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Preliminary Results: The Gaussian Measure and Hermite Polynomials

In this chapter we study the Gaussian measure in $\mathbb{R}^d$ for $d \geq 1$ and several of its properties. Then, we study the problem of the Gaussian measure for balls in $\mathbb{R}^d$, which is crucial in Chapter 4 for studying the associated covering lemmas for that measure. For completeness, we consider Hermite polynomials, which are orthogonal polynomials, with respect to the Gaussian measure, and discuss in detail most of their properties. The interested reader will find the properties and identities of all classical orthogonal polynomials listed in the appendix.

1.1 The Gaussian Measure

The *Gaussian measure* in $\mathbb{R}$ is given by[1]

$$\gamma_1(dx) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx, \tag{1.1}$$

where e^{-x^2} is called the *Gaussian weight.*

The fact that γ_1 is a probability measure is based on the following famous computation; using polar coordinates and Fubini's theorem,

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2} dx\right)^2 &= \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2} dx\, dy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = 2\pi\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^{\infty} = \pi.\end{aligned}$$

[1] In probability theory, it is usual to consider the standard Gaussian probability, defined as $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. Nevertheless, in the context of the theory of orthogonal polynomials, it is more common to use (1.1) and we are going to follow that normalization (see [262]). The formulas differ only by constants.

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The *Fourier transform* of γ_1 (characteristic function in probability terminology) is given by

$$\begin{aligned}\widehat{\gamma_1}(\xi) &= \int_{\mathbb{R}} e^{-i\xi y}\gamma_1(dy) = \int_{\mathbb{R}} e^{-i\xi y}\frac{e^{-y^2}}{\sqrt{\pi}}dy \\ &= \frac{e^{-\xi^2/4}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-(y+i\xi/2)^2}dy = \frac{e^{-\xi^2/4}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-y^2}dy = e^{-\xi^2/4}; \end{aligned} \tag{1.2}$$

thus, the Gaussian measure is ‘essentially’ (up to a constant) its own Fourier transform. Moreover, that integral is uniformly convergent in any disk $D = \{x : |x| \leq r\}$, $r > 0$ and is bounded in that region. Therefore, according to the dominated convergence theorem, we can differentiate an arbitrary number of times, obtaining,

$$\frac{d^n}{dx^n}e^{-x^2} = \frac{(-2i)^n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-y^2}y^n e^{-2ixy}\,dy. \tag{1.3}$$

The *Gaussian distribution function* Φ is defined as

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-y^2}dy. \tag{1.4}$$

In other words, Φ is just the cumulative distribution function of the measure γ_1. It is well known that, unfortunately, there is not a closed form of it. An important estimate of the rate of decrease of the function $1-\Phi$ can be obtained simply using integration by parts, for $x > 0$,

$$\frac{1}{2\sqrt{\pi}}\left(\frac{1}{x} - \frac{1}{2x^3}\right)e^{-x^2} \leq 1 - \Phi(x) \leq \frac{1}{2\sqrt{\pi}x}e^{-x^2}. \tag{1.5}$$

The *Gaussian measure* in $\mathbb{R}^d$ is defined as the product measure

$$\gamma_d(dx) = \frac{1}{\pi^{d/2}}e^{-|x|^2}dx = \frac{1}{\sqrt{\pi}}e^{-x_1^2}dx_1 \otimes \frac{1}{\sqrt{\pi}}e^{-x_2^2}dx_2 \otimes \cdots \otimes \frac{1}{\sqrt{\pi}}e^{-x_d^2}dx_d. \tag{1.6}$$

Being a product of probability measures, it is clear that γ_d is a probability measure in $\mathbb{R}^d$. On the other hand, γ_d is radially symmetric. There is likely no other non-trivial probability measure that satisfies both properties. From the fact that the Gaussian measure in $\mathbb{R}^d$ is a product measure, a technique called *tensorization* has been developed, which consists in obtaining d-dimensional estimates from those of the one-dimensional estimate.

It is clear that the Gaussian measure is highly concentrated near the origin and decays exponentially at infinity, for all $d \geq 1$. That behavior is very far from the invariance by translation of the Lebesgue measure; therefore, there is a big difference between it and the Lebesgue measure. For instance, any argument in classical analysis that uses the translation invariant property of the Lebesgue measure is totally useless in the Gaussian case. On the other hand, the Gaussian measure is invariant by rotation, so we can take advantage of that property.

Finally, even though in probability theory the Gaussian measures form a whole family of probability measures (with different means and variances), γ_d is the only Gaussian measure considered in this book.[2]

1.2 Estimates for the Gaussian Measure of Balls in $\mathbb{R}^d$ and the Doubling Condition

We need to estimate the Gaussian measure of balls in $\mathbb{R}^d$ to obtain covering lemmas and other estimates, for instance, but this is not trivial at all, because, as we have already said, the Gaussian measure is a probability measure, highly concentrated around the origin, with exponential decay at infinity, invariant by rotation around the origin, and not translation invariant.

First, we consider a partition $\mathtt{P}$ of $\mathbb{R}$, obtained by B. Muckenhoupt, in [194] Lemma 2, because in such a partition there is a seminal idea about how to measure balls (or cubes) using the Gaussian measure.

- First, divide the interval $[0,2]$ into the subintervals $[0,1]$ and $[1,2]$ of length one.
- Then, for $n \geq 1$, divide the interval $[2^n, 2^{n+1})$, into 2^{n+1} intervals of length 2^{-n}.
- Finally, consider the mirror images of these intervals for the interval $(-\infty, 0]$.

Then, the elements of the partition in the complement of $[-2,2]$ are of the form $I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n})$, where $n \geq 1$ and k are integers such that $2^{2n} \leq |k| < 2^{2n+1}$.

Proposition 1.1. *The partition*

$$\mathtt{P} = \{[-2,-1],[-1,0],[0,1],[1,2]\} \cup \{I_{k,n} : n \geq 1,\ 2^{2n} \leq |k| \leq 2^{2n+1}\} \tag{1.7}$$

satisfies the following properties:

i) Any compact subset of $\mathbb{R}$ intersects only a finite number of the intervals of the partition $\mathtt{P}$.

ii) An interval in the partition is no more than twice as long as the adjacent intervals. Furthermore, for any $I \in \mathtt{P}$, if $x \in I$, then $1 \wedge \frac{1}{|x|} = \min\{1, \frac{1}{|x|}\}$ is not greater than the length of the interval.

iii) Finally, there exists a constant C, independent of n and k such that

$$\frac{\sup_{x \in I_{k,n}} e^{-x^2}}{\inf_{x \in I_{k,n}} e^{-x^2}} \leq C.$$

Proof.

i) Immediate, because any compact set is bounded.

[2] For more details on Gaussian measures, see for example the book by V. I. Bogachev ([33, Chapter 1]).

ii) That an interval in the partition is no more than twice as long as the adjacent intervals is clear by construction. Now, if $|x| \leq 1$, then[3]

$$1 \wedge \frac{1}{|x|} = 1 = |[0,1]|.$$

If $|x| > 1$, then for the case $x \in [1,2]$, $1 \wedge \frac{1}{|x|} = \frac{1}{|x|} < 1 = |[1,2]|$. The case $x \in [-2,-1]$ is totally analogous. Now, assuming that $x \in I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n}], n \geq 1$ then, $1 \wedge \frac{1}{|x|} = \frac{1}{|x|}$ and

$$|I_{k,n}| = \frac{1}{2^n} = \frac{2^n}{2^{2n}} \geq \frac{2^n}{k} \geq \frac{1}{|x|}.$$

Moreover,

$$|I_{k,n}| = \frac{1}{2^n} = \frac{2}{2^{n+1}} \leq 2\frac{2^{n-1}}{2^{2n}} \leq 2\frac{2^{n-1}}{k+1} \leq \frac{2}{|x|}.$$

iii) By symmetry, let us consider only intervals in $[0,\infty)$. For the first two intervals $[0,1]$ and $[1,2]$, e^{-y^2} varies by a factor of e^{-1}. Now, if $x \geq 2$, assuming $x \in I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n}]$, then, as e^{-x^2} is a decreasing function,

$$\frac{\sup_{x \in I_{k,n}} e^{-x^2}}{\inf_{x \in I_{k,n}} e^{-x^2}} = e^{\frac{(k+1)^2-k^2}{2^{2n}}} = e^{\frac{2k+1}{2^{2n}}} \leq e^{\frac{2^{2n+2}+1}{2^{2n}}} \leq e^{4+\frac{1}{2^{2n}}} \leq e^5,$$

hence, the inequality holds with $C = e^5$. □

Observe that *iii)* shows a very important characteristic of this partition: the Gaussian weight e^{-x^2} is essentially constant at each interval in the partition P. Thus, the exponential decay is controlled; therefore, at each interval of P, the Gaussian measure is equivalent to the Lebesgue measure. Then, usual estimates using the Lebesgue measure can be made, at least locally, instead of working with the Gaussian measure. This technique was used initially by B. Muckenhoupt to obtain certain estimates for singular operators with respect to the Gaussian measure. This idea, as we are going to see later, is the key to a technique that consists in defining a *local region* and then splitting operators into a local and a global part. This will be discussed in more depth later, in Chapter 4.

We can obtain a partition of $\mathbb{R}^d$ of d-dimensional rectangles having the same properties listed in Lemma 1.1 simply by considering Cartesian products of partitions $\mathtt{P}_i$ in each variable x_i as before. This partition can be refined, splitting the rectangles into cubes. A similar partition of $\mathbb{R}^d$ was considered by P. Sjögren in [247]. Also, a similar idea is considered in the work of J. Mass, J. Van Neerven, and P. Portal in [169] on Whitney decomposition. This is discussed in detail in Chapter 4 as well.

The Gaussian measure of any ball $B(x,r)$ in $\mathbb{R}^d$ can be easily estimated, depending of the center of the ball, by using polar coordinates.

[3]Here we are using the convention that $1 \wedge \frac{1}{0} = 1$.

Lemma 1.2. *Let $r > 0$, and $d \geq 2$.*

i) The Gaussian measure of a ball in $\mathbb{R}^d$ centered at the origin, $B(0,r)$, is bounded above by

$$\gamma_d(B(0,r)) \leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d. \tag{1.8}$$

ii) For any $x \in \mathbb{R}^d$,

$$\gamma_d(B(x,r)) \leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d e^{2r|x|} e^{-|x|^2}, \tag{1.9}$$

where ω_{d-1} is the (surface) measure of the unit (hyper)-sphere S^{d-1} in $\mathbb{R}^d$.

Proof.

i) Using polar coordinates, $y = \rho\xi$, for $\rho > 0$, and $\xi \in S^{d-1}$, i.e., $|y| = \rho|\xi| = \rho$, we have

$$\begin{aligned}\gamma_d(B(0,r)) &= \int_{B(0,r)} \frac{e^{-|y|^2}}{\pi^{d/2}}\, dy = \int_{S^{d-1}} \int_0^r \frac{e^{-\rho^2}}{\pi^{d/2}} \rho^{d-1} d\rho\, d\sigma \\ &\leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^{d-2} \int_0^r e^{-\rho^2} 2\rho\, d\rho = \frac{\omega_{d-1}}{2\pi^{d/2}} r^{d-2}(1-e^{-r^2}) \leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d,\end{aligned}$$

as $1 - e^{-t} \leq t$, for any $t \geq 0$.

ii) For any $x \in \mathbb{R}^d$, as $|y|^2 \leq ((y-x)+x)^2 = |y-x|^2 + 2\langle x, y-x\rangle + |x|^2$,

$$\begin{aligned}\gamma_d(B(x,r)) &= \int_{B(x,r)} \frac{e^{-|y|^2}}{\pi^{d/2}}\, dy = \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{B(x,r)} e^{-|y-x|^2} e^{-2\langle x,y-x\rangle}\, dy \\ &\leq \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{B(x,r)} e^{-|y-x|^2} e^{2|x||y-x|}\, dy \\ &\leq \frac{e^{-|x|^2}}{\pi^{d/2}} e^{2r|x|} \int_{B(x,r)} e^{-|y-x|^2}\, dy = \frac{e^{-|x|^2}}{\pi^{d/2}} e^{2r|x|} \int_{B(0,r)} e^{-|y|^2}\, dy \\ &\leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d e^{2r|x|} e^{-|x|^2}. \qquad \square\end{aligned}$$

To extend the idea in Proposition 1.1 to higher dimensions, we define a family of admissible balls.

Definition 1.3. *The family of admissible balls[4] in $\mathbb{R}^d$, with parameter $a, b > 0$, is defined as*

$$\mathscr{B}_{a,b} = \left\{ B(x,r) : x \in \mathbb{R}^d,\ 0 < r < a \wedge \frac{b}{|x|} \right\}. \tag{1.10}$$

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[4] Admissible balls are sometimes also called *hyperbolic balls*.

In particular, if $a = b$, the family of admissible balls with parameter a is defined as

$$\mathscr{B}_a = \left\{B(x,r) : x \in \mathbb{R}^d,\ 0 < r < a\left(1 \wedge \frac{1}{|x|}\right)\right\}$$
$$= \{B(x,r) : x \in \mathbb{R}^d,\ 0 < r < am(x)\}, \tag{1.11}$$

where

$$m(x) = 1 \wedge \frac{1}{|x|}. \tag{1.12}$$

$m(x)$ is called the admissibility function.

Observe that, trivially, $m(x) \leq 1$ and $m(x) \leq \frac{1}{|x|}$. Observe also that admissible balls need to be very small when their center is far from the origin.

For admissible balls in $\mathscr{B}_{a,b}$ the Gaussian weight $e^{-|y|^2}$ is essentially constant. More precisely, we have the following estimates:

Lemma 1.4. *For $a, b > 0$, if $|x-y| < a \wedge \frac{b}{|x|}$, then*

$$e^{-a^2} e^{-2b} e^{-|x|^2} \leq e^{-|y|^2} \leq e^{2b} e^{-|x|^2}. \tag{1.13}$$

Therefore, for admissible balls, $B = B(x,r) \in \mathscr{B}_{a,b}$, their Gaussian measures can be estimated as:

$$\gamma_d(B) = \frac{1}{\pi^{d/2}} \int_B e^{-|y|^2} dy \sim C_d e^{-|x|^2} \left(a \wedge \frac{b}{|x|}\right)^d. \tag{1.14}$$

In particular, for $a > 0$, if $|x-y| < a\left(1 \wedge \frac{1}{|x|}\right) = am(x)$, then

$$e^{-a^2} e^{-2a} e^{-|x|^2} \leq e^{-|y|^2} \leq e^{2a} e^{-|x|^2}; \tag{1.15}$$

therefore, if $B = B(x,r) \in \mathscr{B}_a$,

$$\gamma_d(B) = \frac{1}{\pi^{d/2}} \int_B e^{-|y|^2} dy \sim C_d e^{-|x|^2} a^d \left(1 \wedge \frac{1}{|x|}\right)^d = C_d e^{-|x|^2} a^d m(x)^d. \tag{1.16}$$

Thus, for admissible balls, their Gaussian measure is essentially a multiple (which depends on the center) of their Lebesgue measure.

Proof. Simply by triangle inequality,

$$e^{-|y|^2} = e^{-|x-(x-y)|^2} \leq e^{-|x|^2} e^{2|x||x-y|} e^{-|x-y|^2} \leq e^{2b} e^{-|x|^2},$$

and

$$e^{-|y|^2} = e^{-|x+(y-x)|^2} \geq e^{-|x|^2} e^{-2|x||y-x|} e^{-|y-x|^2} \geq e^{-a^2} e^{-2b} e^{-|x|^2}. \qquad \square$$

On the other hand, J. Maas, J. van Neerven, and P. Portal [169] obtained the following lemma, using an idea similar to the one contained in Lemma 1.4.

Lemma 1.5. *Let $a, A > 0$ be given.*

i) If $|x-y| < At$ and $t \le am(x)$, then $t \le a(1+aA)m(y)$.
ii) If $|x-y| < Am(x)$, then $m(x) \le (1+A)m(y)$ and $m(y) \le 2(1+A)m(x)$.

Proof.

i) We have three cases:

- If $|y| \le 1$, then $m(y) = 1$, and

$$t \le am(x) \le a = am(y) \le a(1+aA)m(y).$$

- If $1 < |y| \le 1+aA$, then $m(y) \ge 1/(1+aA)$ and

$$t \le am(x) \le a \le a(1+aA)m(y).$$

- If $|y| > 1+aA > 1$, then $m(y) = \frac{1}{|y|}$ and

$$t \le am(x) \le \frac{a}{|x|} \le \frac{a}{|y|-At} \le \frac{a}{|y|-aA} \le \frac{a(1+aA)}{|y|} = a(1+aA)m(y).$$

ii) Put $t' = m(x)$. Then $|x-y| < At'$; therefore, we can apply *i*) with $a = 1$ to get that $t' \le (1+A)m(y)$. This gives the first estimate. To obtain the second one we consider three cases:

- If $|x| \le 1$, then $2(1+A)m(x) \ge 1 \ge m(y)$.
- If $|1 \le |x| \le 2A$, (i.e., $A \ge 1/2$) then

$$2(1+A)m(x) \ge \frac{2(1+A)}{2A} \ge 1 \ge m(y).$$

- If $|x| \ge 1$ and $|x| \ge 2A$, then $|y| \ge |x| - \frac{A}{|x|} \ge |x| - \frac{1}{2} \ge \frac{|x|}{2}$; thus,

$$m(y) \le 2m(x) \le 2(1+A)m(x). \qquad \square$$

Part *i*) of Lemma 1.5 says, among other things, that if we have $B(x,r) \in \mathscr{B}_a$ and if $|x-y| < Ar$, then $B(y,r) \in \mathscr{B}_c$ for some constant $c = c_{a,A}$, which depends only on a and A. Additionally, using part *ii*), we get the following estimate, similar to (1.13): if $|x-y| < am(x)$,

$$e^{-a^2} e^{-2a} e^{-|x|^2} \le e^{-|y|^2} \le e^{a^2(1+a)^2} e^{2a(1+a)} e^{-|x|^2}, \tag{1.17}$$

because, as $|x|m(x) \le 1$, we have

$$|y|^2 \le (|x| + |x-y|)^2 \le (|x| + am(x))^2 \le |x|^2 + 2a + a^2,$$

and, as $m(x) \le (1+a)m(y)$,

$$\begin{aligned}|x|^2 &\le (|y| + |x-y|)^2 \le (|y| + am(x))^2 \\ &\le (|y| + a(1+a)m(y))^2 \le |y|^2 + 2a(1+a) + a^2(1+a)^2\end{aligned}$$

(see J. Teuwen [265, Lemma 2], see also G. Mauceri, S. Meda [174, Lemma 2.1 i)]).

The main results of classical harmonic analysis in $\mathbb{R}^d$, which are done with respect to the Lebesgue measure, were later extended for other classes of measures. The initial and probably the most important one is the class of doubling measures. Recall that a Borel measure μ in $\mathbb{R}^d$ is a *doubling measure* if a constant $C > 0$ exists, depending only on the dimension d, such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)), \tag{1.18}$$

for any $x \in \mathbb{R}^d$ and $r > 0$.

The meaning of this condition is that the mass that μ gives to the annulus $2B \setminus B$ is controlled by a constant times the mass of B. The opposite of that means that $\mu(B)$ is much less than $\mu(2B \setminus B)$, and therefore that μ rarefies at B. All the classical notions of harmonic analysis can be extended almost immediately to doubling measures (see for instance [254] or [275]).

As the Gaussian measure γ_d is a probability measure, it is not a doubling measure, for more details see Appendix 10.3. Thus, there is no constant $C > 0$, independent of $x \in \mathbb{R}^d$, and $r > 0$ such that

$$\gamma_d(B(x,2r)) \leq C\gamma_d(B(x,r)),$$

for all $x \in \mathbb{R}^d$ and $r > 0$, i.e., the doubling condition does not hold for all possible balls in $\mathbb{R}^d$. Therefore the classical results of harmonic analysis cannot be extended directly to the case of Gaussian harmonic analysis. Nevertheless, G. Mauceri and S. Meda in a seminal paper [174] observed that if we control the radius appropriately, the Gaussian measure is doubling; more precisely, the Gaussian measure is doubling if we restrict it to the family of admissible balls $\mathscr{B}_a$. Thus, we can adapt the classical arguments, at least in some regions. The doubling condition for the Gaussian measure is therefore a local condition, and is a contained in the following result (see G. Mauceri and S. Meda's paper [174, Proposition 2.1]).

Theorem 1.6. *Let $a,\tau > 0$. For each ball $B = B(c_B, r_B) \in \mathscr{B}_a$, consider the set B^*_τ which is the union of all balls $B' = B(c_{B'}, r_{B'})$, which intersects B and such that $r_{B'} \leq \tau r_B$, then the following inequalities hold.*

i) If $\sigma^*_{a,\tau} = \sup_{B\in\mathscr{B}_a} \frac{\gamma_d(B^*_\tau)}{\gamma_d(B)}$ then

$$\sigma^*_{a,\tau} \leq (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}. \tag{1.19}$$

ii) (Doubling property) There exists a constant $C = C_{a,\tau,d} > 1$ depending only on a,τ and the dimension d, such that for any ball $B' = B(x_{B'}, r_{B'})$ having a non-empty intersection with B and such that $r_{B'} \leq \tau r_B$, then

$$\gamma_d(B') \leq C\gamma_d(B).$$

In particular, this implies that there exists a constant $C = C_d > 1$ *such that for all* $\tau > 1$ *and all* $B = B(x_B, r_B) \in \mathscr{B}_a$

$$\gamma_d(B(x_B, \tau r_B)) \leq C\gamma_d(B(x_B, r_B)). \tag{1.20}$$

Proof.

i) First of all, observe that $B_\tau^* \subset B(c_B, (2\tau+1)r_B) \in \mathscr{B}_{(2\tau+1)a}$ and therefore, using both sides of inequality (1.15) with the parameter $(2\tau+1)a$, we get

$$\begin{aligned}\gamma_d(B_\tau^*) &= \frac{1}{\pi^{d/2}}\int_{B_\tau^*} e^{-|y|^2}dy \leq \frac{1}{\pi^{d/2}}e^{2(2\tau+1)a}e^{-|c_B|^2}|B_\tau^*| \\ &\leq \frac{1}{\pi^{d/2}}e^{2(2\tau+1)a}e^{-|c_B|^2}|B(c_B,(2\tau+1)r_B)| \\ &= \frac{1}{\pi^{d/2}}e^{2(2\tau+1)a}e^{-|c_B|^2}(2\tau+1)^d|B(c_B,r_B)|,\end{aligned}$$

and

$$\gamma_d(B) \geq \frac{1}{\pi^{d/2}}e^{-(2\tau+1)^2a^2}e^{-2(2\tau+1)a}e^{-|c_B|^2}|B(c_B,r_B)|.$$

Thus,

$$\sigma_{a,\tau}^* \leq (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}.$$

ii) As B' is one of the terms in the union that forms B_τ^* then *ii*) follows immediately from *i*) as

$$\begin{aligned}\gamma_d(B') \leq \gamma_d(B_\tau^*) &= \frac{\gamma_d(B_\tau^*)}{\gamma_d(B)}\gamma_d(B) \leq \sigma_{a,\tau}^*\gamma_d(B) \\ &\leq (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}\gamma_d(B) = C\gamma_d(B),\end{aligned}$$

with $C = (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}$. This estimate of C could be improved as $B' \subset B(c_B, (2\tau+1)r_B)$, using inequality (1.15) with the parameter $(2\tau+1)a$. □

J. Maas, J. van Neerven, and P. Portal proved, in [168], that there is also a family of cubes in $\mathbb{R}^d$ such that the Gaussian measure is a doubling measure on them (see Lemma 1.17).

Observe that, because the Gaussian measure is not a doubling measure, the measure space $(\mathbb{R}^d, |\cdot|, \gamma_d)$ is not a space of homogeneous type; thus, there is no overlap between Gaussian harmonic analysis and harmonic analysis of spaces of homogeneous type.

Moreover, the Gaussian measure is trivially a d-dimensional measure[5] in $\mathbb{R}^d$, because, for any $x \in \mathbb{R}^d$, and $r > 0$,

$$\gamma_d(B(x,r)) = \frac{1}{\pi^{d/2}} \int_{B(x,r)} e^{-|x|^2} dx \leq \frac{1}{\pi^{d/2}} |B(x,r)| = C_d r^d.$$

As we have mentioned before, classical harmonic analysis, which was extended initially to doubling measures, has been extended to the case of *s-dimensional measures* (see, for instance, Tolsa [274]). Nevertheless, Gaussian harmonic analysis is not part of that theory because, as was mentioned before, there is another component of it, which is the Ornstein–Uhlenbeck and associated operators.

Going back to the problem of the Gaussian measure of balls, we can still get an estimate for the Gaussian measure of non-admissible balls if they do not contain the origin. That estimate was obtained by L. Forzani in [83], but in this case the estimate does not depend upon the center of the ball but rather on the closest point to the origin.

Proposition 1.7. *(Forzani) Let B a ball in $\mathbb{R}^d$, with radius $r > 0$, which does not contain the origin, and let x_0 denote the point of B whose distance to the origin is minimal, i.e., $d(B,0) = |x_0|$. Then, there exists a constant $C_d > 0$, depending only on the dimension d, such that*

$$\gamma_d(B) \leq C_d \frac{e^{-|x_0|^2}}{|x_0|} \left(\frac{r}{|x_0|} \right)^{(d-1)/2}. \tag{1.21}$$

Moreover, if $r_B > \frac{C}{|x_0|}$, $C > 1$, the opposite inequality is also true; therefore,

$$\gamma_d(B) \sim C_d \frac{e^{-|x_0|^2}}{|x_0|} \left(\frac{r}{|x_0|} \right)^{(d-1)/2}. \tag{1.22}$$

Proof. Let us write $B = B(x,r)$. It is enough to consider that $|x_0| > 1$ because otherwise a constant would do a better job than the estimate. Consider Π_0 the hyperplane orthogonal to x_0 whose distance to the origin is precisely $|x_0|$, that is,

$$\Pi_0 = \{x \in \mathbb{R}^d : \langle x, x_0 \rangle = |x_0|^2\},$$

and consider the hyperspace $\Pi_0^+ = \{x \in \mathbb{R}^d : \langle x, x_0 \rangle > |x_0|^2\}$. Then, any $y \in \Pi_0^+$ can be written as $y = (\xi + |x_0|)\frac{x_0}{|x_0|} + \nu$, with $\xi > 0$ and $\langle \nu, x_0 \rangle = 0$. In particular, we have $x = (r + |x_0|)\frac{x_0}{|x_0|}$.

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[5] A Borel measure is s-dimensional in $\mathbb{R}^d$ if it satisfies the following growth condition:

$$\mu(B(x,r)) \leq C r^s,$$

for some constant C and for all $x \in \mathbb{R}^d$, and $r > 0$.

Observe that $y \in B$ if and only if $\xi \in (0,2r)$ and $|v| < \sqrt{2r\xi - \xi^2}$, because $\max_{y\in B}|y| = 2r + |x_0|$, and according to the Pythagorean theorem,

$$\begin{aligned}|y-x|^2 &= |(\xi + |x_0|)\frac{x_0}{|x_0|} + v - (r+|x_0|)\frac{x_0}{|x_0|}|^2 = |(\xi - r)\frac{x_0}{|x_0|} + v|^2\\ &= (\xi - r)^2 + |v|^2 = \xi^2 - 2\xi r + r^2 + |v|^2 < r^2.\end{aligned}$$

Then, we have

$$\begin{aligned}\gamma_d(B) &= \frac{1}{\pi^{d/2}}\int_B e^{-|y|^2}dy\\ &= C_d e^{-|x_0|^2}\int_0^{2r} e^{-2\xi|x_0|}\, e^{-\xi^2}\Big(\int_{\{v\in\mathbb{R}^{d-1}:\,|v|<\sqrt{2r\xi-\xi^2}\}} e^{-|v|^2}\,dv\Big)\,d\xi\\ &\le C_d e^{-|x_0|^2}\int_0^{2r} e^{-2\xi|x_0|}\,(2r\xi - \xi^2)^{(d-1)/2}\,d\xi\\ &\le C_d e^{-|x_0|^2} r^{(d-1)/2}\int_0^{2r} e^{-2\xi|x_0|}(2\xi)^{(d-1)/2}d\xi\\ &\le C_d\frac{e^{-|x_0|^2}}{|x_0|}\Big(\frac{r}{|x_0|}\Big)^{(d-1)/2}\int_0^{4r|x_0|} e^{-t}t^{(d-1)/2}dt \le C_d\frac{e^{-|x_0|^2}}{|x_0|}\Big(\frac{r}{|x_0|}\Big)^{(d-1)/2}.\end{aligned}$$

Now, if $r > \frac{C}{|x_0|}$, $C > 1$, let us define

$$R(x_0,r) = \Big\{y = (\xi + |x_0|)\frac{x_0}{|x_0|} + v : \xi \in \Big[\frac{1}{2|x_0|},\frac{1}{|x_0|}\Big], \langle v, x_0\rangle = 0, |v| < \frac{1}{2}\sqrt{\frac{r}{|x_0|}}\Big\}.$$

We will prove that $R(x_0,r)\subset B$. Given $y\in R(x_0,r)$ it is enough to prove that if $\xi\in[\frac{1}{2|x_0|},\frac{1}{|x_0|}]$ then $2r\xi - \xi^2 > \frac{r}{4|x_0|}$ because, in that case,

$$\sqrt{2r\xi - \xi^2} > \frac{1}{2}\sqrt{\frac{r}{|x_0|}} > |v|.$$

Observe that the expression $2r\xi - \xi^2$, as a function of ξ, in the interval $\Big[\frac{1}{2|x_0|},\frac{1}{|x_0|}\Big]$ attains its minimum at $\frac{1}{2|x_0|}$, and as $\frac{1}{|x_0|} < \frac{r}{C} < r$, we get

$$2r\xi - \xi^2 \ge \frac{r}{|x_0|} - \frac{1}{2|x_0|^2} > \frac{r}{4|x_0|},$$

and, clearly, $\xi\in(0,2r)$. Now, if $y\in R(x_0,r)$

$$|y|^2 = \xi^2 + 2\xi|x_0| + |x_0|^2 + |v|^2 < \frac{1}{|x_0|^2} + 2 + |x_0|^2 + \frac{r}{4|x_0|} < |x_0|^2 + \tilde{C}.$$

Hence, $e^{-|y|^2} \ge e^{-\tilde{C}}e^{-|x_0|^2}$ and therefore

$$\gamma_d(B) \ge \gamma_d(R(x_0,r)) = \frac{1}{\pi^{d/2}}\int_{R(x_0,r)} e^{-|y|^2}dy \ge e^{-\tilde{C}}\,\frac{e^{-|x_0|^2}}{|x_0|}\Big(\frac{r}{|x_0|}\Big)^{(d-1)/2}. \qquad \square$$

Another version of inequality (1.21) (see [83, Lemma 4.3]) is the following. There exists a constant C depending on d such that for all $x \in \mathbb{R}^d \setminus \{0\}$, $r \in (1/2, 1)$ and $s \in (0, 1/2)$ the following inequality holds:

$$\gamma_d\left(B\left(\frac{x}{r}, \frac{|x|}{r}s\right)\right) \leq C\, s^{(d-1)/2} \exp\left(-\frac{|x|^2}{r^2}(1-s)^2\right)\frac{1}{|x|}. \tag{1.23}$$

This follows immediately from (1.21) taking $\frac{|x|}{r}s$ as the radius, and then $x_0 = \frac{x}{r} - \frac{x}{r}s = \frac{x}{r}(1-s)$.

As we see in Chapter 4, Lemma 4.16, a similar estimate, can be used to prove the $L^p(\gamma_d)$ boundedness, $1 < p < \infty$, for the non-centered Hardy–Littlewood maximal function with respect to the Gaussian measure, obtained in [90]. Moreover, we see in Chapter 4 how these estimates of the Gaussian measure of balls are important in the proof of some covering lemmas.

1.3 Hermite Polynomials

Hermite Polynomials in One Variable

For completeness, we study in detail the *Hermite polynomials.*. Additionally, in Appendix B, we list the properties for all classical orthogonal polynomials. The standard reference in orthogonal polynomial theory is G. Szegő [262].

The *Hermite polynomials* in $\mathbb{R}$, $\{H_n\}_{n\geq 0}$, can be defined (up to a multiplicative constant) as the orthogonal polynomials associated with the Gaussian measure γ_1. Therefore, they are obtained from the canonical polynomial (monomials) base

$$\{1, x, x^2, \cdots, x^n, \cdots\}$$

by using the Gram–Schmidt method, with respect to the inner product in $L^2(\gamma_1)$[6] (see G. Szegő [262] and E. Hille [134]). Thus, if $m \neq n$

$$\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, d\gamma_1(x) = 0. \tag{1.24}$$

The Gram–Schmidt method determines the polynomials up to a constant; thus, for normalization we set

$$\int_{-\infty}^{+\infty} [H_n(x)]^2\, d\gamma_1(x) = 2^n n!. \tag{1.25}$$

[6] In probability theory, another family of Hermite polynomials is used, which is orthogonal with respect to the standard Gaussian measure $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Observe that by using the Gram–Schmidt method, given $n \in \mathbb{N}$,

$$\int_{-\infty}^{+\infty} P(x)\, H_n(x)\, d\gamma_1(x) = 0, \tag{1.26}$$

for any polynomial P such that $\deg(P) \leq n-1$.

Even though this definition is probably the most straightforward, it is not the easiest to handle as it gives us a recursive formula for H_n, but not an explicit expression.

Alternatively, the Hermite polynomials can be defined using Rodrigues' formula:

$$H_0(x) = 1 \tag{1.27}$$

and for $n > 1$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{1.28}$$

One of the advantages of this definition is precisely that it is easy to get explicit expressions of H_n, because the formula itself is not difficult to handle. Observe that according to (1.28), we get the first polynomials easily.

$$\begin{aligned}
H_1(x) &= (-1)^1 e^{x^2} \frac{d}{dx}(e^{-x^2}) = -e^{x^2}(-2x)e^{-x^2} = 2x,\\
H_2(x) &= (-1)^2 e^{x^2} \frac{d^2}{dx^2}(e^{-x^2}) = e^{x^2}(4x^2-2)e^{-x^2} = 4x^2-2,\\
H_3(x) &= (-1)^3 e^{x^2} \frac{d^3}{dx^3}(e^{-x^2}) = -e^{x^2}(-8x^3+12x)e^{-x^2} = 8x^3-12x,\\
H_4(x) &= (-1)^4 e^{x^2} \frac{d^4}{dx^4}(e^{-x^2}) = e^{x^2}(16x^4-48x^2+12)e^{-x^2} = 16x^4-48x^2+12.
\end{aligned}$$

Then, we can prove, by induction, that

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!} \frac{(2x)^{n-2k}}{(n-2k)!}. \tag{1.29}$$

where $[n/2]$ is the integer part of $n/2$, i.e., the largest integer not greater that $n/2$. Nevertheless, we provide a simpler proof of this formula later, using the generating function (see Proposition 1.9).

Also, from (1.3), using Rodrigues' formula, we get the following *integral representation* of H_n,

$$H_n(x) = \frac{(-2i)^n e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2ixy} y^n e^{-y^2}\, dy. \tag{1.30}$$

Let us prove that Rodrigues' formula actually gives the same polynomials as those obtained using the Gram–Schmidt method. To do so, we need to prove that those polynomials are orthogonal with respect to the Gaussian measure, i.e., that they satisfy (1.24) and to the normalization condition (1.25).

First of all, observe that trivially, because $H_0(x) = 1$,

$$\int_{-\infty}^{\infty} H_0(y)\,\gamma_1(dy) = 1,$$

as γ_1 is a probability measure. Moreover, if $n \geq 1$, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} H_n(x)H_0(x)\,d\gamma_1(x) &= \int_{-\infty}^{+\infty} H_n(x)\,d\gamma_1(x) = (-1)^n \int_{-\infty}^{+\infty} e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \frac{1}{\sqrt{\pi}} e^{-x^2} dx \\ &= \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n}{dx^n}(e^{-x^2})\,dx = 0,\end{aligned}$$

simply by integrating by parts.

Now, we need to consider the case $n, m \geq 1$, $m \neq n$. Without loss of generality, assume that $n > m > 0$. Then, using Rodrigues' formula,

$$\begin{aligned}\int_{-\infty}^{+\infty} H_m(x)H_n(x)\,d\gamma_1(x) &= \int_{-\infty}^{+\infty} H_m(x)(-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})\,\frac{1}{\sqrt{\pi}} e^{-x^2} dx \\ &= \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} H_m(x) \frac{d^n}{dx^n}(e^{-x^2})\,dx \\ &= \frac{(-1)^{2n}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n}{dx^n}(H_m(x)) e^{-x^2}\,dx = 0,\end{aligned}$$

by integrating by parts n times, as $n > m$. For the case $n = m$, first observe that, from the explicit expression of H_n (1.29),

$$H_n^{(n)}(x) = 2^n n!, \tag{1.31}$$

then, integrating by parts n times

$$\begin{aligned}\int_{-\infty}^{+\infty} [H_n(x)]^2\,d\gamma_1(x) &= (-1)^n \int_{-\infty}^{+\infty} [e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})]H_n(x)\,\frac{1}{\sqrt{\pi}} e^{-x^2} dx \\ &= \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n}{dx^n}(e^{-x^2})H_n(x)\,dx = \frac{(-1)^{2n}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n H_n(x)}{dx^n}\,e^{-x^2} dx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} 2^n n!\,e^{-x^2}\,dx = 2^n n!.\end{aligned}$$

Hence, as we claimed, the Gram–Schmidt method and Rodrigues' formula give rise to the same family of orthogonal polynomials.

The Hermite polynomials have a simple generating function, as we see in the following proposition.

Proposition 1.8. *The generating function*[7] *of the Hermite polynomials is given by*

$$G(x,y) = e^{2xy-y^2} = e^{-(x-y)^2+x^2}, \tag{1.32}$$

i.e., $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n = e^{2xy-y^2} = e^{-(x-y)^2+x^2}$.

Proof. Observe that from (1.30) and (1.2) we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n &= \sum_{n=0}^{\infty} \frac{(-2i)^n e^{x^2}}{n!\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2} r^n e^{2ixr}\, dr\, y^n = \frac{e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2} \sum_{n=0}^{\infty} \frac{(-2iry)^n}{n!} e^{2ixr}\, dr \\ &= \frac{e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2} e^{2i(x-y)r}\, dr = e^{x^2} e^{-(x-y)^2} = e^{2xy-y^2} = G(x,y). \qquad \square\end{aligned}$$

Moreover, the Hermite polynomials are the only polynomials that satisfy that relation; hence, they can also be defined using $G(x,y)$ as follows:

$$H_n(x) = \frac{\partial^n}{\partial y^n} G(x,y)|_{y=0} = \frac{\partial^n}{\partial y^n}(e^{2xy-y^2})|_{y=0} = e^{-(x-y)^2+x^2}|_{y=0}. \tag{1.33}$$

Hence, using (1.33) we may easily obtain the first five Hermite polynomials:

$$\begin{aligned}H_0(x) &= G(x,y)|_{y=0} = 1, \\ H_1(x) &= \frac{\partial}{\partial y} G(x,y)|_{y=0} = 2(x-y)G(x,y)|_{y=0} = 2x, \\ H_2(x) &= \frac{\partial^2}{\partial y^2} G(x,y)|_{y=0} = (4(x-y)^2-2)G(x,y)|_{y=0} = 4x^2-2 \\ H_3(x) &= \frac{\partial^3}{\partial y^3} G(x,y)|_{y=0} = (8(x-y)^3-12(x-y))G(x,y)|_{y=0} = 8x^3-12x, \\ H_4(x) &= \frac{\partial^4}{\partial y^4} G(x,y)|_{y=0} = (16(x-y)^4-48(x-y)^2+12)G(x,y)|_{y=0} = 16x^4-48x^2+12.\end{aligned}$$

From Rodrigues' formula we directly obtain (1.33):

$$\begin{aligned}H_n(x) &= (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n}(e^{-(x-y)^2})|_{y=0} \\ &= e^{x^2} \frac{\partial^n}{\partial y^n}(e^{-(x-y)^2})|_{y=0} = \frac{\partial^n}{\partial y^n}(e^{2xy-y^2})|_{y=0} = \frac{\partial^n}{\partial y^n} G(x,y)|_{y=0}.\end{aligned}$$

[7] The *generating function* of a family of orthogonal polynomials $\{P_n\}$ is a function $G(x,y)$ such that $\{P_n(x)\}$ are the coefficients of the Taylor expansion of $G(\cdot,y)$ around $y=0$.

Additionally, $G(x,y)$ can be extended analytically as

$$G(x,z)=e^{2xz-z^2},$$

for $x,z\in\mathbb{C}$. Then, from (1.33), using Cauchy's integral formula, we get the following integral representation of H_n,

$$H_n(x) = \left.\frac{\partial^n G(x,z)}{\partial z^n}\right|_{z=0} = \frac{n!}{2\pi i}\oint_C \frac{G(x,\zeta)}{\zeta^{n+1}}d\zeta = \frac{n!}{2\pi i}\oint_C \frac{e^{2x\zeta-\zeta^2}}{\zeta^{n+1}}d\zeta, \quad (1.34)$$

where C is any curve around the origin.

Now, let us prove the following properties of Hermite polynomials.

Proposition 1.9. *For any $n\geq 1$, $H_n(x)$ satisfies the following properties:*

i) Recursive relation:

$$H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x)=0. \quad (1.35)$$

ii) Derivative:

$$H_n'(x)=2nH_{n-1}(x). \quad (1.36)$$

iii) Differential equation:

$$H_n''(x)-2xH_n'(x)+2nH_n(x)=0. \quad (1.37)$$

Thus, the n-th Hermite polynomial is a polynomial solution of the Hermite equation *with parameter n, i.e., the Hermite polynomials are polynomial solutions of the Hermite equation, or equivalently H_n is an eigenfunction of the one-dimensional* Ornstein–Uhlenbeck operator,[8] $L=\frac{1}{2}\frac{d^2}{dx}-x\frac{d}{dx}$*, with eigenvalue $-n$, that is,*

$$LH_n(x)=\frac{1}{2}\frac{d^2}{dx}H_n(x)-x\frac{d}{dx}H_n(x)=-nH_n(x). \quad (1.38)$$

iv)

$$H_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(y)(2x)^k=\sum_{k=0}^{n}\binom{n}{k}H_k(y)(2x)^{n-k}. \quad (1.39)$$

v) Explicit formula:

$$H_n(x)=n!\sum_{k=0}^{[n/2]}\frac{(-1)^k}{k!}\frac{(2x)^{n-2k}}{(n-2k)!}. \quad (1.40)$$

[8] It is also known as the *harmonic oscillator operator*. Its generalization to $\mathbb{R}^d$ is considered in detail in Section 2.1 of Chapter 2.

vi) Mehler's formula[9]*:*

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} r^n = \frac{1}{(1-r^2)^{1/2}} e^{-\frac{r^2(y^2+x^2)-2rxy}{1-r^2}}, \ |r|<1. \tag{1.41}$$

Mehler's formula allows us to express the Abel summability of Hermite series in integral form.[10]

Proof.

i) Observe that the generating function $G(x,y)$ satisfies the differential equation

$$\frac{\partial G}{\partial y} - 2(x-y)G = 0.$$

Then, substituting in (1.32) the Taylor series of $G(x,y)$, we get

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \frac{H_n(x)}{n!} n y^{n-1} - (2x-2y) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n \\ &= \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} y^{n-1} - 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n + 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^{n+1} \\ &= \sum_{n=0}^{\infty} [H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x)] \frac{y^n}{n!}. \end{aligned}$$

Equating term by term, we get the two-term recurrent relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$

for each $n \geq 1$.

ii) Observe that the generating function $G(x,y)$ also satisfies the following differential equation

$$\frac{\partial G}{\partial x} - 2yG = 0.$$

Again, substituting (1.32) the Taylor series of $G(x,y)$, we get

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} y^n - 2y \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} y^n - 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^{n+1} \\ &= \sum_{n=0}^{\infty} [H_n'(x) - 2nH_{n-1}(x)] \frac{y^n}{n!}. \end{aligned}$$

Equating term by term, we get (1.36).

[9] This formula was found by F. G. Mehler in 1866 [183] and, according to E. A. Hille, *"rediscovered by almost everybody working in the field"* (see [134]).

[10] For more on this, see the definition of the Ornstein–Uhlenbeck semigroup in Chapter 2.

iii) Using (1.36) to eliminate $H_{n-1}(x)$ from the recursive relation (1.35), we get

$$H_{n+1}(x) - 2xH_n(x) + H_n'(x) = 0.$$

By differentiating, and using (1.36), we obtain

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

iv) To prove the result, we need to use *Cauchy's product.*[11] Then

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(x+y)\frac{r^n}{n!} &= e^{2(x+y)r-r^2} = e^{2xr}e^{2yr-r^2} = \sum_{n=0}^{\infty}\frac{(2xr)^n}{n!}\sum_{n=0}^{\infty}H_n(y)\frac{r^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2xr)^k}{k!}H_{n-k}(y)\ \frac{r^{n-k}}{(n-k)} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(y)(2x)^k\right)\frac{r^n}{n!}.\end{aligned}$$

Equating the coefficients, (1.39) follows.

v) Taking $y = 0$ in (1.39), we get

$$H_n(x) = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}(0)(2x)^k = n!\sum_{k=0}^{n}\frac{1}{k!(n-k)!}H_k(0)(2x)^{n-k}. \qquad (1.42)$$

Now, taking $x = 0$ in (1.32), we get

$$e^{-y^2} = \sum_{k=0}^{\infty}H_k(0)\frac{y^k}{k!}.$$

But as

$$e^{-y^2} = \sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{k!}\frac{y^{2k}}{(2k)!},$$

we can conclude that

$$H_{2k+1}(0) = 0 \text{ and } H_{2k}(0) = (-1)^k\frac{(2k)!}{k!}, \qquad (1.43)$$

for any $k \leq [n/2]$. Therefore, from (1.42),

[11] Recall that given two convergent series, $\sum a_n$ and $\sum b_n$, if at least one is absolutely convergent, then its *Cauchy product* is defined as $\sum c_n$, where $c_n = \sum_{k=0}^{n} a_k\ b_{n-k}$, and it is also absolutely convergent and its sum the product of the two series.

$$H_n(x) = n!\sum_{k=0}^{n}\frac{1}{k!(n-k)!}H_k(0)(2x)^{n-k} = n!\sum_{k=0}^{[n/2]}\frac{1}{(2k)!(n-2k)!}H_{2k}(0)(2x)^{n-2k}$$
$$= n!\sum_{k=0}^{[n/2]}\frac{(-1)^k}{k!}\frac{(2x)^{n-2k}}{(n-2k)!}.$$

Hence, (1.40) follows.[12]

vi) Observe that by the properties of the Gaussian measure, we get the following identity

$$\int_{-\infty}^{\infty} e^{-a^2x^2-2bx}dx = \frac{\sqrt{\pi}}{a}e^{b^2/a^2}, \tag{1.44}$$

because, by completing the square

$$\int_{-\infty}^{\infty} e^{-a^2x^2-2bx}dx = e^{b^2/a^2}\int_{-\infty}^{\infty} e^{-a^2x^2-2bx-b^2/a^2}dx = e^{b^2/a^2}\int_{-\infty}^{\infty} e^{-(ax+b/a)^2}dx$$
$$= \frac{e^{b^2/a^2}}{a}\int_{-\infty}^{\infty} e^{-y^2}dy = \frac{\sqrt{\pi}}{a}e^{b^2/a^2}.$$

Using the integral representation (1.30) and (1.44) we get, for $|r| < 1$,

$$\sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)}{2^n n!}r^n = \frac{e^{x^2+y^2}}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-s^2-t^2}\sum_{n=0}^{\infty}\frac{(-2str)^n}{n!}e^{2iys}e^{2ixt}\,ds\,dt$$
$$= \frac{e^{x^2+y^2}}{\pi}\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} e^{-s^2-2(-iy+tr)s}\,ds\Big)e^{-t^2+2ixt}\,dt$$
$$= \frac{e^{x^2+y^2}}{\pi}\int_{\mathbb{R}}\sqrt{\pi}e^{(-iy+tr)^2}e^{-t^2+2ixt}\,dt$$
$$= \frac{e^{x^2}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-(1-r^2)t^2}e^{-2i(ry-x)t}\,dt$$
$$= \frac{e^{x^2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{(1-r^2)^{1/2}}e^{-(ry-x)^2/1-r^2} = \frac{1}{(1-r^2)^{1/2}}e^{-\frac{r^2(y^2+x^2)-2rxy}{1-r^2}}.$$

Hence, (1.41) holds. The kernel

$$M_r(x,y) = \frac{1}{(1-r^2)^{1/2}}e^{-\frac{r^2(y^2+x^2)-2rxy}{1-r^2}} = \frac{1}{(1-r^2)^{1/2}}e^{-\frac{|y-rx|^2}{1-r^2}}e^{y^2} \tag{1.45}$$

is called *Mehler's kernel*. □

[12] The explicit formula can also be obtained by solving (1.37) using power series expansions around zero, as $x = 0$ is an ordinary point of the Hermite equation.

Additionally, using the integral representation (1.30) and the formula of their generating function, we can get the following *integral representation* for Mehler's kernel,

$$M_r(x,y) = \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} r^n = \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!} \frac{(-2i)^n e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{\xi^2} \xi^n e^{2i\xi y}\, d\xi\, r^n \tag{1.46}$$
$$= \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\xi^2} e^{2i\xi y} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} (-i\xi r)^n d\xi = \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2i\xi y - \xi^2} e^{-(x+i\xi r)^2 + x^2} d\xi.$$

The following estimate for Hermite polynomials is useful in what follows (see G. Szegő [262, (8.22.8)]). There exists a constant $C > 0$, independent of n, such that

$$|H_n(x)| e^{-x^2/2} \leq C (2^n n!)^{1/2}, \tag{1.47}$$

for all $n \geq 0$. A proof of this fact can be found in [134].

Moreover, using the formula of the generating function (1.32), the estimate (1.47) and (1.39), it is possible to get an analytic proof of the orthogonality of the Hermite polynomials $\{H_n\}_n$. First, observe that

$$\int_{-\infty}^{+\infty} H_m(x)\, e^{-(x-y)^2}\, dx = \int_{-\infty}^{+\infty} \Big(\sum_{n=0}^{\infty} H_m(x)\, H_n(x)\, \frac{y^n}{n!} \Big) e^{-x^2}\, dx \tag{1.48}$$

To interchange the series with the integral on the right-hand side of (1.48), we need to find an integrable bound for the series to apply the dominated convergence theorem; indeed, by applying the inequality (1.47), we obtain

$$\left| \sum_{n=0}^{\infty} H_m(x)\, H_n(x)\, \frac{y^n}{n!}\, e^{-x^2} \right| \leq C \sum_{n=0}^{\infty} \frac{(\sqrt{2}|y|)^n}{\sqrt{n!}} |H_m(x)| e^{-x^2/2} \;\in\; L^1(dx),$$

and thus

$$\int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \Big(H_m(x)\, H_n(x)\, \frac{y^n}{n!} \Big) e^{-x^2}\, dx = \sum_{n=0}^{\infty} \Big(\int_{-\infty}^{+\infty} H_m(x)\, H_n(x) e^{-x^2}\, dx \Big) \frac{y^n}{n!}.$$

On the other hand, by making the change of variables $u = x - y$ and using (1.39), the left-hand side of (1.48) can be written as

$$\int_{-\infty}^{+\infty} H_m(x)\, e^{-(x-y)^2}\, dx = \int_{-\infty}^{+\infty} H_m(u+y)\, e^{-u^2}\, du \tag{1.49}$$
$$= \sum_{k=0}^{m} \binom{m}{k} \int_{-\infty}^{+\infty} H_{m-k}(u)\, e^{-u^2}\, du\, (2y)^k = \sqrt{\pi}\, (2y)^m,$$

by the orthogonality property (1.24). Thus, (1.48) can be rewritten as

$$\sqrt{\pi}\, 2^m\, y^m = \sum_{n=0}^{\infty} \Big(\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, e^{-x^2}\, dx \Big) \frac{y^n}{n!},$$

which implies (1.24),

$$\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, d\gamma_1(x) = 0,$$

for $n \neq m$ and also (1.25)

$$\int_{-\infty}^{+\infty} [H_m(x)]^2\, d\gamma_1(x) = 2^m m!,$$

for $m \geq 0$.

Thus, we know that the Hermite polynomials $\{H_n\}_n$ are linearly independent in $L^2(\gamma_1)$. Now, we shall see that they are also complete.

Proposition 1.10. *The Hermite polynomials form a* complete orthogonal system *in* $L^2(\gamma_1)$.

Proof. Assume $f \in L^2(\gamma_1)$ such that it is orthogonal to H_n for each $n \in \mathbb{N} \cup \{0\}$. Then, the function $f(x)e^{-x^2}$, which is in $L^1(\mathbb{R})$, is orthogonal to each H_n for each $n \geq 0$, and therefore orthogonal to each polynomial, as $\{H_n\}$ is an algebraic basis of the set of all polynomials with real coefficients $\mathscr{P}(\mathbb{R})$. Then, by considering the Fourier transform of $g(x) = f(x)e^{-x^2}$, we have

$$\widehat{g}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-x^2} e^{-ix\zeta} dx = \sum_k \int_{-\infty}^{\infty} f(x) \frac{(-ix\zeta)^k}{k!} e^{-x^2} dx = 0,$$

according to the assumption. The change of order between the integral and the series is justified because the series can be dominated by $e^{|x||\zeta|}$. Hence, the Fourier transform is identically zero; therefore, $f = 0$ almost everywhere □

Moreover, polynomials are dense $L^p(\gamma_d)$ for $1 \leq p < \infty$ (see Theorem 10.7).

As we have already mentioned, Hermite polynomials play a central role in the context of Gaussian harmonic analysis. They are also the building blocks for the eigenfunctions of the harmonic oscillator in quantum mechanics (see for instance [186]).

We denote by h_n the *normalized Hermite polynomial* of degree n, i.e.,

$$h_n(x) = \frac{H_n(x)}{(2^n n!)^{1/2}}. \tag{1.50}$$

It is immediate, then that, up to a constant, the normalized Hermite polynomials satisfy relations similar to those that are satisfied by the Hermite polynomials, for example

$$h_n'(x) = \sqrt{2n}\, h_{n-1}(x), \tag{1.51}$$

and

$$h_n''(x) - 2xh_n'(x) + 2nh_n(x) = 0. \tag{1.52}$$

For a function $f \in L^1(\gamma_1)$, its k-th *Fourier–Hermite coefficient* is defined as

$$\widehat{f}_\gamma(k) = \int_{-\infty}^{\infty} f(y)h_k(y)\gamma_1(dy) = \langle f, h_k \rangle_{\gamma_1}. \tag{1.53}$$

Then, its Hermite expansion is given by

$$f = \sum_{k=0}^{\infty} \widehat{f}_\gamma(k) h_k, \tag{1.54}$$

and its n-th partial sum is

$$S_n f = \sum_{k=0}^{n} \widehat{f}_\gamma(k) h_k. \tag{1.55}$$

Using a standard argument, we can get an integral representation for the partial sums

$$S_n f(x) = \int_{-\infty}^{\infty} D_n(x,y) f(y)\, \gamma_1(dy),$$

where $D_n(x,y)$ is called the *Dirichlet–Szegő's kernel.*

According to the *Christoffel–Darboux formula,* see (10.20), we get the following representation of $D_n(x,y)$

$$D_n(x,y) = \sum_{k=0}^{n} h_k(x)h_k(y) = \left(\frac{n+1}{2}\right)^{1/2} \frac{h_{n+1}(x)h_n(y) - h_n(x)h_{n+1}(y)}{x-y}. \tag{1.56}$$

Hermite Polynomials in d Variables

Now, let us consider the *Hermite polynomials in d variables* $\{\mathbf{H}_\nu\}_\nu$.

Definition 1.11. *For the multi-index $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}_0^d$, the Hermite polynomial in d variables $\mathbf{H}_\nu$ is defined in tensorial form,*[13] *that is to say, $\mathbf{H}_\nu$ is defined as the tensor product of one-dimensional Hermite polynomials,*

$$\mathbf{H}_\nu(x) = \prod_{i=1}^{d} H_{\nu_i}(x_i), \tag{1.57}$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $H_{\nu_i}(x_i)$ is the Hermite polynomial of degree $\nu_i \geq 0$ in the variable x_i.

[13] There are other possibilities for extending the Hermite polynomials to several variables (see for instance [71]), but the tensorial extension is the one that has been used extensively in the theory.

From the way in which the Hermite polynomials in d variables are defined, they inherit several properties from the Hermite polynomials in one variable.

Proposition 1.12. *(Properties of the Hermite polynomials in d variables)*
The Hermite polynomials in d variables satisfy the following properties:

i) Rodrigues' formula: for $x \in \mathbb{R}^d$, we have

$$\mathbf{H}_\nu(x) = (-1)^{|\nu|} e^{|x|^2} \partial^\nu \left(e^{-|x|^2}\right). \tag{1.58}$$

ii) Generating function: for $x, y \in \mathbb{R}^d$, we have

$$e^{2\langle x,y\rangle - |y|^2} = \sum_\nu \mathbf{H}_\nu(x) \frac{y^\nu}{\nu!} = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \mathbf{H}_\nu(x) \frac{y^\nu}{\nu!}. \tag{1.59}$$

iii) Derivative:

$$\frac{\partial \mathbf{H}_\nu}{\partial x_i}(x) = 2\nu_i \mathbf{H}_{\nu - \mathbf{e}_i}, \tag{1.60}$$

where $\mathbf{e}_i$, is the i-th element of the canonical basis of $\mathbb{R}^d$.

iv) Orthogonality relation:

$$\int_{\mathbb{R}^d} \mathbf{H}_\nu(x)\, \mathbf{H}_\eta(x)\, \gamma_d(dx) = \, 2^{|\nu|}\, \nu!\, \delta_{\nu\eta}. \tag{1.61}$$

v) Explicit formula:

$$\mathbf{H}_\nu(x) = \sum_{2\eta \le \nu} \binom{\nu}{2\eta} (-1)^{|\eta|} \frac{(2\eta)!}{\eta!} (2x)^{\nu - 2\eta} \tag{1.62}$$

vi) The Mehler's formula in d dimensions:[14]

$$\sum_{|\nu| \ge 0} \frac{\mathbf{H}_\nu(x)\mathbf{H}_\nu(y)}{2^{|\nu|}\nu!} r^\nu = \frac{1}{(1-r^2)^{d/2}} e^{-\frac{r^2(|y|^2+|x|^2) - 2r\langle x,y\rangle}{1-r^2}}, \tag{1.63}$$

for $|r| < 1$.

Proof. Most of these properties are straightforward, because of the tensorial definition of the Hermite polynomials $\{\mathbf{H}_\nu\}$ and the fact that γ_d is a product measure. □

From (1.47) we can get for fixed $x \in \mathbb{R}^d$,

$$|\mathbf{H}_\nu(x)| \le C_{\nu,x} \nu!, \tag{1.64}$$

where $C_{\nu,x}$ depends on ν (a product of Gamma functions evaluated on ν_i) and x..

[14] For more on this, see the definition of the Ornstein–Uhlenbeck semigroup in Chapter 2.

We see in Chapter 2 that the Hermite polynomials in d variables are eigenfunctions of the Ornstein–Uhlenbeck differential operator L with corresponding eigenvalues $-|\nu| = -\sum_{i=1}^{d} \nu_i$, i.e.,

$$L\mathbf{H}_\nu = -|\nu|\mathbf{H}_\nu. \tag{1.65}$$

Definition 1.13. *The normalized Hermite polynomials in d variables $\{\mathbf{h}_\nu\}_\nu$ are the tensor products of one-dimensional normalized Hermite polynomials, that is,*

$$\mathbf{h}_\nu(x) = \prod_{i=1}^{d} h_{\nu_i}(x_i),$$

where $h_{\nu_i}(x_i)$ is the normalized Hermite polynomial of degree $\nu_i \geq 0$ in the variable x_i.

Therefore,

$$\mathbf{h}_\nu(x) = \frac{\mathbf{H}_\nu(x)}{\|\mathbf{H}_\nu\|_{2,\gamma}} = \frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}.$$

From (1.65), it is immediately seen that the normalized Hermite polynomials $\mathbf{h}_\nu$ are also eigenfunctions of the Ornstein–Uhlenbeck operator,

$$L\mathbf{h}_\nu = -|\nu|\mathbf{h}_\nu.$$

For $f \in L^2(\gamma_d)$, its Fourier–Hermite expansion is given by

$$f = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \widehat{f}_\gamma(\nu)\mathbf{h}_\nu, \tag{1.66}$$

where

$$\widehat{f}_\gamma(\nu) = \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(y)\mathbf{h}_\nu(y)\gamma_d(dy), \tag{1.67}$$

is the *Fourier–Hermite coefficient* associated with the polynomial $\mathbf{h}_\nu$.

Proposition 1.14. *i) The Hermite polynomials in d variables $\{\mathbf{H}_\nu\}_\nu$, form an algebraic basis of $\mathscr{P}(\mathbb{R}^d)$, the set of all polynomials with real coefficients in d variables, that is*

$$\mathscr{P}(\mathbb{R}^d) = span(\{\mathbf{h}_\nu : |\nu| \geq 0\}).$$

ii) Let $\mathscr{C}_k$ be the closed subspace of $L^2(\gamma_d)$ generated by $\{\mathbf{h}_\nu : |\nu| = k\}$, that is

$$\mathscr{C}_k = \overline{span(\{\mathbf{h}_\nu : |\nu| = k\})}^{L^2(\gamma_d)} \tag{1.68}$$

then $\mathscr{C}_k$ is a subspace of dimension $\binom{k+n-1}{k}$. Moreover, $\{\mathscr{C}_k\}$ is an orthogonal decomposition of $L^2(\gamma_d)$, called Wiener chaos or the Wiener–Ito decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{k=0}^{\infty} \mathscr{C}_k. \tag{1.69}$$

Proof.

i) Trivially, from (1.62), it is clear that

$$\text{span}\Big(\Big\{\mathbf{h}_\nu : |\nu| \geq 0\Big\}\Big) \subset \mathscr{P}(\mathbb{R}^d).$$

But we can prove that

$$(2x)^\nu = \sum_{2\eta \leq \nu} \binom{\nu}{\eta} (-1)^{|\eta|} \frac{(2\eta)!}{\eta!} H_{\nu-2\eta}(x). \tag{1.70}$$

Then, as $\{1, x, x^2, \cdots, x^n, \cdots\}$ is the canonical basis of $\mathscr{P}(\mathbb{R}^d)$, we immediately get the other inclusion.

ii) For the fact that the dimension of $\mathscr{C}_k$ is $\binom{k+n-1}{k}$ corresponds to the typical problem of combinations of multi-sets, see [36, Chapter 3, §3.5]. Now, the fact that the subspaces $\mathscr{C}_k$ and $\mathscr{C}_{k'}$ are orthogonal if $k \neq k'$ follows directly from the orthogonality of the Hermite polynomials. From Proposition 1.10, it can be shown that $\{\mathbf{H}_\nu\}_\nu$, is complete in $L^2(\gamma_d)$; the orthogonal decomposition of $L^2(\gamma_d)$ follows immediately from there. □

The Wiener chaos decomposition has an interesting probabilistic interpretation in terms of stochastic integrals obtained by K. Ito, but this is beyond the scope of the book (see for instance [288] or [218]).

Definition 1.15. *For each k, let $\mathbf{J}_k : L^2(\gamma_d) \to \mathscr{C}_k$ be the orthogonal projection of $L^2(\gamma_d)$ onto $\mathscr{C}_k$, which is continuous and (formally) self-adjoint on $L^2(\gamma_d)$. Then, the Hermite expansion of $f \in L^2(\gamma_d)$ can be written as*

$$f = \sum_{k=0}^{\infty} \mathbf{J}_k f = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu, \tag{1.71}$$

where

$$\mathbf{J}_k f = \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu. \tag{1.72}$$

Moreover, as we prove later, as a consequence of the hypercontractivity of the Ornstein–Uhlenbeck semigroup, the projection $\mathbf{J}_k$ (restricted to the polynomials) can be extended continuously to $L^p(\gamma_d)$ for $1 < p < \infty$.

In this book, we study only harmonic analysis with respect to Hermite polynomial expansions; thus, considerations of results related to Hermite functions or other classical orthogonal polynomials or orthogonal functions are beyond its scope. For more information on the latter, we refer to the next section.

1.4 Notes and Further Results

1. In [168] J. Maas, J. van Neerven, and P. Portal have another lemma, along the same lines as Lemma 1.5.

Lemma 1.16. *Let $a, A > 0$ be given. If $B(x,r) \in \mathscr{B}_a$ and $B(y,r') \in \mathscr{B}_A$ have a non-empty intersection, then*

$$|x-y| < k\min\{m(x), m(y)\},$$

where $k = k_{a,A} = \max\{2a\max\{a+A,1\}+A, 2A\max\{a+A,1\}+a\}$.

Proof. We have three cases:

- If $|y| \le 1$, then $m(x) \le 1 = m(y)$.
- If $|y| > 1$ and $|y| \le 2(a+A)$, then

$$m(x) \le 1 \le 2(a+A)\frac{1}{|y|} = 2(a+A)m(y).$$

- If $|y| > 1$ and $|y| = C(a+A)$, with $C > 2$, then

$$|x| \ge |y| - r - r' \ge |y| - a - A = (C-1)(a+A),$$

and therefore,

$$m(x) \le \frac{1}{|x|} \le \frac{C}{C-1}\frac{1}{C(a+A)} = \frac{C}{C-1}\frac{1}{|y|} = \frac{C}{C-1}m(y) \le 2m(y).$$

Hence, in each of these cases,

$$|x-y| \le r + r' \le am(x) + Am(y) \le (2a\max\{a+A,1\}+A)m(y).$$

By symmetry, the same argument yields

$$|x-y| \le (2A\max\{a+A,1\}+a)m(x),$$

and the result follows. □

2. In [174, Proposition 2.1 iii)], G. Mauceri and S. Meda also proved that if $B, B' \in \mathscr{B}_a$, $B \cap B' \neq \emptyset$ and $\gamma_d(B') \le 2\gamma_d(B)$, then

$$r_{B'} \le (2e^{8a+a^2})^{1/d} r_B. \tag{1.73}$$

Because using inequality (1.15) we get

$$\gamma_d(B') \ge \frac{1}{\pi^{d/2}}e^{-|c_{B'}|^2}e^{-2a-a^2}|B'| \quad \text{and} \quad \gamma_d(B) \le \frac{1}{\pi^{d/2}}e^{-|c_B|^2}e^{2a}|B|.$$

Thus, the assumption $\gamma_d(B') \le 2\gamma_d(B)$, implies that

$$e^{-|c_{B'}|^2}e^{-2a-a^2}|B'| \le 2e^{-|c_{B'}|^2}e^{2a}|B|.$$

Therefore, because the Gaussian density is a radially decreasing function, the ball B' satisfying the assumptions and with maximal radius is that of volume $2\gamma_d(B)$ such that $|c_{B'}| \ge |c_B|$ and c_B and $c_{B'}$ are collinear with the origin. In this case $|c_{B'}| - |c_B| = r_{B'} + r_B$, so that

$$\begin{aligned}\left(\frac{r_{B'}}{r_B}\right)^d &\le 2e^{|c_{B'}|^2-|c_B|^2}e^{4a+a^2} = 2e^{(|c_{B'}|+|c_B|)(|c_{B'}|-|c_B|)}e^{4a+a^2} \\ &\le 2e^{2a+|c_{B'}|r_B+|c_B|r_{B'}}e^{4a+a^2} \le 2e^{8a+a^2}.\end{aligned}$$

3. In [168], J. Maas, J. van Neerven, and P. Portal proved that the Gaussian measure satisfies the doubling property on a family of admissible cubes Δ^{γ}.

Lemma 1.17. *For $\alpha > 0$, let αQ be the cube with the same center as Q that has a side length α times the side length of Q. Then, there exists a constant $C = C_{\alpha,d}$ depending only on α and the dimension d, such that for any cube $Q \in \Delta^{\gamma}$,*[15] *we have*

$$\gamma_d(\alpha Q) \leq C\gamma_d(Q). \tag{1.74}$$

Proof. Without loss of generality we may assume that $\alpha > 1$. Let $Q \in \Delta^{\gamma}_{k,l}$ with center y and side-length $2s$, and let $B = B(y,s)$. Then, $B \subset Q$, and moreover $\alpha Q \subset \alpha\sqrt{d}B$. Now, if $|y| > 1$

$$2s = \frac{\text{diam(Q)}}{\sqrt{d}} = 2^{-k-l} \leq 2^{-l} \leq \frac{\sqrt{d}}{|y|} = \sqrt{d}\, m(y),$$

where $m(y) = 1 \wedge \frac{1}{|y|}$ is the admissibility function. If $|y| \leq 1$,

$$2s = \frac{\text{diam(Q)}}{\sqrt{d}} = 2^{-k-l} \leq 1 \leq \sqrt{d}\, m(y),$$

thus, $B \in \mathscr{B}_{\sqrt{d}/2}$. Using the doubling property of the Gaussian measure on $\mathscr{B}_{\sqrt{d}/2}$, see Proposition 1.6, there exists $C = C(\alpha, d)$ such that

$$\gamma_d(\alpha Q) \leq \gamma_d(\alpha\sqrt{d}B) \leq C\gamma_d(B) \leq C\gamma_d(Q). \qquad \square$$

4. The *Hermite functions* are defined as

$$\Psi_0(x) = 1$$

and, for $n \geq 1$,

$$\Psi_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}). \tag{1.75}$$

Therefore, it is clear from Rodrigues formula (1.28) that

$$\Psi_n(x) = H_n(x)e^{\frac{-x^2}{2}};$$

hence, $\{\Psi_n\}_{n\geq 0}$ is an orthogonal system with respect to the Lebesgue measure, that is

$$\int_{-\infty}^{\infty} \Psi_n(x)\Psi_m(x)dx = 0,$$

if $n \neq m$. Moreover, their properties can be easily deduced from the corresponding properties of the Hermite polynomials. In particular, the Hermite functions $\{\psi_n\}_n$ are eigenfunctions of the *Hermite operator*

[15] See the definition of Δ^{γ} and $\Delta^{\gamma}_{k,l}$ in (4.6), Chapter 4.

$$H = -\frac{d^2}{dx^2} + x^2, \tag{1.76}$$

associated with the eigenvalues $\{(2n+1)\}$, i.e.,

$$-\frac{d^2\Psi_n(x)}{dx^2} + x^2\Psi_n(x) = (2n+1)\Psi_n(x).$$

Observe that

$$H = \frac{1}{2}\Big[\Big(-\frac{d}{dx}+x\Big)\Big(\frac{d}{dx}+x\Big) + \Big(\frac{d}{dx}+x\Big)\Big(-\frac{d}{dx}+x\Big)\Big] = \frac{1}{2}(AA^* + A^*A),$$

where $A = (-\frac{d}{dx}+x)$ and $A^* = (\frac{d}{dx}+x)$. A and * are called the *creation and annihilation operators* in quantum mechanics (see [270]).

We define the *normalized Hermite functions* as

$$\psi_n(x) = \frac{\Psi_n(x)}{(\pi^{1/2}2^n n!)^{1/2}}. \tag{1.77}$$

They can also be written in the form

$$\psi_n(x) = h_n(x)\frac{e^{-\frac{x^2}{2}}}{\pi^{1/2}}. \tag{1.78}$$

The paper of A. González Dominguez [114] is an important early reference to the modern study of Hermite functions.

Additionally, by induction and taking the Fourier transform, we can see that the Hermite functions are eigenfunctions of the Fourier transform; see for instance [149] or [270, Lemma 1.1.3.].

The *Hermite functions in d-variables of order* $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}_0^d$, $\overline{\Psi}_\nu$, are defined as the tensor products of Hermite functions in one variable,

$$\overline{\Psi}_\nu(x) = \prod_{i=1}^{d} \Psi_{\nu_i}(x_i),$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $\Psi_{\nu_i}(x_i)$ is the ν_i-Hermite function in the variable x_i.
Analogously, the *normalized Hermite functions in d- variables of order* $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}_0^d$ are defined as the tensor products of Hermite functions in one variable,

$$\overline{\psi}_\nu(x) = \prod_{i=1}^{d} \psi_{\nu_i}(x_i),$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $\psi_{\nu_i}(x_i)$ is the normalized ν_i-Hermite function in the variable x_i.

Observe that defining, for each $1 < p < \infty$, the map $\Xi_d^{(p)} : L^p(\gamma_d) \to L^p(\mathbb{R}^d)$ as

$$\Xi_d^{(p)} f(x) = f(x) \pi^{-(d/2)p} e^{-|x|^2/p}, \tag{1.79}$$

then, $\Xi_d^{(p)}$ is clearly an isometric isomorphism. In particular, $\Xi_1^{(2)} H_n$ is a multiple of Ψ_n or, equivalently, $\Xi_1^{(2)}(h_n) = \psi_n$. Analogously, $\Xi_d^{(2)} \mathbf{H}_\alpha$ is a multiple of $\overline{\Psi}_\alpha$.

5. In spite of the fact that Hermite polynomials are dense in $L^p(\gamma_d)$ for $1 \leq p < \infty$, in [230], H. Pollard proved that $S_n f \to f$ in $L^p(\gamma_1)$, that is

$$\int_{-\infty}^{\infty} |S_n f(x) - f(x)|^p \gamma_1(dx) \to 0,$$

as $n \to \infty$, if and only if $p = 2$ using the fact that the Hermite polynomials are a limiting case of the ultraspherical polynomials (see 10.67). But $p = 2$ is a trivial case from the Hilbert space theory. Pollard's counterexample is the following: given $1 < p < 2$, let us consider the function

$$f(x) = e^{cx^2}, \tag{1.80}$$

with $\frac{1}{2} < c < \frac{1}{p}$. Then, $f \in L^p(\gamma_d)$. It can be shown that for any $k \in \mathbb{N}$,

$$\widehat{f}_H(2k+1) = 0 \quad \text{and} \quad \widehat{f}_H(2k) = M\left(\frac{c}{1-c}\right)^k \frac{1}{4^k k!}.$$

then,

$$\widehat{f}_H(2k) \int_{-\infty}^{\infty} |H_{2k}(x)|^p e^{-x^2} dx \geq \frac{M}{(2k+1)^{1/2}} \left(\frac{c}{1-c}\right)^k \int_{(2k+1)^{1/2}\pi}^{2(2k+1)^{1/2}\pi} |\cos x| dx$$
$$\geq M\left(\frac{c}{1-c}\right)^{(2k+1)^{1/2}};$$

therefore,

$$\limsup_{k\to\infty} \hat{f}_H(2k) \int_{-\infty}^{\infty} |H_{2k}(x)|^p e^{-x^2} dx = \infty.$$

For more details see [230].

6. The other families of classical orthogonal polynomials, the *Jacobi polynomials* and the *Laguerre polynomials* are considered briefly in Appendix B. Similar to the Hermite case, *Jacobi functions* and *Laguerre functions* can also be defined. For more information see [53] or [262].

7. There is a more general class of Hermite polynomials, $\{H_n^\mu\}$ the *generalized Hermite polynomials*. They were defined by G. Szëgo in [262] (see problem 25, p. 380) and studied in detail by T. S. Chihara in his Ph.D. thesis [54]. They are defined as being orthogonal polynomials with respect to the measure

$$d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx, \tag{1.81}$$

with $\mu > -1/2$. When $\mu = 0$ these polynomials coincide, up to a constant, with the classical Hermite polynomials.
Nevertheless, these polynomials are not classical polynomials as they satisfy a second-order differential-difference equation (10.47) instead of a second-order differential equation, i.e., they are eigenfunctions of the differential-difference operator (10.48)

$$L_\mu = \frac{1}{2}\frac{d^2}{dx^2} + (\frac{\mu}{x} - x)\frac{d}{dx} - \mu\frac{I - \tilde{I}}{2x^2}.$$

For more details, see Appendix and [54].