{1}{2}\frac{\partial^2}{\partial x_i^2}\prod_{j=1}^{d} H_{\nu_j}(x_j) - x_i\frac{\partial}{\partial x_i}\prod_{j=1}^{d} H_{\nu_j}(x_j)\Big] \\
&= \sum_{i=1}^{d}\prod_{j=1, j\neq i}^{d} H_{\nu_j}(x_j)\Big[\frac{1}{2}\frac{\partial^2 H_{\nu_i}}{\partial x_i}(x_i) - x_i\frac{\partial H_{\nu_i}}{\partial x_i}(x_i)\Big] \\
&= \sum_{i=1}^{d}\prod_{j=1, j\neq i}^{d} H_{\nu_j}(x_j)L_iH_{\nu_i}(x_i) = \sum_{i=1}^{d}(-\nu_i)\prod_{j=1}^{d} H_{\nu_j}(x_j) \\
&= \sum_{i=1}^{d}(-\nu_i)\mathbf{H}_\nu(x) = -|\nu|\mathbf{H}_\nu(x).
\end{aligned} \tag{2.6}$$

Thus, the Hermite polynomials in d-variables, $\{\mathbf{H}_\nu\}_\nu$ are eigenfunctions of L with corresponding eigenvalues $\lambda_\nu = -|\nu| = -\sum_{i=1}^d \nu_i$, i.e.,

$$L\mathbf{H}_\nu = \lambda_\nu\mathbf{H}_\nu = -|\nu|\mathbf{H}_\nu, \tag{2.7}$$

and the normalized Hermite polynomials $\mathbf{h}_\nu$ are also eigenfunctions of the Ornstein–Uhlenbeck operator, with the same corresponding eigenvalue,

$$L\mathbf{h}_\nu = \lambda_\nu\mathbf{h}_\nu = -|\nu|\mathbf{h}_\nu.$$

Moreover, consider the eigenvalue problem

$$Lu = \lambda u \tag{2.8}$$

with boundary condition

$$u(x) = O(|x|^k), \text{ for some } k \geq 0 \quad \text{as } |x| \to \infty.$$

Then, the set of eigenvalues is the set of negative integers and the eigenfunctions corresponding to $\lambda = -n$ are d-dimensional Hermite polynomials of degree ν, $\mathbf{H}_\nu$, such that $|\nu| = n$.

Hence, the $L^2(\gamma_d)$ spectrum of L is $\{\cdots, -2, -1, 0\}$. This coincides with the $L^p(\gamma_d)$-spectrum for $1 < p < \infty$.[1] Then, the spectral decomposition of L is given by

$$Lf = \sum_{k=0}^{\infty} (-k)\mathbf{J}_k f, \tag{2.9}$$

where, as before see Definition 1.15, $\mathbf{J}_k f = \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu$. Then, the domain of L is given by

$$D(L) = \{f \in L^2(\gamma_d) : \sum_{k=0}^{\infty} k^2 ||\mathbf{J}_k f||_{2,\gamma}^2 < \infty\}, \tag{2.10}$$

and the spectral decomposition (2.9) is well defined for any $f = \sum_{k=0}^{\infty} \mathbf{J}_k f \in D(L)$.

For $i = 1, 2, \cdots, d$ let us consider the differential operators

$$\partial_\gamma^i = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i}. \tag{2.11}$$

∂_γ^i is neither symmetric nor antisymmetric in $L^2(\gamma_d)$. In fact, its formal $L^2(\gamma_d)$-adjoint[2] is

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}} e^{x_i^2} \frac{\partial}{\partial x_i}(e^{-x_i^2} I) = \sqrt{2} x_i I_d - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i}, \tag{2.12}$$

where I_d is the identity in $\mathbb{R}^d$, because, simply by integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}^d} (\partial_\gamma^i f)(x) g(x) \frac{e^{-|x|^2}}{\pi^{d/2}}\, dx &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} \Big[\frac{\partial f}{\partial x_i}(x)\Big] g(x) \frac{e^{-|x|^2}}{\pi^{d/2}}\, dx \\
&= -\frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} f(x) \frac{\partial}{\partial x_i}\Big[g(x)\frac{e^{-|x|^2}}{\pi^{d/2}}\Big]\, dx \\
&= \int_{\mathbb{R}^d} f(x) \Big[\sqrt{2} x_i g(x) - \frac{1}{\sqrt{2}} \frac{\partial g}{\partial x_i}(x)\Big] \frac{e^{-|x|^2}}{\pi^{d/2}}]\, dx \\
&= \int_{\mathbb{R}^d} f(x) ((\partial_\gamma^i)^* g)(x) \frac{e^{-|x|^2}}{\pi^{d/2}}\, dx.
\end{aligned}$$

[1] The $L^1(\gamma_d)$-spectrum of L is the closed right half plane. We will prove this in detail later (see Theorem 2.7, see also E. B. Davies [65, Theorem 4.3.5]).

[2] In $L^2(\mathbb{R}^d)$, $\frac{\partial}{\partial x_i}$ is antisymmetric, by integration by parts.

Observe that $(\partial_\gamma^i)^*$ can be written as

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}} e^{|x|^2} (\partial_i e^{-|x|^2} I).$$

Moreover, it is easy to see that

$$(-L) = \sum_{i=1}^{d} (\partial_\gamma^i)^* \partial_\gamma^i. \tag{2.13}$$

Observe that the commutator $[\partial_\gamma^i, (\partial_\gamma^i)^*]$, is the identity;[3]

$$\begin{aligned}
[\partial_\gamma^i, (\partial_\gamma^i)^*] f(x) &= \partial_\gamma^i (\partial_\gamma^i)^* f(x) - (\partial_\gamma^i)^* \partial_\gamma^i f(x) \\
&= \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i} \Big(\sqrt{2} x_i f(x) - \frac{1}{\sqrt{2}} \frac{\partial f(x)}{\partial x_i} \Big) - \Big(\sqrt{2} x_i I - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i} \Big) \Big(\frac{1}{\sqrt{2}} \frac{\partial f(x)}{\partial x_i} \Big) \\
&= f(x) + \sqrt{2} x_i \frac{\partial f(x)}{\partial x_i} - \frac{1}{2} \frac{\partial^2 f(x)}{\partial x_i^2} - x_i \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \frac{\partial^2 f(x)}{\partial x_i^2} \\
&= f(x).
\end{aligned}$$

Reversing the order in (2.13), we get another second-order differential operator, which will be denoted as $\overline{L}$,

$$(-\overline{L}) = \sum_{i=1}^{d} \partial_\gamma^i (\partial_\gamma^i)^* = (-L) + dI = -\frac{1}{2} \Delta_x + \langle x, \nabla_x \rangle + dI, \tag{2.14}$$

and therefore,

$$\overline{L} = L - dI = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle - dI. \tag{2.15}$$

We will call $\overline{L}$ the *alternative Ornstein–Uhlenbeck* operator. The Hermite polynomials $\{\mathbf{H}_\nu\}_\nu$ are also eigenfunctions of $\overline{L}$, with eigenvalues $\overline{\lambda}_\nu = -(|\nu| + 1)$, i.e.,

$$\overline{L} \mathbf{H}_\nu = (\lambda_\nu - 1) \mathbf{H}_\nu = -(|\nu| + 1) \mathbf{H}_\nu, \tag{2.16}$$

The differential operators ∂_γ^i are considered the "natural" notions of (partial) derivatives for the Gaussian case, and we call it simply the *Gaussian partial derivatives*. Nevertheless, as we already know, there is another notion of Gaussian differentiation, namely, $(\partial_\gamma^i)^*$. The operators $\partial_\gamma^i, (\partial_\gamma^i)^*$ are called the *creation and annihilation operators* in quantum mechanics.[4]

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[3]Recall that, the commutator of two operators A, B is defined as $[A, B] = AB - BA$.

[4]In [210] there is a general analysis of this decomposition for orthogonal polynomials and functions, which is highly recommended.

Thus, the notion of (partial) differentiation in Gaussian harmonic analysis is, up to a constant, the same as in the classical case. These facts are important later on when we discuss the Riesz transforms for the Gaussian measure in Chapter 9.

There are several results in Gaussian harmonic analysis that can be obtained by what is called the *tensorization* argument, see [20, 284], which implies that it is enough to prove only the case $d = 1$ because the case $d > 1$ follows immediately by the tensor product structure.

In this case, the *square field operator* in $\mathbb{R}^d$ is given by

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - gLf - fLg) = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} = \frac{1}{2}\langle \nabla_x f, \nabla_x g\rangle. \quad (2.17)$$

Consider the infinitesimal generator O of an operator semigroup $\{\mathscr{T}_t\}$, and symmetric with respect to the measure μ, the *Dirichlet form* associated with O is defined as

$$\mathscr{E}_\mu(f) = \lim_{t\to 0}\frac{\langle f - \mathscr{T}_t f, f\rangle_\mu}{t} = \langle -Of, f\rangle_\mu = -\int_E f(Of)\,d\mu. \quad (2.18)$$

Then, by symmetry, it can be proved that

$$\mathscr{E}_\mu(f) = \int_E \Gamma(f,f)\,d\mu. \quad (2.19)$$

for $f \in L^2(\mu)$; see [120, 284].

Hence, the *Dirichlet form* associated with the Ornstein–Uhlenbeck operator L and the Gaussian measure γ_d is given by

$$\mathscr{E}_\gamma(f)(x) = \int_{\mathbb{R}^d} \Gamma(f,f)(x)\,\gamma_d(dx) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x f(x)|^2\,\gamma_d(dx). \quad (2.20)$$

This can be obtained simply using integration by parts, as

$$\int_{\mathbb{R}^d} \langle \nabla_x f(x), \nabla_x g(x)\rangle \gamma_d(dx) = 2\int_{\mathbb{R}^d} f(x)(-L)g(x)\gamma_d(dx), \quad (2.21)$$

for $f, g \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz class. In particular, this implies that $(-L)$ is positive definite and that the Ornstein–Uhlenbeck operator is (formally) self-adjoint in $L^2(\gamma_d)$

$$\int_{\mathbb{R}^d} Lf(x)g(x)\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\gamma_d(dx). \quad (2.22)$$

Therefore, as already mentioned L is the "symmetric Laplacian" in this context and the Gaussian measure γ_d is the natural measure for studying the operators associated with the operator L.

In addition, the *iterated square field operator* $\Gamma_2(f,g)$, in this case, is given by

$$\Gamma_2(f,g) = \frac{1}{2}[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)]. \tag{2.23}$$

Finally, in [168, Lemma 4.1] J. Maas, J. van Neerven, and P. Portal obtained a Gaussian version of the *parabolic Caccioppoli inequality*. We consider here only the real version.

Theorem 2.1. *Let* $v : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ *be a* $C^{1,2}$*-function such that* $v(\cdot,t) \in C_b^2(\mathbb{R}^d)$ *for all* $t > 0$, *and suppose that*

$$\frac{\partial v}{\partial t} = Lv$$

on $I(x_0,t_0,2r) := B(x_0,2cr) \times [t_0 - 4r^2, t_0 + 4r^2]$, *for some* $r \in (0,1)$, $0 < C_0 \le c \le C_1 < \infty$, *and* $t_0 > 4r^2$. *Then*

$$\int_{I(x_0,t_0,r)} |\nabla_x v(x,t)|^2 \gamma_d(dx)\,dt \le C\frac{1+r|x_0|}{r^2}\int_{I(x_0,t_0,2r)} |v(x,t)|^2 \gamma_d(dx)\,dt, \tag{2.24}$$

with C *depending only on the dimension* d, C_0, *and* C_1.

Proof. Let $\eta \in C^\infty(\mathbb{R}^d \times (0,\infty))$ be a cut-off function such that $0 \le \eta \le 1$ on $\mathbb{R}^d \times (0,\infty)$, $\eta \equiv 1$ on $I(x_0,t_0,r)$, $\eta \equiv 0$ on the complement of $I(x_0,t_0,2r)$, and

$$\|\nabla_x \eta\|_\infty \lesssim \frac{1}{r}, \quad \|\frac{\partial \eta}{\partial t}\|_\infty \lesssim \frac{1}{r^2}, \quad \|\Delta \eta\|_\infty \lesssim \frac{1}{r^2}$$

with the implied constants depending only on d, C_0 and C_1. Then, in view of

$$\|x \cdot \nabla_x \eta\|_\infty \lesssim (|x_0| + 2r)\cdot \frac{C'}{r},$$

and recalling that $0 < r < 1$, we have

$$\|L\eta\|_\infty \lesssim \frac{1}{r^2} + \frac{1}{r}|x_0| + 1 \lesssim \frac{1+r|x_0|}{r^2}, \tag{2.25}$$

where the implied constants depend only on d, C_0, C_1. By integrating the identity

$$|\eta \nabla_x v|^2 = \langle \eta \nabla_x v, \eta \nabla_x v\rangle = \langle (v\nabla_x \eta - \nabla_x(v\eta)), (v\nabla_x \eta - \nabla_x(v\eta))\rangle,$$

and then using the fact that

$$\begin{aligned}\int_{I(x_0,t_0,2r)} \eta^2 \langle \nabla_x(v\eta), \nabla_x(v\eta)\rangle\, d\gamma_d\, dt &\le \int_0^\infty \int_{\mathbb{R}^d} \langle \nabla_x(v\eta), \nabla_x(v\eta)\rangle\, d\gamma_d\, dt \\ &= 2\int_0^\infty \int_{\mathbb{R}^d} v\eta(-L)(v\eta)\, d\gamma_d\, dt \\ &= -2\int_{I(x_0,t_0,2r)} v\eta L(v\eta)\, d\gamma_d\, dt,\end{aligned}$$

According to (2.21), we obtain

$$\begin{aligned}\int_{I(x_0,t_0,r)}|\nabla_x v|^2\,d\gamma_d\,dt &\le \int_{I(x_0,t_0,2r)}\eta^2|\eta\nabla_x v|^2\,d\gamma_d\,dt\\ &\le \int_{I(x_0,t_0,2r)}\eta^2|v\nabla_x\eta|^2\,d\gamma_d\,dt\\ &\quad +2\Big|\int_{I(x_0,t_0,2r)}v\eta^2\langle\nabla_x(v\eta),\nabla_x\eta\rangle\,d\gamma_d\,dt\Big|\\ &\quad +2\Big|\int_{I(x_0,t_0,2r)}v\eta L(v\eta)\,d\gamma_d\,dt\Big|.\end{aligned}\tag{2.26}$$

For the first term on the right-hand side we have the estimate

$$\int_{I(x_0,t_0,2r)}\eta^2|v\nabla_x\eta|^2\,d\gamma_d\,dt \lesssim \frac{1}{r^2}\int_{I(x_0,t_0,2r)}|v|^2\,d\gamma_d\,dt.$$

For the second term we have, by (2.25),

$$\begin{aligned}\Big|\int_{I(x_0,t_0,2r)}2v\eta^2\langle\nabla_x(v\eta),\nabla_x\eta\rangle\,d\gamma_d\,dt\Big| &= \frac{1}{2}\Big|\int_{I(x_0,t_0,2r)}\langle\nabla_x(v\eta)^2,\nabla_x\eta^2\rangle\,d\gamma_d\,dt\Big|\\ &\le \Big|\int_{\mathbb{R}^d}(v\eta)^2L\eta^2\,d\gamma_d\,dt\Big|\\ &\lesssim \frac{1+r|x_0|}{r^2}\int_{I(x_0,t_0,2r)}|v|^2\,d\gamma_d\,dt\end{aligned}$$

where we used the fact that η^2 satisfies the same assumptions as η to apply (2.25) to η^2. To estimate the third term on the right-hand side of (2.26), we substitute the identity

$$L(v\eta) = \eta Lv + vL\eta - \langle\nabla_x v,\nabla_x\eta\rangle = \eta\frac{\partial v}{\partial t} + vL\eta - \langle\nabla_x v,\nabla_x\eta\rangle$$

and estimate each of the resulting integrals:

$$\begin{aligned}\Big|\int_{I(x_0,t_0,2r)}v\eta^2\frac{\partial v}{\partial t}\,d\gamma dt\Big| &= \frac{1}{2}\Big|\int_{I(x_0,t_0,2r)}\eta^2\frac{\partial v^2}{\partial t}\,d\gamma_d\,dt\Big| = \frac{1}{2}\Big|\int_{I(x_0,t_0,2r)}v^2\frac{\partial\eta^2}{\partial t}\,d\gamma_d\,dt\Big|\\ &= \Big|\int_{I(x_0,t_0,2r)}v^2\eta\frac{\partial\eta}{\partial t}\,d\gamma_d\,dt\Big| \lesssim \frac{1}{r^2}\int_{I(x_0,t_0,2r)}|v|^2\,d\gamma_d\,dt,\end{aligned}$$

$$\Big|\int_{I(x_0,t_0,2r)}v^2\eta L\eta\,d\gamma_d\,dt\Big| \lesssim \frac{1+r|x_0|}{r^2}\int_{I(x_0,t_0,2r)}|v|^2\,d\gamma_d\,dt,$$

and

$$\begin{aligned}\Big|\int_{I(x_0,t_0,2r)} v\eta\,\langle\nabla_x v,\nabla_x\eta\rangle\,d\gamma_d\,dt\Big| &= \frac{1}{4}\Big|\int_{I(x_0,t_0,2r)}\langle\nabla_x v^2, t\nabla_x\eta^2\rangle\,d\gamma_d\,dt\Big|\\ &= \frac{1}{4}\Big|\int_{\mathbb{R}^d} v^2 L\eta^2\,d\gamma_d\,dt\Big|\\ &\lesssim \frac{1+r|x_0|}{r^2}\int_{I(x_0,t_0,2r)}|v|^2\,d\gamma_d\,dt. \qquad \square\end{aligned}$$

2.2 Definition and Basic Properties of the Ornstein–Uhlenbeck Semigroup

Now, we consider the *Ornstein–Uhlenbeck semigroup*. On $L^2(\gamma_d)$ the closure of the Ornstein–Uhlenbeck operator L generates an operator semigroup.

Definition 2.2. *The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is the semigroup of operators generated in $L^2(\gamma_d)$ by the Ornstein–Uhlenbeck operator L as infinitesimal generator, i.e., formally $T_t = e^{-tL}$. In view of the spectral theorem, for $f=\sum_{k=0}^{\infty}\mathbf{J}_k f \in L^2(\gamma_d)$ and $t\geq 0$, T_t is defined as*

$$T_t f = \sum_{\nu} e^{-t|\nu|}\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu = \sum_{k=0}^{\infty} e^{-tk}\sum_{|\nu|=k}\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu = \sum_{k=0}^{\infty} e^{-tk}\mathbf{J}_k f, \tag{2.27}$$

where $\mathbf{J}_k f = \sum_{|\nu|=k}\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu$ is the orthogonal projection of $L^2(\gamma_d)$ onto $\mathscr{C}_k$.

The Ornstein–Uhlenbeck semigroup have the following representations.

Proposition 2.3. *(C. P. Calderón- B. Muckenhoupt)*

i) If $f\in L^2(\gamma_d)$, then $\sum_{k=0}^{\infty} e^{-tk}\mathbf{J}_k f(x)$ converges absolutely to $T_t f(x)$ almost everywhere (a.e.)γ_d.
ii) For any $1\leq p<,2$ there exists a function $f\in L^p(\gamma_d)$ and $t\geq 0$ such that $\sum_{k=0}^{\infty} e^{-tk}\mathbf{J}_k f(x)$ diverges for all x.
iii) For any $t>0$ the integral representation for T_t is given by

$$T_t f(x) = \frac{1}{(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy). \tag{2.28}$$

Proof.

i) Observe that for each multi-index ν, $|\nu|>0$, according to (1.64) and the Cauchy–Schwartz inequality, we have

$$|\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu(x)| \leq C_{\nu,x}\nu!||f||_{2,\gamma} = C'_{\nu,x}||f||_{2,\gamma}.$$

Therefore, the sequence $\{\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu(x)\}$ is bounded for each x; thus using to the Weierstrass M-test, the series $\sum_{k=0}^{\infty} e^{-tk}\mathbf{J}_k f(x)$ converges absolutely for any x. Because $L^2(\gamma_d)\subset L^1(\gamma_d)$, then, according to the first part, $T_t f(x)$ has the expansion $\sum_{k=0}^{\infty} e^{-tk}\mathbf{J}_k f(x)$, and this must be the limit a.e.

ii) Using the multiplicative character of the Gaussian measure γ_d, it is enough to consider the case $d = 1$. According to Pollard's counterexample [230], for $1 \leq p < 2$, there exists a function $f \in L^p(\gamma_1)$ such that

$$\limsup_{k\to\infty}(\langle f, \mathbf{h}_k\rangle_{\gamma_d}|H_k(x)|)^{1/k}$$

is a fixed number greater than 1, for any x. Therefore, for t close enough to zero (i.e., e^{-t} close enough to 1), the expansion of $T_t f$ diverges for any x.

iii) Using again (1.64), the Cauchy–Schwartz inequality and Stirling's formula, we get for $|\nu| = k$

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-tk}|f(y)||\mathbf{h}_\nu(y)||\mathbf{h}_\nu(x)|\gamma_d(dy) &\leq \|f\|_{2,\gamma}\Big(\int_{\mathbb{R}^d} e^{-2tk}|\mathbf{h}_\nu(y)|^2|\mathbf{h}_\nu(x)|^2\gamma_d(dy)\Big)^{1/2}\\ &\leq \|f\|_{2,\gamma}e^{-tk}C_{\nu,x}(\nu!)^{1/2}\Big(\int_{\mathbb{R}^d}|\mathbf{h}_\nu(y)|^2\gamma_d(dy)\Big)^{1/2}\\ &\leq C_{\nu,x}\|f\|_{2,\gamma}e^{-tk}.\end{aligned}$$

Then, using this, Lebesgue's dominated convergence theorem and the d-dimensional Mehler's formula (10.24), for $r = e^{-t}$ we get

$$\begin{aligned}T_t f(x) &= \sum_{|\nu|\geq 0} e^{-t|\nu|}\Big[\int_{\mathbb{R}^d} f(y)\mathbf{h}_\nu(y)\gamma_d(dy)\Big]\mathbf{h}_\nu(x)\\ &= \int_{\mathbb{R}^d}\Big(\sum_{|\nu|\geq 0} e^{-t|\nu|}\mathbf{h}_\nu(x)\mathbf{h}_\nu(y)\Big)f(y)\gamma_d(dy)\\ &= \frac{1}{(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy). \qquad \square\end{aligned}$$

Note that the integral representation (2.28), obtained initially for $f \in L^2(\gamma_d)$, also makes sense for $f \in L^p(\gamma_d)$, $1 \leq p < \infty$, by using Hölder's inequality. Therefore, $\{T_t\}_{t\geq 0}$ can be extended as a family of operators in $L^p(\gamma_d)$. Also note that, taking $r = e^{-t}$, (2.27) is equivalent to the Abel summability of the Hermite expansion of f

$$T_r f = \sum_{k=0}^{\infty} r^k \mathbf{J}_k f.$$

Using this approach, B. Muckenhoupt [193], considered the so-called Poisson integral for the Hermite expansion for $d = 1$, and also C.P. Calderón [44] for the case $d \geq 1$.

The kernel

$$M_t(x,y) = \frac{1}{(1-e^{-2t})^{d/2}} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}}, \tag{2.29}$$

is called *Mehler's kernel*[5].

The integral representation of T_t can be written in several equivalent forms. The first one provides the link between the Ornstein–Uhlenbeck semigroup and the heat semigroup,

$$\begin{aligned} T_t f(x) &= \frac{1}{(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy) \\ &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} f(y)dy, \quad t>0. \end{aligned} \tag{2.30}$$

Observe that now we are integrating with respect to the Lebesgue measure. The alternative expression,

$$M_t(x,y) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}, \tag{2.31}$$

allows us to establish a connection between Mehler's kernel and the *heat kernel*

$$k_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}. \tag{2.32}$$

Using $\{\mathscr{T}_t\}_{t\geq 0}$, the *heat semigroup*[6]

$$\mathscr{T}_t f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4t}} f(y)\,dy,\ t>0,$$

we have the following representation of the Ornstein–Uhlenbeck semigroup

$$T_t f(x) = (k_{(1-e^{-2t})/4} * f)(e^{-t}x) = \delta_{e^{-t}}[k_{(1-e^{-2t})/4} * f](x) = \delta_{e^{-t}}\mathscr{T}_{(1-e^{-2t})/4} f(x),$$

where δ_a is the *dilation operator* by a, defined by

$$\delta_a f(x) = f(ax). \tag{2.33}$$

Thus, the Ornstein–Uhlenbeck semigroup is, after a dilation on the variable x, a reparametrization of the heat semigroup; therefore, it is not a convolution semigroup. More precisely, before taking the convolution with the properly reparametrized heat kernel, a dilation by e^{-t} is applied in the variable x. Because of this dilatation, none

[5] We have already encountered this kernel in Chapter 1, (1.41)

[6] See Appendix 10.5 for more details.

of the methods used in the study of classical semigroups can be applied to this semigroup. Nevertheless, F. Weissler [292], who denotes this semigroup as the *Hermite semigroup,*[7] establishes another explicit relation between the Ornstein–Uhlenbeck and the heat semigroups,

Theorem 2.4. *Let* $1 \leq p,q \leq \infty$*,* $t \geq 0$*, and* $\zeta \geq 0$*.*[8] *Then,*

$$T_t = (\zeta e^t)^{d/2} \pi^{(1/2p-1/2q)d} (\Xi_d^{(q)})^{-1} \mathfrak{M}_\beta \delta_\zeta \mathscr{T}_{\zeta(1-e^{-2t})/4e^{-t}} \mathfrak{M}_\alpha \Xi_d^{(p)}, \tag{2.34}$$

where

$$\alpha = \frac{1}{1-e^{-2t}} - \frac{1}{p} - \frac{e^{-t}}{\zeta(1-e^{-2t})},$$

$$\beta = \frac{1}{1-e^{-2t}} - \frac{1}{q'} - \frac{\zeta e^{-t}}{1-e^{-2t}},$$

$\Xi_d^{(p)} : L^p(\gamma_d) \to L^p(\mathbb{R}^d)$ *is the isometric isomorphism defined, for any* $1 < p < \infty$*, as*

$$\Xi_d^{(p)} f(x) = f(x) \pi^{-d/2p} e^{-|x|^2/p}, \tag{2.35}$$

$\mathfrak{M}_\alpha$ *is the* multiplication operator *defined as*

$$\mathfrak{M}_\alpha f(x) = e^{\alpha |x|^2} f(x),$$

and finally δ_a *is the dilation operator, as defined in (2.33).*

Using this relation, Weissler succeeded in not only extending the Ornstein–Uhlenbeck semigroup holomorphically to the half-plane $\mathrm{Re}z \geq 0$, where the heat semigroup is holomorphic but he was also able to obtain additional information on the continuity of both semigroups (for more details see [292]). We discuss later in this chapter the holomorphic Ornstein–Uhlenbeck semigroup in more detail (see page 49).

Observe that

$$M_t(x,y) = M_t(y,x) e^{|x|^2-|y|^2}.$$

Through the change of variables $u = \frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}}$, we get an alternative representation of T_t

$$T_t f(x) = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} f(\sqrt{1-e^{-2t}}u + e^{-t}x) e^{-|u|^2} du. \tag{2.36}$$

This representation of the Ornstein–Uhlenbeck semigroup allows us to extend it to a space of infinite dimensions, where the Gaussian measure, unlike the Lebesgue measure, is well defined (see P. A. Meyer [187]).

[7] We refer to another semigroup as the Hermite semigroup, see point 10. in Section 2.5, page 70.

[8] Actually, Weissler defines it for $z \in \mathbb{C}$ such that $\mathrm{Re}z \geq 0$ and $\mathrm{Re}(\zeta e^z) \geq 0$, see [292, Theorem 1].

One problem of the kernel (2.31) is that it does not reflect the symmetry of Mehler's kernel. An alternative symmetric representation of (2.29) is given by

$$M_t(x,y) = \frac{1}{(1-e^{-2t})^{d/2}} \exp\left(\frac{1}{2}\frac{|x+y|^2}{e^t+1} - \frac{1}{2}\frac{|x-y|^2}{e^t-1}\right), \tag{2.37}$$

which has been used in several papers about the Ornstein–Uhlenbeck semigroup, (see for instance [249] and [104]). In [265], J. Teuwen has an alternative symmetric representation:

$$M_t(x,y) = \frac{\exp(-\frac{e^{2t}|x-y|^2}{1-e^{2t}})}{(1-e^{-t})^{d/2}} \frac{\exp(2e^{-t}\frac{\langle x,y\rangle}{1+e^t})}{(1+e^{-t})^{d/2}}. \tag{2.38}$$

The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ in $\mathbb{R}^d$ is a *Markov operator semigroup* in $L^p(\gamma_d), 1 \leq p \leq \infty$, i.e., a positive conservative symmetric diffusion semigroup, strongly L^p-continuous in $L^p(\gamma_d), 1 \leq p \leq \infty$, with the Ornstein–Uhlenbeck operator L as its infinitesimal generator (see [23, 20] or [284]). Its properties can be obtained directly from the general theory of Markov semigroups (see [20] or [284]). Nevertheless, because the Ornstein–Uhlenbeck semigroup is of such great importance and serves as a "model" for Markov semigroups associated with classical orthogonal polynomials, we are going to give detailed analytic proof of its properties using its integral representation (2.28).

Theorem 2.5. *The family of operators $\{T_t : t \geq 0\}$ satisfies the following properties:*

i) Semigroup property:

$$T_{t_1+t_2} = T_{t_1} \circ T_{t_2},\; t_1, t_2 \geq 0.$$

ii) Positivity and conservative properties:

$$T_t f \geq 0, \quad \text{for} \quad f \geq 0,\, t \geq 0,$$

and

$$T_t 1 = 1.$$

iii) Contractivity property:

$$||T_t f||_{p,\gamma} \leq ||f||_{p,\gamma},$$

for all $t \geq 0$, and $1 \leq p \leq \infty$.

iv) Strong $L^p(\gamma_d)$-continuity property: The mapping $t \to T_t f$ is continuous from $[0,\infty)$ to $L^p(\gamma_d)$, for $1 \leq p < \infty$ and $f \in L^p(\gamma_d)$.

v) Symmetry property: T_t is a self-adjoint operator in $L^2(\gamma_d)$:

$$\int_{\mathbb{R}^d} T_t f(x) g(x) \gamma_d(dx) = \int_{\mathbb{R}^d} f(x) T_t g(x) \gamma_d(dx), t \geq 0. \tag{2.39}$$

In particular, the Gaussian measure γ_d is the invariant measure for $\{T_t\}_{t\geq 0}$,

$$\int_{\mathbb{R}^d} T_t f(x) \gamma_d(dx) = \int_{\mathbb{R}^d} f(x) \gamma_d(dx), t \geq 0. \tag{2.40}$$

vi) Infinitesimal generator: the Ornstein–Uhlenbeck operator L is the infinitesimal generator of $\{T_t : t \geq 0\}$,

$$\lim_{t\to 0} \frac{T_t f - f}{t} = Lf. \tag{2.41}$$

Proof.

i) To prove the semigroup property, we use integral representation (2.30)[9] as follows. Let $f \in L^1(\gamma_d)$, by Fubini's theorem we have

$$\begin{aligned} T_t(T_s f)(x) &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} \\ &\quad \times \Big(\frac{1}{\pi^{d/2}(1-e^{-2s})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|z-e^{-s}y|^2}{1-e^{-2s}}} f(z)dz \Big) dy \\ &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}\pi^{d/2}(1-e^{-2s})^{d/2}} \\ &\quad \times \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \exp\Big[-\Big(\frac{|y-e^{-t}x|^2}{1-e^{-2t}} + \frac{|z-e^{-s}y|^2}{1-e^{-2s}} \Big) \Big] dy \Big) f(z) dz. \end{aligned}$$

Taking the change of variables $u = y - e^s z$ in the exponent, we get,

$$\begin{aligned} &-\frac{|y-e^{-t}x|^2}{1-e^{-2t}} - \frac{|z-e^{-s}y|^2}{1-e^{-2s}} \\ &= -\frac{|y-e^{-t}x|^2}{1-e^{-2t}} - \frac{e^{-2s}|y-e^{s}z|^2}{1-e^{-2s}} = -\frac{|u+e^s z - e^{-t}x|^2}{1-e^{-2t}} - \frac{e^{-2s}|u|^2}{1-e^{-2s}} \\ &= -\frac{(1-e^{-2s})|u - e^s(e^{-(t+s)}x - z)|^2 - (1-e^{-2t})e^{-2s}|u|^2}{(1-e^{-2t})(1-e^{-2s})} \\ &= -\frac{(1-e^{-2s})(|u|^2 - 2\langle u, e^s(e^{-(t+s)}x - z)\rangle + e^{2s}|e^{-(t+s)}x - z|^2)}{(1-e^{-2t})(1-e^{-2s})} \\ &\qquad - \frac{(1-e^{-2t})e^{-2s}|u|^2}{(1-e^{-2t})(1-e^{-2s})} \\ &= -\frac{e^{2s}|e^{-(t+s)}x - z|^2}{1-e^{-2t}} + \frac{2e^s\langle u, e^{-(t+s)}x - z\rangle}{1-e^{-2t}} - \frac{(1-e^{-2(t+s)})|u|^2}{(1-e^{-2t})(1-e^{-2s})}. \end{aligned}$$

But, the last two terms of the latter expression can be rewritten as

$$\begin{aligned} &\frac{2e^s\langle u, e^{-(t+s)}x - z\rangle}{1-e^{-2t}} - \frac{(1-e^{-2(t+s)})|u|^2}{(1-e^{-2t})(1-e^{-2s})} \\ &= -\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})} \Big[\frac{2e^s\langle u, e^{-(t+s)}x - z\rangle . (1-e^{-2s})}{1-e^{-2(t+s)}} - |u|^2 \Big] \\ &= -\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})} \\ &\quad \times \Big[\Big| u - \frac{e^s(1-e^{-2s})(e^{-(t+s)}x - z)}{1-e^{-2(t+s)}} \Big|^2 - \frac{e^{2s}(1-e^{-2s})^2}{(1-e^{-2(t+s)})^2} |e^{-(t+s)}x - z|^2 \Big]. \end{aligned}$$

[9] For alternative proofs, see point 4. in Notes and Further Results.

Then, we have,

$$
\begin{aligned}
-\frac{|y-e^{-t}x|^2}{1-e^{-2t}} - \frac{|z-e^{-s}y|^2}{1-e^{-2s}} &= -\frac{e^{2s}|e^{-(t+s)}x-z|^2}{1-e^{-2t}} \\
&\quad \times \frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})}\Big|u - \frac{e^{s}(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}\Big|^2 \\
&\qquad + \frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}\frac{e^{2s}(1-e^{-2s})^2}{(1-e^{-2(t+s)})^2}|e^{-(t+s)}x-z|^2 \\
&= -\frac{e^{2s}|e^{-(t+s)}x-z|^2}{1-e^{-2t}} - \frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})} \\
&\quad \times\Big|u - \frac{e^{s}(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}\Big|^2 \\
&\qquad + \frac{e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}|e^{-(t+s)}x-z|^2.
\end{aligned}
$$

Now, taking the change of variables $w = u - \dfrac{e^{s}(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \exp\left(-\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})}\Big|u - \frac{e^{s}(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}\Big|^2\right) du \\
&= \int_{\mathbb{R}^d} \exp\Big(-\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})}|w|^2\Big) dw \\
&= \frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}} \int_{\mathbb{R}^d} e^{-|v|^2} dv \\
&= \pi^{d/2}\frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}}.
\end{aligned}
$$

With another change of variables, $v = \sqrt{\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}}\, w$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \exp\Big[-\Big(\frac{|y-e^{-t}x|^2}{1-e^{-2t}} + \frac{|z-e^{-s}y|^2}{1-e^{-2s}}\Big)\Big] dy \\
&\quad = \pi^{d/2}\exp\Big[\Big(-\frac{e^{2s}}{1-e^{-2t}} + \frac{e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}\Big)|e^{-(t+s)}x-z|^2)\Big] \\
&\quad\quad \times \frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}},
\end{aligned}
$$

but as

$$-\frac{e^{2s}}{1-e^{-2t}}+\frac{e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}=\frac{-(1-e^{-2(t+s)})e^{2s}+e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}$$
$$=\frac{e^{-2t}-1}{(1-e^{-2t})(1-e^{-2(t+s)})}=-\frac{1}{1-e^{-2(t+s)}},$$

we get,

$$\begin{aligned}T_t(T_sf)(x) &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}\pi^{d/2}(1-e^{-2s})^{d/2}}\\ &\times\int_{\mathbb{R}^d}\Big(\exp(\frac{-|e^{-(t+s)}x-z|^2}{1-e^{-2(t+s)}}\Big)\pi^{d/2}\frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}}f(z)dz\\ &= \frac{1}{\pi^{d/2}(1-e^{-2(t+s)})^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|e^{-(t+s)}x-z|^2}{1-e^{-2(t+s)}}}f(z)dz=T_{t+s}f(x).\end{aligned}$$

ii) The conservative property follows immediately by a simple change of variables $u=\frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}}$, the translation invariance property of the Lebesgue measure, and the fact that γ_d is a probability measure:

$$T_t1=\frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}1dy=\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}e^{-|u|^2}du=1.$$

For the positivity of T_t, if $f\geq 0$,

$$T_tf(x)=\frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d}f(y)e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}dy\geq 0,$$

as the kernel is positive.

iii) Because the Ornstein–Uhlenbeck semigroup is not a convolution semigroup, this property cannot be obtained using the theory of approximations of the identity, as in the case of the classical semigroups (see Appendix 10.5). Nevertheless, it can be obtained using Jensen's inequality:

$$\begin{aligned}|T_tf(x)|^p &\leq \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}}|f(y)|^pe^{-|y|^2}dy.\\ &= T_t(|f|^p)(x),\end{aligned}$$

Then, according to *v*),

$$||T_tf||^p_{p,\gamma_d}\leq\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}T_t(|f|^p)(x)e^{-|x|^2}dx=||f||^p_{p,\gamma_d}.$$

Therefore, T_t is a contraction in $L^p(\gamma_d), 1 \leq p < \infty$. The case $p = \infty$ follows immediately because $T_t 1 = 1$, according to *ii)*. Alternatively, this can also be obtained by using interpolation and duality.

iv) We need to prove that $T_t f \to T_{t_0} f$ in $L^p(\gamma_d)$ as $t \to t_0$. Again, this is not a consequence of the general theory of approximations of the identity. According to the semigroup property, it is enough to prove that $T_t f \to f$ in $L^p(\gamma_d)$ as $t \to 0$. Observe that $L^p(\gamma_d)$ is not closed under translation;[10] thus, it does not make sense to speak of continuity in norm $L^p(\gamma_d)$ and hence, this type of argument cannot be used either. The alternative proof below is an extension to d-dimensions of the proof in [193].

$$
\begin{aligned}
|T_t f(x) - f(x)| & \\
\leq \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} & \int_{|x-y|<\delta} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} |f(y)-f(x)| e^{-|y|^2} dy \\
+ \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} & \int_{|x-y|\geq\delta} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} |f(y)-f(x)| e^{-|y|^2} dy.
\end{aligned}
$$

Let f be a function defined in $\mathbb{R}^d$, continuous with compact support, and let $\varepsilon > 0$ and $\delta > 0$ be such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Now, according to *iii*), it is clear that the first integral is less than ε. Now, if y belongs to the support of f, $|x-y| > \delta$ and $0 \leq 1 - e^{-t} < \delta \frac{e^{-t}}{2} \max\{|y| : y \in \operatorname{supp} f\}$. Then,

$$
\exp\Big(- \frac{|e^{-t}(x-y) - y(1-e^{-t})|^2}{1-e^{-2t}} + |y|^2 \Big) \leq \exp\Big(- \frac{e^{-2t}\delta^2}{4(1-e^{-2t})} + \max\{|y|^2 : y \in \operatorname{supp} f\} \Big).
$$

The second integral is less than

$$
\frac{2||f||_{\infty,\gamma}}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\operatorname{supp} f} \exp\Big(- \frac{e^{-2t}\delta^2}{4(1-e^{-2t})} + \max\{|y|^2 : y \in \operatorname{supp} f\} \Big) e^{-|y|^2} dy,
$$

and this tends to zero as $t \to t_0$. Thus, $T_t f \to f$ uniformly in x as $t \to 0$. The general case follows from the density of the continuous functions with compact support in $L^p(\gamma_d)$ for $1 \leq p < \infty$ and using *iii*).

v) To prove (2.39), using Fubini's theorem, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} T_t f(x) g(x) \gamma_d(dx) \\
&= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} e^{(-\frac{(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e-2t})} f(y) e^{-|y|^2} dy \Big) g(x) \frac{1}{\pi^{d/2}} e^{-|x|^2} dx
\end{aligned}
$$

[10]Consider, for $d = 1$, $f(x) = \frac{1}{|x|} e^{|x|/2} \chi_{B(0,1)}(x)$ and its translations.

$$= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} e^{(-\frac{(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}})} g(x) e^{-|x|^2} dx \Big) f(y) \frac{1}{\pi^{d/2}} e^{-|y|^2} dy$$
$$= \int_{\mathbb{R}^d} f(y) T_t g(y) \gamma_d(dy).$$

The invariance property follows immediately from (2.39) and the conservative property, taking $g \equiv 1$.

vi) Let $f \in C_b^2(\mathbb{R}^d)$, that is, a continuous function with bounded derivatives up to the second order. Then, using (2.36), we have

$$\Big(\frac{T_t f - f}{t}\Big)(x) - Lf(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \Big[f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(x) \Big] e^{-|y|^2} dy$$
$$-\frac{1}{2}\sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}(x) + \sum_{j=1}^d x_j \frac{\partial f}{\partial x_j}(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \Big[f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(x)$$
$$-\frac{t}{2}\sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}(x) \Big] e^{-|y|^2} dy + \sum_{j=1}^d x_j \frac{\partial f}{\partial x_j}(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \Big[f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(e^{-t}x)$$
$$-t\sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}(x) y_k^2 \Big] e^{-|y|^2} dy + \Big(\frac{f(e^{-t}x) - f(x)}{t} + \sum_{j=1}^d x_j \frac{\partial f}{\partial x_j}(x) \Big).$$

Now, using the Taylor expansion of order 2 for f, for some θ, with $0 \le \theta \le 1$,

$$f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(e^{-t}x)$$
$$= \sum_{k=1}^d \sqrt{1-e^{-2t}}\, y_k \frac{\partial f}{\partial x_k}(e^{-t}x) + \frac{1}{2}\sum_{i,j=1}^d (1-e^{-2t}) y_i y_j$$
$$\times \frac{\partial^2 f}{\partial x_i \partial x_j}\Big(\theta e^{-t}x + (1-\theta)\sqrt{1-e^{-2t}}y\Big).$$

Then, according to the symmetry of $e^{-|y|^2}$, we have

$$\Big(\frac{T_t f - f}{t}\Big)(x) - Lf(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \Big[\sum_{k=1}^d \sqrt{1-e^{-2t}}\, y_k \frac{\partial f}{\partial x_k}(e^{-t}x)$$
$$+\frac{1}{2}\sum_{i,j=1}^d (1-e^{-2t}) y_i y_j \frac{\partial^2 f}{\partial x_i \partial x_j}\Big(\theta e^{-t}x + (1-\theta)\sqrt{1-e^{-2t}}y\Big)$$

$$-t\sum_{k=1}^{d}\frac{\partial^2 f}{\partial x_k^2}(x)y_k^2\Big]e^{-|y|^2}dy+\left(\frac{f(e^{-t}-f(x)}{t}+\sum_{j=1}^{d}x_j\frac{\partial f}{\partial x_j}(x)\right)$$

$$=\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}\sum_{k=1}^{d}\Big[\frac{1}{2}\left(\frac{1-e^{-2t}}{t}\right)\frac{\partial^2 f}{\partial x_k^2}\left(\theta e^{-t}x+(1-\theta)\sqrt{1-e^{-2t}}y\right)$$

$$-\frac{\partial^2 f}{\partial x_k^2}(x)\Big]y_k^2e^{-|y|^2}dy+\left(\frac{f(e^{-t}x)-f(x)}{t}+\sum_{j=1}^{d}x_j\frac{\partial f}{\partial x_j}(x)\right).$$

Thus,

$$\Big|\Big(\frac{T_tf-f}{t}\Big)(x)-Lf(x)\Big|$$

$$\leq\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}\sum_{k=1}^{d}\Big[\frac{1}{2}\Big|\frac{1-e^{-2t}}{t}\Big|\Big|\frac{\partial^2 f}{\partial x_k^2}(\theta e^{-t}x+(1-\theta)\sqrt{1-e^{-2t}}y)$$

$$-\frac{\partial^2 f}{\partial x_k^2}(x)\Big|e^{-|y|^2}\Big]dy+\Big|\frac{f(e^{-t}x)-f(x)}{t}+\sum_{j=1}^{d}x_j\frac{\partial f}{\partial x_j}(x)\Big|.$$

Then, using Lebesgue's dominated convergence theorem, we conclude that each of these terms tends to zero as $t\to 0$. □

Also, each operator of the Ornstein–Uhlenbeck semigroup is compact.

Lemma 2.6. *For each $t>0$, the operator T_t is compact.*

Proof. Because T_t is given by

$$T_tf=\sum_{k=1}^{\infty}e^{-kt}\mathbf{J}_kf,\quad t>0,$$

we can consider the following sequence of compact operators:

$$T_t(n)f=\sum_{k=1}^{n}e^{-kt}\mathbf{J}_kf,\quad t>0.$$

Then,

$$\|T_tf-T_t(n)f\|_{2,\gamma}^2=\sum_{k=n+1}^{\infty}\|e^{-kt}\mathbf{J}_kf\|_{2,\gamma}^2\leq e^{-2nt}\|f\|_{2,\gamma}^2.$$

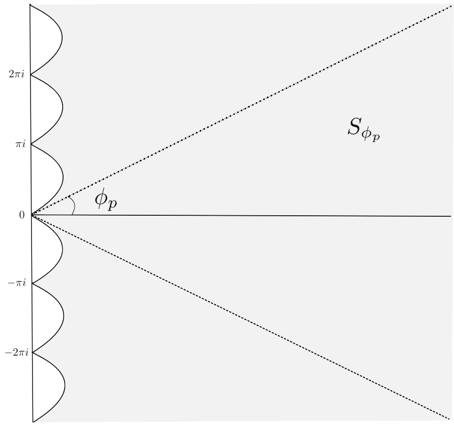

Fig. 2.1. Epperson region $\mathbf{E}_p$.

Therefore, the sequence of compact operators $\{T_t(n)\}$ converges in $L^2(\gamma_d)$-norm to T_t for all $t > 0$. Then, from e) of [62, Theorem A.3.22], we can conclude the compactness of T_t. □

The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ can be extended to complex values of the parameter t. For any $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 0$, the operator $T_z = e^{-zL}$, defined spectrally, is bounded on $L^2(\gamma_d)$. It is given by the kernel, using the representation (2.37), replacing t by z,

$$M_z(x,y) = \frac{1}{(1-e^{-2z})^{d/2}} \exp\left(\frac{1}{2}\frac{|x+y|^2}{e^z+1} - \frac{1}{2}\frac{|x-y|^2}{e^z-1}\right).$$

The function $t \mapsto T_t$ has an holomorphic continuation to a distribution-valued function $z \mapsto T_z$, which is holomorphic in $\operatorname{Re} z > 0$ and continuous in $\operatorname{Re} z \geq 0$. The family of continuous operators $\{T_z : \operatorname{Re} z \geq 0\}$ defined from the space of distributions $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$, then satisfies

$$T_{z+i\pi}(x) = T_z f(-x), \quad T_{\bar{z}} f(x) = \overline{T_z \overline{f(x)}}. \tag{2.42}$$

J. B. Epperson [74] proved that the operator T_z, extends to a bounded operator on $L^p(\gamma_d)$, $1 \leq p \leq \infty$, if and only if $z \in \mathbf{E}_p$, where

$$\mathbf{E}_p := \{z = x+iy : |\sin y| \leq \tan\phi_p \sinh x\}, \quad \phi_p = \arccos|2/p-1|. \tag{2.43}$$

The extension T_z to $L^p(\gamma_d)$ is actually a contraction.

The set $\mathbf{E}_p$ is a closed $i\pi$-periodic subset of the right half-plane, which is called *Epperson's region*, (see Figure 2.1). Each $\mathbf{E}_p$ is a closed subset of the closed right half-plane and periodic with period $i\pi$. Notice the symmetry $\phi_p = \phi_{p'}$ and $\mathbf{E}_p = \mathbf{E}_{p'}$, where p' is the conjugate exponent. Also, we have $\mathbf{E}_p \subset \mathbf{E}_q$ if $1 < p < q < 2$. Furthermore, $\mathbf{E}_p$ depends monotonically on p on either side of 2. The extreme cases are $\mathbf{E}_2 = \{z : \operatorname{Re} z \geq 0\}$ and $\mathbf{E}_1 = \{x + ik\pi : x \geq 0, k \in \mathbb{Z}\}$.

The map $z \to T_z$ from $\mathbf{E}_p$ to the Banach algebra of bounded operators on $L^p(\gamma)$ is continuous in the strong operator topology, and its restriction to the interior of $\mathbf{E}_p$ is holomorphic (see also [249]). Additionally, the holomorphic Ornstein–Uhlenbeck semigroup can be extended to infinite dimensions (see [167]).

Let us prove now that the $L^1(\gamma_d)$-spectrum of L is the closed right half-plane (see E. B. Davies [65, Theorem 4.3.5])

Theorem 2.7. *The $L^1(\gamma_d)$-spectrum of L is the closed right half-plane $\{z : Re\, z \geq 0\}$. Indeed, every z with $Re\, z > 0$ is an eigenvalue of L with multiplicity two.*

Proof. First of all, according to the tensorization argument, it is enough to consider the case $d = 1$. Let us consider the harmonic oscillator operator

$$H_1 f = \frac{1}{2}\left(-\frac{d^2 f}{dx^2} + x^2 f - f\right),$$

with domain in $\mathscr{S}(\mathbb{R}) \in L^2(\mathbb{R})$. It is easy to see, using Mehler's formula, that the semigroup generated by H_1, $\{e^{-tH_1}\}_{t\geq 0}$ has kernel

$$K_t(x,y) = \frac{1}{\pi^{1/2}(1-e^{-2t})} \exp\left(\frac{4xye^{-t} - (x^2+y^2)(1+e^{-2t})}{2(1-e^{-2t})}\right), t > 0,\ x,y \in \mathbb{R}.$$

Consider the isometric isomorphism, $\Xi_1^{(2)} : L^2(\gamma_1) \to L^2(\mathbb{R})$ defined in (2.35), for $d = 1$ and $p = 2$,

$$\Xi_1^{(2)} f(x) = f(x)\pi^{-1/4} e^{-|x|^2/2},$$

and consider

$$\tilde{L} = \Xi_1^{(2)} L (\Xi_1^{(2)})^{-1}.$$

Hence, the operator $\tilde{L}$ on $L^1(\mathbb{R})$ has the same spectrum as L. The kernel of the semigroup generated by $\tilde{L}$, $\{\tilde{T}_t\}_{t\geq 0} = \{e^{-t\tilde{L}}\}_{t\geq 0}$ is

$$\begin{aligned}
\tilde{M}_t(x,y) &= \frac{e^{-|x|^2}}{\pi^{1/2}} M_t(x,y) = \frac{e^{-|x|^2}}{\pi^{1/2}} \frac{1}{(1-e^{-2t})^{1/2}} \exp\left(-\frac{e^{-2t}(|x|^2+|y|^2) - 2e^{-t} x\cdot y}{1-e^{-2t}}\right) \\
&= \frac{e^{-|x|^2/2}}{\pi^{1/4}} K_t(x,y) \frac{e^{-|y|^2/2}}{\pi^{1/4}} \\
&= \frac{1}{\pi^{1/2}(1-e^{-2t})^{1/2}} e^{-\frac{|x-e^{-t}y|^2}{1-e^{-2t}}}
\end{aligned}$$

Taking the Fourier transform $\mathscr{F}$ from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$, it follows that

$$\mathscr{F}(\tilde{T}_t) f(\zeta) = e^{-(1-e^{-2t})\zeta^2/4} f(e^{-t}\zeta).$$

Let us consider $f_z^+(x)$ and $f_z^-(x)$ the $L^1(\mathbb{R})$-functions, whose Fourier transforms are $\chi_{[0,\infty)}(\zeta)|\zeta|^z e^{-\zeta^2/4}$ and $\chi_{(-\infty,0]}(\zeta)|\zeta|^z e^{-\zeta^2/4}$ respectively. Then, for any z, with $\mathrm{Re}\, z > 0$

$$\mathscr{F}(e^{-tH_1}f_z^+)(\zeta) = \chi_{[0,\infty)}(\zeta)e^{-(1-e^{-2t})\zeta^2/4}e^{-zt}|\zeta|^z e^{-e^{-2t}\zeta^2/4} = e^{-zt}\mathscr{F}(f_z^+)(\zeta),$$

and analogously,

$$\mathscr{F}(e^{-tH_1}f_z^-)(\zeta) = \chi_{(-\infty,0]}(\zeta)e^{-(1-e^{-2t})\zeta^2/4}e^{-zt}|\zeta|^z e^{-e^{-2t}\zeta^2/4} = e^{-zt}\mathscr{F}(f_z^-)(\zeta).$$

Hence, according to the uniqueness of the Fourier transform,

$$e^{-tH_1}f_z^+(x) = e^{-zt}f_z^+, \text{ and } e^{-tH_1}f_z^-(x) = e^{-zt}f_z^-.$$

Now, because the spectrum of L is a closed subset of $\{z : \mathrm{Re}\, z \geq 0\}$, as $\{T_t\}_{t\geq 0}$ is a strongly continuous contraction semigroup, we get the conclusion. □

Definition 2.8. *The maximal function of the Ornstein–Uhlenbeck semigroup* $\{T_t\}_{t\geq 0}$ *or maximal Ornstein–Uhlenbeck function, is defined as*

$$T^* f(x) = \sup_{t>0} |T_t f(x)|. \tag{2.44}$$

In Chapter 4, Theorems 4.19 and 4.20, we study the boundedness properties of T^*, proving that it is bounded in $L^p(\gamma_d)$ for $1 < p \leq \infty$, and that it is of weak type $(1,1)$ with respect to the measure γ_d. Also, other versions of maximal functions are study in detail in Chapter 4.

In 1969, C. P. Calderón [44] proved that the *multiparametric Ornstein–Uhlenbeck maximal function*

$$\mathbf{T}^* f(x) = \sup_{\substack{0<t_1<\infty \\ 0<t_2<\infty \\ \cdots \\ 0<t_d<\infty}} \left[\frac{1}{\pi^{d/2}} \prod_{i=1}^{d} \frac{1}{(1-e^{-2t_i})^{1/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t_i}x|^2}{1-e^{-2t_i}}} f(y)dy \right], \tag{2.45}$$

is $L^p(\gamma_d)$-bounded, $1 < p < \infty$. From this result, the $L^p(\gamma_d)$-boundedness for the one-parameter maximal operator T^* also follows.

The maximal function for the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$ can also be considered:

$$\Gamma_p^* f(w) = \sup_{z\in \mathbf{E}_p} |T_z f(w)|, \tag{2.46}$$

where $\mathbf{E}_p$ is Epperson's region defined in (2.43).

In particular, Γ_1^* is the maximal operator of the Ornstein–Uhlenbeck semigroup which, as we are going to see in Chapter 4 is of weak type $(1,1)$ and of strong type (p,p) for each $1 < p < \infty$.

According to the periodicity properties of the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$, we may restrict the parameter z to the set $\mathbf{F}_p$

$$\mathbf{F}_p = \{z \in \mathbf{E}_p : 0 \leq \operatorname{Im} z \leq \pi/2\} \tag{2.47}$$

Consider the map, $\tau : \{\zeta \in \mathbb{C} : |\zeta| \leq 1, |\arg \zeta| \leq \pi/2\} \to \mathbb{C} \cup \{\infty\}$ introduced in [105]:

$$\tau(\zeta) = \begin{cases} \log \frac{1+\zeta}{1-\zeta}, & \text{if } \zeta \neq 1 \\ \infty, & \text{if } \zeta = 1, \end{cases} \tag{2.48}$$

where $\log \omega$ is real when $\omega > 0$; hence τ is real-valued in the interval $[0,1)$. Notice that $\tau((\overline{\zeta})^{-1}) = \overline{\tau(\zeta)} + i\pi$, which means that τ makes reflection in the unit circle $|\zeta| = 1$ correspond to reflection in the line $\operatorname{Im} z = i\pi/2$. Combined with the periodicity and symmetry of T_z, we get

$$\overline{T_{\tau(t^{-1}e^{i\phi})} f(x)} = T_{\tau(te^{i\phi})} \overline{f}(-x).$$

Moreover, τ is a homeomorphism of its domain onto the half-strip $\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \geq 0, |\operatorname{Im} \zeta| \leq \pi/2\}$ mapping the sector

$$S_{\phi_p} := \{\zeta \in \mathbb{C} : |\zeta| \leq 1, |\arg \zeta| \leq \phi_p\} \tag{2.49}$$

onto the set $\mathbf{F}_p \cup \{\infty\}$. In particular, if $1 < p < 2$, then τ maps $S_{\phi_p} \setminus [1,\infty)$ onto the interior of $\mathbf{E} \cap \{z \in \mathbb{C} : |\operatorname{Im}| z < \pi/2\}$ and the ray $[0, e^{i\phi_p}\infty)$ onto $\partial \mathbf{E}_p \cap \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$ (see Figure 2.1). Additionally, if $\zeta \neq 1$,

$$M_{\tau(\zeta)}(x,y) = \frac{(1+\zeta)^d}{(4\zeta)^{d/2}} \exp\left(\frac{|x|^2+|y|^2}{2} - \frac{1}{4}\left(\zeta|x+y|^2 + \frac{1}{\zeta}|x-y|^2\right)\right),$$

because

$$1 - e^{2z} = \frac{4\zeta}{(1+\zeta)^2}, \ \frac{1}{2}\frac{1}{e^z+1} = \frac{1}{4} - \frac{\zeta}{4}, \text{ and} - \frac{1}{2}\frac{1}{e^z-1} = \frac{1}{4} - \frac{1}{4\zeta}.$$

We define $M_{\tau(1)}(x,y) = 1$, for all x,y.

Several estimates for Γ_p^* are given by J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea [104]. The simplest result establish that Γ_q^* is bounded on $L^p(\gamma_d)$ if $|\frac{1}{q} - \frac{1}{2}| > |\frac{1}{p} - \frac{1}{2}|$. This means that for $f \in L^p(\gamma_d)$, the supremum of $|T_z f(x)|$ is taken for $z \in \mathbf{E}_q \subset \mathbf{E}_p$.

For the case $1 < p < 2$, it was proved in [104] that Γ_p^* is not $L^p(\gamma_d)$-bounded, not even of weak type (p,p) with respect to the Gaussian measure. The unboundedness on $L^p(\gamma_d)$ here occurs along the whole boundary of $\mathbf{E}_p$.

$$\Gamma_{\varepsilon,p}^* f(w) = \sup_{z \in \mathbf{E}_p, d(z, i\pi\mathbb{Z}) \geq \varepsilon} |T_z f(w)|, \tag{2.50}$$

is of weak type (p,p) with respect to the Gaussian measure, for any $\varepsilon > 0$. Then, P. Sjögren [249] proved that for $2 < p < \infty$ Γ_p^* is not $L^p(\gamma_d)$-bounded, but $\Gamma_{\varepsilon,p}^*$ is $L^p(\gamma_d)$-bounded; therefore it is of weak type (p,p) with respect to γ, for any $\varepsilon > 0$. Finally, for $p = 2$, the situation is rather different: Γ_2^* is not of weak type $(2,2)$ with respect to the Gaussian measure (see [104]).

According to the Banach principle, it is known (see [107] or [275, Theorem 6.1]), that the study of this maximal operator is a key tool for investigating the almost everywhere convergence of $\{T_t\}_{t\geq 0}$,

$$T_0 f(x) \lim_{t\to 0^+} T_t f(x) = f(x) \quad a.e.\ x \in \mathbb{R}^d \tag{2.51}$$

(see Theorem 4.46), and also

$$T_\infty f(x) := \lim_{t\to\infty} T_t f(x) = \int_{\mathbb{R}^d} f(y)\, \gamma_d(dy) \quad a.e.\ x \in \mathbb{R}^d, \tag{2.52}$$

for all $f \in L^1(\gamma_d)$. This implies it for all $f \in L^p(\gamma_d)$, $1 \leq p \leq \infty$, as $L^q(\gamma_d) \subset L^p(\gamma_d)$ for $p \leq q$. Thus, unlike the classical case of the heat semigroup, the Ornstein–Uhlenbeck semigroup does not decay at infinity. This property expresses the ergodicity of the semigroup. The details of this proof and its generalization to non-tangential convergence are given in Chapter 4.

Proposition 2.9. *If $f \in L^p(\gamma_d)$, $u(x,t) = T_t f(x)$ is a $C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ solution of the parabolic equation*

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta_x u - \langle x, \nabla_x u\rangle = Lu, \quad x \in \mathbb{R}^d, t > 0, \tag{2.53}$$

with boundary condition $u(x,0) = f(x), x \in \mathbb{R}^d$.

Thus, $u(x,t) = T_t f(x)$ is the solution of a boundary value problem.

Proof. According to the general semigroup theory, given the fact that L is the infinitesimal generator of $\{T_t : t \geq 0\}$, we get

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial T_t f(x)}{\partial t} = LT_t f(x) = Lu(x,t).$$

Yet, this can also be shown explicitly:

$$\begin{aligned} Lu(x,t) &= \frac{2e^{-t}}{\pi^{d/2}(1-e^{-2t})^{d/2+1}} \int_{\mathbb{R}^d} \Big[\frac{de^{-t}}{2} + \frac{e^{-t}|y-e^{-t}x|^2}{1-e^{-2t}} - < (y-e^{-t}x), x > \Big] \\ &\qquad \times \exp\Big(-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}\Big) f(y)dy \\ &= \frac{\partial u(x,t)}{\partial t}. \end{aligned}$$

The boundary condition follows from (2.51). □

Now, from the fact that L is the infinitesimal generator of $\{T_t\}_{t\geq 0}$, using the semigroup property, we can easily get that

$$\frac{dT_t}{dt} = LT_t. \tag{2.54}$$

In [106], G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, and B. Viviani find optimal integrability conditions to guarantee the existence of solutions of (2.53).

Moreover, for the study of Hardy spaces in Chapter 7 we need to consider *higher order derivatives* of the Ornstein–Uhlenbeck semigroup,

$$\frac{d^kT_t}{dt^k} = L^kT_t \tag{2.55}$$

We get a closed expression for the integral representation of these derivatives, determining explicitly the kernels M_t^k such that

$$(L^kT_t)f(x) = \int_{\mathbb{R}^d} M_t^k(x,y)f(y)\gamma_d(dy), \tag{2.56}$$

Observe that, for $\nu \in \mathbb{N}_0$

$$\begin{aligned}
(L^kT_t)\mathbf{h}_\nu(x) &= |\nu|^k e^{-t|\nu|}\mathbf{h}_\nu(x) = |\nu|^k e^{-t|\nu|} h_{\nu_1}(x_1)\cdots h_{\nu_d}(x_d)\\
&= \sum_{|\eta|=k} \binom{k}{\eta_1,\eta_2,\cdots,\eta_d} \nu_1^{\eta_1}\cdots\nu_d^{\eta_d} e^{-t\eta_1}\cdots\, e^{-t\eta_d} h_{\nu_1}(x_1)\cdots h_{\nu_d}(x_d)\\
&= \sum_{|\eta|=k} \binom{k}{\eta_1,\eta_2,\cdots,\eta_d} L_1^{\eta_1}T_t^1 h_{\nu_1}(x_1)\cdots L_d^{\eta_d}T_t^{\eta_d} h_{\nu_d}(x_d),
\end{aligned} \tag{2.57}$$

where, as in (2.5), L_i denotes the one-dimensional Ornstein–Uhlenbeck operator, in the i-th variable, and $\{T_t^i\}_{t\geq 0}$ is the one-dimensional Ornstein–Uhlenbeck semigroup, in the i-th variable. Here, we follow J. Teuwen's paper [266], and it should be consulted for full details of the proof.

Theorem 2.10. *Let L be the Ornstein–Uhlenbeck operator in $L^2(\gamma_d)$, $t > 0$, and $N \geq 0$. The integral kernel M_t^k of L^kT_t is given by*

$$\begin{aligned}
M_t^k(x,y) = (-1)^k M_t(x,y) \sum_{|\eta|=k} \binom{k}{\eta_1,\eta_2,\cdots,\eta_d} \prod_{i=0}^{d} \sum_{n_i=0}^{\eta_i} \sum_{l_i=0}^{n_i} 2^{-m_i} \left\{ {\eta_i \atop n_i} \right\} \binom{n_i}{l_i}\\
\times\Big(-\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^{2n_i-l_i} H_{l_i}(x_i) H_{2n_i-l_i}\Big(\frac{y_i - x_i e^{-t}}{\sqrt{1-e^{-2t}}}\Big),
\end{aligned} \tag{2.58}$$

where $\left\{ {n \atop m} \right\}$ are Stirling numbers of the second kind.[11]

[11]For $n \geq m$ non-negative integers, the Stirling number of the second kind $\left\{ {n \atop m} \right\}$ is defined as the number of partitions of a set of n elements into m non-empty subsets, see [36].

Proof. From (2.57) and the tensorization argument, it is enough to consider only the case $d = 1$. Observe that

$$(L^k T_t)f(x) = L^k(T_t f)(x) = L^k\Big(\int_{\mathbb{R}^d} M_t(x,y)f(y)\gamma_d(dy)\Big) = \int_{\mathbb{R}^d} L^k M_t(x,y)f(y)\gamma_d(dy);$$

hence, $M_t^k(x,y) = L^k M_t(x,y)$. Therefore, using the integral representation of Mehler's kernel (1.46) we get

$$\begin{aligned} M_t^k(x,y) &= L^k\Big(\frac{e^{y^2}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2i\xi y-\xi^2} e^{-(x+i\xi e^{-t})^2+x^2} dx\Big) \\ &= \frac{e^{y^2}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2i\xi y-\xi^2} L^k e^{(x+i\xi e^{-t})^2+x^2} d\xi. \end{aligned}$$

Now, observe that

$$\begin{aligned} L^k e^{-(x-t)^2+x^2} &= (-1)^k\Big(t\frac{\partial}{\partial t}\Big)^k e^{-(x-t)^2+x^2} = (-1)^k\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix} t^n e^{x^2}\frac{\partial^k}{\partial t^k}e^{-(x-t)^2} \\ &= (-1)^k\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix} t^n e^{x^2}(-1)^k\frac{\partial^k}{\partial x^k}e^{-(x-t)^2} \\ &= (-1)^k e^{-(x-t)^2+x^2}\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix} t^n H_n(x-t), \end{aligned}$$

by using Rodrigues' formula and [266, Lemma 1].[12] Therefore, using (1.39)

$$\begin{aligned} M_t^k(x,y) &= (-1)^k\frac{e^{x^2+y^2}}{\sqrt{\pi}}\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix}\int_{\mathbb{R}} e^{2i\xi y-\xi^2}(i\xi e^{-t})^k e^{-(x+i\xi e^{-t})^2} H_n(x+i\xi e^{-t})d\xi \\ &= (-1)^k\frac{e^{x^2+y^2}}{\sqrt{\pi}}\sum_{n=0}^{k}\sum_{l=0}^{n}\begin{Bmatrix}k\\n\end{Bmatrix}\binom{n}{l} H_n(x)2^{n-l}\int_{\mathbb{R}} e^{2i\xi y-\xi^2} e^{-(x+i\xi e^{-t})^2}(i\xi e^{-t})^{2n-l}d\xi. \end{aligned}$$

Thus, it remains to compute the inner integral. For each $m \in \mathbb{N}$, using again the integral representation of the Hermite polynomials (1.30), and the change of variables $\eta = \sqrt{1-e^{-2t}}\,\xi$, we have

[12]Teuwen notices that his Lemma 1 is a particular case of a result in Weyl algebras, and depends only on the fact that the commutator $[t,\partial_t] = -1$.

$$\begin{aligned}
\frac{e^{x^2+y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} & e^{2i\xi y-\xi^2} e^{-(x+i\xi e^{-t})^2} (i\xi e^{-t})^m d\xi \\
&= \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2i(y-xe^{-t})\xi} e^{-(1-e^{-2t})\xi^2} (i\xi e^{-t})^m d\xi \\
&= \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2i\xi\sqrt{1-e^{-2t}}[(y-xe^{-t})/\sqrt{1-e^{-2t}}]} e^{-(1-e^{-2t})\xi^2} (i\xi e^{-t})^m d\xi \\
&= \frac{(-1)^m e^{y^2}}{2^m\sqrt{\pi}} \frac{(-2i)^m e^{-tm}}{(\sqrt{1-e^{-2t}})^{m+1}} \int_{\mathbb{R}} e^{2i\eta[(y-xe^{-t})/\sqrt{1-e^{-2t}}]} e^{-\eta^2} \eta^m d\eta \\
&= \frac{e^{-(y-xe^{-t})^2/(1-e^{-2t})} e^{y^2}}{\sqrt{1-e^{-2t}}} 2^{-m} \Big(\frac{-e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^m H_m\Big(\frac{y-xe^{-t}}{\sqrt{1-e^{-2t}}}\Big) \\
&= M_t(x,y) 2^{-m} \Big(\frac{-e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^m H_m\Big(\frac{y-xe^{-t}}{\sqrt{1-e^{-2t}}}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_t^k(x,y) &= (-1)^k M_t(x,y) \sum_{n=0}^{k} \sum_{l=0}^{n} 2^{-n} \begin{Bmatrix} k \\ n \end{Bmatrix} \binom{n}{l} \Big(\frac{-e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^{2n-l} \\
& \quad H_n(x) H_{2n-l}\Big(\frac{y-xe^{-t}}{\sqrt{1-e^{-2t}}}\Big). \qquad \Box
\end{aligned}$$

Another ingredient that is needed for the study of Hardy spaces in Chapter 7 is the following Gaussian version of A. P. Calderón's reproducing formula; see [231].

Lemma 2.11. *(Portal) For all $n \in \mathbb{N}$ and $a, \alpha > 0$, there exists $C > 0$ such that for all $f \in L^2(\gamma_d)$*

$$f(x) = C \int_0^\infty (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} f(x) \, \frac{dt}{t} + \int_{\mathbb{R}^d} f(x) \gamma_d(dx), \tag{2.59}$$

in L^2 sense.

Proof. As this is a formula in $L^2(\gamma_d)$, it is enough to prove (2.59) for the Hermite polynomials, as they are an orthonormal basis for $L^2(\gamma_d)$.

If $\nu = \mathbf{0}$, then as $\mathbf{H_0} = 1$, and $L\mathbf{H_0} = 0$, the right-hand side equals

$$C \int_0^\infty (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} 1 \, \frac{dt}{t} + \int_{\mathbb{R}^d} 1 \, d\gamma_d = C \cdot 0 + 1 = \mathbf{H_0}.$$

Let us assume now that $\nu \neq \mathbf{0}$. For these $\mathbf{H}_\nu$, the last integral in (2.59) is zero according to orthogonality. As $\mathbf{H}_\nu$ is an eigenfunction with eigenvalue of L, then

$$L^{N+1} \mathbf{H}_\nu = (-1)^{N+1} |\nu|^{N+1} \mathbf{H}_\nu.$$

Hence, we obtain for $x \in \mathbb{R}^d$

$$\int_0^\infty (t^2L)^{N+1} T_{(1+a)t^2/\alpha} \mathbf{H}_\nu(x) \frac{dt}{t} = \int_0^\infty (t^2L)^{N+1} e^{-(1+a)t^2|\nu|/\alpha} \mathbf{H}_\nu(x) \frac{dt}{t}$$
$$= (-1)^{N+1} |\nu|^{N+1} \mathbf{H}_\nu(x) \int_0^\infty t^{2(N+1)} e^{-(1+a)t^2|\nu|/\alpha} \frac{dt}{t}$$
$$= \frac{N!}{2} \left(\frac{\alpha}{(1+a)} \right)^{N+1} \mathbf{H}_\nu(x) = C\, \mathbf{H}_\nu(x).$$

Therefore, $C = C_N = \frac{2}{N!}(\frac{1+a}{\alpha})^{N+1}$ is the right constant. □

Another version of A. P. Calderón's reproducing formula was obtained in [164], and is discussed in Chapter 8 (see Theorem 8.31).

2.3 The Hypercontractivity Property for the Ornstein–Uhlenbeck Semigroup and the Logarithmic Sobolev Inequality

The Ornstein–Uhlenbeck semigroup is not only a contraction semigroup but it is also *hypercontractive*. The *hypercontractivity property* of $\{T_t\}_{t\geq 0}$ was initially proved by E. Nelson [204] in the context of quantum field theory, and it has been studied extensively in the literature.

Definition 2.12. *Given a semigroup of contractions* $\{\mathtt{T}_t\}_{t\geq 0}$ *defined in* $L^p(E,\mu)$, *with* $1 \leq p \leq \infty$, *the semigroup* $\{\mathtt{T}_t\}_{t\geq 0}$ *satisfies the* hypercontractivity property *if for each initial condition* $1 < p < \infty$ *there exists an strictly increasing function* $q : \mathbb{R}^+ \to [p,\infty)$, $q(0) = p$ *such that*

$$||\mathtt{T}_t f||_{q(t),\mu} \leq ||f||_{p,\mu}, \textit{ for all } f \in L^p(E,\mu), t \geq 0.$$

The function q is called the contraction function.

We are going to prove in detail that the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is hypercontractive, having contraction function

$$q(t) = 1 + e^{2t}(p-1) > p.$$

Thus, we will prove the following inequality:

$$||T_t f||_{q(t),\gamma} \leq ||f||_{p,\gamma}, \tag{2.60}$$

for all $f \in L^p(\gamma_d)$ and $t \geq 0$.
We will first prove that the Ornstein–Uhlenbeck operator satisfies the *logarithmic Sobolev inequality.*

Theorem 2.13. *The Ornstein–Uhlenbeck operator L satisfies the* logarithmic Sobolev inequality*: for any $f \in L^2(\gamma_d)$ with $\nabla_x f$ (in the weak sense) belonging to $L^2(\gamma_d)$,*

$$\int_{\mathbb{R}^d} |f(x)|^2 \log|f(x)|\gamma_d(dx) \leq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x f(x)|^2 \gamma_d(dx) + ||f||^2_{2,\gamma_d} \log ||f||_{2,\gamma}, \quad (2.61)$$

or, equivalently,

$$\int_{\mathbb{R}^d} |f(x)|^2 \log|f(x)|\gamma_d(dx) - \Big(\int_{\mathbb{R}^d} |f(x)|^2\gamma_d(dx)\Big)\log\Big(\int_{\mathbb{R}^d} |f(x)|\gamma_d(dx)\Big) \leq \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x f(x)|^2 \gamma_d(dx).$$

Proof. To prove (2.61), we will follow Adams and Clarke's proof [4], which is one of the simplest proofs for this inequality[13]. We begin by making a series of reductions. In the first place, it is enough to prove the logarithmic Sobolev inequality in the case $d = 1$. Then, the general case can be obtained by induction in d. In addition, observe that (2.61) is homogeneous with respect to rescaling of f; thus, we may assume that $||f||_{2,\gamma} = 1$. Moreover, we may assume that $||f'||_{2,\gamma} < \infty$ because, otherwise, there is nothing to prove. The change $f(t) = g(t)e^{t^2/2}$ implies the following equivalent formulation of the inequality:

$$\int_{\mathbb{R}} \left(\frac{1}{2}\left|g'(t)\right|^2 - |g(t)|^2 \log|g(t)|\right) dt \geq \frac{\sqrt{\pi}}{2}, \text{ provided } \int_{\mathbb{R}} |g(t)|^2\, dt = \sqrt{\pi}. \quad (2.62)$$

As $\left|(|g|')\right| \leq |g'|$ a.e., we may assume that g is a non-negative real-valued function. It is enough to consider only the case $g(t) > 0$ for all $t \in \mathbb{R}$; this can be justified by a simple argument of density. Finally, it is convenient to split (2.62) into two half-line problems, each of them equivalent to

$$\int_0^\infty \left(\frac{1}{2}\left(g'(t)\right)^2 - (g(t))^2 \log(g(t))\right) dt \geq \frac{\sqrt{\pi}}{4}, \text{ provided } \int_0^\infty (g(t))^2\, dt = \frac{\sqrt{\pi}}{2}. \quad (2.63)$$

For $s, r > 0$, let $V(s,r) = \left[v(s,r)s^2 + r\left(1 - v(s,r)^2 - 2\log s\right)\right]/2$, where $v(s,r) = h^{-1}\left(r/s^2\right)$, and h is given by

$$h(t) = e^{t^2}\int_t^\infty e^{-\tau^2}\, d\tau.$$

It is easy to see that h is strictly decreasing in $\mathbb{R}$ and $\left(h^{-1}\right)'(t) = \left\{2th^{-1}(t) - 1\right\}^{-1}$. The partial derivatives of V are:

$$V_s = vs, \text{ and } V_r = -\left(v^2/2\right) - \log s.$$

[13] For another simple proof see [219].

If $U(s,u) = \left(u^2/2\right) - s^2 \log s$, then,

$$V_s u - V_r s^2 + U(s,u) = \frac{1}{2}(u + vs)^2 \geq 0, \text{ for } s > 0,\ u \in \mathbb{R}. \tag{2.64}$$

Therefore, if g satisfies the inequalities

$$g(t) > 0,\ \int_0^\infty (g(t))^2\, dt = \frac{\sqrt{\pi}}{2},\ \int_0^\infty \left(g'(t)\right)^2 dt < \infty, \tag{2.65}$$

it then follows from (2.64) (setting $s = g(t),\ r = \int_t^\infty (g(\tau))^2\, d\tau,\ u = g'(t)$) that

$$\frac{d}{dt} V\left(g(t), \int_t^\infty (g(\tau))^2\, d\tau\right) = V_s g'(t) - V_r (g(t))^2 \geq -U\left(g(t), g'(t)\right)$$

and

$$\begin{aligned}\int_0^\infty U\left(g(t), g'(t)\right) dt &\geq -\int_0^\infty \frac{d}{dt} V\left(g(t), \int_t^\infty (g(\tau))^2\, d\tau\right) dt \\ &\geq V\left(g(0), \frac{\sqrt{\pi}}{2}\right) - \liminf_{t\to\infty} V\left(g(t), \int_t^\infty (g(\tau))^2\, d\tau\right).\end{aligned}$$

As h^{-1} is decreasing and $h^{-1}\left(\sqrt{\pi}/2s^2\right) = 0$ only for $s = 1$, we conclude that

$$V\left(s, \sqrt{\pi}/2\right) \geq V\left(1, \sqrt{\pi}/2\right) = \sqrt{\pi}/4, \text{ for all } s > 0.$$

The inequality (2.63) would be shown if the following claim holds: if g satisfies (2.65), then

$$\liminf_{t\to\infty} V\left(g(t), \int_t^\infty (g(\tau))^2\, d\tau\right) \leq 0.$$

To prove the claim, we use the fact that $h(\tau) < 1/\tau$ for all $\tau > 0$. Then, $h^{-1}(t) < 1/t$ for all $t > 0$, and $v(s,r)s^2 < s^4/r$. Similarly, $h(\tau) < \sqrt{\pi}e^{\tau^2}$ for $\tau \leq 0$ implies $h^{-1}(t) \leq -\sqrt{\log\left(t/\sqrt{\pi}\right)}$ for $t \geq \sqrt{\pi}$ and therefore, setting $t = r/s^2$, we get

$$(v(s,r))^2 \geq \log r - \log s^2 - \log\sqrt{\pi}$$

for $\sqrt{\pi}s^2 \leq r$. Evidently, $(v(s,r))^2 \geq 0$ if $r < \sqrt{\pi}s^2$ and consequently,

$$r\left(1 - v^2 - 2\log s\right) \leq \max\left\{r\left(1 + \log\sqrt{\pi} - \log r\right), \sqrt{\pi}s^2\left(1 - \log s^2\right)\right\}$$

for all $r, s > 0$. Hence,

$$V(s,r) \leq \frac{s^4}{2r} + \frac{1}{2}\max\left\{r\left(1 + \log\sqrt{\pi} - \log r\right), \sqrt{\pi}s^2\left(1 - \log s^2\right)\right\}. \tag{2.66}$$

If g satisfies (2.65), then taking $s = g(t)$,

$$r = \int_t^\infty (g(\tau))^2 d\tau, \text{ and } \varepsilon = \int_t^\infty \left(g'(\tau)\right)^2 d\tau$$

both terms tend to zero as $t \to \infty$. Moreover, according to Hölder's inequality

$$s^4 = (g(t))^4 \leq \left(2\int_t^\infty g(\tau)\left|g'(\tau)\right| d\tau\right)^2 \leq 4r\varepsilon.$$

From (2.66), it follows that $\liminf_{t\to\infty} V(s,r) \leq 0$. □

In [119], L. Gross proved the following striking result:

Theorem 2.14. *The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is hypercontractive, with contraction function $q(t) = 1 + e^{2t}(p-1) > p$, if and only if the Ornstein–Uhlenbeck operator L satisfies the logarithmic Sobolev inequality (2.61).*

To prove this theorem, we essentially follow Gross' argument (see [119] and [120]). For this we need the following technical (but key) lemma. We are going to formulate it in great generality for any probability space $(E, \mathscr{B}, \mu)$, which will be useful in what follows. Of course, in our case, the probability space is simply $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \gamma_d)$.

Lemma 2.15. *Let $(E, \mathscr{B}, \mu)$ be a probability space. Let us take $1 < p < \infty$, $\varepsilon > 0$ and $q > p$ and let s be a real function, continuously differentiable from $[0,\varepsilon)$ to $(1,\infty)$ such that $s(0) = p$, and let f be a function continuously differentiable from $[0,\varepsilon)$ to $L^q(\mu)$ with $f(0) = v \neq 0$. Then, $||f(t)||_{s(t)}$ is differentiable at $t = 0$ and*

$$\begin{aligned} &\frac{d}{dt}||f(t)||_{s(t)}\Big|_{t=0} \qquad (2.67)\\ &\quad = ||v||_p^{1-p}\left[p^{-1}s'(0)\left(\int_E |v|^p \log|v|d\mu - ||v||_p^p \log||v||_p\right) + \mathrm{Re}\langle f'(0), v_p\rangle_\mu\right], \end{aligned}$$

where, $v_p = (\operatorname{sgn} v)|v|^{p-1}$.

Proof. Let $g : [0,\varepsilon) \to \mathbb{C}$ be a continuously differentiable function. Then, we have

$$\begin{aligned} \frac{d}{dt}|g(t)|^{s(t)} &= \left[s'(t)\log|g(t)| + s(t)\frac{1}{|g(t)|^2}\mathrm{Re}g'(t)\overline{g(t)}\right]|g(t)|^{s(t)} \\ &= s'(t)|g(t)|^{s(t)}\log|g(t)| + s(t)\frac{1}{|g(t)|^2}\mathrm{Re}g'(t)\overline{\frac{g(t)}{|g(t)|}|g(t)|^{s(t)-1}} \\ &= s'(t)|g(t)|^{s(t)}\log|g(t)| + s(t)\frac{1}{|g(t)|^2}\mathrm{Re}g'(t)\overline{g(t)_{s(t)}}. \end{aligned}$$

This holds even when $g(t) = 0$ for some t, because $s(t) > 1$.

Let us take $g(t) = f(t)(x)$(formally), integrate it with respect to μ, and interchange the order of the integration and differentiation. Then,

$$\frac{d}{dt}\int_E |f(t)(x)|^{s(t)}\mu(dx) = \int_E s'(t)|f(t)(x)|^{s(t)}\log|f(t)(x)|\mu(dx) + s(t)\mathrm{Re}\langle f'(t), f_{s(t)}\rangle_\mu.$$

Then, if $V(t) = \int_E |f(t)(x)|^{s(t)}\mu(dx)$, we have

$$\begin{aligned}\frac{d}{dt}||f(t)||_{s(t)} &= \frac{d}{dt}V(t)^{s(t)}\\ &= \frac{1}{s(t)}\Big[\frac{V(t)^{s(t)-1}}{V(t)}\Big]V'(t) - \frac{s'(t)}{s^2(t)}V(t)^{s(t)-1}\log V(t).\end{aligned}$$

The second chain of equalities needs justification, because $f(t)(x)$ is not necessarily differentiable in the variable t for a.e. x (for details see Gross [120, page 63]). Then, taking $t = 0$ (2.67) follows. □

Now, we are ready to prove Theorem 2.14

Proof. First of all, consider the number operator

$$N = 2(-L) = -\Delta_x + 2\langle x, \nabla_x\rangle$$

which is the Dirichlet form for γ_d, i.e.,

$$\int_{\mathbb{R}^d}\langle\nabla_x f(x), \nabla_x g(x)\rangle\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Ng(x)\gamma_d(dx),$$

and consider the semigroup $\{e^{-tN}\}_{t\geq 0}$ generated by N.[14]

Let us assume that (2.61) holds. Then, we can obtain, for each $p > 1$, the Sobolev logarithmic inequality in $L^p(\gamma_d)$

$$\int_{\mathbb{R}^d}|f(x)|^p\log|f(x)|\gamma_d(dx) \leq c(p)\mathrm{Re}\langle Nf(t), f_p\rangle_{\gamma_d} + ||f||^p_{p,\gamma_d}\log||f||_{p,\gamma}, \quad (2.68)$$

with $c(p) = \frac{p}{4(p-1)}$ and $f_p = (\mathrm{sgn}\, f)|f|^{p-1}$. In Gross's notation [120] this means that N is a *Sobolev generator* in $(0,\infty)$.

The outline of this argument is as follows: assume that $p > 1$ and let f be a non-negative bounded function in the domain of N in $L^2(\gamma_d)$. Then, replacing f by $f^{p/2}$ in (2.61), we get

$$\begin{aligned}\frac{p}{2}\int_{\mathbb{R}^d}|f(x)|^p\log|f(x)|\gamma_d(dx) &\leq \frac{1}{2}\int_{\mathbb{R}^d}|\nabla_x f^{p/2}(x)|^2\gamma_d(dx)\\ &\quad + \frac{1}{2}\int_{\mathbb{R}^d}|f(x)|^p\gamma_d(dx)\Big(\log\int_{\mathbb{R}^d}|f(x)|^p\gamma_d(dx)\Big).\end{aligned}$$

[14] This semigroup is simply $\{T_{2t}\}_{t\geq 0}$, the Ornstein–Uhlenbeck semigroup with parameter $2t$.

Now, if f is bounded and smooth, we have

$$|\nabla_x(f(x)^{p/2})|^2 = (p/2)^2(f(x)^{p/2-1})^2|\nabla_x f(x)|^2,$$

and also

$$\langle \nabla_x f(x), \nabla_x(f(x)^{p-1})\rangle = (p-1)f(x)^{p-2}|\nabla_x f(x)|^2.$$

Therefore,

$$|\nabla_x(f(x)^{p/2})|^2 = \Big[\frac{(p/2)^2}{(p-1)}\Big]\langle \nabla_x f(x), \nabla_x(f(x)^{p-1})\rangle,$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla_x f^{p/2}(x)|^2 \gamma_d(dx) &= \Big[\frac{p^2}{4(p-1)}\Big]\int_{\mathbb{R}^d} \langle \nabla_x f(x), \nabla_x(f(x)^{p-1})\rangle \gamma_d(dx)\\ &= \Big[\frac{p^2}{4(p-1)}\Big]\langle Nf, f^{p-1}\rangle_\gamma,\end{aligned}$$

thus proving (2.68).

The set where these computations make sense can be justified from the fact that e^{-tN} is a contractive and positive semigroup in $L^\infty(\gamma_d)$ (see [120]).

Now, let g be a non-negative function in $C_0^\infty(\mathbb{R})$ with support in $(0,\infty)$ and let $u \in L^\infty(\gamma_d)$. Then, $h := \int_0^\infty g(s)(e^{-sN}u)ds$ exists as a Riemann integral in $L^p(\gamma_1), 1 < p < \infty$. If $f(t) = e^{-tN}h$, $t \geq 0$, f is a positive and differentiable function in $L^p(\gamma_d)$ for all $1 < p < \infty$. Then, according to Lemma 2.15 the function $\alpha(t) = ||f(t)||_{1+(p-1)e^{4t},\gamma_d}$ is differentiable in $(0,\infty)$; therefore,

$$\begin{aligned}\frac{d\alpha(t)}{dt} &= \frac{d}{dt}||f(t)||_{1+(p-1)e^{4t},\gamma}\\ &= ||f(t)||_{p,\gamma}^{1-p}\Big[c(p)^{-1}\Big(\int_{\mathbb{R}^d}|f(t)|^p \log|f(t)|d\gamma_d - ||f(t)||_{p,\gamma}^p \log||f(t)||_{p,\gamma}\Big)\\ &\qquad -\mathrm{Re}\langle Nf(t), f(t)_p\rangle_\gamma\Big] \leq 0.\end{aligned}$$

Thus, $\frac{d}{dt}\log\alpha(t) \leq 0$ and $\log\alpha(t) \leq \log\alpha(0) = \log||h||_{p,\gamma}$, i.e.,

$$||e^{-tN}h||_{1+(p-1)e^{4t},\gamma} \leq ||h||_{p,\gamma}. \tag{2.69}$$

Recall that an approximation of the identity is a sequence of functions $\{h_n\}$ that converges to the Dirac delta function (see the Appendix 10.5). Then, for each $t \geq 0$ and for any element of an approximation of the identity, so that the corresponding sequence $\{h_n\}$ converges to u in $L^p(\gamma_d)$-norm, and $e^{-tN}h_n$ converges to $e^{-tN}u$ in $L^p(\gamma_d)$-norm, and also almost everywhere. Applying the previous inequality (2.69) to h_n and using Fatou's lemma, we have

$$||e^{-tN}u||_{1+(p-1)e^{4t},\gamma} \leq ||u||_{p,\gamma}.$$

As $L^\infty(\gamma_d)$ is dense in $L^p(\gamma_d)$, we can again apply Fatou's lemma; thus, the inequality (2.69) holds for any $h \in L^p(\gamma_d)$.

Finally, given that $T_t h = e^{tL}h = e^{-(t/2)N}h$, the previous inequality is equivalent to

$$||T_t h||_{1+(p-1)e^{2t},\gamma} = ||e^{tL}h||_{1+(p-1)e^{2t},\gamma} = ||e^{-(t/2)N}h||_{1+(p-1)e^{4(t/2)},\gamma} \leq ||h||_{p,\gamma};$$

hence, $\{T_t\}_{t\geq 0}$ satisfies (2.60).

Conversely, let us assume that the semigroup $\{T_t\}_{t\geq 0}$ is strongly continuous on $L^p(\gamma_d)$, $1 < p < \infty$, and that it is hypercontractive (2.60). Let $\mathscr{D}$ be the linear hull of the set of functions $h := \int_0^\infty g(s)(e^{-sN}u)ds$, with g a non-negative function in $C_0^\infty(\mathbb{R})$ with support in $(0,\infty)$ and $u \in L^\infty(\gamma_d)$, as was considered in the first part of the proof. Let h be a non-zero element in $\mathscr{D}$ and set $f(t) = e^{-tN}h$, $t \in (0,\infty)$. Then, for each t, we have

$$\frac{||f(t)||_{1+(p-1)e^{4t},\gamma} - ||f(0)||_{p,\gamma}}{t} \leq ||h||_{p,\gamma}\Big(\frac{1-1}{t}\Big) = 0,$$

according to the hypercontractivity property (2.60) and the fact that $e^{-tN}h = T_{2t}h$. By Lemma 2.15, we can take the limit as $t \downarrow 0$ in the preceding inequality to get

$$||h||_{p,\gamma_d}^{1-p}\Big[p^{-1}4(p-1)\Big(\int_{\mathbb{R}^d}|h|^p \log|h|d\gamma_d - ||h||_{p,\gamma_d}^p \log||h||_{p,\gamma}\Big) - \mathrm{Re}\langle Nh, h_p\rangle_{\gamma_d}\Big] \leq 0.$$

Multiplying by $\frac{p}{4(p-1)}||h||_{p,\gamma_d}^{p-1}$, we obtain (2.61).

Now, because $\mathscr{D}$ is dense in the domain of the infinitesimal generator of $\{e^{-Nt}\}_{t\geq 0}$ in $L^p(\gamma_d)$, for any f there exists a sequence $\{h_n\}$ in $\mathscr{D}$ such that $h_n \to f$ in the graph norm and a.e. γ_d. As $x^p \log x$ is bounded from below in $[0,\infty)$, we can use Fatou's lemma on the left side of (2.61). For the right side we observe that the mapping $f \to f_p$ is continuous from $L^p(\gamma_d)$ to $L^{p'}(\gamma_d)$; thus, the right-hand side is a continuous function of f in the graph norm. Therefore, because (2.61) holds for each h_n, it holds for any f. □

The hypercontractivity of the Ornstein–Uhlenbeck semigroup can also be obtained using the curvature-dimension inequalities. In 1984, D. Bakry and M. Emery [21] developed a criterion (sufficient condition) for a Markov diffusion semigroup to satisfy the hypercontractivity property, which is the famous *Bakry–Emery criterion*. This criterion is given in terms of the iterated square field operator Γ_2,

$$\Gamma_2(f,g) = \frac{1}{2}\Big[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(f,Lg)\Big],$$

for every $f,g \in \mathscr{A}$, the standard algebra (an "appropriated class" of functions). The Bakry–Emery criterion has evolved to what is now known as *curvature-dimension inequalities*, which allows us to study the local structure of the generator L and has important applications in differential geometry.

Definition 2.16. *An operator L is said to satisfy a curvature-dimension inequality $CD(\rho,n)$ if*

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{n}(Lf)^2, \tag{2.70}$$

for any $f \in \mathscr{A}$. Here, $\rho \in \mathbb{R}$ is called the curvature and $n \in [1,\infty]$ the dimension.

It can be proved (see for instance [19, 284]) that if an inequality $CD(\rho,\infty)$ holds for some $\rho > 0$, then the invariant measure μ must be finite; moreover, a logarithmic Sobolev inequality holds. Observe that for the Gaussian case, when $d = 1$ (2.17) and (2.23) become

$$\Gamma(f,f)(x) = \frac{1}{2}(f'(x))^2 \quad \text{and} \quad \Gamma_2(f,f)(x) = \frac{1}{4}(f''(x))^2 + \frac{1}{2}(f'(x))^2.$$

Then, trivially, we have a curvature-dimension inequality with $n = \infty$ and constant ρ

$$\frac{1}{4}(f''(x))^2 + \frac{1}{2}(f'(x))^2 \geq \frac{\rho}{2}(f'(x))^2,$$

if and only if $\rho \leq 1$. The extension for higher dimensions follows simply by the tensorization argument.

The original hypercontractive estimates of the Ornstein–Uhlenbeck semigroup were obtained by E. Nelson [204] and were later extended to the complex case, for suitable values of z, by F.B. Weissler [292] and J.B. Epperson [74].

2.4 Applications of the Hypercontractivity Property

One of the first consequences of the hypercontractivity property for the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is that the orthogonal projections $\mathbf{J}_k$ onto the (closed) subspaces $\mathscr{C}_k$ of the Wiener Chaos are $L^p(\gamma_d)$-continuous for $1 < p < \infty$:

Corollary 2.17. *For any $k \in \mathbb{N}$, $\mathbf{J}_k|_{\mathscr{P}(\mathbb{R}^d)}$, the restriction of $\mathbf{J}_k$ to the polynomials $\mathscr{P}(\mathbb{R}^d)$, has an extension, which will also be denoted as $\mathbf{J}_k$, to a bounded operator in $L^p(\gamma_d)$, i.e.,*

$$||\mathbf{J}_k f||_{p,\gamma} \leq C_{p,k}||f||_{p,\gamma}. \tag{2.71}$$

Proof. If $p > 2$, taking $t_0 > 0$ such that $p = e^{2t_0} + 1$, according to the hypercontractivity property of $\{T_t\}$, we have

$$||T_{t_0} f||_{p,\gamma} \leq ||f||_{2,\gamma}.$$

In particular, from Hölder's inequality,

$$||T_{t_0}\mathbf{J}_k f||_{p,\gamma} \leq ||\mathbf{J}_k f||_{2,\gamma} \leq ||f||_{2,\gamma} \leq ||f||_{p,\gamma}.$$

Now, because $T_{t_0} f = \sum_{k=0}^{\infty} e^{-t_0 k}\mathbf{J}_k f$, we have $T_{t_0}\mathbf{J}_k f = e^{-t_0 k}\mathbf{J}_k f$; therefore,

$$||T_{t_0}\mathbf{J}_k f||_{p,\gamma} = e^{-t_0 k}||\mathbf{J}_k f||_{p,\gamma}.$$

Thus, we have

$$||\mathbf{J}_k f||_{p,\gamma} \leq e^{t_0 k}||f||_{p,\gamma}.$$

The case $1 < p < 2$ is obtained by duality from the previous one. Let p' be the conjugated exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$, $p' > 2$. Then, because the projection $\mathbf{J}_k$ is a self-adjoint operator, using Hölder's inequality, we get

$$\begin{aligned}
||\mathbf{J}_k f||_{p,\gamma} &= \sup_{||g||_{p',\gamma}\le 1} \left| \int_{-\infty}^{\infty} \mathbf{J}_k f\, g\, d\gamma_d \right| = \sup_{||g||_{p',\gamma}\le 1} \left| \int_{-\infty}^{\infty} f\, \mathbf{J}_k g\, d\gamma_d \right| \\
&\le \sup_{||g||_{p',\gamma}\le 1} ||f||_{p,\gamma} ||\mathbf{J}_k g||_{p',\gamma} \le \sup_{||g||_{p',\gamma}\le 1} ||f||_{p,\gamma} C ||g||_{p',\gamma} \le C ||f||_{p,\gamma},
\end{aligned}$$

where $C = e^{t_0 k}$, with $t_0 > 0$ such that $p' = e^{2t_0} + 1$. □

The next lemma is useful for the proof of P.A. Meyer's multiplier theorem , which is also a consequence of the hypercontractivity property.

Lemma 2.18. *If* $1 < p < \infty$, *for each* $n \in \mathbb{N}$, *there exists a constant* C_n *such that*

$$||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} \le C_n e^{-tn} ||f||_{p,\gamma}. \tag{2.72}$$

Proof. Again, by duality, it is enough to consider the case $p > 2$. Let t_0 be such that $p = e^{2t_0} + 1$. Then, using the hypercontractivity property and Parseval's identity, we get for $t > t_0$,

$$\begin{aligned}
&||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||^2_{p,\gamma} \\
&= ||T_{t_0} T_{t-t_0}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||^2_{p,\gamma} \\
&\le ||T_{t-t_0}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||^2_{2,\gamma} \\
&= ||\sum_{k=n}^{\infty} e^{-(t-t_0)k} \mathbf{J}_k f||^2_{2,\gamma} = \sum_{k=n}^{\infty} e^{-2(t-t_0)k} ||\mathbf{J}_k f||^2_{2,\gamma} \\
&= \sum_{k=0}^{\infty} e^{-2(t-t_0)(k+n)} ||\mathbf{J}_{k+n} f||^2_{2,\gamma} \le e^{-2(t-t_0)n} \sum_{k=0}^{\infty} ||\mathbf{J}_{k+n} f||^2_{2,\gamma} \\
&\le e^{-2(t-t_0)n} ||f||_{2,\gamma} \le C_n e^{-2tn} ||f||^2_{p,\gamma},
\end{aligned}$$

where $C_n = e^{2t_0 n}$.

Now, if $t < t_0$, because T_t is a contraction,

$$\begin{aligned}
||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} &\le ||(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} \\
&\le (1 + \sum_{k=0}^{n-1} e^{kt_0}) ||f||_{p,\gamma} \le (n+1) e^{nt_0} ||f||_{p,\gamma} \\
&\le C_n e^{-nt_0} ||f||_{p,\gamma} \le C_n e^{-nt} ||f||_{p,\gamma},
\end{aligned}$$

with $C_n = (n+1)e^{nt_0}$. □

Finally, let us consider *potential operators*,

$$U_n f = \int_0^\infty T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f\, dt. \tag{2.73}$$

According to Minkowski's integral inequality and Lemma 2.18, we have

$$||U_n f||_{p,\gamma} \leq \frac{C}{n} ||f||_{p,\gamma}, \quad \text{for } 1 < p < \infty. \tag{2.74}$$

Let us also consider the following operators associated with U_n

$$U_{n,m} f = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f\, dt. \tag{2.75}$$

Then, again according to Minkowski's integral inequality and Lemma 2.18 we have

$$\begin{aligned} ||U_{n,m} f||_{p,\gamma} &\leq \frac{1}{(m-1)!} \int_0^\infty t^{m-1} ||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma}\, dt \\ &\leq \frac{C}{(m-1)!} \int_0^\infty t^{m-1} e^{-tn} dt ||f||_{p,\gamma} \leq \frac{C}{n^m} ||f||_{p,\gamma}; \end{aligned}$$

hence,

$$||U_{n.m} f||_{p,\gamma} \leq \frac{C}{n^m} ||f||_{p,\gamma}, \tag{2.76}$$

for all $n, m \in \mathbb{N}$.

Moreover, if $f \in \mathscr{C}_k$, i.e., $\mathbf{J}_k f = f$, and $k \geq n$,

$$U_n f = \int_0^\infty T_t \mathbf{J}_k f dt = \int_0^\infty e^{-kt} f dt = \frac{1}{k} f,$$

and similarly,

$$U_{n,m} f = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} T_t \mathbf{J}_k f dt = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-kt} f dt = \frac{1}{k^m} f.$$

A very important consequence of the hypercontractivity of the Ornstein–Uhlenbeck semigroup is P. A. Meyer's multiplier theorem (see Theorem 6.2 in Chapter 6).

2.5 Notes and Further Results

1. The definition of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ using the integral representation (2.28) coincides with that obtained using the general theory of Markov semigroups, taking as transition probabilities

$$P_t(x, dy) = \sum_{|\nu|\geq 0} e^{-t|\nu|} \mathbf{h}_\nu(x) \mathbf{h}_\nu(y) \gamma_1(dy) = \frac{1}{(1 - e^{-2t})^{d/2}} e^{-\frac{e^{-2t}(|y|^2 + |x|^2) - 2e^{-t}\langle x, y\rangle}{1 - e^{-2t}}} dy,$$

according to Mehler's formula (10.24). For more details, see for instance [20] or [284]. This is the link to the theory of Markov processes in probability. It is well known that Brownian motion $\{B_t\}_{t\geq 0}$ is associated with the heat semigroup $\{\mathscr{T}_t\}_{t\geq 0}$. Similarly, we have the *Ornstein–Uhlenbeck process* $\{X_t\}_{t\geq 0}$ with transition probabilities $\{P_t\}_{t\geq 0}$, which is associated with the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$. The process $\{X_t\}_{t\geq 0}$ describes the speed of a particle moving in a fluid with viscosity against a resisting force that is proportional to its speed (see Breiman [35, Chapter 6]). Hence, $\{X_t\}_{t\geq 0}$ can be obtained using the following formula to construct its finite-dimensional distributions:

$$\mu^x\{X_{t_1} \in E_1, X_{t_2} \in E_2, \; \cdots, X_{t_k} \in E_k\} = \int_{E_k} \cdots \int_{E_2} \int_{E_1} P_{t_1}(x, dy_1) P_{t_2 - t_1}(y_1, dy_2) \cdots P_{t_k - t_{k-1}}(y_{k-1}, dy_k).$$

It is known that the process can be obtained from the semigroup using (2.77) and that the semigroup $\{T_t\}_{t\geq 0}$ can be represented in terms of the Markov process $\{X_t\}_{t\geq 0}$ as

$$T_t f(x) = \mathbf{E}[f(X_t)|X_0 = x], \, f \in L^\infty(\gamma_d). \tag{2.77}$$

Using this representation, the properties of the semigroup can be proved using probabilistic methods. Moreover, (2.52) expresses the stationarity and ergodicity of the process.

2. The Ornstein–Uhlenbeck semigroup can also be introduced formally, following S. Bochner [32], as a solution to the equation (2.53), as follows: let $f \in L^2(\gamma_d)$ with Hermite expansion $\sum_{|\nu|\geq 0} a_\nu \mathbf{H}_\nu$; therefore, $\sum_{|\nu|\geq 0} (a_\nu)^2 < \infty$. Then, formally, Lf has the expansion

$$Lf = -\sum_{|\nu|\geq 0} |\nu| a_\nu \mathbf{H}_\nu,$$

if $\sum_{|\nu|\geq 0} |\nu|^2 (a_\nu)^2 < \infty$.

Now, let $u(x,t)$ be a solution of (2.53) with Hermite expansion $\sum_{|\nu|\geq 0} a_\nu(t) \mathbf{H}_\nu$; therefore,

$$\sum_{|\nu|\geq 0} (a_\nu(t))^2 < \infty.$$

Thus, Lu and $\frac{\partial u}{\partial t}$ have Hermite expansions

$$-\sum_{|\nu|\geq 0} |\nu| a_\nu(t) \mathbf{H}_\nu, \quad \text{and} \quad \sum_{|\nu|\geq 0} a'_\nu(t) \mathbf{H}_\nu$$

respectively, and then, assuming that

$$\sum_{|\nu|\geq 0} |\nu|^2 (a_\nu(t))^2 < \infty \quad \text{and} \quad \sum_{|\nu|\geq 0} (a'_\nu(t))^2 < \infty,$$

we conclude by the uniqueness of the Hermite expansions that

$$-|\nu|a_\nu(t) = a'_\nu(t),$$

or equivalently

$$a_\nu(t) = a_\nu e^{-|\nu|t}.$$

Thus, we get the expansion (2.27), and again by uniqueness, we conclude that necessarily $u(x,t) = T_t f(x)$.

3. S. Pérez [221] provided another way to see that $u(x,t) = T_t f(x)$ is a solution of (2.53). It consists of looking for an appropriate dilation that, using the Fourier transform, gives us u as a solution of a differential equation that is easier to solve. Let $w(x,t) = u(e^t x,t)$, then

$$w_t(x,t) = e^t \langle x, \nabla_x u(e^t x,t)\rangle + u_t(e^t x,t),$$

$$\nabla_x w(x,t) = e^t \nabla_x u(e^t x,t), \text{ and } \Delta_x w(x,t) = e^{2t} \Delta_x u(e^t x,t).$$

Thus, w satisfies a variant of the heat equation

$$w_t(x,t) = \frac{1}{2} e^{-2t} \Delta_x w(x,t).$$

Then, applying the Fourier transform (in the x variable), we obtain that $\hat{w}$ satisfies the ordinary differential equation

$$\hat{w}'(\xi,t) = -2\pi^2 e^{-2t} |\xi|^2 \hat{w}(\xi,t),$$

whose solution is

$$\hat{w}(\xi,t) = e^{-\pi^2(1-e^{-2t})|\xi|^2} \hat{f}(\xi)$$

and its inverse Fourier transform is given by

$$w(x,t) = C_d \int_{\mathbb{R}^d} \frac{e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}}{(1-e^{-2t})^{d/2}} f(y)dy.$$

4. To prove the semigroup property of $\{T_t\}_{t\geq 0}$, there is an analogous proof to that given for the heat semigroup in the Appendix using the Fourier transform (see proof of Theorem 10.15*i*)). Nevertheless, this would prove the result only for functions in $\mathscr{S}(\mathbb{R}^d)$, Schwartz's space of test functions. Also, because the set of polynomials $\mathscr{P}(\mathbb{R}^d)$ is dense in $L^p(\gamma_d)$, see Corollary 10.12, we can also prove the semigroup property by means of the representation (2.27).

$$(T_{t_1} \circ T_{t_2})f = T_{t_1}(\sum_{k=0}^{\infty} e^{-t_2 k} \mathbf{J}_k f) = \sum_{k=0}^{\infty} e^{-(t_1+t_2)k} \mathbf{J}_k f = T_{t_1+t_2} f.$$

5. The *translated Ornstein–Uhlenbeck semigroups* $\{T_t^{(\kappa)}\}_{t\geq 0}$, $\kappa \geq 0$, see [122] and [124], are defined formally as

$$T_t^{(\kappa)} = e^{-\kappa t} T_t, \tag{2.78}$$

which means that $T_t^{(\kappa)} \mathbf{h}_\nu = e^{-t(|\nu|+\kappa)} \mathbf{h}_\nu$. Thus, they are in fact a type of translation of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$, It can be shown that the infinitesimal generator of $\{T_t^{(\kappa)}\}_{t\geq 0}$ is $L - \kappa I_d$.
In particular, for $\kappa = 1$, we get that the translated Ornstein–Uhlenbeck semigroup $\{T_t^{(1)}\}_{t\geq 0}$ has infinitesimal generator $\overline{L}$, the alternative Ornstein–Uhlenbeck operator (2.15).
Clearly, if $f \geq 0$,

$$T_t^{(\kappa)} f \leq T_t f,$$

for $t \geq 0$. These semigroups and their subordinated semigroups are useful in the study of Littlewood–Paley–Stein functions (see [122]). This is discussed later in Chapter 5.

6. D. Bárcenas, H. Leyva, and W. Urbina in [26] studied the controllability of the following controlled Ornstein–Uhlenbeck equation:

$$z(t) = \frac{1}{2}\Delta z - \langle x, \nabla z \rangle + \sum_{n=1}^{\infty} \sum_{|\nu|=n} u_\nu(t) \big\langle b, \mathbf{h}_\nu \big\rangle_{\gamma_d} \mathbf{h}_\nu, \;\; t > 0, \;\; x \in \mathbb{R}^d, \tag{2.79}$$

where $\mathbf{h}_\nu$ is the normalized Hermite polynomial, $b \in L^2(\gamma_d)$, and the control u is in $L^2(0, t_1; l_2(\gamma_d))$, with $l_2(\gamma_d)$ the Hilbert space of the Fourier–Hermite coefficient,

$$l_2(\gamma_d) = \Big\{ U = \{\{U_\nu\}_{|\nu|=n}\}_{n\geq 1} : \;\; U_\beta \in \mathbb{C}, \;\; \sum_{n=1}^{\infty} \sum_{|\nu|=n} |U_\nu|^2 < \infty \Big\}.$$

Then, if for all $\nu = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathbb{N}^d$

$$\big\langle b, h_\nu \big\rangle_{\gamma_d} = \int_{\mathbb{R}^d} b(x) h_\nu(x) \gamma_d(dx) \neq 0,$$

then the system is approximately controllable on $[0, t_1]$ for some t_1, i.e., for all $z_0, z_1 \in \mathbb{Z}$ and $\varepsilon > 0$, there exists a control $u \in L^2([0, t_1]; l_2(\gamma_d))$ such that the solution $z(t)$ given by (2.79) satisfies

$$\|z(t_1) - z_1\| \leq \varepsilon.$$

Moreover, the system can never be exactly controllable, i.e., there exist $z_0, z_1 \in \mathbb{Z}$ such that for all control $u \in L^2([0, t_1]; l_2(\gamma_d))$ the solution $z(t)$ of (2.79) corresponding to u satisfies $z(t_1) = z_1$. The fact that $\{T_t\}_{t\geq 0}$ is a compact semigroup, proved in Lemma 2.6, is crucial here.

7. The classical *Sobolev inequality* states that for any function $f \in L^2(\mathbb{R}^d)$ with $\nabla_x f \in L^2(\mathbb{R}^d)$, in the weak sense, we have $f \in L^p(\mathbb{R}^d)$ for $\frac{1}{p} = (\frac{1}{2} - \frac{1}{n})$. Thus,

$$||f||_p \leq C_d \int_{\mathbb{R}^d} |\nabla_x f(x)|^2 dx.$$

The logarithmic Sobolev inequality (2.61) generalizes the classical *Sobolev inequality* for the Gaussian measure.
The Gaussian measure can be defined in a space of infinite dimension, unlike the Lebesgue measure, and as the inequality (2.61) is independent of the dimension, it can be extended to the infinite dimensional case. Moreover, observe that in the classical Sobolev inequality $p \to 2$ as $n \to \infty$ and, consequently, there is loss of information in this inequality when the dimension increases toward infinity.

8. It follows from (2.61) that if f and $\nabla_x f \in L^2(\gamma_d)$, then f belongs to the Orlicz space $L^2 \log L(\gamma_d)$. Moreover, it is easy to prove that there exists a function f such that the right hand side of (2.61) is finite, but f does not belong to $L^2 \log L \log\log L(\gamma_d)$ (see [119]). In that sense, the inequality is optimal and the constants are also the best possible.

9. In [7], A. Amenta and J. Teuwen studied $L^p - L^q$ off-diagonal estimates for the Ornstein–Uhlenbeck semigroup. For sufficiently large t (quantified in terms of p and q), these estimates hold in an unrestricted sense. This would suggest potential generalizations to perturbations of the Ornstein–Uhlenbeck operator, whose heat semigroups need not have nice kernels. Moreover, for sufficiently small t, by using direct estimates of Mehler's kernel, it is shown that the estimates fail when restricted to maximal admissible balls and sufficiently small annuli.

10. S. Thangavelu [270], K. Stempak, and J. L. Torrea [259], among several others, have developed an analogous theory for Hermite functions $\{\overline{\Psi}_\nu\}$ in $\mathbb{R}^d$ which are eigenfunctions of the *Hermite operator*

$$H = -\Delta_x + |x|^2,$$

with eigenvalue $\lambda_\nu = -(2|\nu| + d)$.
Then, the *Hermite semigroup* $\{\Upsilon_t = e^{-tH}\}_{t\geq 0}$ can be defined in $L^p(\mathbb{R}^d)$. The Hermite semigroup leads to analogous results in classical harmonic analysis with respect to the Lebesgue measure, which will not be considered here (for more details see, for instance, [259, 267, 268] and [270]).[15]

[15] It is important to observe that the one-dimensional Hermite expansions only converge in L^p-norm for $p = 2$ (see [230]), but expansions in Hermite functions converge in L^p-norm for $\frac{4}{3} < p < 4$.

11. For $\alpha > -1, \beta > -1$, consider the one-dimensional *Jacobi differential operator*, a second-order diffusion operator defined as

$$\mathscr{L}^{\alpha,\beta} = -(1-x^2)\frac{d^2}{dx^2} - (\beta - \alpha - (\alpha+\beta+2)x)\frac{d}{dx}, \tag{2.80}$$

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_k$ can be defined as orthogonal polynomials with respect to the Jacobi (or beta) measure $\mu_{\alpha,\beta}$ in $(-1,1)$

$$\mu_{\alpha,\beta}(dx) = \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\chi_{(-1,1)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx, \tag{2.81}$$

and they are eigenfunctions of $\mathscr{L}^{\alpha,\beta}$ with corresponding eigenvalues $\lambda_n^{\alpha+\beta} = n(n+\alpha+\beta+1)$.

Observe that if we choose $\delta_{\alpha,\beta} = \sqrt{1-x^2}\frac{d}{dx}$, and consider its formal $L^2(\mu_{\alpha,\beta})$-adjoint,

$$\delta^*_{\alpha,\beta} = -\sqrt{1-x^2}\frac{d}{dx} + \Big[(\alpha+\frac{1}{2})\sqrt{\frac{1+x}{1-x}} - (\beta+\frac{1}{2})\sqrt{\frac{1-x}{1+x}}\Big]I,$$

then $\mathscr{L}^{\alpha,\beta} = \delta^*_{\alpha,\beta}\delta_{\alpha,\beta}$. The differential operator $\delta_{\alpha,\beta}$ is considered the "natural" notion of derivative in the Jacobi case.

The square field operator is given by

$$\begin{aligned}
\Gamma^{\alpha,\beta}(f,g)(x) &= \frac{1}{2}\Big[(1-x^2)\frac{d^2(fg)}{dx^2}(x) + (\beta-\alpha+1-(\alpha+\beta+2)x)\frac{d(fg)}{dx}(x) \\
&\quad -(1-x^2)f(x)\frac{d^2g}{dx^2}(x) - (\beta-\alpha+1-(\alpha+\beta+2)x)f(x)\frac{dg}{dx}(x) \\
&\quad -(1-x^2)g(x)\frac{d^2f}{dx^2}(x) - (\beta-\alpha+1-(\alpha+\beta+2)x)g(x)\frac{df}{dx}(x)\Big] \\
&= (1-x^2)\frac{df}{dx}(x)\frac{dg}{dx}(x),
\end{aligned}$$

and

$$\Gamma^{\alpha,\beta}(f)(x) = \Gamma^{\alpha,\beta}(f,f)(x) = (1-x^2)\Big(\frac{df}{dx}(x)\Big)^2. \tag{2.82}$$

Moreover, the iterated square field operator is given by

$$\begin{aligned}
\Gamma_2^{\alpha,\beta}(f,g)(x) &= 2(1-x^2)^2\frac{d^2f}{dx^2}(x)\frac{d^2g}{dx^2}(x) \\
&\quad -2x(1-x^2)\Big(\frac{d^2f}{dx^2}(x)\frac{dg}{dx}(x) + \frac{df}{dx}(x)\frac{d^2g}{dx^2}(x)\Big) \\
&\quad +\Big((1-x^2)(2\alpha+2\beta+3) \\
&\quad -2x(\beta-\alpha+1-(\alpha+\beta+2)x\Big)\frac{df}{dx}(x)\frac{dg}{dx}(x).
\end{aligned}$$

The operator semigroup associated with the Jacobi polynomials can be defined, in $\mathbb{R}$, for positive or bounded measurable Borel functions of $(-1,1)$, as

$$T_t^{\alpha,\beta} f(x) = \int_{-1}^{1} p^{\alpha,\beta}(t,x,y) f(y) \mu_{\alpha,\beta}(dy), \tag{2.83}$$

where

$$p^{\alpha,\beta}(t,x,y) = \sum_k \frac{1}{\hat{h}_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y) e^{-k(k+\alpha+\beta+1)t}, \; x,y \in [-1,1],$$

$t > 0$ and

$$\hat{h}_k^{(\alpha,\beta)} = \frac{1}{(2k+\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}.$$

Different from the cases of the Hermite or Laguerre polynomials, the kernel $p^{\alpha,\beta}(t,x,y)$ does not correspond to the kernel of Abel summability for the Jacobi series because the eigenvalues $\lambda_n^{\alpha,\beta}$ are not n, but $n(n+\alpha+\beta)$, i.e., they are not linearly distributed. W.N. Bailey obtained the following representation for the kernel of Abel summability for the Jacobi series, also called the Jacobi–Poisson integral,

$$\begin{aligned} &\sum_k \frac{1}{\hat{h}_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)}(\cos\theta) P_k^{(\alpha,\beta)}(\cos\phi) r^k \\ &\quad = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{1-r}{(1+r)^{\alpha+\beta+2}} \\ &\quad\quad \times F_4\Big(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \alpha+1, \beta+1; \\ &\quad\quad\quad \Big(\frac{2\sin(\theta/2)\sin(\phi/2)}{r^{1/2}+r^{-1/2}}\Big)^2, \Big(\frac{2\cos(\theta/2)\cos(\phi/2)}{r^{1/2}+r^{-1/2}}\Big)^2\Big), \end{aligned}$$

$|r| < 1$ and $\alpha,\beta > 1$, and F_4 is Appell's hypergeometric function of two variables,

$$F_4(a_1,a_2;b_1,b_2;x,y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}}{(b_1)_m(b_2)_n m!n!} x^m y^n,$$

where $(a)_k$ is the Pochhammer symbol, $(a)_k = a(a+1)\cdots(a+k-1)$. This formula was first stated in 1935 without proof in Bailey's tract [15]. The proof is a consequence of Watson's formula for hypergeometric functions (see [290]), and was published later in [16].

An explicit representation of $p^{\alpha,\beta}(t,x,y)$ was obtained by G. Gasper in 1973 [99, 100], which is an analog of Bailey's formula

$$
\begin{aligned}
p^{\alpha,\beta}(t,x,y) &= \sum_k \frac{1}{\hat{h}_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y) (\lambda_n^{\alpha,\beta})^k \\
&= \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \sum_{n,m=0}^{\infty} \frac{\left(\frac{\alpha+\beta+3}{2}\right)_{m+n}\left(\frac{\alpha+\beta+2}{2}\right)_{m+n}}{m!n!(\alpha+1)_m(\beta+1)_n} \\
&\quad \times [(1-x)(1-y)]^m[(1+x)(1+y)]^n \\
&\quad \times \sum_{k=0}^{\infty} (-1)^m \frac{(2m+2n+\alpha+\beta+1)_k \left(m+n+\frac{\alpha+\beta+3}{2}\right)_k}{k!\left(m+n+\frac{\alpha+\beta+1}{2}\right)_k} e^{-t\lambda_{m+n+k}}.
\end{aligned}
$$

Additionally, in [217], A. Nowak, P. Sjögren, and T. Z. Szarek obtained an integral representation for $p^{\alpha,\beta}(t,x,y)$ valid for all admissible-type parameters $\alpha,\beta > -1$. Finally, in [215, Theorem A], A. Nowak and P. Sjögren, without using an explicit form of $p^{\alpha,\beta}(t,x,y)$, obtained sharp estimates of it, giving the order of magnitude for $\alpha,\beta \geq -1/2$. Previously, only its non-negativity had been proved (see [11, Chapter 2]).

$\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is called the *Jacobi semigroup*, or *Jacobi heat semigroup*, in $\mathbb{R}$, and it can be proved that it is a Markov semigroup (see [213, 214] and references therein). The generalization of the Jacobi operator and the Jacobi semigroup in $\mathbb{R}^d$ is straightforward according to the tensorization argument (see [20, 284]).

Additionally, the Jacobi operator satisfies a *Sobolev inequality*, which implies that it satisfies a tight logarithmic Sobolev inequality; therefore, the Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is hypercontractive, with contraction function

$$q(t) = 1 + (q(0)-1)e^{4t/C}$$

(for details see Bakry's paper [18, page 33–34], [20, 19], or [284]). Moreover, as a consequence of the asymptotic relations among the Jacobi polynomials and the Hermite and Laguerre polynomials (see [262], (5.3.4) and (5.6.3)), from the Sobolev inequality for the Jacobi operator we can obtain the logarithmic Sobolev inequality for the Ornstein–Uhlenbeck and Laguerre operators; see [20, 284].

12. As has been already mentioned, in the Jacobi setting, because of the non-linearity in n of the eigenvalues $\lambda_n^{\alpha+\beta} = n(n+\alpha+\beta+1)$, the Jacobi semigroup does not coincide with the Abel summability for Jacobi expansions, which is an important difference compared with the Hermite and Laguerre cases. The Abel summability for the Jacobi expansions has been studied extensively in the literature (see for instance [41, 46] and [47] and the references therein).

13. For $\alpha > -1$, consider the one-dimensional *Laguerre differential operator*

$$\mathscr{L}^{\alpha} = -x\frac{d^2}{dx^2} - (\alpha+1-x)\frac{d}{dx}. \tag{2.84}$$

The Laguerre polynomials $\{L_n^\alpha\}$ can be defined as orthogonal polynomials associated with the Gamma measure on $(0,\infty)$,

$$\mu_\alpha(dx) = \chi_{(0,\infty)}(x)\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}dx, \tag{2.85}$$

and they are eigenfunctions of $\mathscr{L}^\alpha$ with corresponding eigenvalues $\lambda_k = k$. Observe that if we choose $\delta_\alpha = \sqrt{x}\frac{d}{dx}$, and consider its formal $L^2(\alpha)$-adjoint,

$$\delta_\alpha^* = -\sqrt{x}\frac{d}{dx} + \Big[\frac{\alpha+1/2}{\sqrt{x}} + \sqrt{x}\Big]I$$

then $\mathscr{L}^\alpha = \delta_\alpha^*\delta_\alpha$. The differential operator δ_α is considered the natural notion of derivative in the Laguerre case.

The operator semigroup associated with the Laguerre polynomials can be defined for positive or bounded measurable Borel functions of $(0,\infty)$, as

$$T_t^\alpha f(x) = \int_0^\infty p^\alpha(t,x,y)f(y)\mu_\alpha(dy), \tag{2.86}$$

where, according to the Hille–Hardy formula (10.35),

$$\begin{aligned} p^\alpha(t,x,y) &= \sum_k \frac{\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+1)} L_k^\alpha(x)L_k^\alpha(y)e^{-kt} \\ &= \frac{1}{1-e^{-t}} e^{-\frac{(x+y)e^{-t}}{1-e^{-t}}} (-xye^{-t})^{-\alpha/2} I_\alpha\Big(\frac{2\sqrt{xye^{-t}}}{1-e^{-t}}\Big), \end{aligned}$$

where $I_\alpha(x)$ is the modified Bessel function of the first kind of order α. This identity was found in 1926 by E. Hille [135] and independently rediscovered by G.H. Hardy [130] (see also G.N. Watson [291]).

In this case, the square field operator is given by

$$\begin{aligned} \Gamma^\alpha(f,g)(x) = \frac{1}{2}\Big[&x\frac{d^2(fg)}{dx^2}(x) + (\alpha+1-x)\frac{d(fg)}{dx}(x) - xf(x)\frac{d^2g}{dx^2}(x) \\ &-(\alpha+1-x)f(x)\frac{dg}{dx}(x) - xg(x)\frac{d^2f}{dx^2}(x) - (\alpha+1-x)g(x)\frac{df}{dx}(x)\Big] \\ = x&\frac{df}{dx}(x)\frac{dg}{dx}(x), \end{aligned}$$

and

$$\Gamma^\alpha(f)(x) = \Gamma^\alpha(f,f)(x) = x\Big(\frac{df}{dx}(x)\Big)^2. \tag{2.87}$$

Moreover, the iterated square field operator is given by

$$\Gamma_2^{\alpha}(f,g)(x) = \frac{1}{2}\Big[x\frac{d^2f}{dx^2}(x)\frac{dg}{dx}(x) + x\frac{df}{dx}(x)\frac{d^2g}{dx^2}(x) + 2x^2\frac{d^2f}{dx^2}(x)\frac{d^2g}{dx^2}(x) + (\alpha+1+x)\frac{df}{dx}(x)\frac{dg}{dx}(x)\Big]. \quad (2.88)$$

$\{T_t^{\alpha}\}_{t\geq 0}$ is called the *Laguerre semigroup*, or *Laguerre heat semigroup*. It can be proved that it is a Markov semigroup (see [208] and [193] and the references therein). The generalization of the Laguerre operator and the Laguerre semigroup in $\mathbb{R}^d$ is straightforward according to the tensorization argument (see [20, 284]). Here, again, the semigroup $\{T_t^{\alpha}\}$ coincides with the Abel summability for Laguerre expansions.

Moreover, the Laguerre operator satisfies a tight logarithmic Sobolev inequality; therefore, the Laguerre semigroup $\{T_t^{\alpha}\}$ is hypercontractive, see [20, 284]. The hypercontractivity of the Laguerre semigroup was initially proved by A. Korzeniowski and D. Stroock in [152].

14. The fact that the Jacobi and Laguerre semigroups are hypercontractive allows us to obtain similar applications to those obtained in Section 2.4 for the Ornstein–Uhlenbeck semigroup (see for instance [24, 25, 117, 284], and the references therein).

15. Additionally, an operator semigroup can be defined for the generalized Hermite polynomials $\{H_n^{\mu}\}$, which, as we know, are eigenfunctions of the operator

$$L_{\mu} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{\mu}{x} - x\right)\frac{d}{dx} - \mu\frac{I-\tilde{I}}{2x^2}.$$

Using Mehler's formula (10.46), this semigroup can be written as

$$T_t^{\mu}f(x) = \int_{-\infty}^{\infty} p^{\mu}(t,x,y)f(y)|y|^{2\mu}e^{-|y|^2}dy, \quad (2.89)$$

where

$$p^{\mu}(t,x,y) = \frac{1}{(1-e^{-2t})^{\mu+1/2}}e^{-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}}e_{\mu}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right),$$

for f a positive or bounded measurable function on $(-\infty,\infty)$.
$\{T_t^{\mu}\}_{t\geq 0}$ is called the *generalized Ornstein–Uhlenbeck semigroup*, and it is easy to see that it is also a Markov semigroup with generator L_{μ}; see [20].

The weak type $(1,1)$ inequality, in addition to its L^p-boundedness for $p>1$, with respect to the measure λ of the maximal operator associated with this semigroup, was proved in [30]. Those results were extended to higher dimensions in [92]. Further research into this semigroup and the operators associated with it are particular cases of a more general theory for the Dunkl Ornstein–Uhlenbeck operator (see [212]).

16. For the Hermite, Laguerre, and Jacobi functions in analogous form as above, we can define operator semigroups, usually called *heat diffusion semigroups* (see for instance [270, 259] and the references therein).

17. An unexpected application of the hypercontractivity property of the Ornstein–Uhlenbeck semigroup has been found in several works on non-linear partial differential equations of evolution type (see for instance N. Tzvetkov [277]).

18. The boundedness of the Ornstein–Uhlenbeck semigroup on variable $L^{p(\cdot)}$ Gaussian spaces has been studied in [192] by J. Moreno, E. Pineda, and W. Urbina.