

# Chapter 12

## Continuity Properties of Multilinear Localization Operators on Modulation Spaces



Nenad Teofanov

**Abstract** We introduce multilinear localization operators in terms of the short-time Fourier transform and multilinear Weyl pseudodifferential operators. We prove that such localization operators are in fact Weyl pseudodifferential operators whose symbols are given by the convolution between the symbol of the localization operator and the multilinear Wigner transform. To obtain such interpretation, we use the kernel theorem for the Gelfand–Shilov space  $\mathcal{S}^{(1)}(\mathbb{R}^d)$  and its dual space of tempered ultra-distributions  $\mathcal{S}'^{(1)}(\mathbb{R}^{2d})$ . Furthermore, we study the continuity properties of the multilinear localization operators on modulation spaces. Our results extend some known results when restricted to the linear case.

### 12.1 Introduction

Multilinear localization operators were first introduced in [8], and their continuity properties are formulated in terms of modulation spaces. The key point is the interpretation of these operators as multilinear Kohn–Nirenberg pseudodifferential operators. The multilinear pseudodifferential operators were already studied in the context of modulation spaces in [1]; see also a more recent contribution [24] where such approach is strengthened and applied to the bilinear and trilinear Hilbert transforms.

Our approach is related to Weyl pseudodifferential operators instead, with another (Weyl) correspondence between the operator and its symbol. Both correspondences are particular cases of the so-called  $\tau$ –pseudodifferential operators,  $\tau \in [0, 1]$ . For  $\tau = 1/2$  we obtain Weyl operators, while for  $\tau = 0$  we recapture Kohn–Nirenberg operators. We refer to [7, 10] for the recent contribution in that context (see also the references given there).

The Weyl correspondence provides an elegant interpretation of localization operators as Weyl pseudodifferential operators. This is given by the formula that contains

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N. Teofanov (✉)

Department of Mathematics and Informatics, University of Novi Sad, Novi Sad, Serbia  
e-mail: [nenad.teofanov@dmf.uns.ac.rs](mailto:nenad.teofanov@dmf.uns.ac.rs)

the Wigner transform which is, together with the short-time Fourier transform, the main tool in our investigations. We refer to [17, 41] for more details on the Wigner transform.

In signal analysis, different localization techniques are used to describe signals which are as concentrated as possible in general regions of the phase space. This motivated I. Daubechies to address these questions by introducing certain localization operators in the pioneering contribution [14]. Afterward, Cordero and Grochenig made an essential contribution in the context of time–frequency analysis [6]. Among other things, their results emphasized the role played by modulation spaces in the study of localization operators.

In this paper, we first recall the basic facts on modulation spaces in Sect. 12.2. Then, in Sect. 12.3, following the definition of bilinear localization operators given in [33] we introduce multilinear localization operators, Definition 12.2. Then we define the multilinear Weyl pseudodifferential operators and give their weak formulation in terms of the multilinear Wigner transform (Lemma 12.2). By using the kernel theorem for Gelfand–Shilov spaces, Theorem 12.1, we prove that the multilinear localization operators can be interpreted as multilinear Weyl pseudodifferential operators in the same way as in the linear case, Theorem 12.5.

In Sect. 12.4 we first recall two results from [9]: (multilinear version of) sharp integral bounds for the Wigner transform, Theorem 12.6, and continuity properties of pseudodifferential operators on modulation spaces, Theorem 12.8. These results, in combination with the convolution estimates for modulation spaces from [38], Theorem 12.3, are then used to prove the main result of the continuity properties of multilinear localization operators on modulation spaces, Theorem 12.9.

**Notation.** The Schwartz space of rapidly decreasing smooth functions is denoted by  $\mathcal{S}(\mathbb{R}^d)$ , and its dual space of tempered distributions is denoted by  $\mathcal{S}'(\mathbb{R}^d)$ . We use the brackets  $\langle f, g \rangle$  to denote the extension of the inner product  $\langle f, g \rangle = \int f(t)g(t)dt$  on  $L^2(\mathbb{R}^d)$  to any pair of dual spaces. The Fourier transform is normalized to be

$$\hat{f}(\omega) = \mathcal{F} f(\omega) = \int f(t)e^{-2\pi i t\omega} dt.$$

The involution  $f^*$  is  $f^*(\cdot) = \overline{f(-\cdot)}$ , and the convolution of  $f$  and  $g$  is given by  $f * g(x) = \int f(x - y)g(y)dy$ , when the integral exists.

We denote by  $\langle \cdot \rangle^s$  the polynomial weights

$$\langle (x, \omega) \rangle^s = (1 + |x|^2 + |\omega|^2)^{s/2}, \quad (x, \omega) \in \mathbb{R}^{2d}, \quad s \in \mathbb{R},$$

and  $\langle x \rangle = \langle 1 + |x|^2 \rangle^{1/2}$ , when  $x \in \mathbb{R}^d$ .

We use the notation  $A \lesssim B$  to indicate that  $A \leq cB$  for a suitable constant  $c > 0$ , whereas  $A \asymp B$  means that  $c^{-1}A \leq B \leq cA$  for some  $c \geq 1$ .

**The Gelfand–Shilov space and Weyl pseudodifferential operators.** The Gelfand–Shilov-type space of analytic functions  $\mathcal{S}^{(1)}(\mathbb{R}^d)$  is given by

$$f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \iff \sup_{x \in \mathbb{R}^d} |f(x)e^{h \cdot |x|}| < \infty \text{ and } \sup_{\omega \in \mathbb{R}^d} |\hat{f}(\omega)e^{h \cdot |\omega|}| < \infty, \quad \forall h > 0.$$

Any  $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$  can be extended to a holomorphic function  $f(x + iy)$  in the strip  $\{x + iy \in \mathbb{C}^d : |y| < T\}$  some  $T > 0$ , [18, 25]. The dual space of  $\mathcal{S}^{(1)}(\mathbb{R}^d)$  will be denoted by  $\mathcal{S}^{(1)'}(\mathbb{R}^d)$ .

The space  $\mathcal{S}^{(1)}(\mathbb{R}^d)$  is nuclear, and we will use the following kernel theorem in the context of  $\mathcal{S}^{(1)}(\mathbb{R}^d)$ .

**Theorem 12.1** *Let  $\mathcal{L}_b(\mathcal{A}, \mathcal{B})$  denote the space of continuous linear mappings between the spaces  $\mathcal{A}$  and  $\mathcal{B}$  (equipped with the topology of bounded convergence). Then the following isomorphisms hold:*

1.  $\mathcal{S}^{(1)}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^{(1)}(\mathbb{R}^{d_2}) \cong \mathcal{S}^{(1)}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\mathcal{S}^{(1)'}(\mathbb{R}^{d_1}), \mathcal{S}^{(1)}(\mathbb{R}^{d_2})),$
2.  $\mathcal{S}^{(1)'}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^{(1)'}(\mathbb{R}^{d_2}) \cong \mathcal{S}^{(1)'}(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\mathcal{S}^{(1)}(\mathbb{R}^{d_1}), \mathcal{S}^{(1)'}(\mathbb{R}^{d_2})).$

Theorem 12.1 is a special case of [31, Theorem 2.5], see also [27], so we omit the proof. We refer to the classical reference [40] for kernel theorems and nuclear spaces, and in particular to Theorem 51.6 and its corollary related to  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^d)$ , which will be used later on.

By the isomorphisms in Theorem 12.1 2. it follows that for a given kernel distribution  $k(x, y)$  on  $\mathbb{R}^{d_1+d_2}$  we may associate a continuous linear mapping  $k$  of  $\mathcal{S}^{(1)}(\mathbb{R}^{d_2})$  into  $\mathcal{S}^{(1)'}(\mathbb{R}^{d_1})$  as follows:

$$\langle k_\varphi, \phi \rangle = \langle k(x, y), \phi(x)\varphi(y) \rangle, \quad \phi \in \mathcal{S}^{(1)}(\mathbb{R}^{d_1}),$$

which is commonly written as  $k_\varphi(\cdot) = \int k(\cdot, y)\varphi(y)dy$ . The correspondence between  $k(x, y)$  and  $k$  is an isomorphism and this fact will be used in the proof of Theorem 12.5.

Let  $\sigma \in \mathcal{S}^{(1)}(\mathbb{R}^{2d})$ . Then the Weyl pseudodifferential operator  $L_\sigma$  with the Weyl symbol  $\sigma$  can be defined as the oscillatory integral:

$$L_\sigma f(x) = \iint \sigma\left(\frac{x+y}{2}, \omega\right) f(y) e^{2\pi i(x-y)\cdot\omega} dy d\omega, \quad f \in \mathcal{S}^{(1)}(\mathbb{R}^d).$$

This definition extends to each  $\sigma \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$ , so that  $L_\sigma$  is a continuous mapping from  $\mathcal{S}^{(1)}(\mathbb{R}^d)$  to  $\mathcal{S}^{(1)'}(\mathbb{R}^d)$ , cf. [19, Lemma 14.3.1] If

$$W(f, g)(x, \omega) = \int f\left(x + \frac{t}{2}\right) \overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i\omega t} dt, \quad f, g \in \mathcal{S}^{(1)}(\mathbb{R}^d), \tag{12.1.1}$$

denotes the Wigner transform, also known as the cross-Wigner distribution, then the following formula holds:

$$\langle L_\sigma f, g \rangle = \langle \sigma, W(g, f) \rangle, \quad f, g \in \mathcal{S}^{(1)}(\mathbb{R}^d),$$

for each  $\sigma \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$ ; see e.g., [16, 19, 41].

### 12.2 Modulation Spaces

In this section, we collect some facts on modulation spaces which will be used in Sect. 12.4. First, we introduce the short-time Fourier transform in terms of duality between the Gelfand–Shilov space  $\mathcal{S}^{(1)}(\mathbb{R}^d)$  and its dual space of tempered ultra-distributions  $\mathcal{S}^{(1)' }(\mathbb{R}^{2d})$  as follows.

The short-time Fourier transform (STFT in the sequel) of  $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$  with respect to the window  $g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus 0$  is defined by

$$V_g f(x, \omega) = \langle f, M_\omega T_x g \rangle = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt, \tag{12.2.1}$$

where the translation operator  $T_x$  and the modulation operator  $M_\omega$  are given by

$$T_x f(\cdot) = f(\cdot - x) \text{ and } M_\omega f(\cdot) = e^{2\pi i \omega \cdot} f(\cdot) \quad x, \omega \in \mathbb{R}^d. \tag{12.2.2}$$

The map  $(f, g) \mapsto V_g f$  from  $\mathcal{S}^{(1)}(\mathbb{R}^d) \otimes \mathcal{S}^{(1)}(\mathbb{R}^d)$  to  $\mathcal{S}^{(1)}(\mathbb{R}^{2d})$  extends uniquely to a continuous operator from  $\mathcal{S}^{(1)' }(\mathbb{R}^d) \otimes \mathcal{S}^{(1)' }(\mathbb{R}^d)$  to  $\mathcal{S}^{(1)' }(\mathbb{R}^{2d})$  by duality, cf. [12, Theorem 4.1], [37, proposition 1.8].

Moreover, for a fixed  $g \in \mathcal{S}^{(1)}(\mathbb{R}^d) \setminus 0$  the following characterization holds:

$$f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \iff V_g f \in \mathcal{S}^{(1)}(\mathbb{R}^{2d}).$$

We recall the notation from [33] related to the bilinear case. For given  $\varphi_1, \varphi_2, f_1, f_2 \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ , we put

$$\begin{aligned} V_{\varphi_1 \otimes \varphi_2}(f_1 \otimes f_2)(x, \omega) &= \int_{\mathbb{R}^{2d}} f_1(t_1) f_2(t_2) \overline{M_{\omega_1} T_{x_1} \varphi_1(t_1) M_{\omega_2} T_{x_2} \varphi_2(t_2)} dt_1 dt_2 \\ &= \int_{\mathbb{R}^{2d}} (f_1 \otimes f_2)(t) \overline{M_{\omega_1} T_{x_1} \varphi_1 \otimes M_{\omega_2} T_{x_2} \varphi_2}(t) dt, \end{aligned} \tag{12.2.3}$$

where  $x = (x_1, x_2), \omega = (\omega_1, \omega_2), t = (t_1, t_2), x_1, x_2, \omega_1, \omega_2, t_1, t_2 \in \mathbb{R}^d$ .

To give an interpretation of multilinear operators in the weak sense we note that, if  $\mathbf{f} = (f_1, f_2, \dots, f_n)$  and  $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_n), f_j, \varphi_j \in \mathcal{S}^{(1)}(\mathbb{R}^d), j = 1, 2, \dots, n$ , then (12.2.3) becomes

$$V_{\boldsymbol{\varphi}} \mathbf{f}(x, \omega) = \int_{\mathbb{R}^{nd}} \mathbf{f}(t) \prod_{j=1}^n \overline{M_{\omega_j} T_{x_j} \varphi_j(t_j)} dt, \tag{12.2.4}$$

see also (12.3.1) for the notation.

We refer to [23, 30–32, 37] for more details on STFT in other spaces of Gelfand–Shilov type. Since we restrict ourselves to weighted modulation spaces with polynomial weights in this paper, we proceed by using the duality between  $\mathcal{S}$  and  $\mathcal{S}'$  instead of the more general duality between  $\mathcal{S}^{(1)}$  and  $\mathcal{S}^{(1)'}$ . Related results in the

framework of subexponential and superexponential weights can be found in, e.g., [11, 12, 31, 37], and we leave the study of multilinear localization operators in that case for a separate contribution.

Modulation spaces [15, 19] are defined through decay and integrability conditions on STFT, which makes them suitable for time–frequency analysis, and for the study of localization operators in particular. They are defined in terms of weighted mixed-norm Lebesgue spaces.

In general, a weight  $w(\cdot)$  on  $\mathbb{R}^d$  is a nonnegative and continuous function. The weighted Lebesgue space  $L_w^p(\mathbb{R}^d)$ ,  $p \in [1, \infty]$ , is the Banach space with the norm

$$\|f\|_{L_w^p} = \|fw\|_{L^p} = \left( \int |f(x)|^p w(x)^p dx \right)^{1/p},$$

and with the usual modification when  $p = \infty$ . When  $w(x) = \langle x \rangle^t$ ,  $t \in \mathbb{R}$ , we use the notation  $L_t^p(\mathbb{R}^d)$  instead.

Similarly, the weighted mixed-norm Lebesgue space  $L_w^{p,q}(\mathbb{R}^{2d})$ ,  $p, q \in [1, \infty]$ , consists of (Lebesgue) measurable functions on  $\mathbb{R}^{2d}$  such that

$$\|F\|_{L_w^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |F(x, \omega)|^p w(x, \omega)^p dx \right)^{q/p} d\omega \right)^{1/q} < \infty.$$

where  $w(x, \omega)$  is a weight on  $\mathbb{R}^{2d}$ .

In particular, when  $w(x, \omega) = \langle x \rangle^t \langle \omega \rangle^s$ ,  $s, t \in \mathbb{R}$ , we use the notation  $L_w^{p,q}(\mathbb{R}^{2d}) = L_{s,t}^{p,q}(\mathbb{R}^{2d})$ .

Now, modulation space  $M_{s,t}^{p,q}(\mathbb{R}^d)$  consists of distributions whose STFT belongs to  $L_{s,t}^{p,q}(\mathbb{R}^{2d})$ :

**Definition 12.1** Let  $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ ,  $s, t \in \mathbb{R}$ , and  $p, q \in [1, \infty]$ . The *modulation space*  $M_{s,t}^{p,q}(\mathbb{R}^d)$  consists of all  $f \in \mathcal{S}'(\mathbb{R}^d)$  such that

$$\|f\|_{M_{s,t}^{p,q}} \equiv \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_\phi f(x, \omega) \langle x \rangle^t \langle \omega \rangle^s|^p dx \right)^{q/p} d\omega \right)^{1/q} < \infty$$

(with obvious interpretation of the integrals when  $p = \infty$  or  $q = \infty$ ).

In special cases, we use the usual abbreviations:  $M_{0,0}^{p,p} = M^p$ ,  $M_{t,t}^{p,p} = M_t^p$ , etc.

For the consistency, and according to (12.2.4), we denote by  $\mathcal{M}_{s,t}^{p,q}(\mathbb{R}^{nd})$  the set of  $\mathbf{f} = (f_1, f_2, \dots, f_n)$ ,  $f_j \in \mathcal{S}'(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ , such that

$$\|\mathbf{f}\|_{\mathcal{M}_{s,t}^{p,q}} \equiv \left( \int_{\mathbb{R}^{nd}} \left( \int_{\mathbb{R}^{nd}} |V_\varphi \mathbf{f}(x, \omega) \langle x \rangle^t \langle \omega \rangle^s|^p dx \right)^{q/p} d\omega \right)^{1/q} < \infty, \quad (12.2.5)$$

where  $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ ,  $\varphi_j \in \mathcal{S}(\mathbb{R}^d) \setminus 0$ ,  $j = 1, 2, \dots, n$ , is a given  $n$ -tuple of window functions.

The kernel theorem for  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^d)$  (see [40]) implies that there is an isomorphism between  $\mathcal{M}_{s,t}^{p,q}(\mathbb{R}^{nd})$  and  $M_{s,t}^{p,q}(\mathbb{R}^{nd})$  (which commutes with the operators from (12.2.2)). This allows us to identify  $\mathbf{f} \in \mathcal{M}_{s,t}^{p,q}(\mathbb{R}^{nd})$  with (its isomorphic image)  $F \in M_{s,t}^{p,q}(\mathbb{R}^{nd})$  (and vice versa). We will use this identification whenever convenient and without further explanation.

*Remark 12.1* The original definition of modulation spaces given in [15] deals with more general *submultiplicative* weights. We restrict ourselves to the weights of the form  $w(x, \omega) = \langle x \rangle^t \langle \omega \rangle^s$ ,  $s, t \in \mathbb{R}$ , since the convolution and multiplication estimates which will be used later on are formulated in terms of weighted spaces with such polynomial weights. As already mentioned, weights of exponential type growth are used in the study of Gelfand–Shilov spaces and their duals in cf. [11, 23, 30, 37]. We refer to [20] for a survey on the most important types of weights commonly used in time–frequency analysis.

The following theorem lists some basic properties of modulation spaces. We refer to [15, 19] for the proof.

**Theorem 12.2** *Let  $p, q, p_j, q_j \in [1, \infty]$  and  $s, t, s_j, t_j \in \mathbb{R}$ ,  $j = 1, 2$ . Then:*

1.  $M_{s,t}^{p,q}(\mathbb{R}^d)$  are Banach spaces, independent of the choice of  $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ ;
2. if  $p_1 \leq p_2, q_1 \leq q_2, s_2 \leq s_1$  and  $t_2 \leq t_1$ , then

$$\mathcal{S}(\mathbb{R}^d) \subseteq M_{s_1,t_1}^{p_1,q_1}(\mathbb{R}^d) \subseteq M_{s_2,t_2}^{p_2,q_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d);$$

3.  $\cap_{s,t} M_{s,t}^{p,q}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$ ,  $\cup_{s,t} M_{s,t}^{p,q}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d)$ ;
4. For  $p, q \in [1, \infty)$ , the dual of  $M_{s,t}^{p,q}(\mathbb{R}^d)$  is  $M_{-s,-t}^{p',q'}(\mathbb{R}^d)$ , where  $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$ .

Modulation spaces include the following well-known function spaces:

1.  $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ , and  $M_{t,0}^2(\mathbb{R}^d) = L_t^2(\mathbb{R}^d)$ ;
2. The Feichtinger algebra:  $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$ ;
3. Sobolev spaces:  $M_{0,s}^2(\mathbb{R}^d) = H_s^2(\mathbb{R}^d) = \{f \mid \hat{f}(\omega) \langle \omega \rangle^s \in L^2(\mathbb{R}^d)\}$ ;
4. Shubin spaces:  $M_s^2(\mathbb{R}^d) = L_s^2(\mathbb{R}^d) \cap H_s^2(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$ , cf. [28].

To deal with duality when  $pq = \infty$  we observe that, by a slight modification of [1, Lemma 2.2] the following is true.

**Lemma 12.1** *Let  $L^0(\mathbb{R}^{2nd})$  denote the space of bounded, measurable functions on  $\mathbb{R}^{2nd}$  which vanish at infinity and put*

$$\begin{aligned} \mathcal{M}^{0,q}(\mathbb{R}^{nd}) &= \{\mathbf{f} \in \mathcal{M}^{\infty,q}(\mathbb{R}^{nd}) \mid V_\phi \mathbf{f} \in L^0(\mathbb{R}^{2nd})\}, \quad 1 \leq q < \infty, \\ \mathcal{M}^{p,0}(\mathbb{R}^{nd}) &= \{\mathbf{f} \in \mathcal{M}^{p,\infty}(\mathbb{R}^{nd}) \mid V_\phi \mathbf{f} \in L^0(\mathbb{R}^{2nd})\}, \quad 1 \leq p < \infty, \\ \mathcal{M}^{0,0}(\mathbb{R}^{nd}) &= \{\mathbf{f} \in \mathcal{M}^{\infty,\infty}(\mathbb{R}^{nd}) \mid V_\phi \mathbf{f} \in L^0(\mathbb{R}^{2nd})\}, \end{aligned}$$

equipped with the norms of  $\mathcal{M}^{\infty,q}$ ,  $\mathcal{M}^{p,\infty}$  and  $\mathcal{M}^{\infty,\infty}$  respectively. Then,

1.  $\mathcal{M}^{0,q}$  is  $\mathcal{M}^{\infty,q}$ -closure of  $\mathcal{S}$  in  $\mathcal{M}^{\infty,q}$ , hence is a closed subspace of  $\mathcal{M}^{\infty,q}$ . Likewise for  $\mathcal{M}^{p,0}$  and  $\mathcal{M}^{0,0}$ .
2. The following duality results hold for  $1 \leq p, q < \infty$ :  $(\mathcal{M}^{0,q})' = \mathcal{M}^{1,q'}$ ,  $(\mathcal{M}^{p,0})' = \mathcal{M}^{p',1}$ , and  $(\mathcal{M}^{0,0})' = \mathcal{M}^{1,1}$ .

From now on, we will use these duality relations in the cases  $p = \infty$  and/or  $q = \infty$  without further explanations.

For the results on multiplication and convolution in modulation spaces and in weighted Lebesgue spaces, we first introduce the *Young functional*:

$$\mathbf{R}(p) = \mathbf{R}(p_0, p_1, p_2) \equiv 2 - \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{p_2}, \quad p = (p_0, p_1, p_2) \in [1, \infty]^3. \tag{12.2.6}$$

When  $\mathbf{R}(p) = 0$ , the Young inequality for convolution reads as

$$\|f_1 * f_2\|_{L^{p'_0}} \leq \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad f_j \in L^{p_j}(\mathbb{R}^d), \quad j = 1, 2.$$

The following theorem is an extension of the Young inequality to the case of weighted Lebesgue spaces and modulation spaces when  $0 \leq \mathbf{R}(p) \leq 1/2$ .

**Theorem 12.3** *Let  $s_j, t_j \in \mathbb{R}$ ,  $p_j, q_j \in [1, \infty]$ ,  $j = 0, 1, 2$ . Assume that  $0 \leq \mathbf{R}(p) \leq 1/2$ ,  $\mathbf{R}(q) \leq 1$ ,*

$$0 \leq t_j + t_k, \quad j, k = 0, 1, 2, \quad j \neq k, \tag{12.2.7}$$

$$0 \leq t_0 + t_1 + t_2 - d \cdot \mathbf{R}(p), \text{ and} \tag{12.2.8}$$

$$0 \leq s_0 + s_1 + s_2, \tag{12.2.9}$$

with strict inequality in (12.2.8) when  $\mathbf{R}(p) > 0$  and  $t_j = d \cdot \mathbf{R}(p)$  for some  $j = 0, 1, 2$ .

Then  $(f_1, f_2) \mapsto f_1 * f_2$  on  $C_0^\infty(\mathbb{R}^d)$  extends uniquely to a continuous map from

1.  $L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$  to  $L^{p'_0}_{-t_0}(\mathbb{R}^d)$ ;
2.  $M^{p_1, q_1}_{s_1, t_1}(\mathbb{R}^d) \times M^{p_2, q_2}_{s_2, t_2}(\mathbb{R}^d)$  to  $M^{p'_0, q'_0}_{-s_0, -t_0}(\mathbb{R}^d)$ .

For the proof, we refer to [38]. It is based on the detailed study of an auxiliary three-linear map over carefully chosen regions in  $\mathbb{R}^d$  (see Sects. 3.1 and 3.2 in [38]). This result extends multiplication and convolution properties obtained in [26]. Moreover, the sufficient conditions from Theorem 12.3 are also necessary in the following sense.

**Theorem 12.4** *Let  $p_j, q_j \in [1, \infty]$  and  $s_j, t_j \in \mathbb{R}$ ,  $j = 0, 1, 2$ . Assume that at least one of the following statements hold true:*

1. The map  $(f_1, f_2) \mapsto f_1 * f_2$  on  $C_0^\infty(\mathbb{R}^d)$  is continuously extendable to a map from  $L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$  to  $L^{p'_0}_{-t_0}(\mathbb{R}^d)$ ;

2. The map  $(f_1, f_2) \mapsto f_1 * f_2$  on  $C_0^\infty(\mathbb{R}^d)$  is continuously extendable to a map from  $M_{s_1, t_1}^{p_1, q_1}(\mathbb{R}^d) \times M_{s_2, t_2}^{p_2, q_2}(\mathbb{R}^d)$  to  $M_{-s_0, -t_0}^{p'_0, q'_0}(\mathbb{R}^d)$ ;

Then (12.2.7) and (12.2.8) hold true.

### 12.3 Multilinear Localization Operators

In this section, we introduce multilinear localization operators in Definition 12.2 and show that they can be interpreted as particular Weyl pseudodifferential operators, Theorem 12.5. We also introduce multilinear Weyl pseudodifferential operators and prove their connection to the multilinear Wigner transform in Lemma 12.2. This is done in the context of the duality between  $\mathcal{S}^{(1)}(\mathbb{R}^d)$  and  $\mathcal{S}^{(1)' }(\mathbb{R}^d)$  and carried out verbatim to the duality between  $\mathcal{S}(\mathbb{R}^d)$  and  $\mathcal{S}'(\mathbb{R}^d)$  in the next section.

The localization operator  $A_a^{\varphi_1, \varphi_2}$  with the symbol  $a \in L^2(\mathbb{R}^{2d})$  and with windows  $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$  can be defined in terms of the short-time Fourier transform (12.2.1) as follows:

$$A_a^{\varphi_1, \varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x, \omega) V_{\varphi_1} f(x, \omega) M_\omega T_x \varphi_2(t) dx d\omega, \quad f \in L^2(\mathbb{R}^d).$$

To define multilinear localization operators, we slightly abuse the notation (as it is done in, e.g., [24]) so that  $\mathbf{f}$  will denote both the vector  $\mathbf{f} = (f_1, f_2, \dots, f_n)$  and the tensor product  $\mathbf{f} = f_1 \otimes f_2 \otimes \dots \otimes f_n$ . This will not cause confusion, since the meaning of  $\mathbf{f}$  will be clear from the context.

For example, if  $t = (t_1, t_2, \dots, t_n)$ , and  $F_j = F_j(t_j)$ ,  $t_j \in \mathbb{R}^d$ ,  $j = 1, 2, \dots, n$ , then

$$\prod_{j=1}^n F_j(t_j) = F_1(t_1) \cdot F_2(t_2) \cdot \dots \cdot F_n(t_n) = F_1(t_1) \otimes F_2(t_2) \otimes \dots \otimes F_n(t_n) = \mathbf{F}(t). \tag{12.3.1}$$

**Definition 12.2** Let  $f_j \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ , and  $\mathbf{f} = (f_1, f_2, \dots, f_n)$ . The multilinear localization operator  $A_a^{\varphi, \phi}$  with symbol  $a \in \mathcal{S}^{(1)' }(\mathbb{R}^{2nd})$  and windows

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \text{ and } \phi = (\phi_1, \phi_2, \dots, \phi_n), \quad \varphi_j, \phi_j \in \mathcal{S}^{(1)}(\mathbb{R}^d), \quad j = 1, 2, \dots, n,$$

is given by

$$A_a^{\varphi, \phi} \mathbf{f}(t) = \int_{\mathbb{R}^{2nd}} a(x, \omega) \prod_{j=1}^n (V_{\varphi_j} f_j(x_j, \omega_j) M_{\omega_j} T_{x_j} \phi_j(t_j)) dx d\omega, \tag{12.3.2}$$

where  $x_j, \omega_j, t_j \in \mathbb{R}^d$ ,  $j = 1, 2, \dots, n$ , and  $x = (x_1, x_2, \dots, x_n)$ ,  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ ,  $t = (t_1, t_2, \dots, t_n)$ .



*Remark 12.2* When  $n = 2$  in Definition 12.2 we obtain the bilinear localization operators studied in [33]. (There is a typo in [33, Definition 1]; the integration in (9) should be taken over  $\mathbb{R}^{4d}$ .)

Let  $\mathcal{R}$  denote the trace mapping that assigns to each function  $F$  defined on  $\mathbb{R}^{nd}$  a function defined on  $\mathbb{R}^d$  by the formula

$$\mathcal{R} : F \mapsto F \Big|_{t_1=t_2=\dots=t_n}, \quad t_j \in \mathbb{R}^d, \quad j = 1, 2, \dots, n.$$

Then  $\mathcal{R}A_a^{\varphi,\phi}$  is the multilinear operator given in [8, Definition 2.2].

By (12.2.4) it follows that the weak definition of (12.3.2) is given by

$$\langle A_a^{\varphi,\phi} \mathbf{f}, \mathbf{g} \rangle = \langle aV_\varphi \mathbf{f}, V_\phi \mathbf{g} \rangle = \langle a, \overline{V_\varphi \mathbf{f}} V_\phi \mathbf{g} \rangle, \tag{12.3.3}$$

and  $f_j, g_j \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ . The brackets can be interpreted as duality between a suitable pair of dual spaces. Thus,  $A_a^{\varphi,\phi}$  is well-defined continuous operator from  $\mathcal{S}^{(1)}(\mathbb{R}^{nd})$  to  $(\mathcal{S}^{(1)})'(\mathbb{R}^{nd})$ .

Next, we introduce a class of multilinear Weyl pseudodifferential operators ( $\Psi$ DO for short) and use the Wigner transform to prove appropriate interpretation of multilinear localization operators as multilinear Weyl pseudodifferential operators, Theorem 12.5.

Recall that in [8], multilinear localization operators are introduced in connection to Kohn–Nirenberg  $\Psi$ DOs instead.

By analogy with the bilinear Weyl pseudodifferential operators given in [33], we define the multilinear Weyl pseudodifferential operator as follows:

$$L_\sigma(\mathbf{f})(x) = \int_{\mathbb{R}^{2nd}} \sigma\left(\frac{x+y}{2}, \omega\right) \mathbf{f}(y) e^{2\pi i \mathcal{S}(x-y) \cdot \omega} dy d\omega, \quad x \in \mathbb{R}^{nd}, \tag{12.3.4}$$

where  $\sigma \in \mathcal{S}^{(1)'}(\mathbb{R}^{2nd})$ ,  $\mathbf{f}(y) = \prod_{j=1}^n f_j(y_j)$ ,  $f_j \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ . Here  $\mathcal{S}$  denotes the identity matrix in  $nd$ , that is,  $\mathcal{S}(x-y) \cdot \omega = \sum_{j=1}^n (x_j - y_j) \omega_j$ .

Similarly, the bilinear Wigner transform from [33] extends to

$$W(\mathbf{f}, \mathbf{g})(x, \omega) = \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \left( f_j\left(x_j + \frac{t_j}{2}\right) \overline{g_j\left(x_j - \frac{t_j}{2}\right)} \right) e^{-2\pi i \mathcal{S} \omega t} dt, \tag{12.3.5}$$

where  $f_j, g_j \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ ,  $x_j, \omega_j, t_j \in \mathbb{R}^d$ ,  $j = 1, 2, \dots, n$ , and  $x = (x_1, x_2, \dots, x_n)$ ,  $\omega = (\omega_1, \omega_2, \dots, \omega_n)$ ,  $t = (t_1, t_2, \dots, t_n)$ .

It is easy to see that  $W(\mathbf{f}, \mathbf{g}) \in \mathcal{S}^{(1)}(\mathbb{R}^{2nd})$ , when  $\mathbf{f}, \mathbf{g} \in \mathcal{S}^{(1)}(\mathbb{R}^{nd})$ .

**Lemma 12.2** *Let  $\sigma \in \mathcal{S}^{(1)}(\mathbb{R}^{2nd})$  and  $f_j, g_j \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ . Then  $L_\sigma$  given by (12.3.4) extends to a continuous map from  $\mathcal{S}^{(1)}(\mathbb{R}^{nd})$  to  $(\mathcal{S}^{(1)})'(\mathbb{R}^{nd})$ .*

$$\langle L_\sigma \mathbf{f}, \mathbf{g} \rangle = \langle \sigma, W(\mathbf{g}, \mathbf{f}) \rangle.$$

*Proof* The proof follows by the straightforward calculation:

$$\begin{aligned} \langle \sigma, W(\mathbf{g}, \mathbf{f}) \rangle &= \int_{\mathbb{R}^{2nd}} \sigma(x, \omega) W(\mathbf{f}, \mathbf{g})(x, \omega) dx d\omega \\ &= \int_{\mathbb{R}^{3nd}} \sigma(x, \omega) \prod_{j=1}^n \left( f_j(x_j + \frac{t_j}{2}) \overline{g_j(x_j - \frac{t_j}{2})} \right) e^{-2\pi i \mathcal{S} \omega t} dt dx d\omega \\ &= \int_{\mathbb{R}^{6d}} \sigma\left(\frac{u+v}{2}, \omega\right) \prod_{j=1}^n \left( f_j(v_j) \overline{g_j(u_j)} \right) e^{-2\pi i \mathcal{S} (u-v)\omega} du dv d\omega \\ &= \langle \sigma\left(\frac{u+v}{2}, \omega\right) \mathbf{f}(v) e^{2\pi i \mathcal{S} (u-v)\omega}, \mathbf{g}(u) \rangle = \langle L_\sigma \mathbf{f}, \mathbf{g} \rangle, \end{aligned}$$

where we used  $W(\mathbf{g}, \mathbf{f}) = \overline{W(\mathbf{f}, \mathbf{g})}$  and the change of variables  $u = x + \frac{t}{2}$ ,  $v = x - \frac{t}{2}$ . This extends to each  $\sigma \in \mathcal{S}^{(1)'}(\mathbb{R}^{2nd})$ , since  $W(\mathbf{f}, \mathbf{g}) \in \mathcal{S}^{(1)}(\mathbb{R}^{2nd})$  when  $f_j, g_j \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ . □

The so-called Weyl connection between the set of linear localization operators and Weyl  $\Psi$ DOs is well known; we refer to, e.g., [4, 16, 32]. The corresponding Weyl connection in bilinear case is established in [33, Theorem 4]. The proof is quite technical and based on the kernel theorem for Gelfand–Shilov spaces (see, e.g., [27, 31, 39]) and direct calculations. Since the proof of the following Theorem 12.5 is its straightforward extension, here we only sketch the main ideas. The conclusion of Theorem 12.5 is that any multilinear localization operator can be viewed as a particular multilinear Weyl  $\Psi$ DOs, as expected.

**Theorem 12.5** *Let there be given  $a \in \mathcal{S}^{(1)'}(\mathbb{R}^{2d})$  and let  $\phi = (\phi_1, \phi_2, \dots, \phi_n)$ ,  $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ ,  $\phi_j, \varphi_j \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ . Then the localization operator  $A_a^{\phi, \varphi}$  is the Weyl pseudodifferential operator with the Weyl symbol*

$$\sigma = a * W(\phi, \varphi) = a * \left( \prod_{j=1}^n W(\phi_j, \varphi_j) \right).$$

Therefore, if  $\mathbf{f} = (f_1, f_2, \dots, f_n)$ ,  $\mathbf{g} = (g_1, g_2, \dots, g_n)$ ,  $f_j, g_j \in \mathcal{S}^{(1)'}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ , then

$$\langle A_a^{\phi, \varphi} \mathbf{f}, \mathbf{g} \rangle = \langle L_{a * W(\phi, \varphi)} \mathbf{f}, \mathbf{g} \rangle.$$

*Proof* The formal expressions given below are justified due to the absolute convergence of the involved integrals and the standard interpretation of oscillatory integrals in distributional setting. We refer to [33, Sect. 5] for this and for detailed calculations.

The calculations from the proof of [33, Theorem 4] yield the following kernel representation of (12.3.3):

$$\langle A_a^{\phi, \varphi} \mathbf{f}, \mathbf{g} \rangle = \langle k, \prod_{j=1}^n \overline{f_j} \otimes \prod_{j=1}^n g_j \rangle,$$

where the kernel  $k = k(t, s)$  is given by

$$k(t, s) = \int_{\mathbb{R}^{2nd}} a(x, \omega) \prod_{j=1}^n \overline{M_{\omega_j} T_{x_j} \varphi_j(t)} \cdot \prod_{j=1}^n M_{\omega_j} T_{x_j} \varphi_j(s) dx d\omega, \quad (12.3.6)$$

$t = (t_1, t_2, \dots, t_n), s = (s_1, s_2, \dots, s_n), t_j, s_j \in \mathbb{R}^d, j = 1, 2, \dots, n.$

To calculate the convolution  $a * (\prod_{j=1}^n W(\phi_j, \varphi_j)) = a * W(\phi, \varphi)$ , we use  $W(g, f) = \overline{W(f, g)}$ , the commutation relation  $T_x M_\omega = e^{-2\pi i x \cdot \omega} M_\omega T_x$ , and the covariance property of the Wigner transform:

$$W(T_{x_j} M_{\omega_j} \phi_j, T_{x_j} M_{\omega_j} \varphi_j)(p_j, q_j) = W(\phi_j, \varphi_j)(p_j - x_j, q_j - \omega_j), \quad j = 1, 2, \dots, n.$$

Let  $p = (p_1, p_2, \dots, p_n), q = (q_1, q_2, \dots, q_n), p_j, q_j \in \mathbb{R}^d, j = 1, 2, \dots, n.$  Then,

$$a * W(\phi, \varphi)(p, q) = \int_{\mathbb{R}^{2nd}} a(x, \omega) \times \left( \int_{\mathbb{R}^{nd}} \prod_{j=1}^n M_{\omega_j} T_{x_j} \phi_j(p_j + \frac{t_j}{2}) \cdot \prod_{j=1}^n \overline{M_{\omega_j} T_{x_j} \varphi_j(p_j - \frac{t_j}{2})} e^{-2\pi i q \cdot t} dt \right) dx d\omega, \quad (12.3.7)$$

where  $q \cdot t$  denotes the scalar product of  $q, t \in \mathbb{R}^d$ , cf. [33, Sect. 5].

Therefore,

$$\begin{aligned} \langle L_{a * W(\phi, \varphi)} \mathbf{f}, \mathbf{g} \rangle &= \langle a * \prod_{j=1}^n W(\phi_j, \varphi_j), W(\mathbf{g}, \mathbf{f}) \rangle = \int_{\mathbb{R}^{2nd}} a(x, \omega) \times \\ &\int_{\mathbb{R}^{nd}} \left( \int_{\mathbb{R}^{nd}} \prod_{j=1}^n M_{\omega_j} T_{x_j} \phi_j(p_j + \frac{t_j}{2}) \cdot \prod_{j=1}^n \overline{M_{\omega_j} T_{x_j} \varphi_j(p_j - \frac{t_j}{2})} \times \right. \\ &\left. \prod_{j=1}^n f_j(p_j - \frac{t_j}{2}) \cdot \prod_{j=1}^n \overline{g_j(p_j + \frac{t_j}{2})} dt \right) dp dx d\omega, \end{aligned}$$

Finally, after performing the change of variables we obtain

$$\langle L_{a * W(\phi, \varphi)} \mathbf{f}, \mathbf{g} \rangle = \langle k, \prod_{j=1}^n \overline{f_j} \otimes \prod_{j=1}^n g_j \rangle,$$

where the kernel  $k$  is given by (12.3.6). The theorem now follows from the uniqueness of the kernel representation, Theorem 12.1. □

### 12.4 Continuity Properties of Localization Operators

We first recall the sharp estimates of the modulation space norm for the cross-Wigner distribution given in [9]. There it is shown that the sufficient conditions for the continuity of the cross-Wigner distribution on modulation spaces are also necessary (in the unweighted case). Related results can be found elsewhere, e.g., in [32, 34, 35]. In many situations, such results overlap. For example, Proposition 10 in [33] coincides with certain sufficient conditions from [9, Theorem 1.1] when restricted to  $\mathbf{R}(\mathbf{p}) = 0$ ,  $t_0 = -t_1$ , and  $t_2 = |t_0|$ .

**Theorem 12.6** *Let there be given  $s \in \mathbb{R}$  and  $p_i, q_i, p, q \in [1, \infty]$ , such that*

$$p \leq p_i, q_i \leq q, \quad i = 1, 2 \tag{12.4.1}$$

and

$$\min \left\{ \frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q_1} + \frac{1}{q_2} \right\} \geq \frac{1}{p} + \frac{1}{q}. \tag{12.4.2}$$

If  $f, g \in \mathcal{S}(\mathbb{R}^d)$ , then the map  $(f, g) \mapsto W(f, g)$  where  $W$  is the cross-Wigner distribution given by (12.1.1) extends to sesquilinear continuous map from  $M_{|s|}^{p_1, q_1}(\mathbb{R}^d) \times M_s^{p_2, q_2}(\mathbb{R}^d)$  to  $M_{s,0}^{p, q}(\mathbb{R}^{2d})$  and

$$\|W(f, g)\|_{M_{s,0}^{p, q}} \lesssim \|f\|_{M_{|s|}^{p_1, q_1}} \|g\|_{M_s^{p_2, q_2}}. \tag{12.4.3}$$

Viceversa, if there exists a constant  $C > 0$  such that

$$\|W(f, g)\|_{M^{p, q}} \lesssim \|f\|_{M^{p_1, q_1}} \|g\|_{M^{p_2, q_2}}.$$

then (12.4.1) and (12.4.2) must hold.

*Proof* We omit the proof which is given in [9, Sect. 3] and recall here only the main formulas which highlight its most important parts.

The first formula is the well-known relation between the Wigner transform and the STFT (see [19, Lemma 4.3.1]):

$$W(f, g)(x, \omega) = 2^d e^{4\pi i x \cdot \omega} V_{\overline{g}} f(2x, 2\omega), \quad f, g \in \mathcal{S}(\mathbb{R}^d).$$

To estimate the modulation space norm of  $W(f, g)(x, \omega)$ , we fix  $\psi_1, \psi_2 \in \mathcal{S}(\mathbb{R}^d) \setminus 0$  and use the fact that modulation spaces are independent on the choice of the window function from  $\mathcal{S}(\mathbb{R}^{2d}) \setminus 0$ , Theorem 12.2 1. By choosing the window to be  $W(\psi_1, \psi_2)$ , after some calculations we obtain:

$$\begin{aligned} (V_{W(\psi_1, \psi_2)} W(g, f))(z, \xi) &= e^{-2\pi i z_2 \xi_2} \overline{V_{\psi_1} f} \left( z_1 + \frac{\xi_2}{2}, z_2 - \frac{\xi_1}{2} \right) V_{\psi_2} g \left( z_1 - \frac{\xi_2}{2}, z_2 + \frac{\xi_1}{2} \right), \end{aligned}$$

cf. the proof of [19, Lemma 14.5.1 (b)]. Consequently (cf. [9, Sect. 3]),

$$\begin{aligned} \|W(g, f)\|_{M_{s,0}^{p,q}} &\asymp \left( \int_{\mathbb{R}^{2d}} (|V_{\psi_1} f|^p * |V_{\psi_2} g^*|^p)^{q/p}(\zeta_2, -\zeta_1) \langle (\zeta_2, -\zeta_1) \rangle^{sq} d\zeta \right)^{1/q} \\ &= \| |V_{\psi_1} f|^p * |V_{\psi_2} g^*|^p \|_{L_{ps,0}^{q/p}}. \end{aligned}$$

Then one proceeds with a careful case study to obtain (12.4.3) when (12.4.1) and (12.4.2) hold true. We refer to [9] for details. □

From the inspection of the proof of Theorem 12.6 given in [9, Sect. 3], the definition of  $W(\mathbf{f}, \mathbf{g})$  given by (12.3.5), and the use of the kernel theorem, we conclude the following.

**Corollary 12.1** *Let the assumptions of Theorem 12.6 hold. If  $\mathbf{f} = (f_1, f_2, \dots, f_n)$ ,  $\mathbf{g} = (g_1, g_2, \dots, g_n)$  and  $f_j, g_j \in \mathcal{S}(\mathbb{R}^d)$ ,  $j = 1, 2, \dots, n$ , then the map  $(\mathbf{f}, \mathbf{g}) \mapsto W(\mathbf{f}, \mathbf{g})$ , where  $W$  is the cross-Wigner distribution given by (12.3.5) extends to a continuous map from  $\mathcal{M}_{|s|}^{p_1, q_1}(\mathbb{R}^d) \times \mathcal{M}_s^{p_2, q_2}(\mathbb{R}^d)$  to  $\mathcal{M}_{s,0}^{p,q}(\mathbb{R}^{2d})$ , where the modulation spaces are given by (12.2.5).*

Next, we give an extension of [19, Theorem 14.5.2] and [33, Theorem 14] to the multilinear Weyl  $\Psi$ DOs. Recall, if  $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$  is the Weyl symbol of  $L_\sigma$ , then [19, Theorem 14.5.2] says that  $L_\sigma$  is bounded on  $M^{p,q}(\mathbb{R}^d)$ ,  $1 \leq p, q \leq \infty$ . This result has a long history starting from the Calderon–Vaillancourt theorem on boundedness of the pseudodifferential operators with smooth and bounded symbols on  $L^2(\mathbb{R}^d)$ , [5]. It is generalized by Sjöstrand in [29] where  $M^{\infty,1}$  is used as appropriate symbol class. Sjöstrand’s results were thereafter extended in [19, 21, 22, 34–36]. Moreover, we refer to [1–3] for the multilinear Kohn–Nirenberg  $\Psi$ DOs and the recent contribution [10] related to  $\tau$ - $\Psi$ DOs (these include both Kohn–Nirenberg (when  $\tau = 0$ ) and Weyl operators (when  $\tau = 1/2$ )).

The following fact related to symbols  $\sigma \in M^{\infty,1}(\mathbb{R}^{2nd})$  is a straightforward extension of [33, Theorem 14].

**Theorem 12.7** *Let  $\sigma \in M^{\infty,1}(\mathbb{R}^{2nd})$  and let  $L_\sigma$  be given by (12.3.4). The operator  $L_\sigma$  is bounded from  $\mathcal{M}^{p,q}(\mathbb{R}^{nd})$  to  $\mathcal{M}^{p,q}(\mathbb{R}^{nd})$ ,  $1 \leq p, q \leq \infty$ , with a uniform estimate  $\|L_\sigma\|_{op} \leq \|\sigma\|_{M^{\infty,1}}$  for the operator norm.*

On the other hand, Theorem 12.7 is a special case of [9, Theorem 5.1.] if  $L_\sigma$  is a linear operator. Here, we give the multilinear version of [9, Theorem 5.1.].

**Theorem 12.8** *Let there be given  $s \geq 0$  and  $p_i, q_i, r_i, p, q \in [1, \infty]$ , such that*

$$q \leq \min\{p'_1, q'_1, p_2, q_2\} \tag{12.4.4}$$

and

$$\min \left\{ \frac{1}{p_1} + \frac{1}{p'_2}, \frac{1}{q_1} + \frac{1}{q'_2} \right\} \geq \frac{1}{p'} + \frac{1}{q'}. \tag{12.4.5}$$

Then the operator  $L_\sigma$  given by (12.3.4) with symbol  $\sigma \in M_{s,0}^{p,q}(\mathbb{R}^{2nd})$ , from  $\mathcal{S}(\mathbb{R}^{nd})$  to  $\mathcal{S}'(\mathbb{R}^{nd})$ , extends uniquely to a bounded operator from  $\mathcal{M}_{s,0}^{p_1,q_1}(\mathbb{R}^{nd})$  to  $\mathcal{M}_{s,0}^{p_2,q_2}(\mathbb{R}^{nd})$ , with the estimate

$$\|L_\sigma \mathbf{f}\|_{\mathcal{M}_{s,0}^{p_2,q_2}} \lesssim \|\sigma\|_{M_{s,0}^{p,q}} \|\mathbf{f}\|_{\mathcal{M}_{s,0}^{p_1,q_1}}. \tag{12.4.6}$$

In particular, when  $\sigma \in M^{\infty,1}(\mathbb{R}^{2nd})$  we have  $\|L_\sigma\|_{op} \leq \|\sigma\|_{M^{\infty,1}}$  for the operator norm.

Vice versa, if (12.4.6) holds for  $s = 0$ , and for every  $\mathbf{f} \in \mathcal{S}(\mathbb{R}^{nd})$ ,  $\sigma \in \mathcal{S}'(\mathbb{R}^{2nd})$ , then (12.4.1) and (12.4.2) must be satisfied.

*Proof* The proof is a straightforward extension of the proof of [9, Theorem 5.1.], and we give it here for the sake of completeness.

When  $\mathbf{f} \in \mathcal{M}_{s,0}^{p_1,q_1}(\mathbb{R}^{nd})$  and  $\mathbf{g} \in \mathcal{M}_{s,0}^{p'_2,q'_2}(\mathbb{R}^{nd})$ , their Wigner transform  $W(\mathbf{g}, \mathbf{f}) = \overline{W(\mathbf{f}, \mathbf{g})}$  belongs to  $M_{-s,0}^{p',q'}$  since the conditions (12.4.1) and (12.4.2) of Theorem 12.6 are transferred to (12.4.4) and (12.4.5), respectively.

Now, Lemma 12.2 and the duality of modulation spaces give

$$\begin{aligned} |\langle L_\sigma \mathbf{f}, \mathbf{g} \rangle| &= |\langle \sigma, W(\mathbf{g}, \mathbf{f}) \rangle| \leq \|\sigma\|_{M_{s,0}^{p,q}} \|W(\mathbf{f}, \mathbf{g})\|_{M_{-s,0}^{p',q'}} \\ &\leq C \|\mathbf{f}\|_{\mathcal{M}_{s,0}^{p_1,q_1}} \|\mathbf{g}\|_{\mathcal{M}_{s,0}^{p'_2,q'_2}}, \end{aligned}$$

for some constant  $C > 0$  (and we used the fact that modulation spaces are closed under the complex conjugation).

We refer to [13, Theorem 1.1.] for the necessity of conditions (12.4.4) and (12.4.5) (in linear case). □

Next, we combine different results established so far to obtain an extension of [33, Theorem 15]. More precisely, we use the relation between the Weyl pseudodifferential operators and the localization operators (Lemma 12.5), the convolution estimates for modulation spaces (Theorem 12.3), and boundedness of pseudodifferential operators (Theorem 12.8) to obtain continuity results for  $A_a^{\varphi,\phi}$  for different choices of windows and symbols.

**Theorem 12.9** *Let there be given  $s \geq 0$  and  $p_i, q_i, p, q \in [1, \infty]$ ,  $i = 0, 1, 2$  such that (12.4.4) and (12.4.5) hold. Moreover, let  $q_0 \leq q$ , and*

$$p_0 \geq p \quad \text{if} \quad p \geq 2, \quad \text{and} \quad \frac{2p}{2-p} \geq p_0 \geq p \quad \text{if} \quad 2 > p \geq 1. \tag{12.4.7}$$

If  $\varphi \in \mathcal{M}_{2s,0}^{r_1}(\mathbb{R}^{nd})$ ,  $\phi \in \mathcal{M}_{2s,0}^{r_2}(\mathbb{R}^{nd})$ , where  $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$ , and  $a \in M_{s_0,t_0}^{p_0,q_0}(\mathbb{R}^{2nd})$  with  $s_0 \geq -s$ , and  $t_0 \geq d \left( \frac{1}{p} - \frac{1}{p_0} \right)$  with the strict inequality when  $p_0 = p$ , then  $A_a^{\varphi,\phi}$  is continuous from  $\mathcal{M}_{s,0}^{p_1,q_1}(\mathbb{R}^{nd})$  to  $\mathcal{M}_{s,0}^{p_2,q_2}(\mathbb{R}^{nd})$  with

$$\|A_a^{\varphi,\phi}\|_{op} \lesssim \|a\|_{M_{s_0,t_0}^{p_0,q_0}} \|\varphi\|_{\mathcal{M}_{2s,0}^{r_1}} \|\phi\|_{\mathcal{M}_{2s,0}^{r_2}}.$$

*Proof* We first estimate  $W(\phi, \varphi)$ . If  $\varphi \in \mathcal{M}_{2s,0}^{r_1}(\mathbb{R}^{nd})$ ,  $\phi \in \mathcal{M}_{2s,0}^{r_2}(\mathbb{R}^{nd})$ , with  $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$ , then Corollary 12.1 implies that

$$W(\phi, \varphi) \in \mathcal{M}_{2s,0}^{1,\infty}(\mathbb{R}^{2nd}).$$

Now, we use the result of Theorem 12.3 2. The Young functional (12.2.6) becomes  $R(p) = R(p', p_0, 1)$ , and the condition  $R(p) \in [0, 1/2]$  is equivalent to (12.4.7), while  $R(q) = R(q', q_0, \infty) \leq 1$  is equivalent to  $q_0 \leq q$ . Furthermore, (12.2.9) transfers to  $s_0 \geq -s$ , while (12.2.7) and (12.2.8) are equivalent to  $t_0 \geq d \left( \frac{1}{p} - \frac{1}{p_0} \right)$  with the strict inequality when  $p_0 = p$ . Therefore, the conditions of by Theorem 12.3 2 are fulfilled, and we obtain

$$a * W(\phi, \varphi) \in M_{s_0,t_0}^{p_0,q_0}(\mathbb{R}^{2nd}) * \mathcal{M}_{2s,0}^{1,\infty}(\mathbb{R}^{2nd}) \subset M_{s,0}^{p,q}(\mathbb{R}^{2nd}).$$

Finally, by Theorem 12.7 with  $\sigma = a * W(\phi, \varphi)$ , it follows that

$$\|A_a^{\varphi,\phi}\|_{op} = \|L_\sigma\|_{op} \leq \|\sigma\|_{M_{s,0}^{p,q}} \leq \|a\|_{M_{s_0,t_0}^{p_0,q_0}} \|\varphi\|_{\mathcal{M}_{2s,0}^{r_1}} \|\phi\|_{\mathcal{M}_{2s,0}^{r_2}},$$

and the Theorem is proved. □

In particular, we recover (the linear case treated in) [9, Theorem 5.2] when  $r_1 = r_2 = r$ ,  $t_0 = 0$ ,  $s_0 = -s$ ,  $p_0 = p$  (i.e.,  $R(p', p_0, 1) = 0$ ), and  $q_0 = q$  (i.e.,  $R(q', q_0, \infty) = 1$ ). Therefore, by [9, Remark 5.3], we obtain an extension of [6, Theorem 3.2] and [35, Theorem 4.11] for this particular choice of weights.

Note that conditions  $R(p', p_0, 1) \in (0, 1/2]$  which extend the possible choices of the Lebesgue parameters beyond the usual Young condition  $R(p', p_0, 1) = 0$  must be compensated by an additional condition to the weights, expressed by  $t_0 \geq d \left( \frac{1}{p} - \frac{1}{p_0} \right)$ .

Another result concerning the boundedness of (bilinear) localization operators on unweighted modulation spaces is given by [33, Theorem 15]. There we used different type of estimates, leading to the result which partially overlaps with Theorem 12.9. For example, both results give the same continuity property when the symbol  $a$  belongs to  $a \in M^{\infty,1}(\mathbb{R}^{2nd})$ .

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