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Wilfredo Urbina-Romero

Gaussian Harmonic Analysis

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Gracias a la vida que me ha dado tanto...

This book is dedicated to:

Eugene Fabes,
un corazón de oro...

Julio Urbina y Aura Romero,
por darme la vida y su ejemplo.

Leonardo, Emiliano & Ximena
mis hijos queridos...
siempre en mi corazón!

Foreword

Special functions with their magical properties occur naturally in several branches of analysis, such as differential equations, Fourier analysis, representation theory, and mathematical physics. Most of these functions occurring in the context of Lie groups are eigenfunctions of the underlying Laplacians and consequently they are expressible as hypergeometric functions. Thus, we encounter Bessel functions on Euclidean spaces, Jacobi polynomials on compact Riemannian symmetric spaces, and Jacobi functions on non-compact Riemannian symmetric spaces. In the context of nilpotent Lie groups, especially on Heisenberg groups, we come across Hermite and Laguerre polynomials. The special functions play a fundamental role in harmonic analysis on Lie groups and no detailed analysis on such groups can be carried out without a deep understanding of the special functions involved. It is therefore no exaggeration to say that harmonic analysis on Lie groups is inseparably intertwined with analysis of special functions.

Orthogonal polynomials occurring as special functions in Lie groups are even more special, as they form an orthogonal basis for the L^2 space of the functions on the groups. The best-known example is the case of trigonometric polynomials, leading to the theory of Fourier series. But other orthogonal polynomials have been well studied in the literature: in an influential 1965 paper, B. Muckenhoupt and E. Stein studied expansions in terms of ultraspherical polynomials, in which they investigated a long list of problems that can be studied in the case of other orthogonal systems as well. In 1970, Muckenhoupt wrote a series of papers dealing with expansions in terms of Hermite and Laguerre polynomials. Earlier works by H. Pollard (1948) and R. Askey and S. Wainger (1965) had already brought out certain unexpected behaviors of expansions in terms of Hermite and Laguerre polynomials, which made them the topic of several interesting works that followed.

Hermite functions, which are of the form $H_n(x)e^{-\frac{1}{2}x^2}$ where H_n are the Hermite polynomials, are ubiquitous in analysis; for instance, they occur as eigenfunctions of the simple harmonic oscillator, thus playing a central role in quantum mechanics, and they are also eigenfunctions of the Fourier transform, a fact that has been used to develop a different way of treating Fourier transforms, Schwartz functions, and tempered distributions. But the Hermite polynomials are also expressible as (complex) moments of the normal distribution with Gaussian density:

$$H_n(x) = \frac{(-2i)^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (y+ix)^n e^{-y^2} dy.$$

This makes a connection between Hermite polynomials and probability theory that lies at the base of the so-called Gaussian harmonic analysis, the main theme of this monograph. Suitably normalized, the Hermite functions form an orthonormal basis for $L^2(\mathbb{R}, dx)$ taken with the Lebesgue measure. The resulting theory of Hermite function expansions is fairly straightforward and has been the subject matter of several interesting works with connections to analysis on Heisenberg groups.

On the other hand, Hermite polynomials H_n under suitable normalization form an orthonormal basis for $L^2(\mathbb{R}, \gamma_1)$ taken with respect to the Gaussian measure $\gamma_1(dx) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx$. More generally, we can look at multi-dimensional Hermite polynomials $H_\nu(x) = \Pi_{j=1}^d H_{\nu_j}(x_j)$, $\nu \in \mathbb{N}^d$ in the space $L^2(\mathbb{R}^d, \gamma_d)$ with $\gamma_d(dx) = \frac{1}{(\sqrt{\pi})^d} e^{-|x|^2} dx$. (The measure space $(\mathbb{R}^d, \gamma_d)$ has an added advantage over the Lebesgue measure space because now an infinite dimensional analog is possible. Thus, Gaussian harmonic analysis has a considerable overlap with Malliavin calculus.) The polynomials H_ν are eigenfunctions of the Ornstein–Uhlenbeck operator $L = \frac{1}{2}\Delta_x - x \cdot \nabla_x$ with eigenvalues $|\nu| = \sum_{j=1}^d \nu_j$. Consequently, the natural analogs of classical operators from Fourier analysis, such as Riesz transforms, spectral multipliers, maximal functions, Ornstein–Uhlenbeck semigroup, Littlewood–Paley–Stein theory of g-functions and singular integrals, all make sense in this new setting. A systematic treatment of these topics is presented very nicely in this monograph.

As the Hermite functions are eigenfunctions of the simple harmonic oscillator $H = -\Delta_x + |x|^2$, also known as the Hermite operator, the theory of Hermite function expansions is the spectral theory of H, and many of the resulting operators can be controlled by means of translation-invariant operators. However, a similar remark cannot be made of the Hermite polynomial expansions, which are related to the spectral theory of the Ornstein–Uhlenbeck operator L. This is mainly because the underlying measure $\gamma_d(dx)$ is not doubling, which renders the theory of maximal functions and Calderón–Zygmund singular integrals on homogeneous spaces rather useless. Thus, it became clear that new covering lemmas and decompositions of functions adapted to the Gaussian measure have to be developed to deal with maximal functions and singular integrals. It is therefore not surprising that the celebrated works of P. A. Meyer on spectral multipliers for L and of P. Sjögren on the weak-type $(1,1)$ inequality for the maximal function associate to the Ornstein–Uhlenbeck semigroup e^{-tL} are highly non-trivial and technical. The hypercontractive estimate for e^{-tL} established by E. Nelson has played an important role in the proof of the spectral multiplier theorem. (In this monograph, the author has presented the beautiful proof of L. Gross using the logarithmic Sobolev inequality.) Apart from the topics mentioned above, this monograph also deals with various function spaces, such as Hardy, Besov–Lipschitz, and Trieber–Lizorkin spaces in the context of Gaussian measures and topics such as Riesz potentials and fractional derivatives.

In this monograph, the author has collected and presented in a readable fashion the hard work of several eminent harmonic analysts, including the author himself and his collaborators, performed over half a century or so. Without any doubt, this is a monograph written by an analyst for analysts in which the main characters are "concrete personalities" and there is no place for "abstract arguments." Anyone who is fond of hard analysis and estimates will enjoy reading this monograph. As far as we are aware, there are no books dealing exclusively with Gaussian harmonic analysis, except for possibly two other less well-known monographs by the author, written in Spanish ([280, 281]). This long-awaited monograph thus fulfils the need for such a book and the author is highly commended for producing this fine piece of work.

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Preface

Classical harmonic analysis dates back to the beginning of the nineteenth century and has its roots in the study of Fourier series, the Fourier transform, and the development of tools needed to understand them. In the twentieth century it underwent an exponential expansion, creating quite a few new branches. Gaussian harmonic analysis is one of those new branches.

Classical harmonic analysis is formulated using as its reference measure the *Lebesgue measure* in $\mathbb{R}^d$; thus, it is formulated in the measure space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \mathrm{m})$. Gaussian harmonic analysis is formulated using the *Gaussian probability measure* in $\mathbb{R}^d$,

$$\gamma_d(dx) = \frac{1}{\pi^{d/2}} e^{-|x|^2} dx;$$

Thus, it is formulated in the probability space $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \gamma_d)$.

In Gaussian harmonic analysis, we want to get analogs to the classical notions (semigroups, covering lemmas, maximal functions, Littlewood–Paley functions, spectral multipliers, fractional integrals and fractional derivatives, singular integrals, etc.) with respect to the Gaussian measure.

A second component of classical harmonic analysis is the *Laplace operator*,

$$\Delta_x = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}.$$

Using integration by parts, it is easy to see that the Laplace operator is (formally) self-adjoint on $L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \Delta_x f(x) g(x)\, dx = \int_{\mathbb{R}^d} f(x) \Delta_x g(x)\, dx.$$

In Gaussian harmonic analysis, the second component is the *Ornstein–Uhlenbeck second-order differential operator*,

$$L = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle,$$

where $\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d}\right)$ is the gradient. Again, by integration by parts, it is easy to see that L is (formally) a self-adjoint operator on $L^2(\gamma_d)$, i.e.,

$$\int_{\mathbb{R}^d} Lf(x)g(x)\, \gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\, \gamma_d(dx).$$

Thus, L plays the role of the symmetric Laplacian in the Gaussian context and the Gaussian measure γ_d is the natural measure with which to study the Ornstein–Uhlenbeck operator L and its associated operators. For reasons that will become clear later, these two components are complementary.

Finally, there is a third component of Gaussian harmonic analysis: the Hermite polynomials. They form a family of orthogonal polynomials with respect to the Gaussian measure and are also eigenfunctions of the Ornstein–Uhlenbeck operator L. In other words, the operator is diagonal with respect to the Hermite polynomial family. Precisely for that reason, Gaussian harmonic analysis in $L^2(\gamma_d)$ is relatively straightforward. Nevertheless, in $L^p(\gamma_d)$ it is a lot more difficult, and kernel techniques are crucial.

Gaussian harmonic analysis has experienced important developments over the last 50 years and our work tries to review most of them, for the first time in book form. Also, it has been an important source of new problems, and a model upon which harmonic analysis with respect to other orthogonal expansions and harmonic analysis of generalized Laplacian have been developed. There are several motivations for the study of Gaussian harmonic analysis:

- First, to extend the classical results obtained in Fourier analysis, that is, harmonic analysis of trigonometric expansions, to orthogonal polynomial expansions. In 1965, B. Muckenhoupt and E. Stein published a seminal article on Gegenbauer (or ultraspherical) orthogonal polynomial expansions (see [199]). Then, in late 1969, B. Muckenhoupt published two articles on Hermite and Laguerre orthogonal polynomial expansions ([193, 194]). Finally, also in 1969, C. P. Calderón published two articles on the Abel summability of Hermite and Laguerre polynomial expansions ([44] and [45]), which treated the d-dimensional case for both families as well. The whole area of harmonic analysis of orthogonal polynomial expansions developed from there.
- Another motivation can be found in probability theory. Malliavin calculus was developed to study the regularity of solutions for stochastic differential equations. Malliavin calculus happens to be harmonic analysis in infinite dimensional spaces with respect to the Gaussian measure, using probabilistic methods. The books by D. Nualart [218] and P. Malliavin's [171, 172] are the main references (see also [33, 189, 260] and [288]). It is important to note that there is an important overlap between the topics of Gaussian harmonic analysis and Malliavin calculus. Nevertheless, the scope of the results, and the methods used in the proofs, are completely different. In Gaussian harmonic analysis only analytic methods in finite dimensions are considered, what might be called "kernel techniques," whereas in Malliavin calculus, the main tools are among others Itô's formula and the Burkholder–Davis–Gundy inequality. For this reason, it would be a mistake

to think that Gaussian harmonic analysis is simply the Malliavin calculus in finite dimensions.

- Additionally, there is the work of D. Bakry and M. Emery on semigroup theory, hypercontractivity, and geometric applications (see for instance [19, 21], and [22]).
- A final motivation is based on considerations in quantum mechanics, more specifically on the second quantization (see for instance A. Messiah [186] and E. Nelson [204]).

In this book, we are going to consider only the first motivation; that is, to extend harmonic analysis to Hermite polynomial expansions using analytic methods in the tradition of the Calderón–Zygmund school of harmonic analysis. In other words, we consider Gaussian harmonic analysis only from the point of view of doing harmonic analysis for non-trigonometric orthogonal expansions using analytic methods, mostly kernel techniques. There are other powerful techniques that are beyond the scope of this book; for example, operator theoretic techniques (see for instance J. M. A. M. van Neerven et al [137]), functional and geometric techniques (see for instance D. Bakry [20]), and Bellman function techniques (see for instance A. Carbonaro, O. Dragicević [49]). We try to indicate the connections between Gaussian harmonic analysis and other connected fields, points of view, and alternative techniques. Gaussian harmonic analysis is perhaps the most frequently studied point of view, among all possible harmonic analysis done for orthogonal expansions, precisely because of its place at the intersection of several fields. Thus, Gaussian harmonic analysis may serve as a good introduction to other cases. Additionally, a deep understanding of this theory may help the reader to gain insights into related problems in other non-Euclidean settings.

In their article, B. Muckenhoupt and E. Stein [199] studied the case of harmonic analysis for ultraspherical or Gegenbauer polynomial expansions, including topics such as the Poisson integral, the conjugate function, H^p spaces, Littlewood–Paley theory, multiplier theory, and Riesz potentials. Thus, for ultraspherical polynomial expansions they were able to develop almost all the important topics of harmonic analysis, in one dimension. In the case of Gaussian harmonic analysis, the theory is far from complete. There are still several important gaps, even though advances have been made in recent years. There are two types of problems. The first involves completing results that are not even known in the one-dimensional case. The second involves obtaining analytic proofs with constants independent of dimension. Such proofs should exist because, as we know from Malliavin calculus thanks to probabilistic techniques, all the operators and semigroups considered make perfect sense in infinite dimensions.

This book is intended for a very diverse audience, from graduate students to researchers working in a broad spectrum of areas in analysis (including, but not limited to, real analysis, harmonic analysis, orthogonal polynomial theory, approximation theory, functional analysis, and partial differential equations). Readers will be able to learn more about Gaussian harmonic analysis in particular and/or harmonic analysis with respect to orthogonal expansions in general. Our goal is to provide an updated exposition, as self-contained as possible, of all the topics in Gaussian harmonic analysis that have so far been mostly scattered in research papers and sections of books. Thus, we have tried to provide full details of most of the crucial results of the theory. Nevertheless, to avoid an extremely lengthy exposition, sometimes we skip some technical details, trying instead to give the main ideas behind them. Hence, it is not as self-contained as we would like it to be, but full references are provided for the interested reader. Detailed proofs of hard-to-find results are also given in full detail. The requirements for the reader of this book are a basic knowledge of real analysis and classical harmonic analysis, including the Calderón–Zygmund theory. As references, we can mention the books of A. Zygmund and R. Wheeden [294] and E. Stein [252]. Also, some knowledge of basic orthogonal polynomial theory would be convenient; the main references are the books by T. S Chihara [53] or G. Szegő [262].

The scheme of the book is as follows. It consists of nine chapters and one appendix. Chapter 1 focuses on preliminary results of the Gaussian probability measure. Because the Gaussian measure is highly concentrated near the origin, precise estimates of the Gaussian measure of balls are needed. They are discussed in Chapter 1. We see that the Gaussian measure is a doubling measure for a special family of balls, even though it is not a doubling measure for all balls in $\mathbb{R}^d$; this important conclusion was obtained by G. Mauceri and S. Meda in [174]. Additionally, Hermite polynomials, which are orthogonal with respect to γ_d, are studied there in detail. In Chapter 2, we study the Ornstein–Uhlenbeck operator, the Ornstein–Uhlenbeck semigroup, and its main properties. In particular, we consider one of its most important properties, the hypercontractivity property. In Chapter 3 we study the Poisson–Hermite semigroup, its basic properties, the characterization of $\frac{\partial^2}{\partial t^2}+L$-harmonic functions, and the conjugate Poisson–Hermite semigroup, which will be important in Chapter 9. In Chapter 4, we study covering lemmas in $\mathbb{R}^d$, several maximal functions with respect to the Gaussian measure, and also the behavior of Calderón–Zygmund operators with respect to the Gaussian measure. Chapter 5 is devoted to the Gaussian Littlewood–Paley–Stein theory, which has important applications in Chapters 6, 7, and 9. In Chapter 6, we consider spectral multiplier operators with respect to the Gaussian measure and their boundedness properties. In Chapter 7, we consider function spaces with respect to the Gaussian measure, including the most important spaces used in analysis (Lebesgue spaces, Sobolev spaces, Tent spaces, Hardy spaces, bounded mean oscillation (BMO) spaces, Lipschitz spaces, Besov–Lipschitz spaces and Triebel–Lizorkin spaces). In Chapter 8, we study Gaussian fractional integrals and fractional derivatives, analyzing their regularity on some of the spaces considered in Chapter 7, and finally, in Chapter 9, we study what is probably one of the most important topics in Gaussian harmonic analysis: Gaussian singular integrals. In the Appendix, we have included several topics: the first is the Gamma function and related functions. The second contains the main properties and formulas of all the classical orthogonal polynomials. In the third, we consider doubling measures in a general setting. In the fourth, we study the classical semigroups in analysis (the heat and the Poisson semigroups); this makes it easier to compare them with the Ornstein–Uhlenbeck and Poisson–Hermite semigroups considered in

Chapters 2 and 3. The fifth topic is interpolation theory. The sixth is Hardy's inequalities, which are used extensively in several chapters, especially in Chapters 7 and 8. The seventh is Natanson's lemma and some of its generalizations, which is a basic tool for the initial analytic proofs of the boundedness for the Gaussian Riesz transforms by B. Muckenhoupt [194], C. P. Calderón [44], and W. Urbina [278]. The final topic is forward differences, which are needed in the study of fractional derivatives in Chapter 8, when we study unbounded indexes.

This book is based on two previous monographs, [280] and [281], written when I was full professor at Universidad Central de Venezuela in Caracas, Venezuela, before the country started to fall apart. The present work also has some overlap with a monograph published by the French Mathematical Society (SMF) in 2012, on semigroups for classical polynomials [281] based on a course given in a CIMPA school in Mérida, Venezuela in 2006. It summarizes my own research, the research of my students, and the research of many others in the area. It also summarizes countless talks that I have had with friends and collaborators, in addition to seminars given in several universities: Universidad Central de Venezuela in Caracas; Universidad de los Andes in Mérida, Venezuela; University of Minnesota in Minneapolis; Rutgers University at New Brunswick; Temple University in Philadelphia; Instituto Venezolano de Investigaciones Científicas in Altos de Pipe, Venezuela; Universidad Autónoma de Madrid; Universidad de la Habana; Universidad de Matanzas, Cuba; Universidad de la Rioja; Universidad de Zaragoza; Universidad Simón Bolívar in Caracas; Universidad del País Vasco in Bilbao; Universidad de Málaga; Universidad de Sevilla; Universidad de Valencia; Universidad Carlos III de Madrid; Université Paris V; Université de Angers; Universidad de la Laguna, Tenerife; Universidad Nacional del Litoral, Santa Fé, Argentina; Universidad Nacional de La Plata, Argentina; University of Kansas, University of New Mexico; University of Arizona; New Mexico State University, Las Cruces; University of California at Santa Barbara; CIMAT in Guanajuato, México; DePaul University in Chicago; Florida International University in Miami; University of Georgia in Statesboro; and Roosevelt University. I am very thankful for the opportunities I have had to speak about these topics to so many different audiences, and for the interest they raised.

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Wilfredo Urbina-Romero

Contents

1

Preliminary Results: The Gaussian Measure and Hermite Polynomials

In this chapter we study the Gaussian measure in $\mathbb{R}^d$ for $d \geq 1$ and several of its properties. Then, we study the problem of the Gaussian measure for balls in $\mathbb{R}^d$, which is crucial in Chapter 4 for studying the associated covering lemmas for that measure. For completeness, we consider Hermite polynomials, which are orthogonal polynomials, with respect to the Gaussian measure, and discuss in detail most of their properties. The interested reader will find the properties and identities of all classical orthogonal polynomials listed in the appendix.

1.1 The Gaussian Measure

The *Gaussian measure* in $\mathbb{R}$ is given by[1]

$$\gamma_1(dx) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx, \tag{1.1}$$

where e^{-x^2} is called the *Gaussian weight.*

The fact that γ_1 is a probability measure is based on the following famous computation; using polar coordinates and Fubini's theorem,

$$\begin{aligned}\Big(\int_{\mathbb{R}} e^{-x^2} dx\Big)^2 &= \int_{\mathbb{R}} e^{-x^2} dx \int_{\mathbb{R}} e^{-y^2} dy = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2} dx\, dy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = 2\pi\Big(-\frac{1}{2}e^{-r^2}\Big)\Big|_0^{\infty} = \pi.\end{aligned}$$

[1] In probability theory, it is usual to consider the standard Gaussian probability, defined as $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. Nevertheless, in the context of the theory of orthogonal polynomials, it is more common to use (1.1) and we are going to follow that normalization (see [262]). The formulas differ only by constants.

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4_1

The *Fourier transform* of γ_1 (characteristic function in probability terminology) is given by

$$\begin{aligned}\widehat{\gamma_1}(\xi) &= \int_{\mathbb{R}} e^{-i\xi y}\gamma_1(dy) = \int_{\mathbb{R}} e^{-i\xi y}\frac{e^{-y^2}}{\sqrt{\pi}}dy \\ &= \frac{e^{-\xi^2/4}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-(y+i\xi/2)^2}dy = \frac{e^{-\xi^2/4}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-y^2}dy = e^{-\xi^2/4}; \end{aligned}\tag{1.2}$$

thus, the Gaussian measure is 'essentially' (up to a constant) its own Fourier transform. Moreover, that integral is uniformly convergent in any disk $D = \{x : |x| \leq r\}$, $r > 0$ and is bounded in that region. Therefore, according to the dominated convergence theorem, we can differentiate an arbitrary number of times, obtaining,

$$\frac{d^n}{dx^n}e^{-x^2} = \frac{(-2i)^n}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-y^2}y^n e^{-2ixy}\,dy. \tag{1.3}$$

The *Gaussian distribution function* Φ is defined as

$$\Phi(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-y^2}dy. \tag{1.4}$$

In other words, Φ is just the cumulative distribution function of the measure γ_1. It is well known that, unfortunately, there is not a closed form of it. An important estimate of the rate of decrease of the function $1-\Phi$ can be obtained simply using integration by parts, for $x > 0$,

$$\frac{1}{2\sqrt{\pi}}\left(\frac{1}{x} - \frac{1}{2x^3}\right)e^{-x^2} \leq 1-\Phi(x) \leq \frac{1}{2\sqrt{\pi}x}e^{-x^2}. \tag{1.5}$$

The *Gaussian measure* in $\mathbb{R}^d$ is defined as the product measure

$$\gamma_d(dx) = \frac{1}{\pi^{d/2}}e^{-|x|^2}dx = \frac{1}{\sqrt{\pi}}e^{-x_1^2}dx_1 \otimes \frac{1}{\sqrt{\pi}}e^{-x_2^2}dx_2 \otimes \cdots \otimes \frac{1}{\sqrt{\pi}}e^{-x_d^2}dx_d. \tag{1.6}$$

Being a product of probability measures, it is clear that γ_d is a probability measure in $\mathbb{R}^d$. On the other hand, γ_d is radially symmetric. There is likely no other non-trivial probability measure that satisfies both properties. From the fact that the Gaussian measure in $\mathbb{R}^d$ is a product measure, a technique called *tensorization* has been developed, which consists in obtaining d-dimensional estimates from those of the one-dimensional estimate.

It is clear that the Gaussian measure is highly concentrated near the origin and decays exponentially at infinity, for all $d \geq 1$. That behavior is very far from the invariance by translation of the Lebesgue measure; therefore, there is a big difference between it and the Lebesgue measure. For instance, any argument in classical analysis that uses the translation invariant property of the Lebesgue measure is totally useless in the Gaussian case. On the other hand, the Gaussian measure is invariant by rotation, so we can take advantage of that property.

Finally, even though in probability theory the Gaussian measures form a whole family of probability measures (with different means and variances), γ_d is the only Gaussian measure considered in this book.[2]

1.2 Estimates for the Gaussian Measure of Balls in $\mathbb{R}^d$ and the Doubling Condition

We need to estimate the Gaussian measure of balls in $\mathbb{R}^d$ to obtain covering lemmas and other estimates, for instance, but this is not trivial at all, because, as we have already said, the Gaussian measure is a probability measure, highly concentrated around the origin, with exponential decay at infinity, invariant by rotation around the origin, and not translation invariant.

First, we consider a partition P of $\mathbb{R}$, obtained by B. Muckenhoupt, in [194] Lemma 2, because in such a partition there is a seminal idea about how to measure balls (or cubes) using the Gaussian measure.

- First, divide the interval $[0,2]$ into the subintervals $[0,1]$ and $[1,2]$ of length one.
- Then, for $n \geq 1$, divide the interval $[2^n, 2^{n+1})$, into 2^{n+1} intervals of length 2^{-n}.
- Finally, consider the mirror images of these intervals for the interval $(-\infty, 0]$.

Then, the elements of the partition in the complement of $[-2,2]$ are of the form $I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n})$, where $n \geq 1$ and k are integers such that $2^{2n} \leq |k| < 2^{2n+1}$.

Proposition 1.1. *The partition*

$$\mathtt{P} = \{[-2,-1],[-1,0],[0,1],[1,2]\} \cup \{I_{k,n} : n \geq 1,\ 2^{2n} \leq |k| \leq 2^{2n+1}\} \qquad (1.7)$$

satisfies the following properties:

i) Any compact subset of $\mathbb{R}$ *intersects only a finite number of the intervals of the partition* P.

ii) An interval in the partition is no more than twice as long as the adjacent intervals. Furthermore, for any $I \in \mathtt{P}$, *if* $x \in I$, *then* $1 \wedge \frac{1}{|x|} = \min\{1, \frac{1}{|x|}\}$ *is not greater than the length of the interval.*

iii) Finally, there exists a constant C, independent of n and k such that

$$\frac{\sup_{x \in I_{k,n}} e^{-x^2}}{\inf_{x \in I_{k,n}} e^{-x^2}} \leq C.$$

Proof.

i) Immediate, because any compact set is bounded.

[2]For more details on Gaussian measures, see for example the book by V. I. Bogachev ([33, Chapter 1]).

ii) That an interval in the partition is no more than twice as long as the adjacent intervals is clear by construction. Now, if $|x| \le 1$, then[3]

$$1 \wedge \frac{1}{|x|} = 1 = |[0,1]|.$$

If $|x| > 1$, then for the case $x \in [1,2]$, $1 \wedge \frac{1}{|x|} = \frac{1}{|x|} < 1 = |[1,2]|$. The case $x \in [-2,-1]$ is totally analogous. Now, assuming that $x \in I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n}], n \ge 1$ then, $1 \wedge \frac{1}{|x|} = \frac{1}{|x|}$ and

$$|I_{k,n}| = \frac{1}{2^n} = \frac{2^n}{2^{2n}} \ge \frac{2^n}{k} \ge \frac{1}{|x|}.$$

Moreover,

$$|I_{k,n}| = \frac{1}{2^n} = \frac{2}{2^{n+1}} \le 2\frac{2^{n-1}}{2^{2n}} \le 2\frac{2^{n-1}}{k+1} \le \frac{2}{|x|}.$$

iii) By symmetry, let us consider only intervals in $[0,\infty)$. For the first two intervals $[0,1]$ and $[1,2]$, e^{-y^2} varies by a factor of e^{-1}. Now, if $x \ge 2$, assuming $x \in I_{k,n} = [\frac{k}{2^n}, \frac{k+1}{2^n}]$, then, as e^{-x^2} is a decreasing function,

$$\frac{\sup_{x\in I_{k,n}} e^{-x^2}}{\inf_{x\in I_{k,n}} e^{-x^2}} = e^{\frac{(k+1)^2-k^2}{2^{2n}}} = e^{\frac{2k+1}{2^{2n}}} \le e^{\frac{2^{2n+2}+1}{2^{2n}}} \le e^{4+\frac{1}{2^{2n}}} \le e^5,$$

hence, the inequality holds with $C = e^5$. □

Observe that *iii)* shows a very important characteristic of this partition: the Gaussian weight e^{-x^2} is essentially constant at each interval in the partition $\mathtt{P}$. Thus, the exponential decay is controlled; therefore, at each interval of $\mathtt{P}$, the Gaussian measure is equivalent to the Lebesgue measure. Then, usual estimates using the Lebesgue measure can be made, at least locally, instead of working with the Gaussian measure. This technique was used initially by B. Muckenhoupt to obtain certain estimates for singular operators with respect to the Gaussian measure. This idea, as we are going to see later, is the key to a technique that consists in defining a *local region* and then splitting operators into a local and a global part. This will be discussed in more depth later, in Chapter 4.

We can obtain a partition of $\mathbb{R}^d$ of d-dimensional rectangles having the same properties listed in Lemma 1.1 simply by considering Cartesian products of partitions $\mathtt{P}_i$ in each variable x_i as before. This partition can be refined, splitting the rectangles into cubes. A similar partition of $\mathbb{R}^d$ was considered by P. Sjögren in [247]. Also, a similar idea is considered in the work of J. Mass, J. Van Neerven, and P. Portal in [169] on Whitney decomposition. This is discussed in detail in Chapter 4 as well.

The Gaussian measure of any ball $B(x,r)$ in $\mathbb{R}^d$ can be easily estimated, depending of the center of the ball, by using polar coordinates.

[3]Here we are using the convention that $1 \wedge \frac{1}{0} = 1$.

Lemma 1.2. *Let $r > 0$, and $d \geq 2$.*

i) The Gaussian measure of a ball in $\mathbb{R}^d$ centered at the origin, $B(0,r)$, is bounded above by

$$\gamma_d(B(0,r)) \leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d. \tag{1.8}$$

ii) For any $x \in \mathbb{R}^d$,

$$\gamma_d(B(x,r)) \leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d e^{2r|x|} e^{-|x|^2}, \tag{1.9}$$

where ω_{d-1} is the (surface) measure of the unit (hyper)-sphere S^{d-1} in $\mathbb{R}^d$.

Proof.

i) Using polar coordinates, $y = \rho\xi$, for $\rho > 0$, and $\xi \in S^{d-1}$, i.e., $|y| = \rho|\xi| = \rho$, we have

$$\begin{aligned}
\gamma_d(B(0,r)) &= \int_{B(0,r)} \frac{e^{-|y|^2}}{\pi^{d/2}}\, dy = \int_{S^{d-1}} \int_0^r \frac{e^{-\rho^2}}{\pi^{d/2}} \rho^{d-1} d\rho\, d\sigma \\
&\leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^{d-2} \int_0^r e^{-\rho^2} 2\rho\, d\rho = \frac{\omega_{d-1}}{2\pi^{d/2}} r^{d-2}(1 - e^{-r^2}) \leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d,
\end{aligned}$$

as $1 - e^{-t} \leq t$, for any $t \geq 0$.

ii) For any $x \in \mathbb{R}^d$, as $|y|^2 \leq ((y-x)+x)^2 = |y-x|^2 + 2\langle x, y-x\rangle + |x|^2$,

$$\begin{aligned}
\gamma_d(B(x,r)) &= \int_{B(x,r)} \frac{e^{-|y|^2}}{\pi^{d/2}}\, dy = \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{B(x,r)} e^{-|y-x|^2} e^{-2\langle x, y-x\rangle}\, dy \\
&\leq \frac{e^{-|x|^2}}{\pi^{d/2}} \int_{B(x,r)} e^{-|y-x|^2} e^{2|x||y-x|}\, dy \\
&\leq \frac{e^{-|x|^2}}{\pi^{d/2}} e^{2r|x|} \int_{B(x,r)} e^{-|y-x|^2}\, dy = \frac{e^{-|x|^2}}{\pi^{d/2}} e^{2r|x|} \int_{B(0,r)} e^{-|y|^2}\, dy \\
&\leq \frac{\omega_{d-1}}{2\pi^{d/2}} r^d e^{2r|x|} e^{-|x|^2}. \qquad \square
\end{aligned}$$

To extend the idea in Proposition 1.1 to higher dimensions, we define a family of admissible balls.

Definition 1.3. *The family of admissible balls*[4] *in $\mathbb{R}^d$, with parameter $a, b > 0$, is defined as*

$$\mathscr{B}_{a,b} = \left\{ B(x,r) : x \in \mathbb{R}^d,\ 0 < r < a \wedge \frac{b}{|x|} \right\}. \tag{1.10}$$

---

[4] Admissible balls are sometimes also called *hyperbolic balls*.

In particular, if $a = b$, the family of admissible balls with parameter a is defined as

$$\mathscr{B}_a = \left\{B(x,r) : x \in \mathbb{R}^d,\ 0 < r < a\left(1 \wedge \frac{1}{|x|}\right)\right\}$$
$$= \{B(x,r) : x \in \mathbb{R}^d,\ 0 < r < am(x)\}, \tag{1.11}$$

where

$$m(x) = 1 \wedge \frac{1}{|x|}. \tag{1.12}$$

$m(x)$ is called the admissibility function.

Observe that, trivially, $m(x) \leq 1$ and $m(x) \leq \frac{1}{|x|}$. Observe also that admissible balls need to be very small when their center is far from the origin.

For admissible balls in $\mathscr{B}_{a,b}$ the Gaussian weight $e^{-|y|^2}$ is essentially constant. More precisely, we have the following estimates:

Lemma 1.4. *For $a, b > 0$, if $|x - y| < a \wedge \frac{b}{|x|}$, then*

$$e^{-a^2}e^{-2b}e^{-|x|^2} \leq e^{-|y|^2} \leq e^{2b}e^{-|x|^2}. \tag{1.13}$$

Therefore, for admissible balls, $B = B(x,r) \in \mathscr{B}_{a,b}$, their Gaussian measures can be estimated as:

$$\gamma_d(B) = \frac{1}{\pi^{d/2}}\int_B e^{-|y|^2}dy \sim C_d e^{-|x|^2}\left(a \wedge \frac{b}{|x|}\right)^d. \tag{1.14}$$

In particular, for $a > 0$, if $|x - y| < a\left(1 \wedge \frac{1}{|x|}\right) = am(x)$, then

$$e^{-a^2}e^{-2a}e^{-|x|^2} \leq e^{-|y|^2} \leq e^{2a}e^{-|x|^2}; \tag{1.15}$$

therefore, if $B = B(x,r) \in \mathscr{B}_a$,

$$\gamma_d(B) = \frac{1}{\pi^{d/2}}\int_B e^{-|y|^2}dy \sim C_d e^{-|x|^2}a^d\left(1 \wedge \frac{1}{|x|}\right)^d = C_d e^{-|x|^2}a^d m(x)^d. \tag{1.16}$$

Thus, for admissible balls, their Gaussian measure is essentially a multiple (which depends on the center) of their Lebesgue measure.

Proof. Simply by triangle inequality,

$$e^{-|y|^2} = e^{-|x-(x-y)|^2} \leq e^{-|x|^2}e^{2|x||x-y|}e^{-|x-y|^2} \leq e^{2b}e^{-|x|^2},$$

and

$$e^{-|y|^2} = e^{-|x+(y-x)|^2} \geq e^{-|x|^2}e^{-2|x||y-x|}e^{-|y-x|^2} \geq e^{-a^2}e^{-2b}e^{-|x|^2}. \qquad \square$$

On the other hand, J. Maas, J. van Neerven, and P. Portal [169] obtained the following lemma, using an idea similar to the one contained in Lemma 1.4.

Lemma 1.5. *Let $a, A > 0$ be given.*

i) If $|x-y| < At$ and $t \le am(x)$, then $t \le a(1+aA)m(y)$.
ii) If $|x-y| < Am(x)$, then $m(x) \le (1+A)m(y)$ and $m(y) \le 2(1+A)m(x)$.

Proof.

i) We have three cases:

- If $|y| \le 1$, then $m(y) = 1$, and

$$t \le am(x) \le a = am(y) \le a(1+aA)m(y).$$

- If $1 < |y| \le 1+aA$, then $m(y) \ge 1/(1+aA)$ and

$$t \le am(x) \le a \le a(1+aA)m(y).$$

- If $|y| > 1+aA > 1$, then $m(y) = \frac{1}{|y|}$ and

$$t \le am(x) \le \frac{a}{|x|} \le \frac{a}{|y|-At} \le \frac{a}{|y|-aA} \le \frac{a(1+aA)}{|y|} = a(1+aA)m(y).$$

ii) Put $t' = m(x)$. Then $|x-y| < At'$; therefore, we can apply *i*) with $a = 1$ to get that $t' \le (1+A)m(y)$. This gives the first estimate. To obtain the second one we consider three cases:

- If $|x| \le 1$, then $2(1+A)m(x) \ge 1 \ge m(y)$.
- If $|1 \le |x| \le 2A$, (i.e., $A \ge 1/2$) then

$$2(1+A)m(x) \ge \frac{2(1+A)}{2A} \ge 1 \ge m(y).$$

- If $|x| \ge 1$ and $|x| \ge 2A$, then $|y| \ge |x| - \frac{A}{|x|} \ge |x| - \frac{1}{2} \ge \frac{|x|}{2}$; thus,

$$m(y) \le 2m(x) \le 2(1+A)m(x). \qquad \square$$

Part *i*) of Lemma 1.5 says, among other things, that if we have $B(x,r) \in \mathscr{B}_a$ and if $|x-y| < Ar$, then $B(y,r) \in \mathscr{B}_c$ for some constant $c = c_{a,A}$, which depends only on a and A. Additionally, using part *ii*), we get the following estimate, similar to (1.13): if $|x-y| < am(x)$,

$$e^{-a^2}e^{-2a}e^{-|x|^2} \le e^{-|y|^2} \le e^{a^2(1+a)^2}e^{2a(1+a)}e^{-|x|^2}, \tag{1.17}$$

because, as $|x|m(x) \le 1$, we have

$$|y|^2 \le (|x| + |x-y|)^2 \le (|x| + am(x))^2 \le |x|^2 + 2a + a^2,$$

and, as $m(x) \le (1+a)m(y)$,

$$\begin{aligned} |x|^2 &\le (|y| + |x-y|)^2 \le (|y| + am(x))^2 \\ &\le (|y| + a(1+a)m(y))^2 \le |y|^2 + 2a(1+a) + a^2(1+a)^2 \end{aligned}$$

(see J. Teuwen [265, Lemma 2], see also G. Mauceri, S. Meda [174, Lemma 2.1 i)]).

The main results of classical harmonic analysis in $\mathbb{R}^d$, which are done with respect to the Lebesgue measure, were later extended for other classes of measures. The initial and probably the most important one is the class of doubling measures. Recall that a Borel measure μ in $\mathbb{R}^d$ is a *doubling measure* if a constant $C > 0$ exists, depending only on the dimension d, such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)), \tag{1.18}$$

for any $x \in \mathbb{R}^d$ and $r > 0$.

The meaning of this condition is that the mass that μ gives to the annulus $2B \setminus B$ is controlled by a constant times the mass of B. The opposite of that means that $\mu(B)$ is much less than $\mu(2B \setminus B)$, and therefore that μ rarefies at B. All the classical notions of harmonic analysis can be extended almost immediately to doubling measures (see for instance [254] or [275]).

As the Gaussian measure γ_d is a probability measure, it is not a doubling measure, for more details see Appendix 10.3. Thus, there is no constant $C > 0$, independent of $x \in \mathbb{R}^d$, and $r > 0$ such that

$$\gamma_d(B(x,2r)) \leq C\gamma_d(B(x,r)),$$

for all $x \in \mathbb{R}^d$ and $r > 0$, i.e., the doubling condition does not hold for all possible balls in $\mathbb{R}^d$. Therefore the classical results of harmonic analysis cannot be extended directly to the case of Gaussian harmonic analysis. Nevertheless, G. Mauceri and S. Meda in a seminal paper [174] observed that if we control the radius appropriately, the Gaussian measure is doubling; more precisely, the Gaussian measure is doubling if we restrict it to the family of admissible balls $\mathscr{B}_a$. Thus, we can adapt the classical arguments, at least in some regions. The doubling condition for the Gaussian measure is therefore a local condition, and is a contained in the following result (see G. Mauceri and S. Meda's paper [174, Proposition 2.1]).

Theorem 1.6. *Let $a, \tau > 0$. For each ball $B = B(c_B, r_B) \in \mathscr{B}_a$, consider the set B^*_τ which is the union of all balls $B' = B(c_{B'}, r_{B'})$, which intersects B and such that $r_{B'} \leq \tau r_B$, then the following inequalities hold.*

i) If $\sigma^*_{a,\tau} = \sup_{B\in\mathscr{B}_a} \frac{\gamma_d(B^*_\tau)}{\gamma_d(B)}$ then

$$\sigma^*_{a,\tau} \leq (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}. \tag{1.19}$$

ii) (Doubling property) There exists a constant $C = C_{a,\tau,d} > 1$ depending only on a, τ and the dimension d, such that for any ball $B' = B(x_{B'}, r_{B'})$ having a non-empty intersection with B and such that $r_{B'} \leq \tau r_B$, then

$$\gamma_d(B') \leq C\gamma_d(B).$$

In particular, this implies that there exists a constant $C = C_d > 1$ such that for all $\tau > 1$ and all $B = B(x_B, r_B) \in \mathscr{B}_a$

$$\gamma_d(B(x_B, \tau r_B)) \leq C\gamma_d(B(x_B, r_B)). \tag{1.20}$$

Proof.

i) First of all, observe that $B^*_\tau \subset B(c_B, (2\tau+1)r_B) \in \mathscr{B}_{(2\tau+1)a}$ and therefore, using both sides of inequality (1.15) with the parameter $(2\tau+1)a$, we get

$$\begin{aligned}\gamma_d(B^*_\tau) &= \frac{1}{\pi^{d/2}}\int_{B^*_\tau} e^{-|y|^2}dy \leq \frac{1}{\pi^{d/2}}e^{2(2\tau+1)a}e^{-|c_B|^2}|B^*_\tau| \\ &\leq \frac{1}{\pi^{d/2}}e^{2(2\tau+1)a}e^{-|c_B|^2}|B(c_B,(2\tau+1)r_B)| \\ &= \frac{1}{\pi^{d/2}}e^{2(2\tau+1)a}e^{-|c_B|^2}(2\tau+1)^d|B(c_B,r_B)|,\end{aligned}$$

and

$$\gamma_d(B) \geq \frac{1}{\pi^{d/2}}e^{-(2\tau+1)^2a^2}e^{-2(2\tau+1)a}e^{-|c_B|^2}|B(c_B,r_B)|.$$

Thus,

$$\sigma^*_{a,\tau} \leq (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}.$$

ii) As B' is one of the terms in the union that forms B^*_τ then *ii*) follows immediately from *i*) as

$$\begin{aligned}\gamma_d(B') \leq \gamma_d(B^*_\tau) &= \frac{\gamma_d(B^*_\tau)}{\gamma_d(B)}\gamma_d(B) \leq \sigma^*_{a,\tau}\gamma_d(B) \\ &\leq (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}\gamma_d(B) = C\gamma_d(B),\end{aligned}$$

with $C = (2\tau+1)^d e^{4a(2\tau+1)+a^2(2\tau+1)^2}$. This estimate of C could be improved as $B' \subset B(c_B, (2\tau+1)r_B)$, using inequality (1.15) with the parameter $(2\tau+1)a$. □

J. Maas, J. van Neerven, and P. Portal proved, in [168], that there is also a family of cubes in $\mathbb{R}^d$ such that the Gaussian measure is a doubling measure on them (see Lemma 1.17).

Observe that, because the Gaussian measure is not a doubling measure, the measure space $(\mathbb{R}^d, |\cdot|, \gamma_d)$ is not a space of homogeneous type; thus, there is no overlap between Gaussian harmonic analysis and harmonic analysis of spaces of homogeneous type.

Moreover, the Gaussian measure is trivially a d-dimensional measure[5] in $\mathbb{R}^d$, because, for any $x \in \mathbb{R}^d$, and $r > 0$,

$$\gamma_d(B(x,r)) = \frac{1}{\pi^{d/2}} \int_{B(x,r)} e^{-|x|^2} dx \leq \frac{1}{\pi^{d/2}} |B(x,r)| = C_d r^d.$$

As we have mentioned before, classical harmonic analysis, which was extended initially to doubling measures, has been extended to the case of *s-dimensional measures* (see, for instance, Tolsa [274]). Nevertheless, Gaussian harmonic analysis is not part of that theory because, as was mentioned before, there is another component of it, which is the Ornstein–Uhlenbeck and associated operators.

Going back to the problem of the Gaussian measure of balls, we can still get an estimate for the Gaussian measure of non-admissible balls if they do not contain the origin. That estimate was obtained by L. Forzani in [83], but in this case the estimate does not depend upon the center of the ball but rather on the closest point to the origin.

Proposition 1.7. *(Forzani) Let B a ball in $\mathbb{R}^d$, with radius $r > 0$, which does not contain the origin, and let x_0 denote the point of B whose distance to the origin is minimal, i.e., $d(B,0) = |x_0|$. Then, there exists a constant $C_d > 0$, depending only on the dimension d, such that*

$$\gamma_d(B) \leq C_d \frac{e^{-|x_0|^2}}{|x_0|} \left(\frac{r}{|x_0|} \right)^{(d-1)/2}. \tag{1.21}$$

Moreover, if $r_B > \frac{C}{|x_0|}$, $C > 1$, the opposite inequality is also true; therefore,

$$\gamma_d(B) \sim C_d \frac{e^{-|x_0|^2}}{|x_0|} \left(\frac{r}{|x_0|} \right)^{(d-1)/2}. \tag{1.22}$$

Proof. Let us write $B = B(x,r)$. It is enough to consider that $|x_0| > 1$ because otherwise a constant would do a better job than the estimate. Consider Π_0 the hyperplane orthogonal to x_0 whose distance to the origin is precisely $|x_0|$, that is,

$$\Pi_0 = \{x \in \mathbb{R}^d : \langle x, x_0 \rangle = |x_0|^2\},$$

and consider the hyperspace $\Pi_0^+ = \{x \in \mathbb{R}^d : \langle x, x_0 \rangle > |x_0|^2\}$. Then, any $y \in \Pi_0^+$ can be written as $y = (\xi + |x_0|)\frac{x_0}{|x_0|} + \nu$, with $\xi > 0$ and $\langle \nu, x_0 \rangle = 0$. In particular, we have $x = (r + |x_0|)\frac{x_0}{|x_0|}$.

[5] A Borel measure is s-dimensional in $\mathbb{R}^d$ if it satisfies the following growth condition:

$$\mu(B(x,r)) \leq C r^s,$$

for some constant C and for all $x \in \mathbb{R}^d$, and $r > 0$.

Observe that $y \in B$ if and only if $\xi \in (0,2r)$ and $|v| < \sqrt{2r\xi - \xi^2}$, because $\max_{y\in B}|y| = 2r + |x_0|$, and according to the Pythagorean theorem,

$$\begin{aligned}|y-x|^2 &= |(\xi + |x_0|)\frac{x_0}{|x_0|} + v - (r+|x_0|)\frac{x_0}{|x_0|}|^2 = |(\xi - r)\frac{x_0}{|x_0|} + v|^2\\ &= (\xi - r)^2 + |v|^2 = \xi^2 - 2\xi r + r^2 + |v|^2 < r^2.\end{aligned}$$

Then, we have

$$\begin{aligned}\gamma_d(B) &= \frac{1}{\pi^{d/2}}\int_B e^{-|y|^2}dy\\ &= C_d e^{-|x_0|^2}\int_0^{2r} e^{-2\xi|x_0|}\, e^{-\xi^2}\Big(\int_{\{v\in\mathbb{R}^{d-1}:\,|v|<\sqrt{2r\xi-\xi^2}\}} e^{-|v|^2}\,dv\Big)\,d\xi\\ &\le C_d e^{-|x_0|^2}\int_0^{2r} e^{-2\xi|x_0|}\,(2r\xi - \xi^2)^{(d-1)/2}\,d\xi\\ &\le C_d e^{-|x_0|^2} r^{(d-1)/2}\int_0^{2r} e^{-2\xi|x_0|}(2\xi)^{(d-1)/2}d\xi\\ &\le C_d\frac{e^{-|x_0|^2}}{|x_0|}\Big(\frac{r}{|x_0|}\Big)^{(d-1)/2}\int_0^{4r|x_0|} e^{-t}t^{(d-1)/2}dt \le C_d\frac{e^{-|x_0|^2}}{|x_0|}\Big(\frac{r}{|x_0|}\Big)^{(d-1)/2}.\end{aligned}$$

Now, if $r > \frac{C}{|x_0|}$, $C > 1$, let us define

$$R(x_0,r) = \Big\{y = (\xi + |x_0|)\frac{x_0}{|x_0|} + v : \xi \in \Big[\frac{1}{2|x_0|}, \frac{1}{|x_0|}\Big], \langle v, x_0\rangle = 0, |v| < \frac{1}{2}\sqrt{\frac{r}{|x_0|}}\Big\}.$$

We will prove that $R(x_0,r) \subset B$. Given $y \in R(x_0,r)$ it is enough to prove that if $\xi \in [\frac{1}{2|x_0|}, \frac{1}{|x_0|}]$ then $2r\xi - \xi^2 > \frac{r}{4|x_0|}$ because, in that case,

$$\sqrt{2r\xi - \xi^2} > \frac{1}{2}\sqrt{\frac{r}{|x_0|}} > |v|.$$

Observe that the expression $2r\xi - \xi^2$, as a function of ξ, in the interval $\Big[\frac{1}{2|x_0|}, \frac{1}{|x_0|}\Big]$ attains its minimum at $\frac{1}{2|x_0|}$, and as $\frac{1}{|x_0|} < \frac{r}{C} < r$, we get

$$2r\xi - \xi^2 \ge \frac{r}{|x_0|} - \frac{1}{2|x_0|^2} > \frac{r}{4|x_0|},$$

and, clearly, $\xi \in (0,2r)$. Now, if $y \in R(x_0,r)$

$$|y|^2 = \xi^2 + 2\xi|x_0| + |x_0|^2 + |v|^2 < \frac{1}{|x_0|^2} + 2 + |x_0|^2 + \frac{r}{4|x_0|} < |x_0|^2 + \tilde{C}.$$

Hence, $e^{-|y|^2} \ge e^{-\tilde{C}}e^{-|x_0|^2}$ and therefore

$$\gamma_d(B) \ge \gamma_d(R(x_0,r)) = \frac{1}{\pi^{d/2}}\int_{R(x_0,r)} e^{-|y|^2}dy \ge e^{-\tilde{C}}\,\frac{e^{-|x_0|^2}}{|x_0|}\Big(\frac{r}{|x_0|}\Big)^{(d-1)/2}. \qquad \square$$

Another version of inequality (1.21) (see [83, Lemma 4.3]) is the following. There exists a constant C depending on d such that for all $x \in \mathbb{R}^d \setminus \{0\}$, $r \in (1/2, 1)$ and $s \in (0, 1/2)$ the following inequality holds:

$$\gamma_d\left(B\left(\frac{x}{r}, \frac{|x|}{r}s\right)\right) \leq C\, s^{(d-1)/2} \exp\left(-\frac{|x|^2}{r^2}(1-s)^2\right)\frac{1}{|x|}. \tag{1.23}$$

This follows immediately from (1.21) taking $\frac{|x|}{r}s$ as the radius, and then $x_0 = \frac{x}{r} - \frac{x}{r}s = \frac{x}{r}(1-s)$.

As we see in Chapter 4, Lemma 4.16, a similar estimate, can be used to prove the $L^p(\gamma_d)$ boundedness, $1 < p < \infty$, for the non-centered Hardy–Littlewood maximal function with respect to the Gaussian measure, obtained in [90]. Moreover, we see in Chapter 4 how these estimates of the Gaussian measure of balls are important in the proof of some covering lemmas.

1.3 Hermite Polynomials

Hermite Polynomials in One Variable

For completeness, we study in detail the *Hermite polynomials.*. Additionally, in Appendix B, we list the properties for all classical orthogonal polynomials. The standard reference in orthogonal polynomial theory is G. Szegő [262].

The *Hermite polynomials* in $\mathbb{R}$, $\{H_n\}_{n\geq 0}$, can be defined (up to a multiplicative constant) as the orthogonal polynomials associated with the Gaussian measure γ_1. Therefore, they are obtained from the canonical polynomial (monomials) base

$$\{1, x, x^2, \cdots, x^n, \cdots\}$$

by using the Gram–Schmidt method, with respect to the inner product in $L^2(\gamma_1)$[6] (see G. Szegő [262] and E. Hille [134]). Thus, if $m \neq n$

$$\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, d\gamma_1(x) = 0. \tag{1.24}$$

The Gram–Schmidt method determines the polynomials up to a constant; thus, for normalization we set

$$\int_{-\infty}^{+\infty} [H_n(x)]^2\, d\gamma_1(x) = 2^n n!. \tag{1.25}$$

[6] In probability theory, another family of Hermite polynomials is used, which is orthogonal with respect to the standard Gaussian measure $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Observe that by using the Gram–Schmidt method, given $n \in \mathbb{N}$,

$$\int_{-\infty}^{+\infty} P(x)\, H_n(x)\, d\gamma_1(x) = 0, \tag{1.26}$$

for any polynomial P such that $\deg(P) \leq n-1$.

Even though this definition is probably the most straightforward, it is not the easiest to handle as it gives us a recursive formula for H_n, but not an explicit expression.

Alternatively, the Hermite polynomials can be defined using Rodrigues' formula:

$$H_0(x) = 1 \tag{1.27}$$

and for $n > 1$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{1.28}$$

One of the advantages of this definition is precisely that it is easy to get explicit expressions of H_n, because the formula itself is not difficult to handle. Observe that according to (1.28), we get the first polynomials easily.

$$\begin{aligned}
H_1(x) &= (-1)^1 e^{x^2} \frac{d}{dx}(e^{-x^2}) = -e^{x^2}(-2x)e^{-x^2} = 2x,\\
H_2(x) &= (-1)^2 e^{x^2} \frac{d^2}{dx^2}(e^{-x^2}) = e^{x^2}(4x^2-2)e^{-x^2} = 4x^2-2,\\
H_3(x) &= (-1)^3 e^{x^2} \frac{d^3}{dx^3}(e^{-x^2}) = -e^{x^2}(-8x^3+12x)e^{-x^2} = 8x^3-12x,\\
H_4(x) &= (-1)^4 e^{x^2} \frac{d^4}{dx^4}(e^{-x^2}) = e^{x^2}(16x^4-48x^2+12)e^{-x^2} = 16x^4-48x^2+12.
\end{aligned}$$

Then, we can prove, by induction, that

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!} \frac{(2x)^{n-2k}}{(n-2k)!}. \tag{1.29}$$

where $[n/2]$ is the integer part of $n/2$, i.e., the largest integer not greater that $n/2$. Nevertheless, we provide a simpler proof of this formula later, using the generating function (see Proposition 1.9).

Also, from (1.3), using Rodrigues' formula, we get the following *integral representation* of H_n,

$$H_n(x) = \frac{(-2i)^n e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2ixy} y^n e^{-y^2}\, dy. \tag{1.30}$$

Let us prove that Rodrigues' formula actually gives the same polynomials as those obtained using the Gram–Schmidt method. To do so, we need to prove that those polynomials are orthogonal with respect to the Gaussian measure, i.e., that they satisfy (1.24) and to the normalization condition (1.25).

First of all, observe that trivially, because $H_0(x) = 1$,

$$\int_{-\infty}^{\infty} H_0(y)\,\gamma_1(dy) = 1,$$

as γ_1 is a probability measure. Moreover, if $n \geq 1$, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} H_n(x)H_0(x)\,d\gamma_1(x) &= \int_{-\infty}^{+\infty} H_n(x)\,d\gamma_1(x) = (-1)^n \int_{-\infty}^{+\infty} e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) \frac{1}{\sqrt{\pi}} e^{-x^2} dx \\ &= \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n}{dx^n}(e^{-x^2})\,dx = 0,\end{aligned}$$

simply by integrating by parts.

Now, we need to consider the case $n, m \geq 1$, $m \neq n$. Without loss of generality, assume that $n > m > 0$. Then, using Rodrigues' formula,

$$\begin{aligned}\int_{-\infty}^{+\infty} H_m(x)H_n(x)\,d\gamma_1(x) &= \int_{-\infty}^{+\infty} H_m(x)(-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})\ \frac{1}{\sqrt{\pi}} e^{-x^2} dx \\ &= \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} H_m(x) \frac{d^n}{dx^n}(e^{-x^2})\,dx \\ &= \frac{(-1)^{2n}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n}{dx^n}(H_m(x)) e^{-x^2}\,dx = 0,\end{aligned}$$

by integrating by parts n times, as $n > m$. For the case $n = m$, first observe that, from the explicit expression of H_n (1.29),

$$H_n^{(n)}(x) = 2^n n!, \tag{1.31}$$

then, integrating by parts n times

$$\begin{aligned}\int_{-\infty}^{+\infty} [H_n(x)]^2\,d\gamma_1(x) &= (-1)^n \int_{-\infty}^{+\infty} [e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})]H_n(x)\ \frac{1}{\sqrt{\pi}} e^{-x^2} dx \\ &= \frac{(-1)^n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n}{dx^n}(e^{-x^2}) H_n(x)\,dx = \frac{(-1)^{2n}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{d^n H_n(x)}{dx^n}\ e^{-x^2} dx \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} 2^n n!\, e^{-x^2}\,dx = 2^n n!.\end{aligned}$$

Hence, as we claimed, the Gram–Schmidt method and Rodrigues' formula give rise to the same family of orthogonal polynomials.

The Hermite polynomials have a simple generating function, as we see in the following proposition.

Proposition 1.8. *The generating function*[7] *of the Hermite polynomials is given by*

$$G(x,y) = e^{2xy-y^2} = e^{-(x-y)^2+x^2}, \tag{1.32}$$

i.e., $\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n = e^{2xy-y^2} = e^{-(x-y)^2+x^2}$.

Proof. Observe that from (1.30) and (1.2) we get

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n &= \sum_{n=0}^{\infty} \frac{(-2i)^n e^{x^2}}{n!\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2} r^n e^{2ixr}\, dr\, y^n = \frac{e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2} \sum_{n=0}^{\infty} \frac{(-2iry)^n}{n!} e^{2ixr}\, dr \\ &= \frac{e^{x^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2} e^{2i(x-y)r}\, dr = e^{x^2} e^{-(x-y)^2} = e^{2xy-y^2} = G(x,y). \qquad \square\end{aligned}$$

Moreover, the Hermite polynomials are the only polynomials that satisfy that relation; hence, they can also be defined using $G(x,y)$ as follows:

$$H_n(x) = \frac{\partial^n}{\partial y^n} G(x,y)|_{y=0} = \frac{\partial^n}{\partial y^n} (e^{2xy-y^2})|_{y=0} = e^{-(x-y)^2+x^2}|_{y=0}. \tag{1.33}$$

Hence, using (1.33) we may easily obtain the first five Hermite polynomials:

$$\begin{aligned}H_0(x) &= G(x,y)|_{y=0} = 1, \\ H_1(x) &= \frac{\partial}{\partial y} G(x,y)|_{y=0} = 2(x-y)G(x,y)|_{y=0} = 2x, \\ H_2(x) &= \frac{\partial^2}{\partial y^2} G(x,y)|_{y=0} = (4(x-y)^2-2)G(x,y)|_{y=0} = 4x^2-2 \\ H_3(x) &= \frac{\partial^3}{\partial y^3} G(x,y)|_{y=0} = (8(x-y)^3-12(x-y))G(x,y)|_{y=0} = 8x^3-12x, \\ H_4(x) &= \frac{\partial^4}{\partial y^4} G(x,y)|_{y=0} = (16(x-y)^4-48(x-y)^2+12)G(x,y)|_{y=0} = 16x^4-48x^2+12.\end{aligned}$$

From Rodrigues' formula we directly obtain (1.33):

$$\begin{aligned}H_n(x) &= (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}) = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n}(e^{-(x-y)^2})|_{y=0} \\ &= e^{x^2} \frac{\partial^n}{\partial y^n}(e^{-(x-y)^2})|_{y=0} = \frac{\partial^n}{\partial y^n}(e^{2xy-y^2})|_{y=0} = \frac{\partial^n}{\partial y^n} G(x,y)|_{y=0}.\end{aligned}$$

[7] The *generating function* of a family of orthogonal polynomials $\{P_n\}$ is a function $G(x,y)$ such that $\{P_n(x)\}$ are the coefficients of the Taylor expansion of $G(\cdot,y)$ around $y=0$.

Additionally, $G(x,y)$ can be extended analytically as

$$G(x,z) = e^{2xz - z^2},$$

for $x, z \in \mathbb{C}$. Then, from (1.33), using Cauchy's integral formula, we get the following integral representation of H_n,

$$H_n(x) = \left. \frac{\partial^n G(x,z)}{\partial z^n} \right|_{z=0} = \frac{n!}{2\pi i} \oint_C \frac{G(x,\zeta)}{\zeta^{n+1}} d\zeta = \frac{n!}{2\pi i} \oint_C \frac{e^{2x\zeta - \zeta^2}}{\zeta^{n+1}} d\zeta, \quad (1.34)$$

where C is any curve around the origin.

Now, let us prove the following properties of Hermite polynomials.

Proposition 1.9. *For any $n \geq 1$, $H_n(x)$ satisfies the following properties:*

i) Recursive relation:

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0. \quad (1.35)$$

ii) Derivative:

$$H_n'(x) = 2nH_{n-1}(x). \quad (1.36)$$

iii) Differential equation:

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \quad (1.37)$$

Thus, the n-th Hermite polynomial is a polynomial solution of the Hermite equation *with parameter n, i.e., the Hermite polynomials are polynomial solutions of the Hermite equation, or equivalently H_n is an eigenfunction of the one-dimensional* Ornstein–Uhlenbeck operator,[8] $L = \frac{1}{2}\frac{d^2}{dx} - x\frac{d}{dx}$, *with eigenvalue $-n$, that is,*

$$LH_n(x) = \frac{1}{2}\frac{d^2}{dx}H_n(x) - x\frac{d}{dx}H_n(x) = -nH_n(x). \quad (1.38)$$

iv)

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} H_{n-k}(y)(2x)^k = \sum_{k=0}^{n} \binom{n}{k} H_k(y)(2x)^{n-k}. \quad (1.39)$$

v) Explicit formula:

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!} \frac{(2x)^{n-2k}}{(n-2k)!}. \quad (1.40)$$

[8] It is also known as the *harmonic oscillator operator*. Its generalization to $\mathbb{R}^d$ is considered in detail in Section 2.1 of Chapter 2.

vi) Mehler's formula[9]*:*

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} r^n = \frac{1}{(1-r^2)^{1/2}} e^{-\frac{r^2(y^2+x^2)-2rxy}{1-r^2}}, \ |r|<1. \tag{1.41}$$

Mehler's formula allows us to express the Abel summability of Hermite series in integral form.[10]

Proof.

i) Observe that the generating function $G(x,y)$ satisfies the differential equation

$$\frac{\partial G}{\partial y} - 2(x-y)G = 0.$$

Then, substituting in (1.32) the Taylor series of $G(x,y)$, we get

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} \frac{H_n(x)}{n!} n y^{n-1} - (2x-2y) \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n \\ &= \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} y^{n-1} - 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n + 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^{n+1} \\ &= \sum_{n=0}^{\infty} [H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x)] \frac{y^n}{n!}. \end{aligned}$$

Equating term by term, we get the two-term recurrent relation

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0,$$

for each $n \geq 1$.

ii) Observe that the generating function $G(x,y)$ also satisfies the following differential equation

$$\frac{\partial G}{\partial x} - 2yG = 0.$$

Again, substituting (1.32) the Taylor series of $G(x,y)$, we get

$$\begin{aligned} 0 &= \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} y^n - 2y \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^n = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} y^n - 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} y^{n+1} \\ &= \sum_{n=0}^{\infty} [H_n'(x) - 2nH_{n-1}(x)] \frac{y^n}{n!}. \end{aligned}$$

Equating term by term, we get (1.36).

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[9]This formula was found by F. G. Mehler in 1866 [183] and, according to E. A. Hille, *"rediscovered by almost everybody working in the field"* (see [134]).

[10]For more on this, see the definition of the Ornstein–Uhlenbeck semigroup in Chapter 2.

iii) Using (1.36) to eliminate $H_{n-1}(x)$ from the recursive relation (1.35), we get

$$H_{n+1}(x) - 2xH_n(x) + H_n'(x) = 0.$$

By differentiating, and using (1.36), we obtain

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0.$$

iv) To prove the result, we need to use *Cauchy's product.*[11] Then

$$\begin{aligned}\sum_{n=0}^{\infty} H_n(x+y)\frac{r^n}{n!} &= e^{2(x+y)r-r^2} = e^{2xr}e^{2yr-r^2} = \sum_{n=0}^{\infty}\frac{(2xr)^n}{n!}\sum_{n=0}^{\infty}H_n(y)\frac{r^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(2xr)^k}{k!}H_{n-k}(y)\,\frac{r^{n-k}}{(n-k)} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}H_{n-k}(y)(2x)^k\right)\frac{r^n}{n!}.\end{aligned}$$

Equating the coefficients, (1.39) follows.

v) Taking $y = 0$ in (1.39), we get

$$H_n(x) = \sum_{k=0}^{n}\binom{n}{k}H_{n-k}(0)(2x)^k = n!\sum_{k=0}^{n}\frac{1}{k!(n-k)!}H_k(0)(2x)^{n-k}. \tag{1.42}$$

Now, taking $x = 0$ in (1.32), we get

$$e^{-y^2} = \sum_{k=0}^{\infty}H_k(0)\frac{y^k}{k!}.$$

But as

$$e^{-y^2} = \sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{k!}\frac{y^{2k}}{(2k)!},$$

we can conclude that

$$H_{2k+1}(0) = 0 \text{ and } H_{2k}(0) = (-1)^k\frac{(2k)!}{k!}, \tag{1.43}$$

for any $k \le [n/2]$. Therefore, from (1.42),

[11] Recall that given two convergent series, $\sum a_n$ and $\sum b_n$, if at least one is absolutely convergent, then its *Cauchy product* is defined as $\sum c_n$, where $c_n = \sum_{k=0}^{n} a_k\, b_{n-k}$, and it is also absolutely convergent and its sum the product of the two series.

$$H_n(x) = n!\sum_{k=0}^{n}\frac{1}{k!(n-k)!}H_k(0)(2x)^{n-k} = n!\sum_{k=0}^{[n/2]}\frac{1}{(2k)!(n-2k)!}H_{2k}(0)(2x)^{n-2k}$$
$$= n!\sum_{k=0}^{[n/2]}\frac{(-1)^k}{k!}\frac{(2x)^{n-2k}}{(n-2k)!}.$$

Hence, (1.40) follows.[12]

vi) Observe that by the properties of the Gaussian measure, we get the following identity

$$\int_{-\infty}^{\infty} e^{-a^2x^2-2bx}dx = \frac{\sqrt{\pi}}{a}e^{b^2/a^2}, \tag{1.44}$$

because, by completing the square

$$\int_{-\infty}^{\infty} e^{-a^2x^2-2bx}dx = e^{b^2/a^2}\int_{-\infty}^{\infty} e^{-a^2x^2-2bx-b^2/a^2}dx = e^{b^2/a^2}\int_{-\infty}^{\infty} e^{-(ax+b/a)^2}dx$$
$$= \frac{e^{b^2/a^2}}{a}\int_{-\infty}^{\infty} e^{-y^2}dy = \frac{\sqrt{\pi}}{a}e^{b^2/a^2}.$$

Using the integral representation (1.30) and (1.44) we get, for $|r| < 1$,

$$\sum_{n=0}^{\infty}\frac{H_n(x)H_n(y)}{2^n n!}r^n = \frac{e^{x^2+y^2}}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-s^2-t^2}\sum_{n=0}^{\infty}\frac{(-2str)^n}{n!}e^{2iys}e^{2ixt}\,ds\,dt$$
$$= \frac{e^{x^2+y^2}}{\pi}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-s^2-2(-iy+tr)s}\,ds\right)e^{-t^2+2ixt}\,dt$$
$$= \frac{e^{x^2+y^2}}{\pi}\int_{\mathbb{R}}\sqrt{\pi}e^{(-iy+tr)^2}e^{-t^2+2ixt}\,dt$$
$$= \frac{e^{x^2}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-(1-r^2)t^2}e^{-2i(ry-x)t}\,dt$$
$$= \frac{e^{x^2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{(1-r^2)^{1/2}}e^{-(ry-x)^2/1-r^2} = \frac{1}{(1-r^2)^{1/2}}e^{-\frac{r^2(y^2+x^2)-2rxy}{1-r^2}}.$$

Hence, (1.41) holds. The kernel

$$M_r(x,y) = \frac{1}{(1-r^2)^{1/2}}e^{-\frac{r^2(y^2+x^2)-2rxy}{1-r^2}} = \frac{1}{(1-r^2)^{1/2}}e^{-\frac{|y-rx|^2}{1-r^2}}e^{y^2} \tag{1.45}$$

is called *Mehler's kernel*. □

[12] The explicit formula can also be obtained by solving (1.37) using power series expansions around zero, as $x = 0$ is an ordinary point of the Hermite equation.

Additionally, using the integral representation (1.30) and the formula of their generating function, we can get the following *integral representation* for Mehler's kernel,

$$M_r(x,y) = \sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} r^n = \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!} \frac{(-2i)^n e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{\xi^2} \xi^n e^{2i\xi y} \, d\xi \, r^n \tag{1.46}$$
$$= \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\xi^2} e^{2i\xi y} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} (-i\xi r)^n d\xi = \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2i\xi y - \xi^2} e^{-(x+i\xi r)^2 + x^2} d\xi.$$

The following estimate for Hermite polynomials is useful in what follows (see G. Szegő [262, (8.22.8)]). There exists a constant $C > 0$, independent of n, such that

$$|H_n(x)| e^{-x^2/2} \le C (2^n n!)^{1/2}, \tag{1.47}$$

for all $n \ge 0$. A proof of this fact can be found in [134].

Moreover, using the formula of the generating function (1.32), the estimate (1.47) and (1.39), it is possible to get an analytic proof of the orthogonality of the Hermite polynomials $\{H_n\}_n$. First, observe that

$$\int_{-\infty}^{+\infty} H_m(x) \, e^{-(x-y)^2} \, dx = \int_{-\infty}^{+\infty} \Big(\sum_{n=0}^{\infty} H_m(x) \, H_n(x) \, \frac{y^n}{n!} \Big) e^{-x^2} \, dx \tag{1.48}$$

To interchange the series with the integral on the right-hand side of (1.48), we need to find an integrable bound for the series to apply the dominated convergence theorem; indeed, by applying the inequality (1.47), we obtain

$$\left| \sum_{n=0}^{\infty} H_m(x) \, H_n(x) \, \frac{y^n}{n!} \, e^{-x^2} \right| \le C \sum_{n=0}^{\infty} \frac{(\sqrt{2}|y|)^n}{\sqrt{n!}} |H_m(x)| e^{-x^2/2} \ \in \ L^1(dx),$$

and thus

$$\int_{-\infty}^{+\infty} \sum_{n=0}^{\infty} \Big(H_m(x) \, H_n(x) \, \frac{y^n}{n!} \Big) e^{-x^2} \, dx = \sum_{n=0}^{\infty} \Big(\int_{-\infty}^{+\infty} H_m(x) \, H_n(x) e^{-x^2} \, dx \Big) \frac{y^n}{n!}.$$

On the other hand, by making the change of variables $u = x - y$ and using (1.39), the left-hand side of (1.48) can be written as

$$\int_{-\infty}^{+\infty} H_m(x) \, e^{-(x-y)^2} \, dx = \int_{-\infty}^{+\infty} H_m(u+y) \, e^{-u^2} \, du \tag{1.49}$$
$$= \sum_{k=0}^{m} \binom{m}{k} \int_{-\infty}^{+\infty} H_{m-k}(u) \, e^{-u^2} \, du \, (2y)^k = \sqrt{\pi} \, (2y)^m,$$

by the orthogonality property (1.24). Thus, (1.48) can be rewritten as

$$\sqrt{\pi} \, 2^m \, y^m = \sum_{n=0}^{\infty} \Big(\int_{-\infty}^{+\infty} H_m(x) \, H_n(x) \, e^{-x^2} \, dx \Big) \frac{y^n}{n!},$$

which implies (1.24),

$$\int_{-\infty}^{+\infty} H_m(x)\, H_n(x)\, d\gamma_1(x) = 0,$$

for $n \neq m$ and also (1.25)

$$\int_{-\infty}^{+\infty} [H_m(x)]^2\, d\gamma_1(x) = 2^m m!,$$

for $m \geq 0$.

Thus, we know that the Hermite polynomials $\{H_n\}_n$ are linearly independent in $L^2(\gamma_1)$. Now, we shall see that they are also complete.

Proposition 1.10. *The Hermite polynomials form a* complete orthogonal system *in* $L^2(\gamma_1)$.

Proof. Assume $f \in L^2(\gamma_1)$ such that it is orthogonal to H_n for each $n \in \mathbb{N} \cup \{0\}$. Then, the function $f(x)e^{-x^2}$, which is in $L^1(\mathbb{R})$, is orthogonal to each H_n for each $n \geq 0$, and therefore orthogonal to each polynomial, as $\{H_n\}$ is an algebraic basis of the set of all polynomials with real coefficients $\mathscr{P}(\mathbb{R})$. Then, by considering the Fourier transform of $g(x) = f(x)e^{-x^2}$, we have

$$\widehat{g}(\zeta) = \int_{-\infty}^{\infty} f(x)e^{-x^2} e^{-ix\zeta} dx = \sum_k \int_{-\infty}^{\infty} f(x)\frac{(-ix\zeta)^k}{k!} e^{-x^2} dx = 0,$$

according to the assumption. The change of order between the integral and the series is justified because the series can be dominated by $e^{|x||\zeta|}$. Hence, the Fourier transform is identically zero; therefore, $f = 0$ almost everywhere □

Moreover, polynomials are dense $L^p(\gamma_d)$ for $1 \leq p < \infty$ (see Theorem 10.7).

As we have already mentioned, Hermite polynomials play a central role in the context of Gaussian harmonic analysis. They are also the building blocks for the eigenfunctions of the harmonic oscillator in quantum mechanics (see for instance [186]).

We denote by h_n the *normalized Hermite polynomial* of degree n, i.e.,

$$h_n(x) = \frac{H_n(x)}{(2^n n!)^{1/2}}. \tag{1.50}$$

It is immediate, then that, up to a constant, the normalized Hermite polynomials satisfy relations similar to those that are satisfied by the Hermite polynomials, for example

$$h_n'(x) = \sqrt{2n} h_{n-1}(x), \tag{1.51}$$

and

$$h_n''(x) - 2xh_n'(x) + 2nh_n(x) = 0. \tag{1.52}$$

For a function $f \in L^1(\gamma_1)$, its k-th *Fourier–Hermite coefficient* is defined as

$$\widehat{f}_\gamma(k) = \int_{-\infty}^{\infty} f(y)h_k(y)\gamma_1(dy) = \langle f, h_k \rangle_{\gamma_1}. \tag{1.53}$$

Then, its Hermite expansion is given by

$$f = \sum_{k=0}^{\infty} \widehat{f}_\gamma(k)h_k, \tag{1.54}$$

and its n-th partial sum is

$$S_n f = \sum_{k=0}^{n} \widehat{f}_\gamma(k)h_k. \tag{1.55}$$

Using a standard argument, we can get an integral representation for the partial sums

$$S_n f(x) = \int_{-\infty}^{\infty} D_n(x,y) f(y)\, \gamma_1(dy),$$

where $D_n(x,y)$ is called the *Dirichlet–Szegő's kernel.*

According to the *Christoffel–Darboux formula,* see (10.20), we get the following representation of $D_n(x,y)$

$$D_n(x,y) = \sum_{k=0}^{n} h_k(x)h_k(y) = \left(\frac{n+1}{2}\right)^{1/2} \frac{h_{n+1}(x)h_n(y) - h_n(x)h_{n+1}(y)}{x-y}. \tag{1.56}$$

Hermite Polynomials in d Variables

Now, let us consider the *Hermite polynomials in d variables* $\{\mathbf{H}_\nu\}_\nu$.

Definition 1.11. *For the multi-index $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}_0^d$, the Hermite polynomial in d variables $\mathbf{H}_\nu$ is defined in tensorial form,*[13] *that is to say, $\mathbf{H}_\nu$ is defined as the tensor product of one-dimensional Hermite polynomials,*

$$\mathbf{H}_\nu(x) = \prod_{i=1}^{d} H_{\nu_i}(x_i), \tag{1.57}$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $H_{\nu_i}(x_i)$ is the Hermite polynomial of degree $\nu_i \geq 0$ in the variable x_i.

[13] There are other possibilities for extending the Hermite polynomials to several variables (see for instance [71]), but the tensorial extension is the one that has been used extensively in the theory.

From the way in which the Hermite polynomials in d variables are defined, they inherit several properties from the Hermite polynomials in one variable.

Proposition 1.12. *(Properties of the Hermite polynomials in d variables)*
The Hermite polynomials in d variables satisfy the following properties:

i) Rodrigues' formula: for $x \in \mathbb{R}^d$, we have

$$\mathbf{H}_\nu(x) = (-1)^{|\nu|} e^{|x|^2} \partial^\nu \left(e^{-|x|^2} \right). \tag{1.58}$$

ii) Generating function: for $x, y \in \mathbb{R}^d$, we have

$$e^{2\langle x,y\rangle - |y|^2} = \sum_\nu \mathbf{H}_\nu(x) \frac{y^\nu}{\nu!} = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \mathbf{H}_\nu(x) \frac{y^\nu}{\nu!}. \tag{1.59}$$

iii) Derivative:

$$\frac{\partial \mathbf{H}_\nu}{\partial x_i}(x) = 2\nu_i \mathbf{H}_{\nu - \mathbf{e}_i}, \tag{1.60}$$

where $\mathbf{e}_i$, is the i-th element of the canonical basis of $\mathbb{R}^d$.

iv) Orthogonality relation:

$$\int_{\mathbb{R}^d} \mathbf{H}_\nu(x)\, \mathbf{H}_\eta(x)\, \gamma_d(dx) = \, 2^{|\nu|}\, \nu!\, \delta_{\nu\eta}. \tag{1.61}$$

v) Explicit formula:

$$\mathbf{H}_\nu(x) = \sum_{2\eta \le \nu} \binom{\nu}{2\eta} (-1)^{|\eta|} \frac{(2\eta)!}{\eta!} (2x)^{\nu - 2\eta} \tag{1.62}$$

vi) The Mehler's formula in d dimensions:[14]

$$\sum_{|\nu| \ge 0} \frac{\mathbf{H}_\nu(x) \mathbf{H}_\nu(y)}{2^{|\nu|} \nu!} r^\nu = \frac{1}{(1-r^2)^{d/2}} e^{-\frac{r^2(|y|^2 + |x|^2) - 2r\langle x,y\rangle}{1-r^2}}, \tag{1.63}$$

for $|r| < 1$.

Proof. Most of these properties are straightforward, because of the tensorial definition of the Hermite polynomials $\{\mathbf{H}_\nu\}$ and the fact that γ_d is a product measure. □

From (1.47) we can get for fixed $x \in \mathbb{R}^d$,

$$|\mathbf{H}_\nu(x)| \le C_{\nu,x} \nu!, \tag{1.64}$$

where $C_{\nu,x}$ depends on ν (a product of Gamma functions evaluated on ν_i) and x..

[14] For more on this, see the definition of the Ornstein–Uhlenbeck semigroup in Chapter 2.

We see in Chapter 2 that the Hermite polynomials in d variables are eigenfunctions of the Ornstein–Uhlenbeck differential operator L with corresponding eigenvalues $-|\nu| = -\sum_{i=1}^{d} \nu_i$, i.e.,

$$L\mathbf{H}_\nu = -|\nu|\mathbf{H}_\nu. \tag{1.65}$$

Definition 1.13. *The normalized Hermite polynomials in d variables $\{\mathbf{h}_\nu\}_\nu$ are the tensor products of one-dimensional normalized Hermite polynomials, that is,*

$$\mathbf{h}_\nu(x) = \prod_{i=1}^{d} h_{\nu_i}(x_i),$$

where $h_{\nu_i}(x_i)$ is the normalized Hermite polynomial of degree $\nu_i \geq 0$ in the variable x_i.

Therefore,

$$\mathbf{h}_\nu(x) = \frac{\mathbf{H}_\nu(x)}{\|\mathbf{H}_\nu\|_{2,\gamma}} = \frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}.$$

From (1.65), it is immediately seen that the normalized Hermite polynomials $\mathbf{h}_\nu$ are also eigenfunctions of the Ornstein–Uhlenbeck operator,

$$L\mathbf{h}_\nu = -|\nu|\mathbf{h}_\nu.$$

For $f \in L^2(\gamma_d)$, its Fourier–Hermite expansion is given by

$$f = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \widehat{f}_\gamma(\nu)\mathbf{h}_\nu, \tag{1.66}$$

where

$$\widehat{f}_\gamma(\nu) = \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(y)\mathbf{h}_\nu(y)\gamma_d(dy), \tag{1.67}$$

is the *Fourier–Hermite coefficient* associated with the polynomial $\mathbf{h}_\nu$.

Proposition 1.14. *i) The Hermite polynomials in d variables $\{\mathbf{H}_\nu\}_\nu$, form an algebraic basis of $\mathscr{P}(\mathbb{R}^d)$, the set of all polynomials with real coefficients in d variables, that is*

$$\mathscr{P}(\mathbb{R}^d) = span(\{\mathbf{h}_\nu : |\nu| \geq 0\}).$$

ii) Let $\mathscr{C}_k$ be the closed subspace of $L^2(\gamma_d)$ generated by $\{\mathbf{h}_\nu : |\nu| = k\}$, that is

$$\mathscr{C}_k = \overline{span(\{\mathbf{h}_\nu : |\nu| = k\})}^{L^2(\gamma_d)} \tag{1.68}$$

then $\mathscr{C}_k$ is a subspace of dimension $\binom{k+n-1}{k}$. Moreover, $\{\mathscr{C}_k\}$ is an orthogonal decomposition of $L^2(\gamma_d)$, called Wiener chaos or the Wiener–Ito decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{k=0}^{\infty} \mathscr{C}_k. \tag{1.69}$$

Proof.

i) Trivially, from (1.62), it is clear that

$$\operatorname{span}\Big(\Big\{\mathbf{h}_\nu : |\nu| \geq 0\Big\}\Big) \subset \mathscr{P}(\mathbb{R}^d).$$

But we can prove that

$$(2x)^\nu = \sum_{2\eta \leq \nu} \binom{\nu}{\eta} (-1)^{|\eta|} \frac{(2\eta)!}{\eta!} H_{\nu-2\eta}(x). \tag{1.70}$$

Then, as $\{1, x, x^2, \cdots, x^n, \cdots\}$ is the canonical basis of $\mathscr{P}(\mathbb{R}^d)$, we immediately get the other inclusion.

ii) For the fact that the dimension of $\mathscr{C}_k$ is $\binom{k+n-1}{k}$ corresponds to the typical problem of combinations of multi-sets, see [36, Chapter 3, §3.5]. Now, the fact that the subspaces $\mathscr{C}_k$ and $\mathscr{C}_{k'}$ are orthogonal if $k \neq k'$ follows directly from the orthogonality of the Hermite polynomials. From Proposition 1.10, it can be shown that $\{\mathbf{H}_\nu\}_\nu$, is complete in $L^2(\gamma_d)$; the orthogonal decomposition of $L^2(\gamma_d)$ follows immediately from there. □

The Wiener chaos decomposition has an interesting probabilistic interpretation in terms of stochastic integrals obtained by K. Ito, but this is beyond the scope of the book (see for instance [288] or [218]).

Definition 1.15. *For each k, let* $\mathbf{J}_k : L^2(\gamma_d) \to \mathscr{C}_k$ *be the orthogonal projection of* $L^2(\gamma_d)$ *onto* $\mathscr{C}_k$*, which is continuous and (formally) self-adjoint on* $L^2(\gamma_d)$*. Then, the Hermite expansion of* $f \in L^2(\gamma_d)$ *can be written as*

$$f = \sum_{k=0}^{\infty} \mathbf{J}_k f = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu, \tag{1.71}$$

where

$$\mathbf{J}_k f = \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu. \tag{1.72}$$

Moreover, as we prove later, as a consequence of the hypercontractivity of the Ornstein–Uhlenbeck semigroup, the projection $\mathbf{J}_k$ (restricted to the polynomials) can be extended continuously to $L^p(\gamma_d)$ for $1 < p < \infty$.

In this book, we study only harmonic analysis with respect to Hermite polynomial expansions; thus, considerations of results related to Hermite functions or other classical orthogonal polynomials or orthogonal functions are beyond its scope. For more information on the latter, we refer to the next section.

1.4 Notes and Further Results

1. In [168] J. Maas, J. van Neerven, and P. Portal have another lemma, along the same lines as Lemma 1.5.

Lemma 1.16. *Let $a, A > 0$ be given. If $B(x,r) \in \mathscr{B}_a$ and $B(y,r') \in \mathscr{B}_A$ have a non-empty intersection, then*

$$|x-y| < k\min\{m(x), m(y)\},$$

where $k = k_{a,A} = \max\{2a\max\{a+A,1\}+A, 2A\max\{a+A,1\}+a\}$.

Proof. We have three cases:

- If $|y| \leq 1$, then $m(x) \leq 1 = m(y)$.
- If $|y| > 1$ and $|y| \leq 2(a+A)$, then

$$m(x) \leq 1 \leq 2(a+A)\frac{1}{|y|} = 2(a+A)m(y).$$

- If $|y| > 1$ and $|y| = C(a+A)$, with $C > 2$, then

$$|x| \geq |y| - r - r' \geq |y| - a - A = (C-1)(a+A),$$

and therefore,

$$m(x) \leq \frac{1}{|x|} \leq \frac{C}{C-1}\frac{1}{C(a+A)} = \frac{C}{C-1}\frac{1}{|y|} = \frac{C}{C-1}m(y) \leq 2m(y).$$

Hence, in each of these cases,

$$|x-y| \leq r + r' \leq am(x) + Am(y) \leq (2a\max\{a+A,1\}+A)m(y).$$

By symmetry, the same argument yields

$$|x-y| \leq (2A\max\{a+A,1\}+a)m(x),$$

and the result follows. □

2. In [174, Proposition 2.1 iii)], G. Mauceri and S. Meda also proved that if $B, B' \in \mathscr{B}_a$, $B \cap B' \neq \emptyset$ and $\gamma_d(B') \leq 2\gamma_d(B)$, then

$$r_{B'} \leq (2e^{8a+a^2})^{1/d} r_B. \tag{1.73}$$

Because using inequality (1.15) we get

$$\gamma_d(B') \geq \frac{1}{\pi^{d/2}} e^{-|c_{B'}|^2} e^{-2a-a^2}|B'| \quad \text{and} \quad \gamma_d(B) \leq \frac{1}{\pi^{d/2}} e^{-|c_B|^2} e^{2a}|B|.$$

Thus, the assumption $\gamma_d(B') \leq 2\gamma_d(B)$, implies that

$$e^{-|c_{B'}|^2} e^{-2a-a^2}|B'| \leq 2e^{-|c_{B'}|^2} e^{2a}|B|.$$

Therefore, because the Gaussian density is a radially decreasing function, the ball B' satisfying the assumptions and with maximal radius is that of volume $2\gamma_d(B)$ such that $|c_{B'}| \geq |c_B|$ and c_B and $c_{B'}$ are collinear with the origin. In this case $|c_{B'}| - |c_B| = r_{B'} + r_B$, so that

$$\begin{aligned}\left(\frac{r_{B'}}{r_B}\right)^d &\leq 2e^{|c_{B'}|^2-|c_B|^2} e^{4a+a^2} = 2e^{(|c_{B'}|+|c_B|)(|c_{B'}|-|c_B|)} e^{4a+a^2} \\ &\leq 2e^{2a+|c_{B'}|r_B+|c_B|r_{B'}} e^{4a+a^2} \leq 2e^{8a+a^2}.\end{aligned}$$

3. In [168], J. Maas, J. van Neerven, and P. Portal proved that the Gaussian measure satisfies the doubling property on a family of admissible cubes Δ^{γ}.

   Lemma 1.17. *For $\alpha > 0$, let αQ be the cube with the same center as Q that has a side length α times the side length of Q. Then, there exists a constant $C = C_{\alpha,d}$ depending only on α and the dimension d, such that for any cube $Q \in \Delta^{\gamma}$,[15] we have*
$$\gamma_d(\alpha Q) \leq C\gamma_d(Q). \tag{1.74}$$

   Proof. Without loss of generality we may assume that $\alpha > 1$. Let $Q \in \Delta^{\gamma}_{k,l}$ with center y and side-length $2s$, and let $B = B(y,s)$. Then, $B \subset Q$, and moreover $\alpha Q \subset \alpha\sqrt{d}B$. Now, if $|y| > 1$
$$2s = \frac{\text{diam(Q)}}{\sqrt{d}} = 2^{-k-l} \leq 2^{-l} \leq \frac{\sqrt{d}}{|y|} = \sqrt{d}\, m(y),$$
   where $m(y) = 1 \wedge \frac{1}{|y|}$ is the admissibility function. If $|y| \leq 1$,
$$2s = \frac{\text{diam(Q)}}{\sqrt{d}} = 2^{-k-l} \leq 1 \leq \sqrt{d}\, m(y),$$
   thus, $B \in \mathscr{B}_{\sqrt{d}/2}$. Using the doubling property of the Gaussian measure on $\mathscr{B}_{\sqrt{d}/2}$, see Proposition 1.6, there exists $C = C(\alpha, d)$ such that
$$\gamma_d(\alpha Q) \leq \gamma_d(\alpha\sqrt{d}B) \leq C\gamma_d(B) \leq C\gamma_d(Q). \qquad \square$$

4. The *Hermite functions* are defined as
$$\Psi_0(x) = 1$$
   and, for $n \geq 1$,
$$\Psi_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-x^2}). \tag{1.75}$$
   Therefore, it is clear from Rodrigues formula (1.28) that
$$\Psi_n(x) = H_n(x)e^{\frac{-x^2}{2}};$$
   hence, $\{\Psi_n\}_{n\geq 0}$ is an orthogonal system with respect to the Lebesgue measure, that is
$$\int_{-\infty}^{\infty} \Psi_n(x)\Psi_m(x)dx = 0,$$
   if $n \neq m$. Moreover, their properties can be easily deduced from the corresponding properties of the Hermite polynomials. In particular, the Hermite functions $\{\psi_n\}_n$ are eigenfunctions of the *Hermite operator*

[15] See the definition of Δ^{γ} and $\Delta^{\gamma}_{k,l}$ in (4.6), Chapter 4.

$$H = -\frac{d^2}{dx^2} + x^2, \tag{1.76}$$

associated with the eigenvalues $\{(2n+1)\}$, i.e.,

$$-\frac{d^2\Psi_n(x)}{dx^2} + x^2\Psi_n(x) = (2n+1)\Psi_n(x).$$

Observe that

$$H = \frac{1}{2}\Big[\Big(-\frac{d}{dx}+x\Big)\Big(\frac{d}{dx}+x\Big) + \Big(\frac{d}{dx}+x\Big)\Big(-\frac{d}{dx}+x\Big)\Big] = \frac{1}{2}(AA^* + A^*A),$$

where $A = (-\frac{d}{dx}+x)$ and $A^* = (\frac{d}{dx}+x)$. A and * are called the *creation and annihilation operators* in quantum mechanics (see [270]).

We define the *normalized Hermite functions* as

$$\psi_n(x) = \frac{\Psi_n(x)}{(\pi^{1/2}2^n n!)^{1/2}}. \tag{1.77}$$

They can also be written in the form

$$\psi_n(x) = h_n(x)\frac{e^{-\frac{x^2}{2}}}{\pi^{1/2}}. \tag{1.78}$$

The paper of A. González Dominguez [114] is an important early reference to the modern study of Hermite functions.

Additionally, by induction and taking the Fourier transform, we can see that the Hermite functions are eigenfunctions of the Fourier transform; see for instance [149] or [270, Lemma 1.1.3.].

The *Hermite functions in d-variables of order* $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}_0^d$, $\overline{\Psi}_\nu$, are defined as the tensor products of Hermite functions in one variable,

$$\overline{\Psi}_\nu(x) = \prod_{i=1}^d \Psi_{\nu_i}(x_i),$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $\Psi_{\nu_i}(x_i)$ is the ν_i-Hermite function in the variable x_i.
Analogously, the *normalized Hermite functions in d- variables of order* $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}_0^d$ are defined as the tensor products of Hermite functions in one variable,

$$\overline{\psi}_\nu(x) = \prod_{i=1}^d \psi_{\nu_i}(x_i),$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $\psi_{\nu_i}(x_i)$ is the normalized ν_i-Hermite function in the variable x_i.

Observe that defining, for each $1 < p < \infty$, the map $\Xi_d^{(p)} : L^p(\gamma_d) \to L^p(\mathbb{R}^d)$ as

$$\Xi_d^{(p)} f(x) = f(x)\pi^{-(d/2)p} e^{-|x|^2/p}, \tag{1.79}$$

then, $\Xi_d^{(p)}$ is clearly an isometric isomorphism. In particular, $\Xi_1^{(2)} H_n$ is a multiple of Ψ_n or, equivalently, $\Xi_1^{(2)}(h_n) = \psi_n$. Analogously, $\Xi_d^{(2)} \mathbf{H}_\alpha$ is a multiple of $\overline{\Psi}_\alpha$.

5. In spite of the fact that Hermite polynomials are dense in $L^p(\gamma_d)$ for $1 \leq p < \infty$, in [230], H. Pollard proved that $S_n f \to f$ in $L^p(\gamma_1)$, that is

$$\int_{-\infty}^{\infty} |S_n f(x) - f(x)|^p \gamma_1(dx) \to 0,$$

as $n \to \infty$, if and only if $p = 2$ using the fact that the Hermite polynomials are a limiting case of the ultraspherical polynomials (see 10.67). But $p = 2$ is a trivial case from the Hilbert space theory. Pollard's counterexample is the following: given $1 < p < 2$, let us consider the function

$$f(x) = e^{cx^2}, \tag{1.80}$$

with $\frac{1}{2} < c < \frac{1}{p}$. Then, $f \in L^p(\gamma_d)$. It can be shown that for any $k \in \mathbb{N}$,

$$\widehat{f}_H(2k+1) = 0 \quad \text{and} \quad \widehat{f}_H(2k) = M\Big(\frac{c}{1-c}\Big)^k \frac{1}{4^k k!}.$$

then,

$$\begin{aligned}\widehat{f}_H(2k) \int_{-\infty}^{\infty} |H_{2k}(x)|^p e^{-x^2} dx &\geq \frac{M}{(2k+1)^{1/2}} \Big(\frac{c}{1-c}\Big)^k \int_{(2k+1)^{1/2}\pi}^{2(2k+1)^{1/2}\pi} |\cos x| dx \\ &\geq M\Big(\frac{c}{1-c}\Big)^{(2k+1)^{1/2}};\end{aligned}$$

therefore,

$$\limsup_{k\to\infty} \hat{f}_H(2k) \int_{-\infty}^{\infty} |H_{2k}(x)|^p e^{-x^2} dx = \infty.$$

For more details see [230].

6. The other families of classical orthogonal polynomials, the *Jacobi polynomials* and the *Laguerre polynomials* are considered briefly in Appendix B. Similar to the Hermite case, *Jacobi functions* and *Laguerre functions* can also be defined. For more information see [53] or [262].

7. There is a more general class of Hermite polynomials, $\{H_n^\mu\}$ the *generalized Hermite polynomials*. They were defined by G. Szëgo in [262] (see problem 25, p. 380) and studied in detail by T. S. Chihara in his Ph.D. thesis [54]. They are defined as being orthogonal polynomials with respect to the measure

$$d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx, \tag{1.81}$$

with $\mu > -1/2$. When $\mu = 0$ these polynomials coincide, up to a constant, with the classical Hermite polynomials.
Nevertheless, these polynomials are not classical polynomials as they satisfy a second-order differential-difference equation (10.47) instead of a second-order differential equation, i.e., they are eigenfunctions of the differential-difference operator (10.48)

$$L_\mu = \frac{1}{2}\frac{d^2}{dx^2} + (\frac{\mu}{x} - x)\frac{d}{dx} - \mu\frac{I - \tilde{I}}{2x^2}.$$

For more details, see Appendix and [54].

2

The Ornstein–Uhlenbeck Operator and the Ornstein–Uhlenbeck Semigroup

In this chapter we are going to define and study the Ornstein–Uhlenbeck operator and the Ornstein–Uhlenbeck semigroup. They are analogous, in the Gaussian harmonic analysis, to the Laplacian and the heat semigroup in the classical case. Then, we study an important property of the Ornstein–Uhlenbeck semigroup, the hypercontractivity property, and some of its applications.

2.1 The Ornstein–Uhlenbeck Operator

In the classical case, we consider the Laplacian differential operator $\Delta_x = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ in $\mathbb{R}^d$ and the eigenvalue problem

$$\Delta_x u = \lambda u \tag{2.1}$$

with boundary condition

$$u(x) = O(1), \quad \text{as } |x| \to \infty.$$

Then, the set of eigenvalues of this problem consists of all non-positive real numbers, and given $\lambda < 0$ the eigenfunctions corresponding to λ are

$$e^{i\langle \cdot, y\rangle}, \quad |y|^2 = -\lambda. \tag{2.2}$$

The *Ornstein–Uhlenbeck operator* in $\mathbb{R}^d$ is a second-order differential operator defined as

$$L = \frac{1}{2}\Delta_x - \langle x, \nabla_x\rangle = \sum_{i=1}^{d}\left[\frac{1}{2}\frac{\partial^2}{\partial x_i^2} - x_i\frac{\partial}{\partial x_i}\right], \tag{2.3}$$

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where $\nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d})$ is the gradient, and Δ_x is the Laplace operator defined in the space of test functions $C_0^\infty(\mathbb{R}^d)$ of smooth functions with compact support on $\mathbb{R}^d$. The operator L has a self-adjoint extension to $L^2(\gamma_d)$, that is also denoted as L, that is,

$$\int_{\mathbb{R}^d} Lf(x)g(x)\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\gamma_d(dx); \tag{2.4}$$

thus, L is the natural *"symmetric" Laplacian* in this context.

The Ornstein–Uhlenbeck operator L can also be written as

$$L = \sum_{i=1}^{d} L_i, \tag{2.5}$$

where $L_i = \frac{1}{2}\partial_i^2 - x_i\partial_i$, $i = 1, \cdots d$, is the one-dimensional Ornstein–Uhlenbeck operator in the i-th variable. Hence, for $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$ and $\nu = (\nu_, \cdots, \nu_d)$ a multi-index,

$$\begin{aligned} L\mathbf{H}_\nu(x) &= \sum_{i=1}^{d} \Big[\frac{1}{2}\frac{\partial^2 \mathbf{H}_\nu}{\partial x_i}(x) - x_i \frac{\partial \mathbf{H}_\nu(x)}{\partial x_i}(x)\Big] \\ &= \sum_{i=1}^{d} \Big[\frac{1}{2}\frac{\partial^2}{\partial x_i^2}\prod_{j=1}^{d} H_{\nu_j}(x_j) - x_i\frac{\partial}{\partial x_i}\prod_{j=1}^{d} H_{\nu_j}(x_j)\Big] \\ &= \sum_{i=1}^{d} \prod_{j=1, j\neq i}^{d} H_{\nu_j}(x_j)\Big[\frac{1}{2}\frac{\partial^2 H_{\nu_i}}{\partial x_i}(x_i) - x_i\frac{\partial H_{\nu_i}}{\partial x_i}(x_i)\Big] \\ &= \sum_{i=1}^{d} \prod_{j=1, j\neq i}^{d} H_{\nu_j}(x_j) L_i H_{\nu_i}(x_i) = \sum_{i=1}^{d}(-\nu_i)\prod_{j=1}^{d} H_{\nu_j}(x_j) \\ &= \sum_{i=1}^{d}(-\nu_i)\mathbf{H}_\nu(x) = -|\nu|\mathbf{H}_\nu(x). \end{aligned} \tag{2.6}$$

Thus, the Hermite polynomials in d-variables, $\{\mathbf{H}_\nu\}_\nu$ are eigenfunctions of L with corresponding eigenvalues $\lambda_\nu = -|\nu| = -\sum_{i=1}^{d} \nu_i$, i.e.,

$$L\mathbf{H}_\nu = \lambda_\nu \mathbf{H}_\nu = -|\nu|\mathbf{H}_\nu, \tag{2.7}$$

and the normalized Hermite polynomials $\mathbf{h}_\nu$ are also eigenfunctions of the Ornstein–Uhlenbeck operator, with the same corresponding eigenvalue,

$$L\mathbf{h}_\nu = \lambda_\nu \mathbf{h}_\nu = -|\nu|\mathbf{h}_\nu.$$

Moreover, consider the eigenvalue problem

$$Lu = \lambda u \tag{2.8}$$

with boundary condition

$$u(x) = O(|x|^k), \text{ for some } k \geq 0 \quad \text{as } |x| \to \infty.$$

Then, the set of eigenvalues is the set of negative integers and the eigenfunctions corresponding to $\lambda = -n$ are d-dimensional Hermite polynomials of degree ν, $\mathbf{H}_\nu$, such that $|\nu| = n$.

Hence, the $L^2(\gamma_d)$ spectrum of L is $\{\cdots, -2, -1, 0\}$. This coincides with the $L^p(\gamma_d)$-spectrum for $1 < p < \infty$.[1] Then, the spectral decomposition of L is given by

$$Lf = \sum_{k=0}^{\infty} (-k)\mathbf{J}_k f, \tag{2.9}$$

where, as before see Definition 1.15, $\mathbf{J}_k f = \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu$. Then, the domain of L is given by

$$D(L) = \{f \in L^2(\gamma_d) : \sum_{k=0}^{\infty} k^2 ||\mathbf{J}_k f||_{2,\gamma}^2 < \infty\}, \tag{2.10}$$

and the spectral decomposition (2.9) is well defined for any $f = \sum_{k=0}^{\infty} \mathbf{J}_k f \in D(L)$.

For $i = 1, 2, \cdots, d$ let us consider the differential operators

$$\partial_\gamma^i = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i}. \tag{2.11}$$

∂_γ^i is neither symmetric nor antisymmetric in $L^2(\gamma_d)$. In fact, its formal $L^2(\gamma_d)$-adjoint[2] is

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}} e^{x_i^2} \frac{\partial}{\partial x_i}(e^{-x_i^2} I) = \sqrt{2} x_i I_d - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i}, \tag{2.12}$$

where I_d is the identity in $\mathbb{R}^d$, because, simply by integration by parts,

$$\begin{aligned}
\int_{\mathbb{R}^d} (\partial_\gamma^i f)(x) g(x) \frac{e^{-|x|^2}}{\pi^{d/2}} \, dx &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} \left[\frac{\partial f}{\partial x_i}(x)\right] g(x) \frac{e^{-|x|^2}}{\pi^{d/2}} \, dx \\
&= -\frac{1}{\sqrt{2}} \int_{\mathbb{R}^d} f(x) \frac{\partial}{\partial x_i} \left[g(x) \frac{e^{-|x|^2}}{\pi^{d/2}}\right] dx \\
&= \int_{\mathbb{R}^d} f(x) \left[\sqrt{2} x_i g(x) - \frac{1}{\sqrt{2}} \frac{\partial g}{\partial x_i}(x)\right] \frac{e^{-|x|^2}}{\pi^{d/2}}] \, dx \\
&= \int_{\mathbb{R}^d} f(x) ((\partial_\gamma^i)^* g)(x) \frac{e^{-|x|^2}}{\pi^{d/2}} \, dx.
\end{aligned}$$

[1] The $L^1(\gamma_d)$-spectrum of L is the closed right half plane. We will prove this in detail later (see Theorem 2.7, see also E. B. Davies [65, Theorem 4.3.5]).

[2] In $L^2(\mathbb{R}^d)$, $\frac{\partial}{\partial x_i}$ is antisymmetric, by integration by parts.

Observe that $(\partial_\gamma^i)^*$ can be written as

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}} e^{|x|^2}(\partial_i e^{-|x|^2} I).$$

Moreover, it is easy to see that

$$(-L) = \sum_{i=1}^{d} (\partial_\gamma^i)^* \partial_\gamma^i. \tag{2.13}$$

Observe that the commutator $[\partial_\gamma^i, (\partial_\gamma^i)^*]$, is the identity;[3]

$$\begin{aligned}
[\partial_\gamma^i, (\partial_\gamma^i)^*] f(x) &= \partial_\gamma^i (\partial_\gamma^i)^* f(x) - (\partial_\gamma^i)^* \partial_\gamma^i f(x) \\
&= \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i} \Big(\sqrt{2} x_i f(x) - \frac{1}{\sqrt{2}} \frac{\partial f(x)}{\partial x_i} \Big) - \Big(\sqrt{2} x_i I - \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_i} \Big) \Big(\frac{1}{\sqrt{2}} \frac{\partial f(x)}{\partial x_i} \Big) \\
&= f(x) + \sqrt{2} x_i \frac{\partial f(x)}{\partial x_i} - \frac{1}{2} \frac{\partial^2 f(x)}{\partial x_i^2} - x_i \frac{\partial f(x)}{\partial x_i} + \frac{1}{2} \frac{\partial^2 f(x)}{\partial x_i^2} \\
&= f(x).
\end{aligned}$$

Reversing the order in (2.13), we get another second-order differential operator, which will be denoted as $\overline{L}$,

$$(-\overline{L}) = \sum_{i=1}^{d} \partial_\gamma^i (\partial_\gamma^i)^* = (-L) + dI = -\frac{1}{2}\Delta_x + \langle x, \nabla_x \rangle + dI, \tag{2.14}$$

and therefore,

$$\overline{L} = L - dI = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle - dI. \tag{2.15}$$

We will call $\overline{L}$ the *alternative Ornstein–Uhlenbeck* operator. The Hermite polynomials $\{\mathbf{H}_\nu\}_\nu$ are also eigenfunctions of $\overline{L}$, with eigenvalues $\overline{\lambda}_\nu = -(|\nu| + 1)$, i.e.,

$$\overline{L}\mathbf{H}_\nu = (\lambda_\nu - 1)\mathbf{H}_\nu = -(|\nu| + 1)\mathbf{H}_\nu, \tag{2.16}$$

The differential operators ∂_γ^i are considered the "natural" notions of (partial) derivatives for the Gaussian case, and we call it simply the *Gaussian partial derivatives*. Nevertheless, as we already know, there is another notion of Gaussian differentiation, namely, $(\partial_\gamma^i)^*$. The operators $\partial_\gamma^i, (\partial_\gamma^i)^*$ are called the *creation and annihilation operators* in quantum mechanics.[4]

[3] Recall that, the commutator of two operators A, B is defined as $[A, B] = AB - BA$.

[4] In [210] there is a general analysis of this decomposition for orthogonal polynomials and functions, which is highly recommended.

Thus, the notion of (partial) differentiation in Gaussian harmonic analysis is, up to a constant, the same as in the classical case. These facts are important later on when we discuss the Riesz transforms for the Gaussian measure in Chapter 9.

There are several results in Gaussian harmonic analysis that can be obtained by what is called the *tensorization* argument, see [20, 284], which implies that it is enough to prove only the case $d = 1$ because the case $d > 1$ follows immediately by the tensor product structure.

In this case, the *square field operator* in $\mathbb{R}^d$ is given by

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - gLf - fLg) = \frac{1}{2}\sum_{i=1}^{d} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_i} = \frac{1}{2}\langle \nabla_x f, \nabla_x g\rangle. \quad (2.17)$$

Consider the infinitesimal generator O of an operator semigroup $\{\mathscr{T}_t\}$, and symmetric with respect to the measure μ, the *Dirichlet form* associated with O is defined as

$$\mathscr{E}_\mu(f) = \lim_{t\to 0} \frac{\langle f - \mathscr{T}_t f, f\rangle_\mu}{t} = \langle -Of, f\rangle_\mu = -\int_E f(Of)\, d\mu. \quad (2.18)$$

Then, by symmetry, it can be proved that

$$\mathscr{E}_\mu(f) = \int_E \Gamma(f,f)\, d\mu. \quad (2.19)$$

for $f \in L^2(\mu)$; see [120, 284].

Hence, the *Dirichlet form* associated with the Ornstein–Uhlenbeck operator L and the Gaussian measure γ_d is given by

$$\mathscr{E}_\gamma(f)(x) = \int_{\mathbb{R}^d} \Gamma(f,f)(x)\, \gamma_d(dx) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x f(x)|^2\, \gamma_d(dx). \quad (2.20)$$

This can be obtained simply using integration by parts, as

$$\int_{\mathbb{R}^d} \langle \nabla_x f(x), \nabla_x g(x)\rangle \gamma_d(dx) = 2\int_{\mathbb{R}^d} f(x)(-L)g(x)\gamma_d(dx), \quad (2.21)$$

for $f, g \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz class. In particular, this implies that $(-L)$ is positive definite and that the Ornstein–Uhlenbeck operator is (formally) self-adjoint in $L^2(\gamma_d)$

$$\int_{\mathbb{R}^d} Lf(x)g(x)\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Lg(x)\gamma_d(dx). \quad (2.22)$$

Therefore, as already mentioned L is the "symmetric Laplacian" in this context and the Gaussian measure γ_d is the natural measure for studying the operators associated with the operator L.

In addition, the *iterated square field operator* $\Gamma_2(f,g)$, in this case, is given by

$$\Gamma_2(f,g) = \frac{1}{2}[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(g,Lf)]. \tag{2.23}$$

Finally, in [168, Lemma 4.1] J. Maas, J. van Neerven, and P. Portal obtained a Gaussian version of the *parabolic Caccioppoli inequality*. We consider here only the real version.

Theorem 2.1. *Let $v : \mathbb{R}^d \times (0,\infty) \to \mathbb{R}$ be a $C^{1,2}$-function such that $v(\cdot,t) \in C_b^2(\mathbb{R}^d)$ for all $t > 0$, and suppose that*

$$\frac{\partial v}{\partial t} = Lv$$

on $I(x_0,t_0,2r) := B(x_0,2cr) \times [t_0 - 4r^2, t_0 + 4r^2]$, for some $r \in (0,1)$, $0 < C_0 \le c \le C_1 < \infty$, and $t_0 > 4r^2$. Then

$$\int_{I(x_0,t_0,r)} |\nabla_x v(x,t)|^2 \gamma_d(dx)\,dt \le C\frac{1+r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v(x,t)|^2 \gamma_d(dx)\,dt, \tag{2.24}$$

with C depending only on the dimension d, C_0, and C_1.

Proof. Let $\eta \in C^\infty(\mathbb{R}^d \times (0,\infty))$ be a cut-off function such that $0 \le \eta \le 1$ on $\mathbb{R}^d \times (0,\infty)$, $\eta \equiv 1$ on $I(x_0,t_0,r)$, $\eta \equiv 0$ on the complement of $I(x_0,t_0,2r)$, and

$$\|\nabla_x \eta\|_\infty \lesssim \frac{1}{r}, \quad \|\frac{\partial \eta}{\partial t}\|_\infty \lesssim \frac{1}{r^2}, \quad \|\Delta \eta\|_\infty \lesssim \frac{1}{r^2}$$

with the implied constants depending only on d, C_0 and C_1. Then, in view of

$$\|x \cdot \nabla_x \eta\|_\infty \lesssim (|x_0| + 2r) \cdot \frac{C'}{r},$$

and recalling that $0 < r < 1$, we have

$$\|L\eta\|_\infty \lesssim \frac{1}{r^2} + \frac{1}{r}|x_0| + 1 \lesssim \frac{1+r|x_0|}{r^2}, \tag{2.25}$$

where the implied constants depend only on d, C_0, C_1. By integrating the identity

$$|\eta \nabla_x v|^2 = \langle \eta \nabla_x v, \eta \nabla_x v\rangle = \langle (v\nabla_x \eta - \nabla_x(v\eta)), (v\nabla_x \eta - \nabla_x(v\eta))\rangle,$$

and then using the fact that

$$\begin{aligned}\int_{I(x_0,t_0,2r)} \eta^2 \langle \nabla_x(v\eta), \nabla_x(v\eta)\rangle\, d\gamma_d\, dt &\le \int_0^\infty \int_{\mathbb{R}^d} \langle \nabla_x(v\eta), \nabla_x(v\eta)\rangle\, d\gamma_d\, dt \\ &= 2\int_0^\infty \int_{\mathbb{R}^d} v\eta(-L)(v\eta)\, d\gamma_d\, dt \\ &= -2\int_{I(x_0,t_0,2r)} v\eta L(v\eta)\, d\gamma_d\, dt,\end{aligned}$$

According to (2.21), we obtain

$$\begin{aligned}\int_{I(x_0,t_0,r)} |\nabla_x v|^2 \, d\gamma_d \, dt &\le \int_{I(x_0,t_0,2r)} \eta^2 |\eta \nabla_x v|^2 \, d\gamma_d \, dt \\ &\le \int_{I(x_0,t_0,2r)} \eta^2 |v \nabla_x \eta|^2 \, d\gamma_d \, dt \\ &\quad + 2\Big| \int_{I(x_0,t_0,2r)} v\eta^2 \langle \nabla_x (v\eta), \nabla_x \eta \rangle \, d\gamma_d \, dt \Big| \\ &\quad + 2\Big| \int_{I(x_0,t_0,2r)} v\eta L(v\eta) \, d\gamma_d \, dt \Big|.\end{aligned} \tag{2.26}$$

For the first term on the right-hand side we have the estimate

$$\int_{I(x_0,t_0,2r)} \eta^2 |v \nabla_x \eta|^2 \, d\gamma_d \, dt \lesssim \frac{1}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma_d \, dt.$$

For the second term we have, by (2.25),

$$\begin{aligned}\Big| \int_{I(x_0,t_0,2r)} 2v\eta^2 \langle \nabla_x (v\eta), \nabla_x \eta \rangle \, d\gamma_d \, dt \Big| &= \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} \langle \nabla_x (v\eta)^2, \nabla_x \eta^2 \rangle \, d\gamma_d \, dt \Big| \\ &\le \Big| \int_{\mathbb{R}^d} (v\eta)^2 L\eta^2 \, d\gamma_d \, dt \Big| \\ &\lesssim \frac{1 + r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma_d \, dt\end{aligned}$$

where we used the fact that η^2 satisfies the same assumptions as η to apply (2.25) to η^2. To estimate the third term on the right-hand side of (2.26), we substitute the identity

$$L(v\eta) = \eta L v + v L\eta - \langle \nabla_x v, \nabla_x \eta \rangle = \eta \frac{\partial v}{\partial t} + v L\eta - \langle \nabla_x v, \nabla_x \eta \rangle$$

and estimate each of the resulting integrals:

$$\begin{aligned}\Big| \int_{I(x_0,t_0,2r)} v\eta^2 \frac{\partial v}{\partial t} \, d\gamma dt \Big| &= \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} \eta^2 \frac{\partial v^2}{\partial t} \, d\gamma_d \, dt \Big| = \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} v^2 \frac{\partial \eta^2}{\partial t} \, d\gamma_d \, dt \Big| \\ &= \Big| \int_{I(x_0,t_0,2r)} v^2 \eta \frac{\partial \eta}{\partial t} \, d\gamma_d \, dt \Big| \lesssim \frac{1}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma_d \, dt,\end{aligned}$$

$$\Big| \int_{I(x_0,t_0,2r)} v^2 \eta L\eta \, d\gamma_d \, dt \Big| \lesssim \frac{1 + r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma_d \, dt,$$

and

$$\Big| \int_{I(x_0,t_0,2r)} v\eta \langle \nabla_x v, \nabla_x \eta \rangle \, d\gamma_d \, dt \Big| = \frac{1}{4} \Big| \int_{I(x_0,t_0,2r)} \langle \nabla_x v^2, t\nabla_x \eta^2 \rangle \, d\gamma_d \, dt \Big|$$
$$= \frac{1}{4} \Big| \int_{\mathbb{R}^d} v^2 L\eta^2 \, d\gamma_d \, dt \Big|$$
$$\lesssim \frac{1 + r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma_d \, dt. \qquad \square$$

2.2 Definition and Basic Properties of the Ornstein–Uhlenbeck Semigroup

Now, we consider the *Ornstein–Uhlenbeck semigroup*. On $L^2(\gamma_d)$ the closure of the Ornstein–Uhlenbeck operator L generates an operator semigroup.

Definition 2.2. *The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is the semigroup of operators generated in $L^2(\gamma_d)$ by the Ornstein–Uhlenbeck operator L as infinitesimal generator, i.e., formally $T_t = e^{-tL}$. In view of the spectral theorem, for $f = \sum_{k=0}^{\infty} \mathbf{J}_k f \in L^2(\gamma_d)$ and $t \geq 0$, T_t is defined as*

$$T_t f = \sum_{\nu} e^{-t|\nu|} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu = \sum_{k=0}^{\infty} e^{-tk} \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu = \sum_{k=0}^{\infty} e^{-tk} \mathbf{J}_k f, \qquad (2.27)$$

where $\mathbf{J}_k f = \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu$ is the orthogonal projection of $L^2(\gamma_d)$ onto $\mathscr{C}_k$.

The Ornstein–Uhlenbeck semigroup have the following representations.

Proposition 2.3. *(C. P. Calderón- B. Muckenhoupt)*

i) *If $f \in L^2(\gamma_d)$, then $\sum_{k=0}^{\infty} e^{-tk} \mathbf{J}_k f(x)$ converges absolutely to $T_t f(x)$ almost everywhere (a.e.)γ_d.*

ii) *For any $1 \leq p <, 2$ there exists a function $f \in L^p(\gamma_d)$ and $t \geq 0$ such that $\sum_{k=0}^{\infty} e^{-tk} \mathbf{J}_k f(x)$ diverges for all x.*

iii) *For any $t > 0$ the integral representation for T_t is given by*

$$T_t f(x) = \frac{1}{(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y) \gamma_d(dy). \qquad (2.28)$$

Proof.

i) Observe that for each multi-index ν, $|\nu| > 0$, according to (1.64) and the Cauchy–Schwartz inequality, we have

$$|\langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu(x)| \leq C_{\nu,x} \nu! ||f||_{2,\gamma} = C'_{\nu,x} ||f||_{2,\gamma}.$$

Therefore, the sequence $\{\langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu(x)\}$ is bounded for each x; thus using to the Weierstrass M-test, the series $\sum_{k=0}^{\infty} e^{-tk} \mathbf{J}_k f(x)$ converges absolutely for any x. Because $L^2(\gamma_d) \subset L^1(\gamma_d)$, then, according to the first part, $T_t f(x)$ has the expansion $\sum_{k=0}^{\infty} e^{-tk} \mathbf{J}_k f(x)$, and this must be the limit a.e.

ii) Using the multiplicative character of the Gaussian measure γ_d, it is enough to consider the case $d = 1$. According to Pollard's counterexample [230], for $1 \leq p < 2$, there exists a function $f \in L^p(\gamma_1)$ such that

$$\limsup_{k\to\infty}(\langle f, \mathbf{h}_k\rangle_{\gamma_d}|H_k(x)|)^{1/k}$$

is a fixed number greater than 1, for any x. Therefore, for t close enough to zero (i.e., e^{-t} close enough to 1), the expansion of $T_t f$ diverges for any x.

iii) Using again (1.64), the Cauchy–Schwartz inequality and Stirling's formula, we get for $|\nu| = k$

$$\begin{aligned}\int_{\mathbb{R}^d} e^{-tk}|f(y)||\mathbf{h}_\nu(y)||\mathbf{h}_\nu(x)|\gamma_d(dy) &\leq \|f\|_{2,\gamma}\Big(\int_{\mathbb{R}^d} e^{-2tk}|\mathbf{h}_\nu(y)|^2|\mathbf{h}_\nu(x)|^2\gamma_d(dy)\Big)^{1/2}\\ &\leq \|f\|_{2,\gamma}e^{-tk}C_{\nu,x}(\nu!)^{1/2}\Big(\int_{\mathbb{R}^d}|\mathbf{h}_\nu(y)|^2\gamma_d(dy)\Big)^{1/2}\\ &\leq C_{\nu,x}\|f\|_{2,\gamma}e^{-tk}.\end{aligned}$$

Then, using this, Lebesgue's dominated convergence theorem and the d-dimensional Mehler's formula (10.24), for $r = e^{-t}$ we get

$$\begin{aligned}T_t f(x) &= \sum_{|\nu|\geq 0} e^{-t|\nu|}\Big[\int_{\mathbb{R}^d} f(y)\mathbf{h}_\nu(y)\gamma_d(dy)\Big]\mathbf{h}_\nu(x)\\ &= \int_{\mathbb{R}^d}\Big(\sum_{|\nu|\geq 0} e^{-t|\nu|}\mathbf{h}_\nu(x)\mathbf{h}_\nu(y)\Big)f(y)\gamma_d(dy)\\ &= \frac{1}{(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy). \qquad \square\end{aligned}$$

Note that the integral representation (2.28), obtained initially for $f \in L^2(\gamma_d)$, also makes sense for $f \in L^p(\gamma_d)$, $1 \leq p < \infty$, by using Hölder's inequality. Therefore, $\{T_t\}_{t\geq 0}$ can be extended as a family of operators in $L^p(\gamma_d)$. Also note that, taking $r = e^{-t}$, (2.27) is equivalent to the Abel summability of the Hermite expansion of f

$$T_r f = \sum_{k=0}^{\infty} r^k \mathbf{J}_k f.$$

Using this approach, B. Muckenhoupt [193], considered the so-called Poisson integral for the Hermite expansion for $d = 1$, and also C.P. Calderón [44] for the case $d \geq 1$.

The kernel

$$M_t(x,y) = \frac{1}{(1-e^{-2t})^{d/2}} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}}, \tag{2.29}$$

is called *Mehler's kernel*[5].

The integral representation of T_t can be written in several equivalent forms. The first one provides the link between the Ornstein–Uhlenbeck semigroup and the heat semigroup,

$$\begin{aligned} T_t f(x) &= \frac{1}{(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy) \\ &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} f(y)dy, \quad t>0. \end{aligned} \tag{2.30}$$

Observe that now we are integrating with respect to the Lebesgue measure. The alternative expression,

$$M_t(x,y) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}, \tag{2.31}$$

allows us to establish a connection between Mehler's kernel and the *heat kernel*

$$k_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t}. \tag{2.32}$$

Using $\{\mathscr{T}_t\}_{t\geq 0}$, the *heat semigroup*[6]

$$\mathscr{T}_t f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4t}} f(y)\,dy,\ t>0,$$

we have the following representation of the Ornstein–Uhlenbeck semigroup

$$T_t f(x) = (k_{(1-e^{-2t})/4} * f)(e^{-t}x) = \delta_{e^{-t}}[k_{(1-e^{-2t})/4} * f](x) = \delta_{e^{-t}} \mathscr{T}_{(1-e^{-2t})/4} f(x),$$

where δ_a is the *dilation operator* by a, defined by

$$\delta_a f(x) = f(ax). \tag{2.33}$$

Thus, the Ornstein–Uhlenbeck semigroup is, after a dilation on the variable x, a reparametrization of the heat semigroup; therefore, it is not a convolution semigroup. More precisely, before taking the convolution with the properly reparametrized heat kernel, a dilation by e^{-t} is applied in the variable x. Because of this dilatation, none

[5] We have already encountered this kernel in Chapter 1, (1.41)

[6] See Appendix 10.5 for more details.

of the methods used in the study of classical semigroups can be applied to this semigroup. Nevertheless, F. Weissler [292], who denotes this semigroup as the *Hermite semigroup,*[7] establishes another explicit relation between the Ornstein–Uhlenbeck and the heat semigroups,

Theorem 2.4. *Let* $1 \leq p,q \leq \infty$*,* $t \geq 0$*, and* $\zeta \geq 0$.[8] *Then,*

$$T_t = (\zeta e^t)^{d/2} \pi^{(1/2p-1/2q)d} (\Xi_d^{(q)})^{-1} \mathfrak{M}_\beta \delta_\zeta \mathscr{T}_{\zeta(1-e^{-2t})/4e^{-t}} \mathfrak{M}_\alpha \Xi_d^{(p)}, \tag{2.34}$$

where

$$\alpha = \frac{1}{1-e^{-2t}} - \frac{1}{p} - \frac{e^{-t}}{\zeta(1-e^{-2t})},$$

$$\beta = \frac{1}{1-e^{-2t}} - \frac{1}{q'} - \frac{\zeta e^{-t}}{1-e^{-2t}},$$

$\Xi_d^{(p)} : L^p(\gamma_d) \to L^p(\mathbb{R}^d)$ *is the isometric isomorphism defined, for any* $1 < p < \infty$*, as*

$$\Xi_d^{(p)} f(x) = f(x) \pi^{-d/2p} e^{-|x|^2/p}, \tag{2.35}$$

$\mathfrak{M}_\alpha$ *is the* multiplication operator *defined as*

$$\mathfrak{M}_\alpha f(x) = e^{\alpha|x|^2} f(x),$$

and finally δ_a *is the dilation operator, as defined in (2.33).*

Using this relation, Weissler succeeded in not only extending the Ornstein–Uhlenbeck semigroup holomorphically to the half-plane $\mathrm{Re} z \geq 0$, where the heat semigroup is holomorphic but he was also able to obtain additional information on the continuity of both semigroups (for more details see [292]). We discuss later in this chapter the holomorphic Ornstein–Uhlenbeck semigroup in more detail (see page 49).

Observe that

$$M_t(x,y) = M_t(y,x) e^{|x|^2-|y|^2}.$$

Through the change of variables $u = \frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}}$, we get an alternative representation of T_t

$$T_t f(x) = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} f(\sqrt{1-e^{-2t}}u + e^{-t}x) e^{-|u|^2} du. \tag{2.36}$$

This representation of the Ornstein–Uhlenbeck semigroup allows us to extend it to a space of infinite dimensions, where the Gaussian measure, unlike the Lebesgue measure, is well defined (see P. A. Meyer [187]).

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[7] We refer to another semigroup as the Hermite semigroup, see point 10. in Section 2.5, page 70.

[8] Actually, Weissler defines it for $z \in \mathbb{C}$ such that $\mathrm{Re} z \geq 0$ and $\mathrm{Re}(\zeta e^z) \geq 0$, see [292, Theorem 1].

One problem of the kernel (2.31) is that it does not reflect the symmetry of Mehler's kernel. An alternative symmetric representation of (2.29) is given by

$$M_t(x,y) = \frac{1}{(1-e^{-2t})^{d/2}} \exp\Big(\frac{1}{2}\frac{|x+y|^2}{e^t+1} - \frac{1}{2}\frac{|x-y|^2}{e^t-1}\Big), \tag{2.37}$$

which has been used in several papers about the Ornstein–Uhlenbeck semigroup, (see for instance [249] and [104]). In [265], J. Teuwen has an alternative symmetric representation:

$$M_t(x,y) = \frac{\exp(-\frac{e^{2t}|x-y|^2}{1-e^{2t}})}{(1-e^{-t})^{d/2}} \frac{\exp(2e^{-t}\frac{\langle x,y\rangle}{1+e^t})}{(1+e^{-t})^{d/2}}. \tag{2.38}$$

The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ in $\mathbb{R}^d$ is a *Markov operator semigroup* in $L^p(\gamma_d), 1 \leq p \leq \infty$, i.e., a positive conservative symmetric diffusion semigroup, strongly L^p-continuous in $L^p(\gamma_d), 1 \leq p \leq \infty$, with the Ornstein–Uhlenbeck operator L as its infinitesimal generator (see [23, 20] or [284]). Its properties can be obtained directly from the general theory of Markov semigroups (see [20] or [284]). Nevertheless, because the Ornstein–Uhlenbeck semigroup is of such great importance and serves as a "model" for Markov semigroups associated with classical orthogonal polynomials, we are going to give detailed analytic proof of its properties using its integral representation (2.28).

Theorem 2.5. *The family of operators $\{T_t : t \geq 0\}$ satisfies the following properties:*

i) Semigroup property:

$$T_{t_1+t_2} = T_{t_1} \circ T_{t_2},\ t_1, t_2 \geq 0.$$

ii) Positivity and conservative properties:

$$T_t f \geq 0, \quad \textit{for} \quad f \geq 0,\ t \geq 0,$$

and

$$T_t 1 = 1.$$

iii) Contractivity property:

$$||T_t f||_{p,\gamma} \leq ||f||_{p,\gamma},$$

for all $t \geq 0$, and $1 \leq p \leq \infty$.

iv) Strong $L^p(\gamma_d)$-continuity property: The mapping $t \to T_t f$ is continuous from $[0,\infty)$ to $L^p(\gamma_d)$, for $1 \leq p < \infty$ and $f \in L^p(\gamma_d)$.

v) Symmetry property: T_t is a self-adjoint operator in $L^2(\gamma_d)$:

$$\int_{\mathbb{R}^d} T_t f(x) g(x) \gamma_d(dx) = \int_{\mathbb{R}^d} f(x) T_t g(x) \gamma_d(dx),\ t \geq 0. \tag{2.39}$$

In particular, the Gaussian measure γ_d is the invariant measure for $\{T_t\}_{t\geq 0}$,

$$\int_{\mathbb{R}^d} T_t f(x) \gamma_d(dx) = \int_{\mathbb{R}^d} f(x) \gamma_d(dx),\ t \geq 0. \tag{2.40}$$

vi) *Infinitesimal generator: the Ornstein–Uhlenbeck operator L is the infinitesimal generator of $\{T_t : t \geq 0\}$,*

$$\lim_{t\to 0} \frac{T_t f - f}{t} = Lf. \tag{2.41}$$

Proof.

i) To prove the semigroup property, we use integral representation (2.30)[9] as follows. Let $f \in L^1(\gamma_d)$, by Fubini's theorem we have

$$\begin{aligned} T_t(T_s f)(x) &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} \\ &\quad \times \Big(\frac{1}{\pi^{d/2}(1-e^{-2s})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|z-e^{-s}y|^2}{1-e^{-2s}}} f(z)dz \Big) dy \\ &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}\pi^{d/2}(1-e^{-2s})^{d/2}} \\ &\quad \times \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \exp\Big[-\big(\frac{|y-e^{-t}x|^2}{1-e^{-2t}} + \frac{|z-e^{-s}y|^2}{1-e^{-2s}} \big) \Big] dy \Big) f(z)dz. \end{aligned}$$

Taking the change of variables $u = y - e^s z$ in the exponent, we get,

$$\begin{aligned} &-\frac{|y-e^{-t}x|^2}{1-e^{-2t}} - \frac{|z-e^{-s}y|^2}{1-e^{-2s}} \\ &\quad = -\frac{|y-e^{-t}x|^2}{1-e^{-2t}} - \frac{e^{-2s}|y-e^s z|^2}{1-e^{-2s}} = -\frac{|u+e^s z-e^{-t}x|^2}{1-e^{-2t}} - \frac{e^{-2s}|u|^2}{1-e^{-2s}} \\ &\quad = -\frac{(1-e^{-2s})|u-e^s(e^{-(t+s)}x-z)|^2 - (1-e^{-2t})e^{-2s}|u|^2}{(1-e^{-2t})(1-e^{-2s})} \\ &\quad = -\frac{(1-e^{-2s})(|u|^2 - 2\langle u, e^s(e^{-(t+s)}x-z)\rangle + e^{2s}|e^{-(t+s)}x-z|^2)}{(1-e^{-2t})(1-e^{-2s})} \\ &\qquad\qquad - \frac{(1-e^{-2t})e^{-2s}|u|^2}{(1-e^{-2t})(1-e^{-2s})} \\ &\quad = -\frac{e^{2s}|e^{-(t+s)}x-z|^2}{1-e^{-2t}} + \frac{2e^s\langle u, e^{-(t+s)}x-z\rangle}{1-e^{-2t}} - \frac{(1-e^{-2(t+s)})|u|^2}{(1-e^{-2t})(1-e^{-2s})}. \end{aligned}$$

But, the last two terms of the latter expression can be rewritten as

$$\begin{aligned} &\frac{2e^s\langle u, e^{-(t+s)}x-z\rangle}{1-e^{-2t}} - \frac{(1-e^{-2(t+s)})|u|^2}{(1-e^{-2t})(1-e^{-2s})} \\ &= -\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})} \Big[\frac{2e^s\langle u, e^{-(t+s)}x-z\rangle.(1-e^{-2s})}{1-e^{-2(t+s)}} - |u|^2 \Big] \\ &= -\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})} \\ &\quad \times \Big[\Big| u - \frac{e^s(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}} \Big|^2 - \frac{e^{2s}(1-e^{-2s})^2}{(1-e^{-2(t+s)})^2} |e^{-(t+s)}x-z|^2 \Big]. \end{aligned}$$

[9] For alternative proofs, see point 4. in Notes and Further Results.

Then, we have,

$$
\begin{aligned}
-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}-\frac{|z-e^{-s}y|^2}{1-e^{-2s}} &= -\frac{e^{2s}|e^{-(t+s)}x-z|^2}{1-e^{-2t}}\\
&\quad\times\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})}\Big|u-\frac{e^s(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}\Big|^2\\
&\qquad+\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}\frac{e^{2s}(1-e^{-2s})^2}{(1-e^{-2(t+s)})^2}|e^{-(t+s)}x-z|^2\\
&= -\frac{e^{2s}|e^{-(t+s)}x-z|^2}{1-e^{-2t}}-\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}\\
&\quad\times\Big|u-\frac{e^s(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}\Big|^2\\
&\qquad+\frac{e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}|e^{-(t+s)}x-z|^2.
\end{aligned}
$$

Now, taking the change of variables $w=u-\frac{e^s(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\exp\left(-\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})}\Big|u-\frac{e^s(1-e^{-2s})(e^{-(t+s)}x-z)}{1-e^{-2(t+s)}}\Big|^2\right)du\\
&=\int_{\mathbb{R}^d}\exp\Big(-\frac{1-e^{-2(t+s)}}{(1-e^{-2t})(1-e^{-2s})}|w|^2\Big)dw\\
&=\frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}}\int_{\mathbb{R}^d}e^{-|v|^2}dv\\
&=\pi^{d/2}\frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}}.
\end{aligned}
$$

With another change of variables, $v=\sqrt{\frac{(1-e^{-2(t+s)})}{(1-e^{-2t})(1-e^{-2s})}}\,w$ we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\exp\Big[-\Big(\frac{|y-e^{-t}x|^2}{1-e^{-2t}}+\frac{|z-e^{-s}y|^2}{1-e^{-2s}}\Big)\Big]dy\\
&=\pi^{d/2}\exp\Big[\Big(-\frac{e^{2s}}{1-e^{-2t}}+\frac{e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}\Big)|e^{-(t+s)}x-z|^2\Big)\Big]\\
&\quad\times\frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}},
\end{aligned}
$$

but as

$$-\frac{e^{2s}}{1-e^{-2t}}+\frac{e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}=\frac{-(1-e^{-2(t+s)})e^{2s}+e^{2s}(1-e^{-2s})}{(1-e^{-2t})(1-e^{-2(t+s)})}$$
$$=\frac{e^{-2t}-1}{(1-e^{-2t})(1-e^{-2(t+s)})}=-\frac{1}{1-e^{-2(t+s)}},$$

we get,

$$T_t(T_sf)(x) = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}\pi^{d/2}(1-e^{-2s})^{d/2}}$$
$$\times\int_{\mathbb{R}^d}\Big(\exp(\frac{-|e^{-(t+s)}x-z|^2}{1-e^{-2(t+s)}}\Big)\pi^{d/2}\frac{(1-e^{-2t})^{d/2}(1-e^{-2s})^{d/2}}{(1-e^{-2(t+s)})^{d/2}}f(z)dz$$
$$=\frac{1}{\pi^{d/2}(1-e^{-2(t+s)})^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|e^{-(t+s)}x-z|^2}{1-e^{-2(t+s)}}}f(z)dz=T_{t+s}f(x).$$

ii) The conservative property follows immediately by a simple change of variables $u=\frac{y-e^{-t}x}{\sqrt{1-e^{-2t}}}$, the translation invariance property of the Lebesgue measure, and the fact that γ_d is a probability measure:

$$T_t1=\frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}1dy=\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}e^{-|u|^2}du=1.$$

For the positivity of T_t, if $f\geq 0$,

$$T_tf(x)=\frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d}f(y)e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}dy\geq 0,$$

as the kernel is positive.

iii) Because the Ornstein–Uhlenbeck semigroup is not a convolution semigroup, this property cannot be obtained using the theory of approximations of the identity, as in the case of the classical semigroups (see Appendix 10.5). Nevertheless, it can be obtained using Jensen's inequality:

$$|T_tf(x)|^p\leq\frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}}|f(y)|^pe^{-|y|^2}dy.$$
$$=T_t(|f|^p)(x),$$

Then, according to *v*),

$$||T_tf||^p_{p,\gamma_d}\leq\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}T_t(|f|^p)(x)e^{-|x|^2}dx=||f||^p_{p,\gamma_d}.$$

Therefore, T_t is a contraction in $L^p(\gamma_d), 1 \leq p < \infty$. The case $p = \infty$ follows immediately because $T_t 1 = 1$, according to *ii)*. Alternatively, this can also be obtained by using interpolation and duality.

iv) We need to prove that $T_t f \to T_{t_0} f$ in $L^p(\gamma_d)$ as $t \to t_0$. Again, this is not a consequence of the general theory of approximations of the identity. According to the semigroup property, it is enough to prove that $T_t f \to f$ in $L^p(\gamma_d)$ as $t \to 0$. Observe that $L^p(\gamma_d)$ is not closed under translation;[10] thus, it does not make sense to speak of continuity in norm $L^p(\gamma_d)$ and hence, this type of argument cannot be used either. The alternative proof below is an extension to d-dimensions of the proof in [193].

$$
\begin{aligned}
&|T_t f(x) - f(x)| \\
&\quad \leq \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{|x-y|<\delta} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} |f(y)-f(x)| e^{-|y|^2} dy \\
&\quad + \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{|x-y|\geq\delta} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} |f(y)-f(x)| e^{-|y|^2} dy.
\end{aligned}
$$

Let f be a function defined in $\mathbb{R}^d$, continuous with compact support, and let $\varepsilon > 0$ and $\delta > 0$ be such that if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Now, according to *iii*), it is clear that the first integral is less than ε. Now, if y belongs to the support of f, $|x-y| > \delta$ and $0 \leq 1 - e^{-t} < \delta \frac{e^{-t}}{2} \max\{|y| : y \in \operatorname{supp} f\}$. Then,

$$
\begin{aligned}
&\exp\Big(- \frac{|e^{-t}(x-y) - y(1-e^{-t})|^2}{1-e^{-2t}} + |y|^2 \Big) \\
&\qquad\qquad \leq \exp\Big(- \frac{e^{-2t}\delta^2}{4(1-e^{-2t})} + \max\{|y|^2 : y \in \operatorname{supp} f\} \Big).
\end{aligned}
$$

The second integral is less than

$$
\frac{2||f||_{\infty,\gamma}}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\operatorname{supp} f} \exp\Big(- \frac{e^{-2t}\delta^2}{4(1-e^{-2t})} + \max\{|y|^2 : y \in \operatorname{supp} f\} \Big) e^{-|y|^2} dy,
$$

and this tends to zero as $t \to t_0$. Thus, $T_t f \to f$ uniformly in x as $t \to 0$. The general case follows from the density of the continuous functions with compact support in $L^p(\gamma_d)$ for $1 \leq p < \infty$ and using *iii*).

v) To prove (2.39), using Fubini's theorem, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^d} T_t f(x) g(x) \gamma_d(dx) \\
&= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} e^{(-\frac{(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e-2t})} f(y) e^{-|y|^2} dy \Big) g(x) \frac{1}{\pi^{d/2}} e^{-|x|^2} dx
\end{aligned}
$$

---

[10] Consider, for $d = 1$, $f(x) = \frac{1}{|x|} e^{|x|/2} \chi_{B(0,1)}(x)$ and its translations.

$$= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{(-\frac{(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e-2t})} g(x) e^{-|x|^2} dx \right) f(y) \frac{1}{\pi^{d/2}} e^{-|y|^2} dy$$
$$= \int_{\mathbb{R}^d} f(y) T_t g(y) \gamma_d(dy).$$

The invariance property follows immediately from (2.39) and the conservative property, taking $g \equiv 1$.

vi) Let $f \in C_b^2(\mathbb{R}^d)$, that is, a continuous function with bounded derivatives up to the second order. Then, using (2.36), we have

$$\left(\frac{T_t f - f}{t}\right)(x) - Lf(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \left[f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(x) \right] e^{-|y|^2} dy$$
$$- \frac{1}{2} \sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}(x) + \sum_{j=1}^d x_j \frac{\partial f}{\partial x_j}(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \Big[f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(x)$$
$$- \frac{t}{2} \sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}(x) \Big] e^{-|y|^2} dy + \sum_{j=1}^d x_j \frac{\partial f}{\partial x_j}(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \Big[f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(e^{-t}x)$$
$$- t \sum_{k=1}^d \frac{\partial^2 f}{\partial x_k^2}(x) y_k^2 \Big] e^{-|y|^2} dy + \left(\frac{f(e^{-t}x) - f(x)}{t} + \sum_{j=1}^d x_j \frac{\partial f}{\partial x_j}(x) \right).$$

Now, using the Taylor expansion of order 2 for f, for some θ, with $0 \leq \theta \leq 1$,

$$f(\sqrt{1-e^{-2t}}y + e^{-t}x) - f(e^{-t}x)$$
$$= \sum_{k=1}^d \sqrt{1-e^{-2t}} y_k \frac{\partial f}{\partial x_k}(e^{-t}x) + \frac{1}{2} \sum_{i,j=1}^d (1-e^{-2t}) y_i y_j$$
$$\times \frac{\partial^2 f}{\partial x_i \partial x_j} \left(\theta e^{-t}x + (1-\theta)\sqrt{1-e^{-2t}}y \right).$$

Then, according to the symmetry of $e^{-|y|^2}$, we have

$$\left(\frac{T_t f - f}{t}\right)(x) - Lf(x)$$
$$= \frac{1}{t\pi^{d/2}} \int_{\mathbb{R}^d} \Big[\sum_{k=1}^d \sqrt{1-e^{-2t}} y_k \frac{\partial f}{\partial x_k}(e^{-t}x)$$
$$+ \frac{1}{2} \sum_{i,j=1}^d (1-e^{-2t}) y_i y_j \frac{\partial^2 f}{\partial x_i \partial x_j} \left(\theta e^{-t}x + (1-\theta)\sqrt{1-e^{-2t}}y \right)$$

$$-t\sum_{k=1}^{d}\frac{\partial^2 f}{\partial x_k^2}(x)y_k^2\Big]e^{-|y|^2}dy+\left(\frac{f(e^{-t}-f(x)}{t}+\sum_{j=1}^{d}x_j\frac{\partial f}{\partial x_j}(x)\right)$$

$$=\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}\sum_{k=1}^{d}\Big[\frac{1}{2}\left(\frac{1-e^{-2t}}{t}\right)\frac{\partial^2 f}{\partial x_k^2}\left(\theta e^{-t}x+(1-\theta)\sqrt{1-e^{-2t}}y\right)$$

$$-\frac{\partial^2 f}{\partial x_k^2}(x)\Big]y_k^2e^{-|y|^2}dy+\left(\frac{f(e^{-t}x)-f(x)}{t}+\sum_{j=1}^{d}x_j\frac{\partial f}{\partial x_j}(x)\right).$$

Thus,

$$\Big|\Big(\frac{T_tf-f}{t}\Big)(x)-Lf(x)\Big|$$

$$\leq\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}\sum_{k=1}^{d}\Big[\frac{1}{2}\Big|\frac{1-e^{-2t}}{t}\Big|\Big|\frac{\partial^2 f}{\partial x_k^2}(\theta e^{-t}x+(1-\theta)\sqrt{1-e^{-2t}}y)$$

$$-\frac{\partial^2 f}{\partial x_k^2}(x)\Big|e^{-|y|^2}\Big]dy+\Big|\frac{f(e^{-t}x)-f(x)}{t}+\sum_{j=1}^{d}x_j\frac{\partial f}{\partial x_j}(x)\Big|.$$

Then, using Lebesgue's dominated convergence theorem, we conclude that each of these terms tends to zero as $t\to 0$. □

Also, each operator of the Ornstein–Uhlenbeck semigroup is compact.

Lemma 2.6. *For each $t>0$, the operator T_t is compact.*

Proof. Because T_t is given by

$$T_tf=\sum_{k=1}^{\infty}e^{-kt}\mathbf{J}_kf,\quad t>0,$$

we can consider the following sequence of compact operators:

$$T_t(n)f=\sum_{k=1}^{n}e^{-kt}\mathbf{J}_kf,\quad t>0.$$

Then,

$$\|T_tf-T_t(n)f\|_{2,\gamma}^2=\sum_{k=n+1}^{\infty}\|e^{-kt}\mathbf{J}_kf\|_{2,\gamma}^2\leq e^{-2nt}\|f\|_{2,\gamma}^2.$$

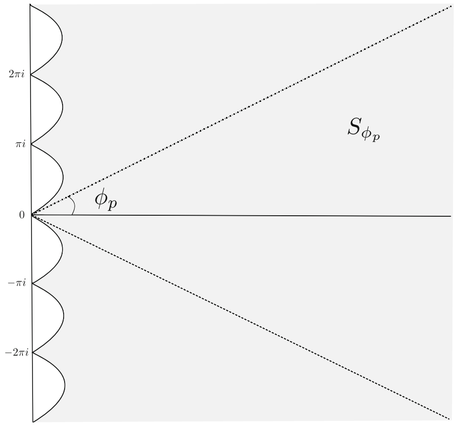

Fig. 2.1. Epperson region $\mathbf{E}_p$.

Therefore, the sequence of compact operators $\{T_t(n)\}$ converges in $L^2(\gamma_d)$-norm to T_t for all $t > 0$. Then, from e) of [62, Theorem A.3.22], we can conclude the compactness of T_t. □

The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ can be extended to complex values of the parameter t. For any $z \in \mathbb{C}$ with $\mathrm{Re}\, z \geq 0$, the operator $T_z = e^{-zL}$, defined spectrally, is bounded on $L^2(\gamma_d)$. It is given by the kernel, using the representation (2.37), replacing t by z,

$$M_z(x,y) = \frac{1}{(1-e^{-2z})^{d/2}} \exp\left(\frac{1}{2}\frac{|x+y|^2}{e^z+1} - \frac{1}{2}\frac{|x-y|^2}{e^z-1}\right).$$

The function $t \mapsto T_t$ has an holomorphic continuation to a distribution-valued function $z \mapsto T_z$, which is holomorphic in $\mathrm{Re}\, z > 0$ and continuous in $\mathrm{Re}\, z \geq 0$. The family of continuous operators $\{T_z : \mathrm{Re}\, z \geq 0\}$ defined from the space of distributions $\mathscr{D}(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$, then satisfies

$$T_{z+i\pi}(x) = T_z f(-x), \quad T_{\bar{z}} f(x) = \overline{T_z \overline{f(x)}}. \tag{2.42}$$

J. B. Epperson [74] proved that the operator T_z, extends to a bounded operator on $L^p(\gamma_d)$, $1 \leq p \leq \infty$, if and only if $z \in \mathbf{E}_p$, where

$$\mathbf{E}_p := \{z = x+iy : |\sin y| \leq \tan\phi_p \sinh x\}, \quad \phi_p = \arccos|2/p-1|. \tag{2.43}$$

The extension T_z to $L^p(\gamma_d)$ is actually a contraction.

The set $\mathbf{E}_p$ is a closed $i\pi$-periodic subset of the right half-plane, which is called *Epperson's region*, (see Figure 2.1). Each $\mathbf{E}_p$ is a closed subset of the closed right half-plane and periodic with period $i\pi$. Notice the symmetry $\phi_p = \phi_{p'}$ and $\mathbf{E}_p = \mathbf{E}_{p'}$, where p' is the conjugate exponent. Also, we have $\mathbf{E}_p \subset \mathbf{E}_q$ if $1 < p < q < 2$. Furthermore, $\mathbf{E}_p$ depends monotonically on p on either side of 2. The extreme cases are $\mathbf{E}_2 = \{z : \mathrm{Re}\, z \geq 0\}$ and $\mathbf{E}_1 = \{x+ik\pi : x \geq 0, k \in \mathbb{Z}\}$.

The map $z \to T_z$ from $\mathbf{E}_p$ to the Banach algebra of bounded operators on $L^p(\gamma)$ is continuous in the strong operator topology, and its restriction to the interior of $\mathbf{E}_p$ is holomorphic (see also [249]). Additionally, the holomorphic Ornstein–Uhlenbeck semigroup can be extended to infinite dimensions (see [167]).

Let us prove now that the $L^1(\gamma_d)$-spectrum of L is the closed right half-plane (see E. B. Davies [65, Theorem 4.3.5])

Theorem 2.7. *The $L^1(\gamma_d)$-spectrum of L is the closed right half-plane $\{z : Re\, z \geq 0\}$. Indeed, every z with $Re\, z > 0$ is an eigenvalue of L with multiplicity two.*

Proof. First of all, according to the tensorization argument, it is enough to consider the case $d = 1$. Let us consider the harmonic oscillator operator

$$H_1 f = \frac{1}{2}\left(-\frac{d^2 f}{dx^2} + x^2 f - f\right),$$

with domain in $\mathscr{S}(\mathbb{R}) \in L^2(\mathbb{R})$. It is easy to see, using Mehler's formula, that the semigroup generated by H_1, $\{e^{-tH_1}\}_{t\geq 0}$ has kernel

$$K_t(x,y) = \frac{1}{\pi^{1/2}(1-e^{-2t})} \exp\left(\frac{4xye^{-t} - (x^2+y^2)(1+e^{-2t})}{2(1-e^{-2t})}\right), t > 0,\ x,y \in \mathbb{R}.$$

Consider the isometric isomorphism, $\Xi_1^{(2)} : L^2(\gamma_1) \to L^2(\mathbb{R})$ defined in (2.35), for $d = 1$ and $p = 2$,

$$\Xi_1^{(2)} f(x) = f(x)\pi^{-1/4} e^{-|x|^2/2},$$

and consider

$$\tilde{L} = \Xi_1^{(2)} L (\Xi_1^{(2)})^{-1}.$$

Hence, the operator $\tilde{L}$ on $L^1(\mathbb{R})$ has the same spectrum as L. The kernel of the semigroup generated by $\tilde{L}$, $\{\tilde{T}_t\}_{t\geq 0} = \{e^{-t\tilde{L}}\}_{t\geq 0}$ is

$$\begin{aligned}
\tilde{M}_t(x,y) &= \frac{e^{-|x|^2}}{\pi^{1/2}} M_t(x,y) = \frac{e^{-|x|^2}}{\pi^{1/2}} \frac{1}{(1-e^{-2t})^{1/2}} \exp\left(-\frac{e^{-2t}(|x|^2+|y|^2) - 2e^{-t} x\cdot y}{1-e^{-2t}}\right) \\
&= \frac{e^{-|x|^2/2}}{\pi^{1/4}} K_t(x,y) \frac{e^{-|y|^2/2}}{\pi^{1/4}} \\
&= \frac{1}{\pi^{1/2}(1-e^{-2t})^{1/2}} e^{-\frac{|x-e^{-t}y|^2}{1-e^{-2t}}}
\end{aligned}$$

Taking the Fourier transform $\mathscr{F}$ from $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$, it follows that

$$\mathscr{F}(\tilde{T}_t) f(\zeta) = e^{-(1-e^{-2t})\zeta^2/4} f(e^{-t}\zeta).$$

Let us consider $f_z^+(x)$ and $f_z^-(x)$ the $L^1(\mathbb{R})$-functions, whose Fourier transforms are $\chi_{[0,\infty)}(\zeta)|\zeta|^z e^{-\zeta^2/4}$ and $\chi_{(-\infty,0]}(\zeta)|\zeta|^z e^{-\zeta^2/4}$ respectively. Then, for any z, with $Re\, z > 0$

$$\mathscr{F}(e^{-tH_1}f_z^+)(\zeta) = \chi_{[0,\infty)}(\zeta)e^{-(1-e^{-2t})\zeta^2/4}e^{-zt}|\zeta|^z e^{-e^{-2t}\zeta^2/4} = e^{-zt}\mathscr{F}(f_z^+)(\zeta),$$

and analogously,

$$\mathscr{F}(e^{-tH_1}f_z^-)(\zeta) = \chi_{(-\infty,0]}(\zeta)e^{-(1-e^{-2t})\zeta^2/4}e^{-zt}|\zeta|^z e^{-e^{-2t}\zeta^2/4} = e^{-zt}\mathscr{F}(f_z^-)(\zeta).$$

Hence, according to the uniqueness of the Fourier transform,

$$e^{-tH_1}f_z^+(x) = e^{-zt}f_z^+, \text{ and } e^{-tH_1}f_z^-(x) = e^{-zt}f_z^-.$$

Now, because the spectrum of L is a closed subset of $\{z : \mathrm{Re}\, z \geq 0\}$, as $\{T_t\}_{t\geq 0}$ is a strongly continuous contraction semigroup, we get the conclusion. □

Definition 2.8. *The maximal function of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ or maximal Ornstein–Uhlenbeck function, is defined as*

$$T^* f(x) = \sup_{t>0} |T_t f(x)|. \tag{2.44}$$

In Chapter 4, Theorems 4.19 and 4.20, we study the boundedness properties of T^*, proving that it is bounded in $L^p(\gamma_d)$ for $1 < p \leq \infty$, and that it is of weak type $(1,1)$ with respect to the measure γ_d. Also, other versions of maximal functions are study in detail in Chapter 4.

In 1969, C. P. Calderón [44] proved that the *multiparametric Ornstein–Uhlenbeck maximal function*

$$\mathbf{T}^* f(x) = \sup_{\substack{0<t_1<\infty \\ 0<t_2<\infty \\ \cdots \\ 0<t_d<\infty}} \left[\frac{1}{\pi^{d/2}} \prod_{i=1}^{d} \frac{1}{(1-e^{-2t_i})^{1/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t_i}x|^2}{1-e^{-2t_i}}} f(y)dy \right], \tag{2.45}$$

is $L^p(\gamma_d)$-bounded, $1 < p < \infty$. From this result, the $L^p(\gamma_d)$-boundedness for the one-parameter maximal operator T^* also follows.

The maximal function for the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$ can also be considered:

$$\Gamma_p^* f(w) = \sup_{z\in \mathbf{E}_p} |T_z f(w)|, \tag{2.46}$$

where $\mathbf{E}_p$ is Epperson's region defined in (2.43).

In particular, Γ_1^* is the maximal operator of the Ornstein–Uhlenbeck semigroup which, as we are going to see in Chapter 4 is of weak type $(1,1)$ and of strong type (p,p) for each $1 < p < \infty$.

According to the periodicity properties of the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$, we may restrict the parameter z to the set $\mathbf{F}_p$

$$\mathbf{F}_p = \{z \in \mathbf{E}_p : 0 \leq \ \operatorname{Im} z \leq \pi/2\} \tag{2.47}$$

Consider the map, $\tau : \{\zeta \in \mathbb{C} : |\zeta| \leq 1, |\arg \zeta| \leq \pi/2\} \to \mathbb{C} \cup \{\infty\}$ introduced in [105]:

$$\tau(\zeta) = \begin{cases} \log \frac{1+\zeta}{1-\zeta}, & \text{if } \zeta \neq 1 \\ \infty, & \text{if } \zeta = 1, \end{cases} \tag{2.48}$$

where $\log \omega$ is real when $\omega > 0$; hence τ is real-valued in the interval $[0,1)$. Notice that $\tau((\overline{\zeta})^{-1}) = \overline{\tau(\zeta)} + i\pi$, which means that τ makes reflection in the unit circle $|\zeta| = 1$ correspond to reflection in the line $\operatorname{Im} z = i\pi/2$. Combined with the periodicity and symmetry of T_z, we get

$$\overline{T_{\tau(t^{-1}e^{i\phi})} f(x)} = T_{\tau(te^{i\phi})} \overline{f}(-x).$$

Moreover, τ is a homeomorphism of its domain onto the half-strip $\{\zeta \in \mathbb{C} : \operatorname{Re} \zeta \geq 0, |\operatorname{Im} \zeta| \leq \pi/2\}$ mapping the sector

$$S_{\phi_p} := \{\zeta \in \mathbb{C} : |\zeta| \leq 1, |\arg \zeta| \leq \phi_p\} \tag{2.49}$$

onto the set $\mathbf{F}_p \cup \{\infty\}$. In particular, if $1 < p < 2$, then τ maps $S_{\phi_p} \setminus [1,\infty)$ onto the interior of $\mathbf{E} \cap \{z \in \mathbb{C} : \ |\operatorname{Im}| z < \pi/2\}$ and the ray $[0, e^{i\phi_p}\infty)$ onto $\partial \mathbf{E}_p \cap \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$ (see Figure 2.1). Additionally, if $\zeta \neq 1$,

$$M_{\tau(\zeta)}(x,y) = \frac{(1+\zeta)^d}{(4\zeta)^{d/2}} \exp\Big(\frac{|x|^2+|y|^2}{2} - \frac{1}{4}\Big(\zeta|x+y|^2 + \frac{1}{\zeta}|x-y|^2\Big)\Big),$$

because

$$1 - e^{2z} = \frac{4\zeta}{(1+\zeta)^2}, \ \frac{1}{2}\frac{1}{e^z+1} = \frac{1}{4} - \frac{\zeta}{4}, \text{ and} -\frac{1}{2}\frac{1}{e^z-1} = \frac{1}{4} - \frac{1}{4\zeta}.$$

We define $M_{\tau(1)}(x,y) = 1$, for all x,y.

Several estimates for Γ_p^* are given by J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea [104]. The simplest result establish that Γ_q^* is bounded on $L^p(\gamma_d)$ if $|\frac{1}{q} - \frac{1}{2}| > |\frac{1}{p} - \frac{1}{2}|$. This means that for $f \in L^p(\gamma_d)$, the supremum of $|T_z f(x)|$ is taken for $z \in \mathbf{E}_q \subset \mathbf{E}_p$.

For the case $1 < p < 2$, it was proved in [104] that Γ_p^* is not $L^p(\gamma_d)$-bounded, not even of weak type (p,p) with respect to the Gaussian measure. The unboundedness on $L^p(\gamma_d)$ here occurs along the whole boundary of $\mathbf{E}_p$.

$$\Gamma_{\varepsilon,p}^* f(w) = \sup_{z \in \mathbf{E}_p, d(z, i\pi\mathbb{Z}) \geq \varepsilon} |T_z f(w)|, \tag{2.50}$$

is of weak type (p,p) with respect to the Gaussian measure, for any $\varepsilon > 0$. Then, P. Sjögren [249] proved that for $2 < p < \infty$ Γ_p^* is not $L^p(\gamma_d)$-bounded, but $\Gamma_{\varepsilon,p}^*$ is $L^p(\gamma_d)$-bounded; therefore it is of weak type (p,p) with respect to γ, for any $\varepsilon > 0$. Finally, for $p = 2$, the situation is rather different: Γ_2^* is not of weak type $(2,2)$ with respect to the Gaussian measure (see [104]).

According to the Banach principle, it is known (see [107] or [275, Theorem 6.1]), that the study of this maximal operator is a key tool for investigating the almost everywhere convergence of $\{T_t\}_{t\geq 0}$,

$$T_0 f(x) \lim_{t\to 0^+} T_t f(x) = f(x) \quad a.e. \;\; x \in \mathbb{R}^d \tag{2.51}$$

(see Theorem 4.46), and also

$$T_\infty f(x) := \lim_{t\to\infty} T_t f(x) = \int_{\mathbb{R}^d} f(y)\, \gamma_d(dy) \quad a.e. \;\; x \in \mathbb{R}^d, \tag{2.52}$$

for all $f \in L^1(\gamma_d)$. This implies it for all $f \in L^p(\gamma_d)$, $1 \leq p \leq \infty$, as $L^q(\gamma_d) \subset L^p(\gamma_d)$ for $p \leq q$. Thus, unlike the classical case of the heat semigroup, the Ornstein–Uhlenbeck semigroup does not decay at infinity. This property expresses the ergodicity of the semigroup. The details of this proof and its generalization to non-tangential convergence are given in Chapter 4.

Proposition 2.9. *If $f \in L^p(\gamma_d)$, $u(x,t) = T_t f(x)$ is a $C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ solution of the parabolic equation*

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{2}\Delta_x u - \langle x, \nabla_x u\rangle = Lu, \quad x \in \mathbb{R}^d, t > 0, \tag{2.53}$$

with boundary condition $u(x,0) = f(x), x \in \mathbb{R}^d$.

Thus, $u(x,t) = T_t f(x)$ is the solution of a boundary value problem.

Proof. According to the general semigroup theory, given the fact that L is the infinitesimal generator of $\{T_t : t \geq 0\}$, we get

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial T_t f(x)}{\partial t} = LT_t f(x) = Lu(x,t).$$

Yet, this can also be shown explicitly:

$$\begin{aligned} Lu(x,t) &= \frac{2e^{-t}}{\pi^{d/2}(1-e^{-2t})^{d/2+1}} \int_{\mathbb{R}^d} \Big[\frac{de^{-t}}{2} + \frac{e^{-t}|y-e^{-t}x|^2}{1-e^{-2t}} - < (y-e^{-t}x), x > \Big] \\ &\qquad\qquad \times \exp\Big(-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}\Big) f(y)dy \\ &= \frac{\partial u(x,t)}{\partial t}. \end{aligned}$$

The boundary condition follows from (2.51). □

Now, from the fact that L is the infinitesimal generator of $\{T_t\}_{t\geq 0}$, using the semigroup property, we can easily get that

$$\frac{dT_t}{dt} = LT_t. \tag{2.54}$$

In [106], G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, and B. Viviani find optimal integrability conditions to guarantee the existence of solutions of (2.53).

Moreover, for the study of Hardy spaces in Chapter 7 we need to consider *higher order derivatives* of the Ornstein–Uhlenbeck semigroup,

$$\frac{d^kT_t}{dt^k} = L^kT_t \tag{2.55}$$

We get a closed expression for the integral representation of these derivatives, determining explicitly the kernels M_t^k such that

$$(L^kT_t)f(x) = \int_{\mathbb{R}^d} M_t^k(x,y)f(y)\gamma_d(dy), \tag{2.56}$$

Observe that, for $\nu \in \mathbb{N}_0$

$$\begin{aligned}(L^kT_t)\mathbf{h}_\nu(x) &= |\nu|^k e^{-t|\nu|}\mathbf{h}_\nu(x) = |\nu|^k e^{-t|\nu|} h_{\nu_1}(x_1)\cdots h_{\nu_d}(x_d)\\ &= \sum_{|\eta|=k}\binom{k}{\eta_1,\eta_2,\cdots,\eta_d}\nu_1^{\eta_1}\cdots\nu_d^{\eta_d}e^{-t\eta_1}\cdots\ e^{-t\eta_d}h_{\nu_1}(x_1)\cdots h_{\nu_d}(x_d)\\ &= \sum_{|\eta|=k}\binom{k}{\eta_1,\eta_2,\cdots,\eta_d}L_1^{\eta_1}T_t^1h_{\nu_1}(x_1)\cdots L_d^{\eta_d}T_t^{\eta_d}h_{\nu_d}(x_d), \end{aligned}\tag{2.57}$$

where, as in (2.5), L_i denotes the one-dimensional Ornstein–Uhlenbeck operator, in the i-th variable, and $\{T_t^i\}_{t\geq 0}$ is the one-dimensional Ornstein–Uhlenbeck semigroup, in the i-th variable. Here, we follow J. Teuwen's paper [266], and it should be consulted for full details of the proof.

Theorem 2.10. *Let L be the Ornstein–Uhlenbeck operator in $L^2(\gamma_d)$, $t>0$, and $N\geq 0$. The integral kernel M_t^k of L^kT_t is given by*

$$\begin{aligned}M_t^k(x,y) &= (-1)^k M_t(x,y)\sum_{|\eta|=k}\binom{k}{\eta_1,\eta_2,\cdots,\eta_d}\prod_{i=0}^{d}\sum_{n_i=0}^{\eta_i}\sum_{l_i=0}^{n_i}2^{-m_i}\left\{{\eta_i \atop n_i}\right\}\binom{n_i}{l_i}\\ &\quad\times\Big(-\frac{e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^{2n_i-l_i}H_{l_i}(x_i)H_{2n_i-l_i}\Big(\frac{y_i-x_ie^{-t}}{\sqrt{1-e^{-2t}}}\Big),\end{aligned}\tag{2.58}$$

where $\left\{{n \atop m}\right\}$ are Stirling numbers of the second kind.[11]

[11]For $n\geq m$ non-negative integers, the Stirling number of the second kind $\left\{{n \atop m}\right\}$ is defined as the number of partitions of a set of n elements into m non-empty subsets, see [36].

Proof. From (2.57) and the tensorization argument, it is enough to consider only the case $d = 1$. Observe that

$$(L^k T_t)f(x) = L^k(T_t f)(x) = L^k\Big(\int_{\mathbb{R}^d} M_t(x,y)f(y)\gamma_d(dy)\Big) = \int_{\mathbb{R}^d} L^k M_t(x,y)f(y)\gamma_d(dy);$$

hence, $M_t^k(x,y) = L^k M_t(x,y)$. Therefore, using the integral representation of Mehler's kernel (1.46) we get

$$\begin{aligned} M_t^k(x,y) &= L^k\Big(\frac{e^{y^2}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2i\xi y-\xi^2} e^{-(x+i\xi e^{-t})^2+x^2}dx\Big) \\ &= \frac{e^{y^2}}{\sqrt{\pi}}\int_{\mathbb{R}} e^{2i\xi y-\xi^2} L^k e^{(x+i\xi e^{-t})^2+x^2}d\xi. \end{aligned}$$

Now, observe that

$$\begin{aligned} L^k e^{-(x-t)^2+x^2} &= (-1)^k\Big(t\frac{\partial}{\partial t}\Big)^k e^{-(x-t)^2+x^2} = (-1)^k\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix} t^n e^{x^2}\frac{\partial^k}{\partial t^k}e^{-(x-t)^2} \\ &= (-1)^k\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix} t^n e^{x^2}(-1)^k\frac{\partial^k}{\partial x^k}e^{-(x-t)^2} \\ &= (-1)^k e^{-(x-t)^2+x^2}\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix} t^n\, H_n(x-t), \end{aligned}$$

by using Rodrigues' formula and [266, Lemma 1].[12] Therefore, using (1.39)

$$\begin{aligned} M_t^k(x,y) &= (-1)^k\frac{e^{x^2+y^2}}{\sqrt{\pi}}\sum_{n=0}^{k}\begin{Bmatrix}k\\n\end{Bmatrix}\int_{\mathbb{R}} e^{2i\xi y-\xi^2}(i\xi e^{-t})^k e^{-(x+i\xi e^{-t})^2}\, H_n(x+i\xi e^{-t})d\xi \\ &= (-1)^k\frac{e^{x^2+y^2}}{\sqrt{\pi}}\sum_{n=0}^{k}\sum_{l=0}^{n}\begin{Bmatrix}k\\n\end{Bmatrix}\binom{n}{l}\, H_n(x)2^{n-l}\int_{\mathbb{R}} e^{2i\xi y-\xi^2}e^{-(x+i\xi e^{-t})^2}(i\xi e^{-t})^{2n-l}d\xi. \end{aligned}$$

Thus, it remains to compute the inner integral. For each $m \in \mathbb{N}$, using again the integral representation of the Hermite polynomials (1.30), and the change of variables $\eta = \sqrt{1-e^{-2t}}\,\xi$, we have

[12] Teuwen notices that his Lemma 1 is a particular case of a result in Weyl algebras, and depends only on the fact that the commutator $[t,\partial_t] = -1$.

$$\begin{aligned}
\frac{e^{x^2+y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2i\xi y-\xi^2} e^{-(x+i\xi e^{-t})^2} (i\xi e^{-t})^m d\xi \\
&= \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2i(y-xe^{-t})\xi} e^{-(1-e^{-2t})\xi^2} (i\xi e^{-t})^m d\xi \\
&= \frac{e^{y^2}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{2i\xi\sqrt{1-e^{-2t}}[(y-xe^{-t})/\sqrt{1-e^{-2t}}]} e^{-(1-e^{-2t})\xi^2} (i\xi e^{-t})^m d\xi \\
&= \frac{(-1)^m e^{y^2}}{2^m\sqrt{\pi}} \frac{(-2i)^m e^{-tm}}{(\sqrt{1-e^{-2t}})^{m+1}} \int_{\mathbb{R}} e^{2i\eta[(y-xe^{-t})/\sqrt{1-e^{-2t}}]} e^{-\eta^2} \eta^m d\eta \\
&= \frac{e^{-(y-xe^{-t})^2/(1-e^{-2t})} e^{y^2}}{\sqrt{1-e^{-2t}}} 2^{-m} \Big(\frac{-e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^m H_m\Big(\frac{y-xe^{-t}}{\sqrt{1-e^{-2t}}}\Big) \\
&= M_t(x,y) 2^{-m} \Big(\frac{-e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^m H_m\Big(\frac{y-xe^{-t}}{\sqrt{1-e^{-2t}}}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
M_t^k(x,y) &= (-1)^k M_t(x,y) \sum_{n=0}^{k} \sum_{l=0}^{n} 2^{-n} \left\{ \begin{matrix} k \\ n \end{matrix} \right\} \binom{n}{l} \Big(\frac{-e^{-t}}{\sqrt{1-e^{-2t}}}\Big)^{2n-l} \\
& \quad H_n(x) H_{2n-l}\Big(\frac{y-xe^{-t}}{\sqrt{1-e^{-2t}}}\Big). \qquad \square
\end{aligned}$$

Another ingredient that is needed for the study of Hardy spaces in Chapter 7 is the following Gaussian version of A. P. Calderón's reproducing formula; see [231].

Lemma 2.11. *(Portal) For all $n \in \mathbb{N}$ and $a, \alpha > 0$, there exists $C > 0$ such that for all $f \in L^2(\gamma_d)$*

$$f(x) = C \int_0^\infty (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} f(x) \, \frac{dt}{t} + \int_{\mathbb{R}^d} f(x) \gamma_d(dx), \tag{2.59}$$

in L^2 sense.

Proof. As this is a formula in $L^2(\gamma_d)$, it is enough to prove (2.59) for the Hermite polynomials, as they are an orthonormal basis for $L^2(\gamma_d)$.

If $\nu = \mathbf{0}$, then as $\mathbf{H_0} = 1$, and $L\mathbf{H_0} = 0$, the right-hand side equals

$$C \int_0^\infty (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} 1 \, \frac{dt}{t} + \int_{\mathbb{R}^d} 1 \, d\gamma_d = C \cdot 0 + 1 = \mathbf{H_0}.$$

Let us assume now that $\nu \neq \mathbf{0}$. For these $\mathbf{H}_\nu$, the last integral in (2.59) is zero according to orthogonality. As $\mathbf{H}_\nu$ is an eigenfunction with eigenvalue of L, then

$$L^{N+1}\mathbf{H}_\nu = (-1)^{N+1} |\nu|^{N+1} \mathbf{H}_\nu.$$

Hence, we obtain for $x \in \mathbb{R}^d$

$$\int_0^\infty (t^2L)^{N+1} T_{(1+a)t^2/\alpha} \mathbf{H}_\nu(x) \frac{dt}{t} = \int_0^\infty (t^2L)^{N+1} e^{-(1+a)t^2|\nu|/\alpha} \mathbf{H}_\nu(x) \frac{dt}{t}$$
$$= (-1)^{N+1}|\nu|^{N+1}\mathbf{H}_\nu(x) \int_0^\infty t^{2(N+1)} e^{-(1+a)t^2|\nu|/\alpha} \frac{dt}{t}$$
$$= \frac{N!}{2}\left(\frac{\alpha}{(1+a)}\right)^{N+1} \mathbf{H}_\nu(x) = C\,\mathbf{H}_\nu(x).$$

Therefore, $C = C_N = \frac{2}{N!}(\frac{1+a}{\alpha})^{N+1}$ is the right constant. □

Another version of A. P. Calderón's reproducing formula was obtained in [164], and is discussed in Chapter 8 (see Theorem 8.31).

2.3 The Hypercontractivity Property for the Ornstein–Uhlenbeck Semigroup and the Logarithmic Sobolev Inequality

The Ornstein–Uhlenbeck semigroup is not only a contraction semigroup but it is also *hypercontractive*. The *hypercontractivity property* of $\{T_t\}_{t\geq 0}$ was initially proved by E. Nelson [204] in the context of quantum field theory, and it has been studied extensively in the literature.

Definition 2.12. *Given a semigroup of contractions $\{\mathtt{T}_t\}_{t\geq 0}$ defined in $L^p(E,\mu)$, with $1 \leq p \leq \infty$, the semigroup $\{\mathtt{T}_t\}_{t\geq 0}$ satisfies the* hypercontractivity property *if for each initial condition $1 < p < \infty$ there exists an strictly increasing function $q : \mathbb{R}^+ \to [p,\infty)$, $q(0) = p$ such that*

$$||\mathtt{T}_t f||_{q(t),\mu} \leq ||f||_{p,\mu}, \textit{ for all } f \in L^p(E,\mu), t \geq 0.$$

The function q is called the contraction function.

We are going to prove in detail that the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is hypercontractive, having contraction function

$$q(t) = 1 + e^{2t}(p-1) > p.$$

Thus, we will prove the following inequality:

$$||T_t f||_{q(t),\gamma} \leq ||f||_{p,\gamma}, \tag{2.60}$$

for all $f \in L^p(\gamma_d)$ and $t \geq 0$.
We will first prove that the Ornstein–Uhlenbeck operator satisfies the *logarithmic Sobolev inequality.*

Theorem 2.13. *The Ornstein–Uhlenbeck operator L satisfies the* logarithmic Sobolev inequality*: for any $f \in L^2(\gamma_d)$ with $\nabla_x f$ (in the weak sense) belonging to $L^2(\gamma_d)$,*

$$\int_{\mathbb{R}^d} |f(x)|^2 \log|f(x)| \gamma_d(dx) \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_x f(x)|^2 \gamma_d(dx) + ||f||_{2,\gamma_d}^2 \log ||f||_{2,\gamma}, \quad (2.61)$$

or, equivalently,

$$\int_{\mathbb{R}^d} |f(x)|^2 \log|f(x)| \gamma_d(dx) - \left(\int_{\mathbb{R}^d} |f(x)|^2 \gamma_d(dx) \right) \log \left(\int_{\mathbb{R}^d} |f(x)| \gamma_d(dx) \right) \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla_x f(x)|^2 \gamma_d(dx).$$

Proof. To prove (2.61), we will follow Adams and Clarke's proof [4], which is one of the simplest proofs for this inequality[13]. We begin by making a series of reductions. In the first place, it is enough to prove the logarithmic Sobolev inequality in the case $d = 1$. Then, the general case can be obtained by induction in d. In addition, observe that (2.61) is homogeneous with respect to rescaling of f; thus, we may assume that $||f||_{2,\gamma} = 1$. Moreover, we may assume that $||f'||_{2,\gamma} < \infty$ because, otherwise, there is nothing to prove. The change $f(t) = g(t)e^{t^2/2}$ implies the following equivalent formulation of the inequality:

$$\int_{\mathbb{R}} \left(\frac{1}{2} \left| g'(t) \right|^2 - |g(t)|^2 \log|g(t)| \right) dt \geq \frac{\sqrt{\pi}}{2}, \text{ provided } \int_{\mathbb{R}} |g(t)|^2 dt = \sqrt{\pi}. \quad (2.62)$$

As $\left| (|g|') \right| \leq |g'|$ a.e., we may assume that g is a non-negative real-valued function. It is enough to consider only the case $g(t) > 0$ for all $t \in \mathbb{R}$; this can be justified by a simple argument of density. Finally, it is convenient to split (2.62) into two half-line problems, each of them equivalent to

$$\int_0^\infty \left(\frac{1}{2} \left(g'(t) \right)^2 - (g(t))^2 \log(g(t)) \right) dt \geq \frac{\sqrt{\pi}}{4}, \text{ provided } \int_0^\infty (g(t))^2 dt = \frac{\sqrt{\pi}}{2}. \quad (2.63)$$

For $s, r > 0$, let $V(s,r) = \left[v(s,r)s^2 + r\left(1 - v(s,r)^2 - 2 \log s\right) \right]/2$, where $v(s,r) = h^{-1}\left(r/s^2\right)$, and h is given by

$$h(t) = e^{t^2} \int_t^\infty e^{-\tau^2} d\tau.$$

It is easy to see that h is strictly decreasing in $\mathbb{R}$ and $\left(h^{-1}\right)'(t) = \left\{2th^{-1}(t) - 1\right\}^{-1}$. The partial derivatives of V are:

$$V_s = vs, \text{ and } V_r = -\left(v^2/2\right) - \log s.$$

[13] For another simple proof see [219].

If $U(s,u) = \left(u^2/2\right) - s^2 \log s$, then,

$$V_s u - V_r s^2 + U(s,u) = \frac{1}{2}(u + vs)^2 \geq 0, \text{ for } s > 0,\ u \in \mathbb{R}. \tag{2.64}$$

Therefore, if g satisfies the inequalities

$$g(t) > 0,\ \int_0^\infty (g(t))^2 dt = \frac{\sqrt{\pi}}{2},\ \int_0^\infty \left(g'(t)\right)^2 dt < \infty, \tag{2.65}$$

it then follows from (2.64) (setting $s = g(t),\ r = \int_t^\infty (g(\tau))^2 d\tau,\ u = g'(t)$) that

$$\frac{d}{dt} V\left(g(t), \int_t^\infty (g(\tau))^2 d\tau\right) = V_s g'(t) - V_r (g(t))^2 \geq -U\left(g(t), g'(t)\right)$$

and

$$\begin{aligned}\int_0^\infty U\left(g(t), g'(t)\right) dt &\geq -\int_0^\infty \frac{d}{dt} V\left(g(t), \int_t^\infty (g(\tau))^2 d\tau\right) dt \\ &\geq V\left(g(0), \frac{\sqrt{\pi}}{2}\right) - \liminf_{t\to\infty} V\left(g(t), \int_t^\infty (g(\tau))^2 d\tau\right).\end{aligned}$$

As h^{-1} is decreasing and $h^{-1}\left(\sqrt{\pi}/2s^2\right) = 0$ only for $s = 1$, we conclude that

$$V\left(s, \sqrt{\pi}/2\right) \geq V\left(1, \sqrt{\pi}/2\right) = \sqrt{\pi}/4, \text{ for all } s > 0.$$

The inequality (2.63) would be shown if the following claim holds: if g satisfies (2.65), then

$$\liminf_{t\to\infty} V\left(g(t), \int_t^\infty (g(\tau))^2 d\tau\right) \leq 0.$$

To prove the claim, we use the fact that $h(\tau) < 1/\tau$ for all $\tau > 0$. Then, $h^{-1}(t) < 1/t$ for all $t > 0$, and $v(s,r)s^2 < s^4/r$. Similarly, $h(\tau) < \sqrt{\pi}e^{\tau^2}$ for $\tau \leq 0$ implies $h^{-1}(t) \leq -\sqrt{\log\left(t/\sqrt{\pi}\right)}$ for $t \geq \sqrt{\pi}$ and therefore, setting $t = r/s^2$, we get

$$(v(s,r))^2 \geq \log r - \log s^2 - \log\sqrt{\pi}$$

for $\sqrt{\pi}s^2 \leq r$. Evidently, $(v(s,r))^2 \geq 0$ if $r < \sqrt{\pi}s^2$ and consequently,

$$r\left(1 - v^2 - 2\log s\right) \leq \max\left\{r\left(1 + \log\sqrt{\pi} - \log r\right), \sqrt{\pi}s^2\left(1 - \log s^2\right)\right\}$$

for all $r, s > 0$. Hence,

$$V(s,r) \leq \frac{s^4}{2r} + \frac{1}{2}\max\left\{r\left(1 + \log\sqrt{\pi} - \log r\right), \sqrt{\pi}s^2\left(1 - \log s^2\right)\right\}. \tag{2.66}$$

If g satisfies (2.65), then taking $s = g(t)$,

$$r = \int_t^{\infty} (g(\tau))^2 \, d\tau, \text{ and } \varepsilon = \int_t^{\infty} \left(g'(\tau)\right)^2 d\tau$$

both terms tend to zero as $t \to \infty$. Moreover, according to Hölder's inequality

$$s^4 = (g(t))^4 \leq \left(2 \int_t^{\infty} g(\tau) \left|g'(\tau)\right| d\tau\right)^2 \leq 4r\varepsilon.$$

From (2.66), it follows that $\liminf_{t\to\infty} V(s,r) \leq 0$. □

In [119], L. Gross proved the following striking result:

Theorem 2.14. *The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is hypercontractive, with contraction function $q(t) = 1 + e^{2t}(p-1) > p$, if and only if the Ornstein–Uhlenbeck operator L satisfies the logarithmic Sobolev inequality (2.61).*

To prove this theorem, we essentially follow Gross' argument (see [119] and [120]). For this we need the following technical (but key) lemma. We are going to formulate it in great generality for any probability space $(E, \mathscr{B}, \mu)$, which will be useful in what follows. Of course, in our case, the probability space is simply $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d), \gamma_d)$.

Lemma 2.15. *Let $(E, \mathscr{B}, \mu)$ be a probability space. Let us take $1 < p < \infty$, $\varepsilon > 0$ and $q > p$ and let s be a real function, continuously differentiable from $[0, \varepsilon)$ to $(1, \infty)$ such that $s(0) = p$, and let f be a function continuously differentiable from $[0, \varepsilon)$ to $L^q(\mu)$ with $f(0) = v \neq 0$. Then, $||f(t)||_{s(t)}$ is differentiable at $t = 0$ and*

$$\frac{d}{dt}||f(t)||_{s(t)}\Big|_{t=0} \tag{2.67}$$
$$= ||v||_p^{1-p}\left[p^{-1}s'(0)\left(\int_E |v|^p \log|v| d\mu - ||v||_p^p \log||v||_p\right) + \mathrm{Re}\langle f'(0), v_p\rangle_\mu\right],$$

where, $v_p = (\operatorname{sgn} v)|v|^{p-1}$.

Proof. Let $g : [0, \varepsilon) \to \mathbb{C}$ be a continuously differentiable function. Then, we have

$$\begin{aligned}\frac{d}{dt}|g(t)|^{s(t)} &= \left[s'(t)\log|g(t)| + s(t)\frac{1}{|g(t)|^2}\mathrm{Re}g'(t)\overline{g(t)}\right]|g(t)|^{s(t)} \\ &= s'(t)|g(t)|^{s(t)}\log|g(t)| + s(t)\frac{1}{|g(t)|^2}\mathrm{Re}g'(t)\overline{\frac{g(t)}{|g(t)|}|g(t)|^{s(t)-1}} \\ &= s'(t)|g(t)|^{s(t)}\log|g(t)| + s(t)\frac{1}{|g(t)|^2}\mathrm{Re}g'(t)\overline{g(t)_{s(t)}}.\end{aligned}$$

This holds even when $g(t) = 0$ for some t, because $s(t) > 1$.

Let us take $g(t) = f(t)(x)$(formally), integrate it with respect to μ, and interchange the order of the integration and differentiation. Then,

$$\frac{d}{dt}\int_E |f(t)(x)|^{s(t)}\mu(dx) = \int_E s'(t)|f(t)(x)|^{s(t)}\log|f(t)(x)|\mu(dx) + s(t)\mathrm{Re}\langle f'(t), f_{s(t)}\rangle_\mu.$$

Then, if $V(t) = \int_E |f(t)(x)|^{s(t)}\mu(dx)$, we have

$$\begin{aligned}\frac{d}{dt}||f(t)||_{s(t)} &= \frac{d}{dt}V(t)^{s(t)} \\ &= \frac{1}{s(t)}\Big[\frac{V(t)^{s(t)-1}}{V(t)}\Big]V'(t) - \frac{s'(t)}{s^2(t)}V(t)^{s(t)-1}\log V(t).\end{aligned}$$

The second chain of equalities needs justification, because $f(t)(x)$ is not necessarily differentiable in the variable t for a.e. x (for details see Gross [120, page 63]). Then, taking $t = 0$ (2.67) follows. □

Now, we are ready to prove Theorem 2.14

Proof. First of all, consider the number operator

$$N = 2(-L) = -\Delta_x + 2\langle x, \nabla_x\rangle$$

which is the Dirichlet form for γ_d, i.e.,

$$\int_{\mathbb{R}^d}\langle\nabla_x f(x), \nabla_x g(x)\rangle\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)Ng(x)\gamma_d(dx),$$

and consider the semigroup $\{e^{-tN}\}_{t\geq 0}$ generated by N.[14]

Let us assume that (2.61) holds. Then, we can obtain, for each $p > 1$, the Sobolev logarithmic inequality in $L^p(\gamma_d)$

$$\int_{\mathbb{R}^d}|f(x)|^p\log|f(x)|\gamma_d(dx) \leq c(p)\mathrm{Re}\langle Nf(t), f_p\rangle_{\gamma_d} + ||f||^p_{p,\gamma_d}\log||f||_{p,\gamma}, \quad (2.68)$$

with $c(p) = \frac{p}{4(p-1)}$ and $f_p = (\mathrm{sgn}\, f)|f|^{p-1}$. In Gross's notation [120] this means that N is a *Sobolev generator* in $(0,\infty)$.

The outline of this argument is as follows: assume that $p > 1$ and let f be a non-negative bounded function in the domain of N in $L^2(\gamma_d)$. Then, replacing f by $f^{p/2}$ in (2.61), we get

$$\begin{aligned}\frac{p}{2}\int_{\mathbb{R}^d}|f(x)|^p\log|f(x)|\gamma_d(dx) \leq &\frac{1}{2}\int_{\mathbb{R}^d}|\nabla_x f^{p/2}(x)|^2\gamma_d(dx) \\ &+\frac{1}{2}\int_{\mathbb{R}^d}|f(x)|^p\gamma_d(dx)\Big(\log\int_{\mathbb{R}^d}|f(x)|^p\gamma_d(dx)\Big).\end{aligned}$$

[14] This semigroup is simply $\{T_{2t}\}_{t\geq 0}$, the Ornstein–Uhlenbeck semigroup with parameter $2t$.

Now, if f is bounded and smooth, we have

$$|\nabla_x(f(x)^{p/2})|^2 = (p/2)^2(f(x)^{p/2-1})^2|\nabla_x f(x)|^2,$$

and also

$$\langle \nabla_x f(x), \nabla_x(f(x)^{p-1})\rangle = (p-1)f(x)^{p-2}|\nabla_x f(x)|^2.$$

Therefore,

$$|\nabla_x(f(x)^{p/2})|^2 = \Big[\frac{(p/2)^2}{(p-1)}\Big]\langle \nabla_x f(x), \nabla_x(f(x)^{p-1})\rangle,$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla_x f^{p/2}(x)|^2 \gamma_d(dx) &= \Big[\frac{p^2}{4(p-1)}\Big]\int_{\mathbb{R}^d} \langle \nabla_x f(x), \nabla_x(f(x)^{p-1})\rangle \gamma_d(dx)\\ &= \Big[\frac{p^2}{4(p-1)}\Big]\langle Nf, f^{p-1}\rangle_\gamma,\end{aligned}$$

thus proving (2.68).

The set where these computations make sense can be justified from the fact that $e^{-tN}$is a contractive and positive semigroup in $L^\infty(\gamma_d)$ (see [120]).

Now, let g be a non-negative function in $C_0^\infty(\mathbb{R})$ with support in $(0,\infty)$ and let $u \in L^\infty(\gamma_d)$. Then, $h := \int_0^\infty g(s)(e^{-sN}u)ds$ exists as a Riemann integral in $L^p(\gamma_1), 1 < p < \infty$. If $f(t) = e^{-tN}h$, $t \geq 0$, f is a positive and differentiable function in $L^p(\gamma_d)$ for all $1 < p < \infty$. Then, according to Lemma 2.15 the function $\alpha(t) = ||f(t)||_{1+(p-1)e^{4t},\gamma_d}$ is differentiable in $(0,\infty)$; therefore,

$$\begin{aligned}\frac{d\alpha(t)}{dt} &= \frac{d}{dt}||f(t)||_{1+(p-1)e^{4t},\gamma}\\ &= ||f(t)||_{p,\gamma}^{1-p}\Big[c(p)^{-1}\Big(\int_{\mathbb{R}^d}|f(t)|^p\log|f(t)|d\gamma_d - ||f(t)||_{p,\gamma}^p\log||f(t)||_{p,\gamma}\Big)\\ &\qquad -\mathrm{Re}\langle Nf(t), f(t)_p\rangle_\gamma\Big] \leq 0.\end{aligned}$$

Thus, $\frac{d}{dt}\log\alpha(t) \leq 0$ and $\log\alpha(t) \leq \log\alpha(0) = \log||h||_{p,\gamma}$, i.e.,

$$||e^{-tN}h||_{1+(p-1)e^{4t},\gamma} \leq ||h||_{p,\gamma}. \tag{2.69}$$

Recall that an approximation of the identity is a sequence of functions $\{h_n\}$ that converges to the Dirac delta function (see the Appendix 10.5). Then, for each $t \geq 0$ and for any element of an approximation of the identity, so that the corresponding sequence $\{h_n\}$ converges to u in $L^p(\gamma_d)$-norm, and $e^{-tN}h_n$ converges to $e^{-tN}u$ in $L^p(\gamma_d)$-norm, and also almost everywhere. Applying the previous inequality (2.69) to h_n and using Fatou's lemma, we have

$$||e^{-tN}u||_{1+(p-1)e^{4t},\gamma} \leq ||u||_{p,\gamma}.$$

As $L^\infty(\gamma_d)$ is dense in $L^p(\gamma_d)$, we can again apply Fatou's lemma; thus, the inequality (2.69) holds for any $h \in L^p(\gamma_d)$.

Finally, given that $T_t h = e^{tL}h = e^{-(t/2)N}h$, the previous inequality is equivalent to

$$||T_t h||_{1+(p-1)e^{2t},\gamma} = ||e^{tL}h||_{1+(p-1)e^{2t},\gamma} = ||e^{-(t/2)N}h||_{1+(p-1)e^{4(t/2)},\gamma} \leq ||h||_{p,\gamma};$$

hence, $\{T_t\}_{t\geq 0}$ satisfies (2.60).

Conversely, let us assume that the semigroup $\{T_t\}_{t\geq 0}$ is strongly continuous on $L^p(\gamma_d)$, $1 < p < \infty$, and that it is hypercontractive (2.60). Let $\mathscr{D}$ be the linear hull of the set of functions $h := \int_0^\infty g(s)(e^{-sN}u)ds$, with g a non-negative function in $C_0^\infty(\mathbb{R})$ with support in $(0,\infty)$ and $u \in L^\infty(\gamma_d)$, as was considered in the first part of the proof. Let h be a non-zero element in $\mathscr{D}$ and set $f(t) = e^{-tN}h$, $t \in (0,\infty)$. Then, for each t, we have

$$\frac{||f(t)||_{1+(p-1)e^{4t},\gamma} - ||f(0)||_{p,\gamma}}{t} \leq ||h||_{p,\gamma}\Big(\frac{1-1}{t}\Big) = 0,$$

according to the hypercontractivity property (2.60) and the fact that $e^{-tN}h = T_{2t}h$. By Lemma 2.15, we can take the limit as $t \downarrow 0$ in the preceding inequality to get

$$||h||_{p,\gamma_d}^{1-p}\Big[p^{-1}4(p-1)\Big(\int_{\mathbb{R}^d} |h|^p \log|h| d\gamma_d - ||h||_{p,\gamma_d}^p \log||h||_{p,\gamma}\Big) - \mathrm{Re}\langle Nh, h_p\rangle_{\gamma_d}\Big] \leq 0.$$

Multiplying by $\frac{p}{4(p-1)}||h||_{p,\gamma_d}^{p-1}$, we obtain (2.61).

Now, because $\mathscr{D}$ is dense in the domain of the infinitesimal generator of $\{e^{-Nt}\}_{t\geq 0}$ in $L^p(\gamma_d)$, for any f there exists a sequence $\{h_n\}$ in $\mathscr{D}$ such that $h_n \to f$ in the graph norm and a.e. γ_d. As $x^p \log x$ is bounded from below in $[0,\infty)$, we can use Fatou's lemma on the left side of (2.61). For the right side we observe that the mapping $f \to f_p$ is continuous from $L^p(\gamma_d)$ to $L^{p'}(\gamma_d)$; thus, the right-hand side is a continuous function of f in the graph norm. Therefore, because (2.61) holds for each h_n, it holds for any f. □

The hypercontractivity of the Ornstein–Uhlenbeck semigroup can also be obtained using the curvature-dimension inequalities. In 1984, D. Bakry and M. Emery [21] developed a criterion (sufficient condition) for a Markov diffusion semigroup to satisfy the hypercontractivity property, which is the famous *Bakry–Emery criterion*. This criterion is given in terms of the iterated square field operator Γ_2,

$$\Gamma_2(f,g) = \frac{1}{2}\Big[L\Gamma(f,g) - \Gamma(f,Lg) - \Gamma(f,Lg)\Big],$$

for every $f,g \in \mathscr{A}$, the standard algebra (an "appropriated class" of functions). The Bakry–Emery criterion has evolved to what is now known as *curvature-dimension inequalities*, which allows us to study the local structure of the generator L and has important applications in differential geometry.

Definition 2.16. *An operator L is said to satisfy a curvature-dimension inequality $CD(\rho,n)$ if*

$$\Gamma_2(f,f) \geq \rho\Gamma(f,f) + \frac{1}{n}(Lf)^2, \tag{2.70}$$

for any $f \in \mathscr{A}$. Here, $\rho \in \mathbb{R}$ is called the curvature and $n \in [1,\infty]$ the dimension.

It can be proved (see for instance [19, 284]) that if an inequality $CD(\rho,\infty)$ holds for some $\rho > 0$, then the invariant measure μ must be finite; moreover, a logarithmic Sobolev inequality holds. Observe that for the Gaussian case, when $d = 1$ (2.17) and (2.23) become

$$\Gamma(f,f)(x) = \frac{1}{2}(f'(x))^2 \quad \text{and} \quad \Gamma_2(f,f)(x) = \frac{1}{4}(f''(x))^2 + \frac{1}{2}(f'(x))^2.$$

Then, trivially, we have a curvature-dimension inequality with $n = \infty$ and constant ρ

$$\frac{1}{4}(f''(x))^2 + \frac{1}{2}(f'(x))^2 \geq \frac{\rho}{2}(f'(x))^2,$$

if and only if $\rho \leq 1$. The extension for higher dimensions follows simply by the tensorization argument.

The original hypercontractive estimates of the Ornstein–Uhlenbeck semigroup were obtained by E. Nelson [204] and were later extended to the complex case, for suitable values of z, by F.B. Weissler [292] and J.B. Epperson [74].

2.4 Applications of the Hypercontractivity Property

One of the first consequences of the hypercontractivity property for the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ is that the orthogonal projections $\mathbf{J}_k$ onto the (closed) subspaces $\mathscr{C}_k$ of the Wiener Chaos are $L^p(\gamma_d)$-continuous for $1 < p < \infty$:

Corollary 2.17. *For any $k \in \mathbb{N}$, $\mathbf{J}_k|_{\mathscr{P}(\mathbb{R}^d)}$, the restriction of $\mathbf{J}_k$ to the polynomials $\mathscr{P}(\mathbb{R}^d)$, has an extension, which will also be denoted as $\mathbf{J}_k$, to a bounded operator in $L^p(\gamma_d)$, i.e.,*

$$||\mathbf{J}_k f||_{p,\gamma} \leq C_{p,k}||f||_{p,\gamma}. \tag{2.71}$$

Proof. If $p > 2$, taking $t_0 > 0$ such that $p = e^{2t_0} + 1$, according to the hypercontractivity property of $\{T_t\}$, we have

$$||T_{t_0} f||_{p,\gamma} \leq ||f||_{2,\gamma}.$$

In particular, from Hölder's inequality,

$$||T_{t_0}\mathbf{J}_k f||_{p,\gamma} \leq ||\mathbf{J}_k f||_{2,\gamma} \leq ||f||_{2,\gamma} \leq ||f||_{p,\gamma}.$$

Now, because $T_{t_0} f = \sum_{k=0}^{\infty} e^{-t_0 k}\mathbf{J}_k f$, we have $T_{t_0}\mathbf{J}_k f = e^{-t_0 k}\mathbf{J}_k f$; therefore,

$$||T_{t_0}\mathbf{J}_k f||_{p,\gamma} = e^{-t_0 k}||\mathbf{J}_k f||_{p,\gamma}.$$

Thus, we have

$$||\mathbf{J}_k f||_{p,\gamma} \leq e^{t_0 k}||f||_{p,\gamma}.$$

The case $1 < p < 2$ is obtained by duality from the previous one. Let p' be the conjugated exponent of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$, $p' > 2$. Then, because the projection $\mathbf{J}_k$ is a self-adjoint operator, using Hölder's inequality, we get

$$\begin{aligned}
||\mathbf{J}_k f||_{p,\gamma} &= \sup_{||g||_{p',\gamma}\le 1} \left| \int_{-\infty}^{\infty} \mathbf{J}_k f \, g \, d\gamma_d \right| = \sup_{||g||_{p',\gamma}\le 1} \left| \int_{-\infty}^{\infty} f \, \mathbf{J}_k g \, d\gamma_d \right| \\
&\le \sup_{||g||_{p',\gamma}\le 1} ||f||_{p,\gamma} ||\mathbf{J}_k g||_{p',\gamma} \le \sup_{||g||_{p',\gamma}\le 1} ||f||_{p,\gamma} C ||g||_{p',\gamma} \le C ||f||_{p,\gamma},
\end{aligned}$$

where $C = e^{t_0 k}$, with $t_0 > 0$ such that $p' = e^{2t_0} + 1$. □

The next lemma is useful for the proof of P.A. Meyer's multiplier theorem , which is also a consequence of the hypercontractivity property.

Lemma 2.18. *If $1 < p < \infty$, for each $n \in \mathbb{N}$, there exists a constant C_n such that*

$$||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma} \le C_n e^{-tn} ||f||_{p,\gamma}. \tag{2.72}$$

Proof. Again, by duality, it is enough to consider the case $p > 2$. Let t_0 be such that $p = e^{2t_0} + 1$. Then, using the hypercontractivity property and Parseval's identity, we get for $t > t_0$,

$$\begin{aligned}
&||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma}^2 \\
&\quad = ||T_{t_0} T_{t-t_0}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma}^2 \\
&\quad \le ||T_{t-t_0}(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{2,\gamma}^2 \\
&\quad = ||\sum_{k=n}^{\infty} e^{-(t-t_0)k} \mathbf{J}_k f||_{2,\gamma}^2 = \sum_{k=n}^{\infty} e^{-2(t-t_0)k} ||\mathbf{J}_k f||_{2,\gamma}^2 \\
&\quad = \sum_{k=0}^{\infty} e^{-2(t-t_0)(k+n)} ||\mathbf{J}_{k+n} f||_{2,\gamma}^2 \le e^{-2(t-t_0)n} \sum_{k=0}^{\infty} ||\mathbf{J}_{k+n} f||_{2,\gamma}^2 \\
&\quad \le e^{-2(t-t_0)n} ||f||_{2,\gamma} \le C_n e^{-2tn} ||f||_{p,\gamma}^2,
\end{aligned}$$

where $C_n = e^{2t_0 n}$.

Now, if $t < t_0$, because T_t is a contraction,

$$\begin{aligned}
||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma} &\le ||(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1})f||_{p,\gamma} \\
&\le (1 + \sum_{k=0}^{n-1} e^{kt_0}) ||f||_{p,\gamma} \le (n+1) e^{nt_0} ||f||_{p,\gamma} \\
&\le C_n e^{-nt_0} ||f||_{p,\gamma} \le C_n e^{-nt} ||f||_{p,\gamma},
\end{aligned}$$

with $C_n = (n+1)e^{nt_0}$. □

Finally, let us consider *potential operators*,

$$U_n f = \int_0^\infty T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f \, dt. \tag{2.73}$$

According to Minkowski's integral inequality and Lemma 2.18, we have

$$||U_n f||_{p,\gamma} \leq \frac{C}{n} ||f||_{p,\gamma}, \quad \text{for } 1 < p < \infty. \tag{2.74}$$

Let us also consider the following operators associated with U_n

$$U_{n,m} f = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f \, dt. \tag{2.75}$$

Then, again according to Minkowski's integral inequality and Lemma 2.18 we have

$$\begin{aligned} ||U_{n,m} f||_{p,\gamma} &\leq \frac{1}{(m-1)!} \int_0^\infty t^{m-1} ||T_t(I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} \, dt \\ &\leq \frac{C}{(m-1)!} \int_0^\infty t^{m-1} e^{-tn} dt ||f||_{p,\gamma} \leq \frac{C}{n^m} ||f||_{p,\gamma}; \end{aligned}$$

hence,

$$||U_{n.m} f||_{p,\gamma} \leq \frac{C}{n^m} ||f||_{p,\gamma}, \tag{2.76}$$

for all $n, m \in \mathbb{N}$.

Moreover, if $f \in \mathscr{C}_k$, i.e., $\mathbf{J}_k f = f$, and $k \geq n$,

$$U_n f = \int_0^\infty T_t \mathbf{J}_k f dt = \int_0^\infty e^{-kt} f dt = \frac{1}{k} f,$$

and similarly,

$$U_{n,m} f = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} T_t \mathbf{J}_k f dt = \frac{1}{(m-1)!} \int_0^\infty t^{m-1} e^{-kt} f dt = \frac{1}{k^m} f.$$

A very important consequence of the hypercontractivity of the Ornstein–Uhlenbeck semigroup is P. A. Meyer's multiplier theorem (see Theorem 6.2 in Chapter 6).

2.5 Notes and Further Results

1. The definition of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t \geq 0}$ using the integral representation (2.28) coincides with that obtained using the general theory of Markov semigroups, taking as transition probabilities

$$P_t(x, dy) = \sum_{|\nu| \geq 0} e^{-t|\nu|} \mathbf{h}_\nu(x) \mathbf{h}_\nu(y) \gamma_1(dy) = \frac{1}{(1 - e^{-2t})^{d/2}} e^{-\frac{e^{-2t}(|y|^2 + |x|^2) - 2e^{-t}\langle x, y \rangle}{1 - e^{-2t}}} dy,$$

according to Mehler's formula (10.24). For more details, see for instance [20] or [284]. This is the link to the theory of Markov processes in probability. It is well known that Brownian motion $\{B_t\}_{t\geq 0}$ is associated with the heat semigroup $\{\mathscr{T}_t\}_{t\geq 0}$. Similarly, we have the *Ornstein–Uhlenbeck process* $\{X_t\}_{t\geq 0}$ with transition probabilities $\{P_t\}_{t\geq 0}$, which is associated with the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$. The process $\{X_t\}_{t\geq 0}$ describes the speed of a particle moving in a fluid with viscosity against a resisting force that is proportional to its speed (see Breiman [35, Chapter 6]). Hence, $\{X_t\}_{t\geq 0}$ can be obtained using the following formula to construct its finite-dimensional distributions:

$$\mu^x\{X_{t_1} \in E_1, X_{t_2} \in E_2, \cdots, X_{t_k} \in E_k\} = \int_{E_k} \cdots \int_{E_2} \int_{E_1} P_{t_1}(x, dy_1) P_{t_2-t_1}(y_1, dy_2) \cdots P_{t_k-t_{k-1}}(y_{k-1}, dy_k).$$

It is known that the process can be obtained from the semigroup using (2.77) and that the semigroup $\{T_t\}_{t\geq 0}$ can be represented in terms of the Markov process $\{X_t\}_{t\geq 0}$ as

$$T_t f(x) = \mathbf{E}[f(X_t)|X_0 = x],\ f \in L^\infty(\gamma_d). \tag{2.77}$$

Using this representation, the properties of the semigroup can be proved using probabilistic methods. Moreover, (2.52) expresses the stationarity and ergodicity of the process.

2. The Ornstein–Uhlenbeck semigroup can also be introduced formally, following S. Bochner [32], as a solution to the equation (2.53), as follows: let $f \in L^2(\gamma_d)$ with Hermite expansion $\sum_{|\nu|\geq 0} a_\nu \mathbf{H}_\nu$; therefore, $\sum_{|\nu|\geq 0}(a_\nu)^2 < \infty$. Then, formally, Lf has the expansion

$$Lf = -\sum_{|\nu|\geq 0} |\nu| a_\nu \mathbf{H}_\nu,$$

if $\sum_{|\nu|\geq 0} |\nu|^2 (a_\nu)^2 < \infty$.

Now, let $u(x,t)$ be a solution of (2.53) with Hermite expansion $\sum_{|\nu|\geq 0} a_\nu(t)\mathbf{H}_\nu$; therefore,

$$\sum_{|\nu|\geq 0} (a_\nu(t))^2 < \infty.$$

Thus, Lu and $\frac{\partial u}{\partial t}$ have Hermite expansions

$$-\sum_{|\nu|\geq 0} |\nu| a_\nu(t)\mathbf{H}_\nu, \quad \text{and} \quad \sum_{|\nu|\geq 0} a'_\nu(t)\mathbf{H}_\nu$$

respectively, and then, assuming that

$$\sum_{|\nu|\geq 0} |\nu|^2 (a_\nu(t))^2 < \infty \quad \text{and} \quad \sum_{|\nu|\geq 0} (a'_\nu(t))^2 < \infty,$$

we conclude by the uniqueness of the Hermite expansions that

$$-|\nu| a_\nu(t) = a_\nu'(t),$$

or equivalently

$$a_\nu(t) = a_\nu e^{-|\nu| t}.$$

Thus, we get the expansion (2.27), and again by uniqueness, we conclude that necessarily $u(x,t) = T_t f(x)$.

3. S. Pérez [221] provided another way to see that $u(x,t) = T_t f(x)$ is a solution of (2.53). It consists of looking for an appropriate dilation that, using the Fourier transform, gives us u as a solution of a differential equation that is easier to solve. Let $w(x,t) = u(e^t x, t)$, then

$$w_t(x,t) = e^t \langle x, \nabla_x u(e^t x,t)\rangle + u_t(e^t x,t),$$

$$\nabla_x w(x,t) = e^t \nabla_x u(e^t x,t), \text{ and } \Delta_x w(x,t) = e^{2t} \Delta_x u(e^t x,t).$$

Thus, w satisfies a variant of the heat equation

$$w_t(x,t) = \frac{1}{2} e^{-2t} \Delta_x w(x,t).$$

Then, applying the Fourier transform (in the x variable), we obtain that $\hat{w}$ satisfies the ordinary differential equation

$$\hat{w}'(\xi,t) = -2\pi^2 e^{-2t} |\xi|^2 \hat{w}(\xi,t),$$

whose solution is

$$\hat{w}(\xi,t) = e^{-\pi^2(1-e^{-2t})|\xi|^2} \hat{f}(\xi)$$

and its inverse Fourier transform is given by

$$w(x,t) = C_d \int_{\mathbb{R}^d} \frac{e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}}{(1-e^{-2t})^{d/2}} f(y) dy.$$

4. To prove the semigroup property of $\{T_t\}_{t\geq 0}$, there is an analogous proof to that given for the heat semigroup in the Appendix using the Fourier transform (see proof of Theorem 10.15*i*)). Nevertheless, this would prove the result only for functions in $\mathscr{S}(\mathbb{R}^d)$, Schwartz's space of test functions. Also, because the set of polynomials $\mathscr{P}(\mathbb{R}^d)$ is dense in $L^p(\gamma_d)$, see Corollary 10.12, we can also prove the semigroup property by means of the representation (2.27).

$$(T_{t_1} \circ T_{t_2}) f = T_{t_1}(\sum_{k=0}^{\infty} e^{-t_2 k} \mathbf{J}_k f) = \sum_{k=0}^{\infty} e^{-(t_1+t_2)k} \mathbf{J}_k f = T_{t_1+t_2} f.$$

5. The *translated Ornstein–Uhlenbeck semigroups* $\{T_t^{(\kappa)}\}_{t\geq 0}$, $\kappa \geq 0$, see [122] and [124], are defined formally as

$$T_t^{(\kappa)} = e^{-\kappa t} T_t, \tag{2.78}$$

which means that $T_t^{(\kappa)}\mathbf{h}_\nu = e^{-t(|\nu|+\kappa)}\mathbf{h}_\nu$. Thus, they are in fact a type of translation of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$, It can be shown that the infinitesimal generator of $\{T_t^{(\kappa)}\}_{t\geq 0}$ is $L - \kappa I_d$.
In particular, for $\kappa = 1$, we get that the translated Ornstein–Uhlenbeck semigroup $\{T_t^{(1)}\}_{t\geq 0}$ has infinitesimal generator $\overline{L}$, the alternative Ornstein–Uhlenbeck operator (2.15).
Clearly, if $f \geq 0$,

$$T_t^{(\kappa)} f \leq T_t f,$$

for $t \geq 0$. These semigroups and their subordinated semigroups are useful in the study of Littlewood–Paley–Stein functions (see [122]). This is discussed later in Chapter 5.

6. D. Bárcenas, H. Leyva, and W. Urbina in [26] studied the controllability of the following controlled Ornstein–Uhlenbeck equation:

$$z(t) = \frac{1}{2}\Delta z - \langle x, \nabla z\rangle + \sum_{n=1}^{\infty}\sum_{|\nu|=n} u_\nu(t)\big\langle b, \mathbf{h}_\nu\big\rangle_{\gamma_d}\mathbf{h}_\nu, \quad t>0, \quad x \in \mathbb{R}^d, \tag{2.79}$$

where $\mathbf{h}_\nu$ is the normalized Hermite polynomial, $b \in L^2(\gamma_d)$, and the control u is in $L^2(0,t_1; l_2(\gamma_d))$, with $l_2(\gamma_d)$ the Hilbert space of the Fourier–Hermite coefficient,

$$l_2(\gamma_d) = \Big\{U = \{\{U_\nu\}_{|\nu|=n}\}_{n\geq 1} : \;\; U_\beta \in \mathbb{C}, \;\; \sum_{n=1}^{\infty}\sum_{|\nu|=n} |U_\nu|^2 < \infty\Big\}.$$

Then, if for all $\nu = (\nu_1, \nu_2, \ldots, \nu_d) \in \mathbb{N}^d$

$$\big\langle b, h_\nu\big\rangle_{\gamma_d} = \int_{\mathbb{R}^d} b(x) h_\nu(x)\gamma_d(dx) \neq 0,$$

then the system is approximately controllable on $[0,t_1]$ for some t_1, i.e., for all $z_0, z_1 \in \mathbb{Z}$ and $\varepsilon > 0$, there exists a control $u \in L^2([0,t_1]; l_2(\gamma_d))$ such that the solution $z(t)$ given by (2.79) satisfies

$$\|z(t_1) - z_1\| \leq \varepsilon.$$

Moreover, the system can never be exactly controllable, i.e., there exist $z_0, z_1 \in \mathbb{Z}$ such that for all control $u \in L^2([0,t_1]; l_2(\gamma_d))$ the solution $z(t)$ of (2.79) corresponding to u satisfies $z(t_1) = z_1$. The fact that $\{T_t\}_{t\geq 0}$ is a compact semigroup, proved in Lemma 2.6, is crucial here.

7. The classical *Sobolev inequality* states that for any function $f \in L^2(\mathbb{R}^d)$ with $\nabla_x f \in L^2(\mathbb{R}^d)$, in the weak sense, we have $f \in L^p(\mathbb{R}^d)$ for $\frac{1}{p} = (\frac{1}{2} - \frac{1}{n})$. Thus,

$$||f||_p \leq C_d \int_{\mathbb{R}^d} |\nabla_x f(x)|^2 dx.$$

The logarithmic Sobolev inequality (2.61) generalizes the classical *Sobolev inequality* for the Gaussian measure.
The Gaussian measure can be defined in a space of infinite dimension, unlike the Lebesgue measure, and as the inequality (2.61) is independent of the dimension, it can be extended to the infinite dimensional case. Moreover, observe that in the classical Sobolev inequality $p \to 2$ as $n \to \infty$ and, consequently, there is loss of information in this inequality when the dimension increases toward infinity.

8. It follows from (2.61) that if f and $\nabla_x f \in L^2(\gamma_d)$, then f belongs to the Orlicz space $L^2 \log L(\gamma_d)$. Moreover, it is easy to prove that there exists a function f such that the right hand side of (2.61) is finite, but f does not belong to $L^2 \log L \log \log L(\gamma_d)$ (see [119]). In that sense, the inequality is optimal and the constants are also the best possible.

9. In [7], A. Amenta and J. Teuwen studied $L^p - L^q$ off-diagonal estimates for the Ornstein–Uhlenbeck semigroup. For sufficiently large t (quantified in terms of p and q), these estimates hold in an unrestricted sense. This would suggest potential generalizations to perturbations of the Ornstein–Uhlenbeck operator, whose heat semigroups need not have nice kernels. Moreover, for sufficiently small t, by using direct estimates of Mehler's kernel, it is shown that the estimates fail when restricted to maximal admissible balls and sufficiently small annuli.

10. S. Thangavelu [270], K. Stempak, and J. L. Torrea [259], among several others, have developed an analogous theory for Hermite functions $\{\overline{\Psi}_\nu\}$ in $\mathbb{R}^d$ which are eigenfunctions of the *Hermite operator*

$$H = -\Delta_x + |x|^2,$$

with eigenvalue $\lambda_\nu = -(2|\nu| + d)$.
Then, the *Hermite semigroup* $\{\Upsilon_t = e^{-tH}\}_{t \geq 0}$ can be defined in $L^p(\mathbb{R}^d)$. The Hermite semigroup leads to analogous results in classical harmonic analysis with respect to the Lebesgue measure, which will not be considered here (for more details see, for instance, [259, 267, 268] and [270]).[15]

[15] It is important to observe that the one-dimensional Hermite expansions only converge in L^p-norm for $p = 2$ (see [230]), but expansions in Hermite functions converge in L^p-norm for $\frac{4}{3} < p < 4$.

11. For $\alpha > -1, \beta > -1$, consider the one-dimensional *Jacobi differential operator*, a second-order diffusion operator defined as

$$\mathscr{L}^{\alpha,\beta} = -(1-x^2)\frac{d^2}{dx^2} - (\beta - \alpha - (\alpha+\beta+2)x)\frac{d}{dx}, \qquad (2.80)$$

The Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_k$ can be defined as orthogonal polynomials with respect to the Jacobi (or beta) measure $\mu_{\alpha,\beta}$ in $(-1,1)$

$$\mu_{\alpha,\beta}(dx) = \frac{1}{2^{\alpha+\beta+1}B(\alpha+1,\beta+1)}\chi_{(-1,1)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx, \quad (2.81)$$

and they are eigenfunctions of $\mathscr{L}^{\alpha,\beta}$ with corresponding eigenvalues $\lambda_n^{\alpha+\beta} = n(n+\alpha+\beta+1)$.

Observe that if we choose $\delta_{\alpha,\beta} = \sqrt{1-x^2}\frac{d}{dx}$, and consider its formal $L^2(\mu_{\alpha,\beta})$-adjoint,

$$\delta^*_{\alpha,\beta} = -\sqrt{1-x^2}\frac{d}{dx} + \Big[(\alpha+\frac{1}{2})\sqrt{\frac{1+x}{1-x}} - (\beta+\frac{1}{2})\sqrt{\frac{1-x}{1+x}}\Big]I,$$

then $\mathscr{L}^{\alpha,\beta} = \delta^*_{\alpha,\beta}\delta_{\alpha,\beta}$. The differential operator $\delta_{\alpha,\beta}$ is considered the "natural" notion of derivative in the Jacobi case.

The square field operator is given by

$$\begin{aligned}
\Gamma^{\alpha,\beta}(f,g)(x) &= \frac{1}{2}\Big[(1-x^2)\frac{d^2(fg)}{dx^2}(x) + (\beta-\alpha+1-(\alpha+\beta+2)x)\frac{d(fg)}{dx}(x) \\
&\quad -(1-x^2)f(x)\frac{d^2g}{dx^2}(x) - (\beta-\alpha+1-(\alpha+\beta+2)x)f(x)\frac{dg}{dx}(x) \\
&\quad -(1-x^2)g(x)\frac{d^2f}{dx^2}(x) - (\beta-\alpha+1-(\alpha+\beta+2)x)g(x)\frac{df}{dx}(x)\Big] \\
&= (1-x^2)\frac{df}{dx}(x)\frac{dg}{dx}(x),
\end{aligned}$$

and

$$\Gamma^{\alpha,\beta}(f)(x) = \Gamma^{\alpha,\beta}(f,f)(x) = (1-x^2)\Big(\frac{df}{dx}(x)\Big)^2. \qquad (2.82)$$

Moreover, the iterated square field operator is given by

$$\begin{aligned}
\Gamma_2^{\alpha,\beta}(f,g)(x) &= 2(1-x^2)^2\frac{d^2f}{dx^2}(x)\frac{d^2g}{dx^2}(x) \\
&\quad -2x(1-x^2)\Big(\frac{d^2f}{dx^2}(x)\frac{dg}{dx}(x) + \frac{df}{dx}(x)\frac{d^2g}{dx^2}(x)\Big) \\
&\quad +\Big((1-x^2)(2\alpha+2\beta+3) \\
&\quad -2x(\beta-\alpha+1-(\alpha+\beta+2)x\Big)\frac{df}{dx}(x)\frac{dg}{dx}(x).
\end{aligned}$$

The operator semigroup associated with the Jacobi polynomials can be defined, in $\mathbb{R}$, for positive or bounded measurable Borel functions of $(-1,1)$, as

$$T_t^{\alpha,\beta} f(x) = \int_{-1}^{1} p^{\alpha,\beta}(t,x,y) f(y) \mu_{\alpha,\beta}(dy), \tag{2.83}$$

where

$$p^{\alpha,\beta}(t,x,y) = \sum_k \frac{1}{\hat{h}_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)}(x) P_k^{(\alpha,\beta)}(y) e^{-k(k+\alpha+\beta+1)t},\ x,y \in [-1,1],$$

$t > 0$ and

$$\hat{h}_k^{(\alpha,\beta)} = \frac{1}{(2k+\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta+1)\Gamma(k+1)\Gamma(k+\alpha+\beta+1)}.$$

Different from the cases of the Hermite or Laguerre polynomials, the kernel $p^{\alpha,\beta}(t,x,y)$ does not correspond to the kernel of Abel summability for the Jacobi series because the eigenvalues $\lambda_n^{\alpha,\beta}$ are not n, but $n(n+\alpha+\beta)$, i.e., they are not linearly distributed. W.N. Bailey obtained the following representation for the kernel of Abel summability for the Jacobi series, also called the Jacobi–Poisson integral,

$$\begin{aligned}
&\sum_k \frac{1}{\hat{h}_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)}(\cos\theta) P_k^{(\alpha,\beta)}(\cos\phi) r^k \\
&\quad = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{1-r}{(1+r)^{\alpha+\beta+2}} \\
&\qquad \times F_4\Big(\frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2}; \alpha+1, \beta+1; \\
&\qquad\qquad \Big(\frac{2\sin(\theta/2)\sin(\phi/2)}{r^{1/2}+r^{-1/2}}\Big)^2, \Big(\frac{2\cos(\theta/2)\cos(\phi/2)}{r^{1/2}+r^{-1/2}}\Big)^2\Big),
\end{aligned}$$

$|r| < 1$ and $\alpha, \beta > 1$, and F_4 is Appell's hypergeometric function of two variables,

$$F_4(a_1,a_2;b_1,b_2;x,y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}}{(b_1)_m (b_2)_n m! n!} x^m y^n,$$

where $(a)_k$ is the Pochhammer symbol, $(a)_k = a(a+1)\cdots(a+k-1)$. This formula was first stated in 1935 without proof in Bailey's tract [15]. The proof is a consequence of Watson's formula for hypergeometric functions (see [290]), and was published later in [16].

An explicit representation of $p^{\alpha,\beta}(t,x,y)$ was obtained by G. Gasper in 1973 [99, 100], which is an analog of Bailey's formula

$$p^{\alpha,\beta}(t,x,y)=\sum_k \frac{1}{\hat{h}_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y)(\lambda_n^{\alpha,\beta})^k$$

$$=\frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}\sum_{n,m=0}^{\infty}\frac{\left(\frac{\alpha+\beta+3}{2}\right)_{m+n}\left(\frac{\alpha+\beta+2}{2}\right)_{m+n}}{m!n!(\alpha+1)_m(\beta+1)_n}$$
$$\times[(1-x)(1-y)]^m[(1+x)(1+y)]^n$$
$$\times\sum_{k=0}^{\infty}(-1)^m\frac{(2m+2n+\alpha+\beta+1)_k\left(m+n+\frac{\alpha+\beta+3}{2}\right)_k}{k!\left(m+n+\frac{\alpha+\beta+1}{2}\right)_k}e^{-t\lambda_{m+n+k}}.$$

Additionally, in [217], A. Nowak, P. Sjögren, and T. Z. Szarek obtained an integral representation for $p^{\alpha,\beta}(t,x,y)$ valid for all admissible-type parameters $\alpha,\beta>-1$. Finally, in [215, Theorem A], A. Nowak and P. Sjögren, without using an explicit form of $p^{\alpha,\beta}(t,x,y)$, obtained sharp estimates of it, giving the order of magnitude for $\alpha,\beta\geq -1/2$. Previously, only its non-negativity had been proved (see [11, Chapter 2]).

$\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is called the *Jacobi semigroup*, or *Jacobi heat semigroup*, in $\mathbb{R}$, and it can be proved that it is a Markov semigroup (see [213, 214] and references therein). The generalization of the Jacobi operator and the Jacobi semigroup in $\mathbb{R}^d$ is straightforward according to the tensorization argument (see [20, 284]).

Additionally, the Jacobi operator satisfies a *Sobolev inequality*, which implies that it satisfies a tight logarithmic Sobolev inequality; therefore, the Jacobi semigroup $\{T_t^{\alpha,\beta}\}_{t\geq 0}$ is hypercontractive, with contraction function

$$q(t)=1+(q(0)-1)e^{4t/C}$$

(for details see Bakry's paper [18, page 33–34], [20, 19], or [284]). Moreover, as a consequence of the asymptotic relations among the Jacobi polynomials and the Hermite and Laguerre polynomials (see [262], (5.3.4) and (5.6.3)), from the Sobolev inequality for the Jacobi operator we can obtain the logarithmic Sobolev inequality for the Ornstein–Uhlenbeck and Laguerre operators; see [20, 284].

12. As has been already mentioned, in the Jacobi setting, because of the non-linearity in n of the eigenvalues $\lambda_n^{\alpha+\beta}=n(n+\alpha+\beta+1)$, the Jacobi semigroup does not coincide with the Abel summability for Jacobi expansions, which is an important difference compared with the Hermite and Laguerre cases. The Abel summability for the Jacobi expansions has been studied extensively in the literature (see for instance [41, 46] and [47] and the references therein).

13. For $\alpha>-1$, consider the one-dimensional *Laguerre differential operator*

$$\mathscr{L}^{\alpha}=-x\frac{d^2}{dx^2}-(\alpha+1-x)\frac{d}{dx}. \tag{2.84}$$

The Laguerre polynomials $\{L_n^\alpha\}$ can be defined as orthogonal polynomials associated with the Gamma measure on $(0,\infty)$,

$$\mu_\alpha(dx) = \chi_{(0,\infty)}(x)\frac{x^\alpha e^{-x}}{\Gamma(\alpha+1)}dx, \tag{2.85}$$

and they are eigenfunctions of $\mathscr{L}^\alpha$ with corresponding eigenvalues $\lambda_k = k$. Observe that if we choose $\delta_\alpha = \sqrt{x}\frac{d}{dx}$, and consider its formal $L^2(\alpha)$-adjoint,

$$\delta_\alpha^* = -\sqrt{x}\frac{d}{dx} + \Big[\frac{\alpha+1/2}{\sqrt{x}} + \sqrt{x}\Big]I$$

then $\mathscr{L}^\alpha = \delta_\alpha^*\delta_\alpha$. The differential operator δ_α is considered the natural notion of derivative in the Laguerre case.

The operator semigroup associated with the Laguerre polynomials can be defined for positive or bounded measurable Borel functions of $(0,\infty)$, as

$$T_t^\alpha f(x) = \int_0^\infty p^\alpha(t,x,y)f(y)\mu_\alpha(dy), \tag{2.86}$$

where, according to the Hille–Hardy formula (10.35),

$$\begin{aligned} p^\alpha(t,x,y) &= \sum_k \frac{\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+1)}L_k^\alpha(x)L_k^\alpha(y)e^{-kt} \\ &= \frac{1}{1-e^{-t}}e^{-\frac{(x+y)e^{-t}}{1-e^{-t}}}(-xye^{-t})^{-\alpha/2}I_\alpha\Big(\frac{2\sqrt{xye^{-t}}}{1-e^{-t}}\Big), \end{aligned}$$

where $I_\alpha(x)$ is the modified Bessel function of the first kind of order α. This identity was found in 1926 by E. Hille [135] and independently rediscovered by G.H. Hardy [130] (see also G.N. Watson [291]).

In this case, the square field operator is given by

$$\begin{aligned} \Gamma^\alpha(f,g)(x) = \frac{1}{2}\Big[&x\frac{d^2(fg)}{dx^2}(x) + (\alpha+1-x)\frac{d(fg)}{dx}(x) - xf(x)\frac{d^2g}{dx^2}(x) \\ &-(\alpha+1-x)f(x)\frac{dg}{dx}(x) - xg(x)\frac{d^2f}{dx^2}(x) - (\alpha+1-x)g(x)\frac{df}{dx}(x)\Big] \\ = &x\frac{df}{dx}(x)\frac{dg}{dx}(x), \end{aligned}$$

and

$$\Gamma^\alpha(f)(x) = \Gamma^\alpha(f,f)(x) = x\Big(\frac{df}{dx}(x)\Big)^2. \tag{2.87}$$

Moreover, the iterated square field operator is given by

$$\Gamma_2^{\alpha}(f,g)(x) = \frac{1}{2}\Big[x\frac{d^2 f}{dx^2}(x)\frac{dg}{dx}(x) + x\frac{df}{dx}(x)\frac{d^2 g}{dx^2}(x) + 2x^2\frac{d^2 f}{dx^2}(x)\frac{d^2 g}{dx^2}(x) + (\alpha+1+x)\frac{df}{dx}(x)\frac{dg}{dx}(x)\Big]. \quad (2.88)$$

$\{T_t^{\alpha}\}_{t\geq 0}$ is called the *Laguerre semigroup*, or *Laguerre heat semigroup*. It can be proved that it is a Markov semigroup (see [208] and [193] and the references therein). The generalization of the Laguerre operator and the Laguerre semigroup in $\mathbb{R}^d$ is straightforward according to the tensorization argument (see [20, 284]). Here, again, the semigroup $\{T_t^{\alpha}\}$ coincides with the Abel summability for Laguerre expansions.

Moreover, the Laguerre operator satisfies a tight logarithmic Sobolev inequality; therefore, the Laguerre semigroup $\{T_t^{\alpha}\}$ is hypercontractive, see [20, 284]. The hypercontractivity of the Laguerre semigroup was initially proved by A. Korzeniowski and D. Stroock in [152].

14. The fact that the Jacobi and Laguerre semigroups are hypercontractive allows us to obtain similar applications to those obtained in Section 2.4 for the Ornstein–Uhlenbeck semigroup (see for instance [24, 25, 117, 284], and the references therein).

15. Additionally, an operator semigroup can be defined for the generalized Hermite polynomials $\{H_n^{\mu}\}$, which, as we know, are eigenfunctions of the operator

$$L_{\mu} = \frac{1}{2}\frac{d^2}{dx^2} + \left(\frac{\mu}{x} - x\right)\frac{d}{dx} - \mu\frac{I-\tilde{I}}{2x^2}.$$

Using Mehler's formula (10.46), this semigroup can be written as

$$T_t^{\mu} f(x) = \int_{-\infty}^{\infty} p^{\mu}(t,x,y) f(y) |y|^{2\mu} e^{-|y|^2} dy, \quad (2.89)$$

where

$$p^{\mu}(t,x,y) = \frac{1}{(1-e^{-2t})^{\mu+1/2}} e^{-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}} e_{\mu}\left(\frac{2xye^{-t}}{1-e^{-2t}}\right),$$

for f a positive or bounded measurable function on $(-\infty,\infty)$.
$\{T_t^{\mu}\}_{t\geq 0}$ is called the *generalized Ornstein–Uhlenbeck semigroup*, and it is easy to see that it is also a Markov semigroup with generator L_{μ}; see [20].

The weak type $(1,1)$ inequality, in addition to its L^p-boundedness for $p > 1$, with respect to the measure λ of the maximal operator associated with this semigroup, was proved in [30]. Those results were extended to higher dimensions in [92]. Further research into this semigroup and the operators associated with it are particular cases of a more general theory for the Dunkl Ornstein–Uhlenbeck operator (see [212]).

16. For the Hermite, Laguerre, and Jacobi functions in analogous form as above, we can define operator semigroups, usually called *heat diffusion semigroups* (see for instance [270, 259] and the references therein).

17. An unexpected application of the hypercontractivity property of the Ornstein–Uhlenbeck semigroup has been found in several works on non-linear partial differential equations of evolution type (see for instance N. Tzvetkov [277]).

18. The boundedness of the Ornstein–Uhlenbeck semigroup on variable $L^{p(\cdot)}$ Gaussian spaces has been studied in [192] by J. Moreno, E. Pineda, and W. Urbina.

3

The Poisson–Hermite Semigroup

In this chapter, we consider the Poisson–Hermite semigroup, which is the semigroup subordinated to the Ornstein–Uhlenbeck semigroup. This is analogous to the classical case in which the Poisson semigroup is obtained by subordination of the heat semigroup (for more details see the Appendix). Then, we study the characterization of the $\frac{\partial^2}{\partial t^2}+L$-harmonic functions, the generalized Poisson–Hermite semigroups, and the conjugated Poisson–Hermite semigroup which, as in the classical case, is closely related to the notion of singular integrals.

3.1 Definition and Basic Properties

We define the *Poisson–Hermite semigroup* as the semigroup subordinated to the Ornstein–Uhlenbeck semigroup using *Bochner's subordination formula*,[1]

$$e^{-\lambda} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\lambda^2/4u} du, \tag{3.1}$$

(see E. Stein [252]). Thus, making the change of variables $r = e^{-t^2/4u}$, we have

$$\begin{aligned} P_t f(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{(t^2/4u)} f(x) du \\ &= \frac{1}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{\exp\left(\frac{-|y-e^{-t^2/4u}x|^2}{1-e^{-t^2/2u}}\right)}{(1-e^{-t^2/2u})^{d/2}} du f(y) dy \qquad (3.2) \\ &= \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 t \frac{\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r} f(y) dy. \end{aligned}$$

[1] In [133], C. Herz considered more general subordination relations between semigroups.

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Then,

$$P_t f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\,dy, \tag{3.3}$$

with what we will call the *Poisson–Hermite kernel*,

$$p(t,x,y) = \frac{1}{\pi^{(d+1)/2}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{\exp\left(\frac{-|y-e^{-t^2/4u}x|^2}{1-e^{-t^2/2u}}\right)}{(1-e^{-t^2/2u})^{d/2}} du \tag{3.4}$$

$$= \frac{1}{4\pi^{(d+1)/2}} \int_0^1 t \frac{\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r}, \tag{3.5}$$

using the change of variables $r = e^{-t^2/4u}$. Moreover, making the change of variables $v = 1-r^2$, we get

$$p(t,x,y) = \frac{1}{8\pi^{(d+1)/2}} \int_0^1 \frac{e^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})^{3/2}} \frac{e^{-\frac{|y-\sqrt{1-v}x|^2}{v}}}{v^{d/2}} \frac{dv}{1-v}. \tag{3.6}$$

The subordination of the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$ can be expressed alternatively in the following way. Let $\mu_t^{(1/2)}$ be the Borel measure on $[0,\infty)$ whose Laplace transform satisfies

$$\int_0^\infty e^{-\lambda s} \mu_t^{(1/2)}(ds) = e^{-\sqrt{\lambda}t}.$$

It is easy to check that the family of measures $\{\mu_t^{(1/2)}\}_{t\geq 0}$ forms a convolution semigroup [81]. Moreover, using Bochner's subordination formula (3.1) (with $\lambda = t\sqrt{\alpha}$ and the change of variables $s = \frac{t^2}{4u}$), it yields the following explicit expression of the measure $\mu_t^{(1/2)}$:

$$\mu_t^{(1/2)}(du) = \frac{t}{2\sqrt{\pi}} e^{-t^2/4u} u^{-3/2} du. \tag{3.7}$$

Then, P_t can be defined by

$$P_t f(x) = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(ds). \tag{3.8}$$

The Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$ is a strongly continuous, symmetric, conservative semigroup of positive contractions in $L^p(\gamma_d)$, $1 \leq p < \infty$, with infinitesimal generator $(-L)^{1/2}$. More precisely,

Theorem 3.1. *The family of operators $\{P_t\}_{t\geq 0}$ satisfies the following properties:*

i) Semigroup property:

$$P_{t_1+t_2} = P_{t_1} \circ P_{t_2},\ t_1, t_2 \geq 0.$$

ii) Positivity and conservative property:

$$P_t f \geq 0, \quad \textit{for} \quad f \geq 0,\, t \geq 0,$$

and

$$P_t 1 = 1.$$

iii) Contractivity property:

$$||P_t f||_{p,\gamma} \leq ||f||_{p,\gamma}, \quad t \geq 0,\, 1 \leq p \leq \infty.$$

iv) Strong $L^p(\gamma_d)$-continuity property: The mapping $t \to P_t f$ is continuous from $[0,\infty)$ to $L^p(\gamma_d)$, for $1 \leq p < \infty$ and $f \in L^p(\gamma_d)$.

v) Symmetry property: P_t is a self-adjoint operator in $L^2(\gamma_d)$, i.e.,

$$\int_{\mathbb{R}^d} P_t f(x) g(x) \gamma_d(dx) = \int_{\mathbb{R}^d} f(x) P_t g(x) \gamma_d(dx),\, t \geq 0. \tag{3.9}$$

vi) Infinitesimal generator: $(-L)^{1/2}$ is the infinitesimal generator of $\{P_t : t \geq 0\}$, that is to say,

$$\lim_{t\to 0} \frac{P_t f - f}{t} = (-L)^{1/2} f. \tag{3.10}$$

Proof. These results can be obtained immediately from Theorem 2.5 using Bochner's subordination formula (3.1). □

As the Poisson–Hermite semigroup is subordinated to the Ornstein–Uhlenbeck semigroup and, therefore, $(-L)^{1/2}$ is its infinitesimal generator, we conclude that P_t can be defined in the spectral sense as $e^{-t(-L)^{1/2}}$. Therefore,

$$P_t \mathbf{h}_\nu = e^{-t\sqrt{|\alpha|}} \mathbf{h}_\nu. \tag{3.11}$$

Proposition 3.2. *(B. Muckenhoupt)*

i) If f has a Hermite expansion $f = \sum_{k=0}^{\infty} \mathbf{J}_k f$, then for all $t \geq 0$, $P_t f$ has a Hermite expansion

$$P_t f = \sum_{k=0}^{\infty} e^{-t\sqrt{k}} \mathbf{J}_k f. \tag{3.12}$$

ii) If $f \in L^2(\gamma_d)$ then $\sum_{k=0}^{\infty} e^{-t\sqrt{k}} \mathbf{J}_k f(x)$ converges absolutely to $P_t f(x)$ almost everywhere (a.e.) x.

Proof.

i) By arguments analogous to those given in Proposition 2.3, and using Bochner's subordination formula (3.1), we obtain

$$\int_{\mathbb{R}^d} P_t f(x)\mathbf{h}_\nu(x)\gamma_d(dx) = \int_{\mathbb{R}^d}\left(\int_0^1 T(t,r)T_{(-\log r)}f(x)dr\right)\mathbf{h}_\nu(x)\gamma_d(dx)$$
$$= \int_0^1 \int_{\mathbb{R}^d} T_{(-\log r)}f(x)\mathbf{h}_\nu(x)\gamma_d(dx)T(t,r)dr$$
$$= \langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\int_0^1 r^\nu T(t,r)dr = e^{-t\sqrt{\nu}}\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}.$$

ii) As the sequence $\{\langle f,\mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu(x)\}_\nu$ is bounded for each x, by the Weierstrass M-test, the series $\sum_{k=0}^{\infty} e^{-t\sqrt{k}}\mathbf{J}_k f(x)$ converges absolutely for each x. Given that $L^2(\gamma_d)\subset L^1(\gamma_d)$, then according to *i*), $P_t f(x)$ has an expansion $P_t f(x) = \sum_{k=0}^{\infty} e^{-t\sqrt{k}}\mathbf{J}_k f(x)$; this must be the limit a.e. □

B. Muckenhoupt obtained this result for $d=1$ (see [193]). It was extended to higher dimensions by C. P. Calderón [44] .

To study higher-order Gaussian Littlewood–Paley g functions and the Gaussian Besov–Lipschitz and Triebel–Lizorkin spaces and Riesz transform, we need some results for the k-th derivatives of the Poisson–Hermite semigroup $\frac{\partial^k P_t f(x)}{\partial t^k}$. Let us consider explicitly their expressions. First, recall that $p(t,x,y)$ can be written as

$$p(t,x,y) = \frac{1}{2\pi^{(d+1)/2}}\int_0^1 t\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}.$$

Therefore, using Rodrigues' formula (1.28),

$$\frac{\partial p(t,x,y)}{\partial t} = \frac{1}{2\pi^{(d+1)/2}}\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\left(1-\frac{t^2}{2(-\log r)}\right)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}.$$
$$= \frac{1}{2^2\pi^{(d+1)/2}}\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\left(2-4\frac{t^2}{4(-\log r)}\right)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}.$$
$$= -\frac{1}{2^2\pi^{(d+1)/2}}\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}H_2\left(\frac{t}{2\sqrt{-\log r}}\right)\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}. \tag{3.13}$$

where H_2 is the Hermite polynomial of order 2.

Moreover, by induction, again using Rodrigues' formula (1.28) and the three-term recurrence relation of the Hermite polynomials (10.19), it can be proved that, for $k>1$

$$\frac{\partial^k p(t,x,y)}{\partial t^k} = C_d\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{H_{k+1}\left(\frac{t}{2(-\log r)^{1/2}}\right)}{(-\log r)^{\frac{k-1}{2}}}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r},$$

where H_{k+1} is the Hermite polynomial of order $k+1$.

On the other hand, for $j = 1, \cdots, d$,

$$\begin{aligned}\frac{\partial p(t,x,y)}{\partial x_j} &= \frac{1}{\pi^{(d+1)/2}} \int_0^1 t \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \frac{(y_j - rx_j)}{(1-r^2)^{(d+1)/2}} \exp\left(\frac{-|y-rx|^2}{1-r^2}\right) dr \\ &= \int_0^1 t \frac{\exp\left(t^2/4\log r\right)}{(-\log r)} \omega(r) \frac{(y_j - rx_j)}{(1-r^2)^{(d+3)/2}} \exp\left(\frac{-|y-rx|^2}{1-r^2}\right) dr,\end{aligned} \tag{3.14}$$

where $\omega(r) = C_d(\frac{1-r^2}{-\log r})^{1/2}$ is a Lipschitz function on $[0,1]$, and

$$\begin{aligned}\frac{\partial^{|\beta|} p(t,x,y)}{\partial x_1^{\beta_1} \cdots x_d^{\beta_d}} &= \frac{(-1)^{|\beta|}}{2\pi^{(d+1)/2}} \int_0^1 t \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} r^{|\beta|} \mathbf{H}_\beta\left(\frac{y-rx}{(1-r^2)^{1/2}}\right) \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r} \\ &= \int_0^1 t \frac{\exp\left(t^2/4\log r\right)}{(-\log r)} \omega(r) r^{|\beta|} \mathbf{H}_\beta\left(\frac{y-rx}{(1-r^2)^{1/2}}\right) \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{(d+1)/2}} \frac{dr}{r}.\end{aligned} \tag{3.15}$$

Now, we will also need the following technical result about the L^1-norm of the derivatives of the kernel $p(t,x,y)$.

Lemma 3.3. *If $p(t,x,y)$ is the Poisson–Hermite kernel, then*

$$\int_{\mathbb{R}^d} \left| \frac{\partial p(t,x,y)}{\partial t} \right| dy \leq \frac{C}{t}, \tag{3.16}$$

where C is a constant independent of x and t. Moreover, for any positive integer k, we have

$$\int_{\mathbb{R}^d} \left| \frac{\partial^k p(t,x,y)}{\partial t^k} \right| dy \leq \frac{C}{t^k}. \tag{3.17}$$

Proof. Let us first prove (3.16). Using Tonelli's theorem, using the fact that

$$\frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} dy = 1,$$

we have

$$\begin{aligned}\int_{\mathbb{R}^d} \left| \frac{\partial p(t,x,y)}{\partial t} \right| dy &\leq \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \left| 1 + \frac{t^2}{2\log r} \right| \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r} dy \\ &= \frac{1}{2\pi^{(d+1)/2}} \int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \left| 1 + \frac{t^2}{2\log r} \right| \int_{\mathbb{R}^d} \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} dy \frac{dr}{r} \\ &= \frac{1}{2\pi^{1/2}} \int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \left| 1 + \frac{t^2}{2\log r} \right| \frac{dr}{r}.\end{aligned}$$

Thus, what we need to prove is

$$\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \left|1+\frac{t^2}{2\log r}\right| \frac{dr}{r} \le \frac{C}{t}. \tag{3.18}$$

Making the change of variables $s=-\log r$, we get

$$\begin{aligned}\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \left|1+\frac{t^2}{2\log r}\right| \frac{dr}{r} &= \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\left|1-\frac{t^2}{2s}\right| ds \\ &\le \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} ds + \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\frac{t^2}{2s} ds\end{aligned}$$

Now, making the change of variables $v=\frac{t^2}{4s}$, $ds=-\frac{t^2}{4v^2}dv$, we get

$$\begin{aligned}\int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} ds &= \int_0^\infty e^{-v}\left(\frac{t^2}{4v}\right)^{-3/2}\frac{t^2}{4v^2}dv = \int_0^\infty e^{-v}\frac{(4v)^{3/2}}{t^3}\frac{t^2}{4v^2}dv \\ &= \frac{C}{t}\int_0^\infty e^{-v}v^{-1/2}dv = \frac{C\Gamma(1/2)}{t} = \frac{C'}{t}\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}}\frac{t^2}{4s} ds &= 2\int_0^\infty e^{-v}\left(\frac{t^2}{4v}\right)^{-3/2} v\frac{t^2}{4v^2}dv = 2\int_0^\infty e^{-v}\frac{(4v)^{3/2}}{t^3}v\frac{t^2}{4v^2}dv \\ &= \frac{C}{t}\int_0^\infty e^{-v}v^{1/2}dv = \frac{C\Gamma(3/2)}{t} = \frac{C'}{t}.\end{aligned}$$

For the proof of the general case (3.17), we use induction. As the case $k=1$ is already proved, let us assume that (3.17) holds for certain k and prove that it also holds for $k+1$. According to the semigroup property, and taking $u=t+s$, we have

$$\begin{aligned}\frac{\partial^{k+1}p(u,x,y)}{\partial u^{k+1}} &= \frac{\partial}{\partial s}\frac{\partial^k}{\partial t^k}p(t+s,x,y) = \frac{\partial}{\partial s}\frac{\partial^k}{\partial t^k}\left[\int_{\mathbb{R}^d} p(s,x,v)p(t,v,y)dv\right] \\ &= \int_{\mathbb{R}^d}\frac{\partial p(s,x,v)}{\partial s}\frac{\partial^k p(t,v,y)}{\partial t^k}dv.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\frac{\partial^{k+1}p(u,x,y)}{\partial u^{k+1}}\right|dy &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\frac{\partial p(s,x,v)}{\partial s}\right|\left|\frac{\partial^k p(t,v,y)}{\partial t^{ks}}\right|dvdy \\ &\le \int_{\mathbb{R}^d}\left|\frac{\partial p(s,x,v)}{\partial s}\right|\int_{\mathbb{R}^d}\left|\frac{\partial^k p(t,v,y)}{\partial t^k}\right|dydv \le \frac{C}{s}\frac{C}{t^k}.\end{aligned}$$

Finally, taking $s=t=u/2$, the case $k+1$ is proved. □

Using the representation of the Poisson–Hermite semigroup (3.8) using the one-sided stable measure

$$\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds = g(t,s)ds,$$

we can rephrase the result of Lemma 3.3 in terms of $\mu_t^{(1/2)}$ as follows. First, for any $k \in \mathbb{N}$, the notation $\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)$ denotes

$$\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds) := \frac{\partial^k g(t,s)}{\partial t^k} ds.$$

Then, by induction, it can be seen that

$$\frac{\partial^k \mu_t^{(1/2)}}{\partial t^k}(ds) = \left(\sum_{\substack{i\in\mathbb{Z}, j\in\mathbb{N}, \\ 0\le j\le k, 2j-i=k}} a_{i,j} \frac{t^i}{s^j} \right) \mu_t^{(1/2)}(ds) \tag{3.19}$$

where $\{a_{i,j}\}$ is a (finite) set of constants.

Moreover, using the change of variables $u = \frac{t^2}{4s}$, it is easy to see that given $k \in \mathbb{N}$ and $t > 0$

$$\int_0^{+\infty} \frac{1}{s^k} \mu_t^{\frac{1}{2}}(ds) = \frac{C_k}{t^{2k}}, \tag{3.20}$$

and then, if $k \in \mathbb{N}$ and $t > 0$

$$\int_0^{+\infty} \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)} \right| (ds) \le \frac{C_k}{t^k}. \tag{3.21}$$

Additionally a pointwise estimate of the k-th derivative of the Poisson–Hermite semigroup is needed in what follows.

Lemma 3.4.

$$\left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| \le C_k T^* f(x) t^{-k}, \tag{3.22}$$

where $T^* f$ is the maximal Ornstein–Uhlenbeck function.

Proof. Using (3.21) and the dominated convergence theorem, we have

$$\begin{aligned} \left| \frac{\partial^k P_t f(x)}{\partial t^k} \right| &= \left| \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) \right| \le \int_0^{+\infty} |T_s f(x)| \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) \right| \\ &\le \int_0^{+\infty} T^* f(x) \left| \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds) \right| \le C_k T^* f(x) t^{-k}. \qquad \square \end{aligned}$$

Now, we need an estimate of the $L^p(\gamma_d)$-norms of the derivatives of the Poisson–Hermite semigroup.

Lemma 3.5. *Suppose $f \in L^p(\gamma_d)$, then for any integer k, the function* $\left|\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right|_{p,\gamma}$ *is a non-increasing function of t, for* $0 < t < +\infty$*. Moreover,*

$$\left|\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right|_{p,\gamma} \leq C\|f\|_{p,\gamma}\, t^{-k},\ t > 0. \tag{3.23}$$

Proof. Let us consider first the case $k = 0$. Let us fix $t_1, t_2 > 0$, then by using the semigroup property, we get

$$u(x,t_1+t_2) = P_{t_1+t_2} f(x) = P_{t_1}(P_{t_2} f(x)) = P_{t_1}(u(x,t_2))$$

Therefore, by definition of P_t, Jensen's inequality and the invariance of γ_d

$$\begin{aligned}\int_{\mathbb{R}^d} |u(x,t_1+t_2)|^p \gamma_d(dx) &= \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} p(t_1,x,y)u(y,t_2)dy\right|^p \gamma_d(dx)\\ &\leq \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} p(t_1,x,y)|u(y,t_2)|^p dy\Big)\gamma_d(dx)\\ &= \int_{\mathbb{R}^d} P_{t_1}(|u(x,t_2)|^p)\gamma_d(dx) = \int_{\mathbb{R}^d} |u(x,t_2)|^p \gamma_d(dx).\end{aligned}$$

Thus,

$$\|P_{t_1+t_2} f\|_{p,\gamma} \leq \|P_{t_2} f\|_{p,\gamma}.$$

Now, we prove the general case $k > 0$. Differentiating the identity

$$u(x,t_1+t_2) = P_{t_1}(u(x,t_2))$$

k-times with respect to t_2, we get

$$\frac{\partial^k u(x,t_1+t_2)}{\partial (t_1+t_2)^k} = P_{t_1}\Big(\frac{\partial^k u(x,t_2)}{\partial t_2^k}\Big)$$

and then we use an analogous argument to the one above.

To prove (3.23), we again use the representation of the Poisson–Hermite semigroup with a one-sided stable measure (3.8), and differentiating it k-times with respect to t, we get

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds).$$

Thus, using Minkowski's integral inequality, the contractive property of the Ornstein–Uhlenbeck semigroup and inequality (3.21), we get for $t > 0$

$$\begin{aligned}\left|\left|\frac{\partial^k P_t f}{\partial t^k}\right|\right|_{p,\gamma} &\leq \int_0^{+\infty} \left|\left|T_s f \frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)\right|\right| = \int_0^{+\infty} \|T_s f\|_{p,\gamma} \left|\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)\right|\\ &\leq \|f\|_{p,\gamma} \int_0^{+\infty} \left|\frac{\partial^k}{\partial t^k}\mu_t^{(1/2)}(ds)\right| \leq \frac{C_k}{t^k}\|f\|_{p,\gamma}. \qquad \square\end{aligned}$$

Definition 3.6. *The maximal function of the Poisson–Hermite semigroup or Poisson–Hermite maximal function* $\{P_t\}_{t\geq 0}$ *is defined as*

$$P^* f(x) = \sup_{t>0} |P_t f(x)|. \tag{3.24}$$

In Theorem 4.28 of Chapter 4, we study the boundedness properties of P^*, proving that it is bounded in $L^p(\gamma_d)$ for $1 < p \leq \infty$, and it is of weak type $(1,1)$ with respect to the measure γ_d. Moreover, from the boundedness property of P^*, it follows that

$$P_0 f(x) = \lim_{t\to 0^+} P_t f(x) = f(x) \quad a.e.\ x \in \mathbb{R}^d, \tag{3.25}$$

and

$$P_\infty f(x) := \lim_{t\to\infty} P_t f(x) = \int_{\mathbb{R}^d} f(y)\, \gamma_d(dy) \quad a.e.\ x \in \mathbb{R}^d, \tag{3.26}$$

for all $f \in L^p(\gamma_d)$, $1 \leq p \leq \infty$; see Theorem 4.46,. Observe that this says that the Poisson–Hermite semigroup does not decay at infinity, i.e., it is not true that $P_t \to 0$ as $t \to \infty$, unless $\int_{\mathbb{R}^d} f(y)\, \gamma_d(dy) = 0$. In this case, one can obtain a precise estimate of the decay, as it is proved in the following result.

Lemma 3.7. *The Poisson–Hermite semigroup* $\{P_t\}_{t>0}$ *has exponential decay on* $\mathscr{C}_0^\perp = \bigoplus_{k=1}^\infty \mathscr{C}_k$. *More precisely, if* $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$,

$$|P_t f(x)| \leq C_{d,f}(d + |x|)e^{-t}. \tag{3.27}$$

Proof. As $\{P_t\}_{t>0}$ is a strongly continuous semigroup, we have

$$\lim_{t\to 0^+} P_t f(x) = f(x) \tag{3.28}$$

and according to the hypothesis, because we are assuming that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$,

$$\lim_{t\to\infty} P_t f(x) = 0. \tag{3.29}$$

Let us prove that

$$\left| \frac{\partial}{\partial t} P_t f(x) \right| \leq C_{d,f}(d + |x|)e^{-t}.$$

As

$$\frac{\partial T_t f}{\partial t}(x) = L(T_t f)(x),$$

differentiating in (2.36), we have

$$\nabla_x (T_t f)(x) = \left(e^{-t} T_t \left(\frac{\partial f}{\partial x_1} \right)(x), \ldots, e^{-t} T_t \left(\frac{\partial f}{\partial x_d} \right)(x) \right)$$

and

$$\triangle_x (T_t f)(x) = \sum_{j=1}^d e^{-2t} T_t \left(\frac{\partial^2 f}{\partial x_j^2} \right)(x).$$

Therefore, taking $f \in C_b^2(\mathbb{R}^d)$ and using (3.2), we have that

$$\frac{\partial P_t f}{\partial t}(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{2u} L(T_{t^2/4u} f)(x) du.$$

Carrying on the computations, as in [122], we get

$$\begin{aligned} \left| \frac{\partial P_t f}{\partial t}(x) \right| &\leq C_d \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} \left[\sum_{j=1}^{d} \frac{e^{-t^2/2u}}{2} + |x_j| e^{-t^2/4u} \right] f(u) du \\ &\leq C_{d,f}(d+|x|)e^{-t}. \end{aligned}$$

Then,

$$|P_t f(x)| \leq \int_t^\infty \left| \frac{\partial}{\partial s} P_s f(x) \right| ds \leq C_{d,f}(d+|x|)e^{-t}. \qquad \square$$

On the other hand, because the Poisson–Hermite semigroup is the subordinated semigroup of the Ornstein–Uhlenbeck semigroup, it is easy to see that it is also hypercontractive.

Additionally, we have the following result.

Proposition 3.8. *If $f \in L^p(\gamma_d)$, $u(x,t) = P_t f(x)$ is a $C^\infty(\mathbb{R}_+^{d+1})$ solution of the elliptic equation,*[2]

$$\frac{\partial^2 u}{\partial t^2}(x,t) + Lu = 0, \quad x \in \mathbb{R}^d, t > 0, \tag{3.30}$$

with boundary condition $u(x,0) = f(x), x \in \mathbb{R}^d$.

Proof. By the general theory of semigroups, given that $(-L)^{1/2}$ is the infinitesimal generator of $\{P_t\}_{t\geq 0}$, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2}(x,t) &= \frac{\partial}{\partial t}\left[\frac{\partial P_t f}{\partial t}(x)\right] = \frac{\partial}{\partial t}[(-L)^{1/2} P_t f(x)] \\ &= (-L)^{1/2}\left[\frac{\partial P_t f}{\partial t}(x)\right] = (-L)^{1/2}[(-L)^{1/2} P_t f(x)] = -Lu(x,t). \end{aligned}$$

Alternatively, if we assume first that $f \in L^2(\gamma_d)$, because the sequence $\{\langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu(x)\}_{\nu \geq 0}$ is bounded for each x, we know that

$$P_t f(x) = \sum_{k=0}^{\infty} e^{-t\sqrt{k}} \mathbf{J}_k f(x) = \sum_{k=0}^{\infty} e^{-t\sqrt{k}} \sum_{|\nu|=k} f_H(\nu) \mathbf{h}_\nu(x)$$

converges absolutely for each x; therefore, we can differentiate term by term. Now, because the Hermite polynomials are eigenfunctions of L, we have

[2]Sometimes called the wave equation (see for instance [59]).

$$\frac{\partial^2 P_t f}{\partial t^2}(x)+LP_t f(x)=\sum_{k=0}^{\infty}\frac{e^{-t\sqrt{k}}}{(2^k k!)^{1/2}}\sum_{|\nu|=k} f_H(\nu)[k\mathbf{H}_\nu(x)-k\mathbf{H}_\nu(x)]=0.$$

Differentiation under the integral sign is justified by showing that the derivatives of the kernel are bounded in y for each (t,x) in a neighborhood of (t_0,x_0), and this is easy to check by estimating the derivatives of $T(t,r)M_{(-\log r)}(x,y)$ and integrating with respect to r. The boundary condition holds by (3.25). □

Therefore, $u(x,t)=P_t f(x)$ satisfies:

$$2\frac{\partial^2 u}{\partial t^2}(x,t)+\Delta_x u(x,t)-2\langle x,\bigtriangledown_x u(x,t)\rangle=0, \tag{3.31}$$

and we will say that u is $\frac{\partial^2}{\partial t^2}+L$*-harmonic*. Moreover, $u(x,t)=P_t f(x)$, which can also be called the *Poisson–Hermite integral*, can be thought of as the $\frac{\partial^2}{\partial t^2}+L$-harmonic extension of f in $\mathbb{R}^d$ to the upper half-plane $\mathbb{R}_+^{(n+1)}$.

In [106], G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, and B. Viviani find optimal integrability conditions to guarantee the existence of solutions of (3.31).

3.2 Characterization of $\frac{\partial^2}{\partial t^2}+L$-Harmonic Functions

In the classical case, it is well known that Δ-harmonic functions on the disc $\mathbb{D}$, and in the case of the semiplane $\mathbb{R}_+^{d+1}$, are characterized by being the Poisson integral of $L^p(\mathbb{R}^d)$-functions, $1<p\leq\infty$, see for instance, [252, Chapter VII, §1] and [299, Vol I Chapter VII, 7].

In his famous paper [199], B. Muckenhoupt and E. Stein defined the notion of Poisson integral for the case of the ultraspherical expansions and then they gave the corresponding characterization of functions that are Poisson integrals of L^p-functions in that case.

In Gaussian harmonic analysis, the analogous problem is the characterization of $\frac{\partial^2}{\partial t^2}+L$-harmonic functions on the half-plane $\mathbb{R}_+^{d+1}$ that are Poisson–Hermite integrals of functions $\mathbb{R}^d$. This was studied by L. Forzani and W. Urbina in [94]. Let us start with the bounded case. The proof of this result essentially follows, with the necessary variations, the classical proof that can be found in Stein's book [252].

Theorem 3.9. *Given a function u defined in $\mathbb{R}_+^{d+1}$, u is $\frac{\partial^2}{\partial t^2}+L$-harmonic and bounded if and only if u is the Poisson–Hermite integral of a function in $L^\infty(\gamma_d)$.*

Proof. It is enough to prove the sufficient condition, because the necessary condition is immediate, as the Poisson–Hermite integral of a bounded function is $\frac{\partial^2}{\partial t^2}+L$-harmonic and bounded. Now, assume that u is a $\frac{\partial^2}{\partial t^2}+L$-harmonic function such that

$|u| \leq M$ in $\mathbb{R}^{d+1}_+$. For each $k \in \mathbb{N}$ set $f_k(x) = u(x, 1/k)$ and let $u_k(x,t)$ be the Poisson–Hermite integral of f_k. Let us consider

$$\Delta_k(x,t) = u(x,t+1/k) - u_k(x,t).$$

It is enough to prove that $\Delta_k \equiv 0$ because, assuming that, we have

$$u(x,t+1/k) = u_k(x,t) = \int_{\mathbb{R}^d} p(t,x,y) f_k(y)\, \gamma_d(dy)$$

and hence, by the boundedness condition

$$||f_k||_{L^\infty(\gamma)} = ||u(\cdot, 1/k)||_{L^\infty(\gamma)} \leq M < \infty.$$

Thus, $\{f_k\}$ is a bounded sequence in $L^\infty(\gamma_d) = (L^1(\gamma_d))^*$, and then, according to the Bourbaki–Alaoglu theorem, there is an $f \in L^\infty(\gamma_d)$ and a subsequence $\{f_{k'}\}$ such that $f_{k'} \to f$ in the weak* topology, that is,

$$\int_{\mathbb{R}^d} f_{k'}(y)\phi(y)\gamma_d(dy) \longrightarrow \int_{\mathbb{R}^d} f(y)\phi(y)\gamma_d(dy),$$

for all $\phi \in L^1(\gamma_d)$.

For a fixed $(x,t) \in \mathbb{R}^{d+1}_+$, choosing $\phi(\cdot) = p(t,x,\cdot)$, in the limit, we have that

$$u(x,t) = \int_{\mathbb{R}^d} p(t,x,y) f(y)\, \gamma_d(dy).$$

Then, to prove that $\Delta_k \equiv 0$; define, for $\varepsilon > 0$, the auxiliary function

$$U(x,t) = \Delta_k(x,t) + 2M\varepsilon t + \varepsilon h(x,t),$$

where $h(x,t) = e^{-2t}(\frac{2}{n}|x|^2 - 1) + 1$ is strictly positive, radial in x, and $\frac{\partial^2}{\partial t^2} + L$-harmonic function.

$U(x,t)$ is clearly $\frac{\partial^2}{\partial t^2} + L$-harmonic on $\mathbb{R}^{d+1}_+$ and continuous on $\overline{\mathbb{R}^{d+1}_+}$. We restrict our attention to the bounded domain $\Sigma = \{(x,t) : 0 < t < 1/\varepsilon, |x| < R\}$, where R is sufficiently large, to be chosen later. Then, on its boundary,

$$\begin{aligned}\partial\Sigma &= \{(x,0) : |x| < R\} \cup \{(x, 1/\varepsilon) : |x| < R\} \cup \{(x,t) : 0 < t < 1/\varepsilon, |x| = R\} \\ &= \partial\Sigma_1 \cup \partial\Sigma_2 \cup \partial\Sigma_3,\end{aligned}$$

we have the following two conditions:

- On $\partial\Sigma_1$, $\Delta_k(x,0) = 0$ and

$$U(x,0) = \varepsilon h(x,0) \geq 0.$$

- On $\partial\Sigma_2$,

$$U(x, 1/\varepsilon) = \Delta_k(x, 1/\varepsilon) + 2M + \varepsilon h(x, 1/\varepsilon) \geq 0$$

  since $|\Delta_k(x,t)| \leq 2M$.

- Finally, on $\partial\Sigma_3$, because $\Delta_k(x,t)$ is bounded and $h(x,t)$ is radially increasing in x, $U(x,t)$ is positive for R big enough (note that R depends on ε).

Then, by using the maximum principle,[3] we get that $U(x,t) \geq 0$ in the region Σ and this implies that for all $(x,t) \in \Sigma$

$$\Delta_k \geq -\varepsilon(2Mt + h(x,t)).$$

By a similar argument, considering $-\Delta_k$ instead of Δ_k, we get that for all $(x,t) \in \Sigma$

$$\Delta_k \leq \varepsilon(2Mt + h(x,t)).$$

Now, consider an arbitrary point $(x,t) \in \mathbb{R}^{d+1}_+$. For any ε small enough $(x,t) \in \Sigma$; thus, we can get both inequalities for $\Delta_k(x,t)$ and, therefore, $\Delta_k(x,t) = 0$. □

The characterization result, mentioned above, is the following theorem (see [94]).

Theorem 3.10. *Given a function u defined in $\mathbb{R}^{d+1}_+$, u is $\frac{\partial^2}{\partial t^2}+L$-harmonic and uniformly $L^p(\gamma_d)$-bounded, $1 \leq p < \infty$, that is to say*

$$sup_{t>0}||u(\cdot,t)||_{p,\gamma} \leq M, \tag{3.32}$$

if and only if u is the Poisson–Hermite integral of a function in $L^p(\gamma_d)$, if $p > 1$. In the case p = 1, u is the Poisson–Hermite integral of a measure μ as above.

In the classical case, the analogous result of Theorem 3.10 is simply a corollary of the corresponding result of Theorem 3.9, but that is not the case here. The proof of Theorem 3.10 is a combination of the classical proof and specific estimates for the Gaussian measure. One of the necessary ingredients is the following result, which first appeared in [87].

Theorem 3.11. *Let us consider the operators*

$$L_1 u = \frac{\partial^2 u}{\partial t^2} + Lu, \text{ and } L_2 u = L_1 u - 2u. \tag{3.33}$$

If u satisfies $L_1 u = 0$ or $L_2 u = 0$, then:

i) Mean value inequality. There exists a constant C, dependent only on dimension, such that

$$|u(x,t)| \leq \frac{C}{|B((x,t),r)|}\int_{B((x,t),r)} |u(y,s)|dyds, \tag{3.34}$$

for $r \leq t$, and $t \leq m(x)$, where, as before, $m(x) = 1 \wedge \frac{1}{|x|}$ is the admissibility function. Thus, the mean value inequality is valid for radii that are small enough.

ii) If $u \geq 0$ in $B((x,t),2r)$, then

$$u(z,l) \approx \frac{1}{|B((x,t),r)|}\int_{B((x,t),r)} u(y,s)dyds, \tag{3.35}$$

for any $(z,l) \in B((x,t),r)$, with $r \leq t$ and $t \leq m(x)$.

[3] The weak maximum principle on bounded domains can be applied here as L is a uniformly elliptic differential operator with continuous coefficients.

iii) Harnack's inequality. There exists a constant $C > 0$ such that if $u \geq 0$ in $B((x,t),2r)$

$$\sup_{B((x,t),r)} u \leq C \inf_{B((x,t),r)} u \tag{3.36}$$

if $r \leq t$ and $t \leq m(x)$.

Proof. For each $(x_0,t_0) \in \mathbb{R}^{d+1}_+$, $x_0 \neq 0, |x_0| > 1$, set $B = B\Big((x_0,t_0), \frac{1}{|x_0|}\Big)$. Let us define on B the transformation

$$x = x_0 + \frac{1}{|x_0|}x',$$

$$t = t_0 + \frac{1}{|x_0|}t'.$$

Then $(x,y) \in B$ if and only if $(x',y') \in B((0,0),1)$. Define the function

$$U(x',t') = u\left(x_0 + \frac{1}{|x_0|}x', t_0 + \frac{1}{|x_0|}t'\right).$$

The function U satisfies the equation

$$\Delta_{x',t'}U - 2\frac{1}{|x_0|}\left(x_0 + \frac{1}{|x_0|}x'\right)\nabla_{x'}U = 0$$

and because $(x',t') \in B((0,0),1)$, then $\frac{1}{|x_0|}\left(x_0 + \frac{1}{|x_0|}x'\right)$ is bounded by a constant. Given that the (classical) mean value inequality is still true for differential operators with bounded first-order coefficients (see D. Gilbarg, N. S. Trudinger [113], page 244), we have

$$U(0,0) \leq \frac{1}{s^{d+1}}\int_{B((0,0),s)} U(x',t')dx'dt'$$

for all $s \leq 1$.

Now, according to the definition of U, the latter inequality can be rewritten as

$$u(x_0,t_0) \leq \frac{1}{s^{d+1}}\int_{B((0,0),s)} u\left(x_0 + \frac{1}{|x_0|}x', t_0 + \frac{1}{|x_0|}t'\right)dx'dt'$$

$$= \frac{|x_0|^{d+1}}{s^{d+1}}\int_{B((x_0,y_0),\frac{s}{|x_0|})} u(x,t)dxdt.$$

Hence, to obtain the inequality, if $t_0 < \frac{1}{|x_0|}$, take $s = |x_0|t_0$ and if $t_0 > \frac{1}{|x_0|}$, $s = 1$.

To prove (3.35) and (3.36) we use, as before, the results we know for classical positive solutions (see D. Gilbarg, N. S. Trudinger [113, pages 244–250]). □

We are now ready to prove Theorem 3.10.

Proof. The necessary condition is immediate because the Poisson–Hermite integral of a $L^p(\gamma_d)$ function is $\frac{\partial^2}{\partial t^2}+L$-harmonic and $L^p(\gamma_d)$-bounded. We then just have to prove the sufficient condition.

For each $(x,t)\in\mathbb{R}^{d+1}_+$, consider an admissible ball $B((x,t),r)$ radius $r\le t$, and $t\le m(x)$, because, as we already know the values of Gaussian density $e^{-|y|^2}$ are equivalents for points (y,s) on that ball, it is clear that

$$B((x,t),r)\subset\Big\{(y,s):t-r<s<t+r\Big\},$$

and $|B((x,t),r)|=Cr^{d+1}$; therefore, using these facts, the mean value inequality (3.34) and Hölder's inequality, we get, for $1\le p<\infty$,

$$\begin{aligned}|u(x,t)|^p&\le\frac{C}{|B((x,t),r)|}\int_{B((x,t),r)}|u(y,s)|^p dy\,ds\\&\le\frac{Ce^{|x|^2}}{r^{d+1}}\int_{t-r}^{t+r}\Big(\int_{\mathbb{R}^d}|u(y,s)|^p\gamma_d(dy)\Big)ds.\end{aligned}$$

Thus, according to the $L^p(\gamma_d)$-boundedness

$$|u(x,t)|\le Cr^{-d/p}e^{|x|^2/p},$$

with $r\le t$ and $t\le m(x)$.

As before, consider, for each $k\in\mathbb{N}$, $f_k(x)=u(x,1/k)$, $u_k(x,t)$ its Poisson–Hermite integral and

$$\Delta_k(x,t)=u(x,t+1/k)-u_k(x,t).$$

According to the weak compactness argument, it is again enough to prove that

$$\Delta_k\equiv 0.$$

Observe that, according to the previous inequality,

$$\begin{aligned}|u(x,t+1/k)|&\le C\left(\left(t+\frac{1}{k}\right)1\wedge\frac{1}{|x|}\right)^{-d/p}e^{|x|^2/p}\\&\le C(k\vee 1\vee|x|)^{d/p}e^{|x|^2/p}.\end{aligned}$$

Now, consider the auxiliary function

$$U(x,t)=\Delta_k+2C\varepsilon(k^2+|x|^2)^d e^{|x|^2/p}t+\varepsilon h(x,t),$$

where h is as in the proof of Theorem 3.9. Then, $U(x,t)$ is clearly $\frac{\partial^2}{\partial t^2}+L$-subharmonic on $\mathbb{R}^{d+1}_+$ and continuous on $\overline{\mathbb{R}^{d+1}_+}$. Thus, according to an analogous argument to that of the proof of Theorem 3.9, to apply the maximum principle on the bounded domain

$$\Sigma = \left\{(x,t) : 0 < t < 1/\varepsilon, |x| < R\right\}$$

we get that $U(x,t) \geq 0$ in the region Σ; thus, this implies for all $(x,t) \in \Sigma$

$$\Delta_k \geq -\varepsilon \left(2C(k^2 + |x|^2)^d e^{|x|^2/p} + h(x,t)\right).$$

Analogously, considering $-\Delta_k$ instead of Δ_k, we get that for all $(x,t) \in \Sigma$

$$\Delta_k \leq \varepsilon \left(2C(k^2 + |x|^2)^d e^{|x|^2/p} + h(x,t)\right).$$

Now, consider an arbitrary point $(x,t) \in \mathbb{R}^{d+1}_+$. For any $\varepsilon > 0$ small enough $(x,t) \in \Sigma$; thus, we can get both inequalities for $\Delta_k(x,t)$, i.e., $\Delta_k(x,t) = 0$. Therefore, for $p > 1$, there exist $f \in L^p(\gamma_d)$ and a subsequence $\{f_{k'}\}$ such that $f_{k'} \to f$ in the weak* topology. Thus, $u(x,t)$ is the Poisson–Hermite integral of that f.

For $p = 1$ there exists a measure μ, such that $e^{-|y|^2}\mu(dy)$ is a finite measure, and a subsequence $\{f_{k'}\}$ such that $f_{k'} \to \mu$ in the weak* topology; therefore $u(x,t)$ is the Poisson–Hermite integral of μ. □

3.3 Generalized Poisson–Hermite Semigroups

The Poisson–Hermite semigroup can be generalized to a family of semigroups obtained from the Ornstein–Uhlenbeck semigroup, by using the *generalized subordination formula*. Let $\mu_t^{(\alpha)}$ be the Borel measure on $[0,\infty)$ such that its Laplace transform satisfies

$$\int_0^\infty e^{-\lambda s} \mu_t^{(\alpha)}(ds) = e^{-\lambda^\alpha t}, 0 < \alpha < 1. \tag{3.37}$$

The measures $\mu_t^{(\alpha)}$ are probability measures, which are known as *one-sided stable measures* in $[0,\infty)$ *of order* α; moreover, for each α fixed, $\{\mu_t^{(\alpha)}\}_{t\geq 0}$ is a convolution semigroup (see [81]).

Definition 3.12. *The generalized Poisson–Hermite semigroup of order α, $\{P_t^\alpha\}_{t\geq 0}$ is defined as*

$$P_t^\alpha f(x) = \int_0^\infty T_s f(x) \mu_t^{(\alpha)}(ds). \tag{3.38}$$

The proof that $\{P_t^\alpha\}_{t\geq 0}$ is a strongly continuous, symmetric, conservative semigroup of positive contractions on $L^p(\gamma_d)$, $1 \leq p < \infty$, with infinitesimal generator $(-L)^\alpha$ can be obtained by adapting the proof for the case $\alpha = 1/2$. Hence, formally

$$P_t^\alpha = e^{-(-L)^\alpha t},$$

which means that for any ν multi-index,

$$P_t^\alpha \mathbf{h}_\nu = e^{-t|\nu|^\alpha} \mathbf{h}_\nu, \tag{3.39}$$

and, therefore, if $f = \sum_{k=0}^\infty \mathbf{J}_k f$,

$$P_t^\alpha f = \sum_{k=0}^\infty e^{-tk^\alpha} \mathbf{J}_k f.$$

Again, these semigroups turn out to be hypercontractive; therefore,

Lemma 3.13. *If* $1 < p < \infty$

$$||P_t^\alpha (I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} \leq e^{-tn^\alpha} ||f||_{p,\gamma}. \tag{3.40}$$

Proof. From Lemma 2.18, we have

$$||T_t (I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} \leq e^{-tn} ||f||_{p,\gamma}.$$

Then, using (3.37) and Minkowski's integral inequality, we get

$$\begin{aligned}
||P_t^\alpha (I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} &\leq \left|\left| \int_0^\infty T_t (I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f \, \mu_t^\alpha (ds) \right|\right|_{p,\gamma} \\
&\leq \int_0^\infty ||T_t (I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f||_{p,\gamma} \, \mu_t^\alpha (ds) \\
&\leq \int_0^\infty e^{-nt} ||f||_{p,\gamma} \, \mu_t^\alpha (ds) \leq C e^{-n^\alpha t} ||f||_{p,\gamma}.
\end{aligned}$$
□

Now, if instead of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$, in formula (2.73), we use the generalized Poisson–Hermite semigroups, $\{P_t^\alpha\}_{t\geq 0}$, we get *generalized potential operators*

$$U_{n,\alpha} f = \int_0^\infty P_t^\alpha (I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f \,; dt, \tag{3.41}$$

and obtain similar $L^p(\gamma_d)$ estimates, as in (2.74), using Lemma 3.13 and Minkowski's integral inequality,

$$\|U_{n,\alpha} f\|_{p,\gamma} \leq \int_0^\infty \|P_t^\alpha (I - \mathbf{J}_0 - \mathbf{J}_1 - \ldots - \mathbf{J}_{n-1}) f\|_{p,\gamma} \, dt \leq C \frac{1}{n^\alpha} \|f\|_{p,\gamma}. \tag{3.42}$$

In particular, if $f \in \mathscr{C}_k$ with $k \geq n$,

$$U_{n,\alpha} f = \int_0^\infty P_t^\alpha f dt = \frac{1}{n^\alpha} f.$$

These results will be key in the proof of Meyer's multiplier theorem (see Theorem 6.2).

3.4 Conjugate Poisson–Hermite Semigroup

The investigation of conjugacy for discrete and continuous non-trigonometric orthogonal expansions was initiated and extensively studied in the seminal article by B. Muckenhoupt and E. M. Stein [199]. B. Muckenhoupt introduced in [194] the conjugate Hermite expansions for dimension $d = 1$. According to (3.31), we know that given $f \in L^1(\gamma_1)$, if $u(x,t) = P_t f(x)$, then $u(x,t)$ satisfies

$$2\frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\partial^2 u}{\partial x^2}(x,t) - 2x\frac{\partial u}{\partial x}(x,t) = 0, \tag{3.43}$$

or equivalently,

$$2\frac{\partial^2 u}{\partial t^2}(x,t) + e^{x^2}\frac{\partial}{\partial x}\left(e^{-x^2}\frac{\partial u}{\partial x}(x,t)\right) = 0.$$

B. Muckenhoupt introduced the Gaussian *conjugate function* v of u by considering the *Gaussian Cauchy–Riemann* equations,

$$\begin{aligned}\frac{\partial u}{\partial x}(x,t) &= -\frac{\partial v}{\partial t}(x,t)\\ \frac{\partial u}{\partial t}(x,t) &= e^{x^2}\frac{\partial}{\partial x}(e^{-x^2}v(x,t)).\end{aligned} \tag{3.44}$$

Then, the function $v(x,t)$ must be defined as

$$v(x,t) = \int_{-\infty}^{\infty} Q(t,x,y) f(y)\,dy,\; t > 0, \tag{3.45}$$

where

$$Q(t,x,y) = \frac{\sqrt{2}}{\pi}\int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \exp\left(\frac{t^2}{4\log r}\right)\frac{y-rx}{(1-r^2)^2}\exp\left(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2}\right)dr. \tag{3.46}$$

Observe that (3.46) can be obtained from (3.4), for $d = 1$, differentiating with respect to x, integrating with respect to t, using the fact that Q must tend to 0 as $t \to \infty$ and multiplying by -1, i.e.,

$$Q(t,x,y) = -\int_t^{\infty}\frac{\partial p(s,x,y)}{\partial x}ds. \tag{3.47}$$

By construction v satisfies the first Cauchy–Riemann equation. Additionally, it is easy to see that v satisfies,

$$2\frac{\partial^2 v}{\partial t^2}(x,t) + \frac{\partial^2 v}{\partial x^2}(x,t) - 2x\frac{\partial v}{\partial x}(x,t) = -2v(x,t), \tag{3.48}$$

which is equivalent to

$$2\frac{\partial^2 v}{\partial t^2}(x,t) + \frac{\partial}{\partial x}\left[e^{x^2}\frac{\partial(e^{-x^2}v(x,t))}{\partial x}\right] = 0.$$

Now, because u satisfies (3.43), i.e., it is a $\frac{\partial^2}{\partial t^2} + L$-harmonic, but v does not, then it seems that probably this is not the best notion of conjugacy. More on the problem of notions of conjugacy for orthogonal polynomials can be found at [39].

Definition 3.14. *The conjugate Poisson–Hermite integral of f, is defined as*

$$P_t^c f(x) = v(x,t).$$

Therefore,

$$P_t^c f(x) = -\int_t^\infty \frac{\partial P_s f}{\partial x}(x)ds. \tag{3.49}$$

In [194], B. Muckenhoupt proved that $P_t^c f$ is bounded on $L^p(\gamma_1)$, $1 < p < \infty$ and as we see later in Chapter 9, if $t \to 0$, $P_t^c f$ tends to the *Gaussian Hilbert transform* $\mathscr{H} f$, in L^p-norm and a.e.

In his doctoral dissertation, R. Scotto [244] extended Muckenhoupt's notion of conjugacy in higher dimensions, $d > 1$, considering the Gaussian Cauchy–Riemann equations in $\mathbb{R}^d$:

$$\begin{aligned}
\frac{\partial u}{\partial x_j}(x,t) &= -\frac{\partial v_j}{\partial t}(x,t),\ j = 1,\ldots,d \\
\frac{\partial v_i}{\partial x_j}(x,t) &= \frac{\partial v_j}{\partial x_i}(x,t),\ i,j = 1,\ldots,d \\
\frac{\partial u}{\partial t}(x,t) &= \frac{1}{2}\sum_{j=1}^d e^{|x|^2}\frac{\partial}{\partial x_j}(e^{-|x|^2}v_j(x,t)).
\end{aligned} \tag{3.50}$$

From these relations, R. Scotto defined a system of conjugates,

$$(u(x,t), v_1(x,t), v_2(x,t), \ldots, v_d(x,t)).$$

Again, following Muckenhoupt's argument, the functions $v_i(x,t)$ verify that the first equation of (3.50); thus,

Definition 3.15. *The i-th conjugate Poisson kernel of f, is defined as*

$$P_{i,t}^c f = v_i(x,t),\ i = 1,\ldots,d.$$

Therefore,

$$P_{i,t}^c f = \int_{\mathbb{R}^d} Q_i(t,x,y) f(y)\, dy,\ t > 0, \tag{3.51}$$

where

$$\begin{aligned} Q_j(t,x,y) &= -\int_t^\infty \frac{\partial p}{\partial x_j}(s,x,y)ds \\ &= \frac{1}{\pi^{(d+1)/2}}\int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} \exp\left(\frac{t^2}{4\log r}\right)\frac{y_j - rx_j}{(1-r^2)^{(d+3)/2}} \\ &\qquad\qquad \times \quad \exp\left(\frac{-r^2(|x|^2+|y|^2)+2r\langle x,y\rangle}{1-r^2}\right)dr, \end{aligned}$$

Thus,

$$P^c_{i,t}f(x) = -\int_t^\infty \frac{\partial P_s f}{\partial x_i}(x)ds, \tag{3.52}$$

for any $i = 1,\dots,d$.

Thus, again following Muckenhoupt [194], we have the following result:

Proposition 3.16.

i) *$Q_i(t,x,y)$ is a bounded function in y, for any $i = 1,\dots,d$.*
ii) *If $f \in L^1(\gamma_d)$, then for any $i = 1,\dots,d$ $P^c_{i,t}f$ exists for any $t > 0$ and they verify an analogous equation as (3.48),*

$$\frac{\partial^2 v}{\partial t^2}(x,t) + Lv(x,t) = -v(x,t), \tag{3.53}$$

and the Gaussian Cauchy–Riemann equations (3.50).
iii) *If f has a Hermite expansion $f = \sum_{k=0}^\infty \sum_{|\nu|=k} \hat{f}_H(\nu)\mathbf{h}_\nu$, then, for any $t \geq 0$, $P^c_{i,t}f$ has a Hermite expansion*

$$P^c_{i,t}f = -\sum_{k=1}^\infty \sum_{|\nu|=k} \hat{f}_H(\nu)e^{-t\sqrt{|\nu|}}\sqrt{\frac{2}{|\nu|}}\nu_i\mathbf{h}_{\nu-\mathbf{e}_i}, \tag{3.54}$$

where $\mathbf{e_i}$ is the unitary vector with zeros in all j-coordinates $j \neq i$ and one in the i-th coordinate. These series are called conjugate Poisson series.
iv) *If $f \in L^2(\gamma_d)$ and $t > 0$, the series (3.54) converges a.e.*

Proof.

i) Let $i = 1,\dots,d$ fixed. Considering the cases $0 \leq r < 1/2$ and $1/2 \leq t < 1$, and replacing $-\log r$ by a multiple of $1-r$, in the second case it can be proved that

$$\left(\frac{1-r^2}{-\log r}\right)^{1/2} \frac{\exp\left(\frac{t^2}{4\log r}\right)}{(1-r^2)^{(d+3)/2}} < C\left(1+\frac{1}{x^{d+4}}\right).$$

Then, we get that $Q_i(t,x,y)$ is bounded.

ii) If $f \in L^1(\gamma_d)$ then $P^c_{i,t} f$ is well defined by *i*). The differentiation under the integral sign can be done as it can be proved that all the kernels are properly bounded (for more details see [244]).

$Q_i(t,x,y)$ satisfies the first equation of (3.50) by construction; therefore $P^c_{i,t} f(x)$ will verify it too. According to an analogous argument to that done for v verifying (3.48), we have that $P^c_{i,t} f(x)$ satisfies (3.53), because, as $u(x,t) = P_t f(x)$ satisfies (3.30), it follows that $\frac{\partial P^c_{i,t} f}{\partial x_i}(x)$ verifies

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial P^c_{i,t} f}{\partial x_i}\right)(x) + L\frac{\partial P^c_{i,t} f}{\partial x_i}(x) = \frac{\partial P^c_{i,t} f}{\partial x_i}(x).$$

The second equation of (3.50) is satisfied immediately, as

$$\frac{\partial P^c_{i,t} f}{\partial x_j}(x) = -\int_t^\infty \frac{\partial^2 P_s f}{\partial x_i \partial x_j}(x) ds = \frac{\partial P^c_{j,t} f}{\partial x_i}(x).$$

Finally, the last equation of (3.50) is satisfied, because

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^d e^{|x|^2}\frac{\partial}{\partial x_i}(e^{-|x|^2} P^c_{i,t} f(x)) &= \sum_{i=1}^d \left[-x_i P^c_{i,t} f(x) + \frac{1}{2}\frac{\partial P^c_{i,t} f}{\partial x_i}(x)\right] \\
&= -\int_t^\infty \sum_{i=1}^d \left[-x_i \frac{\partial P_s f}{\partial x_i}(x) + \frac{1}{2}\frac{\partial^2 P_s f}{\partial x_i^2}(x)\right] ds \\
&= -\int_t^\infty L P_s f(x) ds = -\int_t^\infty \frac{\partial^2 P_s f}{\partial s^2}(x) ds \\
&= \frac{\partial P_t f}{\partial t}(x).
\end{aligned}$$

iii) Following an analogous argument as in Proposition 2.3, we can prove that

$$Q_i(t,x,y) = -\sum_{k=1}^\infty \sum_{|\nu|=k} \hat{f}_H(\nu) e^{-t\sqrt{|\nu|}} \sqrt{\frac{2}{|\nu|}} \nu_i \mathbf{h}_{\nu-\mathbf{e}_i}.$$

and

$$\int_{\mathbb{R}^d} Q_i(t,x,y) \mathbf{h}_{\nu-\mathbf{e}_i}(x) \gamma_d(dx) = e^{-t\sqrt{|\nu|}} \mathbf{h}_\nu(y),$$

and from there, using Fubini's theorem, we can prove that $P^c_{i,t} f$ has the expansion (3.54).

iv) It can be proved by an analogous argument to that in the proof of Proposition 2.3. □

3.5 Notes and Further Results

1. Following B. Muckenhoupt [193], the Poisson–Hermite kernel can also be written as
$$p(t,x,y) = \int_0^1 U(t,r) M_{(-\log r)}(x,y)\,dr,$$
where $M_t(x,y)$ is Mehler's kernel, and
$$U(t,r) = \frac{1}{2\pi^{1/2}} \frac{t\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{1}{r}.$$
P_t can also be written as
$$P_t f(x) = \int_0^1 U(t,r) T_{(-\log r)} f(x)\,dr. \tag{3.55}$$
Observe that the definition of the Poisson–Hermite semigroup given here, for $d = 1$, differs from that in [193] by a constant, because in that case
$$T(t,r) = \frac{1}{(2\pi)^{1/2}} \frac{t\exp(t^2/2\log r)}{(-\log r)^{3/2}} \frac{1}{r},$$
which implies, essentially, similar relations, but with different constants.

2. Similar to the case of the Ornstein–Uhlenbeck semigroup, for the Jacobi semigroup and the Laguerre semigroup, using Bochner's subordination formula (3.1), we can define the *Jacobi–Poisson semigroup* $\{P_t^{\alpha,\beta}\}_{t\geq 0}$and the *Laguerre–Poisson semigroup* $\{P_t^{\alpha}\}_{t\geq 0}$, in addition to their conjugate semigroups (see for instance [213] and [209]). In an expository and very interesting paper [276], J. L. Torrea considers the semigroup theory as a tool for developing harmonic analysis for general differential second operators, based on the seminal papers of B. Muckenhoupt and E. Stein [199, 193] and [194].

3. Associated with the family of translated semigroups $\{T_t^{(\kappa)}\}_{t\geq 0}$, defined in (2.78), we have their *subordinated semigroups* $\{P_t^{(\kappa)}\}_{t\geq 0}$, defined by using the Bochner subordination formula; these are referred to as the *translated Poisson–Hermite semigroups*. Therefore,
$$P_t^{(\kappa)} \mathbf{h}_\nu = e^{-t\sqrt{|\nu|+\kappa}} \mathbf{h}_\nu. \tag{3.56}$$
Moreover, $P_t^{(\kappa)} f \leq P_t f$ for any $t \geq 0$ and $f \geq 0$. These translated semigroups are important in Chapter 5 and in Chapter 9.

4

Covering Lemmas, Gaussian Maximal Functions, and Calderón–Zygmund Operators

Maximal functions are among the most important operators in harmonic analysis and are some of those that are most studied. Averaging is an important operation in analysis and to understand and simplify its study, maximal functions are introduced. Moreover, for any limit process such as almost sure convergence, there is a maximal function that controls it; therefore, the study of their properties is crucial.

In this chapter, we study covering lemmas, the Hardy–Littlewood maximal function with respect to the Gaussian measure, and its variants. Covering lemmas are needed to establish the boundedness properties of the Hardy–Littlewood maximal function, among other reasons. Additionally, we study in detail the maximal functions of the Ornstein–Uhlenbeck and Poisson–Hermite semigroups, and their non-tangential versions. As a consequence, we get results on the non-tangential convergence for the Ornstein–Uhlenbeck and the Poisson–Hermite semigroups. Finally, we consider Calderón–Zygmund operators and their behavior with respect to the Gaussian measure.

4.1 Covering Lemmas with Respect to the Gaussian Measure

As we have already mentioned, the Gaussian measure is highly concentrated near the origin and decays exponentially toward infinity. This behavior, which is far from the invariance by translation of the Lebesgue measure, makes it difficult to obtain good covering lemmas. We first consider the Besicovitch covering lemma, one of the most basic covering lemmas in harmonic analysis, which is more powerful than the classical Vitali's covering lemma, because it is independent of the subjacent measure. In other words, it works fine for any Borel measure and that is the main reason why we want to study it here. For completeness, we give proof of it, even though it is well known.

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Lemma 4.1. *(Besicovitch) Let E be a bounded subset of $\mathbb{R}^d$. Let $\mathscr{F}$ be a family of balls covering E such that, for every $x \in E$, there is a ball $B_x = B(x, r(x)) \in \mathscr{F}$. Then, there exits an at most countable family $\{B_i\} = \{B(x_i, r(x_i))\}_i$ and a constant $C > 0$, dependent only on dimension, such that:*

i) $E \subset \bigcup_{i=1}^{\infty} B_i$.
ii) The family $\{B_i\}$ has bounded overlaps

$$\sum_{i=1}^{\infty} \chi_{B_i}(x) \leq C, \tag{4.1}$$

for all $x \in E$.

Proof. Let $\alpha = \sup\{r(x) : x \in E\}$. If $\alpha = +\infty$ as E is a bounded set, there exists $r > 0$ such that $E \subset B(0, r)$. Choose $x_0 \in E$ such that $r(x_0) > 2r$, then $E \subset B(x_0, r(x_0))$, as if $x \in E$ then

$$|x - x_0| \leq |x| + |x_0| < 2r < r(x_0).$$

Now, let us assume that $\alpha < +\infty$. Set

$$E^1 = E_1^1 = \left\{x \in E : \alpha/2 < r(x) \leq \alpha\right\}.$$

Take $x_1 \in E_1$, and $B_1^1 = B(x_1, r(x_1))$. Assuming that we have constructed the sets $E_1^1, E_2^1, \cdots, E_{k-1}^1$ and selected the balls $B_1^1, B_2^1, \ldots, B_{k-1}^1$, we define $E_k^1 = E^1 - \cup_{i=1}^{k-1} B_i^1$, take $x_k \in E_k^1$, and set $B_k^1 = B(x_k, r(x_k))$. The selection process stops obtaining a finite covering $\{B_i^1\}_{i=1}^{m_1}$ of E_1. In fact, by construction $B\left(x_i^1, \frac{\alpha}{2^i}\right) \cap B\left(x_j^1, \frac{\alpha}{2^j}\right) = \emptyset$ for $i \neq j$ and are all contained in $\{x : \ d(x, E) < \alpha\}$, which is a bounded set.

Assuming that we already have constructed the sets $E^1, \cdots, E^k$, we define

$$E^l = E_1^l = \left\{x \in E : \frac{\alpha}{2^l} < r(x) \leq \frac{\alpha}{2^{l-1}}\right\} - \left(\cup_{j=1}^{l-1} \cup_{i=1}^{m_j} B_i^j\right)$$

and we iterate the construction done for E^1, getting a finite covering $\{B_k^l\}_k$ of E^l. Therefore, $\{B_k^l\}_{l,k}$ by construction satisfies property i). Now, it remains to be proved that it also has bounded overlaps. It is enough to prove that

ii-1) $\sum_i \chi_{B_i^j}(x) \leq C_1$.
ii-2) $\sum_l \chi_{B_j^l}(x) \leq C_2$

with C_1, C_2 independent of j, because then,

$$\sum_{i=1}^{\infty} \chi_{B_i}(x) \leq \sum_{\{k:\, \exists i\ x \in B_i^k\}} \sum_{\{i:\, y \in B_i^k\}} \chi_{B_i^k}(x) \leq C_1 \left(\#\{k : \ \exists i \ \ y \in B_i^k\}\right) \leq C_1\, C_2.$$

- If $x_0 \in \cap_{i=1}^{N} B_i^j = \cap_{i=1}^{N} B(x_i^j, r(x_i^j))$, then we have that the balls in the family $\left\{B\left(x_i^j, \frac{\alpha}{2^{j+1}}\right)\right\}_{i=1}^{N}$ are disjoint and $B\left(x_i^j, \frac{\alpha}{2^{j+1}}\right) \subset B\left(x_0; 3\frac{\alpha}{2^{j+2}}\right)$. Therefore, as

$$M\,\omega_n\,\frac{\alpha^d}{2^{(j+1)n}} = \sum_{i=1}^{M}\left|B\left(x_i^j,\frac{\alpha}{2^{j+1}}\right)\right| = \left|\cup_{i=1}^{N} B\left(x_i^j,\frac{\alpha}{2^{j+1}}\right)\right|$$
$$\leq \left|B\left(x_0,3\frac{\alpha}{2^{j+1}}\right)\right| = 3^d\omega_n d\,\frac{\alpha^d}{2^{(j+1)n}},$$

then we get that $M \leq 3^d$.

- If $x_0 \in \cap_{i=1}^{N} B_{j_i}^{k_i} = \cap_{i=1}^{N} B_i$ with $k_1 < k_2 < \ldots . < k_N, N = N(x_0)$. For $x \in \mathbb{R}^d$, define $T_r(x) = \frac{x-x_0}{r} + x_0,\ r > 0$. By construction,

$$T_{r_k}(x_k) \notin \bigcup_{i=1}^{L} T_{r_k} B_i$$

where r_k is the radius of B_k. As $r_k \leq r_i$ we have $T_{r_i}B_i \subset T_{r_k}B_k$; hence,

$$T_{r_k}(x_k) \notin \bigcup_{i=1}^{k-1} B(\frac{x_i}{r_i};1).$$

In other words, the balls $\left\{B\left(\frac{x_i}{r_i},\frac{1}{2}\right)\right\}_{i=1,\cdots,k-1}$ are disjoint, and because

$$\bigcup_{i=1}^{N} B\left(\frac{x_i}{r_i},\frac{1}{2}\right) \subset B(0,3),$$

we are done. □

There is also a version of the Besicovitch covering lemma for cubes with sides parallel to the axis (see for instance [294, Theorem 10.45]).

The following covering lemma, obtained by L. Forzani (see [83] or [5]), is somehow halfway between the Besicovitch and Wiener covering lemmas[1] and, as we see later, it is the key result to prove that a family of generalized maximal functions are of weak type $(1,1)$ with respect to the Gaussian measure (see Theorem 4.18).

Lemma 4.2. *Let $E = \{x_\alpha :\ \alpha \in I\}$ be a subset of $\mathbb{R}^d \backslash \overline{B}(0,2\zeta)$, with $\zeta > 2$ fixed, and I be a finite set of indices. For each $x \in E$ a number $r = r(x) \in \left(\frac{3}{4}, 1-\frac{\zeta^2}{|x|^2}\right)$ is given. Let $B_j := B\left(\frac{x_j}{r_j},\frac{|x_j|}{r_j}(1-r_j)\right)$ and $B_j^\nu := B\left(\frac{x_j}{r_j},\nu\rho_j\right)$, with $\nu \geq 1$ and $\rho_j = \sqrt{1-r_j}$, and let $\delta_j = \frac{r_j}{|x_j|(1-r_j)}\min\left\{\frac{1}{|x_j|},\sqrt{1-r_j}\right\} = \frac{r_j}{|x_j|^2\rho_j^2}$. Then, there exists a positive constant C, dependent only on dimension, and a subset J of I such that*

i) $E \subset \bigcup_{j\in J}(1+\delta_j)B_j$.

[1] In the sense that you cover the centers $x_\alpha \in E$, as in the Besicovitch covering lemma, the covering obtained is of dilated balls as in the Wiener covering lemma, but they have bounded overlaps as in the Besicovitch covering lemma.

ii) $\sum_{j\in J}\chi_{B_j^\nu} \leq C\nu^{2d}$.

Proof. Let $I_1 = I$, $\alpha_1 \in I_1$ such that $|x_{\alpha_1}| = \min\{|x_\alpha| : \alpha \in I_1\}$. Let $x_1 = x_{\alpha_1}$ and $B_1 = B_{\alpha_1}$.

Assuming that the set of indexes $I_1, \dots I_{k-1}$, the points $x_1, \dots, x_{k-1}$, and the balls $B_1, \dots, B_{k-1}$ have been chosen, we define

$$I_k = \left\{\alpha \in I : x_\alpha \notin \cup_{j=1}^{k-1}(1+\delta_j)B_j\right\}.$$

Choose $\alpha_k \in I_k$ such that $|x_{\alpha_k}| = \min\,\{|x_\alpha| : \alpha \in I_k\}$. Let $x_k = x_{\alpha_k}$ and $B_k = B_{\alpha_k}$. Set $J = \{\alpha_1, \dots, \alpha_N\}$ where N is the first integer for which $I_{N+1} = \emptyset$. Then, *i)* is immediate.

Before proving *ii)*, let us make several remarks.

- x_j was chosen so that $x_j \notin (1+\delta_s)B_s$, for all $s < j$; hence,

$$\frac{|x_s|^2}{r_s^2} + |x_j|^2 - 2|x_j|\frac{|x_s|}{r_s}\cos\left\langle \frac{x_s}{r_s}, x_j\right\rangle = \left|\frac{x_s}{r_s} - x_j\right|^2 \geq R_s^2(1+\delta_s)^2, \quad (4.2)$$

  where $R_s = \frac{|x_s|}{r_s}(1-r_s)$.
- $|x_j| \geq |x_s|$ for $s < j$, i.e., $|x_j|$ is increasing with j.
- Finally, we prove

$$\left|\frac{x_s}{r_s} - \frac{x_j}{r_j}\right|^2 \geq \frac{1}{r_j}\left[\frac{|x_s|^2}{r_s^2 r_j}(r_j - r_s)^2 + 2\frac{(1-r_s)}{r_s}\right] \geq \theta^2 \max{}^2\{\rho_j, \rho_s\} \qquad (4.3)$$

  for $s < j$ and some $\theta > 0$.

In fact, using the two previous inequalities, and because $R_s^2\delta_s = \frac{(1-r_s)}{r_s}$,

$$\begin{aligned}
\left|\frac{x_s}{r_s} - \frac{x_j}{r_j}\right|^2 &= \frac{|x_s|^2}{r_s^2} + \frac{|x_j|^2}{r_j^2} - 2\frac{|x_j|}{r_j}\frac{|x_s|}{r_s}\cos\left\langle \frac{x_s}{r_s}, \frac{x_j}{r_j}\right\rangle \\
&\geq \frac{|x_s|^2}{r_s^2} + \frac{|x_j|^2}{r_j^2} + \frac{1}{r_j}\left[R_s^2(1+\delta_s)^2 - |x_j|^2 - \frac{|x_s|^2}{r_s^2}\right] \\
&= \frac{1}{r_j}R_s^2(1+\delta_s)^2 - \frac{1}{r_j}\left[-|x_j|^2\left(\frac{1-r_j}{r_j}\right) + \frac{|x_s|^2}{r_s^2}(1-r_j)\right] \\
&\geq \frac{1}{r_j}\left[R_s^2(1+\delta_s)^2 + |x_s|^2(1-r_j)\left[\frac{1}{r_j} - \frac{1}{r_s^2}\right]\right] \\
&\geq \frac{1}{r_j}\left[\frac{|x_s|^2}{r_s^2}(1-r_s)^2 + 2\frac{(1-r_s)}{r_s} + |x_s|^2(1-r_j)\left[\frac{1}{r_j} - \frac{1}{r_s^2}\right]\right] \\
&= \frac{1}{r_j}\left[\frac{|x_s|^2}{r_s^2 r_j}(r_j - r_s)^2 + 2\frac{(1-r_s)}{r_s}\right] \geq \theta^2\max{}^2\{\rho_j, \rho_s\}.
\end{aligned}$$

To obtain the latter inequality we consider two cases:

- $\rho_s^2 \geq \frac{1}{2}\rho_j^2$. Because by definition $\rho_s^2 = (1-r_s)$, and the non-negativity of the first term, the inequality then follows.
- $\rho_j^2 \geq 2\rho_s^2$. We have that $(r_j - r_s)^2 = (\rho_j^2 - \rho_s^2)^2 \geq \frac{1}{4}\rho_j^4$. Using the fact that $|x_s|\rho_s \geq \xi$, the inequality follows. Recall that, according to the hypothesis, $r_s \leq 1 - \frac{\xi^2}{|x_s|^2}$.

To prove the second case we define, for $\kappa > 0$ fixed,

$$I_1 = \left\{ j : j \in J \text{ and } \nu\rho_j \geq \kappa \right\}, \quad I_2 = \left\{ j : j \in J \text{ and } \frac{R_j}{2} < \nu\rho_j < \kappa \right\}$$
$$I_3 = \left\{ j : j \in J \text{ and } \nu\rho_j \leq \frac{R_j}{2} \right\}.$$

We prove the inequality

$$\sum_{j \in I_i} \chi_{B_j^\nu}(y) \leq C\nu^{2d}, \tag{4.4}$$

for $y \in \mathbb{R}^d$ and $i = 1,2,3$, from which (ii) follows.

Consider $I_1(y) = \left\{ j \in I_1 : \; y \in B_j^\nu \right\}$. To obtain the desired estimate, we only need to find a sequence of pairwise disjoint measurable sets $\{S_j\}_{j \in I_1(y)}$ such that

- $S_j \subset B(y, C\nu)$,
- $|S_j| \geq \frac{C}{\nu^d}$.

The case $i = 1$ in (4.4) follows immediately from these conditions.

Defining $S_j = B\left(\frac{x_j}{r_j}; \frac{\theta}{2}\rho_j\right)$, $j \in I_1(y)$. The second condition above is trivial, as $j \in I_1$, which implies $\rho_j \geq \frac{\kappa}{\nu}$. To get the first condition, let us take $h \in S_j$. As $y \in B_j^\nu$, we have

$$|h - y| \leq \left| h - \frac{x_j}{r_j} \right| + \left| \frac{x_j}{r_j} - y \right| \leq \frac{\theta}{2}\rho_j + \nu\rho_j \leq C\nu.$$

That $\{S_j\}_{j \in I_1(y)}$ is a family of pairwise disjoint sets follows from (4.3).

Now, consider $I_2(y) = \left\{ j \in I_2 : \; y \in B_j^\nu \right\}$. To obtain the desired estimate, we only need to find a sequence of pairwise disjoint measurable sets $\{S_j\}_{j \in I_2(y)}$ such that

- $S_j \subset B(y, C\nu^2)$.
- $|S_j| \simeq 1$ for some constant C.

The case $i = 2$ in (4.4) follows immediately from these conditions.

Define $S_j = B\left(y + (\frac{x_j}{r_j} - y)\frac{|x_j|}{r_j}, C\right)$, $j \in I_2(y)$. Therefore, the second condition above is immediate. Let us now prove the first condition. Take $h \in S_j$. Then, using the fact that $y \in B_j^\nu$ and $\frac{R_j}{2} \leq \nu\rho_j$ or, equivalently, $\rho_j \frac{|x_j|}{r_j} \leq 2\nu$, we get

$$|h-y| \leq C + \left|\frac{x_j}{r_j} - y\right| \frac{|x_j|}{r_j} \leq C + \nu\rho_j \frac{|x_j|}{r_j} \leq C + \nu^2 \leq C\nu^2.$$

To prove that the sets of the family $\{S_j\}_{j \in I_2(y)}$ are pairwise disjoint we use the following facts:

- $\left|\frac{x_s}{r_s} - \frac{x_j}{r_j}\right| \geq \theta\rho_s$.
- $|x_s|\rho_s \geq \sqrt{\zeta}$.
- $\left|\frac{x_j}{r_j} - y\right| \leq \nu\rho_j \leq \kappa \ (j \in I_2)$.
- $\left|\frac{|x_s|}{r_s} - \frac{|x_j|}{r_j}\right| \leq \nu\rho_s + \nu\rho_j \leq 2\kappa$.

So,

$$\begin{aligned}\left|\left(\frac{x_j}{r_j} - y\right)\frac{|x_j|}{r_j} - \left(\frac{x_s}{r_s} - y\right)\frac{|x_s|}{r_s}\right| &\geq \frac{|x_s|}{r_s}\left|\frac{x_j}{r_j} - \frac{x_s}{r_s}\right| - \left|\frac{x_j}{r_j} - y\right|\left|\frac{|x_s|}{r_s} - \frac{|x_j|}{r_j}\right| \\ &\geq \sqrt{\zeta}\frac{\theta}{r_s} - 2\kappa^2 \geq C,\end{aligned}$$

after choosing ζ and κ in a convenient way.

Finally, consider $I_3(y) = \left\{ j \in I_3 : \ y \in B_j^\nu \right\}$. To obtain the desired estimate, we need to find a sequence of pairwise disjoint measurable sets $\{S_j\}_{j \in I_3(y)}$ such that

- $S_j \subset B(y, C\nu\rho_\tau)$.
- $|S_j| \geq C\rho_\tau$ for some constant C, where $\tau = \min\{j : \ j \in I_3(y)\}$.
  The case $i = 3$ in (4.4) follows from these conditions.

Define $S_j = B\left(\frac{x_j}{r_j}, \frac{\theta}{2}\rho_j\right)$, $j \in I_3(y)$. It is enough to prove that

$$\frac{1}{2}\rho_\tau < \rho_j < 2\rho_\tau, \tag{4.5}$$

for all $j \in I_3(y)$. From the inequality (4.5), we get the conditions above. That the sets of the family $\{S_j\}_{j \in I_3(y)}$ are disjoint follows from (4.3). Let us prove then (4.5). From (4.3)

$$\left|\frac{x_j}{r_j} - \frac{x_\tau}{r_\tau}\right|^2 \geq \frac{1}{r_j}\left[\frac{|x_\tau|^2}{r_\tau^2 r_j}(r_j - r_\tau)^2 + 2\frac{(1-r_\tau)}{r_\tau}\right],$$

then

$$\left|\frac{x_j}{r_j} - \frac{x_\tau}{r_\tau}\right|^2 \geq \frac{|x_\tau|^2}{r_\tau^2}(\rho_j^2 - \rho_\tau^2)^2,$$

and, as $\tau, j \in I_3(y)$, we have $\frac{|x_\tau|}{r_\tau} \geq 2\frac{\nu}{\rho_\tau}$. Therefore,

$$(\nu\rho_j + \nu\rho_\tau)^2 \geq \left|\frac{x_j}{r_j} - \frac{x_\tau}{r_\tau}\right|^2$$

$$\geq \frac{|x_\tau|^2}{r_\tau^2}(\rho_j^2 - \rho_\tau^2)^2 \geq 4\frac{\nu^2}{\rho_\tau^2}(\rho_j^2 - \rho_\tau^2)^2$$

Thus, $1 \geq 2\frac{|\rho_j - \rho_\tau|}{\rho_\tau}$. This inequality is equivalent to $|\rho_j - \rho_\tau| \leq \frac{1}{2}\rho_\tau$, which in turn is equivalent to (4.5). □

There is another tool that gives us a decomposition of $\mathbb{R}^d$ into a family of admissible balls (and therefore the values of the Gaussian density are equivalents) with bounded overlaps, which is useful for studying the local parts of Gaussian Littlewood–Paley functions and Gaussian Riesz transforms. This decomposition was used by R. Scotto in his doctoral dissertation [244], in [77], and also by S. Pérez in [221] (see also J. García-Cuerva, G. Mauceri, P. Sjögren, J. L. Torrea [105] and F. Soria and G. Weiss [251]).

Lemma 4.3. *There exists a strictly increasing sequence of positive real numbers $\{\alpha_k\}_k$, such a family of disjoint balls $\{B_k^j\}_{k\in\mathbb{N},1\leq j\leq N_k}$ can be obtained that satisfies the following properties:*

i) If $\tilde{B}_k^j = 2B_k^j$, the countable collection $\mathscr{F} = \{B(0,\alpha_1), \{\tilde{B}_k^j\}_{j,k}\}$ is a covering of $\mathbb{R}^d$.
ii) $\mathscr{F}$ has a bounded overlaps property.
iii) The center c_k^j of B_k^j satisfies $|c_k^j| = (\alpha_{k+1} + \alpha_k)/2$.
iv) $diam(B_k^j) = \alpha_{k+1} - \alpha_k$.
v) Every ball $B \in \mathscr{F}$ is contained in an admissible ball; therefore, for any pair $x, y \in B$, $e^{-|x|^2} \sim e^{-|y|^2}$ with constants independent of B.
vi) There exists a uniform positive constant C_d such that, if $x \in B \in \mathscr{F}$, then $B(x) \subset C_d B := \hat{B}$. Moreover, the collection $\hat{\mathscr{F}} = \{\hat{B}\}_{B\in\mathscr{F}}$ also satisfies the properties ii) and v).

Proof. Given that $\alpha_1 \geq 2R$, with $R > 1$ fixed, define recursively

$$\alpha_{k+1} := \alpha_k + \frac{R}{\alpha_k},\ k \in \mathbb{N}.$$

The sequence $\{\alpha_k\}_k$ is strictly increasing and tends to infinity as $k \to \infty$.

Set $l_0 := \alpha_0$ and $l_k := \alpha_{k+1} - \alpha_k$, $k \geq 1$, then $l_{k+1} < l_k < 2l_{k+1}$, and let, for any $k \geq 1$, the annulus

$$S_k = \{x \in \mathbb{R}^d : \alpha_k \leq |x| < \alpha_{k+1}\}.$$

Let $B_k^1, B_k^2, \cdots, B_k^{N_k}$ a maximal disjoint family of balls contained in S_k and such that their diameter is l_k, i.e., $\mathrm{diam}(B_k^j) = l_k$, $1 \leq j \leq N_k$. If c_k^j is the center of B_k^j, then $B_k^j = B(c_k^j, l_k/2)$ and $|c_k^j| = \alpha_k + \frac{l_k}{2}$. Then, it is clear that $S_k \subset \bigcup_{j=1}^{N_k} 2B_k^j$, where $2B$ denotes the ball with the same center of B and twice the radius. Now, $\mathscr{F}$ is defined as follows: $B \in \mathscr{F}$ if and only if $B = B(0,\alpha_1)$ or $B = 2B_k^j$ for some j,k. Clearly, $\mathscr{F}$ is a covering of $\mathbb{R}^d$. Now, if $x \in \bigcap_{j=1}^l 2B_k^j$, then $l \leq 4^d$ as $\bigcup_{j=1}^l 2B_k^j \subset B(x, 2l_k)$ (see Figure 4.1).

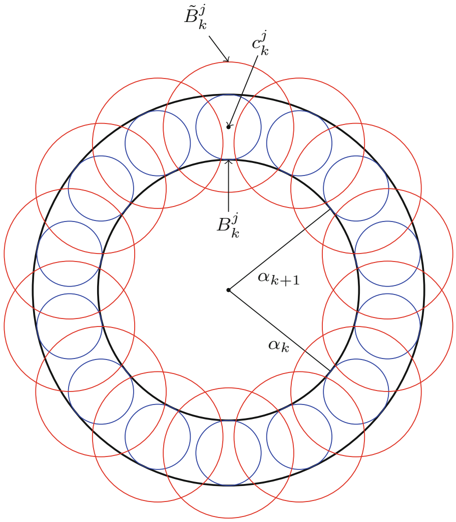

Fig. 4.1. Annulus S_k, the balls B_k^j, and the balls $\tilde{B}_k^j$.

Now, for each $k \geq$, let the annulus $\tilde{S}_k := \left\{x \in \mathbb{R}^d : \alpha_k - \frac{l_k}{2} \leq |x| < \alpha_{k+1} + \frac{l_k}{2}\right\}$. Then, $\bigcup_{j=1}^{N_k} 2B_k^j \subset \tilde{S}_k$. For $k > j+2$, $\alpha_{j+1} + \frac{l_j}{2} < \alpha_k - \frac{l_k}{2}$, and for $k < j-2$, $\alpha_k - \frac{l_k}{2} < \alpha_{j+1} + \frac{l_j}{2}$. This implies that $\tilde{S}_j \cap \tilde{S}_k = \emptyset$ for $|k-j| > 2$; thus, the family $\mathscr{F}$ has the bounded overlaps property. Now, we need to prove that every ball $B \in \mathscr{F}$ is an admissible ball. If $B = B(0, \alpha_1)$, the result is immediate. If $B = 2B(c_k^j, l_k/2) = B(c_k^j, l_k) = B(c_k^j, R/\alpha_k)$ for some j,k, then $B \subset B(c_k^j, 2R/|c_k^j|) \in \mathscr{B}_{2R}$.

Finally, for *vi*) it is enough to take $C_d = 2d+1$. □

Another important tool of Euclidean harmonic analysis is the Whitney decomposition lemma (see E. Stein's book [252, Chapter VI.1] for details). This technique allows an open set O to be covered with dyadic cubes (or balls) whose sizes are proportional to their distance to the complement of O. In the Gaussian case, we run into the problem that admissible cubes become very small at large distances from the origin. As a consequence, the distance of such a cube to the exterior of a given open set is typically much larger than the size of the cube; thus, it may seem that the Whitney covering lemma is useless as a tool in the Gaussian setting. Nevertheless, in [169] J. Mass, J. Van Neerven, and P. Portal were able to adapt it to the Gaussian setting we call it a *Gaussian Whitney covering*, and it is used, in a crucial way, to define Gaussian tent spaces, as we see later in Chapter 7.

For $m \in \mathbb{Z}$, let Δ_m be the set of dyadic cubes at scale m, i.e.,

$$\Delta_m = \{2^{-m}(x + [0,1)^d) : x \in \mathbb{Z}^d\}.$$

For the Gaussian measure, the idea is to use, on every scale, cubes whose diameter depends upon another parameter $l \geq 0$, which keeps track of the distance from the cube to the origin, similar to the idea in Proposition 1.7. More precisely, define the *layers*

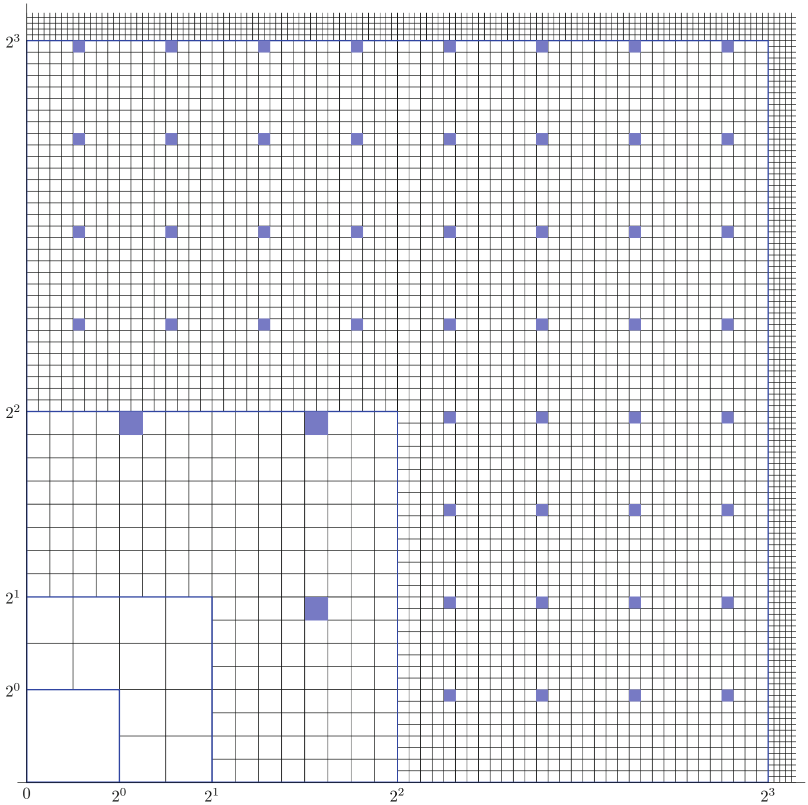

Fig. 4.2. Subdivision of the layers $L_0, L_1, L_2, \cdots$ into cubes of $\Delta^{\gamma}_{0,l}$. Shaded are the (5,8)-cubes in the layers L_2 and L_3 corresponding to the choice $\kappa = 3$.

$$L_0 = [-1,1)^d,\ L_l = [-2^l, 2^l)^d \backslash [-2^{l-1}, 2^{l-1})^d,$$

for $l \geq 1$, and define for $k \in \mathbb{Z}$ and $l \geq 0$

$$\Delta^{\gamma}_{k,l} = \{Q \in \Delta_{l+k} : Q \subset L_l\}, \quad \Delta^{\gamma}_k = \bigcup_{l \geq 0} \Delta^{\gamma}_{k,l}, \quad \Delta^{\gamma} = \bigcup_{k \geq 0} \Delta^{\gamma}_k. \tag{4.6}$$

Fix an integer $\kappa \geq 1$. For each $l \geq \lceil \frac{\kappa+1}{2} \rceil$, the layer L_l is a disjoint union of $2^{\kappa d}$ cubes in $\Delta^{\gamma}_{-\kappa,l}$, each of which is the disjoint union of $2^{\kappa d}$ cubes from $\Delta^{\gamma}_{0,l}$. Each cube can be labeled $i = (i_1, \cdots, i_d) \in \{1, \cdots, 2^{\kappa}\}^d$ (see Figure 4.2), where $d = 2$, $\kappa = 3$, and the shade cubes are the cubes from $\Delta^{\gamma}_{0,l}$ with label $i = (5,8)$, for $l = 2,3$.

Note that, if $k \leq -2l$, then $\Delta^{\gamma}_{k,l} = \emptyset$. Also, if $Q \in \Delta^{\gamma}_{k,l}$, then Q has sidelength 2^{-k-l}, its center c_Q has norm $2^{l-1} \leq |c_Q| \leq 2^l \sqrt{d}$ and its diameter satisfies

$$\operatorname{diam}(Q) = 2^{-k-l}\sqrt{d} \leq 2^{-k} d\, m(c_Q). \tag{4.7}$$

Lemma 4.4. *If a ball $B(x,r) \in \mathscr{B}_a$ intersects a cube $Q \in \Delta_0^\gamma$ with center c_Q, then*

$$r \leq 2a(a+d)m(c_Q).$$

Proof. We consider two cases:

- First, if $|c_Q| \geq 2(a+d)$, we notice that

$$r \leq \frac{a}{|x|} \leq \frac{a}{|c_Q| - (r + dm(c_Q)/2)} \leq \frac{a}{|c_Q| - (a + d/2)} \leq \frac{2a}{|c_Q|} = 2am(c_Q).$$

The first inequality follows, because $\mathrm{diam}(Q) \leq dm(c_Q)$ according to (4.7); the second follows because $m(c_Q) \leq 1$ and $r \leq am(x) \leq a$; the third follows from the assumption we made; and the final identity follows by noting that $|c_Q| \geq 2d \geq 1$.

- Second, if $|c_Q| \leq 2(a+d)$, then together with $1 \leq 2(a+d)$, we obtain

$$1 \leq 2(a+d)m(c_Q), \text{ and } r \leq a \leq 2a(a+d)m(c_Q). \qquad \square$$

In [168, Lemma 2.5] J. Maas, J. van Neerven, & P. Portal also have the following covering lemma,

Lemma 4.5. *Let $E \subseteq \mathbb{R}^d$ be a non-empty set, let $\alpha, \beta, \eta > 0$ be fixed, and let*

$$O_a := \{x \in \mathbb{R}^d : 0 < d(x,E) \leq am(x)\}.$$

There exists a sequence $(x_k)_{k\geq 1}$ in O_α with the following properties:

i) $O_\alpha \subseteq \bigcup_{k\geq 1} B(x_k, \beta \cdot d(x_k,E))$.

ii) $\sum_{k\geq 1} \gamma_d(B(x_k, \delta\, d(x_k,E))) \leq C\gamma_d(O_{2\alpha})$ *with C depending only on a, b, c, and d.*

Proof. Let $\delta := \min\{\frac{1}{2}, \beta\}$ and set $O := O_\alpha$ and $O' := O_{2\alpha}$ for simplicity. We use a Whitney covering of O' by disjoint cubes Q_k such that

$$\tfrac{1}{4}\delta d(Q_k, O'^c) \leq \mathrm{diam}\,(Q_k) \leq \delta d(Q_k, O'^c);$$

(see [252, VI.1]). We discard the cubes that do not intersect O and relabel the remaining sequence of cubes as $\{Q_k\}_{k\geq 1}$ with centers $\{c_k\}_{k\geq 1}$. For each $k \geq 1$ pick $x_k \in O \cap Q_k$.

To check that the balls $B(c_k, \mathrm{diam}\,(Q_k))$ are admissible, we use the fact that $\delta \leq \frac{1}{2}$ to obtain

$$|c_k - x_k| \leq \tfrac{1}{2}\mathrm{diam}\,(Q_k) \leq \tfrac{1}{4}d(Q_k, O'^c) \leq \tfrac{1}{4}d(x_k,E) \leq \tfrac{1}{4}\alpha\, m(x_k).$$

Now, part *ii*) of Lemma 1.5 then shows that $m(x_k) \leq (1 + \frac{\alpha}{4})m(c_k)$. It follows that the balls $B(c_k, \mathrm{diam}(Q_k))$ are admissible.

Next, $\mathrm{diam}\,(Q_k) \leq \delta d(Q_k, O'^c) \leq \beta d(x_k,E)$, so *i*) follows from

$$O \subseteq \bigcup_{k\geq 1} Q_k \subseteq \bigcup_{k\geq 1} B(x_k, \mathrm{diam}\,(Q_k)) \subseteq \bigcup_{k\geq 1} B(x_k, \beta d(x_k,F)).$$

To prove *ii*), we claim that for all $x \in O$,

$$d(x,F) \le 3\max\{1,\alpha\} d(x,O'^c).$$

To prove the claim, we fix $x \in O$ and pick an arbitrary $y \in O'^c$. Setting $\varepsilon := \frac{1}{3}\min\{1,\frac{1}{\alpha}\}$ we need to prove that

$$|x-y| \ge \varepsilon d(x,E).$$

From $y \notin O'$, we know that either $d(y,E) \ge 2\alpha m(y)$ or $d(y,E) = 0$. In the latter case, we have $y \in \overline{E}$; hence, $\varepsilon d(x,E) \le d(x,E) \le |x-y|$. Therefore, in what follows, we may assume that $d(y,E) \ge 2\alpha m(y)$. From $x \in O$, we know that $d(x,E) \le \alpha m(x)$. Suppose, for a contradiction, that $|x-y| < \varepsilon d(x,E)$. Then, $|x-y| < \varepsilon\alpha m(x)$ and therefore $m(x) \le (1+\varepsilon\alpha)m(y)$, again according to *ii*) of Lemma 1.5. Also, for all $e \in E$, we have

$$|x-y| \ge |y-e| - |e-x| \ge 2\alpha m(y) - |e-x|.$$

Minimizing over e, this gives $|x-y| \ge 2\alpha m(y) - d(x,E)$. As $\varepsilon d(x,E) also > |x-y|$, we find that

$$\alpha m(y) < \frac{1}{2}(1+\varepsilon)d(x,E) \le \frac{1}{2}(1+\varepsilon)\alpha m(x).$$

It follows that $m(y) < \frac{1}{2}(1+\varepsilon)m(x)$, and in combination with the inequality $m(x) \le (1+\varepsilon\alpha)m(y)$ we get

$$2 < (1+\varepsilon)(1+\varepsilon\alpha).$$

On the other hand, recalling that $\varepsilon = \frac{1}{3}\min\{1,\frac{1}{\alpha}\}$ we get that

$$(1+\varepsilon)(1+\varepsilon\alpha) \le (1+\frac{1}{3})(1+\frac{1}{3}) = \frac{16}{9} < 2.$$

This contradicts the previous inequality and the claim is proved. Combining the estimate

$$d(x_k,O'^c) \le d(Q_k,O'^c) + \operatorname{diam}(Q_k) \le \left(1+\frac{4}{\delta}\right)\operatorname{diam}(Q_k)$$

with the claim, we obtain

$$d(x_k,E) \le 3\max\{1,\alpha\}d(x_k,O'^c) \le 3\left(1+\frac{4}{\delta}\right)\max\{1,\alpha\}\operatorname{diam}(Q_k).$$

Recalling the inequality $|c_k - x_k| \le \frac{1}{4}d(x_k,E)$ proved before, and then using the doubling property in combination with the above inequality, we obtain

$$\begin{aligned}\sum_{k\ge1}\gamma(B(x_k,\eta d(x_k,E))) &\le \sum_{k\ge1}\gamma(B(c_k,(\eta+\tfrac{1}{4})d(x_k,E))) \\ &\le C\sum_{k\ge1}\gamma(B(c_k,\operatorname{diam}(Q_k))) \le C\sum_{k\ge1}\gamma(Q_k) \le C\gamma(O'). \qquad \square\end{aligned}$$

For a set $E \subset \mathbb{R}^d$, define

$$E + \mathscr{C}_a = \{x \in \mathbb{R}^d : x \text{ is the center of a ball } B \in \mathscr{B}_a \text{ such that } B \cap E \neq \emptyset\}. \quad (4.8)$$

Lemma 4.6. *Given $p \geq 0$, $l \geq p+1$ integers, and $Q \in \Delta^{\gamma}_{0,l}$, if a ball $B = B(c_B, r_B) \in \mathscr{B}_{2^p}$ intersects Q, then $c_B \in L_{l-1} \cup L_l \cup L_{l+1}$.*

Proof. If we had $c_b \in L_{l-m}$ for some $2 \leq m \leq l$, then $r_B \leq 2^p \leq 2^{l-1}$, in case $m = l$; or $r_B \leq \frac{2^p}{|c_B|} \leq \frac{2^{l-1}}{2^{l-m-1}} = 2^m \leq 2^{l-1}$, in the case $2 \leq m \leq l-1$. On the other hand, the distance between layers L_l and L_{l-m} is at least $2^{l-1} + 2^{l-2} + \cdots + 2^{l-m+1}$. Similarly, it can be seen that $x_b \notin L_{l+m}$ for any $m \geq 2$. □

Lemma 4.7. *Fix non-negative integers $p \geq 0$ and $\kappa > p+4$. Let $i \in \{1, 2, \cdots, 2^\kappa\}^d$ and let $Q_1 \in \Delta^{\gamma}_{0,l_1}$ and $Q_2 \in \Delta^{\gamma}_{0,l_2}$ be two distinct cubes with the same label i in the layers L_{l_1} and L_{l_2} with $l_{l_1}, l_{l_2} \geq \max\{5, p+1, \lceil \frac{\kappa}{2} \rceil\}$. Then,*

$$d(Q_1 + \mathscr{C}_{2^p}, Q_2 + \mathscr{C}_{2^p}) > 0.$$

Proof. We first consider only the case that $Q_1 \in L_l$ and $Q_2 \in L_{l+1}$. The case in which both cubes lie in the same layer or that they are in more than one layer apart can be handled with cruder estimates.

The center of a ball $B = B(x_B, r_B) \in \mathscr{B}_{2^p}$ intersecting the layer L_l satisfies $|c_B| \geq 2^{l-1} - r_B \geq 2^{l-1} - 2^p/|r_B|$, which, using Lemma 4.6, implies that $|c_B| \geq 2^{l-1} - 2^{p-l+2}$. Therefore, $r_B \leq 2^p/(2^{l-1} - 2^{p-l+2})$. For $j = 1, 2$ let $B_j = B(c_{B_j}, r_{B_j}) \in \mathscr{B}_{2^p}$ intersecting Q_j. It follows that

$$r_{B_1} \leq \frac{1}{2^{l-p-1} - 2^{-l+2}}, \quad r_{B_2} \leq \frac{1}{2^{l-p} - 2^{-l+1}}.$$

The cubes Q_1 and Q_2 are separated by at least $2^\kappa - 1$ cubes in $\Delta^{\gamma}_{0,l}$ or $\Delta^{\gamma}_{0,l+1}$; thus, the distance between Q_1 and Q_2 is at least $(2^\kappa - 1)/2^{l+1}$. Hence, using that $l \geq p+1$ and $l \geq 5$, we get

$$\begin{aligned}
d(Q_1 + \mathscr{C}_{2^p}, Q_2 + \mathscr{C}_{2^p}) &\geq \frac{2^\kappa - 1}{2^{l+1}} - \left(\frac{1}{2^{l-p-1} - 2^{-l+2}} + \frac{1}{2^{l-p} - 2^{-l+1}}\right) \\
&\geq \frac{2^\kappa - 1}{2^{l+1}} - \left(\frac{1}{2^{l-p-1} - 2^{-p+2}} + \frac{1}{2^{l-p} - 2^{-p+1}}\right) \\
&= \frac{2^\kappa - 1}{2^{l+1}} - 2^p\left(\frac{2}{2^l - 8} + \frac{1}{2^l - 2}\right) \\
&= \frac{2^\kappa - 1}{2^{l+1}} - 2^p \frac{3}{2^l - 8} \geq \frac{2^\kappa - 1}{2^{l+1}} - 2^p \frac{8}{2^{l+1}} = \frac{2^\kappa - 2^{p+3} - 1}{2^{l+1}},
\end{aligned}$$

and the right-hand side is strictly positive as $\kappa \geq p+4$. □

Let us fix $p \geq 4$ and $\kappa \geq p+4$. Note that all $l \geq p+1$ satisfy the assumptions of Lemma 4.7.

Definition 4.8. *Let us define the set* $E^{(i)}_{p,\kappa}$ *as the union of all cubes in* $\bigcup_{l\geq p+1}\Delta^{\gamma}_{0,l}$ *with label in* $i\in\{1,\cdots,2^{\kappa}\}^{d}$.

Definition 4.9. *Let* $\lambda>0$, *a set* $E\subset\mathbb{R}^{d}$ *is said to be an* admissible λ-Whitney set *if for all* $x\in E$ *we have*

$$d(x,E^{c})\leq\lambda\, d\,m(x). \tag{4.9}$$

It is clear that subsets of admissible λ-Whitney sets are also admissible λ-Whitney.

Theorem 4.10. *For* $p\geq 4$ *and* $\kappa\geq p+4$, *we have:*

i) $Q+\mathscr{C}_{2^{p}}$, *with* $Q\in\Delta^{\gamma}_{0,l}$ *and* $l=0,1,\cdots,p$, *is* $(2^{2p+1}\sqrt{d})$*-admissible Whitney.*
ii) $E^{(i)}_{p,\kappa}+\mathscr{C}_{2^{p}}$, *with* $i\in\{1,\cdots,2^{\kappa}\}^{d}$, *is* $(2^{p+3}\sqrt{d})$*-admissible Whitney.*

Proof. According to Lemma 4.7 to prove *i*) and *ii*), it is enough to prove that $Q+\mathscr{C}_{2^{p}}$ is admissible Whitney for any cube $Q\in\Delta^{\gamma}_{0,l}$, for $l\geq 0$ arbitrary.

- First, let $l\in\{1,\cdots,p\}$. If $Q\in\Delta^{\gamma}_{0,l}$ let us take a ball $B=B(c_B,r_B)\in\mathscr{B}_{2^{p}}$ intersecting Q, then $|r_B|\leq 2^{p}$; therefore,

$$Q+\mathscr{C}_{2^{p}}\subset\left\{x\in\mathbb{R}^{d}:d(x,Q)\leq 2^{p}\right\}.$$

  Let $y\in Q+\mathscr{C}_{2^{p}}$ be given. If $y\in Q$, then the distance of y to the complement of $Q+\mathscr{C}_{2^{p}}$ is at most $\frac{1}{2}+2^{p}$, and $m(y)\geq\frac{1}{2^{p}\sqrt{n}}$, as $y\in L_0\cup\cdots\cup L_p$. If $y\notin Q$, then the distance of y to the complement of $Q+\mathscr{C}_{2^{p}}$ is at most 2^{p}, and $m(y)\geq\frac{1}{2^{p+1}\sqrt{d}}$, as $y\in L_0\cup\cdots\cup L_{p+1}$; in both cases (4.9) holds.
- Then, let $l\geq p+1$ and $Q\in\Delta^{\gamma}_{0,l}$ given take a ball $B=B(c_B,r_B)\in\mathscr{B}_{2^{p}}$ intersecting Q. Using Lemma 4.6 we get $|r_B|\leq 2^{p}|c_B|^{-1}\leq 2^{p-l+2}$. It follows that

$$Q+\mathscr{C}_{2^{p}}\subset\left\{x\in\mathbb{R}^{d}:d(x,Q)\leq 2^{p-l+2}\right\}.$$

  Now, given $y\in Q+\mathscr{C}_{2^{p}}$, if $y\in Q$, then the distance of y to the complement of $Q+\mathscr{C}_{2^{p}}$ is at most $2^{-l-1}+2^{p-l+2}$, and $m(y)\geq\frac{1}{2^{l}\sqrt{n}}$, as $y\in L_l$. If $y\notin Q$, then the distance of y to the complement of $Q+\mathscr{C}_{2^{p}}$ is at most 2^{-p-l+2}, and $m(y)\geq\frac{1}{2^{l+1}\sqrt{d}}$, because $y\in L_{l-1}\cup L_l\cup L_{l+1}$; in both cases (4.9) holds. □

Corollary 4.11. *There exists a constant* N, *dependent only on* p *and the dimension* d, *such that every open set in* $\mathbb{R}^{d}$ *can be covered by* N *admissible open* $2^{2p+1}\sqrt{n}$*-Whitney sets.*

An explicit bound of N is obtained by taking $\kappa=p+4$ and counting the number of sets involved in Theorem 4.10, which can be estimated by $2^{d}(1+2^{\kappa d}+\cdots+2^{(p-1)\kappa d})+2^{\kappa d}$.

The next result is an immediate consequence of the classical Whitney covering lemma (see [252, Chapter VI]). The cubes that are picked up from the Euclidean proof are automatically admissible on a suitable scale (which depends only on d) as we start from an admissible Whitney set.

Lemma 4.12. *Let $\lambda > 0$ and $E \subset \mathbb{R}^d$ be an open admissible λ-Whitney set. There exists a constant ρ, depending only on λ and d, a countable family of disjoint cubes $\{Q_n\}_{n\geq 1}$ in Δ^γ, and a family of functions $\{\phi_n\}_{n\geq 1} \subset C_0^\infty(\mathbb{R}^d)$ such that:*

i) $\bigcup_n Q_n = E$.
ii) *For all $n \in \mathbb{N}$, $diam(Q_n) \leq d(Q_n, E^c) \leq \rho diam(Q_n)$.*
iii) *For all $n \in \mathbb{N}$, $supp(\phi_n) \subset Q_n^*$ where Q_n^* denotes the cube with the same center as Q_n, but its side length is ρ times the side length of Q_n.*
iv) *For all $n \in \mathbb{N}$, and all $x \in Q_n$ $\frac{1}{\rho} \leq \phi_n(x) \leq 1$.*
v) *For all $x \in E$, $\sum_n \phi(x) = 1$.*

This result is used to study Gaussian tent spaces $T^{1,q}(\gamma_d)$ in Chapter 7 (see Theorem 7.8).

4.2 Hardy–Littlewood Maximal Function with Respect to the Gaussian Measure and Its Variants

We define the centered *Gaussian Hardy–Littlewood maximal function* using the standard definition of the Hardy–Littlewood maximal function with respect to a general Borel measure in $\mathbb{R}^d$ (see, for instance, E. Stein [254, Chapter 1, §3], L. Grafakos [118, Chapter 2, §2] or A. Torchinski [275, Chapter 9, §1]).

Definition 4.13. *The centered Gaussian Hardy–Littlewood maximal function is defined as*

$$\mathscr{M}_\gamma f(x) = \sup_{r>0} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)| \gamma_d(dy), \tag{4.10}$$

for $f \in L^1_{loc}(\gamma_d)$, $x \in \mathbb{R}^d$.

Observe that $\mathscr{M}_\gamma f$ is defined on balls centered at x. It is easy to see, using classical arguments, that $\mathscr{M}_\gamma f$ is lower continuous (and therefore measurable) and satisfies

- $\mathscr{M}_\gamma f$ is non-negative, $0 \leq \mathscr{M}_\gamma f \leq \infty$.
- $\mathscr{M}_\gamma f$ is sublinear, that is

$$\mathscr{M}_\gamma(f_1 + f_2) \leq \mathscr{M}_\gamma f_1 + \mathscr{M}_\gamma f_2, \quad \text{and} \quad \mathscr{M}_\gamma(cf) = |c| M_{\gamma_d} f.$$

The boundedness properties of the Hardy–Littlewood maximal function with respect to a Borel measure are well known, but for the sake of completeness, we explicitly establish them for the Gaussian measure.

Theorem 4.14. *The (centered) Gaussian Hardy–Littlewood maximal function $\mathscr{M}_\gamma$ satisfies the following properties:*

i) *$\mathscr{M}_\gamma$ is of weak type $(1,1)$ with respect to the Gaussian measure, i.e., there exists a constant C, depending only on the dimension d, such that for any $f \in L^1(\gamma_d)$*

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \mathscr{M}_\gamma f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda}\|f\|_{1,\gamma} \tag{4.11}$$

for any $\lambda > 0$.

ii) *For $1 < p \leq \infty$, $\mathscr{M}_\gamma$ is $L^p(\gamma_d)$-bounded, i.e., there exists a constant A_p depending only on p and the dimension d, such that*

$$\|\mathscr{M}_\gamma f\|_{p,\gamma} \leq A_p\|f\|_{p,\gamma}. \tag{4.12}$$

Proof.

i) Let $\lambda > 0$ and take $E_\lambda = \Big\{x \in \mathbb{R}^d : \mathscr{M}_\gamma f(x) > \lambda\Big\}$. Considering A a bounded Borel set in $\mathbb{R}^d$, take $x \in E_\lambda \cap A$. Next, there exists $r = r(x) > 0$ such that

$$\frac{1}{\gamma_d(B(x,r))}\int_{B(x,r)} |f(y)|\gamma_d(dy) > \lambda.$$

Then, using Besicovitch covering lemma, there exists an at most countable family $\{B_i = B(x_i, r(x_i))\}_i$, $x_i \in E_\lambda$ and a constant C, depending only on the dimension d, such that

$$E_\lambda \cap A \subset \bigcup_i B_i, \quad \frac{1}{\gamma_d(B_i)}\int_{B_i} |f(y)|\gamma_d(dy) > \lambda, \quad \text{and} \quad \sum_i \chi_{B_i}(x) \leq C.$$

Thus, according to the subadditivity of γ_d and the bounded overlaps property of the covering $\{B_i\}$, we get

$$\begin{aligned}\gamma_d(E_\lambda \cap A) &\leq \gamma_d\Big(\bigcup_i B_i\Big) \leq \sum_i \gamma_d(B_i) \leq \frac{1}{\lambda}\sum_i \int_{B_i} |f(y)|\gamma_d(dy) \\ &\leq \frac{C}{\lambda}\int_{\cup_i B_i} |f(y)|\gamma_d(dy) \leq \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|\gamma_d(dy).\end{aligned}$$

Now, using the continuity from below of γ_d, taking $A \uparrow \mathbb{R}^d$, we get (4.11).

ii) Observe that the case $p = \infty$ is trivial, because

$$\|\mathscr{M}_\gamma f\|_{\infty,\gamma} = \|\mathscr{M}_\gamma f\|_\infty \leq \|f\|_{\infty,\gamma}.$$

Then, using the Marcinkiewicz interpolation theorem (see Theorem 10.24), from i) we get (4.12). □

Observe that we have used Besicovitch covering lemma in a crucial way. For doubling measures, we could instead use Vitali's covering lemma or the Calderón–Zygmund decomposition (see for instance [275, pages 223–225]).

The centered Gaussian Hardy–Littlewood maximal function on cubes, can also be defined as $\mathscr{M}_\gamma^Q f(x)$,

$$\mathscr{M}_\gamma^Q f(x) = \sup_{Q(x)} \frac{1}{\gamma_d(Q(x))} \int_{Q(x)} |f(y)|\gamma_d(dy), \tag{4.13}$$

where $Q(x)$ is any cube with the sides parallel to the axis with the center at x.

$\mathscr{M}_\gamma^Q$ has similar boundedness properties as $\mathscr{M}_\gamma$ and the proof is entirely analogous, using the Besicovitch covering lemma for cubes. Nevertheless, unlike the case of the Lebesgue measure, these two functions are not equivalent, i.e., there are no constants A_d, B_d depending only on dimension, such that for any $f \in L^1_{loc}(\gamma_d)$ and $x \in \mathbb{R}^d$,

$$A_d \mathscr{M}_\gamma^Q f(x) \leq \mathscr{M}_\gamma f(x) \leq B_d \mathscr{M}_\gamma^Q f(x),$$

because for the Gaussian measure, we do not have good control of the measures of the balls and cubes, independent of the center.

Additionally, we can also define non-centered *Gaussian Hardy–Littlewood maximal functions*, as

$$\tilde{\mathscr{M}}_\gamma f(x) = \sup_{r>0} \frac{1}{\gamma_d(B(z,r))} \int_{B(z,r)} |f(y)|\gamma_d(dy), \tag{4.14}$$

where $B(z,r)$ is any ball with center at $z \in \mathbb{R}^d$, which contains x, and

$$\tilde{\mathscr{M}}_\gamma^Q f(x) = \sup_{Q} \frac{1}{\gamma_d(Q)} \int_{Q} |f(y)|\gamma_d(dy), \tag{4.15}$$

where Q is any cube with sides parallel to the axis that contains x. Clearly, for any $f \in L^1_{loc}(\gamma_d)$ and $x \in \mathbb{R}^d$,

$$\mathscr{M}_\gamma f(x) \leq \tilde{\mathscr{M}}_\gamma f(x), \quad \text{and} \quad \mathscr{M}_\gamma^Q f(x) \leq \tilde{\mathscr{M}}_\gamma^Q f(x),$$

It is well known that, in one dimension, the Hardy–Littlewood maximal function with respect to any non-negative Borel measure μ, M_μ, is always of weak type $(1,1)$ as can be seen for instance in E. Stein's book [254, Chapter I §3], P. Sjögren [246], or A. Garsia [107]. For dimension $d > 1$, and if μ is a doubling measure, then M_μ is also of weak type $(1,1)$ (see [275, Chapter IX]). Nevertheless, P. Sjögren, [246], proved that $\tilde{\mathscr{M}}_{\gamma_d}$ and $\tilde{\mathscr{M}}_{\gamma_d}^Q$ are not of weak type $(1,1)$ with respect to $\gamma_d, d > 1$. Therefore, they cannot be equivalent to $\mathscr{M}_\gamma$ and $\mathscr{M}_\gamma^Q$. Let us see this argument in detail. The argument is by contradiction: take $d = 2$, and assume the weak type $(1,1)$ for $\tilde{\mathscr{M}}_\gamma$. According to a limiting argument, we would get a weak type $(1,1)$ estimate for the maximal function $\tilde{\mathscr{M}}_\gamma \mu$ for finite measures μ. Take μ to be a Dirac delta mass located at $(0, a+1)$ for $a > 0$ big enough. Take B_1, a unit ball with its center at $(c_0, a+1)$, with $|c_0| < 1$. B_1 is contained in the parabolic region

$$\{(x,y) \in \mathbb{R}^2 : y > a + (x-c_0)^2/2\},$$

as if $(x-c_0)^2 + ((y-a)-1)^2 = (x-c_0)^2 + (y-a)^2 - 2(y-a) + 1 < 1$, then

$$(x-c_0)^2 < (x-c_0)^2 + (y-a)^2 < 2(y-a).$$

Thus, using (1.5),

$$\begin{aligned}\gamma_2(B_1) &\leq C \int_{-1}^{1} \int_{a+x^2/2}^{\infty} e^{-y^2} dy\, dx \leq \frac{C}{a} \int_{-1}^{1} e^{-(a+x^2/2)^2} dx \\ &\leq \frac{C}{a} e^{-a^2} \int_{-1}^{1} e^{-ax^2} dx \leq \frac{C}{a\sqrt{a}} e^{-a^2}.\end{aligned}$$

Hence, $\tilde{\mathscr{M}}_\gamma \mu$ is at least $Ca\sqrt{a}e^{a^2}$ in the set $\{(x,y) \in \mathbb{R}^2 : |x| < 1, a < y < a+2\}$ but, again, using (1.5), this set has Gaussian measure of at least $Ca^{-1}e^{-a^2}$. Taking $a \to \infty$ this leads us to a contradiction. □

This counter-example can be extended to higher dimensions $d > 1$, as is done in [138], using Proposition 1.7. Taking a Dirac delta mass $\delta_{\overline{a}}$ located at $\overline{a} = (a+1, 0, \cdots, 0)$, for $a > 0$, and consider B_1 a unit ball with its center at $(a+1, 0, \cdots, 0)$. Taking a big enough such that (1.22) can be used, where $r = 1$ and $x_0 = (a, 0, \cdots, 0)$, then

$$\gamma_d(B_1) \leq Ce^{-a^2} a^{-(d+1)/2},$$

for a certain constant C that depends only on the dimension d but not on a. Let

$$E_\lambda = \left\{ x : \mathscr{M}_\gamma \delta_{\overline{a}}(x) \geq \frac{1}{\gamma_d(B_1)} \right\},$$

and consider the set

$$D = \left\{ (x, x') \in \mathbb{R}^d : a < x_1 < a + \frac{1}{a}, |x'| < 1, \text{ where } x' = (x_2, \cdots, x_d) \right\}.$$

Clearly $D \subset E_\lambda$, and

$$\gamma_d(D) = \frac{1}{\pi^{d/2}} \int_a^{a+\frac{1}{a}} \int_{|x'|<1} e^{-x_1^2} e^{-|x'|^2} dx_1\, dx' \sim e^{-a^2}/a.$$

Assuming that the weak $(1,1)$ inequality is true, then we would have

$$\gamma_d(E_\lambda) = \gamma_d\left(\left\{ x : \mathscr{M}_\gamma \delta_{\overline{a}}(x) \geq \frac{1}{\gamma_d(B_1)} \right\} \right) \leq C\gamma_d(B_1) \leq Ce^{-a^2}\, a^{-(d+1)/2};$$

therefore,

$$e^{-a^2}/a \sim \gamma_d(D) \leq Ce^{-a^2}\, a^{-(d+1)/2}.$$

This inequality is only true for a big enough if $d = 1$.

Also, it can be proved that $\tilde{\mathscr{M}}^Q_{\gamma_d}$ are not of weak type $(1,1)$, for $d=2$. Again, by contradiction: taking $\delta_{(a,a)}$ a Dirac delta mass located at (a,a) for $a>0$ big enough and consider (two-dimensional) cubes with a lower left vertex at $(a-x_0, a-y_0)$, $x_0, t_0>0$, $x_0+t_0=1$. Then, we get that $\tilde{\mathscr{M}}^Q_{\gamma_d}\delta_{(a,a)}$ is at least $Ca^2e^{a^2-a}$ in the union of these cubes, whose Gaussian measure is at least $Ca^{-1}e^{-a^2+a}$. Taking $a\to\infty$, this leads us again to a contradiction.

As the non-centered Gaussian Hardy–Littlewood maximal functions are not weak $(1,1)$, the interpolation argument used in the proof of Theorem 4.14 for the strong type (p,p), $p>1$, of centered Gaussian Hardy–Littlewood maximal functions cannot be applied. Thus, a natural question is whether or not $\tilde{\mathscr{M}}_\gamma$ and $\tilde{\mathscr{M}}^Q_\gamma$ are of strong type (p,p), $p>1$, for $d>1$. In [90], L. Forzani, R. Scotto, P. Sjögren, and W. Urbina gave a positive answer for $\tilde{\mathscr{M}}_\gamma$.

Theorem 4.15. *The non-centered Gaussian Hardy–Littlewood maximal function $\tilde{\mathscr{M}}_\gamma$ is a bounded operator on $L^p(\gamma_d)$ for $p>1$, that is, there exists a constant $C=C(d,p)$ such that for $f\in L^p(\gamma_d)$,*

$$\|\tilde{\mathscr{M}}_\gamma f\|_{p,\gamma_d}\le C\|f\|_{p,\gamma_d}.$$

Let us denote $S^{d-1}_r=\left\{x\in\mathbb{R}^d : |x|=r\right\}$ and $S^{d-1}=S^{d-1}_1$, and let $d\sigma$ be the area measure on S^{d-1}. The spherical maximal function

$$M^e f(h)=\sup_{R>0}\frac{1}{\sigma(|z'-h|\le R)}\int_{|z'-h|\le R}|f(z')|\,d\sigma(z'),\qquad h\in S^{d-1},$$

is bounded on $L^p(d\sigma)$. One can extend M^e to functions defined in $\mathbb{R}^d$ by using polar coordinates $x=\rho x'$, with $x'\in S^{d-1}$, and applying M^e in the x' variable. Then, M^e is bounded on $L^p(\gamma_d)$.

To prove Theorem 4.15, we need the following technical lemma, which is a variation of (1.22), in the second part of Lemma 1.7.

Lemma 4.16. *Let B be a closed ball in $\mathbb{R}^d$ of radius r. Denote by x_0 the point of B whose distance to the origin is minimal. Assume that $|x_0|\ge 1$ and that $r\ge 1/|x_0|$. Then, for all $x,\ y\in B$*

$$\gamma_d(B)\ge C\,\frac{e^{-|x_0|^2}}{|x_0|}\left(1\wedge\frac{|y-x|^2}{|x_0|(|x|\vee|y|-|x_0|)}\right)^{\frac{d-1}{2}}. \qquad (4.16)$$

Proof. Consider the hyperplane orthogonal to x_0 whose distance from the origin is $|x_0|+t$, with $1/(2|x_0|)<t<1/|x_0|$. Its intersection with B is a $(d-1)-$dimensional ball whose radius is at least $C\sqrt{rt}\ge C\sqrt{r/2|x_0|}$. Integrating the Gaussian density first along this $(d-1)$ -dimensional ball and then in t, we get

$$\gamma_d(B)\ge\int_{1/(2|x_0|)}^{1/|x_0|}e^{-(|x_0|+t)^2}dt\int_{|v|<C\sqrt{r/|x_0|}}e^{-|v|^2}dv,$$

where v is a $(d-1)$-dimensional variable. The inner integral here is at least

$$C\min\{1,(r/|x_0|)^{(d-1)/2}\},$$

and $e^{-(|x_0|+t)^2} \geq Ce^{-|x_0|^2}$ for these t's; therefore,

$$\gamma_d(B) \geq C\frac{e^{-|x_0|^2}}{|x_0|}\left(1 \wedge \left(\frac{r}{|x_0|}\right)^{\frac{d-1}{2}}\right). \tag{4.17}$$

To estimate r from below, we let z be the center of B and w the projection of x onto the line passing through 0, x_0 and z. Write $h = |x-w|$ and $a = |w-x_0|$. Applying the Pythagorean theorem twice, we get

$$|x-z|^2 - (r-a)^2 = h^2 = |x-x_0|^2 - a^2.$$

As $|x-z| \leq r$, we conclude that $2ar \geq |x-x_0|^2$. Clearly, $a \leq |x| - |x_0|$ so that

$$r \geq \frac{|x-x_0|^2}{2(|x|-|x_0|)} \geq \frac{|x-x_0|^2}{2(|x| \vee |y| - |x_0|)}.$$

As x and y are arbitrary points of B, the same argument also implies

$$r \geq \frac{|y-x_0|^2}{2(|x| \vee |y| - |x_0|)}.$$

Using the triangle inequality we conclude that $2|x-x_0| \vee |y-x_0| \geq |x-y|$, and so

$$r \geq \frac{|x-y|^2}{8(|x| \vee |y| - |x_0|)}.$$

Combining this with (4.17), we obtain the inequality (4.16). □

We are ready to prove Theorem 4.15.

Proof. We assume that $d \geq 2$, as we have already mentioned the case $d = 1$ is well known (see for example [246]). Take $0 \leq f \in L^p(\gamma_d)$ and $x \in \mathbb{R}^d$. For any ball B containing x, we must estimate the average

$$\overline{f}(B) = \frac{1}{\gamma_d(B)}\int_B f(y)\,\gamma_d(dy).$$

Let r and x_0 be defined as in Lemma 4.16, $|x_0| > 1$. We first consider small balls B, and denote by $\mathscr{M}_0 f(x)$ the supremum of $\overline{f}(B)$ taken only over balls B containing x and verifying $r < 1 \wedge |x_0|^{-1}$. Split $\mathbb{R}^d$ into rings $S_k = \{x : \sqrt{k-1} \leq |x| < \sqrt{k}\}$, $k = 1,2,\ldots$. The width of S_k is no larger than $1/\sqrt{k}$; thus, the Gaussian density is of constant order of magnitude in each R_k. Using Lebesgue measure arguments, we can easily estimate the $L^p(\gamma_d)$ norm of $\mathscr{M}_0 f$ in S_k in terms of the $L^p(\gamma_d)$ norm of f in $\cup\{S_{k'} : |k'-k| \leq C\}$. This takes care of small balls.

Consider now balls B with $r \geq 1 \wedge |x_0|^{-1}$. Observe first that the case $|x_0| < 2$ is simple, because $\gamma_d(B) \geq C$ and thus

$$\overline{f}(B) \leq C \int f(y)\gamma_d(dy) \leq C\|f\|_{p,\gamma}.$$

The corresponding part of $\mathscr{M} f$ thus satisfies the $L^p(\gamma_d)$ estimate.

It remains to consider the operator

$$\overline{\mathscr{M}} f(x) = \sup_B \overline{f}(B),$$

the supremum taken over balls B containing x and with the property that $r \geq |x_0|^{-1}$ and $|x_0| \geq 2$. Let B be such a ball, and observe that it satisfies the hypotheses of Lemma 4.16.

For each $\rho \geq 1$ such that S_ρ^{d-1} intersects B, let $y_\rho \in S_\rho^{d-1} \cap \partial B$ be such that $|y_\rho - x| = \sup_{z \in B \cap S_\rho^{d-1}} |z - x|$. Write $x' = x/|x|$.

For each $z' \in S^{d-1}$ such that $\rho z' \in B$ we have

$$|x' - z'| = \frac{1}{\rho}|\rho x' - \rho z'| \leq \frac{1}{\rho}\left(|x - \rho z'| + |\rho - |x||\right) \leq \frac{2}{\rho}|y_\rho - x|; \tag{4.18}$$

and trivially $|x' - z'| \leq 2$.

Because of (4.18) and the definition of M^e,

$$\begin{aligned}
\overline{f}(B) &= \int_{|x_0|}^{|x_0|+2r} \frac{1}{\gamma_d(B)} \int_{S^{d-1}} \chi_B(\rho z') f(\rho z') d\sigma(z')\, \rho^{d-1} e^{-\rho^2}\, d\rho \\
&\leq \int_{|x_0|}^{|x_0|+2r} \frac{1}{\gamma_d(B)} \int_{|z'-x'| \leq 2\left(1 \wedge \frac{|y_\rho - x|}{\rho}\right)} f(\rho z') d\sigma(z')\, \rho^{d-1} e^{-\rho^2}\, d\rho \\
&\leq C \int_{|x_0|}^{|x_0|+2r} \frac{\left[1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{d-1}\right]}{\gamma_d(B)} \mathscr{M}^e f(\rho x') \rho^{d-1} e^{-\rho^2}\, d\rho \\
&\leq C \int_{|x_0|}^{|x_0|+2r} |x_0| e^{|x_0|^2} \left[1 \vee \left(\frac{|x_0|(\rho \vee |x| - |x_0|)}{|x - y_\rho|^2}\right)^{\frac{d-1}{2}}\right] \\
&\qquad \times \left[1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{d-1}\right] \mathscr{M}^e f(\rho x') \rho^{d-1} e^{-\rho^2}\, d\rho,
\end{aligned} \tag{4.19}$$

where we applied Lemma 4.16 with $y = y_\rho$ to get the last inequality. Write $M = \rho \vee |x|$ and $m = \rho \wedge |x|$, so that $|x_0| \leq m \leq M$. Now we need to prove the following claim, to conclude, from (4.19), that

$$\overline{f}(B) \leq C \int_1^\infty m e^{m^2} \left(\frac{1}{m^2} \vee \frac{M - m}{m}\right)^{\frac{d-1}{2}} \mathscr{M}^e f(\rho x') \rho^{d-1} e^{-\rho^2}\, d\rho. \tag{4.20}$$

<u>Claim:</u> For $|x_0| < \rho < |x_0| + 2r$ and some $C > 0$,

$$e^{|x_0|^2}\left[1 \vee \left(\frac{|x_0|(M-|x_0|)}{|x-y_\rho|^2}\right)^{\frac{d-1}{2}}\right]\left[1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{d-1}\right] \leq Ce^{m^2}\left(\frac{1}{m^2} \vee \frac{M-m}{m}\right)^{\frac{d-1}{2}}. \tag{4.21}$$

The proof of the claim is as follows: assume first that

$$\left(\frac{|x_0|(M-|x_0|)}{|x-y_\rho|^2}\right)^{\frac{d-1}{2}} \leq 1. \tag{4.22}$$

Then,

$$e^{|x_0|^2}\left[1 \vee \left(\frac{|x_0|(M-|x_0|)}{|x-y_\rho|^2}\right)^{\frac{d-1}{2}}\right]\left[1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{d-1}\right] \leq e^{|x_0|^2}(|x-y_\rho|/\rho)^{d-1}.$$

The angles at x_0 of the triangles $0x_0x$ and $0x_0y_\rho$ are obtuse, so that $|x|^2 \geq |x_0|^2 + |x - x_0|^2$ and $|y_\rho|^2 \geq |x_0|^2 + |y_\rho - x_0|^2$. But

$$|x-y_\rho| \leq |x-x_0| + |y_\rho - x_0|,$$

and this implies that

$$\begin{aligned}|x-y_\rho|^2 \leq 4\max\left\{|x-x_0|^2, |y_\rho - x_0|^2\right\} &\leq 4\max\left\{|x|^2 - |x_0|^2, |y_\rho|^2 - |x_0|^2\right\} \\ &= 4(M^2 - |x_0|^2).\end{aligned}$$

If $|x| \leq 2\rho$, this last quantity is at most $12\rho(M-|x_0|)$, and then

$$e^{|x_0|^2}\left[1 \vee \left(\frac{|x_0|(M-|x_0|)}{|x-y_\rho|^2}\right)^{\frac{d-1}{2}}\right]\left[1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{d-1}\right] \leq Ce^{|x_0|^2}\left(\frac{M-|x_0|}{\rho}\right)^{\frac{d-1}{2}}.$$

In the case $|x| > 2\rho$, we simply observe that

$$e^{|x_0|^2}\left[1 \vee \left(\frac{|x_0|(M-|x_0|)}{|x-y_\rho|^2}\right)^{\frac{d-1}{2}}\right]\left[1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{d-1}\right] \leq Ce^{|x_0|^2},$$

whereas the right-hand side is at least Ce^{m^2}. This case of the lemma is thus trivial.

Assume now that (4.22) is false. Then

$$e^{|x_0|^2}\left[1 \vee \left(\frac{|x_0|(M-|x_0|)}{|x-y_\rho|^2}\right)^{\frac{d-1}{2}}\right]\left[1 \wedge \left(\frac{|y_\rho - x|}{\rho}\right)^{d-1}\right] \leq e^{|x_0|^2}\frac{(|x_0|(M-|x_0|))^{\frac{d-1}{2}}}{\rho^{d-1}}$$

and we arrive again at (4.2). Thus, it only remains to see that (4.2) implies (4.21). This would follow from the estimate

$$e^{|x_0|^2 - m^2}(M-|x_0|)^{\frac{d-1}{2}} \leq C((1/m) \vee (M-m))^{\frac{d-1}{2}}. \tag{4.23}$$

To prove (4.23), we use the fact that

$$(M-|x_0|)^{\frac{d-1}{2}} \le C\left((M-m)^{\frac{d-1}{2}} + (m-|x_0|)^{\frac{d-1}{2}}\right)$$

and, when $m-|x_0|>1/m$, also

$$e^{|x_0|^2-m^2} = e^{-(m-|x_0|)(m+|x_0|)} \le \frac{C}{(m-|x_0|)^{\frac{d-1}{2}} m^{\frac{d-1}{2}}}.$$

Hence, (4.23) and inequality (4.21) follow.

Now, we split the integral in (4.20) into five integrals taken over the following intervals

$$\begin{aligned} I_1 &= \left[1,\frac{|x|}{2}\right],\quad I_2=\left(\frac{|x|}{2},|x|-\frac{1}{|x|}\right],\ I_3=\left(|x|-\frac{1}{|x|},|x|+\frac{1}{|x|}\right],\\ I_4 &= \left(|x|+\frac{1}{|x|},\frac{5}{4}|x|\right],\ I_5=\left(\frac{5}{4}|x|,+\infty\right). \end{aligned}$$

Let for $i=1,\dots,5$ we consider

$$\mathscr{M}_i f(x) = \int_{I_i} m e^{m^2}\left(\frac{1}{m^2}\vee\frac{M-m}{m}\right)^{\frac{d-1}{2}} M^e f(\rho x')\rho^{d-1}e^{-\rho^2}\,d\rho.$$

Then,

$$\tilde{\mathscr{M}}_\gamma f(x) \le C\sum_{i=1}^{5}\mathscr{M}_i f(x).$$

- Bound for $\mathscr{M}_1 f$. It is easy to see that

$$\mathscr{M}_1 f(x) \le\ |x|^d\int_1^{|x|/2} M^e f(\rho x')\,d\rho.$$

Hölder's inequality and the $L^p(d\sigma)$ boundedness of $\mathscr{M}^e$ imply

$$\begin{aligned} \|\mathscr{M}_1 f\|_{p,\gamma}^p &\le \int_1^{+\infty}\int_{S^{d-1}}\left(s^d\int_1^{s/2} M^e f(\rho x')d\rho\right)^p d\sigma(x')s^{d-1}e^{-s^2}ds\\ &\le \int_1^{+\infty}\int_{S^{d-1}} s^{dp}\int_1^{s/2}|M^e f(\rho x')|^p\rho^{d-1}e^{-\rho^2}d\rho\\ &\qquad\times\left(\int_1^{s/2}\rho^{-(d-1)\frac{p'}{p}}e^{\frac{p'}{p}\rho^2}d\rho\right)^{\frac{p}{p'}} d\sigma(x')s^{d-1}e^{-s^2}ds\\ &\le \left(\int_1^{+\infty}s^C e^{-\frac{3}{4}s^2}ds\right)\|f\|_{p,\gamma}^p \le C l f\|_{p,\gamma}^p. \end{aligned}$$

- Bound for $\mathscr{M}_2 f$. Making the change of variables $\rho = |x| - \frac{t}{|x|}$, we get

$$\mathscr{M}_2 f(x) \leq |x|^{\frac{d+1}{2}} \int_{|x|/2}^{|x|-\frac{1}{|x|}} (|x|-\rho)^{\frac{d-1}{2}} M^e f(\rho x')\, d\rho$$
$$\leq \int_1^{|x|^2/2} t^{\frac{d-1}{2}} M^e f\left(\left(|x| - \frac{t}{|x|}\right)x'\right)\, dt.$$

From Minkowski's integral inequality and the $L^p(d\sigma)$ boundedness of $\mathscr{M}^e$, we obtain

$$\|\mathscr{M}_2 f\|_{p,\gamma} \leq \int_1^{+\infty} t^{\frac{d-1}{2}} \left\| M^e f\left(\left(|\cdot| - \frac{t}{|\cdot|}\right)x'\right) \chi_{\{1\leq t\leq \frac{|x|^2}{2}\}} \right\|_{p,\gamma} dt$$
$$\leq \int_1^{+\infty} t^{\frac{d-1}{2}} \left[\int_{S^{d-1}} \int_{\sqrt{2t}}^{+\infty} f\left(\left(s - \frac{t}{s}\right)x'\right)^p s^{d-1} e^{-s^2} ds d\sigma(x')\right]^{\frac{1}{p}} dt.$$

Let us make the change of variables $\rho = s - t/s$, observing that $s \leq 2\rho$,

$$-s^2 = -\rho^2 - 2t + t^2/s^2 \leq -\rho^2 - 3t/2$$

and $d\rho/ds \geq 1$. Thus,

$$\|\mathscr{M}_2 f\|_{p,\gamma} \leq C \int_1^{+\infty} t^{\frac{d-1}{2}} \left[\int_{S^{d-1}} \int_{\sqrt{t/2}}^{+\infty} |f(\rho x')|^p \rho^{d-1} e^{-\rho^2} e^{-3t/2} d\rho d\sigma(x')\right]^{\frac{1}{p}} dt$$
$$\leq C\, \|f\|_{p,\gamma} \left(\int_1^{+\infty} t^{\frac{d-1}{2}} e^{-\frac{3t}{2p}} dt\right) \leq C\, \|f\|_{p,\gamma}.$$

- Bound for $\mathscr{M}_3 f$. Let us consider the Borel measure $d\mu = \rho^{d-1} e^{-\rho^2}\, d\rho$ on $\mathbb{R}_+$. We have

$$\mathscr{M}_3 f(x) \leq C|x| \int_{|x|-1/|x|}^{|x|+1/|x|} M^e f(\rho x') d\rho$$
$$\leq C(\mu(|x| - 1/|x|, |x| + 1/|x|))^{-1} \int_{|x|-1/|x|}^{|x|+1/|x|} M^e f(\rho x') d\mu(\rho).$$

Let M_μ denote the one-dimensional centered maximal operator defined in terms of μ, acting in the ρ variable. Then,

$$\mathscr{M}_3 f(x) \leq C M_\mu(M^e f(|x|x')).$$

But, as we have mentioned before, M_μ is bounded on $L^p(d\mu)$ (see [193] or [246]). Then, the $L^p(\gamma_d)$ boundedness of $\mathscr{M}_3$ follows.

- Bound for $\mathscr{M}_4 f$. Making the change of variables $\rho = |x| + \frac{t}{|x|}$, we have

$$\mathscr{M}_4 f(x) \leq C|x|^{\frac{d+1}{2}} e^{|x|^2} \int_{|x|+\frac{1}{|x|}}^{\frac{5}{4}|x|} (\rho - |x|)^{\frac{d-1}{2}} M^e f(\rho x')\, e^{-\rho^2}\, d\rho$$
$$\leq C \int_1^{\frac{|x|^2}{4}} t^{\frac{d-1}{2}} M^e f\left(\left(|x| + \frac{t}{|x|}\right)x'\right) e^{-2t}\, e^{-\frac{t^2}{|x|^2}} dt.$$

Minkowski's integral inequality implies that

$$\|\mathscr{M}_4 f\|_{p,\gamma} \leq C \int_1^{+\infty} t^{\frac{d-1}{2}} \left\| M^e f\left(\left(|x| + \frac{t}{|x|}\right)x'\right) e^{-\frac{t^2}{|x|^2}} \chi_{\{1 \leq t \leq \frac{|x|^2}{4}\}} \right\|_{p,\gamma} e^{-2t} dt.$$

But M^e is bounded on $L^p(\sigma_d)$, so that

$$\|M^e f\left(\left(|x| + \frac{t}{|x|}\right)x'\right) e^{-\frac{t^2}{|x|^2}} \chi_{\{1 \leq t \leq \frac{|x|^2}{4}\}} \|_{p,\gamma}$$
$$\leq C \int_{2\sqrt{t}}^{\infty} \int_{S^{d-1}} \left| f\left(\left(|x| + \frac{t}{|x|}\right)x'\right) e^{-\frac{t^2}{s^2}} \right|^p d\sigma(x') s^{d-1} e^{-s^2} ds.$$

Analogous to the case of $\mathscr{M}_2$, we make the change of variables $\rho = s + t/s$ and observe that $s \leq \rho$ and $-s^2 = -\rho^2 + 2t + t^2/s^2$ and $d\rho/ds \geq 1/2$. As $e^{-pt^2/s^2} e^{t^2/s^2} < 1$, it follows that the above double integral is at most

$$C \int_{S^{d-1}} \int_1^{+\infty} |f(\rho x')|^p \rho^{d-1} e^{-\rho^2}\, d\rho\, d\sigma(x')\, e^{2t} \leq C\|f\|_{p,\gamma}^p e^{2t}.$$

Thus,

$$\|\mathscr{M}_4 f\|_{p,\gamma} \leq C \int_1^{+\infty} t^{\frac{d-1}{2}} \|f\|_{p,\gamma} e^{\frac{2t}{p}} e^{-2t} dt \;\leq\; C\|f\|_{p,\gamma}.$$

- Bound for $\mathscr{M}_5 f$. Observe that

$$\mathscr{M}_5 f(x) \leq |x|^{\frac{3-n}{2}} e^{|x|^2} \int_{5/4|x|}^{+\infty} M^e f(\rho x')\, \rho^{\frac{d-1}{2}} \rho^{d-1} e^{-\rho^2} d\rho.$$

Taking the L^p norm and then applying Hölder's inequality, we get

$$\|\mathscr{M}_5 f\|_{p,\gamma}^p \leq \int_1^{+\infty} \int_{S^{d-1}} \frac{e^{ps^2}}{s^{p\frac{d-3}{2}}} \left(\int_{5s/4}^{+\infty} M^e f(\rho x') \rho^{\frac{3(d-1)}{2}} e^{-\rho^2} d\rho \right)^p d\sigma(x') s^{d-1} e^{-s^2} ds$$
$$\leq \int_1^{+\infty} \int_{S^{d-1}} \frac{e^{ps^2}}{s^{p\frac{d-3}{2}}} \int_0^{+\infty} |M^e f(\rho x')|^p \rho^{d-1} e^{-\rho^2} d\rho$$
$$\times \left(\int_{5s/4}^{+\infty} \rho^{(\frac{p'}{2}+1)(d-1)} e^{-\rho^2} d\rho \right)^{\frac{p}{p'}} d\sigma(x') s^{d-1} e^{-s^2} ds$$
$$\leq \|f\|_{p,\gamma}^p \left(\int_1^{+\infty} s^C e^{(p-1)s^2} e^{-(p-1)(\frac{5}{4}s)^2} ds \right) \leq C\|f\|_{p,\gamma}^p. \qquad \square$$

On the other hand, in her doctoral dissertation [83], L. Forzani (see also [5]) considered a whole class of *generalized Gaussian maximal functions*, proving that they are of weak type $(1,1)$. This result has several applications, one of which, as we see in the next section, is that the Ornstein–Uhlenbeck maximal function is a particular case of them; therefore, its weak type $(1,1)$ follows from the general result. It is also used for the $L^p(\gamma_d)$ boundedness of the non-tangential maximal function in Section 4.6, and also in Section 9.3 to prove the weak type $(1,1)$ of the alternative Gaussian Riesz transforms.

Definition 4.17. *Let $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a non-increasing function, such that*

$$S = \sum_{v \geq 1} \Phi\left(\frac{1}{2}(v-1)\right) v^{2d} < \infty.$$

Define the Φ-maximal function $\mathscr{M}_\Phi$ as

$$\mathscr{M}_\Phi f(x) = \sup_{0<r<1} \frac{1}{\gamma_d((1+\delta)B\left(\frac{x}{r}, \frac{|x|}{r}(1-r)\right)} \int_{\mathbb{R}^d} \Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right) |f(y)|\, \gamma_d(dy), \tag{4.24}$$

where $\delta = \delta_{r,x} = \frac{r}{|x|(1-r)} \min\left\{\frac{1}{|x|}, \sqrt{1-r}\right\}$.

Let us prove now that $\mathscr{M}_\Phi$ is of weak type $(1,1)$ with respect to the Gaussian measure,

Theorem 4.18. *(Forzani) There exists a constant C dependent only on S and the dimension d, such that for all $\lambda > 0$, and $f \in L^1(d\gamma_d)$, we have*

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : \mathscr{M}_\Phi f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda} ||f||_{1,\gamma}. \tag{4.25}$$

Proof. We consider only $r > \frac{3}{4}$, as the maximal operator is γ_d-weak type $(1,1)$ for $0 < r \leq \frac{3}{4}$ (see [83]). Let us denote with the same letter $\mathscr{M}_\Phi$ the maximal operator restricted to the interval $\frac{3}{4} < r < 1$, with $\mathscr{M}_\Phi^1$ the maximal operator for $\frac{3}{4} < r < 1 - \frac{\zeta^2}{|x|^2}$ and $\mathscr{M}_\Phi^2$ the corresponding one for $1 - \frac{\zeta^2}{|x|^2} < r < 1$ (ζ is the constant chosen in Lemma 4.2).

First, we prove that for $|x| < 2\zeta$,

$$\mathscr{M}_\Phi f(x) \leq C M_{\gamma_d} f(x),$$

where M_{γ_d} is the centered Gaussian Hardy–Littlewood maximal function; see Definition 4.13. Indeed, denoting $R_{x,r} = \frac{r}{|x|(1-r)} \min\left\{\frac{1}{|x|}, \sqrt{1-r}\right\}$, for $|x| \leq 2\zeta$, we have

$$\begin{aligned}\mathscr{M}_\Phi f(x) &= \sup_{3/4<r<1} \frac{1}{\gamma_d((1+\delta)B\left(\frac{x}{r}, R_{x,r}\right)} \int_{\mathbb{R}^d} \Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right) |f(y)|\, \gamma_d(dy) \\ &= C \sup_{3/4<r<1} \frac{e^{|x|^2}}{|B(x,R_{x,r})|} \sum_{v=0}^{\infty} \int_{vR_{x,r} \leq |y-\frac{x}{r}| \leq (v+1)R_{x,r}} \Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right) |f(y)| \gamma_d(dy)\end{aligned}$$

$$\leq C \sum_{\nu=0}^{\infty} \sup_{3/4<r<1} \Phi(\nu/8\zeta)(\nu+2)^d$$
$$\times \sup_{3/4<r<1} \frac{1}{\gamma_d(B(x,(\nu+2)R_{x,r}))} \int_{B(x,(\nu+2)R_{x,r})} |f(y)|\,\gamma_d(dy)$$
$$\leq CM_{\gamma_d} f(x).$$

For $|x| \geq 2\zeta$, because $\mathscr{M}_\Phi f(x) \leq \mathscr{M}^1_\Phi f(x) + \mathscr{M}^2_\Phi f(x)$ and the γ_d-weak type $(1,1)$ of $\mathscr{M}_\Phi$ follows once we prove that $\mathscr{M}^1_\Phi$ and $\mathscr{M}^2_\Phi$ are γ_d-weak type $(1,1)$.

To prove the γ_d-weak type $(1,1)$ of $\mathscr{M}^1_\Phi$, it is enough to prove that

$$\gamma_d(E_N^{1,\lambda}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(y)|\,d\gamma_d(y),$$

with constant C independent of N and f, where

$$E_N^{1,\lambda} = \left\{x \in \mathbb{R}^d :\ |x| \geq 2\zeta \ \text{ and } \ \mathscr{M}^1_\Phi f(y) > \lambda\right\} \cap B(0,N).$$

For each $x \in E_N^{1,\lambda}$, there exists $r = r(x) \in \left(\frac{3}{4}, 1 - \frac{\zeta^2}{|y|^2}\right)$ such that

$$\frac{1}{\gamma_d\left((1+\delta)B(\frac{x}{r}, \frac{|x|}{r}(1-r))\right)} \int_{\mathbb{R}^d} \Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right)|f(y)|\gamma_d(dy) \geq \lambda. \tag{4.26}$$

For every $x \in E_N^{1,\lambda}$ we have that $|x|\frac{1-r}{r}$ is bounded above and below by positive numbers, and the centers $\frac{x}{r}$ are in a bounded subset of $\mathbb{R}^d$. Hence, there exists $\varepsilon > 0$ such that for all $0 < \alpha < 1$,

$$\gamma_d\left(B\left(\frac{x}{r}, (1+\alpha)|x|\frac{1-r}{r} + \varepsilon\right)\right) \leq 2\gamma_d\left(B\left(\frac{x}{r}, (1+\alpha)|x|\frac{1-r}{r}\right)\right)$$

for all $x \in E_N^{1,\lambda}$. Let A be a subset of $E_N^{1,\lambda}$ which is a maximal set with the property $|x - \bar{x}| > \frac{\varepsilon}{2}$ for $x \neq \bar{x}, x \in A, \bar{x} \in A$. As $E_N^{1,\lambda}$ is bounded, A is a finite set $A = \{x_1, \ldots, x_L\}$. If we apply Lemma 4.2 to the set A, we get a family of balls $\left\{B_j = B\left(\frac{x_j}{r_j}, |x_j|\frac{1-r_j}{r_j}\right)\right\}_{j \in J \subset \{1,\ldots,L\}}$ such that $A \subset \cup_{j\in J}(1+\delta_j)B_j$ and *ii*) follows. Thus,

$$E_N^{1,\lambda} \subset \bigcup_{j\in J} B\left(\frac{x_j}{r_j}, (1+\delta_j)\frac{|x_j|}{rj}(1-r_j) + \varepsilon\right).,$$

and then

$$\gamma_d(E_N^{1,\lambda}) \leq 2\sum_{j\in J} \gamma_d\left(B\left(\frac{x_j}{r_j}, (1+\delta_j)|x_j|\frac{1-r_j}{r_j}\right)\right).$$

From (4.26), and the fact that Φ is a non-increasing function such that

$$\sum_{v\geq 1}\Phi\left(\frac{1}{2}(v-1)\right)v^{2n}<\infty,$$

we have, using *ii*) of Lemma 4.2, that

$$\begin{aligned}
\gamma_d(E_N^{1,\lambda}) &\leq 2\sum_{j\geq 1}\gamma_d((1+\delta_j)B_j) \leq \frac{C}{\lambda}\sum_{j\geq 1}\int_{\mathbb{R}^d}\Phi\left(\frac{|x_j-r_jy|}{\sqrt{1-r_j^2}}\right)|f(z)|\,d\gamma(y)\\
&\leq \frac{C}{\lambda}\sum_{j\geq 1}\sum_{v\geq 1}\int_{B\left(\frac{x_j}{r_j},v\rho_j\right)\setminus B\left(\frac{x_j}{r_j},(v-1)\rho_j\right)}\Phi\left(\frac{|x_j-r_jy|}{\sqrt{1-r_j^2}}\right)|f(z)|\,\gamma_d(dy)\\
&= \frac{C}{\lambda}\sum_{j\geq 1}\sum_{v\geq 1}\Phi\Big(\frac{1}{2}(v-1)\Big)\int_{B_j^v}|f(y)|\,\gamma_d(dy)\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^d}\sum_{v\geq 1}\Phi\Big(\frac{1}{2}(v-1)\Big)\sum_j\chi_{B_j^v}(y)|f(y)|\,\gamma_d(dy)\\
&\leq \frac{C}{\lambda}\int_{\mathbb{R}^d}\sum_{v\geq 1}\Phi\Big(\frac{1}{2}(v-1)\Big)v^{2n}|f(y)|\,\gamma_d(dy) \leq \frac{C}{\lambda}\int_{\mathbb{R}^d}|f(y)|\,\gamma_d(dy).
\end{aligned}$$

Finally, we prove that $\mathscr{M}_\Phi^2$ is of γ_d-weak type $(1,1)$. First, let us observe that if $r>1-\frac{\zeta^2}{|x|^2}$ then, for all $y\in(1+\delta)B\Big(\frac{x}{r},\frac{|x|}{r}(1-r)\Big)=B\Big(\frac{x}{r},\frac{|x|}{r}(1-r)+\sqrt{1-r}\Big)$, the values of $e^{-|y|^2}$ are equivalent. Now, let us define

$$E_N^{2,\lambda}=\Big\{x\in\mathbb{R}^d:\ |x|\geq 2\zeta \text{ and } \mathscr{M}_\Phi^2 f(y)>\lambda\Big\}\cap B(0,N).$$

The γ_d-weak type $(1,1)$ for $\mathscr{M}_\Phi^2$ follows once we prove the inequality

$$\gamma_d(E_N^{2,\lambda})\leq\frac{C}{\lambda}\int_{\mathbb{R}^d}|f(y)|\,\gamma_d(dy), \tag{4.27}$$

with constant C independent of N and f.

For each $x\in E_N^{2,\lambda}$, we have $\gamma_d((1+\delta)B(\frac{y}{r},\frac{|y|}{r}(1-r)))\simeq e^{-|y|^2}(1-r)^{n/2}$. To prove (4.27), we divide the region of integration into two parts: one given by $|y-x|<2\frac{C}{|x|r}$ and the other one by $|y-x|>2\frac{C}{|x|r}$.

For the first region, we have

$$\begin{aligned}
&\frac{e^{|x|^2}}{(1-r)^{d/2}}\int_{|y-x|<2\frac{C}{|x|r}}\Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right)|f(y)|\,\gamma_d(dy)\\
&\qquad\leq C\,\frac{e^{|x|^2}}{(1-r)^{d/2}}\int_{|y-x|<c\frac{\sqrt{1-r}}{r}}|f(y)|\,\gamma_d(dy)
\end{aligned}\tag{4.28}$$

$$+\frac{e^{|x|^2}}{(1-r)^{d/2}}\int_{c\frac{\sqrt{1-r}}{r}<|y-x|<2\frac{C}{|x|r}}\Phi\left(C\frac{|x-y|}{\sqrt{1-r^2}}\right)|f(y)|\,\gamma_d(dy)$$
$$\leq C\,\mathscr{M}^1_\gamma f(x),$$

where $\mathscr{M}^a_\gamma$ is the truncated Hardy–Littlewood maximal function, for $a>0$, given in (4.101).

The first inequality follows from the fact that Φ is bounded and

$$|x-ry|\geq r|y-x|-(1-r)|x|\geq\frac{r}{2}|y-x|.$$

The second inequality follows from the fact that Φ is a Lebesgue integrable, non-increasing function; hence, it is a good approximation of the identity.

For the second region, we have that $|x-ry|>C|x-y|$. Therefore,

$$\frac{1}{(1-r^2)^{d/2}}\Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right)\leq\frac{1}{|x-y|^d}\left(\frac{|x-y|}{\sqrt{1-r^2}}\right)^d\Phi\left(\frac{C|x-y|}{\sqrt{1-r^2}}\right)$$
$$\leq C\frac{(\sqrt{1-r^2})^d}{|x-y|^{2d}}\leq\frac{C}{|x|^d|x-y|^{2d}},$$

as $\Phi(\frac{1}{2}(\nu-1))\leq S$. Then,

$$\frac{e^{|x|^2}}{(1-r)^{d/2}}\int_{|x-y|>2\frac{C}{|x|r}}\Phi\left(\frac{|x-ry|}{\sqrt{1-r^2}}\right)|f(y)|\,\gamma_d(dy)$$
$$\leq C'\frac{e^{|x|^2}}{|x|^d}\int_{|x-y|>2\frac{C}{|x|r}}\frac{f(y)}{|x-y|^{2d}}\,\gamma_d(dy),$$

but

$$\frac{e^{|x|^2}}{|x|^d}\int_{|x-y|>2\frac{c}{|x|r}}\frac{f(y)}{|x-y|^{2d}}\,\gamma_d(dy)\in L^1(\gamma_d).$$

Therefore, the weak type $(1,1)$ of $\mathscr{M}^2_\Phi$ follows. □

4.3 The Maximal Functions of the Ornstein–Uhlenbeck and Poisson–Hermite Semigroups

The Continuity Properties of the Ornstein–Uhlenbeck Maximal Function

As we already saw in Chapter 2, the *maximal function for the Ornstein–Uhlenbeck semigroup* or simply the *Ornstein–Uhlenbeck maximal function* is defined, for any $f\in L^1(\gamma_d)$, as

$$\begin{aligned} T^*f(x) &= \sup_{t>0}|T_t f(x)| = \sup_{t>0}\left|\int_{\mathbb{R}^d} M_t(x,y)f(y)dy\right| \\ &= \sup_{0<r<1}\frac{1}{\pi^{d/2}(1-r^2)^{d/2}}\left|\int_{\mathbb{R}^d} e^{-\frac{|y-rx|^2}{1-r^2}}f(y)dy\right|. \end{aligned} \tag{4.29}$$

It is also called *Mehler's maximal transform*, but we reserve that for another operator (see 4.35).

As in the classical case, this maximal function is important as it controls the almost everywhere (a.e.) convergence, i.e., if it is $L^p(\gamma_d)$ bounded for some $1<p<\infty$, then there is almost everywhere convergence for $f\in L^p(\gamma_d)$. Let us prove, then, the $L^p(\gamma_d)$-boundedness of T^*.

Theorem 4.19. *For $1<p<\infty$, the Ornstein–Uhlenbeck maximal function T^* is $L^p(\gamma_d)$-bounded, that is, there exists a constant C_p dependent only on p and the dimension d, such that*

$$\|T^*f\|_{p,\gamma}\leq C_p\|f\|_{p,\gamma}. \tag{4.30}$$

Proof. As the Ornstein–Uhlenbeck semigroup is not a convolution semigroup, the boundedness of T^*f in $L^p(\gamma_d)$, $1<p<\infty$ cannot be obtained, as in the classical case, from the boundedness of the Hardy–Littlewood maximal function.

In the case $d=1$, for r and y fixed, the maximum of Mehler's kernel is attained at $x=y/r$. Thus, the centered maximal operator does not seem to be the best average maximal function to be used to get the γ_d-weak type $(1,1)$ property. The boundedness of T^* can be proved directly by using Natanson's lemma (see Appendix Lemma 10.27). To apply Natanson's lemma, B. Muckenhoupt defines a kernel $L(r,x,y)$, which is a modification of Mehler's kernel in the interval $[y,y/r]$, as

$$L(r,x,y)=\begin{cases} P(r,y/r,y) & \text{if } x\in[y,y/r],\\ P(r,x,y) & \text{if } x\notin[y,y/r].\end{cases}$$

Then, Muckenhoupt proves that $L(r,x,y)$ still satisfies (10.103) and (10.104) (see Lemma 3 of [193]).

Nevertheless, in general, for any $d\geq 1$, because the Ornstein–Uhlenbeck semigroup is a Markov semigroup, the boundedness of T^* in $L^p(\gamma_d)$ can be proved using the maximal theorem, as $\{T_t\}$ is a Markov semigroup, which is based on the Hopf–Dunford–Schwartz ergodic theorem (see E. Stein [253, page 73]). □

The case $p=1$ is highly non-trivial and it was an open problem until 1982, when P. Sjögren [247] proved that T^* is of γ_d-weak type (1,1).

Theorem 4.20. *The Ornstein–Uhlenbeck maximal function T^* is of weak type $(1,1)$ with respect to the Gaussian measure, that is, there exists a constant C dependent only on the dimension d, such that for any $f\in L^1(\gamma_d)$*

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : T^* f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda}\|f\|_{1,\gamma}. \tag{4.31}$$

for any $\lambda > 0$.

There are several proofs of this result.

- First of all, the initial proof was obtained by P. Sjögren. His proof does not use pointwise estimates by means of average maximal operators or covering lemmas, such as Besicovitch or Wiener, which are basic standard tools in classical harmonic analysis because, as we have already said, the Gaussian measure is not a doubling measure. His arguments are very original, but too technical to be given in detail here. Nevertheless, we give a sketch of his proof. Consider the region N_R defined as

$$N_R = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x| \leq R \text{ and } |y| \leq R, \text{ or } |y| \geq R/2 \text{ and } |x-y| \leq R/|y|\right\}.$$

For a fixed $x \in \mathbb{R}^d$, the x-section of N_R is given by $N_R^x = \{y \in \mathbb{R}^d : (x,y) \in N_R\}$. Using N_R^x, Sjögren considers a *local part* and a *global part* of the operator T^*, defined for $f \in L^1(\gamma_d)$ as

$$T_L^* f(x) = \sup_{0<r<1} \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} \int_{N_R^x} e^{-\frac{|y-rx|^2}{1-r^2}} f(y)dy, \tag{4.32}$$

and

$$T_G^* f(x) = \sup_{0<r<1} \int_{(N_R^x)^c} \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} e^{-\frac{|y-rx|^2}{1-r^2}} f(y)dy. \tag{4.33}$$

Sjögren proves that both operators, T_L^* and T_G^*, map $L^1(\gamma_d)$ on $L^{1,\infty}(\gamma_d)$. For the local part T_L^*, he uses the fact that the Gaussian measure and the Lebesgue measure are equivalent on N_R^x, according to similar arguments to those given in Section 4.1; thus, classical estimates for the Hardy–Littlewood maximal function can be used.

The argument to bound the global operator T_G^* is very original and deeply interesting. Unfortunately, it is too technical; therefore, trying to give full details would be outside the scope of this book (for more information, see [247]). Consider *Mehler's maximal kernel* defined as

$$\mathscr{K}^*(x,y) = \sup_{t>0} M_t(x,y) = \sup_{0<r<1} \frac{1}{\pi^{\frac{d}{2}}(1-r^2)^{\frac{d}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}}, \tag{4.34}$$

and the operator defined by the kernel $\mathscr{K}^*(x,y)$ that we call *Mehler's maximal transform*,

$$\mathfrak{T}^* f(x) = \int_{\mathbb{R}^d} \mathscr{K}^*(x,y) f(y)dy. \tag{4.35}$$

What is needed is to estimate the Gaussian measure of the level set is

$$\Big\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\Big\}.$$

More precisely, we want to get, for $f \in L^1(\gamma_d),\ f \geq 0$, the inequality

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} f(x)\gamma_d(dx).$$

It would be enough to prove that

$$\int_{\{x\in\mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}} \mathscr{K}^*(x,y)\gamma_d(dx) \leq Ce^{-|y|^2}, \tag{4.36}$$

because then, using Fubini's theorem, we would get

$$\begin{aligned}
\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\Big\}\Big) &\leq \int_{\{x\in\mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}} \frac{1}{\lambda} \int \mathscr{K}^*(x,y) f(y) dy \gamma_d(dx) \\
&= \frac{1}{\lambda} \int f(y) \int_{\{x\in\mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}} \mathscr{K}^*(x,y)\gamma_d(dx) dy \\
&\leq \frac{C}{\lambda} \int f(y)\gamma_d(dy).
\end{aligned}$$

Unfortunately, the inequality (4.36) is not true; thus, the argument fails. Nevertheless, the argument can be "rescued" if a subset E of $\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}$ can be constructed such that the inequality

$$\int_E \mathscr{K}^*(x,y)\gamma_d(dx) \leq Ce^{-|y|^2}, \tag{4.37}$$

holds. Hence, E must be small, such that

$$\gamma_d(E) \leq C\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\Big\}\Big),$$

which implies that E cannot be too small. Then,

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\Big\}\Big) \leq C'\gamma_d(E);$$

thus, we can repeat the argument above. To construct the set E, $\mathbb{R}^d$ is divided into cubes Q_i centered at x_i, $i = 1, 2, \ldots$ such that their diameters verify

$$cm(x_i) = c\Big(1 \wedge \frac{1}{|x_i|}\Big) \leq \text{diam } (Q_i) \leq C\Big(1 \wedge \frac{1}{|x_i|}\Big) = Cm(x_i).$$

The set $\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}$ is the union of cubes of this type. The problem is that there may be cubes that are very close together. Hence, to construct E, an inductive selection argument is needed to select the cubes so that there are not too many cubes too close to each other, and so that (4.37) holds. For each cube

Q_j, we need to consider its *forbidden region* F_j, which is defined to be the union of cubes $Q_i, i > j$, with non-empty intersection with the set $Q_j + K_j$, where K_j is the cone and

$$\left\{x \in \mathbb{R}^d : \alpha(x,y) \leq \pi/4 \quad \text{for some} \quad y \in Q_j\right\},$$

where $\alpha(x,y)$ is the angle between x and y. It can be proved that

$$\gamma_d(F_j) \leq C\gamma_d(Q_j), \tag{4.38}$$

as the Gaussian density falls exponentially.

For the first step of the construction of E we select Q_1 if and only if it intersects the set $\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}$. In the i-th step, Q_i is chosen if it intersects the set $\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}$ and it is not in the forbidden region F_j for any cube Q_j already selected. Then, by construction, the set $\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\}$ is contained in the union of the selected cubes Q_j and their corresponding forbidden regions F_j; hence, using (4.38), we get

$$\begin{aligned}\gamma_d\left(\left\{x \in \mathbb{R}^d : \mathfrak{T}^* f(x) > \lambda\right\}\right) &\leq \sum_j \gamma_d(Q_j) + \sum_j \gamma_d(F_j) \\ &\leq C\sum_j \gamma_d(Q_j) = C\gamma_d(E).\end{aligned}$$

Moreover, it can be proved that (4.36) holds, because basically, taking any straight line in the direction of $y \in \mathbb{R}^d$, we get a selected cube Q_j and the rest of the line is in the forbidden region F_j (for more details see [247]).

The scheme of Sjögren's proof was later used by several authors to obtain positive results for other operators associated with the Ornstein–Uhlenbeck semigroup, such as the Riesz transforms (see [77, 86]) or the Littlewood–Paley square function (see [87]). In all these cases, the proof follows his arguments very closely. Thus, we can conclude that there is a close connection between all these operators and Mehler's maximal transform.

- Another proof was given by T. Menárguez, S. Pérez, and F. Soria (see [184, 185] and [220]). They obtained a more geometric proof as an alternative to Sjögren's, which in some sense simplifies it, although it is still very involved. The main idea is to use polar coordinates, which turns out to be very natural, as the Gaussian measure is invariant by rotations (around the origin), obtaining a Vitali's covering type for conic regions.

  Again, similar to Sjögren's proof, they split T^* into a *local part* and a *global part*, but the regions considered in this case are different and somehow simpler. Given $x \in \mathbb{R}^d$, the *local part* of the operator T^* is its restriction to the admissible ball

$$B_h(x) = B(x, C_d\, m(x)) = \{y \in \mathbb{R}^d : |y - x| < C_d\, m(x)\},$$

and we have seen that the Gaussian density is essentially constant on admissible balls; see (4.102). The *global part* of the operator T^* is its restriction to the complement of $B_h(x)$. Thus,

$$\begin{aligned} T^* f(x) &= C_d \sup_{0<t<1} \left(\int\limits_{|x-y|<d\, m(x)} M_t(x,y)|f(y)|dy + \int\limits_{|x-y|\geq d\, m(x)} M_t(x,y)|f(y)|dy \right) \\ &\leq T_L^* f(x) + T_G^* f(x), \end{aligned}$$

where $T_L^* f(x) = T^*(f\chi_{B_h(\cdot)})(x)$ is the *local part* and $T_G^* f(x) = T^*(f\chi_{B_h^c(\cdot)})(x)$ is the *global part* of T^*.

This scheme is used in general for operators associated with the Ornstein–Uhlenbeck operator. We split them into a *local part* and a *global part.* Because the Gaussian measure and the Lebesgue measure are equivalent on the local region, the local part is controlled by the corresponding classical operators. The difficult problem is to control the global part of those operators, even though they are positive.

Similar to Sjögren's proof, the local part T_L^* is weak $(1,1)$ as it is bounded by the (classical) Hardy–Littlewood maximal function.

Theorem 4.21. *T_L^* is of weak type $(1,1)$ with respect to the Gaussian measure γ_d,*

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : T_L^* f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda} \int_{\mathbb{R}^d} |f(y)|\gamma_d(dy)$$

Proof. As mentioned above, it is enough to bound T_L^* by the (classical) Hardy–Littlewood maximal function

$$T_L^* f(x) \leq C_d M f(x).$$

Let us denote K_t the reparametrized Mehler's kernel, with $t = 1 - r^2$,

$$K_t(x,y) = \frac{C_d}{t^{d/2}} e^{-\frac{|y-\sqrt{1-t}x|^2}{t}}. \tag{4.39}$$

Observe that if $|x-y| \leq \frac{d}{|x|}$, then $|x-y||x| \leq d$; therefore,

$$\begin{aligned} |y-\sqrt{1-t}x|^2 &\geq \left(|y-x| - |x|(1-\sqrt{1-t})\right)^2 \\ &\geq |y-x|^2 - 2|x||y-x| \frac{t}{1+\sqrt{1-t}} \geq |y-x|^2 - 2dt. \end{aligned}$$

Hence,

$$T_L^* f(x) = C_d \sup_{0<t<1} \int K_t(x,y)|f(y)|dy$$
$$\leq C_d \sup_{0<t<1} \frac{e^{2d}}{t^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{t}} |f(y)|dy \leq C_d M f(x).$$

To prove the weak type $(1,1)$ inequality for T^*, we use Lemma 4.3, considering a family $\mathscr{F} = \{B(0,\alpha), \tilde{B}_j^k\}_{k,j}$ of admissible balls, with bounded overlaps, that covers $\mathbb{R}^d$, and $\hat{B} := C_d B$, $B \in \mathscr{F}$. Therefore,

$$T_L^* f(x) \leq T_L^*(f\chi_{\hat{B}(0,\alpha)})(x) + \sum_{k,j} T_L^*(f\chi_{\hat{B}_j^k})(x) = \sum_{B\in\mathscr{F}} T_L^*(f\chi_{\hat{B}})(x)\chi_B(x).$$

Then, using the argument given above, and using the weak type inequality for the non-centered Hardy–Littlewood maximal function $\tilde{M}$,

$$\gamma_d\Big(\Big\{x\in\mathbb{R}^d : T_L^* f(x) > \lambda\Big\}\Big) \leq \sum_{B\in\mathscr{F}, B\subset S_k} \gamma_d\Big(\Big\{x\in B : \tilde{M}(f\chi_B)(x) > \lambda\Big\}\Big)$$
$$\sim \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k}\Big|\Big\{x\in B : \tilde{M}(f\chi_B)(x) > \lambda\Big\}\Big|$$
$$\leq C \sum_{B\in\mathscr{F}, B\subset S_k} \frac{e^{-\alpha_k}}{\lambda}\int_B |f(x)|dx$$
$$\sim \sum_{B\in\mathscr{F}, B\subset S_k} \int_B |f(x)|\gamma_d(dx) \sim \int_{\mathbb{R}^d} |f(x)|\gamma_d(dx),$$

where we have used the fact that the family $\mathscr{F}$ has the finite overlapping property.

Observe that, with an analogous argument, we can prove, without using interpolation theory, that T_L^* is a bounded operator in $L^p(\gamma_d)$, as for $f \in L^p(\gamma_d)$

$$\|T_L^* f\|_{p,\gamma} \sim \sum_{k=0}^{\infty}\sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k}\int_B |T_L^* f(x)|^p dx \leq \sum_{k=0}^{\infty} e^{-\alpha_k}\int_B |\tilde{M} f(x)|^p dx$$
$$\leq C\sum_{k=0}^{\infty}\sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k}\int_B |f(x)|^p dx \sim \sum_{k=0}^{\infty}\sum_{B\in\mathscr{F}, B\subset S_k}\int_B |f(x)|^p \gamma_d(dx)$$
$$\sim \|f\|_{p,\gamma}. \qquad \square$$

Now, we want to prove the boundedness of the global part T_G^*. Following P. Sjögren's idea, [247] we need to estimate $\mathscr{K}^*$ the supremum of Mehler's kernel (4.34). S. Pérez, in her doctoral dissertation (see [220] or [185]), gets an estimation of the kernel that is very interesting and allows a unified treatment of the global part of several operators related to the Ornstein–Uhlenbeck operator, as we are going to see later. The scheme of her proof is based on the following results:

– Mehler's maximal kernel (4.34) is equivalent to the Gaussian maximal kernel $\overline{\mathscr{K}}$, (4.40), proved in Proposition 4.23.
– Prove that the maximal Gaussian operator $\overline{T}$ associated with $\overline{\mathscr{K}}$, is of weak type $(1,1)$ with respect to the Gaussian measure γ_d (see Theorem 4.24). Its proof is long and very technical and we additionally need Lemmas 4.25, 4.26, and 4.27.

Definition 4.22. *Let us define the Gaussian maximal kernel $\overline{\mathscr{K}}$ as*

$$\overline{\mathscr{K}}(x,y) = \begin{cases} e^{-|y|^2}, & \text{if } \langle x,y\rangle \leq 0 \\ \left(\frac{|x+y|}{|x-y|}\right)^{d/2} e^{-\frac{|y|^2-|x|^2}{2}} e^{-\frac{|x-y||x+y|}{2}}, & \text{if } \langle x,y\rangle > 0. \end{cases} \tag{4.40}$$

Then we have

Proposition 4.23. *If $|x-y| \geq C_d\, m(x)$, then we have*

$$\mathscr{K}^*(x,y) \sim \overline{\mathscr{K}}(x,y).$$

It is easy to see that if $y \in B_h(x)$, i.e., $|x-y| \leq C_d\, m(x)$, we have

$$\mathscr{K}^*(x,y) \sim \frac{1}{|x-y|^d},$$

which corresponds to the action of the (classical) Hardy–Littlewood maximal function on one Dirac delta. This shows the close connection that exists between the operators T^* and $\tilde{\mathscr{M}}$ in the local case.

Proof. To simplify the notation, in what follows we denote

$$a = a(x,y) := |x|^2 + |y|^2,\; b = b(x,y) := 2\langle x,y\rangle,$$

and

$$u(t) = u(t;x,y) := \frac{|y-\sqrt{1-t}x|^2}{t} = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |x|^2.$$

This operator was first introduced by W. Urbina in [278] and later used by S. Pérez extensively.

Observe that, if $d \geq 2$ and $|x-y| \geq C_d\, m(x)$, then $a = |x|^2 + |y|^2 \geq \frac{d}{2}$.
We need to study the function

$$\varphi(t) = \frac{1}{t^{d/2}} e^{-u(t)}, \tag{4.41}$$

for fixed x,y.
Consider its derivative,

$$\varphi'(t) = -\left[\frac{d}{2t} + u'(t)\right] \frac{e^{-u(t)}}{t^{d/2}} = \frac{e^{-u(t)}}{t^{d/2}} \frac{(2a-dt)\sqrt{1-t}-(2-t)b}{2t^2\sqrt{1-t}}.$$

As the factor $2a - dt$ is positive for all $0 < t < 1$, and as $d \geq 2$ and $|x-y| \geq C_d\, m(x)$, we have

$$2a = 2(|x|^2 + |y|^2) \geq |x-y|^2 \geq d^2 m(x)^2 \geq d^2\left(1 \wedge \frac{1}{a}\right);$$

therefore $a \geq d/2$.

- Case #1: $b \leq 0$.
  If $b \leq 0$, then φ' is positive and the function φ is strictly increasing. Therefore,

$$\sup_{0\leq t\leq 1} K_t(x,y) = \sup_{0\leq t\leq 1} \frac{1}{t^{d/2}} e^{-u(t)} \leq C_d \varphi(1) = K_1(x,y) = C_d e^{-a+|x|^2} = C_d e^{-|y|^2}.$$

- Case #2: $b > 0$.
  If $b > 0$, then $|x+y| > |x|$. Moreover, $|x+y||x-y| \geq d$ if $y \in B_h^c(x)$, as

$$|x+y|^2 = |x|^2 + 2\langle x,y\rangle + |y|^2 \geq |x|^2 - 2\langle x,y\rangle + |y|^2 = |x-y|^2.$$

  Thus, $|x+y| \geq |x-y|$ and

$$|x+y|^2 = |x|^2 + 2\langle x,y\rangle + |y|^2 \geq |x|^2,$$

  so $|x+y| > |x|$.

  Therefore, if $|x| \leq 1$

$$|x-y| \geq d\left(1 \wedge \frac{1}{|x|}\right) = d,$$

$$|x-y||x+y| \geq |x-y||x-y| \geq d^2 \geq d.$$

  Hence, $|x-y||x+y| \geq d$. If $|x| > 1$

$$|x-y| \geq d\left(1 \wedge \frac{1}{|x|}\right) = \frac{d}{|x|}, \text{ so } |x||x-y| \geq d$$

  which implies $|x-y||x+y| \geq |x-y||x| \geq d$.

  Thus, in any case, if $b > 0$

$$\sqrt{a^2-b^2} = |x+y||x-y| \geq d. \tag{4.42}$$

  Therefore, $a \geq d$ and this information is useful to find the critical points of φ, as $\varphi'(t) = 0$ if and only if $u'(t) + \frac{d}{2t} = 0$, i.e.,

$$-\frac{2a\sqrt{1-t} - (2-t)b}{2t^2\sqrt{1-t}} + \frac{d}{2t} = 0$$

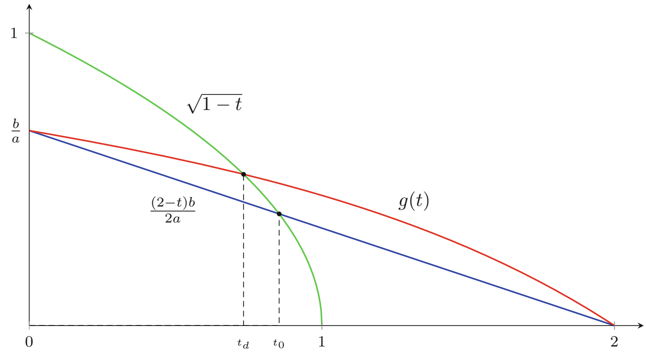

Fig. 4.3.

or equivalently,

$$\sqrt{1-t} = g(t), \text{ where } g(t) = \frac{(2-t)b}{2a-dt}. \tag{4.43}$$

Observe that the denominator of $g(t)$ is always positive. Moreover, it is easy to see that g is concave and decreasing on $[0,1]$ with x-intercept at $x = 2$ and y-intercept at $y = \frac{b}{a} < 1$. Because $\sqrt{1-t}$ is also concave and decreasing on $[0,1]$ with x-intercept at $x = 1$ and y-intercept $y = 1$, there is a unique solution to equation (4.43), which corresponds to the intersection of the two curves, that is denoted as t_d; see Figure 4.3. As φ is increasing up to t_d and then decreasing, we conclude that

$$\sup_{0\leq t\leq 1} K_t(x,y) = K_{t_d}(x,y).$$

Unfortunately, t_d is a solution of a third-order equation, it has a very complicated explicit expression. Hence, we are going to approximate its value by the positive solution of the following quadratic equation

$$\sqrt{1-t} = \frac{(2-t)b}{2a}.$$

This corresponds to finding the point where the function $e^{-u(t)}$ attains its supremum. As the graph of $(2-t)b/(2a)$ is a straight line passing through the points $(0, b/a)$ and $(2,0)$, which coincides with the intersection points of g with the axes, and g is concave, we must have $t_d \leq t_0$. Moreover, t_0 is actually a very good approximation of t_d, as we prove that they are equivalent in the region $|x-y| \geq 2d\, m(x)$. To prove that indeed t_d and t_0 are equivalent, let us look at the tangent line to the graph of the function g at the point $t = 0$, that is,

$$z(t) = -g'(0)t + g(0) = (a-d)b/(2a^2)t + b/a = -pt + q,$$

where $p = -g'(0) = (a-d)b/(2a^2)$ and $q = g(0) = b/a$. The concavity of the graph implies that the positive solution of the equation $\sqrt{1-t} = z(t)$, denoted by t_d', satisfies $t_d' \le t_d \le t_0$, as it can be seen in Figure 4.4.

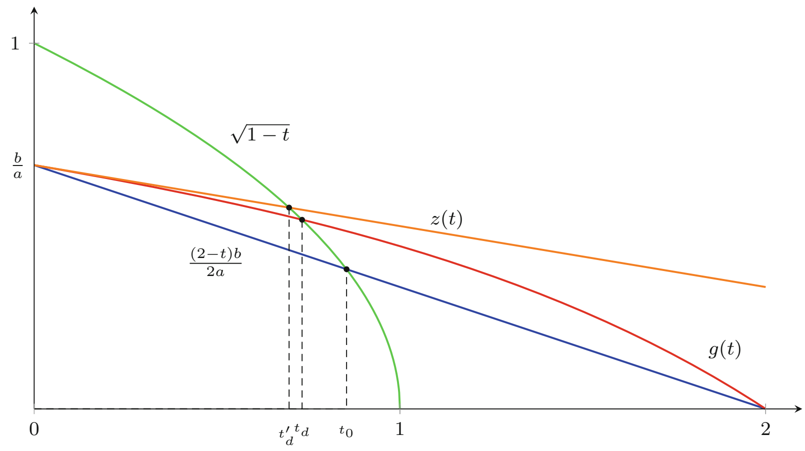

Fig. 4.4.

Moreover, from (4.42), it follows that

$$t_d' = 2\frac{1-q^2}{(1-2pq)+\sqrt{(1-2pq)^2+4p^2(1-q^2)}} \ge \frac{1-q^2}{\sqrt{1-4pq+4p^2}}$$
$$= \frac{\frac{a^2-b^2}{a^2}}{\sqrt{\frac{a^2-b^2}{a^2}+\frac{d^2b^2}{a^4}}} = \frac{a^2-b^2}{\sqrt{(a^2-b^2)a^2+d^2a^2}} \ge \frac{\sqrt{a^2-b^2}}{2a} \ge \frac{1}{4}t_0.$$

This implies that $\varphi(t_d) \sim \varphi(t_0)$ as

$$\varphi(t_0) \le \sup_{0<t<1} \varphi(t) = \varphi(t_d) = \frac{e^{-u(t_d)}}{t_d^{d/2}} \le \frac{e^{-u(t_0)}}{t_d^{d/2}} \le 2^d \frac{e^{-u(t_0)}}{t_0^{d/2}} = 2^d\varphi(t_0). \tag{4.44}$$

Hence,

$$K_{t_0}(x,y) \le K_{t_d}(x,y) = \frac{1}{t_d^{d/2}} e^{-u(t_d)}$$
$$\le \frac{1}{t_d^{d/2}} e^{-u(t_1)} \le C_d \frac{1}{t_0^{d/2}} e^{-u(t_0)} = C_d K_{t_0}(x,y),$$

It is easy to get the explicit expression of t_0, as it is the positive solution of the quadratic equation

$$b^2t^2 + 4(a^2-b^2)t - 4(a^2-b^2) = 0,$$

and then,

$$t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}. \tag{4.45}$$

Therefore, as

$$u(t_0) = \frac{\sqrt{a^2-b^2}}{2} + \frac{a}{2} - |x|^2 = \frac{|y|^2-|x|^2}{2} + \frac{\sqrt{a^2-b^2}}{2}$$

and

$$t_0 \sim \frac{\sqrt{a^2-b^2}}{a} \sim \frac{\sqrt{a-b}}{\sqrt{a+b}} = \frac{|x-y|}{|x+y|},$$

we conclude that

$$K_{t_0}(x,y) \sim \left(\frac{|x+y|}{|x-y|}\right)^{d/2} e^{-\frac{|x+y||x-y|}{2}} e^{-\frac{|y|^2-|x|^2}{2}}. \qquad \square$$

Observe that in case #1, $b = 2\langle x,y\rangle \leq 0$, if $a = |x|^2+|y|^2 \leq d/2$, then

$$K_t(x,y) \leq h(2a/d) \sim \frac{e^{|x|^2}}{a^{d/2}} \sim \frac{1}{|x|^d+|y|^d} e^{|x|^2}$$

As the linear operator $Tf(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x|^d+|y|^d} dy$ is of weak type $(1,1)$ with respect to the Lebesgue measure (see [251]), the same argument used for the Hardy–Littlewood maximal operator above shows that the (strong) local part

$$Sf(x) = \int\limits_{\substack{|x-y|\leq d\,m(x)\\ \langle x,y\rangle \leq 0}} \mathscr{K}^*(x,y)|f(y)|dy$$

is also of weak type $(1,1)$ with respect to the Gaussian measure.

As we have already mentioned, the fundamental observation by P. Sjögren in [247], to study the global part, is to consider Mehler's maximal transform (4.35).

In Proposition 4.23, we have shown that $\mathscr{K}^*(x,y) \sim \overline{\mathscr{K}}(x,y)$ if $|x-y| > d\,m(x)$. Thus, it is enough to prove the weak type $(1,1)$, with respect to the Gaussian measure, of the operator associated with this latter kernel.

Theorem 4.24. *The maximal Gaussian operator defined as*

$$\overline{T}f(x) = \int\limits_{\mathbb{R}^d} \overline{\mathscr{K}}(x,y) f(y)dy \tag{4.46}$$

is of weak type $(1,1)$ *with respect to the Gaussian measure* γ_d.

Proof. The idea is to use geometric arguments similar to Vitali's covering lemma. The problem here is that the Gaussian measure is not doubling; therefore, some replacement for the notion of the "double of a set" needs to be found. Observe that the Gaussian measure can be written in polar coordinates as

$$\gamma_d(dx) = \frac{1}{\pi^{d/2}} e^{-|x|^2} dx = \frac{1}{\pi^{d/2}} e^{-r^2} r^{n-1} dr d\sigma,$$

where $d\sigma$ denotes the surface measure on the unit sphere, which is indeed a doubling measure. Therefore, the dilations are done with respect to the angular variable.

Without loss of generality we assume that $f \geq 0$. First, it is easy to see that given $x \in \mathbb{R}^d$ considering $\overline{T}$ restricted to the region $\{y \in \mathbb{R}^d : \langle x, y\rangle \leq 0\}$ is strong $(1,1)$ continuous because

$$\begin{aligned}\int_{\mathbb{R}^d} \int_{\langle x,y\rangle<0} \overline{\mathscr{K}}(x,y) f(y) dy e^{-|x|^2} dx &= \int_{\mathbb{R}^d} e^{-|y|^2} f(y) dy \int_{\langle x,y\rangle<0} e^{-|x|^2} dx dy \\ &\leq C \int_{\mathbb{R}^d} e^{-|y|^2} f(y) dy.\end{aligned}$$

Hence, we assume $\langle x, y\rangle > 0$ for the rest of the proof.

The local part $\overline{T}_L$ is of strong type $(1,1)$, because, as we have mentioned before, on admissible balls, the Gaussian density is essentially constant. Therefore, if $y \in \mathbb{R}^d$ such that $|x-y| \leq C_d m(x)$, then $e^{-|y|^2} \sim e^{-|x|^2}$; therefore,

$$\begin{aligned}\overline{T}_L f(x) &= \int_{\mathbb{R}^d} \int_{|x-y|\leq d m(x)} \overline{\mathscr{K}}(x,y) f(y) dy e^{-|x|^2} dx \\ &\leq C_d \int_{\mathbb{R}^d} \int_{|x-y|\leq d m(x)} \overline{\mathscr{K}}(x,y) dx f(y) e^{-|y|^2} dy \\ &= C_d \int_{\mathbb{R}^d} \int_{|x-y|\leq d m(x)} \left(\frac{|x+y|}{|x-y|}\right)^{d/2} e^{-\frac{|y|^2-|x|^2}{2}} e^{-\frac{|x-y||x+y|}{2}} dx f(y) e^{-|y|^2} dy\end{aligned}$$

Thus, it is enough to show that the inner integral is bounded uniformly on y. Now, the exponential part of $\overline{\mathscr{K}}(x,y)$ is controlled by $e^{|x|^2+|y|^2}$, as

$$||y|^2 - |x|^2| \leq |x-y||x+y|,$$

which in this case is bounded by a constant. On the other hand,

$$|x+y| \leq |x-y| + 2|y| \leq C_d + +2|y| \leq C_d(1 \vee |y|),$$

so the strong type follows from

$$\int_{|x-y|\le d\,m(x)} \overline{\mathscr{K}}(x,y)dx \le C_d \int_{|x-y|\le d\,m(y)} \left(\frac{|x+y|}{|x-y|}\right)^{d/2} dx$$
$$= C_d(1\vee|y|^{d/2}) \int_{|z|\le dm(y)} \frac{1}{|z|^{d/2}}dz$$
$$= C_d(1\wedge|y|^{d/2})(1\vee|y|)^{d/2} = C_d.$$

The argument above shows in general that if $\overline{\mathscr{K}}(x,y)e^{|y|^2-|x|^2}$ is uniformly integrable in y with respect to the Lebesgue measure over a given region, then Fubini's theorem implies the strong type $(1,1)$ with respect to γ_d of $\overline{T}$ when restricted to that region.

Now, we prove the weak $(1,1)$ continuity of the global part, $\overline{T}_G$,

$$\overline{T}_G f(x) = \int_{|x-y|\ge d\,m(x)} \overline{\mathscr{K}}(x,y)f(y)dy, \tag{4.47}$$

that is,

$$\gamma_d\left(\left\{x\in\mathbb{R}^d : \overline{T}_G f(x) > \lambda\right\}\right) \le \frac{C}{\lambda}\int_{\mathbb{R}^d}|f(y)|\gamma_d(dy).$$

To prove this, we are going to consider several regions, which are somehow generalizations to $\mathbb{R}^d$ of the regions considered by B. Muckenhoupt in the case $d=1$ in [194] (see Figure 4.5). We see that in some of those regions the operator is actually of strong type $(1,1)$ with respect to the Gaussian measure γ_d.

– First, given $x\in\mathbb{R}^d$ with $|x|<10$, let us consider the region $\{y\notin B_h(x) : \langle x,y\rangle > 0\}$. There, the operator is of strong type $(1,1)$, because if $|x-y|\ge c$, it follows that either $C\ge|x+y|\ge|x-y|\ge c$ or $|x+y|\sim|x-y|\sim|y|$. Thus, we get
$$\overline{\mathscr{K}}(x,y) \le Ce^{-|y|^2-|x|^2} \le Ce^{-|y|^2}.$$
– Let us consider, for $x\in\mathbb{R}^d$ with $|x|\ge 10$, the region
$$\{y\notin B_h(x) : \langle x,y\rangle > 0, |x|\ge|y|\}.$$
Then we have
$$\overline{\mathscr{K}}(x,y) \le C_d|x|^d e^{-\frac{|x-y||x|}{2}} e^{|x|^2-|y|^2}.$$
Hence, if $|x-y|>1$,
$$\overline{\mathscr{K}}(x,y)e^{|y|^2-|x|^2} \le C_d|x|^d e^{-c|x|},$$
which is an integrable function on $\mathbb{R}^d$. Thus,

$$\int_{\mathbb{R}^d} \int_{\substack{\langle x,y\rangle>0\\ |x-y|>1,|x|>10,|x|>|y|}} \overline{\mathscr{K}}(x,y)f(y)dye^{-|x|^2}dx$$

$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |x|^d e^{-c|x|} dx f(y) e^{-|y|^2} dy \leq C_d ||f||_{1,\gamma}$$

Now, if $|x-y| \leq 1$, then $|x| \sim |y|$, so taking the change of variables $(x-y)|y|=v$, we get

$$\int_{\mathbb{R}^d} \int_{\substack{\langle x,y\rangle>0 \\ |x-y|\leq 1, |x|>10, |x|>|y|}} \overline{\mathscr{K}}(x,y) f(y) dy e^{-|x|^2} dx$$
$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y|^d e^{-\frac{|x-y||x|}{2}} dx f(y) e^{-|y|^2} dy$$
$$\leq C_d \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-\frac{|v|}{2}} dv f(y) e^{-|y|^2} dy \leq C_d ||f||_{1,\gamma}.$$

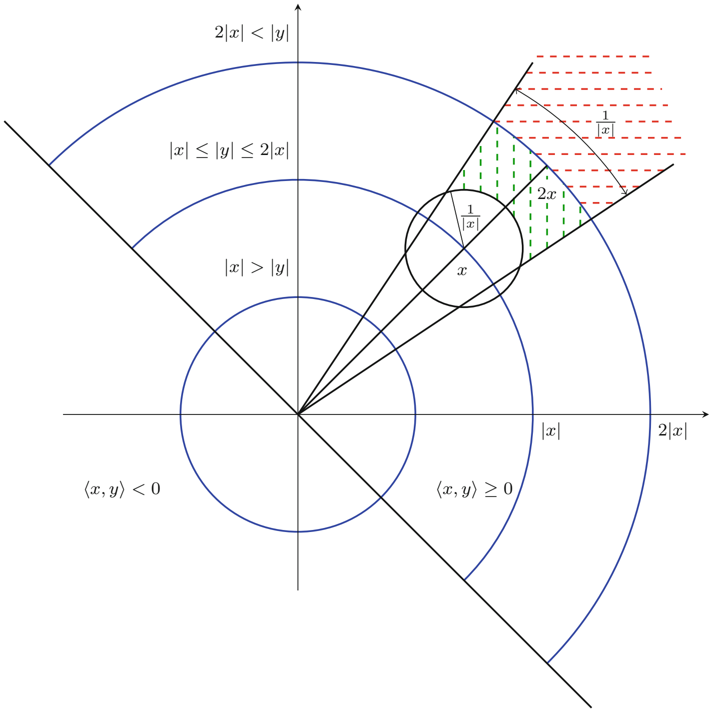

Fig. 4.5.

– For $x \in \mathbb{R}^d$ with $|x| \geq 10$, it remains to be considered the region

$$\left\{ y \notin B_h(x) : \langle x,y\rangle > 0, |x| \leq |y| \right\}.$$

This is the most problematic region, but even here there are some sub-regions that give the strong type $(1,1)$ for the corresponding operator. Here, the fact that we are working with the kernel $\overline{\mathscr{K}}$ becomes crucial. Let $\alpha := \alpha(x,y) = \sin\angle(x,y)$, where $\angle(x,y) \in [0,\pi]$ denotes the shortest angle between the vectors x and y (if $\langle x,y\rangle > 0$, we have $\angle(x,y) \in [0,\pi/2]$). Using the identity

$$\begin{aligned}|x-y|^2|x+y|^2 &= (|x|^2+|y|^2-2\langle x,y\rangle)(|x|^2+|y|^2+2\langle x,y\rangle)\\ &= (|x|^2+|y|^2)^2-4\langle x,y\rangle^2 = (|y|^2-|x|^2)^2+4|x|^2|y|^2\alpha^2(x,y)\\ &= (|x|^2+|y|^2)^2-4\langle x,y\rangle^2 = 4(A^2+|x|^2|y|^2\alpha^2(x,y)),\end{aligned}$$

where

$$A = A(x,y) := (|y|^2-|x|^2)/2. \tag{4.48}$$

In this region, we obtain

$$\begin{aligned}\overline{\mathscr{K}}(x,y) &= \left(\frac{|x+y|}{|x-y|}\right)^{d/2} e^{A-\sqrt{A^2+\alpha^2(|x|^2+|y|^2)}} e^{|x|^2-|y|^2}\\ &= \frac{|x+y|^d}{(|x-y||x+y|)^{d/2}} \exp\Big(-\frac{\alpha^2|x|^2|y|^2}{A+\sqrt{A^2+\alpha^2(|x|^2+|y|^2)}}\Big) e^{|x|^2-|y|^2}\\ &\leq 2^d\Big(\frac{|y|^2}{|x-y||x+y|}\Big)^{d/2}\exp\Big(-\frac{\alpha^2|x|^2|y|^2}{|x-y|x+y|}\Big) e^{|x|^2-|y|^2}.\end{aligned} \tag{4.49}$$

This implies that we need to divide this region into sub-regions, depending on the behavior of $\alpha(x,y)$ and the size of $|x|$ and $|y|$.

Observe that if we consider, for $\varepsilon > 0$, the region

$$\mathtt{B}_\varepsilon = \Big\{(x,y) \in \mathbb{R}^{2d} : \langle x,y\rangle > 0, |x| \leq |y|, |y-x| \geq C_d(1\wedge 1/|x|), \alpha(x,y) > \varepsilon\Big\},$$

and $\overline{T}_\varepsilon$ the operator associated with the kernel $\overline{\mathscr{K}}_\varepsilon = \overline{\mathscr{K}}\chi_{\mathtt{B}_\varepsilon}$, then $\overline{T}_\varepsilon$ is of strong type $(1,1)$, because for $(x,y) \in \mathtt{B}_\varepsilon$ we have $1/2 \leq |y|^2(|x-y||x+y|) < C(1+|x|)^2$; hence,

$$\int \overline{\mathscr{K}}_\varepsilon e^{|y|^2-|x|^2} \leq C,$$

uniformly in y.

A close look at the previous argument shows that what we really need is $(1+|x|)^d e^{-(\alpha^2|x|^2)/2} \in L^1(\mathbb{R}^d)$. This is the case, for example, if $\alpha(x,y) \geq (4(d+1)\ln(1+x))^{1/2}/|x|$. With this remark as a motivation, observe that given a positive function $\alpha_0 : [0,\infty) \to [0,1]$, the set

$$\Gamma_{\alpha_0}(x) = \{y : \langle x,y\rangle > 0, |x| \leq |y|, \alpha(x,y) \leq \alpha_0(|x|)\},$$

represents a (truncated) light cone, centered at x, with angular opening $\alpha_0(|x|)$. Assume that α_0 is a non-increasing function and define $\Gamma^*_{\alpha_0}(x) =$

$\Gamma_{3\alpha_0}(x)$ if $3\alpha_0(|x|) \leq 1$ and $\Gamma^*_{\alpha_0}(x) = \{y : \langle x, y\rangle > 0, |x| \leq |y|\}$ otherwise; hence, $\Gamma^*_{\alpha_0}(x)$ is essentially a dilation of $\Gamma_{\alpha_0}(x)$ in the angular sense.
Then, the collection of cones $\{\Gamma_{\alpha_0}(x)\}$ satisfies the following properties:

- If $\Gamma_{\alpha_0}(x) \cap \Gamma_{\alpha_0}(x') \neq \emptyset$ and $|x| < |x'|$ we have $\Gamma_{\alpha_0}(x') \subset \Gamma^*_{\alpha_0}(x)$.
- There exists a constant C_d dependent only on dimension such that

$$\gamma_d(\Gamma^*_{\alpha_0}(x)) \leq C_d \gamma_d(\Gamma_{\alpha_0}(x)),$$

as

$$\gamma_d(\Gamma_{\alpha_0}(x)) \leq C_d \int_{|x|}^{\infty} e^{-r^2} r^{d-1} (\alpha_0(r))^{d-1} dr \sim C_d (|x|\alpha_0(|x|))^{d-1} e^{-|x|^2}.$$

Thus, $\Gamma^*_{\alpha_0}(x)$ is precisely the needed notion of "double of a set" for $\Gamma_{\alpha_0}(x)$; therefore, we have finally obtained a "doubling condition" for the Gaussian measure γ_d on the sets $\{\Gamma_{\alpha_0}(x)\}$. Therefore, we can apply the usual Vitali's covering lemma and prove that the operator

$$\mathtt{T}_{\alpha_0} f(x) = \frac{1}{\gamma_d(\Gamma_{\alpha_0}(x))} \int_{\Gamma_{\alpha_0}(x)} |f(y)| e^{-|y|^2} dy, \tag{4.50}$$

is of weak type $(1,1)$ with respect to γ_d, because any compact subset of $\{x : \mathtt{T}_{\alpha_0} f(x) > \lambda\}$ can be covered by a finite union of cones $\{\Gamma^*_{\alpha_0}(x_i)\}$ that are dilations of a sub-collection of disjoint cones $\{\Gamma_{\alpha_0}(x_i)\}_{i \in \mathscr{J}}$ and, therefore,

$$\begin{aligned}\gamma_d(K) &\leq \sum_{i \in \mathscr{J}} \gamma_d(\Gamma^*_{\alpha_0}(x_i)) = C_d \sum_{i \in \mathscr{J}} \gamma_d(\Gamma_{\alpha_0}(x_i)) \\ &\leq \frac{C_d}{\lambda} \sum_{i \in \mathscr{J}} \int_{\Gamma^*_{\alpha_0}(x_i)} f(y) e^{-|y|^2} dy \leq \frac{C_d}{\lambda} \int_{\mathbb{R}^d} f(y) e^{-|y|^2} dy.\end{aligned}$$

For $|x| \geq 10$, $k = 1, 2$, $l = 1, 2, \cdots, [|x|^k] + 1$ and $\alpha_0(x) = \frac{l}{|x|^k} \wedge 1$, we write $\Gamma^k_l(x)$

$$\Gamma^k_l(x) = \left\{ y : \langle x, y\rangle > 0, |x| \leq |y|, \alpha(x, y) \leq l/|x|^k \right\} \tag{4.51}$$

and $\mathtt{T}^k_l$

$$\mathtt{T}^k_l f(x) = \frac{1}{\gamma_d(\Gamma^k_l(x))} \int_{\Gamma^k_l(x)} |f(y)| e^{-|y|^2} dy, \tag{4.52}$$

instead of $\Gamma_{\alpha_0}(x)$ and $\mathtt{T}_{\alpha_0}$ respectively (see Figure 4.6).
We know that each $\mathtt{T}^k_l$ is of weak type $(1,1)$ with respect to the Gaussian measure, with a constant independent of l. The idea now is to majorize our operator $\overline{T}$, when restricted to the remaining region under study, by a linear combination (with rapidly decreasing coefficients) of those operators $\mathtt{T}^k_l$. To that end, let us study separately the two operators $\overline{T}_1$ and $\overline{T}_2$ defined by the restrictions of $\mathscr{K}(x, y)$ over the regions

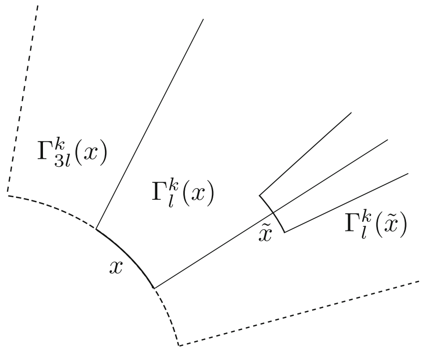

Fig. 4.6.

$$\mathtt{B}_1 = \Big\{(x,y) \in \mathbb{R}^{2d} : y \notin B_h(x), \langle x,y\rangle > 0, |x| \le |y| \text{ and } \alpha(x,y) > 1/|x| \\ \text{or } |x| \le 2|y|, \alpha(x,y) \le 1/|x|\Big\},$$

$$\mathtt{B}_2 = \Big\{(x,y) \in \mathbb{R}^{2d} : y \notin B_h(x) : \langle x,y\rangle > 0, |y|/2 \le |x| < |y|, \alpha(x,y) \le 1/|x|\Big\}.$$

To finish the proof of Theorem 4.24, we need the following lemmas (see [185, Lemma 2.6, 2.7, and 2.8]):

Lemma 4.25. *For some* $\delta > 0$

$$\overline{T}^{(1)} f(x) \le C \sum_{l\ge 1}^{\infty} e^{-\delta l} \mathtt{T}^1_l f(x), \tag{4.53}$$

Proof. For any $s,m \in \mathbb{R}^+$, there exists a constant C_m such that $e^{-s} < C_m s^{-m} e^{-s/2}$. Taking $s = (\alpha^2|x|^2|y|^2)/(|x-y|\,|x+y|)$, $m = d/2$, as $|x-y|\,|x+y| \le 2|y|^2$, we get from (4.49)

$$\begin{aligned}\overline{\mathscr{K}}(x,y) &\le C\Big(\frac{1}{\alpha|x|}\Big)^d \exp\Big(-\frac{\alpha^2|x|^2|y|^2}{2|x-y|x+y|}\Big) e^{|x|^2-|y|^2} \\ &\le C\Big(\frac{1}{\alpha|x|}\Big)^d e^{-(\alpha^2|x|^2)/4} e^{|x|^2-|y|^2}.\end{aligned}$$

On the other hand, when $|y| > 2|x|$ and $\langle x,y\rangle > 0$ we have the inequality$|y|^2 \le 2|x-y|x+y|$ and then, $\overline{\mathscr{K}}(x,y) \le e^{|x|^2-|y|^2}$. Therefore, if we split the integral over the regions where $m \le \alpha|x| < m+1$, for $m = 0,1,\cdots$ we get

$$\overline{T}^{(1)} f(x) \le C_d e^{|x|^2} \int_{\Gamma^1_1(x)} e^{-|y|^2} dy + C_m \sum_{m\ge 1} \frac{1}{m} \frac{e^{-m^2/4}}{\gamma_d(\Gamma^1_{m+1}(x))} \int_{\Gamma^1_{m+1}(x)\setminus\Gamma^1_m(x)} f(y) e^{-|y|^2} dy \tag{4.54}$$

$$\leq C\mathtt{T}_1^1 f(x) + C\sum_{m\geq 1} e^{-m^2/4}\mathtt{T}_{m+1}^1 f(x), \tag{4.55}$$

where we have used

$$\gamma_d(\Gamma_m^1(x)) \sim \left(\frac{m}{|x|}\right)^{d-1}(1+|x|)^{d-2}e^{-|x|^2} \leq C_d m^{d-1}e^{-|x|^2},$$

because $|x| > 1$. □

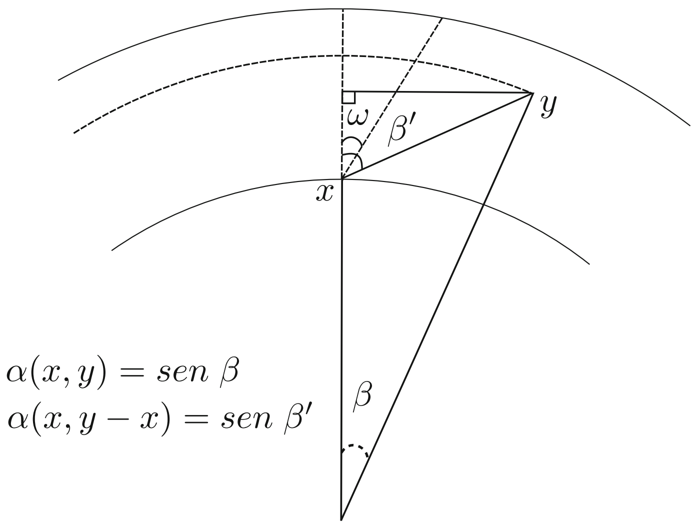

Fig. 4.7.

Lemma 4.26.

$$\overline{T}^{(2)} f(x) \leq C\sum_{l\geq 2}^{\infty} e^{-\delta l}\mathtt{T}_l^2 f(x) + C\tilde{\mathscr{A}} f(x), \tag{4.56}$$

where $\tilde{\mathscr{A}} f$ is a "weighted average" operator defined as

$$\tilde{\mathscr{A}} f(x) = \frac{1}{\gamma_d(\Lambda(x))}\int_{\Lambda(x)} G(x,y) f(y) e^{-|y|^2} dy,$$

where $G(x,y) = A^{-d/2}e^{\alpha^2|y|^4/16A}$, with $A = A(x,y)$ defined as in (4.48), $\overline{\Gamma_1^1}(x) := \{y : (x,y) \in \mathtt{B}_2\}$, and

$$\Lambda(x) = \overline{\Gamma_1^1}(x) \cap \left\{y : |x| < |y| < 2|y|\,\alpha(x, y-x) < \frac{1}{100}\right\},$$

are convex bodies (whose shape resembles that of a pencil).

Proof. As,

$$\overline{T}^{(2)}f(x) = \int_{\overline{\Gamma_1^1}(x)} f(x)\overline{\mathscr{K}}(x,y)dy,$$

with

$$\overline{\mathscr{K}}(x,y) \leq C\frac{|x|^d}{(A+\alpha|x|^2)^{d/2}}\exp\Big(-\frac{\alpha^2|x|^2|y|^2}{2(A+\alpha|x|^2)}\Big)e^{|x|^2-|y|^2},$$

for $y \in \overline{\Gamma_1^1}(x)$, as $|x-y|\,|x+y| = 2\sqrt{A+\alpha|x|^2|y|^2} \sim A+\alpha|x|^2$. Now, we analyze the cases $A > \alpha|x|^2$, and $A \leq \alpha|x|^2$. To do that, we consider the convex bodies $\Lambda(x)$. When $y \in \overline{\Gamma_1^1}(x) \setminus \Lambda(x)$, we have

$$|y-x| = (\alpha(x,y)/\alpha(x,y-x))|y| \leq 100\alpha(x,y)|y|,$$

(see Figure 4.7) and, as $|x-y||x+y| \geq d$ and $|x| \sim |y|$ we obtain from (4.49),

$$\overline{\mathscr{K}}(x,y) \leq C|x|^d e^{-(\alpha|x|^2)/200} e^{|x|^2-|y|^2}.$$

Splitting the region of integration according to the different values of α, $l \leq \alpha|x|^2 < l+1$, we have for some $\delta > 0$

$$\begin{aligned}\int_{\overline{\Gamma_1^1}(x)\setminus\Lambda(x)} \overline{\mathscr{K}}(x,y)f(y)dy &\leq \frac{C}{\gamma_d(\Gamma_{10}^2(x))}\int_{\Gamma_{10}^2(x)} f(x)e^{-|y|^2}dy \\ &\quad + C\sum_{l>10}|x|^d e^{|x|^2}e^{-l\delta}\int_{\Gamma_l^2(x)} f(x)e^{-|y|^2}dy, \\ &\leq C\sum_{l\geq 2} l^{d-1}e^{-l\delta}\frac{1}{\gamma_d(\Gamma_l^2(x)}\int_{\Gamma_l^2(x)} f(x)e^{-|y|^2}dy\end{aligned}$$

where we have used $\gamma_d(\Gamma_l^2(x)) \sim (l/|x|^2)^{d-2}e^{-|x|^2} \leq C_d(l^{d-1}/|x|)^d)e^{-|x|^2}$, for every $l = 1,2,\cdots$

Now, we need to study the properties of $\overline{T}$ over $\Lambda(x)$. To do that, we study the properties of the operator

$$\tilde{\mathscr{A}}f(x) = \frac{1}{\gamma_d(\Lambda(x))}\int_{\Lambda(x)} G(x,y)f(y)e^{-|y|^2}dy.$$

Noting that $\gamma_d(\Lambda(x)) \sim |x|^{-d}e^{-|x|^2}$, and that $A \geq \alpha|x|^2$ if $y \in \Lambda(x)$, it follows from (4.49),

$$\int_{\Lambda(x)} \overline{\mathscr{K}}(x,y)f(y)dy \leq C\tilde{\mathscr{A}}f(x).$$

Finally, we need to prove the weak type $(1,1)$ with respect to γ_d of $\tilde{\mathscr{A}}$. This proof is difficult, as it faces the problem that the family $\{\Lambda(x)\}$ does not have the "doubling property," so we cannot find dilations of $\Lambda(x)$ with essentially the same Gaussian measure.

Lemma 4.27. *The weighted average operator* $\tilde{\mathscr{A}}$ *is of weak type* $(1,1)$ *with respect to* γ_d.

Proof. Let

$$\Lambda^*(x) = \Big\{ y : |y| > |x|, \langle x, y-x \rangle > 0, \alpha(x, y-x) < 99/100 \Big\}.$$

Then,

$$\gamma_d(\Lambda^*(x)) \sim \gamma_d(\Lambda(x)) \sim |x|^d e^{-|x|^2}.$$

If we also $\Lambda^{**}(x) = \Lambda^*(x(1-3/|x|^2))$, we have $\gamma_d(\Lambda^{**}(x)) \sim |x|^d e^{-|x|^2}$ too. Given $\lambda > 0$ and D, a compact subset of the level set $\{x : \tilde{\mathscr{A}} f(x) > \lambda\}$, we can find $x_1, x_2, \cdots, x_N \in D$ such that

$$D \subset \bigcup_j \Lambda^{**}(x_j), \text{ and } x_k \notin \Lambda^{**}(x_j) \text{ if } j \neq k. \tag{4.57}$$

This is somehow similar to the one used by Sjögren for the "forbidden regions." Observe that the family $\{\Gamma_1^2(x_j)\}$ is disjoint. With these assumptions we are able to prove the following generalization of the usual Vitali's covering lemma: instead of asking for the bounded overlap of the collection $\{\Lambda(x_j)\}$, that is, $\sum_{j \in S(y)} \chi_{\Lambda(x_j)}(y) \leq C$ with $S(y) := \{j : y \in \Lambda(x_j)\}$, we show that

$$\sum_{j \in S(y)} G(x_j, y) \leq C. \tag{4.58}$$

Once this is done, we deduce easily the weak type $(1,1)$ for the operator $\tilde{\mathscr{A}}$ with respect to γ_d because

$$\gamma_d(D) \leq C \sum_j \gamma_d(\Lambda(x_j)) \leq \frac{C}{\lambda} \int f(y) e^{-|y|^2} \sum_{j \in S(y)} G(x_j, y)\, dy \leq \frac{C}{\lambda} f(y) e^{-|y|^2}\, dy.$$

To prove (4.58), we fix y and make the following geometric remark: if we call $A_0 := \max\{A_k = (|y|^2 - |x_k|^2)/2 : k \in S(y)\}$, then we obtain, from (4.57) that $\frac{1}{2} A_0 \leq A_j \leq A_0$, for all $j \in S(y)$ (see Figure 4.8). Therefore, writing $\alpha_j = \alpha(x_j, y)$, we get for some $\delta > 0$

$$\sum_{j \in S(y)} G(x_j, y) \leq \sum_{k \geq 0} \frac{1}{A_0^{d/2}} \sum_{\{j \in S(y) : (A_0 k)^{1/2}/|y|^2 \leq \alpha_j < (A_0(k+1))^{1/2}/|y|^2\}} e^{-\delta k}.$$

On the other hand, the number of disjoint "circles" of radius $1/|y|^2$ over the unit sphere of $\mathbb{R}^d$ whose angular distance to a fixed point on it lies between $(A_0 k)^{1/2}/|y|^2$ and $(A_0(k+1))^{1/2}/|y|^2$ is not larger than

$$\left(|y|^2\right)^{d-1} \left[\frac{(A_0(k+1))^{1/2}}{|y|^2} - \frac{(A_0 k)^{1/2}}{|y|^2} \right] \sim (A_0)^{(d-1)/2} (k+1)^{(d-3)/2}.$$

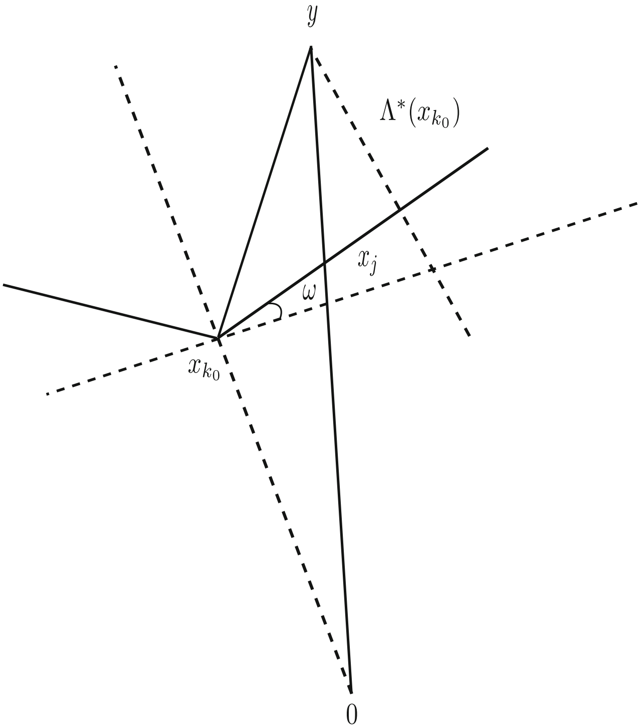

Fig. 4.8.

Thus, as $d < |x+y|\,|x-y| \leq 2(A+\alpha|x|^2) \leq 4A$ whenever $y \in \Lambda(x)$, we conclude that A is bounded below; therefore,

$$\sum_{j \in S(y)} G(x_j, y) \leq \sum_{k \geq 0} \frac{(k+1)^{(d-3)/2}}{A_0^{1/2}} e^{-\delta k} \leq C.$$

This finishes the proof of the lemma. □

Thus, the weak type $(1,1)$ of $\tilde{\mathscr{A}}$ together with (4.53) and (4.56), from the adding-up condition on $L^{1,\infty}$ (see E. M. Stein and N. J. Weiss [257]), finally gives us that the operator $\overline{T}$ is of weak type $(1,1)$ with respect to γ_d; thus, we conclude the proof of Theorem 4.24 and therefore of Theorem 4.20. □

As we have seen in the proof above, one of the main differences between the proofs of S. Pérez and P. Sjögren is the crucial use of polar coordinates. This not only simplifies some of the argument, but also provides a geometrical approach that is very natural for the Gaussian measure, which happens to be a unifying method that can be used to study other operators associated with the Gaussian measure.

- In [91] L. Forzani, E. Harboure, and R. Scotto give a rewritten version of the kernel $\overline{\mathscr{K}}$ sketching a simpler proof than the original one given by T. Menárgez, S. Pérez, and F. Soria in [185, 223], using a clever technique introduced in [104].

- A third proof was obtained by L. Forzani as a consequence of Theorem 4.18, majorizing T^* with an appropriate maximal function (see [5]). Simply choose $\Phi(t) = \frac{1}{\pi^{n/2}} e^{-t^2}$, then from (1.22) we get

$$\gamma_d\Big((1+\delta)B\Big(\frac{x}{r}, \frac{|x|}{r}(1-r)\Big)\Big) \leq C e^{-|x|^2}(1-r)^{\frac{d}{2}}.$$

  Then, for any $f \in L^1(\gamma_d)$

$$T^* f(x) \leq \mathscr{M}_\Phi f(x);$$

  therefore the γ_d-weak type $(1,1)$ of T^* follows. □

- There is yet another proof of the weak $(1,1)$ of T^*, obtained by J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. L. Torrea in [104, Theorem 3.2, Theorem 4.3], using the holomorphic Ornstein–Uhlenbeck semigroup (and therefore complex variables techniques). The proof is actually simpler than the ones given above, but as the techniques are completely different from those considered throughout the book, we do not provide further details.

- Finally, in [37] T. Bruno, combining the ideas of T. Menárguez, S. Pérez, and F. Soria [185] and [223] with those of J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. L. Torrea in [104] gives a different shorter and simpler proof of the weak type $(1,1)$ of the global part of T^* using the kernel

$$\tilde{K}(x,y) = e^{|x|^2-|y|^2}\left(\frac{|x+y|}{|x-y|}\right)^{d/2} e^{-\frac{|x|^2}{2}+\frac{|x|^2}{2}-\frac{|x-y||x+y|}{2}}\Psi(x,y)\chi_G(x,y), \qquad (4.59)$$

  where

$$\Psi(x,y) = \max\{1, \frac{1}{(|x-y||x+y|)^{d/2}}\}, \text{ and } G = N_1^c,$$

  where $N_1 = \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| < \frac{1}{1+|x|+|y|}\right\}$ (see (4.63)). Bruno proves (see [37, Proposition 3.3]) that, for every $(x,y) \in G$

$$\overline{\mathscr{K}}(x,y) \leq C\tilde{K}(x,y).$$

  The kernel $\tilde{K}(x,y)$ controls only from above $\overline{\mathscr{K}}(x,y)$.[2] This greatly simplifies the proofs, because the weak type $(1,1)$ of the operator associated with $\tilde{K}(x,y)$ can be easily deduced by a kernel obtained by J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. L. Torrea in [104] (see [37, Lemma 3.5]).

[2]Except in certain regions where they are equivalent (see [37, Remark 3.4]).

As mentioned in Chapter 2, the boundedness properties of the maximal function for the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$, Γ^*_p

$$\Gamma^*_p f(z) = \sup_{z \in \mathbf{E}_p} |T_z f(x)|,$$

sharp weak type and strong type estimates have been studied by J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea in [104] and by P. Sjögren in [249] using a similar technique of splitting the corresponding operators into local and global parts.

The Continuity Properties of the Poisson–Hermite Maximal Function

The proof of the continuity properties of the *Poisson–Hermite maximal function*,

$$P^* f(x) = \sup_{t>0} |P_t f(x)|,$$

is far simpler.

Theorem 4.28. *For the Poisson–Hermite maximal function P^*, we have*

i) For $1 < p < \infty$ the Poisson–Hermite maximal function P^* is of strong type (p, p) with respect to the Gaussian measure, that is, there exists a constant A_p dependent only on p and the dimension d, such that

$$\|P^* f\|_{p,\gamma} \leq A_p \|f\|_{p,\gamma}. \tag{4.60}$$

ii) The Poisson–Hermite Maximal function P^* is of weak type $(1,1)$ with respect to the Gaussian measure, that is, there exists a constant C dependent only on the dimension d, such that for any $f \in L^1(\gamma_d)$

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : P^* f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda} \|f\|_{1,\gamma} \tag{4.61}$$

for any $\lambda > 0$.

Proof.

i) From the general theory of the semigroup, it follows that P^* is $L^p(\gamma_d)$-bounded for $1 < p < \infty$, with respect to the Gaussian measure, and the constant is independent of the dimension (see [253]). Moreover, this can also be obtained from the fact that the Poisson–Hermite semigroup is a subordinated semigroup (see C. Herz [133]).

ii) On the other hand, from the fact that the Poisson–Hermite semigroup is a subordinate semigroup, it can be deduced immediately that P^* satisfies a weak type $(1,1)$ with respect to the Gaussian measure, because

$$
\begin{aligned}
|P_t f(x)| &\leq \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \left| T_{(t^2/4u)} f(x) \right| du \\
&\leq \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}}\, du\, T^* f(x) = T^* f(x).
\end{aligned}
$$

Hence,

$$P^* f(x) = \sup_{t>0} |P_t f(x)| \leq T^* f(x);$$

therefore, using Theorem 4.20

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : P^* f(x) > \lambda\Big\}\Big) \leq \gamma_d\Big(\Big\{x \in \mathbb{R}^d : T^* f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda} \|f\|_{1,\gamma}. \qquad \square$$

4.4 The Local and Global Regions

The technique in Gaussian harmonic analysis of splitting the kernels, such as Mehler's kernel into a *local and a global part* is well known and goes back to B. Muckenhoupt's paper [193] in dimension one, but there it was considered implicitly. In higher dimensions $d > 1$, this was first used explicitly by W. Urbina in [278], where the local region was given by "admissible rectangles"; for $x \in \mathbb{R}^d$ fix, the local region

$$\mathscr{A}_x := \left\{ y \in \mathbb{R}^d : |y_i - x_i| < 1 \wedge \frac{1}{|x_i|}, \text{ for all } i = 1, 2, \cdots, d \right\},$$

and the global part is simply $\mathscr{A}_x^c$.

The basic idea behind the notion of local region, regardless of the specific definition, is that the corresponding local part of the operator behaves in a similar manner to a classical operator, usually a Calderón–Zygmund operator, from the fact that, on those regions, the Gaussian measure is equivalent to a multiple of the Lebesgue measure. The problem is then reduced to study the global part of the operator, that is to say, the operator restricted to the global region, which is usually a positive operator and/or has nice decay properties. The study of the global part is unfortunately, most of the time, highly non-trivial.

For the proof of the weak type $(1,1)$ of the Ornstein–Uhlenbeck maximal operator (see 4.3), P. Sjögren considered the following local region,

$$N_R := \Big\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x| \leq R \text{ and } |y| \leq R, \text{ or } |y| \geq R/2 \text{ and } |x-y| \leq R/|y| \Big\}, \tag{4.62}$$

and $N_{R,x}$ means the section of N_R at level $x \in \mathbb{R}^d$,

$$N_{R,x} := \Big\{ y \in \mathbb{R}^d : |x| \leq R \text{ and } |y| \leq R, \text{ or } |y| \geq R/2 \text{ and } |x-y| \leq R/|y| \Big\}.$$

R. Scotto, in his doctoral dissertation [244] (see also [77]), considered the same local region.

The local region considered by T. Menárguez, S. Pérez, and F. Soria is given by

$$N = \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \leq C_d m(x) \right\}$$

and the section for $x \in \mathbb{R}^d$, is given by

$$N_x = \left\{ y \in \mathbb{R}^d : (x,y) \in N \right\} = B(x, C_d m(x)) = B_h(x),$$

the admissible ball. The global region, for fixed $x \in \mathbb{R}^d$ is simply the complement of $B_h(x)$.

In [102] and [103], J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea consider for each $t > 0$ the local region N_t, the neighborhood of the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$N_\delta = \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| < \frac{\delta}{1+|x|+|y|} \right\}, \tag{4.63}$$

$\delta > 0$.

Finally, P. Portal in [231] considered the following local region, for all $a > 0$,

$$N_a := \left\{ (x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \leq a m(x) \right\}. \tag{4.64}$$

The global region is then the complement of N_a. Furthermore, Portal also defines the local region $N_a(B)$

$$N_a(B) := \left\{ y \in \mathbb{R}^d : |c_B - y| \leq a m(c_B) \right\}.$$

4.5 Calderón–Zygmund Operators and the Gaussian Measure

A very important generalization of the arguments developed by S. Pérez and F. Soria to bound the local part of T^* are the tools developed to control the local part of Calderón–Zygmund operators (see [223] and [221]). We are going to study in this section a generalization of Calderón–Zygmund's theory adapted to the Gaussian measure γ_d. This extension is very useful in Chapter 5 for the $L^p(\gamma_d)$-boundedness of Gaussian Littlewood–Paley g functions and in Chapter 9 for the $L^p(\gamma_d)$-boundedness of Gaussian singular integrals.

First of all, let us recall the definition of a Calderón–Zygmund operator.[3]

Definition 4.29. *We say that a C^1 function $K(x,y)$, defined off the diagonal of $\mathbb{R}^d \times \mathbb{R}^d$, i.e., $x \neq y$, is a Calderón–Zygmund kernel provided that the following conditions are satisfied:*

i) $|K(x,y)| \leq \frac{C}{|x-y|^d}$.

[3] For more on the classical theory, see E. Stein [252, Chap II], J. Duoandikoetxea [72, Chapter 5], L. Grafakos [118, Chapter 4] or A. Torchinski [275, Chapter XI].

ii) $|\partial_y K(x,y)| \leq \frac{C}{|x-y|^{d+1}}$.

Associated with K we define the operator T by means of the formula

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(x,y)f(y)d = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K(x,y)f(y)dy.$$

with $f \in C_0^\infty(\mathbb{R}^d)$. We say that T is a Calderón–Zygmund operator if T admits a continuous extension to $L^2(\mathbb{R}^d)$.

We are now going to prove that in general, the local part of a singular integral of Calderón–Zygmund type has the expected boundedness properties with respect to the Gaussian measure. This theorem was obtained by S. Pérez (see [221, Theorem 2.1]).

Theorem 4.30. *Let $K(x,y)$ be a kernel that satisfies the Calderón–Zygmund conditions for $|x-y| \leq m(x)$, and consider the integral operator T with kernel K,*

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(x,y)f(y)dy. \tag{4.65}$$

Taking a covering of $\mathbb{R}^d$ $\mathscr{F} = \{B(0,\alpha), \tilde{B}_j^k\}_{k,j}$ of admissible balls, with bounded overlaps,[4] *we can define the "local part" of T as*

$$T_L f(x) = T_{L,\mathscr{F}} f(x) = \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} T(f\chi_{\hat{B}})(x)\chi_B(x). \tag{4.66}$$

Then, if T bounded on $L^p(\gamma_d)$ for some $1 < p < \infty$, T_L is of weak type $(1,1)$ with respect to γ_d.

Given a ball $B \subset \mathbb{R}^d$, let us denote by T_B, the restriction of T to the ball B,

$$T_B f(x) = p.v. \int_B K(x,y)f(y)dy.$$

To prove the theorem, we need the following technical lemma (see S. Pérez [221, Lemma 2.2])

Lemma 4.31. *Let T be a Calderón–Zygmund operator with kernel K. Given $x_0 \in \mathbb{R}^d$ and consider an admissible ball centered at $x \in \mathbb{R}^d$, $B_h(x) = \{y : |y-x| \leq C_d m(x)\}$, then there exists a constant C such that,*

$$\left|\left\{x \in B : |T_{B_h(\cdot)} f(x)| > \lambda\right\}\right| \leq \frac{C}{\lambda} \int_{B_h(x)} |f(y)|dy,$$

for every $f \in L^1(\mathbb{R}^d)$.

[4] The existence of that family is obtained from Lemma 4.3.

Proof. Without loss of generality, we may assume that the support of f is contained in B_h; therefore, we would have $T_{B_h}f(x) = Tf(x)$. We also assume that the $L^1(\gamma_d)$ norm of f is one, $\|f\|_{1,\gamma} = 1$. Set $r = C_d m(x) = C_d(1 \wedge 1/|x|)$.

Recall the classical Calderón–Zygmund decomposition. Given a non-negative function $f \in L^1(B)$ and $\lambda > 0$, there exists a sequence $\{Q_j\}_{j\in\mathscr{A}}$ of cubes whose interiors are mutually disjoint, such that

i) $f(x) \leq \lambda$ for almost everywhere on $(\cup Q_j)^c$.
ii) $|\bigcup_j Q_j| \leq \frac{1}{\lambda}\|f\|_1$.
iii) $\lambda \leq \frac{1}{|Q_j|}\int\limits_{Q_j} f(x)dx \leq 2^d\lambda$.

Associated with this collection, we can write the function f as the sum of two functions b and g given by

$$g(x) = \begin{cases} f(x) & x \in (\cup_j Q_j)^c \\ \frac{1}{|Q_j|}\int\limits_{Q_j} f(y)dy & x \in Q_j, \end{cases}$$

called the "good part" and

$$b(x) = \sum_j b_j(x) \quad \text{with} \quad b_j(x) = \left(f - \frac{1}{|Q_j|}\int\limits_{Q_j} f(y)dy \right)\chi_{Q_j}(x),$$

called the "bad part."

- Case #1: If $r < (1/\lambda)^{1/d}$.
  In this case, the lemma holds trivially because

$$\left|\left\{x \in B : |Tf(x)| \geq \lambda\right\}\right| \leq Cr^d \leq C\frac{\|f\|_1}{\lambda}.$$

- Case #2: If $r \geq (1/\lambda)^{1/d}$. Observe that, according to *ii*), $|Q_j| < 1/\lambda$, for all j; thus, the side length of all the cubes Q_j is smaller than or equal to $(1/\lambda)^{1/d}$. Therefore, all the cubes Q_j are contained in B', a ball with its center at x_0 and its radius $(1+\sqrt{d})r$. From this point onward, the proof follows the classical one. According to the usual argument, it is enough to establish separately the estimates for the level set of height $\lambda/2$ for both b and g. Using Chebyshev's inequality and from the boundedness of T on $L^p(\gamma_d)$, we easily obtain the estimate for Tg (as $g \leq C\lambda$)

$$\begin{aligned}\left|\left\{x \in B : Tg(x) > \lambda/2\right\}\right| &\leq \frac{C}{\lambda^p}\int_B |T(g)(x)|^p dx \leq \frac{Ce^{|x_0|^2}}{\lambda^p}\int_B |T(g)(x)|^p\gamma(x)dx \\ &\leq \frac{Ce^{|x_0|^2}}{\lambda^p}\int_{B'} |g(x)|^p\gamma(x)dx \leq C\frac{\lambda^{p-1}}{\lambda^p}\int_{\mathbb{R}^d} |g(x)|dx\end{aligned}$$

$$\leq C\frac{1}{\lambda}\|f\|_1.$$

Now, let us estimate Tb. Let y_j be the center of Q_j and consider the cube Q'_j is the cube dilated by 2 and concentric with Q_j. When $x \in \cup_j Q'_j$, then

$$\left|\bigcup_j Q'_j\right| \leq \frac{C}{\lambda}\int_{\mathbb{R}^d}|f(y)|dy.$$

Otherwise, the mean value zero of each b_j implies

$$\begin{aligned}\left|\left\{x \in B\backslash\bigcup_j Q'_j : |Tb(x)| > \lambda/2\right\}\right| &\leq \frac{C}{\lambda}\sum_l \int_{B\backslash\cup Q'_j}\left|\int_{Q_l}K(x,y)b_l(y)dy\right|dx\\ &= \frac{C}{\lambda}\sum_l \int_{B\backslash\cup Q'_j}\left|\int_{Q_l}(K(x,y)-K(x,y_l))b_l(y)dy\right|dx\\ &\leq \frac{C}{\lambda}\sum_l \int_{Q_l}|b_l(y)|\int_{B\backslash Q'_l}|K(x,y)-K(x,y_l)|dxdy\\ &\leq \frac{C}{\lambda}\sum_l \int_{Q_l}|b_l(y)|\int_{(Q'_l)^c}\frac{|y-y_l|}{|x-y_l|^{d+1}}dxdy\\ &\leq \frac{C}{\lambda}\sum_l \int_{Q_l}|b_l(y)|dy \leq \frac{C}{\lambda}\int_{\mathbb{R}^d}|f(y)|dy,\end{aligned}$$

where we have used the fact that if $y, y_l \in Q_l$ and y' is any point in the segment $\overline{yy_l}$, then $y' \in Q_l \subset B'$. In particular, if $x \in B$, then $|x-y'| \leq (2+\sqrt{d})r \leq C_1(1 \wedge 1/|x|)$, for some positive constant $C_1 \sim C_d$; therefore, the assumption about the decay of the gradient of K establishes the lemma. □

Let us now prove Theorem 4.30.

Proof. According to the definition of T_L and the fact that $\mathscr{F}$ is a covering of $\mathbb{R}^d$ of admissible balls with bounded overlaps, we have, using Lemma 4.31, that

$$\begin{aligned}\gamma_d\left(\left\{x : |T_L f(x)| > \lambda\right\}\right) &\leq \sum_{k=0}^{\infty}\sum_{B\in\mathscr{F},B\subset S_k}\gamma_d(\{x \in B : |T_B f(x)| > \lambda\})\\ &\sim \frac{1}{\pi^{d/2}}\sum_{k=0}^{\infty}\sum_{B\in\mathscr{F},B\subset S_k}e^{-\alpha_k^2}|\{x \in B : |T_B f(x)| > \lambda\}|\\ &\leq \frac{1}{\pi^{d/2}}\sum_{k=0}^{\infty}\sum_{B\in\mathscr{F},B\subset S_k}e^{-\alpha_k^2}\frac{C}{\lambda}\int_B |f(y)|dy\\ &\sim \frac{C}{\lambda}\sum_{B\in\mathscr{F},B\subset S_k}\int_B |f(y)|\gamma_d(dy) \leq \frac{C}{\lambda}\|f\|_{1,\gamma_d}.\end{aligned}$$

Moreover,

$$\|T_L f\|_{p,\gamma} \sim \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k^2} \int_B |T_B f(x)|^p dx \leq \sum_{k=0}^{\infty} e^{-\alpha_k^2} \int_B |Mf(x)|^p dx$$
$$\leq C \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k^2} \int_B |f(x)|^p dx \sim \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} \int_B |f(x)|^p \gamma_d(dx)$$
$$\sim \|f\|_{p,\gamma} \qquad \square.$$

In particular, as is well known in the classical Calderón–Zygmund theory (see for instance E. Stein [252, Chapter II] or J. Duoandikoetxea [72, Chapter 4]), that if we consider a *singular integral of convolution type*, i.e., the kernel $\mathscr{K}$ is given as

$$\mathscr{K}(x) = \frac{\Omega(x)}{|x|^d}, \tag{4.67}$$

where Ω is a C^1 homogeneous function of degree zero with mean zero over the unit (hyper)-sphere $S^{d-1} \subset \mathbb{R}^d$,

$$\int_{S^{d-1}} \Omega(x')d\sigma(x') = 0,$$

and the operator T is defined by convolution with $\mathscr{K}$, then T is a Calderón–Zygmund operator. In this case, we have the following result, obtained by S. Pérez [221, Proposition 4.3]:

Theorem 4.32. *Let T be the Calderón–Zygmund operator defined by convolution with $\mathscr{K} = \frac{\Omega(x)}{|x|^d}$, with Ω as above,*

$$Tf(x) = p.v. \int_{\mathbb{R}^d} \mathscr{K}(x-y)f(y)dy = p.v. \int_{\mathbb{R}^d} \frac{\Omega(x-y)}{|x-y|^d} f(y)dy.$$

Then, its local part T_L is bounded in $L^p(\gamma_d)$, $1 < p < \infty$, and of weak type $(1,1)$ with respect to γ_d.

Proof. Using Lemma 4.3 again, we can take a family $\mathscr{F} = \{B(0,\alpha), \tilde{B}_j^k\}_{k,j}$ of admissible balls, with bounded overlaps, that covers $\mathbb{R}^d$, and set $\hat{B} := C_d B$. Given an operator T, we can define its "local part" as

$$T_L f(x) = T_{L,\mathscr{F}} f(x) = \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} T(f\chi_B)(x)\chi_B(x).$$

Consider the truncated maximal operator,

$$\mathscr{N} f(x) = \sup_{0,\varepsilon<N} \left| \int_{\varepsilon\leq|x-y|\leq N} \mathscr{K}(x-y)f(y)dy \right|. \tag{4.68}$$

It is well known; see for instance E. Stein [252, Chap II, § 4], that $\mathscr{N}$ is $L^p(\mathbb{R}^d)$-bounded,$1 < p < \infty$, and weak $(1,1)$ with respect to the Lebesgue measure. From the definition of T_L, we have $|T_B g(x)| \leq \mathscr{N} g(x)$ for any function g and any admissible ball $B \in \mathscr{F}$. Then, using the fact that the family $\mathscr{F}$ is a covering of $\mathbb{R}^d$ of admissible balls with bounded overlaps, making a similar argument than in the proof of Theorem 4.30, we have the strong type (p,p),

$$\begin{aligned}\|T_L f\|_{p,\gamma} &\sim \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k^2} \int_B |T_B f(x)|^p dx \leq \sum_{k=0}^{\infty} e^{-\alpha_k^2} \int_B |\mathscr{N} f(x)|^p dx \\ &\leq C \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k^2} \int_B |f(x)|^p dx \sim \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset B} \int_B |f(x)|^p \gamma_d(dx) \\ &\sim \|f\|_{p,\gamma},\end{aligned}$$

and also, the weak type $(1,1)$,

$$\begin{aligned}\gamma_d\Big(\Big\{x : |T_L f(x)| > \lambda\Big\}\Big) &\leq \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} \gamma_d\Big(\Big\{x\in B : |T_B f(x)| > \lambda\Big\}\Big) \\ &\sim \frac{1}{\pi^{d/2}} \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k^2} \Big|\Big\{x\in B : |T_B f(x)| > \lambda\Big\}\Big| \\ &\leq \frac{1}{\pi^{d/2}} \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k^2} \Big|\Big\{x\in B : |\mathscr{N} f(x)| > \lambda\Big\}\Big| \\ &\leq \frac{1}{\pi^{d/2}} \sum_{k=0}^{\infty} \sum_{B\in\mathscr{F}, B\subset S_k} e^{-\alpha_k^2} \frac{C}{\lambda} \int_B |f(y)| dy \\ &\sim \frac{C}{\lambda} \sum_{B\in\mathscr{F}, B\subset S_k} \int_B |f(y)| \gamma_d(dy) \leq \frac{C}{\lambda} \|f\|_{1,\gamma}. \qquad \square\end{aligned}$$

Now, we extend Theorem 4.32 by considering a condition on the "size" of the operator. This argument is similar to the one obtained by F. Soria and G. Weiss in [251] and is obtained by S. Pérez [221, Theorem 5.1]. We use it for Calderón–Zygmund operators, but it can be used in more general situations, in particular, for vector valued operators.

Theorem 4.33. *Let T be a sublinear operator of weak type $(1,1)$ with respect to the Lebesgue measure, i.e., there exists a constant C such that for any $\lambda > 0$*

$$\Big|\Big\{x\in\mathbb{R}^d : |Tf(x)| \geq \lambda\Big\}\Big| \leq \frac{C}{\lambda} \|f\|_1 \tag{4.69}$$

for any $f \in L^1(\mathbb{R}^d)$, and let T_0 be the operator defined as

$$T_0 f(x) = T(f\chi_{B_h(\cdot)})(x),$$

where $B_h(x) = \Big\{y\in\mathbb{R}^d : |y-x| < dm(x)\Big\}$. Then, if T satisfies

$$|Tf(x)| \leq C\int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^d} dy, \ \textit{for } x \notin \textit{supp} f, \tag{4.70}$$

T_0 is of weak type $(1,1)$ with respect to the Gaussian measure. Moreover, if T is $L^p(\mathbb{R}^d)$ bounded for some $1 < p < \infty$, then T_0 is also $L^p(\gamma_d)$ bounded for the same p.

Proof. We use Lemma 4.3, considering a family $\mathscr{F} = \{B(0,\alpha), \tilde{B}^k_j\}_{k,j}$ of admissible balls, with bounded overlaps, that covers $\mathbb{R}^d$, and setting $\hat{B} = C_d B$, $B \in \mathscr{F}$, where $C_d > d$. T_0 can be bounded as,

$$|T_0 F(x)| \leq \sum_{B\in\mathscr{F}} |T_0(f\chi_{\hat{B}(\cdot)})(x)|\chi_B(x).$$

Let T_1 be defined as

$$T_1 f(x) = T(f\chi_{B^c_h(\cdot)})(x).$$

We also have

$$|T_0(f\chi_{\hat{B}(\cdot)})(x)| \leq |Tf(f\chi_{\hat{B}(\cdot)})(x)| + |T_1(f\chi_{\hat{B}(\cdot)})(x)|.$$

Now, for $x \in B \in \mathscr{F}$, using condition (4.70) and the fact that if $x \in B$ then $m(x) \sim m(c_B)$ where c_B is the center of B, we get

$$\begin{aligned}
|T_1(f\chi_{\hat{B}(\cdot)})(x)| &\leq \int_{\mathbb{R}^d} \frac{|(f\chi_{\hat{B}(x)}\chi_{B^c_h(x)})(y)|}{|x-y|^d} dy \\
&\leq \int_{dm(x)\leq|y-x|<C_d m(c_b)} \frac{|(f\chi_{\hat{B}(x)})(y)|}{|x-y|^d} dy \\
&\leq \frac{C}{m(c_b)^d} \int_{|y-x|<C_d m(c_b)} |(f\chi_{\hat{B}(x)})(y)| dy \leq CM(f\chi_{\hat{B}(\cdot)})(x).
\end{aligned}$$

We conclude that,

$$|T_0 f(x)| \leq C\sum_{B\in\mathscr{F}} \Big(|T(f\chi_{\hat{B}(\cdot)})(x)| + M(f\chi_{\hat{B}(\cdot)})(x)\Big)\chi_B(x).$$

The theorem follows from the boundedness properties of T with respect to the Lebesgue measure using the same arguments used the proof of Theorem 4.32. □

Finally, there is an extension of these results to the vector valued case, which follows the same steps with minor changes (see [87]).

Theorem 4.34. *Let $(A_1, |\cdot|_1)$ and $(A_2, |\cdot|_2)$ be two separable Banach spaces. Let T be a bounded linear transformation from $L^p_\gamma(\mathbb{R}^d, A_1)$ to $L^p_\gamma(\mathbb{R}^d, A_2)$ for some $1 < p < \infty$ defined as*

$$Tf(x) = p.v. \int_{\mathbb{R}^d} K(x,y) f(y) dy$$

where K defined on the set $\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x \neq y\}$ takes its values in $\mathscr{B}(A_1, A_2)$ and satisfies that for every constant $C_1 > 0$ there exists another constant $C_2 > 0$, so that whenever $|x-y| \leq C_1 m(x)$,

i) $|K(x,y)|_{\mathscr{B}} \leq \frac{C_2}{|x-y|^d}$.
ii) $\int_{|z-x|\geq 2|x-y|} |K(z,x) - K(z,y)|_{\mathscr{B}}\, dz \leq C_2$.

Let $T_L f(x) = T(f\chi_{B_h(\cdot)})(x)$. Then, for every $f \in L^1_\gamma(\mathbb{R}^d, A_1)$ and every $\lambda > 0$, there exists $C > 0$ such that

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : |T_L f(x)|_2 > \lambda\Big\}\Big) \leq \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|_1 d\gamma(y).$$

Proof. Let $\mathscr{F} = \{B_j\}_{j=1}^\infty$ be a sequence of admissible balls $B_j = B(x_j, dm(x_j))$ such that $\mathbb{R}^d = \bigcup_{j=1}^\infty B_j$ and $\sum_{j=1}^\infty \chi_{B_j^*}(x) \leq C$ for all $x \in \mathbb{R}^d$ where $B_j^* = 2B_j$ is such that $\bigcup_{x\in B_j} B_h(x) \subseteq B_j^*$ (see Lemma 4.3). For $x \in B_j$,

$$|T_L f(x)|_2 \leq |T(f\chi_{B_j^*})(x)|_2 + |T(f\chi_{B_j^*\setminus B_h(\cdot)})(x)|_2, \tag{4.71}$$

and

$$|T(f\chi_{B_j^*\setminus B_h(\cdot)})(x)|_2 \leq \int_{B_j^*\setminus B_h(x)} |K(x,y)|_{\mathscr{B}}\, |f(y)|_1\, dy \leq C_2 \int_{B_j^*\setminus B_h(\cdot)} \frac{|f(y)|_1}{|x-y|^d}\, dy,$$

where the last inequality comes from condition *i*). For $y \in B_j^* \setminus B_h(x)$, we have

$$d\, m(x) < |x-y| \leq 2d\, m(x_j).$$

On B_j, $|x| \sim |x_j|$. Therefore,

$$|T(f\chi_{B_j^*\setminus B_h(\cdot)})(x)|_2 \leq CM(\chi_{B_j^*}|f|_1)(x), \tag{4.72}$$

where M is the Hardy–Littlewood maximal function with respect to the Lebesgue measure.

Taking into account that the balls B_j cover $\mathbb{R}^d$ together with inequality (4.71) and the fact that the Gaussian density $e^{-|x|^2}$ is on constant order of magnitude on each B_j, we have

$$\begin{aligned}\gamma_d\Big(\Big\{x \in \mathbb{R}^d : |T_L f(x)|_2 > \lambda\Big\}\Big) &\leq \sum_{j=1}^{+\infty} \gamma_d\Big(\Big\{x \in B_j : |T(f\chi_{B_h(\cdot)})(x)|_2 > \lambda\Big\}\Big) \\ &\sim \sum_{j=1}^{+\infty} e^{-|x_j|^2}\Big|\Big\{x \in B_j : |T(f\chi_{B_j^*})(x)|_2 > \ \lambda/2\Big\}\Big| + \\ &+ \sum_{j=1}^{+\infty} e^{-|x_j|^2}\Big|\Big\{x \in B_j : |T(f\chi_{B_j^*\setminus B_h(\cdot)})(x)|_2 > \lambda/2\Big\}\Big|\end{aligned}$$

From (4.72), and the fact that M is of weak type $(1,1)$ with respect to the Lebesgue measure, we have

$$\left|\left\{x \in B_j : |T(f\chi_{B_j^*\setminus B_h(\cdot)})(x)|_2 > \lambda/2\right\}\right| \leq \left|\left\{x \in \mathbb{R}^d : M(|f|_1\chi_{B_j^*})(x) > \lambda/2C\right\}\right|$$
$$\leq \frac{C}{\lambda}\int_{B_j^*}|f(y)|_1\, dy.$$

It remains to prove that

$$\left|\left\{x \in B_j : |T(f\chi_{B_j^*})(x)|_2 > \lambda/2\right\}\right| \leq \frac{C}{\lambda}\int_{B_j^*}|f(y)|_1\, dy.$$

Minor changes in the proof of the Calderón–Zygmund decomposition, where we have to replace $f \geq 0$ by $|f|_1$ to get the sequence of cubes $\{Q_k\}_{k=1}^{\infty}$, such that

1. $|f(x)|_1 \leq \lambda$ almost everywhere $x \notin \bigcup_k Q_k$.
2. $|\cup Q_k| \leq \frac{1}{\lambda}\int_{B_j^*}|f(x)|_1\, dx$.
3. $\lambda < \frac{1}{|Q_k|}\int_{Q_k}|f(x)|_1\, dx \leq 2^d\lambda$.

Define

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_k Q_k \\ \frac{1}{|Q_k|}\int_{Q_k} f(y)dy & \text{if } x \in Q_k, \end{cases} \quad \text{and } \; b(x) = f(x) - g(x).$$

Then, following the steps written there (and whenever we find an absolute value, this must be changed by $|\cdot|_1$ or $|\cdot|_2$, whichever corresponds), and replacing the condition on the gradient of the kernel with the Hormänder condition ii) of the hypothesis, we get

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : |T_L f(x)|_2 > \lambda\right\}\right) \leq \frac{C}{\lambda}\sum_{j=1}^{+\infty} e^{-|x_j|^2}\int_{B_j^*}|f(y)|_1\, dy \leq \frac{C}{\lambda}\int_{\mathbb{R}^d}|f(y)|_1\, \gamma_d(dy).$$

where this latter inequality was obtained because the Gaussian density is essentially constant over each B_j^* and that the sequence $\{B_j^*\}_{j=1}^{\infty}$ has a bounded overlap. □

Additionally, we have several technical results that are needed to bound the local or the global part of certain operators. First, we consider the following technical result, obtained S. Pérez [221, Lemma 3.1], which will be crucial for bounding the local part of several operators later. We use the notation of Proposition 4.23,

$$a = a(x,y) := |x|^2 + |y|^2,\; b = b(x,y) := 2\langle x, y\rangle,$$

$$u(t) = u(t;x,y) := \frac{|y-\sqrt{1-t}x|^2}{t} = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |x|^2,$$

$$t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \sim \frac{\sqrt{a^2-b^2}}{a} \sim \frac{\sqrt{a-b}}{\sqrt{a+b}} = \frac{|x-y|}{|x+y|},$$

and

$$u_0 = u(t_0) = \frac{|y|^2 - |x|^2}{2} + \frac{|x+y||x-y|}{2}.$$

Lemma 4.35. *For every $\eta \geq 0$, there exists a constant C such that if $|x-y| \leq C_d m(x)$, we have*

$$\int_0^1 (u(t))^\eta \frac{e^{-u(t)}}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}} \leq \frac{C}{|x-y|^{d+1}}.$$

Proof. As $u(t)$ is always positive for any $0 < \delta < 1$, the function $(u(t))^\eta e^{-(1-\delta)u(t)}$ is uniformly bounded on $(0,1)$ for all η. Then, it is enough to show that for some $0 < \delta < 1$, we have

$$\int_0^1 \frac{e^{-\delta u(t)}}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}} \leq \frac{C}{|y-x|^{\frac{d+1}{2}}}.$$

Actually, this is true for any $\delta > 0$. The reason is that whenever $|x-y| \leq C_d(1 \wedge 1/|x|)$,

$$u(t) \geq \frac{(|y-x| - |x|(1-\sqrt{1-t}))^2}{t} \geq \frac{|y-x|^2}{t} - 2\frac{|x||x-y|}{1+\sqrt{1-t}} \geq \frac{|y-x|^2}{t} - 2C_d.$$

Therefore, the change of variables $\frac{|y-x|^2}{t} = s$ gives the desired estimate,

$$\begin{aligned}\int_0^1 \frac{e^{-\delta u(t)}}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}} &\leq e^{\delta 2C_1} \int_0^1 \frac{\exp\left(-\frac{\delta|y-x|^2}{t}\right)}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}} \\ &\leq \frac{C_d}{|y-x|^{d+1}} \int_{|y-x|^2}^\infty e^{-\delta s} \frac{s^{d/2}}{\sqrt{s-|y-x|^2}} ds = \frac{C_d}{|x-y|^{d+1}}.\end{aligned}$$

□

Now, to bound the global part of certain operators later on, we need the following technical results, also obtained by S. Pérez and F. Soria. We use the same notation as for Proposition 4.23. The first result is a generalization of [223, Lemma 2.3] and [221, Lemma 4.1]

Lemma 4.36. *For every $0 \leq \eta \leq 1$ and $\nu > 0$, there exists a constant C such that if $\langle x,y \rangle > 0$ and $|x-y| > C_d m(x)$, we have,*

$$\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}\sqrt{1-t}} \leq C \frac{e^{-\nu u_0}}{t_0^{1/2}}. \tag{4.73}$$

Proof. The change of variables $s = u(t) - u_0$ is one-to-one over the intervals $(0, t_0)$ and $(t_0, 1)$. Writing t as a function of s, we observe that if $\beta(s) = 2s + a + \sqrt{a^2 - b^2}$, then t satisfies the equation

$2\sqrt{1-t}b = 2a - \beta(s)t$, or, equivalently, $(\beta(s))^2\frac{t^2}{4} - (a\beta(s)-b^2)t + (a^2-b^2) = 0.$

The discriminant of this second-order equation is given by

$$(a\beta(s)-b^2)^2 - (\beta(s))^2(a^2-b^2) = 4b^2(s^2+s\sqrt{a^2-b^2}).$$

Therefore, t is given, over the intervals $(0,t_0)$ and $(t_0,1)$, by the two solutions of this equation, i.e.,

$$\begin{aligned} v(s) &= 2\frac{a\beta(s)-b^2-\sqrt{(a\beta(s)-b^2)^2-(\beta(s))^2(a^2-b^2)}}{(\beta(s))^2} \\ &= 2\frac{a\beta(s)-b^2-2b\sqrt{s^2+s\sqrt{a^2-b^2}}}{(\beta(s))^2}, \end{aligned}$$

and

$$\begin{aligned} w(s) &= 2\frac{a\beta(s)-b^2+\sqrt{(a\beta(s)-b^2)^2-(\beta(s))^2(a^2-b^2)}}{(\beta(s))^2} \\ &= 2\frac{a\beta(s)-b^2+2b\sqrt{s^2+s\sqrt{a^2-b^2}}}{(\beta(s))^2}. \end{aligned}$$

We split the integral at t_0 and set $h(t) = \frac{1}{|u'(t)|t^2\sqrt{1-t}}$. Then, using that $\sqrt{v(s)} \leq \sqrt{w(s)}$, we get

$$\begin{aligned} &\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}\sqrt{1-t}} \\ &\qquad \leq e^{-\nu u_0} \int_0^{u(0^+)-u_0} (s+u_0)^{\eta/2} e^{-\nu s} h(v(s))\sqrt{v(s)}ds \\ &\qquad\qquad + e^{-\nu u_0} \int_0^{u(1^-)-u_0} (s+u_0)^{\eta/2} e^{-\nu s} h(w(s))\sqrt{w(s)}ds \\ &\qquad \leq Ce^{-\nu u_0} \int_0^{\infty} (s^{\eta/2}+u_0^{\eta/2}) e^{-\nu s}[h(v(s))+h(w(s))]\sqrt{w(s)}ds. \end{aligned}$$

To estimate the sum $h(v(s))+h(w(s))$, observe that

$$2|u'(t)|t^2\sqrt{1-t} = |2a\sqrt{1-t}-(2-t)b|, \text{ so } h(t) = \frac{2b}{|2(a^2-b^2)-(a\beta(s)-b^2)t|}.$$

After some calculations, we obtain

$$h(v(s)) = \frac{a\beta(s)-b^2+2b\sqrt{s^2+s\sqrt{a^2-b^2}}}{2(a^2-b^2)\sqrt{s^2+s\sqrt{a^2-b^2}}}$$

and

$$h(w(s)) = \frac{a\beta(s) - b^2 - 2b\sqrt{s^2 + s\sqrt{a^2-b^2}}}{2(a^2-b^2)\sqrt{s^2+s\sqrt{a^2-b^2}}}.$$

Thus, from the definition of $\beta(s)$, we have

$$h(v(s))+h(w(s)) \leq \frac{2a}{a^2-b^2} + \frac{2a}{(a^2-b^2)^{1/2}\sqrt{s}(a^2-b^2)^{1/4}} \leq \frac{C}{t_0}\frac{1}{(a^2-b^2)^{1/4}}\left(1+\frac{1}{\sqrt{s}}\right),$$

because $t_0 \sim \frac{\sqrt{a^2-b^2}}{a}$ and $a^2 - b^2 > d \geq 1$, in the global region and $b > 0$.

The estimate of $\sqrt{w(s)}$ is analogous with the additional fact that $\beta(s) \geq a$,

$$w(s) \leq 4\frac{a\beta(s) - b^2}{(\beta(s))^2} \leq C\frac{as + a\sqrt{a^2-b^2}}{(\beta(s))^2} \leq C\frac{s+\sqrt{a^2-b^2}}{a} \leq Ct_0(s+1).$$

Adding up these estimates, again using that $a^2 - b^2 > d \geq 1$, and $u_0 \leq \sqrt{a^2-b^2}$, we get

$$\begin{aligned}
&\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}\sqrt{1-t}} \\
&\qquad \leq C\frac{e^{-\nu u_0}}{t_0^{1/2}} \frac{1}{(a^2-b^2)^{1/4}} \int_0^\infty (s^{\eta/2} + u_0^{\eta/2}) e^{-\nu s}\left(\sqrt{s} + \frac{1}{\sqrt{s}}\right) ds \\
&\qquad \leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\left(1 + \frac{u_0^{\eta/2}}{(a^2-b^2)^{1/4}}\right)\int_0^\infty e^{-\nu s}\left(s + \frac{1}{\sqrt{s}}\right) ds \\
&\qquad \leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}. \qquad \square
\end{aligned}$$

Using the same estimates as in the previous lemma, we get this other technical result (see [223, Lemma 4.3]).

Lemma 4.37. *For every $\eta \geq 2$ and $\nu > 0$, there exists a constant C such that if $\langle x, y\rangle > 0$ and $|x-y| > C_d m(x)$, we have*

$$\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}} \leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\left(1 + u_0^{(\eta-1)/2}\left(\frac{b}{a} + \frac{1}{a^2-b^2}\right)\right). \tag{4.74}$$

Proof. We consider again the change of variables $s = u(t) - u_0$ and setting $h(t) = \frac{1}{|u'(t)|t^2\sqrt{1-t}}$, we get, using that $v(s) \leq w(s)$; therefore, $\sqrt{1-w(s)} \leq \sqrt{1-v(s)}$,

$$\begin{aligned}
&\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}} \\
&\qquad \leq Ce^{-\nu u_0}\int_0^\infty (s^{\eta/2} + u_0^{\eta/2}) e^{-\nu s} h(v(s))\sqrt{1-v(s)}\sqrt{v(s)} ds \\
&\qquad\quad + Ce^{-\nu u_0}\int_0^\infty (s^{\eta/2} + u_0^{\eta/2}) e^{-\nu s} h(w(s))\sqrt{1-w(s)}\sqrt{w(s)} ds \\
&\qquad \leq Ce^{-\nu u_0}\int_0^\infty (s^{\eta/2} + u_0^{\eta/2}) e^{-\nu s}(h(v(s)) + h(w(s)))\sqrt{1-v(s)}\sqrt{w(s)} ds.
\end{aligned}$$

Using the same estimates as in Lemma 4.36, we get

$$\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}} \leq C \frac{e^{-\nu u_0}}{t_0^{1/2}} \frac{1}{(a^2-b^2)^{1/4}}$$
$$\int_0^\infty (s^{\eta/2}+u_0^{\eta/2}) e^{-\nu s} \left(\sqrt{s}+\frac{1}{\sqrt{s}}\right) \sqrt{1-\nu(s)}\,ds.$$

On the other hand, $t = v(s)$ is the solution of $2\sqrt{1-t}b = 2a - \beta(s)t$. Hence, as $\beta(s) > a$, we get

$$\begin{aligned}\sqrt{1-v(s)} &= \frac{b^2+2b\sqrt{s^2+s\sqrt{a^2-b^2}}}{b\beta(s)}\\ &\leq \frac{2}{a}\left(b+(s+s^{1/2})(a^2-b^2)^{1/4}\right) \leq C(1+s)\left(\frac{b}{a}+(\frac{1}{(a^2-b^2)^{1/4}}\right).\end{aligned}$$

Therefore, adding all these estimates,

$$\begin{aligned}\int_0^1 (u(t))^{\eta/2} e^{-\nu u(t)} \frac{dt}{t^{3/2}} &\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\left(1+u_0^{(\eta-2)/2}\left(\frac{b}{a}+\frac{1}{(a^2-b^2)^{1/4}}\right)\right)\\ &\qquad\times\int_0^\infty \left(s^{(m+3)/2}+\frac{1}{s^{1/2}}\right)e^{-\eta s}ds\\ &\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\left(1+u_0^{(\eta-2)/2}\left(\frac{b}{a}+\frac{1}{(a^2-b^2)^{1/4}}\right)\right). \qquad \square\end{aligned}$$

Finally, we have the following lemma,

Lemma 4.38. *If* $|x-y| \geq C_d\left(1 \wedge \frac{1}{|x|}\right) = C_d m(x)$ *then*

$$\int_0^1 (u(t))^{1/2} e^{-u(t)} \frac{dt}{t^{\frac{d}{2}+1}\sqrt{1-t}} \sim \overline{\mathscr{K}}(x,y), \tag{4.75}$$

where $\overline{\mathscr{K}}$ *is the Gaussian maximal kernel defined in (4.40).*

Proof. We know that $u(t)$ is strictly decreasing in $(0,t_0)$, and strictly increasing in $(t_0,1)$, with

$$t_0 = \frac{2\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}} \sim \frac{\sqrt{a^2-b^2}}{a},$$

as $u'(t) = -\frac{2a\sqrt{1-t}-(2-t)b}{2t^2\sqrt{1-t}}$. Then, we need to analyze two cases.

- Case #1: $b \leq 0$. We prove that

$$\int_0^1 (u(t))^{1/2} e^{-u(t)} \frac{dt}{t^{\frac{d}{2}+1}\sqrt{1-t}} \leq Ce^{-|y|^2}.$$

Given that b is non-positive, we have

$$\frac{a}{t} - |x|^2 \leq u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t} b - |x|^2 \leq \frac{2a}{t}. \tag{4.76}$$

Thus,

$$\int_0^1 (u(t))^{1/2} e^{-u(t)} \frac{at}{t^{\frac{d}{2}+1}\sqrt{1-t}} \leq Ce^{-|y|^2} \int_0^1 e^{(-\frac{a}{t}+a)} \left(\frac{2a}{t}\right)^{1/2} \frac{dt}{t^{\frac{d}{2}+1}\sqrt{1-t}}.$$

It is enough to prove that this last integral is uniformly bounded in a. Making the change of variables $a(\frac{1}{t}-1) = s$ and using that $a > d/2 \geq 1/2$, if $|x-y| \geq C_d \left(1 \wedge \frac{1}{|x|}\right)$, we obtain

$$\begin{aligned} \int_0^1 (u(t))^{1/2} e^{-u(t)} \frac{dt}{t^{\frac{d}{2}+1}\sqrt{1-t}} &\leq C\frac{e^{-|y|^2}}{a^{d/2}} \int_0^\infty e^{-s}(s+a)^{d/2} \frac{ds}{\sqrt{s}} \\ &\leq Ce^{-|y|^2} \int_0^\infty e^{-s}(2s+1)^{d/2} \frac{ds}{\sqrt{s}} \leq Ce^{-|y|^2}. \end{aligned}$$

- Case #2: $b > 0$. Consider again

$$u_0 = u(t_0) = \frac{|y|^2 - |x|^2}{2} + \frac{\sqrt{a^2-b^2}}{2} \leq (a^2-b^2)^{1/2}.$$

If $d \geq 2$, considering the function $\varphi(t) = \frac{1}{t^{d/2}} e^{-u(t)}$ used in Proposition 4.23, we know that

$$\varphi(t) = \frac{e^{-u(t)}}{t^{d/2}} \leq \frac{e^{-u(t_0)}}{t^{d/2}} = \varphi(t_0); \tag{4.77}$$

therefore,

$$\begin{aligned} \frac{e^{-\frac{d-2}{d}u(t)}}{t^{\frac{d-2}{2}}} &= \left(\frac{e^{-u(t)}}{t^{\frac{d}{2}}}\right)^{\frac{d-2}{d}} = (\varphi(t))^{\frac{d-2}{d}} \leq (\varphi(t_d))^{\frac{d-2}{d}} \\ &\leq C(\varphi(t_0))^{\frac{d-2}{d}} = C\left(\frac{e^{-u_0}}{t_0^{\frac{d}{2}}}\right)^{\frac{d-2}{d}} = \frac{e^{-\frac{d-2}{d}u_0}}{t_0^{\frac{d-2}{2}}}, \end{aligned} \tag{4.78}$$

and we know that $\varphi(t_d) \sim \varphi(t_0)$ (see 4.44). Now, according to Lemma 4.36, taking $\eta = 1$, we have for any $\nu > 0$ the following inequality:

$$\int_0^1 u^{1/2}(t) e^{-\nu u(t)} \frac{dt}{t^2\sqrt{1-t}} \leq C_\nu \frac{e^{-\nu u_0}}{t_0}. \tag{4.79}$$

Hence, taking $\nu = d/2$, we get the desired inequality as

$$C_d \frac{e^{-\frac{d-2}{d}u_0}}{t_0^{\frac{d-2}{2}}} \int_0^1 (u(t))^{1/2} e^{-\frac{d}{2}u(t)} \frac{dt}{t^2\sqrt{1-t}} \leq C_d \frac{e^{-u_0}}{t_0^{d/2}} \sim \overline{\mathscr{K}}(x,y). \qquad \square$$

4.6 The Non-tangential Maximal Functions for the Ornstein–Uhlenbeck and Poisson–Hermite Semigroups

We consider now the non-tangential maximal functions for the Ornstein–Uhlenbeck and Poisson–Hermite semigroups. In the classical case, the cones with vertex at $x \in \mathbb{R}^d$, and aperture $a > 0$ are defined as

$$\Gamma_a(x) = \Big\{(y,t) \in \mathbb{R}^{d+1}_+ : |y-x| < at\Big\}. \tag{4.80}$$

Then, the classical non-tangential maximal function associated with the heat semigroup (see E. Stein [252, Chapter VII]), is defined as

$$\mathscr{T}_a^* f(x) = \sup_{(y,t)\in\Gamma_a(x)} |\mathscr{T}_t(y)|. \tag{4.81}$$

$\mathscr{T}_a^*$ is bounded almost everywhere by the classical Hardy–Littlewood maximal function, i.e.,

$$\mathscr{T}_a^* f(x) \le CMf(x).$$

Therefore, the boundedness properties of $\mathscr{T}_a^*$ follow immediately from those of $Mf(x)$.

In the Gaussian case, again because Gaussian harmonic analysis is local, we need to consider *Gaussian or admissible cones*, with arbitrary aperture $A > 0$ and a cut-off parameter $a > 0$, which are defined as

$$\Gamma_\gamma^{A,a}(x) = \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \ |y-x| < At,\ t < a\left(1 \wedge \frac{1}{|x|}\right) = am(x)\right\}, \tag{4.82}$$

where A, $a > 0$.

As a consequence of the definition of $\Gamma_\gamma^{A,a}(x)$, we have that $|x-y||x| < At|x| < aA$ and if $(y,t) \in \Gamma_\gamma^{A,a}(x)$, then $|x| \sim |y|$; therefore, $e^{-|x|^2} \sim e^{-|y|^2}$.

In particular, for the case $A = 1$, we have

$$\Gamma_\gamma^{a}(x) = \left\{(y,t) \in \mathbb{R}^{d+1}_+ : \ |y-x| < t,\ t < a\left(1 \wedge \frac{1}{|x|}\right) = am(x)\right\}, \tag{4.83}$$

for $a > 0$, and if $a = 1$, we simply write $\Gamma_\gamma(x)$ instead of $\Gamma_\gamma^a(x)$.

The Non-tangential Ornstein–Uhlenbeck Maximal Function

Definition 4.39. *The non-tangential maximal function associated with the Ornstein–Uhlenbeck semigroup is defined as*

$$\mathscr{T}_\gamma^*(A,a) f(x) = \sup_{(y,t)\in\Gamma_\gamma^{A,a}(x)} |\, T_{t^2} f(y)\,|\,. \tag{4.84}$$

If $A = a = 1$ we write simply $\mathscr{T}_\gamma^*(1,1) = \mathscr{T}_\gamma^*$ and similarly $\mathscr{P}_\gamma^*(A,a) = \mathscr{P}_\gamma^*$.

These functions, and the Gaussian square functions, were defined initially in 1994 by L. Forzani and E. Fabes for the case $A = a = 1$[5]. They are Gaussian analogs to the sublinear operators, which in the classical case, are the cornerstones of the real variable theory of H^1. Their $L^p(\gamma_d)$-boundedness was shown by L. Forzani, R. Scotto, and W. Urbina in [87].

Now, the boundedness properties of $\mathscr{T}^*$, are immediate consequences of the following inequality, obtained by J. Teuwen in [265].

Theorem 4.40. *Let $A, a > 0$. For all $x \in \mathbb{R}^d$ and $f \in L^1(\gamma_d)$ then*

$$\mathscr{T}_\gamma^*(A,a)f(x) \leq C\mathscr{M}_\gamma f(x), \tag{4.85}$$

where the constant $C = C_{A,a,d}$ is dependent only on A, a, and d.

This result extends a previous result obtained by E. Pineda and W. Urbina in [225], which is Theorem 4.49 at the end of this chapter; not only by enlarging the family of Gaussian cones in which the non-tangential maximal function is defined, but also allowing an arbitrary aperture $A > 0$ and a cut-off parameter $a > 0$ without any additional technicalities. As we are going to see, this additional generality is very important, and it has been used by P. Portal in [231] for his definition of Gaussian Hardy spaces. Moreover, J. Teuwen's result is not only an extension of Theorem 4.49, but its proof also happens to be simpler and shorter.

To prove Theorem 4.40, we need the following off-diagonal kernel estimates on annuli of Mehler's kernel. We decompose $\mathbb{R}^d$ into disjoint annuli. In what follows, we fix $x \in \mathbb{R}^d$, constants $A, a \geq 1$, a pair $(y,t) \in \Gamma_\gamma^{A,a}(x)$. The annuli $C_k := C_k(B(y,t))$ are given by:

$$C_k := \begin{cases} B(y,2t), & k = 0, \\ B(y,2^{k+1}t) \setminus B(y,2^k t), & k \geq 1. \end{cases}$$

Whenever $\xi \in C_k$, we get, for $k \geq 1$,

$$2^k t < |y - \xi| < 2^{k+1} t. \tag{4.86}$$

On C_k, we have the following bound for $M_{t^2}(y,\cdot)$:

Lemma 4.41. *For all $\xi \in C_k$ and $k \geq 1$, we have*

$$M_{t^2}(y,\xi) \leq \frac{e^{|y|^2}}{(1-e^{-2t^2})^{d/2}}\, e^{2^{k+1}t|y|}\, e^{-\frac{4^k}{2e^{2t^2}}}. \tag{4.87}$$

[5] L. Forzani and E. Fabes defined $\mathscr{T}_\gamma^*$ on "parabolic" Gaussian cones, which are unnecessary. The definition given here using Gaussian cones $\Gamma_\gamma^{A,a}(x)$ is equivalent because we are considering the Ornstein–Uhlenbeck semigroup in parameter t^2.

Proof. Considering Mehler's kernel in the form (2.38),

$$M_{t^2}(y,\xi) = \frac{\exp\left(-\frac{e^{-2t^2}|y-\xi|^2}{1-e^{-2t^2}}\right)}{(1-e^{-t^2})^{d/2}} \frac{\exp\left(2e^{-t^2}\frac{<y,\xi>}{1+e^{-t^2}}\right)}{(1+e^{-t^2})^{d/2}}.$$

For the first exponential, using the inequality $1-e^{-s} \leq s$, $s \geq 0$, and (4.86), we have,

$$\exp\left(-\frac{e^{-2t^2}|y-\xi|^2}{1-e^{-2t^2}}\right) \leq \exp\left(-\frac{4^k}{e^{2t^2}}\frac{t^2}{1-e^{-2t^2}}\right) \leq \exp\left(-\frac{4^k}{2e^{2t^2}}\right).$$

On the other hand, if $0 \leq s \leq 1$ trivially, we have $2s \leq 1+s$. Then, using that estimate and (4.86), we have for the second exponential of Mehler's kernel

$$\exp\left(2e^{-t^2}\frac{<y,\xi>}{1+e^{-t^2}}\right) \leq e^{|<y,\xi>|} \leq e^{|<y,\xi-y>|}e^{|y|^2} \leq e^{2^{k+1}t|y|}e^{|y|^2}.$$

Combining these two inequalities we get the required estimate. □

Now, we are ready to prove Theorem 4.40.

Proof. Let $x \in \mathbb{R}^d$ be fixed, and take $(y,t) \in \Gamma_\gamma^{A,a}(x)$, then

$$|T_{t^2}f(y)| \leq \sum_{k=0}^{\infty}\int_{C_k} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi).$$

As $t \leq am(x) \leq a$ and, according to Lemma 1.5, $t|y| \leq 1+aA$, then from (4.85) and Lemma 1.5, we conclude for $\xi \in C_k$ and $k \geq 1$

$$\begin{aligned} M_{t^2}(y,\xi) &\leq \frac{e^{|y|^2}}{(1-e^{-2t^2})^{d/2}}\, e^{2^{k+1}t|y|}\, e^{-\frac{4^k}{2e^{2t^2}}} \leq \frac{e^{|y|^2}}{(1-e^{-2t^2})^{d/2}}\, e^{2^{k+1}(1+aA)}\, e^{-\frac{4^k}{2e^{2t^2}}} \\ &= \frac{e^{|y|^2}}{(1-e^{-2t^2})^{d/2}}\lambda_k \sim \frac{e^{|x|^2}}{(1-e^{-2t^2})^{d/2}}\lambda_k. \end{aligned} \tag{4.88}$$

with $\lambda_k = e^{2^{k+1}(1+aA)}\, e^{-\frac{4^k}{2e^{2t^2}}}$.

Now, using (4.86) we have

$$|x-\xi| \leq |x-y|+|y-\xi| \leq (2^{k+1}+A)t.$$

Let K be the smallest integer such that $2^{k+1} \geq A$ whenever $k \geq K$. Then $C_k \subset B(x,2^{k+2}t)$ for $k \geq K$ and $C_k \subset B(x,2At)$ for $k < K$.

Set

$$D_k = D_k(x) := \begin{cases} B(x,2^{k+2}t) & \text{if } k \geq K \\ B(x,2At) & \text{if } k < K. \end{cases}$$

Then, using (4.88), we get,

$$\begin{aligned}\int_{C_k} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi) &\le C_{a,A}\lambda_k \frac{e^{|x|^2}}{(1-e^{-2t^2})^{d/2}}\int_{C_k}|f(\xi)|\gamma_d(d\xi)\\ &\le C_{a,A}\lambda_k \frac{e^{|x|^2}}{(1-e^{-2t^2})^{d/2}}\int_{D_k}|f(\xi)|\gamma_d(d\xi)\\ &\le C_{a,A}\lambda_k \frac{e^{|x|^2}}{(1-e^{-2t^2})^{d/2}}\gamma_d(D_k)M_{\gamma_d}f(x).\end{aligned}$$

Thus, using Lemma 1.2, if $k \ge K$

$$\gamma_d(D_k)e^{|x|^2} \le C\frac{\omega_{d-1}}{2\pi^{d/2}}2^{d(k+1)}t^d e^{2^{k+3}t|x|} \le C2^{dk}t^d e^{2^{k+3}a},$$

as $t \le a$ and $t|x| \le am(x)|x| \le a$. If $k < K$,

$$\gamma_d(D_k)e^{|x|^2} \le C\frac{\omega_{d-1}}{2\pi^{d/2}}(2A)^d t^d e^{2At|x|} \le Ct^d e^{2Aa}.$$

Additionally, as $t \le a$ and the fact that the function $s/(1-e^{-s})$ is increasing, we obtain,

$$\frac{t^d}{1-e^{-2t^2}} \le \frac{a^d}{1-e^{-2a^2}} \le C_a.$$

Therefore, if $k \ge K$

$$\int_{C_k} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi) \le C_{a,A}\lambda_k 2^{kd}t^d e^{2^{k+2}a}\mathscr{M}_\gamma f(x),$$

and if $k < K$

$$\int_{C_k} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi) \le C_{a,A}\lambda_k t^d e^{2Aa}\mathscr{M}_\gamma f(x).$$

Similarly, if $\xi \in B(x,2t)$ then

$$\int_{B(x,2t)} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi) \le C_{a,A}\mathscr{M}_\gamma f(x).$$

Finally, putting together all these estimates, we get

$$\begin{aligned}|T_{t^2}f(y)| &\le \int_{B(x,2t)} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi) + \sum_{k=1}^{K-1}\int_{C_k} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi)\\ &\quad + \sum_{k=K}^{\infty}\int_{C_k} M_{t^2}(y,\xi)|f(\xi)|\gamma_d(d\xi)\\ &\le C_{A,a,d}\left[1+\sum_{k=1}^{K-1} e^{2^{k+1}(1+aA)}\, e^{-\frac{4^k}{2e^{e^{2t^2}}}} + \sum_{k=K}^{\infty} 2^{kd}e^{2^{k+1}(1+2a+aA)}\, e^{-\frac{4^k}{2e^{2t^2}}}\right]\mathscr{M}_\gamma f(x),\end{aligned}$$

valid for all $(y,t) \in \Gamma_\gamma^{A,a}(x)$. The series on the right-hand side converges; thus, we conclude the proof. □

Corollary 4.42. *i) There exists a constant C, dependent only on dimension d, such that for all $\lambda > 0$, if $f \in L^1(\gamma_d)$, then*

$$\gamma_d\Big(\Big\{y \in \mathbb{R}^d : \mathscr{T}_\gamma^*(A,a)f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda}||f||_{1,\gamma} \tag{4.89}$$

ii) There exists a constant C, dependent only on p and the dimension d, such that, if $f \in L^p(\gamma_d)$, $1 < p \leq \infty$, then

$$||\mathscr{T}_\gamma^*(A,a)f||_{p,\gamma} \leq C||f||_{p,\gamma} \tag{4.90}$$

Proof. It is enough to prove (4.89), the weak type $(1,1)$ of $\mathscr{T}_\gamma^*(A,a)$, since the strong type on L^∞ is trivial; therefore, using the Marcinkiewicz interpolation theorem 10.24, we obtain the strong type (p,p), $1 < p < +\infty$, (4.90). But the weak type $(1,1)$ is immediate from the properties of the Gaussian Hardy–Littlewood function, Theorem 4.14, and (4.85). □

In [168], J. Maas, J. Van Neerven, and P. Portal consider an *"averaged version" of the non-tangential Ornstein -Uhlenbeck maximal function* defined as follows

$$\Upsilon_\gamma^*(A,a)f(x) = \sup_{(y,t)\in\Gamma_\gamma^{A,a}(x)} \left(\frac{1}{\gamma_d(B(y,At))}\int_{B(y,At)} |T_{t^2}f(z)|^2\gamma_d(dz)\right)^{1/2}, \tag{4.91}$$

for $f \in C_0(\mathbb{R}^d)$.

The additional averaging adds some technical difficulties, but as we see later, such averaging can be helpful in the Hardy space theory and its applications. For these maximal functions, they prove a change of aperture for the Gaussian cones $\Gamma_\gamma^{A,a}(x)$ appearing in their definition (see [168, Theorem 3.1]), in the spirit of one of the key results of R. Coifman, Y. Meyer, and E. Stein in [55].

Theorem 4.43. *There exists a constant D, dependent only on A, and a and the dimension d, such that for all $f \in L^1(\gamma_d)$ and $\lambda > 0$*

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \Upsilon_\gamma^*(A,a)f(x) > \lambda\Big\}\Big) \leq \gamma_d\Big(\Big\{x \in \mathbb{R}^d : \Upsilon_\gamma^*(1,C_{A,a})f(x) > D\lambda\Big\}\Big), \tag{4.92}$$

with $C_{A,a}$ a constant dependent only on A, and a. By interpolation, this inequality implies

$$\|\Upsilon_\gamma^*(A,a)f\|_{p,\gamma} \leq \|\Upsilon_\gamma^*(1,C_{A,a})f\|_{p,\gamma}.$$

For details of the proof see [168, Theorem 3.1].

The Non-tangential Poisson–Hermite Maximal Function

Definition 4.44. *The non-tangential maximal function associated with the Poisson semigroup is defined as*

$$\mathscr{P}_\gamma^*(A,a)f(x) = \sup_{(y,t)\in\Gamma_\gamma^{A,a}(x)} |P_t f(y)|, \tag{4.93}$$

The analogous results for the non-tangential Poisson–Hermite maximal function $\mathscr{P}_\gamma^*(A,a)$ follow basically by subordination. This function was first introduced by E. Fabes and L. Forzani in an unpublished manuscript [84].

Theorem 4.45. *(Forzani - Fabes) For the non-tangential Poisson–Hermite maximal function $\mathscr{P}_\gamma^*(A,a)$, we have the following boundedness properties.*

i) There exists a constant C, dependent only on the dimension d, such that for all $\lambda > 0$, if $f \in L^1(\gamma_d)$, then

$$\gamma_d\Big(\Big\{y \in \mathbb{R}^d : \ \mathscr{P}_\gamma^*(A,a)f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda}||f||_{1,\gamma}. \tag{4.94}$$

ii) There exists a constant C, dependent only on p and the dimension d, such that, if $f \in L^p(\gamma_d)$, and $1 < p \leq \infty$, then

$$||\mathscr{P}_\gamma^*(A,a)f||_{p,\gamma} \leq C||f||_{p,\gamma}. \tag{4.95}$$

Proof. Again, according to the interpolation argument, it is enough to prove (4.94). We know from the subordination formula (3.1) that P_t can be written as,

$$P_t f(x) = \frac{t}{\sqrt{\pi}}\int_0^\infty T_s f(x) e^{-t^2/4s} s^{-3/2} ds = \frac{t}{\sqrt{\pi}}\int_0^\infty T_{u^2} f(x) e^{-t^2/4u^2} u^{-2} ds,$$

after the change of variables $s = u^2$. Then, taking $\psi_t(v) = \frac{t}{\sqrt{\pi}} e^{-\frac{1}{4v}} v^{-1}$, we can write

$$P_t f(x) = t^{-1}\int_0^\infty \psi_t\Big(\frac{s^2}{t^2}\Big) T_{s^2} f(x) ds.$$

Given $x \in \mathbb{R}^d$, let $(y,t) \in \Gamma_\gamma^{A,a}(x)$. Then

$$\begin{aligned} P_t f(y) &= t^{-1}\int_0^{|x-y|/A} \psi_t\Big(\frac{s^2}{t^2}\Big) T_{s^2} f(y) ds + t^{-1}\int_{|x-y|/A}^{am(x)} \psi_t\Big(\frac{s^2}{t^2}\Big) T_{s^2} f(y) ds \\ &\qquad + t^{-1}\int_{am(x)}^{\infty} \psi_t\Big(\frac{s^2}{t^2}\Big) T_{s^2} f(y) ds \\ &= (I) + (II) + (III). \end{aligned}$$

Using the definition of $\mathscr{T}_\gamma^*(A,a)$, we have[6]

[6] Observe, that $t^{-1}\int_0^\infty \psi\Big(\frac{s^2}{t^2}\Big) ds = P_t 1 = 1.$

$$(II) \leq C\mathscr{T}_\gamma^*(A,a)f(x)t^{-1}\int_{|x-y|/A}^{am(x)} \psi_t\Big(\frac{s^2}{t^2}\Big)ds$$
$$\leq C\mathscr{T}_\gamma^*(A,a)f(x)t^{-1}\int_0^\infty \psi_t\Big(\frac{s^2}{t^2}\Big)ds = C\mathscr{T}_\gamma^*(A,a)f(x).$$

To bound (III), we use similar arguments to those in the proof of (II);

$$(III) = Ct^{-1}\int_{am(x)}^{+\infty} \psi_t\Big(\frac{s^2}{t^2}\Big)T_{s^2}f(y)ds,$$
$$\leq Ct^{-1}\int_{am(x)}^{+\infty} \psi_t\Big(\frac{s^2}{t^2}\Big)\Big[\sup_{|x-y|<am(x)<s} T_{s^2}f(y)\Big]\,ds \leq C\mathscr{T}_\gamma^*(A,a)f(x)$$

Finally, we bound (I), which is the most difficult one,

$$(I) = t^{-1}\int_0^{|x-y|/A} \psi_t\Big(\frac{s^2}{t^2}\Big)\Big(\frac{e^{|y|^2}}{(1-e^{-s^2})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|e^{-s^2}u-y|^2}{1-e^{-2s^2}}}f(u)\gamma_d(du)\Big)ds$$
$$= Ct^{-1}\int_0^{|x-y|/A} \psi_t\Big(\frac{s^2}{t^2}\Big)\mathscr{M}_s f(y)ds,$$

where

$$\mathscr{M}_s f(y) = \frac{e^{|y|^2}}{(1-e^{-s^2})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|e^{-s^2}u-y|^2}{1-e^{-2s^2}}}f(u)\gamma_d(du). \tag{4.96}$$

We prove that

$$\mathscr{M}_s f(y) \leq \mathscr{M}_\Phi f(x),$$

for $0<s<|x-y|/A$, for some Φ as in Theorem 4.18. In fact, given $|x-y|<At$, and $t<am(x)$, we have from (1.15)

$$\mathscr{M}_s f(y) = \frac{e^{|y|^2}}{(1-e^{-2s^2})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{|e^{-s^2}u-y|^2}{1-e^{-2s^2}}}f(u)\gamma_d(du)$$
$$\leq C\frac{e^{|x|^2}}{(1-e^{-2s^2})^{d/2}}\sum_{v\geq 1}\int_{B\left(xe^{2t^2},v\sqrt{1-e^{-2t^2}}\right)\setminus B\left(xe^{2t^2},(v-1)\sqrt{1-e^{-2t^2}}\right)} e^{-\frac{|e^{-s^2}u-y|^2}{1-e^{-2s^2}}}$$
$$\times f(u)\gamma_d(du).$$

Let $u\in B\Big(xe^{2t^2},v\sqrt{1-e^{2t^2}}\Big)\setminus B\Big(xe^{2t^2},(v-1)\sqrt{1-e^{-2t^2}}\Big)$, then for $v\geq 6$

$$|e^{-s^2}u-y| \geq e^{-s^2}|u-xe^{2t^2}| - e^{-s^2}|x-y| - e^{-s^2}|x|(e^{2t^2}-1) - |y|(1-e^{-s^2})$$
$$\geq C\big((v-1)\sqrt{1-e^{-2t^2}} - 2\sqrt{1-e^{-2t^2}} - |x||x-y|^2 - |y||x-y|^2\big)$$
$$\geq C((v-1)\sqrt{1-e^{-2t^2}} - 2\sqrt{1-e^{-2t^2}} - 2|x-y|) \geq C(v-1)\sqrt{1-e^{-2t^2}}.$$

Moreover, because $0<\sqrt{1-e^{-s^2}}<cs<c|x-y|^2/A<ct<c(1-e^{-2t^2})^{1/2}$, we have for $u\in B\Big(xe^{2t^2},v\sqrt{1-e^{-2t^2}}\Big)\setminus B\Big(xe^{2t^2},(v-1)\sqrt{1-e^{-2t^2}}\Big)$,

$$\frac{|e^{-s^2}u-y|^2}{1-e^{-s^2}} \geq c(v-1)^2.$$

Therefore, for $0 < s < |x-y|/A$,

$$\mathscr{M}_s f(y) \leq \mathscr{M}_{\Phi} f(x),$$

with $\Phi(x) = \sum_{v\geq 1} e^{-C(v-1)^2} \chi_{[v-1,v)}(x)$. As a consequence of this

$$(I) \leq C\mathscr{M}_{\Phi} f(x) t^{-1} \int_0^{|x-y|/A} \psi_t\Big(\frac{s^2}{t^2}\Big) ds \leq C\mathscr{M}_{\Phi} f(x).$$

Hence, as each term of $P_t f(x)$ is bounded by an operator that is of weak type $(1,1)$, we get (4.94). □

4.7 Radial and Non-tangential Convergence of the Ornstein–Uhlenbeck and Poisson–Hermite Semigroups

Now, by the usual argument, the almost everywhere radial and non-radial convergence of $\{T_t\}$ and $\{P_t\}$ can be proved, from the boundedness properties of T^*, P^*, $\mathscr{T}^*_\gamma(A,a)$ and $\mathscr{P}^*_\gamma(A,a)$, obtained in previous sections.

Theorem 4.46. *The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ and the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$ have (radial) convergence almost everywhere, that is to say*

i) For any $f \in L^1(\gamma_d)$, $\{T_t f\}$ converges almost everywhere to f as $t \to 0^+$,

$$\lim_{t\to 0^+} T_t f(x) = f(x) \quad \textit{almost everywhere } \; x \in \mathbb{R}^d.$$

ii) For any $f \in L^1(\gamma_d)$, $\{P_t f\}$ converges almost everywhere to f as $t \to 0^+$,

$$\lim_{t\to 0^+} P_t f(x) = f(x) \quad \textit{almost everywhere } \; x \in \mathbb{R}^d.$$

Proof. The proof follows a very classical argument in harmonic analysis. We prove *i*), the case of the Poisson–Hermite semigroup, *ii*) that it is completely analogous. Set

$$\Omega f(x) := \Big| \limsup_{t\to 0^+} T_t f(x) - \liminf_{t\to 0^+} T_t f(x) \Big|.$$

If $f \in C_0(\mathbb{R}^d)$ continuous with compact support, then it is easy to see that

$$\lim_{t\to 0^+} T_t f(x) = f(x),$$

uniformly; thus, $\Omega f(x) = 0$. Now, for the general case, $f \in L^1(\gamma_d)$ by writing[7] $f = f_1 + f_2$ with $f_1 \in C_0(\mathbb{R}^d)$ and f_2 such that $\|f_2\|_{1,\gamma} < \varepsilon$. But as $T_t f(x) = T_t f_1(x) + T_t f_2(x)$ and $\Omega f(x) \leq \Omega f_1(x) + \Omega f_2(x)$ then

[7] Here we are using the fact that $C_0(\mathbb{R}^d)$ is dense in $L^1(\gamma_d)$; we discuss this in detail in Chapter 7.

$$\begin{aligned}\Omega f(x) &= \left|\limsup_{t\to 0^+} T_t f(x) - \liminf_{t\to 0^+} T_t f(x)\right| \leq \left|\limsup_{t\to 0^+} T_t f_2(x) - \liminf_{t\to 0^+} T_t f_2(x)\right| \\ &\leq 2T^* f_2(x).\end{aligned}$$

Hence,

$$\Omega f(x) \leq 2T^* f_2(x);$$

therefore, given $\varepsilon > 0$, according to Theorem 4.20,

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \Omega f(x) > \varepsilon\Big\}\Big) \leq \gamma_d\Big(\Big\{x \in \mathbb{R}^d : T^* f_2(x) > \varepsilon\Big\}\Big) \leq C\|f_2\|_{1,\gamma} \leq C\varepsilon.$$

Thus, $\Omega f(x) = 0$. a.e. □

In [106] G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, and B. Viviani found optimal integrability conditions to guarantee almost everywhere (radial) convergence of the Ornstein–Uhlenbeck semigroup and the Poisson–Hermite semigroups[8] (see also [2]).

As the Ornstein–Uhlenbeck and the Poisson–Hermite semigroups are strong $L^p(\gamma_d)$-continuous for $1 \leq p < \infty$ (see Theorem 2.5 *iv* and Theorem 3.1 *iv*), we also have

$$\|T_t f - f\|_{p,\gamma} \to 0, \text{ as } t \to 0^+, \text{ and } \|P_t f - f\|_{p,\gamma} \to 0, \text{ as } t \to 0^+$$

Thus, in the Gaussian case, $\{T_t\}_{t\geq 0}$ and $\{P_t\}_{t\geq 0}$ have the properties of an *approximation of the identity,* but they are not obtained by convolution.

Now, we want to study the non-tangential convergence for the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$, in the following sense,

$$\lim_{\substack{(y,t)\to x,\\ (y,t)\in\Gamma_\gamma(x)}} T_{t^2} f(y) = f(x), \text{ almost everywhere } x \in \mathbb{R}^d,$$

and the non-tangential convergence for the Poisson–Hermite semigroup $\{P_t\}$, in the following sense,

$$\lim_{\substack{(y,t)\to x\in\mathbb{R}^d,\\ (y,t)\in\Gamma_\gamma(x)}} P_t f(y) = f(x), \text{ almost everywhere } x \in \mathbb{R}^d.$$

In [225], E. Pineda and W. Urbina study the non-tangential convergence for the Ornstein–Uhlenbeck semigroup using the non-tangential maximal function associated with the Ornstein–Uhlenbeck semigroup using the $\Gamma_\gamma^{p,T}(x)$ cones (see also (4.108). Their argument can be adapted for Gaussian cones Γ_γ (i.e., cones with $A = a = 1$).

[8] The Ornstein–Uhlenbeck case is one among several other operators that are studied there, which also includes the classical Laplacian and the Hermite operators.

Theorem 4.47. *For any $f \in L^1(\gamma_d)$, $\{T_t f\}_{t\geq 0}$ converges to f non- tangentially as $t \to 0^+$,*

$$\lim_{\substack{(y,t)\to x,\\ (y,t)\in\Gamma_\gamma(x)}} T_{t^2} f(y) = f(x), \text{ almost everywhere } x \in \mathbb{R}^d. \tag{4.97}$$

Proof. The proof is a modification of the proof of Theorem 4.46. Considering

$$\Omega f(x) = \lim_{\alpha\to 0^+}\left[\sup_{(y,t)\in\Gamma_\gamma(x)} |\, T_{t^2} f(y) - f(x)\,|\right],$$

and let us set $f(x) = f(x)\chi_{(0,k)}(x) + f(x)(\chi_{(0,k)^c}(x)) = f_1(x) + f_2(x)$, for $k \in \mathbb{N}$ fixed.

Let us prove that

$$\Omega f(x) \leq C_d \mathscr{M}_\gamma f_2(x),$$

for almost everywhere $|\, x\,| \leq k-1$. Let x be a Lebesgue point for $f \in L^1(\gamma_d)$, i.e., x satisfies

$$\lim_{r\to 0^+} \frac{1}{\gamma_d(B(x,r))}\int_{B(x,r)} |\, f(u) - f(x)\,|\, \gamma_d(du) = 0.$$

Then, given $\varepsilon > 0$ there exists $0 < \delta < 1$ such that

$$\frac{1}{\gamma_d(B(x,r))}\int_{B(x,r)} |\, f(u) - f(x)\,|\, \gamma_d(du) < \varepsilon,$$

for $0 < r < \delta$. Defining g as

$$g(u) = \begin{cases} f(u) - f(x) & \text{if } |u-x| < \delta \\ 0 & \text{if } |u-x| \geq \delta \end{cases},$$

we get that g depends on x and $\mathscr{M}_\gamma g(x) < \varepsilon$. On the other hand, we can write

$$T_{t^2} f(y) - f(x) = (u_1(y,t) - f_1(x)) + (u_2(y,t) - f_2(x)),$$

where

$$u_i(y,t) = \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\int_{\mathbb{R}^d} \exp\left(-\frac{|\, e^{-t^2}y - u\,|^2}{1-e^{-2t^2}}\right) f_i(u)du, \;\; \text{with } i = 1,2.$$

Then, we get

$$\begin{aligned} u_1(y,t) - f_1(x) \;=\; & \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\int_{|x-u|<\delta} \exp\Big(-\frac{|\, e^{-t^2}y - u\,|^2}{1-e^{-2t^2}}\Big)(f_1(u) - f_1(x))du \\ & + \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\int_{|x-u|\geq\delta} \exp\Big(-\frac{|\, e^{-t^2}y - u\,|^2}{1-e^{-2t^2}}\Big)(f_1(u) - f_1(x))du. \end{aligned}$$

Now, if $| x |\leq k-1$, $| u-x |< \delta$ implies

$$| u |=| u-x+x |\leq| u-x | + | x |< \delta+k-1 < 1+k-1 = k$$

and then, $f_1(u) = f(u) \wedge f_1(x) = f(x)$. Therefore, if $(y,t) \in \Gamma_\gamma(x)$,

$$\begin{aligned}
&\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\left|\int_{|x-u|\leq\delta} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big)(f_1(u)-f_1(x))du\right|\\
&\quad= \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\left|\int_{|x-u|\leq\delta} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big)(f(u)-f(x))du\right|\\
&\quad= \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\left|\int_{\mathbb{R}^d} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big)g(u)du\right|\\
&\quad\leq \mathscr{T}^*_\gamma g(x) \leq C_d \mathscr{M}_\gamma g(x) \leq C_d\varepsilon,
\end{aligned}$$

by using Theorem 4.40.

Observe that if $(y,t) \in \Gamma_\gamma(x)$ and taking $t < \delta/2$, then $| u-x |> \delta$ implies

$$\delta <| u-x |\leq| u-y | + | y-x |;$$

therefore,

$$\delta <| u-y | + | y-x |<| u-y | +t \leq| u-y | + \frac{\delta}{2};$$

thus, $| u-y |> \frac{\delta}{2}$. Hence,

$$\begin{aligned}
&\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\left|\int_{|u-x|>\delta} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big)(f_1(u)-f_1(x))du\right|\\
&\quad\leq \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\int_{|u-x|>\delta} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big) | f_1(u) | du\\
&\qquad+\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} | f_1(x) | \int_{|u-x|>\delta} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big)du\\
&\quad\leq \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\int_{|u-y|>\frac{\delta}{2}} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big) | f_1(u) | du\\
&\qquad+\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} | f_1(x) | \int_{|u-x|>\delta} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big)du.
\end{aligned}$$

Now, we have

$$\begin{aligned}
&\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\int_{|u-y|>\frac{\delta}{2}} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big) | f_1(u) | du\\
&\quad= \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}}\int_{|u-y|>\frac{\delta}{2},|u|<k} \exp\Big(-\frac{| e^{-t^2}y-u |^2}{1-e^{-2t^2}}\Big) | f(u) | du
\end{aligned}$$

$$\leq \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} e^{k^2} \int_{|u-y|>\frac{\delta}{2},|u|<k} \exp\left(-\frac{|e^{-t^2}y-u|^2}{1-e^{-2t^2}}\right) |f(u)| e^{-|u|^2} du.$$

Then, for $0 < t^2 < \log\left(\frac{4k+2\delta}{4k+\delta}\right)$, $|u-y| > \frac{\delta}{2}, |u| < k$ implies that

$$\begin{aligned}
|e^{-t^2}y-u| &= |e^{-t^2}y - e^{-t^2}u + e^{-t^2}u - u| = |e^{-t^2}(y-u) - (u - e^{-t^2}u)| \\
&\geq e^{-t^2}|y-u| - |u - e^{-t^2}u| = e^{-t^2}|y-u| - (1-e^{-t^2})|u| \\
&\geq e^{-t^2}\frac{\delta}{2} - k(1-e^{-t^2}) = e^{-t^2}\left(\frac{\delta}{2}+k\right) - k.
\end{aligned}$$

But, as $0 < t^2 < \log\left(\frac{4k+2\delta}{4k+\delta}\right)$, then $e^{-t^2} > \frac{4k+\delta}{4k+2\delta}$. Hence,

$$\begin{aligned}
e^{-t^2}\left(\frac{\delta}{2}+k\right) - k &> \frac{4k+\delta}{4(2k+\delta)}(2k+\delta) - k \\
&= \frac{4k+\delta}{4} - k = \frac{4k+\delta-4k}{4} = \frac{\delta}{4}.
\end{aligned}$$

Therefore, $|u-y| > \frac{\delta}{2}$, $|u| < k$ implies $|e^{-t^2}y-u| > \frac{\delta}{4}$; hence,

$$\begin{aligned}
&\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} \int_{|u-y|>\frac{\delta}{2}} \exp\left(-\frac{|e^{-t^2}y-u|^2}{1-e^{-2t^2}}\right) |f_1(u)| du \\
&\leq \frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} e^{k^2} \int_{|u-y|>\frac{\delta}{2},|u|<k} e^{-\frac{\delta^2}{16(1-e^{-2t^2})}} |f(u)| e^{-|u|^2} du \\
&\leq \frac{e^{-\frac{\delta^2}{16(1-e^{-2t^2})}+k^2}}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} \int_{\mathbb{R}^d} |f(u)| e^{-|u|^2} du = \frac{e^{-\frac{\delta^2}{16(1-e^{-2t^2})}+k^2}}{(1-e^{-2t^2})^{d/2}} \|f\|_{1,\gamma}.
\end{aligned}$$

On the other hand, taking the change of variables $s = u - e^{-t^2}y$, we have

$$\begin{aligned}
&\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} |f_1(x)| \int_{|u-x|>\delta} \exp\left(-\frac{|e^{-t^2}y-u|^2}{1-e^{-2t^2}}\right) du \\
&= \frac{|f_1(x)|}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} \int_{|x-s-e^{-t^2}y|>\delta} e^{\frac{-|s|^2}{1-e^{-2t^2}}} ds \\
&= \frac{|f(x)|}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} \int_{|x-s-e^{-t^2}y|>\delta} e^{\frac{-|s|^2}{1-e^{-2t^2}}} ds,
\end{aligned}$$

because $f_1(x) = f(x)$ as $|x| \leq k-1 < k$.

Thus, taking $0 < t^2 < \log\left(\dfrac{k-1-\delta/2}{k-1-3\delta/4}\right)$, $|\, x-s-e^{-t^2}y \,| > \delta$ implies

$$|\, s \,| = |\, s-x+e^{-t^2}y+x-e^{-t^2}y \,| \geq |\, s-x+e^{-t^2}y \,| - |\, e^{-t^2}y-x \,|,$$

but

$$|\, e^{-t^2}y-x \,| = |\, e^{-t^2}y-e^{-t^2}x+e^{-t^2}x-x \,| \leq e^{-t^2}\,|\, y-x \,| + (1-e^{-t^2})\,|\, x \,|$$

$$\leq e^{-t^2}t+(1-e^{-t^2})(k-1).$$

Hence, because $t < \dfrac{\delta}{2}$,

$$|\, s-x+e^{-t^2}y \,| - |\, e^{-t^2}y-x \,| > \delta - e^{-t^2}t-(1-e^{-t^2})(k-1)$$

$$\geq \delta-(k-1)+\left(k-1-\frac{\delta}{2}\right)e^{-t^2},$$

and as $0 < t^2 < \log\left(\dfrac{k-1-\delta/2}{k-1-3\delta/4}\right)$, then $e^{-t^2} > \dfrac{k-1-3\delta/4}{k-1-\delta/2}$. Hence,

$$\begin{aligned}|\, s \,| &> \delta-(k-1)+(k-1-\delta/2)e^{-t^2} \geq \delta-(k-1)+k-1-3\delta/4 \\ &= \delta-3\delta/4 = \frac{\delta}{4}.\end{aligned}$$

Then, $|\, x-s-e^{-t^2}y \,| > \delta$ implies $|\, s \,| > \dfrac{\delta}{4}$ if $0 < t^2 < \log\left(\dfrac{k-1-\delta/2}{k-1-3\delta/4}\right)$. Therefore, taking $w = \dfrac{s}{\sqrt{1-e^{-2t^2}}}$,

$$\begin{aligned}\frac{1}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} \int_{|u-y|>\frac{\delta}{2}} \exp\left(-\frac{|\, e^{-t^2}y-u \,|^2}{1-e^{-2t^2}}\right) |\, f_1(x) \,|\, du \\ \leq \frac{|\, f(x) \,|}{\pi^{d/2}(1-e^{-2t^2})^{d/2}} \int_{|s|>\frac{\delta}{4}} e^{\frac{-|s|^2}{1-e^{-2t^2}}}\, ds \\ = \frac{|\, f(x) \,|}{\pi^{d/2}} \int_{|w|>\frac{\delta}{4\sqrt{1-e^{-2t^2}}}} e^{-|w|^2} dw.\end{aligned}$$

Now, because $|\, x \,| \leq k-1 < k$, then $f_2(x) = 0$. Hence,

$$|\, u_2(y,t)-f_2(x) \,| = |\, u_2(y,t) \,| \leq \mathscr{T}^* f_2(x) \leq C_d \mathscr{M}_\gamma f_2(x)$$

for $(y,t) \in \Gamma_\gamma(x)$. Therefore,

$$|\, u(y,t)-f(x) \,| \leq |\, u_1(y,t)-f_1(x) \,| + |\, u_2(y,t)-f_2(x) \,|$$

$$
\begin{aligned}
&= |\, u_1(y,t) - f_1(x)\,| + |\, u_2(y,t)\,| \\
&\leq C_d\varepsilon + \frac{e^{\frac{-\delta}{16(1-e^{-2t^2})}+k^2}}{(1-e^{-2t^2})^{d/2}}\|f\|_{1,\gamma} \\
&\qquad + \frac{|\, f(x)\,|}{\pi^{d/2}}\int_{|w|>\frac{\delta}{4\sqrt{1-e^{-2t^2}}}} e^{-|w|^2}dw + C_d\mathscr{M}_\gamma f_2(x),
\end{aligned}
$$

if $(y,t) \in \Gamma_\gamma(x)$ and

$$
0 < t^2 < \min\left\{\log\left(\frac{4k+2\delta}{4k+\delta}\right), \log\left(\frac{k-1-\delta/2}{k-1-3\delta/4}\right), m(x)\right\} = \Lambda,
$$

by using Theorem 4.40. Thus, taking supremum on $(y,t) \in \Gamma_\gamma(x)$, $0 < t < \alpha < \Lambda$ and then taking $\alpha \to 0^+$ we obtain,

$$
\Omega f(x) \leq C_d(\varepsilon + \mathscr{M}_\gamma f_2(x))
$$

for all $\varepsilon > 0$ and almost every x with $|\, x\,| \leq k-1$. Thus,

$$
\Omega f(x) \leq C_d\mathscr{M}_\gamma f_2(x).
$$

Using the weak (1,1) boundedness of $\mathscr{M}_\gamma$, we obtain

$$
\begin{aligned}
&\gamma_d\left(\left\{x \in \mathbb{R}^d : |\, x\,| \leq k-1, \Omega f(x) > \varepsilon\right\}\right) \\
&\leq \gamma_d\left(\left\{x \in \mathbb{R}^d : |\, x\,| \leq k-1, C_d\mathscr{M}_\gamma f_2(x) > \varepsilon\right\}\right) \leq C'/\varepsilon\|f_2\|_{1,\gamma}.
\end{aligned}
$$

Given $\varepsilon > 0$, let us take for k sufficiently large such that $\|f_2\|_{1,\gamma} \leq \varepsilon^2/2C'$. We get,

$$
\gamma_d\left(\left\{x \in \mathbb{R}^d : |\, x\,| \leq k-1, \Omega f(x) > \varepsilon\right\}\right) \leq \varepsilon/2.
$$

Finally, observe that taking k big enough, we can make

$$
\gamma_d\left(\left\{x \in \mathbb{R}^d : |\, x\,| > k-1, \Omega f(x) > \varepsilon\right\}\right) \leq \gamma_d\left(\left\{x \in \mathbb{R}^d : |\, x\,| > k-1\right\}\right) \leq \varepsilon/2,
$$

and that implies that for any $\varepsilon > 0$,

$$
\gamma_d\left(\left\{x \in \mathbb{R}^d : \Omega f(x) > \varepsilon\right\}\right) \leq \varepsilon.
$$

Hence, $\Omega f(x) = 0$ almost everywhere. □

4.8 Notes and Further Results

1. Let us prove in detail that $\mathscr{M}_\gamma$ is not bounded in $L^1_{loc}(\gamma_d)$, i.e., given $f \in L^1_{loc}(\gamma_d)$, $\mathscr{M}_\gamma f \notin L^1(\gamma_d)$.
   - First, if $f = \chi_E$, where $E \subset \mathbb{R}^d$ is measurable and bounded, then let us see that
   $$\mathscr{M}_\gamma \chi_E(x) \geq C_d \frac{\gamma_d(E)}{|x|^d},$$
   for $|x|$ is big enough. As E is bounded, there exists $R > 0$ such that $E \subset B(0,R)$. Taking $x \notin B(0,R)$, i.e., $|x| > R$. Then, $E \subset B(x, 2|x|)$, and
   $$\begin{aligned}\mathscr{M}_\gamma \chi_E(x) &= \sup_{r>0} \frac{\gamma_d(E \cap B(x,r))}{\gamma_d(B(x,r))} \geq \frac{\gamma_d(E \cap B(x,2|x|))}{\gamma_d(B(x,|x|))} \\ &= \frac{\gamma_d(E)}{\gamma_d(B(x,r))} \geq \frac{C_d \gamma_d(E)}{r^d} = \frac{C_d \gamma_d(E)}{2^d |x|^d} = C_d \frac{\gamma_d(E)}{|x|^d},\end{aligned}$$
   as γ_d is a d-dimensional measure.
   - Now, let us prove that if $f \in L^1_{loc}(\gamma_d)$, $f \neq 0$ in a set of positive measure, then there exists C such that
   $$\mathscr{M}_\gamma f(x) \geq \frac{C}{|x|^d}, \tag{4.98}$$
   for $|x|$ big enough. As in the classical case, set $F = \Big\{x \in \mathbb{R}^d : f(x) \neq 0\Big\}$. Then,
   $$F = \bigcup_{m \in \mathbb{N}} F_m = \bigcup_{m \in \mathbb{N}} \Big\{x \in \mathbb{R}^d : |f(x)| > 1/m, \text{ and } |x| < m\Big\}.$$
   As $F_m \subset F_{m+1}$, for all $m \in \mathbb{N}$, then, according to the lower continuity property of γ_d,
   $$\gamma_d(F) = \lim_{m \to \infty} \gamma_d(F_m).$$
   As $\gamma_d(F) > 0$, then there exists $M \in \mathbb{N}$ such that $\gamma_d(F_M) > 0$. Hence, F_M is a bounded set with a positive measure. Let $R > 0$, fixed. Then using the previous claim
   $$\begin{aligned}M_{\gamma_d} f(x) &\geq \frac{1}{\gamma_d(B(x,R))} \int_{F \cap B(x,R)} |f(y)| \gamma_d(dy) \\ &\geq \frac{1}{\gamma_d(B(x,R))} \int_{F_M \cap B(x,R)} |f(y)| \gamma_d(dy) \\ &> \frac{1}{M} \frac{\gamma_d(F_M \cap B(x,R))}{\gamma_d(B(x,R))} = \frac{1}{M} M_{\gamma_d} \chi_{F_M}(x) \geq \frac{C_d}{M} \frac{\gamma_d(F_M)}{|x|^d} = \frac{C}{|x|^d},\end{aligned}$$
   for $|x|$ big enough.

- Finally, from (4.98), as $|x|^d e^{-|x|^2} \notin L^1(\mathbb{R}^d)$, then $\mathscr{M}_\gamma f \notin L^1(\gamma_d)$.

2. We can also define for $a, b > 0$ the (a,b)-*truncated centered Hardy–Littlewood maximal function* as

$$M^{a,b} f(x) = \sup_{0<r<a\wedge \frac{b}{|x|}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \qquad (4.99)$$

for $f \in L^1_{loc}(\mathbb{R}^d)$, and the *Gaussian* (a,b)-*truncated centered Hardy–Littlewood maximal function*,

$$\begin{aligned} \mathscr{M}_\gamma^{a,b} f(x) &= \sup_{0<r<a\wedge \frac{b}{|x|}} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)| \gamma_d(dy) \\ &= \sup_{B(x,r)\in \mathscr{B}_{a,b}} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)| \gamma_d(dy), \qquad (4.100) \end{aligned}$$

$f \in L^1_{loc}(\gamma_d)$, i.e., we are taking the supremum over admissible balls $B(x,r) \in \mathscr{B}_{a,b}$. In particular, taking $a = b$, we get,

$$\begin{aligned} \mathscr{M}_\gamma^{a} f(x) &= \sup_{0<r<am(x)} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)| \gamma_d(dy) \\ &= \sup_{B(x,r)\in \mathscr{B}_{a}} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)| \gamma_d(dy), \qquad (4.101) \end{aligned}$$

which is called the truncated centered *Gaussian Hardy–Littlewood maximal function*.

Observe that, for any $f \in L^1_{loc}(\gamma_d)$,

$$\mathscr{M}_\gamma^{a,b} f(x) \leq \mathscr{M}_\gamma f(x),$$

for any $x \in \mathbb{R}^d$, which then implies that $\mathscr{M}_\gamma^{a,b}$ and $\mathscr{M}_\gamma^{a}$ are of weak type $(1,1)$, and strong type (p,p) for $1 < p < \infty$ with respect to the Gaussian measure γ_d.

Moreover, on $L^1_{loc}(\gamma_d)$, $M^{a,b}$ and $\mathscr{M}_\gamma^{a,b}$ are equivalent, as given an admissible ball $B(x,r) = \{y \in \mathbb{R}^d : |y-x| < r, 0 < r < a \wedge \frac{b}{|x|}\} \in \mathscr{B}_{a,b}$, we know, according to Lemma 1.4 that the Gaussian density is essentially constant. Then, as $0 < r < 1 \wedge \frac{1}{|x|}$,

$$\frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)| \gamma_d(dy) \geq \frac{e^{-2b} e^{|x|^2}}{|B(y,r)|} \int_{B(x,r)} |f(y)| e^{-a^2} e^{-2b} e^{-|x|^2} dy$$

$$= \frac{e^{-a^2}e^{-4b}}{|B(y,r)|}\int_{B(x,r)}|f(y)|\,dy,$$

and

$$\frac{1}{\gamma_d(B(x,r))}\int_{B(x,r)}|f(y)|\,\gamma_d(dy) \le \frac{e^{a^2}e^{2b}e^{|x|^2}}{|B(y,r)|}\int_{B(x,r)}|f(y)|\,e^{2b}e^{-|x|^2}dy$$
$$= \frac{e^{a^2}e^{4b}}{|B(y,r)|}\int_{B(x,r)}|f(y)|\,dy.$$

Hence, for any $f \in L^1_{loc}(\gamma_d)$ and $x \in \mathbb{R}^d$,

$$C_{a,b}^{-1}M^{a,b}f(x) \le \mathscr{M}_\gamma^{a,b}f(x) \le C_{a,b}M^{a,b}f(x),$$

where $C_{a,b} = e^{a^2}e^{4b}$. Additionally, observe that for the same reasons, $\mathscr{M}_\gamma^a$ can be written as

$$\mathscr{M}_\gamma^a f(x) = C_d \sup_{0<r<am(x)} \frac{e^{|x|^2}}{r^d}\int_{B(x,r)} f(y)\gamma_d(dy). \tag{4.102}$$

Finally, observe that, for any $f \in L^1_{loc}(\gamma_d)$ and $x \in \mathbb{R}^d$,

$$\mathscr{M}_\gamma f(x) \le \mathscr{M}_\gamma^{a,b}f(x) + \sup_{r\ge a\wedge \frac{b}{|x|}} \frac{1}{\gamma_d(B(x,r))}\int_{B(x,r)}|f(y)|\gamma_d(dy)$$
$$\le \mathscr{M}_\gamma^{a,b}f(x) + \frac{1}{\gamma_d\Big(B\Big(x, a\wedge\frac{b}{|x|}\Big)\Big)}\int_{\mathbb{R}^d}|f(y)|\gamma_d(dy)$$
$$\le \mathscr{M}_\gamma^{a,b}f(x) + C_d\Big(\frac{1}{a}\vee\frac{|x|}{b}\Big)^d e^{|x|^2}\|f\|_{1,\gamma},$$

and as $M^{a,b}$ and $\mathscr{M}_\gamma^{a,b}$ are equivalent, we also have

$$\mathscr{M}_\gamma f(x) \le C_{a,b}M^{a,b}f(x) + C_d\Big(\frac{1}{a}\vee\frac{|x|}{b}\Big)^d e^{|x|^2}\|f\|_{1,\gamma}.$$

See also [174, Theorem 3.1].

3. A more general result than [246] was obtained by A. Vargas in [285], who got necessary and sufficient conditions for radial, strictly positive measures μ with support on all $\mathbb{R}^d$, for which the corresponding Hardy–Littlewood maximal operator M_μ is of weak type $(1,1)$. The condition is that the measure μ is doubling "away from cero." Also, in [250] P. Sjögren and F. Soria proved estimates for the maximal operator associated with a wide class of monotone decreasing and radial measures μ. Additionally, in [139] (see also [138]), A. Infante and F. Soria extended the result obtained in [250] for the case of rotation invariant measures that are increasing along rays, in addition to results on general measures.

4. In [48], C. P. Calderón, A. S. Coré, and W. Urbina obtained, in one dimension, a special type of covering for open sets in $\mathbb{R}$ that has the same type of building principle as the classical Whitney decomposition. Observe that the Gaussian distribution (1.4), $\Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2} dy$ is a bijection between $\mathbb{R}$ and $(0,1)$. Then, considering the distance $d(x,y) = \left| \int_x^y e^{-t^2} dt \right|$, and using the same arguments as in [48], it seems possible to obtain from the classical Whitney decomposition a Gaussian version of it, and also a Gaussian Calderón–Zygmund decomposition. Its generalization to higher dimensions also seems possible.

5. The interest in studying the boundedness properties for maximal operators associated with a non-doubling measure was renewed after the development of the Calderón–Zygmund theory in this non-doubling setting by, among others, J. Tolsa [274] and J. Verdera [287]. For the particular case of Gaussian-like measures, the motivation comes from the study of the properties of the Ornstein–Uhlenbeck semigroup and operators associated with it.

6. In [128], E. Harboure, J. L. Torrea, and B. Viviani characterize the weights w for which both the centered Gaussian Hardy–Littlewood maximal function $\mathscr{M}_\gamma$ and the maximal function for the Ornstein–Uhlenbeck semigroup T^* are well defined for every function in $L^p(w\gamma_d)$ and their Gaussian means converge almost everywhere. The condition is that $w^{-\frac{1}{p-1}} \in L^1(\gamma_d)$. This condition is also necessary and sufficient for the existence of a weight v such that $\mathscr{M}_\gamma$ and T^* are bounded from $L^p(w\gamma_d)$ into $L^p(v\gamma_d)$. Using J. L. Rubio de Francia's classical approach (see [101]), they prove the result by obtaining some vector value inequalities for the operator under consideration. Moreover, the weight v whose existence is guaranteed satisfies $\|v\|_{s/(p-s),\gamma} < \infty$ for every $0 < s < 1$ (see [128, Theorem 2.12]).

7. In 1988, C. Gutierrez and W. Urbina [125] came back to the problem of pointwise estimates for T^* and proved that

Theorem 4.48. *(Gutiérrez–Urbina) For $f \in L^1(\gamma_d)$, we have*

$$T^* f(x) \leq C_d \mathscr{M}_\gamma f(x) + (2 \vee |x|)^d e^{|x|^2} ||f||_{1,\gamma}. \tag{4.103}$$

Proof. We may assume $f \geq 0$. Set

$$u_0(x,s) = \frac{1}{\pi^{d/2} s^{d/2}} \int_{\mathbb{R}^d} \exp\Big(- \frac{|x-y|^2}{s} \Big) f(y) dy,$$

the heat semigroup, up to a constant. Taking $s = 1 - e^{-2t}$, then

$$\begin{aligned} T_t f(x) &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} \exp\Big(- \frac{|y - e^{-t}x|^2}{1-e^{-2t}} \Big) f(y) dy \\ &= \frac{1}{\pi^{d/2} s^{d/2}} \int_{\mathbb{R}^d} \exp\Big(- \frac{|y - \sqrt{1-s}x|^2}{s} \Big) f(y) dy = u_0(\sqrt{1-s}\, x, s); \end{aligned}$$

therefore,

$$T^* f(x) = \sup_{0<t<1} |u_0(\sqrt{1-t}\,x,t)|.$$

Let

$$T_1^* f(x) = \sup_{0<t<\left(\frac{1}{|x|}\wedge\frac{1}{2}\right)^2} |u_0(\sqrt{1-t}\,x,t)|,$$

and

$$T_2^* f(x) = \sup_{\left(\frac{1}{|x|}\wedge\frac{1}{2}\right)^2 \leq t<1} |u_0(\sqrt{1-t}\,x,t)|.$$

Given $x \in \mathbb{R}^d$, let

$$\Gamma_\gamma^p(x) = \left\{(y,t) : |y-x| \leq \sqrt{t}, 0 < t < \left(\frac{1}{|x|}\wedge\frac{1}{2}\right)^2\right\},$$

be a truncated parabolic cone and let us consider the non-tangential maximal function for the heat semigroup

$$U^* f(x) = \sup_{(y,t)\in\Gamma^p(x)} |u_0(y,t)|. \tag{4.104}$$

We show that

$$T_1^* f(x) \leq U^* f(x) \leq C_d M_{\gamma_d} f(x), \tag{4.105}$$

and

$$T_2^* f(x) \leq (2\vee|x|)^d e^{|x|^2} ||f||_{1,\gamma}. \tag{4.106}$$

The first inequality in (4.105) follows because, if $0 < t < \left(\frac{1}{|x|}\wedge\frac{1}{2}\right)^2$, then

$$|\sqrt{1-t}\,x - x| = (1-\sqrt{1-t})|x| < \frac{1-\sqrt{1-t}}{\sqrt{t}} < \sqrt{t};$$

therefore, $(\sqrt{1-t}\,x,t) \in \Gamma_\gamma^p(x)$.
For the second inequality, set $a_0 = 0$ and $a_j = j^{1/2}$ for $j \in \mathbb{N}$. We can write

$$\begin{aligned} u_0(y,t) &= \frac{1}{\pi^{d/2}t^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|y-u|^2}{t}\right) f(u)du \\ &= \frac{1}{\pi^{d/2}t^{d/2}} \sum_{j=1}^{\infty} \int_{a_{j-1}t^{1/2}\leq|y-u|<a_j t^{1/2}} \exp\left(-\frac{|y-u|^2}{t}\right) f(u)du. \end{aligned}$$

If $(y,t) \in \Gamma_\gamma^p(x)$ and $|y-u| < a_j t^{1/2}$, then

$$|x-u| \leq |y-u| + |y-u| \leq (1+a_j)t^{1/2}$$

and, consequently,

$$u_0(y,t) \le \frac{1}{\pi^{d/2}s^{d/2}} \sum_{j=1}^{\infty} e^{-a_{j-1}^2} \int_{|x-u|<(a_j+1)t^{1/2}} f(u)du.$$

We have $|u|^2 = |u-x|^2 + 2\langle x,(u-x)\rangle + |x|^2$. Hence,

$$\begin{aligned}
\int_{|x-u|<(a_j+1)t^{1/2}} f(u)du &\le \exp\Big(|x|^2 + 2|x|(1+a_j)s^{1/2} + (1+a_j)^2 t\Big) \\
&\qquad \times \int_{|y-u|<(a_j+1)t^{1/2}} f(u)e^{-|u|^2}du \\
&\le \exp\Big(|x|^2 + 2|x|(1+a_j)t^{1/2} + (1+a_j)^2 s\Big) \\
&\qquad \times \mathscr{M}_\gamma f(x) \int_{|x-u|<(a_j+1)t^{1/2}} e^{-|u|^2}du \\
&\le \exp\Big(2|x|(1+a_j)t^{1/2} + (1+a_j)^2 t\Big) \\
&\qquad \times \mathscr{M}_\gamma f(x) \int_{|x-u|<(a_j+1)t^{1/2}} e^{-|u-x|^2+2|x||u-x|}du \\
&\le \exp\Big(4|x|(1+a_j)t^{1/2} + (1+a_j)^2 t\Big) \\
&\qquad \times \mathscr{M}_\gamma f(x) \int_{|x-u|<(a_j+1)t^{1/2}} e^{-|u|^2}du \\
&\le C_d(1+a_j)^d t^{d/2} \exp\Big(4|x|(1+a_j)t^{1/2} + (1+a_j)^2 t\Big) \\
&\qquad \times \mathscr{M}_\gamma f(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
u_0(y,s) \le \frac{C_d}{\pi^{d/2}s^{d/2}} \Big[\sum_{j=1}^{\infty} e^{-a_{j-1}^2}(1+a_j)^d s^{d/2} \\
\times \exp\Big(4|x|(1+a_j)t^{1/2} + (1+a_j)^2 t\Big) \Big] \mathscr{M}_\gamma f(x).
\end{aligned}$$

But if $(y,t) \in \Gamma_\gamma^p(x)$

$$\begin{aligned}
&\sum_{j=1}^{\infty} e^{-a_{j-1}^2}(1+a_j)^d t^{d/2} \exp\Big(4|x|(1+a_j)t^{1/2} + (1+a_j)^2 t\Big) \\
&\qquad \le \sum_{j=1}^{\infty} e^{-a_{j-1}^2}(1+a_j)^d t^{d/2} \exp\big(4(1+a_j) + (1+a_j)^2/4\big) < \infty.
\end{aligned}$$

Hence,

$$u_0(y,s) \le C_d \mathscr{M}_\gamma f(x).$$

To prove (4.106), observe that if $t > t'$ and $x, x' \in \mathbb{R}^d$, then for any $u \in \mathbb{R}^d$ we have

$$\frac{|y-u|^2}{s} - \frac{|x-u|^2}{t} \le \frac{|y-x|^2}{s-t}.$$

Thus,

$$u(y,s) \le u(x,s)\left(\frac{s}{t}\right)^{d/2} e^{\frac{|x-y|^2}{(t-s)}}, \tag{4.107}$$

for $0 < t < s$. In particular,

$$u_0(\sqrt{1-t}\,x,t) \le u_0(0,1)\left(\frac{1}{t}\right)^{d/2} e^{|x|^2},$$

for $0 < t < 1$. If $t \ge \left(\frac{1}{|x|} \wedge \frac{1}{2}\right)^2$, then $1/t \le (2 \wedge |x|)^2$ and as

$$u_0(0,1) = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|u|^2} f(u)du = ||f||_{1,\gamma},$$

then (4.106) follows.

8. The inequality (4.103) implies, in particular, that $T^* f < \infty$ almost everywhere. Unfortunately, this inequality only allows us to prove the weak type $(1,1)$ inequality of T^* with respect to γ_d in the case $d = 1$, because of the second term (bad term). But for the case $d = 1$, Muckenhoupt's proof [193] is more direct and easier. Moreover, it is easy to see that the estimation done in Theorem 4.48 of $T_2^* f(x)$, is not good enough. Observe that if instead of using the inequality (4.107), we simply use the fact that the exponential is less than one,

$$\begin{aligned} u_0(\sqrt{1-s}x,s) &= \frac{1}{\pi^{d/2}s^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|y-\sqrt{1-s}x|^2}{s}\right) f(y)dy \\ &= \frac{e^{|x|^2}}{\pi^{d/2}s^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|\sqrt{1-s}y-x|^2}{s}\right) f(y)e^{-|y|^2}dy \\ &\le (2 \vee |x|)^d e^{|x|^2} \int_{\mathbb{R}^d} f(y)\frac{e^{-|y|^2}}{\pi^{d/2}}dy = (2 \vee |x|)^d e^{|x|^2} ||f||_{1,\gamma}, \end{aligned}$$

if $s > \left(\frac{1}{|x|} \wedge \frac{1}{2}\right)^2$.

Therefore, the problem is to improve the estimate of $T_2^* f(x)$. If we decompose $u_0(\sqrt{1-s}x,s)$ as

$$\begin{aligned} u_0(\sqrt{1-s}x,s) &= \frac{1}{\pi^{d/2}s^{d/2}} \int_{|x-y|<\frac{1}{|x|}\wedge\frac{1}{2}} \exp\left(-\frac{|y-\sqrt{1-s}x|^2}{s}\right) f(y)dy \\ &\quad + \frac{1}{\pi^{d/2}s^{d/2}} \int_{|x-y|\ge\frac{1}{|x|}\wedge\frac{1}{2}} \exp\left(-\frac{|y-\sqrt{1-s}x|^2}{s}\right) f(y)dy \\ &= (I) + (II). \end{aligned}$$

The first integral can be bounded, using an analogous argument as before, by

$$(I) \leq (2 \vee |x|)^d e^{|x|^2} \int_{|x-y|<\frac{1}{|x|}\wedge\frac{1}{2}} f(y) \frac{e^{|x|^2}}{\pi^{d/2} s^{d/2}} dy.$$

Now, as we know, the values of the exponential $e^{-|y|^2}$ in an admissible ball $B\Big(x, \frac{1}{|x|} \wedge \frac{1}{2}\Big)$ are equivalent; therefore,

$$\gamma_d\Big(B\Big(x, \frac{1}{|x|} \wedge \frac{1}{2}\Big)\Big) = \int_{|x-y|<\frac{1}{|x|}\wedge\frac{1}{2}} f(y) \frac{e^{-|y|^2}}{\pi^{d/2}} dy = C_d \left(\frac{1}{|x|} \wedge \frac{1}{2}\right)^d e^{-|x|^2}.$$

Hence, the first integral is bounded by the truncated Hardy–Littlewood maximal function $\mathscr{M}_\gamma^{1/2,1} f$, which we already know is bounded by the Gaussian Hardy–Littlewood maximal function $\mathscr{M}_\gamma f$. Thus, the second integral is the problematic one. Observe that it can be written as

$$(II) = \frac{1}{\pi^{d/2} s^{d/2}} \int_{|x-y|\geq(1/2)\wedge(1/|x|)} \exp\Big(-\frac{(1-s)(|y|^2+|x|^2) - 2\sqrt{1-s}\langle x,y\rangle}{s}\Big) \times e^{-|y|^2} f(y) dy,$$

and this integral can be divided into two integrals, one where $\langle x,y\rangle \geq 0$ and the other one where $\langle x,y\rangle < 0$. The latter is less than or equal to $(2 \vee |x|)^d ||f||_{1,\gamma}$, just by bounding the exponential term by one. As the term $(2 \vee |x|)^d \in L^1(\gamma_d)$, then, according to Chebyshev's inequality, we get that this integral behaves correctly for the weak $(1,1)$ with respect to γ_d.

For the first integral, we divide the region

$$\Big\{y \in \mathbb{R}^d : |x-y| \geq (1/2) \wedge (1/|x|), \langle x,y\rangle \geq 0\Big\}$$

into two parts: $|y| < \sqrt{1-s}|x|$ and $|y| \geq \sqrt{1-s}|x|$.

The integral on the first region is again of weak type $(1,1)$ because using Chebyshev's inequality, we get

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \int_{\substack{|x-y|\geq((1/2)\wedge(1/|x|))\\ \langle x,y\rangle\geq 0, |y|<\sqrt{1-s}|x|}} \exp\Big(-\frac{(1-s)(|y|^2+|x|^2) - 2\sqrt{1-s}\langle x,y\rangle}{s}\Big) \times e^{-|y|^2} f(y) dy > \lambda\Big\}\Big)$$

$$\leq \frac{1}{\lambda} \frac{1}{(\pi s)^{d/2}} \int_{\mathbb{R}^d} \int_{\substack{|x-y|\geq((1/2)\wedge(1/|x|))\\ \langle x,y\rangle\geq 0, |y|<\sqrt{1-s}|x|}} \exp\Big(-\frac{(1-s)(|y|^2+|x|^2) - 2\sqrt{1-s}\langle x,y\rangle}{s}\Big) \times e^{-|y|^2} f(y) dy \frac{e^{-|x|^2}}{\pi^{d/2}} dx.$$

But, as $\varepsilon > 0, \langle x,y\rangle \leq \frac{\varepsilon}{2}|x|^2 + \frac{1}{2\varepsilon}|y|^2$. Then, taking $\varepsilon = \sqrt{1-s}$, and using the inequality $|y| < \sqrt{1-s}|x|$ we get

$$\frac{1}{\lambda}\frac{1}{\pi^{d/2}s^{d/2}}\int_{\mathbb{R}^d}\int_{\substack{|x-y|\geq((1/2)\wedge(1/|x|))\\ \langle x,y\rangle\geq 0, |y|<\sqrt{1-s}|x|}} e^{-|y|^2} f(y)dy\frac{e^{-s|x|^2}}{\pi^{d/2}}dx$$

$$\leq \frac{1}{\lambda}\frac{1}{s^{d/2}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\frac{e^{-|y|^2}}{\pi^{d/2}}dy\frac{e^{-s|x|^2}}{\pi^{d/2}}dx \leq \frac{1}{\lambda}(2\vee|x|)^d\frac{e^{-s|x|^2}}{\pi^{d/2}}dx||f||_{1,\gamma}.$$

Therefore, the problematic region that remains to be studied is

$$\left\{y\in\mathbb{R}^d : |x-y|\geq(1/2)\wedge(1/|x|), \langle x,y\rangle\geq 0, |y|\geq\sqrt{1-s}|x|\right\}.$$

The proof of S. Pérez [221] is a substantial improvement of the arguments above, using very cleverly the geometry of the problem to obtain pointwise inequalities to get the weak type $(1,1)$.

9. As a corollary of Theorem 4.48, using inequality (4.103), we can prove the non-tangential convergence of

$$u_0(y,s) = \frac{1}{\pi^{d/2}s^{d/2}}\int_{\mathbb{R}^d}\exp\left(-\frac{|x-y|^2}{s}\right)f(y)dy,$$

which is a modification of the heat semigroup, for *truncated parabolic Gaussian cones*,

$$\Gamma_\gamma^{p,T}(x) = \left\{(y,t) : |y-x|\leq t^{1/2},\ 0<t<\left(\frac{1}{|x|}\wedge\frac{1}{2}\right)^2\right\}. \tag{4.108}$$

10. E. Pineda and W. Urbina's result regarding the non-tangential maximal function is the following:

Theorem 4.49. *Consider the truncated non-tangential maximal function associated with the Ornstein–Uhlenbeck semigroup, defined as*

$$\mathscr{T}_T^* f(x) = \sup_{(y,t)\in\Gamma_\gamma^{p,T}(x)} |\,T_t f(y)\,|\,. \tag{4.109}$$

Then, there exists a constant C_d such that

$$\mathscr{T}_T^* f(x) \leq C_d\mathscr{M}_\gamma f(x), \tag{4.110}$$

for any $f\in L^1(\gamma_d)$ and for all $x\in\mathbb{R}^d$.

For the proof, see [225, Lemma 1.1].

11. Using Forzani's result, Theorem 4.18, an alternative proof of Corollary 4.42 can be given (see for instance [281]). It is enough to prove that there exists a function Φ that satisfies the conditions in that result such that

$$\mathscr{T}_\gamma^*(A,a)f(x) \leq \mathscr{M}_\Phi f(x). \tag{4.111}$$

In fact, given $x \in \mathbb{R}^d$ by the definition of $\mathscr{T}_\gamma^*(A,a)f(x)$, let $y \in \mathbb{R}^d$ such that $|x-y| < At$, $t < am(x)$. Then, if $1 > r = e^{-t^2} > e^{-a^2} = 2C$ we have that $|x-y| < C\sqrt{1-r^2} \wedge m(x)$; therefore, again using the auxiliary function $\mathscr{M}_s$ defined in (4.96), and taking $r = e^{-t^2}$, we get

$$\begin{aligned}
\mathscr{M}_r f(y) &= \frac{e^{|y|^2}}{(1-r^2)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|ru-y|^2}{1-r^2}} f(u)\gamma_d(du) \\
&\leq C\frac{e^{|x|^2}}{(1-r^2)^{d/2}} \sum_{v\geq 1} \int_{B(\frac{x}{r}, v\sqrt{1-r^2})\setminus B(\frac{x}{r},(v-1)\sqrt{1-r^2})} e^{-\frac{|ru-y|^2}{1-r^2}} f(u)\gamma_d(du) \\
&\leq C\frac{e^{|x|^2}}{(1-r^2)^{d/2}} \int_{\mathbb{R}^d} \left(\sum_{v\geq 1} e^{-c(v-1)^2} \chi_{B(\frac{x}{r}, v\sqrt{1-r^2})\setminus B(\frac{x}{r},(v-1)\sqrt{1-r^2})}(u) \right) \\
&\qquad\qquad \times f(u)\gamma_d(du) \\
&\leq C\mathscr{M}_\Phi f(x)
\end{aligned}$$

where $\Phi(x) = \sum_{v\geq 1} e^{-C(v-1)^2} \chi_{[v-1,v)}(x)$.
The second inequality is a consequence of the fact that if $|x-y| < c\sqrt{1-r} \wedge m(x)$ and $z \in B\left(\frac{x}{r}, v\sqrt{1-r^2}\right) \setminus B\left(\frac{x}{r},(v-1)\sqrt{1-r^2}\right)$, then we have

$$\begin{aligned}
|rz-y| &\geq r|z-\frac{x}{r}| - |x-y| \\
&\geq rv\sqrt{1-r^2} - c\sqrt{1-r^2} \geq c(v-1)\sqrt{1-r^2}.
\end{aligned}$$

12. There is another proof of Theorem 4.45, using Theorem 4.28 (see [87]). Using Harnack's inequality (3.36), given $x \in \mathbb{R}^d$ we have, for all $(y,t) \in \Gamma_\gamma^{A,a}(x)$, i.e., $y \in B((x,t), A\,am(x))$, that $P_t(y) \leq CP_t(x)$; hence,

$$\mathscr{P}_\gamma^*(A,a)f(x) = \sup_{(y,t)\in\Gamma_\gamma^{A,a}(x)} |P_t f(y)| \leq C \sup_{t>0} |P_t f(x)| = P^* f(x).$$

Thus the weak type $(1,1)$ of the Poisson-Hermite maximal function follows from the weak type $(1,1)$ of P^* which was obtained in Theorem 4.28.

13. As a corollary of Theorem 4.48, using inequality (4.103), we can prove the almost everywhere (radial) convergence of the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ in $L^1(\gamma_d)$ in addition to the almost everywhere convergence for $f \in L^1(\gamma_d)$ as $t \to 0$ (see [125]).

14. Using a general statement for families of linear operators, we can get a simpler proof of the non-tangential convergence, both for the Ornstein–Uhlenbeck semigroup and for the Poisson–Hermite semigroup (see [225]). This result is a generalization of Theorem 2.2 of J. Duoandikoetxea's book [72].

Theorem 4.50. *Let $\{\Gamma_t\}_{t\geq 0}$ be a family of linear operators on $L^p(\mathbb{R}^d,\mu)$ and for any $x\in\mathbb{R}^d$, let $\Lambda(x)$ be a subset of $\mathbb{R}^{d+1}_+$ such that x is an accumulation point of $\Lambda(x)$. Define*

$$\Gamma^* f(x) = \sup\{|\Gamma_t f(y)| : (y,t)\in\Lambda(x)\},$$

for $f\in L^p(\mathbb{R}^d,\mu)$ and $x\in\mathbb{R}^d$. If Γ^* is weak (p,q), then the set

$$S=\left\{f\in L^p(\mathbb{R}^d,\mu) : \lim_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} \Gamma_t f(y) = f(x) \text{ almost everywhere}\right\}$$

is closed in $L^p(\mathbb{R}^d,\mu)$.

Proof. Let us consider a sequence (f_n) in S such that $f_n\to f$ in $L^p(\mathbb{R}^d,\mu)$. Then,

$$|\Gamma_t f(y) - f(x)| - |\Gamma_t f_n(y) - f_n(x)| \leq |\Gamma_t(f-f_n)(y) - (f(x)-f_n(x))|,$$

and this implies that, for each $n\in\mathbb{N}$, and for almost every $x\in\mathbb{R}^d$

$$\begin{aligned}\limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t f(y) - f(x)| &\leq \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t(f-f_n)(y) - (f(x)-f_n(x))| \\ &\leq \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t(f-f_n)(y)| + \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |f(x)-f_n(x)| \\ &\leq \Gamma^*(f-f_n)(x) + |f(x)-f_n(x)|.\end{aligned}$$

On the other hand, if we know that $a\leq b+c$, then $a>\lambda$ implies $b>\frac{\lambda}{2}\vee c>\frac{\lambda}{2}$.

Hence, given $\lambda>0$ and $n\in\mathbb{N}$, as $\limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t f(y) - f(x)| > \lambda$, we have

$$\Gamma^*(f-f_n)(x) > \frac{\lambda}{2} \vee |f(x)-f_n(x)| > \frac{\lambda}{2} \text{ almost everywhere}$$

and this implies that, given $\lambda>0$,

$$\begin{aligned}&\mu\left(\left\{x : \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t f(y) - f(x)| > \lambda\right\}\right) \\ &\qquad \leq \mu\left(\left\{x : \Gamma^*(f-f_n)(x) > \frac{\lambda}{2}\right\}\right) + \mu\left(\left\{x : |f(x)-f_n(x)| > \frac{\lambda}{2}\right\}\right)\end{aligned}$$

$$\leq \Big(\frac{2C}{\lambda}\|f-f_n\|_p\Big)^q + \Big(\frac{2}{\lambda}\|f-f_n\|_p\Big)^p,$$

for all $n \in \mathbb{N}$. Therefore,

$$\mu\Big(\Big\{x: \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t f(y) - f(x)| > \lambda\Big\}\Big) = 0$$

and, as this is true for all $\lambda > 0$, we get that

$$\mu\Big(\Big\{x: \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t f(y) - f(x)| > 0\Big\}\Big) = 0.$$

Now,

$$\Big\{x: \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t f(y) - f(x)| > 0\Big\} = \bigcup_{n=1}^{\infty} \Big\{x: \limsup_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} |\Gamma_t f(y) - f(x)| > \frac{1}{n}\Big\}.$$

Then,

$$\lim_{\substack{(y,t)\to x,\\ (y,t)\in\Lambda(x)}} \Gamma_t f(y) = f(x) \text{ almost everywhere}$$

and then $f \in S$. Therefore, S is a closed set in $L^p(\mathbb{R}^d, \mu)$. □

15. As a consequence of Theorem 4.50, we can alternatively get a proof of the non-tangential convergence for the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ and the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$.

Corollary 4.51. *The Ornstein–Uhlenbeck semigroup* $\{T_t\}_{t\geq 0}$ *and the Poisson–Hermite semigroup* $\{P_t\}_{t\geq 0}$ *verify*

$$\lim_{\substack{(y,t)\to x,\\ (y,t)\in\Gamma_\gamma^p(x)}} T_t f(y) = f(x) \textit{ almost everywhere } x \in \mathbb{R}^d,$$

and

$$\lim_{\substack{(y,t)\to x,\\ (y,t)\in\Gamma_\gamma^p(x)}} P_t f(y) = f(x) \textit{ almost everywhere } x \in \mathbb{R}^d.$$

Proof. Let us discuss the proof for the Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$. The proof for the Poisson–Hermite semigroup $\{P_t\}_{t>0}$ is totally similar. It is immediate that for any given polynomial $f(x) = \sum_{k=0}^{n} \mathbf{J}_k f(x)$, because

$$T_t f(y) = T_t\Big(\sum_{k=0}^{n} \mathbf{J}_k f(y)\Big) = \sum_{k=0}^{n} e^{-tk} \mathbf{J}_k f(y),$$

we have the non-tangential convergence,

$$\lim_{\substack{(y,t)\to x, \\ (y,t)\in\Gamma_\gamma^p(x)}} T_t f(y) = f(x),$$

for all $x \in \mathbb{R}^d$. Now, considering the set

$$S = \left\{ f \in L^p(\gamma_d) : \lim_{\substack{(y,t)\to x, \\ (y,t)\in\Gamma_\gamma^p(x)}} T_t f(y) = f(x) \text{ almost everywhere} \right\},$$

corresponding to the Ornstein–Uhlenbeck semigroup, then the polynomials are in S. From the previous result, because the non-tangential maximal function for the Ornstein–Uhlenbeck semigroup $\mathscr{T}_\gamma^* f$ is weak $(1,1)$ with respect to the Gaussian measure, we get that the set S is closed in $L^p(\gamma_d)$, and as the polynomials are dense in $L^p(\gamma_d)$ then $S = L^p(\gamma_d)$. □

16. Regarding *approximations of the identity* for the Gaussian case, we have that, using the functional calculus of L, that is discussed in Chapter 6, we can get an approximation of the identity by taking any bounded holomorphic function ϕ on the sector of holomorphy that decays polynomially at infinity and that is equal to one at 0. The functions $\phi_t(z) = e^{-t|z|}$ and $\phi_t(z) = e^{-tz^2}$, which generate P_t and T_t, are good examples, but there are many more, such as $\phi_t(z) = \frac{1}{(1+tz^2)}$. The difference in the Euclidean case is that any radial convolution is in the functional calculus of the Laplacian operator Δ, which means that geometric approximations of the identity are appropriately related to the Laplacian operator. Attempts to use a similar geometric approach to the Gaussian setting have not been successful, because the modifications to classical approximations of the identity tend to be adapted to the Gaussian case only in the local region.

17. For the maximal function of the Laguerre semigroup T_α^*, the weak-type $(1,1)$ estimate with respect to the Gamma measure was obtained for $d = 1$ by B. Muckenhoupt in [195] using direct estimates, and for $d \geq 2$ by U. Dinger in [68]. She proves it first when α is an integer or half integer, taking advantage of the relationship between Laguerre and Hermite polynomials (see G. Szegő's book [262, (5.6.1)]) and using the fact that the maximal function of the Ornstein–Uhlenbeck semigroup is of weak type $(1,1)$ with respect to the Gaussian measure. Then, for general α, taking $r = e^{-t}$, she proves the result for the supremum restricted to the intervals $0 < r < 1/2$ and $1/2 \leq r < 1$ separately.

18. Additionally, in [238], E. Sasso proves weak-type and strong-type estimates for a class of maximal operators associated with the holomorphic Laguerre semigroup on the half-line $\mathbb{R}_+$. A complete and readable proof of this result was given by A. Nowak, P. Sjögren, and T. Z. Szarek in [216].

5

Littlewood–Paley–Stein Theory with Respect to the Gaussian Measure

Littlewood–Paley–Stein theory is an important area in harmonic analysis, with a great number of applications, as Littlewood–Paley functions are very useful in the proof of the L^p boundedness of singular integral operators, and in the characterization of Hardy spaces. E. Stein, in his beautiful monograph [253] showed how the classical notions of the Littlewood–Paley theory could be extended well beyond the Euclidean setting and also showed explicitly its link to the martingale theory in probability.

In this chapter, we study the Littlewood–Paley–Stein theory for the Gaussian measure. In 1976, P. A. Meyer [188] introduced some Littlewood–Paley functions with respect to the Gaussian measure using probabilistic methods (see also D. Stroock [260]). The study of Littlewood–Paley functions with respect to the Gaussian measure from an analytic point of view started in 1994 with C. Gutiérrez's paper [122], where the Gaussian Littlewood–Paley g_γ function was introduced, among other variants, and their $L^p(\gamma_d)$-boundedness properties were studied. The same year, E. Fabes and L. Forzani considered an area function for the Gaussian measure S_γ^a. Since then, several other Littlewood–Paley functions have been introduced. In this chapter, we study some of them, their $L^p(\gamma_d)$-boundedness properties, and some of their variants.

5.1 The Gaussian Littlewood–Paley g Function and Its Variants

As already stated, the study of Littlewood–Paley–Stein theory for the Gaussian measure started in 1994 with a paper by C. Gutiérrez [122].[1] Analogous to the classical case and following E. Stein's monograph [253], he introduces, the (first-order) Gaussian Littlewood–Paley g-function, g_γ and some variants.

[1]Gutiérrez actually considers a more general case of Gaussian measures involving positive-definite symmetric matrix B, but the case when B is the identity matrix is the only one that we are interested in.

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4_5

Definition 5.1. *(Gutierrez) The Gaussian Littlewood–Paley–Stein g-function is defined as*

$$g_\gamma(f)(x) = \left(\int_0^\infty |t\nabla P_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}, \tag{5.1}$$

where $\nabla = \left(\frac{\partial}{\partial t}, \frac{1}{\sqrt{2}}\nabla_x\right)$ *is the (total) gradient.*

Observe that g_γ can also be written as

$$g_\gamma(f)(x) = \left(\int_0^\infty |\nabla P_t(f)(x)|^2\, t\, dt\right)^{1/2}.$$

As in the classical case (see E. Stein [252, IV, §, page 82]), this operator can be considered as a vector valued singular integral, defined on a given Hilbert space. Let $\mathbb{H}_1 = \mathbb{R}$ and let $\mathbb{H}_2$ be the direct sum of $(d+1)$-copies of

$$L^2((0,\infty), \frac{dt}{t}) = \{h : (0,\infty) \to \mathbb{R} : \int_0^\infty |h(t)|^2 \frac{dt}{t} < \infty\},$$

with norm $\|h\|_{L^2((0,\infty),dt/t)} = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t}\right)^{1/2}$ and for $\mathbf{h} = (h_1, h_2, \cdots, h_{d+1}) \in \mathbb{H}_2$ let

$$|\mathbf{h}|_2 = \left(\sum_{k=1}^{d+1} \|h_k\|_{L^2}^2\right)^{1/2}$$

Consider the singular kernel,

$$\begin{aligned}\mathbf{K}(t,x,y) &= \left(t\frac{\partial p}{\partial t}(t,x,y), \frac{t}{\sqrt{2}}\frac{\partial p}{\partial x_1}(t,x,y), \ldots, \frac{t}{\sqrt{2}}\frac{\partial p}{\partial x_d}(t,x,y)\right) \\ &= \Big(K_0(t,x,y), K_1(t,x,y), \ldots, K_d(t,x,y)\Big)\end{aligned} \tag{5.2}$$

then, the Gaussian Littlewood–Paley function g_γ can be written as

$$g_\gamma(f)(x) = \left|\int_{\mathbb{R}^d} \mathbf{K}(\cdot,x,y) f(y) dy\right|_2. \tag{5.3}$$

Observe that if $f \geq 0$ is a bounded smooth function whose first and second derivatives are also bounded, then,

$$g_\gamma(f)(x)^2 \leq C(1+|x|)^2 \int_0^\infty te^{-2t} dt \leq C(1+|x|)^2, \tag{5.4}$$

and, therefore, $g_\gamma(f) \in L^p(\gamma_d)$ for $p > 0$ for this case. In general, we have the following result.

Theorem 5.2. *(Gutiérrez) For* $1 < p < \infty$ *there exists a constant* C_p*, depending only on p, such that, for any* $f \in L^p(\gamma_d)$

$$||g_\gamma(f)||_{p,\gamma} \leq C_p ||f||_{p,\gamma}. \tag{5.5}$$

To prove this result, we need the following lemmas,

Lemma 5.3. *Let u be a non-negative solution of the equation*

$$L_1 u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2} + Lu(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{1}{2}\Delta u(x,t) - \langle x, \nabla_x u(x,t)\rangle = 0. \quad (5.6)$$

If $p \geq 1$, then

$$L_1 u^p = p(p-1)u^{p-2}|\nabla_x u|^2.$$

Proof. Differentiating

$$\begin{aligned}
\frac{\partial^2 u^p(x,t)}{\partial t^2} &= p(p-1)u^{p-2}\frac{\partial u(x,t)}{\partial t} + pu^{p-1}\frac{\partial^2 u(x,t)}{\partial t^2},\\
\frac{\partial u^p(x,t)}{\partial x_i} &= pu^{p-1}\frac{\partial u(x,t)}{\partial x_i},\\
\frac{\partial^2 u^p(x,t)}{\partial x_i \partial x_j} &= p(p-1)u^{p-2}(\frac{\partial u(x,t)}{\partial x_i})^2 + pu^{p-1}\frac{\partial^2 u(x,t)}{\partial x_i \partial x_j}.
\end{aligned}$$

The lemma follows immediately. □

Lemma 5.4. *Let F be a smooth function in $\mathbb{R}^{d+1}_+$, such that $F(x,t) \geq 0$ and $L_1 F(x,t) \geq 0$. Suppose that there are constants M and C, such that for all $R > 0$, we have*

$$|F(x,t)| \leq M, \quad (5.7)$$

$$\sup\{|\nabla_x F(x,t)| : |x| \leq R, \, 0 < t < R\} \leq M, \quad (5.8)$$

$$R|\frac{\partial F}{\partial t}(x,R)| \leq C|x|\phi(R), \quad (5.9)$$

where $\phi(R) \to 0$ as $R \to \infty$, and such that $\int_0^\infty \frac{\phi(t)}{t} dt < \infty$. Then,

$$\int_0^\infty \int_{\mathbb{R}^d} tL_1 F(x,t)\gamma_d(dx)\,dt = \int_{\mathbb{R}^d} F(x,0)\gamma_d(dx) - \int_{\mathbb{R}^d} F(x,\infty))\gamma_d(dx). \quad (5.10)$$

Proof. The scheme of the proof is analogous to the proof of Lemma 3, page 50 of Stein's monograph [253], but some technicalities have to be overcome.
Let

$$D_R = \{(x,t) \in \mathbb{R}^d : |x| \leq R, \, 0 \leq t \leq R\}, \quad C_1(R) = \{(x,0) \in \mathbb{R}^d : |x| \leq R\},$$

$$C_2(R) = \{(x,R) \in \mathbb{R}^d : |x| \leq R\}, \quad C_3(R) = \{(x,t) \in \mathbb{R}^d : |x| = R, \, 0 \leq t \leq R\}.$$

Then,

$$\partial D_R = C_1(R) \cup C_2(R) \cup C_3(R),$$

and

$$\int\int_{D_R} tL_1F(x,t)\gamma_d(dx)\,dt = \int\int_{D_R} t(\frac{\partial^2 F(x,t)}{\partial t^2} + \frac{1}{2}\Delta_x F(x,t))\gamma_d(dx)\,dt - \int\int_{D_R} t\langle x, \nabla_x F(x,t)\rangle \gamma_d(dx)\,dt = (I) - (II).$$

Now, using the product rule of differentiation and the divergence theorem

$$\begin{aligned}
(II) &= \int\int_{D_R} t\langle x, \nabla_x F(x,t)\rangle \gamma_d(dx)\,dt = \frac{1}{\pi^{d/2}}\int\int_{D_R} t\Big(\sum_{i=1}^{d} x_i \frac{\partial F}{\partial x_i}\Big) e^{-|x|^2} dx\,dt \\
&= -\frac{1}{2\pi^{d/2}}\int\int_{D_R} t\Big(\sum_{i=1}^{d} \frac{\partial e^{-|x|^2}}{\partial x_i}\frac{\partial F}{\partial x_i}(x,t)\Big) dx\,dt \\
&= -\frac{1}{2\pi^{d/2}}\int\int_{D_R} t\Big(\sum_{i=1}^{d} \frac{\partial}{\partial x_i}(F(x,t)e^{-|x|^2})\Big) dx\,dt \\
&\qquad + \frac{1}{2\pi^{d/2}}\int\int_{D_R} t\Delta_x(e^{-|x|^2})F(x,t)dx\,dt \\
&= -\frac{1}{2\pi^{d/2}}\sum_{i=1}^{d}\int\int_{\partial D_R} tF(x,t)\frac{\partial e^{-|x|^2}}{\partial x_i}\eta_i\, d\sigma(x,t) \\
&\qquad + \frac{1}{2\pi^{d/2}}\int\int_{D_R} t\Delta_x(e^{-|x|^2})F(x,t)dx\,dt \\
&= -\frac{1}{2\pi^{d/2}}\int\int_{\partial D_R} tF(x,t)\frac{\partial e^{-|x|^2}}{\partial \eta}\, d\sigma(x,t) \\
&\qquad + \frac{1}{2\pi^{d/2}}\int\int_{D_R} t\Delta_x(e^{-|x|^2})F(x,t)dx\,dt,
\end{aligned}$$

where $\eta = (\eta_1, \eta_2, \cdots, \eta_{d+1})$ denotes the unit outward normal to ∂D_R. Then, as $\partial D_R = C_1(R) \cup C_2(R) \cup C_3(R)$, splitting the first integral into three integrals, it can be proved that it tends to zero as $R \to \infty$ (for more details see [122, page 117]). We then have, using the divergence theorem again,

$$\begin{aligned}
&\lim_{R\to\infty}\int\int_{D_R} tL_1F(x,t)\gamma_d(dx)\,dt \\
&= \lim_{R\to\infty}\Big[\int\int_{D_R} t(\frac{\partial^2 F(x,t)}{\partial t^2} + \frac{1}{2}\Delta_x F(x,t))\gamma_d(dx)\,dt - \frac{1}{2\pi^{d/2}}\int\int_{D_R} t\Delta_x(e^{-|x|^2})F(x,t)dx\,dt\Big] \\
&= \lim_{R\to\infty}\frac{1}{\pi^{d/2}}\Big[\int\int_{D_R} t(\frac{\partial^2 F(x,t)}{\partial t^2} + \frac{1}{2}\Delta_x F(x,t))e^{-|x|^2} - (\frac{\partial^2 (te^{-|x|^2})}{\partial t^2} + \Delta_x(et^{-|x|^2})F(x,t))dx\,dt\Big] \\
&= \lim_{R\to\infty}\frac{1}{\pi^{d/2}}\Big[\int\int_{\partial D_R} te^{-|x|^2}(\triangle\, F(x,t)\cdot\eta') - F(x,t)(\triangle\,(te^{-|x|^2})\cdot\eta')\, d\sigma(x,t)\Big],
\end{aligned}$$

where $\eta' = (\frac{1}{2}\eta_1, \frac{1}{2}\eta_2, \cdots, \frac{1}{2}\eta_d, \eta_{d+1})$ denotes the unit outward normal to ∂D_R.

Using (5.9), $\phi(R) \to 0$, the mean value theorem and the integrability condition on ϕ, we conclude that $\lim_{R\to\infty} F(x,R) = F(x,\infty)$ exists for each x. Then, again as $\partial D_R = C_1(R) \cup C_2(R) \cup C_3(R)$, splitting the last integral into three integrals, it can be proved that tends to

$$\int_{\mathbb{R}^d} F(x,0)\rangle \gamma_d(dx) - \int_{\mathbb{R}^d} F(x,\infty)\rangle \gamma_d(dx)$$

as $R \to \infty$ (for more details see [122, page 118]). This completes the proof of the lemma. □

The hypothesis $L_1 F(x,t) \geq 0$ in Lemma 5.4 can be replaced by

$$\int_0^\infty \int_{\mathbb{R}^d} t|L_1 F(x,t)|\gamma_d(dx)dt < \infty. \tag{5.11}$$

In that case, the proof is the same, with the exception that when we take the limit, we need to use Lebesgue's dominated convergence.

We are ready to prove Theorem 5.2.

Proof. First, we shall apply Lemma 5.4 in the case $F(x,t) = u(x,t)^p, p > 1$, $u(x,t) = P_t f(x)$, where $f \geq 0$a bounded smooth function whose first and second derivatives are also bounded. We need to verify that such a $F(x,t)$ satisfies the hypothesis of Lemma 5.4. Set $v(x,t) = T_t f(x)$, then

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta_x v - \langle x, \nabla_x v\rangle.$$

According to the subordination formula, we know that

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} v\left(x, \frac{t^2}{4u}\right) du,$$

then

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{\partial}{\partial t}\left(v\left(x, \frac{t^2}{4u}\right)\right) du \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} \left[\frac{1}{2}\Delta_x v\left(x, \frac{t^2}{4u}\right) - \langle x, \nabla_x v\left(x, \frac{t^2}{4u}\right)\rangle\right] du.\end{aligned}$$

Now, let us prove that

$$\nabla_x v(x,t) = e^{-t} T_t(\nabla_x f)(x),$$

and

$$\Delta_x v(x,t) = e^{-2t} T_t(\Delta_x f)(x).$$

This follows by using the representation (2.36) of the Ornstein–Uhlenbeck semigroup

$$v(x,t) = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} f(e^{-t}x - (1-e^{-2t})^{1/2}y)e^{-|y|^2} dy,$$

and differentiating with respect to x_i. Therefore,

$$\frac{\partial u}{\partial t}(x,t) = \frac{1}{4\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/2u} T_{t^2/4u}(\Delta_x f)(x) du$$

$$- \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/4u} \langle x, T_{t^2/4u}(\nabla_x f)(x)\rangle du$$

Using the change of variables $s = \frac{t^2}{2u}$ and the identity $\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2/4u} du = e^{-t}$, we get,

$$|t \frac{1}{4\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/2u} T_{t^2/4u}(\Delta_x f)(x) du| \leq C \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t^2}{u} e^{-t^2/2u} du = Cte^{-t\sqrt{2}}.$$

Also,

$$|t \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t}{u} e^{-t^2/4u} \langle x, T_{t^2/4u}(\nabla_x f)(x)\rangle du| \leq C \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{t^2}{u} |x| e^{-t^2/4u} du = Ct|x|e^{-t}.$$

Therefore,

$$t \left| \frac{\partial u}{\partial t}(x,t) \right| \leq Ct(1+|x|)e^{-t},$$

and because $u(x,t) \leq C$, then conditions $i)$ and $iii)$ of Lemma 5.4 for $F(x,t) = u^p(x,t)$ hold. To show condition $ii)$, we have

$$\nabla_x u(x,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \nabla_x v\left(x, \frac{t^2}{4u}\right) du,$$

and because the first derivatives of f are bounded, we obtain

$$|\nabla_x u(x,t)| \leq Ce^{-t},$$

which implies condition $ii)$.

Moreover, we know from inequality (5.4), that $g_\gamma(f) \in L^p(\gamma_d)$ for $p > 0$. Now, we prove

$$||g_\gamma(f)||_{p,\gamma} \leq C_p ||f||_{p,\gamma}.$$

It is done in several cases,

- Case #1: $1 < p \leq 2$. Again, let $f \geq 0$ be a bounded smooth function whose first and second derivatives are also bounded. If $\varepsilon > 0$, let $f_\varepsilon = \varepsilon + f$, then as the Ornstein–Uhlenbeck semigroup is conservative, $T_t(f_\varepsilon) = \varepsilon + T_t f$; therefore, $P_t(f_\varepsilon) = \varepsilon + P_t f > 0$. From Lemma 5.3, we get

$$(g_\gamma(f)(x))^2 = g_\gamma(f_\varepsilon)(x)^2 = \int_0^\infty t|\nabla(P_t f_\varepsilon)|^2 dt = \frac{1}{p(p-1)}\int_0^\infty t(P_t f_\varepsilon(x))^{2-p} L_1(P_t f_\varepsilon)^p dt$$
$$\leq \frac{1}{p(p-1)} P^* f_\varepsilon(x)^{2-p} \int_0^\infty t L_1(P_t f)^p dt.$$

Then,

$$\|g_\gamma(f)\|_{p,\gamma}^p = \int_{\mathbb{R}^d} g_\gamma(f)(x)^p \gamma_d(dx) \leq \frac{1}{(p(p-1))^{p/2}} \int_{\mathbb{R}^d} P^* f_\varepsilon(x)^{(1-\frac{p}{2})p} \left(\int_0^\infty t L_1(P_t f)^p dt\right)^{p/2}$$
$$\leq \frac{1}{(p(p-1))^{p/2}} \left(\int_{\mathbb{R}^d} P^* f_\varepsilon(x)^p \gamma_d(dx)\right)^{1-\frac{p}{2}} \left(\int_{\mathbb{R}^d}\int_0^\infty t L_1(P_t f)^p \gamma_d(dx) dt\right)^{p/2}$$

$$\leq \frac{A_p}{(p(p-1))^{p/2}} \|f_\varepsilon\|_{p,\gamma}^{(1-\frac{p}{2})p} \left(\int_{\mathbb{R}^d}\int_0^\infty t L_1(P_t f)^p \gamma(x) dx dt\right)^{p/2}$$
$$\leq A'_p \|f_\varepsilon\|_{p,\gamma}^{(1-\frac{p}{2})p} \left(\int_{\mathbb{R}^d} f(x)^p \gamma(x) dx\right)^{p/2} = A'_p \|f_\varepsilon\|_{p,\gamma}^{(1-\frac{p}{2})p} \|f\|_{p,\gamma}^{p^2/2},$$

using Hölder's inequality, Lemma 5.4 and the L^p-boundedness of the maximal function P^*, (4.60). Taking $\varepsilon \to 0$, the inequality follows.

- Case #2: $p \geq 4$. Again, let $f \geq 0$ be a bounded smooth function whose first and second derivatives are also bounded. Recall that for any $t > 0$, T_t is a self-adjoint operator (see 2.39). We claim that

$$|P_t f(x)| \leq (P_t(f^2)(x))^{1/2}. \tag{5.12}$$

In fact,

$$P_t f(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} \int_{\mathbb{R}^d} M_{t^2/4u}(x,y) f(y) dy du$$
$$= \int_{\mathbb{R}^d} \widetilde{M_t}(x,y) f(y) dy,$$

where

$$\widetilde{M_t}(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} M_{t^2/4u}(x,y)du.$$

Observe that

$$\int_{\mathbb{R}^d} \widetilde{M_t}(x,y)dy = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \int_{\mathbb{R}^d} M_{t^2/4u}(x,y)dydu = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} 1du = 1.$$

and therefore, using the Cauchy–Schwartz inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^d} \widetilde{M_t}(x,y)f(y)dy &\leq \left(\int_{\mathbb{R}^d} f^2 \widetilde{M_t}(x,y)f^2(y)dy \right)^{1/2} \left(\int_{\mathbb{R}^d} \widetilde{M_t}(x,y)dy \right)^{1/2} \\ &= \left(\int_{\mathbb{R}^d} \widetilde{M_t}(x,y)f^2(y)dy \right)^{1/2} \end{aligned}$$

and from there (5.12) follows.

Let $u(x,t) = P_t f(x)$; according to the semigroup property of $\{P_t\}$, we have

$$\frac{\partial u}{\partial t}(x,s+t) = P_s\left(\frac{\partial u}{\partial t}(\cdot,t)\right)(x).$$

Additionally we have,

$$\frac{\partial T_t f}{\partial x_i}(x) = e^{-t} T_t \left(\frac{\partial f}{\partial y_i}\right)(x).$$

Hence,

$$\frac{\partial P_t f}{\partial x_i}(x) = P_t^{(1)} \left(\frac{\partial f}{\partial y_i}\right)(x),$$

and

$$\frac{\partial u}{\partial x_i}(x,t+s) = P_t^{(1)} \left(\frac{\partial u}{\partial x_i}(\cdot,s)\right)(x),$$

where $\{P_t^{(1)}\}$ is the translated Poisson–Hermite semigroup subordinated to the translated Ornstein–Uhlenbeck semigroup $\{T_t^{(1)}\}$ defined in (2.78). Therefore,

$$\begin{aligned} \nabla_x u(x,t) = \Bigg(P_{t/2}\left(\frac{\partial u}{\partial t}(\cdot,t/2)\right)(x), \frac{1}{\sqrt{2}} P_{t/2}^{(1)} \left(\frac{\partial u}{\partial x_1}(\cdot,t/2)\right)(x), \ldots \\ , \frac{1}{\sqrt{2}} P_{t/2}^{(1)} \left(\frac{\partial u}{\partial x_n}(\cdot,t/2)\right)(x) \Bigg). \end{aligned}$$

Let $\phi \geq 0$ and smooth, then

$$\begin{aligned}\int_{\mathbb{R}^d} g_\gamma(f)(x)^2\phi(x)\gamma_d(dx) &= \int_{\mathbb{R}^d}\int_0^\infty t|\nabla_x u(x,t)|^2\phi(x)\gamma_d(dx)dt\\ &\leq \int_{\mathbb{R}^d}\int_0^\infty t\Big[|P_{t/2}\left(\frac{\partial u}{\partial t}\right)(\cdot,t/2)(x)|^2\\ &\qquad +\frac{1}{2}\sum_{i=1}^n |P^{(1)}_{t/2}\left(\frac{\partial u}{\partial x_i}(\cdot,t/2)\right)(x)|^2\Big]\phi(x)\gamma_d(dx)dt\\ &\leq \int_{\mathbb{R}^d}\int_0^\infty t\Big[P_{t/2}\left(\left(\frac{\partial u}{\partial t}(\cdot,t/2)\right)^2\right)(x)\\ &\qquad +\frac{1}{2}\sum_{i=1}^n P_{t/2}\left(\left(\frac{\partial u}{\partial x_i}(\cdot,t/2)\right)^2\right)(x)\Big]\phi(x)\gamma_d(dx)dt,\end{aligned}$$

because $P^{(1)}_t f \leq P_t f$, if $f \geq 0$. Moreover, because for any $t > 0$, P_t is a self-adjoint operator (3.9), we obtain

$$\int_{\mathbb{R}^d} g_\gamma(f)(x)^2\phi(x)\gamma_d(dx) \leq 4\int_0^\infty t\int_{\mathbb{R}^d} |\nabla_x u(x,t)|^2\phi(x,t)\gamma_d(dx)dt,$$

where $\phi(x,t) = P_t\phi(x)$. According to Lemma 5.3 with $p = 2$, the right-hand side of the last inequality equals

$$J = 2\int_0^\infty t\int_{\mathbb{R}^d} L_1 u^2(x,t)\phi(x,t)\gamma_d(dx)dt.$$

Because $L_1(u^2\phi) = (L_1\phi)u^2 + 2\langle\nabla_x\phi,\nabla_x(u^2)\rangle$ and $L_1\phi(x,t) = 0$,

$$\phi L_1(u^2) = L_1(u^2\phi) - 2\langle\nabla_x\phi,\nabla_x(u^2)\rangle.$$

We claim that

$$\int_{\mathbb{R}^{d+1}_+} t|L_1(u^2\phi)|\gamma_d(dx)dt$$

is finite and then $F = u^2\phi$ satisfies the hypothesis of Lemma 5.4. Assume that $\phi \geq 0$ is a bounded smooth function whose first and second derivatives are also bounded. We have,

$$u^2\phi \geq 0, \quad |u^2\phi| \leq M, \quad |\nabla_x(u^2\phi)| \leq C$$

and because $(u^2\phi)_t = 2uu_t\phi + u^2\phi_t$, we obtain

$$t|(u^2\phi)_t| \leq ct|u_t| + ct\phi_t \leq ct(1+|x|)e^{-t}.$$

Also,

$$|L_1(u^2\phi)| \leq c|L_1 u^2| + 2|\nabla_x \phi||\nabla_x(u^2)|.$$

According to Lemma 5.4 $\int_{\mathbb{R}^{d+1}_+} t|L_1 u^2|\gamma_d(dx)dt < \infty$. Also, using the Cauchy–Schwartz inequality and the inequality (5.4), we have

$$\int_0^\infty t|\nabla_x \phi||\nabla_x u^2|dt \leq C g_\gamma(f)(x) g(\phi)(x) \leq c(1+|x|)^2.$$

This proves the claim. Therefore, according to condition (5.11), we write

$$\begin{aligned} J &= 2\int_0^\infty t \int_{\mathbb{R}^d} L_1(u^2(x,t)\phi(x,t))\gamma_d(dx)dt - 4\int_0^\infty \int_{\mathbb{R}^d} t\langle|\nabla_x \phi|, \nabla_x(u^2)\rangle \gamma_d(dx)dt \\ &= I - II, \end{aligned}$$

and by Lemma 5.3,

$$I \leq 2\int_{\mathbb{R}^d} f(x)^2 \phi(x)\gamma_d(dx).$$

Also, using the Cauchy–Schwartz inequality

$$II \leq 8\int_{\mathbb{R}^d} P^* f(x) g_\gamma(f)(x) g(\phi)(x)\gamma_d(dx).$$

Then, we obtain

$$\begin{aligned} \int_{\mathbb{R}^d} g_\gamma(f)(x)^2 \phi(x)\gamma_d(dx) \leq 2\int_{\mathbb{R}^d} f(x)^2 \phi(x)\gamma_d(dx) \\ + 8\int_{\mathbb{R}^d} P^* f(x) g_\gamma(f)(x) g(\phi)(x)\gamma_d(dx). \end{aligned}$$

Because $p \geq 4$, let us consider $\frac{1}{q} + \frac{2}{p} = 1$, and ϕ such that $\|\phi\|_{q,\gamma} \leq 1$, then, by Hölder's inequality, the $L^p(\gamma_d)$ boundedness of P^* and the $L^p(\gamma_d)$ boundedness of g, for $q \leq 2$ we get

$$\int_{\mathbb{R}^d} g_\gamma(f)^2(x)\phi(x)\gamma_d(dx) \leq c\|f\|_{p,\gamma}^2 + c\|g_\gamma(f)\|_{p,\gamma}\|f\|_p,$$

for all $\phi \geq 0, \|\phi\|_{q,\gamma} \leq 1$. Therefore,

$$\|g_\gamma(f)^2\|_{p/2,\gamma} \leq c\|f\|_{p,\gamma}^2 + c\|g_\gamma(f)\|_{p,\gamma}\|f\|_p,$$

which implies the result.

- Case #3: $2 \leq p < 4$. The result follows by the Riesz–Thorin interpolation theorem, Theorem 10.21.

The inequality (5.20) has been proved for $f \geq 0$, a bounded smooth function whose first and second derivatives are also bounded. For a general function f, using the density of those functions on $L^p(\gamma_d)$, we write f as a difference of its positive and negative part $f = f^+ - f^-$ and, approximating each part by an increasing sequence of bounded smooth function, whose first and second derivatives are also bounded, we get (5.1) in full generality. □

It is easy to see that the proof is still true for Hilbert space valued functions. This observation is crucial for the proof of Theorem 5.13.

Additionally, *time and spatial Gaussian Littlewood–Paley* g functions can be defined.

Definition 5.5.

$$g_{t,\gamma}(f)(x) = \left(\int_0^\infty \left| t \frac{\partial P_t f}{\partial t}(x) \right|^2 \frac{dt}{t} \right)^{1/2}, \tag{5.13}$$

and

$$g_{x,\gamma}(f)(x) = \left(\int_0^\infty |t \nabla_x P_t f(x)|^2 \frac{dt}{t} \right)^{1/2}. \tag{5.14}$$

Observe that $g_\gamma^2 = g_{t,\gamma}^2 + g_{x,\gamma}^2$. Moreover, because these functions are bounded pointwise by g_γ then, from Theorem 5.2, it is immediate that they are also $L^p(\gamma_d)$-bounded, $1 < p < \infty$. Moreover, we have

Theorem 5.6. *For* $1 < p < \infty$, *there exist constants* $C_p, C'_p, C"_p$ *such that,*

$$C_p \|f\|_{p,\gamma} \leq \|g_\gamma(f)\|_{p,\gamma}, \tag{5.15}$$

$$C'_p \|f\|_{p,\gamma} \leq \|g_{t,\gamma}(f)\|_{p,\gamma}, \tag{5.16}$$

and

$$C"_p \|f\|_{p,\gamma} \leq \|g_{x,\gamma}(f)\|_{p,\gamma}, \tag{5.17}$$

Proof. Let us prove (5.16). Following E. Stein's proof [252, Chapter IV, §1], observe first that

$$\|g_{t,\gamma}(f)\|_{2,\gamma} = \frac{1}{2} \|f\|_{p2,\gamma},$$

for $f \in L^2(\gamma_d)$. Let us check this identity on the Hermite polynomials $\{H_\nu\}$,

$$\begin{aligned}
\|g_{t,\gamma}(H_\nu)\|_{2,\gamma}^2 &= \int_{\mathbb{R}^d} \left(\int_0^\infty \left| t \frac{\partial P_t H_\nu}{\partial t}(x) \right|^2 \frac{dt}{t} \right) \gamma_d(dx) \\
&= \int_{\mathbb{R}^d} \left(\int_0^\infty \left| t \frac{d}{dt}(e^{-\sqrt{|\nu|}t}) H_\nu(x) \right|^2 \frac{dt}{t} \right) \gamma_d(dx) \\
&= \int_{\mathbb{R}^d} \left(\int_0^\infty t(-\sqrt{|\nu|})^2 e^{-2\sqrt{|\nu|}t} dt \right) |H_\nu(x)|^2 \gamma_d(dx) \\
&= |\nu| \int_0^\infty t e^{-2\sqrt{|\nu|}t} dt \int_{\mathbb{R}^d} |H_\nu(x)|^2 \gamma_d(dx) = \frac{1}{4} \|H_\nu\|_{2,\gamma}^2.
\end{aligned}$$

Then, by polarization, we get

$$4\int_{\mathbb{R}^d}\Big(\int_0^{\infty} t\frac{\partial P_t f}{\partial t}(x)\frac{\partial P_t h}{\partial t}(x)dt\Big)\gamma_d(dx) = \int_{\mathbb{R}^d} f(x)h(x)\gamma_d(dx),$$

for $f,h \in L^2(\gamma_d)$. This identity leads, using the Cauchy–Schwarz inequality, to the inequality

$$\frac{1}{4}\Big|\int_{\mathbb{R}^d} f(x)g(x)\gamma_d(dx)\Big| \leq \int_{\mathbb{R}^d} g_{t,\gamma}(f)(x)g_{t,\gamma}(g)(x)\gamma_d(dx).$$

Now, for $f \in L^2(\gamma_d)\cap L^p(\gamma_d)$, $1<p<\infty$, and q such that $\frac{1}{p}+\frac{1}{q}=1$, we get by the duality argument, and using Hölder's inequality

$$\begin{aligned}\|f\|_{p,\gamma} &= \sup_{h\in L^2(\gamma_d)\cap L^q(\gamma_d), \|h\|_{q,\gamma}\leq 1}\Big|\int_{\mathbb{R}^d} f(x)h(x)\gamma_d(dx)\Big| \\ &\leq 4\sup_{h\in L^2(\gamma_d)\cap L^q(\gamma_d), \|h\|_{q,\gamma}\leq 1}\Big|\int_{\mathbb{R}^d} g_{t,\gamma}(f)(x)g_{t,\gamma}(h)(x)\gamma_d(dx)\Big| \\ &\leq 4\sup_{h\in L^2(\gamma_d)\cap L^q(\gamma_d), \|h\|_{q,\gamma}\leq 1}\|g_{t,\gamma}(f)\|_{p,\gamma}\|g_{t,\gamma}(h)\|_{q,\gamma} \leq C_p\|g_{t,\gamma}(f)\|_{p,\gamma}.\end{aligned}$$

The passage to the general case is provided by a standard limiting argument. □

Also, in [122], C. Gutiérrez considered the following Gaussian Littlewood–Paley functions, associated with the *translated Poison–Hermite semigroup* $\{P_t^{(1)}\}_{\{t\geq 0\}}$ (see 3.56).

Definition 5.7.

$$g_{+,\gamma}^{(1)}(f)(x) = \Big(\int_0^{\infty}(|t\nabla P_t^{(1)}f(x)|^2 + (tP_t^{(1)}f(x))^2)\,\frac{dt}{t}\Big)^{1/2}, \tag{5.18}$$

$$g_{t,\gamma}^{(1)}(f)(x) = \Big(\int_0^{\infty}\Big|t\frac{\partial P_t^{(1)}f}{\partial t}(x)\Big|^2\frac{dt}{t}\Big)^{1/2}. \tag{5.19}$$

For those functions, Gutiérrez proves the following result.

Theorem 5.8. *For $1<p<\infty$, there exist constants C_p and C_p', depending only on p, such that for all $f\in L^p(\gamma_d)$*

$$||g_{+,\gamma}^{(1)}(f)||_{p,\gamma} \leq C_p||f||_{p,\gamma}, \tag{5.20}$$

and

$$||f||_{p,\gamma} \leq C_p'||g_{t,\gamma}^{(1)}(f)||_{p,\gamma}. \tag{5.21}$$

Proof. The proof of (5.20) is essentially analogous to that of Theorem 5.2, but working with the translated Poisson–Hermite semigroup $\{P_t^{(1)}\}_t$ and the operator

$$L_2u(x,t) = \frac{\partial^2 u(x,t)}{\partial t^2} + Lu(x,t) - u(x,t) = L_1u(x,t) - u(x,t)$$

(for details see Gutiérrez's article [122, Theorem 2]).

To prove the inequality (5.21), we know that the operator

$$\overline{L} = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle - I_d$$

has as eigenfunctions the Hermite polynomials $\mathbf{h}_\nu$ with eigenvalues $|\nu|+1$. Then, given $f \in L^2(\gamma_d)$ with Hermite expansion $f = \sum_\nu \hat{f}_H(\nu)\mathbf{h}_\nu$, for each $t > 0$, the translated Poisson operator $u(x,t) = P_t^{(1)} f(x)$ has expansion

$$\sum_\nu e^{-t\sqrt{|\nu|+1}} \hat{f}_H(\nu)\mathbf{h}_\nu;$$

therefore,

$$\frac{\partial u}{\partial t} \sim -\sum_\nu \sqrt{|\nu|+1} e^{-t\sqrt{|\nu|+1}} \hat{f}_H(\nu)\mathbf{h}_\nu.$$

Now, according to Parseval's identity, we have

$$\int_{\mathbb{R}^d} \left|\frac{\partial u(x,t)}{\partial t}\right|^2 \gamma_d(dx) = -\sum_\alpha (|\nu|+1)e^{-2t\sqrt{|\nu|+1}} (\hat{f}_H(\nu))^2;$$

hence,

$$\begin{aligned}\int_{\mathbb{R}^d} (g_{t,\gamma}^{(1)}(f)(x))^2 \gamma_d(dx) &= \int_0^\infty t \int_{\mathbb{R}^d} \left|\frac{\partial u(x,t)}{\partial t}\right|^2 \gamma_d(dx)dt \\ &= \sum_\nu \Big(\int_0^\infty t(|\nu|+1)e^{-2t\sqrt{|\nu|+1}} dt\Big)(\hat{f}_H(\nu))^2 = \frac{1}{4}\sum_\nu (\hat{f}_H(\nu))^2.\end{aligned}$$

Then,

$$4||g_{t,\gamma}^{(1)}(f)||_{2,\gamma}^2 = \sum_\nu (\hat{f}_H(\nu))^2 = ||f||_{2,\gamma}^2$$

and using the polarization argument in the last equality, we obtain

$$4\int_0^\infty t \int_{\mathbb{R}^d} \frac{\partial u_1(x,t)}{\partial t} \frac{\partial u_2(x,t)}{\partial t} \gamma_d(dx)dt = \int_{\mathbb{R}^d} f_1(x) f_2(x)\gamma_d(dx)dt,$$

where $u_i(x,t) = P_t^{(1)} f_i(x), f_i \in L^2(\gamma_d), i = 1,2.$

If $f \in L^2(\gamma_d) \cap L^p(\gamma_d)$, then, by duality, using Cauchy–Schwartz and Hölder's inequalities, we get

$$\begin{aligned}
\|f\|_{p,\gamma} &= \sup_{h \in L^2(\gamma_d) \cap L^q(\gamma_d), \|h\|_{q,\gamma} \leq 1} \Big| \int_{\mathbb{R}^d} f(x) h(x) \gamma_d(dx) \Big| \\
&= 4 \sup_{h \in L^2(\gamma_d) \cap L^q(\gamma_d), ||h||_{q,\gamma} \leq 1} \Big| \int_0^\infty t \int_{\mathbb{R}^d} \frac{\partial u_f(x,t)}{\partial t} \frac{\partial u_h(x,t)}{\partial t} \gamma_d(dx) dt \Big| \\
&\leq 4 \sup_{h \in L^2(\gamma_d) \cap L^q(\gamma_d), ||h||_{q,\gamma} \leq 1} \int_{\mathbb{R}^d} g_{t,\gamma}^{(1)}(f)(x) g_{t,\gamma}^{(1)}(h)(x) \gamma_d(dx) \\
&\leq 4 \sup_{h \in L^2(\gamma_d) \cap L^q(\gamma_d), ||h||_{q,\gamma} \leq 1} ||g_{t,\gamma}^{(1)}(f)||_{p,\gamma} ||g_{t,\gamma}^{(1)}(h)||_{q,\gamma} \\
&\leq C_p ||g_1^{(1)}(f)||_{p,\gamma},
\end{aligned}$$

using (5.20). □

In his doctoral dissertation, R. Scotto [244] proved the weak type $(1,1)$ for the function g_γ.

Theorem 5.9. *(R. Scotto) There exists a constant $C = C_d$ dependent only on dimension, such that for any function $f \in L^1(\gamma_d)$*

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : g_\gamma(f)(x) > \lambda\Big\}\Big) \leq \frac{C_d}{\lambda} \|f\|_{1,\gamma} \tag{5.22}$$

Scotto proved this result by adapting the technique developed by Sjögren for the maximal function of the Ornstein–Uhlenbeck semigroup, splitting the operator into a local part and a global part using as the local region N_R defined in (4.62). First, Scotto considers the representation of g_γ (5.3) and studies the components of the kernel $K(x,y)$, see [245]. From (3.13) we have,

$$\begin{aligned}
K_0(t,x,y) &= t \frac{\partial p(t,x,y)}{\partial t} \qquad (5.23) \\
&= \frac{t}{2\pi^{(d+1)/2}} \int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \Big(1 - \frac{t^2}{2(-\log r)}\Big) \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r}. \\
&= -\frac{1}{2^2 \pi^{(d+1)/2}} \int_0^1 \varphi_1(t,r) \frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r}.
\end{aligned}$$

where,

$$\varphi_1(t,r) = C_d \frac{t \exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} H_2\Big(\frac{t}{2\sqrt{-\log r}}\Big),$$

and H_2 is the Hermite polynomial of order 2.

For the function $\varphi_1(t,r)$, we have

$$|\varphi_1(t,r)| \leq \frac{C_d}{(-\log r)^{3/2}} \exp\left(t^2/c\log r\right),$$

because $|x|e^{|x|^2} \leq Ce^{-|x|^2/c} \leq C$ for some $c>0$, and

$$\int_0^1 \varphi_1(t,r)\frac{dr}{r} = 0,$$

for all $t>0$, because taking the change of variables $u = \frac{t}{2\sqrt{(-\log r)}}$, integrating by parts and using the orthogonality of the Hermite polynomials, we get

$$\int_0^1 \varphi_1(t,r)\frac{dr}{r} = C_d\int_0^{+\infty} H_2(u)e^{-u^2}du = 0.$$

Let $\tau_1(t,r) := \int_0^r \varphi_1(t,s)\frac{ds}{s}$, then, for every $0<r<1$,

$$\|\tau_1(\cdot,r)\|^2_{L^2((0,\infty),dt/t)} = \int_0^\infty \left(\int_0^r \varphi_1(t,s)\frac{ds}{s}\right)^2\frac{dt}{t},$$

Splitting the integral in t into the sum of the integral over the intervals $[0,(-\log r)^{1/2}]$ and $[(-\log r)^{1/2},\infty)$ and using previous estimates, it is easy to prove that $\|\tau_1(\cdot,r)\|^2_{L^2((0,\infty),dt/t)}$ is finite and independent of r.

Now, because $\frac{\partial\tau_1}{\partial r}(t,r) = \frac{\varphi(t,r)}{r}$, using integration by parts we get

$$K_0(t,x,y) = \int_0^1 \left[\frac{2(xr-y)(x-ry)}{(1-r^2)^{(d+4)/2}} - \frac{rd}{(1-r^2)^{(d+2)/2}}\right]\exp\left(\frac{-|y-rx|^2}{(1-r^2)}\right)\tau_1(t,r)\,dr.$$

On the other hand, for $j=1,\cdots,d$, using (3.14), we get

$$\begin{aligned} K_j(t,x,y) &= \frac{t}{\sqrt{2}}\frac{\partial p(t,x,y)}{\partial x_j}(x,y) \qquad (5.24)\\ &= \frac{t}{2\sqrt{2}\pi^{(d+1)/2}}\int_0^1 \frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{(y_j-rx_j)}{(1-r^2)^{(d+2)/2}}\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)\frac{dr}{r}\\ &= C_d\int_0^1 t\frac{\exp\left(t^2/4\log r\right)}{(-\log r)}\omega(r)\frac{(y_j-rx_j)}{(1-r^2)^{(d+3)/2}}\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)dr, \end{aligned}$$

Next, Scotto uses the vector representation of g_γ (5.3), and decomposes it into a local part and a global part.

$$\begin{aligned} g_\gamma(f)(x) &= |\int_{\mathbb{R}^d} K(x,y)f(y)dy|_2\\ &= |\int_{N_R^x} K(x,y)f(y)dy|_2 + |\int_{\mathbb{R}^d\setminus N_R^x} K(x,y)f(y)dy|_2\\ &= g_{L,\gamma}(f)(x) + g_{G,\gamma}(f)(x). \end{aligned}$$

Then, he follows P. Sjögren's proof of Theorem 4.20. The argument is very long and technical (for details see [244]; see also [77]). Nevertheless, we discuss later in more detail a simplified version of the proof for the weak type $(1,1)$ for the higher order Gaussian Littlewood–Paley g functions (see Theorem 5.14), which happens to be essentially analogous to this one.

Alternatively, the weak type $(1,1)$ of g_γ function can be proved using S. Pérez and F. Soria's approach ([221] and [223]); considering as the local region $B_h(x) = B(x, d\,m(x))$ and splitting g_γ into a local part,

$$g_{L,\gamma} f(x) = g_\gamma(f\chi_{B_h(\cdot)})(x)$$

and a global part

$$g_{G,\gamma} = g_\gamma(f\chi_{B_h^c(\cdot)})(x)$$

Theorem 5.10. *(Pérez-Soria)*

i) The local part $g_{L,\gamma}$ of g_γ is of weak type $(1,1)$ with respect to the Gaussian measure, that is, there exists a constant C_d such that

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : g_{L,\gamma} f(x) > \lambda\right\}\right) \leq \frac{C_d}{\lambda}\|f\|_{1,\gamma}, \tag{5.25}$$

and $L^p(\mathbb{R}^d)$-bounded, for $1 < p < \infty$, that is, for each $1 < p < \infty$, there exists a constant $C_{d,p}$ such that

$$\|g_{L,\gamma} f\|_{p,\gamma} \leq C_{d,p}\|f\|_{p,\gamma}. \tag{5.26}$$

ii) For the global part $g_{G,\gamma} f$ we have,

$$g_{G,\gamma}(f)(x) \leq \int_{\{y:|x-y|\geq C_d m(x)\}} \overline{\mathscr{K}}(x,y)|f(y)|dy, \tag{5.27}$$

where $\overline{\mathscr{K}}(x,y)$ is the Gaussian maximal kernel defined in (4.40). Consequently,

$$g_{G,\gamma}(f)(x) \leq \overline{T} f(x) = \int_{\mathbb{R}^d} \overline{\mathscr{K}}(x,y) f(y) dy$$

where $\overline{T}$ is the maximal Gaussian operator defined in (4.46); therefore $g_{G,\gamma}$ is of weak type $(1,1)$ with respect to the Gaussian measure.

Proof. Again, we are going to use (5.3), the vector representation of g_γ.

From (3.6), we have

$$\begin{aligned}\frac{\partial p(t,x,y)}{\partial t} &= C_d \int_0^1 \left(1 + \frac{t^2}{2\log\sqrt{1-v}}\right) \frac{e^{\frac{t^2}{4\log\sqrt{1-s}}}}{(-\log\sqrt{1-v})^{3/2}} \frac{e^{-\frac{|y-\sqrt{1-v}x|^2}{v}}}{v^{d/2}} \frac{dv}{1-v} \\ &= C_d \int_0^1 \left(1 + \frac{t^2}{2\log\sqrt{1-v}}\right) \frac{e^{\frac{t^2}{4\log\sqrt{1-s}}}}{(-\log\sqrt{1-v})^{3/2}} \frac{e^{-u(v)}}{v^{d/2}} \frac{dv}{1-v},\end{aligned}$$

and, by integration by parts, we have

$$\begin{aligned}\frac{\partial p(t,x,y)}{\partial t} &= C_d \int_0^1 \frac{\partial}{\partial s}\Big(\int_0^s \Big(1+\frac{t^2}{2\log\sqrt{1-v}}\Big)\frac{e^{\frac{t^2}{4\log\sqrt{1-s}}}}{(-\log\sqrt{1-v})^{3/2}}\frac{dv}{1-v}\Big)\frac{e^{-u(s)}}{s^{d/2}}\,ds\\ &= C_d \int_0^1 \Big(\int_0^s \Big(1+\frac{t^2}{2\log\sqrt{1-v}}\Big)\frac{e^{\frac{t^2}{4\log\sqrt{1-s}}}}{(-\log\sqrt{1-v})^{3/2}}\frac{dv}{1-v}\Big)\frac{e^{-u(s)}}{s^{d/2}}\Big(u'(s)+\frac{d}{2s}\Big)\,ds\\ &= \int_0^1 \Psi(t,s)\frac{e^{-u(s)}}{s^{d/2}}\Big(\frac{d}{2s}-\frac{2(|x|^2+|y|^2)\sqrt{1-s}-2(2-s)\langle x,y\rangle}{2s^2\sqrt{1-s}}\Big)ds,\end{aligned}$$

with $\Psi(t,s) = C_d \int_0^s \Big(1+\frac{t^2}{2\log\sqrt{1-v}}\Big)\frac{e^{\frac{t^2}{4\log\sqrt{1-s}}}}{(-\log\sqrt{1-v})^{3/2}}\frac{dv}{1-v}$. Here, we have used the fact that

$$\int_0^1 \Big(1+\frac{t^2}{2\log\sqrt{1-v}}\Big)\frac{e^{\frac{t^2}{4\log\sqrt{1-s}}}}{(-\log\sqrt{1-v})^{3/2}}\frac{dv}{1-v} = 0;$$

which can also be proved by integration by parts (see [245]). Thus,

$$K_0(t,x,y) = t\int_0^1 \Psi(t,s)\frac{e^{-u(s)}}{s^{d/2}}\Big(\frac{d}{2s}-\frac{2(|x|^2+|y|^2)\sqrt{1-s}-2(2-s)\langle x,y\rangle}{2s^2\sqrt{1-s}}\Big)ds.$$

Using the similar arguments done by R. Scotto (see above and [245] or [223]), it can be proved that the $\mathbb{H}_2$-norm

$$|\Psi(t,\cdot)|_2 = C_d\Big(\int_0^\infty t\Big[\int_0^s \Big(1+\frac{t^2}{2\log\sqrt{1-v}}\Big)\frac{e^{\frac{t^2}{4\log\sqrt{1-s}}}}{(-\log\sqrt{1-v})^{3/2}}\frac{dv}{1-v}\Big]^2 ds\Big)^{1/2} \le C, \tag{5.28}$$

uniformly in $0 \le t \le 1$.

On the other hand, again using (3.6), we have

$$\frac{\partial p(t,x,y)}{\partial x_j} = -C_d\int_0^1 \frac{e^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})^{3/2}}(y_j-\sqrt{1-v}x_j)\frac{e^{-\frac{|y-\sqrt{1-v}x|^2}{v}}}{v^{d/2+1}}\frac{dv}{\sqrt{1-v}};$$

therefore,

$$K_j(t,x,y) = -C_d\int_0^1 t\frac{e^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})^{3/2}}(y_j-\sqrt{1-v}x_j)\frac{e^{-\frac{|y-\sqrt{1-v}x|^2}{v}}}{v^{d/2+1}}\frac{dv}{\sqrt{1-v}}.$$

Now, let us first consider *ii*). Using Minkowski's inequality and (5.28), we have

$$|K_0(\cdot,x,y)|_2 \le C\int_0^1 \frac{e^{-u(s)}}{s^{d/2}}\Big|u'(s)+\frac{d}{2s}\Big|\,ds = C\int_0^1 \Big|\varphi'(s)\Big|ds,$$

where $\varphi(s)$ was considered in the proof of Proposition 4.23 (see 4.41). Using similar arguments to those used there (see also [223]), we can conclude that

$$|K_0(\cdot,x,y)|_2 \sim \overline{\mathscr{K}}(x,y).$$

Now, for the partial derivatives in the spatial variable, observe that, for $j = 1,\cdots,d$

$$\begin{aligned} |K_j(t,x,y)| &= C_d\Big|\int_0^1 t\frac{e^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})^{3/2}}(y_j-\sqrt{1-v}x_j)\frac{e^{-\frac{|y-\sqrt{1-v}x|^2}{v}}}{v^{d/2+1}}\frac{dv}{\sqrt{1-v}}\Big| \\ &\le C_d\int_0^1 t\frac{e^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})}|u(v)|^{1/2}\frac{e^{-u(v)}}{v^{d/2+1}}\frac{dv}{\sqrt{1-v}}. \end{aligned}$$

It is easy to check that, for any $v \in (0,1)$

$$\int_0^\infty t\Big(\frac{te^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})}\Big)^2 dt \le C.$$

Therefore, using Minkowski's inequality

$$\begin{aligned} \Big(\int_0^\infty \Big|K_j(t,x,y)f(x)|^2\frac{dt}{t}\Big)^{1/2} &\le \int_{B_h^c(x)} |K_j(\cdot,x,y)|_2|f(y)|dy \\ &\le \int_{B_h^c(x)}\Big(\int_0^1 |u(v)|^{1/2}\frac{e^{-u(v)}}{v^{d/2+1}}\frac{dv}{\sqrt{1-v}}\Big)|f(y)|dy. \end{aligned}$$

Then, using Lemma 4.38 we get that this is bounded by $\overline{\mathscr{K}}(x,y)$. Finally, again using Minkowski's inequality,

$$\begin{aligned} g_{G,\gamma}(f)(x) &\le C\int_{B_h^c(x)}\Big(|K_0(\cdot,x,y)|_2+\sum_{j=1}^d |K_j(\cdot,x,y)|_2\Big)|f(y)|dy \\ &\le C_d\int_{B_h^c(x)}\overline{\mathscr{K}}(x,y)|f(y)|dy \le C_d\int_{\mathbb{R}^d}\overline{\mathscr{K}}(x,y)|f(y)|dy = \overline{T}|f|(x) \end{aligned}$$

and from Theorem 4.24, we know that $\overline{T}$ is of weak type $(1,1)$ with respect to the Gaussian measure γ_d.

Now, let us prove *i*). Consider the vector valued kernel

$$\tilde{\mathbf{K}}(t,x) = (\tilde{K}_0(t,x),\tilde{K}_1(t,x),\cdots,\tilde{K}_d(t,x)),$$

whose components are given by

$$\tilde{K}_0(t,x) = t\int_0^1 \Psi(t,s)\frac{e^{-|x|^2/s}}{s^{d/2}}\Big(\frac{d}{2s}-\frac{|x|^2}{2s^2\sqrt{1-s}}\Big)ds,$$

and, for $j = 1, \cdots, d$,

$$\tilde{K}_j(t,x) = \int_0^1 t \frac{e^{\frac{t^2}{4\log\sqrt{1-v}}}}{(-\log\sqrt{1-v})^{3/2}} \frac{-x_j e^{-\frac{|x|^2}{v}}}{v^{d/2+1}} \frac{dv}{\sqrt{1-v}}.$$

It is easy to check, because $|\Psi(t,\cdot)|_2 \leq C$, that

$$|\widehat{\tilde{\mathbf{K}}}(t,\cdot)|_2 \leq C \text{ and } |\nabla_x \tilde{\mathbf{K}}(t,\cdot)|_2 \leq \frac{C}{|x|^{n+1}}.$$

Thus, from the classical Calderón–Zygmund theory (see for instance E. Stein [252, Chapter II]), the operator defined as,

$$\Theta f(x) = |(\tilde{\mathbf{K}}(t,\cdot) * f)(x)|_2 = \Big| \int_{\mathbb{R}^d} \tilde{\mathbf{K}}(t, x-y) f(y) dy \Big|_2,$$

is a Calderón–Zygmund operator and therefore of weak type $(1,1)$ and $L^p(\mathbb{R}^d)$-bounded, $1 < p < \infty$ with respect to the Lebesgue measure. Also, it is not difficult to see that

$$|(\tilde{\mathbf{K}}(t,\cdot) * f)(x)|_2 \leq \frac{C}{|x|^n},$$

Therefore, Θ satisfies the required boundedness properties of Theorem 4.33 because using Minkowski's inequality

$$\Theta f(x) = |(\tilde{\mathbf{K}}(t,\cdot) * f)(x)|_2 \leq \sum_{i=0}^{d} \int_{\mathbb{R}^d} |\tilde{K}_i(\cdot, x-y)|_2 |f(y)| dy \leq C \int_{\mathbb{R}^d} \frac{|f(y)|}{|x-y|^d} dy.$$

Hence, it is of weak type $(1,1)$ and $L^p(\gamma_d)$-bounded, for $1 < p < \infty$.

Now, because

$$\begin{aligned} g_{L,\gamma}(f)(x) &= \Big| \int_{B_h(x)} \mathbf{K}(\cdot, x, y) f(y) dy \Big|_2 \\ &\leq \Big| \int_{B_h(x)} \tilde{\mathbf{K}}(\cdot, x-y) f(y) dy \Big|_2 + \Big| \int_{B_h(x)} (\mathbf{K}(\cdot, x, y) - \tilde{\mathbf{K}}(\cdot, x-y)) f(y) dy \Big|_2. \end{aligned}$$

To conclude the proof, we have that it can be proved that the operator defined by

$$\tilde{\Theta} f(x) = \Big| \int_{B_h(x)} (\mathbf{K}(\cdot, x, y) - \tilde{\mathbf{K}}(\cdot, x-y)) f(y) dy \Big|_2$$

is bounded by an operator with kernel $\frac{1+|x|^2}{|x-y|^{d-1/2}}$ and therefore $L^p(\gamma_d)$-bounded for $1 \leq p < \infty$, using similar arguments to those done in Theorem 9.17. □

5.2 The Higher Order Gaussian Littlewood–Paley g Functions

In this section, we study the higher order Gaussian Littlewood–Paley–Stein functions. We see that they are $L^p(\gamma)$ bounded for any $1 < p < \infty$ and for the case $p = 1$, we see that some of them are of weak type $(1,1)$ but others are not.

They were defined by C. Gutiérrez, C. Segovia, and J. L. Torrea in their article [124]. Their pointwise definition is as follows

Definition 5.11. *For $f \in L^1(\gamma_d)$, the* higher order time Gaussian Littlewood–Paley g *function is defined as*

$$g^k_{t,\gamma}(f)(x) = \left(\int_0^{+\infty} \left| t^k \frac{\partial^k P_t f}{\partial t^k}(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \tag{5.29}$$

and the higher order spatial Gaussian Littlewood–Paley g *function is defined as*

$$g^k_{x,\gamma}(f)(x) = \left(\int_0^{+\infty} |t^k \nabla^k_x P_t f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \tag{5.30}$$

with $\nabla^k_x = \left(\frac{\partial^{|\beta|}}{\partial x_{\beta_1} \cdots \partial x_{\beta_k}} \right)_{\beta \in \Gamma_k}$ *the gradient operator of order k, and the norm $|\quad|$, which appears in the integral of g^k_2, is the Euclidean norm in $\mathbb{R}^{d^k}$.*

Additionally, for $\mathbf{f}(x) = (f_\beta(x))_{\beta \in \Lambda_k}$, we define the vector version of the higher order time Gaussian Littlewood–Paley *g function, as*

$$\mathbf{g}^k_{t,\gamma}(\mathbf{f})(x) = \Big(\int_0^\infty \sum_{\beta \in \Lambda_k} \Big| t^k \frac{\partial^k P^{(k)}_t f_\beta}{\partial t^k}(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}, \tag{5.31}$$

where for $\beta = (\beta_1, \cdots, \beta_k)$, and $P^{(k)}_t$ is the translated Poisson–Hermite semigroup (3.56).

As before, it is clear that $g^k_{t,\gamma}$ can be written as

$$g^k_{t,\gamma}(f)(x) = \Big(\int_0^\infty t^{2k-1} \Big| \frac{\partial^k P_t f}{\partial t^k}(x) \Big|^2 dt \Big)^{1/2}.$$

Nevertheless, the former representation is more convenient, because it allows a vector interpretation, using the space $L^2((0,\infty), \frac{dt}{t})$. Also, we have

Proposition 5.12. *For any $k \in \mathbb{N}$, we have the following pointwise inequality*

$$g^k_{t,\gamma}(f)(x) \le g^{k+1}_{t,\gamma}(f)(x). \tag{5.32}$$

Proof. The proof is by induction. For $k = 1$, according to the fundamental theorem of calculus and the Cauchy–Schwartz inequality

$$\Big|\frac{\partial P_t f}{\partial t}(x)\Big| \leq \int_t \Big|\frac{\partial^2 P_s f}{ds^2}\Big| ds = \int_t s\Big|\frac{\partial^2 P_s f}{ds^2}\Big|\frac{ds}{s} \leq \Big(\int_t s^2\Big|\frac{\partial^2 P_s f}{ds^2}\Big|^2 ds\Big)^{1/2} t^{-1/2}.$$

Hence, using Fubini's theorem

$$\begin{aligned} g_{t,\gamma}(f)(x) &= \left(\int_0^{+\infty} t\left|\frac{\partial P_t f}{\partial t}(x)\right|^2 dt\right)^{\frac{1}{2}} \leq \left(\int_0^{+\infty}\Big(\int_t s^2\Big|\frac{\partial^2 P_s f}{ds^2}\Big|^2 ds\Big) dt\right)^{\frac{1}{2}} \\ &= \left(\int_0^{+\infty} s^3\left|\frac{\partial^2 P_s f}{\partial s^2}(x)\right|^2 ds\right)^{\frac{1}{2}} = g^2_{t,\gamma}(f)(x). \end{aligned}$$

To prove the induction hypothesis, the argument is completely analogous. □

In [124], C. Gutiérrez, C. Segovia, and J. L. Torrea prove the following:

Theorem 5.13. *For $1 < p < \infty$, the Littlewood–Paley functions $g^k_{t,\gamma}$, $g^k_{x,\gamma}$ are $L^p(\gamma_d)$-bounded, that is, there exists a constant C_p such that for all $f \in L^p(\gamma_d)$,*

$$||g^k_{t,\gamma}(f)||_{p,\gamma} \leq C_p||f||_{p,\gamma}, \tag{5.33}$$

and there exists a constant C'_p such that for all $f \in L^p(\gamma_d)$,

$$||g^k_{x,\gamma}(f)||_{p,\gamma} \leq C'_p||f||_{p,\gamma}; \tag{5.34}$$

there also exists a constant $C"_p$ such that for all $f \in L^p(\gamma_d)$,

$$||\,|\mathbf{f}|\,||_{p,\gamma} \leq C_p||\mathbf{g}^k_{t,\gamma}(\mathbf{f})||_{p,\gamma}. \tag{5.35}$$

Proof. Let us prove (5.33) using induction in k. If $k = 1$, it is clear, by definition, that $g^1_{t,\gamma} = g_{t,\gamma}$; thus,

$$||g^1_{t,\gamma}(f)||_{p,\gamma} \leq ||g_\gamma(f)||_{p,\gamma};$$

therefore, using Theorem 5.2, we get

$$||g^1_{t,\gamma}(f)||_{p,\gamma} \leq C_p||f||_{p,\gamma}.$$

Thus,

$$\Big\|\Big(\int_0^\infty \Big|t\frac{\partial P_t f}{\partial t} f\Big|^2 \frac{dt}{t}\Big)^{1/2}\Big\|_{p,\gamma} \leq C_p||f||_{p,\gamma}. \tag{5.36}$$

Considering the vector inequality (see [124, Corollary, Theorem 2]):

$$\begin{aligned} &\Big(\int_{\mathscr{X}}\Big(\int_{\mathscr{M}}\int_\Omega |\tilde{T}f(x,\omega,\mu)|^2 d\mu d\omega\Big)^{p/2} d\rho(x)\Big)^{1/p} \\ &\qquad \leq C_p||T||_p\Big(\int_{\mathscr{X}}\Big(\int_{\mathscr{M}} |f(x,\mu)|^2 d\mu\Big)^{1/2} d\rho(x)\Big)^{1/p}, \end{aligned}$$

with

$$\mathscr{X} = \mathbb{R}^d, d\rho(x) = \gamma_d(dx), \mathscr{M} = (0,\infty), \mu = t, d\mu = \frac{dt}{t}, \Omega = (0,\infty), \omega = s, d\omega = \frac{ds}{s}$$

and considering $T = S_1 = t\frac{\partial P_t}{\partial t} : L^p(\gamma_d) \to L^p_{L^2((0,\infty),\frac{dt}{t})}(\gamma_d)$ then

$$\tilde{S}_1 f(x,t,s) =: L^p_{L^2((0,\infty),\frac{dt}{t})}(\gamma_d) \to L^p_{L^2((0,\infty),\frac{dt}{t})}(\gamma_d)$$

defined as

$$\tilde{S}_1 f(x,t,s) = s\frac{\partial P_s f}{\partial s}(x,t),$$

for any $f \in L^2((0,\infty),\frac{dt}{t})$. We have,

$$\Big\|\Big(\int_0^\infty\int_0^\infty \Big|s\frac{\partial P_s h}{\partial s}(x,t)\Big|^2\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}\Big\|_{p,\gamma} \le C_p\Big\|\Big(\int_0^\infty |h(x,t)|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{p,\gamma}. \quad (5.37)$$

Assuming now that (5.36) holds up to a given k. Taking

$$h(x,t) = S_k f(x,t) = t^k\frac{\partial^k P_t f}{\partial t^k}(x).$$

and applying (5.37) we obtain

$$\begin{aligned}\Big\|\Big(\int_0^\infty\int_0^\infty \Big|s\frac{\partial P_s}{\partial s}\Big(t^k\frac{\partial^k P_t f}{\partial t^k}(x)\Big)\Big|^2\frac{dt}{t}\frac{ds}{s}\Big)^{1/2}\Big\|_{p,\gamma} &\le C_p\Big\|\Big(\int_0^\infty \Big|t^k\frac{\partial^k P_t f}{\partial t^k}(x)\Big|^2\frac{dt}{t}\Big)^{1/2}\Big\|_{p,\gamma}\\ &\le C_{p,k}||f||_{p,\gamma}.\end{aligned}$$

Then, taking the change of variables $u = t+s$ and using Fubini's theorem

$$\begin{aligned}\int_0^\infty\int_0^\infty \Big|s\frac{\partial P_s}{\partial s}\Big(t^k\frac{\partial^k P_t f}{\partial t^k}(x)\Big)\Big|^2\frac{dt}{t}\frac{ds}{s} &= \int_0^\infty\int_0^\infty \Big|st^k\frac{\partial}{\partial s}\frac{\partial^k P_{s+t} f}{\partial t^k}(x)\Big|^2\frac{dt}{t}\frac{ds}{s}\\ &= \int_0^\infty\int_0^\infty \Big|st^k\frac{\partial^{k+1} P_u f}{\partial u^{k+1}}(x)\Big|_{u=t+s}\Big|^2\frac{dt}{t}\frac{ds}{s}\\ &= \int_0^\infty\int_s^\infty \Big|s(u-s)^k\frac{\partial^{k+1}}{\partial u^{k+1}}P_u f(x)\Big|^2\frac{du}{u-s}\frac{ds}{s}\\ &= \int_0^\infty\int_0^u \Big|s(u-s)^k\frac{\partial^{k+1} P_u f}{\partial u^{k+1}}(x)\Big|^2\frac{ds}{s}\frac{du}{u-s}\\ &= \frac{1}{2k(2k+1)}\int_0^\infty \Big|u^{k+1}\frac{\partial^{k+1} P_u f}{\partial u^{k+1}}(x)\Big|^2\frac{du}{u}\\ &= \frac{1}{2k(2k+1)}(g^{k+1}_{t,\gamma}(f)(x))^2.\end{aligned}$$

This implies

$$||g^{k+1}_{t,\gamma}(f)||_{p,\gamma} \le 2k(2k+1)C_p||f||_{p,\gamma} = C_{p,k}||f||_{p,\gamma}.$$

The proofs of the $L^p(\gamma_d)$-boundedness for the function $g^k_{x,\gamma}$, and of the inequality (5.35), are also by induction on k, and uses another vector inequality (for details see [124, Theorem 2]) □.

On the other hand, they proved the opposite inequality from $g^k_{t,\gamma}(f)(\cdot)$.

Theorem 1. *Given* $1 < p < \infty$, *let* $k \geq 1$, *there exists a constant* $B_{p,k} > 0$ *such that for every polynomial* f, *we have*

$$\|f\|_{p,\gamma_d} \leq B_{p,k} \left\| g^k_{t,\gamma}(f) \right\|_{p,\gamma_d}. \tag{5.38}$$

In [87], R. Scotto studies the weak type $(1,1)$ for the $g^k_{t,\gamma}$ and $g^k_{x,\gamma}$ functions in detail.

Theorem 5.14. *(Scotto) There exists a constant C, depending on d and k, such that for every* $f \in L^1(d\gamma)$ *and every* $\lambda > 0$,

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : g^k_{t,\gamma} f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda}||f||_{1,\gamma}. \tag{5.39}$$

For $k = 1$ *or* $k = 2$, $g^k_{2,\gamma}$ *satisfies,*

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : g^k_{x,\gamma} f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda}||f||_{1,\gamma}, \tag{5.40}$$

but if $k > 2$ *then* $g^k_{x,\gamma}$ *need not satisfy* (5.40).

Before proving this theorem, let us make the following remarks:

1. As in the case of g_γ, the operator $g^k_{1,\gamma}$ can be viewed as a vector valued singular integral operator(see [252]). Let $A_1 = \mathbb{C}$ be the set of complex numbers and

$$A_2 = L^2((0,+\infty), \frac{dt}{t}),$$

the space of $\mathbb{C}$-valued measurable functions, which are square integrable on $(0,+\infty)$ with respect to the measure $\frac{dt}{t}$. For $h \in A_2$, let

$$|h|_2 = \Big(\int_0^{+\infty} |h(t)|^2 \frac{dt}{t}\Big)^{1/2}.$$

Let $\mathscr{B}(A_1, A_2)$ be the set of bounded linear transformations from A_1 to A_2; it can be identified with A_2. Thus,

$$g^k_{t,\gamma} f(x) = \left| p.v. \int_{\mathbb{R}^d} K^k_0(\cdot, x, y) f(y) dy \right|_2,$$

where,

$$\begin{aligned}
K^k_0(t,x,y) &= \int_0^1 t^k \frac{\partial^k p(t,x,y)}{\partial t^k} \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \\
&= \frac{t^k}{2\pi^{(d+1)/2}} \int_0^1 \frac{e^{(t^2/4\log r)}}{(-\log r)^{3/2}} \frac{H_{k+1}\left(\frac{t}{2(-\log r)^{1/2}}\right)}{(-\log r)^{\frac{k-1}{2}}} \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \\
&= \int_0^1 \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \varphi_k(t,r) \frac{dr}{r}
\end{aligned}$$

where H_{k+1} is the Hermite polynomial of degree $k+1$, and

$$\varphi_k(t,r) = C_d \frac{t^k \exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}} \frac{H_{k+1}\left(\frac{t}{2(-\log r)^{1/2}}\right)}{(-\log r)^{\frac{k-1}{2}}}.$$

For the function $\varphi_k(t,r)$, we then have

$$|\varphi_k(t,r)| \leq \frac{C_d}{t} \frac{e^{\frac{t^2}{8\log r}}}{(-\log r)^{3/2}},$$

for $t>0$, $0<r<1$, and

$$\int_0^1 \varphi_k(t,r)\frac{dr}{r} = 0,$$

for all $t>0$, which follows from the change of variables $u = \frac{t}{2\sqrt{-\log r}}$ and using the orthogonality property of the Hermite polynomials, or equivalently integrating by parts $k-1$ times. Indeed,

$$\begin{aligned}\int_0^1 \varphi_k(t,r)\frac{dr}{r} &= C_k \int_0^{+\infty} u^{k-1} H_{k+1}(u) e^{-u^2} du \\ &= C_k(-1)^{k+1} \int_0^{+\infty} u^{k-1} \frac{d^{k+1}}{du^{k+1}}\left(e^{-u^2}\right) du \\ &= C_k(k-1)! \int_0^{+\infty} H_2(u) e^{-u^2} du = 0.\end{aligned}$$

Now, integrating by parts

$$K_0^k(t,x,y) = \int_0^1 \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \varphi_k(t,r)\frac{dr}{r} \tag{5.41}$$

$$= \int_0^1 \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{\partial}{\partial r}\left(\int_0^r \varphi_k(t,s)\frac{ds}{s}\right) dr = -\int_0^1 \frac{\partial}{\partial r}\left(\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\right) \tau_k(t,r)dr \tag{5.42}$$

where $\tau_k(t,r) = \int_0^r \varphi_k(t,s)\frac{ds}{s}$. Hence,

$$K_0^k(t,x,y) = \int_0^1 \left[\frac{2(rx-y).(x-ry)}{(1-r^2)^{\frac{d+4}{2}}} - \frac{rd}{(1-r^2)^{\frac{d+2}{2}}}\right] e^{-\frac{|rx-y|^2}{1-r^2}} \tau_k(t,r)dr,$$

2. $\tau_k(\cdot,r) \in A_2$ and $|\tau_k(.,r)|_2$ is bounded by a constant independent of r. Indeed,

$$\begin{aligned}|\tau_k(\cdot,r)|_2 &= \int_0^{+\infty}\left(\int_0^r \varphi_k(t,s)\frac{ds}{s}\right)^2 \frac{dt}{t} \\ &= \int_0^{(-\log r)^{1/2}}\left(\int_0^r \varphi_k(t,s)\frac{ds}{s}\right)^2 \frac{dt}{t} \\ &\quad + \int_{(-\log r)^{1/2}}^{+\infty}\left(-\int_r^1 \varphi_k(t,s)\frac{ds}{s}\right)^2 \frac{dt}{t}\end{aligned}$$

where in the inner integral of the second term we use remark 1 to replace $\int_0^r \varphi_k(t,s)ds/s$ by $-\int_r^1 \varphi_k(t,s)ds/s$. Then, we use remark 1 5.2 above to bound $|\varphi_k(t,s)|$ in the inner integrals of both terms. Once this is done, we make the change of variables $-\log r = t^2 v$ and then $|\tau_k(\cdot,r)|_2$ turns out to be bounded by

$$C\left[\int_0^{(-\log r)^{1/2}}\left(\int_{\frac{-\log r}{t^2}}^{+\infty} v^{-3/2}\,dv\right)^2\frac{dt}{t}+\int_{(-\log r)^{1/2}}^{+\infty}\left(\int_0^{\frac{-\log r}{t^2}} dv\right)^2\frac{dt}{t}\right],$$

which is a constant independent of r.

3. For every $f \in L^1(d\gamma)$, $P_t f(x)$ turns out to be a smooth function

$$|\nabla_x^k P_t f(x)|^2 = \sum_{\substack{1\le \beta_j\le d\\ 1\le j\le k}}\left|\frac{\partial^k}{\partial x_{\beta_1}\cdots\partial x_{\beta_k}}P_t f(x)\right|^2 = C\sum_{|\alpha|=k}|\partial^\alpha P_t f(x)|^2$$

where $\alpha=(\alpha_1,\ldots,\alpha_d)$ is a multi-index with non-negative integer entries, $|\alpha| = \alpha_1+\cdots+\alpha_d$, and $\partial^\alpha = \frac{\partial^k}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$. Thus

$$g_{x,\gamma}^k f(x) = C\left(\int_0^{+\infty}\sum_{|\alpha|=k}\left|t^k\partial^\alpha P_t f(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}.$$

4. The operator $g_{x,\gamma}^k$ can also be viewed as a vector-valued singular integral operator with $A_1=\mathbb{C}$ and A_2 the direct sum of $\binom{k+d-1}{d-1}$ copies of $L^2((0,+\infty),dt/t)$. Let

$$|h|_2 = \left(\int_0^{+\infty}\sum_{|\alpha|=k}|h_\alpha(t)|^2\frac{dt}{t}\right)^{1/2}$$

for $h=(h_\alpha)_{|\alpha|=k}\in A_2$. Here, $\mathscr{B}(A_1,A_2)$ can also be identified with A_2. Thus,

$$g_{x,\gamma}^k f(x) = \left|p.v.\int_{\mathbb{R}^d}K^k(\cdot,x,y)f(y)dy\right|_2$$

where, by (3.15), $K^k(t,x,y) = (K_\nu^k(t,x,y))_{|\nu|=k}$, with

$$K_\nu^k(t,x,y) = C_{d,k}\int_0^1 \eta_k(t,r)\;\omega_k I\,\mathbf{H}_\nu\left(\frac{rx-y}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d+2}{2}}}dr$$

with

$$\eta_k(t,r) = \frac{t^{k+1}e^{\frac{t^2}{4\log r}}}{(-\log r)^{\frac{k+1}{2}}},\;\omega_k(r) = r^{k-1}\left(\frac{-\log r}{1-r^2}\right)^{\frac{k-2}{2}}.$$

5. By direct computation, we can easily see that $|\eta_k(\cdot,r)|_{L^2((0,+\infty),dt/t)}$ is bounded by a constant independent of r. In addition, $\omega_1(r)$ is a bounded function on $(0,1)$, and for $k \geq 2$, $\omega_k(r) \leq Cr$ on the same interval.

We are ready to prove Theorem 5.14.

Proof. $g^k_{t,\gamma}f$ can be bounded as

$$\begin{aligned} g^k_{t,\gamma}f(x) &\leq g^k_{t,L}f(x) + g^k_{t,G}f(x) \\ &= \left| p.v. \int_{N_x} K^k_0(\cdot,x,y)f(y)dy \right|_2 \\ &\qquad + \int_{\mathbb{R}^d \setminus N_x} |K^k_0(\cdot,x,y)|_2 \; |f(y)| \, dy. \end{aligned}$$

Similarly, for $k=1$ or $k=2$,

$$\begin{aligned} g^k_{x,\gamma}f(x) &\leq g^k_{x,L}f(x) + g^k_{2,G} \\ &= \left| p.v. \int_{N_x} K^k(\cdot,x,y)f(y)dy \right|_2 \\ &\qquad + \int_{\mathbb{R}^d \setminus N_x} |K^k(\cdot,x,y)|_2 \; |f(y)| \, dy. \end{aligned}$$

Analogous to the proof of Theorem 5.10, the kernels of $g^k_{t,G}$, $|K^k_0(\cdot,x,y)|_2$, and of $g^1_{x,G}$, $|K^1(\cdot,x,y)|_2$, on $\mathbb{R}^d \setminus N_x$, can be bounded by the maximal Gaussian kernel (4.40)

$$\overline{\mathscr{K}}(x,y) = \begin{cases} e^{-|y|^2} & \text{if } \langle x,y\rangle \leq 0 \\ \left(\frac{|x+y|}{|x-y|}\right)^2 e^{-\frac{|y|^2-|x|^2}{2} - \frac{|x-y||x+y|}{2}} & \text{if } \langle x,y\rangle > 0. \end{cases}$$

Indeed, by applying Minkowski's integral inequality to K^k_0 and remark 2 above, we get

$$|K^k_0(\cdot,x,y)|_2 \leq C \int_0^1 \left| \frac{\partial}{\partial r} \left(\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \right) \right| dr.$$

This last term is bounded by $\overline{\mathscr{K}}(x,y)$ as was done in [220, pages 47–49], after the change of variables $t = 1-r^2$. Now, applying Minkowski's integral inequality to K^1 and remark 5 above, we get

$$|K^1(\cdot,x,y)|_2 \leq C \int_0^1 \frac{|rx-y|}{(1-r^2)^{\frac{d+3}{2}}} e^{-\frac{|rx-y|^2}{1-r^2}} dr.$$

The right-hand side of this inequality is also bounded by $\overline{\mathscr{K}}(x,y)$ (see [220, page 39]). Therefore $g^k_{1,G}f(x)$ and $g^1_{2,G}f(x)$ are bounded by the operator (4.46)

$$\overline{T}f(x) = \int_{\mathbb{R}^d} \overline{\mathscr{K}}(x,y)\, |f(y)| dy,$$

which is of weak type (1,1) with respect to γ_d, as was proved in Theorem 4.24 (see also S. Pérez in [220, page 25]).

$|K^2(\cdot,x,y)|_2$, on the other hand, is bounded by a *2-modified maximal Gaussian kernel*

$$\overline{\mathscr{K}}_2(x,y) = \begin{cases} \overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle \le 0 \\ \left((|x+y||x-y|)^{\frac{1}{2}} \frac{|x||y|}{|x|^2+|y|^2} + 1\right) \overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle > 0 \end{cases} \tag{5.43}$$

where $\overline{\mathscr{K}}$ is the Gaussian maximal kernel defined in (4.40). This estimate can also be found in [220, page 52], once we have applied Minkowski's integral inequality and remark 5 above to $|K^2(\cdot,x,y)|_2$. Therefore $g^2_{2,G}f(x)$ is bounded by the 2-modified maximal operator

$$\overline{T}_2 f(x) = \int_{\mathbb{R}^d} \overline{\mathscr{K}}_2(x,y)\, f(y)\, dy, \tag{5.44}$$

which, similar to $\overline{T}$, is also of weak type $(1,1)$, as can be seen in [220, page 56].

It remains to be proved that the local parts of these operators are also of weak type (1,1) with respect to γ_d. This follows once we check that the kernels of these operators satisfy conditions *i*) and *ii*) of Theorem 4.34.

For $y \in N_x$, we have

- $$|\langle x,(x-y)\rangle| \le |x|\; |x-y| \le C, \text{ and } |x| \sim |y|. \tag{5.45}$$
- For $0<r<1$, $e^{-\frac{|rx-y|^2}{c(1-r)}} = e^{-\frac{|x-y|^2}{c(1-r)}} e^{-\frac{(1-r)|x|^2}{c}} e^{\frac{2x.(x-y)}{c}} \le Ce^{-\frac{|x-y|^2}{c(1-r)}} e^{-\frac{(1-r)|x|^2}{c}}$.
- $|rx-y|\; |x-ry| = |x-y-(1-r)x|\; |x-y+(1-r)y| \le |x-y|^2 + (1-r)(|x|+|y|)|x-y| + (1-r)^2|x|\; |y| \le C(|x-y|^2 + (1-r) + (1-r)^2|x|^2)$.

1. Condition *i*): By applying Minkowski's integral inequality to kernel K^k_0 and (5.41), we have

   $$|K^k_0(\cdot,x,y)|_2 \le C\int_0^1 \left[\frac{|rx-y|\; |x-ry|}{(1-r^2)^{\frac{d+4}{2}}} + \frac{1}{(1-r^2)^{\frac{d+2}{2}}}\right] e^{-\frac{|rx-y|^2}{2(1-r)}}\, dr.$$

   The right-hand side of this inequality can be bounded, using the remarks above, by

   $$C\int_0^1 \left[\frac{|x-y|^2+(1-r)+(1-r)^2|x|^2}{(1-r)^{\frac{d+4}{2}}} + \frac{1}{(1-r)^{\frac{d+2}{2}}}\right] e^{-\frac{|x-y|^2}{2(1-r)}} e^{-\frac{(1-r)|x|^2}{2}}\, dr$$
   $$\le C\int_0^1 \left(\frac{|x-y|^2}{(1-r)^{\frac{d+4}{2}}} + \frac{1}{(1-r)^{\frac{d+2}{2}}}\right) e^{-\frac{|x-y|^2}{2(1-r)}}\, dr$$

By the change of variables $u = |x-y|/(1-r)^{1/2}$, the integral on the right-hand side of the above inequality becomes

$$\frac{C}{|x-y|^d}\int_{|x-y|}^{+\infty}(u^{d+1}+u^{d-1})e^{-u^2/2}\,du \le \frac{C}{|x-y|^d}.$$

By applying Minkowski's integral inequality to kernel K^k and remark 5 above, we get, for any $k\in\mathbb{N}$

$$|K^k(\cdot,x,y)|_2 \le C\sum_{|\alpha|=k}\int_0^1\left|H_\nu\left(\frac{rx-y}{\sqrt{1-r^2}}\right)\right|e^{-\frac{|rx-y|^2}{2(1-r^2)}}\frac{e^{-\frac{|rx-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+2}{2}}}\,dr$$
$$\le C\int_0^1\frac{e^{-\frac{|rx-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+2}{2}}}\,dr \le \frac{C}{|x-y|^d}.$$

2. Condition *ii*): To verify Hörmander's condition *ii*) of Theorem 4.34, it will be enough to check that both $\left|\frac{\partial K^0}{\partial y_j}(\cdot,x,y)\right|_2$ and $\left|\frac{\partial K^k_\alpha}{\partial y_j}(\cdot,x,y)\right|_2$ are bounded by $\frac{C}{|x-y|^{d+1}}$. Indeed,

$$\nabla_y K_0(t,x,y) = 2\int_0^1\Big[\left(\frac{2|rx-y|^2+(1-r)(rx-y).(x+y)}{(1-r^2)^{\frac{d+4}{2}}}-\frac{rd}{(1-r^2)^{\frac{d+2}{2}}}\right)\frac{rx-y}{1-r^2}$$
$$-\left(\frac{1+r}{(1-r^2)^{\frac{d+4}{2}}}(rx-y)+\frac{1-r}{(1-r^2)^{\frac{d+4}{2}}}(x+y)\right)\Big]e^{-\frac{|rx-y|^2}{1-r^2}}\tau_k(t,r)\,dr,$$

and

$$\nabla_y K^k_\alpha(t,x,y) = C\int_0^1\eta_k(t,r)\omega(r)\Big[\frac{\alpha}{\sqrt{1-r^2}}H_{\alpha-1}\left(\frac{rx-y}{\sqrt{1-r^2}}\right)$$
$$+\frac{rx-y}{1-r^2}H_\alpha\left(\frac{rx-y}{\sqrt{1-r^2}}\right)\Big]\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d+2}{2}}}\,dr,$$

where $\alpha-1$ means the d-dimensional vector $(\alpha_1-1,\ldots,\alpha_d-1)$.

Thus, applying Minkowski's integral inequality and remarks 2 and 5 above to $\nabla_y K^0$ and $\nabla_y K^k$ respectively, then remarks (5.45), and finally the change of variables $u=|x-y|/(1-r)^{1/2}$, we get

$$|\ |\nabla_y K_0(\cdot,x,y)|\ |_2 \le C\int_0^1\Big[\left(\frac{|x-ry|}{(1-r^2)^{1/2}}\right)^3+(1-r)^{\frac{1}{2}}|x|\left(\frac{|rx-y|^2}{1-r^2}+1\right)$$
$$\frac{|rx-y|}{(1-r^2)^{1/2}}\Big]e^{-\frac{(1-r)|x|^2}{4}}e^{-\frac{|rx-y|^2}{2(1-r^2)}}\frac{e^{-\frac{|x-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+3}{2}}}\,dr$$
$$\le C\int_0^1\frac{e^{-\frac{|x-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+3}{2}}} \le \frac{C}{|x-y|^{d+1}},$$

and

$$| \ |\nabla_y K^k(\cdot,x,y)| \ |_2 \leq C \int_0^1 \frac{e^{-\frac{|x-y|^2}{4(1-r)}}}{(1-r)^{\frac{d+3}{2}}} \leq \frac{C}{|x-y|^{d+1}}.$$

According to Theorem 4.34, the local operators $g^k_{t,L}$ and $g^k_{x,L}$ are of weak type (1,1) with respect to γ_d for all k.

To see that g^k_x for $k > 2$ need not satisfy the weak type (1,1) inequality, we refer to [102], where they show that the higher order Riesz transforms need not be of weak type (1,1) with respect to γ_d if their order is greater than 2 (see the proof of Theorem 9.10). Take $|y|$ large and $y_i \geq |y|, \ \ i = 1,\ldots,d$, and $C > 0$ so that

$$H_\nu\left(\frac{y-rx}{\sqrt{1-r^2}}\right) > C|y|^{|\nu|}.$$

Then, define

$$\mathbf{J} = \left\{\xi\frac{y}{|y|} + v : \ \frac{1}{2}|y| < \xi < \frac{3}{4}|y|, v \perp y, \ |v| < 1\right\},$$

for $|\nu| = k$ and $x \in \mathbf{J}$

$$K^k_\nu(t,x,y) \geq ct^{k+1}e^{-ct^2}|y|^k e^{\xi^2-|y|^2}\int_{1/4}^{3/4} e^{-c(\xi-r|y|)^2}\,dr \geq ct^{k+1}e^{-ct^2}|y|^{k-1}e^{\xi^2-|y|^2}.$$

Now, by taking $0 \leq f \in L^1(\gamma_d)$ to be close to an approximation of the identity near to the point mass δ, with $L^1(\gamma_d)$ norm equals to 1, we conclude that

$$g^k_x f(x) \geq c|y|^k e^{\xi^2} \geq c|y|^k e^{\left(\frac{|y|}{2}\right)^2}$$

for $x \in \mathbf{J}$. Because $\gamma_d(\mathbf{J}) \geq c|y|^{-1}e^{-\left(\frac{|y|}{2}\right)^2}$, then the $L^{1,\infty}(\gamma_d)$ quasi-norm of $g^k_x f$ is at least $c|y|^{k-2} \to \infty$ as $|y| \to \infty$ when $k > 2$, which means that the weak type (1,1) inequality is not satisfied. □

5.3 The Gaussian Lusin Area Function

The area function and the area function of higher order have significant applications in the classical case of the Lebesgue measure, because a harmonic function on $\mathbb{R}^{d+1}_+$, the existence of non-tangential limits is equivalent to the finiteness of the S operator. These area functions are very important in the characterization of Hardy spaces and they are related to the atomic decomposition of Hardy spaces. Therefore, the study of the analogous notions in the Gaussian case is important and interesting. In the Gaussian context, these operators were first introduced by E. Fabes and L. Forzani, in an unpublished manuscript. They considered the Gaussian counterpart of the Lusin area function.

Definition 5.15. *(Fabes–Forzani) The Gaussian Lusin area function S_γ^a is defined for $f \in L^1(\gamma_d)$, as*

$$S_\gamma^a f(x) = \left(\int_{\Gamma_\gamma^a(x)} |t \nabla_x P_t f(y)|^2 \left(t^{-d} \vee |x|^d \vee 1 \right) dy \frac{dt}{t} \right)^{1/2}, \tag{5.46}$$

where, as before, $\Gamma_\gamma^a(x) = \left\{ (y,t) \in \mathbb{R}_+^{d+1} : |y-x| < t, t < am(x) \right\}$ is a Gaussian cone with aperture $a > 0$ and vertex at $x \in \mathbb{R}^d$. If $a = 1$, we simply denote it as S_γ.

The main results for this area function appeared in [87], and are the following.

Theorem 5.16. *(Fabes–Forzani)*

i) There exists a constant C such that for every $x \in \mathbb{R}^d$,

$$g_{x,\gamma}(f)(x) \leq C S_\gamma^a f(x), \tag{5.47}$$

for $f \in L^1(\gamma_d)$.

ii) If $1 < p < \infty$, then there exists a constant A_p such that

$$||S_\gamma^a f||_{p,\gamma} \leq A_p ||f||_{p,\gamma}, \tag{5.48}$$

for $f \in L^p(\gamma_d)$.

Proof. The proof of the result follows that of E. Stein ([252, Chapter IV, §1 pages 86–94]).

i) For f smooth enough $\nabla_y u(y,t) = \nabla_y P_t f(y)$ is a solution of $L_2 u = 0$; then, applying the mean value inequality, Theorem 3.11 *ii*), that there exists a constant C, dependent only on dimension, such that

$$|\nabla_x u(x,t)| \leq (t^{-d-1} \vee |x|^{d+1} \vee 1) \int_{B((x,t),C_1(t \wedge a(1 \wedge \frac{1}{|x|})))} |\nabla_y u(y,s)|^2 dy\, ds$$

where C_1 is such that $B\Big((x,t), C_1\Big(t \wedge a\Big(1 \wedge \frac{1}{|x|}\Big)\Big)\Big) \subset \Gamma_\gamma^a(x)$. Then, after using the Cauchy–Schwartz inequality, in the definition of $g_{2,\gamma}(f)$, we get

$$(g_{2,\gamma}(f))^2(x) \leq C \int_0^\infty t (t^{-d-1} \vee |x|^{d+1} \vee 1) \int_{B((x,t),C_1(t \wedge a(1 \wedge \frac{1}{|x|})))} |\nabla_y u(y,s)|^2 dy\, ds\, dt.$$

Now, if $(y,s) \in B\Big((x,t), C_1\Big(t \wedge a\Big(1 \wedge \frac{1}{|x|}\Big)\Big)\Big)$, then $(y,s) \in \Gamma_\gamma^a(x)$, and

$$t - C_1\Big(t \wedge a\Big(1 \wedge \frac{1}{|x|}\Big)\Big) < s < t + C_1\Big(t \wedge a\Big(1 \wedge \frac{1}{|x|}\Big)\Big)$$

Then, using Fubini's theorem,

$$(g_{2,\gamma}(f))^2(x) \leq C\int_{\Gamma_\gamma^a(x)} |\nabla_y u(y,s)|^2 \int_{|s-t|<C_1(t\wedge a(\frac{1}{|x|}\wedge 1))} t(t^{-d-1}\vee |x|^{d+1}\vee 1)ds\,dy\,dt$$
$$\leq C\int_{\Gamma_\gamma^a(x)} |\nabla_y u(y,s)|^2 s(s^{-d}\vee 1\vee |x|^d)ds\,dy = C(S_\gamma^a)^2(f)(x),$$

as we wanted to prove.

ii) To prove the L^p inequality, we consider two cases:

- $p \geq 2$. Let ϕ be a positive function in $\mathbb{R}^d$. Interchanging integrals in the definition of the area function we have

$$\int_{\mathbb{R}^d} (S_\gamma^a(f))^2(x)\phi(x)\gamma_d(dx)$$
$$\leq \int_{\mathbb{R}^d}\int_{B((x,t),t\wedge a(1\wedge\frac{1}{|x|}))}\int_0^\infty t|\nabla_y u(y,t)|^2(t^{-d}\vee|x|^d\vee 1)dtdy\phi(x)\gamma_d(dx)$$
$$\leq \int_{\mathbb{R}^d}(g_{2,\gamma}(f)(y))^2\mathscr{M}_\gamma^a\phi(y)\gamma_d(dy),$$

  where $\mathscr{M}_\gamma^a$ is the Gaussian truncated Hardy–Littlewood maximal function defined in (4.101). Because $\mathscr{M}_\gamma^a$ is of strong type (p,p) for $p>1$ and of weak type $(1,1)$ with respect to the Gaussian measure γ_d, applying the $L^p(\gamma_d)$-boundedness of $g_{2,\gamma}(f)$ in the last inequality, we have (5.48) for $p \geq 2$.
- $1<p<2$. If f is smooth enough and positive, we can use the fact that

$$L_1(P_tf)^p = C_p(P_tf)^{p-2}|\nabla u|^2$$

  to have

$$(S_\gamma^a(f))^2(x) \leq C_p(\mathscr{P}^*(1,a)f(x))^{2-p}I^{*,a}(x), \tag{5.49}$$

  where $\mathscr{P}^*(1,a)f(x)$ is the non-tangential maximal function associated with the Poisson–Hermite semigroup (see 4.93), and

$$I^{*,a}(x) = \int_{\Gamma_\gamma^a(x)} tL_1(P_tf(y))^p(t^{-d}\vee|x|^d\vee 1)dydt.$$

  In [122], C. Gutiérrez proved that

$$\int_0^\infty\int_{\mathscr{R}^d} tL_1(P_tf)^p(y)dt\gamma_d(dy) \leq \int_{\mathbb{R}^d}|f(y)|^p\gamma_d(dy)$$

  and as $\left|B\left(x,t\wedge a\left(1\wedge\frac{1}{|x|}\right)\right)\right| \approx C(t^d\wedge|x|^{-d}\wedge 1)$, we get

$$\int_{\mathbb{R}^d} I^{*,a}(x)\gamma_d(dx) \leq C||f||_{L^p(\gamma_d)}^p, \tag{5.50}$$

because

$$\int_{\mathbb{R}^d} I^{*,a}(x)\gamma_d(dx)$$
$$= \int\int_{\mathbb{R}^{d+1}_+} \Big(e^{|y|^2} \int_{|x-y|<t\wedge a(1\wedge \frac{1}{|x|})} (t^{-d} \vee |x|^d \vee 1) e^{-|x|^2} dx \Big)$$
$$\times t L_1 (P_t f)^p(y) \gamma_d(dy)\, dt.$$

Hence, using this latter expression and Theorem 5.10, we get

$$\int_{\mathbb{R}^d} I^{*,a}(x)\gamma_d(dx) \leq C \int\int_{\mathbb{R}^{d+1}_+} t L_1 (P_t f)^p(y)\gamma_d(dy)\, dt \leq \int_{\mathbb{R}^d} f(y)\gamma_d(dy).$$

Now, let us prove (5.48) for this case. Inequality (5.49) tells us that

$$\int_{\mathbb{R}^d} |S_\gamma(f)(x)|^p \gamma_d(dx)$$
$$\leq C \int_{\mathbb{R}^d} \Big(\mathscr{P}^*(1,a) f(x) \Big)^{\frac{(2-p)p}{2}} I^{*,a}(x)^{\frac{p}{2}} \gamma_d(dx)$$
$$\leq C \Big(\int_{\mathbb{R}^d} \Big(\mathscr{P}^*(1,a) f(x) \Big)^p \gamma_d(dx) \Big)^{\frac{2-p}{2}} \Big(\int_{\mathbb{R}^d} (I^{*,a}(x)) \gamma_d(dx) \Big)^{p/2}$$
$$\leq C \int_{\mathbb{R}^d} |f(x)|^p \gamma_d(dx),$$

where we have used Hölder's inequality with exponent $r = \frac{2}{2-p}$, its conjugate index r' and Theorem 4.28, *ii*). □

In [168], J. Maas, J. Van Neerven, and P. Portal introduce the following *Gaussian admissible conical square function*:

Definition 5.17. *The Gaussian admissible conical square function* $\mathscr{S}^a_\gamma$ *is defined for* $f \in L^1(\gamma_d)$ *as*

$$\mathscr{S}_{a,\gamma} f(x) = \Big(\int_{\Gamma^{1,a}_\gamma(x)} \frac{1}{\gamma_d(B(y,t))} |t T_{t^2} f(y)|^2 \gamma_d(dy) \frac{dt}{t} \Big)^{1/2}, \tag{5.51}$$

where, as before, $\Gamma^{1,a}_\gamma(x) = \Big\{ (y,t) \in \mathbb{R}^{d+1}_+ : |y-x| < t < am(x) \Big\}$ *is a Gaussian cone with aperture* $1 > 0$ *and vertex at* $x \in \mathbb{R}^d$.

Then, they prove that this admissible conical square function is controlled by "average" non-tangential maximal function (4.91).

Theorem 5.18. *The Gaussian 'admissible conical square function* $\mathscr{S}_\gamma$ *is controlled by the "average" non-tangential maximal function in the following sense: there exists a constant* $a' \geq 1$ *and a constant* C *such that*

$$\|\mathscr{S}_{a,\gamma} f\|_{1,\gamma} \leq C \|\Upsilon^*_\gamma(1,a') f\|_{1,\gamma}. \tag{5.52}$$

To obtain this result, they follow the arguments given in the proof of the square function estimates in Hardy spaces for the classical case by C. Fefferman and E. M. Stein in their famous paper [79]. They need to use the covering obtained in Lemma 4.5, Lemma 1.5, the parabolic Caccioppoli inequality, Theorem 2.1, and Theorem 4.43, about the change of aperture for the admissible cone appearing in the definition of the non-tangential maximal function. By using the truncated cones, we are only averaging over admissible balls in the definition of the operators. The idea is, of course, to exploit the doubling property of the Gaussian measure on these balls. This makes the operators "admissible." Unfortunately, they are not local, in the sense that their kernels are not supported in the local region. Moreover, they cannot be written as sums of local operators. This is due to a lack of off-diagonal estimates, which is a crucial difference between the Ornstein–Uhlenbeck semigroup and the heat semigroup. The proof is long and hard (for more details see [168, Theorem 1.1]).

5.4 Notes and Further Results

1. We can also define the "total" Gaussian Lusin area function S^a_γ for $f \in L^1(\gamma_d)$, as
$$S^a_{T,\gamma} f(x) = \left(\int_{\Gamma^a_\gamma(x)} |t\nabla P_t f(y)|^2 \left(t^{-d} \vee |x|^d \vee 1\right) dy \frac{dt}{t} \right)^{1/2}, \tag{5.53}$$
where, as before, $\Gamma^a_\gamma(x) = \left\{ (y,t) \in \mathbb{R}^{d+1}_+ : |y-x| < t, t < am(x) \right\}$ is a Gaussian cone with opening $a > 0$ and vertex at $x \in \mathbb{R}^d$, and $\nabla = \left(\frac{\partial}{\partial t}, \frac{1}{\sqrt{2}} \nabla_x \right)$.
In that case, with the same proof as in Theorem 5.16, we can prove that there exists a constant C such that for every $x \in \mathbb{R}^d$,
$$g_\gamma(f)(x) \leq C S^a_{T,\gamma} f(x), \tag{5.54}$$
for $f \in L^1(\gamma_d)$, and
$$||S^a_{T,\gamma} f||_{p,\gamma} \leq A_p \, ||f||_{p,\gamma}, \tag{5.55}$$
for $f \in L^p(\gamma_d)$.
Observe that, using the same notation as for the Littlewood–Paley g functions
$$S^a_\gamma f(x) = S^a_{x,T,\gamma} f(x).$$
For more details see [87].

2. A classical Littlewood–Paley function that has not been generalized to the Gaussian case is the function g^*_λ, which is defined as
$$g^*_\lambda(x) = \Big(\int_0^\infty \int_{\mathbb{R}^d} \big(\frac{t}{|y|+t}\big)^{\lambda d} |\nabla u(x-y,t)|^2 t^{1-d} dy \, dt \Big)^{1/2}.$$

The problem with the generalization of this function is find an appropriate approximation of identity. The inner integral in the classical g^*_λ function is the convolution of the gradient of the Poisson integral u with a family of approximations of identity. In the Gaussian case it is not entirely clear what to do in that direction. A possible definition, that looks familiar because of what is done for the Gaussian area function, suggested by L. Forzani, is the following,

$$g^*_{\gamma,\lambda}(x) = \Big(\int_0^\infty \int_{\mathbb{R}^d} (\frac{(t \wedge \frac{1}{|x|} \wedge 1)}{|x-y| + (t \wedge \frac{1}{|x|} \wedge 1)})^{\lambda d} |\nabla P_t f(y)|^2 t\, (t^{-d} \vee |x|^d \vee 1) dy\, dt\Big)^{1/2}. \tag{5.56}$$

Nevertheless, such a function is problematic, and more research in this direction is needed.

3. Following the same scheme formulated by E. Stein in [253], the main motivation of C. Gutiérrez's article [122], of considering Gaussian Littlewood–Paley functions, is to prove the $L^p(\gamma_d)$ boundedness, $1 < p < \infty$, for the Riesz transform. An important advantage of this argument is that the constants obtained are independent of dimension. This is discussed in more detail in Chapter 9.

4. In [122], C. Gutiérrez, also considers the following Gaussian Littlewood–Paley functions, associated with the translated semigroups $\{P_t^{(k)}\}_t$

$$g^{(k)}_{+,\gamma}(f)(x) = \Big(\int_0^\infty t(|\nabla P_t^{(k)} f(x)|^2 + (P_t^{(k)} f(x))^2 dt\Big)^{1/2}, \tag{5.57}$$

and C. Gutiérrez, C. Segovia, and J. L. Torrea in [124] also consider Gaussian Littlewood–Paley functions of order $k \geq 1$, with respect to the spatial variable x where the k-th derivation is done in x (but not in the variable t).

5. Following C. Gutiérrez's article [122], C. Gutiérrez, C. Segovia, and J. L. Torrea in [124] consider higher order Gaussian Littlewood–Paley functions to get the $L^p(\gamma_d)$ boundedness of Gaussian higher order Riesz transforms. Again, an important advantage of this proof is that the constants are independent of dimension (see Chapter 9).

6. The Jacobi–Littlewood–Paley g function can be defined as

$$g^{(\alpha,\beta)} f(x) = \Big(\int_0^\infty t|\nabla_{(\alpha,\beta)} P_t^{\alpha,\beta} f(x)|^2 dt\Big)^{1/2}, \tag{5.58}$$

where $\nabla_{(\alpha,\beta)} = \left(\frac{\partial}{\partial t}, \delta_{\alpha,\beta}\right) = \left(\frac{\partial}{\partial t}, \sqrt{1-x^2}\frac{\partial}{\partial x}\right)$
The L^p-continuity of the Jacobi–Littlewood–Paley g-function $g(\alpha;\beta)$, was proved by A. Nowak and P. Sjögren in [213].

Theorem 5.19. *Assume that* $1 < p < \infty$ *and* $\alpha, \beta \in [-1/2, \infty)^d$. *There exists a constant* c_p *such that*

$$\|g^{(\alpha,\beta)} f\|_{p,(\alpha,\beta)} \leq c_p \|f\|_{p,(\alpha,\beta)}. \tag{5.59}$$

7. The Laguerre–Littlewood–Paley g function can be defined as

$$g^{\alpha} f(x) = \Big(\int_0^{\infty} t |\nabla_{\alpha} P_t^{\alpha} f(x)|^2 dt \Big)^{1/2} \tag{5.60}$$

where $\nabla_{\alpha} = \left(\frac{\partial}{\partial t}, \delta_{\alpha}\right) = \left(\frac{\partial}{\partial t}, \sqrt{x}\frac{\partial}{\partial x}\right)$ and $\{P_t^{\alpha}\}$ is the Poisson–Laguerre semigroup, i.e., the subordinated semigroup to the Laguerre semigroup (for more information see [279]).

The L^p continuity of the Laguerre–Littlewood–Paley g function was proved by A. Nowak in [208].

Theorem 5.20. *Assume that* $1 < p < \infty$ *and* $\alpha \in [1/2, \infty)^d$. *There exists a constant* c_p *such that*

$$\|g^{\alpha} f\|_{p,\alpha} \leq c_p \|f\|_{p,\alpha}. \tag{5.61}$$

8. The main motivation of A. Nowak's article [209] on Littlewood–Paley g functions for the Laguerre case, and also A. Nowak and P. Sjögren's article on Littlewood–Paley g functions for the Jacobi case was, following E. Stein monograph [253], to study the L^p boundedness of the corresponding Riesz transforms.

9. In [202], E. Navas and W. Urbina develop a transference method to obtain the L^p-boundedness, $1 < p < \infty$ of the Gaussian–Littlewood–Paley g function and the L^p-boundedness of the Laguerre–Littlewood–Paley g function from the L^p-continuity of the Jacobi–Littlewood–Paley g function, in dimension one, using the well-known asymptotic relations between Jacobi polynomials and other classical orthogonal polynomials (10.64) and (10.67) (see also [262, (5.3.4),(5.6.3)]).
   i) For Hermite polynomials,

$$\lim_{\lambda \to \infty} \lambda^{-n/2} C_n^{\lambda}(x/\sqrt{\lambda}) = \frac{H_n(x)}{n!},$$

   where $\{C_n^{\lambda}(x)\}$ are the Gegenbauer polynomials defined as

$$C_n^{\lambda}(x) = \frac{\Gamma(\lambda + 1/2)\Gamma(n + 2\lambda)}{\Gamma(2\lambda)\Gamma(n + \lambda + 1/2)} P_n^{(\lambda - 1/2, \lambda - 1/2)}(x).$$

   ii) For Laguerre polynomials,

$$\lim_{\beta \to \infty} P_n^{(\alpha,\beta)}(1 - 2x/\beta) = L_n^{\alpha}(x).$$

Both relations hold uniformly in every closed interval of $\mathbb{R}$.

10. In [165], I. López and W. Urbina, following R. Wheeden and C. Segovia [293], consider some Gaussian area functions of higher order. $S^{k,a}_{2,\gamma}$, which generalize the function S_γ. They are defined as

$$S^{k,a}_{2,\gamma} f(x) = \left(\int_{\Gamma^a_\gamma(x,\delta)} \left(t \wedge 1 \wedge |x|^{-1} \right)^{2k-1+d} \left| \nabla^k P_t(f)(y) \right|^2 t^{-d} \left(t^{-d} \vee |x|^d \vee 1 \right) dy\, dt \right)^{1/2},$$

for $k \geq 1$ and $a > 0$, where $\Gamma^a_\gamma(x)$ is a Gaussian cone with aperture $a > 0$ and ∇^k_x is the k-th gradient. (in the spatial variable x). If $k = 1$

$$S^{1,a}_\gamma f(x) \leq S^a_\gamma f(x).$$

They also define an associated Littlewood–Paley g-type function of order k as

$$\widetilde{g}^k_\gamma(f)(x) = \left(\int_0^\infty t^{-(d+1)} \left(t \wedge 1 \wedge |x|^{-1} \right)^{2k+d} \left| \nabla^k P_t(f)(x) \right|^2 dt \right)^{1/2}, \quad (5.62)$$

proving the $L^p(\gamma_d)$-boundedness for $1 < p < \infty$ of the function $S^{k,a}_{2,\gamma}$ and $\widetilde{g}^k_\gamma$.

Theorem 5.21. *Suppose $f \in L^p(\gamma_d)$. Then,*
i) If $1 < p < \infty$, then there exists a constant $C_{p,\delta,k,d} > 0$ such that

$$\left\| S^{k,a}_{2,\gamma}(f)(\cdot, \delta) \right\|_{p,\gamma_d} \leq C_{p,\delta,k,d} \|f\|_{p,\gamma_d}. \quad (5.63)$$

ii) There exists a constant $C_{k,\delta,d} > 0$ such that for every $y \in \mathbb{R}^d$

$$\widetilde{g}^k_\gamma(f)(y) \leq C_{k,\delta,d} S^{k,a}_{2,\gamma}(f)(y, \delta). \quad (5.64)$$

The $L^p(\gamma_d)$ boundedness of $S^{k,a}_{2,\gamma}$ is obtained simply by proving the following pointwise estimate

$$S^{k,\alpha}_\gamma f(x) \leq S^{k,\beta}_\gamma f(x),$$

$0 < \alpha < \beta$ (see [165, Lemma 2.3]), and applying (5.16). For the inequality (5.64), the arguments are similar to those in Theorem 5.16.

11. The Littlewood–Paley theory for the Jacobi semigroup was done, in the case $d = 1$ by W. Connett and A. Schwartz in [58]. For $d > 1$ it was done by A. Nowak and P. Sjögren in [213], to study the L^p continuity of the Jacobi–Riesz transforms for more, see Section 9.5 in Chapter 9.

12. The Littlewood–Paley theory for the Laguerre semigroup for $d > 1$ was obtained by A. Nowak in [209] to study the L^p continuity of the Laguerre–Riesz transforms for more, see Section 9.5 in Chapter 9.

13. Some of the Littlewood–Paley estimates obtained in this chapter, at least for the time derivative, can also be obtained from the functional calculus of L that will be studied in the next chapter (see for instance [137, Theorem 10.4.16]). This shows that something more general is at play. Also, this implies that changing the square functions involving the subordinated semigroup $\{P_t\}_{t\geq 0}$ to square functions involving $\{T_t\}_{t\geq 0}$ or some other functional of L is possible and easy.

6

Spectral Multiplier Operators with Respect to the Gaussian Measure

In this chapter, we study spectral multiplier operators for Hermite polynomial expansions. First, we consider Meyer's multiplier theorem, which is one of the most basic and most useful results for Hermite expansions. Then, we consider spectral multipliers of Laplace transform type. In both cases, we prove their boundedness in $L^p(\gamma_d)$, for $1 < p < \infty$. For the case of spectral multipliers of Laplace transform type, we also study the boundedness in the case $p = 1$. Finally, we discuss the fact that the Ornstein–Uhlenbeck operator has a bounded holomorphic functional calculus.

6.1 Gaussian Spectral Multiplier Operators

Definition 6.1. *Given a bounded function $m : \mathbb{N}_0 \to \mathbb{C}$. According to the spectral theorem, we may form the operator $m(L)$[1] defined for any $f \in L^2(\gamma_d)$ [2]*

$$m(L)f = \sum_{k=0}^{\infty} m(k)\mathbf{J}_k f = \sum_{k=0}^{\infty} m(k) \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu. \tag{6.1}$$

Observe that $m(L)$ is trivially bounded in $L^2(\gamma_d)$, as

$$\|m(L)f\|_{2,\gamma} = \sum_{k=0}^{\infty} |m(k)|^2 \|\mathbf{J}_k f\|_{2,\gamma} \leq \|m\|_\infty \sum_{k=0}^{\infty} \|\mathbf{J}_k f\|_{2,\gamma} = \|m\|_\infty \|f\|_{2,\gamma}.$$

We call $m(L)$ the *spectral multiplier operator* associated with the *spectral multiplier m*.

[1]Formally speaking, it should be denoted by $m(-L)$ because of (2.7); for simplicity we just write it as $m(L)$.

[2]Alternatively, we could define $m(L)$ on the set of polynomials in d-variables, $\mathscr{P}(\mathbb{R}^d)$, as they have finite Hermite expansion $f = \sum_{k=0}^{\infty} \mathbf{J}_k f = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu \rangle_{\gamma_d} \mathbf{h}_\nu$.

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4_6

Moreover, because $m(L)$ is well defined in $\mathscr{P}(\mathbb{R}^d)$, and we know that $\mathscr{P}(\mathbb{R}^d)$ is dense in $L^p(\gamma_d)$ for any $1 \leq p < \infty$, the multiplier operator $m(L)$ is densely defined in $L^2(\gamma_d)$ with domain

$$\mathscr{D}_m = \left\{ f \in L^2(\gamma_d) : \sum_{k=0}^{\infty} |m(k)|^2 \|J_k f\|_{2,\gamma} < \infty \right\}.$$

The basic problem of the multiplier theory is to determine the conditions on the spectral multiplier m such that the spectral multiplier operator $m(L)$, initially defined in $L^2(\gamma_d) \cap L^p(\gamma_d)$, has a bounded extension on $L^p(\gamma_d), 1 < p < \infty$; in other words, when we can find a constant $C_p > 0$ dependent only on p such that

$$\|m(L)f\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma}, \tag{6.2}$$

for all $f \in L^p(\gamma_d)$.
We also want to consider under which conditions $m(L)$ is of weak type $(1,1)$ with respect to the Gaussian measure; in other words, when we can find a constant $C > 0$ dependent only on p such that

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : m(L)f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda} \|f\|_{1,\gamma} \tag{6.3}$$

for any $f \in L^p(\gamma_1)$.

6.2 Meyer's Multipliers

One of the most basic results in Gaussian multiplier theory was obtained by P. A. Meyer in [189] (see also [288] and [218]), using in a fundamental way the hypercontractivity property of the Ornstein–Uhlenbeck semigroup. Therefore, the multiplier theory for Hermite expansions and the hypercontractivity property of the Ornstein–Uhlenbeck semigroup are closely related.

Theorem 6.2. *(Meyer) Given a function h, holomorphic in a neighborhood of the origin, and let m be a spectral multiplier such that $m(k) = h\left(\frac{1}{k^\alpha}\right)$, for some $\alpha > 0$ and $k \geq n_0$, for some $n_0 \geq 0$, then the spectral multiplier operator $m(L)$ admits an $L^p(\gamma_d)$-bounded extension for any $1 < p < \infty$. Moreover, its $L^p(\gamma_d)$-norm does not depend on the dimension.*

Proof. Using Corollary 2.17, Lemma 2.18 and the inequality (3.42), the proof is almost immediate. Let us decompose $m(L)$ into its finite and infinite parts.

$$m(L)f = \sum_{k=0}^{n_0-1} m(k)\,\mathbf{J}_k f + \sum_{k=n_0}^{\infty} m(k)\,\mathbf{J}_k f = m_1(L)f + m_2(L)f.$$

Using Corollary 2.17, we know that $\mathbf{J}_n$ is $L^p(\gamma_d)$-bounded; therefore $m_1(L)$ is $L^p(\gamma_d)$-bounded,

$$\|m_1(L)f\|_{p,\gamma} \le \sum_{k=0}^{n_0-1} m(k)\|\mathbf{J}_k f\|_{p,\gamma} \le C_p\|f\|_{p,\gamma}.$$

Thus, it is enough to prove that m_2 is $L^p(\gamma_d)$-bounded,

$$\|m_2(L)f\|_{p,\gamma} \le C_p\|f\|_{p,\gamma}.$$

Using the generalized potential operators (3.41) and the inequality (3.42), then, as $h(x) = \sum_{n=0}^{\infty} a_n x^n$,

$$\begin{aligned} m_2(L)f &= \sum_{k=n_0}^{\infty} m(k)\mathbf{J}_k f = \sum_{k=n_0}^{\infty}\Big(\sum_{n=0}^{\infty} a_n \frac{1}{k^{\alpha n}}\Big)\mathbf{J}_k f \\ &= \sum_{n=0}^{\infty} a_n\Big(\sum_{k=n_0}^{\infty}\frac{1}{k^{\alpha n}}\mathbf{J}_k f\Big) = \sum_{n=0}^{\infty} a_n (U_{n_0,\alpha} f)^n. \end{aligned}$$

Using the $L^p(\gamma_d)$-boundedness of $U_{n_0,\alpha}$ n-times, we get

$$\|m_2(L)f\|_{p,\gamma} \le \sum_{k=1}^{\infty} |a_n| \|(U_{n_0,\alpha} f)^n\|_{p,\gamma} \le C\Big(\sum_{n=0}^{\infty}|a_n|\frac{1}{n_0^{\alpha n}}\Big)\|f\|_{p,\gamma} = C\|f\|_{p,\gamma}. \qquad \square$$

Definition 6.3. *A spectral multiplier operator $m(L)$ is called Meyer's multiplier if it satisfies the hypothesis of Theorem 6.2, i.e., there exists a function h holomorphic in a neighborhood of the origin such that*

$$m(k) = h\Big(\frac{1}{k^{\alpha}}\Big), \tag{6.4}$$

for some $\alpha > 0$ and $k \ge n_0$, for some $n_0 \ge 0$.

We see in Chapter 8 that the Gaussian Riesz potentials are the simplest Meyer's multipliers possible (see 8.5), and that the Gaussian Bessel potentials are not Meyer's multipliers, but the composition of two Meyer's multipliers (see 8.19). On the other hand, the Ornstein–Uhlenbeck and the Poisson–Hermite operators and their variations are Gaussian multipliers but are not Meyer's multipliers. Finally, as we are going to see in Chapter 9, the Gaussian Riesz transforms are not Gaussian multipliers, different than the Riesz transforms in the classical case.

6.3 Gaussian Laplace Transform Type Multipliers

Following E. Stein [253, Chapter 4], let us consider *Laplace type multipliers*.

Definition 6.4. *A function $m : (0,\infty) \to \mathbb{C}$ is said to be of Laplace transform type if and only if*

$$m(k) = k\int_0^{\infty} \phi(t)e^{-tk}dt, \quad k > 0, \tag{6.5}$$

where $\phi : (0,\infty) \to \mathbb{C}$, is a bounded measurable function.

Observe that taking the change of variables $r = e^{-t}$, we see that m can be rewritten as

$$m(k) = k\int_0^1 \psi(t) r^k \frac{dr}{r}, \quad k > 0, \tag{6.6}$$

where $\psi(r) = \phi(-\log r)$.

Definition 6.5. *A spectral multiplier operator $m(L)$ is said to be a Laplace transform type multiplier, if the spectral multiplier m is a function of Laplace transform type. Then, $m(L)$ can be written as*

$$m(L)f(x) = \sum_{k=0}^{\infty} m(\sqrt{k})\mathbf{J}_k f = \sum_{k=0}^{\infty} m(\sqrt{k}) \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu\rangle_{\gamma_d} \mathbf{h}_\nu, \tag{6.7}$$

for a function f with Hermite expansion $f = \sum_{k=0}^{\infty} \mathbf{J}_k f = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \langle f, \mathbf{h}_\nu\rangle_{\gamma_d} \mathbf{h}_\nu$.

Observe that if we ask the function ϕ to be not only bounded but integrable, then we can get the following easy result:

Proposition 6.6. *If $m : (0,\infty) \to \mathbb{C}$ is a spectral multiplier of Laplace transform type function such that ϕ is bounded and integrable, then the spectral multiplier operator $m(L)$ is a $L^p(\gamma_d)$-bounded operator, for $1 < p < \infty$.*

Proof. Let $f = \sum_{k=0}^{\infty} \mathbf{J}_k f$, then

$$\begin{aligned} m(L)f &= \sum_{k=0}^{\infty} \phi(\sqrt{k})\mathbf{J}_k f = \sum_{k=0}^{\infty} \Big(\int_0^{\infty} \phi(t) e^{-\sqrt{k}t} dt\Big) \mathbf{J}_k f = \int_0^{\infty} \Big[\sum_{k=0}^{\infty} e^{-\sqrt{k}t} \mathbf{J}_k f\Big] \phi(t) dt \\ &= \int_0^{\infty} P_t f\, \phi(t) dt. \end{aligned}$$

Therefore, using Minkowski's integral inequality, and the $L^p(\gamma_d)$-boundedness of the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$,

$$||m(L)(f)||_{p,\gamma} = \left\| \int_0^{\infty} P_t f\, \phi(t) dt \right\|_{p,\gamma} \leq \int_0^{\infty} ||P_t f||_{p,\gamma} |\phi(t)| dt \leq C_p\, ||f||_{p,\gamma}. \qquad \square$$

Now for the general case, using the Littlewood–Paley theory, following E. Stein [253, Chapter II], we get

Theorem 6.7. *Given a Laplace transform type spectral multiplier m, the spectral multiplier operator $m(L)$ has a $L^p(\gamma_d)$-bounded extension, for $1 < p < \infty$.*

Proof. The proof is given here for the case of the Poisson–Hermite semigroup for completeness, but it is still valid in far more general settings, as is clear from E. Stein's monograph [253].[3]

We need to prove the following identity,

$$m(L)(f)(x) = -\int_0^{\infty} \frac{\partial P_t f}{\partial t}(x) \phi(t) dt. \tag{6.8}$$

[3] In fact, nowadays this theorem is known as *Stein's universal multiplier theorem.*

For it suffices to check the identity for the normalized Hermite polynomials $\{\mathbf{h}_\nu\}$,

$$\int_0^\infty \frac{\partial P_t \mathbf{h}_\nu}{\partial t}(x)\phi(t)dt = \int_0^\infty \frac{d}{dt}(e^{-\sqrt{|\nu|}t})\mathbf{h}_\nu(x)dt\ \phi(t)$$
$$= -\sqrt{|\nu|}\int_0^\infty e^{-\sqrt{|\nu|}t}\phi(t)dt\ \mathbf{h}_\nu(x) = -m(\sqrt{|\nu|})\ \mathbf{h}_\nu(x).$$

Now,

$$P_{t_1}(m(L)f)(x) = -\int_0^\infty P_{t_1}\Big(\frac{\partial P_t f}{\partial t}(x)\Big)\phi(t)dt = -\int_0^\infty \frac{\partial P_{t+t_1} f}{\partial t}(x)\phi(t)dt.$$

Hence,

$$\frac{\partial P_{t_1}(m(L)f)}{\partial t_1}(x) = -\frac{\partial}{\partial t_1}\Big(\int_0^\infty P_{t_1}\Big(\frac{\partial P_t f}{\partial t}(x)\Big)\phi(t)dt\Big) = -\int_0^\infty \frac{\partial^2 P_{t+t_1} f}{\partial t^2}(x)\phi(t)dt,$$

thus, as ϕ is bounded, using the Cauchy–Schwartz inequality

$$\Big|\frac{\partial P_{t_1}(m(L)f)}{\partial t_1}(x)\Big| \le \int_0^\infty \Big|\frac{\partial^2 P_{t+t_1} f}{\partial t^2}(x)\Big||\phi(t)|dt \le M\int_0^\infty \Big|\frac{\partial^2 P_{t+t_1} f}{\partial t^2}(x)\Big|dt$$
$$= M\int_{t_1}^\infty s\Big|\frac{\partial^2 P_s f}{\partial s^2}(x)\Big|\frac{ds}{s} \le M\Big(\int_{t_1}^\infty s^2\Big|\frac{\partial^2 P_s f}{\partial s^2}(x)\Big|^2 ds\Big)^{1/2} t_1^{-1/2}.$$

Therefore, according to the same argument used in the proof of Proposition 5.12, we have using Fubini's theorem

$$g_{t,\gamma}((m(L)f)(x) = \left(\int_0^\infty t_1\left|\frac{\partial P_{t_1} m(L)f}{\partial t_1}(x)\right|^2 dt_1\right)^{1/2}$$
$$\le C\left(\int_0^\infty\left(\int_{t_1}^\infty s^2\left|\frac{\partial^2 P_s f}{\partial s^2}(x)\right|^2 ds\right)dt_1\right)^{1/2}$$
$$= C\left(\int_0^\infty s^3\left|\frac{\partial^2 P_s f}{\partial s^2}(x)\right|^2 ds\right)^{1/2} = Cg^2_{t,\gamma}f(x).$$

Now, using Theorem 5.6 and Definition 5.7, we get

$$C_p'\|m(L)f\|_{p,\gamma} \le \|g_{t,\gamma}((m(L)f)(x)\| \le C\|g^2_{t,\gamma}f\|_{p,\gamma} \le C_p\|f\|_{p,\gamma}. \qquad \square$$

In particular, the imaginary powers $(-L)^{i\lambda}$ arising from $\phi(t) = \frac{t^{-i\lambda}}{\Gamma(1-i\lambda)}$ admits a $L^p(\gamma_d)$-bounded extension for any $1 < p < \infty$, because

$$\lambda^{-i\alpha} = \frac{\lambda}{\Gamma(1-i\alpha)}\int_0^\infty e^{-\lambda s}s^{-i\alpha}ds.$$

Theorem 6.7, is a weak version of *the Marcinkiewicz multiplier theorem* in the Euclidean case for the d-dimensional torus $\mathbb{T}^d$. The link is that if ϕ is of Laplace type then

$$|x^k \phi(x)| \leq C_k$$

for any $k \geq 0$, which is a particular case of the Marcinkiewicz condition, then $\phi(|x|)$ is a multiplier in $L^p(\mathbb{T}^d)$, $1 < p < \infty$.

6.4 Functional Calculus for the Ornstein–Uhlenbeck Operator

Now, we are going to discuss the fact that the Ornstein–Uhlenbeck operator has a bounded holomorphic functional calculus. In [105] J. García-Cuerva, G. Mauceri, S. Meda, and P. Sjögren, J. L. Torrea proved that for Gaussian multipliers if $p \neq 2$ there is no reasonable non-holomorphic functional calculus in $L^p(\gamma_d)$ for L. In particular, they proved that there is not an analog of the classical Hörmander multiplier theorem. In fact, for each $p \neq 2$, there exists a spectral multiplier m_p such that $m_p(L)$ does not extend to a bounded operator on $L^p(\gamma_d)$, which is a restriction of a holomorphic function in a neighborhood of $\mathbb{R}^d_+$, which satisfies the conditions

$$\sup_{x>0} |x^j \partial_j m_p(x)| < \infty,$$

for all $j \in \mathbb{N}$.

Moreover, in [103], J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea proved that a spectral multiplier operator $m(L)$ of Laplace type is also of weak type $(1,1)$ with respect to the Gaussian measure. Given a spectral multiplier m is of Laplace transform type, then $m(L)$ is a continuous operator from the space of test functions to the space of distributions on $\mathbb{R}^d$; thus, it has a distributional kernel. Let us prove that, off the diagonal, this kernel has a density K_ψ with respect to the measure $\gamma_d(dx) \otimes dy$, which satisfies the standard Calderón–Zygmund estimates in a suitable neighborhood of the diagonal (see [103, Lemma 2.1 and Theorem 2.2]). Consider the operator $r^L, 0 \leq r < 1$, whose integral kernel

$$M_r(x,y) = \frac{1}{\pi^{d/2}(1-r^2)^{d/2}} e^{-\frac{|y-rx|^2}{1-r^2}},$$

may be obtained from Mehler's kernel by the change of variables $t = -\log r$. Thus

$$r^L f(x) = \int_{\mathbb{R}^d} M_r(x,y) f(y) dy,$$

for all test functions f. As Mehler's kernel satisfies the heat equation $\partial_t M_t(x,y) = LM_t(x,y)$ (see (2.53)), the kernel $M_r(x,y)$ satisfies the transformed equation $r\partial_r M_r(x,y) = -LM_r(x,y)$. If $\psi \in L^\infty(\mathbb{R}^d)$, define

$$K_\psi(x,y) = \int_0^1 \psi(r) \partial_r M_r(x,y) dr.$$

For $t > 0$, the local region N_t defined in (4.63) is the neighborhood of the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$.

Lemma 6.8. *If $x \neq y$, the integral defining K_ψ is absolutely convergent. Moreover, for $t > 0$ and each pair of multi-indices $\alpha, \beta \in \mathbb{N}^d$, there exists a constant C such that*

$$|\partial_x^\alpha \partial_y^\beta K_\psi(x,y)| \leq C \frac{\|\psi\|_\infty}{|x-y|^{d+|\alpha|+|\beta|}} \tag{6.9}$$

for all $(x,y) \in N_t$, $x \neq y$.

Proof. Using Rodrigues' formula for the Hermite polynomials (1.28), we have

$$\partial_x^\alpha \partial_y^\beta M_r(x,y) = \frac{(-r)^{|\alpha|}}{\pi^{d/2}(1-r^2)^{d/2+|\alpha|+|\beta|}} H_{\alpha+\beta}\left(\frac{xr-y}{\sqrt{1-r^2}}\right) e^{(1-r^2)^{d/2}}. \tag{6.10}$$

An elementary computation shows that the function $r \mapsto \partial_x^\alpha \partial_y^\beta M_r(x,y)$ is the product of the positive function,

$$\frac{1}{\pi^{d/2}} \frac{1}{(1-r^2)^{d/2+|\alpha|+|\beta|}} e^{-\frac{|y-rx|^2}{1-r^2}},$$

and a polynomial in r of degree at most $2|\alpha| + |\beta| + 3$, whose coefficients depend on x and y. Hence, as a function of r, it changes sign a finite number of times and there exists a constant C such that

$$\int_0^1 |\psi(r)| |\partial_r \partial_x^\alpha \partial_y^\beta M_r(x,y)| dr \leq C \|\psi\|_\infty \max_{0<r<1} |\partial_x^\alpha \partial_y^\beta M_r(x,y)|,$$

for all $x, y \in \mathbb{R}^d$. According to (6.10), we have that

$$\max_{0<r<1} |\partial_x^\alpha \partial_y^\beta M_r(x,y)| \leq \frac{C}{(1-r^2)^{(d+|\alpha|+|\beta|)/2}} \exp\left(-c_0 \frac{|y-rx|^2}{1-r^2}\right)$$

for some positive constant c_0. Because, in the local region N_t,

$$|rx-y|^2 \geq |x-y|^2 - 2(1-r)|x||x-y| \geq |x-y|^2 - 2(1-r)t,$$

the right-hand side of the previous inequality can be estimated by

$$C(t)(1-r^2)^{(d+|\alpha|+|\beta|)/2} \exp\left(-c_0 \frac{|y-rx|^2}{1-r^2}\right) \leq C|x-y|^{(d+|\alpha|+|\beta|)},$$

for all $(x,y) \in N_t$. □

Using this lemma, we can obtain the following representation of $m(L)$ in terms of $K_\psi(x,y)$.

Theorem 6.9. *Given a spectral multiplier m of Laplace transform type given by the formula* (6.6)*, then the spectral multiplier operator has the following integral representation*

$$m(L) = \int_0^1 \psi(r) L r^L \frac{dr}{r} \tag{6.11}$$

where the integral converges on the weak operator topology of $L^2(\gamma_d)$*. Moreover, f is a test function,*

$$m(L) f(x) = \int_0^1 K_\psi(x,y) f(y) dy, \tag{6.12}$$

for all x in the support of f.

Proof.

$$\begin{aligned}
\langle m(L)f, g\rangle_\gamma &= \sum_{k=1}^\infty m(k)\langle \mathbf{J}_k f, g\rangle_\gamma = \sum_{k=1}^\infty k \int_0^\infty \phi(t) e^{-tk} dt \langle \mathbf{J}_k f, g\rangle_\gamma \\
&= \sum_{k=1}^\infty k \int_0^1 \psi(r) r^k \frac{dr}{r} \langle \mathbf{J}_k f, g\rangle_\gamma = \int_0^1 \psi(r) \sum_{k=1}^\infty k r^k \langle \mathbf{J}_k f, g\rangle_\gamma \frac{dr}{r} \\
&= \int_0^1 \psi(r) \langle L r^L f, g\rangle_\gamma \frac{dr}{r} = \int_0^1 \langle \int_0^1 \psi(r) L r^L f \frac{dr}{r}, g\rangle_\gamma
\end{aligned}$$

where we have used that $\sum_{k=1}^\infty |\langle \mathbf{J}_k f, g\rangle_\gamma| \leq \|f\|_{2,\gamma}\|g\|_{2,\gamma}$; thus, we may interchange the order of summation and integration. Therefore, we have obtained (6.11).

To compute the kernel of the spectral multiplier operator $m(L)$, assume that f and g are test functions on $\mathbb{R}^d$. Then

$$\begin{aligned}
\langle L r^L f, g\rangle_\gamma = \langle r^L f, Lg\rangle_\gamma &= \iint M_r(x,y) f(y) dy\, \overline{Lg}\gamma_d(dx) \\
&= \langle M_r \gamma_d(dx) \otimes dy, L(\overline{g} \otimes f)\rangle
\end{aligned}$$

Here, $\langle\cdot,\cdot\rangle$ denotes the pairing between distributions and test functions on $\mathbb{R}^d \times \mathbb{R}^d$ and $M_r\gamma_d(dx) \otimes dy$ is the distribution whose density with respect to the measure $\gamma_d(dx) \otimes dy$ is M_r. As the operator L is symmetric with respect to the Gaussian measure,

$$\begin{aligned}
\langle r^L f, Lg\rangle_\gamma &= \langle L(M_r)\gamma_d(dx) \otimes dy, \overline{g} \otimes f\rangle \\
&= \iint r\partial_r M_r(x,y)\overline{g}(x) f(y) dy \gamma_d(dx)
\end{aligned}$$

Thus, using (6.11),

$$\langle m(L)f, g\rangle_\gamma = \int_0^1 \psi(r) \iint \partial_r M_r(x,y)\overline{g}(x) f(y)\gamma_d(dx) dy dr.$$

If f and g have disjoint supports, the triple integral in the identity above is absolutely convergent because of the previous lemma. Thus, using Fubini's theorem

$$\langle m(L)f,g\rangle_\gamma = \iint K_\psi(x,y)f(y)dy\overline{g}(x)\gamma_d(dx).$$

This proves that K_ψ is the restriction to the complement of the diagonal of the kernel of $m(L)$, i.e., we have proved that off the diagonal, $m(L)$ has density K_ψ with respect to the measure $\gamma_d(dx)\otimes dy$ □

Now, it can be proved that a spectral multiplier operator $m(L)$ of Laplace type is also of weak type $(1,1)$ with respect to the Gaussian measure [103, Theorem 3.8]. The proof uses these two previous results. The operator is split, as usual, into a local part and a global part, using in this case the local region R_t(4.63). J. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea improved the treatment of the local part by making a smooth truncation and reducing the estimates to the general Calderón–Zygmund theory. Then, the global part is immediately bounded by the maximal Mehler's kernel used by P. Sjögren, in [247] (for more details, we refer the reader to [103, Section 3]).

Additionally, in [103], they also investigate how to define the multiplier operator in terms of its kernel, as a limit of truncated integrals. In particular, we see under what conditions the multiplier is given by a principal value integral. Boundedness is also proved for the maximal multiplier operator, via a vector-valued version of the estimates. The result applies, in particular, to the imaginary powers of $(-L)$, $(-L)^{i\lambda}$. Here, the growth of the operator (quasi-)norm for large imaginary powers is of special interest. As $-L$ has a non-trivial kernel, to define imaginary powers, it is first needed to restrict $-L$ to the orthogonal complement of the kernel. This amounts to considering $L^{i\alpha}\Pi_0$, where, as before, $\Pi_0 = I - \mathbf{J}_0$. The weak type $(1,1)$ constant of $L^{i\alpha}\Pi_0$ increases at most exponentially as $|\alpha|\to\infty$. They proved that this estimate cannot be improved to polynomial growth.

On the other hand, the assumption that a spectral multiplier m of Laplace type implies that m can be extended to a holomorphic function on the half-plane $\{z\in\mathbb{C} : \mathrm{Re}\, z > 0\}$, which is bounded on every sector $S_\theta = \{z\in\mathbb{C} : |\arg z| < \theta\}, 0<\theta<\pi/2$ (see Figure 2.1). As the spectrum of the Ornstein–Uhlenbeck L on $L^1(\gamma_d)$ is the closed right half-plane (see Theorem 2.7), it is natural to impose a holomorphy condition on the multiplier m if we want the operator $m(L)$ to be defined on $L^1(\gamma_d)$. Nevertheless, because the spectrum of $-L$ on $L^p(\gamma_d)$, for $1<p<\infty$, is the set $\mathbb{N}_0$ of non-negative integers, it seems too restrictive to require holomorphy of the multiplier m to obtain the $L^p(\gamma_d)$-boundedness of $m(L)$. In [182], S. Meda gave a sufficient condition for the existence of a non-holomorphic functional calculus for the generator A of a symmetric contraction semigroup on $L^p(M), 1<p<\infty$, where M is a σ-finite measure space.

If we fix $p\in(1,\infty)$, as we have mentioned before, it is important to determine the minimal regularity conditions of the spectral multiplier m, which imply that the spectral multiplier operator $m(L)$ is bounded in $L^p(\gamma_d)$. These conditions are sometimes best expressed in terms of Banach spaces of holomorphic functions. If $\theta \in (0, \pi/2)$, consider the open sector $S_\theta = \{z \in \mathbb{C} : |\arg z| < \theta\}$, and denote by $H^\infty(S_\theta)$ the space of bounded holomorphic functions on S_θ. A consequence of an abstract result by M. Cowling [59, Theorem 2] is that if $\theta > \pi|\frac{1}{q} - \frac{1}{2}|$, the spectral multiplier m is bounded and there exists $\tilde{m} \in H^\infty(S_\theta)$ such that $m(k) = \tilde{m}(k), k = 1, 2, 3, \cdots$, then $m(L)$ extends to a bounded operator on $L^q(\gamma_d)$.

Moreover, in [105] it is shown that requiring holomorphy of a spectral multiplier m, in a sector of angle smaller than $\phi_p^* = \arcsin|\frac{2}{p} - 1|$, is not sufficient for the boundedness of $m(L)$ on $L^p(\gamma_d)$. Observe that $\phi_p^* \to \pi/2$ as $p \to 1$ is in line, with the fact, already mentioned, that the spectrum of L on $L^1(\gamma_d)$ is the (closed) right half-plane (see Theorem 2.7). Furthermore, the $L^1(\gamma_d)$-boundedness of dilation-invariant spectral multiplier operators $m(L)$ was characterized in [131, Theorem 3.5].

Finally, let us mention the main result in [105, Theorem 1] by J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, and J. L. Torrea, which is an improvement, in the finite dimensional case of Cowling's result. Using the notation introduced in Chapter 2, the statement of the theorem is roughly as follows: for every $p \in (1, \infty)$, $p \neq 2$, and consider the sector $S_{\phi_p^*} := \{z \in \mathbb{C} : |\arg z| < \phi_p^*\}$. If m is a bounded holomorphic function on $S_{\phi_p^*}$ whose boundary values on $\partial S_{\phi_p^*}$ satisfy suitable *Hörmander-type conditions*, then the spectral multiplier $m(L)$ extends to a bounded operator on $L^p(\gamma_d)$ and hence to $L^q(\gamma_d)$ for all q such that $|\frac{1}{q} - \frac{1}{2}| \leq |\frac{1}{p} - \frac{1}{2}|$.

To establish the theorem, we first need the following notation. Suppose that J is a non-negative integer and that $\theta \in (0, \pi/2)$. Denote by $H^\infty(S_\theta; J)$ the Banach space of all $m \in H^\infty(S_\theta)$ for which a *Hörmander condition* of order J holds: there exists a constant C such that

$$\sup_{R>0} \int_R^{2R} |x^j \partial_j m(e^{\pm i\theta} x)|^2 \frac{dx}{x} \leq C^2, \text{ for } j = 0, 1, \cdots, J. \tag{6.13}$$

$H^\infty(S_\theta; J)$ is endowed with the norm

$$\|m\|_{\theta,J} = \inf\left\{C : \sup_{R>0} \int_R^{2R} |x^j \partial_j m(e^{\pm i\theta} x)|^2 \frac{dx}{x} \leq C^2, \text{ for } j = 0, 1, \cdots, J\right\}.$$

Note that (6.13) implies that

$$\sup_{z \in S_\theta} |m(z)| \leq 2C,$$

if $J > 0$.

Theorem 6.10. *Let $1 < p < \infty$, $p \neq 2$, let $m : \mathbb{N} \to \mathbb{C}$ be a bounded function, and assume that there exists a bounded holomorphic function $\tilde{M}$ in $S_{\phi_p^*}$, such that*

$$\tilde{M}(k) = m(k), \quad k = 1, 2, 3, \cdots$$

then,

i) If $\tilde{M} \in H^\infty(S_{\phi_p^*};4)$, *then* $m(L)$ *extends to a bounded operator on* $L^p(\gamma_d)$*; hence, on* $L^q(\gamma_d)$ *for all* q *such that* $|\frac{1}{q}-\frac{1}{2}| \leq |\frac{1}{p}-\frac{1}{2}|$.

ii) If $\tilde{M} \in H^\infty(S_{\phi_p^*})$, *and* $|\frac{1}{q}-\frac{1}{2}| < |\frac{1}{p}-\frac{1}{2}|$, *then* $m(L)$ *extends to a bounded operator on* $L^q(\gamma_d)$.

A significant feature of Theorem 6.10 is that the number of derivatives in the Hörmander condition in *i*) is independent of the dimension. However, the estimates depend strongly on dimension; thus, they fail to give a result for the infinite dimensional case, but Cowling's result holds in the infinite dimensional case. Also, the theorem may be sharpened using $H^\infty(S_{\phi_p^*};J)$, for J non-integer.

Moreover, the size of the region of holomorphy, measured by the aperture of the cone, cannot be reduced, as is proved in the following result:

Theorem 6.11. *Let* $1 < p < \infty$, $p \neq 2$, *and* $\theta < \phi_p^*$. *Then, there exists a function* m, *which decays exponentially at infinity and belongs to* $H^\infty(S_{\phi_p^*};J)$, *for* J *for every positive integer* J, *such that* $m(L)$ *does not extend to a bounded operator on* $L^p(\gamma_d)$.

For details of the proofs of Theorem 6.10 and 6.11 we refer the reader to [105, Section 3]. They use an abstract multiplier result for generators of holomorphic semigroups, which is a variant of an earlier result by S. Meda (see [182] or [60]).

6.5 Notes and Further Results

1. In [260], D. Stroock also considers the case of spectral multipliers m, being the Laplace transform of a measure μ in $[0,\infty)$ such that, for some integer N

$$\int_0^\infty e^{-Nt}\nu(dt) < \infty,$$

and then m is defined as

$$m(k) = \begin{cases} 0, & \text{if } 0 \leq k < N-1 \\ \lambda \int_0^\infty e^{-\lambda t}\nu(dt), & \text{if } k \geq N. \end{cases}$$

The proof that the spectral multiplier operator $m(L)$ has an extension to $L^p(\gamma_d)$ is completely analogous to that of Theorem 6.7.

2. Some other examples of spectral multipliers, whose spectral multiplier operator $m(L)$ is $L^p(\gamma_d)$-bounded, but that are not Meyer's multipliers, are:
   - Let us consider the *even part projection* multiplier operator

$$m_e(L)f = \sum_{k=0}^\infty \mathbf{J}_{2k} f = \sum_{k=0}^\infty m(1/k)\mathbf{J}_k f$$

where

$$m(x) = \begin{cases} 1, & \text{if } x = \frac{1}{2n}, n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $m_e(L)$ is a $L^p(\gamma_d)$-multiplier because the even part of f, $f_e(x) = \frac{f(x)-f(-x)}{2} = \sum_{k=0}^{\infty} \mathbf{J}_{2k} f = m_e(L)f$. Therefore,

$$\|m_e(L)f\|_{p,\gamma} = \|f_e\|_{p,\gamma} \leq \|f\|_{p,\gamma},$$

but m_e is not a Meyer's multiplier.

- Analogously, we can consider the *odd part projection* multiplier operator,

$$m_o(L) = \sum_{k=0}^{\infty} \mathbf{J}_{2k+1} f.$$

Since $f = m_e(L)f + m_o(L)f$, and we know that $m_e(L)$ is $L^p(\gamma_d)$-bounded, then we conclude that $m_o(L)$ is a $L^p(\gamma_d)$-multiplier, which is not a Meyer's multiplier either.

- Let us consider the spectral multiplier operator

$$m_-(L)f = \sum_{k=0}^{\infty} (-1)^k \mathbf{J}_k f.$$

As

$$-f(-x) = 2f_p(x) - f(x) = \sum_{k=0}^{\infty} \mathbf{J}_{2k} f - \sum_{k=0}^{\infty} \mathbf{J}_2 f = m_-(L)f,$$

then it is clear that

$$||m_-(L)f||_{p,\gamma_d} = ||f||_{p,\gamma_d},$$

but $m_-(L)f$ is not a Meyer's multiplier.

3. Meyer's theorem admits an extension to spectral multiplier operators of the form

$$m(L)f(x) = \sum_{k=0}^{\infty} m(k,x) \mathbf{J}_k f(x), \tag{6.14}$$

where $f = \sum_{k=0}^{\infty} \mathbf{J}_k f$. The same proof carries over, if m admits an expansion of the form

$$m(t,x) = \sum_n a_n(x) t^n,$$

$|a_n(x)| \leq M_n$ and $\sum_{n=0}^{\infty} |M_n| \frac{1}{n_0^{\delta n}} < \infty$. Operators of the form (6.14) are in a sense *pseudo-differential operators* in the Gaussian context, and require further analysis and study.

4. An open question, as far as we know, is what is the boundedness property of Meyer's multiplier operators for the case of $p = 1$?

5. In his seminal article [28], W. Beckner proved, among other things, that the hypercontractivity property for the Ornstein–Uhlenbeck semigroup is a consequence of Young's generalized inequality, which itself is obtained from an inequality for multipliers of Hermite expansions. In fact, Beckner proved the continuity $L^p(\gamma_d) - L^{p'}(\gamma_d)$ of the operators T_t, but with a purely imaginary parameter $t = i\sqrt{p-1}$, something that is closely related to Weissler's representation [292] given in (2.34) and the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$. Moreover, the proof is quite interesting by itself, using in a decisive way the classical central limit theorem (CLT). Beckner makes clear the intimate relationship between classical harmonic analysis and Gaussian harmonic analysis, because, for example, the multiplier result allows him to obtain the best constant in the Haussdorff–Young inequality for the Fourier transform on $\mathbb{R}^d$.

6. In B. Muckenhoupt's monograph [198], he uses transplantation theorems to get spectral multiplier theorems for Jacobi expansions. This idea could be explored for the Hermite expansions, but to do that we would need to work with the whole family of generalized Hermite polynomials $\{H_n^\mu\}$ (see note 4 in Chapter 1; see also T. Chihara [54]).

7. In [148], M. Kemppainen studies a method of decomposing a spectral multiplier operators $m(L)$ into three parts according to the notion of admissibility, which quantifies the doubling behavior of the underlying Gaussian measure. He proves that the above-mentioned admissible decomposition is bounded in $L^p(\gamma_d)$ for $1 < p \leq 2$ in a certain sense involving the Gaussian conical square function. The proof relates admissibility to E. Nelson's hypercontractivity theorem in a novel way.

8. In [147], M. Kemppainen studies a class of spectral multiplier operators $m(L)$, defined using spectral multipliers m such that,

$$m(\lambda) = \int_0^\infty e^{-\lambda t}(t\lambda)^2 \phi(t) \frac{dt}{t},\ \lambda \geq 0,$$

where $\phi : (0,\infty) \to \mathbb{C}$ is twice continuously differentiable, satisfying

$$\sup_{0<t<\infty} (|\phi(t)| + t|\phi'(t)| + t^2|\phi''(t)|) + \int_1^\infty (|\phi'(t)| + t|\phi''(t)|)dt < \infty,$$

and finds a sufficient condition for the integrability of $m(L)$ in terms of the admissible conical square function (5.51) and a maximal function using a decomposition method presented in [231].

9. In [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, & W. Urbina established a version of P. A. Meyer's multiplier theorem for the Laguerre case, because, as we have mentioned in Chapter 2, point 12. of Section 2.5, the Laguerre semigroup is also hypercontractive.

10. In [236] E. Sasso obtains a multiplier theorem for spectral multipliers of Laplace transform type in the Laguerre case, proving that they are of weak type $(1,1)$ for the Gamma measure.

11. In [49], A. Carbonaro and O. Dragicević have an impressive result, using Bellman function techniques. It provides an alternative to the results in [105], but is also valid in infinite dimensions.

7

Function Spaces with Respect to the Gaussian Measure

One of the main goals of functional spaces is to interpret and quantify the smoothness of functions. In this chapter, we discuss the analogs of classical functional spaces with respect to the Gaussian measure. We see that almost all classical spaces with respect to the Lebesgue measure have an analog for the Gaussian measure; nevertheless, we see that in some cases, for instance, Hardy spaces, the analogs to classical spaces are still incomplete and/or imperfect. On the other hand, most of the time, even if the spaces look similar, most of the proofs are different, mainly because the Gaussian measure is not invariant by translation, which implies the need for completely different techniques.

7.1 Gaussian Lebesgue Spaces $L^p(\gamma_d)$

The Gaussian Lebesgue spaces have been used implicitly in previous chapters for the study of continuity properties of the Ornstein–Uhlenbeck semigroup, the Poisson–Hermite semigroup, and maximal functions. For completeness, we are including them in this chapter.

Definition 7.1. *For $1 \leq p < \infty$, the Gaussian Lebesgue space $L^p(\gamma_d)$ is defined as*

$$L^p(\gamma_d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f \text{ is a measurable function and } \int_{\mathbb{R}^d} |f(x)|^p \gamma_d(dx) < \infty \right\} \tag{7.1}$$

and the L^p-norm is given by

$$\|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^d} |f(x)|^p \gamma_d(dx) \right)^{1/p}. \tag{7.2}$$

Using analogous arguments, as in the classical case, it can be proved that the normed space $(L^p(\gamma_d), \|\cdot\|_{p,\gamma})$ is a Banach space for $1 \leq p < \infty$, that is, $L^p(\gamma_d)$ is a complete space (see for instance [263, Theorem 7.3]).

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4_7

As the Gaussian measure is a probability measure, using Hölder's inequality, we have for $1 \leq p < q$,

$$L^q(\gamma_d) \subset L^p(\gamma_d). \tag{7.3}$$

Additionally, from Theorem 10.7, we know that the family of polynomials with real coefficients is not only contained in $L^p(\gamma_d)$, $1 \leq p < \infty$, but is also dense there.

Thus, the Gaussian Lebesgue spaces $L^p(\gamma_d)$ are very different from the classical Lebesgue space $L^p(\mathbb{R}^d)$ theory with respect to the Lebesgue measure, because if $f \in L^p(\mathbb{R}^d)$, then $f(x) \to 0$ as $|x| \to \infty$, but for $f \in L^p(\gamma_d)$, we may have $f(x) \to \infty$ as $|x| \to \infty$, as long as it grows no faster than $e^{\delta |x|^2/p}$ with $\delta < 1$.

Observe that for any $1 \leq p < \infty$, the space $L^p(\gamma_d)$ is not closed under translations. For instance, in dimension one and $p = 1$, taking the function $f(x) = e^{|x|^2 - |x|}$, then it is clear that $f \in L^1(\gamma_1)$, but it is easy to see that

$$\tau_1 f(x) = f(x+1) = e^{|x+1|^2 - |x+1|} \notin L^1(\gamma_1).$$

Finally, because the Gaussian measure is trivially absolutely continuous with respect to the Lebesgue measure, with the Radon–Nikodym derivative the Gaussian weight, $\frac{d\gamma_d}{dx} = e^{-|x|^2}$, then

$$L^\infty(\gamma_d) = L^\infty(\mathbb{R}^d).$$

7.2 Gaussian Sobolev Spaces $L^p_\beta(\gamma_d)$

Sobolev spaces in the classical case are used to measure the regularity of solutions of partial differential equations (PDEs). Gaussian Sobolev spaces were introduced in the context of Malliavin calculus (see for instance P. Malliavin [172], D. Nualart [218] or S. Watanabe [288]). They play a fundamental role in it because they are used as a scale to measure the regularity of solutions of stochastic differential equations (see [218]). Moreover, similar to the classical case, Gaussian Sobolev spaces are particular cases of Gaussian Besov spaces; therefore, Besov spaces are a "better scale" to measure the regularity of functions.

Definition 7.2. *Given $\beta \geq 0$ and $1 \leq p < \infty$, the Gaussian Sobolev space of order β, $L^p_\beta(\gamma_d)$, is defined as the completion of the set of polynomials $\mathscr{P}(\mathbb{R}^d)$ with respect to the norm*

$$\|f\|_{p,\beta} := \left\| (I-L)^{\beta/2} f \right\|_{p,\gamma}. \tag{7.4}$$

Therefore, the set of polynomials in $\mathbb{R}^d$, $\mathscr{P}(\mathbb{R}^d)$ is trivially a dense set in these spaces. The spaces $L^p_\beta(\gamma_d)$, are also called *potential spaces* (see [145]).

In the classical case, Sobolev spaces appear naturally in partial differential equations to measure the integrability of partial derivatives of a given function, A. P. Calderón proved that Sobolev spaces can be characterized using the integrability of the derivatives. We are going to see that the same holds in the Gaussian case, i.e., fractional derivatives D_β can be used to characterize $L^p_\beta(\gamma_d)$ (see Theorem 8.8). A probabilistic proof of this fact was given by Sugita in [261].

Moreover, from the definition given of the Gaussian Sobolev spaces, $L^p_\alpha(\gamma_d)$, we see they can be characterized as the image of the Gaussian Lebesgue spaces under Gaussian Bessel potentials (see 8.21) Proposition 8.6. They can also be characterized using Riesz fractional derivatives (see Theorem 8.8). Additionally, as an application of the Littlewood–Paley functions $g^k_{x,\gamma}$ and $g^k_{t,\gamma}$, a characterization of Gaussian Sobolev spaces, $L^p_\beta(\gamma_d)$ for $1 < p < \infty$ can also be provided (see Section 9.5 in Chapter 9.

Finally, we have the following *Gaussian Sobolev embeddings*,

Proposition 7.3. *Gaussian Sobolev spaces satisfy*

i) If $p < q$ *then* $L^q_\beta(\gamma_d) \subset L^p_\beta(\gamma_d)$ *for each* $\beta \geq 0$.
ii) If $0 \leq \alpha_1 < \beta_2$ *then* $L^p_{\beta_2}(\gamma_d) \subset L^p_{\beta_1}(\gamma_d)$ *for each* $1 < p < \infty$.

Moreover, the embeddings in i) and ii) are continuous

Proof. Claim *i)* is an immediate consequence of Hölder's inequality.

For claim *ii)*, let f be a polynomial and let us consider $g = (1-L)^{-\beta_2/2} f$, then

$$(1-L)^{(\beta_1-\beta_2)/2} g = (1-L)^{\beta_1/2} f.$$

Using Meyer's multiplier theorem, Theorem 6.2, we can conclude that there exists $C > 0$, such that

$$\|f\|_{p,\beta_1} \leq C \|f\|_{p,\beta_2}.$$

□

7.3 Gaussian Tent Spaces $T^{1,q}(\gamma_d)$

In 1985, R. Coifman, Y. Meyer, and E. M. Stein [55], introduced the *tent spaces* T^p_q with respect to the Lebesgue measure, as the space of functions $F : \mathbb{R}^{d+1}_+ :\to \mathbb{R}$ such that,

$$J_q(f)(x) = \Big(\int_{\Gamma(x)} |F(y,t)|^q dy \frac{dt}{t^{d+1}} \Big)^{1/q} \in L^p(\mathbb{R}^d),$$

where $\Gamma(x) = \Big\{ (y,t) \in \mathbb{R}^{d+1}_+ : |x-y| < t \Big\}$, $1 < q < \infty$, and

$$\|F\|_{q,p} = \| J_q(f)\|_p.$$

In 2012, J. Mass, J. Van Neerven, and P. Portal [169] introduced *Gaussian tent spaces* as follows. Let

$$D := \{(x,t) \in \mathbb{R}^d \times (0,\infty) : t < m(x)\},$$

where as usual, $m(x) = 1 \wedge \frac{1}{|x|}$, is the admissibility function. Note that a point $(x,t) \in \mathbb{R}^d \times (0,\infty)$ belongs to $\overline{D}$ if and only if $B(x,t) \in \mathscr{B}_1$.

Definition 7.4. *The Gaussian tent space $T^{1,q}(\gamma_d)$ is the completion of $C_0(D)$ with respect to the norm,*

$$\|F\|_{T^{1,q}(\gamma)} := \|JF\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(D,\gamma_d\times\frac{dt}{t}))}, \tag{7.5}$$

where

$$(JF(x))(y,t) := \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))^{1/q}}F(y,t), \quad F \in C_0(D), \tag{7.6}$$

that is,

$$\|F\|_{T^{1,q}(\gamma)} = \int_{\mathbb{R}^d}\left(\int\int_{\Gamma_x^1(\gamma_d)}\frac{1}{\gamma_d(B(y,t))}|(JF(x))(y,t)|^q\gamma_d(dy)\frac{dt}{t}\right)^{1/q}\gamma_d(dx),$$

where, $\Gamma_x^1(\gamma_d) = \left\{(y,t) \in \mathbb{R}^d \times (0,\infty) : |y-x| < t < m(x)\right\}$ is a Gaussian cone with $a = 1$, see (4.83).

In [169], J. Mass, J. Van Neerven, and P. Portal obtained an atomic decomposition for $T^{1,q}(\gamma_d)$. As in the Euclidean case, this atomic decomposition turns out to be very useful, because using an atomic decomposition, we only have to check results for atoms and then the rest follows easily. First, let us see what a Gaussian *tent* is:

Definition 7.5. *For a measurable set $E \subset \mathbb{R}^d$ and a real number $a > 0$, we define the tent with aperture α over E by*

$$\mathsf{T}_\alpha(E) = \{(y,t) \in \mathbb{R}^{d+1}_+ : d(y,E^c) \geq \alpha t\}. \tag{7.7}$$

Now, let us define a *Gaussian atom.*

Definition 7.6. *Given $\alpha > 0$ a function $A : D \to \mathbb{C}$ is called a $T^{1,q}(\gamma_d)$ α-atom if there exists a ball B in $\mathscr{B}_\alpha$ such that*

i) A is supported in $\mathsf{T}_1(B) \cap D$, i.e.,

$$supp(A) \subset \{(y,t) \in D : \ t \leq d(y,B^c)\}.$$

ii) $\displaystyle\int\int_D |A(y,t)|^q\,\gamma_d(dy)\frac{dt}{t} \leq \frac{1}{\gamma_d(B)^{q/q'}}$, *where* $\frac{1}{q}+\frac{1}{q'} = 1$.

Lemma 7.7. *If A is a $T^{1,q}(\gamma_d)$ α-atom, then $A \in T^{1,q}(\gamma_d)$ and $\|A\|_{T^{1,q}(\gamma)} \leq 1$.*

Proof. Let A be a $T^{1,q}(\gamma_d)$ α-atom supported in $\mathsf{T}_1(B)\cap D$, for some $B\in\mathscr{B}_\alpha$. If $(y,t)\in\mathsf{T}_1(B)\cap D$ and $x\in B(y,t)$, then $x\in B$. Then, using this fact, Hölder's inequality, and Fubini's theorem, we obtain,

$$\begin{aligned}\int_{\mathbb{R}^d}\Big(\int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|A(y,t)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\ &= \int_{\mathbb{R}^d}\Big(\int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|A(y,t)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\chi_B(x)\gamma_d(dx)\\ &\le \Big(\int_{\mathbb{R}^d}\int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|A(y,t)|^q\gamma_d(dy)\frac{dt}{t}\gamma_d(dx)\Big)^{1/q}\gamma_d(B)^{1/q'}\\ &= \Big(\int\int_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|A(y,t)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(B)^{1/q'}\le 1.\end{aligned}$$

The set D admits a locally finite cover with tents $\mathsf{T}_1(B)$ based at balls $B\in\mathscr{B}_\alpha$ if and only if $\alpha>1$; this explains the condition $\alpha>1$ in the next theorem, which establishes an *atomic decomposition* of $T^{1,q}(\gamma_d)$.

Theorem 7.8. *(Mass, Van Neerven, and Portal) For all $F\in T^{1,q}(\gamma_d)$ and $\alpha>1$, there exists a sequence $(\lambda_n)_{n\ge1}\in\ell^1$ and a sequence of $T^{1,q}(\gamma)$ α-atoms $\{A_n\}_{n\ge1}$ such that*

i) $F=\sum_{n=1}^\infty \lambda_n A_n$.
ii) $\sum_{n=1}^\infty|\lambda_n|\le C\|f\|_{T^{1,q}(\gamma)}$, *for some constant independent of f.*

The proof of this result follows the lines of the classic counterpart in [55]; however, we can only use the doubling property of γ_d for admissible balls. That is why we need the Gaussian Whitney covering (see Theorem 4.10). Before we start with the proof, we need some notations and auxiliary results. Given a measurable set $E\subseteq\mathbb{R}^d$ and a real number $\alpha>0$, we define

$$R_\alpha(E)=\{(y,t)\in\mathbb{R}^d\times(0,\infty):\ d(y,E)<\alpha t\}=T^c_\alpha(E^c).$$

We also put, for any measurable set $E\subseteq\mathbb{R}^d$ and real number $\beta>0$,

$$E^{[\beta]}=\left\{x\in\mathbb{R}^d:\frac{\gamma_d(E\cap B)}{\gamma_d(B)}\ge\beta\ \text{ for all } B\in\mathscr{B}_{\frac{3}{2}} \text{ with center } x\right\}.$$

We call $E^{[\beta]}$ the set of points of admissible β-density of E. Note that $E^{[\beta]}$ is a closed subset of $\mathbb{R}^d$ contained in $\overline{E}$.

Lemma 7.9. *For all $\eta\in(\frac12,1)$ there exists an $\overline{\eta}\in(0,1)$ such that, for all measurable sets $E\subseteq\mathbb{R}^d$ and all non-negative measurable functions F on D, there exists a constant $C>0$ such that*

$$\iint_{R_{1-\eta}(E^{[\overline{\eta}]})\cap D}F(y,t)\,\gamma_d(dy)\,\frac{dt}{t}\le C\int_E\left(\iint_D\frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}F(y,t)\,\gamma_d(dy)\,\frac{dt}{t}\right)\gamma_d(dx).$$

Proof. First, let $\overline{\eta} \in (0,1)$ be arbitrary and fixed. Let $(y,t) \in R_{1-\eta}(E^{[\overline{\eta}]}) \cap D$. Note that $(y,t) \in D$ implies $B(y,t) \in \mathscr{B}_1$. There exists $x \in E^{[\overline{\eta}]}$ such that $|y-x| < (1-\eta)t$. Notice first that, because $t \leq m(x)$, we have $|x| \leq (1-\eta)t + \frac{1}{t} \leq \frac{1}{2} + \frac{1}{t} \leq \frac{3}{2}\frac{1}{t}$. Thus, we have that $t \in (0, \frac{3}{2}m(x))$. Moreover, $B(x,\eta t) \subseteq B(y,t) \subseteq B(x, \frac{3}{2}t)$, and thus $B(y,t) \in \mathscr{B}_1$, $B(x,t) \in \mathscr{B}_{\frac{3}{2}}$, and $\gamma_d(B(x,t)) \sim \gamma_d(B(y,t))$ by repeated application of Theorem 1.6 *ii*), the doubling property on admissible balls. Therefore, we have

$$\begin{aligned}\gamma_d(E \cap B(y,t)) &\geq \gamma_d(E \cap B(x,t)) - \gamma_d(B(x,t) \cap B(y,t)^c) \\ &\geq \overline{\eta}\gamma_d(B(x,t)) - \gamma_d(B(x,t)) + \gamma_d(B(x,t) \cap B(y,t)) \\ &\geq (\overline{\eta} - 1)\gamma_d(B(x,t)) + \gamma_d(B(x,\eta t)).\end{aligned}$$

Now, picking $\overline{\eta}$ close enough to 1 and using the doubling property, we obtain a constant $c = c(\eta, n) \in (0,1)$ such that

$$\gamma_d(E \cap B(y,t)) \geq c\gamma_d(B(x,t)).$$

Therefore, there exists a constant $c' = c'(\eta, n) > 0$ such that

$$\gamma_d(E \cap B(y,t)) \geq c'\gamma_d(B(y,t)),$$

for all $(y,t) \in R_{1-\eta}(E^{[\overline{\eta}]}) \cap D$. Finally,

$$\begin{aligned}\int_E \left(\iint_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} F(y,t)\, \gamma_d(dy) \frac{dt}{t} \right) \gamma_d(dx) &= \iint_D \frac{\gamma_d(E \cap B(y,t))}{\gamma_d(B(y,t))} F(y,t)\, \gamma_d(dy)\, \frac{dt}{t} \\ &\geq c' \iint_{R_{1-\eta}(E^{[\overline{\eta}]}) \cap D} F(y,t)\, \gamma_d(dy)\, \frac{dt}{t}. \qquad \square\end{aligned}$$

Lemma 7.10. *If a function $F \in T^{1,q}(\gamma_d)$ admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for some $\alpha > 1$, then it admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for all $\alpha > 1$.*

Proof. Suppose that $F \in T^{1,q}(\gamma_d)$ admits a decomposition in terms of $T^{1,q}(\gamma_d)$ β-atoms for some $\beta > 1$. We will show that f admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for any $\alpha > 1$. This is immediate if $\alpha \geq \beta$, because in this case any $T^{1,q}(\gamma_d)$ β-atom is a $T^{1,q}(\gamma_d)$ α-atom as well.

Therefore, let us assume that $1 < \alpha < \beta$. We claim that it suffices to show that there exists an integer N, depending only upon α, β, and the dimension d, such that if $B \in \mathscr{B}_\beta$, then $\mathsf{T}_1(B) \cap D$ can be covered by at most N tents of the form $\mathsf{T}_1(B')$ with $B' = B(c', r') \in \mathscr{B}_\alpha$ satisfying $r' = \alpha m(c')$.

To prove the claim, it clearly suffices to consider the case that F is a $T^{1,q}(\gamma_d)$ β-atom having support in $\mathsf{T}_1(B) \cap D$ for some ball $B \in \mathscr{B}_\beta$, with center c and radius $r = \beta m(c)$. Let $\{\mathsf{T}_1(B'_1), \cdots, \mathsf{T}_1(B'_N)\}$ be a covering of $\mathsf{T}_1(B)$, where each B'_j, $j = 1, \cdots N$, is a ball in $\mathscr{B}_\alpha$ with center c_j, radius $r_j = \alpha m(c_j)$, and intersecting B. For $x \in \mathsf{T}_1(B) \cap D$ we set

$$n(x) := \#\{1 \leq j \leq N : x \in \mathsf{T}_1(B'_1)\}, \text{ and } F_j(x) = \frac{F(x)}{n(x)} \chi_{\mathsf{T}_1(B'_1)}(x).$$

Then it follows that $F = \sum_{j=1}^{N} F_j$. Moreover, each F_j is a $T^{1,q}(\gamma_d)$ α-atom, because F_j is supported in $\mathsf{T}_1(B_j) \cap D$ and

$$\|F_j\|_{L^q(D,\gamma_d dt/t)} \leq \|F\|_{L^q(D,\gamma_d dt/t)} \leq \gamma_d(B)^{-1/q'} \leq C\gamma_d(B'_j)^{-1/q'}.$$

To obtain the latter estimate, we pick an arbitrary $b \in B'_j \cup B$ and use Lemma 1.5 *ii*) to conclude that

$$m(c_j) \leq (1+\alpha)m(b) \leq 2(1+\alpha)(1+\beta)m(c),$$

and then we estimate,

$$r_j = \alpha m(c_j) \leq 2\alpha(1+\alpha)(1+\beta)m(c) = 2\frac{\alpha}{\beta}(1+\alpha)(1+\beta)r.$$

Using the doubling property, Theorem 1.6, we conclude $\gamma_d(B_j) \leq C\gamma_d(B)$. It follows that $F = \sum_{j=1}^{N} F_j$ is a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms, which proves the claim.

Fix $R \geq 1+\beta$ large enough such that $\alpha(R-\beta)/(R-\beta+\alpha) > 1$. The set $\{(y,t) \in D : |y| \leq R+1\}$ can be covered with finitely many sets – their number depending only upon R, d and α – of the form $\mathsf{T}_1(B')$ with $B' = B(c',r') \in \mathscr{B}_\alpha$ and $r' = \alpha m(c')$.

Take a ball $B = B(c,r) \in \mathscr{B}_\beta$ with $|c| \geq R$ and choose $\delta \in (0,1)$ small enough such that $(1-\delta)\alpha(R-\beta)/(R-\beta+\alpha) > 1$. Observe that if $x \in B$, then $|x| \geq R-\beta \geq 1$, and therefore $m(x) = \frac{1}{|x|}$. Let us define

$$C_B := \{(x,t) \in B \times (0,\infty)\}.$$

Noting that $\mathsf{T}_1(B) \cap D \subset C_B$, it remains to cover C_B with N tents $\mathsf{T}_1(B')$ based on balls $B' \in \mathscr{B}_\alpha$ where the number N depends on α, β, and d only. To do so, let us start picking $c' \in B$, and let $r' = \alpha m(c') = \frac{\alpha}{|c'|}$ and $B' = B(c',r')$. If $(x,t) \in C_B$ is such that $|x-c'| \leq \delta r'$, then

$$\begin{aligned} d(x,(B')^c) &= d(c',(B')^c) - |x-c'| \geq (1-\delta)r' = (1-\delta)\frac{\alpha}{|c'|} \\ &\geq (1-\delta)\frac{\alpha}{|x|+|x-c'|} \geq m(x)(1-\delta)\left(\frac{\alpha|x|}{|x|+\alpha}\right) \\ &\geq m(x)(1-\delta)\frac{\alpha(R-\beta)}{R-\beta+\alpha} \geq m(x) \geq t. \end{aligned}$$

Here, we have used the monotonicity of the function $t \to t/(t+\alpha)$.

We have proved that the point $(x,t) \in C_B$ belongs to $\mathsf{T}_1(B')$ whenever $|x-c'| \leq \delta r'$. Using that $(|c|+\beta)r \leq (|c|+\beta)\frac{\beta}{|c|} \leq \beta+\beta^2$, we have

$$r' = \alpha m(c') \geq \frac{\alpha}{|c|+\beta} \geq \frac{\alpha}{\beta+\beta^2}r.$$

This implies that B can be covered with N balls $B' = B(c',\delta r')$ as above, with N dependent only on α, β, and d. The union of the N sets $\mathsf{T}_1(B') \cap D$ then covers C_B, thus completing the proof of the lemma. □

We are ready to prove Theorem 7.8.

Proof. Using Lemma 7.10, it suffices to prove that each $F \in T^{1,q}(\gamma_d)$ admits a decomposition in terms of $T^{1,q}(\gamma_d)$ α-atoms for some $\alpha > 0$.

Recall that the disjoint sets $A^{(i)}_{p,\kappa}$ have been introduced in Definition 4.8. We shall apply Theorem 4.10 for $p = 4$ and $\kappa = 8$ (the reason for this choice is the constant $16 = 2^4$ produced in the argument below). As

$$\Big(\bigcup_{0\le l\le 4} L_l\Big) \cup \Big(\bigcup_{i\in\{1,\dots,8\}^d} A^{(i)}_{4,8}\Big) = \mathbb{R}^d,$$

we may write

$$f = f\chi_{\{\|Jf\|_2>0\}} = \sum_{0\le l\le 4}\sum_{Q\in\Delta^{\gamma}_{0,l}} f\chi_{Q\cap\{\|Jf\|_2>0\}} + \sum_{i\in\{1,\dots,8\}^d} f\chi_{A^{(i)}_{4,8}\cap\{\|Jf\|_2>0\}}, \tag{7.8}$$

where $f\chi_{\{\|Jf\|_2>0\}}(x,t) := f(x,t)\chi_{\{\|Jf\|_2>0\}}(x)$ and

$$\{\|Jf\|_2 > 0\} := \{x\in\mathbb{R}^d : \|Jf(x)\|_{L^2(D,d\gamma_d\frac{dt}{t})} > 0\}.$$

The first equality in (7.8) is justified as follows. For all $x_0 \in V := \{\|Jf\|_2 = 0\}$ we have $\chi_{B(y,t)}(x_0)f(y,t) = 0$ for almost all $(y,t)\in D$; therefore, using Fubini's theorem, for almost all $y\in\mathbb{R}^d$, we have

$$\chi_{B(y,t)}(x_0)f(y,t) = 0 \ \text{ for almost all } t > 0.$$

Fix $\delta > 0$ arbitrary. Then, for almost all $y \in B(x_0,\delta)$ we have $f(y,t) = 0$ for almost all $t \ge \delta$. Applying again Fubini's theorem, this implies that $f(y,t) = 0$ for almost all $(y,t) \in (B(x_0,\delta)\times[\delta,\infty))\cap D$. Taking the union over all rational $\delta > 0$, it follows that $f \equiv 0$ almost everywhere on $\Gamma_x := \{(y,t)\in D : \ |x-y| < t\}$ the "admissible cone" over x. If K is any compact set contained in V, then by taking the union over a countable dense set of points $x\in K$, it follows that $f(y,t) = 0$ almost everywhere on the "admissible cone" over K. Finally, using the inner regularity of the Lebesgue measure on $\mathbb{R}^d$, it follows that $f(y,t) = 0$ almost everywhere on the "admissible cone" over V. In particular, this proves the first identity in (7.8).

To prove the theorem it suffices to prove that each of the summands on the right-hand side of (7.8) has an atomic decomposition. In view of Theorem 4.10 for $p = 4$ and $\kappa = 8$ it suffices to prove that

$$g := f\chi_{W\cap\{\|Jf\|_2>0\}}$$

has an atomic decomposition for every measurable set W in $\mathbb{R}^d$ such that $W + \mathscr{C}_{16}$ is admissible $2^9\sqrt{d}$-Whitney.

Given $k\in\mathbb{Z}$, let us define

$$O_k := \{\|Jf\|_2 > 2^k\}$$

and $F_k := O_k^c$. Fix an arbitrary $\eta \in (\frac{1}{2},1)$ and let $\overline{\eta}$ be as in Lemma 7.9. With abuse of notation we let $O_k^{[\overline{\eta}]} := (F_k^{[\overline{\eta}]})^c$, where $F_k^{[\overline{\eta}]}$ denotes the set of points of admissible $\overline{\eta}$-density of F_k. We claim that $O_k^{[\bar{\eta}]}$ is contained in $W + \mathscr{C}_{16}$ (see (4.8)).

To prove the claim, we fix $x \in O_k^{[\bar{\eta}]}$, and check that $x \in W + \mathscr{C}_2$. Indeed, as $Jg(x)$ does not vanish almost everywhere on D, we can find a set $D' \subset D$ of positive measure such that for almost all $(y,t) \in D'$

$$\chi_{B(y,t)}(x) g(y,t) = \chi_{B(y,t)}(x) f(y,t) \chi_{W \cap \{\|Jf\|_2 > 0\}}(y) \neq 0.$$

For those points, we have $y \in W$, $|x-y| < t$ and $t < m(y)$, so $t < 2m(x)$, using Lemma 1.5 *i*). Thus, $B(x,t)$ belongs to $\mathscr{B}_2$ and intersects W; thus, $x \in W + \mathscr{C}_2$.

As x is not a point of admissible $\overline{\eta}$-density of F_k, there is a ball $B \in \mathscr{B}_{\frac{3}{2}}$ with center x such that $\gamma_d(F_k \cap B) < \bar{\eta} \gamma_d(B)$. This is only possible if B intersects $O_k = F_k^c$. As O_k is contained in $W + \mathscr{C}_2$, this means that B intersects $W + \mathscr{C}_2$. Fix an arbitrary $x' \in B \cap (W + \mathscr{C}_2)$ and let $B' \in \mathscr{C}_2$ be any admissible ball centered at x' and intersecting W. From $x' \in B$ and $B \in \mathscr{B}_{\frac{3}{2}}$, it follows that $|x - x'| < \frac{3}{2} m(x)$. Also, because $B' \in \mathscr{B}_2$ and intersects W, $d(x', W) < 2m(x)$, it follows that $d(x, W) < \frac{3}{2} m(x) + 2m(x')$. Using Lemma 1.5 *ii*), we have $m(x') < 5m(x)$, and therefore $\operatorname{dist}(x, W) \le 16 m(x)$. This proves the claim.

For each $N \ge 1$ define $g_N(y,t) := \chi_{\{|y| \le N\}} \chi_{\{|g| \le N\}} \chi_{(\frac{1}{N}, \infty)}(t) g(y,t)$. Clearly, $g_N \in T^{q,q}(\gamma_d)$ and, by dominated convergence, $\lim_{N \to \infty} g_N = g$ in $T^{1,q}(\gamma_d)$. Defining the sets $F_{k,N}$, $O_{k,N}$, $F_{k,N}^{[\overline{\eta}]}$, $O_{k,N}^{[\overline{\eta}]}$ in the same way as above, Lemma 7.9 gives that

$$\iint_{R_{1-\eta}(F_{k,N}^{[\overline{\eta}]}) \cap D} |g_N(y,t)|^q \gamma_d(dy) \frac{dt}{t}$$
$$\le C \int_{F_{k,N}} \left(\iint_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))} |g_N(y,t)|^q \gamma_d(dy) \frac{dt}{t} \right) \gamma_d(dx) \le C \|g_N\|_{T^{q,q}(\gamma_d)}^q.$$

As $k \to -\infty$, the middle term tends to 0; therefore, the support of $f g_N$ is contained in the union $\bigcup_{k \in \mathbb{Z}} \mathsf{T}_{1-\eta}(O_{k,N}^{[\overline{\eta}]}) \cap D$. Clearly, $O_{k,N} \subseteq O_k$ implies $\mathsf{T}_{1-\eta}(O_{k,N}^{[\overline{\eta}]}) \subseteq \mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]})$; therefore, a limiting argument shows that the support of g is contained in the union $\bigcup_{k \in \mathbb{Z}} \mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]}) \cap D$.

Choose cubes $(Q_k^m)_{m \in \mathbb{N}}$ and functions $(\phi_k^m)_{m \in \mathbb{N}}$ as in Lemma 4.12, applied to the open sets $O_k^{[\overline{\eta}]}$ which are contained in $W + \mathscr{C}_8$. Define for $(y,t) \in D$,

$$b_k^m(y,t) := (\chi_{\mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]})}(y,t) - \chi_{\mathsf{T}_{1-\eta}(O_{k+1}^{[\overline{\eta}]})}(y,t)) \phi_k^m(y) f(y,t),$$
$$\mu_k^m := \iint_D |b_k^m(y,t)|^q \gamma_d(dy) \frac{dt}{t},$$

and put

$$\lambda_k^m := (\gamma_d(Q_k^m))^{\frac{1}{q'}} (\mu_k^m)^{\frac{1}{q}}, \quad a_k^m(y,t) := \frac{b_k^m(y,t)}{\lambda_k^m}.$$

Then,

$$g = \sum_{k \in \mathbb{Z}} \sum_{m \in \mathbb{N}} \lambda_k^m a_k^m.$$

Let C be a constant to be determined later and denote by $(Q_k^m)^{**}$ the cube that has the same center as Q_k^m, but side length multiplied by C. Let us further denote by δ_k^m the length of the diagonal of Q_k^m and by c_k^m its center. We now show that $\mathrm{supp}(a_k^m) \subseteq \mathsf{T}_1((Q_k^m)^{**})$. We have

$$\mathrm{supp}(a_k^m) \subseteq \mathsf{T}_{1-\eta}(O_k^{[\overline{\eta}]}) \cap \{(y,t) \in D : \; y \in (Q_k^m)^*\},$$

where $(Q_k^m)^*$ is as in Lemma 4.12. Therefore, for $(y,t) \in \mathrm{supp}(a_k^m)$, we have $d(y, F_k^{[\overline{\eta}]}) \geq (1-\eta)t$ and $y \in (Q_k^m)^*$. For $z \notin (Q_k^m)^{**}$ this gives

$$d(y,z) \geq d(z,c_k^m) - d(y,c_k^m) \geq \Big(\frac{C}{2\sqrt{n}} - \frac{\rho}{2}\Big)\delta_k^m, \tag{7.9}$$

where $\rho = \rho_{2^{10}\sqrt{d},d}$ is the constant from Lemma 4.12. Moreover, using property *ii)* in Lemma 4.12,

$$d(c_k^m, F_k^{[\overline{\eta}]}) \leq (\rho + \frac{1}{2})\delta_k^m.$$

For $u \in F_k^{[\overline{\eta}]}$ such that $d(c_k^m, u) \leq (\rho + \frac{1}{2})\delta_k^m + \varepsilon$, this gives

$$(1-\eta)t \leq d(y, F_k^{[\overline{\eta}]}) \leq d(y,u) \leq d(y,c_k^m) + d(c_k^m,u) \leq \frac{3\rho+1}{2}\delta_k^m + \varepsilon. \tag{7.10}$$

Upon taking $C = 2\sqrt{n}(\frac{\rho}{2} + \frac{3\rho+1}{2(1-\eta)})$, from (7.9) and (7.10) and letting $\varepsilon \downarrow 0$, we infer that

$$d(y,z) \geq \frac{3\rho+1}{2(1-\eta)}\delta_k^m \geq t.$$

This means that $(y,t) \in \mathsf{T}_1((Q_k^m)^{**})$, thus proving the claim: $\mathrm{supp}(a_k^m) \subseteq \mathsf{T}_1((Q_k^m)^{**})$.

Using the definitions of λ_k^m and a_k^m together with the doubling property for admissible balls, we also get that

$$\iint_D |a_k^m(y,t)|^q \gamma_d(dy)\frac{dt}{t} \leq \frac{1}{\gamma_d(Q_k^m)^{\frac{q}{q'}}} \leq C\frac{1}{\gamma_d((Q_k^m)^{**})^{\frac{q}{q'}}}.$$

Up to a multiplicative constant, the a_k^m are thus $T^{1,q}(\gamma_d)$ α-atoms for some $\alpha = \alpha(C,n) > 0$. To get the norm estimates, we first use Lemma 7.9. Noting that $(y,t) \in \mathsf{T}_1((Q_k^m)^{**})$ and $x \in B(y,t)$ imply $x \in (Q_k^m)^{**}$, we obtain

$$\begin{aligned}
\mu_k^m &\leq \iint_{R_{1-\eta}(F_{k+1}^{[\overline{\eta}]})\cap D} \chi_{\mathsf{T}_1((Q_k^m)^{**})}(y,t)|f(y,t)|^q \gamma_d(dy)\frac{dt}{t} \\
&\leq C\int_{F_{k+1}} \left(\iint_D \frac{\chi_{B(y,t)}(x)\chi_{\mathsf{T}_1((Q_k^m)^{**})}(y,t)}{\gamma(B(y,t))} |f(y,t)|^q \gamma_d(dy)\frac{dt}{t}\right)\gamma_d(dx) \\
&\leq C\int_{F_{k+1}\cap(Q_k^m)^{**}} \|Jf(x)\|^q_{L^q(D,\gamma_d\frac{dt}{t}))}\gamma_d(dx) \\
&\leq C2^{2(k+1)}\gamma_d((Q_k^m)^{**}) \leq C2^{2k}\gamma_d(Q_k^m).
\end{aligned}$$

This then gives

$$\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\lambda_k^m = \sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\sqrt{\mu_k^m\gamma_d(Q_k^m)} \le C\sum_{k\in\mathbb{Z}} 2^k\gamma_d(O_k^{[\overline{\eta}]}).$$

Because $x \in O_k^{[\overline{\eta}]}$ implies $\mathscr{M}_\gamma^{\frac{3}{2}}(\chi_{O_k})(x) > 1-\overline{\eta}$ i.e.,

$$O_k^{[\overline{\eta}]} \subset \{x \in W:\ \mathscr{M}_\gamma^{\frac{3}{2}}(\chi_{O_k})(x) > 1-\overline{\eta}\},$$

the weak type $(1,1)$ of the truncated centered Gaussian Hardy–Littlewood maximal function $\mathscr{M}_\gamma^{\frac{3}{2}}$ defined by using only $\mathscr{B}_{\frac{3}{2}}$-balls (see (4.101), gives that

$$(1-\overline{\eta})\gamma(O_k^{[\overline{\eta}]}) \le C\gamma(O_k)$$

and thus

$$(1-\overline{\eta})\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{N}}\lambda_k^m \lesssim \sum_{k\in\mathbb{Z}} 2^k\gamma_d(O_k) \lesssim \int_0^\infty \gamma_d(x\in\mathbb{R}^d:\ \|Jf(x)\|_q > s)\,ds = \|f\|_{T^{1,q}(\gamma_d)}.$$

□

As an application of the atomic decomposition, we prove a result on change of aperture of the cones. The proof is different from the classical one (see [55]), because the result is derived directly from the atomic decomposition.

Definition 7.11. *For $\alpha > 0$, the Gaussian tent space $T_\alpha^{1,q}(\gamma_d)$ with aperture α is the completion of $C_0(D)$ with respect to the norm,*

$$\|f\|_{T_\alpha^{1,q}(\gamma_d)} = \|J_\alpha f\|_{L^1(\mathbb{R}^d,\gamma_d);L^q(D,\gamma_d\frac{dt}{t}))}, \tag{7.11}$$

where

$$(J_\alpha f(x))(y,t) := \frac{\chi_{B(y,\alpha t)}(x)}{\gamma_d(B(y,t))^{1/q}} f(y,t), \quad f \in C_0(D). \tag{7.12}$$

Theorem 7.12. *(Change of aperture) For all $1 < \alpha_0 < \alpha$, we have $T_\alpha^{1,q}(\gamma_d) = T_{\alpha_0}^{1,q}(\gamma_d)$ with equivalent norms.*

Proof. It is clear that $T_\alpha^{1,q}(\gamma_d) \subset T_{\alpha_0}^{1,q}(\gamma_d)$; thus, it suffices to show that $T_{\alpha_0}^{1,q}(\gamma_d) \subset T_\alpha^{1,q}(\gamma_d)$. To get that, it is enough to show that

$$J_\alpha \in L^1(\mathbb{R}^d,\gamma_d; L^q(D,\gamma_d \times \frac{dt}{t})),$$

whenever $f \in T_{\alpha_0}^{1,q}(\gamma_d)$. Observe that $(y,t) \in D$ implies $B(y,t) \in \mathscr{B}_1$; therefore, using the doubling property

$$\begin{aligned}\|J_\alpha f\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(D,\gamma_d\times\frac{dt}{t}))} &= \int_{\mathbb{R}^d}\Big(\iint_D \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|f(y,t)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\ &= \int_{\mathbb{R}^d}\Big(\iint_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t/\alpha))}|f(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\ &\le \int_{\mathbb{R}^d}\Big(\iint_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|f(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\ &= \|J\tilde{f}\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))},\end{aligned}$$

where $\tilde{D} := \{(x,t)\in\mathbb{R}^d\times(0,\infty) : (x,t/\alpha)\in D\}$, and $\tilde{f}(y,t) := f(y,t/\alpha)$. To prove the result, it is enough to show that

$$\|J\tilde{f}\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))} \le C\|J_{\alpha_0}f\|_{L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))}, \tag{7.13}$$

for $f\in T^{1,q}_\alpha(\gamma_d)$.

Suppose a is a $T^{1,q}_\alpha(\gamma_d)$ α_0-atom. Then, a is supported in $\mathsf{T}_1(B)\cap D$ for some ball $B=B(c,r)\in\mathscr{B}_{\alpha_0}$. Then $\tilde{a}(y,t)=a(y,y/\alpha)$ is supported in $\tilde{\mathsf{T}}_1(B)\cap\tilde{D}$ where $\tilde{\mathsf{T}}_1(B) := \{(y,t)\in\mathbb{R}^d\times(0,\infty) : (y,t/\alpha)\in\mathsf{T}_1(B)\}$. Using that $(y,t)\in\tilde{\mathsf{T}}_1(B)$ and $x\in B(y,t)$ imply $x\in B(c,\alpha r)$, the doubling property for admissible balls gives,

$$\begin{aligned}&\int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(dx)\\ &\quad\le \int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\chi_{B(c,\alpha r)}(x)\gamma_d(dx)\\ &\quad\le \int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\chi_{B(c,\alpha r)}(x)\gamma_d(dx)\\ &\quad\le \Big(\int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}} \frac{\chi_{B(y,t)}(x)}{\gamma_d(B(y,t))}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(B(c,\alpha r))^{1/q'}\\ &\quad\le C\Big(\int_{\mathbb{R}^d}\Big(\int\int_{\tilde{D}}|a(y,t/\alpha)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(B(c,r))^{1/q'}\\ &\quad\le C\Big(\int_{\mathbb{R}^d}\Big(\int\int_{D}|a(y,t)|^q\gamma_d(dy)\frac{dt}{t}\Big)^{1/q}\gamma_d(B(c,r))^{1/q'}\le C.\end{aligned}$$

This shows that $J\tilde{a}\in L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))$. Then, using the atomic decomposition, Theorem 7.8, we can conclude that $J\tilde{f}\in L^1(\mathbb{R}^d,\gamma_d;L^q(\tilde{D},\gamma_d\times\frac{dt}{t}))$, for all $f\in T^{1,q}_\alpha(\gamma_d)$. The estimate (7.13) then follows from the closed graph theorem. □

7.4 Gaussian Hardy Spaces $H^1(\gamma_d)$

The real variable theory of Hardy spaces originates from the work of C. Fefferman and E. Stein [79]. There are several equivalent definitions for the Hardy spaces on

$\mathbb{R}^d$, with respect to the Lebesgue measure. We are going to discuss briefly the most important ones, at least for us, here. First, there is the atomic Hardy space $H^1_{\mathrm{at}}(\mathbb{R}^d)$. Here, an atom is a complex-valued function a defined on $\mathbb{R}^d$, which is supported on a cube Q and is such that

$$\int_Q a(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \|a\|_\infty \leq \frac{1}{|Q|}.$$

The atomic space $H^1_{\mathrm{at}}(\mathbb{R}^d)$ is defined by

$$H^1_{\mathrm{at}}(\mathbb{R}^d) := \Big\{ \sum_j \lambda_j a_j \colon a_j \,\text{atoms}, \lambda_j \in \mathbb{C},\, \sum_j |\lambda_j| < \infty \Big\},$$

with norm

$$\|f\|_{H^1_{\mathrm{at}}(\mathbb{R}^d)} := \inf \Big\{ \sum_j |\lambda_j| \colon f = \sum_j \lambda_j a_j \,\sum_j |\lambda_j| < \infty \Big\}.$$

The other relevant characterizations of the classical Hardy space are given using the non-tangential maximal function $\mathscr{T}^*_{NT}$ of the heat semigroup

$$\mathscr{T}^*_{NT} f(x) := \sup_{(y,t)\in\Gamma_x} |\mathscr{T}_{t^2} f(y)|, \tag{7.14}$$

and the conical square function of the heat semigroup

$$\mathscr{S}_{NT} f(x) := \frac{1}{|B(y,t)|} \Big(\int_{\Gamma_x} \Big| t\mathscr{T}_{t^2} f(y) \Big|^2 \mathrm{d}y \frac{dt}{t} \Big)^{\frac{1}{2}}, \tag{7.15}$$

where $\Gamma_x := \big\{ (y,t) \in \mathbb{R}^d \times (0,\infty) \colon |y-x| < t \big\}$ are the usual cones in $\mathbb{R}^{d+1}$ with a vertex at $x \in \mathbb{R}^d$.

The Hardy spaces can then be defined as the completion of the space of compactly supported functions $C_0(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{H^1_{max}} := \|f\|_1 + \|\mathscr{T}^*_{NT} f\|_1,$$

or with respect to the norm

$$\|f\|_{H^1_{quad}} := \|f\|_1 + \|\mathscr{S}_{NT} f\|_1.$$

It can be proved that these norms are equivalent norms.

The Calderón–Zygmund operators are not bounded on $L^1(\mathbb{R}^d)$, but are bounded on weak-L^1, which is not a Banach space. Another characterization of $H^1(\mathbb{R}^d)$ is precisely the subspace of functions $f \in L^1(\mathbb{R}^d)$ such that their Riesz transforms $R_j f$ are also in $L^1(\mathbb{R}^d)$, i.e.,

$$H^1(\mathbb{R}^d) = \Big\{ f \in L^1(\mathbb{R}^d) : R_j f \in L^1(\mathbb{R}^d), j = 1,2,\cdots,d \Big\}.$$

Finally, in 1971, it was proved by C. Fefferman in [78] (see also [79]), that the dual of $H^1(\mathbb{R}^d)$ is $BMO(\mathbb{R}^d)$, the space of functions with bounded mean oscillations introduced by F. John and L. Nirenberg in [144].

In recent years, the theory of Hardy spaces has been extended to a variety of new settings. These developments involve replacing the (Euclidean) Laplacian with a different semigroup generator L, and the space $\mathbb{R}^d$ endowed with the Borel σ-algebra and the Lebesgue measure with a different metric measure space (M,d,μ). Important references include S. Hofmann and S. Mayboroda's work on the Euclidean space, with the Laplacian replaced by a more general divergence form second-order elliptic differential operator with bounded measurable coefficients (see [136] and the Auscher–McIntosh–Russ Hardy spaces of differential forms associated with the Hodge Laplacian on a Riemannian manifold [13]. These results rely heavily on two assumptions: that the measure μ is a doubling measure (see Appendix), and that the semigroup generated by L, $\{e^{tL}\}$, has some appropriate L^2 off-diagonal decay: for $f \in L^2(\mathbb{R}^d)$, there exists a constant C independent of E,F,t and f such that

$$\left\| \chi_E e^{tL}(\chi_F f) \right\|_2 \leq c(1+\frac{d(E,F)}{t})^{-k} \|\chi_f f\|_2,$$

where E,F, are Borel sets in $\mathbb{R}^d$.

Given the success of Hardy space techniques in deterministic partial differential equations, we can expect that a Gaussian analog would similarly have applications to non-linear stochastic partial differential equations and stochastic boundary value problems.

There have been several attempts to define Gaussian Hardy spaces, but the main difficulty has been the fact that the Gaussian measure is not a doubling measure and the Ornstein–Uhlenbeck semigroup does not satisfy the kernel bounds required to apply the non-doubling theory of Tolsa [274]. The first result was obtained in 2007 by G. Mauceri and S. Meda in their seminal paper [174]. Their work is striking precisely because the Gaussian measure is not doubling, but the key to their success relies on the fact that they discovered that the Gaussian measure is a doubling measure when restricted to the class of admissible balls (see Proposition 1.6). The Mauceri–Meda Hardy spaces $H^1_{at}(\gamma_d)$ are defined via an atomic decomposition. An atom is either the constant function 1 or a function supported in an admissible ball $B \in \mathscr{B}_1$ with vanishing integral and satisfying an appropriate size condition. More precisely,

Definition 7.13. *Let $1 < r < \infty$, a $(1,r)$-atom is either the constant function 1, or a function a in $L^1(\gamma_d)$ supported in a ball $B \in \mathscr{B}_1$ with the following properties:*

$$\int_B a(y)\, \gamma_d(dy) = 0, \tag{7.16}$$

and

$$\Big(\frac{1}{\gamma_d(B)} \int_B |a(y)|^r\, \gamma_d(dy)\Big)^{1/r} \leq \frac{1}{\gamma_d(B)}, \tag{7.17}$$

or equivalently,

$$\|a\|_{r,\gamma} \leq \gamma_d(B)^{1/r-1}. \tag{7.18}$$

Then, we have

Definition 7.14. *The atomic Gaussian Hardy space $H^{1,r}_{at}(\gamma_d)$ is the space of all functions f in $L^1(\gamma_d)$ that admit an atomic decomposition of the form*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \tag{7.19}$$

where a_k is a $(1,r)$-atom and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$, with norm

$$\|f\|_{H^{1,r}_{at}(\gamma)} = \inf\Big\{ \sum_{k=1}^{\infty} |\lambda_k| : f = \sum_{k=1}^{\infty} \lambda_k a_k,\ a_k\ (1,r)-\text{atom and } \sum_{k=1}^{\infty} |\lambda_k| < \infty \Big\}. \tag{7.20}$$

By duality with the $BMO(\gamma_d)$ spaces, it can be proved that all Gaussian Hardy spaces $H^{1,r}_{at}(\gamma_d)$ coincide for all $r \in (1,\infty)$ with equivalent norms. Moreover, in [177, Theorem 2.2], G. Mauceri, S. Meda, and P. Sjögren prove that this can be extended to the case $r = \infty$. Thus, we can denote any of them simply by $H^1_{at}(\gamma_d)$ and use any of the equivalent norms. Additionally, the Mauceri–Meda space $H^1_{at}(\gamma_d)$ provides a good endpoint to the L^p scale from the interpolation point of view.

J. Maas, J. van Neerven, and P. Portal in [168] and [169] developed an alternative approach to the theory of Hardy spaces for the Gaussian case. This involved considering adequate dyadic cubes, Whitney-type covering lemmas (which were discussed in Section 4.1), related tent spaces and their atomic decomposition (which were discussed in Section 7.3), and techniques to estimate non-tangential maximal functions and conical square functions (see Section 4.6).

In 2012, P. Portal in [231] gave another characterization of Gaussian Hardy spaces, introducing two new spaces:

Definition 7.15. *i) The (maximal) Gaussian Hardy space, or non-tangential maximal function Hardy space, $H^1_{max,a}(\gamma_d)$ is the completion of the L^2 range of L, $R(L)$,*[1] *with respect to the norm*

$$\|f\|_{H^1_{max,a}(\gamma)} := \|\mathscr{T}^*_\gamma(1,a)f\|_{1,\gamma}, \tag{7.21}$$

where $\mathscr{T}^*_\gamma(1,a)$ is the non-tangential maximal function associated with the Ornstein–Uhlenbeck semigroup (4.84).

[1] In [231], the spaces are defined as completions of $C_0^\infty(\mathbb{R}^d)$. This unfortunate mistake was pointed out in [232]. These spaces, just like other Hardy spaces associated with an operator L, can only be defined on the range of L (where the reproducing formula holds in a L^1 sense). In other situations, this is only a minor technical hindrance. For the Ornstein–Uhlenbeck operator, however, this is critical because of the change of spectrum at $p = 1$.

ii) The (quadratic) Gaussian Hardy space $H^1_{quad,a}(\gamma_d)$ is the completion of the L^2 range of L, $R(L)$, with respect to the norm,

$$\|f\|_{H^1_{quad,a}(\gamma)} := \|f\|_{1,\gamma_d} + \|\mathscr{S}_{a,\gamma} f\|_{1,\gamma}, \tag{7.22}$$

where $\mathscr{S}_{a,\gamma}$ is the "averaged version" of the non-tangential Ornstein–Uhlenbeck maximal function, (4.91).

Then, we have the following crucial result:

Theorem 7.16. *Given $a > 0$, there exists $a' > 0$ such that the norms $\|\cdot\|_{H^1_{quad,a}(\gamma)}$ and $\|\cdot\|_{H^1_{max,a'}(\gamma)}$ are equivalent; therefore,*

$$H^1_{quad,a}(\gamma_d) = H^1_{max,a}(\gamma_d). \tag{7.23}$$

The proof of this result is technically very difficult and long. We give some of the main elements (for full details, see [231, Theorem 1.1]). The proof is based on the Gaussian version of A. P. Calderón's reproducing formula (2.59).

First of all, observe that from Theorem 7.12, we can immediately obtain one of the required inequalities, because

$$\|\mathscr{S}_{a,\gamma} f\|_{1,\gamma} \leq C \|\mathscr{T}^*_\gamma(1,a') f\|_{1,\gamma},$$

for some $C, a' > 0$.[2]

Therefore, to prove Theorem 7.16, we need to prove the reverse inequality.

The (local) part

$$J_1 f(x) := \int_0^{m(x)} (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} f(x) \, \frac{dt}{t}, \tag{7.24}$$

is treated, via atomic decomposition of the tent space $T^{1,q}(\gamma_d)$, leading to the estimate,

$$\|J_1 f\|_{H^1_{max,a'}(\gamma)} \leq C'(\|f\|_{1,\gamma} + \|f\|_{H^1_{quad,a}(\gamma)}). \tag{7.25}$$

The (global) term,

$$J_\infty f(x) := \int_{m(x)}^{\infty} (t^2 L)^{N+1} T_{(1+a)t^2/\alpha} f(x) \, \frac{dt}{t}, \tag{7.26}$$

is very problematic, as the boundedness of the square function norm $\|\mathscr{S}_{a,\gamma}\|_{1,\gamma_d}$ does not give any information about it. Nevertheless, estimates of the Ornstein–Uhlenbeck semigroup give the estimate,

$$\|J_1 f\|_{H^1_{max,a'}(\gamma)} \leq C'' \|f\|_{1,\gamma}. \tag{7.27}$$

[2] Actually, Theorem 4.43 gives a slightly stronger inequality involving $\Upsilon^*_\gamma(1,a')$, the "average" non-tangential maximal function.

Let us look at the main argument of the proof in more detail. Using the Gaussian version of A. P. Calderón's reproducing formula (2.59)

$$f(x) = \int_{\mathbb{R}^d} f(x)\gamma_d(dx) + C\int_0^\infty (t^2L)^{N+1} T_{(1+a)t^2/\alpha} f(x)\,\frac{dt}{t},$$

for $f \in L^2(\gamma_d)$, in L^2-sense, and the atomic decomposition, we can prove the following corollary of Theorem 7.8, for $q=2$. This corollary is the actual underlying identity for proving Theorem 7.16.

Corollary 7.17. *For all $N \in \mathbb{N}, a > 1, b \geq \frac{1}{2}$ and $\alpha > a^2$ there exists $C_1, C_2, C_3, C_4 > 0$, and d sequences of α-atoms $\{A_{n,j}\}_{n\geq 1}$ and numbers $(\lambda_{n,j})_{n\geq 1} \in \ell^1$, such that for all $f \in C_c^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} f(x) = \int_{\mathbb{R}^d} f(y)\gamma_d(dy) &- C_1 \sum_{j=1}^d \sum_{n=1}^\infty \lambda_{n,j} \int_0^2 (t^2L)^N T_{t^2/\alpha}\left(t(\partial_\gamma^j)^* A_{n,j}(x,t)\right)\frac{dt}{t} \\ &+ C_2 \sum_{j=1}^d \sum_{n=1}^\infty \lambda_{n,j} \int_0^2 \chi_{\left[\frac{m(x)}{b},2\right]}(t)(t^2L)^N T_{t^2/\alpha}\left(t(\partial_\gamma^j)^* A_{n,j}(x,t)\right)\frac{dt}{t} \\ &- C_3 \sum_{j=1}^d \int_0^{\frac{m(x)}{b}} (t^2L)^N T_{t^2/\alpha}\left(\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha} f(x)\right)\frac{dt}{t} \\ &+ C_4 \int_{\frac{m(x)}{b}}^\infty (t^2L)^{N+1} T_{(1+a^2)t^2/\alpha} f(x)\,\frac{dt}{t}, \end{aligned} \tag{7.28}$$

and

$$\sum_{j=1}^d \sum_{n=1}^\infty |\lambda_{n,j}| \leq C\|f\|_{H^1_{quad,a}(\gamma)},$$

where $(\partial_\gamma^j)^* = \sqrt{2}x_j I_d - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_j}$, the formal $L^2(\gamma_d)$-adjoint of ∂_γ^j, see (2.12).

Proof. Let us recall that $L = -\sum_{j=1}^d (\partial_\gamma^j)^*\partial_\gamma^j$, see (2.13). Hence, as L and $T_t, t \geq 0$ commute,

$$\begin{aligned} (t^2L)^{N+1} T_{(1+a^2)t^2/\alpha} f(x) &= -\sum_{j=1}^d (t^2L)^N t^2 (\partial_\gamma^j)^* \partial_\gamma^j T_{t^2/\alpha} T_{a^2t^2/\alpha} f(x) \\ &= -\sum_{j=1}^d (t^2L)^N T_{t^2/\alpha} t(\partial_\gamma^j)^* [\chi_D(x,t) + \chi_{D^c}(x,t)] t\partial_\gamma^j T_{a^2t^2/\alpha} f(x). \end{aligned}$$

Set $F_j(x,t) := \chi_D(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha} f(x)$, for $j = 1, \cdots, d$. We need to check that $F_j \in T^{1,2}(\gamma_d)$, i.e., that they have an atomic decomposition.

Using Theorem 1.6 *ii*) we have, taking the change of variables $t = \sqrt{\alpha}s$

$$
\begin{aligned}
\|F_j\|_{T^{1,2}_{\alpha}(\gamma_d)} &\leq C\int_{\mathbb{R}^d}\Big(\iint_{\Gamma^1_x(\gamma_d)}\frac{\chi_D(y,t)}{\gamma_d(B(y,t))}|t\partial^j_\gamma T_{a^2t^2/\alpha}f(y)|^2\gamma_d(dy)\frac{dt}{t}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,t)}\frac{\chi_D(y,t)}{\gamma_d(B(y,t))}|t\partial^j_\gamma T_{a^2t^2/\alpha}f(y)|^2\gamma_d(dy)\frac{du}{u}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,\sqrt{\alpha}s)}\frac{\chi_D(y,\sqrt{\alpha}s)}{\gamma_d(B(y,\sqrt{\alpha}s))}|s\partial^j_\gamma T_{a^2s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,\sqrt{\alpha}s)}\frac{\chi_D(y,\sqrt{\alpha}s)}{\gamma_d(B(y,\sqrt{\alpha}s))}|s\nabla T_{a^2s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,\sqrt{\alpha}s)}\frac{\chi_D(y,\sqrt{\alpha}s)}{\gamma_d(B(y,s))}|s\nabla T_{a^2s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx),
\end{aligned}
$$

as $\gamma_d(B(y,\sqrt{\alpha}s)) \geq \gamma_d(B(y,s))$. Then, by the change of aperture formula, Theorem 7.12, and the change of variables $at = s$, we get

$$
\begin{aligned}
\|F_j\|_{T^{1,2}_{\alpha}(\gamma_d)} &\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,a^2t)}\frac{\chi_D(y,a^2t)}{\gamma_d(B(y,t))}|t\nabla T_{a^2t^2}f(y)|^2\gamma_d(dy)\frac{dt}{t}\Big)^{1/2}\gamma_d(dx)\\
&\leq C\int_{\mathbb{R}^d}\Big(\int_0^\infty\int_{B(x,as)}\frac{\chi_D(y,as)}{\gamma_d(B(y,t))}|s\nabla T_{s^2}f(y)|^2\gamma_d(dy)\frac{ds}{s}\Big)^{1/2}\gamma_d(dx) \leq \|f\|_{H^1_{quad,a}}.
\end{aligned}
$$

Then, using Theorem 7.8, we conclude that

$$
F_j(x,t) = \sum_{n=1}^{\infty}\lambda_{n,j}A_{n,j}(x,t),
$$

with $\sum_{n=1}^{\infty}|\lambda_{n,j}| < \infty$, for $j = 1,\cdots,d$. Hence, using the Gaussian version of A. P. Calderón's reproducing formula

$$
\begin{aligned}
f(x) &= \int_{\mathbb{R}^d}f(y)\gamma_d(dy) + C\int_0^\infty(t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&= \int_{\mathbb{R}^d}f(y)\gamma_d(dy) + C\int_{\frac{m(x)}{b}}^\infty(t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&\quad - C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}} d(t^2L)^N T_{t^2/\alpha}t(\partial^j_\gamma)^*[\chi_D(x,t)+\chi_{D^c}(x,t)]t\partial^j_\gamma T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
&= \int_{\mathbb{R}^d}f(y)\gamma_d(dy) + C\int_{\frac{m(x)}{b}}^\infty(t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&\quad - C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}} (t^2L)^N T_{t^2/\alpha}t(\partial^j_\gamma)^*[\chi_D(x,t)+\chi_{D^c}(x,t)]t\partial^j_\gamma T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
&= \int_{\mathbb{R}^d}f(y)\gamma_d(dy) + C\int_{\frac{m(x)}{b}}^\infty(t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}
\end{aligned}
$$

$$
\begin{aligned}
&-C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*\chi_D(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
&-C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}f(x)\frac{dt}{t}\\
=&\int_{\mathbb{R}^d}f(y)\gamma_d(dy)+C\int_{\frac{m(x)}{b}}^{\infty}(t^2L)^{N+1}T_{(1+a)t^2/\alpha}f(x)\,\frac{dt}{t}\\
&-C\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}\\
&-C\sum_{j=1}^{d}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*\chi_{D^c}(x,t)t\partial_\gamma^j T_{a^2t^2/\alpha}f(x)\frac{dt}{t}
\end{aligned}
$$

It is easy to check that the interchange of the (Bochner) integral with the sum is allowed. Finally, using that $m(x)/b \le 2$, we get

$$
\begin{aligned}
&\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{\frac{m(x)}{b}}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}\\
&\quad=\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{2}(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}\\
&\qquad-\sum_{j=1}^{d}\sum_{n=1}^{\infty}\lambda_{n,j}\int_0^{2}\chi_{\left[\frac{m(x)}{b},2\right]}(t)(t^2L)^N T_{t^2/\alpha}t(\partial_\gamma^j)^*A_{n,j}(x,t)\frac{dt}{t}.
\end{aligned}
$$

This gives (7.28). Thus, we have shown that $\|F_j\|_{T_\alpha^{1,2}(\gamma_d)} \le C\|f\|_{H^1_{quad,a}(\gamma)}$, so

$$\sum_{j=1}^{d}\sum_{n=1}^{\infty}|\lambda_{n,j}| \le C\|f\|_{H^1_{quad,a}(\gamma)}. \qquad \square$$

The proof of Theorem 7.16 uses (7.28) obtained in Corollary 7.17.

For $a>0$, Theorem 7.12 gives that there exists $a'>0$ such that $H^1_{max,a'}(\gamma_d) \subset H^1_{quad,a}(\gamma_d)$. Let us fix a' and pick

$$\alpha > \max\left\{2^{38}, 32e^4, 4\sqrt{a}e^{2a^2}\right\},\ b \ge \max\left\{2e, \sqrt{\frac{32e^4}{(\alpha-32e^4)(1-e^{-2a^2/\alpha})}}\right\},$$

and $N > d/4$. Let $f \in C_c^\infty(\mathbb{R}^d)$ and apply Corollary 7.17. We have

$$\begin{aligned}\|f\|_{H^1_{max,a'}(\gamma)} \leq C = & \left\| \mathscr{T}_\gamma^*(1,a) \left(\int_{\mathbb{R}^d} f(y)\gamma_d(dy) \right) \right\|_{1,\gamma} \\ & + C \sum_{j=1}^{d} \sum_{n=1}^{\infty} |\lambda_{n,j}| \left\| \int_0^2 (t^2 L)^N T_{t^2/\alpha} \left(t (\partial_\gamma^j)^* A_{n,j}(\cdot,t) \right) \frac{dt}{t} \right\|_{H^1_{max,a'}} \\ & + C \sum_{j=1}^{d} \sum_{n=1}^{\infty} |\lambda_{n,j}| \left\| \int_0^2 \chi_{\left[\frac{m(\cdot)}{b},2\right]}(t) (t^2 L)^N T_{t^2/\alpha} \left(t (\partial_\gamma^j)^* A_{n,j}(\cdot,t) \right) \frac{dt}{t} \right\|_{H^1_{max,a'}} \\ & + C \sum_{j=1}^{d} \left\| \int_0^{\frac{m(\cdot)}{b}} (t^2 L)^N T_{t^2/\alpha} \left(\chi_{D^c}(x,t) t \partial_\gamma^j T_{a^2t^2/\alpha} f(\cdot) \right) \frac{dt}{t} \right\|_{H^1_{max,a'}} \\ & + C \left\| \int_{\frac{m(\cdot)}{b}}^{\infty} (t^2 L)^{N+1} T_{(1+a^2)t^2/\alpha} f(\cdot) \frac{dt}{t}) \right\|_{H^1_{max,a'}} + \|f\|_{1,\gamma}.\end{aligned}$$

As the Ornstein–Uhlenbeck semigroup is conservative, i.e., $T_t 1 =, t \geq 0$ then

$$\left\| \mathscr{T}_\gamma^*(1,a) \left(\int_{\mathbb{R}^d} f(y)\gamma_d(dy) \right) \right\|_{1,\gamma} \leq \|f\|_{1,\gamma} \leq \|f\|_{H^1_{quad,a'}(\gamma)}.$$

To bound the rest of the terms above, several estimates of Mehler's kernel (off-diagonal estimates) are needed, in addition to the introduction of the notion of molecules (see Sections 3 and 4 of [231]). Once that is done, we can then bound the remaining terms. Using [231, Proposition 5.5], we get

$$\left\| \int_{\frac{m(\cdot)}{b}}^{\infty} (t^2 L)^{N+1} T_{(1+a^2)t^2/\alpha} f(\cdot) \frac{dt}{t}) \right\|_{H^1_{max,a'}(\gamma)} \leq C \leq \|f\|_{1,\gamma} \leq C \|f\|_{H^1_{quad,a'}(\gamma)}.$$

Now, for $j = 1, \cdots, d$, using [231, Proposition 5.4], we obtain

$$\left\| \int_0^{\frac{m(\cdot)}{b}} (t^2 L)^N T_{t^2/\alpha} \left(\chi_{D^c}(x,t) t \partial_\gamma^j T_{a^2t^2/\alpha} f(\cdot) \right) \frac{dt}{t} \right\|_{H^1_{max,a'}(\gamma)} \leq C \leq \|f\|_{1,\gamma} \leq C \|f\|_{H^1_{quad,a'}(\gamma)}.$$

Applying [231, Proposition 5.3] gives that

$$\left\| \int_0^2 \chi_{\left[\frac{m(\cdot)}{b},2\right]}(t) (t^2 L)^N T_{t^2/\alpha} \left(t (\partial_\gamma^j)^* A_{n,j}(\cdot,t) \right) \frac{dt}{t} \right\|_{H^1_{max,a'}(\gamma)} \leq C,$$

whereas Proposition 4.2 combined with Theorem 4.3 of [231] gives, for $j = 1, \cdots, d$,

$$\left\| \mathscr{T}_\gamma^*(1,a) \left(\int_{\mathbb{R}^d} f(y)\gamma_d(dy) \right) \right\|_{1,\gamma} \leq C.$$

Therefore,

$$\|f\|_{H^1_{max,a'}(\gamma)} \leq C\|f\|_{H^1_{quad,a'}(\gamma)} + C\sum_{j=1}^{d}\sum_{n=1}^{\infty}|\lambda_{n,j}| \leq C\|f\|_{H^1_{quad,a'}(\gamma)}. \qquad \square$$

G. Mauceri and S. Meda proved that the topological dual of $H^{1,r}_{at}(\gamma_d)$ is isomorphic to $BMO(\gamma_d)$. They also proved that the imaginary power of the Ornstein–Uhlenbeck operator, $(-L)^{i\alpha}$ and the adjoint of the Riesz transforms $\mathscr{R}_j^*$ are bounded from $H^{1,r}_{at}(\gamma_d)$ to $L^1(\gamma_d)$. Unfortunately, it was proved by G. Mauceri, S. Meda, and P. Sjögren in [176, Theorem 3.1] that the Riesz transforms $\mathscr{R}_j$ are not bounded from $H^1_{at}(\gamma_d)$ to $L^1(\gamma_d)$ in a dimension greater than one. On the other hand, P. Portal proved, in [231, Theorem 6.1], that the Riesz transforms $\mathscr{R}_j$ are bounded from $H^1_{max}(\gamma_d)$ to $L^1(\gamma_d)$, but it is not known if the imaginary powers of $(-L)$ are bounded there. Also, nothing is known about duality and interpolation for $H^1_{max}(\gamma_d)$. Thus, these spaces are different.

As we have seen, Portal's proof is based on the theory of Gaussian tent spaces $T^{1,2}_{\alpha}(\gamma_d)$. Although these tent spaces are defined using an atomic decomposition, and the equivalence of $H^1_{max}(\gamma_d)$ and $H^1_{quad}(\gamma_d)$ uses the atomic decomposition of $T^{1,2}_{\alpha}(\gamma_d)$ via the Gaussian version of Calderón's reproducing formula, their explicit characterization is not provided in [231]. In [37], T. Bruno introduces a new atomic Gaussian Hardy space $X^1(\gamma_d)$, which is strictly contained in the space $H^1_{at}(\gamma_d)$.

First, we need the following notation,

Definition 7.18. *Let E be a bounded open set and K be a compact set in $\mathbb{R}^d$.*

i) We denote by $q^2(E)$ the space of all functions $f \in L^2(E)$ such that Lf is constant on E, and by $q^2(K)$ the space of all functions on K, which are restriction to K of a function in $q^2(E')$ for some bounded open set, such that $K \subset E'$.
ii) We denote by $h^2(E)$ the space of all functions $f \in L^2(E)$ such that $Lf = 0$ on E, and by $h^2(K)$ the space of all functions on K that are restriction to K of a function in $h^2(E')$ for some bounded open set, such that $K \subset E'$.

The spaces $h^2(E)^{\perp}$ and $q^2(E)^{\perp}$ are the orthogonal complements of $h^2(E)$ and $q^2(E)$ in $L^2(E,\gamma_d)$ respectively. The spaces $h^2(K)^{\perp}$ and $q^2(K)^{\perp}$ are the orthogonal complements of $h^2(K)$ and $q^2(K)$ in $L^2(K,\gamma_d)$ respectively.

Now, following G. Mauceri, S. Meda, and P. Vallarino in [178], we defined the atomic Gaussian Hardy space $X^1(\gamma_d)$.

Definition 7.19. *An X^1-atom is a function $a \in L^2(\gamma_d)$, supported in a ball $B \in \mathscr{B}_1$, with the following properties:*

i) $a \in q^2(\overline{B})^{\perp}$.
ii) $\|a\|_{2,\gamma} \leq \gamma_d(B)^{1/2}$.

Definition 7.20. *The atomic Gaussian Hardy space $X^1(\gamma_d)$ is the space of all functions f in $L^1(\gamma_d)$ that admit an atomic decomposition of the form*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k \tag{7.29}$$

where a_k is a X^1-atom and $\sum_{k=1}^{\infty} |\lambda_k| < \infty$, with norm

$$\|f\|_{X^1(\gamma)} = \inf\Big\{ \sum_{k=1}^{\infty} |\lambda_k| : f = \sum_{k=1}^{\infty} \lambda_k a_k,\ a_k\ X^1\text{-atom and } \sum_{k=1}^{\infty} |\lambda_k| < \infty \Big\}. \tag{7.30}$$

If $B \in \mathscr{B}_1$, the functions in $q^2(\overline{B})$ are referred to as *Gaussian quasi-harmonic functions* in B.

Observe that the space $X^1(\gamma_d)$ is strictly contained in the atomic Gaussian space $H^1_{at}(\gamma_d)$ of Mauceri and Meda. Indeed, the atoms defining $H^1_{at}(\gamma_d)$ are supported on admissible balls of $\mathscr{B}_1$, but have only zero integral, a much weaker condition than being in $q^2(\overline{B})^{\perp}$. The great advantage of the space $X^1(\gamma_d)$ is that T. Bruno proved that the Riesz transforms are bounded from $X^1(\gamma_d)$ to $L^1(\gamma_d)$. However, the understanding of the space $X^1(\gamma_d)$ is far from complete; for instance, it seems that $X^1(\gamma_d)$ is also a subspace of $H^1_{max}(\gamma_d)$.

7.5 Gaussian $BMO(\gamma_d)$ Spaces

In 1961, F. John and L. Nirenberg [144] introduced the space of functions of bounded mean oscillations (BMO) with respect to the Lebesgue measure, as the space of all locally integrable functions on $\mathbb{R}^d$ such that

$$\sup_{Q \in \mathscr{Q}} \frac{1}{|Q|} \int_{\mathbb{R}^d} |f(y) - f_Q| dy < \infty, \tag{7.31}$$

where $\mathscr{Q}$ is the family of all open cubes in $\mathbb{R}^d$ with sides parallel to the coordinate axes, and $f_Q = \frac{1}{|Q|} \int_{\mathbb{R}^d} |f(y)| dy$, the average of f over Q with respect to the Lebesgue measure. It is easy to see that by replacing the family $\mathscr{Q}$ with the family of balls $\mathscr{B}$ in the formula above, we obtain an equivalent norm on BMO.

Extensions of the space of functions of bounded mean oscillations have been considered in the literature. In particular, a theory of functions of bounded mean oscillations that parallels the Euclidean theory has been developed on spaces of homogeneous type by R. Coifman and G. Weiss [56] (see also [170]). As mentioned before, $(\mathbb{R}^d, |\cdot|, \gamma_d)$ is not a space of homogeneous type and the theory of BMO spaces developed in [56] and [170] does not apply to this setting.

More recently, spaces of functions of bounded mean oscillations have been introduced on measured metric spaces not of homogeneous type, specifically on

$(\mathbb{R}^d, |\cdot|, \mu)$, where μ is a (possibly non-doubling) non-negative Radon measure. In particular, X. Tolsa [274] has defined a regular BMO space, $RBMO(\mu)$, whenever μ is a non-negative Radon measure on $\mathbb{R}^d$, which is n-dimensional, i.e., there exists a constant $C > 0$ such that for any ball $B(x,r) \subset \mathbb{R}^d$

$$\mu(B(x,r)) \leq Cr^n,$$

for some $n \in [1,d]$. Tolsa's space enjoys many good properties of BMO of spaces of homogeneous type. In particular, Calderón–Zygmund singular integrals are bounded from $L^\infty(\mu)$ to $RBMO(\mu)$.

As mentioned before, γ_d is trivially a d-dimensional measure. However, $RBMO(\gamma_d)$ is not the appropriate space to study the boundedness on $L^\infty(\gamma_d)$ of Gaussian singular integrals, because the kernel of these operators does not satisfy the standard estimates uniformly in the whole complement of the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$. As we discuss in detail in Chapter 9, the local part of Gaussian singular integrals satisfies the usual estimates of a Calderón–Zygmund operator. In 2007, G. Mauceri and S. Meda in [174] also introduced *Gaussian BMO spaces, $BMO(\gamma_d)$*, as follows:

Definition 7.21. *The Gaussian space of functions of bounded mean oscillations $BMO(\gamma_d)$, is the space of functions $f \in L^1(\gamma_d)$ that satisfy*

$$\sup_{B \in \mathscr{B}_1} \frac{1}{\gamma_d(B)} \int_B |f(x) - f_B^\gamma| \gamma(dx) < \infty, \tag{7.32}$$

where

$$f_B^\gamma = \frac{1}{\gamma_d(B)} \int_B f(x)\gamma_d(dx),$$

the average of f over B. We define

$$\|f\|_*^{\mathscr{B}_1} = \sup_{B \in \mathscr{B}_1} \frac{1}{\gamma_d(B)} \int_B |f(x) - f_B^\gamma| \gamma(dx), \tag{7.33}$$

and the norm in $BMO(\gamma_d)$ is then defined as

$$\|f\|_{BMO(\gamma)} = \|f\|_{1,\gamma} + \|f\|_*^{\mathscr{B}_1}.$$

Observe that by definition $BMO(\gamma_d) \subset L^1(\gamma_d)$. Moreover, it can be proved that $BMO(\gamma_d)$ is a Banach space, and also that if we replace the family $\mathscr{B}_1$ with any other family $\mathscr{B}_a$ in the definition of $BMO(\gamma_d)$, we obtain the same space with an equivalent norm (see [174, Proposition 2.4]),[3]

[3] Also, we obtain the same space with an equivalent norm if instead of $\mathscr{B}_a$, we consider $\mathscr{Q}_a$ the *admissible cubes of parameter a*, i.e., the cubes Q with sides parallel to the axes , with a center at c_q and a side length $l_q \leq am(cQ)$.

We define the (local) sharp function $f^\sharp$ as follows:

Definition 7.22. *Given $f \in L^1(\gamma_d)$, the (local) sharp function $f^\sharp$ is defined as*

$$f^\sharp(x) = \sup_{B\in\mathscr{B}_1, x\in B} \frac{1}{\gamma_d(B)} \int_B |f(y) - f_B^\gamma|\gamma_d(dy). \tag{7.34}$$

Clearly, $f \in BMO(\gamma_d)$ if and only if $f^\sharp \in L^\infty(\gamma_d)$, and $\|f\|_*^{\mathscr{B}_1} = \|f^\sharp\|_{\infty,\gamma}$. Moreover, it is straightforward to prove that $f^\sharp \leq 2\mathscr{M}_\gamma^a 1 f(x)$, for any $x \in \mathbb{R}^d$.

Additionally, G. Mauceri and S. Meda in [174] prove that an inequality of John–Nirenberg type for admissible balls holds for functions in $BMO(\gamma_d)$ (see [174, Proposition 4.1]) and that the topological dual of $H^1_{at}(\gamma_d)$ is isomorphic to $BMO(\gamma_d)$. The proof of this result is modeled over the classical result of Fefferman, although there are several additional difficulties to overcome to adapt the original proof to the Gaussian setting (see [174, Theorem 5.2]).

7.6 Gaussian Lipschitz Spaces $Lip_\alpha(\gamma)$

The standard Euclidean Lipschitz space $Lip_\alpha(\mathbb{R}^n)$ consists of all bounded functions f such that for some $C > 0$

$$|f(y) - f(x)| \leq C|x-y|^\alpha, \quad x, y \in \mathbb{R}^n. \tag{7.35}$$

This characterization is based on the regularity of the functions. It is known that the space $Lip_\alpha(\mathbb{R}^n)$ can also be characterized by convolution with the standard Poisson kernel,

$$q_t(x) = c_n \frac{t}{(t^2 + |x-y|^2)^{(d+1)/2}},$$

see E. Stein [252, Section V. 4. 2], as $f \in Lip_\alpha(\mathbb{R}^n)$ if and only if

$$\left\| \frac{\partial \mathscr{P}_t}{\partial t}(x,y) f \right\|_{L^\infty} \leq Ct^{\alpha-1}, \tag{7.36}$$

for all $t > 0$.

We would like to define Lipschitz spaces associated with the Gaussian measure. Observe that, as mentioned above, the spaces $L^p(\gamma_d)$ are not closed under the action of the classical translation operator; thus, it would not be a good idea to try to define them following the classical definition (7.35). Therefore, we use the Poisson–Hermite semigroup to define Gaussian Lipschitz spaces.

In what follows, we need the technical result about the L^1-norm of the derivatives discussed in Lemma 3.16. From there, we then get the following key result,

Proposition 7.23. *Suppose $f \in L^\infty(\gamma)$ and $\alpha > 0$. Let k and l be two integers both greater than α. The two conditions*

$$\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{\infty,\gamma} \leq A_{\alpha,k} t^{-k+\alpha} \tag{7.37}$$

and

$$\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{\infty,\gamma} \leq A_{\alpha,l} t^{-l+\alpha}, \tag{7.38}$$

are equivalent. Moreover, the smallest $A_{\alpha,k}$ and $A_{\alpha,l}$ holding in the above inequalities, are comparable.

Proof. It suffices to prove that if $k > \alpha$,

$$\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{\infty,\gamma} \leq A_{\alpha,k} t^{-k+\alpha} \tag{7.39}$$

and

$$\Big\|\frac{\partial^{k+1} P_t f}{\partial t^{k+1}}\Big\|_{\infty,\gamma} \leq A_{\alpha,k+1} t^{-(k+1)+\alpha}, \tag{7.40}$$

are equivalent.

Let us assume (7.39). Applying the semigroup property, if $t = t_1 + t_2$, $P_t f = P_{t_1}(P_{t_2} f)$, then using the hypothesis and Lemma 3.3,

$$\begin{aligned}\Big\|\frac{\partial^{k+1} P_t f}{\partial t^{k+1}}\Big\|_{\infty,\gamma} &= \Big\|\frac{\partial P_{t_1}}{\partial t_1}\Big(\frac{\partial^k P_{t_2} f}{\partial t_2^k}\Big)\Big\|_{\infty,\gamma} \leq \Big\|\frac{\partial^k P_{t_2} f}{\partial t_2^k}\Big\|_{\infty,\gamma} \int_{\mathbb{R}^d} \Big|\frac{\partial p(t_1,\cdot,y)}{\partial t_1}\Big| dy \\ &\leq A_{\alpha,k} t_2^{-k+\alpha} C t_1^{-1}.\end{aligned}$$

For $t_1 = t_2 = t/2$ we get (7.40).

Now, assume (7.40). Observe that, again by Lemma 3.3,

$$\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{\infty,\gamma} \leq \|f\|_\infty \int_{\mathbb{R}^d} \Big|\frac{\partial^k p(t,x,y)}{\partial t^k}\Big| dy \leq \frac{C}{t^k}\|f\|_\infty;$$

thus, $\frac{\partial^k P_t f}{\partial t^k} \to 0$ as $t \to \infty$, and then using hypothesis

$$\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{\infty,\gamma} \leq \int_t^\infty \Big\|\frac{\partial^{k+1} P_s f}{\partial s^{k+1}}\Big\|_{\infty,\gamma} ds \leq A_{\alpha,k+1} \frac{t^{-k+\alpha}}{-k+\alpha} = C t^{-k+\alpha}. \qquad \square$$

Now, we can define the Gaussian Lipschitz spaces as follows:

Definition 7.24. *For $\alpha > 0$ let n be the smallest integer greater than α. The Gaussian Lipschitz space $Lip_\alpha(\gamma)$ is defined as the set of L^∞ functions for which there exists a constant A such that*

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_\infty \leq A t^{-n+\alpha}. \tag{7.41}$$

The norm of $f \in Lip_\alpha(\gamma)$ is defined as

$$\|f\|_{Lip_\alpha(\gamma)} := \|f\|_{\infty,\gamma} + A_\alpha(f), \tag{7.42}$$

where $A_\alpha(f)$ is the smallest constant A appearing in (7.41).

Observations 7.25. *For the Gaussian Lipschitz spaces, we have*

i) The definition of $Lip_\alpha(\gamma)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent, according to Proposition 7.23.

ii) Condition (7.41) is of interest for t near zero, because the inequality

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_\infty \leq A t^{-n}, \tag{7.43}$$

which is stronger away from zero, follows for $f \in L^\infty$ immediately from (3.17),

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{\infty,\gamma} \leq \int_{\mathbb{R}^d} \left| \frac{\partial^n p(t,x,y)}{\partial t^n} \right| |f(y)| dy \leq \frac{C}{t^n} \|f\|_\infty.$$

iii) For the completeness of the Gaussian Lipschitz spaces see Lemma 7.35.

We also define, for $\alpha > 0$, homogeneous Gaussian Besov spaces $\dot{B}^{\alpha}_{\infty,\infty}(\gamma)$ as follows:

Definition 7.26. *For $\alpha > 0$, let n be the smallest integer greater than α, then the homogeneous Gaussian Besov space type $\dot{B}^{\alpha}_{\infty,\infty}(\gamma)$ is defined as the set of $L^1(\gamma)$ functions such that (7.41) holds for a constant $B_{\alpha,n}$.*

All these spaces can also be obtained using abstract interpolation theory using the Poisson–Hermite semigroup (see [271] 1.6.5.)

Observe that $Lip_\alpha(\gamma) \subset \dot{B}^{\alpha}_{\infty,\infty}(\gamma)$. There are also inclusion relations among the Gaussian Lipschitz spaces,

Proposition 7.27. *If $0 < \alpha_1 < \alpha_2$, then we have the inclusion*

$$Lip_{\alpha_2}(\gamma) \subset Lip_{\alpha_1}(\gamma).$$

Proof. Take $f \in Lip_{\alpha_2}(\gamma)$ and $n \geq \alpha_2$, then

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{\infty,\gamma} \leq A_\alpha(f) t^{-n+\alpha_2}.$$

If $0 < t < 1$, then $t^{-n+\alpha_2} \leq t^{-n+\alpha_1}$; therefore,

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{\infty,\gamma} \leq A_\alpha(f) t^{-n+\alpha_1}.$$

Now, if $t \geq 1$, then we know from (7.43) that

$$\Big\| \frac{\partial^n P_t f}{\partial t^n} \Big\|_{\infty,\gamma} \leq A_\alpha(f) t^{-n}$$

and as $t^{-n+\alpha_1} > t^{-n}$, we also get in this case

$$\Big\| \frac{\partial^n P_t f}{\partial t^n} \Big\|_{\infty,\gamma} \leq A_\alpha(f) t^{-n+\alpha_1}$$

because $n > \alpha_1$; then, $f \in Lip_{\alpha_1}(\gamma)$. □

Proposition 7.28. *If $f \in Lip_\alpha(\gamma)$ with $0 < \alpha < 1$, then*

$$||P_t f - f||_{\infty,\gamma} \leq A_\alpha(f) t^\alpha. \tag{7.44}$$

Proof. Applying the fundamental theorem of calculus,

$$\begin{aligned} ||P_t f - f||_{\infty,\gamma} &= \Big\| \int_0^t \frac{\partial P_s f}{\partial s} ds \Big\|_\infty \leq \int_0^t \Big\| \frac{\partial P_s f}{\partial s} \Big\|_{\infty,\gamma} ds \\ &\leq A_\alpha(f) \int_0^t s^{-1+\alpha} ds = A_\alpha(f) t^\alpha. \qquad \square \end{aligned}$$

Gaussian Lipschitz spaces were defined by A. E. Gatto and W. Urbina in [109] following E. Stein's approach in [252, Chapter V], using the Poisson–Hermite semigroup. After the given definition of those spaces in this way, it is natural to ask if there is a characterization based on the regularity of the functions involved, as in the classical case. In [159], L. Liu and P. Sjögren have characterized these spaces, for $0 < \alpha < 1$, in terms of a combination of ordinary Lipschitz continuity conditions, giving a positive answer to the question posed. The main result of Liu and Sjögren's paper is the following:

Theorem 7.29. *Let $\alpha \in (0,1)$, an essentially bounded function $f \in Lip_\alpha(\gamma)$ if and only if there exists a constant K such that for all $x, y \in \mathbb{R}^n$,*

$$|f(y) - f(x)| \leq K \min\Big\{ |x-y|^\alpha, \Big(\frac{|x-y|}{1+|x|+|y|}\Big)^{\alpha/2} + ((|x|+|y|)\sin\theta)^\alpha \Big\}, \tag{7.45}$$

after a correction of f on a null set. Here, θ denotes the angle between the vectors x and y; if $x = 0$ or $y = 0$, then θ is understood to be 0.

In one dimension, the inequality becomes,

$$|f(y) - f(x)| \leq K \min\Big\{ |x-y|^\alpha, \Big(\frac{|x-y|}{1+|x|+|y|}\Big)^{\alpha/2} \Big\}. \tag{7.46}$$

This is a combined Lipschitz condition, with exponent α for a short distance $|x-y|$ and exponent $\alpha/2$ with a different coefficient, for a long distance.

As usual in Gaussian harmonic analysis, the two parts of this estimate correspond to the "local part" (for short distance $|x-y|$), in which the estimate coincides with the Euclidean case, and the "global part" corresponding to the long distance (i.e., $|x-y|$ big), in which the effect of the Gaussian measure makes the estimate a little different.

In higher dimensions, the expression $(|x|+|y|)\sin\theta$ describes the "orthogonal component" of the vector $x-y$, because it is the distance from x to the line in the direction x. To make this clearer, Liu and Sjögren state a non-symmetric inequality equivalent to (7.45). For $x,y \in \mathbb{R}^n$ with $x \neq 0$, we decompose y as $y = y_x + y'_x$, where y_x is parallel to x and y'_x orthogonal to x,

$$|f(y)-f(x)| \leq K' \min\Big\{|x-y|^{\alpha}, \Big(\frac{|x-y_x|}{1+|x|}\Big)^{\alpha/2} + |y'_x|\Big\}. \tag{7.47}$$

This inequality means that the combined Lipschitz condition applies in the radial direction, but in the orthogonal direction, the exponent is always α. The equivalence between these two inequalities is valid in any dimension, with a constant $K' > 0$ comparable with K.

The proof of (7.45) relies on very precise pointwise estimates of the Poisson–Hermite kernel $p(t,x,y)$ and its derivatives; for all $t > 0$ and $x,y \in \mathbb{R}^n$,

$$p(t,x,y) \leq C[K_1(t,x,y) + K_2(t,x,y) + K_3(t,x,y) + K_4(t,x,y)], \tag{7.48}$$

where,

$$K_1(t,x,y) = \frac{t}{(t^2+|x-y|^2)^{(n+1)/2}} \exp(-C_1 t(1+|x|)),$$

for some constant C_1,

$$K_2(t,x,y) = \frac{t}{|x|}\Big(t^2 + \frac{|x-y_x|}{|x|} + |y'_x|^2\Big)^{-(n+2)/2} \times \exp\Big(-C_2\frac{(t^2+|y'_x|^2)|x|}{|x-y_x|}\Big)\chi_{\{|x|>1, x\cdot y>0, |x|/2\leq|y_x|<|x|\}};$$

for some constant C_2,

$$K_3(t,x,y) = \min(1,t)\exp(-C_3|y|^2);$$

for some constant C_3, and

$$K_4(t,x,y) = \frac{t}{|y_x|}\Big(\log\frac{|x|}{|y_x|}\Big)^{-3/2} \exp\Big(-C_4\frac{t^2}{\log\frac{|x|}{|y_x|}}\Big)\exp(-C_5|y'_x|^2)\chi_{\{x\cdot y>0, 1<|y_x|<|x|/2\}};$$

for some constant C_4.

Similar estimates are also possible for the derivatives of $p(t,x,y)$, both $\partial_t p(t,x,y)$ and $\partial_{x_i} p(t,x,y)$. Thus,

$$p(t,x,y) + |t\partial_t p(t,x,y)| + |t\partial_{x_i} p(t,x,y)| \\ \leq C[K_1(t,x,y) + K_2(t,x,y) + K_3(t,x,y) + K_4(t,x,y)]. \quad (7.49)$$

Moreover, Liu and Sjögren prove that these estimates are also sharp. For each of the four kernels $K_i(t,x,y)$ there is a set $\tilde{E}_i$ of points (t,x,y) in which $p(t,x,y)$ is equivalent to $K_i(t,x,y)$, but where the other terms are much smaller; thus, none of the four terms can be suppressed in the estimate. The estimates are product of a very deep understanding of the kernel $p(t,x,y)$ and how it compares with the standard Poisson kernel $q_t(x)$ (for more details, we refer the reader to their paper [159]).

The estimates of the Poisson–Hermite kernel $p(t,x,y)$ and its derivatives are of independent interest, and the proof of the main result is almost straightforward once we have those estimates. It would be interesting to know if alternative characterization of the Gaussian Besov–Lipschitz and the Gaussian Triebel–Lizorkin spaces, which are defined in the next two sections, using higher order derivatives of the Poisson–Hermite kernel, can be obtained using similar estimates.

Another open question would be if the characterization of the Gaussian Lipschitz spaces obtained by Liu and Sjögren is related to the notion of translation operator introduced by C. Markett in [173].

In the Euclidean case, as mentioned above, condition (7.36) characterizes the ordinary Lipchitz space only if the functions considered are bounded. Thus, we obtain the *inhomogeneous Lipschitz space*; without the boundedness assumption, we get the larger *homogeneous Lipschitz space.*

In the Gaussian setting, as no homogeneity is involved, the condition (7.41) without the boundedness assumption defines a space that had been considered by L. Liu and P. Sjögren in [160]. It is called the *global Gaussian Lipschitz space*. Using a result by G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, and B. Viviani [106], Liu and Sjögren consider measurable functions f in $\mathbb{R}^d$ with the condition

$$\int_{\mathbb{R}^d} \frac{e^{-|y|^2}}{\sqrt{\ln(e+|y|)}} |f(y)|dy < \infty, \quad (7.50)$$

which according to Theorem 1.1 of [106] guarantees that the $P_t f$ is well defined. Moreover, the same condition ensures that $P_t f(x) \to f(x)$ as $t \to 0$ a.e. $x \in \mathbb{R}^n$. Therefore,

Definition 7.30. *Let $\alpha \in (0,1)$. A measurable function f defined in $\mathbb{R}^n$ and satisfying (7.50) belongs to the global Gaussian Lipschitz space $GLip_\alpha(\gamma)$ if (7.41) holds. The corresponding norm is*

$$\|f\|_{GLip_\alpha(\gamma)} = \inf\{A > 0 : A \text{ satisfies } (7.41)\}.$$

This space is actually a space of equivalence classes, as it consists of functions modulo constants.

A natural question is what continuity condition characterizes these spaces? To answer this, Liu and Sjögren introduce the following distance:

$$d(x,y) = \left| \int_x^y \frac{d\xi}{1+|\xi|} \right| = |\ln(1+|x|) - \operatorname{sgn} xy \ln(1+|y|)|, \quad x,y \in \mathbb{R}, \tag{7.51}$$

with the convention $\operatorname{sgn} 0 = 1$. In several dimensions, we use this distance on the line spanned by x, defining

$$d(x,y) = |\ln(1+|x|) - \operatorname{sgn} < x,y > \ln(1+|y_x|)|, \quad x,y \in \mathbb{R}^n,$$

with y_x as before. The main result in [160] is the following:

Theorem 7.31. *Let $\alpha \in (0,1)$ and let f be a measurable function in $\mathbb{R}^n$. The following conditions are equivalent:*

i) f satisfies condition (7.50) and $f \in GLip_\alpha(\gamma)$.
ii) There exists a positive constant K such that for all $x,y \in \mathbb{R}^n$

$$|f(y) - f(x)| \leq K \min\left\{|x-y|^\alpha, d(x,y_x)^{\alpha/2} + |y'_x|^\alpha\right\}, \quad x,y \in \mathbb{R}^n \tag{7.52}$$

after a correction of f on a null set.

Moreover, the space $GLip_\alpha(\gamma)$ is defined in terms of the distance function d. Indeed, (7.47) implies boundedness, then (7.47) holds if and only if there exists a constant $K" > 0$ such that,

$$|f(y) - f(x)| \leq K \min\left\{1, |x-y|^\alpha, d(x,y_x)^{\alpha/2} + |y'_x|^\alpha\right\},$$

for $x,y \in \mathbb{R}^n$. This also tells us that for bounded functions (7.47) and (7.52) are equivalent.

The condition (7.52) implies only

$$f(x) = O(\ln|x|)^{\alpha/2} \quad \text{as } |x| \to \infty.$$

Liu and Sjögren show that this condition is sharp using a counterexample in Section 7.5.

To obtain (7.52), they need to modify the kernel K_3 to decay for large values of x, refining a few of the previous arguments. The estimates (7.48) and (7.49) remain valid if the kernel $K_3(t,x,y)$ is replaced by

$$\widetilde{K}_3(t,x,y) = \min\left\{1, \frac{t}{[\ln(e+|x|)]^{1/2}}\right\} \exp(-C_3|y|^2)$$

The introduction in (7.51) of the distance d in the context of Gaussian harmonic analysis is an interesting point that may be used in other problems.

After several technical results, analogous estimates can obtained for $f \in GLip_\alpha(\gamma)$ with norm 1:

- For all $i = 1, 2, \cdots n, t > 0$, and $x \in \mathbb{R}^n$,

$$|\partial_{x_i} P_t f(x)| \leq C t^{\alpha-1}.$$

- For all $t > 0$ and $x = (x_1, 0, \cdots, 0) \in \mathbb{R}^n$ with $x_1 \geq 0$,

$$|\partial_{x_i} P_t f(x)| \leq C t^{\alpha-2}(1+x_1)^{-1}.$$

The proof of the main result, Theorem 1.2, follows almost immediately from all the previous estimates.

7.7 Gaussian Besov–Lipschitz Spaces $B^{\alpha}_{p,q}(\gamma_d)$

In the next two sections, we study the Gaussian Besov–Lipschitz and the Gaussian Triebel–Lizorkin spaces. They were introduced initially by E. Pineda in his doctoral dissertation (see [224] and also [226]).

For any $\alpha \geq 0$, we define Gaussian Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$, following E. Stein [252] to define and study the $B^{\alpha}_{p,q}(\gamma_d)$ spaces, using the Poisson–Hermite semigroup. But because the Poisson–Hermite semigroup is not a convolution semigroup, the proofs of the results are totally different to those given there.

As in the case of Gaussian Lipschitz spaces, Besov–Lipschitz spaces can also be obtained as interpolated spaces using interpolation theory for semigroups defined on a Banach space (see for instance Chapter 3 of [38, 112] or [271]).

We use the representation of the Poisson–Hermite semigroup (3.8) in a crucial way, using the one-sided stable measure

$$\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds = g(t,s)ds,$$

and the estimates (3.19), (3.20) and (3.21).

In Chapter 3, we have obtained an estimate of the $L^p(\gamma_d)$-norms of the derivatives of the Poisson–Hermite semigroup (see Lemma 3.5); additionally, we have

Lemma 7.32. *Given $f \in L^p(\gamma_d), \alpha \geq 0$ and k, l integers greater than α, then*

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma} \leq A_k t^{-k+\alpha} \text{ if and only if } \left\| \frac{\partial^l P_t f}{\partial t^l} \right\|_{p,\gamma} \leq A_l t^{-l+\alpha}.$$

Moreover, if $A_k(f), A_l(f)$ are the smallest constants appearing in the above inequalities, then there exist constants $A_{k,l,\alpha}$ and $D_{k,l,\alpha}$ such that

$$A_{k,l,\alpha} A_k(f) \leq A_l(f) \leq C D_{k,l,\alpha} A_k(f),$$

for all $f \in L^p(\gamma_d)$.

Proof. Let us suppose, without loss of generality, that $k \geq l$. We prove the direct implication first. For this, we use again the representation of the Poisson–Hermite semigroup (3.8),

$$P_t f(x) = \int_0^{+\infty} T_s f(x) \mu_t^{(1/2)}(ds).$$

Then, differentiating k-times with respect to t,

$$\frac{\partial^k P_t f(x)}{\partial t^k} = \int_0^{+\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{(1/2)}(ds).$$

Using the identity (3.19), it is easy to prove that for all $m \in \mathbb{N}$

$$\lim_{t \to +\infty} \frac{\partial^m P_t f(x)}{\partial t^m} = 0;$$

therefore, given $n \in \mathbb{N}, n > \alpha$

$$\frac{\partial^n P_t f(x)}{\partial t^n} = -\int_t^{+\infty} \frac{\partial^{n+1} P_s f(x)}{\partial s^{n+1}} ds$$

Thus,

$$\begin{aligned}\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_{p,\gamma} &\leq \int_t^{+\infty} \left\| \frac{\partial^{n+1} P_s f}{\partial s^{n+1}} \right\|_{p,\gamma} ds \leq \int_t^{+\infty} A_{n+1}(f) s^{-(n+1)+\alpha} ds \\ &= \frac{A_{n+1}(f)}{n-\alpha} t^{-n+\alpha}.\end{aligned}$$

Then,

$$A_n(f) \leq \frac{A_{n+1}(f)}{n-\alpha},$$

and as $n > \alpha$ is arbitrary, by using the above result $k-l$ times, we get

$$\begin{aligned}A_l(f) &\leq \frac{A_{l+1}(f)}{l-\alpha} \leq \frac{A_{l+2}}{(l-\alpha)(l+1-\alpha)} \leq \cdots \leq \frac{A_k(f)}{(l-\alpha)(l+1-\alpha)\ldots(k-1-\alpha)} \\ &= D_{k,l,\alpha} A_k(f).\end{aligned}$$

To prove the converse implication, using again the representation of the Poisson–Hermite semigroup (3.8),

$$u(x, t_1 + t_2) = P_{t_1}(P_{t_2} f)(x) = \int_0^{+\infty} T_s(P_{t_2} f)(x) \mu_{t_1}^{(1/2)}(ds).$$

Therefore, taking $t = t_1 + t_2$ and differentiating l times with respect to t_2 and $k-l$ times with respect to t_1, we get

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s\Big(\frac{\partial^l P_{t_2} f(x)}{\partial t_2^l}\Big) \frac{\partial^{k-l}}{\partial t_1^{k-l}} \mu_{t_1}^{(1/2)}(ds). \tag{7.53}$$

Thus, using the inequality (3.21) and the fact that the Ornstein–Uhlenbeck semigroup is a contraction semigroup, we get

$$\begin{aligned}
\Big\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\Big\|_{p,\gamma} &\leq \int_0^{+\infty}\Big\|T_s\Big(\frac{\partial^l P_{t_2}f}{\partial t_2^l}\Big)\Big\|_{p,\gamma}\Big|\frac{\partial^{k-l}\mu_{t_1}^{(1/2)}}{\partial t_1^{k-l}}(ds)\Big| \\
&\leq \Big\|\frac{\partial^l P_{t_2}f}{\partial t_2^l}\Big\|_{p,\gamma}\int_0^{+\infty}\Big|\frac{\partial^{k-l}}{\partial t_1^{k-l}}\mu_{t_1}^{(1/2)}(ds)\Big| \leq C_{k-l}\Big\|\frac{\partial^l}{\partial t_2^l}P_{t_2}f\Big\|_{p,\gamma}t_1^{l-k} \\
&\leq C_{k-l}A_l(f)t_2^{-l+\alpha}t_1^{l-k}.
\end{aligned}$$

Therefore, taking $t_1 = t_2 = \frac{t}{2}$,

$$\Big\|\frac{\partial^k u(\cdot,t)}{\partial t^k}\Big\|_{p,\gamma} \leq C_{k-l}A_l(f)(\frac{t}{2})^{-k+\alpha},$$

and then,

$$A_k(f) \leq \frac{C_{k-l}}{2^{-k+\alpha}}A_l(f).$$

□

The following technical result is crucial for defining Gaussian Besov–Lipschitz spaces:

Lemma 7.33. *Given $\alpha \geq 0$ and k,l integers greater than α. Then,*

$$\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} < \infty$$

if and only if

$$\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha}, D_{k,l,\alpha}$ such that

$$\begin{aligned}
D_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} &\leq \Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\leq A_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}
\end{aligned}$$

Proof. Let us suppose, without loss of generality, that $k \geq l$. We prove the converse implication first; from Lemma 7.32, we have

$$\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma} \leq C_{k-l}\Big\|\frac{\partial^l P_{\frac{t}{2}} f}{\partial(\frac{t}{2})^l}\Big\|_{p,\gamma}(\frac{t}{2})^{l-k}.$$

Thus,

$$\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} \leq \frac{C_{k-l}}{2^{l-k}}\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_{t/2} f}{\partial(\frac{t}{2})^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= A_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(s^{l-\alpha}\Big\|\frac{\partial^l P_s f}{\partial s^l}\Big\|_{p,\gamma}\Big)^q\frac{ds}{s}\Big)^{\frac{1}{q}}$$

with $A_{k,l,\alpha} = C_{k-l}2^{k-\alpha}$.

For the direct implication, given $n \in \mathbb{N}$, $n > \alpha$, using the previous lemma again, we get

$$\Big\|\frac{\partial^n P_t f}{\partial t^n}\Big\|_{p,\gamma} \leq \int_t^{+\infty}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma} ds$$

Therefore, using Hardy's inequality (10.101),

$$\Big(\int_0^{+\infty}\Big(t^{n-\alpha}\Big\|\frac{\partial^n P_t f}{\partial t^n}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \Big(\int_0^{+\infty}\Big(t^{n-\alpha}\int_t^{+\infty}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma} ds\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \Big(\int_0^{+\infty}\Big(\int_t^{+\infty}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma} ds\Big)^q t^{(n-\alpha)q-1}dt\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{n-\alpha}\Big(\int_0^{+\infty}\Big(s^{n+1-\alpha}\Big\|\frac{\partial^{n+1} P_s f}{\partial s^{n+1}}\Big\|_{p,\gamma}\Big)^q\frac{ds}{s}\Big)^{\frac{1}{q}}.$$

Now, as $n > \alpha$ is arbitrary, using the above result $k-l$, times

$$\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big\|\frac{\partial^l P_t f}{\partial t^l}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{l-\alpha}\Big(\int_0^{+\infty}\Big(t^{l+1-\alpha}\Big\|\frac{\partial^{l+1} P_t f}{\partial t^{l+1}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{(l-\alpha).(l+1-\alpha)}\Big(\int_0^{+\infty}\Big(t^{l+2-\alpha}\Big\|\frac{\partial^{l+2} P_t f}{\partial t^{l+2}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\cdots$$
$$\leq D_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$

where $D_{k,l,\alpha} = \dfrac{1}{(l-\alpha).(l+1-\alpha)\dots(k-1-\alpha)}$. □

Following the classical case, we are going to define the Gaussian Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$ or Besov–Lipschitz spaces for Hermite polynomial expansions.

Definition 7.34. *Let $\alpha \geq 0$, k be the smallest integer greater than α, and $1 \leq p,q \leq \infty$. For $1 \leq q < \infty$ the Gaussian Besov–Lipschitz space $B^{\alpha}_{p,q}(\gamma_d)$ is defined as the set of functions $f \in L^p(\gamma_d)$, for which*

$$\left(\int_0^{\infty} (t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma})^q \frac{dt}{t} \right)^{1/q} < \infty. \tag{7.54}$$

The norm of $f \in B^{\alpha}_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{B^{\alpha}_{p,q}} := \|f\|_{p,\gamma} + \left(\int_0^{\infty} (t^{k-\alpha} \left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma})^q \frac{dt}{t} \right)^{1/q} \tag{7.55}$$

For $q = \infty$, the Gaussian Besov–Lipschitz space $B^{\alpha}_{p,\infty}(\gamma_d)$ is defined as the set of functions $f \in L^p(\gamma_d)$ for which exists a constant A, such that

$$\left\| \frac{\partial^k P_t f}{\partial t^k} \right\|_{p,\gamma} \leq A t^{-k+\alpha}$$

and then the norm of $f \in B^{\alpha}_{p,\infty}(\gamma_d)$ is defined as

$$\|f\|_{B^{\alpha}_{p,\infty}} := \|f\|_{p,\gamma} + A_k(f), \tag{7.56}$$

where $A_k(f)$ is the smallest constant A appearing in the above inequality.

In particular, the space $B^{\alpha}_{\infty,\infty}(\gamma_d)$ is the Gaussian Lipschitz space $Lip_{\alpha}(\gamma_d)$.

Lemma 7.33 shows us that we could have replaced k with any other integer l greater than α and that the resulting norms are equivalent. Let us prove now that the Gaussian Besov–Lipschitz spaces are complete.

Lemma 7.35. *For any $\alpha \geq 0$, $1 \leq p,\ q \leq \infty$, the Gaussian Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$ are Banach spaces.*

Proof. To prove the completeness, it is enough to see that if $\{f_n\}$ is a sequence in $B^{\alpha}_{p,q}(\gamma_d)$, such that $\sum_{n=1}^{\infty} \|f_n\|_{B^{\alpha}_{p,q}} < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges in $B^{\alpha}_{p,q}(\gamma_d)$. Because

$$\sum_{n=1}^{\infty} \|f_n\|_{B^{\alpha}_{p,q}} = \sum_{n=1}^{\infty} \left(\|f_n\|_{p,\gamma} + \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k}{\partial t^k} P_t f_n \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \right) < \infty.$$

In particular, this implies that

$$\sum_{n=1}^{\infty} \|f_n\|_{p,\gamma} < \infty, \text{ and } \sum_{n=1}^{\infty} \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k}{\partial t^k} P_t f_n \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} < \infty.$$

But as $L^p(\gamma_d)$ is complete, there exists a function $f \in L^p(\gamma_d)$, such that

$$\sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{a.e.} x.$$

We need to prove that $\sum_{n=1}^{\infty} f_n = f$ in $B^{\alpha}_{p,q}$, i.e. $\lim_{n\to\infty} \| \sum_{i=1}^{n} f_i - f\|_{B^{\alpha}_{p,q}} = 0$.

Given $t > 0$ and $x \in \mathbb{R}^d$, by linearity, $P_t(\sum_{i=1}^{n} f_i(x)) = \int_{\mathbb{R}^d} p(t,x,y) \sum_{i=1}^{n} f_i(y)dy$ and then

$$\lim_{n\to\infty} p(t,x,y) \sum_{i=1}^{n} f_i(y) = p(t,x,y) \sum_{i=1}^{\infty} f_i(y) = p(t,x,y) f(y) \quad \text{a.e.}\, y$$

and for all $n \in \mathbb{N}$

$$|p(t,x,y) \sum_{i=1}^{n} f_i(y)| \leq p(t,x,y) \sum_{i=1}^{n} |f_i(y)| \leq p(t,x,y) g(y) \quad \text{a.e.}$$

As $\int_{\mathbb{R}^d} p(t,x,y)g(y)dy = P_t g(x) < \infty$, i.e., $p(t,x,y)g(y)$ is integrable, we conclude using Lebesgue's dominated convergence theorem, for any $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\lim_{n\to\infty} P_t(\sum_{i=1}^{n} f_i(x)) = \lim_{n\to\infty} \int_{\mathbb{R}^d} p(t,x,y) \sum_{i=1}^{n} f_i(y) = \int_{\mathbb{R}^d} p(t,x,y) f(y)dy = P_t f(x).$$

Similarly, we have, $\lim_{n\to\infty} T_t(\sum_{i=1}^{n} f_i(x)) = T_t f(x)$, for any $t \geq 0$ and $x \in \mathbb{R}^d$, and again using Lebesgue's dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t(f_i(x)) &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t(\sum_{i=1}^{n} f_i(x)) = \lim_{n\to\infty} \int_0^{\infty} T_s(\sum_{i=1}^{n} f_i(x)) \frac{\partial^k}{\partial t^k} \mu_t^{1/2}(ds) \\
&= \int_0^{\infty} \lim_{n\to\infty} T_s(\sum_{i=1}^{n} f_i(x)) \frac{\partial^k}{\partial t^k} \mu_t^{1/2}(ds) \\
&= \int_0^{\infty} T_s f(x) \frac{\partial^k}{\partial t^k} \mu_t^{1/2}(ds) = \frac{\partial^k}{\partial t^k} P_t f(x),
\end{aligned}$$

for any $t \geq 0$ and $x \in \mathbb{R}^d$. Then, for $t > 0$, using Fatou's lemma,

$$\begin{aligned}
\left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p,\gamma}^p &= \int_{\mathbb{R}^d} \left| \frac{\partial^k}{\partial t^k} P_t f \right|^p \gamma_d(dx) \\
&= \int_{\mathbb{R}^d} \left| \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) \right|^p \gamma_d(dx) \\
&\leq \liminf_{n\to\infty} \int_{\mathbb{R}^d} \left| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) \right|^p \gamma_d(dx).
\end{aligned}$$

Thus, for any $t > 0$, by triangle inequality,

$$\left\| \frac{\partial^k}{\partial t^k} P_t f \right\|_{p,\gamma} \leq \liminf_{n\to\infty} \left\| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i \right\|_{p,\gamma} \leq \liminf_{n\to\infty} \sum_{i=1}^{n} \left\| \frac{\partial^k}{\partial t^k} P_t f_i \right\|_{p,\gamma},$$

and again, by triangle inequality,

$$\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\qquad\le\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\qquad\le\liminf_{n\to\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\qquad=\sum_{n=1}^{\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_n\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big)<\infty.
\end{aligned}$$

Then, $f\in B^{\alpha}_{p,q}$.

Let, for each $t>0$,

$$h(t)=t^{k-\alpha}\Big(\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)+\Big\|\frac{\partial^k}{\partial^k}P_t f\Big\|_{p,\gamma}\Big).$$

Then,

$$\begin{aligned}
&\int_0^{+\infty}|h(t)|^q\frac{dt}{t}\\
&\qquad\le\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)+\Big\|\frac{\partial^k}{\partial t^k}P_t f\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\\
&\qquad\le\int_0^{+\infty}\Big(t^{k-\alpha}\Big(2\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)\Big)^q\frac{dt}{t}\\
&\qquad\le 2\liminf_{n\to\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big);
\end{aligned}$$

hence,

$$\begin{aligned}
\Big(\int_0^{+\infty}|h(t)|^q\frac{dt}{t}\Big)^{1/q}&\le 2\liminf_{n\to\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\le 2\liminf_{n\to\infty}\sum_{i=1}^{n}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&=2\sum_{n=1}^{\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_n\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}<\infty.
\end{aligned}$$

Thus, $h\in L^q((0,\infty),\frac{dt}{t})$; therefore,

$$h(t)=t^{k-\alpha}\Big(\liminf_{n\to\infty}\Big(\sum_{i=1}^{n}\Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)+\Big\|\frac{\partial^k}{\partial t^k}P_t f\Big\|_{p,\gamma}\Big)<\infty\quad\text{a.e. } t$$

and this immediately implies

$$\sum_{n=1}^{\infty} \Big\| \frac{\partial^k}{\partial t^k} P_t f_n \Big\|_{p,\gamma} + \Big\| \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma} < \infty \quad \text{a.e.}\, t. \tag{7.57}$$

Let $t > 0$ such that $h(t) < \infty$, we know that for all $x \in \mathbb{R}^d$

$$\lim_{n\to\infty} \Big(\sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big) = \lim_{n\to\infty} \frac{\partial^k}{\partial t^k} P_t \Big(\sum_{i=1}^{n} f_i(x) - f(x) \Big) = 0,$$

Set, for each $x \in \mathbb{R}^d$,

$$H(x) := 2 \sum_{n=1}^{\infty} \Big| \frac{\partial^k}{\partial t^k} P_t f_n(x) \Big|.$$

Then, from the above $H \in L^p(\gamma_d)$ and, therefore, as for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}^d$,

$$\Big| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big| \leq 2 \sum_{i=1}^{\infty} \Big| \frac{\partial^k}{\partial t^k} P_t f_i(x) \Big| = H(x).$$

Then, using Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{n\to\infty} & \Big\| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i - \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma} \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \Big| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big|^p \gamma_d(dx) \\ &= \int_{\mathbb{R}^d} \lim_{n\to\infty} \Big| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i(x) - \frac{\partial^k}{\partial t^k} P_t f(x) \Big|^p \gamma_d(dx) = 0, \end{aligned}$$

and as $h(t) < \infty \quad a.e.t$, we conclude,

$$\lim_{n\to\infty} \Big\| \sum_{i=1}^{n} \frac{\partial^k}{\partial t^k} P_t f_i - \frac{\partial^k}{\partial t^k} P_t f \Big\|_{p,\gamma} = 0, \quad \text{a.e.}\, t.$$

Now, for each $n \in \mathbb{N}$,

$$\begin{aligned} \Big\| \frac{\partial^k}{\partial t^k} P_t \Big(\sum_{i=1}^{n} f_i - f \Big) \Big\|_{p,\gamma} &\leq \sum_{i=1}^{\infty} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} + \liminf_{n\to\infty} \Big(\sum_{i=1}^{n} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} \Big) \\ &= 2 \liminf_{n\to\infty} \Big(\sum_{i=1}^{n} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} \Big). \end{aligned}$$

For each $t > 0$, let $G(t) = \liminf_{n\to\infty} \Big(2t^{k-\alpha} \sum_{i=1}^{n} \Big\| \frac{\partial^k}{\partial t^k} P_t f_i \Big\|_{p,\gamma} \Big)$. Then, using Fatou's lemma and triangle inequality,

$$\Big(\int_0^\infty |G(t)|^q \frac{dt}{t}\Big)^{1/q} \le 2\liminf_{n\to\infty}\Big(\int_0^\infty \Big(t^{k-\alpha}\sum_{i=1}^n \Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$\le 2\sum_{n=1}^\infty \Big(\int_0^\infty \Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t f_n\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} < \infty.$$

Thus, $G \in L^q((0,\infty),\frac{dt}{t})$, so $\liminf_{n\to\infty}\Big(t^{k-\alpha}\sum_{i=1}^n \Big\|\frac{\partial^k}{\partial t^k}P_t f_i\Big\|_{p,\gamma}\Big)^q \frac{1}{t}$ is integrable, and therefore, using Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\Big(\int_0^\infty \Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^n f_i - f\Big)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$= \Big(\int_0^\infty \lim_{n\to\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^n f_i - f\Big)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} = 0.$$

Hence,

$$\lim_{n\to\infty}\Big\|\sum_{i=1}^n f_i - f\Big\|_{B^{\alpha}_{p,q}}$$
$$= \lim_{n\to\infty}\Big(\Big\|\sum_{i=1}^n f_i - f\Big\|_{p,\gamma} + \lim_{n\to\infty}\Big(\int_0^\infty \Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^n f_i - f\Big)\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} = 0.$$

□

Finally, we study some inclusions among the Gaussian Besov–Lipschitz spaces:

Proposition 7.36. *The inclusion $B^{\alpha_1}_{p,q_1}(\gamma_d) \subset B^{\alpha_2}_{p,q_2}(\gamma_d)$ holds if either:*

i) $\alpha_1 > \alpha_2 > 0$ *(q_1 and q_2 need not be related), or*
ii) If $\alpha_1 = \alpha_2$ *and* $q_1 \le q_2$.

Proof. To prove ii), we set $A = \Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{1}{q_1}}$
Now, fixing $t_0 > 0$

$$\int_{\frac{t_0}{2}}^{t_0}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_1}\frac{dt}{t} \le A^{q_1}.$$

Using Lemma 3.5, $\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}$ takes its minimum value at the upper end point ($t = t_0$) of the above integral; thus, we get

$$\Big\|\frac{\partial^k P_{t_0} f}{\partial t^k}\Big\|_{p,\gamma}^{q_1}\int_{\frac{t_0}{2}}^{t_0} t^{(k-\alpha)q_1}\frac{dt}{t} \le A^{q_1}.$$

That is $\left\|\frac{\partial^k P_{t_0} f}{\partial t^k}\right\|_{p,\gamma} \leq CAt_0^{-k+\alpha}$, but because t_0 is arbitrary, then

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} \leq CAt^{-k+\alpha},$$

for all $t > 0$. In other words, $f \in B^{\alpha}_{p,q_1}$ also implies that $f \in B^{\alpha}_{p,\infty}$. Thus, as $q_2 \geq q_1$

$$\begin{aligned}
\int_0^{+\infty} \Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_2} \frac{dt}{t} \\
&\leq \int_0^{+\infty} \Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_2-q_1} \Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_1} \frac{dt}{t} \\
&\leq (CA)^{q_2-q_1} \int_0^{+\infty} \Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_1} \frac{dt}{t} \\
&= (CA)^{q_2-q_1} A^{q_1} = CA^{q_2} < +\infty;
\end{aligned}$$

therefore $f \in B^{\alpha}_{p,q_2}$.

Now, to prove part $i)$, using Lemma 3.5, we have

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} \leq Ct^{-k}, t > 0.$$

Then, given $f \in B^{\alpha_1}_{p,q_1}$, taking again

$$A = \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{1}{q_1}},$$

we get, as in part $ii)$,

$$\left\|\frac{\partial^k P_t f}{\partial t^k}\right\|_{p,\gamma} \leq CAt^{-k+\alpha_1},$$

for all $t > 0$. Thus,

$$\begin{aligned}
\int_0^{+\infty} \Big(t^{k-\alpha_2}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_2} \frac{dt}{t} &= \int_0^{1} \Big(t^{k-\alpha_2}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_2} \frac{dt}{t} \\
&\quad + \int_1^{+\infty} \Big(t^{k-\alpha_2}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^{q_2} \frac{dt}{t} \\
&= (I) + (II).
\end{aligned}$$

Now,

$$\begin{aligned}
(I) &= \int_0^1 t^{(k-\alpha_2)q_2} \Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}^{q_2} \frac{dt}{t} \leq \int_0^1 t^{(k-\alpha_2)q_2} (CA)^{q_2} t^{(\alpha_1-k)q_2} \frac{dt}{t} \\
&= (CA)^{q_2} \int_0^1 t^{(\alpha_1-\alpha_2)q_2} \frac{dt}{t} = CA^{q_2},
\end{aligned}$$

and

$$(II) = \int_1^{+\infty} t^{(k-\alpha_2)q_2} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma}^{q_2} \frac{dt}{t} \leq \int_1^{+\infty} t^{(k-\alpha_2)q_2} C^{q_2} t^{-kq_2} \frac{dt}{t}$$
$$= C^{q_2} \int_1^{+\infty} t^{-\alpha_2 q_2} \frac{dt}{t} = C.$$

Hence,

$$\int_0^{+\infty} \Big(t^{k-\alpha_2} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \Big)^{q_2} \frac{dt}{t} < +\infty;$$

thus, $f \in B^{\alpha_2}_{p,q_2}$. □

7.8 Gaussian Triebel–Lizorkin Spaces $F^{\alpha}_{p,q}(\gamma_d)$

Finally, we define Gaussian Triebel–Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$ for any $\alpha \geq 0$. The following technical result is key for their definition:

Lemma 7.37. *Let $\alpha \geq 0$ and k, l integers such that $k \geq l > \alpha$. Then*

$$\Big\| \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big| \frac{\partial^k}{\partial t^k} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma} < \infty$$

if and only if

$$\Big\| \Big(\int_0^{+\infty} \Big(t^{l-\alpha} \Big| \frac{\partial^l}{\partial t^l} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma} < \infty.$$

Moreover, there exist constants $A_{k,l,\alpha}, D_{k,l,\alpha}$ such that

$$D_{k,l,\alpha} \Big\| \Big(\int_0^{+\infty} \Big(t^{l-\alpha} \Big| \frac{\partial^l}{\partial t^l} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \|_{p,\gamma} \leq \Big\| \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big| \frac{\partial^k}{\partial t^k} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq A_{k,l,\alpha} \Big\| \Big(\int_0^{+\infty} \Big(t^{l-\alpha} \Big| \frac{\partial^l}{\partial t^l} P_t f \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}.$$

Proof. Let $n \in \mathbb{N}$ such that $n > \alpha$. It can be proved that

$$\Big| \frac{\partial^n}{\partial t^n} P_t f(x) \Big| \leq \int_t^{+\infty} \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| ds$$

Then, using Hardy's inequality,

$$\Big(\int_0^{+\infty} \Big(t^{n-\alpha} \Big| \frac{\partial^n}{\partial t^n} P_t f(x) \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \leq \Big(\int_0^{+\infty} \Big(t^{n-\alpha} \int_t^{+\infty} \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{n-\alpha} \Big(\int_0^{+\infty} \Big(s \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| \Big)^q s^{(n-\alpha)q-1} ds \Big)^{\frac{1}{q}}$$
$$= \frac{1}{n-\alpha} \Big(\int_0^{+\infty} \Big(s^{n+1-\alpha} \Big| \frac{\partial^{n+1}}{\partial s^{n+1}} P_s f(x) \Big| \Big)^q \frac{ds}{s} \Big)^{\frac{1}{q}}.$$

Now, as $n > \alpha$ is arbitrary, iterating the previous argument $k-l$ times, we have

$$\begin{aligned}\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l}{\partial t^l}P_tf(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}&\\ &\leq \frac{1}{l-\alpha}\Big(\int_0^{+\infty}\Big(t^{l+1-\alpha}\Big|\frac{\partial^{l+1}}{\partial t^{l+1}}P_tf(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq \frac{1}{(l-\alpha)(l+1-\alpha)}\Big(\int_0^{+\infty}\Big(t^{l+2-\alpha}\Big|\frac{\partial^{l+2}}{\partial t^{l+2}}P_tf(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\dots\\ &\leq C_{k,l,\alpha}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_tf(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\end{aligned}$$

where $C_{k,l,\alpha} = \dfrac{1}{(l-\alpha)(l+1-\alpha)\dots(k-1-\alpha)}$. Thus,

$$D_{k,l,\alpha}\Big\|\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l}{\partial t^l}P_tf\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} \leq \Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_tf\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma},$$

where $D_{k,l,\alpha} = 1/C_{k,l,\alpha}$.

The converse inequality is also obtained by an inductive argument from the case $k = l+1$. Let us recall (7.53),

$$\frac{\partial^k u(x,t)}{\partial t^k} = \int_0^{+\infty} T_s\Big(\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big)\frac{\partial^{k-l}}{\partial t_1^{k-l}}\mu_{t_1}^{(1/2)}(ds),$$

and because, from (3.19), $\dfrac{\partial}{\partial t_1}\mu_{t_1}^{(1/2)}(ds) = \Big(t_1^{-1} - \dfrac{t_1}{2s}\Big)\mu_{t_1}^{(1/2)}(ds)$ we get

$$\begin{aligned}\Big|\frac{\partial^k u(x,t)}{\partial t^k}\Big|&\\ &\leq \int_0^{+\infty} T_s\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)\Big|\Big(t_1^{-1} - \frac{t_1}{2s}\Big)\Big|\mu_{t_1}^{(1/2)}(ds)\\ &\leq t_1^{-1}\int_0^{+\infty} T_s(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|)\mu_{t_1}^{(1/2)}(ds) + \frac{t_1}{2}\int_0^{+\infty} T_s(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|)\frac{1}{s}\mu_{t_1}^{(1/2)}(ds).\end{aligned}$$

Therefore,

$$\begin{aligned}\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}\Big|\frac{\partial^k u(x,t)}{\partial t^k}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}&\\ &\leq C_q\Big[\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\int_0^{+\infty} T_s\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)\mu_{t_1}^{(1/2)}(ds)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\\ &+\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}\frac{t_1}{2}\int_0^{+\infty} T_s\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)\frac{1}{s}\mu_{t_1}^{(1/2)}(ds)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big]\\ &= (I) + (II)\end{aligned}$$

Then, using Minkowski's integral inequality twice (because T_s is an integral transformation with a positive kernel) and the fact that $\mu^{(1/2)}_{t_1}(ds)$ is a probability, we get

$$\begin{aligned}(I) &= C_q\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\int_0^{+\infty}T_s\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)\mu^{(1/2)}_{t_1}(ds)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\\ &\le C_q\int_0^{+\infty}\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(T_s\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\mu^{(1/2)}_{t_1}(ds)\\ &\le C_q\int_0^{+\infty}T_s\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big)\mu^{(1/2)}_{t_1}(ds)\\ &\le C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big)\end{aligned}$$

and, using the same argument for (II) and (3.20), we have

$$\begin{aligned}(II) &\le C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big)\frac{1}{t_1^2}\\ &= C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t_2^{k-\alpha}t_1^{-1}\Big)^q\Big(\Big|\frac{\partial^l P_{t_2}f(x)}{\partial t_2^l}\Big|\Big)^q\frac{dt_2}{t_2}\Big)^{1/q}\Big).\end{aligned}$$

Taking $t_1 = t_2 = \frac{t}{2}$ and changing the variable, we get

$$(I) \le C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big)^q\Big(\Big|\frac{\partial^l P_t f(x)}{\partial t^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big)$$

and

$$(II) \le C_qT^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big)^q\Big(\Big|\frac{\partial^l P_t f(x)}{\partial t^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big).$$

Hence, using the L^p boundedness of T^*

$$\begin{aligned}&\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k u(x,t)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}\\ &\qquad\le C_{q,k,\alpha}\Big\|T^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l P_t f(x)}{\partial u^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big)\Big\|_{p,\gamma}\\ &\qquad\quad+C_q\Big\|T^*\Big(\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l P_t f(x)}{\partial u^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big)\Big\|_{p,\gamma}\Big)\\ &\qquad\le C_{k,\alpha,q}\Big\|\Big(\int_0^{+\infty}\Big(t^{l-\alpha}\Big|\frac{\partial^l P_t f(x)}{\partial t^l}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}.\end{aligned}$$ □

Now, we can introduce the Gaussian Triebel–Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$ following the classical case:

Definition 7.38. *Let $\alpha \geq 0$, k be the smallest integer greater than α, and $1 \leq p, q < \infty$. The Gaussian Triebel–Lizorkin space $F^{\alpha}_{p,q}(\gamma_d)$ is the set of functions $f \in L^p(\gamma_d)$ for which*

$$\left\| \left(\int_0^{\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma} < \infty. \tag{7.58}$$

The norm of $f \in F^{\alpha}_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{F^{\alpha}_{p,q}} := \|f\|_{p,\gamma} + \left\| \left(\int_0^{\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma}. \tag{7.59}$$

Observe that according to Lemma 7.37, the definition of $F^{\alpha,q}_{p}(\gamma_d)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent.

Let us prove now that the Gaussian Triebel–Lizorkin spaces are complete,

Lemma 7.39. *For any $\alpha \geq 0$, $1 \leq p, q < \infty$, the Gaussian Triebel–Lizorkin space $F^{\alpha}_{p,q}(\gamma_d)$ is a Banach space.*

Proof. To prove the completeness, it is enough to see that if (f_n) is a sequence in $F^{\alpha}_{p,q}(\gamma_d)$ such that $\sum_{n=1}^{\infty} \|f_n\|_{F^{\alpha}_{p,q}} < \infty$, then $\sum_{n=1}^{\infty} f_n$ converges in $F^{\alpha}_{p,q}(\gamma_d)$. Since,

$$\sum_{n=1}^{\infty} \|f_n\|_{F^{\alpha}_{p,q}} = \sum_{n=1}^{\infty} \|f_n\|_{p,\gamma} + \left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f_n}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} < \infty.$$

In particular, this implies that

$$\sum_{n=1}^{\infty} \|f_n\|_{p,\gamma} < \infty, \text{ and } \sum_{n=1}^{\infty} \left\| \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f_n}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} \right\|_{p,\gamma} < \infty.$$

But as $L^p(\gamma_d)$ is complete, there exist functions $f, g \in L^p(\gamma_d)$, such that

$$g(x) = \sum_{n=1}^{\infty} |f_n(x)|, \text{ and } \sum_{n=1}^{\infty} f_n(x) = f(x) \quad \text{a.e.}x.$$

Moreover, $\sum_{n=1}^{\infty} f_n = f$ in $L^p(\gamma_d)$. Analogously, there exists $h \in L^p(\gamma_d)$, such that

$$\sum_{n=1}^{\infty} \left(\int_0^{+\infty} \left(t^{k-\alpha} \left| \frac{\partial^k P_t f_n(x)}{\partial t^k} \right| \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} = h(x) \quad \text{a.e.}x,$$

and

$$\sum_{n=1}^{\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f_n}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}=h$$

in $L^p(\gamma_d)$.

We need to prove that $\sum_{n=1}^{\infty} f_n = f$ in $F^{\alpha}_{p,q}$, i.e., $\lim_{n\to\infty}\|\sum_{i=1}^{n} f_i - f\|_{F^{\alpha}_{p,q}} = 0$. Let $h_n(x) = \sum_{i=1}^{n}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f_i(x)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$, then $h(x) = \lim_{n\to\infty} h_n(x)$ a.e.x, and for each x, $\{h_n(x)\}$ is a non-decreasing sequence of real numbers, also $h_n(x)\le h(x)$ a.e. x.

As in the proof of the completeness of the Besov–Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$, we have, using Lebesgue's dominated convergence theorem, for any $t\ge 0$ and $x\in\mathbb{R}^d$,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)=\frac{\partial^k}{\partial t^k}P_t f(x).$$

Now, let us prove that $f\in F^{\alpha}_{p,q}$. In fact, using the triangle inequality and Fatou's lemma,

$$\begin{aligned}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} &= \int_0^{+\infty}\lim_{n\to\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\le \liminf_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\le \liminf_{n\to\infty}\sum_{i=1}^{n}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &= \sum_{n=1}^{\infty}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f_n(x)\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} = h(x)\text{ a.e.}x.\end{aligned}$$

Therefore,

$$\Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}\le\|h\|_{p,\gamma}<\infty.$$

Because for any $t\ge 0$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x)-\frac{\partial^k}{\partial t^k}P_t f(x)\Big| &\le \sum_{i=1}^{n}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big| + \Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\\ &\le \sum_{i=1}^{\infty}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big| + \Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|,\end{aligned}$$

and

$$\int_0^{+\infty} \Big(t^{k-\alpha}\Big(\sum_{i=1}^{\infty}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big| + \Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \sum_{i=1}^{\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} + \int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= h(x) + \int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \leq 2h(x) < \infty \quad \text{a.e.} x,$$

thus, $\Big(t^{k-\alpha}\Big(\sum_{i=1}^{\infty}\Big|\frac{\partial^k}{\partial t^k}P_t f_i(x)\Big| + \Big|\frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)\Big)^q \frac{1}{t}$ is integrable a.e. x, and, therefore, according to Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x) - \frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q \frac{dt}{t}\Big) = 0 \text{ a.e.} x,$$

and,

$$\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i(x) - f(x)\Big)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x) - \frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q \frac{dt}{t}\Big)^{1/q}$$
$$\leq 2h(x),$$

a.e.x, for all $n \in \mathbb{N}$, where $h \in L^p(\gamma_d)$; thus,

$$\lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i(x) - f(x)\Big)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\sum_{i=1}^{n}\frac{\partial^k}{\partial t^k}P_t f_i(x) - \frac{\partial^k}{\partial t^k}P_t f(x)\Big|\Big)^q \frac{dt}{t}\Big) = 0 \quad \text{a.e.} x.$$

Then, again using Lebesgue's dominated convergence theorem,

$$\lim_{n\to\infty}\Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i - f\Big)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}$$
$$\Big\|\lim_{n\to\infty}\int_0^{+\infty}\Big(t^{k-\alpha}|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i - f\Big)|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} = 0.$$

Finally,

$$\lim_{n\to\infty}\Big\|\sum_{i=1}^{n} f_i - f\Big\|_{F^{\alpha}_{p,q}}$$
$$= \lim_{n\to\infty}\left(\Big\|\sum_{i=1}^{n} f_i - f\Big\|_{p,\gamma} + \Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i - f\Big)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}\right)$$
$$= \lim_{n\to\infty}\Big\|\sum_{i=1}^{n} f_i - f\Big\|_{p,\gamma} + \lim_{n\to\infty}\Big\|\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k}{\partial t^k}P_t\Big(\sum_{i=1}^{n} f_i - f\Big)\Big|\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} = 0.$$

□

Observe that using the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley $g_{t,\gamma}$-function (5.13),

$$g_{t,\gamma}(f)(x) = \left(\int_0^{\infty} t\left|\frac{\partial P_t f}{\partial t}\right|^2 dt\right)^{1/2},$$

it can be seen, for $1 < p < \infty$, that

$$L^p(\gamma_d) = F^{0}_{p,2}(\gamma_d).$$

Also, by the trivial identification of the L^p spaces with the Hardy spaces, we have

$$H^p(\gamma_d) = F^{0}_{p,2}(\gamma_d).$$

For Gaussian Triebel–Lizorkin spaces, we have the following inclusion result, which is analogous to Proposition 7.36 *i*:

Proposition 7.40. *The inclusion $F^{\alpha_1}_{p,q_1}(\gamma_d) \subset F^{\alpha_2}_{p,q_2}(\gamma_d)$ holds for $\alpha_1 > \alpha_2 > 0$ and $q_1 \geq q_2$.*

Proof. Let us consider $f \in F_p^{\alpha_1,q_1}(\gamma_d)$. Then,

$$\Big(\int_0^{+\infty}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}}$$
$$= \Big(\int_0^{1}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t} + \int_1^{+\infty}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}}$$
$$\leq \Big(\int_0^{1}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}} + \Big(\int_1^{+\infty}\Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2}\frac{dt}{t}\Big)^{\frac{1}{q_2}}$$
$$= (I) + (II).$$

Let us observe that for the first term I, the result for the case $q_1 = q_2$ is immediate, because, as $t < 1$, $t^{k-\alpha_2} < t^{k-\alpha_1}$ and then

$$(I)^{q_2} \leq \int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}.$$

Now, in the case $q_1 > q_2$, taking $r = \frac{q_1}{q_2}$, $s = \frac{q_1}{q_1 - q_2}$ then $r, s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, then, using Hölder's inequality

$$(I)^{q_2} = \int_0^1 t^{(\alpha_1-\alpha_2)q_2} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2} \frac{dt}{t} \leq \Big(\int_0^1 t^{(\alpha_1-\alpha_2)q_2 s} \frac{dt}{t}\Big)^{\frac{1}{s}} \times \Big(\int_0^1 \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2 r} \frac{dt}{t}\Big)^{\frac{1}{r}}$$
$$= \frac{1}{(\alpha_1-\alpha_2)q_2 s}\Big(\int_0^1 \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{q_2}{q_1}} \leq C\Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{q_2}{q_1}}.$$

Now, for the second term II, using Lemma 3.4, we have

$$(II) = \Big(\int_1^{+\infty} \Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^{q_2} \frac{dt}{t}\Big)^{\frac{1}{q_2}} \leq CT^* f(x)\Big(\int_1^{+\infty} \Big(t^{k-\alpha_2} t^{-k}\Big)^{q_2} \frac{dt}{t}\Big)^{\frac{1}{q_2}}$$
$$= CT^* f(x)\Big(\int_1^{+\infty} t^{-\alpha_2 q_2} \frac{dt}{t}\Big)^{\frac{1}{q_2}} = CT^* f(x).$$

Then, using the $L^p(\gamma_d)$ continuity of T^*, we get

$$\Big\|\Big(\int_0^{+\infty} \Big(t^{k-\alpha_2}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_2} \frac{dt}{t}\Big)^{\frac{1}{q_2}}\Big\|_{p,\gamma}$$
$$\leq C\Big\|\Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{1}{q_1}}\Big\|_{p,\gamma} + C\|T^* f\|_{p,\gamma_d}$$
$$\leq C\Big[\Big\|\Big(\int_0^{+\infty} \Big(t^{k-\alpha_1}|\frac{\partial^k P_t f}{\partial t^k}|\Big)^{q_1} \frac{dt}{t}\Big)^{\frac{1}{q_1}}\Big\|_{p,\gamma} + \|f\|_{p,\gamma}\Big] < +\infty.$$

Thus, $f \in F_p^{\alpha_2,q_2}(\gamma_d)$. □

Observe that the Gaussian Besov–Lipschitz spaces and the Gaussian Triebel–Lizorkin spaces are, by construction, subspaces of $L^p(\gamma_d)$ and the inclusions are trivially continuous.

Additionally, it is clear that for all $t > 0$ and $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial t^k} P_t h_\beta(x) = (-1)^k |\beta|^{k/2} e^{-t\sqrt{|\beta|}} h_\beta(x);$$

therefore,

$$\Big(\int_0^{+\infty} \Big(t^{k-\alpha}\Big\|\frac{\partial^k}{\partial t^k} P_t h_\beta\Big\|_{p,\gamma}\Big)^q \frac{dt}{t}\Big)^{1/q} = \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}}\Big(\Gamma((k-\alpha)q)\Big)^{1/q} \|h_\beta\|_{p,\gamma} < \infty.$$

Thus, $h_\beta \in B^\alpha_{p,q}(\gamma_d)$ and

$$\|h_\beta\|_{B^\alpha_{p,q}} = (1 + \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}}\Big(\Gamma((k-\alpha)q)\Big)^{1/q})\|h_\beta\|_{p,\gamma}.$$

Similarly, $h_\beta \in F^{\alpha}_{p,q}(\gamma_d)$ and

$$\|h_\beta\|_{F^{\alpha}_{p,q}} = (1 + \frac{|\beta|^{\alpha/2}}{q^{k-\alpha}}\Big(\Gamma((k-\alpha)q)\Big)^{1/q})\|h_\beta\|_{p,\gamma} = \|h_\beta\|_{B^{\alpha}_{p,q}}.$$

Therefore, the set of polynomials $\mathscr{P}$ is included in $B^{\alpha}_{p,q}(\gamma_d)$ and in $F^{\alpha}_{p,q}(\gamma_d)$. An open question is to prove whether or not $\mathscr{P}$ is dense in $B^{\alpha}_{p,q}(\gamma_d)$ or $F^{\alpha}_{p,q}(\gamma_d)$.

Also, we have the following inclusion relations between Gaussian Triebel–Lizorkin spaces and Gaussian Besov–Lipschitz spaces:

Proposition 7.41. *Let $\alpha \geq 0$ and $p,q > 1$*

i) If $p = q$, then $F^{\alpha}_{p,p}(\gamma_d) = B^{\alpha}_{p,p}(\gamma_d)$.
ii) If $q > p$, then $F^{\alpha}_{p,q}(\gamma_d) \subset B^{\alpha}_{p,q}(\gamma_d)$.
iii) If $p > q$, then $B^{\alpha}_{p,q}(\gamma_d) \subset F^{\alpha}_{p,q}(\gamma_d)$.

Proof.

i) Using Tonelli's theorem, we trivially have

$$\begin{aligned}\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\Big\|_{p,\gamma} &= \Big(\int_0^{+\infty} t^{(k-\alpha)p}\int_{\mathbb{R}^d}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|^p\gamma_d(dx)\frac{dt}{t}\Big)^{\frac{1}{p}} \\ &= \Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}.\end{aligned}$$

ii) Suppose $q > p$, by Minkowski's integral inequality we then have,

$$\begin{aligned}\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{p/q} &= \Big(\int_0^{\infty} t^{(k-\alpha)q}\Big(\int_{\mathbb{R}^d}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|^p\gamma_d(dx)\Big)^{q/p}\frac{dt}{t}\Big)^{p/q} \\ &\leq \int_{\mathbb{R}^d}\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f(x)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{p/q}\gamma_d(dx).\end{aligned}$$

Therefore,

$$\begin{aligned}\|f\|_{B^{\alpha}_{p,q}} &= \|f\|_{p,\gamma} + \Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} \\ &\leq \|f\|_{p,\gamma} + \Big\|\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} = \|f\|_{F^{\alpha}_{p,q}}.\end{aligned}$$

iii) Finally, if $p > q$, again using Minkowski's integral inequality, we get

$$\begin{aligned}\|f\|_{F^{\alpha}_{p,q}} &= \|f\|_{p,\gamma} + \Big\|\Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} \\ &\leq \|f\|_{p,\gamma} + \Big(\int_0^{\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} = \|f\|_{B^{\alpha}_{p,q}}. \qquad \square\end{aligned}$$

Moreover, Gaussian Sobolev spaces $L^p_\alpha(\gamma_d)$ are contained in some Besov–Lipschitz and Triebel–Lizorkin spaces; therefore, these spaces are "finer scales" for measuring the regularity of functions.

Theorem 7.42. *Let us suppose that* $1 < p < +\infty$ *and* $\alpha > 0$. *Then*

i) $L^p_\alpha(\gamma_d) \subset F^\alpha_{p,2}(\gamma_d)$ *if* $p > 1$.
ii) $L^p_\alpha(\gamma_d) \subset B^\alpha_{p,p}(\gamma_d) = F^\alpha_{p,p}(\gamma_d)$ *if* $p \geq 2$.
iii) $L^p_\alpha(\gamma_d) \subset B^\alpha_{p,2}(\gamma_d)$ *if* $p \leq 2$.

Proof. For the proof of these inclusions, we need to use a characterization of the Gaussian Sobolev spaces, which will be discussed in the next chapter (see 8.21).

i) We have to consider two cases:

i-1) If $\alpha \geq 1$. Suppose $h \in L^p_\alpha(\gamma_d)$ then $h = \mathcal{J}_\alpha f$, $f \in L^p(\gamma_d)$, by the change of variable $u = t+s$, using the fact of the representation of the Bessel potentials (8.20) and Hardy's inequality to get,

$$\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h(x)}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad= \Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big|\frac{\partial^k P_t \mathcal{J}_\alpha f(x)}{\partial t^k}\Big|^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big(\int_0^{+\infty} s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad= \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big(\int_t^{+\infty} (u-t)^{\alpha-1} e^{t-u}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty}\Big(\int_t^{+\infty} u^{\alpha-1}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2 t^{2(k-\alpha)-1}dt\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\frac{1}{k-\alpha}\Big(\int_0^{+\infty}\Big(u^k\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2\frac{du}{u}\Big)^{\frac12}.
\end{aligned}$$

Hence, using the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley $g^k_{t,\gamma}$-function (see Theorem 5.13),

$$\begin{aligned}
\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac12}\Big\|_{p,\gamma} &\leq \frac{1}{\Gamma(\alpha)}\frac{1}{k-\alpha}\Big\|\Big(\int_0^{+\infty}\Big(u^k\Big|\frac{\partial^k P_u f}{\partial u^k}\Big|\Big)^2\frac{du}{u}\Big)^{\frac12}\Big\|_{p,\gamma}\\
&= C_{k,\alpha}\|g_k f\|_{p,\gamma} \leq C_{k,\alpha}\|f\|_{p,\gamma} = C_{k,\alpha}\|h\|_{p,\alpha};
\end{aligned}$$

thus, $h \in F^\alpha_{p,2}(\gamma_d)$.

i-2) If $0 \leq \alpha < 1$. Suppose $h \in L^p_\alpha(\gamma_d)$, then $h = \mathcal{J}_\alpha f$, $f \in L^p(\gamma_d)$, again using (8.20),

$$\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h(x)}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)}\Big(\int_0^{+\infty} s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\frac{dt}{t}\Big)^{\frac12}\\
&\quad\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)-1}\Big[\Big(\int_0^{t} s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\\
&\qquad\qquad + \Big(\int_t^{+\infty} s^\alpha e^{-s}\Big|\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}\Big|\frac{ds}{s}\Big)^2\Big]dt\Big)^{\frac12}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)-1}\Big(\int_0^t s^{\alpha-1}e^{-s}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\Big)^{\frac{1}{2}}\\
&\qquad + \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{2(k-\alpha)-1}\Big(\int_t^{+\infty} s^{\alpha-1}e^{-s}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\Big)^{\frac{1}{2}}\\
&= (I)+(II).
\end{aligned}
$$

Now, because $e^{-s}<1$, $s^{\alpha-1}<t^{\alpha-1}$ as $\alpha<1$, and, using the change of variables $u=t+s$ and Hardy inequality we get,

$$
\begin{aligned}
(II) &\leq \Big(\int_0^{+\infty} t^{2(k-1)-1}\Big(\int_t^{+\infty}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\Big)^{\frac{1}{2}}\\
&= \Big(\int_0^{+\infty} t^{2(k-1)-1}\Big(\int_{2t}^{+\infty}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2 dt\Big)^{\frac{1}{2}}\\
&\leq \Big(\int_0^{+\infty} t^{2(k-1)-1}\Big(\int_t^{+\infty}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|du\Big)^2 dt\Big)^{\frac{1}{2}}\\
&\leq \Big(\int_0^{+\infty}\Big(u\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2 u^{2(k-1)-1}du\Big)^{\frac{1}{2}}.\\
&= \Big(\int_0^{+\infty}\Big|u^k\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2\frac{du}{u}\Big)^{\frac{1}{2}} = g^k_{t,\gamma}f(x).
\end{aligned}
$$

In addition, again using that $e^{-s}<1$, we get

$$
\begin{aligned}
(I)^2 &\leq \int_0^{+\infty} t^{2(k-\alpha)-1}\Big(\int_0^t s^{\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt\\
&= \frac{1}{\alpha^2}\int_0^{+\infty} t^{2k-1}\Big(\frac{\alpha}{t^\alpha}\int_0^t s^{\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|ds\Big)^2 dt
\end{aligned}
$$

Then, as $\alpha>0$ using Jensen's inequality (for the measure $\frac{\alpha}{t^\alpha}s^{\alpha-1}ds$) and Tonelli's theorem,

$$
\begin{aligned}
(I)^2 &\leq \frac{1}{\alpha^2}\int_0^{+\infty} t^{2k-1}\Big(\frac{\alpha}{t^\alpha}\int_0^t s^{\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|^2 ds\Big)dt\\
&\leq \frac{1}{\alpha}\int_0^{+\infty} s^{\alpha-1}\Big(\int_s^{+\infty}(t+s)^{2k-\alpha-1}\Big|\frac{\partial^k P_{t+s}f(x)}{\partial(t+s)^k}\Big|^2 dt\Big)ds,
\end{aligned}
$$

as $2k-\alpha-1>0$. Finally, again using the change of variables $u=t+s$ and the Hardy inequality

$$
\begin{aligned}
(I)^2 &\leq \frac{1}{\alpha}\int_0^{+\infty} s^{\alpha-1}\Big(\int_{2s}^{+\infty} u^{2k-\alpha-1}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2 du\Big)ds\\
&\leq \frac{1}{\alpha}\int_0^{+\infty} s^{\alpha-1}\Big(\int_s^{+\infty} u^{2k-\alpha-1}\Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2 du\Big)ds\\
&\leq \frac{1}{\alpha}\int_0^{+\infty}\Big|u^k\frac{\partial^k P_u f(x)}{\partial u^k}\Big|^2\frac{du}{u} = \frac{1}{\alpha}(g^k_{t,\gamma}f(x))^2.
\end{aligned}
$$

Hence, again using the $L^p(\gamma_d)$-continuity of the Gaussian Littlewood–Paley $g^k_{t,\gamma}$-function,

$$\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t h}{\partial t^k}\Big|\Big)^2\frac{dt}{t}\Big)^{\frac{1}{2}}\Big\|_{p,\gamma}\leq C_{k,\alpha}\|g_k f\|_{p,\gamma}\leq C_{k,\alpha}\|f\|_{p,\gamma}=C_{k,\alpha}\|h\|_{p,\alpha}.$$

Thus, $h\in F^{\alpha}_{p,2}(\gamma_d)$, for $0<\alpha<1$.

ii) Suppose $h\in L^p_\alpha(\gamma_d)$ with $p\geq 2$, then $h=\mathscr{J}_\alpha f$, $f\in L^p(\gamma_d)$. Using the inequality $(a+b)^p\leq C_p(a^p+b^p)$, if $a,b\geq 0, p\geq 1$, we get

$$\begin{aligned}
&\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t\mathscr{J}_\alpha f}{\partial t^k}\Big\|_{p,\gamma}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\qquad\leq\frac{1}{\Gamma(\alpha)}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\int_0^{+\infty}s^\alpha e^{-s}\Big\|\frac{\partial^k P_{t+s}f}{\partial(t+s)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\qquad\leq\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_0^t s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\\
&\qquad\qquad+\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}.
\end{aligned}$$

Using the inequality $(a+b)^{1/p}\leq a^{1/p}+b^{1/p}$ if $a,b\geq 0, p\geq 1$

$$\begin{aligned}
&\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_0^t s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\\
&\qquad\qquad+\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{(k-\alpha)p}\Big(\int_0^t s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\qquad+\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{(k-\alpha)p}\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&=(I)+(II).
\end{aligned}$$

Now, again using Hardy's inequality, because $k>\alpha$ and Lemma 3.5

$$\begin{aligned}
(II)&=\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_{s+t}f}{\partial(s+t)^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty}t^{p(k-\alpha)}\Big(\int_t^{+\infty}s^\alpha\Big\|\frac{\partial^k P_s f}{\partial s^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq\frac{C}{\Gamma(\alpha)}\frac{1}{k-\alpha}\Big(\int_0^{+\infty}\Big(s^\alpha\Big\|\frac{\partial^k}{\partial s^k}P_s f\Big\|_{p,\gamma}\Big)^p s^{(k-\alpha)p-1}ds\Big)^{\frac{1}{p}}\\
&=C_{k,\alpha}\Big(\int_0^{+\infty}\Big(s^k\Big\|\frac{\partial^k}{\partial s^k}P_s f\Big\|_{p,\gamma}\Big)^p\frac{ds}{s}\Big)^{\frac{1}{p}}=C_{k,\alpha}\Big\|\Big(\int_0^{+\infty}\Big|s^k\frac{\partial^k P_s f}{\partial s^k}\Big|^p\frac{ds}{s}\Big)^{\frac{1}{p}}\Big\|_p,
\end{aligned}$$

using Tonelli's theorem.

Now, because $p \geq 2$ using Lemma 3.4, we have

$$\int_0^{+\infty} \Big|u^k \frac{\partial^k P_u f(x)}{\partial u^k}\Big|^p \frac{du}{u} = \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^{p-2} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}$$

$$\leq C\Big(T^* f(x)\Big)^{p-2} \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}.$$

Therefore,

$$\Big\|\Big(\int_0^{+\infty} |u^k \frac{\partial^k P_u f}{\partial u^k}|^p \frac{du}{u}\Big)^{\frac{1}{p}}\Big\|_p^p$$

$$= \int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big|u^k \frac{\partial^k P_u f(x)}{\partial u^k}\Big|^p \frac{du}{u}\Big) \gamma_d(dx)$$

$$\leq C \int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^{p-2} \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2 \frac{du}{u}) \gamma_d(dx)$$

Using Hölder's inequality, with $\theta = \dfrac{2}{p}$, and the $L^p(\gamma_d)$ continuity of T^* and g_k, we have

$$\Big\|\Big(\int_0^{+\infty} \Big|u^k \frac{\partial^k P_u f}{\partial u^k}\Big|^p \frac{du}{u}\Big)^{\frac{1}{p}}\Big\|_p^p$$

$$\leq C \int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^{p-2} \int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k P_u f(x)}{\partial u^k}\Big|\Big)^2 \frac{du}{u}\Big) \gamma_d(dx)$$

$$\leq C\Big(\int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^{(p-2)\cdot\frac{1}{1-\theta}} \gamma_d(dx)\Big)^{1-\theta} \cdot$$

$$\times \Big(\int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}\Big)^{\frac{1}{\theta}} \gamma_d(dx)\Big)^{\theta}$$

$$= C\Big(\int_{\mathbb{R}^d} \Big(\Big(T^* f(x)\Big)^{p} \gamma_d(dx)\Big)^{\frac{p-2}{p}} \cdot$$

$$\times \Big(\int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(u^k \Big|\frac{\partial^k}{\partial u^k} P_u f(x)\Big|\Big)^2 \frac{du}{u}\Big)^{\frac{p}{2}} \gamma_d(dx)\Big)^{\frac{2}{p}}$$

$$= C\|T^* f\|_{p,\gamma}^{p-2} \|g_k f\|_{p,\gamma}^2 \leq C\|f\|_{p,\gamma}^p.$$

Thus,

$$(II) \leq C_{k,\alpha} \|h\|_{p,\alpha}.$$

Now, again using Lemma 3.5 and because $\alpha > 0$

$$
\begin{aligned}
(I) &= \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{p(k-\alpha)}\Big(\int_0^t s^{\alpha}\Big\|\frac{\partial^k}{\partial(s+t)^k}P_{s+t}f\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&\leq \frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^{p(k-\alpha)}\Big(\int_0^t s^{\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^p\frac{dt}{t}\Big)^{\frac{1}{p}}\\
&= \frac{1}{\alpha}\frac{C}{\Gamma(\alpha)}\Big(\int_0^{+\infty} t^k\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}^p\frac{dt}{t}\Big)^{\frac{1}{p}} \leq C_{k,\alpha}\|h\|_{p,\alpha},
\end{aligned}
$$

Thus, $h \in B^{\alpha}_{p,p}(\gamma_d)$, if $p \geq 2$.

iii) This inclusion could be proved using similar arguments as in i) and ii), but it is an immediate consequence of *i*) and of Proposition 7.41 ii). □

In [166], using Theorem 3.2, it is claimed that the Gaussian Sobolev spaces $L^p_\alpha(\gamma_d)$ coincide with the homogeneous Gaussian Triebel–Lizorkin $\dot{F}^{\alpha}_{p,2}$, but the proof of that theorem is wrong because it is assumed that the operator involved is linear; however, it is actually only sublinear.

Now, let us prove some interpolation results for the Gaussian Besov–Lipschitz spaces and for the Gaussian Triebel–Lizorkin Spaces.

Theorem 7.43. *We have the following interpolation results:*

i) For $1 < p_j, q_j < +\infty$ *and* $\alpha_j \geq 0$, *if* $f \in B^{\alpha_j}_{p_j,q_j}(\gamma_d)$, $j = 0,1$, *then* $f \in B^{\alpha}_{p,q}(\gamma_d)$, *where* $\alpha = \alpha_0(1-\theta)+\alpha_1\theta$, *and*

$$\frac{1}{p} = \frac{1}{p_0}(1-\theta)+\frac{\theta}{p_1}, \frac{1}{q} = \frac{1}{q_0}(1-\theta)+\frac{\theta}{q_1},\ 0<\theta<1.$$

ii) For $1 < p_j, q_j < +\infty$ *and* $\alpha_j \geq 0$, *if* $f \in F^{\alpha_j}_{p_j,q_j}(\gamma_d)$, $j = 0,1$, *then* $f \in F^{\alpha}_{p,q}(\gamma_d)$, *where* $\alpha = \alpha_0(1-\theta)+\alpha_1\theta$, *and*

$$\frac{1}{p} = \frac{1}{p_0}(1-\theta)+\frac{\theta}{p_1}, \frac{1}{q} = \frac{1}{q_0}(1-\theta)+\frac{\theta}{q_1},\ 0<\theta<1.$$

Proof. The proof of both results is based on the following interpolation result for $L^p(\gamma_d)$ spaces (actually true for any measure μ) obtained using Hölder's inequality:

For $1 < r_0, r_1 < \infty$ and $\frac{1}{r} = \frac{1}{r_0}(1-\eta)+\frac{\eta}{r_1}, 0<\eta<1$. If $f \in L^{r_j}(\gamma_d)$, $j = 0,1$ then $f \in L^r(\gamma_d)$ and

$$\|f\|_{r,\gamma} \leq \|f\|^{1-\eta}_{r_0,\gamma}\|f\|^{\eta}_{r_1,\gamma}. \tag{7.60}$$

Let us prove *i*). Let k be any integer greater than α_0 and α_1. By using the above result, we get for $\alpha = \alpha_0(1-\theta)+\alpha_1\theta$,

$$\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}$$
$$\leq \int_0^{+\infty}\Big(t^{k-(\alpha_0(1-\theta)+\alpha_1\theta)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{1-\theta}_{p_0,\gamma}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{\theta}_{p_1,\gamma}\Big)^q\frac{dt}{t}$$
$$=\int_0^{+\infty}\Big(t^{(1-\theta)(k-\alpha_0)+\theta(k-\alpha_1)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{1-\theta}_{p_0,\gamma}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|^{\theta}_{p_1,\gamma}\Big)^q\frac{dt}{t}$$
$$=\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_0,\gamma}\Big)^{(1-\theta)q}\Big(t^{k-\alpha_1}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_1,\gamma}\Big)^{\theta q}\frac{dt}{t}.$$

Now, if $\lambda=\dfrac{\theta q}{q_1}$ then $0<\lambda<1$ and $q=(1-\lambda)q_0+\lambda q_1$. Therefore, by using Hölder's inequality again,

$$\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}$$
$$\leq\Big(\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_0,\gamma}\Big)^{q_0}\frac{dt}{t}\Big)^{1-\lambda}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p_1,\gamma}\Big)^{q_1}\frac{dt}{t}\Big)^{\lambda}<\infty;$$

thus $f\in B^{\alpha}_{p,q}(\gamma_d)$.

ii) Analogously, by taking $\beta=\frac{p\theta}{p_1}$, $\lambda=\frac{q\theta}{q_1}$, we have $0<\beta,\lambda<1$ and $p=(1-\beta)p_0+\beta p_1, q=(1-\lambda)q_0+\lambda q_1$. Let k be any integer greater than α_0 and α_1, by using Hölder's inequality we get for $\alpha=\alpha_0(1-\theta)+\alpha_1\theta$,

$$\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}$$
$$=\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{(1-\theta)q}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{\theta q}\frac{dt}{t}$$
$$=\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{(1-\lambda)q_0}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{\lambda q_1}\frac{dt}{t}$$
$$\leq\Big(\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0}\frac{dt}{t}\Big)^{1-\lambda}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\lambda}.$$

Thus,

$$\Big\|\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|^p_{p,\gamma}=\int_{\mathbb{R}^d}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{p}{q}}\gamma_d(dx)$$
$$\leq\int_{\mathbb{R}^d}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0}\frac{dt}{t}\Big)^{\frac{(1-\lambda)p}{q}}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{\lambda p}{q}}\gamma_d(dx)$$
$$=\int_{\mathbb{R}^d}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0}\frac{dt}{t}\Big)^{\frac{(1-\theta)p}{q_0}}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{\theta p}{q_1}}\gamma_d(dx)$$
$$=\int_{\mathbb{R}^d}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_0}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_0}\frac{dt}{t}\Big)^{\frac{(1-\beta)p_0}{q_0}}\Big(\int_0^{+\infty}\Big(t^{k-\alpha_1}\Big|\frac{\partial^k P_t f}{\partial t^k}\Big|\Big)^{q_1}\frac{dt}{t}\Big)^{\frac{\beta p_1}{q_1}}\gamma_d(dx),$$

and then again using Hölder's inequality,

$$\Big\| \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big| \frac{\partial^k P_t f}{\partial t^k} \Big| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}^p$$
$$\leq \Big(\int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_0} \Big| \frac{\partial^k P_t f}{\partial t^k} \Big| \Big)^{q_0} \frac{dt}{t} \Big)^{\frac{p_0}{q_0}} \gamma_d(dx) \Big)^{1-\beta}$$
$$\times \Big(\int_{\mathbb{R}^d} \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1} \Big| \frac{\partial^k P_t f}{\partial t^k} \Big| \Big)^{q_1} \frac{dt}{t} \Big)^{\frac{p_1}{q_1}} \gamma_d(dx) \Big)^{\beta}$$
$$= \Big\| \Big(\int_0^{+\infty} \Big(t^{k-\alpha_0} \Big| \frac{\partial^k P_t f}{\partial t^k} \Big| \Big)^{q_0} \frac{dt}{t} \Big)^{\frac{1}{q_0}} \Big\|_{p_0,\gamma}^{p_0(1-\beta)}$$
$$\times \Big\| \Big(\int_0^{+\infty} \Big(t^{k-\alpha_1} \Big| \frac{\partial^k P_t f}{\partial t^k} \Big| \Big)^{q_1} \frac{dt}{t} \Big)^{\frac{1}{q_1}} \Big\|_{p_1,\gamma}^{p_1\beta} < +\infty.$$

Hence, $f \in F^{\alpha}_{p,q}(\gamma_d)$. □

Finally, we are going to study the continuity properties of the Ornstein–Uhlenbeck semigroup and the Poisson–Hermite semigroup on the Gaussian Besov–Lipschitz and Triebel–Lizorkin spaces. In the next chapter, we consider the boundedness property of other operators on those spaces.

Theorem 7.44. *For The Ornstein–Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ and the Poisson–Hermite semigroup $\{P_t\}_{t\geq 0}$,*

i) Both are bounded on $B^{\alpha}_{p,q}(\gamma_d)$.
ii) Both are bounded on $F^{\alpha}_{p,q}(\gamma_d)$.

Proof.

i) Let us prove the $B^{\alpha}_{p,q}(\gamma_d)$-continuity of P_t for any $t > 0$; the proof for T_t is totally analogous. Using the L^p-continuity of the Poisson–Hermite semigroup, Lebesgue's dominated convergence theorem, and Jensen's inequality, we get

$$\int_{\mathbb{R}^d} \Big| \frac{\partial^k P_t(P_s f)}{\partial t^k}(x) \Big|^p \gamma_d(dx) = \int_{\mathbb{R}^d} \Big| P_s\Big(\frac{\partial^k P_t f}{\partial t^k} \Big)(x) \Big|^p \gamma_d(dx)$$
$$\leq \int_{\mathbb{R}^d} P_s\Big(\Big| \frac{\partial^k P_t f(x)}{\partial t^k} \Big|^p \Big) \gamma_d(dx)$$
$$= \int_{\mathbb{R}^d} \Big| \frac{\partial^k P_t f(x)}{\partial t^k} \Big|^p \gamma_d(dx).$$

Thus,

$$\Big\| \frac{\partial^k P_t(P_s f)}{\partial t^k} \Big\|_{p,\gamma} \leq \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma};$$

therefore,

$$\|P_s f\|_{B^{\alpha}_{p,q}} = \|P_s f\|_{p,\gamma} + \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t(P_s f)}{\partial t^k} \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q}$$
$$\leq \|f\|_{p,\gamma} + \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} = \|f\|_{B^{\alpha}_{p,q}}.$$

ii) Let us prove the $F^{\alpha}_{p,q}$-continuity of P_t for any $t > 0$; the proof for T_t is totally analogous. Using Lebesgue's dominated convergence theorem and Minkowski's integral inequality, we have

$$\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_t(P_s g)}{\partial s^k}(x)\right|\right)^q \frac{ds}{s}\right)^{1/q}$$
$$= \left(\int_0^{\infty}\left(s^{k-\alpha}\left|\int_{\mathbb{R}^d} p(t,x,y)\frac{\partial^k P_s g(y)}{\partial s^k}dy\right|\right)^q \frac{ds}{s}\right)^{1/q}$$
$$\leq \int_{\mathbb{R}^d} p(t,x,y)\left(\int_0^{\infty}(s^{k-\alpha}\left|\frac{\partial^k P_s g(y)}{\partial s^k}\right|\Big)^q \frac{ds}{s}\right)^{1/q} dy$$
$$= P_t\left(\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right)(x).$$

Therefore, by the L^p continuity of P_t we get

$$\Big\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\|_{p,\gamma}$$
$$\leq \|P_t\left(\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right)\Big\|_{p,\gamma}$$
$$\leq \Big\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\Big\|_{p,\gamma}$$

Thus,

$$\|P_t g\|_{F^{\alpha}_{p,q}} = \|P_t g\|_{p,\gamma} + \Big\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\Big\|_{p,\gamma}$$
$$\leq \|g\|_{p,\gamma} + \Big\|\left(\int_0^{\infty}\left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\Big\|_{p,\gamma} = \|g\|_{F^{\alpha}_{p,q}}. \qquad \square$$

7.9 Notes and Further Results

1. In [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, and W. Urbina define and study Sobolev spaces associated with multi-dimensional Laguerre expansions of type α. The result is obtained by means of transference from a Hermite setting using the relationship between Laguerre and Hermite polynomials (see G. Szegő's book [262, (5.6.1)]).

2. In [177], G. Mauceri, S. Meda, and P. Sjögren found a maximal characterization of $H^1_{at}(\gamma_d)$ that unfortunately is only valid for $d = 1$. In the same paper, they give a description of the non-negative functions in $H^1_{at}(\gamma_d)$ and use it to prove that $L^p(\gamma_d) \subset H^1_{at}(\gamma_d)$, for $1 < p \leq \infty$.

3. In 1995, J. Epperson [75] considered Triebel–Lizorkin spaces with respect to the Hermite function expansions. Those spaces are completely different than the spaces that we are considering here, because the reference measure is the Lebesgue measure; therefore it should not be confused with them, because he was working with the Lebesgue measure.

4. In [161], L. Liu and D. Yang consider Gaussian bounded lower oscillation (BLO) spaces $BLO_a(\gamma_d)$, the space of functions with bounded lower oscillation associated with a given class of admissible balls with parameter a.

5. In [166], I. López defines and briefly studies Besov spaces and Triebel–Lizorkin spaces for Hermite and Laguerre expansions. There are some technical problems in the definitions and some gaps in the proofs.

6. More abstract approaches to Besov and Triebel–Lizorkin spaces associated with a general differential operator can be found, for instance, in [154].

7. Hardy spaces for Jacobi expansions have a curious story. The first construction obtained by L. Cafarelli in his doctoral dissertation in 1971, under the direction of C. P. Calderón, [39]. He defined the conjugation as a smooth differential operator, and from there he was able to give a definition of them. Unfortunately, that memoir, which contains very original and novel ideas, for example, the proof that the Jacobi measure is doubling, well before the notion of doubling measure was formulated, was never published. Then, 25 years later, in 1996, Zhongkai Li [157, 158], formulated another definition of Hardy spaces for Jacobi expansions, closely following the work of B. Muckenhoupt and E. Stein [199] in the ultraspherical case.

8. There is a class of spaces that are an intermediate generalization between the classical Lebesgue spaces and the Orlicz spaces; they are the variable Lebesgue spaces, which have been intensively studied over the last 25 years, extending almost all the boundedness properties of classical harmonic analysis operators with respect to the Lebesgue measure (see, for instance, [61] or [66]). For the study of variable Lebesgue spaces with respect to general Radon measures, see [3]. In particular, some results for variable Lebesgue spaces with respect to the Gaussian measure can be found in [63] and [192].

8

Gaussian Fractional Integrals and Fractional Derivatives, and Their Boundedness on Gaussian Function Spaces

In this chapter, we study several important operators in Gaussian harmonic analysis. First, we consider Riesz and Bessel potentials with respect to the Ornstein–Uhlenbeck operator L, and then, Riesz and Bessel fractional derivatives. We study their regularity on Gaussian Lipschitz spaces, on Gaussian Besov–Lipschitz spaces, and on Gaussian Triebel–Lizorkin spaces. The results obtained are essentially similar to the classical results, as mentioned before, the methods of proofs are completely different. The boundedness results for Gaussian Besov–Lipschitz and Triebel–Lizorkin spaces were obtained by A. E. Gatto, E. Pineda, and W. Urbina, and appeared initially in [110] and [111]. These results can be extended to the case of Laguerre and Jacobi expansions by analogous arguments.

8.1 Riesz and Bessel Potentials with Respect to the Gaussian Measure

Gaussian Riesz Potentials

In the classical case, the Riesz potential of order $\beta > 0$ is defined as the negative fractional powers of $-\Delta$,

$$(-\Delta)^{-\beta/2},$$

which means, using Fourier transform, that

$$((-\Delta)^{-\beta/2} f)\hat{}\,(\xi) = (2\pi|\xi|)^{-\beta} \hat{f}(\xi). \tag{8.1}$$

For more details, see [252, 118].

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The Gaussian fractional integrals or Gaussian Riesz potentials can also be defined as negative fractional powers of $(-L)$. However, because the Ornstein–Uhlenbeck operator has eigenvalue 0, the negative powers are not defined on all of $L^2(\gamma_d)$; thus, we need to be more careful with the definition. Let us consider $\Pi_0 f = f - \int_{\mathbb{R}^d} f(y)\gamma_d(dy)$ the $L^2(\gamma_d)$ for $f \in L^2(\gamma_d)$, the orthogonal projection on the orthogonal complement of the eigenspace corresponding to the eigenvalue 0.

Definition 8.1. *The Gaussian fractional integral or Riesz potential of order $\beta > 0$, I_β is defined spectrally as*

$$I_\beta = (-L)^{-\beta/2}\Pi_0, \tag{8.2}$$

which means that for any multi-index ν, $|\nu| > 0$ its action on the Hermite polynomial $\mathbf{H}_\nu$ is given by

$$I_\beta \mathbf{H}_\nu(x) = \frac{1}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x), \tag{8.3}$$

and for $\nu = 0 = (0,\ldots,0)$, $I_\beta(\mathbf{H}_0) = 0$.

By linearity, using the fact that the Hermite polynomials are an algebraic basis of $\mathscr{P}(\mathbb{R}^d)$, I_β can be defined for any polynomial function $f(x) = \sum_\nu \widehat{f}_\gamma(\nu)\mathbf{H}_\nu$ as

$$I_\beta f(x) = \sum_\nu \frac{\widehat{f}_\gamma(\nu)}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x) = \sum_{k\geq 1} \frac{1}{k^{\beta/2}}\mathbf{J}_k f(x). \tag{8.4}$$

and similarly for $f \in L^2(\gamma_d)$.

From (8.4), it is clear that the Gaussian Riesz potentials I_β are the simplest Meyer's multipliers, because in this case

$$m(k) = \frac{1}{k^\beta} = h(\frac{1}{k^\beta}), \tag{8.5}$$

with $h(x) = x$ the identity function.

Proposition 8.2. *The Gaussian Riesz potential I_β, $\beta > 0$, has the following integral representations, for $f \in (\mathbb{R}^d)$ is a polynomial or $f \in C_b^2(\mathbb{R}^d)$,*

$$I_\beta f(x) = \frac{1}{\Gamma(\beta/2)}\int_0^\infty t^{\beta/2-1}T_t(I - \mathbf{J}_0)f(x)\,dt, \tag{8.6}$$

with respect to the Ornstein–Uhlenbeck semigroup, and

$$I_\beta f(x) = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}P_t(I - \mathbf{J}_0)f(x)\,dt, \tag{8.7}$$

with respect to the Poisson–Hermite semigroup, *Proof.* It is enough to prove that (8.6) holds for the Hermite polynomials. By the change of variables $u = |\nu|t$

$$\begin{aligned}\frac{1}{\Gamma(\beta/2)}\int_0^\infty t^{\beta/2-1}(T_t(I-\mathbf{J}_0)\mathbf{H}_\nu)(x)\,dt &= \frac{1}{\Gamma(\beta/2)}\int_0^\infty t^{\beta/2-1}e^{-t|\nu|}\,dt\,\mathbf{H}_\nu(x)\\ &= \frac{1}{\Gamma(\beta/2)}\int_0^\infty \frac{u^{\beta/2-1}}{|\nu|^{\beta/2-1}}e^{-u}\,\frac{du}{|\nu|}\,\mathbf{H}_\nu(x)\\ &= \frac{1}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x).\end{aligned}$$

Then, again as the Hermite polynomials are an algebraic base of the set of polynomials $\mathscr{P}(\mathbb{R}^d)$, the formula holds for any polynomial. It can be proved that (8.6) also holds for $f \in C_b^2(\mathbb{R}^d)$.

Observe that the integral representation (8.7) only means a change of scale, as $I_\beta = [(-L)^{1/2}]^{-\beta}$. Taking the change of variables $u = t\sqrt{|\nu|}$,

$$\begin{aligned}\frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}(P_t(I-\mathbf{J}_0)\mathbf{H}_\nu)(x)\,dt &= \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{-t\sqrt{|\nu|}}\,dt\,\mathbf{H}_\nu(x)\\ &= \frac{1}{\Gamma(\beta)}\int_0^\infty \frac{u^{\beta-1}}{|\nu|^{(\beta-1)/2}}e^{-u}\,\frac{du}{\sqrt{|\nu|}}\,\mathbf{H}_\nu(x)\\ &= \frac{1}{|\nu|^{\beta/2}}\mathbf{H}_\nu(x),\end{aligned}$$

again using that the Hermite polynomials are an algebraic base of the set of polynomials $\mathscr{P}(\mathbb{R}^d)$, □

Following the classical case, in general, we prefer to use the representation of I_β (8.7), using the Poisson–Hermite semigroup. This representation will be crucial later to get several boundedness results to operators associated with L.

On the other hand, let us recall that in the classical case (see [252, Chapter V §1]), Riesz potentials have the following integral representation:

$$(-\Delta)^{-\beta/2}f(x) = C_\beta\int_{\mathbb{R}^d}\frac{f(y)}{|x-y|^{d-\beta}}dy.$$

In the Gaussian case, we can also get an integral representation, as follows:

Theorem 8.3. *The Gaussian Riesz potential* I_β, $\beta > 0$, *has an integral representation,*

$$I_\beta f(x) = \int_{\mathbb{R}^d} N_{\beta/2}(x,y)f(y)dy, \tag{8.8}$$

where the kernel $N_{\beta/2}(x,y)$ *is defined as*

$$N_{\beta/2}(x,y) = \frac{1}{\pi^{d/2}\Gamma(\beta/2)}\int_0^1(-\log r)^{\beta/2-1}\Big(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2}\Big)\frac{dr}{r}. \tag{8.9}$$

Proof. To find the integral representation of I_β, because the negative powers of L do not exist in all of $L^2(\gamma_d)$, we add a small multiple of the identity. Hence, let us consider the operator $(\varepsilon I_d - L)$, where I_d is the identity in $\mathbb{R}^d$ and $\varepsilon > 0$, and let us take its negative powers. The advantage of this trick is that it can be represented as a Laplace transform and this allows us to use the expression for Mehler's kernel $M_t(x,y)$. More precisely, for $\varepsilon > 0$ and $\beta > 0$,

$$(\varepsilon I - L)^{-\beta/2} = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-(\varepsilon I - L)t} dt; \tag{8.10}$$

therefore, the kernel of $(\varepsilon I - L)^{-\beta/2}$ is

$$\begin{aligned} N_{\beta/2,\varepsilon}(x,y) &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} M_t(x,y) dt \\ &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} dt, \end{aligned}$$

because, if $f \in L^1(\gamma_d)$,

$$\begin{aligned} (\varepsilon I - L)^{-\beta/2} f(x) &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-(\varepsilon I - L)t} f(x) dt \\ &= \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^d} \Big(\int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} M_t(x,y) dt \Big) f(x) dy. \end{aligned}$$

As Π_0 is the orthogonal projection of the orthogonal complement of the eigenspace corresponding to the eigenvalue 0, then $\mathbf{J}_0 = I - \Pi_0$, where $\mathbf{J}_0$ is the orthogonal projection on the subspace generated by $\mathbf{H}_0 \equiv 1$ (that is, the constants), and then we have

$$(\varepsilon I - L)^{-\beta/2} \Pi_0 = (\varepsilon I - L)^{-\beta/2} - \varepsilon^{-\beta/2} \mathbf{J}_0.$$

The kernel of $\mathbf{J}_0$ is clearly $\pi^{-d/2} e^{-|y|^2}$ and trivially $\varepsilon^{-\beta} = \int_0^\infty t^{\beta-1} e^{-\varepsilon t} dt$, then the kernel of $(\varepsilon I - L)^{-\beta/2} \Pi_0$ is

$$\frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} e^{-\varepsilon t} \Big(M_t(x,y) - \pi^{-d/2} e^{-|y|^2} \Big) dt.$$

We can take $\varepsilon \to 0$ in the integral above without problems, then

$$I_\beta = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} T_t (I - \mathbf{J}_0) dt.$$

Therefore, the kernel of I_β is given by

$$\begin{aligned} N_{\beta/2}(x,y) &= \frac{1}{\pi^{d/2}\Gamma(\beta/2)} \int_0^\infty t^{\beta/2-1} \Big(\frac{e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}}{(1-e^{-2t})^{d/2}} - e^{-|y|^2} \Big) dt \\ &= \frac{1}{\pi^{d/2}\Gamma(\beta/2)} \int_0^1 (-\log r)^{\beta/2-1} \Big(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \Big) \frac{dr}{r}. \end{aligned} \tag{8.11}$$

taking $r = e^{-t}$. Thus

$$I_\beta f(x) = \int_{\mathbb{R}^d} N_{\beta/2}(x,y) f(y)\,dy.$$

□

In [102], it is proven that these operators are not of weak type $(1,1)$ with respect to γ. On the other hand, the strong type (p,p), for $1 < p < \infty$,

$$||I_\beta||_{p,\gamma_d} \leq C_p ||f||_{p,\gamma_d}, \tag{8.12}$$

follows either directly, from the hypercontractivity property of the Ornstein–Uhlenbeck semigroup, or by applying P. A. Meyer's multiplier theorem, Theorem 6.2.

The classical Riesz potentials are homogeneous (see E. Stein [252, Chapter V (10)]), but it is easy to see that this is not the case for the Gaussian Riesz potentials I_β.

Moreover, it is well-known that the classical Riesz potentials are of strong type (p,q) with $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$, that is to say, the classical Riesz potentials "improve" in the sense that $I_\beta : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ continuously, with $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{d}$. The Gaussian Riesz potentials, however, do not improve integrability. More formally, for any $\beta > 0$ for the Gaussian Riesz potential I_β, there is no $q > p$ such that it sends $L^p(\gamma_d) \to L^q(\gamma_d)$ continuously. This can be proved using the following counterexample, due to L. Forzani and W. Urbina, [87]. For every $a > 0$, let us split I_β as,

$$I_\beta f(x) = I_1 f(x) + I_2 f(x) = \int_{\mathbb{R}^d} N_\beta^1(x,y) f(y)\,dy + \int_{\mathbb{R}^d} N_\beta^2(x,y) f(y)\,dy,$$

where the kernel (8.11) is split into the sum of two parts,

$$N_\beta^1(x,y) = C_{\beta,d} \int_0^{e^{-a}} (-\log r)^{\beta-1} \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \right) \frac{dr}{r}$$

$$N_\beta^2(x,y) = C_{\beta,d} \int_{e^{-a}}^1 (-\log r)^{\beta-1} \left(\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}} - e^{-|y|^2} \right) \frac{dr}{r}.$$

The operator

$$I_1 f(x) = \int_{\mathbb{R}^d} N_\beta^1(x,y) f(y)\,dy$$

can be written as

$$I_1 f(x) = \frac{1}{\Gamma(\beta)} \int_a^{+\infty} t^{\beta-1} T^t \Pi_0 f(x)\,dt,$$

where $T_t = e^{Lt}$ is the Ornstein–Uhlenbeck semigroup (see Chapter 2). Taking into account that T^t is a hypercontractive semigroup, I_1 turns out to be of strong type (p,q), with $q = 1 + (p-1)e^{4t}$.

Additionally, I_2 is an operator defined for every function on $L^p(\gamma_d)$. To prove that it does not improve integrability, it would be enough to show that for every $q > p$ there is a function $f \in L^p(d\gamma)$ such that $I_2 f \notin L^q(\gamma_d)$. Let us take $\frac{1}{q} < c < \frac{1}{p}$ and $f(y) = e^{c|y|^2}\chi_{|y|\geq 1} \in L^p(d\gamma)$.[1] It can be proved (see [86]), that the kernel $N^2_\beta(x,y) \geq Ce^{|x|^2}\frac{e^{|y|^2}}{|y|}$ in the region $\{(x,y): \quad |x| \geq 1, \quad \frac{1}{4}|y|^2 + 1 < |x|^2 < \frac{3}{4}|y|^2\}$. Hence, $I_2(e^{c|x|^2}) \geq \frac{e^{c|x|^2}}{|x|^2}$ for $|x| \geq 1$; therefore, $I_2(e^{c|x|^2}) \notin L^q(\gamma_d)$.

The reason why Gaussian Riesz potentials do not improve integrability is the fact that L satisfies a logarithmic Sobolev inequality and not a Sobolev inequality. Nevertheless, a $L^p \log L(\gamma_d)$ inequality can still be pulled out. Following E. Fabes' suggestion, applying certain techniques used by L. Gross in [119], to prove that hypercontractivity implies a Sobolev logarithmic inequality, we can prove the following result:

Proposition 8.4. *For any $\beta > 0$ the Gaussian Riesz potential I_β maps $L^p(\gamma_d)$ into $L^p \log L(\gamma_d)$ continuously; in other words, the following inequality holds*

$$\int_{\mathbb{R}^d} |I_\beta f(x)|^p \log|I_\beta f(x)|\,\gamma(dx) \leq C\left(\int_{\mathbb{R}^d} |f(x)|^p\, d\gamma + ||f||^p_{p,\gamma}\log||f||_{p,\gamma}\right), \quad (8.13)$$

for each $f \in L^p(\gamma_d)$.

Proof. Indeed, for $\beta > 0$, consider the generalized Poisson–Hermite semigroup $P_t^\beta = e^{-(-L)^\beta t}$, defined in (3.38). Let f be a polynomial, such that $\int_{\mathbb{R}^d} f\, d\gamma = 0$, $I_\beta f \neq 0$, and set $F(t) = P_t^\beta(I_\beta f)$, then for every $t > 0$,

$$\frac{||F(t)||_{1+(p-1)e^{4t},\gamma} - ||F(0)||_{p,\gamma}}{t} \leq \frac{1-1}{t}||I_\beta f||_{p,\gamma} = 0 \quad (8.14)$$

where the above inequality is a consequence of the hypercontractivity of P_t^β. In (8.14) we let $t \to 0^+$ to get

$$\frac{d}{dt}||F(t)||_{1+(p-1)e^{4t},\gamma}\Big|_{t=0} \leq 0 \quad (8.15)$$

Using a lemma proved in [119],

$$\frac{d}{dt}||F(t)||_{1+(p-1)e^{4t},\gamma}\Big|_{t=0} = ||I_\beta f||^{1-p}_{p,\gamma}[p^{-1}4(p-1)(\int_{\mathbb{R}^d}|I_\beta|^p\log|I_\beta|\,d\gamma \quad (8.16)$$
$$- ||I_\beta f||_{p,\gamma}\log||I_\beta f||_{p,\gamma}) + Re\langle F'(0), \operatorname{sgn}(I_\beta f)|I_\beta f|^{p-1}\rangle_\gamma].$$

[1] For $d = 1$ the function f, defined above, is the same as that used by H. Pollard in his famous counterexample in [230].

But $F'(0) = (-L)^{\beta} I_{\beta} f = f$. Now, combining (8.15) and (8.16) we get

$$\int_{\mathbb{R}^d} |I_\beta f(x)|^p \log |I_\beta f(x)| \, d\gamma \leq C(||I_\beta f||_{p,\gamma}^p \log ||I_\beta f||_{p,\gamma} + \langle |f|, |I_\beta f|^{p-1} \rangle_\gamma).$$

By applying Hölder's inequality to the second term of the sum appearing on the right-hand side of the above inequality, and then the $L^p(d\gamma)$ continuity of I_β, we get inequality (8.13). □

Thus, although I_β do not improve in the $L^p(\gamma_d)$ "scale," they do improve in the "logarithmic scale" $L^p(\gamma_d) \log L(\gamma_d)$.

Gaussian Bessel Potentials

Definition 8.5. *The Gaussian Bessel potential of order $\beta > 0$, $\mathscr{J}_\beta$, is defined spectrally as*

$$\mathscr{J}_\beta = (I + \sqrt{-L})^{-\beta}, \tag{8.17}$$

meaning that for the Hermite polynomials we have,

$$\mathscr{J}_\beta \mathbf{H}_\nu(x) = \frac{1}{(1 + \sqrt{|\nu|})^\beta} \mathbf{H}_\nu(x). \tag{8.18}$$

Again, by linearity, $\mathscr{J}_\beta$ can be extended to any polynomial; thus, if $f = \sum_k \mathbf{J}_k f$, then

$$\mathscr{J}_\beta = \sum_k \frac{1}{(1 + \sqrt{|k|})^\beta} \mathbf{J}_k f.$$

From (8.18), it is clear that the Gaussian Bessel potentials $\mathscr{J}_\beta$ are not Meyer's multipliers, but a composition of two Meyer's multipliers, because in this case

$$\frac{1}{(1+\sqrt{k})^\beta} = \left(\frac{1}{\sqrt{k}} + 1\right)^{-\beta} \frac{1}{k^{\beta/2}} = m_1(L)(m_2(L)(k)), \tag{8.19}$$

with $h_1(x) = (1+x)^{-\beta}$ and $h_2(x) = x$.

Using a similar argument to that above (8.7), the Bessel potentials can be represented as

$$\mathscr{J}_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^\beta e^{-t} P_t f(x) \frac{dt}{t} = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^{\beta-1} e^{-t} P_t f(x) dt \tag{8.20}$$

P. A. Meyer's multiplier theorem, Theorem 6.2, shows that $\mathscr{J}_\beta$ is a bounded operator on $L^p(\gamma_d)$, $1 < p < \infty$, and again (8.20) can be extended to $L^p(\gamma_d)$, using the density of the polynomials there.

On the other hand, again using P. A. Meyer's multiplier theorem, Theorem 6.2, we get that the operators

$$\frac{I_\beta}{\mathscr{J}_\beta}, \text{ and } \frac{\mathscr{J}_\beta}{I_\beta}$$

are bounded on every $L^p(\gamma_d), 1 < p < \infty$; because, for instance, for any multi-index ν, $|\nu| > 0$

$$\left(\frac{I_\beta}{\mathscr{J}_\beta}\right)\mathbf{H}_\nu(x) = \left(\frac{(1+\sqrt{|\nu|})^\beta}{|\nu|^{\beta/2}}\right)\mathbf{H}_\nu(x) = \left(\frac{1}{\sqrt{|\nu|}}+1\right)^\beta \mathbf{H}_\nu(x) = h(\frac{1}{|\nu|^{1/2}})\mathbf{H}_\nu(x),$$

with $h(x) = (x+1)^\beta$. These give the relation between the Riesz and Bessel potentials, similar to those in the classical case (see [252, Chapter V. Lemma. 2]).

It is easy to see, from the fact that $\mathscr{J}_\beta$ is a multiplier, that it is also a bijection over the set of polynomials $\mathscr{P}$. Additionally, the Gaussian Sobolev spaces can be characterized in terms of Gaussian Bessel potentials,

Proposition 8.6. *For* $\beta \geq 0$ *and* $1 \leq p < \infty$

$$L^p_\beta(\gamma_d) = \{\mathscr{J}_\beta f : f \in L^p(\gamma_d)\} \tag{8.21}$$

Proof. First of all, observe that $\mathscr{J}_\beta$ maps the family of polynomials $\mathscr{P}(\mathbb{R}^d)$ into itself injectively. Then, as we already know $\mathscr{J}_\beta$ is continuous in $L^p(\gamma_d)$, then we conclude $\mathscr{J}_\beta : L^p(\gamma_d) \to L^p_\alpha(\gamma_d)$ is bijective. □

Moreover, considering the family $\{\mathscr{J}_\beta\}_\beta$ it is easy to see that it is a strongly continuous semigroup on $L^p(\gamma)$, $1 \leq p < \infty$, having as infinitesimal generator $\frac{1}{2}\log(I - L)$.

8.2 Fractional Derivatives with Respect to the Gaussian Measure

Gaussian Riesz Fractional Derivate

In the classical case, fractional derivates for the Laplacian operator are defined as,

$$(-\triangle)^{\beta/2} f(x) = c_\beta \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} \frac{f(x+y)-f(x)}{|y|^{d+\beta}}dy$$

for $0 < \beta < 2$, $c_\beta = \frac{2^\beta \Gamma(d+\beta/2)}{\pi^{d/2}\Gamma(-\beta/2)}$, see [255].

For the case of doubling measures, and more recently for s-dimensional non-doubling measures, this has been generalized by A. E. Gatto, C. Segovia, and S. Vàgi in [108].

On the other hand, observe that

$$\int_{\mathbb{R}^d} \frac{f(x+y)-f(x)}{|y|^{d+\beta}} dy = C_{\beta,d} \int_0^{\infty} t^{-\beta-1} \left(P_t f(x) - f(x)\right) dt, \tag{8.22}$$

where P_t is the classical Poisson semigroup. Then, following the classical case:

Definition 8.7. *The Gaussian Riesz fractional derivative of order $\beta > 0$, D^β is defined spectrally as*

$$D^\beta = (-L)^{\beta/2}, \tag{8.23}$$

meaning that for the Hermite polynomials, we have

$$D^\beta \mathbf{H}_\nu(x) = |\nu|^{\beta/2} \mathbf{H}_\nu(x). \tag{8.24}$$

Thus, by linearity, D^β can be extended to any polynomial (see [164] and [224]).

Now, if f is a polynomial, by the linearity of the operators I_β and D^β, (8.3) and (8.24), we get

$$I_\beta(D^\beta f) = D^\beta(I_\beta f) = \Pi_0 f. \tag{8.25}$$

In the case of $0 < \beta < 1$ we have the following integral representation for f a polynomial,

$$D^\beta f(x) = \frac{1}{c_\beta} \int_0^{\infty} t^{-\beta-1} (I - P_t) f(x) dt, \tag{8.26}$$

where $c_\beta = \int_0^\infty u^{-\beta-1}(1-e^{-u})du$: because for the Hermite polynomials we have, by the change of variables $u = \sqrt{|\nu|}t$,

$$\begin{aligned}
\frac{1}{c_\beta} \int_0^{\infty} t^{-\beta-1} (I - P_t) \mathbf{H}_\nu(x)(x) dt &= \left(\frac{1}{c_\beta} \int_0^{\infty} t^{-\beta-1} \left(e^{-t\sqrt{|\nu|}} - 1 \right) dt \right) \mathbf{H}_\nu(x) \\
&= |\nu|^{\beta/2} \left(\frac{1}{c_\beta} \int_0^{\infty} u^{-\beta-1} \left(e^{-u} - 1 \right) du \right) \mathbf{H}_\nu(x) \\
&= |\nu|^{\beta/2} \mathbf{H}_\nu(x) = D^\beta \mathbf{H}_\nu(x).
\end{aligned}$$

The identity (8.26) is very important in the development of a version of A. P. Calderón's reproduction formula (see Theorem 8.31 below).

Now, if $\beta \geq 1$, let k be the smallest integer greater than β i.e. $k-1 \leq \beta < k$, then the fractional derivative D^β can be represented as

$$D^\beta f = \frac{1}{c_\beta^k} \int_0^{\infty} t^{-\beta-1} (I - P_t)^k f \, dt, \tag{8.27}$$

where $c_\beta^k = \int_0^\infty u^{-\beta-1}(1-e^{-u})^k du$ and f a polynomial function (see [239]).

As was mentioned earlier, fractional derivatives D^β can be used to characterize the Gaussian Sobolev spaces $L^p_\beta(\gamma_d)$. First, we need to extend the fractional derivative operator D^β to all the Gaussian Sobolev spaces $L^p_\beta(\gamma_d)$, $1<p<\infty$. The union of these spaces

$$L_\beta(\gamma_d) := \bigcup_{p>1} L^p_\beta(\gamma_d)$$

is a natural domain of D^β. Observe that the definition of D^β in all the spaces $L^p_\beta(\gamma_d)$, $1<p<\infty$, is based on an application of Meyer's multiplier theorem, Theorem 6.2.

Theorem 8.8. *Let $\beta>0$ and $1<p<\infty$.*

i) If $\{P_n\}_n$ is a sequence of polynomials such that $\lim_{n\to\infty} P_n = f$ in $L^p_\beta(\gamma_d)$, then $\lim_n D^\beta P_n$ exists in $L^p_\beta(\gamma_d)$ and does not depend on the choice of a sequence $\{P_n\}_n$. If $f\in L^p_\beta(\gamma_d)\cap L^r_\beta(\gamma_d)$, then the limit does not depend on the choice of p or r. Thus, the fractional derivative is well defined by

$$D^\beta f = \lim_{n\to\infty} D^\beta P_n \text{ in } L^p_\beta(\gamma_d), \quad as \quad \lim_{n\to\infty} P_n = f \text{ in } L^p_\beta(\gamma_d),$$

$f\in L_\beta(\gamma_d)$, is well defined.

ii) $f\in L^p_\beta(\gamma_d)$ if and only if $D^\beta f\in L^p(\gamma_d)$. Moreover,

$$B_{p,\beta}\,\|f\|_{p,\beta} \le \left\|D^\beta f\right\|_{p,\gamma_d} \le A_{p,\beta}\,\|f\|_{p,\beta}\,. \tag{8.28}$$

Proof. ii) Let f be a polynomial. Then

$$D^\beta f = \sum_{n\ge 0}\left(\frac{n}{1+n}\right)^{\beta/2}\mathbf{J}_n g,$$

where $g=(1-L)^{-\beta/2}f$. Note that g is also a polynomial. Observe that by construction,

$$\|f\|_{p,\beta} = \|g\|_{p,\gamma}.$$

Using Meyer's multiplier theorem, Theorem 6.2, with the holomorphic function $h(z)=(1+z)^{-\beta/2}$, we get

$$\left\|D^\beta f\right\|_{p,\gamma} \le C_1\|g\|_{p,\gamma}.$$

To prove the converse inequality, observe that the polynomial g can be rewritten as

$$g = \sum_{n\ge 0}\left(\frac{n}{1+n}\right)^{\beta/2}\mathbf{J}_n(D^\beta f),$$

and using Meyer's multiplier theorem again we obtain,

$$\|h\|_{p,\gamma} \le C_2\left\|D^\beta f\right\|_{p,\gamma}.$$

Thus, we get (8.28) for polynomials.

i) The completeness of $L^p_\beta(\gamma_d)$ can be proved using (8.28), and the fact that for $r \geq p$ the embedding $L^r_\beta(\gamma_d) \subset L^p_\beta(\gamma_d)$ is continuous. Finally, from there, we can obtain (8.28) for any $f \in L^p_\beta(\gamma_d)$. □

From the previous result and Proposition 7.3, we can immediately obtain a characterization of the Gaussian Sobolev spaces.

Corollary 8.9. *Assume that* $1 < p < \infty$ *and* $\beta > 0$. *Then*

$$L^p_\beta(\gamma_d) = \left\{ f \in L_\beta(\gamma_d) : D^\beta f \in L^p(\gamma_d) \right\}. \tag{8.29}$$

If $\beta = k \in \mathbb{N}$, *then*

$$L^p_k(\gamma_d) = \left\{ f \in L_k(\gamma_d) : D^j f \in L^p(\gamma_d), j \leq k \right\}. \tag{8.30}$$

This characterization of Sobolev spaces is the most common one in the classical case.

Gaussian Bessel Fractional Derivates

We can also define the Gaussian Bessel fractional derivatives, $\mathscr{D}^\beta$.

Definition 8.10. *The Gaussian Bessel fractional derivatives of order* β, $\mathscr{D}^\beta$, *are defined spectrally as*

$$\mathscr{D}^\beta = (I + \sqrt{-L})^\beta, \tag{8.31}$$

which means that for the Hermite polynomials, we have

$$\mathscr{D}^\beta \mathbf{H}_\nu(x) = (1 + \sqrt{|\nu|})^\beta \mathbf{H}_\nu(x); \tag{8.32}$$

thus, by linearity, it can be extended to any polynomial (see [224]).

In the case of $0 < \beta < 1$, we have the following integral representation,

$$\mathscr{D}^\beta f = \frac{1}{c_\beta} \int_0^\infty t^{-\beta-1} (I - e^{-t} P_t) f dt, \tag{8.33}$$

where, as before, $c_\beta = \int_0^\infty u^{-\beta-1}(1 - e^{-u}) du$ and f is a polynomial.

Moreover, if $\beta \geq 1$, let k be the smallest integer greater than β, i.e.. $k - 1 \leq \beta < k$, then we have the following representation of $\mathscr{D}^\beta f$

$$\mathscr{D}^\beta f = \frac{1}{c^k_\beta} \int_0^\infty t^{-\beta-1} (I - e^{-t} P_t)^k f \, dt, \tag{8.34}$$

where $c^k_\beta = \int_0^\infty u^{-\beta-1}(1 - e^{-u})^k du$ and f is a polynomial (see [239]).

8.3 Boundedness of Fractional Integrals and Fractional Derivatives on Gaussian Lipschitz Spaces

The boundedness results in the case of Gaussian Lipschitz spaces initially appeared in A. E. Gatto and W. Urbina's article [109]. First, observe that the Gaussian Riesz potentials are not bounded operators on $L^\infty(\gamma_d)$ and, therefore, not on $Lip_\alpha(\gamma)$ either. Then, to make sense of Riesz potentials on L^∞, we consider, for $\beta > 0$, the *truncated Gaussian Riesz potentials,*

$$I_\beta^T f(x) = \int_0^1 t^{\beta-1} P_t f(x) dt.$$

We want to study the truncated Gaussian Riesz potentials I_β^T on the Gaussian Lipschitz spaces $Lip_\alpha(\gamma_d)$,

Theorem 8.11. *For* $0 < \beta < 1$ *and* $\alpha > 0$, *the Riesz potential of order* β, I_β^T : $Lip_\alpha(\gamma_d) \to Lip_{\alpha+\beta}(\gamma_d)$ *is bounded.*

Proof. Let $f \in Lip_\alpha(\gamma_d)$, i.e., $f \in L^\infty$ such that $\left\| \frac{\partial P_t f}{\partial t} \right\|_{\infty,\gamma_d} \leq A t^{-1+\alpha}$. First, observe that

$$|P_t f(x)| \leq \int_{\mathbb{R}^d} p(t,x,y)|f(y)|dy \leq \|f\|_{\infty,\gamma},$$

that is, $P_t f \in L^\infty$ and then

$$|I_\beta^T f(x)| \leq \int_0^1 t^{\beta-1}|P_t f(x)|dt \leq \int_0^1 t^{\beta-1}\|f\|_{\infty,\gamma} dt = \frac{1}{\beta}\|f\|_{\infty,\gamma}.$$

Therefore, $I_\beta^T f \in L^\infty$. Now, using the semigroup property and Fubini's theorem,

$$P_s I_\beta^T f(x) = \int_{\mathbb{R}^d} p(s,x,y) I_\beta^T f(y) dy = \int_0^1 t^{\beta-1} P_{s+t} f(y) dt = v(x,s).$$

If $\alpha + \beta < 1$, then for $0 \leq s \leq 1$

$$\begin{aligned} \frac{\partial v}{\partial s}(x,s) &= \int_0^1 t^{\beta-1} \frac{\partial}{\partial s} P_{s+t} f(x) dt = \int_0^1 t^{\beta-1} \frac{\partial}{\partial t} P_{s+t} f(x) dt \\ &= \int_0^s t^{\beta-1} \frac{\partial}{\partial t} P_{s+t} f(x) dt + \int_s^1 t^{\beta-1} \frac{\partial}{\partial t} P_{s+t} f(x) dt \\ &= (I) + (II). \end{aligned}$$

Now, for (I), because $t < s$

$$\begin{aligned} |(I)| &\leq \int_0^s t^{\beta-1} |\frac{\partial}{\partial t} P_{s+t} f(x)| dt \leq C \int_0^s t^{\beta-1}(t+s)^{\alpha-1} dt \\ &\leq C s^{\alpha-1} \int_0^s t^{\beta-1} dt = C s^{(\alpha+\beta)-1}, \end{aligned}$$

and, for (II), as $t > s$

$$|(II)| \leq \int_s^1 t^{\beta-1} |\frac{\partial}{\partial t} P_{s+t} f(x)| dt \leq C \int_s^\infty t^{\beta-1}(t+s)^{\alpha-1} dt$$
$$\leq C \int_s^\infty t^{\beta-1} t^{\alpha-1} dt = C s^{(\alpha+\beta)-1}.$$

Thus,

$$\left\| \frac{\partial}{\partial s} I_\beta f \right\|_{\infty,\gamma_d} < C s^{(\alpha+\beta)-1},$$

which implies $I_\beta f \in Lip_{\alpha+\beta}(\gamma_d)$. The general case follows in a similar manner. □

Now, we study the action of the Bessel potentials on the Gaussian Lipschitz spaces $Lip_\alpha(\gamma)$, which is much better than the case of the Riesz potentials:

Theorem 8.12. *Let* $\alpha, \beta > 0$ *then* $\mathscr{J}_\beta$ *is bounded from* $Lip_\alpha(\gamma)$ *to* $Lip_{\alpha+\beta}(\gamma)$.

Proof. Let $f \in Lip_\alpha(\gamma)$ and consider a fixed integer $n > \alpha + \beta$, then

$$\left\| \frac{\partial^n P_t f}{\partial t^n} \right\|_\infty \leq A_\beta(f) t^{-n+\alpha}, \quad t > 0.$$

Using (8.20), the fact that $f \in L^\infty$, and consequently $P_{t+s} f \in L^\infty$, we obtain

$$P_t(\mathscr{J}^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} P_{t+s} f(x)\, ds; \tag{8.35}$$

therefore,

$$\|P_t(\mathscr{J}_\beta f)\|_\infty \leq \|f\|_\infty,$$

i.e. $P_t(\mathscr{J}_\beta f) \in L^\infty$.

Now, we want to verify the Lipschitz condition. Differentiating (8.35), we get

$$\frac{\partial^n P_t(\mathscr{J}_\beta f)(x)}{\partial t^n} = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} \frac{\partial^n P_{t+s} f(x)}{\partial t^n} ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} \frac{\partial^n P_{t+s} f(x)}{\partial (t+s)^n} ds,$$

and this implies

$$\left\| \frac{\partial^n P_t(\mathscr{J}_\beta f)}{\partial t^n} \right\|_\infty \leq \frac{1}{\Gamma(\beta)} \int_0^t s^{\beta-1} e^{-s} \left\| \frac{\partial^n P_{t+s} f}{\partial (t+s)^n} \right\|_\infty ds$$
$$+ \frac{1}{\Gamma(\beta)} \int_t^{+\infty} s^{\beta-1} e^{-s} \left\| \frac{\partial^n P_{t+s} f}{\partial (t+s)^n} \right\|_\infty ds$$
$$= (I) + (II).$$

Because $\beta > 0$ as $t+s > t$,

$$(I) \leq \frac{A_\beta(f)}{\Gamma(\beta)} \int_0^t s^{\beta-1}(t+s)^{-n+\alpha} e^{-s}\, ds$$
$$\leq \frac{A_\beta(f)}{\Gamma(\beta)} t^{-n+\alpha} \int_0^t s^{\beta-1} ds(\gamma) \leq Ct^{-n+\alpha+\beta}\, \|f\|_{Lip_\beta(\gamma)}.$$

On the other hand, because $n > \alpha+\beta$, as $t+s > s$

$$(II) \leq \frac{A_\beta(f)}{\Gamma(\beta)} \int_t^\infty s^{\beta-1} e^{-s}(t+s)^{-n+\alpha}\, ds \leq \frac{A_\beta(f)}{\Gamma(\beta)} \int_t^\infty s^{\beta-1} e^{-s} s^{-n+\alpha}\, ds$$
$$\leq \frac{A_\beta(f)}{\Gamma(\beta)} \int_t^\infty s^{-n+\alpha+\beta-1} ds = CA_\beta(f) t^{-n+\alpha+\beta}.$$

Therefore,

$$\left\| \frac{\partial^n P_t(\mathscr{I}_\beta f)}{\partial t^n} \right\|_\infty \leq CA_\beta(f) t^{-n+\alpha+\beta}, \quad t > 0.$$

Thus, $\mathscr{I}_\beta f \in Lip_{\alpha+\beta}(\gamma)$, and moreover

$$\|\mathscr{I}_\beta f\|_{Lip_{\alpha+\beta}(\gamma)} = \|\mathscr{I}_\beta f\|_{\infty,\gamma} + A_\beta(\mathscr{I}_\beta f)$$
$$\leq \|f\|_{\infty,\gamma} + CA_\beta(f) \leq C\|f\|_{Lip_\beta(\gamma)}.$$

□

Finally, let us study the action of the fractional derivative D^β on the Gaussian Lipschitz spaces.

Theorem 8.13. *For $0 < \beta < \alpha < 1$, the fractional derivate of order β, D^β : $Lip_\alpha(\gamma_d) \to Lip_{\alpha-\beta}(\gamma_d)$ is bounded.*

Proof. Let $f \in Lip_\alpha(\gamma_d)$, i.e., $f \in L^\infty$ such that $\left\| \frac{\partial P_t f}{\partial t} \right\|_{\infty,\gamma} \leq At^{-1+\alpha}$. Observe that using (7.44) and Proposition 7.23, we get

$$|D^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^\infty t^{-\beta-1} |P_t f(x) - f(x)| dt$$
$$= \frac{1}{c_\beta} \int_0^1 t^{-\beta-1} |P_t f(x) - f(x)| dt + \frac{1}{c_\beta} \int_1^\infty t^{-\beta-1} |P_t f(x) - f(x)| dt$$
$$\leq \frac{1}{c_\beta} \int_0^1 t^{-\beta-1} \|P_t f(x) - f(x)\|_{\infty,\gamma} dt + \frac{2\|f\|_{\infty,\gamma}}{c_\beta} \int_1^\infty t^{-\beta-1} dt$$
$$\leq \frac{A_1(f)}{c_\beta} \int_0^1 t^{\alpha-\beta-1} dt + \frac{2\|f\|_{\infty,\gamma}}{c_\beta} \int_1^\infty t^{-\beta-1} dt$$
$$= \frac{A_1(f)}{c_\beta(\alpha-\beta)} + \frac{2\|f\|_{\infty,\gamma}}{\beta c_\beta} \leq C_{\beta,\beta} \|f\|_{Lip_\beta(\gamma)}.$$

Thus, $D^\beta f \in L^\infty(\gamma_d)$. Now, using (8.26), and fixing s, we have

$$\begin{aligned}
\frac{\partial}{\partial s}(P_s D^\beta f(x)) &= \frac{1}{c_\beta}\frac{\partial}{\partial s}\int_0^\infty t^{-\beta-1}[P_{s+t}f(x) - P_s f(x)]dt \\
&= \frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}[\frac{\partial}{\partial s}(P_{s+t}f(x)) - \frac{\partial}{\partial s}P_s f(x)]dt \\
&= \frac{1}{c_\beta}\int_0^s t^{-\beta-1}[\frac{\partial}{\partial s}(P_{s+t}f(x)) - \frac{\partial}{\partial s}P_s f(x)]dt \\
&\quad + \frac{1}{c_\beta}\int_s^\infty t^{-\beta-1}[\frac{\partial}{\partial s}(P_{s+t}f(x)) - \frac{\partial}{\partial s}P_s f(x)]dt \\
&= (I) + (II).
\end{aligned}$$

Using Proposition 7.27, we have

$$\left\|\frac{\partial^2}{\partial u^2}P_u f\right\|_{\infty,\gamma_d} \le A u^{\alpha-2}, \tag{8.36}$$

and then, using the fundamental theorem of calculus, we get

$$\begin{aligned}
|\frac{\partial}{\partial s}(P_{s+t}f(x)) - \frac{\partial}{\partial s}P_s f(x)| &\le \int_s^{s+t}|\frac{\partial^2}{\partial u^2}P_u f(x)|du \le A\int_s^{s+t}u^{\alpha-2}du \\
&\le A\int_s^\infty u^{\alpha-2}du \le \frac{A}{1-\alpha}s^{\alpha-1}.
\end{aligned}$$

Then, as $t < s$,

$$\begin{aligned}
|(I)| &\le \frac{1}{c_\beta}\int_0^s t^{-\beta-1}|\frac{\partial}{\partial s}(P_{s+t}f(x)) - \frac{\partial}{\partial s}P_s f(x)|dt \\
&\le A\frac{s^{-1}}{c_\beta}\int_0^s t^{-\beta-1}s^\alpha dt \le C_{\alpha,\beta}s^{-1}\int_0^s t^{\alpha-\beta-1}dt = C_{\alpha,\beta}\, s^{\alpha-\beta-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|(II)| &\le \frac{1}{c_\beta}\int_s^\infty t^{-\beta-1}|\frac{\partial}{\partial s}(P_{s+t}f(x)) - \frac{\partial}{\partial s}P_s f(x)|dt \\
&\le \frac{As^{\alpha-1}}{(\beta-1)c_\beta}\int_s^\infty t^{-\beta-1}dt = C_{\alpha,\beta}\, s^{\alpha-\beta-1}.
\end{aligned}$$

Therefore,

$$\left\|\frac{\partial}{\partial s}(P_s D^\beta f)\right\|_{\infty,\gamma_d} \le C s^{\alpha-\beta-1},$$

which implies $D^\beta f \in Lip_{\alpha-\beta}(\gamma_d)$. □

8.4 Boundedness of Fractional Integrals and Fractional Derivatives on Gaussian Besov–Lipschitz Spaces

As we discussed in the previous section, in the case of the Lipschitz spaces only a truncated version of the Riesz potentials is bounded from $Lip_\alpha(\gamma_d)$ to $Lip_{\alpha+\beta}(\gamma_d)$. Now, we study the boundedness properties of the Riesz potentials on Besov–Lipschitz spaces, and we see that in this case, the results are better.

Theorem 8.14. *Let $\alpha \geq 0, \beta > 0$, $1 < p < \infty, 1 \leq q \leq \infty$ then I_β is bounded from $B^{\alpha}_{p,q}(\gamma_d)$ into $B^{\alpha+\beta}_{p,q}(\gamma_d)$.*

Proof. Let $k > \alpha+\beta$ a fixed integer, $f \in B^{\alpha}_{p,q}(\gamma_d)$, using the integral representation of Riesz potentials (8.7), the semigroup property of $\{P_t\}_{t\geq 0}$ and the fact that $P_\infty f(x)$ is a constant and the semigroup is conservative, we get

$$\begin{aligned} P_t I_\beta f(x) &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} P_t(P_s f(x) - P_\infty f(x)) ds \\ &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} (P_{t+s} f(x) - P_\infty f(x)) ds. \end{aligned} \tag{8.37}$$

Using the fact that $P_\infty f(x)$ is a constant again, and the chain rule,

$$\begin{aligned} \frac{\partial^k}{\partial t^k}(P_t I_\beta f)(x) &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \frac{\partial^k}{\partial t^k}(P_{t+s} f(x) - P_\infty f(x))\, ds \\ &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} u^{(k)}(x, t+s)\, ds. \end{aligned} \tag{8.38}$$

Then, using Minkowski's integral inequality

$$\Big\| \frac{\partial^k}{\partial t^k} P_t I_\beta f \Big\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \|u^{(k)}(\cdot, t+s)\|_{p,\gamma} ds. \tag{8.39}$$

Hence, if $1 \leq q < \infty$,

$$\begin{aligned} &\Big(\int_0^{+\infty} \Big(t^{k-(\alpha+\beta)} \Big\| \frac{\partial^k}{\partial t^k}(P_t I_\beta f) \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\quad \leq \frac{1}{\Gamma(\beta)} \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big(\int_0^{+\infty} s^{\beta-1} \|u^{(k)}(\cdot, t+s)\|_{p,\gamma} ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\quad \leq C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big(\int_0^{t} s^{\beta-1} \|u^{(k)}(\cdot, t+s)\|_{p,\gamma} ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\qquad + C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big(\int_t^{+\infty} s^{\beta-1} \|u^{(k)}(\cdot, t+s)\|_{p,\gamma} ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} = (I) + (II). \end{aligned}$$

Now, as $\beta > 0$ using Lemma 3.5, and as $t+s>t$,

$$(I) \leq C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big(\int_0^t s^{\beta-1} \|u^{(k)}(\cdot,t)\|_{p,\gamma} ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$= C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma}^q \Big(\frac{t^\beta}{\beta} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$= C'_\beta \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} < +\infty,$$

because $f \in B_p^{\alpha,q}(\gamma_d)$.

On the other hand, as $k > \alpha+\beta$ using Lemma 3.5 again, because $t+s>s$, and Hardy's inequality (10.101), we obtain

$$(II) \leq C_\beta \Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q} \Big(\int_t^{+\infty} s^\beta \|u^{(k)}(\cdot,s)\|_{p,\gamma} \frac{ds}{s} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq \frac{C_\beta}{k-(\alpha+\beta)} \int_0^{+\infty} \Big(s^{k-\alpha} \Big\| \frac{\partial^k P_s f}{\partial s^k} \Big\|_{p,\gamma} \Big)^q \frac{ds}{s} \Big)^{\frac{1}{q}} < +\infty,$$

because $f \in B_{p,q}^\alpha(\gamma_d)$. Therefore, $I_\beta f \in B_{p,q}^{\alpha+\beta}(\gamma_d)$ and, moreover,

$$\|I_\beta f\|_{B_{p,q}^{\alpha+\beta}} = \|I_\beta f\|_{p,\gamma} + \Big(\int_0^{+\infty} \Big(t^{k-(\alpha+\beta)} \Big\| \frac{\partial^k}{\partial t^k} (P_t I_\beta f) \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}}$$
$$\leq C\|f\|_{p,\gamma} + C_{\alpha,\beta} \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \leq C\|f\|_{B_{p,q}^\beta}.$$

Now, if $q=\infty$, (8.39) can be written as

$$\Big\| \frac{\partial^k}{\partial t^k} P_t I_\beta f \Big\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^t s^{\beta-1} \|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds$$
$$+ \frac{1}{\Gamma(\beta)} \int_t^\infty s^{\beta-1} \|u^{(k)}(\cdot,t+s)\|_{p,\gamma} ds$$
$$= (I) + (II).$$

Using that $\beta>0$, Lemma 3.5, as $t+s>t$, and because $f \in B_{p,\infty}^\alpha(\gamma_d)$,

$$(I) \leq \frac{1}{\Gamma(\beta)} \Big\| \frac{\partial^k P_t f}{\partial t^k} \Big\|_{p,\gamma} \int_0^t s^{\beta-1} ds \leq \frac{1}{\Gamma(\beta)} \frac{t^\beta}{\beta} A_k(f) t^{-k+\alpha} = C_\beta A_k(f) t^{-k+\alpha+\beta}.$$

Now, because $k > \alpha+\beta$, using Lemma 3.5, as $t+s > s$, and because $f \in B^{\alpha}_{p,\infty}(\gamma_d)$, we get

$$(II) \leq \frac{1}{\Gamma(\beta)} \int_t^{\infty} s^{\beta-1} \Big\| \frac{\partial^k P_s f}{\partial s^k} \Big\|_{p,\gamma} ds \leq \frac{A_k(f)}{\Gamma(\beta)} \int_t^{\infty} s^{-k+\alpha+\beta-1} ds$$
$$= \frac{A_k(f)}{\Gamma(\beta)} \frac{t^{-k+\alpha+\beta}}{k-(\alpha+\beta)} = C_{k,\alpha,\beta}\, t^{-k+\alpha+\beta}.$$

Therefore,

$$\Big\| \frac{\partial^k}{\partial t^k} P_t I_\beta f \Big\|_{p,\gamma} \leq C A_k(f) t^{-k+\alpha+\beta}, \quad t > 0,$$

and this implies that $I_\beta f \in B^{\alpha+\beta}_{p,\infty}(\gamma_d)$ and $A_k(I_\beta f) \leq C A_k(f)$.

Moreover, as I_β is a bounded operator on $L^p(\gamma_d), 1 < p < \infty$,

$$\|I_\beta f\|_{B^{\alpha+\beta}_{p,\infty}} = \|I_\beta f\|_{p,\gamma} + A_k(I_\beta f) \leq \|f\|_{p,\gamma} + C A_k(f) \leq C \|f\|_{B^{\alpha}_{p,\infty}}. \qquad \square$$

Now, we are going to study the boundedness properties of the Bessel potentials on Besov–Lipschitz spaces.

Theorem 8.15. *Let $\alpha \geq 0$, $1 \leq p,q < \infty$, then for $\beta > 0$,*

i) $\mathscr{J}_\beta$ is bounded on $B^{\alpha}_{p,q}(\gamma_d)$.
ii) Moreover, $\mathscr{J}_\beta$ is bounded from $B^{\alpha}_{p,q}(\gamma_d)$ to $B^{\alpha+\beta}_{p,q}(\gamma_d)$.
iii) Finally, for $q = \infty$, $\mathscr{J}_\beta$ is bounded from $B^{\alpha}_{p,\infty}(\gamma_d)$ into $B^{\alpha+\beta}_{p,\infty}(\gamma_d)$.

Proof.

i) Let us see that $\mathscr{J}_\beta$ is bounded on $B^{\alpha}_{p,q}(\gamma_d)$. Using Lebesgue's dominated convergence theorem, Minkowski's integral inequality, and Jensen's inequality, we have

$$\Big\| \frac{\partial^k P_t}{\partial t^k} \Big(\mathscr{J}_\beta f \Big) \Big\|^q_{p,\gamma_d} = \Big(\int_{\mathbb{R}^d} \Big| \frac{\partial^k P_t}{\partial t^k} \Big(\frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta e^{-s} P_s f(x) \frac{ds}{s} \Big) \Big|^p \gamma_d(dx) \Big)^{\frac{q}{p}}$$
$$\leq \Big(\frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta e^{-s} \Big(\int_{\mathbb{R}^d} \Big| \frac{\partial^k P_t P_s f(x)}{\partial t^k} \Big|^p \gamma_d(dx) \Big)^{\frac{1}{p}} \frac{ds}{s} \Big)^q$$
$$\leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta e^{-s} \Big\| \frac{\partial^k P_t P_s f}{\partial t^k} \Big\|^q_{p,\gamma_d} \frac{ds}{s},$$

and then, using Tonelli's theorem,

$$\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t}{\partial t^k} \Big(\mathscr{J}_\beta f \Big) \Big\|_{p,\gamma_d} \Big)^q \frac{dt}{t}$$
$$\leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\alpha e^{-s} \Big(\int_0^{+\infty} \Big(t^{k-\alpha} \Big\| \frac{\partial^k P_t \Big(P_s f \Big)}{\partial t^k} \Big\|_{p,\gamma_d} \Big)^q \frac{dt}{t} \Big) \frac{ds}{s}$$

$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\alpha}e^{-s}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma_d}\Big)^q\frac{dt}{t}\Big)\frac{ds}{s}$$
$$= \int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma_d}\Big)^q\frac{dt}{t}.$$

Therefore,

$$\|\mathscr{I}_\beta f\|_{B^{\alpha}_{p,q}} = \|\mathscr{I}_\beta f\|_{p,\gamma_d} + \int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t}{\partial t^k}\big(\mathscr{I}_\beta f\big)\Big\|_{p,\gamma_d}\Big)^q\frac{dt}{t}$$
$$\leq \|f\|_{p,\gamma_d} + \int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma_d}\Big)^q\frac{dt}{t} = \|f\|_{B^{\alpha}_{p,q}}.$$

ii) We use the notation $u(x,t) = P_t f(x)$ and $U(x,t) = P_t\mathscr{I}_\beta f(x)$, using the representation (3.8) of P_t we have,

$$U(x,t) = \int_0^{+\infty} T_s(\mathscr{I}_\beta f)(x)\mu_t^{(1/2)}(ds).$$

Therefore,

$$U(x,t_1+t_2) = P_{t_1}(P_{t_2}(\mathscr{I}_\beta f))(x) = \int_0^{+\infty} T_s(P_{t_2}(\mathscr{I}_\beta f))(x)\mu_{t_1}^{(1/2)}(ds).$$

Now, let k,l be integers greater than α,β respectively, by differentiating k times with respect to t_2 and l times with respect to t_1,

$$\frac{\partial^{k+l}U(x,t_1+t_2)}{\partial(t_1+t_2)^{k+l}} = \int_0^{+\infty} T_s\Big(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f)\Big)(x)\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds).$$

Thus,

$$\frac{\partial^{k+l}U(x,t)}{\partial t^{k+l}} = \int_0^{+\infty} T_s\Big(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f)\Big)(x)\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds),$$

if $t = t_1+t_2$ and therefore, using the L^p continuity of T_s and (3.21)

$$\Big\|\frac{\partial^{k+l}U(\cdot,t)}{\partial t^{k+l}}\Big\|_{p,\gamma} \leq \int_0^{+\infty}\Big\|T_s\Big(\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f)\Big)\Big\|_{p,\gamma}\Big|\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds)\Big|$$
$$\leq \int_0^{+\infty}\Big\|\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f)\Big\|_{p,\gamma}\Big|\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds)\Big|$$
$$= \Big\|\frac{\partial^k P_{t_2}}{\partial t_2^k}(\mathscr{I}_\beta f)\Big\|_{p,\gamma}\int_0^{+\infty}\Big|\frac{\partial^l}{\partial t_1^l}\mu_{t_1}^{(1/2)}(ds)\Big|$$
$$\leq C t_1^{-l}\Big\|\frac{\partial^k}{\partial t_2^k}P_{t_2}\mathscr{I}_\beta f\Big\|_{p,\gamma}. \tag{8.40}$$

On the other hand, using the representation of Bessel potential (8.20), we have

$$P_t(\mathscr{I}_\beta f)(x) = \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s} P_{t+s}f(x)\frac{ds}{s}$$

then

$$\begin{aligned}\frac{\partial^k P_t}{\partial t^k}(\mathscr{I}_\beta f)(x) &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\frac{\partial^k P_{t+s}f(x)}{\partial t^k}\frac{ds}{s}\\ &= \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}\frac{ds}{s},\end{aligned}$$

and this implies that

$$\Big\|\frac{\partial^k P_t}{\partial t^k}(\mathscr{I}_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\Big\|\frac{\partial^k P_{t+s}f}{\partial (t+s)^k}\Big\|_{p,\gamma}\frac{ds}{s} < \infty,$$

because $f\in B^\alpha_{p,q}(\gamma_d)$. Now, because the definition of $B^\alpha_{p,q}(\gamma_d)$ is independent on the integer $k>\alpha$ that we can choose, let us take $k>\alpha+\beta$ and $l>\beta$, then $k+l>\alpha+2\beta>\alpha+\beta$; thus, $k+l$ is an integer greater than $\alpha+\beta$. Let us now see that

$$\Big(\int_0^{+\infty}\Big(t^{k+l-(\alpha+\beta)}\Big\|\frac{\partial^{k+l}U(\cdot,t)}{\partial t^{k+l}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} < +\infty.$$

In fact, taking $t_1=t_2=t/2$ in (8.40), we get

$$\begin{aligned}&\Big(\int_0^{+\infty}\Big(t^{k+l-(\alpha+\beta)}\Big\|\frac{\partial^{k+l}U(\cdot,t)}{\partial t^{k+l}}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq C\Big(\int_0^{+\infty}\Big(t^{k+l-(\alpha+\beta)}\Big\|\frac{\partial^k P_{\frac{t}{2}}}{\partial(\frac{t}{2})^k}(\mathscr{I}_\beta f)\Big\|_{p,\gamma}(\frac{t}{2})^{-l}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}\Big(\int_0^{+\infty}s^\beta e^{-s}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\ &\leq \frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty}t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\\ &\qquad+\Big(\int_t^{+\infty}s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}.\end{aligned}$$

Again using that $(a+b)^q\leq C_q(a^q+b^q)$ if $a,b\geq 0, q\geq 1$, but because $(a+b)^{1/q}\leq a^{1/q}+b^{1/q}$ if $a,b\geq 0, q\geq 1$,

$$\begin{aligned}&\frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty}t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\\ &\qquad+\Big(\int_t^{+\infty}s^\beta\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}\end{aligned}$$

$$\leq \frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{1/q}$$
$$+\frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_t^{+\infty} s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}$$
$$= (I)+(II).$$

Now, using Lemma 3.5 and because $\beta > 0$

$$\begin{aligned}(I) &= \frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&\leq \frac{C}{\Gamma(\beta)}\Big[\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_0^t s^{\beta}\Big\|\frac{\partial^k P_{\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&= \frac{C}{\beta\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(t^{k-\alpha}\Big\|\frac{\partial^k P_{\frac{t}{2}}f}{\partial(\frac{t}{2})^k}\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&= C_{\alpha,\beta}\Big(\int_0^{+\infty}\Big(u^{k-\alpha}\Big\|\frac{\partial^k P_u f}{\partial u^k}\Big\|_{p,\gamma}\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}} < +\infty,\end{aligned}$$

because $f \in B_p^{\alpha,q}(\gamma_d)$.

On the other hand, using Hardy inequality, because $k > \alpha+\beta$ and Lemma 3.5, we get

$$\begin{aligned}(II) &= \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_t^{+\infty} s^{\beta}\Big\|\frac{\partial^k P_{s+\frac{t}{2}}f}{\partial(s+\frac{t}{2})^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{(k-(\alpha+\beta))q}\Big(\int_t^{+\infty} s^{\beta}\Big\|\frac{\partial^k P_s f}{\partial s^k}\Big\|_{p,\gamma}\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\\
&\leq \frac{C}{\Gamma(\beta)}\frac{1}{k-(\alpha+\beta)}\int_0^{+\infty}\Big(s^{k-\alpha}\Big\|\frac{\partial^k}{\partial s^k}P_s f\Big\|_{p,\gamma}\Big)^q\frac{ds}{s}\Big)^{\frac{1}{q}} < +\infty\end{aligned}$$

because $f \in B_{p,q}^{\alpha}(\gamma_d)$. Thus, $\mathscr{I}_\beta f \in B_{p,q}^{\alpha+\beta}(\gamma_d)$ and, moreover,

$$\|\mathscr{I}_\beta f\|_{B_{p,q}^{\alpha+\beta}} \leq C_{\alpha,\beta}\|f\|_{B_{p,q}^{\beta}}.$$

iii) Let $k > \alpha+\beta$ a fixed integer, $f \in B_{p,\infty}^{\alpha}(\gamma_d)$, by using the representation of Bessel potential (8.20), we get

$$P_t(\mathscr{I}_\beta f)(x) = \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta}e^{-s}P_{t+s}f(x)\frac{ds}{s};$$

thus, using the chain rule, we obtain

$$\frac{\partial^k}{\partial t^k}P_t(\mathscr{I}_\beta f)(x) = \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta}e^{-s}u^{(k)}(x,t+s)\frac{ds}{s},$$

which implies, using Minkowski's integral inequality,

$$\Big\|\frac{\partial^k}{\partial t^k}P_t(\mathscr{I}_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma}\frac{ds}{s}$$
$$= \frac{1}{\Gamma(\beta)}\int_0^{t} s^\beta e^{-s}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma}\frac{ds}{s}$$
$$+\frac{1}{\Gamma(\beta)}\int_t^{\infty} s^\beta e^{-s}\|u^{(k)}(\cdot,t+s)\|_{p,\gamma}\frac{ds}{s} = (I)+(II).$$

Now, as $\beta > 0$, using Lemma 3.5, as $t+s > t$, and because $f \in B^\beta_{p,\infty}(\gamma_d)$,

$$(I) \leq \frac{1}{\Gamma(\beta)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\int_0^t s^\beta e^{-s}\frac{ds}{s} \leq \frac{1}{\Gamma(\beta)}\Big\|\frac{\partial^k P_t f}{\partial t^k}\Big\|_{p,\gamma}\int_0^t s^{\beta-1}ds$$
$$\leq \frac{1}{\Gamma(\beta)}\frac{t^\beta}{\beta}A_k(f)t^{-k+\alpha} = C_\beta A_k(f)t^{-k+\alpha+\beta}.$$

On the other hand, as $k > \alpha+\beta$ using Lemma 3.5, as $t+s > s$, and because $f \in B^\alpha_{p,\infty}(\gamma_d)$

$$(II) \leq \frac{1}{\Gamma(\beta)}\int_t^\infty s^\beta e^{-s}\Big\|\frac{\partial^k P_s f}{\partial s^k}\Big\|_{p,\gamma}\frac{ds}{s} \leq \frac{A_k(f)}{\Gamma(\beta)}\int_t^\infty s^\beta e^{-s}s^{-k+\alpha}\frac{ds}{s}$$
$$\leq \frac{A_k(f)}{\Gamma(\beta)}\int_t^\infty s^{-k+\alpha+\beta-1}ds = \frac{A_k(f)}{\Gamma(\beta)}\frac{t^{-k+\alpha+\beta}}{k-(\alpha+\beta)} = C_{k,\alpha,\beta}A_k(f)t^{-k+\alpha+\beta}.$$

Therefore,

$$\Big\|\frac{\partial^k}{\partial t^k}P_t(\mathscr{I}_\beta f)\Big\|_{p,\gamma} \leq CA_k(f)t^{-k+\alpha+\beta},$$

then $\mathscr{I}_\beta f \in B^{\alpha+\beta}_{p,\infty}(\gamma_d)$ and $A_k(\mathscr{I}_\beta f) \leq CA_k(f)$. Thus,

$$\Big\|\mathscr{I}_\beta f\Big\|_{B^{\alpha+\beta}_{p,\infty}} = \Big\|\mathscr{I}_\beta f\Big\|_{p,\gamma} + A_k(\mathscr{I}_\beta f) \leq \|f\|_{p,\gamma} + CA_k(f) \leq C\|f\|_{B^\alpha_{p,\infty}}.$$

□

Now, we study the boundedness of the Riesz fractional derivatives and of the Bessel fractional derivatives on Besov–Lipschitz spaces. We use the representation (8.24) of the fractional derivative and Hardy's inequalities. Because they require different techniques, we consider two cases:

- The bounded case, $0 < \beta < \alpha < 1$.
- The unbounded case $0 < \beta < \alpha$.

Let us start with the bounded case for the Riesz derivative:

Theorem 8.16. *Let* $0 < \beta < \alpha < 1$, $1 \leq p < \infty$ *and* $1 \leq q \leq \infty$ *then* D^β *is bounded from* $B^\alpha_{p,q}(\gamma_d)$ *into* $B^{\alpha-\beta}_{p,q}(\gamma_d)$.

Proof. Let $f \in B^{\alpha}_{p,q}(\gamma_d)$, using Hardy's inequality (10.100), with $p = 1$, and the fundamental theorem of calculus,

$$|D^{\beta} f(x)| \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|P_s f(x) - f(x)|ds \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^{s}|\frac{\partial}{\partial r}P_r f(x)|dr\,ds$$
$$\leq \frac{1}{c_\beta \beta}\int_0^{+\infty} r^{1-\beta}|\frac{\partial}{\partial r}P_r f(x)|\frac{dr}{r}. \tag{8.41}$$

Thus, using Minkowski's integral inequality

$$\left\|D^{\beta} f\right\|_{p,\gamma} \leq C_\beta \int_0^{+\infty} r^{1-\beta}\left\|\frac{\partial}{\partial r}P_r f\right\|_{p,\gamma}\frac{dr}{r} < \infty, \tag{8.42}$$

because $f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\beta}_{p,1}(\gamma_d)$, $1 \leq q \leq \infty$ as $\alpha > \beta$, i.e., $D_\beta f \in L^p(\gamma_d)$.

Now, by analogous argument

$$\frac{\partial}{\partial t}P_t(D^{\beta} f)(x) = \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}[\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)]ds$$
$$= \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_t^{t+s} u^{(2)}(x,r)dr\,ds$$

and again, using Minkowski's integral inequality

$$\left\|\frac{\partial}{\partial t}P_t(D^{\beta} f)\right\|_{p,\gamma} \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_t^{t+s}\|u^{(2)}(\cdot,r)\|_{p,\gamma}\,dr\,ds \tag{8.43}$$

Then, if $1 \leq q < \infty$, by (8.43)

$$\int_0^{\infty}\left(t^{1-(\alpha-\beta)}\left\|\frac{\partial}{\partial t}P_t(D_\beta f)\right\|_{p,\gamma}\right)^q\frac{dt}{t}$$
$$\leq C_\beta\int_0^{\infty}\left(t^{1-(\alpha-\beta)}\int_0^{+\infty} s^{-\beta-1}\int_t^{t+s}\|u^{(2)}(\cdot,r)\|_{p,\gamma}\,dr\,ds\right)^q\frac{dt}{t}$$
$$= C_\beta\int_0^{\infty}\left(t^{1-(\alpha-\beta)}\int_0^{t} s^{-\beta-1}\int_t^{t+s}\|u^{(2)}(\cdot,r)\|_{p,\gamma}\,dr\,ds\right)^q\frac{dt}{t}$$
$$+ C_\beta\int_0^{\infty}\left(t^{1-(\alpha-\beta)}\int_t^{+\infty} s^{-\beta-1}\int_t^{t+s}\|u^{(2)}(\cdot,r)\|_{p,\gamma}\,dr\,ds\right)^q\frac{dt}{t}$$
$$= (I) + (II).$$

Now, because $r > t$ using Lemma 3.5 and the fact that $0 < \beta < 1$,

$$(I) \leq C_\beta\int_0^{\infty}\left(t^{1-(\alpha-\beta)}\int_0^{t} s^{-\beta}ds\,\|u^{(2)}(\cdot,r)\|_{p,\gamma}\right)^q\frac{dt}{t}$$
$$= C_{\beta,q}\int_0^{\infty}\left(t^{2-\alpha}\left\|\frac{\partial^2}{\partial r^2}P_r f\right\|_{p,\gamma}\right)^q\frac{dt}{t}.$$

On the other hand, as $r > t$ using Hardy's inequality (10.101), because $(1-\alpha)q > 0$, we get

$$
\begin{aligned}
(II) &\leq C_\beta \int_0^\infty t^{(1-(\alpha-\beta))q} \Big(\int_t^{+\infty} s^{-\beta-1} ds \int_t^\infty \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr \Big)^q \frac{dt}{t} \\
&= C'_\beta \int_0^\infty t^{(1-\alpha)q} \Big(\int_t^\infty \|u^{(2)}(\cdot,r)\|_{p,\gamma}\, dr \Big)^q \frac{dt}{t} \\
&\leq \frac{C'_\beta}{(1-\alpha)} \int_0^\infty \Big(r^{2-\alpha} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} \Big)^q \frac{dr}{r}.
\end{aligned}
$$

Thus,

$$
\Big(\int_0^\infty \Big(t^{1-\alpha+\beta} \Big\| \frac{\partial}{\partial t} P_t D_\beta f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} \leq C \Big(\int_0^\infty \Big(t^{2-\alpha} \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} < \infty,
$$

as $f \in B^\alpha_{p,q}(\gamma_d)$. Then, $D_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$ and

$$
\begin{aligned}
\|D_\beta f\|_{B^{\alpha-\beta}_{p,q}} &= \|D_\beta f\|_{p,\gamma} + \Big(\int_0^\infty \Big(t^{1-\alpha+\beta} \Big\| \frac{\partial}{\partial t} P_t D_\beta f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} \\
&\leq C_1 \|f\|_{B^\alpha_{p,q}} + C_2 \Big(\int_0^\infty \Big(t^{2-\alpha} \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} \Big)^q \frac{dt}{t} \Big)^{1/q} \leq C \|f\|_{B^\alpha_{p,q}}.
\end{aligned}
$$

Therefore, $D_\beta f : B^\alpha_{p,q} \to B^{\alpha-\beta}_{p,q}$ is bounded.

Now if $q = \infty$, inequality (8.43) can be written as

$$
\begin{aligned}
\Big\| \frac{\partial}{\partial t} P_t (D_\beta f) \Big\|_{p,\gamma} &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_t^{t+s} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} dr\, ds \\
&= \frac{1}{c_\beta} \int_0^t s^{-\beta-1} \int_t^{t+s} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} dr\, ds \\
&\quad + \frac{1}{c_\beta} \int_t^{+\infty} s^{-\beta-1} \int_t^{t+s} \Big\| \frac{\partial^2}{\partial r^2} P_r f \Big\|_{p,\gamma} dr\, ds = (I) + (II).
\end{aligned}
$$

Now, using Lemma 3.5, because $r > t$,

$$
\begin{aligned}
(I) &\leq \frac{1}{c_\beta} \int_0^t s^{-\beta} \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} ds = C_\beta \Big\| \frac{\partial^2}{\partial t^2} P_t f \Big\|_{p,\gamma} t^{1-\beta} \\
&\leq C_\beta A(f) t^{-2+\alpha} t^{1-\beta} = C_\beta A(f) t^{-1+\alpha-\beta},
\end{aligned}
$$

and by Lemma 3.5, because $r > t$, and the fact that $f \in B^{\alpha}_{p,\infty}$,

$$
\begin{aligned}
(II) &\leq \frac{1}{c_\beta} \int_t^{+\infty} s^{-\beta-1} \int_t^{\infty} \left\| \frac{\partial^2}{\partial r^2} P_r f \right\|_{p,\gamma} dr\, ds \\
&\leq C_\beta t^{-\beta} \int_t^{\infty} \left\| \frac{\partial^2}{\partial r^2} P_r f \right\|_{p,\gamma} dr \leq C_\beta A(f) t^{-\beta} \int_t^{\infty} r^{-2+\alpha} dr \\
&= C_{\alpha,\beta} A(f) t^{-1+\alpha-\beta}.
\end{aligned}
$$

Thus,

$$
\left\| \frac{\partial}{\partial t} P_t (D_\beta f) \right\|_{p,\gamma} \leq C A(f) t^{-1+\alpha-\beta}, \quad t > 0.
$$

Hence, $D_\beta f \in B^{\alpha-\beta}_{p,\infty}(\gamma_d)$ then $A(D_\beta f) \leq C A(f)$, and

$$
\|D_\beta f\|_{B^{\alpha-\beta}_{p,\infty}} = \|D_\beta f\|_{p,\gamma} + A(D_\beta f) \leq C_1 \|g\|_{B^{\alpha}_{p,\infty}} + C_2 A(f) \leq C \|f\|_{B^{\alpha}_{p,\infty}}.
$$

Therefore, $D_\beta : B^{\alpha}_{p,\infty} \to B^{\alpha-\beta}_{p,\infty}$ is bounded. □

Next, we study the boundedness of the Bessel fractional derivative on Besov–Lipschitz spaces for the bounded case $0 < \beta < \alpha < 1$:

Theorem 8.17. *Let $0 < \beta < \alpha < 1$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then $\mathscr{D}_\beta$ is bounded from $B^{\alpha}_{p,q}(\gamma_d)$ into $B^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in L^p(\gamma_d)$, using the fundamental theorem of calculus we can write,

$$
\begin{aligned}
|\mathscr{D}_\beta f(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} P_s f(x) - f(x)| ds \\
&\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} |P_s f(x) - f(x)| ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} - 1| |f(x)| ds \\
&\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \left| \int_0^s \frac{\partial}{\partial r} P_r f(x) dr \right| ds + \frac{|f(x)|}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \left| -\int_0^s e^{-r} dr \right| ds \\
&\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \left| \frac{\partial}{\partial r} P_r f(x) \right| dr\, ds + \frac{|f(x)|}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s e^{-r} dr\, ds.
\end{aligned}
$$

Now, using Hardy's inequality (10.100), with $p = 1$ in both integrals, we have

$$
|\mathscr{D}_\beta f(x)| \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} \left| \frac{\partial}{\partial r} P_r f(x) \right| \frac{dr}{r} + \frac{\Gamma(1-\beta)}{\beta c_\beta} |f(x)|.
$$

Therefore, according to Minkowski's integral inequality

$$\|\mathscr{D}_\beta f\|_p \leq \frac{1}{\beta c_\beta}\int_0^{+\infty} r^{1-\beta}\Big\|\frac{\partial}{\partial r}P_r f\Big\|_p \frac{dr}{r} + \frac{\Gamma(1-\beta)}{\beta c_\beta}\|f\|_p < C_1\|f\|_{B^{\alpha}_{p,q}} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\beta}_{p,1}(\gamma_d)$, $1 \leq q \leq \infty$ as $\alpha > \beta$, i.e. $D_\beta f \in L^p(\gamma_d)$.

On the other hand, using the fundamental theorem of calculus and, Hardy's inequality (10.100) again, with $p = 1$ in the second integral, we have

$$\begin{aligned}
|\frac{\partial}{\partial t}P_t(\mathscr{D}_\beta f)(x)| &\leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}|e^{-s}\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)|ds \\
&\leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}e^{-s}|\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)|ds \\
&\qquad + \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}|e^{-s}-1||\frac{\partial}{\partial t}P_t f(x)|ds \\
&\leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}\int_t^{t+s}|\frac{\partial^2}{\partial r^2}P_r f(x)|dr\,ds \\
&\qquad + \frac{1}{c_\beta}|\frac{\partial}{\partial t}P_t f(x)|\int_0^{\infty} s^{-\beta-1}\int_0^{s} e^{-r}dr\,ds, \\
&\leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}\int_t^{t+s}|\frac{\partial^2}{\partial r^2}P_r f(x)|dr\,ds + \frac{\Gamma(1-\beta)}{\beta c_\beta}|\frac{\partial}{\partial t}P_t f(x)|.
\end{aligned}$$

Thus, using Minkowski's integral inequality,

$$\Big\|\frac{\partial}{\partial t}P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}\int_t^{t+s}\Big\|\frac{\partial^2}{\partial r^2}P_r f\Big\|_{p,\gamma}dr\,ds + \frac{\Gamma(1-\beta)}{\beta c_\beta}\Big\|\frac{\partial}{\partial t}P_t f\Big\|_{p,\gamma}. \tag{8.44}$$

Then, if $1 \leq q < \infty$, using (8.44) and Minkowski's integral inequality, we get

$$\begin{aligned}
&\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\Big\|\frac{\partial}{\partial t}P_t\mathscr{D}_\beta f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} \\
&\quad \leq \frac{1}{c_\beta}\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_0^{\infty} s^{-\beta-1}\int_t^{t+s}\Big\|\frac{\partial^2}{\partial r^2}P_r f\Big\|_{p,\gamma}dr\,ds\Big)^q\frac{dt}{t}\Big)^{1/q} \\
&\qquad + \frac{\Gamma(1-\beta)}{\beta c_\beta}\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\Big\|\frac{\partial}{\partial t}P_t f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} = (I) + (II).
\end{aligned}$$

For the first term, the argument is the same as that considered in the second part of the proof of Theorem 8.16; thus,

$$(I) \leq C_\beta\Big(\int_0^{\infty}\Big(t^{2-\beta}\Big\|\frac{\partial^2}{\partial t^2}P_t f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \|f\|_{B^{\alpha}_{p,q}} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$, and for the second term trivially

$$(II) \leq C\|f\|_{B^{\alpha-\beta}_{p,q}} \leq C\|f\|_{B^{\alpha}_{p,q}}$$

because $\alpha > \alpha - \beta$ and the inclusion relation given in Proposition 7.36.

Hence, if $1 \leq q < \infty$,

$$\left(\int_0^\infty \left(t^{1-(\alpha-\beta)}\left\|\frac{\partial}{\partial t}P_t(\mathscr{D}_\beta f)\right\|_{p,\gamma}\right)^q \frac{dt}{t}\right)^{1/q} \leq C_2\|f|_{B^{\alpha}_{p,q}},$$

so $\mathscr{D}_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$ and, moreover,

$$\begin{aligned}\|\mathscr{D}_\beta f\|_{B^{\alpha-\beta}_{p,q}} &= \|\mathscr{D}_\beta f\|_{p,\gamma} + \left(\int_0^\infty \left(t^{1-\alpha+\beta}\left\|\frac{\partial}{\partial t}P_t\mathscr{D}_\beta f\|_{p,\gamma}\right)^q \frac{dt}{t}\right)^{1/q}\\ &\leq C_1\|f\|_{B^{\alpha}_{p,q}} + C_2\left(\int_0^\infty \left(t^{2-\alpha}\left\|\frac{\partial^2}{\partial t^2}P_t f\right\|_{p,\gamma}\right)^q \frac{dt}{t}\right)^{1/q} \leq C\|f\|_{B^{\alpha}_{p,q}}.\end{aligned}$$

If $q = \infty$, using the same argument as in Theorem 8.16, inequality (8.44) can be written as

$$\begin{aligned}\left\|\frac{\partial}{\partial t}P_t\mathscr{D}_\beta f\right\|_{p,\gamma} &\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}\int_t^{t+s}\left\|\frac{\partial^2}{\partial r^2}P_r f\right\|_{p,\gamma} dr\,ds + \frac{\Gamma(1-\beta)}{\beta c_\beta}\left\|\frac{\partial}{\partial t}P_t f\right\|_{p,\gamma}\\ &\leq C_{\alpha,\beta}A(f)t^{-1+\alpha-\beta} + \frac{\Gamma(1-\beta)}{\beta c_\beta}A(f)t^{-1+\alpha-\beta} \leq C_{\alpha,\beta}A(f)t^{-1+\alpha-\beta},\end{aligned}$$

for $t > 0$, then, $\mathscr{D}_\beta f \in B^{\alpha-\beta}_{p,\infty}(\gamma_d)$ and $A(\mathscr{D}_\beta f) \leq C_{\alpha,\beta}A(f)$; thus,

$$\|\mathscr{D}_\beta f\|_{B^{\alpha-\beta}_{p,\infty}} = \|\mathscr{D}_\beta f\|_{p,\gamma} + A(\mathscr{D}_\beta f) \leq C_1\|f\|_{B^{\alpha}_{p,\infty}} + C_2 A(f) \leq C\|f\|_{B^{\alpha}_{p,\infty}}.$$

□

We consider now the unbounded case for fractional derivatives (removing the condition that the indexes must be less than 1). To do this, we need to consider forward differences. Remember that for a given function f, the k-th order forward difference of f starting at t with increment s is defined as

$$\Delta_s^k(f,t) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j f(t+(k-j)s).$$

The forward differences have the following properties (see Appendix Lemma 10.30), which will be needed in what follows. For any positive integer k

i) $\Delta_s^k(f,t) = \Delta_s^{k-1}(\Delta_s(f,\cdot),t) = \Delta_s(\Delta_s^{k-1}(f,\cdot),t)$.
ii) $\Delta_s^k(f,t) = \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-2}}^{v_{k-2}+s}\int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)\,dv_k dv_{k-1}\ldots dv_2 dv_1$.

iii) For any positive integer k

$$\frac{\partial}{\partial s}(\Delta_s^k(f,t)) = k\Delta_s^{k-1}(f',t+s), \tag{8.45}$$

and for any integer $j>0$,

$$\frac{\partial^j}{\partial t^j}(\Delta_s^k(f,t)) = \Delta_s^k(f^{(j)},t). \tag{8.46}$$

Observe that using the binomial theorem and the semigroup property of $\{P_t\}$, we have

$$\begin{aligned}(P_t-I)^k f(x) &= \sum_{j=0}^{k}\binom{k}{j}P_t^{k-j}(-I)^j f(x) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j P_t^{k-j}f(x)\\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j P_{(k-j)t}f(x) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j u(x,(k-j)t)\\ &= \Delta_t^k(u(x,\cdot),0),\end{aligned} \tag{8.47}$$

where as usual, $u(x,t) = P_t f(x)$. Additionally, we need the following result:

Lemma 8.18. *Let $f\in L^p(\gamma_d)$, $1\le p<\infty$ and $k,n\in\mathbb{N}$ then*

$$\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma_d} \le s^k\|u^{(k+n)}(\cdot,t)\|_{p,\gamma_d}$$

Proof. From property *ii*) of forward differences (see Lemma 10.30), we have

$$\Delta_s^k(u^{(n)}(x,\cdot),t) = \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-2}}^{v_{k-2}+s}\int_{v_{k-1}}^{v_{k-1}+s} u^{(k+n)}(x,v_k)dv_k dv_{k-1}\ldots dv_2 dv_1,$$

then, using Minkowski's integral inequality k-times and Lemma 3.5,

$$\begin{aligned}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma_d} &\le \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-2}}^{v_{k-2}+s}\int_{v_{k-1}}^{v_{k-1}+s}\|u^{(k+n)}(\cdot,v_k)\|_{p,\gamma_d}dv_k dv_{k-1}\ldots dv_2 dv_1\\ &\le s^k\|u^{(k+n)}(\cdot,t)\|_{p,\gamma_d} = s^k\left\|\frac{\partial^{k+n}}{\partial t^{k+n}}u(\cdot,t)\right\|_{p,\gamma_d}.\end{aligned}$$

□

Let us start studying the boundedness of the Riesz fractional derivative in $B_{p,q}^{\alpha}(\gamma_d)$

Theorem 8.19. *Let $0<\beta<\alpha$, $1\le p<\infty$ and $1\le q\le\infty$ then D^β is bounded from $B_{p,q}^{\alpha}(\gamma_d)$ into $B_{p,q}^{\alpha-\beta}(\gamma_d)$.*

Proof. Let $f\in B_{p,q}^{\alpha}(\gamma_d)$, using (8.47), Hardy's inequality (10.100), $p=1$, the fundamental theorem of calculus, and property iii) of forward differences (see Lemma 10.30), we get

$$|D^{\beta} f(x)| \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(u(x,\cdot),0)|ds \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s |\frac{\partial}{\partial r}\Delta_r^k(u(x,\cdot),0)|dr\,ds$$
$$\leq \frac{1}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}|\frac{\partial}{\partial r}\Delta_r^k(u(x,\cdot),0)|dr = \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}|\Delta_r^{k-1}(u'(x,\cdot),r)|dr.$$

Now, using Minkowski's integral inequality and Lemma 8.18

$$\|D_\beta f\|_{p,\gamma} \leq \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}\big\|\Delta_r^{k-1}(u',r)\big\|_{p,\gamma} dr$$
$$\leq \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{k-\beta}\Big\|\frac{\partial^k}{\partial r^k}P_r f\Big\|_{p,\gamma}\frac{dr}{r} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\beta}_{p,1}(\gamma_d)$, as $\alpha > \beta$. Therefore, $D_\beta f \in L^p(\gamma_d)$.

On the other hand,

$$P_t[(P_s - I)^k f(x)] = P_t(\Delta_s^k(u(x,\cdot),0)) = P_t(\sum_{j=0}^{k}\binom{k}{j}(-1)^j P_{(k-j)s}f(x))$$
$$= \sum_{j=0}^{k}\binom{k}{j}(-1)^j P_{t+(k-j)s}f(x) = \Delta_s^k(u(x,\cdot),t). \tag{8.48}$$

Thus, if n is the smaller integer greater than α, i.e., $n-1 \leq \alpha < n$, then according to Lemma 10.30 *iv*),

$$\frac{\partial^n}{\partial t^n}P_t(D_\beta f)(x) = \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\frac{\partial^n}{\partial t^n}(\Delta_s^k(u(x,\cdot),t)$$
$$= \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Delta_s^k(u^{(n)}(x,\cdot),t)ds;$$

therefore, using Minkowski's integral inequality

$$\Big\|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds. \tag{8.49}$$

Now, if $1 \leq q < \infty$, by (8.49),

$$\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_0^{t} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$+\frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$= (I) + (II).$$

Then, using Lemma 8.18,

$$(I) \leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^{n+k}}{\partial t^{n+k}}P_t f\Big\|_{p,\gamma}\int_0^t s^{k-\beta-1}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$= \frac{1}{c_\beta(k-\beta)}\Big(\int_0^\infty \Big(t^{n+k-\alpha}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$, and using Lemma 3.5

$$(II) \leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t+(k-j)s)\|_{p,\gamma}\Big)ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t)\|_{p,\gamma}\Big)ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$= \frac{2^k}{c_\beta}\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^{n}}{\partial t^{n}}P_t f\Big\|_{p,\gamma}\int_t^{+\infty} s^{-\beta-1}ds\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$= \frac{2^k}{c_\beta \beta}\Big(\int_0^\infty \Big(t^{n-\alpha}\Big\|\frac{\partial^{n}}{\partial t^{n}}P_t f\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$. Therefore, if $1 \leq q < \infty$, $D_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$; moreover,

$$\|D_\beta f\Big\|_{B^{\alpha-\beta}_{p,q}} = \|D_\beta f\|_{p,\gamma} + \Big(\int_0^\infty \Big(t^{n-\alpha+\beta}\Big\|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq C_1\|f\|_{B^{\alpha}_{p,q}} + C_2\|f\|_{B^{\alpha}_{p,q}} \leq C\|f\|_{B^{\alpha}_{p,q}}$$

Thus, $D_\beta f : B^{\alpha}_{p,q} \to B^{\alpha-\beta}_{p,q}$ is bounded.

If $q = \infty$, inequality (8.49) can be written as

$$\|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)\Big\|_{p,\gamma} \leq \frac{1}{c_\beta}\int_0^t s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds$$
$$+\frac{1}{c_\beta}\int_t^{+\infty} s^{-\beta-1}\|\Delta_s^k(u^{(n)},t)\|_{p,\gamma}ds$$
$$= (I) + (II)$$

and then as $f \in B^{\beta}_{p,\infty}$, by Lemma 8.18,

$$(I) \leq \frac{1}{c_\beta}\int_0^t s^{-\beta-1}s^k\|u^{(n+k)}\|_{p,\gamma}ds = C_\beta\Big\|\frac{\partial^{n+k}}{\partial t^{n+k}}P_t f\Big\|_{p,\gamma}t^{k-\beta}$$
$$\leq C_\beta A(f)t^{-n-k+\alpha}t^{k-\beta} = C_\beta A(f)t^{-n+\alpha-\beta},$$

and as above, using Lemma 3.5,

$$\begin{aligned}(II) &\leq \frac{1}{c_\beta}\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t+(k-j)s)\|_{p,\gamma}\Big)ds\\ &\leq C_\beta\int_t^{+\infty} s^{-\beta-1}\Big(\sum_{j=0}^{k}\binom{k}{j}\|u^{(n)}(\cdot,t)\|_{p,\gamma}\Big)ds = C_\beta t^{-\beta}\Big\|\frac{\partial^n}{\partial t^n}P_t f\Big\|_{p,\gamma}\\ &\leq C_\beta A(f)t^{-n+\alpha}t^{-\beta} = C_\beta A(f)t^{-n+\alpha-\beta}.\end{aligned}$$

□

There is an alternative proof of the fact that $D_\beta f \in L^p(\gamma_d)$ without using Hardy's inequality following the same scheme as in the proof of *i*) [109, Theorem 3.5], and using the inclusion $B^{\alpha}_{p,q} \subset B^{\beta+\varepsilon}_{p,\infty}$ with $\beta+\varepsilon < k$.

Now, let us consider the case of the Bessel derivative.

Theorem 8.20. *Let $0<\beta<\alpha$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$ then $\mathscr{D}_\beta$ is bounded from $B^{\alpha}_{p,q}(\gamma_d)$ into $B^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in B^{\alpha}_{p,q}(\gamma_d)$, and set $v(x,t) = e^{-t}u(x,t)$. Then, using Hardy's inequality (10.100), the fundamental theorem of calculus, and property *iii*) of forward differences (see Lemma 10.30),

$$\begin{aligned}|\mathscr{D}_\beta f(x)| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(v(x,\cdot),0)|ds\\ &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s\Big|\frac{\partial}{\partial r}\Delta_r^k(v(x,\cdot),0)\Big|dr\,ds\\ &\leq \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}|\Delta_r^{k-1}(v'(x,\cdot),r)|dr\end{aligned}$$

and this implies, using Minkowski's integral inequality,

$$\|\mathscr{D}_\beta f\|_{p,\gamma_d} \leq \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}\|\Delta_r^{k-1}(v',r)\|_{p,\gamma}dr.$$

Now, using property *ii*) of forward differences (see Lemma 10.30),

$$\|\Delta_r^{k-1}(v',r)\|_{p,\gamma} \leq \int_r^{2r}\int_{v_1}^{v_1+r}\cdots\int_{v_{k-2}}^{v_{k-2}+r}\|v^{(k)}(\cdot,v_{k-1})\|_{p,\gamma}dv_{k-1}dv_{k-2}\ldots dv_2dv_1$$

and using Leibniz's differentiation rule for the product

$$\begin{aligned}\|v^{(k)}(\cdot,v_{k-1})\Big\|_{p,\gamma} &= \Big\|\sum_{j=0}^{k}\binom{k}{j}(e^{-v_{k-1}})^{(j)}u^{(k-j)}(\cdot,v_{k-1})\Big\|_{p,\gamma}\\ &\leq \sum_{j=0}^{k}\binom{k}{j}e^{-v_{k-1}}\|u^{(k-j)}(\cdot,v_{k-1})\|_{p,\gamma}.\end{aligned}$$

Then

$$\begin{aligned}
&\|\Delta_r^{k-1}(v',r)\|_{p,\gamma}\\
&\quad\leq \sum_{j=0}^{k}\binom{k}{j}\int_r^{2r}\int_{v_1}^{v_1+r}\dots\int_{v_{k-2}}^{v_{k-2}+r} e^{-v_{k-1}}\|u^{(k-j)}(\cdot,v_{k-1})\|_{p,\gamma}dv_{k-1}dv_{k-2}\dots dv_2dv_1\\
&\quad\leq \sum_{j=0}^{k}\binom{k}{j} r^{k-1}e^{-r}\|u^{(k-j)}(\cdot,r)\|_{p,\gamma}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\mathscr{D}_\beta f\|_{p,\gamma} &\leq \frac{k}{\beta c_\beta}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k-j)}(\cdot,r)\|_{p,\gamma}dr\\
&= \frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{(k-j)-(\beta-j)-1}e^{-r}\Big\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_r f\Big\|_{p,\gamma}dr\\
&\qquad + \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|P_r f\|_{p,\gamma}dr\\
&\leq \frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{(k-j)-(\beta-j)-1}\Big\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_r f\Big\|_{p,\gamma}dr\\
&\qquad + \frac{k}{\beta c_\beta}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|f\|_{p,\gamma}dr
\end{aligned}$$

Thus,

$$\begin{aligned}
\|\mathscr{D}_\beta f\|_{p,\gamma} \leq \frac{k}{\beta c_\beta}\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{k-j-(\beta-j)}\Big\|\frac{\partial^{k-j}}{\partial r^{k-j}}P_r f\Big\|_{p,\gamma}\frac{dr}{r}\\
+\frac{k\Gamma(k-\beta)}{\beta c_\beta}\|f\|_{p,\gamma}<\infty,
\end{aligned}$$

because $f\in B^{\alpha}_{p,q}(\gamma_d)\subset B^{\beta-j}_{p,1}(\gamma_d)$ as $\alpha>\beta>\beta-j\geq 0$, for $j\in\{0,\dots,k-1\}$, then $\mathscr{D}_\beta f\in L^p(\gamma_d)$.

On the other hand,

$$P_t(e^{-s}P_s-I)^k f(x) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-s(k-j)}u(x,t+(k-j)s).$$

Let n be the smaller integer greater than β, i.e., $n-1\leq\beta<n$, we have

$$\begin{aligned}
\frac{\partial^n}{\partial t^n} P_t(\mathscr{D}_\beta f)(x) &= \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-s(k-j)} u^{(n)}(x,t+(k-j)s)ds\\
&= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-(t+s(k-j))} u^{(n)}(x,t+(k-j)s)ds\\
&= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Delta_s^k(w(x,\cdot),t)ds,
\end{aligned}$$

where $w(x,t) = e^{-t}u^{(n)}(x,t)$. Now, using the fundamental theorem of calculus,

$$\begin{aligned}
\frac{\partial^n}{\partial t^n} P_t(\mathscr{D}_\beta f)(x) &= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Delta_s^k(w(x,\cdot),t)ds\\
&= \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s \frac{\partial}{\partial r}\Delta_r^k(w(x,\cdot),t)drds.
\end{aligned}$$

Then, using Hardy's inequality (10.100) and property *iii*) of forward differences (see Lemma 10.30),

$$\begin{aligned}
\left|\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x)\right| &\le \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s \left|\frac{\partial}{\partial r}\Delta_r^k(w(x,\cdot),t)\right|drds\\
&\le \frac{e^t}{c_\beta\beta}\int_0^{+\infty} r\left|\frac{\partial}{\partial r}\Delta_r^k(w(x,\cdot),t)\right|r^{-\beta-1}dr\\
&= \frac{ke^t}{c_\beta\beta}\int_0^{+\infty} r^{-\beta}|\Delta_r^{k-1}(w'(x,\cdot),t+r)|dr
\end{aligned}$$

and according to Minkowski's integral inequality, we get

$$\left\|\frac{\partial^n}{\partial t^n} P_t(\mathscr{D}_\beta f)\right\|_{p,\gamma} \le \frac{ke^t}{\beta c_\beta}\int_0^{+\infty} r^{-\beta}\|\Delta_r^{k-1}(w',t+r)\|_{p,\gamma}dr.$$

Now, using an analogous argument to that above, Lemma 10.30 and Leibniz's product rule give us

$$\|\Delta_r^{k-1}(w',t+r)\|_{p,\gamma} \le \sum_{j=0}^{k}\binom{k}{j} r^{k-1}e^{-(t+r)}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma},$$

and this implies that

$$\begin{aligned}
\left\|\frac{\partial^n}{\partial t^n} P_t(\mathscr{D}_\beta f)\right\|_{p,\gamma} &\le e^t\frac{k}{c_\beta\beta}\int_0^{+\infty} r^{-\beta}\Big(\sum_{j=0}^{k}\binom{k}{j} r^{k-1}e^{-(t+r)}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}\Big)dr\\
&= \frac{k}{c_\beta\beta}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}dr.
\end{aligned}$$

Thus,

$$\Big\|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma} \leq \frac{k}{c_\beta \beta}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}\,dr.$$

Now, if $1 \leq q < \infty$, using (8.50) we have,

$$\begin{aligned}&\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)}\Big\|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}_\beta f)\Big\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\quad\leq \frac{k}{c_\beta\beta}\sum_{j=0}^{k}\binom{k}{j}\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}.\end{aligned}$$

For each $1 \leq j \leq k$, $0 < \alpha-\beta+k-j \leq \beta$ and using Lemma 3.5

$$\begin{aligned}&(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^\infty r^{k-\beta-1}e^{-r}\|u^{(k+n-j)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\qquad\leq (\int_0^\infty\Big(t^{n-(\alpha-\beta)}\|u^{(n+k-j)}(\cdot,t)\|_{p,\gamma}\int_0^{+\infty} r^{k-\beta-1}e^{-r}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\qquad= \Gamma(k-\beta)(\int_0^\infty\Big(t^{n+(k-j)-(\alpha-\beta+k-j)}\|u^{(n+k-j)}(\cdot,t)\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \infty,\end{aligned}$$

as $f \in B^{\alpha}_{p,q}(\gamma_d) \subset B^{\alpha-\beta+(k-j)}_{p,q}(\gamma_d)$ for any $0 \leq j \leq k$.

Now, for the case $j = 0$,

$$\begin{aligned}&\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(n+k)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\qquad\leq \Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{t} r^{k-\beta-1}e^{-r}\|u^{(n+k)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\qquad\qquad+\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_t^{+\infty} r^{k-\beta-1}e^{-r}\|u^{(n+k)}(\cdot,t+r)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &\qquad= (I)+(II).\end{aligned}$$

Using Lemma 3.5, and $k > \beta$,

$$\begin{aligned}(I) &\leq \Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{t} r^{k-\beta-1}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &= \Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}\int_0^{t} r^{k-\beta-1}dr\Big)^q\frac{dt}{t}\Big)^{1/q}\\ &= \frac{1}{k-\beta}\Big(\int_0^\infty\Big(t^{n+k-\alpha}\|u^{(n+k)}(\cdot,t)\|_{p,\gamma}\Big)^q\frac{dt}{t}\Big)^{1/q} < \infty,\end{aligned}$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$ and $n+k > \alpha$. For the second term, using Lemma 3.5 and Hardy's inequality (10.101)

$$(II) \leq \left(\int_0^{\infty} \left(t^{n-(\alpha-\beta)} \int_t^{+\infty} r^{k-\beta-1} \|u^{(n+k)}(\cdot,r)\Big\|_{p,\gamma} dr\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\leq \frac{1}{n-(\alpha-\beta)} \left(\int_0^{\infty} \left(r^{n+k-\alpha} \|u^{(n+k)}(\cdot,r)\Big\|_{p,\gamma}\right)^q \frac{dr}{r}\right)^{1/q} < \infty,$$

because $f \in B^{\alpha}_{p,q}(\gamma_d)$.

Therefore, $\mathscr{D}_\beta f \in B^{\alpha-\beta}_{p,q}(\gamma_d)$. Moreover,

$$\|\mathscr{D}_\beta f\|_{B^{\alpha-\beta}_{p,q}} = \|\mathscr{D}_\beta f\|_{p,\gamma} + \left(\int_0^{\infty} \left(t^{n-(\alpha-\beta)} \left\|\frac{\partial^n}{\partial t^n} P_t \mathscr{D}_\beta f\right\|_{p,\gamma}\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\leq C_1 \|f\|_{p,\gamma} + \frac{k}{c_\beta \beta} \sum_{j=0}^{k} \binom{k}{j} C_2 \left(\int_0^{\infty} \left(r^{n-\alpha} \left\|\frac{\partial^n}{\partial r^n} P_r f\right\|_{p,\gamma}\right)^q \frac{dr}{r}\right)^{1/q}$$
$$\leq C\|f\|_{B^{\alpha}_{p,q}}$$

Finally, if $q = \infty$, from the inequality (8.50)

$$\left\|\frac{\partial^n}{\partial t^n} P_t(\mathscr{D}_\beta f)\right\|_{p,\gamma} \leq \frac{k}{c_\beta \beta} \sum_{j=0}^{k} \binom{k}{j} \int_0^{+\infty} r^{k-\beta-1} e^{-r} \|u^{(k+n-j)}(\cdot, t+r)\|_{p,\gamma} dr,$$

and then, the argument is essentially similar to the previous case, as in the last part of the proof of Theorem 8.19. □

8.5 Boundedness of Fractional Integrals and Fractional Derivatives on Gaussian Triebel–Lizorkin Spaces

First, we study the boundedness of the Riesz potentials I_β on Gaussian Triebel–Lizorkin spaces.

Theorem 8.21. *Let $\alpha \geq 0, \beta > 0$, $1 < p < \infty, 1 \leq q < \infty$ then I_β is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ into $F^{\alpha+\beta}_{p,q}(\gamma_d)$.*

Proof. Let $k > \alpha+\beta+1$ be an integer fixed and $f \in F^{\alpha}_{p,q}(\gamma_d)$. Using the integral representation of Riesz potentials (8.7), the semigroup property of $\{P_t\}_{t\geq 0}$, and the fact that $P_\infty f(x)$ is a constant, we get

$$P_t(I_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} P_t(P_s f(x) - P_\infty f(x)) ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} (P_{t+s} f(x) - P_\infty f(x)) ds. \tag{8.50}$$

Then, again using that $P_\infty f(x)$ is a constant and the chain rule,

$$\frac{\partial^k}{\partial t^k} P_t(I_\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \frac{\partial^k}{\partial t^k}(P_{t+s}f(x) - P_\infty f(x))ds$$
$$= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} u^{(k)}(x,t+s)ds. \tag{8.51}$$

i) Case $\beta \geq 1$: Using (8.51), the change of variables $r = t+s$, $dr = ds$, and Hardy's inequality (10.101), we have

$$\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}\Big|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$= \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_t^{+\infty} (r-t)^{\beta-1}|u^{(k)}(x,r)|dr\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_t^{+\infty} r^{\beta-1}|u^{(k)}(x,r)|dr\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\frac{1}{(k-(\alpha+\beta))^{1/q}}\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|u^{(k)}(x,r)|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}},$$

and, therefore,

$$\Big\|\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}\Big|\frac{\partial^k P_t(I_\beta f)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}$$
$$\leq C_{k,\alpha,\beta}\Big\|\Big(\int_0^{+\infty}\Big(r^{k-\alpha}\Big|\frac{\partial^k P_r f}{\partial r^k}\Big|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma} < \infty,$$

because $f \in F^\alpha_{p,q}$. By (8.12) and the previous estimate,

$$\|I_\beta f\|_{F^{\alpha+\beta}_{p,q}} \leq C\|f\|_{F^\alpha_{p,q}}.$$

ii) Case $0 < \beta < 1$: again using (8.51),

$$\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}\Big|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}\Big|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty} s^{\beta}|u^{(k)}(x,t+s)|\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_0^{t} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}}$$
$$+\frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_t^{+\infty} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}}$$
$$= (I) + (II).$$

Writing $s^{\beta-1} = s^{\frac{\beta-1}{q}+\frac{\beta-1}{q'}}$, $\frac{1}{q}+\frac{1}{q'}=1$, and using Hölder's inequality in the internal integral,

$$(I) \leq \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} t^{q(k-\alpha)-\beta-1}\int_0^t s^{\beta-1}|u^{(k)}(x,t+s)|^q ds\,dt\Big)^{1/q}.$$

Using the Fubini–Tonelli theorem and using that $q(k-\beta)-\beta-1>0$, as $k>\beta+\beta+1$, we get

$$\begin{aligned}(I) &\leq \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_s^{+\infty} t^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^q dt\,ds\Big)^{1/q}\\ &\leq \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_s^{+\infty} (t+s)^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^q dt\,ds\Big)^{1/q}.\end{aligned}$$

Then, by the change of variables $r=t+s$ and using Hardy's inequality (10.101),

$$\begin{aligned}(I) &\leq \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_{2s}^{+\infty} r^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^q dr\,ds\Big)^{1/q}\\ &\leq \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_{s}^{+\infty} r^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^q dr\,ds\Big)^{1/q}\\ &\leq \frac{C_\beta}{\beta}\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|u^{(k)}(x,r)|\Big)^q\frac{dr}{r}\Big)^{1/q}.\end{aligned}$$

On the other hand, because $\beta<1$, then $t<s$ implies that $s^{\beta-1}<t^{\beta-1}$, and by the change of variables $r=t+s$ and according to Hardy's inequality (10.101), as $k>\alpha+\beta+1>\alpha+1$, we obtain

$$\begin{aligned}(II) &\leq C_\beta\Big(\int_0^{+\infty} t^{q(k-\alpha-1)-1}\Big(\int_t^{+\infty}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}}\\ &\leq C_\beta\Big(\int_0^{+\infty} t^{q(k-\alpha-1)-1}\Big(\int_{2t}^{+\infty}|u^{(k)}(x,r)|dr\Big)^q dt\Big)^{\frac{1}{q}}\\ &\leq C_\beta\frac{1}{(k-\alpha-1)^{1/q}}\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|\frac{\partial^k P_r f(x)}{\partial r^k}|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\Big\|\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}\\ &\qquad\leq C_{k,\alpha,\beta}\Big\|\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|\frac{\partial^k P_r f}{\partial r^k}|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}<\infty.\end{aligned}$$

as $f\in F^{\alpha}_{p,q}$. Then, using (8.12) and the previous estimate, we get

$$\|I_\beta f\|_{F^{\alpha+\beta}_{p,q}} \leq C\|f\|_{F^{\alpha}_{p,q}}.$$

□

Next, we study the boundedness properties of the Bessel potentials on Triebel–Lizorkin spaces.

Theorem 8.22. *Let $\alpha \geq 0$, $1 \leq p,q < \infty$ then for every $\beta > 0$,*

i) *$\mathscr{I}_\beta$ is bounded on $F^{\alpha}_{p,q}(\gamma_d)$.*

ii) *Moreover, $\mathscr{I}_\beta$ is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ to $F^{\alpha+\beta}_{p,q}(\gamma_d)$.*

Proof.

i) Let us prove that $\mathscr{I}_\beta$ is bounded on $F^{\alpha}_{p,q}(\gamma_d)$. Using Lebesgue's dominated convergence theorem, Minkowski's integral inequality, and *iii*), we have

$$\left(\int_0^\infty (s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathscr{I}_\beta g\right)(x)\right|)^q \frac{ds}{s}\right)^{1/q}$$

$$= \left(\int_0^\infty (s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t} P_t g(x)\frac{dt}{t}\right)\right|)^q \frac{ds}{s}\right)^{1/q}$$

$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left(\int_0^\infty (s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}(x)\right|)^q \frac{ds}{s}\right)^{1/q}\frac{dt}{t},$$

then, again using Minkowski's integral inequality, and *iii*)

$$\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathscr{I}_\beta g\right)\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}$$

$$\leq \left\|\frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\frac{dt}{t}\right\|_{p,\gamma}$$

$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s(P_t g)}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}\frac{dt}{t}$$

$$\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} t^\beta e^{-t}\left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}\frac{dt}{t}$$

$$= \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}.$$

Thus,

$$\|\mathscr{I}_\beta g\|_{F^{\beta}_{p,q}} = \|\mathscr{I}_\beta g\|_{p,\gamma} + \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s}{\partial s^k}\left(\mathscr{I}_\beta g\right)\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma}$$

$$\leq \|g\|_{p,\gamma} + \left\|\left(\int_0^\infty \left(s^{k-\alpha}\left|\frac{\partial^k P_s g}{\partial s^k}\right|\right)^q \frac{ds}{s}\right)^{1/q}\right\|_{p,\gamma} = \|g\|_{F^{\alpha}_{p,q}}.$$

ii) Let $k > \alpha+\beta+1$ be a fixed integer, let $f \in F^{\alpha}_{p,q}(\gamma_d)$, and let $h = \mathscr{I}_\beta f$. We consider two cases:

ii-1) If $\beta \geq 1$. Taking the change of variables $u = t+s$ and using Hardy's inequality, we get

$$\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t h(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{1/q}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty} s^{\beta}e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}|\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_t^{+\infty}(u-t)^{\beta-1}|\frac{\partial^k P_u f(x)}{\partial u^k}|du\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(\int_t^{+\infty}u^{\beta-1}|\frac{\partial^k P_u f(x)}{\partial u^k}|du\Big)^q t^{q(k-(\alpha+\beta))-1}dt\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\frac{1}{k-(\alpha+\beta)}\Big(\int_0^{+\infty}\Big(u^{k-\alpha}|\frac{\partial^k P_u f(x)}{\partial u^k}|\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}}.$$

Therefore,

$$\Big\|\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t h}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}$$
$$\leq \frac{1}{\Gamma(\beta)(k-(\alpha+\beta))}\Big\|\Big(\int_0^{+\infty}\Big(u^{k-\alpha}|\frac{\partial^k P_u f}{\partial u^k}|\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}}\Big\|_{p,\gamma}<\infty,$$

because $f\in F^{\alpha}_{p,q}(\gamma_d)$. Thus $\mathscr{I}_\beta f\in F^{\alpha+\beta}_{p,q}(\gamma_d)$.

ii-2) If $0<\beta<1$.

$$\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t h(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty} s^{\beta}e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}|\frac{ds}{s}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_0^{t} s^{\alpha-1}e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}|ds\Big)^q dt\Big)^{\frac{1}{q}}$$
$$+\frac{C}{\Gamma(\beta)}(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_t^{+\infty} s^{\alpha-1}e^{-s}|\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}|ds\Big)^q dt\Big)^{\frac{1}{q}}$$
$$=I+II.$$

Now, $e^{-s}<1$ and as $\beta<1$, then $s^{\beta-1}<t^{\beta-1}$ for $t<s$.
Hence, again by the change of variables $u=t+s$ and using Hardy's inequality, we get

$$II\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-\beta-1)-1}\Big(\int_t^{+\infty}|\frac{\partial^k P_{t+s}f(x)}{\partial (t+s)^k}|ds\Big)^q dt\Big)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-\beta-1)-1}\Big(\int_t^{+\infty}|\frac{\partial^k P_u f(x)}{\partial u^k}|du\Big)^q dt\Big)^{\frac{1}{q}}$$
$$\leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty}\Big(u^{k-\alpha}|\frac{\partial^k P_u f(x)}{\partial u^k}|\Big)^q\frac{du}{u}\Big)^{\frac{1}{q}}$$

On the other hand, using $e^{-s} < 1$ again,

$$I^q \leq \frac{C}{\Gamma(\beta)} \int_0^{+\infty} t^{q(k-(\alpha+\beta))-1} \Big(\int_0^t s^{\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}| ds \Big)^q dt$$
$$= \frac{C}{\Gamma(\beta)\beta^q} \int_0^{+\infty} t^{q(k-\alpha)-1} \Big(\frac{\beta}{t^\beta} \int_0^t s^{\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}| ds \Big)^q dt$$

Now, as $\beta > 0$, $\int_0^t s^{\beta-1} ds = \frac{t^\beta}{\beta}$, then using Jensen's inequality for the probability measure $\frac{\beta}{t^\beta} s^{\beta-1} ds$ and Fubini's theorem, we get

$$I^q \leq \frac{C}{\Gamma(\beta)\beta^q} \int_0^{+\infty} t^{q(k-\alpha)-1} \Big(\frac{\beta}{t^\beta} \int_0^t s^{\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}|^q ds \Big) dt$$
$$= \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_s^{+\infty} t^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}|^q dt \Big) ds$$
$$\leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_s^{+\infty} (t+s)^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_{t+s} f(x)}{\partial (t+s)^k}|^q dt \Big) ds$$

as $q(k-\alpha)-\beta-1 > 0$, because $0 < \beta < 1$. Finally, again taking the change of variables $u = t+s$ and using Hardy's inequality, we get

$$I^q \leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_{2s}^{+\infty} u^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_u f(x)}{\partial u^k}|^q du \Big) ds$$
$$\leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} s^{\beta-1} \Big(\int_s^{+\infty} u^{q(k-\alpha)-\beta-1} |\frac{\partial^k P_u f(x)}{\partial u^k}|^q du \Big) ds$$
$$\leq \frac{C}{\Gamma(\beta)\beta^{q-1}} \int_0^{+\infty} \Big(u^{k-\beta} |\frac{\partial^k P_u f(x)}{\partial u^k}| \Big)^q \frac{du}{u}.$$

Hence,

$$\Big\| \Big(\int_0^{+\infty} \Big(t^{k-(\alpha+\beta)} |\frac{\partial^k P_t h}{\partial t^k}| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq C_{k,\alpha,\beta} \Big\| \Big(\int_0^{+\infty} \Big(u^{k-\alpha} |\frac{\partial^k P_u f}{\partial u^k}| \Big)^q \frac{du}{u} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma} < \infty.$$

Thus, $\mathscr{I}_\beta f \in F_{p,q}^{\alpha+\beta}(\gamma_d)$, for $0 < \beta < 1$.

Therefore, in both cases we have,

$$\|\mathscr{I}_\beta f\|_{F_{p,q}^{\alpha+\beta}} = \|\mathscr{I}_\beta f\|_{p,\gamma} + \| \Big(\int_0^{+\infty} \Big(t^{k-(\alpha+\beta)} |\frac{\partial^k P_t \mathscr{I}_\beta f}{\partial t^k}| \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq C_\beta \|f\|_{p,\gamma} + C_{k,\alpha,\beta} \| \Big(\int_0^{+\infty} \Big(u^{k-\alpha} |\frac{\partial^k P_u}{\partial u^k}| \Big)^q \frac{du}{u} \Big)^{\frac{1}{q}} \Big\|_{p,\gamma}$$
$$\leq C_{k,\alpha,\beta} \|f\|_{F_{p,q}^\alpha}.$$

□

Now, we study the boundedness of the Riesz fractional derivatives and of the Bessel fractional derivatives on Triebel–Lizorkin spaces. Again, because they require different techniques, we consider two cases:

- The bounded case, $0 < \beta < \alpha < 1$.
- The unbounded case $0 < \beta < \alpha$.

Let us start with the bounded case for the Riesz derivative.

Theorem 8.23. *Let $1 \le p,q < \infty$, for $0 < \beta < \alpha < 1$, D^β is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in F^{\alpha}_{p,q}(\gamma_d)$, using the fundamental theorem of calculus, and Hardy's inequality (10.100) with $p = 1$,

$$|D^\beta f(x)| \le \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\Big|P_s f(x) - f(x)\Big| ds$$
$$\le \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s \Big|\frac{\partial}{\partial r}P_r f(x)\Big| dr\,ds \le \frac{1}{c_\beta \beta}\int_0^{+\infty} r^{1-\beta}\Big|\frac{\partial}{\partial r}P_r f(x)\Big|\frac{dr}{r}.$$

Thus,

$$\|D^\beta f\|_{p,\gamma} \le C_\beta \Big\|\int_0^{+\infty} r^{1-\beta}\Big|\frac{\partial}{\partial r}P_r f\Big|\frac{dr}{r}\Big\|_{p,\gamma} \le C_\beta \|f\|_{F^{\alpha}_{p,q}} < \infty, \tag{8.52}$$

because $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$ ($\alpha > \beta$ and $q \ge 1$). Now, using an analogous argument using Hardy's inequality (10.100) with $p = 1$,

$$\Big|\frac{\partial}{\partial t}P_t(D^\beta f)(x)\Big| \le \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)| ds$$
$$\le \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}\int_0^s |u^{(2)}(x,t+r)| dr\,ds \le \frac{1}{c_\beta \beta}\int_0^{+\infty} r^{-\beta}|u^{(2)}(x,t+r)| dr.$$

This implies that

$$\int_0^\infty \Big(t^{1-(\alpha-\beta)}|\frac{\partial}{\partial t}P_t(D_\beta f)(x)|\Big)^q \frac{dt}{t} \le \frac{1}{c_\beta \beta}\int_0^\infty \Big(t^{1-(\alpha-\beta)}\int_0^{+\infty} r^{-\beta}|u^{(2)}(x,t+r)| dr\Big)^q \frac{dt}{t}$$
$$\le C_\beta \int_0^\infty \Big(t^{1-(\alpha-\beta)}\int_0^t r^{-\beta}|u^{(2)}(x,t+r)| dr\Big)^q \frac{dt}{t} \tag{8.53}$$
$$+ C_\beta \int_0^\infty \Big(t^{1-(\alpha-\beta)}\int_t^{+\infty} r^{-\beta}|u^{(2)}(x,t+r)| dr\Big)^q \frac{dt}{t}$$
$$= (I) + (II).$$

Writing $r^{-\beta} = r^{\frac{-\beta}{q}+\frac{-\beta}{q'}}$, $\frac{1}{q}+\frac{1}{q'}=1$, and using Hölder's inequality in the internal integral, we have

$$(I) \leq C_\beta(1-\beta)^{1-q}\int_0^\infty t^{(2-\alpha)q-2+\beta}\int_0^t r^{-\beta}|u^{(2)}(x,t+r)|^q dr\,dt.$$

Then, according to the Fubini–Tonelli theorem, we get

$$(I) \leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{-\beta}\int_r^\infty t^{(2-\alpha)q+\beta-2}|u^{(2)}(x,t+r)|^q dt\,dr.$$

It is easy to prove that $(2-\alpha)q+\beta-2>-1$. We need to study two cases:

Case #1 – if $(2-\alpha)q+\beta-2<0$: as $r<t$ and taking the change of variables $w=t+r$, we have

$$\begin{aligned}(I) &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{(2-\alpha)q-2}\int_r^\infty |u^{(2)}(x,t+r)|^q dt\,dr\\ &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{[(2-\alpha)q-1]-1}\int_{2r}^\infty |u^{(2)}(x,w)|^q dw\,dr\\ &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{[(2-\alpha)q-1]-1}\int_r^\infty |u^{(2)}(x,w)|^q dw\,dr.\end{aligned}$$

Then using Hardy's inequality (10.101) as $(2-\alpha)q-1>0$

$$(I) \leq C_\beta(1-\beta)^{1-q}\frac{1}{(2-\beta)q-1}\int_0^\infty \left(w^{2-\beta}|u^{(2)}(x,w)|\right)^q\frac{dw}{w}.$$

Case #2 – if $(2-\beta)q+\beta-2\geq 0$: taking the change of variables $w=t+r$, we get

$$\begin{aligned}(I) &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{-\beta}\int_r^\infty (t+r)^{(2-\alpha)q+\beta-2}|u^{(2)}(x,t+r)|^q dt\,dr\\ &= C_\beta(1-\beta)^{1-q}\int_0^\infty r^{-\beta}\int_{2r}^\infty w^{(2-\alpha)q+\beta-2}|u^{(2)}(x,w)|^q dw\,dr\\ &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{-\beta}\int_r^\infty w^{(2-\alpha)q+\beta-2}|u^{(2)}(x,w)|^q dw\,dr,\end{aligned}$$

and using Hardy's inequality (10.101),

$$(I) \leq \frac{C_\beta}{(1-\beta)^q}\int_0^\infty \left(w^{2-\alpha}|u^{(2)}(x,w)|\right)^q\frac{dw}{w}.$$

Therefore, in both cases we have

$$(I) \leq C_\beta\int_0^\infty \left(w^{2-\alpha}|u^{(2)}(x,w)|\right)^q\frac{dw}{w}.$$

To estimate (II), observe that $r^{-\beta} < t^{-\beta}$, for $r > t$ and $\beta > 0$, then using the same argument as before to estimate (I) case #1, taking the change of variables $w = t + r$, and using Hardy's inequality (10.101), so that

$$(II) \leq \frac{C_\beta}{1-\alpha} \int_0^\infty \left(w^{2-\alpha} |u^{(2)}(x,w)| \right)^q \frac{dw}{w}.$$

Finally,

$$\left\| \left(\int_0^\infty \left(t^{1-(\alpha-\beta)} |\frac{\partial}{\partial t} P_t(D^\beta f)| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma}$$

$$\begin{aligned} &\leq C \left\| (\int_0^\infty \left(t^{1-(\alpha-\beta)} \int_0^{+\infty} r^{-\beta} |u^{(2)}(\cdot, t+r)| dr \right)^q \frac{dt}{t})^{1/q} \right\|_{p,\gamma} \\ &\leq C \left\| \left(\int_0^\infty \left(w^{2-\beta} |u^{(2)}(\cdot, w)| \right)^q \frac{dw}{w} \right)^{1/q} \right\|_{p,\gamma} < \infty, \end{aligned} \tag{8.54}$$

as $f \in F^\beta_{p,q}(\gamma_d)$. Using the previous estimate and (8.52)

$$\|D^\beta f\|_{F^{\alpha-\beta}_{p,q}} \leq C \|f\|_{F^\beta_{p,q}}. \qquad \square$$

In the following theorem, we study the boundedness of the Bessel fractional derivative on Triebel–Lizorkin spaces for the bounded case $0 < \beta < \alpha < 1$.

Theorem 8.24. *Let $0 < \beta < \alpha < 1$, $1 \leq p, q < \infty$ then $\mathscr{D}^\beta$ is bounded from $F^\alpha_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in L^p(\gamma_d)$, using the fundamental theorem of calculus, we can write

$$\begin{aligned} |\mathscr{D}^\beta f(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} P_s f(x) - f(x)| ds \\ &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} |P_s f(x) - f(x)| ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} - 1| |f(x)| ds \\ &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s \left| \frac{\partial}{\partial r} P_r f(x) \right| dr\, ds + \frac{1}{c_\beta} |f(x)| \int_0^{+\infty} s^{-\beta-1} \int_0^s e^{-r} dr\, ds. \end{aligned}$$

Now, using Hardy's inequality (10.100) with $p = 1$ in both integrals, we have

$$|\mathscr{D}^\beta f(x)| \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} \left| \frac{\partial}{\partial r} P_r f(x) \right| \frac{dr}{r} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) |f(x)|.$$

Thus,

$$|\mathscr{D}^\beta f(x)| \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} |u^{(1)}(x,r)| \frac{dr}{r} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) |f(x)|.$$

Therefore, if $f \in F^{\alpha}_{p,q}(\gamma_d)$, we get

$$\begin{aligned}\|\mathscr{D}^{\beta} f\|_{p,\gamma} &\leq \frac{1}{\beta c_\beta}\Big\| \int_0^{+\infty} r^{1-\beta}|u^{(1)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma} + \frac{1}{\beta c_\beta}\Gamma(1-\beta)\|f\|_{p,\gamma}\\ &\leq C_\beta \|f\|_{F^{\beta}_{p,1}} \leq C'_\beta \|f\|_{F^{\alpha}_{p,q}}, \end{aligned} \tag{8.55}$$

because $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$, as $\alpha > \beta$, and $q \geq 1$.

Using a similar argument to that above, the fundamental theorem of calculus and Hardy's inequality (10.100) with $p = 1$, we get

$$\begin{aligned}\Big|\frac{\partial}{\partial t}P_t(\mathscr{D}^{\beta} f)(x)\Big| &\leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}\Big|e^{-s}\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)\Big| ds\\ &\leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}e^{-s}\Big|\frac{\partial}{\partial t}P_{t+s}f(x) - \frac{\partial}{\partial t}P_t f(x)\Big| ds\\ &\qquad + \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}|e^{-s}-1|\Big|\frac{\partial}{\partial t}P_t f(x)\Big| ds\\ &\leq \frac{1}{c_\beta}\int_0^{\infty} s^{-\beta-1}\int_0^{s} |u^{(2)}(x,t+r)|dr\,ds\\ &\qquad + \frac{1}{c_\beta}|u^{(1)}(x,t)|\int_0^{\infty} s^{-\beta-1}\int_0^{s} e^{-r}dr\,ds,\\ &\leq \frac{1}{\beta c_\beta}\int_0^{\infty} r^{-\beta}|u^{(2)}(x,t+r)|dr + \frac{\Gamma(1-\beta)}{\beta c_\beta}|u^{(1)}(x,t)|.\end{aligned}$$

Therefore,

$$\Big|\frac{\partial}{\partial t}P_t(\mathscr{D}^{\beta} f(x))\Big| \leq \frac{1}{\beta c_\beta}\int_0^{\infty} r^{-\beta}|u^{(2)}(x,t+r)|dr + \frac{\Gamma(1-\beta)}{\beta c_\beta}|u^{(1)}(x,t)|.$$

Then, we have

$$\begin{aligned}&\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\Big|\frac{\partial}{\partial t}P_t(\mathscr{D}^{\beta} f)\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}\\ &\leq \frac{C}{\beta c_\beta}\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_0^{\infty} r^{-\beta}|u^{(2)}(\cdot,t+r)|dr\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}\\ &\qquad + \frac{C}{\beta c_\beta}\Gamma(1-\beta)\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}|u^{(1)}(\cdot,t)|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}.\end{aligned}$$

Now, the first term can be estimated as in the proof of Theorem 3, estimates (8.53) and (8.54), so that

$$\Big\|\Big(\int_0^{\infty}\Big(t^{1-(\alpha-\beta)}\int_0^{\infty} r^{-\beta}|u^{(2)}(\cdot,t+r)|dr\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} \leq C\Big\|\int_0^{\infty}\Big(t^{2-\alpha}\Big|\frac{\partial^2}{\partial t^2}P_t f\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}$$

which is finite as $f \in F^{\alpha}_{p,q}(\gamma_d)$. For the second term, we have

$$\left\| \left(\int_0^\infty \left(t^{1-(\alpha-\beta)} |u^{(1)}(x,t)| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma} \leq C\|f\|_{F^{\alpha-\beta}_{p,q}} \leq C\|f\|_{F^{\alpha}_{p,q}}$$

as $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\alpha-\beta}_{p,q}(\gamma_d)$; thus,

$$\left\| \left(\int_0^\infty \left(t^{1-(\alpha-\beta)} |\frac{\partial}{\partial t} P_t(\mathscr{D}^\beta f)| \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma} \leq C\|f\|_{F^{\alpha}_{p,q}}.$$

Therefore, $\mathscr{D}^\beta f \in F^{\alpha-\beta}_{p,q}(\gamma_d)$ and moreover, using the previous estimate and (8.55)

$$\|\mathscr{D}^\beta f\|_{F^{\alpha-\beta}_{p,q}} \leq C\|f\|_{F^{\alpha}_{p,q}}.$$

□

To study the general case for fractional derivatives (removing the condition that the indexes must be less than 1), we need to consider forward differences again. Also, we need the generalized version of Hardy's inequality (see Theorem 10.26 in the Appendix, and also the following technical results):

Lemma 8.25. *For any positive integer k,*

$$\begin{aligned}\Delta_s^k(f,t) &= \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)dv_k\dots dv_2 dv_1\\ &= \int_0^s\cdots\int_0^s f^{(k)}(t+v_1+\dots+v_k)dv_k\dots dv_1\end{aligned}$$

For the proof of this result, see Lemma 10.30 in the Appendix, or [109] Lemma 3.1, *ii*).

Lemma 8.26. *Let $t \geq 0, \beta > 0$ and let k be the smallest integer greater than β, and let f differentiable up to order k, then*

$$\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds \leq C_{\beta,k}\int_0^{+\infty} w^{k-\beta-1}|f^{(k)}(t+w)|dw$$

where $C_{\beta,k} = \int_0^1\cdots\int_0^1 (v_1+\dots+v_k)^{\beta-k}dv_1\dots dv_k$

Proof. Using Lemma10.26, with $p=1$, and Lemma 8.25 we have,

$$\begin{aligned}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds &\leq \int_0^{+\infty} s^{-\beta-1}\int_0^s\cdots\int_0^s |f^{(k)}(t+v_1+\dots+v_k)|dv_1\dots dv_k ds\\ &\leq \int_0^1\cdots\int_0^1 \Big(\int_0^{+\infty}(s^k|f^{(k)}(t+s(v_1+\dots+v_k)|)s^{-\beta-1}ds\Big)dv_1\dots dv_k\\ &= \int_0^1\cdots\int_0^1 \Big(\int_0^{+\infty}(s^{k-\beta-1}|f^{(k)}(t+s(v_1+\dots+v_k)|)ds\Big)dv_1\dots dv_k\end{aligned}$$

taking $r = s(v_1 + \ldots + v_k)$ then $dr = (v_1 + \ldots + v_k)ds$,

$$\begin{aligned}\int_0^{+\infty} s^{k-\beta-1}|f^{(k)}(t+s(v_1+\ldots+v_k)|ds &= \int_0^{+\infty} r^{k-\beta}|f^{(k)}(t+r)|\frac{dr}{r}(v_1+\ldots+v_k)^{\beta-k}\\ &= \int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr(v_1+\ldots+v_k)^{\beta-k}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds\\ &\leq \int_0^1 \cdots \int_0^1 \Big(\int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr(v_1+\ldots+v_k)^{\beta-k}\Big)dv_1\ldots dv_k\\ &= \Big(\int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr\Big)\int_0^1 \cdots \int_0^1 (v_1+\ldots+v_k)^{\beta-k}dv_1\ldots dv_k\\ &= C_{\beta,k}\Big(\int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr,\end{aligned}$$

where $C_{\beta,k} = \int_0^1 \cdots \int_0^1 (v_1+\ldots+v_k)^{\beta-k}dv_1\ldots dv_k < \infty$. □

We need to use (8.47)

$$(P_s - I)^k f(x) = \Delta_s^k(u(x,\cdot),0),$$

and (8.48)

$$P_t(P_s - I)^k f(x) = \Delta_s^k(u(x,\cdot),t).$$

Let us consider the unbounded case $0 < \beta < \alpha$ for the Riesz derivative,

Theorem 8.27. *Let $0 < \beta < \alpha$, $1 \leq p,q < \infty$, then D^β is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in F^{\alpha}_{p,q}(\gamma_d)$, using (8.47), (8.48) and Lemma 8.26,

$$\begin{aligned}|D_\beta f(x)| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|(P_s-I)^k f(x)|ds\\ &= \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(u(x,\cdot),0)|ds\\ &\leq C_{\beta,k}\int_0^{+\infty} r^{k-\beta-1}|u^{(k)}(x,r)|dr.\end{aligned}$$

Then

$$\|D_\beta f\|_{p,\gamma} \leq C_{\beta,k}\Big\|\int_0^{+\infty} r^{k-\beta}|u^{(k)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma} < \infty,$$

because $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$, ($\alpha > \beta$ and $1 \leq q < \infty$).

Let $n \in \mathbb{N}, n > \alpha$; using Lemma 10.30(8.46) and Lemma 8.26, we get

$$\left|\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x)\right| \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(u^{(n)}(x,\cdot),t)|ds \\ \leq \frac{1}{c_\beta} C_{\beta,k}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr.$$

Therefore,

$$\int_0^\infty \left(t^{n-(\alpha-\beta)}\left|\frac{\partial^n}{\partial t^n}P_t(D_\beta f)(x)\right|\right)^q \frac{dt}{t} \\ \leq C_{\beta,k}\int_0^\infty \left(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr\right)^q \frac{dt}{t}$$

which is inequality (8.53) for $n = k = 1$. The rest of the proof follows the argument used in Theorem 8.23, so that

$$\left(\int_0^\infty \left(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr\right)^q \frac{dt}{t}\right)^{1/q} \\ \leq C\left(\int_0^\infty \left(s^{n+k-\alpha}|u^{(n+k)}(x,s)|\right)^q \frac{ds}{s}\right)^{1/q}, \tag{8.56}$$

taking $L^p(\gamma)$-norm both sides of the inequality, we get the result. □

Finally, the following result extends Theorem 8.24 to the general case $0 < \beta < \alpha$:

Theorem 8.28. *Let $0 < \beta < \alpha$, $1 < p < \infty$ and $1 \leq q < \infty$, then $\mathscr{D}^\beta$ is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof. Let $f \in F^{\alpha}_{p,q}(\gamma_d)$, k be an integer such that $k-1 \leq \beta < k$ and $v(x,r) = e^{-r}u(x,r)$, using Lemma 8.26 and Leibniz's differentiation rule for the product

$$|\mathscr{D}^\beta f(x)| \leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|(e^{-s}P_s - I)^k f(x)|ds = \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(v(x,\cdot),0)|ds \\ \leq C_{\beta,k}\int_0^{+\infty} r^{k-\beta}|v^{(k)}(x,r)|\frac{dr}{r} \leq C_{\beta,k}\left(\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta}e^{-r}|u^{(k-j)}(x,r)|\frac{dr}{r}\right) \\ = C_{\beta,k}\left(\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{k-\beta}e^{-r}|u^{(k-j)}(x,r)|\frac{dr}{r}\right) + C_{\beta,k}\int_0^{+\infty} r^{k-\beta}e^{-r}|u(x,r)|\frac{dr}{r},$$

then

$$\begin{aligned}
\|\mathscr{D}^\beta f\|_{p,\gamma} &\le C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\Big\|\int_0^{+\infty} r^{k-\beta}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\Big)+C_{\beta,k}\Big\|\int_0^{+\infty} r^{k-\beta}e^{-r}|u(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\\
&\le C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\Big\|\int_0^{+\infty} r^{k-\beta}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\Big)+C_{\beta,k}\int_0^{+\infty} r^{k-\beta}e^{-r}\|u(\cdot,r)\|_{p,\gamma}\frac{dr}{r}\\
&\le C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\Big\|\int_0^{+\infty} r^{k-j-(\beta-j)}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\Big\|_{p,\gamma}\Big)+C_{\beta,k}\|f\|_{p,\gamma}\Gamma(k-\beta)\\
&\le C\|f\|_{F^\alpha_{p,q}},
\end{aligned}$$

because $F^\alpha_{p,q}(\gamma_d)\subset F^{\beta-j}_{p,1}(\gamma_d)$, as $\alpha>\beta\ge\beta-j\ge 0$, for $j=0,\dots,k-1$ and $q\ge 1$.

Now, let $n\in\mathbb{N}, n>\alpha$ and $w(x,t)=e^{-t}u^{(n)}(x,t)$, using Lemma 8.26, we get

$$\begin{aligned}
\Big|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^\beta f)(x)\Big| &\le \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(w(x,\cdot),t)|ds\\
&\le e^t C_{\beta,k}\int_0^{+\infty} s^{k-\beta-1}|w^{(k)}(x,t+s)|ds.
\end{aligned}$$

Now, using Leibniz's rule, $w^{(k)}(x,r)=\sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-r}u^{(k+n-j)}(x,r)$ and then

$$|w^{(k)}(x,r)|\le\sum_{j=0}^{k}\binom{k}{j}e^{-r}|u^{(k+n-j)}(x,r)|,$$

for all $r>0$. Thus,

$$\Big|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^\beta f)(x)\Big|\le C_{\beta,k}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} s^{k-\beta-1}e^{-s}|u^{(k+n-j)}(x,t+s)|ds.$$

Therefore,

$$\begin{aligned}
&\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\Big|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^\beta f)(x)\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\\
&\le C_{\beta,k}\sum_{j=0}^{k}\binom{k}{j}\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{k-j-(\beta-j)-1}e^{-s}|u^{(k-j+n)}(x,t+s)|ds\Big)^q\frac{dt}{t}\Big)^{1/q}
\end{aligned}$$

For $0\le j\le k-1$, we have $\beta-j\ge\beta-(k-1)\ge 0$, and taking into account that each term of the above sum is bounded by the left side of the inequality (8.56), with k replaced by $k-j$ and β replaced by $\beta-j$, we get that

$$\Big\|\Big(\int_0^\infty\Big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{k-j-(\beta-j)-1}e^{-s}|u^{(k+n-j)}(x\cdot,t+s)|ds\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma}<C\|f\|_{F^\alpha_{p,q}}$$

for $0 \le j \le k-1$. Unfortunately, the remaining case $j = k$ requires a special argument that uses the following known inequality for the Poisson–Hermite semigroup:

$$\Big|\frac{\partial^n}{\partial t^n} P_t f(x)\Big| \le C T^* f(x) t^{-n} \tag{8.57}$$

(see Lemma 3.4; see also [226, Lemma 1], or [224]). Then

$$\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)} \int_0^{+\infty} s^{k-\beta-1} e^{-s} |u^{(n)}(\cdot, t+s)| ds\Big)^q \frac{dt}{t}\Big)^{1/q}$$

$$\begin{aligned}
&\le C\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)} \int_0^t s^{k-\beta-1} e^{-s} |u^{(n)}(x, t+s)| ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&\quad + C\Big(\int_0^\infty \Big(t^{n-(\alpha-\beta)} \int_t^{+\infty} s^{k-\beta-1} e^{-s} |u^{(n)}(x, t+s)| ds\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&= (I) + (II).
\end{aligned}$$

We first consider the case $k \le \beta$. The term (I) is estimated as term (I) in the proof of Theorem 8.23.

$$(I) \le C\Big(\int_0^\infty \Big(v^{n-(\alpha-k)} |u^{(n)}(x, v)|\Big)^q \frac{dv}{v}\Big)^{1/q}.$$

Because $\beta \ge k-1$, taking the change of variables $v = t+s$, we get

$$\begin{aligned}
(II) &\le C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_t^{+\infty} |u^{(n)}(x, t+s)| ds\Big)^q dt\Big)^{1/q}\\
&= C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_{2t}^{+\infty} |u^{(n)}(x, r)| dr\Big)^q dt\Big)^{1/q}\\
&\le C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_t^{+\infty} |u^{(n)}(x, r)| dr\Big)^q dt\Big)^{1/q}.
\end{aligned}$$

Therefore, using Hardy's inequality (10.101),

$$(II) \le \frac{C}{(n+k-\alpha-1)^{1/q}} \Big(\int_0^\infty \Big(r^{n-(\alpha-k)} |u^{(n)}(x, r)|\Big)^q \frac{dr}{r}\Big)^{1/q},$$

Next, consider the case $k > \alpha$. In this case, using inequality (8.57) and Hardy's inequality (10.100), we have

$$\begin{aligned}
(I) &\le C_n |T^* f(x)| \Big(\int_0^\infty t^{-(\alpha-\beta)q-1} \Big(\int_0^t s^{k-\beta-1} e^{-s} ds\Big)^q dt\Big)^{1/q}\\
&\le C_n |T^* f(x)| \frac{1}{(\alpha-\beta)^{1/q}} \Big(\int_0^\infty s^{(k-\alpha)q-1} e^{-sq} ds\Big)^{1/q}\\
&= C_n |T^* f(x)| \frac{1}{(\alpha-\beta)^{1/q} q^{k-\alpha}} \Big(\Gamma((k-\alpha)q)^{1/q}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(II) &\leq \Big(\int_0^1 t^{(n+k-\alpha-1)q-1}\Big(\int_t^{+\infty} e^{-s}|u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&\quad + \Big(\int_1^{\infty} t^{(n+k-\alpha-1)q-1}\Big(\int_t^{+\infty} e^{-s}|u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&= (III)+(IV).
\end{aligned}
$$

Using the usual argument the change of variables $v = t+s$ and Hardy's inequality (10.101), we get

$$
\begin{aligned}
(III) &\leq \Big(\int_0^1 t^{(n-1)q-1}\Big(\int_t^{+\infty} |u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&\leq \Big(\int_0^{\infty} t^{(n-1)q-1}\Big(\int_t^{+\infty} |u^{(n)}(x,t+s)|ds\Big)^q dt\Big)^{1/q}\\
&= \Big(\int_0^{\infty} t^{(n-1)q-1}\Big(\int_{2t}^{+\infty} |u^{(n)}(x,r)|dr\Big)^q dt\Big)^{1/q}\\
&\leq \Big(\int_0^{\infty} t^{(n-1)q-1}\Big(\int_t^{+\infty} |u^{(n)}(x,r)|dr\Big)^q dt\Big)^{1/q}\\
&\leq \frac{1}{n-1}\Big(\int_0^{\infty}\Big(r^n|u^{(n)}(x,r)|\Big)^q \frac{dr}{r}\Big)^{1/q}.
\end{aligned}
$$

Finally, using inequality (8.57) again, we get

$$
\begin{aligned}
(IV) &\leq \Big(\int_1^{\infty} t^{(n+k-\alpha-1)q-1}\Big(\int_t^{+\infty} e^{-s}C_n|T^*f(x)|t^{-n}ds\Big)^q dt\Big)^{1/q}\\
&= C_n|T^*f(x)|\Big(\int_1^{\infty} t^{(k-\alpha-1)q-1}e^{-tq}dt\Big)^{1/q}\\
&\leq C_n|T^*f(x)|\Big(\int_1^{\infty} t^{(k-\alpha-1)q-1}dt\Big)^{1/q} = C_n|T^*f(x)|\Big(\frac{1}{(\alpha+1-k)q}\Big)^{1/q}.
\end{aligned}
$$

Hence, in both cases, we get that

$$
\Big\|\Big(\int_0^{\infty}\Big(t^{n-(\alpha-\beta)}\Big|\frac{\partial^n}{\partial t^n}P_t(\mathscr{D}^{\beta}f)\Big|\Big)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{p,\gamma} < \infty,
$$

as $f \in F^{\alpha}_{p,q}(\gamma_d)$. Therefore, $\mathscr{D}^{\beta}f \in F^{\alpha-\beta}_{p,q}(\gamma_d)$ and moreover,

$$
\|\mathscr{D}^{\beta}f\|_{F^{\alpha-\beta}_{p,q}} \leq C\|f\|_{F^{\alpha}_{p,q}}. \qquad \square
$$

8.6 Notes and Further Results

1. Observe that the arguments given in the proofs of theorems in this chapter are still valid in the classical case taking the Poisson integral; therefore, they are alternative proofs to those given in E. Stein's book [252].

2. Moreover, if instead of considering the *Ornstein–Uhlenbeck operator* and the *Poisson–Hermite semigroup*, we consider the *Laguerre differential operator* in $\mathbb{R}^d_+$, for $\alpha = (\alpha_1, \cdots, \alpha_d)$ a multi-index,

$$\mathscr{L}^\alpha = \sum_{i=1}^{d} \left[x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right] \tag{8.58}$$

and the corresponding *Poisson–Laguerre semigroup*, or if we consider the *Jacobi differential operator* in $(-1,1)^d$,

$$\mathscr{L}^{\alpha,\beta} = -\sum_{i=1}^{d} \left[(1 - x_i^2) \frac{\partial^2}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial}{\partial x_i} \right], \tag{8.59}$$

and the corresponding *Poisson–Jacobi semigroup* (for more details, we refer the reader to [279]), the arguments are completely analogous. To see this, it is more convenient to use the representation of P_t in terms of the one-sided stable measure $\mu_t^{(1/2)}(ds)$ and to write Lemma 3.3 in those terms (see [225]). In other words, we can define in an analogous manner *Laguerre–Lipschitz spaces* and *Jacobi–Lipschitz spaces,* and prove the corresponding notions of fractional integrals and fractional derivatives (see [117, 25]).

3. Following similar arguments to those given in Chapter 7, we can define in an analogous manner *Laguerre–Besov–Lipschitz spaces* and *Jacobi–Besov–Lipschitz spaces*, in addition to *Laguerre–Triebel–Lizorkin spaces* and *Jacobi–Triebel–Lizorkin spaces,* and then prove that the corresponding notions of fractional integrals and fractional derivatives of corresponding operators $\mathscr{L}^{\alpha,\beta}$ and $\mathscr{L}^\alpha$ behave similarly.

4. In [146], G. E. Karadzhov & M. Milman show that the Gaussian Riesz potentials I_β maps $L^p(\log L)_a$ continuously into $L^p(\log L)_{a+\beta}$, for $1 < p < \infty$ and $a \in \mathbb{R}$. The proof is using extrapolation in an abstract setting. Moreover, their proof is in fact valid for any hypercontractive semigroup (see [146, Theorem 5.7]).

5. We can also consider *alternative Riesz potentials, alternative Bessel potential, alternative Riesz and alternative Bessel fractional derivatives* using the same formulas as before, but with respect to $\overline{L}$, the alternative Ornstein–Uhlenbeck operator (2.14). This case is actually simpler, as 0 is not a eigenvalue of $\overline{L}$. For instance, for $\beta > 0$ the alternative Riesz potential $\overline{I}_\beta$ can be defined as

$$\overline{I}_\beta = (-\overline{L})^{-\beta/2}, \tag{8.60}$$

meaning that any multi-index ν such that $|\nu| > 0$ its action on the Hermite polynomial $\mathbf{H}_\nu$ is

$$\overline{I}_\beta \mathbf{H}_\nu(x) = \frac{1}{(|\nu| + d)^{\beta/2}} \mathbf{H}_\nu(x). \tag{8.61}$$

$\overline{I}_\beta$ has the following integral representation, using the fact that $\overline{L}$ is the infinitesimal generator of the semigroup $\{T_t^{(d)}\}_t = \{e^{-td}T_t\}_t$, the d-translated Ornstein–Uhlenbeck semigroup (2.78),

$$\begin{aligned}\overline{I}_\beta f(x) = (-\overline{L})^{-\beta/2} f(x) &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta-2}{2}} T_t^{(d)} f(x)\, dt \\ &= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta-2}{2}} e^{-dt} T_t f(x)\, dt \\ &= C_\beta e^{|x|^2} \int_{\mathbb{R}^d} \left(\int_0^1 (-\log r)^{\frac{\beta-2}{2}} r^d \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \right) f(y)\, \gamma_d(dy). \\ &= C_\beta \int_{\mathbb{R}^d} \left(\int_0^1 (-\log r)^{\frac{\beta-2}{2}} r^d \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \right) f(y) (dy).\end{aligned} \tag{8.62}$$

The integral representation (8.62) is crucial to getting the $L^p(\gamma_d)$-boundedness results of some of the Gaussian singular integrals considered in Chapter 9.

Similar representations can be found for Bessel potentials and the fractional derivatives associated with $\overline{L}$.

6. In [164], alternate representations of I_β and D_β are obtained.

Proposition 8.29. *Suppose $f \in C_B^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, then*

$$D^\beta f = \frac{1}{\beta c_\beta} \int_0^\infty t^{-\beta} \frac{\partial}{\partial t} P_t f dt, \quad 0 < \beta < 1, \tag{8.63}$$

$$I_\beta f = -\frac{1}{\beta \Gamma(\beta)} \int_0^\infty t^\beta \frac{\partial}{\partial t} P_t f dt, \quad \beta > 0. \tag{8.64}$$

Proof. Let us start proving (8.63). Integrating by parts in (8.26) ,we get

$$\begin{aligned} D_\beta f(x) &= \frac{1}{c_\beta} \lim_{\substack{a\to 0^+ \\ b\to\infty}} \int_a^b t^{-\beta-1} \left(P_t f(x) - f(x) \right) dt \\ &= \frac{1}{c_\beta} \lim_{\substack{a\to 0^+ \\ b\to\infty}} \left\{ \frac{t^{-\beta}}{-\beta} \left(P_t f(x) - f(x) \right) \Big|_a^b + \frac{1}{\beta} \int_a^b t^{-\beta} \frac{\partial}{\partial t} P_t f(x) dt \right\} \\ &= \frac{1}{\beta c_\beta} \int_0^\infty t^{-\beta} \frac{\partial}{\partial t} P_t f(x) dt \end{aligned}$$

because, using (3.28) and (3.29), we have

$$\lim_{b\to\infty} \left(\frac{P_b f(x) - f(x)}{b^\beta} \right) = 0$$

and

$$\lim_{a\to 0^+}\left|\frac{P_a f(x)-f(x)}{a^\beta}\right| \le \lim_{a\to 0^+}\frac{1}{a^\beta}\int_0^a\left|\frac{\partial}{\partial s}P_s f(x)\right|ds$$
$$\le C_{d,f}(d+|x|)\lim_{a\to 0^+}\frac{1-e^{-a}}{a^\beta}=0.$$

Let us prove now (8.64). Again, by integrating by parts, we have

$$\begin{aligned} I_\beta f(x) &= \frac{1}{\Gamma(\beta)}\lim_{\substack{a\to 0^+\\ b\to\infty}}\int_a^b t^{\beta-1}P_t f(x)dt \\ &= \frac{1}{\Gamma(\beta)}\lim_{\substack{a\to 0^+\\ b\to\infty}}\left\{\frac{t^\beta}{\beta}P_t f(x)\Big|_a^b-\frac{1}{\beta}\int_a^b t^\beta\frac{\partial}{\partial t}P_t f(x)dt\right\} \\ &= -\frac{1}{\beta\Gamma(\beta)}\int_0^\infty t^\beta\frac{\partial}{\partial t}P_t f(x)dt, \end{aligned}$$

because, using the previous result

$$\lim_{b\to\infty}\left|P_b f(x)b^\beta\right| \le C_{d,f}(d+|x|)\lim_{b\to\infty}b^\beta e^{-b}=0$$

and

$$\lim_{a\to 0^+}\left|P_a f(x)a^\beta\right|=0. \qquad \square$$

Observe that because the previous proposition holds for $f=\mathbf{H}_\beta$, the Hermite polynomial of order $\beta, |\beta|>0$, then it holds for any non-constant polynomial f such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy)=0$.

By using (3.3) and (8.63), D_β can be expressed explicitly as

$$D_\beta f(x)=\int_{\mathbb{R}^d}K_\beta(x,y)f(y)dy,$$

where,

$$K_\beta(x,y)=C_d\int_0^\infty\int_0^1 t^{-\beta}e^{t^2/4logr}(-logr)^{1/2}\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2}}$$
$$\times\left(\frac{2r^2\left|y-rx\right|^2-2r(1-r^2)\langle y-rx,x\rangle-dr^2(1-r^2)}{(1-r^2)^2}\right)\frac{dr}{r}dt.$$

Now, let us write

$$q_t(x,y)=-t\frac{\partial}{\partial t}\left(\int_0^1 t\frac{\exp\left(t^2/4\log r\right)}{(-\log r)^{3/2}}\frac{\exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}}\frac{dr}{r}\right), \tag{8.65}$$

and define the operator Q_t as

$$Q_t f(x) = -t\frac{\partial}{\partial t}P_t f(x) = \int_{\mathbb{R}^d} q_t(x,y)f(y)dy. \tag{8.66}$$

Following [108] we immediately get from (8.63) and (8.64) the following formulas:

Corollary 8.30. *Suppose $f \in C_B^2(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$. Then, we have*

$$-\beta D_\beta f = \frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}Q_t f dt, 0<\beta<1, \tag{8.67}$$

$$\beta I_\beta = \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}Q_t f dt, \beta>0. \tag{8.68}$$

7. An interesting use of the family $\{Q_t\}$ is that it allows us to give a version of *A. P. Calderón's reproducing formula* for the Gaussian measure; see [164].

Theorem 8.31.
i) Suppose $f \in L^1(\gamma_d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, then we have

$$f = \int_0^\infty Q_t f\frac{dt}{t}. \tag{8.69}$$

ii) Suppose f a polynomial such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, then we have

$$f = C_\beta\int_0^\infty\int_0^\infty t^{-\beta}s^\beta Q_t\left(Q_s f\right)\frac{ds}{s}\frac{dt}{t} \quad 0<\beta<1. \tag{8.70}$$

Also,

$$\int_0^\infty\int_0^\infty t^{-\beta}s^\beta Q_t\left(Q_s f\right)\frac{ds}{s}\frac{dt}{t} = d_\beta\int_0^\infty u\frac{\partial^2}{\partial u^2}P_u f du. \tag{8.71}$$

Formula (8.70) is the aforementioned version of Calderón's reproducing formula for the Gaussian measure.

Proof.
i) Using (3.28) and (3.29) we have,

$$\int_0^\infty Q_t f\frac{dt}{t} = \lim_{\substack{a\to 0^+\\ b\to\infty}}\left(-\int_a^b \frac{\partial}{\partial t}P_t f dt\right) = \lim_{\substack{a\to 0^+\\ b\to\infty}}\left(-P_t f\right)\Big|_a^b = f.$$

ii) Let us prove (8.70), given f, a polynomial such that $\int_{\mathbb{R}^d} f(y)\gamma_d(dy) = 0$, by Corollary 8.30, we have

$$D_\beta\left(I_\beta f\right) = \frac{1}{\beta c_\beta}\int_0^\infty t^{-\beta-1}Q_t\left(I_\beta f\right)dt.$$

Now, using the definition of Q_t and Fubini's theorem, we have

$$Q_t\left(I_\beta f\right) = \frac{1}{\beta\Gamma(\beta)}\int_{\mathbb{R}^d}\int_0^\infty s^{\beta-1}Q_s(f)(y)dsdy.$$

Again, using the definition of Q_s, we obtain

$$f = D_\beta\left(I_\beta f\right) = d_\beta\int_0^\infty\int_0^\infty t^{-\beta-1}s^{\beta-1}Q_t\left(Q_s f\right)dsdt.$$

To show (8.71), we see that from (8.66)

$$Q_t\left(Q_s f\right)(x) = ts\frac{\partial}{\partial t}\frac{\partial}{\partial s}P_{t+s}f(x).$$

But

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}P_{t+s}f(x) = \frac{\partial^2}{\partial u^2}P_u f(x)\Big|_{u=t+s},$$

then

$$\begin{aligned}\int_0^\infty\int_0^\infty t^{-\beta-1}s^{\beta-1}Q_t\left(Q_s f\right)dsdt &= \int_0^\infty\int_0^\infty t^{-\beta}s^{\beta}\frac{\partial^2}{\partial u^2}P_u f\Big|_{u=t+s}dsdt\\ &= d_\beta\int_0^\infty u\frac{\partial^2}{\partial u^2}P_u f du,\end{aligned}$$

where $d_\beta = \frac{B(-\beta+1,\beta+1)}{a_\beta c_\beta}$, $B(-\beta+1,\beta+1)$ being the beta function of parameter $(-\beta+1,\beta+1)$. □

8. Also, in [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, and W. Urbina obtained an analog of A. P. Calderón's reproducing formula for the Laguerre case.

9. Using more abstract approaches to Besov and Triebel–Lizorkin spaces associated with a general differential operator, as in [154], many of the results contained in this chapter would follow from the functional calculus for the Ornstein–Uhlenbeck operator.

9

Singular Integrals with Respect to the Gaussian Measure

Singular integrals are among the most important operators in classical harmonic analysis. They first appear naturally in the proof of the $L^p(\mathbb{T})$ convergence of Fourier series, $1 < p < \infty$, where the notion of the *conjugated function* is needed[1]

$$\tilde{f}(x) = p.v.\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x-y)}{2\tan\frac{y}{2}}dy = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\pi>|y|>\varepsilon}\frac{f(x-y)}{2\tan\frac{y}{2}}dy.$$

This notion was extended to the non-periodic case with the definition of the *Hilbert transform,*

$$Hf(x) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x-y)}{y}dy = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy,$$

and then to $\mathbb{R}^d$, with the notion of the *Riesz transform* (see E. Stein [252, Chap III, §1]),

$$\begin{aligned} R_j f(x) &= p.v.\, C_d\int_{\mathbb{R}^d}\frac{y_i}{|y|^{d+1}}f(x-y)dy \\ &= \lim_{\varepsilon\to 0} C_d\int_{|y|>\varepsilon}\frac{y_j}{|y|^{d+1}}f(x-y)dy, \end{aligned} \tag{9.1}$$

for $j = 1,\cdots,d$, $f \in L^p(\mathbb{R}^d)$ with $C_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}}$. Taking Fourier transform, we get

$$\widehat{(R_j f)}(\zeta) = C_d\int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d}\frac{y_j}{|y|^{d+1}}f(x-y)dy\right]e^{-i<\xi,x>}dx$$

The original version of this chapter was revised. The correction to this chapter is available at https://doi.org/10.1007/978-3-030-05597-4_10

[1] For a detailed study of this problem see, for instance, R. Weeden & A. Zygmund [294, Chapter 12], E. Stein [252, Chapter II, III], J. Duoandikoetxea [72, Chapter 4, 5], L. Grafakos [118, Chapter 4] or A. Torchinski [275, Chapter XI].

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4_9

$$= C_d \int_{\mathbb{R}^d} \frac{y_j}{|y|^{d+1}} \Big[\int_{\mathbb{R}^d} f(x-y) e^{-i<\xi,x>} dx \Big] dy$$

$$= C_d \int_{\mathbb{R}^d} \frac{y_j}{|y|^{d+1}} e^{-i<\xi,y>} \hat{f}(\xi) dy = C_d\, i \frac{\xi_j}{|\xi|} \hat{f}(\xi).$$

Hence,

$$\widehat{(R_j f)}(\zeta) = i \frac{\zeta_j}{|\zeta|} \hat{f}(\zeta);$$

thus, $R_j f$ is a classical multiplier operator, with multiplier $m(y) = C_d\, i \frac{y_j}{|y|}$, and hence

$$R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2} \tag{9.2}$$

where $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator and $(-\Delta)^{-1/2}$ is the (classical) *Riesz potential* of order $1/2$. For more details on this, see E. Stein [252, Chap V].

Moreover, we have seen (see 2.2), $e^{i<\cdot,y>}$, $|y|^2 = -\lambda$, for $\lambda < 0$ are the eigenfunctions of the Laplacian, then,

$$R_j(e^{<\cdot,y>})(x) = -\frac{1}{|y|} \frac{\partial}{\partial x_j} e^{i<x,y>} = -i \frac{y_j}{|y|} e^{i<x,y>} = -i \frac{y_j}{\sqrt{\lambda}} e^{i<x,y>}. \tag{9.3}$$

In their seminal paper [43], A. P. Calderón and A. Zygmund considered a general class of singular operators in $\mathbb{R}^d$, which is nowadays called the Calderón–Zygmund theory.

In this chapter, we consider singular integrals with respect to the Gaussian measure. Singular integrals have been, without any doubt, one of the topics in Gaussian harmonic analysis that have been more extensively researched over the last 40 years. We begin with the study of the Gaussian Riesz transform, then the higher-order Gaussian Riesz transforms, and finally, we consider a fairly general class of Gaussian singular integrals initially studied by W. Urbina in [278] and later extended by S. Pérez in [220]. For completeness, and to facilitate comprehension of the topic, we give full proof of the boundedness properties in each case, even though the Gaussian Riesz transform and higher-order Gaussian Riesz transforms are particular cases of the general class of Gaussian singular integrals that we are going to study in Section 9.4. Additionally, in Section 9.3, we study an alternative class of Riesz transforms introduced by H. Aimar, L. Forzani, and R. Scotto in [5].

9.1 Definition and Boundedness Properties of the Gaussian Riesz Transforms

In analogy with the classical case (9.2), the Gaussian Riesz transforms in $\mathbb{R}^d$ are defined in terms of the Gaussian derivatives and Riesz potentials.

Definition 9.1. *The Gaussian j-th Riesz transform in* $\mathbb{R}^d$ *is defined spectrally, for* $1 \le i \le d$, *as*

$$\mathscr{R}_j = \partial_\gamma^j I_{1/2} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_j}(-L)^{-1/2}, \tag{9.4}$$

where $L = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle$ *is the Ornstein–Uhlenbeck operator,* $I_{1/2}$ *the Gaussian Riesz potential of order* $1/2$, *and* $\partial_i^\gamma = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_i}$ *is the Gaussian partial derivative with respect to the variable* x_i. *The meaning of this is that for any multi-index* ν *such that* $|\nu| > 0$, *its action on the Hermite polynomial* $\mathbf{H}_\nu$ *is*

$$\mathscr{R}_j \mathbf{H}_\nu = \sqrt{\frac{2}{|\nu|}} \nu_j \mathbf{H}_{\nu - \mathbf{e}_j} \tag{9.5}$$

where $\mathbf{e_j}$ *is the unitary vector with zeros in all coordinates except for the j-th coordinate, which is one.*

Observe that (9.5) is the Gaussian analogous to (9.3). Moreover, for the normalized Hermite polynomials $\mathbf{h}_\nu$, we have

$$\mathscr{R}_j \mathbf{h}_\nu = \mathscr{R}_j \Big(\frac{\mathbf{H}_\nu}{(2^{|\nu|}\nu!)^{1/2}} \Big) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}} \sqrt{\frac{2}{|\nu|}} \nu_j \mathbf{H}_{\nu - \mathbf{e_j}} = \mathbf{h}_{\nu - \mathbf{e_j}}. \tag{9.6}$$

From the integral representation of the Riesz potential (8.8), obtained in Theorem 8.3, using the kernel (8.9), we immediately get the kernel of $\mathscr{R}_j$,

$$\begin{aligned} \mathscr{K}_j(x,y) &= \frac{\partial}{\partial x_i} N_{1/2}(x,y) \\ &= \frac{1}{\pi^{d/2}\Gamma(1/2)} \int_0^1 \left(\frac{1-r^2}{-\log r} \right)^{1/2} \frac{y_j - r x_j}{(1-r^2)^{\frac{(d+3)}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}} dr; \end{aligned} \tag{9.7}$$

therefore, we get the integral representation of $\mathscr{R}_i$,

$$\begin{aligned} \mathscr{R}_j f(x) &= p.v. \int_{\mathbb{R}^d} \mathscr{K}_j(x,y) f(y) dy \qquad (9.8) \\ &= p.v. \frac{1}{\pi^{d/2}\Gamma(1/2)} \int_{\mathbb{R}^d} \Big(\int_0^1 \left(\frac{1-r^2}{-\log r} \right)^{1/2} \frac{y_j - r x_j}{(1-r^2)^{\frac{(d+3)}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}} \Big) dr f(y) dy. \end{aligned}$$

In particular, for $d = 1$, the *Gaussian Hilbert transform* is defined spectrally as

$$\mathscr{H} = \partial^\gamma I_{1/2} = \frac{1}{\sqrt{2}} \frac{d}{dx}(-L)^{-1/2}. \tag{9.9}$$

meaning that

$$\mathscr{H} H_n(x) = \frac{1}{\sqrt{2}} \frac{d}{dx}((-L)^{-1/2} H_n(x)) = \frac{1}{\sqrt{2n}} \frac{d}{dx} H_n(x) = \sqrt{2n} H_{n-1}(x).$$

As a particular case of (9.8), we get the following integral representation of $\mathscr{H}$

$$\begin{aligned}\mathscr{H}f(x) &= p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\Big(\int_0^1 (\frac{1-r^2}{-\log r})^{1/2}\frac{y-rx}{(1-r^2)^2} \\ &\qquad \times \exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})\,dr f(y)\Big)\,\gamma_1(dy) \\ &= p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\Big(\int_0^1 (\frac{1-r^2}{-\log r})^{1/2}\frac{y-rx}{(1-r^2)^2}e^{-\frac{-|y-rx|^2}{1-r^2}}\,dr\Big) f(y)\,dy.\end{aligned}$$

Theorem 9.2. *The Gaussian Riesz transforms $\mathscr{R}_j,\ j=1,\cdots,d$ are $L^p(\gamma_d)$ bounded for $1<p<\infty$, that is to say, there exists $C>0$, depending on p, β and dimension d such that*

$$\|\mathscr{R}_j f\|_{p,\gamma} \leq \|f\|_{p,\gamma}, \tag{9.10}$$

for any $f\in L^p(\gamma_d)$.

In 1969, B. Muckenhoupt considered the one-dimensional case of the Gaussian Hilbert transform $\mathscr{H}$, using real analysis methods, based on Natanson's lemma (see Lemma 10.27). Then, in 1984, P. A. Meyer [189] established the $L^p(\gamma_d)$-boundedness of the Gaussian Riesz transforms $\mathscr{R}_j$ with respect to the Gaussian measure $\gamma_d(dx)$ in $\mathbb{R}^d$, for $1<p<\infty$, using probabilistic methods, by considering the Brownian motion and the famous Burkholder–Gundy inequality (see also [82] for a simpler proof of P. A. Meyer's theorem). After these two landmark papers, several other proofs of the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_j$ were obtained. In 1986, R. Gundy [121] got one, also by using the Brownian motion and the notion of background radiation as a stochastic process, and G. Pisier [227] got one by using the method of rotations and transference methods introduced in [57] by R. Coifman and G. Weiss. In both proofs, the estimates are independent of dimension. In 1988, W. Urbina [278], in his doctoral dissertation, got the first proof using real analysis methods in $\mathbb{R}^d$, $d>1$ by studying the kernel directly, extending B. Muckenhoupt's proof to the higher dimensional case, but the constants are strongly dependent on dimension. Then, in 1994, C. Gutiérrez [122] got an alternative proof, using the Littlewood–Paley–Stein theory, with constants independent of dimension. Finally, in 1996, S. Pérez, S. & F. Soria [223] (see also [220]), got an alternative real analysis proof using refined estimates of the kernel, with constants dependent on dimension, by using analog estimates of those they obtained for the maximal function of the Ornstein–Uhlenbeck semigroup.

On the other hand, the weak type $(1,1)$ with respect to γ_d of $\mathscr{R}_j$ was proved by B. Muckenhoupt, in the case $d=1$, in his 1969 paper [194], and R. Scotto proved it for the case $d>1$ in his doctoral dissertation [244] (see also [77]), by using the method developed by P. Sjögren in [247] to prove the weak type $(1,1)$ of T^*, the maximal function of the Ornstein–Uhlenbeck semigroup already discussed in Chapter 4. Also, S. Pérez has an alternative proof of this result (see [220, 221]).

Theorem 9.3. *(Scotto) There exists a constant C such that*

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \mathscr{R}_j f(x) > \lambda\Big\}\Big) \leq \frac{C}{\lambda}||f||_{1,\gamma_d}. \tag{9.11}$$

for all $f \in L^1(\gamma_d)$.

Observe that, in general, if T is a linear operator associated with a given kernel $K(x,y)$, its adjoint with respect to the Gaussian measure has kernel $K\ (x,y) = K(y,x)e^{|x|^2-|y|^2}$. Then, as $\mathscr{K}_j(y,x)e^{|x|^2-|y|^2} = \mathscr{K}_j(x,y)$, it follows that the adjoint of $\mathscr{R}_j$ is also of weak type $(1,1)$ with respect to γ_d.

To prove Theorem 9.2 and Theorem 9.3, we essentially follow the proof given by S. Pérez, S. and F. Soria in [223]. As was done for the Ornstein–Uhlenbeck maximal function T^*, we split the operator $\mathscr{R}_j$ into a local part and a global part. Given $x \in \mathbb{R}^d$, the *local part* of the operator $\mathscr{R}_j$ is its restriction to the admissible ball

$$B_h(x) = B(x, d\,m(x)) = \{y \in \mathbb{R}^d : |y-x| < d\,m(x)\},$$

and we have seen that the Gaussian density is essentially constant on admissible balls (see 4.102). The *global part* of the operator $\mathscr{R}_j$ is its restriction to the complement of $B_h(x)$. Thus,

$$\begin{aligned}\mathscr{R}_j f(x) &= C_d \int\limits_{|x-y|<d\,m(x)} \mathscr{K}_j(x,y) f(y) dy + C_d \int\limits_{|x-y|\geq d\,m(x)} \mathscr{K}_j(x,y)|f(y)|dy \\ &= \mathscr{R}_{j,L} f(x) + \mathscr{R}_{j,G} f(x),\end{aligned}$$

where $\mathscr{R}_{j,L} f(x) = \mathscr{R}_j(f\chi_{B_h(x)})(x)$ is the *local part* and $\mathscr{R}_{j,G} f(x) = \mathscr{R}_j(f\chi_{B_h^c(x)})(x)$ is the *global part* of $\mathscr{R}_j$.

To study the local part of the Gaussian Riesz transform $\mathscr{R}_j$, we use Theorem 4.30, to see that the local part $\mathscr{R}_{j,L}$ corresponds essentially to a classical Calderón–Zygmund singular integral. First, we need to verify the size and smooth conditions (4.29),

$$\begin{aligned}|\nabla_y \mathscr{K}_j(x,y)| &= \left|\nabla_y \left(\int_0^1 \left(\frac{1-r^2}{-\log r}\right)^{1/2} (y_j - rx_j) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}} dr\right)\right| \\ &= \left(\sum_{j=1}^d \left|\int_0^1 \left(\frac{1-r^2}{-\log r}\right) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}} \left(\delta_{i,j} - \frac{2(y_j-rx_j)(y_i-rx_i)}{1-r^2}\right) dr\right|^2\right)^{\frac{1}{2}} \\ &\leq C_d \int_0^1 \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}} dr + C_d \int_0^1 \frac{|y-rx|^2}{1-r^2} \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d+3}{2}}} dr,\end{aligned}$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

Let us recall the notation introduced in Proposition 4.23, given $x,y \in \mathbb{R}^d$ and $t>0$. Writing $a = |x|^2+|y|^2, b = 2\langle x,y\rangle$, $u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |x|^2$. Therefore, taking the change of variables, $t = 1-r^2$

$$|\nabla_y \mathscr{K}_j(x,y)| \le C_d \int_0^1 \frac{e^{-u(t)}}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}} + C_d \int_0^1 u(t) \frac{e^{-u(t)}}{t^{\frac{d+3}{2}}} \frac{dt}{\sqrt{1-t}}.$$

Also, it is easy to see that for the kernel $\mathscr{K}_j$ we have

$$|\mathscr{K}_j(x,y)| \le C_{|\beta|} \int_0^1 (u(t))^{1/2} \frac{e^{-u(t)}}{t^{(d+2)/2}} dt.$$

Therefore, using Lemma 4.35, with exponent $d-1$ instead of d, we get

$$|\mathscr{K}_\beta(x,y)| \le \frac{C}{|x-y|^d},$$

and then, we can apply Theorem 4.30 to the kernel $\mathscr{K}_j$ and the operator determined by it.

The global part $\mathscr{R}_{j,G}$ can be bounded using the following result.

Theorem 9.4. *(Pérez) If* $|x-y| \ge C_d \left(1 \wedge \frac{1}{|x|}\right) = C_d m(x)$*, then, for* $1 \le j \le d$*,*

$$|\mathscr{K}_j(x,y)| \le C_d \overline{\mathscr{K}}(x,y), \tag{9.12}$$

where $\overline{\mathscr{K}}$ *is the Gaussian maximal kernel defined in (4.40).*

Proof. Let $\mathscr{K}(x,y)$ be the kernel defined as

$$\mathscr{K}(x,y) = \int_0^1 \frac{|y-rx|}{(1-r^2)^{(d+3)/2}} e^{-\frac{|y-rx|^2}{1-r^2}} dr. \tag{9.13}$$

Given that $(\frac{1-r^2}{-\log r})^2$ is a bounded function for $0 \le r \le 1$, then,

$$|\mathscr{K}_j(x,y)| \le C_d \mathscr{K}(x,y).$$

Thus, it is enough to prove that

$$\mathscr{K}(x,y) \le \overline{\mathscr{K}}(x,y),$$

when $|x-y| \ge C_d m(x)$. Making the change of variables $t = 1-r^2$, we get

$$\begin{aligned}\mathscr{K}(x,y) &= \frac{1}{2}\int_0^1 \frac{|y-\sqrt{1-t}x|}{t^{1/2}} \frac{1}{t^{d/2}} e^{-\frac{|y-\sqrt{1-t}x|^2}{t}} \frac{dt}{t\sqrt{1-t}} \\ &= \frac{1}{2}\int_0^1 u^{1/2}(t) e^{-u(t)} \frac{dt}{t^{\frac{d}{2}+1}\sqrt{1-t}}.\end{aligned}$$

Then, using Lemma 4.38, we immediately get

$$|\mathscr{K}_j(x,y)| \le C_d \mathscr{K}(x,y) \le C_d \overline{\mathscr{K}}(x,y). \qquad \square$$

From the inequality obtained in Theorem 9.4 and using Theorem 4.24, we immediately get that $\mathscr{R}_{j,G}$ is of weak type $(1,1)$ with respect to the Gaussian measure and with that we conclude the proof of Theorem 9.3. Moreover, observe that in general, if T is the linear operator associated with a kernel $K(x,y)$, its adjoint with respect to the Gaussian measure has kernel $K^*(x,y) = K(y,x)e^{|x|^2-|y|^2}$. As $\overline{\mathscr{K}}(y,x)e^{|x|^2-|y|^2} = \overline{\mathscr{K}}(x,y)$, it follows easily that the adjoint of $\mathscr{R}_j$ is also of weak type $(1,1)$ with respect to the Gaussian measure.

In [37], T. Bruno gives an alternative and simpler proof of Theorem 9.3 (see [37, Theorem 1.1]), proving that $\mathscr{K}_j$, the kernel of the j-th Riesz transform is also bounded by its kernel $\tilde{K}$, (4.59), in the global region, [37, Proposition 3.8], and then apply [37, Lemma 3.5].

As we have mentioned earlier, the main goal of C. Gutiérrez's article [122] is, following Stein's scheme in [253, Chapter IV], to prove Theorem 9.2, using the Littlewood–Paley theory. Let us see the basics of his arguments. First, he gets the following identity:

$$\frac{\partial P_t^{(1)}}{\partial t}(\mathscr{R}_j f)(x) = -\frac{1}{\sqrt{2}}\frac{P_t f}{\partial x_j}(x),\ j = 1,\cdots,d. \tag{9.14}$$

To prove this identity, it is enough to check it for the Hermite polynomials $\{\mathbf{H}_\nu\}$. From (9.5),

$$\begin{aligned}
\frac{\partial P_t^{(1)}}{\partial t}(\mathscr{R}_j \mathbf{H}_\nu)(x) &= \left(\sqrt{\frac{2}{|\nu|}}\nu_j\right)\frac{\partial P_t^{(1)}}{\partial t}(\mathbf{H}_{\nu-\mathbf{e_j}})(x) \\
&= \left(\sqrt{\frac{2}{|\nu|}}\nu_j\right)\frac{\partial}{\partial t}(e^{-\sqrt{|\nu-\mathbf{e_j}|+1}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x)) \\
&= \left(\sqrt{\frac{2}{|\nu|}}\nu_j\right)\frac{\partial}{\partial t}(e^{-\sqrt{|\nu|}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x)) = -\sqrt{2}\nu_j e^{-\sqrt{|\nu|}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x),
\end{aligned}$$

and by (1.60)

$$\frac{\partial P_t \mathbf{H}_\nu}{\partial x_j}(x) = e^{-\sqrt{|\nu|}}\frac{\partial \mathbf{H}_\nu}{\partial x_j}(x) = 2\nu_j e^{-\sqrt{|\nu|}\,t}\mathbf{H}_{\nu-\mathbf{e_j}}(x).$$

Thus,

$$\frac{\partial P_t^{(1)}}{\partial t}(\mathscr{R}_j \mathbf{H}_\nu)(x) = -\frac{1}{\sqrt{2}}\frac{P_t \mathbf{H}_\nu}{\partial x_j}(x),$$

and the formula can be extended immediately to polynomial functions, which are dense in $L^p(\gamma_d)$. Therefore,

$$g_{t,\gamma}^{(1)}(\mathscr{R}_i f)(x) = \Big(\int_0^\infty \Big|t\frac{\partial P_t^{(1)}}{\partial t}(\mathscr{R}_i f)(x)\Big|^2 \frac{dt}{t}\Big)^{1/2} = \frac{1}{\sqrt{2}}\Big(\int_0^\infty \Big|t\frac{P_t f}{\partial x_j}(x)\Big|^2 \frac{dt}{t}\Big)^{1/2}$$
$$\leq \frac{1}{\sqrt{2}} g_\gamma(f)(x).$$

Then, using Theorem 5.2 and Theorem 5.8, we get

$$1/C_p' \|\mathscr{R}_i f\|_{p,\gamma} \leq \|g_{t,\gamma}^{(1)}(\mathscr{R}_i f)\|_{p,\gamma} \leq \frac{1}{\sqrt{2}}\|g_\gamma(f)\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma}.$$

An important advantage of this proof is that the constants C_p, C_p' are independent of dimension.

Finally, the atomic definition of the Gaussian Hardy spaces, given by G. Mauceri and S. Meda in [174], does not provide a fully satisfying theory. Unfortunately, that may not relate to the Ornstein–Uhlenbeck operator as well as classical Hardy spaces relate to the usual Laplacian (see [79]). In particular, G. Mauceri and S. Meda in [174], proved that the imaginary powers of L, $(-L)^{i\alpha}$ and the adjoint of the Riesz transforms $\mathscr{R}_j^*$ are bounded from $H_{at}^{1,r}(\gamma_d)$ to $L^1(\gamma_d)$, but later in [176, Theorem 3.1] G. Mauceri, S. Meda, and P. Sjögren proved that the Riesz transforms $\mathscr{R}_j$ are bounded from L^∞ to the dual of $H_{at}^{1,r}(\gamma_d) = BMO(\gamma_d)$ in any dimension, but they are not bounded from $H_{at}^1(\gamma_d)$ to $L^1(\gamma_d)$ in a dimension greater than one. Thus, their definition does not contain all the machinery that makes Fefferman–Stein [79] so outstanding, and has proven useful in a range of applications, specially in the study of partial differential equations. This was the main reason why J. Maas, J. van Neerven, and P. Portal developed a program to find an alternative definition of the Hardy spaces. In [231, Theorem 6.1], P. Portal proved that the Riesz transforms $\mathscr{R}_j$ are bounded from $H_{max}^1(\gamma_d) = H_{quad}^1(\gamma_d)$ to $L^1(\gamma_d)$, with a similar approach to that in the proof of Theorem 7.16, using an appropriated Calderón reproducing formula (see [231, Lemma 6.2]). More recently, T. Bruno proved that the Riesz transforms are bounded from the atomic Gaussian Hardy space $X^1(\gamma_d)$ to $L^1(\gamma_d)$ (see [37, Theorem 1.2]).

9.2 Definition and Boundedness Properties of the Higher-Order Gaussian Riesz Transforms

In the Gaussian case, the *higher-order Gaussian Riesz transforms* are defined directly.

Definition 9.5. *For $\beta = (\beta_1, \beta_2, \cdots, \beta_d) \in \mathbb{N}_0^d$, the higher order Riesz transforms are defined spectrally as*

$$\mathscr{R}_\beta = \partial_\gamma^\beta (-L)^{-|\beta|/2}, \tag{9.15}$$

where $|\beta| = \sum_{j=1}^d \beta_j$ and $\partial_\beta^\gamma = \frac{1}{2^{|\beta|/2}}\partial_{x_1}^{\beta_1}\cdots\partial_{x_d}^{\beta_d}$. The meaning of this is that for any multi-index ν such that $|\nu| > 0$, its action on the Hermite polynomial $\mathbf{H}_\nu$ is

$$\mathscr{R}_\beta \mathbf{H}_\nu = \Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta} \tag{9.16}$$

if $\beta_i \leq \nu_i$ *for all* $i = 1,2,\cdots,d$, *and zero otherwise.*

Observe that (9.16) follows directly from the definition of $\mathscr{R}_\beta$, because $\mathbf{H}_\nu$ is the eigenfunction of the Ornstein–Uhlenbeck operator $-L$, with eigenvalue $|\nu|$; therefore,

$$(-L)^{-|\beta|/2}\mathbf{H}_\nu = \frac{1}{|\nu|^{|\beta|/2}}\mathbf{H}_\nu.$$

Hence, using (1.57) and (1.36), we get

$$\begin{aligned}
\mathscr{R}_\beta \mathbf{H}_\nu(x) &= \partial_\beta^\gamma(-L)^{-|\beta|/2}\mathbf{H}_\nu(x) = \partial_\beta^\gamma\big(\frac{1}{|\nu|^{|\beta|/2}}\mathbf{H}_\nu(x)\big)\\
&= \frac{1}{2^{|\beta|/2}|\nu|^{|\beta|/2}}\partial_1^{\beta_1}\cdots\partial_d^{\beta_d}\Big(\prod_{i=1}^{d}H_{\nu_i}(x_i)\Big) = \frac{1}{2^{|\beta|/2}|\nu|^{|\beta|/2}}\prod_{i=1}^{d}\big(\partial_i^{\beta_i}H_{\nu_i}(x_i)\big)\\
&= \frac{1}{2^{|\beta|/2}|\nu|^{|\beta|/2}}\prod_{i=1}^{d}\big(2^{\beta_i}[\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]H_{\nu_i-\beta_i}(x_i)\big)\\
&= \frac{2^{|\beta|/2}}{|\nu|^{|\beta|/2}}\prod_{i=1}^{d}\big([\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]H_{\nu_i-\beta_i}(x_i)\big)\\
&= \Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta}(x).
\end{aligned}$$

Observe that this implies that

$$\mathscr{R}_\beta \mathbf{h}_\nu(x) = \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]^{1/2}\mathbf{h}_{\nu-\beta}(x), \tag{9.17}$$

because

$$\begin{aligned}
\mathscr{R}_\beta \mathbf{h}_\nu(x) &= \mathscr{R}_\beta\Big(\frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}\Big) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}}\Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\\
&\qquad\Big[\prod_{i=1}^{d}\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta}(x)\\
&= \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\frac{\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)}{(\nu_i!)^{1/2}}\Big]\frac{\mathbf{H}_{\nu-\beta}(x)}{(2^{|\nu|-|\beta|})^{1/2}}\\
&= \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^{d}\frac{[\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]^{1/2}}{(\nu_i-\beta_i!)^{1/2}}\Big]\frac{\mathbf{H}_{\nu-\beta}(x)}{(2^{|\nu-\beta|})^{1/2}}\\
&= \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\prod_{i=1}^{d}[\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)]^{1/2}\frac{\mathbf{H}_{\nu-\beta}(x)}{(2^{|\nu-\beta|}(\nu-\beta)!)^{1/2}}
\end{aligned}$$

$$= \left(\frac{1}{|\nu|}\right)^{|\beta|/2} \Big[\prod_{i=1}^{d} \nu_i(\nu_i - 1)\cdots(\nu_i - \beta_i + 1)\Big]^{1/2} \mathbf{h}_{\nu-\beta}(x).$$

The higher-order Gaussian Riesz transforms have a kernel given by

$$\mathscr{K}_\beta(x,y) = \partial_\gamma^\beta N_{|\beta|/2}(x,y) \tag{9.18}$$
$$= \frac{1}{\pi^{d/2}\Gamma(|\beta|/2)} \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|} \mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r}.$$

Therefore,

$$\mathscr{R}_\beta f(x) = p.v. \int_{\mathbb{R}^d} \mathscr{K}_\beta(x,y) f(y)dy \tag{9.19}$$
$$= p.v. \frac{1}{\pi^{d/2}\Gamma(|\beta|/2)} \int_{\mathbb{R}^d} \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|} \mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)$$
$$\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r} f(y)dy.$$

Let us study the $L^p(\gamma_d)$ boundedness of these operators, for $1 < p < \infty$,

Theorem 9.6. *The higher-order Gaussian Riesz transforms $\mathscr{R}_\beta$, $|\beta| > 1$ are $L^p(\gamma_d)$ bounded for $1 < p < \infty$, that is, there exists $C > 0$, dependent only on p and dimension such that*

$$\|\mathscr{R}_\beta f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}, \tag{9.20}$$

for any $f \in L^p(\gamma_d)$.

There are several analytic proofs of this result. The first analytic proof was given by W. Urbina in [278] with constants dependent on dimension. A clever proof was given by G. Pisier [227], which combines probabilistic and analytic techniques (method of rotations and transference methods), with constants independent of dimension, but valid only for the case $|\beta|$ odd. In [124], C. Gutiérrez, C. Segovia, and J. L. Torrea obtained a proof, with constants independent of dimension, following the work of C. Gutiérrez in [122], by using the Littlewood–Paley theory, with higher-order Gaussian Littlewood–Paley functions, which were discussed in Chapter 6. In [223], S. Pérez and F. Soria provide an analytic proof, with constants dependent on dimension, with a similar technique to that developed to study the Ornstein–Uhlenbeck maximal function T^* already discussed in Chapter 4. We study their proof in detail. Finally, L. Forzani, R. Scotto, and W. Urbina in [88] have a very simple proof, with constants independent of dimension, based on Meyer's multiplier theorem (Theorem 6.2; see Corollary 9.12).

Proof. As in the case of the Gaussian Riesz transforms, we follow the proof of S. Pérez and F. Soria ([223]). Again, we split these operators into a local part and a global part,

$$\begin{aligned}\mathscr{R}_\beta f(x) &= C_d \int\limits_{|x-y|<d\,m(x)} \mathscr{K}_\beta(x,y) f(y)dy + C_d \int\limits_{|x-y|\geq d\,m(x)} \mathscr{K}_\beta(x,y)|f(y)|dy \\ &= \mathscr{R}_{\beta,L} f(x) + \mathscr{R}_{\beta,G} f(x),\end{aligned}$$

where $\mathscr{R}_{\beta,L} f(x) = \mathscr{R}_\beta(f\chi_{B_h(\cdot)})(x)$ is the *local part* , $\mathscr{R}_{\beta,G} f(x) = \mathscr{R}_\beta(f\chi_{B_h^c(\cdot)})(x)$ is the *global part* of $\mathscr{R}_\beta$, and $B_h = B(x, C_d\, m(x)) = \{y \in \mathbb{R}^d : |y-x| < C_d\, m(x)\}$, is an admissible ball.

I) It has been clear, since W. Urbina's work in [278], that the local part, as in the case of the Gaussian Riesz transforms, corresponds to a classical Calderón–Zygmund singular integral.
Now, we see that the kernel $\mathscr{K}_\beta$ satisfies the decay conditions (4.29) in the local region. Observe that $r^{|\beta|-2}\left(\frac{-\log r}{1-r^2}\right)^{|\beta|-2}$ is bounded for every $r \in (0,1)$ and any $\beta, \geq 2$. We also use the fact that, $|\mathbf{H}_\beta(x)| \leq C|x|^{|\beta|}$. Then,

$$\begin{aligned}&\left|\nabla_y\left(e^{-\frac{|y-rx|^2}{1-r^2}}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\right)\right| \\ &= \Bigg(\sum_{i=1}^{d} e^{-\frac{|y-rx|^2}{1-r^2}}\Bigg| -\frac{2(y_i-rx_i)}{1-r^2}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right) \\ &\quad -\frac{2\beta_i}{\sqrt{1-r^2}}H_{\beta_1}\left(\frac{y_1-rx_1}{\sqrt{1-r^2}}\right)\dots H_{\beta_{i-1}}\left(\frac{y_i-rx_i}{\sqrt{1-r^2}}\right)\dots H_{\beta_d}\left(\frac{y_d-rx_d}{\sqrt{1-r^2}}\right)\Bigg|^2\Bigg)^{\frac{1}{2}} \\ &\leq C_\beta\left(\frac{|y-rx|^{|\alpha|+1}}{(1-r^2)^{\frac{|\beta|+1}{2}}} + \frac{|y-rx|^{|\beta|-1}}{(1-r^2)^{\frac{|\beta|-1}{2}}}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{1}{2}}}.\end{aligned}$$

Again, using the notation of Proposition 4.23, given $x, y \in \mathbb{R}^d$ and $t > 0$, we write $a = |x|^2 + |y|^2, b = 2\langle x, y\rangle$ and $u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t}b - |x|^2$. We can conclude that the above expression is bounded by

$$\int_0^1 \left(u^{\frac{|\alpha|-1}{2}}(t) + u^{\frac{|\alpha|+1}{2}}(t)\right)\frac{e^{-u(t)}}{t^{\frac{d+3}{2}}}dt;$$

therefore, using Lemma 4.35, we have, in the local region,

$$\left|\nabla_y \mathscr{K}_\beta(x,y)\right| \leq \frac{C}{|x-y|^{d+1}}.$$

Also, it is easy to see that for the kernel $\mathscr{K}_\beta$ we have

$$|\mathscr{K}_\beta(x,y)| \leq C_{|\beta|} \int_0^1 (u(t))^{|\beta|/2} \frac{e^{-u(t)}}{t^{(d+2)/2}} dt.$$

Therefore, again using Lemma 4.35, with exponent $d-1$ instead of d, we get

$$|\mathscr{K}_\beta(x,y)| \leq \frac{C}{|x-y|^d}.$$

Therefore, we can apply Theorem 4.30 to $\mathscr{K}_\beta$ and the operator determined by it.

II) For the global part of $\mathscr{R}_\alpha$, we use a generalization of Theorem 9.4.

First, let us consider the following kernel:

Definition 9.7. *For each $m \geq 2$ the* m-modified maximal Gaussian kernel is defined as

$$\overline{\mathscr{K}}_m(x,y) = \begin{cases} (|x+y||x-y|)^{\frac{m-2}{2}} \overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle \leq 0 \\ (|x+y||x-y|)^{\frac{m-2}{2}} \left(|x+y||x-y|)^{\frac{1}{2}} \frac{|x|\,|y|}{|x|^2+|y|^2} + 1 \right) \overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle \geq 0 \end{cases} \tag{9.21}$$

where $\overline{\mathscr{K}}$ is the Gaussian maximal kernel defined in (4.40), and define the m-modified maximal operator

$$\overline{T}_m f(x) = \int_{\mathbb{R}^d} \overline{\mathscr{K}}_m(x,y) f(y) dy. \tag{9.22}$$

Theorem 9.8. *(Pérez–Soria) For the kernel $\mathscr{K}_\beta$ of the Gaussian Riesz transform of order β, $|\beta| \geq 2$. Then, we have*

$$|\mathscr{K}_\beta(x,y)| \leq C \overline{\mathscr{K}}_{|\beta|}(x,y), \tag{9.23}$$

on the region $|x-y| > C_d(1 \wedge 1/|x|)$.

Proof. Observe that the function $r^{|\beta|-2} \left(\frac{-\log r}{1-r^2}\right)^{(|\beta|-2)/2}$ is bounded for any $r \in (0,1)$ and any $\beta \geq 2$. Again, using the fact that $|\mathbf{H}_\beta(x)| \leq C|x|^{|\beta|}$, and making the change of variables $t = 1-r^2$, we get

$$|\mathscr{K}_\beta(x,y)| \leq C \int_0^1 \left| \mathbf{H}_\beta \left(\frac{y-\sqrt{1-t}x}{\sqrt{t}} \right) \right| \frac{e^{-\frac{|y-\sqrt{1-t}x|^2}{t}}}{t^{\frac{d+2}{2}}} dt \leq C \int_0^1 \frac{u^{|\beta|/2}(t) e^{-u(t)}}{t^{\frac{d+2}{2}}} dt.$$

Thus, it is enough to prove that the last integral is bounded by $\mathscr{K}^*_{|\beta|}(x,y)$. We need to analyze two cases:

- Case #1: $b = 2\langle x,y\rangle \leq 0$. In this case, we see that

$$\int_0^1 \frac{u^{|\beta|/2}(t) e^{-u(t)}}{t^{\frac{d+2}{2}}} dt \leq C a^{\frac{|\beta|-2}{2}} e^{-|y|^2},$$

Using the inequality (4.76):

$$\frac{a}{t} - |x|^2 \leq u(t) \leq \frac{2a}{t},$$

from Proposition 4.23, the change of variables $a\left(\frac{1}{t} - 1\right) = s$, and the fact that, in the global region, $a > 1/2$, we obtain,

$$\int_0^1 \frac{u^{|\beta|/2}(t)e^{-u(t)}}{t^{\frac{d+2}{2}}} dt \leq e^{-|y|^2} \int_0^1 \exp\left(-\frac{a}{t} + a\right)\left(\frac{2a}{t}\right)^{|\alpha|/2} \frac{dt}{t^{\frac{d}{2}+1}}$$

$$\leq C_\beta a^{\frac{|\beta|-2}{2}} e^{-|y|^2} \int_0^\infty e^{-s}(2s+1)^{\frac{d+|\beta|-2}{2}} ds \leq C a^{\frac{|\beta|-2}{2}} e^{-|y|^2}.$$

- Case #2: $b = 2\langle x, y\rangle > 0$.
  Using the same argument as in Theorem 9.4, we have that for $d \geq 2$ (4.78) holds,

$$\frac{e^{-\frac{d-2}{d}u(t)}}{t^{\frac{d-2}{2}}} \leq C \frac{e^{-\frac{d-2}{d}u_0}}{t_0^{\frac{d-2}{2}}}.$$

  Then, using Lemma 4.37 for $\nu = 2/d$, we get

$$\int_0^1 u^{|\alpha|/2}(t) e^{-\frac{2u(t)}{d}} \frac{dt}{t^2} \leq \frac{C_d e^{-\frac{2u_0}{2}}}{t_0}\left(u_0^{\frac{|\alpha|-1}{2}} \frac{b}{a} u(t_0)^{\frac{|\alpha|-2}{2}} + 1\right),$$

  because $u_0 \leq |x+y||x-y|, b/a \leq 2|x||y|/(|x|^2+|y|^2)$ and $d \leq |x+y||x-y|$ if $\langle x,y\rangle \geq 0$ and $|x-y| > C_d(1 \wedge 1/|x|)$. □

Similar to the case of the Riesz transforms, the symmetry of the non-exponential factor of the kernel $\overline{\mathscr{K}}_{|\beta|}(x,y)$ allows us to obtain that the adjoint operator to the higher-order Riesz transforms are also of weak type $(1,1)$ with respect to the Gaussian measure, as

$$\overline{\mathscr{K}_{|\alpha|}}^*(x,y) = \overline{\mathscr{K}}_{|\alpha|}(y,x)e^{|y|^2-|x|^2}.$$

As mentioned already, the main goal of C. Gutiérrez, C. Segovia, and J. L. Torrea's article [124, Chapter 4] is, also following Stein's scheme in [253], to prove Theorems 9.2 using higher-order Littlewood–Paley functions. To do so, they first get the following identity: given a multi-index $\beta \in \Gamma_k$ of order k, i.e., $|\beta| = k$ using translated Poisson semigroups $\{P_t^{(k)}\}_{t\geq 0}$ (see 3.56),

$$\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathscr{R}_\beta f)(x) = (-\frac{1}{\sqrt{2}})^k \partial^\beta P_t f(x). \tag{9.24}$$

To prove this identity, it is enough to check it for the Hermite polynomials $\{\mathbf{H}_\nu\}$. From (9.16) and (1.60),

$$\begin{aligned}\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathscr{R}_\beta \mathbf{H}_\nu(x)) &= \Big(\frac{2}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^d \nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathbf{H}_{\nu-\beta})(x)\\ &= \Big(\frac{2}{|\nu|}\Big)^{k/2}\Big[\prod_{i=1}^d \nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\\ &\qquad\times\frac{\partial^k}{\partial t^k}\Big(e^{-\sqrt{|\nu-\beta|+k}\,t}\Big)\mathbf{h}_{\nu-\beta}(x)\\ &= \Big(\frac{2}{|\nu|}\Big)^{k/2}\Big[\prod_{i=1}^d \nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\Big)\frac{\partial^k}{\partial t^k}(e^{-\sqrt{|\nu|}\,t})\mathbf{H}_{\nu-\beta}(x)\\ &= (-\frac{1}{\sqrt{2}})^k e^{-\sqrt{|\nu|}\,t}\Big[\prod_{i=1}^d \nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]\mathbf{H}_{\nu-\beta}(x)\\ &= (-\frac{1}{\sqrt{2}})^k e^{-\sqrt{|\nu|}\,t}(\partial^\beta \mathbf{H}_\nu)(x) = (-\frac{1}{\sqrt{2}})^k(\partial^\beta P_t \mathbf{H}_\nu)(x).\end{aligned}$$

Then, let $\mathscr{R}_k f = (\mathscr{R}_\beta f)_{\beta\in\Lambda_k}$

$$\begin{aligned}\mathbf{g}^k_{t,\gamma}(\mathscr{R}_k f)(x) &= \Big(\int_0^\infty \sum_{\beta\in\Lambda_k}\Big|t^k\frac{\partial^k P_t^{(k)}}{\partial t^k}(\mathscr{R}_\beta f)(x)\Big|^2\frac{dt}{t}\Big)^{1/2}\\ &= C\left(\int_0^{+\infty}\Big|t^k(\partial^\beta P_t f)(x)\Big|^2\frac{dt}{t}\right)^{\frac{1}{2}} = Cg^k_{x,\gamma}f(x).\end{aligned}$$

Therefore, using Theorem 5.13, we get

$$||\,|\mathscr{R}_k f|\,||_{p,\gamma} \le C_p||\mathbf{g}^k_{t,\gamma}(\mathscr{R}_k f)||_{p,\gamma} = C_p||g^k_{x,\gamma}(f)||_{p,\gamma} \le C_p||f||_{p,\gamma}.$$

Thus, we get the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_\beta$, for any β, $|\beta|>1$ with constants independent of dimension.

The Riesz transforms of order 2 are of weak type $(1,1)$ with respect to the Gaussian measure, that is, they map $L^1(\gamma_d)$ into $L^{1,\infty}(\gamma_d)$. This result has been shown, by L. Forzani and R. Scotto for the case $d=1$ in [86], and for general $d>1$ by J. García-Cuerva, G. Mauceri, P. Sjögren and J. L. Torrea in [102], but their proof contains a gap. Additionally, S. Pérez and F. Soria [223] have an alternative proof using the fact that the 2-modified maximal Gaussian kernel $\overline{\mathscr{K}}_2$ bounds the kernels of the Gaussian Riesz transforms of order 2, based on the following result (see [223, Theorem 4.4]):

Theorem 9.9. *The operator $\overline{T}_2$ is of weak type $(1,1)$ with respect to the Gaussian measure.*

The proof of this theorem involves heavily all the arguments used to prove Theorem 4.24, with some slight modifications. In particular, it is important to recall some of the notation and facts:

- Let $\alpha := \alpha(x,y) = \sin\angle(x,y)$, where $\angle(x,y) \in [0,\pi]$ denotes the shortest angle between the vectors x and y if $\langle x,y\rangle > 0$, we have $\angle(x,y) \in [0,\pi/2]$).
- Define for $k = 1,2$ and $l \in \mathbb{N}$

$$\Gamma_l^k(x) = \left\{ y : \langle x,y\rangle > 0, |x| \le |y|, \alpha(x,y) \le l/|x|^k \right\},$$

(see (4.51)).
- Then, for fixed values of k and l, the average operator, defined by

$$\mathtt{T}_l^k f(x) = \frac{1}{\gamma_d(\Gamma_l^k(x))} \int_{\Gamma_l^k(x)} |f(y)| e^{-|y|^2} dy$$

(see (4.52)), is of weak type $(1,1)$ with respect to γ_d.

The arguments follow closely the proof of Lemmas 4.25, 4.26 and 4.27 (see also [185, Lemma 2.6, 2.7 and 2.8]).

Proof. Without loss of generality, we may assume that $f \ge 0$. As the operator $\overline{T}$, defined in (4.46), is of weak type $(1,1)$ with respect to the Gaussian measure (see Theorem 4.24), and $\overline{\mathscr{K}}_2(x,y)$ is dominated by $\overline{\mathscr{K}}(x,y)$ if $\langle x,y\rangle > 0$ and $|x| \le 10$, or on the local region, the operator $\overline{T}_2$ is also of weak type $(1,1)$ with respect to the Gaussian measure on those regions. Thus, it remains to consider the case when we are outside of those regions.

When $|x| > |y|$, as $|x+y||x-y| > d$, the kernel $\overline{\mathscr{K}}_2(x,y)$ satisfies

$$\begin{aligned}
\overline{\mathscr{K}}_2(x,y) &\le \frac{|x+y|^d}{(|x+y||x-y|)^{(d-1)/2}} \exp\left(-\frac{|y|^2 - |x|^2}{2} - \frac{|x+y||x-y|}{2} \right) \\
&\le C|x|^d \exp\left(-\frac{|x||x-y|}{2} \right) e^{|x|^2 - |y|^2}.
\end{aligned}$$

It is easy to check that $\overline{\mathscr{K}}_2(x,y) e^{|y|^2 - |x|^2} \in L^1(\gamma_d)$, uniformly in y; thus, the operator is of strong type (p,p), $1 < p < \infty$ with respect to the Gaussian measure in the global region.

Next, we consider for $\underline{\langle x,y\rangle} > 0$ and $|x| > 10$ two operators defined $\tilde{\mathtt{T}}_1$ and $\tilde{\mathtt{T}}_2$ defined by the restriction of $\overline{\mathscr{K}}_2(x,y)$ to the regions,

$$\begin{aligned}
\mathtt{B}_1 = \Big\{ (x,y) \in \mathbb{R}^{2d} : y \notin B_h(x), \langle x,y\rangle > 0, |x| \le |y| \text{ and } \alpha(x,y) > 1/|x| \\
\text{or } |x| \le 2|y|, \alpha(x,y) \le 1/|x| \Big\}, \\
\mathtt{B}_2 = \Big\{ (x,y) \in \mathbb{R}^{2d} : y \notin B_h(x) : \langle x,y\rangle > 0, |y|/2 \le |x| < |y|, \alpha(x,y) \le 1/|x| \Big\},
\end{aligned}$$

respectively.

On $\mathtt{B}_1$, we have $\overline{\mathscr{K}}_2(x,y) \le C|x|\overline{\mathscr{K}}(x,y)$; therefore $\tilde{\mathtt{T}}_1 f(x) \le \overline{\mathtt{T}}_1 f(x)$, where $\overline{\mathtt{T}}_1$ corresponds to the operator associated with the restriction of $\overline{\mathscr{K}}(x,y)$ on $\mathtt{B}_1$. Now, from the estimate (4.54) we obtain,

$$\tilde{\mathrm{T}}_1 f(x) \leq C\mathrm{T}_1^1 f(x) + C\sum_{m\geq 1} e^{-m^2/4}\mathrm{T}_{m+1}^1 f(x),$$

similar to the proof of Lemma 4.25.

To estimate $\tilde{\mathrm{T}}_2$, we follow the same arguments and notation as in the proof of Lemma 4.26. We have that

$$\overline{\mathscr{K}}_2(x,y) \leq CA^{1/2}\overline{\mathscr{K}}(x,y), \text{ if } y \in \Lambda(x),$$

as $A \geq c\alpha(x,y)|x|^2$, and

$$\overline{\mathscr{K}}_2(x,y) \leq C\alpha(x,y)^{1/2}\overline{\mathscr{K}}(x,y), \text{ if } y \in \overline{\Gamma_1^1(x)} \setminus \Lambda(x),$$

as $A \leq C\alpha(x,y)|x|^2$.

Consider now the average operator,

$$\tilde{\mathscr{A}_2} f(x) = \frac{1}{\gamma_d(\Lambda(x))}\int_{\Lambda(x)} G_2(x,y)f(y)e^{-|y|^2}dy,$$

with $G_2(x,y) = A^{-d/2}G(x,y)$ and $G(x,y) = A^{-d/2}e^{\alpha^2|y|^4/16A}$. Then, we conclude with the same arguments as in the proof of Lemma 4.26, that

$$\tilde{\mathrm{T}}_2 f(x) \leq C\sum_{l\geq 2}^{\infty} e^{-\delta l}\mathrm{T}_l^2 f(x) + C\tilde{\mathscr{A}} f(x).$$

The value of $\delta > 0$ can be chosen as before.

It remains only to show that $\tilde{\mathscr{A}_2}$ is of weak type $(1,1)$ and the proof of that is similar to the proof of Lemma 4.27, replacing $G(x,y)$ by $G_2(x,y)$. □

Moreover, the Gaussian higher-order Riesz transforms $\mathscr{R}_\beta$ are of weak type $(1,1)$ with respect to the Gaussian measure if and only if $|\beta| \leq 2$; equivalently, it can be proved that the result breaks down for $|\beta| > 2$. This is a surprising result, compared with the classical case, and it was initially proved by R. Scotto and L. Forzani in the one-dimensional case in [86]. The case for higher dimensions $d > 1$ was considered by J. L. García-Cuerva, G. Mauceri, P. Sjögren, and J. L. Torrea in [102], even though there are certain technical issues in their proof, and by S. Pérez and F. Soria [223]. This fact implies then that the theory of Gaussian singular integrals is different from the classical Calderón–Zygmund and, in particular, it cannot be developed using interpolation results.

Now, let us discuss the counterexample that Riesz transforms of at least order three are not of weak type $(1,1)$ with respect to the Gaussian measure. This is taken from [102]. The idea of the counterexample is to consider a function $f \in L^1(\gamma_d)$ which is "equivalent" to a point mass at $y \in \mathbb{R}^d$ properly normalized in $L^1(\gamma_d)$, that is to say, $f \sim e^{|y|^2}\delta_y$, for $|y|$ large.

Theorem 9.10. *Let $|\beta| \geq 3$. Then, the Riesz transform $\mathscr{R}_\beta$ is not of weak type $(1,1)$ with respect to the Gaussian measure.*

Proof. Let $y \in \mathbb{R}^d$ with $|y| = \eta$ large and $y_i \geq C\eta,\ i = 1, \cdots, d$. Write $x \in \mathbb{R}^d$ as $x = \xi \frac{y}{\eta} + v$ with $\xi \in \mathbb{R}$ and $v \perp y$. Consider the tubular region

$$\mathbf{J} = \{x \in \mathbb{R}^d : x = \xi \frac{y}{\eta} + v \quad \text{with} \quad \eta/2 < \xi < 3\eta/4, v \perp y,\ |v| < 1\}.$$

It follows that for $x \in \mathbf{J}$, there is a $C > 0$ so that

$$\frac{y_i - rx_i}{\sqrt{1-r^2}} \geq \frac{C\eta}{\sqrt{1-r^2}} \geq C\eta,\ i = 1, \cdots, d. \tag{9.25}$$

Hence,

$$\mathbf{H}_\beta\left(\frac{y - rx}{\sqrt{1-r^2}}\right) > C|y|^\beta.$$

In particular, the integrand in (9.18) is positive for $0 < r < 1$, and observe that

$$e^{-\frac{|y-rx|^2}{1-r^2}} = e^{\xi^2 - \eta^2} e^{-\frac{|\xi - r\eta|^2 + r^2|v|^2}{1-r^2}}, \tag{9.26}$$

so that for $1/4 < r < 3/4$ and $x \in \mathbf{J}$

$$e^{-\frac{|y-rx|^2}{1-r^2}} \geq e^{\xi^2 - \eta^2} e^{-C|\xi - r\eta|^2}. \tag{9.27}$$

These estimates imply that

$$|\mathscr{K}_\beta(x,y)| \geq C_d \eta^\beta e^{\xi^2 - \eta^2} \int_{1/4}^{3/4} e^{-C|\xi - r\eta|^2} dr \geq C_d \eta^{|\beta|-1} e^{\xi^2 - \eta^2}.$$

for $x \in \mathbf{J}$.

Now, let $f \in L^1(\gamma_d)$, $f \geq 0$ be a close approximation of a point mass at y, with norm $\|f\|_{1,\gamma} = 1$. Then, $\mathscr{R}_\beta f(x)$ is close to $e^{\eta^2}\mathscr{K}_\beta(x,y)$ when $x \in \mathbf{J}$. We conclude that

$$\mathscr{R}_\beta f(x) \geq C\eta^{\beta-1} e^{\xi^2} \geq C\eta^{\beta-1} e^{(\eta/2)^2},$$

for $x \in \mathbf{J}$.

On the other hand, because $\gamma_d(\mathbf{J}) \geq \frac{C}{\eta} e^{-(\eta/2)^2}$, and

$$\gamma_d(\mathbf{J}) \leq \gamma_d\left(\left\{x \in \mathbb{R}^d : \mathscr{R}_\beta f(x) > C\eta^{\beta-1} e^{(\eta/2)^2}\right\}\right) \leq C\frac{e^{-(\eta/2)^2}}{\eta^{\beta-1}}.$$

Then,

$$||\mathscr{R}_\beta f||_{1,\infty,\gamma} \geq C\eta^{|\beta|-2} \to \infty,$$

if $\eta \to \infty$, for $|\beta| \geq 3$. $\square$

In [85], L. Forzani, E. Harboure, and R. Scotto give a different and simpler proof of this result for a more general class of Gaussian singular integrals that includes the Gaussian higher-order Riesz transforms, which are discussed later (see Theorem 9.18 and Theorem 9.19).

Additionally, S. Pérez and F. Soria [223, Theorem 4.5] obtained the following result on the boundedness of Gaussian higher-order Riesz transforms of order greater than or equal to 3 on Orlicz spaces, "near" $L^1(\gamma_d)$, using the estimates of the size of the kernel of $\mathscr{R}_\beta$.

Theorem 9.11. *The higher-order Gaussian Riesz transform $\mathscr{R}_\beta$, $|\beta| \geq 3$ is of weak type in the Orlicz space $L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)$. In other words, there exists a constant C such that*

$$\gamma_d\Big(\Big\{x \in \mathbb{R}^d : |\mathscr{R}_{\beta,G} f(x)| \geq \lambda\Big\}\Big) \leq \frac{C}{\lambda}(\|f\|_{L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)} + 1), \tag{9.28}$$

where, as before $\mathscr{R}_{\beta,G} f(x) = \mathscr{R}_\beta(f\chi_{B_h^c(\cdot)})(x)$, is the global part of the Riesz transform $\mathscr{R}_\beta$ and $\|\cdot\|_{L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)}$ denotes the functional associated with the space $L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)$. Thus, $\mathscr{R}_{\beta,G}$ sends the space $L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)$ continuously into $L^{1,\infty}(\gamma_d)$.

Proof. From Theorem 9.8, it is enough to work with the m-modified maximal operator $\overline{T}_m$, as it controls $\mathscr{R}_{\beta,G}$, with $m = |\beta|$. Thus, we will prove that $\overline{T}_m$ satisfies (9.28) for $m \geq 3$. When we restrict ourselves to the region $|x| > |y|$, the usual arguments, which show that $\overline{T}$ or $\overline{T}_2$ are of strong type 1 (see [185, Theorem 2.3] or [223, Theorem 4.4]), tell us that $\overline{T}_m$, is also of strong type 1 in this region. This is easy to see, for $\langle x,y\rangle \leq 0$ and $|x| > |y|$then

$$\overline{\mathscr{K}}_m(x,y) \leq C|x|^m e^{|y|^2},$$

whereas $\langle x,y\rangle > 0$ and $|x| > |y|$ then

$$\begin{aligned}\overline{\mathscr{K}}_m(x,y) &\leq C(|x+y||x-y|)^{\frac{m-1}{2}}|x|^m e^{-\frac{|y|^2-|x|^2}{2}} e^{-\frac{|x-y||x+y|}{2}} e^{|x|^2-|y|^2}\\ &\leq C|x|^m \exp\Big(-\frac{|x|\,|x-y|}{3}\Big)e^{|x|^2-|y|^2}\end{aligned}$$

In both cases, the integral in the variable x is uniformly bounded in y and the strong type $(1,1)$ follows.

For $|x| < |y|$, we use the crude estimate

$$\overline{\mathscr{K}}_m(x,y) \leq C(|x+y||x-y|)^{\frac{m-2}{2}}\overline{\mathscr{K}}_2(x,y).$$

Hence,

$$\int_{|x|<|y|} \overline{\mathscr{K}}_m(x,y)|f(y)|dy \leq C \int_{\mathbb{R}^d} \overline{\mathscr{K}}_2(x,y)|f(y)||y|^{m-2}dy.$$

We use a particular case of Young's inequality: given positive u and v, we have $u \cdot v \leq u(1+\log^+ u) + e^v$, which implies with more generality that

$$u \cdot v \leq \delta^k \Big(u^{1/k}\frac{v^{1/k}}{\delta}\Big)^k \leq C_k \delta^k (u(1+\log^+ u)^k + e^{(k/\delta)v^{1/k}}).$$

Taking $u = |f(y)|$, $v = |y|^{m-2}$ and $k/\delta = 1/2$, we obtain

$$\begin{aligned}\int_{|x|<|y|} \overline{\mathscr{K}}_m(x,y)|f(y)|dy \leq C \int_{\mathbb{R}^d} \overline{\mathscr{K}}_2(x,y)|f(y)|(1+\log^+|f(y)|)^{m-2}\,dy \\ + \int_{\mathbb{R}^d} \overline{\mathscr{K}}_2(x,y)e^{|y|^2/2}dy \\ = \overline{T}_2(|f|(1+\log^+|f|)^{m-2} + e^{|\cdot|^2/2})(x).\end{aligned}$$

Because $\overline{T}_2$ is of weak type $(1,1)$ with respect to the Gaussian measure, as we have seen, we conclude that

$$\begin{aligned}\gamma_d\Big(\Big\{x \in \mathbb{R}^d : \overline{T}_2(|f|(1+\log^+|f|)^{m-2} + e^{|\cdot|^2/2})(x) \geq \lambda\Big\}\Big) \\ \leq \frac{C}{\lambda}\int_{\mathbb{R}^d}\Big[|f(y)|(1+\log^+|f(y)|)^{m-2} + e^{|y|^2/2}\Big]\gamma_d(dy) \\ \leq \frac{C}{\lambda}(\|f\|_{L(1+\log^+ L)^{\frac{|\beta|-2}{2}}(\gamma_d)} + 1). \qquad \square\end{aligned}$$

Finally, as was mentioned before, the $L^p(\gamma_d)$-boundedness, $1 < p < \infty$, of the higher-order Riesz transforms, with constants independent of dimension, can be obtained as a consequence of Meyer's multiplier theorem (Theorem 6.2; see [88]).

Corollary 9.12. *The higher-order Gaussian Riesz transforms $\mathscr{R}_\beta$, $|\beta| > 1$, are $L^p(\gamma_d)$ bounded for $1 < p < \infty$, that is to say, there exists $C > 0$, dependent only on p and β, but not on dimension, such that*

$$\|\mathscr{R}_\beta f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}, \tag{9.29}$$

for any $f \in L^p(\gamma_d)$.

Proof. Given the multi-index $\beta = (\beta_1, \ldots, \beta_d)$, from (9.17), we know that the action of $\mathscr{R}_\beta$ over the normalized Hermite polynomial $\mathbf{h}_\nu$ is given by

$$\mathscr{R}_\beta \mathbf{h}_\nu(x) = \Big(\frac{1}{|\nu|}\Big)^{|\beta|/2}\Big[\prod_{i=1}^d \nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)\Big]^{1/2}\mathbf{h}_{\nu-\beta}(x),$$

with $\beta_i \leq \nu_i$ for all $i = 1, \cdots, d$, otherwise $\mathscr{R}_\beta \mathbf{h}_\nu(x) = 0$.

Now, for the same multi-index β, let us consider the operator

$$\mathscr{R}_1^{\beta_1}\mathscr{R}_2^{\beta_2}\ldots\mathscr{R}_d^{\beta_d},$$

the composition of powers of the Riesz transforms, $\mathscr{R}_1^{\beta_1},\mathscr{R}_2^{\beta_2},\ldots,\mathscr{R}_d^{\beta_d}$. Then,

$$\mathscr{R}_1^{\beta_1}\mathscr{R}_2^{\beta_2}\ldots\mathscr{R}_d^{\beta_d}\mathbf{h}_\nu(x) = \Big[\prod_{i=1}^{d}\frac{\nu_i(\nu_i-1)\cdots(\nu_i-\beta_i+1)}{|\nu|(|\nu|-1)\ldots(|\nu|-\beta_i+1)}\Big]^{1/2}\mathbf{h}_{\nu-\beta}(x).$$

Now, define the multiplier operator T_β as

$$\begin{aligned} T_\beta\mathbf{h}_\nu(x) &= \Big[\frac{\prod_{i=1}^d|\nu|(|\nu|-1)\ldots(|\nu|-\beta_i+1)}{|\nu|^{|\beta|}}\Big]^{1/2}\mathbf{h}_\nu(x)\\ &= \Big[\frac{\prod_{i=1}^d(|\nu|-1)\ldots(|\nu|-(\beta_i-1))}{|\nu|^{|\beta|-d}}\Big]^{1/2}\mathbf{h}_\nu(x)\\ &= \Big[\prod_{i=1}^d\frac{(|\nu|-1)\ldots(|\nu|-(\beta_i-1))}{|\nu|^{\beta_i-1}}\Big]^{1/2}\mathbf{h}_\nu(x)\\ &= \Big[\prod_{i=1}^d(1-\frac{1}{|\nu|})\ldots(1-\frac{(\beta_i-1)}{|\nu|})\Big]^{1/2}\mathbf{h}_\nu(x). \end{aligned}$$

Then, T_β is a Meyer's multiplier (6.4), with multiplier ϕ defined using the function,

$$h(x)=[\prod_{i=1}^d(1-x)\ldots(1-(\beta_i-1)x)]^{1/2}.$$

By construction, T_β satisfies,

$$\mathscr{R}_\beta=\mathscr{R}_1^{\beta_1}\mathscr{R}_2^{\beta_2}\ldots\mathscr{R}_d^{\beta_d}\circ T_\beta \tag{9.30}$$

Therefore, the $L^p(\gamma_d)$ boundedness of $\mathscr{R}_\beta$ is obtained immediately from the $L^p(\gamma_d)$ boundedness of the Riesz transforms $\mathscr{R}_j$ using Meyer's multiplier theorem (Theorem 6.2), where the constant is dependent on p and β, but independent of the dimension d, as long as we have proof of the L^p-boundedness of the (first-order) Riesz transforms with constants independent of dimension[2]. □

9.3 Alternative Gaussian Riesz Transforms

We have mentioned before in Chapter 2, that the Gaussian partial derivatives in $\mathbb{R}^d$, ∂_γ^i are not self-adjoint in $L^2(\gamma_d)$, and its adjoint is given by

[2] It has been mentioned before that there are several proofs of this fact (see, for instance, G. Pisier [227] or C. Gutiérrez [122])

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_i} + \sqrt{2}x_i I_d$$

(see 2.12). The Ornstein–Uhlenbeck operator can be written as

$$(-L) = \sum_{i=1}^{d}(\partial_\gamma^i)^*\partial_\gamma^i.$$

Therefore, there is another "natural" differential operator, the alternative Ornstein–Uhlenbeck operator, (2.14), which is given by

$$(-\overline{L}) = \sum_{i=1}^{d}\partial_\gamma^i(\partial_\gamma^i)^* = (-L) + dI = -\frac{1}{2}\Delta + \langle x, \nabla_x\rangle + dI.$$

H. Aimar, L. Forzani, and R. Scotto in [5] considered the following *alternative Riesz transforms*, by taking the derivatives $(\partial_\gamma^i)^*$ and the operator $(-\overline{L})$,

$$\overline{\mathscr{R}}_j = (\partial_j^\gamma)^*(-\overline{L})^{-1/2}, \tag{9.31}$$

Moreover, we can also consider alternative higher-order Gaussian Riesz transforms, that is, for a multi-index β, $|\beta| \geq 1$ we use the gradient

$$(\partial_\gamma^\beta)^* = \frac{(-1)^{|\beta|}}{2^{|\beta|/2}}e^{|x|^2}(\partial^\beta e^{-|x|^2}I)$$

and the Riesz potentials associated with $\overline{L}$. Then, these new singular integral operators are defined as follows:

Definition 9.13. *The alternative Gaussian Riesz transform $\overline{\mathscr{R}}_\beta$ for $|\beta| \geq 1$ is defined spectrally as*

$$\overline{\mathscr{R}}_\beta f(x) = (\partial_\gamma^\beta)^*(-\overline{L})^{-|\beta|/2}f(x).$$

Thus, the action of $\overline{\mathscr{R}}_\beta$ over the Hermite polynomial $\mathbf{H}_\nu$ is given by

$$\overline{\mathscr{R}}_\beta \mathbf{H}_\nu = \frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}}\mathbf{H}_{\nu+\beta}, \tag{9.32}$$

because, using the fact that the Hermite polynomials $\{\mathbf{H}_\nu\}$ are eigenfunctions of $\overline{L}$,

$$(-\overline{L})^{-|\beta|/2}H_\nu = \frac{1}{(|\nu|+d)^{|\beta|/2}}\mathbf{H}_\nu,$$

and using Rodrigues' formula (1.28), we get

$$\begin{aligned}\overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x) &= (\partial_\gamma^\beta)^*(-\overline{L})^{-|\beta|/2}\mathbf{H}_\nu(x) = \frac{(-1)^{|\beta|}}{(|\nu|+d)^{|\beta|/2}}e^{|x|^2}\partial^\beta(e^{-|x|^2}\mathbf{H}_\nu(x))\\ &= \frac{(-1)^{|\beta+\nu|}}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}}e^{|x|^2}\partial^{\beta+\nu}(e^{-|x|^2}) = \frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}}\mathbf{H}_{\nu+\beta}(x);\end{aligned}$$

therefore,

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+d)^{|\beta|/2}} \Big[\prod_{i=1}^d (\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x), \quad (9.33)$$

because,

$$\begin{aligned}
\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) &= \overline{\mathscr{R}}_\beta\Big(\frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}\Big) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}}\overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x)\\
&= \frac{1}{(2^{|\nu|}\nu!)^{1/2}}\frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}}\mathbf{H}_{\nu+\beta}(x) = \frac{1}{(\nu!)^{1/2}(|\nu|+d)^{|\beta|/2}}\frac{\mathbf{H}_{\nu+\beta}(x)}{2^{|\nu|/2+|\beta|/2}}\\
&= \frac{1}{(|\nu|+d)^{|\beta|/2}}\Big(\frac{(\nu+\beta)!}{\nu!}\Big)^{1/2}\frac{\mathbf{H}_{\nu+\beta}(x)}{(2^{|\nu+\beta|}(\nu+\beta)!)^{1/2}}\\
&= \frac{1}{(|\nu|+d)^{|\beta|/2}}\Big[\prod_{i=1}^d (\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2}\mathbf{h}_{\nu+\beta}(x).
\end{aligned}$$

With an argument analogous to Lemma 8.3, we can get that the alternative higher-order Gaussian Riesz transforms then have the following integral representation

$$\overline{\mathscr{R}}_\beta f(x) = \text{p.v. } e^{|x|^2}\int_{\mathbb{R}^d} \overline{\mathscr{K}}_\beta(x,y) f(y)\,\gamma_d(dy)$$

where

$$\overline{\mathscr{K}}_\beta(x,y) = C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\,dr.$$

Formally, $\overline{\mathscr{K}}_\beta$ is obtained by differentiating with respect to the adjoint of ∂^γ the kernel corresponding to the Riesz potentials associated with $\overline{L}$, (8.62),

$$\begin{aligned}
(-\overline{L})^{-|\beta|/2} f(x) &= \frac{1}{\Gamma(|\beta|/2)}\int_0^\infty t^{\frac{|\beta|-2}{2}} T_t^{(d)} f(x)\,dt\\
&= C_\beta e^{|x|^2}\int_{\mathbb{R}^d}\Big(\int_0^1 (-\log r)^{\frac{|\beta|-2}{2}} r^d \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\frac{dr}{r}\Big) f(y)\,\gamma_d(dy).\\
&= C_\beta \int_{\mathbb{R}^d}\Big(\int_0^1 (-\log r)^{\frac{|\beta|-2}{2}} r^d \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\frac{dr}{r}\Big) f(y)\,dy.
\end{aligned}$$

Similar to Corollary 9.12, the $L^p(\gamma_d)$ boundedness of $\overline{\mathscr{R}}_\beta$, $1<p<\infty$ can be obtained from P. A. Meyer's multiplier theorem (Theorem 6.2).

Corollary 9.14. *The alternative Gaussian Riesz transforms $\overline{\mathscr{R}}_\beta$ are $L^p(\gamma_d)$ bounded for $1<p<\infty$, that is to say, there exists $C>0$, dependent only on p and β, but not on dimension, such that*

$$\|\overline{\mathscr{R}}_\beta f\|_{p,\gamma} \le C\|f\|_{p,\gamma}, \quad (9.34)$$

for any $f\in L^p(\gamma_d)$.

Proof. Given the multi-index $\beta = (\beta_1, \dots, \beta_d)$, from (9.33), we know that the action of $\mathscr{R}_\beta$ over the normalized Hermite polynomial $\mathbf{h}_\nu$ is given by

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+d)^{|\beta|/2}} \Big[\prod_{j=1}^{d} (\nu_j+\beta_j)\cdots(\nu_j+1)\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x).$$

Now, for the same multi-index β, let us consider the operator

$$\overline{\mathscr{R}}_1^{\beta_1} \overline{\mathscr{R}}_2^{\beta_2} \dots \overline{\mathscr{R}}_d^{\beta_d},$$

the composition of powers of the Riesz transforms, $\overline{\mathscr{R}}_1^{\beta_1}, \overline{\mathscr{R}}_2^{\beta_2}, \dots \overline{\mathscr{R}}_d^{\beta_d}$. Then,

$$\begin{aligned}
\overline{\mathscr{R}}_1^{\beta_1} \overline{\mathscr{R}}_2^{\beta_2} \dots \overline{\mathscr{R}}_d^{\beta_d} \mathbf{h}_\nu(x) &= \prod_{j=1}^{d} \Big(\prod_{i=1}^{\beta_j} \Big(\frac{\nu_j+i}{|\nu|+d+(i-1)}\Big)\Big)^{1/2} \mathbf{h}_{\nu+\beta}(x) \\
&= \Big[\prod_{j=1}^{d} \frac{(\nu_j+\beta_j)\cdots(\nu_j+1)}{(|\nu|+d+\beta_j-1)\cdots(|\nu|+d)}\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x)
\end{aligned}$$

Consider the multiplier T_β defined as

$$\begin{aligned}
T_\beta \mathbf{h}_\nu(x) &= \Big[\frac{\prod_{j=1}^{d}(|\nu|+d+\beta_j-1)\cdots(|\nu|+d)}{(|\nu|+d)^{|\beta|}}\Big]^{1/2} \mathbf{h}_\nu(x) \\
&= \Big[\frac{\prod_{j=1}^{d}(|\nu|+d+\beta_j-1)\cdots(|\nu|+2)}{(|\nu|+d)^{|\beta|-d}}\Big]^{1/2} \mathbf{h}_\nu(x) \\
&= \Big[\prod_{j=1}^{d} \Big(\frac{|\nu|+d+\beta_j-1)\cdots(|\nu|+d+1)}{(|\nu|+d)^{\beta_j-1}}\Big)\Big]^{1/2} \mathbf{h}_\nu(x) \\
&= \Big[\prod_{j=1}^{d} \Big(\frac{(|\nu|+d)+(\beta_j-1)}{|\nu|+d}\Big)\cdots\frac{(|\nu|+d)+1}{|\nu|+d}\Big)\Big]^{1/2} \mathbf{h}_\nu(x) \\
&= \Big[\prod_{j=1}^{d} \Big(1+\frac{(\beta_j-1)}{|\nu|+d}\Big)\cdots\Big(1+\frac{1}{|\nu|+d}\Big)\Big]^{1/2} \mathbf{h}_\nu(x)
\end{aligned}$$

By construction, T_β satisfies,

$$\overline{\mathscr{R}}_\beta = \overline{\mathscr{R}}_1^{\beta_1} \overline{\mathscr{R}}_2^{\beta_2} \dots \overline{\mathscr{R}}_d^{\beta_d} \circ T_\beta \tag{9.35}$$

As in the case of the Gaussian Bessel potentials, T_β is the composition of two Meyer's multipliers 6.4), one of the multipliers defined using the function,

$$h(x) = \Big[\prod_{j=1}^{d} (1+x(\beta_j-1))\cdots(1+x)\Big]^{1/2}.$$

Therefore, the $L^p(\gamma_d)$ boundedness of $\overline{\mathscr{R}}_\beta$ is obtained immediately from the $L^p(\gamma_d)$ boundedness of the Riesz transforms $\overline{\mathscr{R}}_j$ using Meyer's multiplier theorem (Theorem 6.2), where the constant is dependent on p and β, but independent of the dimension d, as long as we have proof of the L^p-boundedness of the (first-order) Riesz transforms with constants independent of dimension. □

In [5], H. Aimar, L. Forzani, and R. Scotto obtained a surprising result: the alternative Riesz transforms $\overline{\mathscr{R}}_\beta$ are of weak type $(1,1)$ for all multi-index β, i.e., independently of their orders, which is a contrasting fact with respect to the anomalous behavior of the higher-order Riesz transforms $\mathscr{R}_\beta$.

Theorem 9.15. *For any multi-index β, there exists a constant C dependent only on d and β such that for all $\lambda > 0$, $f \in L^1(\gamma_d)$, we have*

$$\gamma_d\left(\left\{x \in \mathbb{R}^d : \overline{\mathscr{R}}_\beta f(x) > \lambda\right\}\right) \leq \frac{C}{\lambda}\int_{\mathbb{R}^d} |f(y)|\gamma_d(dy),$$

i.e., $\overline{\mathscr{R}}_\beta f$ is of γ_d-weak type $(1,1)$.

Proof. The main feature, to prove this theorem, is to apply Theorem 4.18 with an special Φ. For each $x \in \mathbb{R}^d$, as usual, we write this operator as the sum of two operators that are obtained by splitting $\mathbb{R}^d$ into a local region, $B_h(x) = \{y \in \mathbb{R}^d : |y-x| < C_d m(x)\}$, an admissible ball and its complement $B_h^c(x)$ called the global region. Thus,

$$\overline{\mathscr{R}}_\beta f(x) = \overline{\mathscr{R}}_{\beta,L} f(x) + \overline{\mathscr{R}}_{\beta,G} f(x)$$

where $\overline{\mathscr{R}}_{\beta,L} f(x) = \overline{\mathscr{R}}_\beta(f\chi_{B_h(\cdot)})(x)$ is the local part of $\mathscr{R}_\beta$ and $\overline{\mathscr{R}}_{\beta,G} f(x) = \overline{\mathscr{R}}_\beta(f(\chi^c_{B_h(\cdot)})(x)$ is the global part of $\mathscr{R}_\beta$.

We prove that these two operators are γ_d-weak type $(1,1)$; thus, also $\overline{\mathscr{R}}_\beta$ is weak type $(1,1)$. To prove that $\overline{\mathscr{R}}_{\beta,L}$ is of γ-weak type $(1,1)$, we apply Theorem 4.30. In our case,

$$Tf(x) = p.v. \int_{\mathbb{R}^d} \mathscr{K}(x,y) f(y)dy$$

with

$$\begin{aligned}\mathscr{K}(x,y) &= e^{|x|^2}\overline{\mathscr{K}}_\beta(x,y)e^{-|y|^2}\\ &= C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\,dr.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial \mathscr{K}}{\partial y_j}(x,y) = 2C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}&\left[\frac{-r\beta_j}{\sqrt{1-r^2}}\mathbf{H}_{\beta-\mathbf{e_j}}\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right.\\ &\left.+\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{(rx_j-y_j)}{1-r^2}\right]\frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\,dr.\end{aligned}$$

Now, we show that the hypotheses of Theorem 4.30 are fulfilled for this operator. Thus, we prove that, in the local region $B_h(x)$, we have,

$$|\mathscr{K}(x,y)| \le \frac{C}{|x-y|^d}$$

and

$$|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)| \le \frac{C}{|x-y|^{d+1}}.$$

There exists a constant $C>0$ such that for every $y \in B_h$ $C^{-1} \le e^{|y|^2-|x|^2} \le C$, then

$$|\mathscr{K}(x,y)| \le C|e^{-|x|^2+|y|^2}\mathscr{K}(x,y)| = C|\overline{\mathscr{K}}_\beta(x,y)|$$

and

$$\left|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)\right| \le C\left|e^{-|x|^2+|y|^2}\frac{\partial \overline{\mathscr{K}}}{\partial y_j}(x,y)\right|.$$

On the other hand, on B_h, for any $c>0$,

$$e^{-c\frac{|x-ry|^2}{1-r^2}} = e^{-c\frac{|x-y|^2}{1-r^2}}e^{-c\frac{1-r}{1+r}|y|^2}e^{-c\frac{(x-y)\cdot y}{1-r}} \le Ce^{-c\frac{|x-y|^2}{1-r}};$$

thus, with thisinequality and taking into account that $t^m e^{-ct^2} \le C_m$, for all $t \ge 0$, we get

$$\left|\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right|e^{-\frac{|x-ry|^2}{1-r^2}} \le C\sum_{m=0}^{|\beta|}\left|\frac{x-ry}{\sqrt{1-r^2}}\right|^m e^{-\frac{|x-ry|^2}{2(1-r^2)}}e^{-\frac{|x-ry|^2}{2(1-r^2)}} \le Ce^{-c\frac{|x-y|^2}{1-r}}.$$

Therefore, by combining all the above remarks, on B_h we have,

$$\begin{aligned}|\mathscr{K}(x,y)| &\le C\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}}\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d}{2}+1}}dr\\ &\le C\left[\int_0^{\frac12}(-\log r)^{\frac{|\beta|-2}{2}}dr+\int_{\frac12}^1\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d}{2}+1}}dr\right] \le C\left(1+\frac{1}{|x-y|^d}\right) \le \frac{C}{|x-y|^d}\end{aligned}$$

and

$$\begin{aligned}|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)| &\le C\int_0^1\left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}}\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d+3}{2}}}dr\\ &\le C\left[\int_0^{\frac12}(-\log r)^{\frac{|\beta|-2}{2}}dr+\int_{\frac12}^1\frac{e^{-c\frac{|x-y|^2}{1-r}}}{(1-r)^{\frac{d+3}{2}}}dr\right]\\ &\le C\left(1+\frac{1}{|x-y|^{d+1}}\right) \le \frac{C}{|x-y|^{d+1}}.\end{aligned}$$

Let us prove that the operator $\overline{\mathscr{R}}_\beta$ is bounded on $L^2(\gamma_d)$. Given $f \in L^2(\gamma_d)$, with Hermite expansion $f = \sum_\nu \widehat{f}_\gamma(\nu)\mathbf{h}_\nu = \sum_\nu \langle f, \mathbf{h}_\nu\rangle_\gamma \mathbf{h}_\nu$. Then, because the action of $\overline{\mathscr{R}}_\beta$ over the normalized Hermite polynomial $\mathbf{h}_\nu$ is given by (9.33),

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{2^{|\beta|/2}} \frac{\prod_{j=1}^d [(\nu_j+d)\cdots(\nu_j+\beta_j)]^{\frac{1}{2}}}{(|\nu|+d)^{|\beta|/2}} h_{\nu+\beta}(x).$$

Therefore,

$$\begin{aligned}
\|\overline{\mathscr{R}}_\beta f\|^2_{L^2(d\gamma)} &= \sum_\nu \frac{\prod_{j=1}^d [(\nu_j+1)\cdots(\nu_j+\beta_j)]}{2^{|\beta|}(|\nu|+d)^{|\beta|}} |\widehat{f}_\gamma(\nu)|^2 \\
&\leq \sum_\nu \prod_{j=1}^d (\beta_j+1)^{\beta_j} |\widehat{f}_\gamma(\nu)|^2 \leq (|\beta|+1)^{|\beta|} \sum_\nu |\widehat{f}_\gamma(\nu)|^2 \leq C\|f\|^2_{L^2(\gamma_d)}.
\end{aligned}$$

Therefore, using Theorem 4.30, the γ_d-weak type $(1,1)$ of $\overline{\mathscr{R}}_{\beta,L}$ follows.

To prove that $\overline{\mathscr{R}}_{\beta,G}$ is also γ_d-weak type $(1,1)$, we prove on $\mathbb{R}^d \setminus B_h$,

$$|\overline{\mathscr{R}}_{\beta,G} f(x)| \leq C\mathscr{M}_\Phi f(x), \tag{9.36}$$

with $\Phi(t) = e^{-ct^2}$. Then, using Theorem 4.18, we get the weak type $(1,1)$ inequality for $\overline{\mathscr{R}}_{\beta,G}$.

$$\begin{aligned}
|\overline{\mathscr{K}}_\beta(x,y)| &= \left| \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1} \mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}} dr \right| \\
&\leq C\int_0^{\frac{3}{4}} (-\log r)^{\frac{|\beta|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}} dr \\
&\qquad + C\int_{\frac{3}{4}}^{1-\zeta/|x|^2} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d-1}{2}}} (|x| \vee (1-r^2)^{-\frac{1}{2}}) \frac{dr}{|x|(1-r^2)^{3/2}} \\
&\qquad + C\int_{1-\zeta/|x|^2}^1 \frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d-1}{2}}} (|x| \vee (1-r^2)^{-\frac{1}{2}}) \frac{e^{-\overline{c}\frac{|x-y|^2}{1-r}}}{1-r} dr.
\end{aligned}$$

Hence,

$$|\overline{\mathscr{K}}_\beta(x,y)| \leq C\left(\overline{\mathscr{K}}^1_\beta(x,y) + \overline{\mathscr{K}}^2_\beta(x,y) + \overline{\mathscr{K}}^3_\beta(x,y)\right),$$

where the inequality is obtained by annihilating the Hermite polynomial with part of the exponential, then splitting the unit interval of the integral into three subintervals $[0,3/4]$, $[3/4, 1-\zeta/|x|^2]$, and $[1-\zeta/|x|^2, 1]$ and taking into account that on the second one $|x| \vee (1-r^2)^{-1/2} \geq |x|$, on the third one $|x| \vee (1-r^2)^{-1/2} \geq (1-r^2)^{-1/2}$

and $|x-ry| \geq \bar{c}|x-y|$, and on the last two intervals, the function $-\log r/(1-r^2)$ is bounded by a constant.

Thus, by using the definition of kernels $\overline{\mathscr{K}}_\beta^j$ with $j=1,2,3$, interchanging the order of integration on each operator $\overline{\mathscr{R}}_{\beta,G}{}^j$ with $j=1,2,3$, using Lemma 1.23 and taking $\Phi(t)=e^{-ct^2}$, we get, using Fubini's theorem

$$\overline{\mathscr{R}}_{\beta,G}{}^1 f(x) = e^{|x|^2}\int_{\mathbb{R}^d}\int_0^{\frac{3}{4}}(-\log r)^{\frac{|\beta|-2}{2}}\frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d}{2}}}\,dr\,|f(y)|\,\gamma_d(dy)$$

$$= \int_0^{\frac{3}{4}}(-\log r)^{\frac{|\beta|-2}{2}}e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d}{2}}}|f(y)|\gamma_d(dy)\,dr$$

$$\leq C\int_0^{\frac{3}{4}}(-\log r)^{\frac{|\beta|-2}{2}}dr \quad \mathscr{M}_\Phi f(x) \leq C\,\mathscr{M}_\Phi f(x),$$

$$\overline{\mathscr{R}}_{\beta,G}{}^2 f(x) = e^{|x|^2}\int_{\mathbb{R}^d}\int_{\frac{3}{4}}^{1-\zeta/|x|^2}\frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d-1}{2}}}\,(|x|\vee(1-r^2)^{-\frac{1}{2}})$$
$$\times\frac{dr}{|x|(1-r^2)^{3/2}}|f(y)|\,\gamma_d(dy)$$

$$= \int_{3/4}^{1-\zeta/|x|^2}e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(d-1)/2}}(|x|\vee(1-r^2)^{-1/2})|f(y)|\gamma_d(dy)$$
$$\times\frac{dr}{|x|(1-r^2)^{3/2}}$$

$$\leq C\frac{1}{|x|}\int_{3/4}^{1-\zeta/|x|^2}\frac{dr}{(1-r)^{3/2}} \quad \mathscr{M}_\Phi f(x) \leq C\mathscr{M}_\Phi f(x),$$

and, finally,

$$\overline{\mathscr{R}}_{\beta,G}{}^3 f(x) = e^{|x|^2}\int_{\mathbb{R}^d}\int_{1-\zeta/|x|^2}^1\frac{e^{-c\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d-1}{2}}}\,(|x|\vee(1-r^2)^{-\frac{1}{2}})\frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}\,dr\,|f(y)|\,\gamma_d(dy)$$

$$= \int_{1-\zeta/|x|^2}^1 e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(d-1)/2}}(|x|\vee(1-r^2)^{-1/2})\frac{e^{-\bar{c}\frac{|x-y|^2}{1-r}}}{1-r}$$
$$\times|f(y)|\,\gamma_d(dy)\,dr$$

$$\leq \int_{1-\zeta/|x|^2}^1 e^{|x|^2}\int_{\mathbb{R}^d}\frac{e^{-c\frac{|x-ry|^2}{(1-r^2)}}}{(1-r^2)^{(d-1)/2}}(|x|\vee(1-r^2)^{-1/2})$$
$$\times\frac{1}{|x-y|^2}\,|f(y)|\gamma_d(dy)\,dr$$

$$\leq C|x|^2 \int_{1-\zeta/|x|^2}^{1} dr \;\; \mathscr{M}_{\Phi} f(x) \leq C \mathscr{M}_{\Phi} f(x).$$

Thus, because $|\overline{\mathscr{R}}_{\beta,G} f(x)| \leq C_\beta \sum_{j=1}^{3} \overline{\mathscr{R}}^{j}_{\beta,G} f(x)$, then (9.36) follows. □

9.4 Definition and Boundedness Properties of General Gaussian Singular Integrals

Finally, we define *general Gaussian singular integrals*, generalizing the Gaussian higher-order Riesz transforms. We follow, essentially, the outline developed for them. The first formulation of general Gaussian singular integrals was given by W. Urbina in [278]. Later, S. Pérez [221] extended it. We consider S. Pérez's class, as it is a much larger class.

Definition 9.16. *Given a C^1-function F, satisfying the orthogonality condition*

$$\int_{\mathbb{R}^d} F(x)\gamma_d(dx) = 0, \tag{9.37}$$

and such that for every $\varepsilon > 0$, there exist constants, C_ε y C'_ε such that

$$|F(x)| \leq C_\varepsilon e^{\varepsilon|x|^2} \quad \textit{and} \quad |\nabla F(x)| \leq C'_\varepsilon e^{\varepsilon|x|^2}. \tag{9.38}$$

Then, for each $m \in \mathbb{N}$ the generalized Gaussian singular integral is defined as

$$T_{F,m} f(x) = \int_{\mathbb{R}^d} \int_0^1 \Big(\frac{-\log r}{1-r^2}\Big)^{\frac{m-2}{2}} r^m F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} \frac{dr}{r} f(y)dy. \tag{9.39}$$

$T_{F,m}$ can be written as

$$T_{F,m} f(x) = \int_{\mathbb{R}^d} \mathscr{K}_{F,m}(x,y) f(y) dy,$$

denoting,

$$\begin{aligned} \mathscr{K}_{F,m}(x,y) &= \int_0^1 \Big(\frac{-\log r}{1-r^2}\Big)^{\frac{m-2}{2}} r^{m-1} F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} dr \\ &= \int_0^1 \varphi_m(r) F\Big(\frac{y-rx}{\sqrt{1-r^2}}\Big) \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}} dr \\ &= \int_0^1 \psi_m(t) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt, \end{aligned} \tag{9.40}$$

with $\varphi_m(r) = (\frac{-\log r}{1-r^2})^{\frac{m-2}{2}} r^{m-1}$; and taking the change of variables $t = 1-r^2$, with $\psi_m(t) = \varphi_m(\sqrt{1-t})/\sqrt{1-t}$, and $u(t) = \frac{|\sqrt{1-t}x-y|^2}{t}$.

In [278], instead of condition (9.38), it was asked that F and ∇F would have at most polynomial growth, which of course is a particular case of (9.38). On the other hand, the higher-order Riesz transforms $\mathscr{R}_\beta$ are clearly particular cases of the operators $T_{F,m}$ by simply taking $F = H_\beta$, and $m = |\beta|$.

We will prove that the operator $T_{F,m}$ is a bounded operator in $L^p(\gamma_d)$, $1 < p < \infty$.

Theorem 9.17. *The operators $T_{F,m}$ are $L^p(\gamma_d)$-bounded for $1 < p < \infty$; that is to say, there exists $C > 0$, dependent only on p and on dimension such that*

$$\|T_{F,m}f\|_{p,\gamma} \leq C\|f\|_{p,\gamma}, \tag{9.41}$$

for any $f \in L^p(\gamma_d)$.

Proof. As usual, we split $T_{F,m}$ into its local part and its global part,

$$T_{F,m}f(x) = T_{F,m}(f\chi_{B_h(\cdot)})(x) + T_{F,m}(f\chi_{B_h^c(\cdot)})(x) = T_{F,m,L}f(x) + T_{F,m,G}f(x).$$

I) For the local part $T_{F,m,L}$, we prove that it is always of weak type $(1,1)$. The estimates needed follow from an idea that appeared initially in W. Urbina's article [278], that the local part differs from a Calderón–Zygmund singular integral by an operator that is $L^1(\gamma_d)$-bounded; in other words, the operator is defined by the difference of $T_{F,m}$ and an appropriate approximation of it (which is an operator defined as the convolution with a Calderón–Zygmund kernel) is $L^1(\mathbb{R}^d)$-bounded.

- First, observe that if F satisfies the orthogonality condition (9.37) and (9.38), setting
$$K(x) = \int_0^\infty F\Big(-\frac{x}{t^{1/2}}\Big)e^{-|x|^2/t}\frac{dt}{t^{d/2+1}},$$
then, K is a Calderón–Zygmund kernel of convolution type (see (4.67)), as the integral is absolutely convergent when $x \neq 0$. Taking the change of variables $s = |x|/t^{1/2}$ we get
$$K(x) := \frac{2\int_0^\infty F\Big(-\frac{x}{|x|}s\Big)e^{-s^2}s^{d-1}ds}{|x|^d} = \frac{\Omega(x)}{|x|^d},$$
with Ω homogeneous of degree zero; therefore, K is homogeneous of degree $-d$. Moreover, Ω is C^1 with mean zero on S^{d-1}, because
$$\begin{aligned}\int_{S^{d-1}}\Omega(x')d\sigma(x') &= 2\int_0^\infty\int_{S^{d-1}}F(-x's)d\sigma(x')e^{-s^2}s^{d-1}ds\\ &= 2\int_{\mathbb{R}^d}F(-y)e^{-|y|^2}dy = 0.\end{aligned}$$
Therefore, according to the classical Calderón–Zygmund theory, the convolution operator defined using convolution with the kernel K is continuous in $L^p(\mathbb{R}^d)$, $1 < p < \infty$ and weak type $(1,1)$, with respect to the Lebesgue measure. Therefore, using Theorem 4.32, its local part S_L is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and of weak type $(1,1)$ with respect to γ_d.

- Second, we need to get rid of the function ψ_m. Taking a limit from the right, we can define $\psi(0) := \psi_m(0^+) = 2^{-(m-2)/2}$, then ψ_m is continuous on $[0,1)$. Moreover,
$$|\psi_m(t) - \psi_m(0)| \leq C\frac{t}{\sqrt{1-t}}.$$
Thus, from (9.40), we can write,
$$\begin{aligned}\mathscr{K}_{F,m}(x,y) &= \psi_m(0)\int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt \\ &\quad + \int_0^1 (\psi_m(t)-\psi_m(0))F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt.\end{aligned}$$
Set
$$K_1(x,y) := \int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt,$$
Now, over the local part we know that $u(t) \geq |y-x|^2/t - 2d$, then, using condition (9.38), we get
$$\begin{aligned}&\int_0^1 |\psi_m(t)-\psi_m(0)|\Big|F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\Big|\frac{e^{-u(t)}}{t^{d/2+1}}dt \\ &\qquad \leq \int_0^1 |\psi_m(t)-\psi_m(0)|\Big|F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\Big|\frac{e^{-u(t)}}{t^{d/2+1}}dt \\ &\qquad \leq C\int_0^1 \frac{e^{-(1-\varepsilon)u(t)}}{t^{d/2}}\frac{dt}{\sqrt{1-t}} \leq C\int_0^1 \frac{e^{-\frac{\delta|x-y|^2}{t}}}{t^{d/2}}\frac{dt}{\sqrt{1-t}}.\end{aligned}$$
Set
$$K_2(x) := \frac{e^{-\frac{\delta|x|^2}{t}}}{t^{d/2}}.$$

- Third, we need to control the difference between K_1 and the Calderón–Zygmund kernel K.
<u>Claim</u>
$$|K_1(x,y) - K(x-y)| \leq C\frac{1+|x|^{1/2}}{|x-y|^{d-1/2}} \quad \chi_{\{|x-y|<d(1\wedge 1/|x|)\}}(x,y)$$
Proof of the claim We need to estimate,
$$\begin{aligned}|K_1(x,y)-K(x-y)| &= \Big|\int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt \\ &\quad - \int_0^\infty F\Big(\frac{y-x}{t^{1/2}}\Big)e^{-|x-y|^2/t}\frac{dt}{t^{d/2+1}}\Big|.\end{aligned}$$

Using again the notation of Proposition 4.23, consider t_0 defined in (4.45),

$$t_0 = 2\frac{\sqrt{a^2-b^2}}{a+\sqrt{a^2-b^2}}.$$

Now, if $t \geq t_0$, because

$$t_0 \sim \frac{\sqrt{a^2-b^2}}{a} \sim \frac{\sqrt{a-b}}{\sqrt{a+b}} = \frac{|x-y|}{|x+y|} \wedge 1,$$

and again using that on the local part $u(t) \geq |y-x|^2/t - 2d$, there is a $\delta > 0$ such that,

$$\begin{aligned}
&\Big| \int_{t_0}^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\frac{e^{-u(t)}}{t^{d/2+1}}dt - \int_{t_0}^\infty F\Big(\frac{y-x}{\sqrt{t}}\Big)e^{-|x-y|^2/t}\frac{dt}{t^{d/2+1}}\Big| \\
&\quad \leq \int_{t_0}^1 \Big|F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\Big|\frac{e^{-u(t)}}{t^{d/2+1}}dt + \int_{t_0}^\infty \Big|F\Big(\frac{y-x}{\sqrt{t}}\Big)\Big|e^{-|x-y|^2/t}\frac{dt}{t^{d/2+1}} \\
&\quad \leq C\int_{t_0}^1 \frac{e^{-\delta|x-y|^2/t}}{t^{(d-1)/2}}\frac{dt}{t^{3/2}} \leq C\frac{1}{|x-y|^{d-1}}\frac{1}{t_0^{1/2}} \leq C\frac{1+|x|^{1/2}}{|x-y|^{d-1/2}}.
\end{aligned}$$

For $t \leq t_0$ setting $v(s) = y - \sqrt{1-s}\,x$, we then have

$$\begin{aligned}
&\Big|F\Big(\frac{v(t)}{t^{1/2}}\Big)e^{-\frac{|v(t)|^2}{t}} - F\Big(\frac{v(0)}{t^{1/2}}\Big)e^{-\frac{|v(0)|^2}{t}}\Big| = \Big|\int_0^t \frac{\partial}{\partial s}\Big(F\Big(\frac{v(s)}{t^{1/2}}\Big)e^{-\frac{|v(s)|^2}{t}}\Big)ds\Big| \\
&\quad = \Big|\int_0^t \Big(\Big\langle\frac{v'(s)}{t^{1/2}}, (\nabla F)\Big(\frac{v(s)}{t^{1/2}}\Big)\Big\rangle e^{-\frac{|v(s)|^2}{t}} \\
&\qquad\qquad -2\Big\langle v'(s), \frac{v(s)}{t}\Big\rangle F\Big(\frac{v(s)}{t^{1/2}}\Big)e^{-\frac{|v(s)|^2}{t}}\Big)ds\Big| \\
&\quad \leq \int_0^t \Big|\frac{v'(s)}{t^{1/2}}\Big|\Big|(\nabla F)\Big(\frac{v(s)}{t^{1/2}}\Big)\Big|e^{-\frac{|v(s)|^2}{t}}ds \\
&\qquad\qquad +2\int_0^t \frac{|v'(s)|}{t^{1/2}}\frac{|v(s)|}{t^{1/2}}\Big|F\Big(\frac{v(s)}{t^{1/2}}\Big)\Big|e^{-\frac{|v(s)|^2}{t}}ds.
\end{aligned}$$

Using the hypothesis (9.38) and the fact that in the local part

$$\frac{-|v(s)|^2}{t} \leq \frac{-|x-y|^2}{t} + 2d\frac{s}{t},$$

we get, for some $\delta > 0$,

$$\begin{aligned}
&\int_0^{t_0}\Big|F\Big(\frac{v(t)}{t^{1/2}}\Big)e^{-\frac{|v(t)|^2}{t}} - F\Big(\frac{v(0)}{t^{1/2}}\Big)e^{-\frac{|v(0)|^2}{t}}\Big|\frac{dt}{t^{d/2+1}} \\
&\qquad\qquad \leq \int_0^{t_0}\int_0^t \frac{|v'(s)|}{t^{1/2}}e^{-\delta\frac{|v(s)|^2}{t}}ds\,\frac{dt}{t^{d/2+1}} \\
&\qquad\qquad \leq C|x|\int_0^{t_0}\int_0^t \frac{1}{\sqrt{1-s}}ds\,\frac{1}{t^{1/2}}e^{-\delta\frac{|x-y|^2}{t}}\frac{dt}{t^{d/2+1}}
\end{aligned}$$

$$\leq C|x| \int_0^{t_0} \frac{1}{t^{1/2}} \frac{e^{-\delta \frac{|x-y|^2}{t}}}{t^{d/2}} dt$$
$$\leq C \frac{|x|}{|x-y|^d} \int_0^{t_0} \frac{1}{t^{1/2}} dt \leq C \frac{|x| t_0^{1/2}}{|x-y|^d} \leq C \frac{1+|x|^{1/2}}{|x-y|^{d-1/2}}.$$

Set

$$K_3(x,y) := \frac{1+|x|^{1/2}}{|x-y|^{d-1/2}}.$$

Observe that $K_3(x,y)$ defines a function in the variable x, which is $L^1(\mathbb{R}^d)$, uniformly in the variable y.

Hence, writing $\mathscr{K}_{F,m}(x,y)$ as

$$\begin{aligned}
\mathscr{K}_{F,m}(x,y) &= \int_0^1 \psi_m(t) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt, \\
&= \psi_m(0) \int_0^1 F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt \\
&\qquad + \int_0^1 (\psi_m(t)-\psi_m(0)) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt \\
&= \psi_m(0) \int_0^1 \Big[F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} - F\Big(\frac{y-x}{\sqrt{t}}\Big) \frac{e^{-|x-y|^2/t}}{t^{d/2+1}}\Big] dt \\
&\quad + \psi_m(0) \int_0^1 F\Big(\frac{y-x}{\sqrt{t}}\Big) \frac{e^{-|x-y|^2/t}}{t^{d/2+1}} dt \\
&\qquad + \int_0^1 (\psi_m(t)-\psi_m(0)) F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big) \frac{e^{-u(t)}}{t^{d/2+1}} dt.
\end{aligned}$$

Using the estimates above, we conclude that the local part $T_{F,m,L}$ can be bounded as

$$\begin{aligned}
|T_{F,m,L} f(x)| &= |T_{F,m} f(\chi_{B_h(x)})(x)| = \Big| \int_{B_h(x)} \mathscr{K}_{F,m}(x,y) f(y)\, dy \Big| \\
&\leq C \int_{B_h(x)} K_3(x,y) |f(y)|\, dy + C \Big| p.v. \int_{B_h(x)} K(x,y) f(y)\, dy \Big| \\
&\qquad + \int_{B_h(x)} K_2(x-y) |f(y)|\, dy \\
&= (I) + (II) + (III).
\end{aligned}$$

Using Theorem 4.32, (II) is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and is of weak type $(1,1)$ with respect to γ_d. Thus, it remains to prove that (I) and (III) are also bounded. To do so, we use Lemma 4.3, taking a countable family of admissible balls. $\mathscr{F}$

Now, given $B \in \mathscr{F}$, if $x \in B$ then $B_h(x) \subset \hat{B}$; therefore,

$$(I) = (1+|x|^{1/2}) \sum_{k=0}^{\infty} \int_{2^{-(k+1)}C_d m(x)<|x-y|<2^{-k}C_d m(x)} \frac{|f(y)|\chi_{\hat{B}}}{|x-y|^{d-1/2}} dy$$
$$\leq C_d 2^d \mathscr{M}(f\chi_{\hat{B}})(x)(1+|x|^2)m(x)^{1/2} \sum_{k=0}^{\infty} 2^{-(k+1)/2} \leq LC\mathscr{M}(f\chi_{\hat{B}})(x)(\chi_{B_h(\cdot)})(x).$$

On the other hand, let us consider $\varphi(y) = C_\delta e^{-\delta|y|^2}$, where C_δ is a constant such that $\int_{\mathbb{R}^d} \varphi(y)dy = 1$. φ is a non-increasing radial function, and given $t > 0$, we rescale this function as $\varphi_{\sqrt{t}}(y) = t^{-d/2}\phi(y/\sqrt{t})$, and, because $0 \leq \varphi \in L^1(\mathbb{R}^d)$, $\{\varphi_{\sqrt{t}}\}_{t>0}$ is an approximation of the identity (see the Appendix). Then, because $\int_0^1 (1/\sqrt{1-t})dt < \infty$,

$$(III) = \int_{B_h(x)} K_2(x-y)|f(y)|\, dy = \int_{B_h(x)} \Big(\int_0^1 \varphi_{\sqrt{t}}(x-y)\frac{dt}{\sqrt{1-t}}\Big)|f(y)|dy$$
$$\leq \int_{B_h(x)} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y)\Big)\Big(\int_0^1 \frac{dt}{\sqrt{1-t}}\Big)|f(y)|dy$$
$$\leq C\int_{B_h(x)} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y)\Big)|f(y)|dy.$$

Again, using the family $\mathscr{F}$, if $x \in B$, $B_h(x) \subset \hat{B}$, and then, using a similar argument to previously,

$$(III) = \int_{B_h(x)} K_2(x-y)|f(y)|\, dy \leq C\int_{\mathbb{R}^d} \Big(\sup_{t>0} \varphi_{\sqrt{t}}(x-y)\Big)|f(y)|\chi_{\hat{B}}(y)dy$$

which yields, using Theorem 4 in Stein's book [252, Chapter II §4.],

$$(III) = \int_{B_h(x)} K_2(x-y)|f(y)|\, dy \leq \sum_{B\in\mathscr{F}} \sup_{t>0} \Big|(\varphi_{\sqrt{t}} * |f\chi_{\hat{B}}|)(x)\Big|\chi_B(x)$$
$$\leq \sum_{B\in\mathscr{F}} \mathscr{M}(f\chi_{\hat{B}})(x)\chi_B(x).$$

Therefore, the local part $T_{F,m,L}$ is bounded in $L^p(\gamma_d)$, $1 < p < \infty$ and is of weak type $(1,1)$ with respect to γ_d,

II) Now, for the global part $T_{F,m,G}$, we prove that it is $L^p(\gamma_d)$-bounded for all $1 < p < \infty$ using similar techniques to those used for the Gaussian Riesz transform, to estimate the kernel $\mathscr{K}_{F,m}$. The idea is to exploit the size of the kernel and treat T as a positive operator.
Observe that the function $\varphi_m(r) = (\frac{-\log r}{1-r^2})^{\frac{m-2}{2}} r^{m-1}$ is bounded in $(0,1)$ for any $m \in \mathbb{N}$. Hence, using (9.38), we get

$$|\mathscr{K}_{F,m}(x,y)| \leq \int_0^1 \Big|F\Big(\frac{y-\sqrt{1-t}\,x}{\sqrt{t}}\Big)\Big| \frac{e^{-u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} \leq C_\varepsilon \int_0^1 e^{\varepsilon u(t)} \frac{e^{-u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}},$$

for some $\varepsilon > 0$ to be determined. As before, we consider two cases:

- Case #1: $b = 2\langle x,y\rangle \leq 0$. In this case we use again the inequality (4.76):

$$\frac{a}{t} - |x|^2 \leq u(t) = \frac{a}{t} - \frac{\sqrt{1-t}}{t} b - |x|^2 \leq \frac{2a}{t};$$

thus, the change of variables $s = a(\frac{1}{t} - 1)$ gives

$$\int_0^1 \frac{e^{-(1-\varepsilon)u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} \leq e^{-(1-\varepsilon)|y|^2} \frac{1}{a^{d/2}} \int_0^\infty e^{-(1-\varepsilon)s}(s+a)^{(d-1)/2} \frac{ds}{\sqrt{s}}$$
$$\leq Ce^{-(1-\varepsilon)|y|^2},$$

as $a > 1/2$ over the global region. Therefore, using Hölder's inequality

$$\begin{aligned}
\|T_{F,m,G}f\|_{p,\gamma}^p &\leq \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} |\mathscr{K}_{F,m}(x,y)||f(y)|dy \Big)^p e^{-|x|^2} dx \\
&\leq C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} e^{-(1-\varepsilon)|y|^2} e^{|y|^2/p} e^{-|y|^2/p} |f(y)|dy \Big)^p e^{-|x|^2} dx \\
&\leq C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} e^{-(1-\varepsilon)q|y|^2} e^{q|y|^2/p} dy \Big)^{p/q} e^{-|x|^2} dx \, \|f\|_{p,\gamma}^p,
\end{aligned}$$

where $q = \frac{p}{p-1}$. Now, we select an appropriate $\varepsilon > 0$ so that the above integral is finite. We can see that any $\varepsilon > 0$, with $\varepsilon < 1 - 1/p$, suffices.

- Case #2: $b = 2\langle x,y\rangle > 0$.
We have,

$$|\mathscr{K}_{F,m}(x,y)| \leq \int_0^1 \Big| F\Big(\frac{y - \sqrt{1-t}\,x}{\sqrt{t}}\Big) \Big| \frac{e^{-u(t)}}{t^{d/2+1}} \frac{dt}{\sqrt{1-t}} \leq C_\varepsilon \frac{e^{-(1-\varepsilon)u(t_0)}}{t_0^{d/2}}.$$

When $d = 1$, this inequality follows directly from Lemma 4.36, by taking $\eta = 0$, and $\nu = 1 - \varepsilon$ for $0 < \varepsilon < 1$.
For $d \geq 2$, we use (4.77) and the boundedness of F for ε smaller than $1/d$. Thus, using Lemma 4.36, we have

$$|\mathscr{K}_{F,m}(x,y)| \leq C_\varepsilon \frac{e^{\frac{-d-1}{d} u(t_0)}}{t_0^{(d-1)/2}} \int_0^1 e^{\varepsilon u(t)} \frac{e^{-u(t)/d}}{t^{3/2}} \frac{dt}{\sqrt{1-t}} \leq C_\varepsilon \frac{e^{-(1-\varepsilon)u(t_0)}}{t_0^{d/2}},$$

with $\varepsilon > 0$ to be determined. Then,

$$\begin{aligned}
\|T_{F,m,G}f\|_{p,\gamma}^p &\leq \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} |\mathscr{K}_{F,m}(x,y)||f(y)|dy \Big)^p e^{-|x|^2} dx \\
&\leq C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} \frac{e^{-(1-\varepsilon)u(t_0)}}{t_0^{d/2}} |f(y)|dy \Big)^p e^{-|x|^2} dx \\
&= C \int_{\mathbb{R}^d} \Big(\int_{B_h^c(x)} e^{\frac{|y|^2 - |x|^2}{p}} e^{\varepsilon u(t_0)} \frac{e^{-u(t_0)}}{t_0^{d/2}} |f(y)| e^{-\frac{|y|^2}{p}} dy \Big)^p dx.
\end{aligned}$$

Therefore, it is enough to check that the operator defined using the kernel,

$$\tilde{K}(x,y) = e^{\frac{|y|^2-|x|^2}{p}} e^{\varepsilon u(t_0)} \frac{e^{-u(t_0)}}{t_0^{d/2}} \chi_{B_h^c(x)}(y),$$

is of strong type p with respect to the Lebesgue measure. Using the inequality $||y|^2 - |x|^2| \leq |x+y||x-y|$, and that, as $b > 0$, on the global region, $|x+y||x-y| \geq d$, we have

$$\begin{aligned} e^{\frac{|y|^2-|x|^2}{p}} e^{\varepsilon u(t_0)} \frac{e^{-u(t_0)}}{t_0^{d/2}} &= \frac{1}{t_0^{d/2}} e^{(\frac{1}{p}-\frac{1-\varepsilon}{2})(|y|^2-|x|^2)} e^{-\frac{1-\varepsilon}{2}|x+y||x-y|} \\ &\leq C|x+y|^d e^{-\alpha_p|x+y||x-y|}, \end{aligned}$$

where $\alpha_p = \frac{1-\varepsilon}{2} - |\frac{1}{p} - \frac{1-\varepsilon}{2}|$. Since $p > 1$, we can choose $\varepsilon > 0$ so that $\alpha_p > 0$. Observe that the last expression is symmetric in x and y; therefore, it suffices to prove its integrability with respect to one of the

$$\begin{aligned} \int_{\mathbb{R}^d} |x+y|^d e^{-\alpha_p|x+y||x-y|} dy &\leq C + C\int_{|x-y|<1} |x|^d e^{-\alpha_p|x||x-y|} dy \\ &\quad + C\int_{|x-y|<1} |x+y|^d e^{-\alpha_p|x+y|} dy \\ &\leq C\int_{\mathbb{R}^d} e^{\alpha_p|v|} dv + C_d \int_0^\infty r^{2d-1} e^{-\alpha_p r} dr \leq C. \end{aligned}$$

Observe that, once $p > 1$ is chosen, then the operator defined using the kernel $\tilde{K}(x,y)$ is in fact $L^q(\mathbb{R}^d)$-bounded for $1 \leq q \leq \infty$, but for the proof of the theorem, the case $p = q$ is enough. □

Now, we discuss the results corresponding to the weak type $(1,1)$ for the operators $T_{F,m}$. First of all, observe that condition (9.38) provides a function Φ satisfying the property

iii) $|F(x)| \leq \Phi(|x|)$ for some continuous function $\Phi : [0,\infty) \to [0,\infty)$ for which there exists a $\delta > 0$ with $1-2/d < \delta < 1$, such that $\Phi(t)e^{-(1-\delta)t^2}$ is a non-increasing function for all $t \geq 0$.

Indeed, for $0 < \varepsilon < 2/d$ we set $\Phi(t) = C_\varepsilon e^{\varepsilon t^2}$ and $\delta 1 - \varepsilon$. In what follows, we denote by Φ any function satisfying the property *iii)*. We see that the smaller the function Φ is taken, the better the result that can be obtained. The goal of the following two theorems is to answer the question: what are the precise conditions needed on F and on m to guarantee the weak type $(1,1)$ with respect to the Gaussian measure of $T_{F,m}$? The answer is given in the following two theorems. First, let us consider the *negative result*, which roughly says that if the function $\Phi(t)$ increases at infinity more than t^2, then the operator $T_{F,m}$ fails to be of weak type $(1,1)$ and it is a generalization of what is already known about the behavior on $L^1(\gamma_d)$ of the Gaussian higher-order Riesz transforms.

Theorem 9.18. *Let* $\Omega_t = \left\{z \in \mathbb{R}^d : \min_{1\le i\le d} |z_i| \ge t\right\}$ *and* $\Theta(t) = \frac{\inf_{\Omega_t} F(z)}{t^2}$, *if* $\limsup_{t\to\infty} \Theta(t) = \infty$, *then the operator* $T_{F,m}$ *is not of weak type* $(1,1)$ *with respect to the Gaussian measure.*

Proof. We follow the proof of Theorem 9.10. Again, let $y \in \mathbb{R}^d$ with $|y| = \eta$ large and $y_i \ge C\eta$, $i = 1,\cdots,d$. Write $x \in \mathbb{R}^d$ as $x = \xi\frac{y}{\eta} + v$ with $\xi \in \mathbb{R}$ and $v \perp y$. Consider the tubular region

$$\mathbf{J} = \{x \in \mathbb{R}^d : x = \xi\frac{y}{\eta} + v \quad \text{with} \quad \eta/2 < \xi < 3\eta/4, v \perp y,\ |v| < 1\}.$$

It follows that for $x \in \mathbf{J}$ (9.25) holds; therefore,

$$F\left(\frac{y - rx}{\sqrt{1-r^2}}\right) \ge C\eta^2\Theta(c\eta).$$

Thus, for $x \in \mathbf{J}$ using this estimate and (9.26) and (9.27) we get

$$\begin{aligned}\mathscr{K}_{F,m}(x,y) &\ge C_d\eta^2\Theta(c\eta)\int_{1/4}^{3/4}\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}dr\\ &\ge C_d\eta^2\Theta(c\eta)e^{\xi^2-\eta^2}\int_{1/4}^{3/4}e^{-C|\xi-r\eta|^2}dr \ge C_d\eta\Theta(c\eta)e^{\xi^2-\eta^2}\end{aligned}$$

Now, let $f \in L^1(\gamma_d)$, $f \ge 0$ be a close approximation of a point mass at y, with norm $\|f\|_{1,\gamma} = 1$. Then, for $x \in \mathbf{J}$

$$T_{F,m}f(x) \ge C\eta\Theta(c\eta)e^{\xi^2} \ge C\eta\Theta(c\eta)e^{(\eta/2)^2}.$$

Let us assume that $T_{F,m}$ is of weak type $(1,1)$ with respect to the Gaussian measure. Then,

$$\gamma_d(\mathbf{J}) \le \gamma_d\left(\left\{x \in \mathbb{R}^d : T_{F,m}f(x) > C\eta\Theta(c\eta)e^{(\eta/2)^2}\right\}\right) \le C\frac{e^{-(\eta/2)^2}}{\eta\Theta(c\eta)},$$

but $\gamma_d(\mathbf{J}) \ge \frac{C}{\eta}e^{-(\eta/2)^2}$; therefore, $\Theta(\eta)$ is bounded for η large, which is a contradiction with the assumption on Θ. □

The positive result is contained in the following theorem. To get sufficient conditions on F for the weak type $(1,1)$ of $T_{F,m}$, because the weak type is not true, the natural question is: what weights can be put to get a weak type inequality? From the proof of Theorem 9.11, it is clear that for $|\beta| \ge 3$, the weight should be of the form $w(y) = 1 + |y|^{|\beta|-2}$. Moreover, for every $0 < \varepsilon < |\beta| - 2$, there exists a function $F \in L^1((1+|\cdot|^{\varepsilon})\gamma_d)$ such that $\mathscr{R}_\beta f \notin L^{1,\infty}(\gamma_d)$ (see [86]). The weights w that are considered, to ensure that $T_{F,m}$ is bounded from $L^1(w\gamma_d)$ into $L^{1,\infty}(\gamma_d)$, depend on the function Φ.

Theorem 9.19. *The operator $T_{F,m}$ maps continuously $L^1(w\gamma_d)$ into $L^{1,\infty}(\gamma_d)$ with $w(y) = 1 \vee \max_{1\leq t\leq |y|} \eta(t)$ and*

$$\eta(t) = \begin{cases} \Phi(t)/t \text{ if } & 1 \leq m < 2, \\ \Phi(t)/t^2 \text{ if } & m \geq 2, \end{cases}$$

The proof is long and technical; it is based on the refinement of several inequalities used by S. Pérez in [220], and the application of a technique developed by García-Cuerva et al. in gmst4. For details, see [85, Theorem 2].

As an immediate consequence we get:

Corollary 9.20. *If for t large either $\Phi(t) \leq Ct$ when $1 \leq m < 2$ or $\Phi(t) \leq Ct^2$ when $m \geq 2$, then the operator $T_{F,m}$ is of weak type $(1,1)$ with respect to the Gaussian measure.*

9.5 Notes and Further Results

1. What is known as *Meyer's inequality* in Malliavin calculus is given in the following terms. Given L a self-adjoint, second-order differential operator on $L^2(\mathbb{R}^d, d\mu)$, for some probability measure μ, and suppose that L is the infinitesimal generator of a Markov semigroup, then there exist constants c_p, C_p such that
$$c_p \|L^{1/2} f\|_{p,\mu} \leq \|\nabla f\|_{p,\mu} \leq C_p \|L^{1/2} f\|_{p,\mu},$$
holds for all p, $1 < p < \infty$. Observe that that statement is equivalent to the $L^p(\mu)$-boundedness of the corresponding Riesz transforms.

2. In [194] B. Muckenhoupt introduces, for $d = 1$, the Gaussian Hilbert transform in a different way. He follows the classical definition of the conjugated function as the limit of the conjugated Fourier series, using the Cauchy–Riemann equations. In more detail, he considers the conjugated Poisson–Hermite semigroup $P_t^c f(x)$, based on the Gaussian Cauchy–Riemann equations (see section 3.4). As we know from Chapter 3, the Poisson–Hermite operator P_t on f is defined as
$$P_t f(x) = u(x,t) = \frac{\sqrt{2}}{\pi} \int_{-\infty}^{\infty} p(t,x,y) f(y)\, dy,$$
$t > 0$, where
$$p(t,x,y) = \int_0^1 \frac{t \exp(\frac{t^2}{2\log r})}{r(-\log r)^{3/2}} \frac{\exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})}{(1-r^2)^{1/2}}\, dr,$$
then, $u(x,t)$ satisfies:
$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial^2 u(x,t)}{\partial x^2} - 2x\frac{\partial u(x,t)}{\partial x} = 0,$$

which is equivalent to

$$\frac{\partial^2 u(x,t)}{\partial t^2} + e^{x^2}\frac{\partial}{\partial x}(e^{-x^2}\frac{\partial u(x,t)}{\partial x}) = 0.$$

Then, considering a L-harmonic conjugated v given by the Gaussian Cauchy–Riemann equations (3.44),

$$\begin{aligned}\frac{\partial u(x,t)}{\partial x} &= -\frac{\partial v(x,t)}{\partial t}\\ \frac{\partial u(x,t)}{\partial t} &= e^{x^2}\frac{\partial}{\partial x}(e^{-x^2}\frac{\partial v(x,t)}{\partial x}),\end{aligned}$$

it is easy to see that v can be written as

$$P_t^c f(x) = v(x,t) = \int_{-\infty}^{\infty} Q(t,x,y) f(y)\, dy,$$

$t > 0$, where

$$Q(t,x,y) = \frac{\sqrt{2}}{\pi}\int_0^1 \frac{(y-rx)\exp(\frac{t^2}{2\log r})}{(-\log r)^{1/2}} \frac{\exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})}{(1-r^2)^{3/2}}\, dr.$$

B. Muckenhoupt [194] proved that v is $L^p(\gamma_1)$-bounded for $p > 1$, and for $f \in L^p(\gamma_1)$, $p > 1$, $v(x,t)$ tends to Gaussian Hilbert transform $\mathscr{H}$, as $t \to 0^+$; therefore,

$$\mathscr{H} f(x) = \frac{\sqrt{2}}{\pi}\int_{-\infty}^{\infty}\int_0^1 \frac{(y-rx)}{(-\log r)^{1/2}} \frac{\exp(\frac{-r^2x^2+2rxy-r^2y^2}{1-r^2})}{(1-r^2)^{3/2}}\, dr f(y)\, \gamma_1(dy).$$

The convergence is in $L^p(\gamma_1)$-norm sense, $1 < p < \infty$ and also almost everywhere (a.e.). He also proved the $L^p(\gamma_1)$-boundedness and the weak type $(1,1)$ with respect to the Gaussian measure γ_1 using analytic methods based on Natanson's lemma, see (10.27).

3. In his doctoral dissertation, [244][3] R. Scotto got the extension of this approach to the higher dimensions $d > 1$, by considering the Gaussian Cauchy–Riemann equations in $\mathbb{R}^d$ (3.50),

$$\begin{aligned}\frac{\partial u}{\partial x_j}(x,t) &= -\frac{\partial v_j}{\partial t}(x,t), j = 1,\dots,d\\ \frac{\partial v_i}{\partial x_j}(x,t) &= \frac{\partial v_j}{\partial x_i}(x,t), i,j = 1,\dots,d\\ \frac{\partial u}{\partial t}(x,t) &= \frac{1}{2}\sum_{j=1}^{d} e^{|x|^2}\frac{\partial}{\partial x_j}(e^{-|x|^2} v_j(x,t)).\end{aligned}$$

[3] See also [77].

Then, he defined a system of conjugates

$$(u(x,t), v_1(x,t), v_2(x,t), \ldots, v_d(x,t)),$$

in a similar way to the one-dimensional case, and then, taking $t \to 0^+$, he proved that the system of conjugates converges to the vector where the first coordinate is the function f and the other coordinates are the Gaussian Riesz transforms of f,

$$(f(x), \mathscr{R}_1 f(x), \cdots, \mathscr{R}_d f(x)).$$

The convergence is in $L^p(\gamma_1)$-norm sense, $1 < p < \infty$ and also a.e. For more details, see [244, Chapter] (see also section 3.4).

4. The definition used by B. Muckenhoupt and R. Scotto for the Gaussian Riesz transforms using Cauchy–Riemann equations is, of course, equivalent (up to a constant) to the one given in this chapter. To see this, observe that according to the general semigroup theory, we have

$$-\int_0^\infty P_s ds = (-L)^{-1/2},$$

because, as $(-L)^{1/2}$ is the infinitesimal generator of the Poisson–Hermite semigroup $\{P_t\}$, then, at least formally

$$\begin{aligned} -(-L)^{1/2}\Big(\int_0^\infty P_s ds\Big) &= -\lim_{t\to 0^+} \frac{1}{t}\Big[P_t\Big(\int_0^\infty P_s ds\Big) - \int_0^\infty P_s ds\Big] \\ &= -\lim_{t\to 0^+} \frac{1}{t}\Big[\Big(\int_0^\infty P_{(t+s)} ds\Big) - \int_0^\infty P_s ds\Big] \\ &= -\lim_{t\to 0^+} \frac{1}{t}\Big[\Big(\int_t^\infty P_s ds\Big) - \int_0^\infty P_s ds\Big] \\ &= \lim_{t\to 0^+} \frac{1}{t}\Big[\Big(\int_0^t P_s ds\Big)\Big] = P_0 = I. \end{aligned}$$

Therefore, from (3.45) we know that the conjugated Poisson–Hermite integral can be written as

$$v(x,t) = \int_{-\infty}^\infty Q(t,x,y) f(y)\, dy,$$

and from (3.47), $Q(t,x,y)$ can be written as

$$Q(t,x,y) = -\int_t^\infty \frac{\partial p(s,x,y)}{\partial x} ds = -\frac{\partial}{\partial x}\int_t^\infty p(s,x,y) ds$$

then,

$$\begin{aligned} v(x,t) &= \int_{-\infty}^\infty Q(t,x,y) f(y)\, dy = -\frac{\partial}{\partial x}\int_{-\infty}^\infty \int_t^\infty p(s,x,y) ds f(y)\, dy \\ &= -\frac{\partial}{\partial x}\int_t^\infty \int_{-\infty}^\infty p(s,x,y) f(y)\, dy ds = -\frac{\partial}{\partial x}\int_t^\infty P_s f(x) ds, \end{aligned}$$

formally we get, taking $t \to 0^+$,

$$v(x,t) \to \frac{\partial}{\partial x}(-L)^{1/2} f(x) = \sqrt{2}\mathscr{H} f(x).$$

The argument for higher dimensions is analogous, using (3.51) and (3.52).

5. In [127], E. Harboure, R. A. Macías, M. T. Menárguez, and J. L. Torrea studied the rate of convergence for the family of truncations of the Gaussian Riesz transforms and Hermite–Poisson semigroup through the oscillation and variation operators. More precisely, they search for their $L^p(\gamma_d)$-boundedness properties, by looking at the oscillation and variation operators from a vector-valued point of view.

6. We know that the Gaussian Hilbert transform is defined spectrally as

$$\mathscr{H} = \frac{1}{\sqrt{2}}\frac{d}{dx}(-L)^{-1/2},$$

then,

$$\mathscr{H} H_n(x) = (-L)^{-1/2} H_n(x) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n}}\frac{d}{dx} H_n(x) = \sqrt{2n} H_{n-1}(x),$$

which of course is a particular version of (9.5). Therefore,

$$\mathscr{H} h_n = \mathscr{H}\left(\frac{H_n}{(2^n n!)^{1/2}}\right) = \sqrt{2n}\frac{H_{n-1}}{(2^n n!)^{1/2}} = h_{n-1}. \tag{9.42}$$

Hence, given $f \in L^2(\gamma_1)$ with Hermite expansion $f = \sum_{n=0}^{\infty} \langle f, H_n\rangle_\gamma H_n$, then its Gaussian Hilbert transform is the *conjugated series*

$$\mathscr{H} f = \sum_{n=1}^{\infty} \sqrt{2n} \langle f, H_n\rangle H_{n-1}, \tag{9.43}$$

This fact motivates the study of the Gaussian Hilbert transform $\mathscr{H}$ from the operator theory point of view. These results are contained in M. D. Morán and W. Urbina's article [191]. Let $\mathbb{D}$ be the open unit disk, $\mathbb{T}$ the circumference, consider the square integrable functions in $\mathbb{T}$

$$L^2(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : \int_{-\pi}^{\pi} |f(e^{it})|^2 dt < \infty\}.$$

Let $\{e_n\}_{n\in\mathbb{Z}}$ be the trigonometric system, $e_n(\xi) = \xi^n, \xi \in \mathbb{T}$, which is a complete orthonormal system in $L^2(\mathbb{T})$, and finally let $\mathscr{S} : setL^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the shift operator given by

$$(\mathscr{S} f)(\xi) = \xi f(\xi),$$

for all $\xi \in \mathbb{T}$. For more details on the shift operator, we refer the reader to Nikol'skii [206].

If we consider

$$H^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} : f = \sum_{n=0}^{\infty} a_n z^n \text{ and } \sum_{n=0}^{\infty} |a_n|^2 < \infty\},$$

then we can identify $H^2(\mathbb{D})$ with the subspace of $L^2(\mathbb{T})$ consisting of functions such that

$$\langle f, e_n \rangle = 0, \forall n < 0.$$

The restriction of the bilateral forward shift operator to $H^2(\mathbb{D})$, which, abusing the notation, we also call $\mathscr{S}$, is the *unilateral forward shift*, which leaves the space $H^2(\mathbb{D})$ invariant and

$$\mathscr{S}(\sum_{n=0}^{\infty} a_n z^n) = \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{n=1}^{\infty} a_{n-1} z^n.$$

The main result in this direction is that the Gaussian Hilbert transform is unitary equivalent to the adjoint of the unilateral shift operator acting on $H^2(\mathbb{D})$; thus, we are able to completely characterize the invariant subspaces and the commutant of the Gaussian Hilbert transform. The main results are as follows:

Theorem 9.21. *The Gaussian Hilbert transform $\mathscr{H}$ as an operator on $L^2(\gamma_1)$ is unitarily equivalent to the adjoint of the restriction of the shift operator on $H^2(\mathbb{D})$.*

Proof. Let us consider $\Omega : L^2(\gamma_1) \to H^2(\mathbb{D})$ defined by

$$\Omega(\sum_{n=0}^{\infty} \langle f, h_n \rangle h_n) = \sum_{n=0}^{\infty} \langle f, h_n \rangle e_n.$$

It is easy to see, by Parseval's identity, that Ω is a well-defined operator and unitary, also Ω intertwining $\mathscr{H}$ and $\mathscr{S}^*$, that is, $\Omega\mathscr{H} = \mathscr{S}^*\Omega$:

$$\begin{aligned}
\Omega\mathscr{H}(\sum_{n=0}^{\infty} \langle f, h_n \rangle h_n) &= \Omega\mathscr{H}(\sum_{n=0}^{\infty} (\sqrt{2^n n!})^{-2} \langle f, H_n \rangle H_n) \\
&= \Omega(\sum_{n=1}^{\infty} (\sqrt{2^n n!})^{-2} \sqrt{2n} \langle f, H_n \rangle H_{n-1}) \\
&= \Omega(\sum_{n=1}^{\infty} \langle f, h_n \rangle h_{n-1}) = \sum_{n=1}^{\infty} \langle f, h_n \rangle e_{n-1} \\
&= \mathscr{S}^*(\sum_{n=0}^{\infty} \langle f, h_n \rangle e_n) = \mathscr{S}^*\Omega(\sum_{n=0}^{\infty} \langle f, h_n \rangle h_n). \qquad \square
\end{aligned}$$

There are several consequences of this result. The first completely characterizes the invariant subspaces of the Gaussian Hilbert transform.

Theorem 9.22. *Given the Gaussian Hilbert transform $\mathscr{H}$ and A a proper and closed subspace of $L^2(\gamma_1)$, then, $\mathscr{H}(A) \subset A$ if and only if there exists a sequence of complex numbers $\{a_n\}$ such that $|\sum_{n=0}^{\infty} a_n z^n| = 1$ almost everywhere in $\mathbb{T}$ and*

$$A = \{f = \sum_{n=0}^{\infty} \langle f, h_n \rangle h_n \in L^2(\gamma_1) : \sum_{n \geq k}^{\infty} \langle f, h_n \rangle \overline{a_{n-k}} = 0, \forall k \geq 0\}$$

Proof. We shall prove first that the condition is necessary. Let Ω intertwining $\mathscr{H}$ and $\mathscr{S}^*$ as in the previous theorem. It is clear that $\mathscr{H}(A) \subset A$ if and only if $\mathscr{S}^*(\Omega A) \subset \Omega A$, and this is equivalent to

$$S(H^2(\mathbb{D}) \ominus \Omega A) \subset H^2(\mathbb{D}) \ominus \Omega A.$$

Now $H^2(\mathbb{D}) \ominus \Omega A = \Omega(L^2(\gamma_1) \ominus A) \neq 0$ according to the hypothesis, then $H^2(\mathbb{D}) \ominus \Omega A$ is a non-trivial, closed subspace of $H^2(\mathbb{D})$, then (see [31] or [132]), there exists $\theta \in H^2(\mathbb{D})$ with $|\theta(\xi)| = 1$ for almost all $\xi \in \mathbb{T}$ such that

$$H^2(\mathbb{D}) \ominus \Omega A = \theta H^2(\mathbb{D}),$$

or equivalently

$$A = \Omega^{-1}[H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})] = L^2(\gamma_1) \ominus \Omega^{-1} \theta H^2(\mathbb{D}).$$

Let $\theta(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f = \sum_{n=0}^{\infty} \langle f, h_n \rangle h_n \in A$ if and only if

$$\langle f, \Omega^{-1} \theta u \rangle = 0, \text{for all } u \in H^2(\mathbb{D}).$$

Given that $k \geq 0$, let us take $u = e_k$, then

$$0 = \langle f, \Omega^{-1} \theta e_k \rangle = \langle \Omega f, \theta e_k \rangle = \sum_{n=0}^{\infty} \langle f, h_n \rangle \langle e_n, \theta e_k \rangle,$$

but we have that

$$\langle e_n, \theta e_k \rangle = \begin{cases} \{\overline{a_{n-k}}, & \text{if } k \geq n, \\ 0 & \text{if } k < n, \end{cases}$$

then for all $k \geq 0$, $\sum_{n=0}^{\infty} \langle f, h_n \rangle \overline{a_{n-k}} = 0$.

We shall now prove the sufficiency. Let $\theta(z) = \sum_{n=0}^{\infty} a_n z^n$. Then, given $f \in L^2(\gamma_1)$,

$$f \in A \text{ if and only if for all } k \geq 0 \ \langle \Omega f, \theta e_k \rangle = 0.$$

Thus, if $u \in H^2(\mathbb{D})$, then we have

$$|\langle \Omega f, \theta u \rangle| \leq |\langle \Omega f, \theta(u - \sum_{k=0}^{n} \langle u, e_k \rangle e_k) \rangle| \leq ||\Omega f||_{H^2} ||\theta(u - \sum_{k=0}^{n} \langle u, e_k \rangle e_k)||_{H^2}$$
$$\leq ||f||_{L^2(\gamma_1)} ||u - \sum_{k=0}^{n} \langle u, e_k \rangle e_k||_{H^2},$$

but

$$\lim_{n\to\infty} ||u-\sum_{k=0}^{n}\langle u,e_k\rangle e_k||_{H^2}=0.$$

Therefore, $\langle \Omega f,\theta u\rangle = 0$ for all $f\in A, u\in H^2(\mathbb{D})$; thus, $\Omega A = H^2\ominus \theta H^2(\mathbb{D})$, and then

$$\begin{aligned}\mathscr{H}A &= \Omega^{-1}\Omega\mathscr{H}A=\Omega^{-1}S^*\Omega A\\ &=\Omega^{-1}\mathscr{S}^*(H^2\ominus\theta H^2(\mathbb{D}))\subset \Omega^{-1}(H^2\ominus\theta H^2(\mathbb{D}))=A. \qquad \square\end{aligned}$$

The next result characterizes the commutant of the Gaussian Hilbert transform.

Theorem 9.23. *Let $\mathscr{F}$ be a linear operator on $L^2(\gamma_1)$. If $\mathscr{F}\mathscr{H}=\mathscr{H}\mathscr{F}$ then there exists $g\in H^\infty(\mathbb{D})$ such that*

$$\begin{aligned}\mathscr{F}f &= \mathscr{F}(\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}h_n)\\ &=\sum_{k=0}^{\infty}\sum_{n\ge k}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}\langle g,e_{n-k}\rangle_{H^2(\mathbb{D})}h_k.\end{aligned}$$

Conversely, if this relation holds and $P_0 f=\langle f,h_0\rangle_{L^2(\gamma_1)}h_0$, then

$$\mathscr{F}\mathscr{H}=\mathscr{H}\mathscr{F}(I-P_0).$$

Proof. Let $G=\Omega\mathscr{F}\Omega^{-1}$. It is clear that $G\in L(H^2(\mathbb{D}))$ and

$$\mathscr{S}^*G=\mathscr{S}^*\Omega\mathscr{F}\Omega^{-1}=\Omega\mathscr{H}\mathscr{F}\Omega^{-1}=\Omega\mathscr{F}\mathscr{H}\Omega^{-1}=\Omega\mathscr{F}\Omega^{-1}\mathscr{S}^*=G\mathscr{S}^*;$$

thus,

$$\mathscr{S}^*G=G\mathscr{S}^*,\ \text{ and also } G^*\mathscr{S}=\mathscr{S}G^*.$$

Let $g=G^*e_0$. Then, it is easy to check that $g\in H^\infty(\mathbb{D})$, $G^*u=gu$ for all $u\in\mathbb{D}$ and

$$\begin{aligned}\mathscr{F}f &= \mathscr{F}(\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}h_n)=\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}\mathscr{F}h_n\\ &=\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}\Omega^{-1}\mathscr{F}\Omega h_n=\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}\Omega^{-1}(\sum_{k=0}^{\infty}\langle Ge_n,e_k\rangle e_k)\\ &=\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}\Omega^{-1}(\sum_{k=0}^{\infty}\langle e_n,ge_k\rangle e_k)=\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}\Omega^{-1}(\sum_{k=0}^{n}\overline{\langle g,e_{n-k}\rangle}e_k)\\ &=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\langle f,h_n\rangle_{L^2(\gamma_1)}\overline{\langle g,e_{n-k}\rangle}h_k=\sum_{k=0}^{\infty}(\sum_{k\ge n}\langle f,h_n\rangle_{L^2(\gamma_1)}\overline{\langle g,e_{n-k}\rangle}h_k).\end{aligned}$$

Conversely,

$$\mathscr{F}\mathscr{H}f=\mathscr{F}\mathscr{H}(\sum_{n=0}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}h_n)=\mathscr{F}(\sum_{n=1}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}h_{n-1})$$

$$= \mathscr{T}(\sum_{n=0}^{\infty}\langle f,h_{n+1}\rangle_{L^2(\gamma_1)}h_n) = \sum_{k=0}^{\infty}\sum_{n\geq k}\langle f,h_{n+1}\rangle_{L^2(\gamma_1)}\overline{\langle g,e_{n-k}\rangle}h_k$$

and

$$\begin{aligned}\mathscr{H}\mathscr{T}(I-P_0)f &= \mathscr{H}\mathscr{T}(\sum_{n=1}^{\infty}\langle f,h_n\rangle_{L^2(\gamma_1)}h_n) = \mathscr{H}(\sum_{k=0}^{\infty}\sum_{n\geq k}\langle f,h_{n+1}\rangle_{L^2(\gamma_1)} \\ &\qquad \times\overline{\langle g,e_{n-k}\rangle}h_k) \\ &= \mathscr{H}(\sum_{k=0}^{\infty}\sum_{n\geq k}\langle f,h_n\rangle_{L^2(\gamma_1)}\overline{\langle g,e_{n-k}\rangle}h_k) = \sum_{k=1}^{\infty}\sum_{n\geq k}\langle f,h_n\rangle_{L^2(\gamma_1)} \\ &\qquad \times\overline{\langle g,e_{n-k}\rangle}h_{k-1} \\ &= \sum_{k=0}^{\infty}\sum_{n\geq k+1}\langle f,h_n\rangle_{L^2(\gamma_1)}\overline{\langle g,e_{n-(k+1)}\rangle}h_k = \sum_{k=0}^{\infty}\sum_{n\geq k}\langle f,h_{n+1}\rangle_{L^2(\gamma_1)} \\ &\qquad \times\overline{\langle g,e_{n-k}\rangle}h_k;\end{aligned}$$

thus, $\mathscr{T}\mathscr{H} = \mathscr{H}\mathscr{T}(I-\mathbf{J}_0)$. □

Finally,

Theorem 9.24. *Let A be a (closed) subspace of $L^2(\gamma_1)$ that is invariant for $\mathscr{H}$ and P_A is the orthogonal projection of $L^2(\gamma_1)$ onto A. If $\mathscr{T}$ is a linear operator on A such that*

$$\mathscr{T}(P_A\mathscr{H}^*) = (P_A\mathscr{H}^*)\mathscr{T},$$

then there exists $\mathscr{T}_1$ a linear operator acting on $L^2(\gamma_1)$ such that $\mathscr{T}_1\mathscr{H} = \mathscr{H}\mathscr{T}_1$ and

$$\mathscr{T} = P_A{\mathscr{T}_1}^*|_A.$$

Proof. Recall that $\mathscr{H}(A)\subset A$ implies that $\mathscr{H}(A^\perp)\subset A^\perp$. Set $\mathbf{B} = \Omega A$, then

$$\mathbf{S}(H^2(\mathbb{D})\ominus B)\subset H^2(\mathbb{D})\ominus B.$$

Let T be a linear operator in B given by $T = \Omega\mathscr{T}\Omega^{-1}$, then

$$\begin{aligned}TP_B\mathscr{S}|_B &= \Omega\mathscr{T}\Omega^{-1}P_B\mathscr{S}|_B = \Omega\mathscr{T}P_A\Omega^{-1}\mathscr{S}|_B = \Omega\mathscr{T}P_A\mathscr{H}^*\Omega^{-1}|_B \\ &= \Omega P_A\mathscr{H}^*\mathscr{T}\Omega^{-1}|_B = P_B\Omega\mathscr{H}^*\Omega^{-1}T|_B = P_B\mathscr{S}T|_B.\end{aligned}$$

Thus, using the Sarason generalized interpolation theorem [241], there exists $g\in\mathscr{H}^\infty(\mathbb{D})$ such that $T = P_BM_g|_B$ (i.e., $Tf = P_B(fg)\,\forall f\in B$), and if $\mathscr{T}_1 = \Omega M_g\Omega^{-1}$, then

$$\begin{aligned}P_A\mathscr{T}_1|_A &= P_A\Omega M_g\Omega^{-1}|_A = \Omega^{-1}P_B\Omega\Omega^{-1}M_g\Omega|_A \\ &= \Omega^{-1}P_BM_g\Omega|_A = \Omega^{-1}T\Omega|_A = \mathscr{T},\end{aligned}$$

and

$$\begin{aligned}\mathscr{T}_1\mathscr{H}^* &= \Omega^{-1}M_g\Omega\mathscr{H}^* = \Omega^{-1}M_gS\Omega = \Omega^{-1}SM_g\Omega \\ &= \Omega^{-1}S\Omega\Omega^{-1}M_g\Omega = \Omega^{-1}\Omega\mathscr{H}^*\mathscr{T}_1 = \mathscr{H}^*\mathscr{T}_1.\end{aligned}$$ □

The generalization of these results to higher dimensions is an open problem.

7. G. Pisier's proof of the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_j$, $1 < p < \infty$ is analytic in the sense that it does not use the Brownian motion, but instead uses a variation of the Calderón's method of rotations and methods of transference developed by R. R. Coifman and G. Weiss in [57]. It turns out that the inequalities needed, also include the classical case, are consequences of the one-dimensional results. Using the same method, he is also able to prove the $L^p(\gamma_d)$-boundedness of $\mathscr{R}_\beta$, if $|\beta|$ is odd (see [227]).

8. In [69], O. Dragicevic and A. Volberg get the $L^p(\gamma_d)$-boundedness of the vector of Gaussian Riesz transforms $(\mathscr{R}_1, \cdots, \mathscr{R}_d)$

$$\|\Big(\sum_{j=1}^{d} |\mathscr{R}_j f|^2\Big)^{1/2}\|_{p,\gamma} \leq C_p \|f\|_{p,\gamma},$$

obtained from a dimensionless bilinear estimate of the Littlewood–Paley type, using the Bellman function technique (see [203]). This technique also works in the classical case.

9. The boundedness of the Riesz transforms can be used to obtain the Littlewood–Paley estimates for the spatial gradient; that is to say, the opposite direction of Stein's scheme is also possible.

10. In [174], G. Mauceri and S. Meda also proved that imaginary powers of the Ornstein–Uhlenbeck operator $(-L)^{i\alpha}$ and Riesz transforms $\mathscr{R}_\beta$, of any order $|\beta| > 0$, are bounded from L^∞ to $BMO(\gamma_d)$ (with a bound dependent on the dimension). They also proved that imaginary powers are bounded from $H^1_{at}(\gamma_d)$ to $L^1(\gamma_d)$.

11. In [63], E. Dalmaso and R. Scotto have studied the boundedness of general Gaussian singular integrals in variable $L^{p(\cdot)}$ Gaussian spaces following S. Pérez's approach in [221].

12. The *Jacobi–Riesz transform* can be defined spectrally as

$$R^{\alpha,\beta} = \sqrt{1-x^2}\,\frac{d}{dx}(\mathscr{L}^{\alpha,\beta})^{-1/2}, \tag{9.44}$$

where $(\mathscr{L}^{\alpha,\beta})^{-\nu/2}$ is the Jacobi–Riesz potential of order $\nu/2$. $(\mathscr{L}^{\alpha,\beta})^{-\nu/2}$ can be represented as

$$(\mathscr{L}^{\alpha,\beta})^{-\nu/2} f = \frac{1}{\Gamma(\nu)} \int_0^\infty t^{\nu-1} P_t^{(\alpha,\beta)} f dt;$$

moreover, it is easy to see that for $f \in L^2\left([-1,1], \mu_{(\alpha,\beta)}\right)$, with Laguerre expansion

$$f = \sum_{k=0}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{h_k^{(\alpha,\beta)}} P_k^{(\alpha,\beta)},$$

where

$$h_k^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

then, $(\mathscr{L}^{\alpha,\beta})^{-\nu/2} f$ will have a Jacobi expansion

$$(\mathscr{L}^{\alpha,\beta})^{-\nu/2} f = \sum_{k=0}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{\hat{h}_k^{(\alpha,\beta)}} \lambda_k^{-\nu/2} P_k^{(\alpha,\beta)} \tag{9.45}$$

and since

$$\frac{d}{dx}\left\{P_k^{(\alpha,\beta)}(x)\right\} = \frac{(k+\alpha+\beta+1)}{2} P_{k-1}^{(\alpha+1,\beta+1)}(x), \tag{9.46}$$

then its Jacobi–Riesz transform have an expansion

$$R^{\alpha,\beta} f = \sum_{k=1}^{\infty} \frac{\langle f, P_k^{(\alpha,\beta)} \rangle}{\hat{h}_k^{(\alpha,\beta)}} \lambda_k^{-1/2} \frac{(k+\alpha+\beta+1)}{2} \sqrt{1-x^2} P_{k-1}^{(\alpha+1,\beta+1)}. \tag{9.47}$$

where $\lambda_k = k(k+\alpha+\beta+1)$.

Observe that (9.47) is not a proper Jacobi expansion given the presence of the factor $\sqrt{1-x^2}$. This is different than the Hermite case, and complicates the arguments. The L^p-boundedness of the Riesz–Jacobi transform $R^{\alpha,\beta}$, was proved by Z. Li [157], in the case $d=1$, and by A. Nowak and P. Sjögren, [213, Theorem 5.1] in the case $d \geq 1$,

Theorem 9.25. *Assume that* $1 < p < \infty$ *and* $\alpha, \beta \in [-1/2, \infty)^d$. *There exists a constant* c_p *such that*

$$\|R_i^{\alpha,\beta} f\|_{p,(\alpha,\beta)} \leq c_p \|f\|_{p,(\alpha,\beta)}. \tag{9.48}$$

for all $i = 1, \cdots, d$.

For the particular case of the Gegenbauer polynomials, this result was obtained in the one-dimensional case by B. Muckenhoupt and E. Stein in their seminal article of 1965 [199].

13. The *Laguerre–Riesz transform* can be defined spectrally, as

$$R^{\alpha} = \sqrt{x}\frac{d}{dx}(\mathscr{L}^{\alpha})^{-1/2}; \tag{9.49}$$

therefore for $f \in L^2((0,\infty),\mu_\alpha)$ with Laguerre expansion

$$f = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+1)} \langle f, L_k^\alpha \rangle L_k^\alpha$$

its Laguerre–Riesz transform has expansion

$$R^\alpha f(x) = -\sum_{k=1}^{\infty} \frac{\Gamma(\alpha+1)k!}{\Gamma(k+\alpha+1)} (\sqrt{k})^{-1} \langle f, L_k^\alpha \rangle \sqrt{x} L_{k-1}^{\alpha+1}(x). \quad (9.50)$$

Observe that (9.50) is not a Laguerre expansion given the presence of the factor $\sqrt{x}$. This is different from the Hermite case, and complicates the arguments; thus, the proofs in the Laguerre setting are more involved than that of the Hermite case. The L^p boundedness of the Laguerre–Riesz transform was proved, for the case $d = 1$ by B. Muckenhoupt [196], and the case $d \geq 1$ was proved by A. Nowak [209] using Littlewood–Paley's theory and also following Stein's scheme in [253].

Theorem 9.26. *Assume that* $1 < p < \infty$ *and* $\alpha \in [-1/2,\infty)^d$*. There exists a constant* C_p *such that*

$$\|R_i^\alpha f\|_{p,\alpha} \leq C_p \|f\|_{p,\alpha}. \quad (9.51)$$

for all $i = 1,\cdots,d,$.

14. In [201], E. Navas and W. Urbina develop a transference method to obtain the L^p-boundedness, $1 < p < \infty$ of the Gaussian Riesz transforms $\mathscr{R}_i$, and the L^p-boundedness of the Laguerre–Riesz transform R_i^α from the L^p-boundedness of the Jacobi–Riesz transform $R_i^{\alpha,\beta}$ for the one-dimensional case by using the well-known asymptotic relations between Jacobi polynomials and other classical orthogonal polynomials (10.64) and (10.67) (see also [262, (5.3.4),(5.6.3)]). The transference for the higher dimensional case is open.

15. In [117], P. Graczyk, J. J. Loeb, I. López, A. Nowak, and W. Urbina proved the $L^p(\mu_\alpha)$-boundedness of higher-order Laguerre–Riesz transforms and the weak type $(1,1)$ for the Riesz–Laguerre transform of order 2. However, the methods they used impose substantial restrictions on the admissible values of type multi-index α, because the result is obtained by means of transference from the Hermite setting using the classical relations formulas that relate to the Hermite polynomials and Laguerre polynomials (10.36). This method has been used by many authors, for instance [152, 19, 68, 123], to study different properties of Laguerre semigroups of half-integer type, which are related to Hermite semigroups. They provide a considerable extension of this technique, and show how to transfer higher-order Riesz type operators and certain differential operators. Although the corresponding formulas are rather complex, because their combinatorial component, they shed some light on an interplay between Hermite and Laguerre expansions.

16. In [237], E. Sasso proved that the first-order Riesz–Laguerre transforms associated with the Laguerre semigroup are of weak type $(1,1)$ with respect to the Gamma measure, analogous to the Gaussian case. She also presents a counterexample showing that for the Riesz transforms of order three or higher, the weak type $(1,1)$ estimate fails.

17. In 2006, A. Nowak and K. Stempak in [210] proposed a fairly general and unified approach to the theory of Riesz transforms and conjugacy in the setting of multi-dimensional orthogonal expansions (polynomials and functions), proving their L^2-boundedness under certain conditions. Additionally, they attempt to offer a unified conjugacy scheme that includes definitions of Riesz transforms and conjugate Poisson integrals for a broad class of expansions. The postulated definitions were supported by a good L^2-theory, the existence of Cauchy–Riemann-type equations, and numerous examples in the literature that are covered by the scheme. There is, however, a shortcoming of this unified conjugacy scheme manifested in a lack of symmetry in the decomposition (2.13). Asymmetry of the decomposition of L has, in fact, a deep impact on the whole conjugacy scheme postulated in [210]. Then, in [211], they proposed a symmetrization procedure and consider the resulting symmetrized situation. The construction is motivated to some extent by the setting of the Dunkl harmonic oscillator with the underlying reflection group isomorphic to $\mathbb{Z}_2^d = \{0,1\}^d$, and gives a different notion of conjugacy (for more details see [211]).

18. In 2015, L. Forzani, E. Sasso, and R. Scotto [93] extended Nowak and Stempak's approach to the general case of multi-dimensional orthogonal polynomial expansions, proving the L^p boundedness, $1 < p < \infty$ of those Riesz transforms, with constants independent of dimension.

19. In [296], B. Wróbel derives a scheme to deduce the L^p boundedness of certain d-dimensional Riesz transforms from the L^p boundedness of appropriate one-dimensional Riesz transforms, by using an H^∞ joint functional calculus for strongly commuting operators. Moreover, the L^p bounds obtained are independent of the dimension. The scheme is applied to Riesz transforms connected with orthogonal expansions and discrete Riesz transforms on products of groups with polynomial growth, which of course include the Gaussian case. For the vector case, an explicit Bellman function is used to prove a bilinear embedding theorem for operators associated with general multi-dimensional orthogonal expansions on product spaces and as a consequence the L^p boundedness of the vector of Riesz transforms is obtained (see [297]).

Correction to: Gaussian Harmonic Analysis

Correction to:
W. Urbina-Romero, *Gaussian Harmonic Analysis*,
Springer Monographs in Mathematics,
https://doi.org/10.1007/978-3-030-05597-4

This book was inadvertently published without updating the following corrections:

Chapter 2

Page 34 line 2 ↓ it says

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}} e^{|x|^2} (\partial_\gamma^i e^{-|x|^2}).$$

it should say

$$(\partial_\gamma^i)^* = -\frac{1}{\sqrt{2}} e^{|x|^2} (\partial_i e^{-|x|^2} I).$$

The updated online versions of the chapters can be found at
https://doi.org/10.1007/978-3-030-05597-4_2
https://doi.org/10.1007/978-3-030-05597-4_4
https://doi.org/10.1007/978-3-030-05597-4_8
https://doi.org/10.1007/978-3-030-05597-4_9

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4_10

Page 34 line 12 ↑ it says

$$(-\overline{L}) = \sum_{i=1}^{d} \partial_\gamma^i (\partial_\gamma^i)^* = (-L) + I_d = -\frac{1}{2}\Delta_x + \langle x, \nabla_x \rangle + I_d,$$

it should say

$$(-\overline{L}) = \sum_{i=1}^{d} \partial_\gamma^i (\partial_\gamma^i)^* = (-L) + dI = -\frac{1}{2}\Delta_x + \langle x, \nabla_x \rangle + dI,$$

Page 34 line 10 ↑ it says

$$\overline{L} = L - I_d = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle - I_d.$$

it should say

$$\overline{L} = L - dI = \frac{1}{2}\Delta_x - \langle x, \nabla_x \rangle - dI.$$

Chapter 4

Page 157 line 11 ↑ it says

$$\leq \frac{C}{m(c_b)^d} \int_{|y-x|<C_d m(c_b)} \frac{|(f\chi_{\hat{B}(x)})(y)|}{|x-y|^d} dy \leq CM(f\chi_{\hat{B}(\cdot)})(x).$$

it should say

$$\leq \frac{C}{m(c_b)^d} \int_{|y-x|<C_d m(c_b)} |(f\chi_{\hat{B}(x)})(y)| dy \leq CM(f\chi_{\hat{B}(\cdot)})(x).$$

Page 162 line 3 ↓ it says

$$h(v(s))+h(w(s)) \leq \frac{a}{a^2-b^2} + \frac{2a}{(a^2-b^2)^{1/2}\sqrt{s}(a^2-b^2)^{1/4}} \leq \frac{C}{t_0}\frac{1}{(a^2-b^2)^{1/4}}\left(1+\frac{1}{\sqrt{s}}\right),$$

it should say

$$h(v(s))+h(w(s)) \leq \frac{2a}{a^2-b^2} + \frac{2a}{(a^2-b^2)^{1/2}\sqrt{s}(a^2-b^2)^{1/4}} \leq \frac{C}{t_0}\frac{1}{(a^2-b^2)^{1/4}}\left(1+\frac{1}{\sqrt{s}}\right),$$

Page 162 line 10 ↓ it says

$$\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\frac{1}{(a^2-b^2)^{1/4}}\int_0^\infty (s^{\eta/2}+u_0^{\eta/2})e^{-\nu u(s)}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)ds$$

it should say

$$\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\frac{1}{(a^2-b^2)^{1/4}}\int_0^\infty (s^{\eta/2}+u_0^{\eta/2})e^{-\nu s}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)ds$$

Page 162 line 11 ↓ it says

$$\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\left(1+\frac{u_0^{\eta/2}}{(a^2-b^2)^{1/4}}\right)\int_0^\infty \left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)\,ds$$

it should say

$$\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\left(1+\frac{u_0^{\eta/2}}{(a^2-b^2)^{1/4}}\right)\int_0^\infty e^{-\nu s}\left(s+\frac{1}{\sqrt{s}}\right)\,ds$$

Page 163 line 2–3 ↓ it says

$$\int_0^1 (u(t))^{\eta/2}e^{-\nu u(t)}\frac{dt}{t^{3/2}}\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\frac{1}{(a^2-b^2)^{1/4}}\int_0^\infty (s^{\eta/2}+u_0^{\eta/2})e^{-\nu u(s)}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)ds.$$

it should say

$$\int_0^1 (u(t))^{\eta/2}e^{-\nu u(t)}\frac{dt}{t^{3/2}}\leq C\frac{e^{-\nu u_0}}{t_0^{1/2}}\frac{1}{(a^2-b^2)^{1/4}}\int_0^\infty (s^{\eta/2}+u_0^{\eta/2})e^{-\nu s}\left(\sqrt{s}+\frac{1}{\sqrt{s}}\right)\sqrt{1-\nu(s)}\,ds.$$

Chapter 8

Page 353 line 1 ↑ it says

$$\bar{I}_\beta \mathbf{H}_\nu(x)=\frac{1}{(|\nu|+1)^{\beta/2}}\mathbf{H}_\nu(x).$$

it should say

$$\bar{I}_\beta \mathbf{H}_\nu(x)=\frac{1}{(|\nu|+d)^{\beta/2}}\mathbf{H}_\nu(x).$$

Page 354 line 2 ↓ it says "$\{T_t^{(1)}\}_t=\{e^{-t}T_t\}_t$, the 1-translated"

it should say "$\{T_t^{(d)}\}_t=\{e^{-td}T_t\}_t$, the d-translated"

Page 354 line 4 ↓ it says

$$\bar{I}_\beta f(x) = (-\overline{L})^{-|\beta|/2} f(x) = \frac{1}{\Gamma(|\beta|/2)} \int_0^\infty t^{\frac{|\beta|-2}{2}} T_t^{(1)} f(x)\, dt$$
$$= \frac{1}{\Gamma(|\beta|/2)} \int_0^\infty t^{\frac{|\beta|-2}{2}} e^{-t} T_t f(x)\, dt$$
$$= C_\beta e^{|x|^2} \int_{\mathbb{R}^d} \left(\int_0^1 (-\log r)^{\frac{|\beta|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\, dr \right) f(y)\, \gamma_d(dy).$$

it should say

$$\bar{I}_\beta f(x) = (-\overline{L})^{-\beta/2} f(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta-2}{2}} T_t^{(d)} f(x)\, dt$$
$$= \frac{1}{\Gamma(\beta/2)} \int_0^\infty t^{\frac{\beta-2}{2}} e^{-dt} T_t f(x)\, dt$$
$$= C_\beta e^{|x|^2} \int_{\mathbb{R}^d} \left(\int_0^1 (-\log r)^{\frac{\beta-2}{2}} r^d \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \right) f(y)\, \gamma_d(dy).$$
$$= C_\beta \int_{\mathbb{R}^d} \left(\int_0^1 (-\log r)^{\frac{\beta-2}{2}} r^d \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}} \frac{dr}{r} \right) f(y) (dy).$$

Chapter 9

Page 360 line 6 ↓ it says

$$R_j = \frac{\partial}{\partial x_j} (-L)^{-1/2}$$

it should say

$$R_j = \frac{\partial}{\partial x_j} (-\Delta)^{-1/2}$$

Page 361 line 3 ↑ it says

$$\mathscr{R}_j = \partial_j^\gamma I_{1/2} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_j} (-L)^{-1/2},$$

it should say

$$\mathscr{R}_j = \partial_\gamma^j I_{1/2} = \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_j} (-L)^{-1/2},$$

Page 366 line 3 ↑ it says

$$\mathscr{R}_\beta = \partial_\beta^\gamma (-L)^{-|\beta|/2},$$

it should say

$$\mathscr{R}_\beta = \partial_\gamma^\beta (-L)^{-|\beta|/2},$$

Page 368 lines 3–7 ↓ it says

$$\begin{aligned}\mathscr{K}_\beta(x,y) &= \partial_\beta^\gamma N_{|\beta|/2}(x,y)\\ &= \frac{1}{\pi^{d/2}\Gamma(\beta)}\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}\frac{dr}{r}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{R}_\beta f(x) &= p.v.\int_{\mathbb{R}^d}\mathscr{K}_\beta(x,y)f(y)dy\\ &= p.v.\frac{1}{\pi^{d/2}\Gamma(\beta)}\int_{\mathbb{R}^d}\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\\ &\qquad \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}\frac{dr}{r}f(y)dy.\end{aligned}$$

it should say

$$\begin{aligned}\mathscr{K}_\beta(x,y) &= \partial_\gamma^\beta N_{|\beta|/2}(x,y)\\ &= \frac{1}{\pi^{d/2}\Gamma(|\beta|/2)}\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}\frac{dr}{r}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathscr{R}_\beta f(x) &= p.v.\int_{\mathbb{R}^d}\mathscr{K}_\beta(x,y)f(y)dy\\ &= p.v.\frac{1}{\pi^{d/2}\Gamma(|\beta|/2)}\int_{\mathbb{R}^d}\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{|\beta|}\mathbf{H}_\beta\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\\ &\qquad \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}\frac{dr}{r}f(y)dy.\end{aligned}$$

Page 379 line 7 ↓ it says

$$(-\overline{L}) = \sum_{i=1}^d \partial_\gamma^i(\partial_\gamma^i)^* = (-L) + I_d = -\frac{1}{2}\Delta + \langle x, \nabla_x\rangle + I_d.$$

it should say

$$(-\overline{L}) = \sum_{i=1}^{d} \partial_\gamma^i (\partial_\gamma^i)^* = (-L) + dI = -\frac{1}{2}\Delta + \langle x, \nabla_x \rangle + dI.$$

Page 379 line 12 ↓ it says

$$(\partial_\gamma^\beta)^* = \frac{(-1)^{|\beta|}}{2^{|\beta|/2}} e^{|x|^2} (\partial_\gamma^\beta e^{-|x|^2})$$

it should say

$$(\partial_\gamma^\beta)^* = \frac{(-1)^{|\beta|}}{2^{|\beta|/2}} e^{|x|^2} (\partial^\beta e^{-|x|^2} I)$$

Page 379 line 6 ↑ it says

$$\overline{\mathscr{R}}_\beta \mathbf{H}_\nu = \frac{1}{2^{|\beta|/2} (|\nu| + 1)^{|\beta|/2}} \mathbf{H}_{\nu+\beta},$$

it should say

$$\overline{\mathscr{R}}_\beta \mathbf{H}_\nu = \frac{1}{2^{|\beta|/2} (|\nu| + d)^{|\beta|/2}} \mathbf{H}_{\nu+\beta},$$

Page 379 line 4 ↑ it says

$$(-\overline{L})^{-|\beta|/2} H_\nu = \frac{1}{(|\nu| + 1)^{|\beta|/2}} \mathbf{H}_\nu,$$

it should say

$$(-\overline{L})^{-|\beta|/2} H_\nu = \frac{1}{(|\nu| + d)^{|\beta|/2}} \mathbf{H}_\nu,$$

Page 379 lines 1–2 ↑ it says

$$\begin{aligned} \overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x) &= (\partial_\gamma^\beta)^* (-\overline{L})^{-|\beta|/2} \mathbf{H}_\nu(x) = \frac{(-1)^{|\beta|}}{(|\nu| + 1)^{|\beta|/2}} e^{|x|^2} \partial_\gamma^\beta (e^{-|x|^2} \mathbf{H}_\nu(x)) \\ &= \frac{(-1)^{|\beta+\nu|}}{2^{|\beta|/2} (|\nu| + 1)^{|\beta|/2}} e^{|x|^2} \partial^{\beta+\nu} (e^{-|x|^2}) = \frac{1}{2^{|\beta|/2} (|\nu| + 1)^{|\beta|/2}} \mathbf{H}_{\nu+\beta}(x); \end{aligned}$$

it should say

$$\begin{aligned} \overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x) &= (\partial_\gamma^\beta)^* (-\overline{L})^{-|\beta|/2} \mathbf{H}_\nu(x) = \frac{(-1)^{|\beta|}}{(|\nu| + d)^{|\beta|/2}} e^{|x|^2} \partial^\beta (e^{-|x|^2} \mathbf{H}_\nu(x)) \\ &= \frac{(-1)^{|\beta+\nu|}}{2^{|\beta|/2} (|\nu| + d)^{|\beta|/2}} e^{|x|^2} \partial^{\beta+\nu} (e^{-|x|^2}) = \frac{1}{2^{|\beta|/2} (|\nu| + d)^{|\beta|/2}} \mathbf{H}_{\nu+\beta}(x); \end{aligned}$$

Page 380 line 2 ↓ it says

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+1)^{|\beta|/2}} \Big[\prod_{i=1}^{d} (\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x),$$

it should say

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+d)^{|\beta|/2}} \Big[\prod_{i=1}^{d} (\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x),$$

Page 380 lines 4–7 ↓ it says

$$\begin{aligned}
\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) &= \overline{\mathscr{R}}_\beta \Big(\frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}\Big) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}} \overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x) \\
&= \frac{1}{(2^{|\nu|}\nu!)^{1/2}} \frac{1}{2^{|\beta|/2}(|\nu|+1)^{|\beta|/2}} \mathbf{H}_{\nu+\beta}(x) = \frac{1}{(\nu!)^{1/2}(|\nu|+1)^{|\beta|/2}} \frac{\mathbf{H}_{\nu+\beta}(x)}{2^{|\nu|/2+|\beta|/2}} \\
&= \frac{1}{(|\nu|+1)^{|\beta|/2}} \Big(\frac{(\nu+\beta)!}{\nu!}\Big)^{1/2} \frac{\mathbf{H}_{\nu+\beta}(x)}{(2^{|\nu+\beta|}(\nu+\beta)!)^{1/2}} \\
&= \frac{1}{(|\nu|+1)^{|\beta|/2}} \Big[\prod_{i=1}^{d} (\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x).
\end{aligned}$$

it should say

$$\begin{aligned}
\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) &= \overline{\mathscr{R}}_\beta \Big(\frac{\mathbf{H}_\nu(x)}{(2^{|\nu|}\nu!)^{1/2}}\Big) = \frac{1}{(2^{|\nu|}\nu!)^{1/2}} \overline{\mathscr{R}}_\beta \mathbf{H}_\nu(x) \\
&= \frac{1}{(2^{|\nu|}\nu!)^{1/2}} \frac{1}{2^{|\beta|/2}(|\nu|+d)^{|\beta|/2}} \mathbf{H}_{\nu+\beta}(x) = \frac{1}{(\nu!)^{1/2}(|\nu|+d)^{|\beta|/2}} \frac{\mathbf{H}_{\nu+\beta}(x)}{2^{|\nu|/2+|\beta|/2}} \\
&= \frac{1}{(|\nu|+d)^{|\beta|/2}} \Big(\frac{(\nu+\beta)!}{\nu!}\Big)^{1/2} \frac{\mathbf{H}_{\nu+\beta}(x)}{(2^{|\nu+\beta|}(\nu+\beta)!)^{1/2}} \\
&= \frac{1}{(|\nu|+d)^{|\beta|/2}} \Big[\prod_{i=1}^{d} (\nu_i+\beta_i)(\nu_i+\beta_i-1)\cdots(\nu_i+1)\Big]^{1/2} \mathbf{h}_{\nu+\beta}(x).
\end{aligned}$$

Page 380 line 12 ↓ it says

$$\overline{\mathscr{K}}_\beta(x,y) = C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} \mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\, dr.$$

it should say

$$\overline{\mathscr{K}}_\beta(x,y) = C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1} \mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\, dr.$$

Page 380 lines 10 ↑ it says

$$(-\overline{L})^{-|\beta|/2}f(x) = \frac{1}{\Gamma(|\beta|/2)}\int_0^\infty t^{\frac{|\beta|-2}{2}} T_t^{(1)} f(x)\, dt$$
$$= C_\beta e^{|x|^2}\int_{\mathbb{R}^d}\Big(\int_0^1 (-\log r)^{\frac{|\beta|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\, dr\Big)\, f(y)\, \gamma_d(dy).$$

it should say

$$(-\overline{L})^{-|\beta|/2}f(x) = \frac{1}{\Gamma(|\beta|/2)}\int_0^\infty t^{\frac{|\beta|-2}{2}} T_t^{(d)} f(x)\, dt$$
$$= C_\beta e^{|x|^2}\int_{\mathbb{R}^d}\Big(\int_0^1 (-\log r)^{\frac{|\beta|-2}{2}} r^d \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\, \frac{dr}{r}\Big)\, f(y)\, \gamma_d(dy)$$
$$= C_\beta \int_{\mathbb{R}^d}\Big(\int_0^1 (-\log r)^{\frac{|\beta|-2}{2}} r^d \frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}}}\, \frac{dr}{r}\Big)\, f(y)\, dy.$$

Page 381 line 3 ↓ it says

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+1)^{|\beta|/2}}\Big[\prod_{j=1}^d (\nu_j+\beta_j)\cdots(\nu_j+1)\Big]^{1/2}\mathbf{h}_{\nu+\beta}(x).$$

it should say

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{(|\nu|+d)^{|\beta|/2}}\Big[\prod_{j=1}^d (\nu_j+\beta_j)\cdots(\nu_j+1)\Big]^{1/2}\mathbf{h}_{\nu+\beta}(x).$$

Page 381 lines 7–8 ↓ it says

$$\overline{\mathscr{R}}_1^{\beta_1}\overline{\mathscr{R}}_2^{\beta_2}\ldots\overline{\mathscr{R}}_d^{\beta_d}\mathbf{h}_\nu(x) = \prod_{j=1}^d\Big(\prod_{i=1}^{\beta_j}\Big(\frac{\nu_j+i}{|\nu|+i}\Big)\Big)^{1/2}\mathbf{h}_{\nu+\beta}(x)$$
$$= \Big[\prod_{j=1}^d \frac{(\nu_j+\beta_j)\cdots(\nu_j+1)}{(|\nu|+\beta_j)\cdots(|\nu|+1)}\Big]^{1/2}\mathbf{h}_{\nu+\beta}(x)$$

it should say

$$\overline{\mathscr{R}}_1^{\beta_1}\overline{\mathscr{R}}_2^{\beta_2}\ldots\overline{\mathscr{R}}_d^{\beta_d}\mathbf{h}_\nu(x) = \prod_{j=1}^d\Big(\prod_{i=1}^{\beta_j}\Big(\frac{\nu_j+i}{|\nu|+d+(i-1)}\Big)\Big)^{1/2}\mathbf{h}_{\nu+\beta}(x)$$
$$= \Big[\prod_{j=1}^d \frac{(\nu_j+\beta_j)\cdots(\nu_j+1)}{(|\nu|+d+\beta_j-1)\cdots(|\nu|+d)}\Big]^{1/2}\mathbf{h}_{\nu+\beta}(x)$$

Page 381 lines 10–15 ↓ it says

$$
\begin{aligned}
T_\beta \mathbf{h}_\nu(x) &= \Big[\frac{\prod_{j=1}^d (|\nu|+\beta_j)\cdots(|\nu|+1)}{(|\nu|+1)^{|\beta|}}\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\frac{\prod_{j=1}^d (|\nu|+\beta_j)\cdots(|\nu|+2)}{(|\nu|+1)^{|\beta|-d}}\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{j=1}^d \Big(\frac{|\nu|+\beta_j)\cdots(|\nu|+2)}{(|\nu|+1)^{\beta_j-1}}\Big)\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{j=1}^d \Big(\frac{(|\nu|+1)+(\beta_j-1)}{|\nu|+1}\Big)\cdots\frac{(|\nu|+1)+1)}{|\nu|+1}\Big)\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{j=1}^d \Big(\frac{(|\nu|+1)+(\beta_j-1)}{|\nu|+1}\Big)\cdots\Big(\frac{(|\nu|+1)+1}{|\nu|+1}\Big)\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{j=1}^d \Big(1+\frac{(\beta_j-1)}{|\nu|+1}\Big)\cdots\Big(1+\frac{1}{|\nu|+1}\Big)\Big]^{1/2}\mathbf{h}_\nu(x)
\end{aligned}
$$

it should say

$$
\begin{aligned}
T_\beta \mathbf{h}_\nu(x) &= \Big[\frac{\prod_{j=1}^d (|\nu|+d+\beta_j-1)\cdots(|\nu|+d)}{(|\nu|+d)^{|\beta|}}\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\frac{\prod_{j=1}^d (|\nu|+d+\beta_j-1)\cdots(|\nu|+2)}{(|\nu|+d)^{|\beta|-d}}\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{j=1}^d \Big(\frac{|\nu|+d+\beta_j-1)\cdots(|\nu|+d+1)}{(|\nu|+d)^{\beta_j-1}}\Big)\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{j=1}^d \Big(\frac{(|\nu|+d)+(\beta_j-1)}{|\nu|+d}\Big)\cdots\frac{(|\nu|+d)+1)}{|\nu|+d}\Big)\Big]^{1/2}\mathbf{h}_\nu(x)\\
&= \Big[\prod_{j=1}^d \Big(1+\frac{(\beta_j-1)}{|\nu|+d}\Big)\cdots\Big(1+\frac{1}{|\nu|+d}\Big)\Big]^{1/2}\mathbf{h}_\nu(x)
\end{aligned}
$$

Page 382 line 4 ↑ it says

$$
= C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} \mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\,dr.
$$

it should say

$$
= C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1}\mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}}\,dr.
$$

Page 382 line 2 ↑ it says

$$\frac{\partial \mathscr{K}}{\partial y_j}(x,y) = 2C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} \left[\frac{-r\beta_j}{\sqrt{1-r^2}}\mathbf{H}_{\beta-\mathbf{e_j}}\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right.$$

it should say

$$\frac{\partial \mathscr{K}}{\partial y_j}(x,y) = 2C_\beta \int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} r^{d-1} \left[\frac{-r\beta_j}{\sqrt{1-r^2}}\mathbf{H}_{\beta-\mathbf{e_j}}\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\right.$$

Page 383 line 9 ↓ it says

$$\left|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)\right| \leq C\left|e^{-|x|^2+|y|^2}\frac{\partial \mathscr{K}}{\partial y_j}(x,y)\right|.$$

it should say

$$\left|\frac{\partial \mathscr{K}}{\partial y_j}(x,y)\right| \leq C\left|e^{-|x|^2+|y|^2}\frac{\partial \overline{\mathscr{K}}}{\partial y_j}(x,y)\right|.$$

Page 384 line 4 ↓ it says

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{2^{|\beta|/2}}\frac{\prod_{j=1}^d [(\nu_j+1)\cdots(\nu_j+\beta_j)]^{\frac{1}{2}}}{(|\nu|+1)^{|\beta|/2}} h_{\nu+\beta}(x).$$

it should say

$$\overline{\mathscr{R}}_\beta \mathbf{h}_\nu(x) = \frac{1}{2^{|\beta|/2}}\frac{\prod_{j=1}^d [(\nu_j+d)\cdots(\nu_j+\beta_j)]^{\frac{1}{2}}}{(|\nu|+d)^{|\beta|/2}} h_{\nu+\beta}(x).$$

Page 384 line 6 ↓ it says

$$\|\overline{\mathscr{R}}_\beta f\|^2_{L^2(d\gamma)} = \sum_\nu \frac{\prod_{j=1}^d [(\nu_j+1)\cdots(\nu_j+\beta_j)]}{2^{|\beta|}(|\nu|+1)^{|\beta|}} |\widehat{f}_\gamma(\nu)|^2$$

it should say

$$\|\overline{\mathscr{R}}_\beta f\|^2_{L^2(d\gamma)} = \sum_\nu \frac{\prod_{j=1}^d [(\nu_j+1)\cdots(\nu_j+\beta_j)]}{2^{|\beta|}(|\nu|+d)^{|\beta|}} |\widehat{f}_\gamma(\nu)|^2$$

Page 384 line 10 ↑ it says

$$|\overline{\mathscr{K}}_\beta(x,y)| = \left|\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{|\beta|-2}{2}} \mathbf{H}_\beta\left(\frac{x-ry}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{d}{2}+1}}\,dr\right|$$

$$\leq C\int_0^{\frac{3}{4}} (-\log r)^{\frac{|\beta|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{d}{2}}}\,dr$$

it should say

$$|\overline{\mathscr{K}}_\beta(x,y)| = \left| \int_0^1 \left(\frac{-\log r}{1-r^2} \right)^{\frac{|\beta|-2}{2}} r^{d-1} \mathbf{H}_\beta \left(\frac{x-ry}{\sqrt{1-r^2}} \right) \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\frac{n}{2}+1}} \, dr \right|$$

$$\leq C \int_0^{\frac{3}{4}} (-\log r)^{\frac{|\beta|-2}{2}} \frac{e^{-\frac{|x-ry|^2}{2(1-r^2)}}}{(1-r^2)^{\frac{n}{2}}} \, dr$$

Page 384 line 5 ↑ it says

$$|\overline{\mathscr{K}}_\beta(x,y)| = C \left(\overline{\mathscr{K}}^1_\beta(x,y) + \overline{\mathscr{K}}^2_\beta(x,y) + \overline{\mathscr{K}}^3_\beta(x,y) \right),$$

it should say

$$|\overline{\mathscr{K}}_\beta(x,y)| \leq C \left(\overline{\mathscr{K}}^1_\beta(x,y) + \overline{\mathscr{K}}^2_\beta(x,y) + \overline{\mathscr{K}}^3_\beta(x,y) \right),$$

10

Appendix

10.1 The Gamma Function and Related Functions

Definition 10.1. *The* Gamma function*, denoted by* $\Gamma(z)$*, is defined as*

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \tag{10.1}$$

for $Re\, z > 0$.

It is easy to verify that the Gamma function satisfies the following *functional equation*, using integration by parts:

$$\Gamma(z+1) = z\Gamma(z), \tag{10.2}$$

in particular,

$$\Gamma(n+1) = n!.$$

Moreover,

$$\Gamma(n+1/2) = \frac{(2n)!\sqrt{\pi}}{2^{2n} n!}, \Gamma(-n+1/2) = (-1)^n \frac{2^{2n} n! \sqrt{\pi}}{(2n)!},$$

$$\Gamma(1) = 1, \Gamma(1/2) = \sqrt{\pi}.$$

Stirling's formula gives us asymptotics for the Gamma function and the factorial,

$$\Gamma(z) \sim (2\pi)^{1/2} z^{z-1/2} e^{-z}, \quad z \to \infty, \tag{10.3}$$

and

$$n! \sim \sqrt{2\pi n}\, (\frac{n}{e})^n, \quad n \to \infty. \tag{10.4}$$

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4

The *Beta function* is closely related to the Gamma function, and is defined as follows:

Definition 10.2. *The* Beta function *is defined as*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \tag{10.5}$$

$Re\, z > 0, Re\, w > 0.$

Observe that, by the change of variables $t = 1-u$, we can prove that $B(z,w)$ is symmetric, i.e., $B(z,w) = B(w,z)$. Moreover, by the change of variables $t = \frac{1+u}{2}$, yields

$$B(z,w) = \frac{1}{2^{z+w-1}} \int_{-1}^1 (1+u)^{z-1}(1-u)^{w-1}du.$$

The *relation between the Gamma function and the Beta function* is the following:

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{10.6}$$

Finally,

Definition 10.3. *The Pochhammer symbol* $(\alpha)_n$*, for* $0 \neq \alpha \in \mathbb{R}$ *and* $n \in \mathbb{N}$ *is defined as*

$$(\alpha)_n = \alpha\,(\alpha+1)\,(\alpha+2), \quad n \geq 0 \ldots (\alpha+n-1)\,, \tag{10.7}$$

and

$$(\alpha)_0 = 1,$$

with $\alpha \neq 0$ *a real number.*

Note that $(1)_n = n!$.

From (10.2), we have the following *relation between the Pochhammer symbol and the Gamma function*:

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \tag{10.8}$$

The *binomial coefficients* are defined, for any $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, as

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}; \tag{10.9}$$

therefore

$$\binom{\alpha}{n} = \frac{(\alpha-n+1)_n}{n!} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)\,\Gamma(n+1)}.$$

10.2 Classical Orthogonal Polynomials

The standard reference for the study of classical orthogonal polynomials is G. Szegő's book [262]. The bibliography on this subject is very long; among other references, we should mention T. S. Chihara [54], G. E. Andrews, R. Askey & R. Roy [9], N. N. Lebedev [156], and M. Abramowitz & I. A. Stegun [1].

Hermite Polynomials

The *Hermite polynomials* $\{H_n\}_{n\in\mathbb{N}}$ are defined (up to a multiplicative constant) as the orthogonal polynomials associated with the Gaussian measure in $\mathbb{R}$,

$$\gamma_1(dx) = \frac{e^{-x^2}}{\pi^{d/2}}dx;$$

therefore, they can be obtained from the canonical basis of the polynomials $\{1, x, x^2, \cdots, x^n, \cdots\}$ using the Gram–Schmidt orthogonalization process with respect to the inner product in $L^2(\gamma_1)$. Thus, the *orthogonality property* of the Hermite polynomials with respect to ν is

$$\int_{-\infty}^{\infty} H_n(y)H_m(y)\,\gamma(dy) = 2^n n!\,\delta_{n,m}, \tag{10.10}$$

$n, m = 0, 1, 2, \cdots$, with the *normalization*

$$H_{2n+1}(0) = 0, \quad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}. \tag{10.11}$$

Before the study on Hermite polynomials by C. Hermite (1822–1901) was published, they had been considered by P. L. Chebyshev; moreover, they appear for the first time in Pierre Simon Laplace's famous *Celestial Mechanics*.

Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2}). \tag{10.12}$$

The orthogonality relation can be obtained from Rodrigues' formula using integration by parts.

Differential relations

$$H_n'(x) = 2nH_{n-1}(x), \tag{10.13}$$

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0. \tag{10.14}$$

Thus, the n-th Hermite polynomial is a polynomial solution to the *Hermite equation* with parameter n.

Explicit representation

$$H_n(x) = n! \sum_{k=0}^{[n/2]} \frac{(-1)^k}{k!} \frac{(2x)^{n-2k}}{(n-2k)!}. \tag{10.15}$$

Integral representation

$$H_n(x) = \frac{(-2i)^n e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} t^n e^{-t^2} e^{2ixt}\,dt. \tag{10.16}$$

Using Rodrigues' formula, this representation is obtained from the equality,

$$e^{-x^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{2ixt} dt. \tag{10.17}$$

Generating function

$$\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} r^n = e^{2xr-r^2}. \tag{10.18}$$

As Hermite polynomials are the only polynomials that satisfy this relation, the generating function can be used as an alternative definition. This formula is obtained, almost immediately, by considering Taylor's expansion of the function e^{2xy-y^2} at $y = 0$ and observing that we have

$$H_n(x) = \frac{d^n}{dy^n}(e^{2xy-y^2})|_{y=0},$$

by the Leibniz product formula.

Three-term recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), n \geq 0, \tag{10.19}$$

where $H_0(x) = 1, H_1(x) = 2x$.

This formula can be obtained from Rodrigues' formula for H_{n+1} by using the Leibniz product formula.

Christoffel–Darboux formula

$$\sum_{k=0}^{n} \frac{H_k(x)H_k(y)}{2^k k!} = \frac{1}{2^{n+1} n!} \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{x-y}. \tag{10.20}$$

Using the generating function and the relation (10.17), we can prove *Mehler's formula* (F. G. Mehler, 1866)

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{2^n n!} r^n = \frac{1}{(1-r^2)^{1/2}} e^{-\frac{r^2(x^2+y^2)-2rxy}{1-r^2}}. \tag{10.21}$$

The *normalized Hermite polynomial* of degree n, which is denoted by h_n, is then

$$h_n(x) = \frac{H_n(x)}{(\pi 2^n n!)^{1/2}}. \tag{10.22}$$

It follows that, with different constants, the normalized Hermite polynomials satisfy relations similar to those that are satisfied by the Hermite polynomials, for example,

$$h_n'(x) = \sqrt{2n} h_{n-1}(x),$$

and

$$h_n''(x) - 2xh_n'(x) + 2nh_n(x) = 0.$$

The *Hermite polynomial in d-variables* $\{\mathbf{H}_\nu\}$ for multi-indices $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}^d$, are defined in tensorial form, i.e., $\mathbf{H}_\nu$ is defined as the tensor product of one-dimensional Hermite polynomials,

$$\mathbf{H}_\nu(x) = \prod_{i=1}^{d} H_{\nu_i}(x_i), \tag{10.23}$$

where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $H_{\nu_i}(x_i)$ is the Hermite polynomial of degree α_i in the variable x_i

The normalized Hermite polynomials in d-variables $\{\mathbf{h}_\nu\}$ are the tensor products of one-dimensional normalized Hermite polynomials,

$$\mathbf{h}_\nu(x) = \prod_{i=1}^{d} h_{\nu_i}(x_i),$$

where $h_{\nu_i}(x_i)$ is the normalized Hermite polynomial of degree α_i in the variable x_i; therefore,

$$\mathbf{h}_\nu(x) = \frac{\mathbf{H}_\nu(x)}{(\pi^d 2^{|\nu|} \nu!)^{1/2}},$$

where, as usual, for each multi-index $\nu = (\nu_1, \nu_2, \cdots, \nu_d)$, $\nu! = \prod_{i=1}^{d} \nu_i!$ and $|\nu| = \sum_{i=1}^{d} \nu_i$. Mehler's formula in d dimensions is then

$$\begin{aligned} \sum_{|\alpha| \geq 0} \frac{\mathbf{H}_\nu(x)\mathbf{H}_\nu(y)}{2^{|\nu|}\nu!} r^\nu &= \sum_{|\nu| \geq 0} \mathbf{h}_\nu(x)\mathbf{h}_\nu(y) r^\nu \\ &= \frac{1}{(1-r^2)^{d/2}} e^{-\frac{r^2(|y|^2+|x|^2)-2r\langle x,y\rangle}{1-r^2}}. \end{aligned} \tag{10.24}$$

Laguerre Polynomials

The *Laguerre polynomials* (or Sonine–Laguerre polynomials) of type $\alpha > -1$, $\{L_n^\alpha\}$, are defined (up to a multiplicative constant) as the orthogonal polynomials associated with the Gamma measure on $(0, \infty)$,

$$\nu_\alpha(dx) = \chi_{(0,\infty)}(x) x^\alpha e^{-x} dx;$$

therefore, they can be obtained from the canonical basis of the polynomials $\{1, x, x^2, \cdots, x^n, \cdots\}$ using the Gram–Schmidt orthogonalization process with respect to the inner product in $L^2(\mu_\alpha)$. Therefore, the *orthogonality property* of the Laguerre polynomials of type α with respect to μ_α, is

$$\int_0^\infty L_n^\alpha(y) L_m^\alpha(y)\, \nu_\alpha(dy) = \Gamma(\alpha+1) \binom{n+\alpha}{n} \delta_{n,m} = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}, \tag{10.25}$$

$n, m = 0, 1, 2, \cdots$, with the *normalization*

$$L_n^{\alpha}(0) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!}. \tag{10.26}$$

These polynomials were studied originally by E. N. Laguerre (1834–1886), but they had appeared previously in the works of N. K. Abel, J. L. Lagrange, and P. L. Chebyshev.

Rodrigues' formula

$$L_n^{\alpha}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}(x^{n+\alpha} e^{-x}), \quad n \in \mathbb{N}, \; x > 0. \tag{10.27}$$

Differential relations

$$(L_n^{\alpha}(x))' = -L_{n-1}^{\alpha+1}(x). \tag{10.28}$$

$$x(L_n^{\alpha}(x))'' + (\alpha + 1 - x)(L_n^{\alpha}(x))' + nL_n^{\alpha}(x) = 0. \tag{10.29}$$

Thus, the n-th Laguerre polynomial of type α is a polynomial solution of the *Laguerre differential equation* with parameters α, n.

Explicit representation

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{d} (-1)^k \binom{n+\alpha}{n-k} \frac{x^k}{k!}. \tag{10.30}$$

Integral representation

$$L_n^{\alpha}(x) = \frac{e^x x^{-\alpha/2}}{n!} \int_0^{\infty} t^{n+\alpha/2} J_{\alpha}(2\sqrt{xt}) e^{-t} dt, \tag{10.31}$$

where J_{α} is the Bessel function of order α (see Watson [288]),[1]

$$J_{\alpha}(x) = \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{2^{2\nu+|\alpha|}\Gamma(\nu+1)\Gamma(|\alpha|+\nu+1)} x^{\nu+|\alpha|}, \; \alpha \neq \pm 1/2.$$

Using Rodrigues' formula, this representation is obtained from the identity,

$$x^{n+\alpha} e^{-x} = \int_0^{\infty} (\sqrt{xt})^{n+\alpha} J_{n+\alpha}(2\sqrt{xt}) e^{-t} dt,$$

Generating function

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) r^n = \frac{1}{(1-r)^{\alpha+1}} e^{-\frac{xr}{1-r}}. \tag{10.32}$$

[1] And $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$, $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

The generating function can be obtained from the integral representation and it gives us an alternative definition of the Laguerre polynomials, as they are the only coefficients (depending on x) that verify that identity.

Three-term recurrence relation

$$(n+1)L_{n+1}^{\alpha}(x) = [(2n+\alpha+1)-x]L_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(x), \tag{10.33}$$

with $L_0^{\alpha}(x) = 1$, $L_1^{\alpha}(x) = -x+\alpha+1$.

Christoffel–Darboux formula

$$\sum_{k=0}^{n} \frac{k!}{\Gamma(k+\alpha+1)} L_k^{\alpha}(x)L_k^{\alpha}(y) = \frac{(n+1)!}{\Gamma(n+\alpha+1)} \frac{L_{n+1}(y)L_n(x) - L_{n+1}(x)L_n(y)}{x-y}. \tag{10.34}$$

Hille–Hardy formula

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(n+\alpha+1)} L_n^{\alpha}(x)L_n^{\alpha}(y)r^n = \frac{1}{1-r} e^{-\frac{(x+y)r}{1-r}} (-xyr)^{-\alpha/2} J_{\alpha}\left(\frac{2\sqrt{-xyr}}{1-r}\right), \tag{10.35}$$

with $|r| < 1$, $\alpha > -1$ and $J_{\alpha}(x)$ is the Bessel function of order α.

Relation between Laguerre and Hermite polynomials

$$H_{2n} = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2) \tag{10.36}$$

$$H_{2n+1} = (-1)^n 2^{2n+1} n! x L_n^{1/2}(x^2). \tag{10.37}$$

The *orthonormal Laguerre polynomials* of type α are defined as

$$l_n^{\alpha}(x) = \frac{(n!)^{1/2} L_n^{(\alpha)}(x)}{(\Gamma(n+\alpha+1))^{1/2}}. \tag{10.38}$$

It follows then that, with different constants, the normalized Laguerre polynomials satisfy relations similar to those that are satisfied by the Laguerre polynomials.

Given the multi-indexes $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha \in (-1, \infty)^d$, and $\nu = (\nu_1, \nu_2, \cdots, \nu_d) \in \mathbb{N}^d$ the *Laguerre polynomial* in d-variables of type α and degree ν, which are denoted as $\mathbf{L}_{\nu}^{\alpha}$, is defined for $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$ as the tensor product of one-dimensional Laguerre polynomials,

$$\mathbf{L}_{\nu}^{\alpha}(x) = \prod_{i=1}^{d} L_{\nu_i}^{\alpha_i}(x_i), \tag{10.39}$$

where $L_{\nu_i}^{\alpha}(x_i)$ is the Laguerre polynomial of degree ν_i in the variable x_i; therefore, the normalized Laguerre polynomial $\mathbf{l}_{\nu}^{\alpha}$ is then

$$\mathbf{l}_\nu^\alpha(x) = \frac{(\nu!)^{1/2}\mathbf{L}_\nu^\alpha(x)}{(\prod_{i=1}^d \Gamma(\nu_i+\alpha+1))^{1/2}}.$$

It can also be expressed as the tensor product of one-dimensional normalized Laguerre polynomials,

$$\mathbf{l}_\nu^\alpha(x) = \prod_{i=1}^d l_{\nu_i}^\alpha(x_i),$$

where $l_{\nu_i}^\alpha(x_i)$ is the normalized Laguerre polynomial of degree ν_i in the variable x_i.

The Hille–Hardy formula in d dimensions is then

$$\sum_{|\nu|\geq 0} \frac{\nu!}{\prod_{i=1}^d \Gamma(\nu_i+\alpha+1)} L_\nu^\alpha(x) L_\nu^\alpha(y) r^\nu = \prod_{j=1}^d \frac{1}{1-r} e^{\left(-\frac{(x_j+y_j)r}{1-r}\right)} (x_j y_j r)^{-\alpha_j/2} J_{\alpha_j}\left(\frac{2\sqrt{-r x_j y_j}}{1-r}\right),$$

with $|r|<1$ and $J_\alpha(x)$ is the Bessel function of order α.

Generalized Hermite Polynomials

The generalized Hermite polynomials were defined by G. Szëgo in [262, Problem 25, page 380] as being orthogonal polynomials with respect to the measure

$$d\lambda(x) = d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx,$$

with $\mu > -1/2$.

There are two normalizations of these polynomials that have been considered. On the one hand, T. S. Chihara in [53] (see also [54]) defines them with a normalization so that their leading coefficient (i.e., the coefficient of x^n) is 2^n; on the other hand, M. Rosenblum in [234] uses a different one.

Let H_n^μ denote the generalized Hermite polynomial of degree n, then for n even

$$H_{2m}^\mu(x) = (-1)^m (2m)! \frac{\Gamma(\mu+\frac{1}{2})}{\Gamma(m+\mu+\frac{1}{2})} L_m^{\mu-\frac{1}{2}}(x^2) \tag{10.40}$$

and for n odd

$$H_{2m+1}^\mu(x) = (-1)^m (2m+1)! \frac{\Gamma(\mu+\frac{3}{2})}{\Gamma(m+\mu+\frac{3}{2})} x L_m^{\mu+\frac{1}{2}}(x^2), \tag{10.41}$$

L_m^γ being the Laguerre γ-polynomial of degree m.

Here, we list the first few generalized Hermite polynomials:

$$H_0^\mu(x) = 1, \quad H_1^\mu(x) = (1+2\mu)^{-1} 2x, \quad H_2^\mu(x) = (1+2\mu)^{-1} 4x^2 - 2,$$

$$H_3^\mu(x) = (1+2\mu)^{-1}(3+2\mu)^{-1}24x^3 - (1+2\mu)^{-1}12x,$$
$$H_4^\mu(x) = (1+2\mu)^{-1}(3+2\mu)^{-1}48x^4 - (1+2\mu)^{-1}48x^2 + 12.$$

This class of generalized Hermite polynomials has a rather nice generating function formula that involves the modified Bessel function I_μ. To get such a formula, we need to consider a generalized exponential function e_μ,

$$e_\mu(x) = \Gamma(\mu + \frac{1}{2})(\frac{2}{x})^{\mu-1/2}[I_{\mu-\frac{1}{2}}(x) + I_{\mu+\frac{1}{2}}(x)], \tag{10.42}$$

where I_ν is the modified Bessel function,

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(\nu+k+1)}.$$

Thus,

$$e_\mu(z) = \sum_{m=0}^\infty \frac{z^k}{\gamma_\mu(m)}, \tag{10.43}$$

where $\gamma_\mu(m)$ is a generalized factorial

$$\gamma_\mu(2m) = \frac{2^{2m}m!\Gamma(m+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})} = (2m)!\frac{\Gamma(m+\mu+\frac{1}{2})}{\Gamma(\mu+\frac{1}{2})}\frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})},$$

$$\gamma_\mu(2m+1) = \frac{2^{2m+1}m!\Gamma(m+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})} = (2m)!\frac{\Gamma(m+\mu+\frac{3}{2})}{\Gamma(\mu+\frac{1}{2})}\frac{\Gamma(\frac{1}{2})}{\Gamma(m+\frac{3}{2})}.$$

Then, clearly e_μ is an entire function and indeed $e_0(x) = e^x$. The *generating function* for $\{H_n^\mu\}$ is then

$$e^{-z^2}e_\mu(2xz) = \sum_{k=0}^\infty H_k^\mu(x)\frac{z^k}{k!}. \tag{10.44}$$

The *three-term recurrence relation* for $\{H_n^\mu\}$ can also be obtained

$$2nH_{n-1}^\mu(x) + \frac{\gamma_\mu(n+1)}{(n+1)\gamma_\mu(n)}H_{n+1}^\mu(x) = 2xH_n^\mu(x), \tag{10.45}$$

in addition to *Mehler's formula* for $x, y \in \mathbb{R}$ and $|z| < 1$,

$$\sum_{n=0}^\infty \frac{\gamma_\mu(n)}{2^n(n!)^2}H_n^\mu(x)H_n^\mu(y)z^n = \frac{1}{(1-z^2)^{\mu+1/2}}e^{-\frac{z^2(x^2+y^2)}{1-z^2}}e_\mu\left(\frac{2xyz}{1-z^2}\right). \tag{10.46}$$

Let us note that each generalized Hermite polynomial satisfies the following differential–difference equation (see [53]),

$$(H_n^\mu)''(x) + 2(\frac{\mu}{x} - x)(H_n^\mu)'(x) + 2(n - \mu\frac{\theta_n}{x^2})H_n^\mu(x) = 0, \tag{10.47}$$

with

$$\theta_n = \begin{cases} 1 \text{ if } & n \text{ is odd,} \\ 0 \text{ if } & n \text{ is even.} \end{cases}$$

and $n \geq 0$.
Therefore, by considering the operator

$$L_\mu = \frac{1}{2}\frac{d^2}{dx^2} + (\frac{\mu}{x} - x)\frac{d}{dx} - \mu\frac{I - \tilde{I}}{2x^2}, \tag{10.48}$$

where $If(x) = f(x)$ and $\tilde{I}f(x) = f(-x)$, H_n^μ turns out to be an eigenfunction of L_μ with eigenvalue $-n$. The operator L_μ is one simple example of a *Dunkl operator*.

Jacobi Polynomials

The *Jacobi polynomials*, $\{P_n^{(\alpha,\beta)}\}_{n\in\mathbb{N}}$, are defined (up to a multiplicative constant) as the orthogonal polynomials associated with the Jacobi measure (or beta measure) in $(-1,1)$, defined as $\nu_{\alpha,\beta}$, $\alpha, \beta > -1$,

$$\nu_{\alpha,\beta}(dx) = \omega_{\alpha,\beta}(x)dx = \chi_{(-1,1)}(x)(1-x)^\alpha(1+x)^\beta dx. \tag{10.49}$$

The function $\omega_{\alpha,\beta}$ is called the *Jacobi weight*. Thus, the Jacobi polynomials can be obtained from the canonical basis of the polynomials $\{1, x, x^2, \cdots, x^n, \cdots\}$ using the Gram–Schmidt orthogonalization process with respect to the inner product in $L^2(\mu_{\alpha,\beta})$. Therefore, the *orthogonality property* of the Jacobi polynomials with respect to $\nu_{\alpha,\beta}$, is

$$\int_{-\infty}^{\infty} P_n^{(\alpha,\beta)}(y)P_m^{(\alpha,\beta)}(y)\,\nu_{\alpha,\beta}(dy) = h_n^{(\alpha,\beta)}\delta_{n,m}, \tag{10.50}$$

$n, m = 0, 1, 2, \cdots$, where

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, \tag{10.51}$$

with the *normalization*

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}. \tag{10.52}$$

As we are working in a symmetric interval and the weight $\omega_{\alpha,\beta}$ satisfies

$$\omega_{\alpha,\beta}(-x) = \omega_{\beta,\alpha}(x), \tag{10.53}$$

it is easy to prove that

$$P_n^{(\alpha,\beta)}(-x) = (-1)^d P_n^{(\beta,\alpha)}(x) \tag{10.54}$$

and therefore

$$P_n^{(\alpha,\beta)}(-1) = (-1)^d \binom{n+\beta}{n} = (-1)^d \frac{\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)}. \tag{10.55}$$

These polynomials were studied originally by Karl Gustav Jacob Jacobi (1804–1851), and they were introduced in 1850. We have as particular cases of the Jacobi polynomials:

1. The Legendre polynomials, $\{P_n\}$, when $\alpha = \beta = 0$.
2. The Chebyshev polynomials of the first type, $\{T_n\}$, when $\alpha = \beta = -1/2$.
3. The Chebyshev polynomials of the second type, $\{U_n\}$, when $\alpha = \beta = 1/2$.
4. The Gegenbauer or ultraspherical polynomials, $\{C_n^{\lambda}\}$, when $\alpha = \beta = \lambda - 1/2 > -1/2$.

Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\right\}. \tag{10.56}$$

Differential relations

$$\frac{d}{dx}\left\{P_n^{(\alpha,\beta)}(x)\right\} = \frac{(n+\alpha+\beta+1)}{2}P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{10.57}$$

$$\left(1-x^2\right)y'' + [\beta-\alpha-(\alpha+\beta+2)x]\,y' + n(n+\alpha+\beta+1)\,y = 0. \tag{10.58}$$

Thus, the Jacobi polynomial $P_n^{(\alpha,\beta)}$ is a polynomial solution of the *Jacobi differential equation*, with parameter α, β, n.

Explicit representation

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{1}{2^n}\sum_{k=0}^{\infty}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(x-1)^{n-k}(x+1)^k \\ &= \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\alpha+\beta+k}{k}\left(\frac{x-1}{2}\right)^k \\ &= \binom{n+\alpha}{n}\left[1+\sum_{k=1}^{n}\frac{(-n)_k(n+\alpha+\beta+1)_k}{k!\,(\alpha+1)_k}\left(\frac{1-x}{2}\right)^k\right]. \end{aligned} \tag{10.59}$$

Integral representation

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{2\pi i}\oint\left(1+\frac{x+1}{2}z\right)^{n+\alpha}\left(1+\frac{x-1}{2}z\right)^{n+\beta}z^{-n-1}dz, \tag{10.60}$$

where $x \neq \pm 1$ and the integral is over a closed curve around zero, in the positive sense, such that the points $-2(x\pm 1)^{-1}$ are not in the interior.

Generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, r^n = 2^{\alpha+\beta}\left(1-2xr+r^2\right)^{-\frac{1}{2}}\left(\left(1-2xr+r^2\right)^{\frac{1}{2}}-r+1\right)^{-\alpha} \times\left(\left(1-2xr+r^2\right)^{\frac{1}{2}}+r+1\right)^{-\beta}. \quad (10.61)$$

The generating function can be obtained from the integral representation and gives us an alternative definition of the Jacobi polynomials, as they are the only coefficients (dependent on x) to satisfy that identity.

Three-term recurrence relation

$$\begin{aligned}&2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)\,P_{n+1}^{(\alpha,\beta)}(x)\\&=(2n+\alpha+\beta+1)\left\{(2n+\alpha+\beta+2)(2n+\alpha+\beta)x+\alpha^2-\beta^2\right\}P_n^{(\alpha,\beta)}(x)\\&\quad -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)\,P_{n-1}^{(\alpha,\beta)}(x),\end{aligned} \quad (10.62)$$

with

$$P_0^{(\alpha,\beta)}(x)=1, \text{y } P_1^{(\alpha,\beta)}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta).$$

Christoffel–Darboux formula

$$\sum_{k=0}^{n}\left\{h_k^{(\alpha,\beta)}\right\}^{-1} P_k^{(\alpha,\beta)}(x)\,P_k^{(\alpha,\beta)}(y) = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}\,\frac{\Gamma(n+2)\,\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)} \times\frac{P_{n+1}^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)-P_n^{(\alpha,\beta)}(x)\,P_{n+1}^{(\alpha,\beta)}(y)}{x-y}. \quad (10.63)$$

Asymptotic relations with other classical orthogonal polynomials

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2x/\beta)=L_n^{\alpha}(x). \quad (10.64)$$

The *Gegenbauer polynomials* $\{C_n^{\lambda}\}$ are defined by

$$C_n^{\lambda}(x)=\frac{(2\lambda)_n}{(\lambda+1/2)_n}P_n^{(\lambda-1/2,\lambda-1/2)}(x), \quad (10.65)$$

and their generating function is

$$\sum_{n=0}^{\infty} C_n^{\lambda}(x)r^n=\frac{1}{(1-2xr+r^2)^{\lambda}}. \quad (10.66)$$

The asymptotic relation for the Hermite polynomials,

$$\lim_{\lambda\to\infty} \lambda^{-n/2} C_n^{\lambda}(x/\lambda) = \frac{H_n(x)}{n!}. \tag{10.67}$$

The *Jacobi orthonormal polynomials* are defined as

$$p_n^{(\alpha,\beta)}(x) = \frac{P_n^{(\alpha,\beta)}(x)}{(h_n^{(\alpha,\beta)})^{1/2}}. \tag{10.68}$$

It is immediately clear that the normalized Jacobi polynomials satisfy similar relations to those that are satisfied by the Jacobi polynomials, with different constants.

Given the multi-index $\alpha = (\alpha_1,\dots,\alpha_d), \beta = (\beta_1,\dots,\beta_d) \in \mathbb{R}^d$, $\alpha_i, \beta_i > -1$ and $\nu = (\nu_1,\nu_2,\cdots,\nu_d) \in \mathbb{N}^d$ the *Jacobi polynomial in d-variables* of type (α,β) and degree ν, which is denoted by $\mathbf{P}^{\alpha,\beta}{}_{\nu}$, is defined for $x = (x_1,x_2,\cdots,x_d) \in \mathbb{R}^d$, as the tensor product of one-dimensional Jacobi polynomials,

$$\mathbf{P}^{(\alpha,\beta)}{}_{\nu}(x) = \prod_{i=1}^{d} P_{\nu_i}^{(\alpha,\beta)}(x_i), \tag{10.69}$$

where $P_{\nu_i}^{(\alpha,\beta)}(x_i)$ is the Jacobi polynomial of degree ν_i in the variable x_i, and the normalized Jacobi polynomial $\mathbf{p}^{(\alpha,\beta)}{}_{\nu}$ is then the tensor product of one-dimensional normalized Jacobi polynomials,

$$\mathbf{p}^{(\alpha,\beta)}{}_{\nu}(x) = (h_{\nu}^{(\alpha,\beta)})^{-1/2} \prod_{i=1}^{d} p_{\nu_i}^{(\alpha,\beta)}(x_i), \tag{10.70}$$

where $h_{\nu}^{(\alpha,\beta)} = \prod_{i=1}^{d} h_{\nu_i}^{(\alpha_i,\beta_i)}$, with $h_{\nu_i}^{(\alpha_i,\beta_i)} = \frac{2^{\alpha_i+\beta_i+1}}{2\nu_i+\alpha_i+\beta_i+1} \frac{\Gamma(\nu_i+\alpha_i+1)\Gamma(\nu_i+\beta_i+1)}{\Gamma(\nu_i+1)\Gamma(\nu_i+\alpha_i+\beta_i+1)}$.

10.3 Doubling Measures

Given a metric space (X,d), we denote by $B_\rho(x)$ the open ball and by $\overline{B}_\rho(x)$ the closed ball centered at $x \in X$ with radius $\rho > 0$. With the notation $B(X)$ we mean the collection of all closed balls of X, and with $\mathscr{B}(X)$ the collection of all Borel sets. If $B = B_\rho(x)$ is any ball, we denote with $2B$ the ball with the same center x as B and with the double radius, i.e., $2B = B_{2\rho}(x)$. A measure is said to be *doubling* if there exists a constant $C > 0$ such that the following *doubling condition* holds for every ball $B_\rho(x) \in B(X)$

$$\mu(2B_\rho(x)) = \mu(B_{2\rho}(x)) \le C\mu(B_\rho(x)).$$

We denote by C_D the least constant that satisfies the doubling condition, i.e., we define

$$C_D = \sup_{B\in B(X)} \frac{\mu(2B)}{\mu(B)}.$$

i) There exists a lower bound for the density of the space X; more precisely, if we set $s = \log_2 C_D$, then

$$\frac{\mu(B_\rho(x))}{\mu(B_R(x))} \geq \frac{1}{C_D^2}\left(\frac{\rho}{R}\right)^s, \quad 0 < \rho \leq R < \infty,\ x,y \in X.$$

This means that in some sense, the number $s = \log_2 C_D$, defines a dimension on X; it is called the *homogeneous dimension* of X. We point out that this is not the topological dimension of X (it can be greater), and it depends on μ and on the metric d. It can be proved that, if the metric d is changed, then the homogeneous dimensions may change as well.

ii) A measure μ is finite if and only if the diameter of X is finite. In fact, if $d = diam(X)$ is finite, then trivially, taking an arbitrary ball B_ρ with $\rho > 0$, for n such that $n > d/\rho$, we get

$$u(X) \leq \mu(B_{n\rho}) \leq C_D^n\, \mu(B_\rho).$$

Conversely, assume that $diam(X) = \infty$. Then, fix a point $y \in X$ and two radii ρ, R with $0 < \rho < R/2$. Then, for infinitely many $n \in \mathbb{N}$ there is a ball $B_\rho(x_n)$ contained in the annulus $B_{2^n R}(y) \setminus B_{2^{n-1}R}(y)$ with the property that any point $x \in X$ lies at most in two of such balls. From the observation above, we know that, for every n

$$\mu(B_\rho(x_n)) \geq \frac{1}{C_D^2}\left(\frac{\rho}{R}\right)^s \mu(B_R(y)),$$

from which

$$\mu(X) \geq \frac{1}{2}\sum_n \mu(B_\rho(x_n)) = \infty.$$

As a consequence, even in a finite-dimensional space, probability measures are not doubling.

iii) The doubling condition implies the Lebesgue differentiation theorem and obtains the same $L^p(X,\mu)$-boundedness results for the Hardy–Littlewood maximal function with respect to the measure μ as in the classical case.

For more details on doubling measures in general, see for instance [6].

10.4 Density Theorems for Positive Radon Measures

This section is taken from [89].

Theorem 10.4. *Let μ be a positive Radon measure on $\mathbb{R}^d$, then for $1 \leq p < \infty$, $C_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions on $\mathbb{R}^d$ with compact support is dense in $L^p(\mu)$*

Proof. To prove this theorem, we are going to use a fact from functional analysis. Let B be a Banach space and D a linear subspace of B, then D is dense in B if and only if every bounded linear functional on B that vanishes on D must be the zero functional. In this case, $C_0^\infty(\mathbb{R}^d)$ is a linear subspace of $L^p(\mu)$ for $1 \le p < \infty$. On the other hand, we know that the dual space of $L^p(\mu)$ is identified with $L^{p'}(\mu)$ with $\frac{1}{p} + \frac{1}{p'} = 1$; therefore, let $f \in L^{p'}(\mu)$ be given such that

$$\int_{\mathbb{R}^d} f(x)\, g(x)\, \mu(dx) = 0, \tag{10.71}$$

for all $g \in C_0^\infty(\mathbb{R}^d)$. Thus, we have to prove that $f = 0$ μ-a.e.

According to the proof we are going to give, we can assume without loss of generality that μ is a finite measure (otherwise we write $\mathbb{R}^d = \bigcup_{k=1}^\infty \bar{B}(0,k)$ where $\bar{B}(0,k) = \{x \in \mathbb{R}^d : |x| \le k\}$, and take into account that $\mu(\bar{B}(0,k)) < \infty$ for all k).

For $\lambda > 0$, let

$$\begin{aligned} E_\lambda := \Big\{x \in \mathbb{R}^d : |f(x)| > \lambda\Big\} &= \Big\{x \in \mathbb{R}^d : f(x) > \lambda\Big\} \cup \Big\{x \in \mathbb{R}^d : -f(x) > \lambda\Big\} \\ &=: E_\lambda^1 \cup E_\lambda^2 \end{aligned}$$

If $\mu(E_\lambda^1) > 0$, because μ is a finite Radon measure, then there exists a compact set K and an open set G such that $K \subset E_\lambda^1 \subset G$,

$$\mu(K) > \mu(E_\lambda^1)/2, \quad \text{and} \quad \mu(G \setminus K)^{1/p} < \frac{\lambda \mu(E_\lambda^1)}{2\|f\|_{p'}}.$$

Let g be a $\mathscr{C}_0^\infty(\mathbb{R}^d)$ such that $0 \le g \le 1$, $g = 1$ on K, and $\operatorname{supp}(g) \subset G$, then

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)\, g(x)\, \mu(dx) &= \int_K f(x)\, \mu(dx) + \int_{G\setminus K} f(x)\, g(x)\, \mu(dx) \\ &\ge \lambda\, \mu(K) - \mu(G \setminus K)^{1/p} \|f\|_{p'} > 0, \end{aligned}$$

which is a contradiction with (10.71). Thus, $\mu(E_\lambda^1) = 0$. To prove that $\mu(E_\lambda^2) = 0$, we use the same reasoning as before applied to the function $-f$ instead of f. Hence, $\mu(E_\lambda) = 0$ for all $\lambda > 0$. Therefore, $f = 0$ μ-a.e. □

Corollary 10.5. *For $1 \le p < \infty$, $C_0^\infty(\mathbb{R})$ is a dense subspace of $L^p(\mathbb{R})$ and $L^p(\gamma_1)$.*

Also, we have immediately,

Corollary 10.6. *For $1 \le p < \infty$, $C_0^\infty(\mathbb{R}^d)$ is a dense subspace of $L^p(\mathbb{R}^d)$ and $L^p(\gamma_d)$.*

Also, we want to prove that for $1 \le p < \infty$, $\mathscr{P}(\mathbb{R})$ is a dense subset of $L^p(\gamma_1)$ and $\mathscr{E} = \operatorname{span}\{\psi_n(x) : n \in \mathbb{Z}_0^+\}$ is a dense subset of $L^p(\mathbb{R})$. To this end, we are going to prove a density theorem obtained by C. Berg and J. P. Christensen [29], which says:

Theorem 10.7. *Let μ be a positive Radon measure on $\mathbb{R}$ and suppose that there exists a number $\alpha > 0$ such that*

$$\int_{-\infty}^{+\infty} e^{\alpha|x|}\, \mu(dx) < +\infty. \tag{10.72}$$

Then, the set $\mathscr{P}(\mathbb{R})$ *is dense in* $L^p(\mu)$ *for* $1 \leq p < \infty$.

Proof. Before getting into the proof of this theorem, let us recall that the Fourier transform of an integrable function f, with respect to the Lebesgue measure is defined as

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-2\pi i \xi t}\, f(t)\, dt.$$

On $\mathscr{S}(\mathbb{R})$, the space of test functions on $\mathbb{R}$, the Fourier transform has an inverse, i.e., for $f \in \mathscr{S}(\mathbb{R})$,

$$\check{f}(x) = \int_{-\infty}^{+\infty} e^{2\pi i x \xi}\, f(\xi)\, d\xi$$

is the inverse Fourier transform of f.

Now, the condition (10.72) implies that all moments associated with μ are finite.

Let us observe that $\mathscr{P}(\mathbb{R})$ is a linear subspace of $L^p(\mu)$ for $1 \leq p < \infty$ and the dual space of L^p can be identified with $L^{p'}(\mu)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Thus, let us start with a function $f \in L^{p'}(\mu)$ such that

$$\int_{-\infty}^{+\infty} x^n\, f(x)\, \mu(dx) \;=\; 0 \tag{10.73}$$

for all $n \in \mathbb{N}_0$. We should conclude that $f = 0\ \mu - a.e.$

Now, let us define

$$F(z) = \int_{-\infty}^{+\infty} e^{-2\pi i z x}\, f(x)\, \mu(dx) \quad \text{for} \quad z \in \Omega := \{z \in \mathbb{C} : |\mathrm{Im}(z)| < \alpha/4\pi p\}.$$

Using Hölder's inequality, F is an analytic function in Ω. Indeed, for $z \in \Omega$,

$$F'(z) = \int_{-\infty}^{+\infty} e^{-2\pi i z x}\, (-2\pi i) x f(x)\, \mu(dx)$$

and the integral defining the derivative exists because using Hölder's inequality for the exponents $\frac{1}{2p} + \frac{1}{2p} + \frac{1}{p'} = 1$, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} |x|\, e^{2\pi|\mathrm{Im}\,(z)||x|}\, |f(x)|\, \mu(dx) &\leq \int_{-\infty}^{+\infty} |x|\, e^{\frac{\alpha}{2p}|x|}\, |f(x)|\, \mu(dx) \\ &\leq \left(\int_{-\infty}^{+\infty} |x|^{2p}\, \mu(dx)\right)^{1/2p} \left(\int_{-\infty}^{+\infty} e^{\alpha|x|}\, \mu(dx)\right)^{1/2p} \\ &\quad \left(\int_{-\infty}^{+\infty} |f(x)|^{p'}\, \mu(dx)\right)^{1/p'} < \infty\end{aligned}$$

Similarly, we can calculate out

$$F^{(n)}(z) = (-2\pi i)^n \int_{-\infty}^{+\infty} e^{-2\pi i z x}\, x^n\, f(x)\, \mu(dx)$$

According to hypothesis $F^{(n)}(0) = 0$ for all $n \in \mathbb{N}_0$ because $0 \in \Omega$ and F is holomorphic in Ω, then for all $z \in \Omega$,

$$F(z) = \sum_{n \geq 0} \frac{F^{(n)}(0)}{n!} z^n = 0.$$

In particular, for $z = \xi + 0i \in \Omega$, we get that

$$\mathscr{F}_\mu f(\xi) := \int_{-\infty}^{+\infty} e^{-2\pi i \xi x}\, f(x)\, \mu(dx) \;=\; 0,$$

for all $\xi \in \mathbb{R}$. Now, let $g \in C_0^\infty(\mathbb{R})$ then

$$\begin{aligned} 0 = \int_{-\infty}^{+\infty} \mathscr{F}_\mu f(\xi)\, \check{g}(\xi)\, d\xi &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-2\pi i \xi x}\, \check{g}(\xi)\, d\xi\, f(x)\, \mu(dx) \\ &= \int_{-\infty}^{+\infty} \hat{\check{g}}(x)\, f(x)\, \mu(dx) \\ &= \int_{-\infty}^{+\infty} g(x)\, f(x)\, \mu(dx) \end{aligned}$$

for any $g \in C_0^\infty(\mathbb{R})$. Hence, from the proof of Theorem 10.4 we can conclude that $f = 0$ μ-a.e. □

Corollary 10.8. *For $1 \leq p < \infty$, $\mathscr{P}(\mathbb{R})$ is a dense subspace of $L^p(\gamma_1)$ and so is $\mathscr{E}$ of $L^p(\mathbb{R})$.*

Proof. As $E = \mathscr{P}(\mathbb{R})$, the conclusion follows directly from this theorem with $\mu = \gamma_1$. On the other hand, $\mathscr{E}$ is a linear subspace of $L^p(\mathbb{R})$; thus, to prove that $\mathscr{E}$ is dense in $L^p(\mathbb{R})$, it is enough to show that for a given function $f \in L^{p'}(\mathbb{R})$ such that

$$\int_{-\infty}^{+\infty} f(x)\, \psi_n(x)\, dx = 0,$$

for all $n \in \mathbb{N}_0$, then $f = 0$ μ-a.e. Observe that the above condition is equivalent to

$$\int_{-\infty}^{+\infty} f(x)\, x^n\, e^{-x^2} dx = 0,$$

for all $n \in \mathbb{N}_0$. Now, from the proof of Theorem 10.7 with $\mu(dx) = e^{-x^2} dx$, we get the conclusion. □

Corollary 10.9. *The family $\{h_n\}_{n\geq 0}$ is an orthonormal basis in $L^2(\gamma_1)$ and so is $\{\psi_n\}_{n\geq 0}$ in $L^2(\mathbb{R})$, that is, for $f \in L^2(\gamma_1)$,*

$$f = \sum_{n \geq 0} \langle f, h_n \rangle h_n$$

in $L^2(\gamma_1)$ with $\langle f, h_n \rangle = \int_{-\infty}^{+\infty} f(y)\, h_n(y)\, \gamma_1(y)$; similarly for the basis of Hermite functions.

Proof. This is a result that has to do with the Hilbert space structure. Let H be an infinite dimensional Hilbert space and $\{x_n\}_{n\geq 0}$ be an orthonormal family in H, then $\sum_{n=0}^{\infty}\langle x, x_n\rangle x_n$ is always convergent because using Bessel's inequality for all N,

$$\|\sum_{n=0}^{N}\langle x, x_n\rangle x_n\| \leq \|x\|.$$

Let $y = \sum_{n=0}^{\infty}\langle x, x_n\rangle x_n$, if we want to prove that $y = x$, we should ask the family to satisfy an extra condition: the only vector orthogonal to every x_n is the null vector, i.e., let z be in H such that $\langle z, x_n\rangle = 0$ for all $n \in \mathbb{N}_0$, then $z = 0$. If this is the case, then

$$\langle y - x, x_k\rangle = \sum_{n=0}^{\infty}\langle x, x_n\rangle\langle x_n, x_k\rangle - \langle x, x_k\rangle = 0,$$

for all $k \in \mathbb{N}_0$, then $x = y$. In this case, $\{x_n\}_{n\geq 0}$ is called a basis for H. □

Corollary 10.10. *For $1 \leq p < \infty$, and $f \in L^p(\gamma_1)$ is such that $\langle f, h_n\rangle = 0$ for all $n \geq 0$, then $f = 0$ a.e. Similarly, if $f \in L^p(dx)$ is such that $\langle f, \psi_n\rangle := \int_{-\infty}^{+\infty} f(x)\ \psi_n(x)\ dx = 0$ for all $n \geq 0$, then $f = 0$ a.e.*

Proof. The proof of this corollary follows the same steps as the proof of corollary 10.9. In the case of the Hermite polynomials, we also get that the function $f = 0$ a.e. because the Lebesgue measure is also absolutely continuous with respect to the Gaussian measure γ_1. □

Observe that this theorem might be useful for other orthogonal expansions as well, as in the case of Jacobi or Laguerre polynomials and functions.

In our case, we have two Hilbert spaces, one $H = L^2(\gamma_1)$ with the orthonormal family $\{h_n\}_{n\geq 0}$ and the other one $H = L^2(\mathbb{R})$ with the family $\{\psi_n\}_{n\geq 0}$. If we want to prove that our two families are the basis of the corresponding Hilbert spaces, we have to show that the only function orthogonal to every member of the family has to be the zero function. For the first, let $f \in L^2(\gamma_1)$ be such that

$$\langle f, h_n\rangle = \int_{-\infty}^{+\infty} f(x)\ h_n(x)\ \gamma_1(dx) = 0$$

for all $n \in \mathbb{N}_0$, then $f = 0$, but from (1.70) we get that

$$\int_{-\infty}^{+\infty} f(x)\ x^n\ \gamma_1(dx) = 0$$

for all $n \in \mathbb{N}_0$ and the conclusion follows from the proof of Theorem 10.7 with $\mu = \gamma_1$. On the other hand, for the second, let $f \in L^2(\mathbb{R})$ be such that

$$\langle f, \psi_n\rangle = \int_{-\infty}^{+\infty} f(x)\ \psi_n(x)\ dx = 0$$

for all $n \in \mathbb{N}_0$, but this implies that

$$\int_{-\infty}^{+\infty} f(x)\, x^n\, e^{-x^2/2} dx = 0$$

for all $n \in \mathbb{N}_0$ and again the result follows from the proof of Theorem 10.7 with $\mu = e^{-x^2}\, dx$ and as the Lebesgue measure is absolutely continuous with respect to this measure, we also get that $f = 0$ a.e.

We know that $\{\mathbf{h}_\nu\}$ is an orthonormal basis in $L^2(\gamma_d)$, so it is $\{\overline{\psi}_\nu\}$ in $L^2(\mathbb{R}^d)$.

As in the one-dimensional case, these statements are a consequence of a more general result about density theorems involving polynomials.

Let $\{\mu_j\}_{j=1}^d$ be d positive Radon measures defined on $\mathbb{R}$, and let $\mu^d = \mu_1 \times \cdots \times \mu_d$ be the product measure on $\mathbb{R}^d$ of them. We want to prove under what conditions the μ_j's the subspace $\mathscr{P}(\mathbb{R}^d)$ is dense in $L^p(\mu^d)$.

Theorem 10.11. *Let $\{\mu_j\}_{j=1}^d$ be a family of d positive Radon measures defined on $\mathbb{R}$ satisfying condition (10.72) from Theorem 10.7 and let $\mu^d = \mu_1 \times \cdots \times \mu_d$, then $\mathscr{P}(\mathbb{R}^d)$ is dense in $L^p(\mu^d)$ for $1 \le p < \infty$.*

Proof. For $1 \le p < \infty$, let f be a function in $L^{p'}(\mu^d)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ such that

$$\int_{\mathbb{R}^d} x^\alpha\, f(x)\, \mu^d(dx) = 0, \tag{10.74}$$

for all $\alpha \in \mathbb{N}_0^d$, we want to prove that $f = 0\ \mu^d - a.e.$ As we mentioned before, this statement would imply that $\mathscr{P}(\mathbb{R}^d)$ is dense in $L^p(\mu^d)$.

By using Fubini's theorem, condition (10.74) can be written as

$$\int_{\mathbb{R}} x_d^{\alpha_d} \left(\int_{\mathbb{R}^{d-1}} (x^{d-1})^{\alpha^{d-1}}\, f(x^{d-1}, x_d)\, \mu^{d-1}(dx^{d-1}) \right) \mu_d(dx_d) = 0$$

for all $\alpha_d \in \mathbb{N}_0^+$, where $x^s = (x_1, x_2, \cdots, x_s)$ and $\alpha^s = (\alpha_1, \alpha_2, \cdots, \alpha_s)$, $1 \le s \le d$.

Observe that

$$\int_{\mathbb{R}^{d-1}} (x^{d-1})^{\alpha^{d-1}}\, f(x^{d-1}, x_d)\, \mu^{d-1}(dx^{d-1}) \in L^{p'}(\mu_d(dx_d)).$$

Then, from the proof of Theorem 10.7, there exists a Borel subset $N(\alpha^{d-1})$ of $\mathbb{R}$ with $\mu_d(N(\alpha^{d-1})) = 0$ such that

$$\int_{\mathbb{R}^{d-1}} (x^{d-1})^{\alpha^{d-1}}\, f(x^{d-1}, x_d)\, \mu^{d-1}(dx^{d-1}) = 0 \tag{10.75}$$

for all $x_d \in \mathbb{R} \setminus N(\alpha^{d-1})$. Let $N_d = \bigcup_{\alpha^{d-1} \in \mathbb{N}_0^{d-1}} N(\alpha^{d-1})$, then $\mu_d(N_d) = 0$ and condition (10.75) is satisfied for all $x_n \in \mathbb{R} \setminus N_d$ and for all $\alpha^{d-1} \in (\mathbb{N}_0)^{d-1}$.

Proceeding in this way, we can find one-dimensional Borel sets $N_{d-1}, \cdots, N_1$ such that $\mu_j(N_j) = 0$ and $f(x) = 0$ for all $x \in (\mathbb{R} \setminus N_1 \times \cdots \times \mathbb{R} \setminus N_d) = \mathbb{R}^d \setminus N$ with

$$N = (N_1 \times \mathbb{R}^{d-1}) \cup (\mathbb{R} \times N_2 \times \mathbb{R}^{d-2}) \cup \cdots \cup (\mathbb{R}^{d-1} \times N_d)$$

and $\mu^d(N) = 0$, i.e., $f = 0\ \ \mu^d - a.e.$ □

From the proof of this theorem, we have the following corollaries, which are stated without proofs, as they are similar to those of the one-dimensional case.

Corollary 10.12. *For* $1 \le p < \infty$*,*

$$E = span\{\mathbf{h}_\nu(x) : \nu \in \mathbb{N}_0^d, x \in \mathbb{R}^d\} = \mathscr{P}(\mathbb{R}^d)$$

is a dense subspace of $L^p(\gamma_d)$ *and so is*

$$\mathscr{E} := span\{\overline{\Psi}_\nu(x) : \nu \in \mathbb{N}_0^d, x \in \mathbb{R}^d\}$$

of $L^p(\mathbb{R}^d)$.

Corollary 10.13. *The family* $\{\mathbf{h}_\nu\}_{\nu \in \mathbb{N}_0^n}$ *is an orthonormal basis in* $L^2(\gamma_d)$ *and so is* $\{\overline{\Psi}_\nu\}_{\nu \in \mathbb{N}_0^d}$ *in* $L^2(\mathbb{R}^d)$*, that is, for* $f \in L^2(\gamma_d)$*,*

$$f = \sum_{k \ge 0} \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu$$

in $L^2(\gamma_d)$ *with* $\langle f, \mathbf{h}_\nu \rangle_\gamma = \int_{\mathbb{R}^d} f(y)\, \mathbf{h}_\nu(y)\, \gamma_d(dy)$; *and for* $f \in L^2(\mathbb{R}_d)$*,*

$$f = \sum_{k \ge 0} \sum_{|\nu|=k} \langle f, \overline{\Psi}_\nu \rangle \overline{\Psi}_\nu$$

in $L^2(\mathbb{R}^d)$ *with* $\langle f, \overline{\Psi}_\nu \rangle = \int_{\mathbb{R}^d} f(y)\, \overline{\Psi}_\nu(y)\, dy$.

Corollary 10.14. *For* $1 \le p < \infty$*, and* $f \in L^p(\gamma_d)$ *is such that* $\langle f, \mathbf{h}_\nu \rangle_\gamma = 0$ *for all* $\nu \in \mathbb{N}_0^d$*, then* $f = 0$ *a.e. Similarly, if* $f \in L^p(\mathbb{R}^d)$ *is such that* $\langle f, \overline{\Psi}_\nu \rangle = 0$ *for all* $\nu \in \mathbb{N}_0^d$*, then* $f = 0$ *a.e.*

10.5 Classical Semigroups in Analysis: The Heat and the Poisson Semigroups

The Heat Semigroup

Let us consider the function

$$k(x) = \frac{1}{(4\pi)^{d/2}} e^{\frac{-|x|^2}{4}}, x \in \mathbb{R}^d, \tag{10.76}$$

which is a C^∞ function, integrable, radial, bounded, and such that

$$\int_{\mathbb{R}^d} k(x)dx = 1,$$

and consider the *heat kernel* (or Gauss–Weierstrass kernel),

$$k_t(x) = \frac{1}{t^{d/2}} k\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t}}. \tag{10.77}$$

Then, according to the properties of k, we have that $\{k_t : t > 0\}$ is an approximation of the identity in $\mathbb{R}^d$.[2]

[2] An *approximation of the identity* in $\mathbb{R}^d$, $\{k_\varepsilon\}_{\varepsilon>0}$ is a family of functions in $L^1(\mathbb{R}^d)$, such that

Given $f \in L^p(\mathbb{R}^d), 1 \le p \le \infty$, let us define

$$\mathscr{T}_t f(x) = (k_t * f)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4t}} f(y)\,dy$$
$$= \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x+\sqrt{t}y) e^{\frac{-|y|^2}{4}}\,dy, t > 0. \quad (10.78)$$

Using Young's inequality, we have that $\mathscr{T}_t f$ is well defined for $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$. On the other hand, if $f \in L^2(\mathbb{R}^d)$, given that

$$\int_{\mathbb{R}^d} e^{-4\pi 2t|\xi|^2} e^{-2\pi <x,\xi>} d\xi = \frac{1}{(4\pi)^{d/2}} e^{-|x|^2/4t}$$

(see [256, Theorem 1.13]), we have that $\mathscr{T}_t f$ can be written as

$$\mathscr{T}_t f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{-4\pi 2t|\xi|^2} e^{2\pi <x,\xi>} d\xi. \quad (10.79)$$

The heat semigroup $\{\mathscr{T}_t\}_{t\ge 0}$, also called the Gauss–Weierstrass semigroup, is a positive, conservative, symmetric, convolution semigroup of diffusions, strongly L^p-continuous in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, with infinitesimal generator, the Laplace operator Δ. More precisely,

Theorem 10.15. *The family of operators $\{\mathscr{T}_t\}_{t\ge 0}$ satisfies the following properties:*

i) Semigroup property:

$$\mathscr{T}_{t_1+t_2} = \mathscr{T}_{t_1} \circ \mathscr{T}_{t_2}, t_1, t_2 \ge 0.$$

ii) Positivity and conservative property:

$$\mathscr{T}_t f \ge 0, \quad \textit{for} \quad f \ge 0, t \ge 0,$$

and

$$\mathscr{T}_t 1 = 1.$$

iii) Contractivity property:

$$||\mathscr{T}_t f||_p \le ||f||_p, \quad t \ge 0, 1 \le p \le \infty.$$

- There exists a constant C such that $\|k_\varepsilon\| \le C$ for any $\varepsilon > 0$.
- $\int_{\mathbb{R}^d} k_\varepsilon = 1$ for any $\varepsilon > 0$.
- $\int_{|x|\ge\delta} |k_\varepsilon(x)dx \to 0$.

Observe that, given $\varphi \in L^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \varphi(x)dx = 1$ defining $\varphi_\varepsilon(x) = \varepsilon^{-d}\varphi(x/\varepsilon), \varepsilon > 0$, the family $\{\varphi_\varepsilon\}$ is an approximation of the identity in $\mathbb{R}^d$,

$$\mathscr{T}^* f(x) = \sup_{t>0} |\mathscr{T}_t f(x)| = \sup_{t>0} |k_t * f(x)| \le C_d M f(x).$$

For a detailed study of the approximations of the identity in $\mathbb{R}^d$, we refer to E. Stein [252, Chapter III], J. Duoandikoetxea, [72, Chapter 2] or L. Grafakos [118, Chapter 1].

iv) Strong L^p-continuity property: The mapping $t \to \mathscr{T}_t f$ is continuous from $[0,\infty)$ to $L^p(\mathbb{R}^d)$, for $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^d)$.

v) Symmetry property: $\mathscr{T}_t$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \mathscr{T}_t f(x) g(x) dx = \int_{\mathbb{R}^d} f(x) \mathscr{T}_t g(x) dx, t \ge 0. \tag{10.80}$$

In particular, we have that the Lebesgue measure is the invariant measure for $\{\mathscr{T}_t\}$

$$\int_{\mathbb{R}^d} \mathscr{T}_t f(x) dx = \int_{\mathbb{R}^d} f(x) dx, t \ge 0. \tag{10.81}$$

vi) Infinitesimal generator: the Laplacian operator $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$ is the infinitesimal generator of $\{\mathscr{T}_t : t \ge 0\}$

$$\lim_{t \to 0} \frac{\mathscr{T}_t f - f}{t} = \Delta f. \tag{10.82}$$

Proof.

i) The semigroup property can also be obtained using the Fourier transform for functions in $\mathscr{S}(\mathbb{R}^d)$, using the Schwartz test function space. Using the properties of the Fourier transform with respect to convolutions and dilations, we have

$$\begin{aligned}
(\mathscr{T}_{t_1+t_2} f)^\wedge(\xi) &= [k_{t_1+t_2} * f]^\wedge(\xi) = [k_{t_1+t_2}]^\wedge(\xi) \hat{f}(\xi) \\
&= e^{(-4\pi^2(t_1+t_2)|\xi|^2)} \hat{f}(\xi) = e^{(-4\pi^2 t_1 |\xi|^2)} (e^{(-4\pi^2 t_2)|\xi|^2)} \hat{f}(\xi)) \\
&= \hat{K}_{t_1}(\xi) \hat{K}_{t_2}(\xi) \hat{f}(\xi) = [k_{t_1} * (k_{t_2} * f)]^\wedge(\xi) = (\mathscr{T}_{t_1}(\mathscr{T}_{t_2} f))^\wedge(\xi).
\end{aligned}$$

Now, because $\mathscr{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d), 1 \le p \le \infty$ there exists an extension of this equality to $L^p(\mathbb{R}^d)$.

Alternatively, we can prove the semigroup property directly by a change of variables and by completing squares. Using Fubini's theorem and the change of variables $v = y - u$

$$\begin{aligned}
\mathscr{T}_{t_1}(\mathscr{T}_{t_2} f(x)) &= \frac{1}{(4\pi t_1)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|y-x|^2}{4t_1}} \Big(\frac{1}{(4\pi t_2)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|u-y|^2}{4t_2}} f(u) du \Big) dy \\
&= \frac{1}{(4\pi)^d (t_1 t_2)^{d/2}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{4t_1} - \frac{|u-y|^2}{4t_2}} dy \Big) f(u) du \\
&= \frac{1}{(4\pi)^d (t_1 t_2)^{d/2}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} e^{-\frac{|u-x-v|^2}{4t_1} - \frac{|v|^2}{4t_2}} dy \Big) f(u) du,
\end{aligned}$$

but completing the square,

$$\begin{aligned}
-\frac{|u-x-v|^2}{4t_1} - \frac{|v|^2}{4t_2} &= -\frac{|u-x|^2}{4t_1} + 2\frac{\langle v, (u-x) \rangle}{4t_1} - \frac{|v|^2}{4t_1} - \frac{|v|^2}{4t_2} \\
&= -\frac{|u-x|^2}{4t_1} + \frac{2t_2 \langle v, (u-x) \rangle - (t_1+t_2)|v|^2}{4t_1 t_2}
\end{aligned}$$

$$=-\frac{|u-x|^2}{4t_1}-\frac{t_1+t_2}{4t_1t_2}|v-\frac{t_2}{t_1+t_2}(u-x)|^2-\frac{t_2}{4t_1(t_1+t_2)}|u-x|^2$$
$$=-\frac{|u-x|^2}{4t_1}[1-\frac{t_2}{t_1+t_2}]-\frac{t_1+t_2}{4t_1t_2}|v-\frac{t_2}{t_1+t_2}(u-x)|^2$$
$$=-\frac{|u-x|^2}{4(t_1+t_2)}-\frac{|v-\frac{t_2}{t_1+t_2}(u-x)|^2}{4t_1t_2/(t_1+t_2)}.$$

Therefore, making the change of variables $\omega=\frac{v-\frac{t_2}{t_1+t_2}(u-x)}{2(t_1t_2/(t_1+t_2))^{1/2}}$

$$\begin{aligned}\mathscr{T}_{t_1}(\mathscr{T}_{t_2}f(x)) &= \frac{1}{(4\pi)^d(t_1t_2)^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|u-x|^2}{4(t_1+t_2)}}\Big(\int_{\mathbb{R}^d}e^{-\frac{|v-\frac{t_2}{t_1+t_2}(u-x)|^2}{4t_1t_2/(t_1+t_2)}}dv\Big)f(u)\,du\\ &= \frac{1}{(4\pi)^d(t_1t_2)^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|u-x|^2}{4(t_1+t_2)}}\frac{2^d(t_1t_2)^d}{(t_1+t_2)^{d/2}}\Big(\int_{\mathbb{R}^d}e^{-|\omega|^2}d\omega\Big)f(u)\,du\\ &= \frac{1}{(4\pi(t_1+t_2))^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|u-x|^2}{4(t_1+t_2)}}f(u)\,du=\mathscr{T}_{t_1+t_2}f(x).\end{aligned}$$

ii) The first equality is the conservative property and it follows immediately by a simple change of variables and the translation invariance property of the Lebesgue measure,

$$\frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}e^{-\frac{|y-x|^2}{4t}}1dy=\frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}e^{-|\frac{(y-x)}{\sqrt{4t}}|^2}dy=\frac{1}{\pi^{d/2}}\int_{\mathbb{R}^d}e^{-|y|^2}dy=1.$$

The positivity is clear as the kernel is positive.

iii) Because the heat semigroup is a convolution semigroup, this property follows directly from Young's inequality,

$$||\mathscr{T}_t f||_p=||k_t*f||_p\le||f||_p||k_t||_1=||f||_p.$$

Then, for each $t>0$, $\mathscr{T}_t$ is a contraction in $L^p(\mathbb{R}^d), 1\le p<\infty$. The case $p=\infty$ is trivial, because $\mathscr{T}_t 1=1$, using ii).

iv) We need to prove that $\mathscr{T}_t f\to\mathscr{T}_{t_0}f$ in $L^p(\gamma_d)$ if $t\to t_0$. Using the semigroup property, it is enough to show that $\mathscr{T}_t f\to f$ in $L^p(\gamma_d)$ if $t\to 0$. This is a consequence of the general theory of approximations of the identity (see [252]).

v) To prove (10.80), using Fubini's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\mathscr{T}_t f(x)g(x)dx &= \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}e^{-\frac{|y-x|^2}{4t}}f(y)dy\Big)g(x)dx\\ &= \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}e^{-\frac{|y-x|^2}{4t}}g(x)dx\Big)f(y)dy=\int_{\mathbb{R}^d}f(y)\mathscr{T}_t g(y)dy.\end{aligned}$$

Because the Lebesgue measure is the invariant measure of $\{\mathcal{T}_t\}$, the symmetry property follows immediately, taking $g \equiv 1$ and using the conservative property.

vi) Let $f \in C_b^2(\mathbb{R}^d)$ be a function with continuous and bounded second derivatives, then, using the representation (10.78), we have

$$\begin{aligned}
(\frac{\mathcal{T}_t f - f}{t})(x) - \Delta f(x) &= \frac{1}{t(4\pi)^{d/2}} \int_{\mathbb{R}^d} [f(x+\sqrt{t}y) - f(x)] e^{\frac{-|y|^2}{4}} dy - \sum_{k=1}^{d} \frac{\partial^2 f(x)}{\partial x_k^2} \\
&= \frac{1}{t(4\pi)^{d/2}} \int_{\mathbb{R}^d} [f(x+\sqrt{t}y) - f(x) \\
&\qquad - t \sum_{k=1}^{d} \frac{\partial^2 f(x)}{\partial x_k^2}] e^{\frac{-|y|^2}{4}} dy \\
&= \frac{1}{t(4\pi)^{d/2}} \int_{\mathbb{R}^d} [f(x+\sqrt{t}y) - f(x) \\
&\qquad - \frac{t}{2} \sum_{k=1}^{d} y_k^2 \frac{\partial^2 f(x)}{\partial x_k^2}] e^{\frac{-|y|^2}{4}} dy.
\end{aligned}$$

Now, using Taylor's expansion of order 2 for f, we have that for some θ, with $0 \le \theta \le 1$,

$$f(x+\sqrt{t}y) - f(x) = \sqrt{t} \sum_{k=1}^{d} y_k \frac{\partial f}{\partial x_k}(x) + \frac{t}{2} \sum_{i,j=1}^{d} y_i y_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\theta x + (1-\theta)\sqrt{t}y).$$

Then, according to the symmetry of $e^{-|y|^2/4}$,

$$\begin{aligned}
(\frac{\mathcal{T}_t f - f}{t})(x) - \Delta f(x) &= \frac{1}{2(4\pi)^{d/2}} \int_{\mathbb{R}^d} [\sum_{i,j=1}^{d} y_i y_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\theta x + (1-\theta)\sqrt{t}y) \\
&\qquad - \sum_{k=1}^{d} y_k^2 \frac{\partial^2 f}{\partial x_k^2}(x)] e^{-\frac{|y|^2}{4}} dy \\
&= \frac{1}{2(4\pi)^{d/2}} \int_{\mathbb{R}^d} \sum_{k=1}^{d} y_k^2 [\frac{\partial^2 f}{\partial^2 x_k}(\theta x + (1-\theta)\sqrt{t}y) \\
&\qquad - \frac{\partial^2 f}{\partial x_k^2}(x)] e^{-\frac{|y|^2}{4}} dy.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|(\frac{\mathcal{T}_t f - f}{t})(x) \; - \Delta f(x)| \le \frac{1}{2(4\pi)^{d/2}} \sum_{k=1}^{d} \int_{\mathbb{R}^d} y_k^2 |\frac{\partial^2 f}{\partial^2 x_k}(\theta x + (1-\theta)\sqrt{t}y) \\
- \frac{\partial^2 f}{\partial x_k^2}(x)| e^{-\frac{|y|^2}{4}} dy,
\end{aligned}$$

and using Lebesgue's dominated convergence theorem, we have that these terms go to zero as $t \to 0$.

Observe that, in this case, the square field operator is given by

$$\Gamma(f,g)(x) = \frac{1}{2}[\Delta(fg)(x) - f(x)\Delta g(x) - \Delta f(x)g(x)] =< \nabla f(x), \nabla g(x) >;$$

therefore,

$$\Gamma(f)(x) = \Gamma(f,f)(x) = |\nabla f(x)|^2. \tag{10.83}$$

The maximal function of the heat semigroup is defined as,

$$\mathscr{T}^* f(x) = \sup_{t>0} |\mathscr{T}_t f(x)| \tag{10.84}$$

The maximal function $\mathscr{T}^*$ is weak $(1,1)$ and strong (p,p) $1 < p \leq \infty$ with respect to the Lebesgue measure.

Proposition 10.16. *The maximal function $\mathscr{T}^*$ satisfies*

i) *$\mathscr{T}^*$ is weak $(1,1)$ with respect to the Lebesgue measure, i.e., there exists a constant C, dependent only on the dimension d, such that for each $f \in L^1(\mathbb{R}^d)$*

$$m(\{x \in \mathbb{R}^d : |\mathscr{T}^* f(x)| > \lambda\}) \leq \frac{C}{\lambda}||f||_1. \tag{10.85}$$

for any $\lambda > 0$.

ii) *If $1 < p \leq \infty$ $\mathscr{T}^*$ is strong (p,p) with respect to the Lebesgue measure, i.e., there exists a constant A_p, dependent only on p and on the dimension d, such that for each $f \in L^p(\mathbb{R}^d)$, then $\mathscr{T}^* f \in L^p(\mathbb{R}^d)$ and*

$$||\mathscr{T}^* f||_p \leq A_p ||f||_p. \tag{10.86}$$

Proof. Observe that for $f \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ with $\|f\|_{L^p(\mathbb{R}^d)+L^\infty(\mathbb{R}^d)} \leq 1$, considering $t > 0$ fixed and $x \in \mathbb{R}^d$ taking $a_k = \sqrt{k}, B_k(x) = \{y \in \mathbb{R}^d : |y-x| \leq a_k\sqrt{4t}\}$ and $A_k(x) = B_k(x) \backslash B_{k-1}(x), k \in \mathbb{N}$, then,

$$\begin{aligned}
|\mathscr{T}_t f(x)| &\leq \int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{4t}} |f(y)| dy \leq \sum_{k=1}^{\infty} \int_{A_k(x)} e^{-\frac{|y-x|^2}{4t}} |f(y)| dy \\
&\leq \sum_{k=1}^{\infty} e^{-(a_{k-1})^2} \int_{A_k(x)} |f(y)| dy \leq \sum_{k=1}^{\infty} e^{-(a_{k-1})^2} \int_{B_k(x)} |f(y)| dy.
\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d} e^{-\frac{|y-x|^2}{4t}} |f(y)| dy \leq \sum_{k=1}^{\infty} e^{-(a_{k-1})^2} \int_{B_k(x)} |f(y)| dy$$

$$\leq \sum_{k=1}^{\infty} e^{-(a_{k-1})^2} C_d (a_k \sqrt{4t})^d \frac{1}{|B_k(x)|} \int_{B_k(x)} |f(y)|dy$$
$$= C_d \sum_{k=1}^{\infty} e^{-(k-1)} (\sqrt{4t}k)^d Mf(x),$$

where Mf is the Hardy–Littlewood maximal function with respect to the Lebesgue measure. Hence,

$$|\mathscr{T}_t f(x)| \leq \frac{1}{(\sqrt{4\pi t})^d} C_d \sum_{k=1}^{\infty} e^{1-k} (\sqrt{k}\sqrt{4t})^d \int_{B_k(x)} |f(y)|dy$$
$$= C_d \sum_{k=1}^{\infty} e^{-k} k^{d/2} \mathscr{M} f(x) = C_d \mathscr{M} f(x),$$

as the series $\sum_{k=1}^{\infty} e^{-k} k^{d/2}$ converges. Therefore

$$\mathscr{T}^* f(x) \leq C_d \mathscr{M} f(x),$$

Alternatively, these results can be obtained immediately from the fact that the heat semigroup is a convolution semigroup, which is an approximation of the identity.

Properties *i*) and *ii*) are then consequences of the boundedness properties of $\mathscr{M} f$.

An interesting result obtained by Gian Carlo Rota [233] establishes that, for a very large class of semigroups, the constant A_p for the maximal function of the semigroup does not actually depend on the dimension, using probabilistic results from the martingale theory (see also [253]).

According to the second representation of the heat semigroup (10.78), it is easy to see that

$$\mathscr{T}_t f(x) \to 0, \text{ as } \quad t \to \infty,$$

in other words, the heat semigroup decays to 0 at infinity. Moreover,

Proposition 10.17. *If* $f \in L^p(\mathbb{R}^d)$, $u(x,t) = \mathscr{T}_t f(x)$ *is a* $C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ *solution of the parabolic equation*

$$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \quad x \in \mathbb{R}^d, t > 0, \tag{10.87}$$

with boundary condition $u(x,0) = f(x), x \in \mathbb{R}^d$.

Proof. According to the general semigroup theory, given that Δ is the infinitesimal generator of $\{\mathscr{T}_t : t \geq 0\}$, we have

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial \mathscr{T}_t f}{\partial t}(x) = \Delta \mathscr{T}_t f(x) = \Delta u(x,t).$$

This can also be proved directly:

$$\Delta u(x,t) = \Delta[k_t * f](x) = [\Delta k_t * f](x)$$
$$= \frac{1}{2(4\pi t)^{d/2}} \int_{\mathbb{R}^d} [\frac{|y-x|^2}{2t^2} - \frac{n}{t}] e^{-\frac{|y-x|^2}{4t}} f(y)dy = \frac{\partial u}{\partial t}(x,t).$$

The boundary condition holds according to the properties of the approximations of the identity.

From this result, we say that $u(x,t)$ is a *parabolic extension* of f to the half-space $\mathbb{R}^d \times \mathbb{R}_+$.

On the other hand, the heat semigroup is closely related to the Brownian motion in $\mathbb{R}^d$. The Brownian motion in $\mathbb{R}^d$ is a random process in $\mathbb{R}^d$ such that

i) $B_0 = 0$ a.e.
ii) B_t has Gaussian distribution $N(0, tI_d)$, with covariance function

$$cov(B_t, B_s) = (s \wedge t) I_d,$$

being I_d identity matrix $d \times d$.
iii) If $s < t, B_t - B_s$ is independent of $\mathscr{F}_s = \sigma(\{B_u : u \leq s\})$.

This process describes the motion of a particle in $\mathbb{R}^d$ with no friction and it can be constructed using the functions $p_t(x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}$ as the transition probability density, in the usual formula to construct a Markov process (see [20]). Then, we have that the heat semigroup can be represented by Brownian motion as

$$\mathscr{T}_t f(x) = E[f(B_t)|B_0 = x], \, f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d).$$

The Poisson Semigroup

The Poisson semigroup is obtained from the heat semigroup, using *Bochner's subordination formula*,

$$e^{-\lambda} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-\lambda^2/4u} du$$

(see E. Stein [252]). Thus, we define

$$\mathscr{P}_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathscr{T}_{t^2/4u} f(x) du. \tag{10.88}$$

Therefore, we have the following explicit representation of $\mathscr{P}_t$,

$$\mathscr{P}_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \mathscr{T}_{t^2/4u} f(x) du = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{u^{d/2}}{(\pi t^2)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-u|x-y|^2}{t^2}} f(y)\, dy\, du$$
$$= \frac{1}{t^d \pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^\infty e^{-u(\frac{|x-y|^2}{t^2}+1)} u^{(d-1)/2} du f(y)\, dy$$
$$= \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \frac{1}{(|x-y|^2+t^2)^{(d+1)/2}} f(y)\, dy = (q_t * f)(x),$$

where

$$q_t(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}} \frac{1}{(|x|^2+t^2)^{(d+1)/2}} = \frac{c_d}{(|x|^2+t^2)^{(d+1)/2}}, \tag{10.89}$$

with $c_d = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}}$.

Let us consider the *Poisson kernel*,

$$q(x) = \frac{c_d}{(|x|^2+1)^{(d+1)/2}}, x \in \mathbb{R}^d, \tag{10.90}$$

which is a C^∞ function, integrable, radial, bounded, and such that

$$\int_{\mathbb{R}^d} q_t(x) = 1$$

(see [256, pages 9–10]). Then,

$$q_t(x) = \frac{1}{t^d} q(\frac{x}{t}).$$

According to the properties of q, we have that $\{q_t : t > 0\}$ is an approximation of the identity in $\mathbb{R}^d$.

Observe that, according to Young's inequality, $\mathscr{P}_t f$ is well defined for $f \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$. Moreover, similar to the heat semigroup's case, if $f \in L^2(\mathbb{R}^d)$, given that

$$\int_{\mathbb{R}^d} e^{-2\pi t|\xi|} e^{-2\pi <x,\xi>} d\xi = \frac{c_d}{(|x|^2+t^2)^{(d+1)/2}}$$

(see [256, Theorem 1.14]), we have that $\mathscr{P}_t f$ can be written as

$$\mathscr{P}_t f(x) = \int_{\mathbb{R}^d} \hat{f}(\xi) e^{-2\pi t|\xi|} e^{2\pi <x,\xi>} d\xi. \tag{10.91}$$

This result can be obtained from the analogous representation of the heat semigroup (10.79) and the subordination formula.

The Poisson semigroup $\{\mathscr{P}_t\}_{t\ge 0}$ is a conservative, symmetric, convolution semigroup, strongly L^p-continuous of positive contractions in $L^p(\mathbb{R}^d)$, $1 \le p < \infty$, with infinitesimal generator $(-\Delta)^{1/2}$. More precisely,

Theorem 10.18. *The family of operators $\{\mathscr{P}_t\}_{t\ge 0}$ satisfies the following properties:*

i) Semigroup property:

$$\mathscr{P}_{t_1+t_2} = \mathscr{P}_{t_1} \circ \mathscr{P}_{t_2}, t_1, t_2 \ge 0.$$

ii) Positivity and conservative property:

$$\mathscr{P}_t f \ge 0, \quad \textit{for} \quad f \ge 0, t \ge 0,$$

and

$$\mathscr{P}_1 = 1.$$

iii) Contractivity property:

$$||\mathscr{P}_t f||_p \leq ||f||_p, \quad t \geq 0, 1 \leq p \leq \infty.$$

iv) Strong L^p-continuity property: The mapping $t \to \mathscr{P}_t f$ is continuous from $[0,\infty)$ to $L^p(\mathbb{R}^d))$, for $1 \leq p < \infty$ and $f \in L^p(\mathbb{R}^d)$.

v) Symmetry property: $\mathscr{P}_t$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \mathscr{P}_t f(x) g(x) dx = \int_{\mathbb{R}^d} f(x) \mathscr{P}_t g(x) dx, t \geq 0.$$

vi) Infinitesimal generator: $(-\Delta)^{1/2}$ is the infinitesimal generator of $\{\mathscr{P}_t : t \geq 0\}$,

$$\lim_{t \to 0} \frac{\mathscr{P}_t f - f}{t} = (-\Delta)^{1/2} f. \tag{10.92}$$

Proof. These results can be obtained immediately from Theorem 10.15 for $\{\mathscr{T}_t : t \geq 0\}$ by using the subordination formula.

The proof that the infinitesimal generator of $\{\mathscr{P}_t\}_{t \geq 0}$ is $(-\Delta)^{1/2}$ can be obtained directly using the Fourier transform.

The maximal function of the Poisson semigroup is defined as

$$\mathscr{P}^* f(x) = \sup_{t>0} |\mathscr{P}_t f(x)| \tag{10.93}$$

Again, the maximal function $\mathscr{P}^*$ is weak $(1,1)$ and strong (p,p) $1 < p \leq \infty$ with respect to the Lebesgue measure,

Proposition 10.19. *The maximal function $\mathscr{P}^*$ satisfies*

i) $\mathscr{P}^*$ is weak $(1,1)$ with respect to the Lebesgue measure, i.e., there exists a constant C, dependent only on the dimension d, such that for each $f \in L^1(\mathbb{R}^d)$

$$m\left(\left\{x \in \mathbb{R}^d : |\mathscr{P}^* f(x)| > \lambda\right\}\right) \leq \frac{C}{\lambda} ||f||_1. \tag{10.94}$$

for any $\lambda > 0$.

ii) If $1 < p \leq \infty$ $\mathscr{P}^*$ is strong (p,p) with respect to the Lebesgue measure, i.e., there exists a constant A_p, dependent only on p and on the dimension d, such that for each $f \in L^p(\mathbb{R}^d)$ then $\mathscr{P}^* f \in L^p(\mathbb{R}^d)$ and

$$||\mathscr{P}^* f||_p \leq A_p ||f||_p. \tag{10.95}$$

Proof. These results can be obtained immediately from the fact that the Poisson semigroup is a semigroup generated by an approximation of the identity; therefore,

$$\mathscr{P}^* f(x) = \sup_{t>0} |\mathscr{P}_t f(x)| = \sup_{t>0} |q_t * f(x)| \leq Mf(x),$$

where Mf is the Hardy–Littlewood maximal function with respect to the Lebesgue measure (see E. Stein [252]).

Properties *i*) and *ii*) are then consequences of the properties of Mf.

Observe that, using the representation of the Poisson semigroup, it is easy to see directly that

$$\mathscr{P}_t f(x) \to 0, \text{as } t \to \infty,$$

in other words, the Poisson semigroup also decays to 0 at infinity.

On the other hand,

Proposition 10.20. *If $f \in L^p(\mathbb{R}^d)$, $u(x,t) = \mathscr{P}_t f(x)$ is a $C^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ solution of the elliptic equation*

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \Delta u(x,t) = 0, x \in \mathbb{R}^d, t > 0, \tag{10.96}$$

with boundary condition $u(x,0) = f(x), x \in \mathbb{R}^d$.

Proof. According to the general semigroup theory, given that $(-\Delta)^{1/2}$ is the infinitesimal generator of $\{\mathscr{P}_t : t \geq 0\}$, we have $\frac{\partial \mathscr{P}_t}{\partial t}(x) = (-\Delta)^{1/2}\mathscr{P}_t f(x)$; therefore,

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 \mathscr{P}_t f}{\partial t^2}(x) = (-\Delta)\mathscr{P}_t f(x) = -\Delta u(x,t).$$

This can also be proved directly:

$$\Delta u(x,t) = \Delta[q_t * f](x) = [\Delta q_t * f](x) = -\frac{\partial^2 u}{\partial t^2}(x,t),$$

because q_t is harmonic in the half-space $\mathbb{R}^d \times \mathbb{R}_+$, i.e.,

$$\frac{\partial^2 q_t}{\partial t^2}(x,t) + \Delta q_t = 0.$$

(*Exercise*) The boundary condition holds according to the properties of the approximations of the identity.

From this result, we can say that $u(x,t)$ is a *harmonic extension* of f to the half-space $\mathbb{R}^d \times \mathbb{R}_+$.

10.6 Interpolation Theory

The two most important results in interpolation theory that are used more frequently in harmonic analysis are the Riesz–Thorin and the Marcinkiewicz interpolation theorems (see for instance [256, Chapter 5], [72, Chapter 2], [118, §1.3]or [275, Chapter 4]). The Riesz-Thorin theorem is the basis of the complex method and the Marcinkiewicz theorem is the basis of the real method.

Theorem 10.21. *(Riesz–Thorin interpolation theorem) Let* (X,μ) *and* (Y,ν) *be two measure spaces. Let* T *be a linear operator defined on the set of all simple functions on* X *taking values in the set of measurable functions on* Y*. Let* $1 \le p_0, p_1, q_0, q_1 \le \infty$ *and assume that*

$$\|Tu\|_{L^{q_0}} \le M_0 \|u\|_{L^{p_0}},$$
$$\|Tu\|_{L^{q_1}} \le M_1 \|u\|_{L^{p_1}},$$

for all simple functions u *on* X*. Then, for all* $0 < \theta < 1$ *we have*

$$\|Tu\|_{L^q} \le M_0^{1-\theta} M_1^{\theta} \|u\|_{L^p}$$

for all simple functions u *on* X*, where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{10.97}$$

Using density, T *has a unique extension as a bounded operator from* $L^p(X,\mu)$ *to* $L^q(Y,\nu)$ *for all* p *and* q *as in* (10.97)*.*

Definition 10.22. *Let* (X,μ) *and* (Y,ν) *be two measure spaces, and let* T *be an operator from* $L^p(X,\mu)$ *into the space of measurable functions from* Y *to* $\mathbb{C}$*. We say that* T *is* weak type (p,q)*, if*

$$\nu(\{y \in Y : |Tf(y)| > \lambda\}) \le \left(\frac{C\|f\|_p}{\lambda}\right)^q, \tag{10.98}$$

and we say that it is weak type (p,∞) *if it is a bounded operator from* $L^p(X,\mu)$ *to* $L^\infty(Y,\nu)$*. We say that* T *is* strong type (p,q) *if it is bounded from* $L^p(X,\mu)$ *to* $L^q(Y,\nu)$

The weak-type (p,q) condition can be rewritten using the weak-L^p spaces, $L^{p,\infty}$, which are defined as,

Definition 10.23. *For* $f : X \to \mathbb{R}$ *a measurable function on* X*, the* distribution function *of* f *is the function* $m_f : [0,\infty) \to [0,\infty]$ *defined as follows:*

$$m_f(\lambda) := \mu(\{x \in X : |f(x)| > \lambda\}). \tag{10.99}$$

For $0 < p < \infty$ *the weak space* $L^{p,\infty}$*, is defined as the set of all* μ*-measurable functions* f *such that*

$$\|f\|_{L^{p,\infty}} := \sup\left\{\lambda m_f(\lambda)^{\frac{1}{p}} : \lambda > 0\right\},$$

is finite.

Then, an operator is said to be of *weak type* (p,q) if it maps L^p to weak-L^q. Observe that if an operator T is of strong type (p,q), then it is of weak type (p,q), because if $E_\lambda = \{y \in Y : |Tf(y)| > \lambda\}$, then

$$\nu(E_\lambda) = \int_{E_\lambda} d\nu \le \int_{E_\lambda} |\frac{Tf(y)}{\lambda}|^q d\nu \le \frac{\|Tf\|_q^q}{\lambda^q} \le \left(\frac{C\|f\|_p}{\lambda}\right)^q.$$

Theorem 10.24. *(Marcinkiewicz interpolation theorem) Let (X,μ) and (Y,ν) be measure spaces and let $0 < p_0 < p_1 \leq \infty$. Let T be a sublinear operator from $L^{p_0}(X)+L^{p_1}(X)$ to the space of the measurable functions on Y, that is, weak type (p_0,q_0) and weak type (p_1,q_1),*

$$\nu(\{y\in Y : |Tf(y)| > \lambda\}) \leq \Big(\frac{A_0\|f\|_{p_0}}{\lambda}\Big)^{q_0},$$

and

$$\nu(\{y\in Y : |Tf(y)| > \lambda\}) \leq \Big(\frac{A_1\|f\|_{p_1}}{\lambda}\Big)^{q_1}.$$

Then, T is of strong type (p,p) for all $p_0 < p < p_1$, i.e., and for all f in $L^p(X)$ we have the estimate

$$\|Tf\|_{L^p(Y)} \leq A\|f\|_{L^p(X)},$$

where the constant A only depends on p, p_0, p_1, A_0, and A_1.

For the proof, we refer the reader to [72, 118, 254] or [275].

10.7 Hardy's Inequalities

Hardy's inequalities are very important tools in analysis.

Theorem 10.25. *For $f\geq 0, p\geq 1$ and $r>0$,*

$$\int_0^{+\infty}\Big(\int_0^x f(y)dy\Big)^p x^{-r-1}dx \leq \frac{p}{r}\int_0^{+\infty}(yf(y))^p y^{-r-1}dy, \tag{10.100}$$

and

$$\int_0^{+\infty}\Big(\int_x^{\infty} f(y)dy\Big)^p x^{r-1}dx \leq \frac{p}{r}\int_0^{+\infty}(yf(y))^p y^{r-1}dy, \tag{10.101}$$

For more details (see [299] Vol I (9.16) page 20). A generalization of these inequalities was obtained in [111], and it is used in Chapter 6.

Theorem 10.26. *(Generalized Hardy's inequality) Let $f\geq 0, r>0, p\geq 1$ and $k\in\mathbb{N}$ then*

$$\begin{aligned}&\Big(\int_0^{+\infty}\Big(\int_0^x\cdots\int_0^x f(r_1,\ldots,r_k)\,dr_1\ldots dr_k)\Big)^p x^{-r-1}dx\Big)^{1/p} \\ &\leq \int_0^1\cdots\int_0^1\Big(\int_0^{+\infty}(x^k f(xv_1,\ldots,xv_k))^p x^{-r-1}dx\Big)^{1/p}dv_1\ldots dv_k\end{aligned} \tag{10.102}$$

Proof. Taking $r_1 = xv_1,\ldots,r_k = xv_k$, we get

$$\begin{aligned}&\Big(\int_0^{+\infty}\Big(\int_0^x\cdots\int_0^x f(r_1,\ldots,r_k)dr_1\ldots dr_k)\Big)^p x^{-r-1}dx\Big)^{1/p} \\ &\qquad = \Big(\int_0^{+\infty}\Big(\int_0^1\cdots\int_0^1 f(xv_1,\ldots,xv_k)x^k dv_1\ldots dv_k)\Big)^p x^{-r-1}dx\Big)^{1/p}\end{aligned}$$

$$= \Big(\int_0^{+\infty} \Big(\int_{(0,1)^k} f(xv_1,\ldots,xv_k)x^k dv_1\ldots dv_k)\Big)^p x^{-r-1}dx\Big)^{1/p}$$

Now, consider the spaces $L^p((0,+\infty),x^{-r-1})$ and $L^p((0,1)^k)$. Then, using Minkowski's inequality,

$$\begin{aligned}\Big(\int_0^{+\infty} \Big(\int_{(0,1)^k} f(xv_1,\ldots,xv_k)x^k dv_1\ldots dv_k)\Big)^p x^{-r-1}dx\Big)^{1/p} \\ \leq \int_{(0,1)^k} \Big(\int_0^{+\infty} \big(f(xv_1,\ldots,xv_k)x^k\big)^p x^{-r-1}dx\Big)^{1/p} dv_1\ldots dv_k \\ = \int_0^1 \cdots \int_0^1 \Big(\int_0^{+\infty} \big(x^k f(xv_1,\ldots,xv_k)\big)^p x^{-r-1}dx\Big)^{1/p} dv_1\ldots dv_k.\end{aligned}$$

□

10.8 Natanson's Lemma and Generalizations

The discussion of these results is taken from [47]; the original reference is [200].

Lemma 10.27. *(Natanson)*
Given $-\infty \leq a < b \leq \infty$ and a non-negative kernel $K(x,\cdot) \in L^1(a,b)$, such that $K(x,y)$ is monotone increasing for $a<y<x$ and monotone decreasing for $b>y>x$, and

$$\int_x^b K(x,y)dy = M_1, \quad \int_a^x K(x,y)dy = M_2,$$

where M_1, M_2 are constants independent of x, then, for $f \in L^1(a,b)$ and $f \geq 0$,

$$\int_x^b f(y)K(x,y)dy \leq M_1 f_+^*(x), \tag{10.103}$$

and

$$\int_a^x f(y)K(x,y)dy \leq M_2 f_-^*(x), \tag{10.104}$$

where

$$f_+^*(x) = \sup_{h>0} \frac{1}{|[x,x+h]|}\int_x^{x+h} f(y)dy,$$

and

$$f_-^*(x) = \sup_{h>0} \frac{1}{|[x-h,x]|}\int_{x-h}^{x} f(y)dy,$$

are the one-sided Hardy–Littlewood maximal functions.

Observations 10.28.

i) *Observe that as*

$$|[x,x+h]| = |[x-h,x]| = \frac{1}{2}|[x-h,x+h]|,$$

then

$$f_-^*(x) \le 2f^*(x), \text{ and } f_+^*(x) \le 2f^*(x),$$

where

$$f^*(x) = \sup_{h>0} \frac{1}{|[x-h,x+h]|} \int_{x-h}^{x+h} f(y)dy,$$

is the (centered) Hardy–Littlewood maximal function. *Thus, the conclusion of the lemma can be given in terms of the (centered) Hardy–Littlewood maximal function instead of the one-sided functions, i.e.,*

$$\int_x^b f(y)K(x,y)dy \le 2M_1 f^*(x), \tag{10.105}$$

and

$$\int_a^x f(y)K(x,y)dy \le 2M_2 f^*(x). \tag{10.106}$$

ii) *We can take* $M_1 = \sup_x \int_x^b K(x,y)dy$ *and* $M_2 = \sup_x \int_a^x K(x,y)dy$.

iii) *As is clear from the proof, this lemma is still true if we consider a Borel measure* μ *is* $\mathbb{R}$, *i.e.,* $K(x,y) \in L^1(\mu)$ *such that* $K(x,y)$ *is monotone increasing for* $a < y < x$ *and monotone decreasing for* $b > y > x$, *and*

$$\int_x^b K(x,y)\mu(dy) = M_1, \quad \int_a^x K(x,y)\mu(dy) = M_2,$$

where M_1, M_2 *are constants independent of* x, *then, for* $f \in L^1(\mu)$ *and* $f \ge 0$,

$$\int_x^b f(y)K(x,y)\mu(dy) \le M_1 f_{+,\mu}^*(x), \tag{10.107}$$

and

$$\int_a^x f(y)K(x,y)\mu(dy) \le M_2 f_{-,\mu}^*(x), \tag{10.108}$$

where

$$f_{+,\mu}^*(x) = \sup_{h>0} \frac{1}{\mu([x,x+h])} \int_x^{x+h} f(y)\mu(dy),$$

and

$$f_{-,\mu}^*(x) = \sup_{h>0} \frac{1}{\mu([x-h,x])} \int_{x-h}^{x} f(y)\mu(dy),$$

are the one-sided Hardy–Littlewood maximal functions with respect to the measure μ. *Nevertheless, to get the bound with the centered Hardy–Littlewood maximal function with respect to* μ, *we need the measure* μ *to be a doubling measure.*

Proof. Let us first consider the case that $K(x,\cdot)$ is a step function in y, i.e., it can be written as

$$K(x,y) = \sum_i c_i(x)\chi_{[x,y_i]} + \sum_j d_j(x)\chi_{[y_j,x]}.$$

Then,

$$\int_x^b f(y)K(x,y)dy = \sum_i c_i(x)\int_x^{y_i} f(y)dy = \sum_i c_i(x)\int_x^{y_i} |[x,y_i]|\frac{1}{|[x,y_i]|} f(y)dy$$
$$\leq \sum_i c_i(x)|[x,y_i]|\frac{1}{|[x,y_i]|} f_+^*(x) = \int_x^b K(x,y)dy\, f_+^*(x) \leq M_1 f_+^*(x),$$

and similarly for the estimate of the integral $\int_a^x f(y)K(x,y)dy$

The general case follows by approximating the kernel K by functions. □

There is a more general version of this result, obtained by A. Zygmund,

Lemma 10.29. *(Zygmund)*

Given $-\infty \leq a < b \leq \infty$, a Borel measure μ with support in (a,b) and a kernel $K(r,x,\cdot)$ dependent on a parameter r, such that

$$\int_a^b |K(r,x,y)|\mu(dy) \leq M_1 \tag{10.109}$$

and

$$\int_x^b \mu(x,y)V_2(K(r,x,dy)) \leq M_2, \quad \int_a^x \mu(y,x)V_2(K(r,x,dy)) \leq M_2, \tag{10.110}$$

where M_1, M_2 are constants independent of x and r, and $V_2(K(r,x,\cdot))$ is the (first) variation of the kernel $K(r,x,y)$ in the variable y, i.e.,

$$V_2(K(r,x,\cdot)) = \sup \sum_i |K(r,x,y_i) - K(r,x,y_{i-1})|,$$

where the supremum is taken over all partitions of $[a,b]$ and the integrals are considered in the Lebesgue–Stieltjes sense.

Then, for $f \in L^1(\mu)$,

$$|\int_a^b K(r,x,y)f(y)\mu(dy)| \leq M f_\mu^*(x), \tag{10.111}$$

where M depends only on M_1, M_2 and

$$f_\mu^*(x) = \sup_{x\in I}\frac{1}{\mu(I)}\int_I f(y)\mu(dy),$$

is the non-centered Hardy–Littlewood maximal function for f with respect to the measure μ.

Proof. Using the integration by parts formula for Stieltjes integrals, we have

$$\begin{aligned}\int_x^b K(r,x,y)\mu(dy) &= (\int_x^b \mu(du))K(r,x,b) - \int_x^b (\int_x^y \mu(du))K(r,x,dy) \\ &= \mu(x,b)K(r,x,b) - \int_x^b \mu(x,y)K(r,x,dy).\end{aligned}$$

Therefore, according to hypothesis

$$\begin{aligned}|\mu(x,b)K(r,x,b)| &\leq \int_x^b |K(r,x,y)|\mu(dy) + \int_x^b \mu(x,y)K(r,x,dy) \\ &\leq \int_x^b |K(r,x,y)|\mu(dy) + \int_x^b \mu(x,y)V_2(K(r,x,dy)) \leq M_1 + M_2.\end{aligned}$$

Now, for $f \in L^1(\mu)$, again using the integration by parts formula,

$$\begin{aligned}\int_x^b f(y)K(r,x,y)\mu(dy) &= (\int_x^b f(y)\mu(dy))K(r,x,b) - \int_x^b (\int_x^y f(y)\mu(dy))K(r,x,dy) \\ &= (\int_x^b f(y)\mu(dy))K(r,x,b) - \int_x^b (\int_x^b f(y)\mu(dy))K(r,x,dy) \\ &= (\frac{1}{\mu(x,b)} \int_x^b f(y)\mu(dy))\mu(x,b)K(r,x,b) \\ &\qquad - \int_x^b (\frac{1}{\mu(x,y)} \int_x^b f(y)\mu(dy))\mu(x,y)K(r,x,dy).\end{aligned}$$

Thus,

$$\begin{aligned}|\int_x^b f(y)K(r,x,y)\mu(dy)| &\leq f_\mu^*(x)|\mu(x,b)K(r,x,b)| + f_\mu^*(x)\int_x^b \mu(x,y)V_2(K(r,x,dy)) \\ &\leq (M_1+M_2)f_\mu^*(x) + M_2 f_\mu^*(x) = (M_1+2M_2)f_\mu^*(x).\end{aligned}$$

□

Given a μ measure as before, observe that for a Natanson's kernel $K(r,x,y)$, i.e., $K(r,x,\cdot) \in L^1(\mu)$ and for every x fixed $K(r,x,\cdot)$ is non-decreasing for $y < x$ and is non-increasing for $y > x$, if K then trivially satisfies the conditions of Zygmund's lemma.

10.9 Forward Differences

Let

$$\Delta_s^k(f,t) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(t+(k-j)s)$$

be the k-th order forward difference of f starting at t with increment s.

Here are some results regarding forward differences. These are well-known results (see for instance [95]), and have already been listed in Chapter 6. For the sake of completeness, their proofs are given here.

Lemma 10.30. *The forward differences satisfy the following properties:*

i) For any positive integer k,

$$\Delta_s^k(f,t) = \Delta_s^{k-1}(\Delta_s(f,\cdot),t) = \Delta_s(\Delta_s^{k-1}(f,\cdot),t).$$

ii) For any positive integer k,

$$\Delta_s^k(f,t) = \int_t^{t+s}\int_{v_1}^{v_1+s}\cdots\int_{v_{k-2}}^{v_{k-2}+s}\int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)\,dv_k dv_{k-1}\cdots dv_2 dv_1.$$

iii) For any positive integer k,

$$\frac{\partial}{\partial s}(\Delta_s^k(f,t)) = k\,\Delta_s^{k-1}(f',t+s),$$

and for any integer $j>0$,

$$\frac{\partial^j}{\partial t^j}(\Delta_s^k(f,t)) = \Delta_s^k(f^{(j)},t).$$

Proof.

i) Let us prove the first equality; the second one is totally analogous.

$$\begin{aligned}
\Delta_s^{k-1}(\Delta_s(f,\cdot),t) &= \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\Delta_s(f,t+(k-1-j)s)\\
&= \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j f(t+(k-j)s)\\
&\qquad -\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j f(t+(k-1-j)s)\\
&= f(t+ks)+\sum_{j=1}^{k-1}\binom{k-1}{j}(-1)^j f(t+(k-j)s)\\
&\quad +\sum_{j=0}^{k-2}\binom{k-1}{j}(-1)^{(j+1)} f(t+(k-(j+1))s)+(-1)^k f(t)\\
&= f(t+ks)+\sum_{j=1}^{k-1}\binom{k-1}{j}(-1)^j f(t+(k-j)s)\\
&\quad +\sum_{j=1}^{k-1}\binom{k-1}{j+1}(-1)^j f(t+(k-j)s)+(-1)^k f(t)\\
&= f(t+ks)+\sum_{j=1}^{k-1}[\binom{k-1}{j}+\binom{k-1}{j+1}](-1)^j f(t+(k-j)s)\\
&\qquad +(-1)^k f(t)
\end{aligned}$$

$$= f(t+ks) + \sum_{j=1}^{k-1} \binom{k}{j} (-1)^j f(t+(k-j)s) + (-1)^k f(t)$$
$$= \Delta_s^k(f,t).$$

ii) By induction in k. For $k=1$, using the fundamental theorem of calculus

$$\Delta_s(f,t) = f(t+s) - f(t) = \int_t^{t+s} f'(v)dv.$$

Let us assume that the identity is true for $k-1$,

$$\Delta_s^{k-1}(f,t) = \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1})\, dv_{k-1} \cdots dv_2 dv_1,$$

and let us prove it for k. Using i) and the fundamental theorem of calculus, we get, after performing $k-1$ change of variables,

$$\begin{aligned}
\Delta_s^k(f,t) &= \Delta_s(\Delta_s^{k-1}(f,\cdot),t) = \Delta_s^{k-1}(f,t+s) - \Delta_s^{k-1}(f,t) \\
&= \int_{t+s}^{t+2s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1})\, dv_{k-1} \cdots dv_2 dv_1 \\
&\qquad - \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1})\, dv_{k-1} \cdots dv_2 dv_1 \\
&= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}+s}^{v_{k-2}+2s} f^{(k-1)}(v_{k-1})\, dv_{k-1} \cdots dv_2 dv_1 \\
&\qquad - \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1})\, dv_{k-1} \cdots dv_2 dv_1 \\
&= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots [\int_{v_{k-2}+s}^{v_{k-2}+2s} f^{(k-1)}(v_{k-1})\, dv_{k-1} \\
&\qquad\qquad - \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1})\, dv_{k-1}] \cdots dv_2 dv_1 \cdots dv_2 dv_1 \\
&= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} [f^{(k-1)}(v_{k-1}+s) - f^{(k-1)}(v_{k-1})]\, dv_{k-1} \cdots dv_2 dv_1 \\
&= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)\, dv_k dv_{k-1} \cdots dv_2 dv_1.
\end{aligned}$$

iii) Let us prove (8.45),

$$\begin{aligned}
\frac{\partial}{\partial s}(\Delta_s^k(f,t)) &= D_s\Big(\sum_{j=0}^{k} \binom{k}{j} (-1)^j f(t+(k-j)s)\Big) \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{\partial}{\partial s}(f(t+(k-j)s)) \\
&= \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j (k-j) f'(t+(k-j)s)
\end{aligned}$$

$$= k\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j f'((t+s)+(k-1-j)s)$$
$$= k\Delta_s^{k-1}(f',t+s).$$

Finally, let us prove (8.46)

$$\begin{aligned}\frac{\partial^j}{\partial t^j}(\Delta_s^k(f,t)) &= \frac{\partial^j}{\partial t^j}\Big(\sum_{j=0}^{k}\binom{k}{j}(-1)^j f(t+(k-j)s)\Big)\\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j\frac{\partial^j}{\partial t^j}(f(t+(k-j)s))\\ &= \sum_{j=0}^{k}\binom{k}{j}(-1)^j f^{(j)}(t+(k-j)s)) = \Delta_s^k(f^{(j)},t).\end{aligned}$$

References

1. M. Abramowitz, I. A. Stegun *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* 9th printing. New York. Dover (1972).
2. I. Abu-Falahah, P. R. Stinga, J. L. Torrea, *A note on the almost everywhere convergence to initial data for some evolution equations*, Potential Anal. 40 (2014), no. 2, 195–202. MR315216.
3. T. Adamowicz, P. Harjulehto, P. Hästö. *Maximal Operator in Variable Exponent Lebesgue Spaces on Unbounded Quasimetric Measure Space.* Math. Scand. 116 (2015), no. 1, 5–22.
4. R. Adams, F. Clarke *Gross' Logarithmic Sobolev inequality: a simple proof.* Amer. J. Math. 101 (1979) 1265–1269.
5. H. Aimar, L. Forzani, R. Scotto *On Riesz transforms and maximal functions in the context of Gaussian harmonic analysis*. Trans. Amer. Math. Soc. 359 no. 5 (2007) 2137–2154.
6. L. Ambrosio, M. Miranda Jr., D. Pallara, *Special functions of bounded variation in doubling metric measure spaces*. Calculus of variations: topics from the mathematical heritage of E. De Giorgi, 1–45, Quad. Mat., 14, Dept. Math., Seconda Univ. Napoli, Caserta, (2004).
7. A. Amenta, J. Teuwen $L^p - L^q$ *off-diagonal estimates for the Ornstein–Uhlenbeck semigroup: some positive and negative results.* Bull. Aust. Math. Soc. 96 (2017), no. 1, 154–161. arXiv:1607.01666v1.
8. C. Ané, D. Blacheré, D. Chaifaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Sheffer *Sur les inégalités de Sobolev logarithmiques*. Panoramas et Synthèses 10, Société Mathématique de France. Paris (2002).
9. G. E. Andrews, R. Askey, R. Roy *Special Functions.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge (2000).
10. N. Arcozzi *Riesz transforms on compact Lie groups, spheres and Gauss space.*Ark. Mat. 36 (1998), no. 2, 201–231.

W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4

11. R. Askey *Orthogonal Polynomials and Special Functions.* Regional Conf in Applied Math. 21 SIAM (1975).
12. R. Askey, S. Wainger *Mean convergence of expansions in Laguerre and Hermite series.* Amer. J. Math. 87 (1965) 695–708.
13. P. Auscher, A. McIntosh, E. Russ, *Hardy spaces of differential forms and Riesz transforms on Riemannian manifolds*. J. Geom. Anal. 18 (2008) 192–248.
14. P. Auscher, T. Coulhon, X. T. Duong, S Hofmann *Riesz transform on manifolds and heat kernel regularity.*(English, French summary) Ann. Sci. Ecole Norm. Supp. (4) 37 (2004), no. 6, 911–957.
15. W. N. Bayley *Generalized hypergeometric series.* Cambridge Tracts in Mathematics and Mathematical Physics # 32. New York. (1964).
16. W. N. Bayley *The generating function of Jacobi polynomials.* J. London Math. Soc. 13 (1938), 8–12.
17. D. Bakry *Transformations de Riesz pour les semi-groupes symétriques II. Etude sous la condition* $\Gamma_2 \geq 0$. in Séminaire de probabilités XIX, 1983/84. Lec. Notes in Math. 1123. Springer (1985).
18. D. Bakry *Remarques sur les semi-groupes de Jacobi.* Hommage á P.A. Meyer et J. Neveau, volumen 236, page 23–40. Astérique (1996)
19. D. Bakry *L'hypercontractivité et son utilisation en théorie des semigroupes.* Lectures on Probability Theory, Lec. Notes in Math. 1581. Springer (1988)
20. D. Bakry *Functional inequalities for Markov semigroups*. Probability measures on groups: recent directions and trends, 91–147, Tata Inst. Fund. Res., Mumbai, (2006)
21. D. Bakry, M. Emery *Hypercontractivité de semi-groupes de diffusion.* C.R. Acad. Sci. 299 (série I) (1984) 775–778.
22. D. Bakry, M. Emery *Diffusions hypercontractives.* in Séminaire de probabilités XIX, 1983/84. Lec. Notes in Math. 1123. Springer (1985).
23. D. Bakry, O. Mazet *Characterization of Markov semigroups on* $\mathbb{R}$ *associated to some families of orthogonal polynomials*. Sem. Prob. XXXVII. Lec. Notes in Math 1832 Springer, Berlin (2003) 60–80.
24. C. Balderrama, W. Urbina *Fractional Integration and Fractional Differentiation for Jacobi Expansions.* Divulgaciones Matemáticas. Vol. 15, No. 2 (2007) 93–113.
25. C. Balderrama, W. Urbina *Fractional Integration and Fractional Differentiation for d-dimensional Jacobi Expansions.* Contemporary Mathematics. AMS #471 (2008) 1–14. arXiv: math.AP/0608639.
26. D. Bárcenas, H. Leyva, W. Urbina *Controllability of the Ornstein Uhlenbeck Equation.* IMA Journal of Mathematical Control and Information. 22, no. 3, (2005) 310–320.
27. H. Bateman *Transcendental functions.* Edited by A. Erdérlyi et al. NY (1955).
28. W. Beckner *Inequalities in Fourier Analysis.* Ann. Math. 102 (1975) 159–182.
29. C. Berg, J. P. Reus Christensen *Density questions in the classical theory of moments.* Annales de l'institut Fourier, tome 31, 3 (1981), 99–114.
30. J. Betancor, L. Forzani, R. Scotto, W. Urbina *On the maximal function for the generalized Ornstein–Uhlenbeck semigroup*. Quaest. Math. 30, no. 4, (2007) 471–482.

31. Beurling, A. *On two problems concerning linear operators in Hilbert space.* Acta Math. 81 (1949)
32. S. Bochner *Sturm–Liouville and heat equations whose eigenfunctions are ultraspherical polynomial or associated Bessel functions.* Proceedings of the Conference on differential equations, Univ. Maryland (1955), 23–48.
33. V. I. Bogachev *Gaussian Measures.* AMS Mathematical Surveys and Monographs. Vol 62. Providence, RI. (1998).
34. A. Bonami *Étude des coefficients de Fourier des fonctions de $L^p(G)$.* Ann. Inst. Fourier (Grenoble) 20 fasc. 2, (1971) 335–402.
35. L. Breiman *Probability*. Addison-Wesley. Reading (1968).
36. R. Brualdi *Introduction to Combinatorics.* 4th. ed. Pearson, Upper Saddle River, New Jersey (2004).
37. T. Bruno *Endpoint Results for the Riesz Transform of the Ornstein–Uhlenbeck Operator*. Pre-print. arxiv.org/abs/1801.07214
38. P. L. Butzer, H. Berens *Semi-groups of operators and approximation.* Die Grundkehren der mathematischen Wissenschaften, Ban145. Springer. New York, (1967)
39. L. Cafarelli *Sobre la conjugación y sumabilidad de series de Jacobi.* Ph.D. thesis. Universidad de Buenos Aires, Argentina (1971).
40. L. Cafarelli, C. P. Calderón *Weak type estimates for the Hardy–Littlewood maximal function.* Studia Math.49 (1974) 217–224.
41. L. Cafarelli, C. P. Calderón *On Abel summability of multiple Jacobi series.* Colloquium Mathtematicum 30 (1974) 277–288.
42. A. P. Calderón *Intermediate spaces and interpolation, the complex method.* Studia Math. 24 (1964), 113–190. MR 0167830 (29 #5097)
43. A. P. Calderón & A. Zygmund *Intermediate spaces and interpolation, the complex method.* Studia Math. 24 (1964), 113–190. MR 0167830 (29 #5097)
44. C. P. Calderón *Some remarks on the multiple Weierstrass Transform and Abel summability of multiple Fourier-Hermite series.* Studia Math. 32 (1969) 119–148. MR0262770 (41:7375)
45. C. P. Calderón, *On Abel summability of multiple Laguerre series.* Studia Math. 33 1969 273–294. MR0279523
46. C. P. Calderón, V. Vera de Serio *On Abel summability of Jacobi type series.* Illinois J Math 41 (1997) No. 2, 237–265.
47. C. P. Calderón, W. Urbina *On Abel summability of Jacobi polynomials series, the Watson Kernel and applications.* Illinois Journal of Math, Vol. 57 No. 2, (2013) 343–371.
48. C. P. Calderón, A. S. Coré, W. Urbina *Some new results on H summability of Fourier series.* Quaestiones Mathematicae, Published online: 03 Sep 2018. https://doi.org/10.2989/16073606.2018.1504253.
49. A. Carbonaro, O. Dragicević *Functional Calculus for Generators of Symmetric Contraction Semigroups* Duke Math. J. 166 937–974 (2017) arXiv:1308.1338v2

50. A. Carbonaro, G. Mauceri, S. Meda *H^1 and BMO on certain measured metric spaces.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 8 (2009), no. 3, 543–582. MR2581426 (2010j:42025)
51. A. Carbonaro, G. Mauceri, S. Meda *H^1 and BMO for certain locally doubling metric measure spaces of finite measure.* Colloq. Math. 118 (2010), 13–41. MR2600517 (2011b:42034)
52. A. Carbonaro, G. Mauceri, S. Meda *Comparison of spaces of Hardy type for the Ornstein–Uhlenbeck operator.* Potential Anal. 33 (2010), 85–105. MR2644215 (2011k:42045)
53. T. S Chihara *An Introduction to Orthogonal Polynomials.* Gordon & Breach. New York (1978).
54. T. S. Chihara, *Generalized Hermite Polynomials* Ph.D. thesis, Purdue (1955).
55. R. R. Coifman, Y. Meyer, E. M Stein *Some new function spaces and their applications to harmonic analysis.* J. Funct. Anal. 62 (1985), 304–335. MR791851 (86i:46029)
56. R. R. Coifman, G. Weiss, *Extensions of Hardy spaces and their use in analysis.* Bull. Amer. Math. Soc. 83 (1977), 569–645. MR0447954 (56:264)
57. R. R. Coifman, G. Weiss, *Transference methods in analysis*, Amer. Math. Soc., Providence, RI (1976) MR0481928
58. W. Connett & A. Schwartz *The Littlewood–Paley Theory for Jacobi Expansions* Trans. Amer. Math. Soc. 139 (1969) 231–242
59. M. R. Cowling *Harmonic Analysis on Semigroups.* Annals of Math, Second Series, Vol. 251, (1979), 219–234.
60. M. R. Cowling & S. Meda *Harmonic Analysis and Ultracontractivity.* Trans Amer. Math Soc. 340 (1993), 733–752.
61. D. Cruz-Uribe & A. Fiorenza *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*. Birkhäuser. Springer-Verlag. Heidelberg (2013).
62. R. F. Curtain, H. J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, Tex in Applied Mathematics, Vol. 21. Springer, New York (1995).
63. E. Dalmasso & R. Scotto *Riesz transforms on variable Lebesgue spaces with Gaussian measure,* Integral Transforms and Special Functions, 28:5, 403–420, DOI: 10.1080/10652469.2017.1296835
64. E. B. Davies *One-Parameter Semigroups.* L.M.S. Monographs. Academic Press. London (1980).
65. E. B. Davies *Heat kernels and spectral theory.* Cambridge Tracts in Math. 92. Cambridge University Press. Cambridge (1989).
66. L. Diening, P. Harjulehto, P. Hästö and M. Růžička.*Lebesgue and Sobolev spaces with variable exponents*. Lecture Notes in Mathematics, Vol. 2017 Springer, Heidelberg (2011).
67. J. Diestel, J. J. Uhl Jr *Vector measures.* Amer. Math. Soc., Providence, R.I., (1977), Mathematical Surveys, No. 15. MR 0453964 (56 #12216)
68. U. Dinger *Weak-type* $(1,1)$ *Estimates of the Maximal Function for the Laguerre Semigroup in Finite Dimensions.* Rev. Mat. Iberoam. 8 (1) (1992), 93–120.

69. O. Dragicevic, A. Volberg *Bellman functions and dimensionless estimates of Littlewood–Paley type.* J. Oper. Theory 56 (2006), 167–198. MR2261616 (2007g:42023)
70. N. Dunford, J. T. Schwartz *Linear Operator.* Part I. Interscience. New York (1958).
71. C. Dunkl, Y. Xu *Orthogonal Polynomials of Several Variables.* Cambridge University Press. Cambridge (2001).
72. J. Duoandikoetxea *Fourier Analysis.* Graduate Studies in Math vol 29. AMS. Providence (2001).
73. K.L. Engel, R. Nagel *A Short Course on Operator Semigroups* Universitext. Springer (2006)
74. J. Epperson *The hypercontractive approach to exactly bounding an operator with complex Gaussian kernel.* J. Funct. Anal., **87** (1989),1–30.
75. J. Epperson *Triebel–Lizorkin spaces for Hermite expansions.* Studia Math, **114** (1995),199–209.
76. W. N. Everit, K. H. Kwon, L. L. Littlejohn, R. Wellman *Orthogonal polynomial solutions of linear ordinary differential equations.* J. Comput. Appl. Math. 133 (2001). 85–109.
77. E. Fabes, C. Gutiérrez, R. Scotto *Weak-type estimates for the Riesz transforms associated with the Gaussian measure.* Rev. Mat. Iber. 10 (1994) 229–281. MR1286476 (95m:42023)
78. C. Fefferman, *Characterizations of Bounded Mean Oscillation* Bull. Amer. Math. Soc. 77 (1971) 587–588.
79. C. Fefferman, E. M. Stein *H^p spaces of several variables.* Acta Math. 129 (1972) 137–193.
80. P. Feinsilver *Special functions, probability semigroups and Hamiltonian flows.* Lec. Notes in Math 696 Springer, Berlin (1978).
81. W. Feller *Introduction to Probability Theory and its Applications.* Vol 2. John Wiley and Sons, New York (1971).
82. D. Feyel *Transformations de Hilbert–Riesz gaussiennes*, C. R. Acad. Sci. Paris, Sér. I Math 310 (1990), 653–655.
83. L. Forzani *Lemas de cubrimiento de tipo Besicovitch y su aplicación al estudio del operador maximal de Ornstein–Uhlenbeck (Besicovitch type covering lemmas and their applications to the study of the Ornstein–Uhlenbeck maximal operator).* Ph.D. thesis. Universidad Nacional de San Luis. Argentina (1993).
84. L. Forzani *Unpublished manuscript*. (1994).
85. L. Forzani, E. Harboure, R. Scotto *Weak Type Inequality for a Family of Singular Integral Operators Related with the Gaussian Measure.* Potential Anal no. 31, (2009),103–116.
86. L. Forzani, R. Scotto *The higher order Riesz transform for Gaussian measure need not be of weak type (1,1).* Studia Math. 131 (1998), no. 3, 205–214. MR1644460 (99h:42034)
87. L. Forzani, R. Scotto, W. Urbina *Riesz and Bessel Potentials, the g_k functions and an area function, for the Gaussian measure γ_d.* Rev. Un. Mat. Argentina (UMA) vol 42 (1) (2000) 17–37. MR1852728 (2002h:42022)

88. L. Forzani, R. Scotto, W. Urbina *A simpler proof of the L^p continuity of the higher order Riesz Transform with respect to the Gaussian measure γ_d*. Seminaire de Probabilitès XXXV Lecture Notes in Math 1755. Springer. (2001) 162–166. MR1837285 (2002c:42019)
89. L. Forzani, R. Scotto, W. Urbina *Differential operators and Singular Integrals for Hermite functions and polynomials expansions.* Unpublished manuscript. (2006).
90. L. Forzani, R. Scotto, P. Sjögren, W. Urbina *On the L^p boundedness of the non-centered Gaussian Hardy–Littlewood maximal function*. Proc. Amer. Math. Soc. 130 , no. 1, (2002) 73–79.
91. L. Forzani, E. Harboure, R. Scotto *Harmonic Analysis related to Hermite Expansions* Recent developments in real and harmonic analysis: in honor of Carlos Segovia. Appl. Numer. Harmon. Anal. 76–93. Boston. Birkhauser (2010).
92. L. Forzani, E. Sasso, R. Scotto *Maximal operators associated with generalized Hermite polynomial and function expansions*. Rev. Un. Mat. Argentina 54 (2013), no. 1, 83–107.
93. L. Forzani, E. Sasso, R. Scotto, *L^p boundedness of Riesz transforms for orthogonal polynomials in a general context*. Studia Math. 231 (2015), no. 1, 45–71.
94. L. Forzani, W. Urbina *Poisson–Hermite representation of solutions of the equation* $\frac{\partial^2}{\partial t^2}u(x,t) + \Delta_x u(x,t) - 2x \cdot \nabla_x u(x,t) = 0.$ Approximation, optimization and mathematical economics (Pointe-Pitre, 1999) 109–115, Physica, Heidelberg (2001).
95. T. Fort *Finite Differences and difference Equations in the Real Domain* Oxford Clarendon Press (1948).
96. M. Frazier, B. Jawerth, G. Weiss *Littlewood Paley Theory and the Study of Functions Spaces.* CBMS-Conference Lecture Notes 79. Amer. Math. Soc. Providence RI. (1991).
97. G. Gasper *Positivity and convolution structure for Jacobi series.* Ann. of Math. (2) 93 (1971), 112–118.
98. G. Gasper *Banach algebras for Jacobi series and positivity of a kernel.* Ann. of Math. (2) 95 (1972), 261–280.
99. G. Gasper. Unpublished manuscript.
100. G. Gasper *Nonnegativity of a discrete Poisson kernel for the Hahn polynomials.* Collection of articles dedicated to Salomon Bochner. J. Math. Anal. Appl. 42 (1973), 438–451.
101. J. Garcia-Cuerva, J. L. Rubio de Francia *Weighted norm inequalities and related topics.* North-Holland. Amsterdam (1985).
102. J. García-Cuerva, G. Mauceri, P. Sjögren, J. L. *Higher-order Riesz operators for the Ornstein–Uhlenbeck semigroup.* Potential Anal. 10 (1999), 379–407. MR1698617 (2000j:42024)
103. J. García-Cuerva, G. Mauceri, P. Sjögren, J. L. Torrea *Spectral multipliers for the Ornstein–Uhlenbeck semigroup.* J. Anal. Math. 78 (1999), 281–305. MR1714425 (2000m:42015)

104. J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, J. L. Torrea *Maximal operators for the holomorphic Ornstein–Uhlenbeck semigroup.* J. London Math. Soc. (2) 67 (2003), 1, 219–234.
105. J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren, J. L. Torrea *Functional calculus for the Ornstein–Uhlenbeck operator.* J. Funct. Anal. 183 (2001), no. 2, 413–450. MR1844213 (2002d:47023)
106. G. Garrigós, S. Harzstein, T. Signes, J. L. Torrea, B. Viviani *Pointwise convergence to initial data of heat and Laplace equations*, Trans. Amer. Math. Soc. 368 (2006), no.9, 6575–6600. MR3461043
107. A. Garsia *Topics in Almost Everywhere Convergence.* Murkham, Chicago, (1970)
108. A. E. Gatto, C. Segovia, S. Vági *On Fractional Differentiation and Integration on Spaces of Homogeneous Type*, Rev. Mat. Iberoamericana 12 (1996) 111–145.
109. A. E. Gatto, W. Urbina *On Gaussian Lipschitz spaces and the boundedness of Fractional Integrals and Fractional Derivatives on them.* (with A. Eduardo Gatto). Quaest. Math. 38 (2015), no. 1, 1–25. arXiv:0911.3962
110. A. E. Gatto, E. Pineda, W. Urbina *Riesz Potentials, Bessel Potentials and Fractional Derivatives on Besov–Lipschitz spaces for the Gaussian Measure.* Recent advances in harmonic analysis and applications, dedicate to Konstantin Oskolkov for his 65th birthday. Proc. Math. Stat., 25, Springer. New York, (2013). 105–130. arXiv:1108.0238
111. A. E. Gatto, E. Pineda, W. Urbina *Riesz Potentials, Bessel Potentials and Fractional Derivatives on Triebel–Lizorkin spaces for the Gaussian Measure.* J. Math. Anal. Appl. 432, No 2, 798–818 (2015). arXiv:1108.0238
112. G. Geiss, A. Toivola *On Fractional Smoothness and L_p – Approximation on the Gaussian Space.* Annals of Prob. (2015), Vol. 43, No. 2, 605–638. DOI: 10.1214/13-AOP884
113. D. Gilbarg, N. S. Trudinger *Elliptic Partial Differential Equations of Second Order.* Springer, 2nd Edition, (1983).
114. A. González Domínguez *Contribución a la teoría de funciones de Hille* Ciencia y Técnica (1941) 475–521.
115. E. Görlich, C. Markett *A convolution structure for Laguerre series*. Indag. Math. (N.S.) 44 (1982), 161–171.
116. P. Graczyk, A. Nowak *A composition formula for squares of Hermite polynomials and its generalizations.* C. R. Acad. Sci. Paris Ser. I Vol 338, 11 (2004) 849–852.
117. P. Graczyk, J. J. Loeb, I. López, A. Nowak, W. Urbina *Higher order Riesz transforms, Fractional differentiation and Sobolev spaces for Laguerre expansions.* J. Math. Pures Appl. (9). 84 (2005), no. 3, 375–405.
118. L. Grafakos *Classical Fourier Analysis* GTM 249-50. 2nd. Edition. Springer (2008)
119. L. Gross *Logarithmic Sobolev inequalities.* Amer. J. Math. 97 (1976) 1061–1083.
120. L. Gross *Logarithmic Sobolev Inequalities and Contractivity Properties of Semigroups.* Lec. Notes in Math. 1593 Springer (1993) 54–88.

121. R. Gundy *Sur les Transformations de Riesz pour le semigroup d'Ornstein–Uhlenbeck.* C.R. Acad. Sci. 303 (série I) (1986) 967–970. MR877182 (88c:60108)
122. C. Gutiérrez *On the Riesz transforms for the Gaussian measure.* J. Func. Anal. 120 (1) (1994) 107–134. MR1262249 (95c:35013)
123. C. Gutiérrez, A. Incognito, J. L. Torrea *Riesz transforms, g-functions and multipliers for the Laguerre semigroup.* Houston J. Math. 27 (2001), no. 3, 579–592.
124. C. Gutiérrez, C. Segovia, J. L. Torrea *On higher Riesz transforms for the Gaussian measure.* J. Fourier Anal. Appl. Vol 2 #6 (1996) 583–596. MR1423529 (98m:42017)
125. C. Gutiérrez, W. Urbina *Estimates for the maximal operator of the Ornstein–Uhlenbeck semigroup.* Proc. Amer. Math. Soc.113 (1991) 99–104. MR1068123 (92a:42023)
126. M. de Guzman *Real variable methods in Fourier Analysis.* North-Holland. Amsterdam (1981).
127. E. Harboure, R. A. Maciás, M. T. Menaŕguez, J. L. Torrea. *Oscillation and variation for the Gaussian Riesz transforms and Poisson integral.* Proc. R. Soc. Edinburgh Sect. A 135 (2005), no. 1, 85–104. 42B10 (44A15 47B38)
128. E. Harboure, J. L. Torrea, B. Viviani *On the search for weighted inequalities for operators related to the Ornstein–Uhlenbeck semigroup.* Math. Ann. 318 (2000), no. 2, 341–353. MR 1795566 (2002b:42029)
129. E. Harboure, J. L. Torrea, B. Viviani *Vector valued extensions of operators related to the Ornstein–Uhlenbeck semigroup.* J. Anal. Math. 91 (2003), 1–29. MR2037400 (2004m:47095)
130. G. H. Hardy, *Summation of a series of polynomials of Laguerre* J. London Math. Soc. 7 (1932), 138–139; addendum, 192.
131. W. Hebisch, G. Mauceri, S. Meda *Holomorphy of spectral multipliers of the Ornstein–Uhlenbeck operator.* J. Funct. Anal. 210 (2004), 101–124. MR2052115 (2005g:47024)
132. H. Helson, *Lectures on invariant subspaces.* New York. Academic Spaces (1964).
133. C. Herz *A Maximal Theorem.* Proc. Amer. Math. Soc. 12 (1961) 229–233.
134. E. Hille, *A class of reciprocal functions.* Ann. Math. 27 (1926) 426–464.
135. E. Hille, *On Laguerre's series. I, II, III*, Proc. Nat. Acad. Sci. U.S.A. 12 (1926), 261–269; 348–352.
136. S. Hofmann, S. Mayboroda *Hardy and BMO spaces associated with divergence form elliptic operators.* Math Ann 344 (2009) 37–116.
137. T. Hytönen, J. van Neerven, M. Veraar, L. Weis *Analysis in Banach Spaces* Volume II: Probabilistic methods and operator theory. Ergebnisse der Mathematik. Vol. 67, Springer-Verlag. (2017)
138. A. Infante *Análisis Armónico para mediadas no doblantes: operadores maximales con respecto a la medida Gaussiana.* Ph.D. thesis. Universidad Autónoma de Madrid (2003).

139. A. Infante *On the Maximal Operator Associated with Certain Rotational Invariant Measures.* Acta Math. Sinica, English Series (2010), Vol. 26, No. 6 993–1004.
140. K. Ito *Malliavin's C^∞ Functionals of a Centered Gaussian System.* IMA Preprint Series 327 (1987), University of Minnesota.
141. D. Jackson *Fourier series and orthogonal polynomials.* No. 6 Carus Mathematical Monographs series. MAA (1941). Reprint from Dover Publications, Mineola, NY (2004).
142. S. Janson *On hypercontractivity for multipliers on orthogonal polynomials.* Ark. Math. 21 (1983), 1, 97–110.
143. S. Janson *Gaussian Hilbert spaces.* Cambridge Tracts in Mathematics. Vol 129. Cambridge University Press, Cambridge (1997)
144. F. John and L. Nirenberg *On functions of bounded mean oscillation.* Comm. Pure Appl. Math 14 (1961) 415–426.
145. A. Jonsson and H. Wallin, *Function spaces on subsets of R^n* , Harwood Acad. Publ., London,(1984).
146. G. E. Karadzhov & M. Milman *Extrapolation theory: new results and applications.* J. Approx. Theory 133 (2005), no. 1 28–99 MR2122269.
147. M. Kemppainen *Admissible decomposition for spectral multipliers on Gaussian L^p.* arXiv:1608.03747
148. M. Kemppainen *An L^1-estimate for certain spectral multipliers associated with the Ornstein Uhlenbeck operator.* J. Fourier Anal. Appl, (to appear) (2016). arXiv:1508.06102
149. A. N. Kolmogorov, S. V. Fomin *Elements of the theory of Functions and Functional Analysis.* Dover Publ. Inc, Mineola, NY (1999)
150. T. H. Koornwinder, *The addition formula for Jacobi polynomials. I. Summary of results.* Indag. Math. (N.S.) 34 (1972), 188–191.
151. A. Korzeniowski *On Logarithmic Sobolev Constant for Diffusion Semigroups.* J. Funct. Anal. 71 (1987), 363–370.
152. A. Korzeniowski, D. Stroock *An example in the theory of hypercontractive semigroups.* Proc. AMS 94 (1985), 87–90.
153. H. L. Krall *Certain differential equations for the Tchebycheff polynomials.* Duke Math. J. **4** (1938), 705–718.
154. P. Kunstmann, A. Ullmann *$\mathscr{R}_s$-Sectorial Operators and generalized Triebel–Lizorkin Spaces.* J. Fourier Anal. Appl, (2014), Vol 20, 1, 135–185.
155. D. Lamberton, B. Lapeyre *Introduction au calcul stochastique appliqué à la finance.* Mathématiques et Applications, Ellipses, Paris, 1991.
156. N. N. Lebedev *Special functions and their applications.* Dover Publications, Inc., New York, (1972).
157. Z. K. Li *Conjugated Jacobi Series and Conjugated Functions.* J. Approx Theory, 86, 179–196 (1996).
158. Z. K. Li *Hardy spaces for Jacobi expansions.* Analysis, 16, 27–49 (1996).
159. L. Liu, P. Sjögren *A characterization of the Gaussian Lipschitz space and sharp estimates for the Ornstein-Uhlenbeck Poisson kernel* Revista Mat. Iberoam. 32 (2016), no. 4, 1189–1210.

160. L. Liu, P. Sjögren, *On the global Gaussian Lipschitz space* Proc. Edinb. Math. Soc. 60 (2017), 707–720.
161. L. Liu, D. Yang *BLO spaces associated with the Ornstein–Uhlenbeck operator.* Bull. Sci. Math. 132 (2008), 633–649. MR2474485 (2010c:42045)
162. L. Liu, D. Yang, *Characterizations of BMO associated with Gauss measures via commutators of local fractional integrals.* Israel J. Math. 180 (2010), 285–315. MR2735067 (2011k:42049)
163. L. Liu, D. Yang, *Boundedness of Littlewood–Paley Operators Associated with Gauss Measures.* J. Inequal. Appl. 180 (2010), 285–315. MR2735067 (2011k:42049)
164. I. López, W. Urbina *Fractional differentiation for the Gaussian measure and applications.* Bull. Sci. Math. 128 (2004), no. 7, 587–603
165. I. López, W. Urbina *On some functions of the Littlewood Paley theory the Gaussian measure and applications.* Rev. Un. Mat. Argentina. Vol 45 (2004), no. 2 , 41–53.
166. I. López, *Introduction to the Besov Spaces and Triebel–Lizorkin Spaces for Hermite and Laguerre expansions and some applications.* J. Math Stat. **1** (2005), no. 3, 172–179.
167. J. Maas, J. van Neerven *On analytic Ornstein–Uhlenbeck semigroups in infinite dimensions.* Arch. Math. 89 (2007), 226–236.
168. J. Maas, J. van Neerven, P. Portal *Non-tangential maximal functions and conical square functions with respect to the gaussian measure.* Publ. Mat. 55 (2011), no. 2, 313–341. MR2839445 arXiv:1003.4092.
169. J. Maas, J. van Neerven, P. Portal *Whitney coverings and the tent spaces $T^{1,q}(\gamma)$ for the Gaussian measure.* Ark. Mat. 50 no.2 (2012) 379–395. arXiv:1002.4911v1
170. R. Macías, *Interpolation theorems on generalized Hardy spaces.* Ph.D. thesis, Washington University, (1974).
171. P. Malliavin *Stochastic Analysis* Springer. Berlin (1997)
172. P. Malliavin, H. Airault *Intégration, Analyse of Fourier, Probabilités, Analyse Gaussienne.* Collection Maitrise de Mathematiques Pures. Mason. Paris (1994).
173. C. Markett *The Product Formula and Convolution Structure associated with generalized Hermite Polynomials.* J. Approx. Theory. 73 (1993), 199–217.
174. G. Mauceri, S. Meda *BMO and H^1 for the Ornstein–Uhlenbeck operator.* J. Funct. Anal. 252 (2007), no. 1, 278–313. MR 2357358 (2008m:42024)
175. G. Mauceri, S. Meda, P. Sjögren *Sharp estimates for the Ornstein–Uhlenbeck operator.* Ann. Sc. Norm. Sup. Pisa, Classe di Scienze, Serie V, 3 (2004), n. 3, 447–480. MR2099246 (2005j:47042)
176. G. Mauceri, S. Meda, P. Sjögren *Endpoint estimates for first-order Riesz transforms associated to the Ornstein–Uhlenbeck operator.*, Studia Math. 224 (2014), no. 2, 153–168.
177. G. Mauceri, S. Meda, P. Sjögren *A Maximal Function Characterization of the Hardy Space for the Gauss Measure.* Proc Amer. Math Soc. 141, 5 (2013) 1679–1692.

178. G. Mauceri, S. Meda, M. Vallarino *Sharp endpoint results for imaginary powers and Riesz transforms on certain noncompact manifolds.* Studia Math. 224 (2014) no. 2, 864–891.
179. G. Mauceri, S. Meda, P. Vallarino *Atomic decomposition and operators on Hardy spaces.* Rev. Un. Mat. Argentina (UMA) vol 50 (2009) 15–22.
180. O. Mazet *Classification des semi-groupes de diffusion sur* $\mathbb{R}$ *asocié a une famille de polynomes orthogonaux*. Séminaire de Probabilité, Lecture Notes in Math, vol. 1655, 40–54. Springer (1997).
181. O. Mazet *A characterization of Markov property for semigroups with invariant measure.* Potential Analysis 16 (2002) 279–287.
182. S. Meda *A general multiplier theorem.* Proc. Amer. Soc. 110 (1996) 639–647 .
183. F. G. Mehler *Ueber die Entwicklung einer Function von beliebig vielen variablen nach Laplaceschen Functionen hoherer Ordnung*, J. Reine Angew. Math. 66 (1866), 161–176.
184. T. Menárguez, S. Pérez, F. Soria *Pointwise and norm estimates for operators associated with the Ornstein–Uhlenbeck semigroup* . C. R. Acad. Sci. Paris, t. 326, Série I, Analyse mathématique/Mathematical Analysis (1998), 25–30.
185. T. Menárguez, S. Pérez, F. Soria *The Mehler maximal function: a geometric proof of weak-type 1.* J. London Math. Soc. 61 (2000), no. 3, 846–856. MR1766109 (2001i:42029)
186. A. Messiah *Quantum Mechanics.* Vol. I. North-Holland, Publishing Co., Amsterdam. (1966). MR0129790 (23:B2826)
187. P. A. Meyer *Quelques résultats analytiques sur le semígroupe d'Ornstein–Uhlenbeck en dimension infinie.* Lectures Notes in Contr. and Inform. Sci. 49 (1983) Springer 201–214.
188. P. A. Meyer *Démonstration probabiliste de certaines inéqualités de Littlewood-Paley.* Lectures Notes in Math 511 (1976) Springer 125–183.
189. P. A. Meyer *Note sur les processus d'Ornstein–Uhlenbeck.* Lectures Notes in Math 920 (1982) Springer 95–133. MR770960 (86i:60150)
190. P. A. Meyer *Transformations de Riesz pour les lois Gaussiennes.* Lectures Notes in Math 1059 (1984) Springer 179–193. MR770960 (86i:60150)
191. M. D. Morán, W. Urbina *Some properties of the Gaussian Hilbert Transform.* Acta Científica Venezolana, Vol 49, No 2 (1998) 102–105.
192. J. Moreno, E. Pineda & W. Urbina *On the Ornstein–Uhlenbeck semigroup on variable* L^p *Gaussian spaces and related operators.* pre-print.
193. B. Muckenhoupt, *Poisson Integrals for Hermite and Laguerre expansion.* Trans. Amer. Math. Soc. 139 (1969) 231–242.
194. B. Muckenhoupt, *Hermite conjugate expansions.* Trans. Amer. Math. Soc. 139 (1969) 243–260. MR0249918 (40:3159)
195. B. Muckenhoupt, *Conjugate functions for Laguerre expansions,* Trans. Amer. Math. Soc. 139 (1969), 231–242.
196. B. Muckenhoupt, *Mean Convergence of Hermite and Laguerre Series I.* Trans. Amer. Math. Soc. 147 (1970) 419–431.
197. B. Muckenhoupt, *Mean Convergence of Hermite and Laguerre Series II.* Trans. Amer. Math. Soc. 147 (1970) 433–460.

198. B. Muckenhoupt *Transplantation theorems and multiplier theorems for Jacobi series.* Memoirs Amer. Math. Soc. 356 Providence (1986).
199. B. Muckenhoupt, E. M.Stein *Classical Expansions.* Trans. Amer. Math. Soc. 147 (1965) 17–92.
200. I. P. Natanson, *Theory of functions of a real variable*,Vol. II, Ungar, New York, (1960)
201. E. Navas, W. Urbina *A transference result of the L^p continuity of the Jacobi Riesz transform to the Gaussian and Laguerre Riesz transforms.* J. Fourier Anal. Appl. 19 (2013), no. 5, 910–942
202. E. Navas, W. Urbina *A transference result of the L^p continuity of the Jacobi Littlewood-Paley $g-$function to the Gaussian and Laguerre Littlewood-Paley $g-$function.* Preprint arXiv:1612.05301.
203. F. L. Nazarov, S. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis.* Algebra i Analiz 8 (1996), no. 5, 32–162 (in Russian) translation in St. Petersburg Math. J. 8 (1997), no. 5, 721–824 .
204. E. Nelson *The free Markoff field.* J. Funct. Anal. 12 (1973) 211–227.
205. E. Nelson *Construction of quantum fields from Markoff fields.* J. Funct. Anal. 12 (1973) 97–112.
206. N. K. Nikol'skii *Treatise on the Shift Operator: Spectral Function Theory.* Springer-Verlag. Berlin (1986).
207. J. van Neerven, P. Portal. *Finite speed of propagation and off-diagonal bounds for Ornstein Uhlenbeck operators in infinite dimensions*. (2015). arXiv:1507.02082.
208. A. Nowak *Heat-diffusion and Poisson integrals for Laguerre and special Hermite expansions on weighted L^p spaces.* Studia Math. 158 (2003), 239–268.
209. A. Nowak, *On Riesz transforms for Laguerre expansions*, J. Funct. Anal. 215 (2004), 217–240.
210. A. Nowak, K. Stempak *L^2-Theory of Riesz Transforms for Orthogonal Expansions.* J. Four. Anal. 121 (6) (2006) 675–711.
211. A. Nowak, K. Stempak *A Symmetrized Conjugacy Scheme for Orthogonal Expansions.* Proc. R. Soc. Edinburgh Sect. A 143 (2013), no. 2, 427–443.
212. A. Nowak, L. Roncal, K. Stempak *Riesz Transforms for the Dunkl Ornstein–Uhlenbeck Operator.* Colloq. Math. 118 (2010), no. 2, 669–684.
213. A. Nowak, P. Sjögren *Riesz Transforms for Jacobi expansions.* J. Anal Math. 104 (2008) 341–369.
214. A. Nowak, P. Sjögren *Calderón–Zygmund operators related to Jacobi expansions.* J. Fourier Anal. Appl. 18 (2012), no. 4, 717–749.
215. A. Nowak, P. Sjögren *Sharp estimates of the Jacobi heat kernel.* Studia Math. 218 (2013), 219–244.
216. A. Nowak, P. Sjögren, T. Z. Szarek *Maximal operators of exotic and non-exotic Laguerre and other semigroups* Adv. Math. 318 (2017), 307–354.
217. A. Nowak, P. Sjögren, T. Z. Szarek *Analysis related to all admissible type parameter in the Jacobi setting.* Constr Approx. 41 (2015), 185–218.

218. D. Nualart *The Malliavin Calculus and Related Topics.* 2nd edition. Probability and its Applications (New York). Springer, Berlin, 2006
219. J. M. Pearson *An Elementary Proof of Gross' Logarithmic Sobolev Inequality.* Bull. London Math. Soc. 25 (1993) 463–466.
220. S. Pérez *Estimaciones puntuales y en norma para operadores relacionados con el semigrupo de Ornstein–Uhlenbeck.* Ph.D. thesis. Universidad Autónoma de Madrid (1996)
221. S. Pérez *The local part and the strong type for operators related to the Gauss measure.* J. Geom. Anal. 11 (2001), no. 3, 491–507. MR1857854 (2002h:42027)
222. S. Pérez *Boundedness of Littlewood–Paley g-functions of higher order associated with the Ornstein–Uhlenbeck semigroup.* Indiana Univ. Math. J. 50 (2001), no. 2, 1003–1014. MR1871397 (2002h:42026)
223. S. Pérez, F. Soria *Operators associated with the Ornstein–Uhlenbeck semigroup.* J. London Math. Soc. 61 (2000), 857–871. MR1766110 (2001i:42030)
224. E. Pineda *Tópicos en Análisis Armónico Gaussiano: Comportamiento en la frontera y espacios de funciones para la medida Gaussiana..* PhD thesis, Facultad de Ciencias, UCV, Caracas. (2009).
225. E. Pineda, W. Urbina *Non Tangential convergence for the Ornstein–Uhlenbeck semigroup.* Divulgaciones Matemáticas 16, no. 1 (2007), 107–124. MR 2587011 (2010m:60121)
226. E. Pineda, W. Urbina *Some results on Gaussian Besov–Lipschitz spaces and Gaussian Triebel–Lizorkin spaces.* Journal of Approximation Theory, Volume 161, Issue 2, December (2009), 529–564.
227. G. Pisier *Riesz Transform: a simpler analytic proof of P. A. Meyer inequality. Séminaire de Probabilités, XXII* Lectures Notes in Math 1321. Springer (1988) 485–501. MR960544 (89m:60178)
228. E. Poiani *Mean Césaro summability of Laguerre and Hermite Series.* Trans. Amer. Math. Soc. 173 (1972) 1–31.
229. H. Pollard *Mean Convergence of Orthogonal Series I.* Trans. Amer. Math. Soc. 62 (1947) 433–460.
230. H. Pollard *Mean Convergence of Orthogonal Series II.* Trans. Amer. Math. Soc. 63 (1948) 355–367.
231. P. Portal *Maximal and quadratic Gaussian Hardy spaces.* 2012. Rev. Mat. Iberoam. 30 (2014), no. 1, 79–108 arXiv:1203.1998.
232. Personal communication 2018.
233. G. C. Rotta *An "Alternierende Verfahren" for general positive operators* Bull Amer. Math. Soc. 68 (1962) 95–102.
234. M. Rosenblum *Generalized Hermite polynomials and the Bose like oscillator calculus.* In: Non-self-adjoint operators and related topics (Beer Sheva, 1992), 369–396, Oper. Theory Adv. Appl., 73, Birkhauser, Basel, (1994).
235. O. Rothaus, *Logarithmic Sobolev inequalities and the spectrum of Sturm–Lioville operators.* J. Funct. Anal. 39 (1980) 42–56.
236. E. Sasso *Spectral Multipliers of Laplace Transform type for the Laguerre Operator.* Bull. Austral. Math. Soc. Vol. 69 (2004) 255–266.

237. E. Sasso *Weak Type Estimates for Riesz–Laguerre Transforms* Bull. Austral. Math. Soc. Vol. 75 (2007) 397–408.
238. E. Sasso, *Maximal operators for the holomorphic Laguerre semigroup,* Math. Scand. 97 (2005), 235–265.
239. S. Samko, A. Kilbas, O. Marichev *Singular integrals and differentiability properties of functions.* Gordon and Breach Science Publishers, Philadelphia, 199
240. R. L. Sapiro, *Special functions related to representations of the group $SU(n)$ of class I with respect to $SU(n-1)(n \geq 3)$.* Izv Vyssh. Uchebn. Zaved. Mathematika. no. 4 71 (1968) 97–107.
241. Sarason, D. *Generalized interpolation in H^∞.* Trans. Amer. Math. Soc. 127 (1967) 179–203.
242. O. V. Sarmanov, Z. N. Bratoeva *A generalized symmetric gamma correlation.* Dokl. Akad. Nauk SSSR. 179, (1968) 1276–1278; Soviet Math. Dokl., 9 (1968), 547–550.
243. O. V. Sarmanov *Probabilistic properties of bilinear expansions of Hermite polynomials.* Teor. Verojatnost i Primenen. 12, (1967) 520–531.
244. R. Scotto *Weak type estimates for Singular Integral operators associated with the Ornstein–Uhlenbeck process.* Ph.D. thesis. University of Minnesota (1993)
245. R. Scotto *The Littlewood–Paley g-function associated to the Ornstein–Uhlenbeck process.* Pre-print.
246. P. Sjögren *A remark on the maximal function for measures in $\mathbb{R}^n$.* Amer. J. Math. 105 (1983) 1231–1233. MR0714775 (86a:28003)
247. P. Sjögren *On the maximal function for the Mehler kernel* in: Mauceri G., Ricci F., Weiss G. (eds) Harmonic analysis (Cortona, 1982), Lectures Notes in Math vol 992. Springer, Berlin (1983) 73–82. MR729346 (85j:35031)
248. P. Sjögren *Operators associated with the Hermite semigroup – a survey.* J. Fourier Anal. Appl. 3 (1997), 813–823.
249. P. Sjögren *Maximal operators related to the Ornstein–Uhlenbeck semigroup with time complex parameter.* J. Funct. Anal. 237 (2006), no. 2, 675–688.
250. P. Sjögren, F. Soria *Sharp estimates for the non-centered maximal operator associated to Gaussian and other radial measures.* Adv. Math. 181 (2004), no. 2, 251–275.
251. F. Soria, G. Weiss *A remark on Singular Integrals and power weights.* Indiana Uni. Math 43 (I) (1994) 187–204.
252. E. M. Stein *Singular Integrals and Differentiability Properties of Functions.* Princeton Univ. Press. Princeton (1970) .
253. E. M. Stein *Topics in Harmonic Analysis related to the Littlewood–Paley Theory.* Princeton Univ. Press. Princeton (1970). MR0252961 (40:6176)
254. E. M. Stein *Harmonic Analysis. Real variable methods, orthogonality and oscillatory integrals.* Princeton Math. Series No. 43, Princeton, N.J., (1993). MR1232192 (95c:42002)
255. E. M. Stein *The Characterization of Functions Arising as Potentials I*, Bull. Amer. Math. Soc. 97 (1961) 102–104. II (ibid) 68 (1962) 577–582.
256. E. M. Stein, G. Weiss *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton Univ. Press. Princeton (1975).

257. E. M. Stein, G. Weiss *On the convergence of Poisson Integrals.* Trans. Amer. Math. Soc.140 (1969) 35–54.
258. K. Stempak, *Conjugacy for Jacobi expansions.* Studia Sci. Math. Hung. 44 (2007), 117–130.
259. K. Stempak, J. L. Torrea *Poisson Integrals and Riesz Transforms for Hermite function expansions with weights.* J. Funct. Anal. 202 (2003) 443–472.
260. D. W. Stroock *The Malliavin calculus, a functional analytic approach.* J. Funct. Anal. 44, 212–257 (1981) MR 84d:60076.
261. H. Sugita *On characterization of the Sobolev spaces over abstract Wiener spaces.* J. Math Kioto Univ. 25-5 (1985) 717–725.
262. G. Szegő *Orthogonal Polynomials.* Colloq. Publ. 23. Amer. Math. Soc. Providence (1959).
263. S. J. Taylor, *Introduction to Measure and Integration.* Cambridge University Press, London (1966).
264. J. Teuwen *Gaussian Hardy spaces.* M.Sc. Thesis. Delft University of Technology (2012).
265. J. Teuwen *A note on the Gaussian maximal function.* Indag. Math. (N.S.) 26, no. 1, (2015) 106–112.
266. J. Teuwen *On the Integral Kernel of Derivatives of the Ornstein–Uhlenbeck Semigroup.* Infin. Dimens. Anal. Quantum Probab. Relat. Top. 19 (2016), no. 4, 1650030, 13 pp. arXiv:1509.05702v1
267. S. Thangavelu *Summability of Hermite Expansions I.* Trans. Amer. Math. Soc. 314 (1989) 119–142.
268. S. Thangavelu *Summability of Hermite Expansions II.* Trans. Amer. Math. Soc. 314 (1989) 143–170.
269. S. Thangavelu *On conjugate Poisson integral and Riesz transform for the Hermite and special Hermite expansions.* Colloq. Math. 64 (1993), no. 1, 103–113.
270. S. Thangavelu *Lectures on Hermite and Laguerre Expansions.* Princeton Univ. Press. Princeton (1993).
271. H. Triebel *Interpolation theory, function spaces differential operators.* North Holland (1978).
272. H. Triebel *Theory of function spaces.* Birkhäuser Verlag, Basel (1983).
273. H. Triebel *Theory of function spaces II.* Birkhäuser Verlag, Basel (1992).
274. X. Tolsa *BMO,* H^1*, and Calderón–Zygmund operators for non doubling measures*. Math. Ann. 319 (2001), 89–149.
275. A. Torchinski *Real variable methods in Harmonic Analysis.* Acad. Press. Pure and Applied Math 123. San Diego (1986).
276. J. L. Torrea *Semigroups, a tool to develop Harmonic Analysis for general Laplacians* Advanced Courses of Mathematical Analysis VI: Proceedings of the Sixth International School. F. J. Martín-Reyes, P. Ortega Salvador, M. Lorente, C. González Eds. World Scientific, (2016)
277. N. Tzvetkov *Construction of a Gibbs measure associated to the Benjamin–Ono equation.* Probab. Theory Related Fields 146 (2010), no. 3–4, 481–514.
278. W. Urbina *Singular Integrals with respect to the Gaussian measure.* Scuola Normale Superiore di Pisa. Classe di Science. Serie IV Vol XVII, 4 (1990) 531–567. MR1093708 (92d:42010)

279. W. Urbina *La Transformada de Riesz para la Medida Gaussiana.* Notas de Matemáticas 115. Departamento de Matemática, ULA, Mérida (1991).
280. W. Urbina *Elementos de Análisis Armónico Gaussiano.* Trabajo de Ascenso. Facultad de Ciencias. UCV. (1992).
281. W. Urbina *Análisis Armónico Gaussiano: una visión panorámica.* Facultad de Ciencias. UCV. (1998).
282. W. Urbina *Análisis Armónico Gaussiano: una visión panorámica.* Boletín de la AMV Vol V No (1998) 143–184.
283. W. Urbina *Sobre la propiedad hipercontractiva.* Divulg. Mat. 15 (2007), no. 2, 235–252.
284. W. Urbina *Semigroups of Operators for Classical Orthogonal Polynomials and Functional Inequalities*. CIMPA Workshop Mérida, Venezuela, French Mathematical Society (SMF), Séminaires et Congrès 25 (2012).
285. A. M. Vargas *On the maximal function for rotation invariant measures in* R^n. Studia Math. 110 (1) (1994), 9–17.
286. N. Varopoulus *Hardy–Littlewood theory for semigroups.* J. Funct. Anal. 63 (1985) 240–260.
287. J. Verdera, *The fall of the doubling condition in Calderón–Zygmund*. Publ. Mat., (2002) 275–292.
288. S. Watanabe *Lecture on Stochastic Differential Equations and Malliavin Calculus.* Tata Inst. Fund. Res., Mumbai. Springer (1984).
289. G. N. Watson *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge Univ. Press. Cambridge (1980).
290. G. N. Watson, *The product of two hypergeometric functions*, Proc. London Math. Soc. 20 (1922), 189–195.
291. G.N. Watson, *Notes on generating functions of polynomials: (1) Laguerre polynomials* J. London Math. Soc. 8 (1933), 189–192.
292. F. Weissler *Two-point Inequalities, the Hermite Semigroup, and the Gauss–Weierstrass Semigroup.* J. Funct. Anal. 32 (1979) 102–121.
293. R. Wheeden, C. Segovia *On certain fractional area integrals*. J. Math. Mech. 19 (1969) 247–262.
294. R. Wheeden, A. Zygmund *Measure and Integral. An Introduction to Real Analysis*. Marcel Dekker Inc. New York (1977)
295. D. V. Widder *The Heat Equation*. Academic Press, N.Y. (1976)
296. B. Wróbel *Dimension Free* L^p *Estimates for Single Riesz Transforms via an* H^∞ *Joint Functional Calculus*. J. Func. Anal. 267 (9) (2014) 3332–3350 MR3261112
297. B. Wróbel *Dimension Free* L^p *Estimates for Vectors Riesz Transforms Associated with Orthogonal Expansions*. Anal PDE no 3. (2018) 745–773. MR3261112
298. K. Yosida *Functional Analysis*. Springer. Berlin (1965).
299. A. Zygmund *Trigonometric Series.* 2nd ed. Cambridge Univ. Press. Cambridge (1959).

Glossary of Symbols

- C will always denote a constant, not necessarily the same in each occurrence.
- $\delta_{i,j}$ is the Dirac delta,

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

- $\operatorname{sgn} a$ is the sign of $a \in \mathbb{R}$

$$\operatorname{sgn}\, a = \begin{cases} 1, & \text{if } a \geq 0, \\ -1, & \text{if } a < 0. \end{cases}$$

- $a \wedge b = \min\{a, b\}$.
- $a \vee b = \max\{a, b\}$.
- $[a]$ is the integer part of a, i.e., the largest integer not greater that $a \in \mathbb{R}$.
- $A \subset B$ means A is a subset of B.
- A^c denotes the complement of A

$$A^c = \{\zeta : \zeta \notin A\}.$$

- $A \cup B$ is the union of A and B,

$$A \cup B = \{\zeta : \zeta \in A \text{ or } \zeta \in B\}.$$

- $A \cap B$ is the intersection of A and B,

$$A \cap B = \{\zeta : \zeta \in A \text{ and } \zeta \in B\}.$$

- $A \backslash B$ is the set difference,

$$A \backslash B = \{\zeta : \zeta \in A \text{ and } \zeta \notin B\}.$$

- χ_E denotes the characteristic function of the set E,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

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- $\mathbb{N}$ is the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \cdots\}.$$

- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of non-negative integers.
- $\mathbb{R}^d$ is the d-dimensional real space,

$$\mathbb{R}^d = \Big\{x = (x_1, \cdots, x_d) : x_i \in \mathbb{R},\ i = 1, \cdots, d\Big\}.$$

- Given $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$,

$$|x| = \left(\sum_{i=1}^{d} x_i^2\right)^{1/2}$$

  is its Euclidean norm in $\mathbb{R}^d$.
- Given $x, y \in \mathbb{R}^d$, $\langle x, y\rangle$ is the inner product in $\mathbb{R}^d$,

$$\langle x, y\rangle = \left(\sum_{i=1}^{d} x_i y_i\right)^{1/2}.$$

- $|A|$ is the Lebesgue measure of the Borel set A in $\mathbb{R}^d$, $A \in \mathscr{B}(\mathbb{R}^d)$.
- γ_d is the Gaussian measure in $\mathbb{R}^d$,

$$\gamma_d(dx) = \frac{1}{\pi^{d/2}} e^{-|x|^2} dx.$$

- $\{H_n\}_{n\geq 0}$ are the Hermite orthogonal polynomials in one variable.
- $\{h_n\}_{n\geq 0}$ are the Hermite orthonormal polynomials in one variable

$$h_n(x) = \frac{H_n(x)}{(2^n n!)^{1/2}}.$$

- $\{\mathbf{H}_\nu\}_\nu$ are the Hermite orthogonal polynomials in d variables,

$$\mathbf{H}_\nu(x) = \prod_{i=1}^{d} H_{\nu_i}(x_i),$$

  where $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$, and $H_{\nu_i}(x_i)$ is the Hermite polynomial of degree $\nu_i \geq 0$ in the variable x_i.
- $\{\mathbf{h}_\nu\}_\nu$ are the Hermite orthonormal polynomials in d variables,

$$\mathbf{h}_\nu(x) = \prod_{i=1}^{d} h_{\nu_i}(x_i),$$

  where $h_{\nu_i}(x_i)$ is the normalized Hermite polynomial of degree $\nu_i \geq 0$ in the variable x_i.

- $S_r^{d-1} = \{x \in \mathbb{R}^d : |x| = r\}$ is the hypersphere of radius r in $\mathbb{R}^d$, and in particular, $S^{d-1} = S_1^{d-1}$ is the unit hypersphere.
- $d\sigma$ is the area measure on S^{d-1} and ω_{d-1} is the (surface) measure of S^{d-1}.
- $\beta = (\beta_1, \cdots, \beta_d) \in \mathbb{N}^d$ or $\nu = (\nu_1, \cdots, \nu_d) \in \mathbb{N}^d$ is called a multi-index.
- Λ_k the set of multi-indexes with k coordinates, i.e., $\beta \in \Lambda_k$ if $\beta = (\beta_1, \cdots, \beta_k)$.
- $|\beta| = \sum_{i=1}^d \beta_i$, $\beta! = \prod_{i=1}^d \beta_i!$ and $x^\beta = \prod_{i=1}^d x_i^{\beta_1}$, for $\beta \in \mathbb{N}^d$ multi-index.
- ∂_i is the standard partial derivative, $\partial_i = \frac{\partial}{\partial x_i}$.
- $\nabla_x = (\partial_1, \partial_2, \ldots, \partial_d)$ is the gradient operator.
- $\nabla = \left(\frac{\partial}{\partial t}, \frac{1}{\sqrt{2}}\nabla_x\right)$ is the (total) gradient.
- $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}$ is the partial derivative of order $|\beta|$.
- $\nabla_x^k = (\partial^\beta)_{|\beta|=k} = (\partial_1^{\beta_1} \cdots \partial_d^{\beta_d})_{|\beta|=k}$ is the gradient operator of order k.
- ∂_γ^i the Gaussian derivative

$$\partial_\gamma^I = \frac{1}{\sqrt{2}}\partial_i.$$

- ∂_γ^β is the Gaussian partial derivative of order $|\beta|$

$$\partial_\gamma^\beta = \frac{1}{2^{|\beta|/2}}\partial_1^{\beta_1} \cdots \partial_d^{\beta_d}.$$

- $(\partial_\gamma^i)^*$ is the alternative (adjoint) Gaussian derivative,

$$(\partial_\gamma^i)^* = \sqrt{2}x_i I_d - \frac{1}{\sqrt{2}}\partial_i.$$

- $(\partial_\gamma^\beta)^* = \frac{1}{2^{|\beta|}}(\partial_1^{\beta_1})^* \cdots (\partial_d^{\beta_d})^*$ is the Gaussian alternative partial derivative of order $|\beta|$.
- $m(x)$ is the admissibility function,

$$m(x) = 1 \wedge \frac{1}{|x|}.$$

- $\mathscr{B}_{a,b}$ and $\mathscr{B}_a$ are the families of admissible (or hyperbolic) balls,

$$\mathscr{B}_{a,b} = \left\{B(x,r) : x \in \mathbb{R}^d,\ 0 < r < a \wedge \frac{b}{|x|}\right\},$$

and

$$\mathscr{B}_a = \left\{B(x,r) : x \in \mathbb{R}^d,\ 0 < r < a\left(1 \wedge \frac{1}{|x|}\right)\right\} = \left\{B(x,r) : x \in \mathbb{R}^d,\ 0 < r < am(x)\right\}.$$

- $B_h(x) = \left\{y \in \mathbb{R}^d : |y - x| < C_d m(x)\right\}$ an admissible (or hyperbolic) ball with its center at $x \in \mathbb{R}^d$ and radius $C_d m(x)$ for certain constant C_d dependent only on d, usually $C_d = d$ or $2d$.

- $\mathscr{P}(\mathbb{R}^d)$ is the set of all polynomials with real coefficients in d-variables, $d \geq 1$.
- $C_b^2(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$, with bounded derivatives up to second order.
- $C_0(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$, with compact support.
- $C_0^\infty(\mathbb{R}^d)$ the space of smooth functions on $\mathbb{R}^d$, with compact support.
- $\mathscr{S}(\mathbb{R}^d)$ is the space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, also called the Schwartz space or the space of test functions.
- $L^p(\mathbb{R}^d)$, $1 < p < \infty$, is the classical Lebesgue space: the space of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, such that

$$\int_{\mathbb{R}^d} |f(x)|^p \, dx < \infty.$$

  with norm

$$\|f\|_p = \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{1/p}.$$

- $L^p(\gamma_d)$, $1 < p < \infty$ is the Gaussian Lebesgue space: the space of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$, such that

$$\int_{\mathbb{R}^d} |f(x)|^p \gamma_d(dx) < \infty.$$

  with norm

$$\|f\|_{p,\gamma} = \left(\int_{\mathbb{R}^d} |f(x)|^p \gamma_d(dx) \right)^{1/p}.$$

- $\langle f, g \rangle_\gamma$ is the internal product in $L^2(\gamma_d)$,

$$\langle f, g \rangle_\gamma = \int_{\mathbb{R}^d} f(x) g(x) \gamma_d(dx).$$

- $\widehat{f}_\gamma(\nu)$ is the ν-th Fourier–Hermite coefficient of a function $f \in L^2(\gamma_d)$ with respect to the Hermite polynomial $\mathbf{h}_\nu$,

$$\widehat{f}_\gamma(\nu) = \langle f, \mathbf{h}_\nu \rangle_\gamma.$$

- The Fourier–Hermite expansion of a function $f \in L^2(\gamma_d)$ is given by

$$f = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu.$$

- $\mathscr{C}_k$ is the closed subspace of $L^2(\gamma_d)$ generated by $\{\mathbf{h}_\nu : |\nu| = k\}$,

$$\mathscr{C}_k = \overline{\operatorname{span}\left(\{\mathbf{h}_\nu : |\nu| = k\}\right)}^{L^2(\gamma_d)}.$$

- $\mathbf{J}_k$ is the orthogonal projection of $L^2(\gamma_d)$ onto $\mathscr{C}_k$,

$$\mathbf{J}_k f = \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu,$$

so

$$f = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \langle f, \mathbf{h}_\nu \rangle_\gamma \mathbf{h}_\nu = \sum_{k=0}^{\infty} \sum_{|\nu|=k} \mathbf{J}_k f,$$

which implies the Wiener chaos or Ito–Wiener decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{k=0}^{\infty} \mathscr{C}_k.$$

- Δ_x is the Laplace differential operator,

$$\Delta_x = \sum_{i=1}^{d} \partial_i^2.$$

- $\{\mathscr{T}_t\}_{t\geq 0}$ is the heat semigroup,

$$\mathscr{T}_t f(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{\frac{-|x-y|^2}{4t}} f(y)\, dy.$$

- $\Gamma_a(x)$ is the (classical) cone with vertex at $x \in \mathbb{R}^d$, and aperture $a > 0$

$$\Gamma_a(x) = \Big\{ (y,t) \in \mathbb{R}^{d+1}_+ : |x-y| < at \Big\}.$$

- $\mathscr{T}_a^*$ is the non-tangential maximal function of the heat semigroup,

$$\mathscr{T}_a^* f(x) = \sup_{(y,t)\in\Gamma_a(x)} |\mathscr{T}_{t^2} f(y)|.$$

- $\mathscr{S}_a$ is the conical square function of the heat semigroup,

$$\mathscr{S}_a f(x) = \frac{1}{|B(y,t)|} \Big(\int_{\Gamma_a(x)} \Big| t\, \mathscr{T}_{t^2} f(y) \Big|^2 dy \frac{dt}{t} \Big)^{\frac{1}{2}}.$$

- L is the Ornstein–Uhlenbeck differential operator,

$$L = \frac{1}{2}\Delta - \langle x, \nabla_x \rangle = \sum_{i=1}^{d} [\frac{1}{2}\partial_i^2 - x_i \partial_i].$$

- $\overline{L}$ is the alternative Ornstein–Uhlenbeck differential operator,

$$\overline{L} = L - I_d = -\frac{1}{2}\Delta + \langle x, \nabla_x \rangle - I_d = \sum_{i=1}^{d} [-\frac{1}{2}\partial_i^2 - x_i \partial_i].$$

- $\{T_t\}_{t\geq 0}$ is the Ornstein–Uhlenbeck semigroup,

$$\sum_{\nu}^{\infty} e^{-t|\nu|}\langle f, \mathbf{h}_\nu\rangle_{\gamma_d}\mathbf{h}_\nu = \frac{1}{(1-e^{-2t})^{d/2}}\int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|y|^2+|x|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy).$$

- $M_t(x,y)$ is Mehler's kernel

$$M_t(x,y) = \frac{1}{(1-e^{-2t})^{d/2}} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} = \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}}.$$

- $\mathbf{E}_p$ is the Epperson region

$$\mathbf{E}_p := \{z = x+iy : |\sin y| \leq \tan\phi_p \sinh x\}, \quad \phi_p = \arccos|2/p-1|.$$

- T^* is the Ornstein–Uhlenbeck maximal function,

$$T^* f(x) = \sup_{t>0} |T_t f(x)|.$$

- $\mathbf{T}^*$ is the multiparametric Ornstein–Uhlenbeck maximal function

$$\mathbf{T}^* f(x) = \sup_{\substack{0<t_1<\infty \\ 0<t_2<\infty \\ \cdots \\ 0<t_d<\infty}} \left[\frac{1}{\pi^{d/2}}\prod_{i=1}^{d}\frac{1}{(1-e^{-2t_i})^{1/2}}\int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t_i}x|^2}{1-e^{-2t_i}}} f(y)dy\right].$$

- Γ_p^*, is the maximal function for the holomorphic Ornstein–Uhlenbeck semigroup $\{T_z : \mathrm{Re}\, z \geq 0\}$

$$\Gamma_p^* f(z) = \sup_{z\in\mathbf{E}_p} |T_z f(x)|.$$

- $\mathscr{K}^*(x,y)$ is Mehler's maximal kernel,

$$\mathscr{K}^*(x,y) = \sup_{t>0} M_t(x,y) = \sup_{0<r<1} \frac{1}{\pi^{\frac{d}{2}}(1-r^2)^{\frac{d}{2}}} e^{-\frac{|y-rx|^2}{1-r^2}}.$$

- $\mathbf{T}^*$ is the multiparametric Gaussian maximal operator

$$\mathbf{T}^* f(x) = sup_{\substack{0<t_1<\infty \\ 0<t_2<\infty \\ \cdots \\ 0<t_d<\infty}} \left[\frac{1}{\pi^{d/2}}\prod_{i=1}^{d}\frac{1}{(1-e^{-2t_i})^{1/2}}\int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t_i}x|^2}{1-e^{-2t_i}}} f(y)dy\right].$$

- $\mathfrak{T}^* f$ is the maximal Mehler's transform

$$\mathfrak{T}^* f(x) = \int_{\mathbb{R}^d} \mathscr{K}^*(x,y) f(y)dy.$$

- $\overline{\mathscr{K}}(x,y)$ is the maximal Gaussian kernel,

$$\overline{\mathscr{K}}(x,y) = \begin{cases} e^{-|y|^2}, & \text{if } \langle x,y\rangle \leq 0 \\ \left(\frac{|x+y|}{|x-y|}\right)^{d/2} e^{-\frac{|y|^2-|x|^2}{2}} e^{-\frac{|x-y||x+y|}{2}}, & \text{if } \langle x,y\rangle > 0. \end{cases}$$

- $\overline{T}f$ is the maximal Gaussian operator,

$$\overline{T}f(x) = \int_{\mathbb{R}^d} \overline{\mathscr{K}}(x,y)f(y)dy.$$

- $\overline{\mathscr{K}}_m(x,y)$ is the m-modified maximal Gaussian kernel,

$$\overline{\mathscr{K}}_m(x,y) = \begin{cases} (|x+y||x-y|)^{\frac{m-2}{2}}\overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle \leq 0 \\ (|x+y||x-y|)^{\frac{m-2}{2}}\left(|x+y||x-y|)^{\frac{1}{2}} \frac{|x|\,|y|}{|x|^2+|y|^2}+1\right)\overline{\mathscr{K}}(x,y) & \text{if } \langle x,y\rangle \geq 0 \end{cases}$$

- $\overline{T}_m f$ is the m-modified maximal Gaussian operator,

$$\overline{T}_m f(x) = \int_{\mathbb{R}^d} \overline{\mathscr{K}}_m(x,y)f(y)dy.$$

- $\Gamma_\gamma^{A,a}(x) = \left\{(y,t) \in \mathbb{R}_+^{d+1} : \ |y-x| < At,\ t < a(1 \wedge \frac{1}{|x|}) = am(x)\right\}$, and $\Gamma_\gamma^{a}(x) = \left\{(y,t) \in \mathbb{R}_+^{d+1} : \ |y-x| < t,\ t < a(1 \wedge \frac{1}{|x|}) = am(x)\right\}$ are the Gaussian or admissible cones.
- $\mathscr{T}_\gamma^*(A,a)$ is the non-tangential maximal function associated with the Ornstein–Uhlenbeck semigroup,

$$\mathscr{T}_\gamma^*(A,a)f(x) = \sup_{(y,t)\in\Gamma_\gamma^{A,a}(x)} |\ T_{t^2}f(y)\ | .$$

- $\Upsilon_\gamma^*(A,a)$ is the "averaged version" of the non-tangential maximal function defined as,

$$\Upsilon_\gamma^*(A,a)f(x) = \sup_{(y,t)\in\Gamma_\gamma^{A,a}(x)} \left(\frac{1}{\gamma_d(B(y,At))}\int_{B(y,At)} |T_{t^2}f(z)|^2\gamma_d(dz)\right)^{1/2}.$$

- $\{T_t^{(\kappa)}\}_{t\geq 0}$ is the translated Ornstein–Uhlenbeck semigroup

$$T_t^{(\kappa)} = e^{-\kappa t}T_t.$$

- $\{\mathscr{P}\}_{t\geq 0}$ is the Poisson semigroup,

$$\mathscr{P}_t f(x) = \frac{\Gamma(\frac{d+1}{2})}{\pi^{(d+1)/2}}\int_{\mathbb{R}^d} \frac{1}{(|x-y|^2+t^2)^{(d+1)/2}} f(y)\,dy.$$

- $\{P_t\}_{t\geq 0}$ is the Poisson–Hermite semigroup,

$$P_t f(x) = \frac{1}{\pi^{(n+1)/2}} \int_{\mathbb{R}^d} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{\exp(\frac{-|y-e^{-t^2/4u}x|^2}{1-e^{-t^2/2u}})}{(1-e^{-t^2/2u})^{d/2}} du f(y) dy$$
$$= \frac{1}{2\pi^{(n+1)/2}} \int_{\mathbb{R}^d} \int_0^1 t \frac{\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{\exp(\frac{-|y-rx|^2}{1-r^2})}{(1-r^2)^{d/2}} \frac{dr}{r} f(y) dy.$$

- $p(t,x,y)$ is the Poisson–Hermite kernel,

$$p(t,x,y) = \frac{1}{\pi^{(n+1)/2}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} \frac{\exp(\frac{-|y-e^{-t^2/4u}x|^2}{1-e^{-t^2/2u}})}{(1-e^{-t^2/2u})^{d/2}} du$$
$$= \frac{1}{2\pi^{(n+1)/2}} \int_0^1 t \frac{\exp(t^2/4\log r)}{(-\log r)^{3/2}} \frac{\exp(\frac{-|y-rx|^2}{1-r^2})}{(1-r^2)^{d/2}} \frac{dr}{r}.$$

- P^* is the maximal function of the Poisson–Hermite semigroup or Poisson–Hermite maximal function,

$$P^* f(x) = \sup_{t>0} |P_t f(x)|.$$

- $\mathscr{P}^*_\gamma(A,a)$ is the non-tangential Poisson–Hermite maximal function,

$$\mathscr{P}^*_\gamma(A,a) f(x) = \sup_{(y,t)\in\Gamma^{A,a}_\gamma(x)} |P_t f(y)|.$$

- $\{P_t^{(\kappa)}\}_{t\geq 0}$ is the translated Poisson–Hermite semigroup, that is to say, the subordinated semigroup to $\{T_t^{(\kappa)}\}_{t\geq 0}$.
- Mf is the centered Hardy–Littlewood maximal function with respect to the Lebesgue measure on balls,

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy.$$

- $\mathscr{M}_\gamma f$ is centered Gaussian Hardy–Littlewood maximal function on balls,

$$\mathscr{M}_\gamma f(x) = \sup_{r>0} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)| \gamma_d(dy).$$

- M^Q is the centered Hardy–Littlewood maximal function with respect to the Lebesgue measure on cubes,

$$M^Q f(x) = \sup_{Q(x)} \frac{1}{|Q(x)|} \int_{Q(x)} |f(y)| dy.$$

- $\mathscr{M}_{\gamma}^{Q}$ is the centered Gaussian Hardy–Littlewood maximal function on cubes,

$$\mathscr{M}_{\gamma}^{Q} f(x) = \sup_{Q(x)} \frac{1}{\gamma_d(Q(x))} \int_{Q(x)} |f(y)|\gamma_d(dy).$$

- $\tilde{M}f$ is the non-centered Hardy–Littlewood maximal function with respect to the Lebesgue measure,

$$\tilde{M}f(x) = \sup_{r>0, x\in B(z,r)} \frac{1}{|B(z,r)|} \int_{B(x,r)} |f(y)|dy.$$

- $M^e f$ is the spherical maximal function

$$M^e f(h) = \sup_{R>0} \frac{1}{\sigma(|z'-h| \leq R)} \int_{|z'-h|\leq R} |f(z')|\, d\sigma(z'), \qquad h \in S^{d-1}.$$

- $\tilde{\mathscr{M}}_{\gamma} f$ is the non-centered Gaussian Hardy–Littlewood maximal function,

$$\tilde{\mathscr{M}}_{\gamma} f(x) = \sup_{r>0, x\in B(z,r)} \frac{1}{\gamma_d(B(z,r))} \int_{B(z,r)} |f(y)|\gamma_d(dy).$$

- $M_{a,b}f$ is the (a,b)-truncated centered Hardy–Littlewood maximal function,

$$M_{a,b}f(x) = \sup_{0<r<a\wedge \frac{b}{|x|}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

- $\mathscr{M}_{\gamma}^{a,b} f$ is the Gaussian (a,b)-truncated centered Hardy–Littlewood maximal function

$$\mathscr{M}_{\gamma}^{a,b} f(x) = \sup_{0<r<a\wedge \frac{b}{|x|}} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)|\gamma_d(dy).$$

- $\mathscr{M}_{\Phi} f$ is the generalized Gaussian maximal function,

$$\mathscr{M}_{\Phi} f(x) = \sup_{0<r<1} \frac{1}{\gamma_d((1+\delta)B(\frac{x}{r}, \frac{|x|}{r}(1-r))} \int_{\mathbb{R}^d} \Phi\Big(\frac{|ry-x|}{\sqrt{1-r^2}}\Big) |f(y)|\, \gamma_d(dy),$$

where $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a non-increasing function, such that

$$S = \sum_{v\geq 1} \Phi\Big(\frac{1}{2}(v-1)\Big) v^{2d} < \infty \text{ and } \delta = \delta_{r,x} = \frac{r}{|x|(1-r)} \min\Big\{\frac{1}{|x|}, \sqrt{1-r}\Big\}.$$

- g_{γ} is the Gaussian Littlewood–Paley–Stein g-function,

$$g_{\gamma}(f)(x) = \Big(\int_0^{\infty} |t\nabla P_t(f)(x)|^2 \frac{dt}{t}\Big)^{1/2}.$$

- $g_{t,\gamma}$ is the time Gaussian Littlewood–Paley g function,

$$g_{t,\gamma}(f)(x) = \left(\int_0^\infty \left| t \frac{\partial P_t f}{\partial t}(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

- $g_{x,\gamma}$ is the spatial Gaussian Littlewood–Paley g function

$$g_{x,\gamma}(f)(x) = \left(\int_0^\infty |t \nabla_x P_t f(x)|^2 \frac{dt}{t} \right)^{1/2}.$$

- $g_{+,\gamma}^{(1)}$ is the Gaussian Littlewood–Paley g function for the translated Poisson–Hermite semigroup $\{P_t^{(1)}\}_{\{t\geq 0\}}$.

$$g_{+,\gamma}^{(1)}(f)(x) = \Big(\int_0^\infty (|t \nabla P_t^{(1)} f(x)|^2 + (t P_t^{(1)} f(x))^2) \frac{dt}{t} \Big)^{1/2}.$$

- $g_{t,\gamma}^{(1)}$ is the time Gaussian Littlewood–Paley g function for the translated Poisson–Hermite semigroup $\{P_t^{(1)}\}_{\{t\geq 0\}}$

$$g_{t,\gamma}^{(1)}(f)(x) = \Big(\int_0^\infty \Big| t \frac{\partial P_t^{(1)} f}{\partial t}(x) \Big|^2 \frac{dt}{t} \Big)^{1/2}.$$

- $g_{t,\gamma}^k$ is the higher-order time Gaussian Littlewood–Paley g function

$$g_{t,\gamma}^k(f)(x) = \left(\int_0^{+\infty} \left| t^k \frac{\partial^k P_t f}{\partial t^k}(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

- $g_{x,\gamma}^k$ is the higher-order spatial Gaussian Littlewood–Paley g function

$$g_{x,\gamma}^k(f)(x) = \left(\int_0^{+\infty} |t^k \nabla_x^k P_t f(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

- $\mathbf{g}_{t,\gamma}^k$ is the vector version of the higher-order time Gaussian Littlewood–Paley g function

$$\mathbf{g}_{t,\gamma}^k(\mathbf{f})(x) = \Big(\int_0^\infty \sum_{\beta \in \Lambda_k} \Big| t^k \frac{\partial^k P_t^{(k)} f_\beta}{\partial t^k}(x) \Big|^2 \frac{dt}{t} \Big)^{1/2},$$

  where $\mathbf{f}(x) = (f_\beta(x))_{\beta \in \Lambda_k}$.
- $m(L)$ is a Gaussian spectral multiplier operator,

$$m(L)f = \sum_{k=0}^\infty m(k) \mathbf{J}_k f,$$

  where $f = \sum_{k=0}^\infty \mathbf{J}_k f$ and $m : \mathbb{N}_0 \to \mathbb{R}$.

- I_β is the Gaussian fractional integral or Riesz potential of order β,

$$I_\beta = (-L)^{-\beta/2}\Pi_0,$$

where $\Pi_0 = I - \mathbf{J}_0$.
- $\mathscr{I}_\beta$ is the Gaussian Bessel potential of order $\beta > 0$,

$$\mathscr{I}_\beta = (I + \sqrt{-L})^{-\beta}.$$

- D^β is the Gaussian Riesz fractional derivative of order $\beta > 0$,

$$D^\beta = (-L)^{\beta/2}.$$

- $\mathscr{D}^\beta$ is the Gaussian Bessel fractional derivative of order β,

$$\mathscr{D}^\beta = (I + \sqrt{-L})^{\beta}.$$

- $L^p_\beta(\gamma_d)$ is the Gaussian Sobolev space of order $\alpha > 0$, $1 < p < \infty$.
- $T^{1,q}(\gamma_d)$ is the Gaussian tent space, for $1 < q < \infty$.
- $H^1_{at}(\gamma_d)$ is the (Mauceri–Meda) atomic Gaussian Hardy space.
- $H^1_{max}(\gamma_d)$ and $H^1_{quad}(\gamma_d)$ are the (Portal) maximal and quadratic Gaussian Hardy spaces respectively.
- $BMO(\gamma_d)$ is the Gaussian space of functions of bounded mean oscillations.
- $Lip_\alpha(\gamma_d)$ is the Gaussian Lipschitz space of order $\alpha > 0$.
- $B^\alpha_{p,q}(\gamma_d)$ is the Gaussian Besov–Lipschitz space for $\alpha \geq 0$ and $1 \leq p, q \leq \infty$.
- $F^\alpha_{p,q}(\gamma_d)$ is the Gaussian Triebel–Lizorkin space for $\alpha \geq 0$ and $1 \leq p, q \leq \infty$.
- $\mathscr{R}_j$ is the j-th Gaussian Riesz transform, $1 \leq j \leq d$,

$$\mathscr{R}_j = \partial_{x_j}(-L)^{1/2}.$$

- $\overline{\mathscr{R}}_j$ is the j-th alternative Gaussian Riesz transform, $1 \leq j \leq d$,

$$\overline{\mathscr{R}}_i = (\partial_i^\gamma)^*(-\overline{L})^{-1/2}.$$

- $\mathscr{R}_\beta$, $|\beta| > 0$, is the higher-order Riesz transform,

$$\mathscr{R}_\beta = \partial_\beta^\gamma(-L)^{-|\beta|/2}.$$

- $\overline{\mathscr{R}}_\beta$, $|\beta| > 0$, is the higher order alternative Riesz transform,

$$\overline{\mathscr{R}}_\beta = (\partial_\gamma^\beta)^*(-\overline{L})^{-|\beta|/2}.$$

- $T_{F,m}$ is a general Gaussian singular integral,

$$T_{F,m}f(x) = \int_{\mathbb{R}^d}\int_0^1 \left(\frac{-\log r}{1-r^2}\right)^{\frac{m-2}{2}} r^m F\left(\frac{y-rx}{\sqrt{1-r^2}}\right)\frac{e^{-\frac{|y-rx|^2}{1-r^2}}}{(1-r^2)^{d/2+1}}\frac{dr}{r}f(y)dy,$$

for an appropriate function F.

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W. Urbina-Romero, *Gaussian Harmonic Analysis*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-3-030-05597-4