Chapter 5
The German Speaking Didactic Tradition

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Abstract This chapter gives a historical sketch on the development of the Didactics of Mathematics as practised in German language countries, from the nineteen sixties to the present. Beginning from "Stoffdidaktik", anecdotal teaching episodes and large scale psycho-pedagogical research, Didactics of Mathematics has developed into a well defined scientific discipline. This tradition has embraced a plethora of areas researched via a range of methodological approaches stretching from local case studies to large scale surveys in order to capture broad perspectives on research into teaching and learning mathematics. Enlarged "Stoffdidaktik", and Didactics of Mathematics as a "design science", draws on a large number of classroom studies looking into a wide variety of specific aspects of teaching and learning mathematics. The sometimes politically driven, large-scale studies use sophisticated statistical methods to generate generalizable results on the state of educational systems, especially in terms of the changes and trends over time.

5.1 Introductory Remark

It is impossible to give adequate credit to all important contributions to Didactics of Mathematics (internationally often called mathematics education or research in mathematics education) in the German speaking countries in a short text of twenty pages. This chapter attempts to provide an overview of major contributions to the development of the Didactics of Mathematics in German language countries in past seven decades. As such, this chapter is unavoidably personally coloured and the author sincerely apologises to those who have not been mentioned or been misrepresented.

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This chapter first gives a historical sketch on German-speaking Didactics of Mathematics starting in the nineteen sixties. We then give an overview on the situation of Didactics of Mathematics (or research on mathematics education) in the 21st century until present in these countries. The paper finishes with a personal perspective (Rudolf Sträßer alone) on the future of German speaking Didactics of Mathematics. As an additional perspective from outside the German-speaking countries, three researchers from the Nordic countries (Norway/Sweden), from Poland and from the Czech Republic comment on interactions between research in their own countries and the German-speaking Didactics of Mathematics. A more detailed picture on German-speaking Didactics of Mathematics could be found in a parallel meeting during the Thematic Afternoon of ICME-13 (Hamburg Germany), where a whole afternoon was totally devoted to Mathematical Didactics from the perspective of the host country of the conference. The table of contents of a book, which emerged out of this activity (see Jahnke & Hefendehl-Hebeker, 2019) is added as an appendix at the end of this paper, representing the plans for the book as of June 2018.

5.2 Historical Sketch on German Speaking Didactics of Mathematics

5.2.1 Starting Point in the 1960s

Even if a historical sketch starts with the nineteen sixties, it is more than appropriate to give credit to work in Didactics of Mathematics already existing in these years in the German speaking countries (for another detailed description of Didactics of Mathematics in German speaking countries see e.g. Schubring, 2014, for a text in German nearer to the past see Steiner 1978).

In the nineteen sixties, three types of texts were available on the teaching and learning of mathematics in German speaking countries: First, personal reports from mathematics classrooms, which were given by experienced teachers and personnel from the education administration, often combined with document analysis for curriculum development purposes. A typical exemplar of this genre are the reports for the renewed International Commission on Mathematics Instruction (ICMI) like Drenckhahn (1958) or Behnke and Steiner (1967).

Second, German speaking Didactics of Mathematics at that time was strong in the so-called Subject Matter Didactics (“Stoffdidaktik”). “Stoffdidaktik” can be described as the mathematical analysis of the subject matter to be taught. The purpose of the analysis was to find appropriate ways or even the best (in certain approaches: one and only) way to make a mathematics topic accessible and understandable for students. This work was completed by mathematicians, teacher trainers, textbook authors and mathematics teachers—and often authors of “Stoffdidaktik” had more than one of these roles to play. Two strands of “Stoffdidaktik” can be distinguished, namely the one done by university and “Gymnasium” teachers as separated from the one done by teacher trainers for primary and general education. Even if separated from an institutional point of view, the common basic methodology is mathematics as a scientific discipline together with anecdotal classroom experience. Differences of the two strands come down to different mathematical topics for teaching according to the two—at that time in Germany—clearly different types of schools, namely Gymnasium (at that time targeting a small minority of students heading for university) and “Volksschule” (the then standard type of schooling for eight years aiming at the vast majority of students in general education ending at a student age of about 15 years). Typical exemplars of this genre are Oehl (1962) for “Volksschule” or the famous work by Lietzmann aimed at teaching in Gymnasium. The first edition of Lietzmann was published in 1916, had intermediate editions during the Weimar republic and the Nazi regime, and was re-published with the same title in 1951. The latest amended edition was published in 1985 under the editorship of Jahner (see Jahner 1978/1985).

The third genre is quantitative, mainly comparative studies, which were often completed by scientists from university departments of psychology. This research analysed the psychological preconditions and constraints of the learners. In Germany, these works had a long tradition from Katz (1913) (already sponsored by ICMI) to Strunz (1962) (4th edition), but did not develop into a strong research paradigm in the German speaking countries (for this judgement see Schubring, 2012).

5.2.2 Institutionalisation

In the second half of the nineteen sixties, in connection with the ‘sputnik crisis’, one German societal debate circled around a so-called educational catastrophe (“Bildungskatastrophe”) within the West-German Federal Republic. As a consequence and to foster economical growth, a social and political move made efforts to expand the educational system in the Federal Republic of Germany, prolonging the time of compulsory schooling and especially promoting the scientific school subjects including mathematics. On a societal level, this led to the creation of new universities, the training of more mathematics teachers, more teacher training for mathematics and the academisation of teacher training for primary education teachers. A detailed description of these developments can be found in Schubring (2016, pp. 15ff).

For Didactics of Mathematics as a scientific discipline, this societal change had important and long lasting consequences: For the first time in German history, full professorships in Didactics of Mathematics were created at universities—especially within newly founded institutions. In 1975, professionals in Didactics of Mathematics founded a scientific society for Didactics of Mathematics, the “Gesellschaft für Didaktik der Mathematik (GDM)”, followed by the creation of a new research journal in 1980, the “Journal für Mathematik-Didaktik (JMD)”—still under publication and now (since 2010) with Springer. On the initiative of the Volkswagen Foundation in 1972, a research institute for Didactics of Mathematics was founded at Bielefeld University, the “Institut für Didaktik der Mathematik (IDM)”. This was eventually integrated (1999) into the Faculty of Mathematics at Bielefeld University and gradually turned into a "normal" university institute with a "standard" mission in research and teacher training, but still one of the best libraries specialising in Didactics of Mathematics worldwide. A different indicator of the institutionalisation of German speaking Didactics of Mathematics was the organisation of the third International Conference on Mathematics Education (ICME-3), which was held in Karlsruhe in 1976 with a strong participation from German speaking scholars in the preparation and the activities of this conference (see Athen, 1977 for the original proceedings of the meeting).

5.2.3 The 1970s/1980s: The "Realistic Turn"

Together with the institutionalisation of Didactics of Mathematics as a scientific discipline, research on the teaching and learning of mathematics in schools underwent a major change. Compared to the situation in the nineteen sixties, we can identify two non-convergent developments, which drastically changed the research landscape in West-Germany and Austria: On the one hand, we see a "realistic turn" with more detailed empirical research, less document analysis and anecdotal reports, which is done to develop a description and (if possible: a causal) explanation of what is going on in the teaching and learning of mathematics. This development of Didactics of Mathematics followed a scientific development in Germany, which—at that time—started to favour empirical research over philosophical and historical research in pedagogy. Internally, inside Didactics of Mathematics, this move was a scientific answer to the failure of the 'Modern Math Movement'. Taking set theory as the foundation of Mathematics in schools together with an axiomatic approach to teaching, abstract algebra and a high importance of logic in schools became less prominent at that time. In general, research in the 1970/80 s favoured small scale, qualitative empirical research, looking at transcripts of classroom communication and interviews viewed through a specific theoretical lens. This was done with a focus on the communication process, not so much on the mathematical achievements of the students in order to identify the opportunities and challenges associated with interactions in mathematics classrooms. The move to a more empirical, small scale approach in research is accompanied by a methodological development, which favours qualitative or sometimes linguistic analysis of classroom processes, as opposed to a more traditional experimental research approach. In West-Germany, this movement was initiated by the Bauersfeld group from the Bielefeld Institute and implied a "turn to the everyday classroom" (for a more detailed description of this change of paradigms see Voigt, 1996, pp. 383–388).

To sum up the situation of Didactics of Mathematics in Austria, West-Germany and German-speaking Switzerland, this discipline started as a rather homogeneous field made up of subject matter didactics and classroom studies. In the 1980s, it diversified into a plethora of research with different paradigms and on a variety of aspects of the teaching and learning (process) of mathematics. In 1992, Burscheid, Struve

& Walther analysed the topics of the “publications in West German professional journals and research reports of the IDM”, the research institute at Bielefeld university mentioned above and came to the following important research areas (see Burscheid, Struve, &Walther, 1992, p. 296):

– “empirical research
– subject matter didactics
– applications in mathematics teaching
– historical and philosophical investigations
– methodological aspects of mathematics education
– principles of mathematics education
– the epistemological dimension of mathematics education
– proving.”

Besides the methodological decision to give empirical research a greater importance, research not only in subject matter didactics concentrated on Arithmetic and Calculus—with a relative neglect of topics from Geometry. With ‘Modern Math’ (logically based on set theory) becoming less important, the role of applications in teaching Mathematics came into the centre of one German speaking strand of Didactics of Mathematics with a focus on the modelling circle (see Pollak, 1979). Historical and philosophical investigations served as a basis for enriching classroom teaching of Mathematics with information on its role in culture and history. Principles of mathematics education (like the genetic principle to follow the historic line of development of mathematical topics or the operative principle to learn through close inspection of operations on carefully selected situations) were identified as a way to structure the variety of school mathematics topics to form a coherent whole, if possible showing the general value of Mathematics for education as a whole.

In the 1970/80s, discussion was focused on the idea of a comprehensive theory of Didactics of Mathematics taking into account its epistemological dimensions. In 1974, Hans-Georg Steiner, one of the directors of the institute at Bielefeld University, edited a special issue of Zentralblatt für Didaktik der Mathematik (ZDM) with contributions from Bigalke, Freudenthal, Griesel, Otte and Wittmann (among others), looking into the possibility of a comprehensive theory of Didactics of Mathematics, especially ideas on its subject area, its scientific character and its relation to reference disciplines, especially mathematics, psychology and educational science. No coherent conceptualisation was reached. About ten years later in 1983, a comparable debate in the Journal für Mathematikdidaktik (JMD) with protagonists Burscheid, Bigalke, Fischer and Steiner also did not produce a coherent paradigm for Didactics of Mathematics, but ended with the conclusion that Didactics of Mathematics may never develop into a science based on a single paradigm underpinned by one and only one unifying theoretical approach (for details see the chapter on “Theories of and in mathematics education related to German speaking Countries” in the volume edited by Jahnke et al., 2019). In addition to these discussions, in 1983, Steiner took the initiative of creating an international group on “Theory of Mathematics Education (TME)”, which had a first meeting after ICME-5 in Adelaide in 1984. This international group had four follow-up conferences in different locations, but also could not arrive at a joint unifying conceptualisation of Didactics of Mathematics as a scientific discipline. The international work on a theory of Didactics of Mathematics was complemented by a series of bi-lateral symposia organised by Steiner (see the French-German symposium in 1986, the Italian-German seminars in 1988 and 1992 and the International Symposia on Mathematics Education in Bratislava in 1988 and 1990; for all activities see Biehler & Peter-Koop, 2007, p. 27f). By the late 1990s, this line of discussion began to wane, at least in the German speaking countries, but re-emerged internationally in the 21st century with the meta-theoretical approach of networking theories (for a comprehensive presentation of this movements see Bikner-Ahsbahs & Prediger, 2014).

5.2.4 Didactics in the German Democratic Republic (GDR)

The attentive reader will have noticed that in some places, we employed the wording "West-Germany"—and this was done on purpose. Some reports on the history of Didactics of Mathematics in Germany tend to forget that—because of political reasons and differences—research on the teaching and learning of mathematics developed differently in West-Germany, the Federal Republic of Germany ("Bundesrepublik Deutschland—BRD") and in East-Germany, the German Democratic Republic ("Deutsche Demokratische Republik—DDR"). This was even marked by a different designation of the activity. In East Germany, the science of teaching and learning mathematics was called "Methodik" (direct, but misleading translation: "methodology"). For more detailed information see Henning and Bender (2003). If we follow the basic and informative text by Bruder (2003), four characteristics have to be mentioned for "Methodik" in the German Democratic Republic (GDR):

(1) As usual in a "socialist" country, the science of teaching and learning mathematics was highly controlled by political authorities, which tried to implement a uniform planning for the comprehensive school installed in the country. This implied the use of one and only one textbook in the GDR, accompanied by just one set of teaching aids.
(2) Teaching and learning of mathematics had a systematic disciplinary orientation following the systematisation of the discipline Mathematics, but trying to cope with the teaching reality. Research into teaching and learning picked up the difficulties experienced by teachers in classrooms and focussed on intervention to overcome these difficulties. A theory of teaching/learning played a minimal role in the creation of textbooks, teaching aids and detailed suggestions for the way to teach mathematics.
(3) Periodical repetition and mental training of basics were the backbone of the optimization of instruction and construction of learning environments. The overall learning goal was deeply rooted in the ideas of a developed social personality, i.e. a socialist idea of humankind and a high esteem of mathematics and science in the socialist society. This "Methodik" was highly accepted by teachers

due to the institutionalisation of these ideas by a high proportion of lessons on Didactics of Mathematics at university and in practical training at school during the study and no inconsistencies between pre-service teacher education and in-service teaching experience.

(4) The teaching in classroom was marked by a linear, uniform structure of subjects to be taught. Subjects (like mathematics) had the priority compared to the individual needs of the students—with the consequence that students' differences were taken into account by means of inner differentiation in a uniform educational system. For gifted students, special support was available from Mathematical Olympiads on different levels and from special schools ("Spezialschulen") for students who seemed to be able to cope with higher demands and extended training in areas like sports, music or mathematics.

These ideas and practices of a "socialist" (mathematics) education were left behind with the fall of the German Democratic Republic in 1989. Nevertheless, careful and attentive observers of the educational systems in Germany can still recognise consequences of these traditions in some regions of the former German Democratic Republic such as a higher importance given to mathematics and science in the former German Democratic Republic.

5.2.5 1990s: The PISA Shock

During the 1990s, German Didactics of Mathematics had to cope with the change induced by having only one state in the German country and the gradual adaptation of the East German regions to the educational system of West-Germany. In this period, an external incident drastically changed the public views on teaching and learning mathematics in German-speaking countries. During 1996, the results of the Third International Mathematics and Science Study (TIMSS) were released and showed Germany to be a nation with an average student achievement in grades 7 and 8 if compared to all nations participating in this study. Two years later, the Programme for International Student Assessment (PISA) placed Germany even below average for the results in mathematical literacy inducing what was called in Germany the *PISA-shock*. The realisation that German (mathematics) education in schools was not performing to expectations had numerous political consequences. Most prominent were policy sponsored efforts to enhance the teaching and learning especially of mathematics in general education by means of programs to enhance this educational effort. The "SINUS" study (described further down in Sect. 5.3.3) was the most important effort of this type of government sponsored intervention aiming at professional development of mathematics teachers. Another consequence of the *PISA-shock* were studies devoted to the identification of teaching standards in Germany (heavily drawing on a concept of teaching/learning mathematics with the intention to offer mathematical *Bildung*, e.g., Winter, 1995) and numbers of (regional and national) evaluations aimed at gathering detailed information on the achievement (or not) of students. Politicians also hoped to get pertinent information and ideas how to enhance the knowledge and skills of the population to do better economically (for details see below in Sect. 5.3.3).

Additional activities originating from the *PISA-shock* were large scale projects from the regional education ministries aimed at increasing teaching quality in "normal" classrooms. The regional political entities responsible for education, the „Länder" joined forces to set up the qualitative large scale development study „SINUS" and later its follow-up study „SINUS-transfer" (for the background and detailed information see http://www.sinus-transfer.eu). The ministers of education in the „Länder" of the Federal Republic of Germany agreed upon 11 „modules", that is, 11 topics for enhancing the teaching quality in mathematics and science: developing a task culture, scientific working, learning from mistakes, gaining basic knowledge, cumulative learning, interdisciplinary working, motivating girls and boys, cooperative learning, autonomous learning, progress of competencies, quality assurance (see http://www.sinus-transfer.eu/). The two development projects SINUS and SINUS-transfer aimed at helping the classroom teacher enhance her/his teaching by developing examples of good teaching and were heavily supported by research institutes and didacticians. The most important activity of the projects consisted of the organisation of teacher cooperation and the dissemination of good teaching units developed to illustrate the goals of the 11 modules.

5.3 The 21st Century—At Present

By the 21st century, German speaking Didactics of Mathematics was well-established as a research field. An indication of its strength was the invitation to host a second International Congress on Mathematics Education, following ICME-3, namely ICME-13 in Hamburg in 2016. Being able to structure and manage a congress of about 3500 participants from 107 countries demonstrates the maturity and stability of the German-speaking community of didacticians of Mathematics. In addition, scholars from the German speaking community have a history of support, through their time and expertise, for European and international research efforts in Mathematics Education. Numerous colleagues from German speaking countries serve in the steering bodies of European and international research organisations of Didactics of Mathematics and in the organisation of working groups in research organisations such as the International Conference on Mathematics Education (ICME), the group on Psychology of Mathematics Education (PME), the European Society for Research in Mathematics Education (ERME) or the International Conferences on the Teaching of Mathematical Modelling and Applications (ICTMA)—to name just a few.

The research situation in German speaking Didactics of Mathematics can be characterised by three major strands of research, which for the second case offer a broad variation even within this strand. We distinguish a strand of enlarged Stoffdidaktik, which now also embraces the design of learning environments (sometimes controlled by learning theories; for details see 5.3.1 below). This is different from a plethora of (mostly qualitative) descriptive and for the most part comparative case studies, especially "classroom studies". A third strand is a variety of large scale developmental or evaluation studies, which—following a quantitative or qualitative approach—are sometime deeply influenced by the TIMSS or PISA activities and—for the majority—are also motivated, if not sponsored by (educational) politicians. For a more detailed account on Didactics of Mathematics in German-speaking countries see Jahnke et al. (2019), a book describing the parallel Thematic Afternoon on German-speaking Didactics of Mathematics, which was held in parallel to the afternoon on European traditions. An appendix in the end of this text offers the table of contents of this book.

5.3.1 Stoffdidaktik Enlarged—The Design of Learning Environments

The first major strand of German-speaking Didactics of Mathematics developed out of traditional Stoffdidaktik (based on the mathematical analysis of the subject matter to be taught). In addition to the diligent analysis of mathematical contents, enlarged Stoffdidaktik takes into account additional influences on the subject matter taught, especially the history and epistemology of mathematics, fundamental ideas of mathematics (if visible), and information on the learner and her/his pre-requisites for learning (including 'basic mental models' and beliefs of the learner). This approach also embraces empirical studies on consequences of subject matter innovation and can be distinguished from traditional Stoffdidaktik by its acceptance of a developed cooperation with other disciplines (e.g. Educational Psychology) looking into the teaching and learning of mathematics. On the occasion of ICME-13, the German-speaking research journal Journal für Mathematikdidaktik (JMD) published a supplement totally devoted to modern Stoffdidaktik (see https://link.springer.com/journal/13138/37/1/suppl/page/1) with an introductory description of this development.

Growing out of the search of Stoffdidaktik for the best way to teach mathematics in an understandable way, some German-speaking didacticians make the design of learning environments the defining 'kernel' of research in mathematics education (Didactics of Mathematics). One of the protagonists of this approach, Erich C. Wittmann, came up with the definition of "Didactics of Mathematics as a design science" (see already the title of Wittmann, 1992/1995). Wittmann starts from the distinction of a "core" of mathematics education and "areas related to mathematics education" (like mathematics as a scientific discipline, sociology, pedagogy and general didactics; see Fig. 5.1 on p. 357 of Wittmann 1995). For mathematics education, he identifies "the framework of a design science" as "a promising perspective for fulfilling its tasks and also for developing an unbroken self-concept of mathematics educators" (loc. cit., p. 362). As a design science, "the core of mathematics education concentrates on constructing *artificial objects*, namely teaching units, sets of coherent teaching units and curricula as well as the investigation of their possible effects in different educational "ecologies"" (loc. cit., p. 362f).

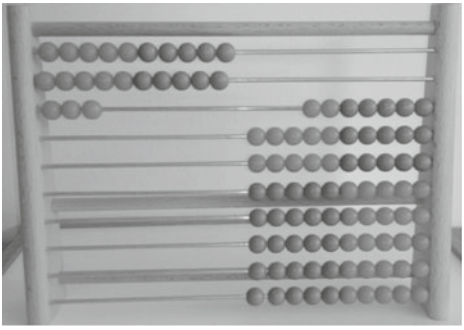
5.3.2 Ongoing Diversification of Classroom Studies

The second major strand of German-speaking Didactics of Mathematics can be globally described as Classroom Studies. This strand is very diverse and often done in cooperation with other disciplines, such as psychology, pedagogy or educational sciences. These studies can be described as mostly qualitative and descriptive case studies, which seek to reconstruct specific aspects of everyday teaching and learning. For the moment, there is no comprehensive overview of this research field, but an impression nonetheless can be gleaned from the following list of typical topics (compiled from the last five years of the Journal für Mathematikdidaktik JMD):

- the use of technology in mathematics teaching and learning
- subject matter analysis—proof and argumentation
- modelling in mathematics classrooms
- the role of language
- early childhood and primary education
- variables of good class management
- gender and teaching & learning mathematics
- textbook research
- history and epistemology of mathematics
- semiotics and mathematics
- teacher training (preparatory and in-service)
- competencies in mathematics.

One way to develop an understanding of *Classroom Studies* is to learn more about their aims, history and ways of working. First of all, *Classroom Studies* can be characterized by their *sociological orientation* on learning mathematics. Often, the focus is on the social dimension of learning processes; not so much to understand one individual case, but to compare a number of cases in order to identify common features and differences on a theoretical level. It is because of this comparative approach that researchers from that strand ascribe a meaning to their results that goes well beyond the individual case.

The strand of *Classroom Studies* was initiated by the works of *Heinrich Bauersfeld* and his colleagues in the 1980s. Their key assumption was that mathematical knowledge is always developed within social interaction (social negotiation of mathematical meaning). First, they observe everyday mathematics classes and collect different kinds of data, such as videos, audiotapes or written products. Second, they transcribe the processes of social interactions. Third, they analyse these transcripts by means of the interaction analysis. This method combines a sociological and a mathematical perspective. It aims at reconstructing the thematic development of a given face-to-face interaction. A very important step in this analytical work, then,



Utterance 1 on 32:

"because here are 3 and here are 2"

Utterance 2 on 15:

"one row of tens and five beads"

Fig. 5.1 The bead frame

is to compare the interpretations of different scenarios and to identify similarities and differences. As a result, the researchers finally describe different types of a phenomenon. And these descriptive results can be used to analyse other lessons or to develop suggestions for improvement.

To illustrate the strand of Classroom Studies, we give a short *example* from a research project about primary students and their development of subject-related language that was presented during the Thematic Afternoon of ICME-13 by Kerstin Tiedemann.

Research Example 1 (by Kerstin Tiedemann):

Material artefacts and classroom communication

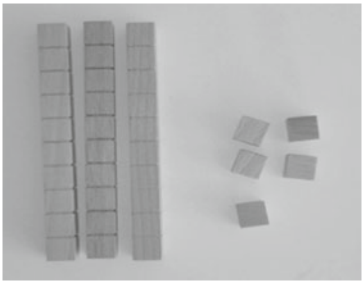
The example is about a young girl named Hanna. She is 9 years old, goes to school in the third grade, and speaks German as her first language. For that reason, her utterances (utterance 1-5) are translated from German to English. Hanna was filmed in discussions with her teacher Britta over a period of three months. In their lessons, Hanna and Britta work with two different didactic materials one after the other: with a bead frame and with Dienes blocks.

First, they work with the bead frame (see Fig. 5.1).

Hanna uses quite unspecific language (Fig. 5.1, utterance 1). She refers only to amounts, but she does not mention what the objects are that she is talking about. For example, she names a twenty-three (23) on the bead frame as a thirty-two (32) and gives the argument: "because this is 3 and this is 2". At this point, tens and ones are exchangeable for Hanna. But, then, in interaction with Britta, she starts to specify her language in relation to the bead frame (utterance 2). She learns to distinguish tens and ones and refers to them as 'rows of tens' and single 'beads'. For example: "1 row of tens and 5 beads."

Later on, Hanna and Britta work with the Dienes blocks (see Fig. 5.2), with another material.

In those lessons—and this is really interesting -, Tiedemann (2017) could reconstruct a surprisingly analogue language development. First, Hanna uses quite unspecific language, refers only to amounts and often mixes up the tens and ones (Fig. 5.2, utterance 3). But then, she starts to specify her language again, this time in relation to the Dienes blocks (utterance 4). She learns to distinguish tens and ones and refers to them as 'bars of tens' and single 'cubes'. It is this specification of language that allows Hanna to compare both materials in the end and to describe their mathematical similarity (utterance 5). With regard to a forty-three



Utterance 3 on 53:

"because here are five and here are three"

Utterance 4 on 25:

"two bars of tens and five little cubes"

Utterance 5 on 42:

"for the four tens, you take rows and not bars, okay?"

Fig. 5.2 Dienes' blocks

(43) on the bead frame, she says in comparison to the Dienes blocks: "For the 4 tens, you take rows and not bars, okay?"

To sum up, Hanna is a prototypical example of a certain type of language development – she begins with an unspecific language use, but then she specifies her language while interacting with Britta and is then able to compare different representations of tens and ones. It is in this process that she shows more and more understanding of the mathematical concept of place value.

5.3.3 Large Scale Comparative Studies

The third major strand in German-speaking Didactics of Mathematics grew in large parts out of the *PISA-shock* mentioned in Sect. 5.2.5. Politicians and researchers turned the deception on the (average or below average) German results in the TIMSS and PISA-studies into efforts to know better and with the help of statistical inquiry about the actual achievements of students when learning mathematics in general education. This created the opportunities and needs for quantitative large scale evaluation studies not only in mathematics, but also in German, English and Science teaching and learning. First, the deceiving results from the TIMSS- and the TIMSS-video-study led to German participation in the PISA-studies, where German didacticians of mathematics took a major role in preparing the research instruments. This was complemented by the creation of an extension of the original PISA-study (called PISA-E), which allowed for an intra-national comparisons of the teaching/learning practice of individual German *Länder* (the political regional entities below the whole nation, which—by constitution—are responsible for education in general). From the PISA-study in 2003 onwards, we can see a heavy involvement of German didacticians of mathematics in such comparative evaluation studies on the teaching and learning of mathematics in German general education (for details see http://archiv.ipn.uni-kiel.de/PISA/pisa2003/fr_reload_eng.html?mathematik_eng.html). To illustrate this development, we present an evaluative study on teacher competence, which was embedded in the PISA process, the COACTIV-study on "teacher competence as a key determinant of instructional quality in mathematics" and COACTIV-R (a continuation of COACTIV from 2007 onwards) on "teacher candidates acquisition of professional competence during teaching practice" (for details see → https://www.mpib-berlin.mpg.de/coactiv/index.htm). This example was also presented during the Thematic Afternoon at ICME-13.

Research example 2 (by Stefan Krauss): The Impact of Professional Knowledge on Student Achievement

The German COACTIV 2003/04 research program (*Co*gnitive *Activ*ation in the Classroom: Professional Competence of Teachers, Cognitively Activating Instruction, and Development of Students Mathematical Literacy) empirically examined a large representative sample of German secondary mathematics teachers whose classes participated in the German PISA study and its longitudinal extension during 2003/04. Since the students of the Grade 9 classes which were tested in PISA 2003 in Germany were examined again in Grade 10 in the following year, the design allowed to analyse the impact of teacher competencies (as assessed by COACTIV) on students' learning gains in mathematics (as assessed by PISA-03 and -04). The COACTIV-project was funded by the German research foundation (DFG) from 2002 to 2006 (directors: Jürgen Baumert, Max-Planck-Institute for Human Development Berlin; Werner Blum, University of Kassel; Michael Neubrand, University of Oldenburg). The rationale of the project and its central results are summarized in the compendium by Kunter, Baumert, Blum, et al. (2013a).

Analyses of structural equation models revealed that the newly developed test on teachers' pedagogical content knowledge (PCK) was highly predictively valid for the lesson quality aspects of cognitive activation and individual learning support. Furthermore, a "black-box"-model for the direct impact of teacher competence on students learning gains (without mediation by aspects of instructional quality, adapted from Kunter et al. 2013b) shows a regression coefficient of 0.62 of PCK on the students' learning gain and can for instance be interpreted in the sense that classes of teachers who scored one standard deviation *below* the PCK-mean of all teachers yielded an average learning gain within one school year (in terms of effect sizes) of about $d = 0.2$ and classes of teachers who scored one standard deviation *above* the PCK-average yielded a learning gain of about $d = 0.5$. Since the average learning gain of all PISA-classes was $d = 0.3$, this demonstrates that a difference of two standard deviations in the PCK of a mathematics teacher can make a difference in the learning gain of the typical amount of what on average is learned in a whole school year. The construction of a psychometric test on the PCK of mathematics teachers meanwhile inspired studies on other school disciplines like German, English, Physics, Latin, Musics and for religious education (see Krauss et al., 2017).

Besides evaluation studies linked to PISA, other major large scale comparative studies were hosted by the International Association for the Evaluation of Educational Achievement (IEA), which had already managed the TIMSS- and TIMSS-video-study. A major activity in German-speaking countries was the Teacher Education and Development Study on Mathematics named TEDS-M (for details see Blömeke et al., 2014, for more information on the study see https://arc.uchicago.edu/reese/projects/teacher-education-and-development-study-mathematics-teds-m). Inspired by ideas from Shulman (1986/1987), the TEDS-M-study looked into the "Policy, Practice, and Readiness to Teach Primary and Secondary Mathematics in 17 Countries" (quote from the subtitle of the technical report Tatto 2013) including Germany. As described by Tatto (2013) "The key research questions focused on the relationships between teacher education policies, institutional practices, and future teachers' mathematics and pedagogy knowledge at the end of their pre-service education." (from the back-cover of Tatto, 2013; for results see Tatto et al., 2012; a detailed presentation of results in German can be found in Blömeke, Kaiser, and Lehmann 2010a, b). In Germany, the international TEDS-M study resulted in several further national follow-up studies, amongst others the study TEDS-LT, that compared the development of the professional knowledge of student teachers in the subjects mathematics, German language and English as first foreign language ("EFL", see Blömeke et al., 2013), or the TEDS-FU study that analysed situational facets of teacher competence (Kaiser et al., 2017). While TEDS-M and TEDS-LT used paper and pencil based assessment of the mathematical content (MCK), mathematical pedagogical content knowledge (MPCK), and general pedagogical knowledge (GPK) of teacher students and teachers, the instruments for measuring teacher competence have been expanded to include also video and online testing in TEDS-FU.

Another consequence of the search for information on students' achievement were a number of assessment studies (so-called Schulleistungsstudien) installed by the majority of the regions (*Länder*) responsible for general education. In nearly all regions detailed studies of the results of teaching and learning mathematics were set up on various school levels. According to regional decisions, they use different methods (like traditional textbook tasks, multiple choice questionnaires and other instruments) and they are not restricted to the school subject mathematics. Examples of these studies are the VERA study with a representative comparison of classes and schools in grade 3, for most regions also in grade 8, looking at least into the two school subjects Mathematics and German (for more information see https://www.iqb.hu-berlin.de/vera). A more local, but more comprehensive study using the VERA framework is the KERMIT-study in Hamburg. From 2012 onwards it looks into the teaching and learning of mathematics (together with other school subjects) in Grades 2, 3, 5, 6, 7, 8 and 9 with the aim of informing teachers about the strengths and weaknesses of their classes, in order to use this information to tailor the teaching to students' needs (for details see http://www.hamburg.de/bsb/kermit/).

5.4 About the Future of German-Speaking Didactics of Mathematics

After preparing the historical sketch on German-speaking Didactics of Mathematics and especially informed by interviews I have made with Lisa Hefendehl-Hebeker and Hans-Georg Weigand, I offer a personal look into what may be the developments and challenges for German-speaking Didactics of Mathematics in the future. Obviously and as a sort of unavoidable kernel, a comprehensive, epistemologically and historically well informed understanding of Mathematics is crucial, including a concept of the vocational and social rule of this scientific discipline. With Didactics of Mathematics focussing on teaching and learning, another crucial point and great challenge is how the human brain works. These two major issues would perhaps give a chance of developing a comprehensive theory of Didactics of Mathematics.

Such a theory also has to model the many different impacts on teaching and learning mathematics, like the impact of the personal development of the student, the impact of the social situation the learner lives in and how the individual brain works apart from invariant development variables. This theory also has to account for the setting of the student within the family s/he comes from, the situation in the classroom, the political situation of the school, and the situation in society at large. Because of the many students with different mother tongues and because we have a new consciousness of the relation between thinking and speaking, the role of language in teaching and learning mathematics will be another major issue in future research into the Didactics of Mathematics, together with the use of technology, which deeply influences mathematics teaching and learning. These areas of research will only be helpful for the everyday classroom if we find out more about characteristics of good classroom management.

In terms of research methodology, the last decades clearly show that empirical research is important, but we should take care to keep the balance between empirical, maybe statistical quantitative research, qualitative investigations and conceptual, theoretical work in Didactics of Mathematics. As for research based on quantitative methods, a mixed methods approach may be appropriate. This balance allows for different approaches from a simplified empirical paradigm. For example subject oriented analysis and design experiments and philosophical and epistemological discussions about questions appear to be effective guidelines for mathematics education today.

Using a more global perspective, one can find different initiatives to systematize or to synthesize different theoretical approaches, from all over the world. In Germany, researchers like Angelika Bikner-Ahsbahs and Susanne Prediger could be named in this respect. It is visible that even today, the overall result is not a unifying theory, but limited local relationships between different approaches. It is clear that we do not, as yet, have an overarching, comprehensive theory.

A different challenge for the future is and will be bringing the results of empirical investigations, including large scale investigations to the school. How to "scale up" (sometimes local, limited) findings, insights and suggestions to bring them to the learner, to the school, to the administration, to politicians—apart from advancing Didactics of Mathematics as a scientific discipline.

5.5 Comments from Critical Friends

5.5.1 Doing Empirical Research Differently: The Nordic and German Cases. A View from the Nordic Countries

(By Barbro Grevholm)

Reasons for Nordic connections to Germany

Germany is a neighbour to the Nordic area and can easily be reached via land from Denmark, or sea from Sweden, Norway, Finland, and Iceland. German language is often the second foreign language, after English, learnt in the Nordic area and it used to be the first foreign language. The linguistic connections are many and in historical times German for long periods was the spoken language in Stockholm, for example. The classical *Bildningsresorna* (in German: Bildungsreisen; an English explanation is very difficult because of the concept of *Bildung*, which does not exist in English) went to Germany for many young persons in the Nordic countries. One famous example is the young Norwegian mathematician Abel.

Some early examples of German influence

In 1920, Salomon Eberhard Henschen, a medical professor in Stockholm, published a book entitled *Klinische und Anatomische Beiträge zur Pathologie des Gehirns. 5. Teil. Uber Aphasie, Amusie und Akalkulie* [Papers on Brain Pathology from Clinical Research and Anatomy, part 5 on aphasia, amusia and acalculia; title translated by RS]. This book can be considered an early contribution to studies on dyscalculia. Another example is the Kassel-Exeter study, which had links to Norway and the KIM-study (for details see Streitlien, Wiik, & Brekke, 2001) and there are also later master studies, where the Kassel-Exeter tasks were used again in Norway. The Hanseatic traditions[1] were common to the countries we speak about and being in Hamburg we are reminded of all the Hanseatic cities in the Nordic and Baltic countries that were linked to German centres of commerce and communication. German textbooks were used in Swedish University studies in mathematics. For example, in the 1960s, books by Knopp, *Funktionentheorie* and *Non-Euclidian Geometry,* were regularly studied in higher education courses. The book *Algebra* by van der Waerden was also part of the doctoral study curriculum. Visits from German researchers also took place. For example, Professor Doktor Emil Artin und Frau Braun (this is how they were introduced to us: Artin with all his titles and Hel Braun just as “Frau” although she was also a mathematician; Hel Braun was the very first and only lady in mathematics that I as a young doctoral student had the opportunity to meet) visited The Mathematical Society in Lund in the 1960s. These are just a few examples, there are many more.

How and why are we doing empirical research differently?

There exist some important differences in the conditions for research in didactics of mathematics. For example, the first professorships in Didactics of Mathematics were created 1992–1993 in Denmark, Finland and Norway and in Sweden in 2001 (at Luleå University of Technology), in contrast to the 60 professorships created in Germany in the 1960s. The first chair in Sweden was held for some years by R. Sträßer from

[1]From Wikipedia: “The Hanseatic League … was a commercial and defensive confederation of merchant guilds and market towns in Northwestern and Central Europe. Growing from a few North German towns in the late 1100s, the league came to dominate Baltic maritime trade for three centuries along the coast of Northern Europe. It stretched from the Baltic to the North Sea and inland during the Late Middle Ages and declined slowly after 1450.”

Germany. The academization of teacher education took place around 1960 but not until in the 1980s was it an explicit demand that teacher education should be research based. A scientific society of Didactics of Mathematics was created in Sweden in 1998 (called Svensk Förening för Matematik Didaktisk Forskning—SMDF) and some years earlier in Denmark and Finland. Norway and Iceland still do not formally have such societies. Thus, it can be argued that the Nordic countries were about 20–30 years behind in the development of Didactics of Mathematics as an area of research studies compared to Germany.

What was it that triggered the development in the Nordic countries? The First Mathematics and Science Study, FIMSS, created a huge debate in Sweden on this topic and the government set up committee for mathematics in school. They published their report in 1986, entitled "Matematik i skolan" [Mathematics in School], and it suggested academic courses in Didactics of Mathematics, positions and revision of the teacher education. A new teacher education started in 1988, where Didactics of Mathematics was introduced and the education became clearly research based. Student teachers were required to carry out a small research study and write a scientifically oriented report, the so called 'examensarbete'. The work called for supervision and research literature.

The International trends also swept over the Nordic countries. First in the 1960s there was a focus on the modern mathematics, followed by a back to basics movement in the 1980s, and more recently the use of ICT and problem solving. The results from TIMSS and PISA are influencing the politicians much and creating public debate in society about school mathematics. Nordic teachers were taught methods of teaching which links to Erich Wittman's view of Didactics of Mathematics as a design science. It was not seen as mere research.

In Sweden, the National Center for Mathematics Teaching (NCM), was created in the end of the 1990s and a similar centre was founded in Trondheim during 2002, the Norwegian Centre for Mathematics Education (NSMO). Again, politicians emphasized the teaching, and research was not included in their agenda. But in Norway a great effort was also given to creating research in Didactics of Mathematics. At the University of Agder (UiA) a master's education started in Didactics of Mathematics in 1994 followed by a doctoral education in 2002. Four professors of Didactics of Mathematics were hired (all women) and asked to build up a research environment and establish doctoral education. This was the first time in the Nordic countries when a group of professors could work together in Didactics of Mathematics at the same university. One of many guest researchers in the mathematics education research group, MERGA, at UiA was S. Rezat from University of Paderborn in Germany.

A huge five-year grant was given to UiA in order to set up a Nordic Graduate School in Mathematics Education, called NoGSME, which started in the beginning of 2004. Most of the Nordic universities with Didactics of Mathematics-students were linked to this Graduate School and it held about 90 doctoral students and 100 supervisors, also from the Baltic countries. Several German scholars were invited to this Graduate School to lecture, participate in summer schools or hold seminars for supervisors and thus the link to Germany was kept alive. The Graduate School was in action between 2004 and 2010 and its network for research on mathematics textbooks continued to be funded from the Nordic Research Academy until 2016 (Grevholm, 2017). In the network for research on textbooks two of our German colleagues were very active during a period of ten years. R Sträßer and S. Rezat created the link to German research on textbooks (Rezat & Strässer, 2013). NoGSME was followed by a fruitful networking collaboration institutionalised in the Nordic Society for Research in Mathematics Education (NoRME), between the Nordic and Baltic countries in the form of common conferences, the Nordic Conferences on Mathematics Education (NORMA-conferences), a common scientific journal, the Nordic Studies in Mathematics Education (NOMAD; started in 1993 and revived in 2004), and joint activities in doctoral education including courses, supervisors' seminars, summer schools in which German scholars took part.

Thus, in the Nordic countries empirical research was most often carried out by single researchers on their own, often with a lack of funding, and it resulted in a fragmented picture of results from the research. The only places where we find a larger group of researchers is at University of Agder in Norway and Umeå University in Sweden. There was no common plan for the empirical studies that were carried out in the Nordic area. The early German studies on Stoffdidaktik and textbooks had no counterpart in the Nordic Universities, and research centres such as the one in Bielefeld did not exist.

The German impact in Didactics of Mathematics in the Nordic countries

In what parts of Didactics of Mathematics can we trace the German influences from collaboration and research?

A few examples of such research areas are mentioned below.

The use of ICT in mathematics teaching and learning

The use of Information and Communication Technology (ICT) and other technological resources in mathematics teaching and learning has been a research interest in Didactics of Mathematics since the 1980s and was another link between Germany and the Nordic area. The works by Dahland (1993, 1998) at Gothenburg University and Hedrén (1990) at Linköping University illustrate studies in early days and the influence from German researchers. Dahland utilises theories in Didactics of Mathematics from the German traditions and discusses them in his dissertation. Dahland and Hedrén had many followers interested in the use of ICT, such as T. Lingefjärd, B. Grevholm, L. Engström, A-B. Fuglestad, C. Bergsten, M. Blomhöj and others. German influence was provided by Blankertz (1987), and later from Berger (1998), R. Sträßer and others.

Students' mathematical learning difficulties and dyscalculia

As mentioned above, publications on dyscalculia were already in progress from the beginning of the 1920s and continue to be of interest, especially in relation to effective pedagogies. The Swedish pedagogue Olof Magne (1967, 1998) was inspired by Henschen and several other German researchers (for example G. Schmitz, F. Padberg), when he studied pupils with difficulties in mathematics and dyscalculia. In his turn, he evoked the interest from other Nordic researchers for learning difficulties in mathematics (e.g., D. Neumann, G. Malmer, C. Ohlin, A. Engström, L. Häggblom, K. Linnanmäki).

Studies on gender and mathematics

The Swedish network on Gender and mathematics (Kvinnor och matematik) was created in 1990 after inspiration from a study group on Women and mathematics at ICME-6 in Budapest and other German colleagues who were active in Hungary. The International Organisation of Women and Mathematics, IOWME, is an ICMI affiliated group. Active members were, for example, Christine Keitel, Erika Schildkamp-Kundiger, Cornelia Niederdrenk-Felgner, Gabriele Kaiser, and later Christine Knipping. The network Women and Mathematics had members from all the Nordic countries. Some German colleagues gave presentations on issues of equity in the conferences of this society (for example Christine Keitel in 1993 and 1996, Kristina Reiss 1993 and Gabriele Kaiser on several occasions).

Teachers' and students' view of mathematics

In 1995, E. Pehkonen in Helsinki and G. Törner in Duisburg started a series of conferences with the theme Current State of Research on Mathematical Beliefs. At the beginning, it was mainly Finnish researchers such as E. Pehkonen, M. Hannula and German researchers such as G. Törner, G. Graumann, P. Berger, B. Rösken who contributed. Later the conferences became more internationally oriented and took place in many different countries. The presentations concern mathematical beliefs, attitudes, emotions and in general views of mathematics.

Problem solving in mathematics education

Another group of Finnish and German researchers in 1999 initiated the ProMath-group, also lead by Erkki Pehkonen. The aim of the ProMath group is to study and examine those mathematical-didactical questions which arise through research on the implementation of open problem solving in school. The group organizes yearly international conferences and publishes proceedings from them. The proceedings from the conferences mirror the recent international development in problem-solving, such as for example use of problem fields and open problems.

Another series of Nordic and Baltic conferences are the NORMA-conferences, started in 1994. They rotate among the five Nordic countries and there is a tradition of German participation (e.g. Graumann, 1995; Steinbring, 2005), thereby contributing with theory and methods from the German Didactics of Mathematics. The conferences held by SMDF ("Matematikdidaktiska forskningsseminarium", called the MADIF-conferences) have also opened a window to German research projects and tendencies. The contributions are often closely linked to teaching and classroom work. German teaching projects like SINUS and DISUM (Didaktische Interventionsformen für einen selbständigkeitsorientierten aufgabengesteuerten Unterricht am Beispiel Mathematik) and use of modelling can serve as examples (Blum & Leiss, 2007; Borromeo Ferri, 2007).

How can German research contribute in the future?

A wish for the future is to forge stronger bonds between Germany and the Nordic countries in Didactics of Mathematics and to develop collaboration in many ways.

Doctoral student exchange could for example be one excellent way to do this. Another could be academic teacher exchange. Guest researchers and visiting scholars are important features. German researchers are welcomed to continue to participate in the Nordic activities like conferences and courses. They are encouraged to publish in NOMAD and create joint research studies with colleagues from the Nordic countries. Such collaboration will be fruitful for all who take part.

5.5.2 Perspectives on Collaborative Empirical Research in Germany and in Poland

(By Edyta Nowinska)

Do Germany and Poland have some joint historical roots in the development of Didactics of Mathematics? Do researchers in Didactics of Mathematics in both countries collaborate on empirical research? What are the perspective on collaborative empirical research in Germany and in Poland in the future?

I have been asked these questions many times by my German colleagues interested in learning the past and the current development in Didactics of Mathematics in Poland. The geographical location of both countries might lead to the presumption of close relations between German and Polish researchers in Didactics of Mathematics. A closer look on the institutional context of their work shows, however, that such relations do not really have the form of a close cooperation, and it also gives some explanations for this fact. In the following, I give a short overview of some historical developments in Didactics of Mathematics in Poland and on the institutional context in which the most Polish researchers in Didactics of Mathematics work. Afterwards I use this background information to explain perspectives on collaborative empirical research in Germany and in Poland.

The overview given by Rudolf Sträßer on the development of German speaking didactics of mathematics points to names of German institutes and researchers who played an essential role in this development. Some of them were known in Poland due to international contacts of the Polish Professor Anna Zofia Krygowska—probably the most important person who contributed to the development of Didactics of Mathematics in Poland at the end of 1950s. She had contacts to Hans-Georg Steiner and to the Bauersfeld group from the Bielefeld *Institut für Didaktik der Mathematik*. As many other researchers in Germany, Krygowska was engaged at this time in establishing Didactics of Mathematics as a scientific discipline. In 1958 the Methods of Teaching Mathematics Department (later called the Department of Didactics of Mathematics) was created within the Faculty of Mathematics and Physics of the Pedagogical University in Cracow and Anna Zofia Krygowska was appointed to a professorship in this department.

In 1982, Krygowska succeeded in establishing the first Polish journal publishing work dealing with didactics of mathematics—Dydaktyka Matematyki ('Didactics of Mathematics') issued as the Fifth Series of *Roczniki Polskiego Towarzystwa Matem-*

atycznego ('Annals of the Polish Mathematical Society'). Since foreign journals publishing such works were not available in Poland, one important goal of the new Polish journal was publishing of English, French and German language articles dealing with Didactics of Mathematics (translated into Polish). The international work of Anna Zofia Krygowska and in her contacts to the German speaking researchers did not have the form of a collaborative empirical research, but the international discourse was important for Krygowska in searching for new ideas for the developing scientific discipline and in making her own ideas for teaching and learning mathematics more precise.

When reflecting on the current perspectives on collaborative empirical research in Germany and in Poland, one has to be conscious about the research tradition in Didactics of Mathematics initiated by Krygowska and about the institutional context of Didactics of Mathematics in Poland.

Stefan Turnau, one of Krygowska's successor describes Krygowska's work as follows: "She always thought that school mathematics should be genuine mathematics, whatever the teaching level. She also praised logical rigour, which view she embodied in her rigorous geometry textbooks. But on the other hand she was prudent enough not to allow Bourbakism to prevail in the curriculum." (1988, pp. 421–422).

Krygowska tried to create a theoretical and methodological base for the new branch of knowledge. On the core of her research were always mathematical concepts and ideas. Her research was characterized by a strong epistemic component—by attention to mathematical meanings and mathematical understandings specific to particular concepts. Her focus was mainly on the didactics of *mathematics* and less *psychology* or *sociology* of teaching and learning mathematics. The research field established by Krygowska was quite homogeneous and this seems to be a specific characteristic of research in Didactics of Mathematics in Poland to date and is also reflected in recent publications written in English. They concentrate on conceptions related to learning functions, limits, proofs, on algebraic thinking, generalization and elementary geometry or vocational education (the dominant method is thereby the case study). This seems to be a quite stable and typical characteristic of research in Didactics of Mathematics in Poland.

This characteristic can be understood if one considers the institutional context of work of Polish researchers in Didactics of Mathematics. Most of them work in institutes for mathematics and their work is evaluated (and measured in points based amongst others on publications in high ranking journals) due to the same criteria as the work of mathematicians working in these institutes. According to these criteria, publications in proceedings issued after international conferences in mathematics education—which are important for initiating and maintaining a collaborative work with other researchers in Didactics of Mathematics—have no value and therefore it is quite difficult to get financial support for the participation in such conferences (the institutional context of some of its consequences for research in Didactics of Mathematics in Poland were discussed in the panel session during the Ninth Congress of the European Society for Research in Mathematics Education in Prague; see the comments written by Marta Pytlak in Jaworski et al. (2015, p. 24). This is the first reason why conducting research related to psychological or social perspectives on teaching and learning mathematics in schools may be very problematic in this context. The second reason, even more important, is the fact that Didactics of Mathematics does not still have the status of a scientific discipline in Poland, despite of the previously mentioned tremendous work of prof. Krygowska and her successors like Ewa Swoboda and the whole Polish society of Didactics of Mathematics researchers. Consequently there is also no well-established tradition in funding empirical research on the quality of teaching and learning of mathematics in schools or on in-service teacher professional development programs or on video-based classroom research. This kind of research does not really match neither the criteria of mathematical nor pedagogical research. Thus, the institutional context has a strong influence on the work of Polish researchers in mathematics. It excludes some research questions which are essentially important for investigating and improving teaching and learning mathematics as being not valuable in the institutional context.

Many individual Polish researchers in Didactics of Mathematics use their private contacts to German researchers to learn and discuss new trends in the development of Didactics of Mathematics. The German conception of mathematics education as a ‘design science’ (Wittmann, 1992) and the construct of ‘substantial learning environments’ have been intensively used in some researchers’ groups in Poland (see Jagoda, 2009), in particular in geometry. One can say that the German Didactics of Mathematics often inspires and enriches the work of individual Polish researchers but this has not been a part of a collaboration as far. If the institutional criteria for evaluating the work of researchers in Didactics of Mathematics in Poland and for funding research projects in Didactics of Mathematics do not change, such a collaboration seems to be possible only for passion and has to cope with difficulties in recognizing its value in the institutional context.

5.5.3 Didaktik der Mathematik and Didaktika Matematiky

(by Nad̆a Vondrová)

The aim of this part is to show concrete examples of how Czech *didaktika matematiky* have been influenced by German *mathematik didaktik* (or better, by individual German researchers).

The connection between the Czech (or previously Czechoslovak) and German didactics must be divided into two periods; prior to the Velvet Revolution in 1989 and after it.

Prior to 1989, the landscape of Czechoslovak teaching of mathematics was similar to that of the German Democratic Republic (see Sect. 5.2.4). There was very little connection with Western Germany, mainly because of few opportunities to travel or to have an access to international journals and books. Only a limited number of people was allowed to travel abroad and attend conferences such as ICME. Many of them were from the Faculty of Mathematics and Physics, Charles University (e.g., Jaroslav Šedivý, Oldřich Odvárko, Leoš Boček, Jiří Mikulčák and Jan Vyšín).

Jan Vyšín was also a member of Wissenschaftlicher Beirat ZDM and both he and Oldřich Odvárko were members of the editorial board of Zentralblatt für Didaktik der Mathematik. On the other hand, some German researchers such as Hans- Georg Steiner or Roland Stowasser came to visit the Faculty of Mathematics and Physics and held lectures there. I must also mention Hans-Georg Steiner's personal efforts to help researchers from Czechoslovakia, which resulted in the authorities allowing the organisation of the International Symposium on Research and Development in Mathematics Education in Bratislava in 1988 (for the proceedings see Steiner & Hejný, 1988). The event opened the door for Czechoslovak researchers to western research and new cooperation.

Western influences became more prominent in former Czechoslovakia at the time of New Math movement. Czech Mathematicians and mathematics educators closely followed the New Math movement abroad, mostly in East Germany but also in Western countries. Some articles were published in Czech journals by Miloš Jelínek and others about New Math abroad which influenced the efforts in former Czechoslovakia.

After the Velvet Revolution, completely new opportunities arose for Czech researchers. In terms of German influence, I must mention conferences like the Tagung für Didaktik der Mathematik organised by the German Gesellschaft für Didaktik der Mathematik (GDM), where Czech researchers such as František Kuřina, Milan Koman, Leoš Boček, Oldřich Odvárko, Marie Tichá, Alena Hošpesová and others were invited to present their work. Many of them were invited to give lectures at German universities (e.g. at the Mathematikdidaktisches Kolloquium at TU Dortmund). This was often accompanied by financial support through Deutsche Forschungsgemeinschaft which especially in the first years of a new regime was an important help.

The empirical turn to everyday classrooms mentioned in Sect. 5.2.3 appeared in the Czech Republic as well under the influence of Western (also German) research (see Sect. 5.2.3). Stoffdidaktik (see Sect. 5.2.1) has always been close to the Czech conception of mathematics education (from which mathematics never disappeared), and nowadays, Stoffdidaktik enlarged (see Sect. 5.3.1) taking into account the history and epistemology of mathematics and fundamental ideas of mathematics can also be found in the work of people such as Ladislav Kvasz.

From among German researchers who markedly influenced research in Czech mathematics education (as documented by publications of Czech authors), I have to name Erich Wittmann and the project Mathe 2000. The idea of mathematics as a design science resonates in Czech research (for example, in the work by František Kuřina), as well as the idea of substantial learning environments. It is often cited, for example, by Alena Hošpesová, Marie Tichá, Nad̆a Vondrová, and mainly Milan Hejný for whom „the design of learning environments indeed is the defining 'kernel' of research in mathematics education" (Hejný, 2012; Hošpesová et al., 2010; Stehlíková, Hejný, & Jirotková, 2011).

Other German researchers whose work is well known in the Czech Republic and used in Czech research include E. Cohors-Fresenborg, K. Hasemann, H. Meissner, G. Muller, P. Scherer, H. Steinbring, B. Wollring, E. Glasserfeld or H. Freudenthal (who had German roots). Obviously, I only mention researchers whose influence in the former Czechoslovakia can clearly be seen in publications and presentations of researchers. Surely, there were others whose work influenced research in mathematics education for individuals and who cannot be listed here.

Finally, I would like to mention common projects of Czech and German researchers which have had an impact on mathematics education both in research and in teaching in the Czech Republic. For example:

Understanding of mathematics classroom culture in different countries (Czech Republic, Germany, Italy; M. Tichá, A. Hošpesová, P. Scherer, H. Steinbring and others): The goal of the project was to improve the quality of continuous in-service education of primary school teachers. The role of a qualified joint reflection which was heavily conceptualised by the German colleagues was stressed in this process.

Motivation via *Natural Differentiation in Mathematics* (Czech Republic, Germany, Netherlands, Poland; G. Krauthausen, P. Scherer, M. Tichá, A. Hošpesová): Wittmann's concept of substantial learning environments was in the centre of the project. The team members developed the idea of a 'new' kind of differentiation starting in the first school years which is expected to contribute to a deeper understanding of what constitutes mathematics learning, by considering the learners' individual personalities and the advantage of learning in groups, as opposed to minimizing the individual differences among the pupils (see Hošpesová et al., 2010).

Implementation of Innovative Approaches to the Teaching of Mathematics (Czech Republic, United Kingdom, Germany, Greece; M. Hejný, D. Jirotková, N. Vondrová, B. Wollring, B. Spindeler and others): The project aimed to promote constructivist teaching approaches in mathematics, to change the role of the teacher in the classroom and to make the pupils more responsible for their learning. The team members developed tasks and trialled them with cooperating teachers (see Hejný et al., 2006). The main contribution of the German colleagues lied in designing elementary mathematics tasks which connected geometry, arithmetic and relations to the real world of pupils within a learning environment of regular polygons.

Communicating Own Strategies in Primary Mathematics Education (Czech Republic, Germany, the United Kingdom; M. Hejný, J. Kratochvílová, B. Wollring, A. Peter-Koop): The project focused on the design of classroom materials to structure working environments which prepare student teachers as well as classroom teachers to motivate, moderate, record and analyse classroom communication about shape, number and structure (see Cockburn, 2007). German colleagues' unique contribution was in their focus on scientific analyses of pupils' artefacts by "student teachers as researchers".

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Appendix

Table of contents of an ICME-13 book on „German-speaking Traditions in Mathematics Education Research“
Editors: Hans Niels Jahnke, Lisa Hefendehl-Hebeker.
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