Applied and Numerical Harmonic Analysis

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Editors

$$\hat{f}(\gamma) = \int f(x) e^{-2\pi i x \gamma} dx$$

Landscapes of Time–Frequency Analysis

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Applied and Numerical Harmonic Analysis

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Dedicated to our Families,
Their continuous support is our strength

ANHA Series Preface

The *Applied and Numerical Harmonic Analysis (ANHA)* book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time–frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis, but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time–frequency and time-scale analysis*
Numerical partial differential equations	*Wavelet theory*

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers, but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time–frequency–scale methods such as wavelet theory. The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

College Park

John J. Benedetto
Series Editor
University of Maryland

Preface

The first international conference *Aspects of Time–Frequency Analysis* "ATFA17" took place during 5–7 June 2017 at the Politecnico di Torino. It was a major international scientific event gathering many of the brightest stars in harmonic analysis and its applications. This meeting was jointly organized by Dipartimento di Matematica (Università di Torino), Dipartimento di Scienze Matematiche (Politecnico di Torino) and the Numerical Harmonic Analysis Group (NuHAG, Vienna).

The Organizing Committee consisted of Paolo Boggiatto, Elena Cordero, Alessandro Oliaro (University di Torino), Maurice de Gosson, Hans Feichtinger (Universität Wien), Enrico Magli, Fabio Nicola, and Anita Tabacco (Politecnico di Torino).

The financial support was granted by local funds from the Dipartimento di Matematica (Università di Torino) and Dipartimento di Scienze Matematiche (Politecnico di Torino); partial support was provided by Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni—GNAMPA (INDAM).

Topics included function spaces, time–frequency analysis and Gabor analysis, sampling theory and compressed sensing, mathematical signal processing, microlocal analysis, pseudodifferential and Fourier integral operators, numerical harmonic analysis, abstract harmonic analysis, and applications of harmonic analysis to quantum mechanics. This wide range of topics illustrates well the broadness of the scope of ATFA17. The given talks formed the heart of the conference and provided ample opportunity for fruitful discussions and led in some cases to new collaborations.

It is our duty and pleasure to thank all participants for their contributions to the conference program.

The present volume gathers written texts from invited participants, our choice covering the full range of the conference topics. It thus reflects well the spirit of ATFA17.

Organizing the volume and reminding late contributors was a challenging process, may Elena Cordero be praised for her patience and tenacity and careful proofreading!

We would also like to thank the Proceedings team for having invested so much time in very dedicated and professional work. The ATFA17 proceeding is a credit to a large group of people, and everybody should be proud of the outcome. The success of this conference means that we can now envisage with confidence the next event ATFA19 to be held in Turin in June 2019. We are sure that it will be as interesting and enjoyable as its predecessor.

Vienna, Austria
August 2018

Maurice de Gosson

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3. GNAMPA—Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni, INDAM, Italy.

One of the successes of ATFA17 (Aspects of Time–Frequency Analysis) has been the dynamic participation of graduate students and postdocs, pure and applied mathematicians, and scientists of other disciplines. We have been fortunate to be able to provide travel and living expenses to many participants due to the funds from the aforementioned institutions.

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Chapter 1
On the Probabilistic Cauchy Theory for Nonlinear Dispersive PDEs

Árpád Bényi, Tadahiro Oh and Oana Pocovnicu

Abstract In this note, we review some of the recent developments in the well-posedness theory of nonlinear dispersive partial differential equations with random initial data.

1.1 Introduction

Nonlinear dispersive partial differential equations (PDEs) naturally appear as models describing wave phenomena in various branches of physics and engineering such as quantum mechanics, nonlinear optics, plasma physics, water waves, and atmospheric sciences. They have received wide attention from the applied science community due to their importance in applications and have also been studied extensively from the theoretical point of view, providing a framework for the development of analytical ideas and tools.

The simplest yet most important examples of nonlinear dispersive PDEs are the following nonlinear Schrödinger equations (NLS):[1]

[1]For conciseness, we restrict our attention to the defocusing case in the following.

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$$\begin{cases} i\partial_t u = \Delta u - |u|^{p-1}u \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathcal{M} \tag{1.1.1}$$

and nonlinear wave equations (NLW):

$$\begin{cases} \partial_t^2 u = \Delta u - |u|^{p-1}u \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathcal{M}, \tag{1.1.2}$$

where $\mathcal{M} = \mathbb{R}^d$ or $\mathbb{T}^d$ and $p > 1$. Over the last several decades, multilinear harmonic analysis has played a crucial role in building basic insights on the study of nonlinear dispersive PDEs, settling questions on the existence of solutions to these equations, their long-time behavior, and singularity formation. Furthermore, in recent years, a remarkable combination of PDE techniques and probability theory has had a significant impact on the field. In this note, we go over some of the recent developments in this direction.

In the classical deterministic well-posedness theory, the main goal is to construct unique solutions for *all* initial data belonging to a certain fixed function space such as the L^2-based Sobolev spaces:

$$H^s(\mathcal{M}) \text{ for (1.1.1)} \quad \text{and} \quad \mathcal{H}^s(\mathcal{M}) := H^s(\mathcal{M}) \times H^{s-1}(\mathcal{M}) \text{ for (1.1.2)}.$$

In practice, however, we are often interested in the typical behavior of solutions. Namely, even if certain pathological behavior occurs, we may be content if we can show that almost all solutions behave well and do not exhibit such pathological behavior. This concept may be formalized in terms of probability. For example, in terms of well-posedness theory, one may consider an evolution equation with random initial data and try to construct unique solutions in an almost sure manner. This idea first appeared in Bourgain's seminal paper [9], where he constructed global-in-time solutions to NLS on $\mathbb{T}$ almost surely with respect to the random initial data distributed according to the Gibbs measure. See Sect. 1.3.2.

Such probabilistic construction of solutions with random initial data also allows us to go beyond deterministic thresholds in certain situations. First, recall that NLS (1.1.1) and NLW (1.1.2) enjoy the following dilation symmetry:

$$u(t,x) \longmapsto u^\lambda(t,x) = \lambda^{-\frac{2}{p-1}} u(\lambda^{-\alpha}t, \lambda^{-1}x), \tag{1.1.3}$$

with $\alpha = 2$ for (1.1.1) and $\alpha = 1$ for (1.1.2). Namely, if u is a solution to (1.1.1) or (1.1.2) on $\mathbb{R}^d$, then u^λ is also a solution to the same equation on $\mathbb{R}^d$ with the rescaled initial data. This dilation symmetry induces the following scaling-critical Sobolev regularity:

$$s_{\text{crit}} = \frac{d}{2} - \frac{2}{p-1} \tag{1.1.4}$$

such that the homogeneous $\dot{H}^{s_{\text{crit}}}(\mathbb{R}^d)$-norm is invariant under the dilation symmetry. This critical regularity s_{crit} provides a threshold regularity for well-posedness and ill-posedness of (1.1.1) and (1.1.2).[2] While there is no dilation symmetry in the periodic setting, the heuristics provided by the scaling argument also plays an important role. On the one hand, there is a good local-in-time theory for (1.1.1) and (1.1.2) (at least when the dimension d and the degree p are not too small). See [6, 62, 79] for the references therein. On the other hand, it is known that (1.1.1) and (1.1.2) are ill-posed in the supercritical regime: $s < s_{\text{crit}}$. See [21, 24, 26, 48, 62, 64, 76, 90]. Regardless of these ill-posedness results, by considering random initial data (see Sect. 1.2), one may still prove almost sure local well-posedness[3] in the supercritical regime. This probabilistic construction of local-in-time solutions was first implemented by Bourgain [10] in the context of NLS and by Burq–Tzvetkov [21] in the context of NLW. In more recent years, there have also been results on almost sure global well-posedness for these equations; see, for example, [22, 27, 47, 52, 63, 65, 79, 84]. See also the lecture note by Tzvetkov [88]. We will discuss some aspects of probabilistic well-posedness in Sect. 1.3.

1.2 On Random Initial Data

In this section, we go over random initial data based on random Fourier series on $\mathbb{T}^d$ and its analogue on $\mathbb{R}^d$.

1.2.1 Random Initial Data on $\mathbb{T}^d$

In the context of nonlinear dispersive PDEs, probabilistic construction of solutions was initiated in an effort to construct well-defined dynamics almost surely with respect to the Gibbs measure for NLS on $\mathbb{T}^d$, $d = 1, 2$ [9, 10, 54]. Before discussing this problem for NLS on $\mathbb{T}^d$, let us consider the following finite dimensional Hamiltonian flow on $\mathbb{R}^{2N}$:

$$\dot{p}_n = \frac{\partial H}{\partial q_n} \qquad \text{and} \qquad \dot{q}_n = -\frac{\partial H}{\partial p_n}, \tag{1.2.1}$$

[2] In fact, there are other critical regularities induced by the Galilean invariance for (1.1.1) and the Lorentzian symmetry for (1.1.2) below which the equations are ill-posed; see [25, 42, 51, 56]. We point out, however, that these additional critical regularities are relevant only when the dimension is low and/or the degree p is small. For example, for NLS (1.1.1) with an algebraic nonlinearity ($p \in 2\mathbb{N} + 1$), the critical regularity induced by the Galilean invariance is relevant (i.e., higher than the scaling-critical regularity s_{crit} in (1.1.4)) only for $d = 1$ and $p = 3$. For simplicity, we only consider the scaling-critical regularities in the following.

[3] Namely, local-in-time existence of unique solutions almost surely with respect to given random initial data.

$n = 1, \ldots, N$, with Hamiltonian $H(p, q) = H(p_1, \ldots, p_N, q_1, \ldots, q_N)$. Liouville's theorem states that the Lebesgue measure $dpdq = \prod_{n=1}^{N} dp_n dq_n$ on $\mathbb{R}^{2N}$ is invariant under the flow. Then, it follows from the conservation of the Hamiltonian $H(p, q)$ that the Gibbs measure:[4]

$$d\rho = Z^{-1} e^{-H(p,q)} dpdq$$

is invariant under the flow of (1.2.1). Recall that NLS (1.1.1) is a Hamiltonian PDE with the following Hamiltonian:

$$H(u) = \frac{1}{2} \int_{\mathscr{M}} |\nabla u|^2 dx + \frac{1}{p+1} \int_{\mathscr{M}} |u|^{p+1} dx. \tag{1.2.2}$$

Moreover, the mass $M(u)$ defined by

$$M(u) = \frac{1}{2} \int_{\mathscr{M}} |u|^2 dx \tag{1.2.3}$$

is conserved under the dynamics of (1.1.1). Then, by drawing an analogy to the finite dimensional case, one may expect that the Gibbs measure:[5]

$$\text{“} d\rho = Z^{-1} e^{-H(u) - M(u)} du \text{”} \tag{1.2.4}$$

is invariant under the flow of (1.1.1). Here, the expression in (1.2.4) is merely formal, where "du" denotes the nonexistent Lebesgue measure on an infinite dimensional phase space.

We first introduce a family of mean-zero Gaussian measures μ_s, $s \in \mathbb{R}$, on periodic distributions on $\mathbb{T}^d$, formally given by

$$d\mu_s = Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z^{-1} \prod_{n \in \mathbb{Z}^d} e^{-\frac{1}{2}\langle n \rangle^{2s} |\widehat{u}(n)|^2} d\widehat{u}(n), \tag{1.2.5}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. As we see in (1.2.7) below, the Gibbs measure ρ is constructed as the Gaussian measure μ_1 with a weight. While the expression $d\mu_s = Z^{-1} \exp(-\frac{1}{2}\|u\|_{H^s}^2) du$ may suggest that μ_s is a Gaussian measure on $H^s(\mathbb{T}^d)$, we need to enlarge a space in order to make sense of μ_s as a countably additive probability measure. In fact, the Gaussian measure μ_s defined above is the induced probability measure under the following map:[6]

---

[4]Hereafter, we use Z to denote various normalizing constants so that the resulting measure is a probability measure provided that it makes sense.

[5]Here, we added the mass in the exponent to avoid a problem at the zeroth frequency in (1.2.8) below.

[6]In the following, we drop the harmless factor of 2π.

$$\omega \in \Omega \longmapsto u^\omega(x) = u(x;\omega) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{in\cdot x}, \tag{1.2.6}$$

where $\{g_n\}_{n\in\mathbb{Z}^d}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathscr{F}, P)$. It is easy to check that the random function u^ω in (1.2.6) lies in

$$H^\sigma(\mathbb{T}^d) \text{ for } \sigma < s - \frac{d}{2} \quad \text{but} \quad \text{not in } H^{s-\frac{d}{2}}(\mathbb{T}^d),$$

almost surely. Moreover, for the same range of σ, μ_s is a Gaussian probability measure on $H^\sigma(\mathbb{T}^d)$ and the triplet $(H^s(\mathbb{T}^d), H^\sigma(\mathbb{T}^d), \mu_s)$ forms an abstract Wiener space. See [39, 49]. Note that, when $s = 1$, the random Fourier series (1.2.6) basically corresponds to the Fourier–Wiener series for the Brownian motion. See [4] for more on this.

By restricting our attention to $\mathbb{T}^d$, we substitute the expressions (1.2.2) and (1.2.3) in (1.2.4). Then, we formally obtain

$$\begin{aligned} d\rho &= Z^{-1} e^{-\frac{1}{p+1}\int |u|^{p+1}} e^{-\frac{1}{2}\|u\|_{H^1}^2} du \\ &= Z^{-1} e^{-\frac{1}{p+1}\int |u|^{p+1}} d\mu_1, \end{aligned} \tag{1.2.7}$$

where μ_1 is as in (1.2.5) with $s = 1$. When $d = 1$, it is easy to see that the Gibbs measure ρ is a well-defined probability measure, absolutely continuous with respect to the Gaussian measure μ_1. In particular, it is supported on $H^\sigma(\mathbb{T})$ for any $\sigma < \frac{1}{2}$. When $d = 2$, a typical element under μ_1 lies in $H^{-\varepsilon}(\mathbb{T}^2) \setminus L^2(\mathbb{T}^2)$ for any $\varepsilon > 0$. As such, the weight $e^{-\frac{1}{p+1}\int |u|^{p+1}}$ in (1.2.7) equals 0 almost surely and hence the expression (1.2.7) for ρ does not make sense as a probability measure. Nonetheless, when $p \in 2\mathbb{N}+1$, one can apply a suitable renormalization (the Wick ordering) and construct the Gibbs measure ρ associated with the Wick-ordered Hamiltonian such that ρ is absolutely continuous to μ_1.[7] See [10, 70, 71] for more on the renormalization in the two-dimensional case. This shows that when $d = 1, 2$, it is of importance to study the dynamical property of NLS (1.1.1) with the random initial data u_0^ω distributed according to the Gaussian measure μ_1, namely given by the random Fourier series (1.2.6) with $s = 1$:

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x} \in H^{1-\frac{d}{2}-\varepsilon}(\mathbb{T}^d) \setminus H^{1-\frac{d}{2}}(\mathbb{T}^d) \tag{1.2.8}$$

almost surely for any $\varepsilon > 0$. In [10], Bourgain studied the (renormalized) cubic NLS on $\mathbb{T}^2$ with the random initial data (1.2.8). Recalling that the two-dimensional

[7] When $d \geq 3$, it is known that the Gibbs measure ρ can be constructed only for $d = 3$ and $p = 3$. In this case, the resulting Gibbs measure ρ is not absolutely continuous with respect to the Gaussian measure μ_1. See [2] for the references therein, regarding the construction of the Gibbs measure (the Φ_3^4 measure) in the real-valued setting.

cubic NLS is L^2-critical, we see that this random initial data u_0^ω lies slightly below the critical regularity.[8] Nonetheless, by combining the deterministic analysis (the Fourier restriction norm method introduced in [8]) with the probabilistic tools, in particular, the probabilistic Strichartz estimates (Lemma 1.3), he managed to prove almost sure local well-posedness with respect to the random initial data u_0^ω in (1.2.8).

In the context of the cubic NLW on a three-dimensional compact Riemannian manifold $\mathscr{M}$, Burq–Tzvetkov [21] considered the Cauchy problem with a more general class of random initial data. In particular, given a rough initial data $(u_0, u_1) \in \mathcal{H}^s(\mathscr{M})$ with $s < s_{\text{crit}} = \frac{1}{2}$, they introduced a randomization (u_0^ω, u_1^ω) of the given initial data (u_0, u_1) and proved almost sure local well-posedness with respect to this randomization. For simplicity, we discuss this randomization for a single function u_0 on $\mathbb{T}^d$ in the following. Fix $u_0 \in H^s(\mathbb{T}^d)$. Then, we define a randomization u_0^ω of u_0 by setting

$$u_0^\omega(x) := \sum_{n \in \mathbb{Z}^d} g_n(\omega) \widehat{u}_0(n) e^{in\cdot x}, \tag{1.2.9}$$

where $\widehat{u}_0(n)$ denotes the Fourier coefficient of u_0 and $\{g_n\}_{n\in\mathbb{Z}^d}$ is a sequence of independent mean-zero complex-valued random variables with bounded moments up to a certain order.[9] Note that the random Fourier series in (1.2.6) and (1.2.8) can be viewed as a randomization of the particular function u_0 with the Fourier coefficient $\langle n \rangle^{-s}$ by independent standard Gaussian random variables $\{g_n\}_{n\in\mathbb{Z}^d}$. The main point of the randomization (1.2.9) is that while the randomized function u_0^ω does not enjoy any smoothing in terms of differentiability, it enjoys a gain of integrability (Lemma 1.2).

1.2.2 Probabilistic Strichartz Estimates

In this subsection, we discuss the effect of the randomization (1.2.9) introduced in the previous subsection. For simplicity, we further assume that the probability distributions $\mu_n^{(1)}$ and $\mu_n^{(2)}$ of the real and imaginary parts of g_n in (1.2.9) are independent and satisfy

$$\left| \int_{\mathbb{R}} e^{\gamma x} d\mu_n^{(j)}(x) \right| \le e^{c\gamma^2} \tag{1.2.10}$$

for all $\gamma \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$. Note that (1.2.10) is satisfied by standard complex-valued Gaussian random variables, Bernoulli random variables, and any random vari-

[8] In terms of the Besov spaces $B^\sigma_{p,\infty}$, $p < \infty$, we see that u_0^ω in (1.2.8) lies almost surely in the critical spaces $B^0_{2,\infty}$. See [4].

[9] In the real-valued setting, we also need to impose that $g_{-n} = \overline{g_n}$ so that, given a real-valued function u_0, the resulting randomization u_0^ω remains real-valued. A similar comment applies to the randomization (1.2.13) introduced for functions on $\mathbb{R}^d$.

ables with compactly supported distributions. Under this extra assumption (1.2.10), we have the following estimate. See [21] for the proof.

Lemma 1.1 *Assume* (1.2.10). *Then, there exists* $C > 0$ *such that*

$$\Big\| \sum_{n\in\mathbb{Z}^d} g_n(\omega) c_n \Big\|_{L^p(\Omega)} \le C\sqrt{p}\|c_n\|_{\ell^2_n(\mathbb{Z}^d)}$$

for all $p \ge 2$ *and* $\{c_n\} \in \ell^2(\mathbb{Z}^d)$.

When $\{g_n\}_{n\in\mathbb{Z}^d}$ is a sequence of independent standard Gaussian random variables, Lemma 1.1 follows from the Wiener chaos estimate (see Lemma 1.4 below) with $k = 1$.

Given $u_0 \in H^s(\mathbb{T}^d)$, it is easy to see that its randomization u_0^ω in (1.2.9) lies in $H^s(\mathbb{T}^d)$ almost surely. One can also show that, under some non-degeneracy condition, there is no smoothing upon randomization in terms of differentiability; see, for example, Lemma B.1 in [21]. The main point of the randomization (1.2.9) is its improved integrability.

Lemma 1.2 *Given* $u_0 \in L^2(\mathbb{T}^d)$, *let* u_0^ω *be its randomization defined in* (1.2.9), *satisfying* (1.2.10). *Then, given finite* $p \ge 2$, *there exist* $C, c > 0$ *such that*

$$P\Big(\|u_0^\omega\|_{L^p} > \lambda\Big) \le Ce^{-c\lambda^2\|u_0\|_{L^2}^{-2}}$$

for all $\lambda > 0$. *In particular,* u_0^ω *lies in* $L^p(\mathbb{T}^d)$ *almost surely.*

Such gain of integrability is well known for random Fourier series; see, for example, [3, 46, 78]. The proof of Lemma 1.2 is standard and follows from Minkowski's integral inequality, Lemma 1.1, and Chebyshev's inequality. See [5, 21, 27]. By a similar argument, one can also establish the following probabilistic improvement of the Strichartz estimates.

Lemma 1.3 *Given* u_0 *on* $\mathbb{T}^d$, *let* u_0^ω *be its randomization defined in* (1.2.9), *satisfying* (1.2.10). *Then, given finite* $q \ge 2$ *and* $2 \le r \le \infty$, *there exist* $C, c > 0$ *such that*

$$P\Big(\|e^{-it\Delta}u_0^\omega\|_{L^q_t L^r_x([0,T]\times\mathbb{T}^d)} > \lambda\Big) \le C\exp\bigg(-c\frac{\lambda^2}{T^{\frac{2}{q}}\|u_0\|_{H^s}^2}\bigg)$$

for all $T > 0$ *and* $\lambda > 0$ *with* (i) $s = 0$ *if* $r < \infty$ *and* (ii) $s > 0$ *if* $r = \infty$.

By setting $\lambda = T^\theta\|u_0\|_{L^2}$, we have

$$\|e^{-it\Delta}u_0^\omega\|_{L^q_t L^r_x([0,T]\times\mathbb{T}^d)} \lesssim T^\theta\|u_0\|_{L^2(\mathbb{T}^d)} \tag{1.2.11}$$

outside a set of probability at most $C\exp\big(-cT^{2\theta-\frac{2}{q}}\big)$. Note that, as long as $\theta < \frac{1}{q}$, this probability can be made arbitrarily small by letting $T \to 0$. We can interpret

(1.2.11) as the probabilistic improvement of the usual Strichartz estimates, where the indices q and r satisfy certain relations[10] and the resulting estimates come with possible loss of derivatives. See [16, 43]. In Lemma 1.3, we only stated the probabilistic Strichartz estimates for the Schrödinger equation. Similar probabilistic Strichartz estimates also hold for the wave equation. See [21, 22, 79].

On the one hand, the probabilistic Strichartz estimates in Lemma 1.3 allow us to exploit the randomization at the linear level. On the other hand, the following Wiener chaos estimate ([83, Theorem I.22]) allows us to exploit the randomization at a multilinear level. See also [86, Proposition 2.4].

Lemma 1.4 *Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a sequence of independent standard Gaussian random variables. Given $k \in \mathbb{N}$, let $\{P_j\}_{j\in\mathbb{N}}$ be a sequence of polynomials in $\bar{g} = \{g_n\}_{n\in\mathbb{Z}^d}$ of degree at most k. Then, for $p \geq 2$, we have*

$$\Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^p(\Omega)} \leq (p-1)^{\frac{k}{2}} \Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^2(\Omega)}.$$

This lemma is a direct corollary to the hypercontractivity of the Ornstein–Uhlenbeck semigroup due to Nelson [59]. It allows us to prove the following probabilistic improvement of Young's inequality. Such a probabilistic improvement was essential in the probabilistic construction of solutions to the (renormalized) cubic NLS on $\mathbb{T}^d$, $d = 1, 2$, in a low regularity setting [10, 27]. For simplicity, we consider a trilinear case.

Lemma 1.5 *Let $a_n, b_n, c_n \in \ell^2(\mathbb{Z}^d; \mathbb{C})$. Given a sequence $\{g_n\}_{n\in\mathbb{Z}^d}$ of independent standard complex-valued Gaussian random variables, define $a_n^\omega = g_n a_n$, $b_n^\omega = g_n b_n$, and $c_n^\omega = g_n c_n$, $n \in \mathbb{Z}^d$. Then, given $\varepsilon > 0$, there exists a set $\Omega_\varepsilon \subset \Omega$ with $P(\Omega_\varepsilon^c) < \varepsilon$ and $C_\varepsilon > 0$ such that*[11]

$$\big\| a_n^\omega * b_n^\omega * c_n^\omega \big\|_{\ell^2} \leq C_\varepsilon \|a_n\|_{\ell^2} \|b_n\|_{\ell^2} \|c_n\|_{\ell^2} \tag{1.2.12}$$

for all $\omega \in \Omega_\varepsilon$.

The proof of Lemma 1.5 is immediate from the following tail estimate:

$$P\bigg(\Big| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big| > \lambda \bigg) \leq C \exp\bigg(-c \Big\| \sum_{j\in\mathbb{N}} P_j(\bar{g}) \Big\|_{L^2(\Omega)}^{-\frac{2}{k}} \lambda^{\frac{2}{k}} \bigg),$$

which is a consequence of Lemma 1.4 and Chebyshev's inequality. Note that Young's inequality (without randomization) only yields

$$\big\| a_n * b_n * c_n \big\|_{\ell^2} \leq \|a_n\|_{\ell^2} \|b_n\|_{\ell^1} \|c_n\|_{\ell^1}.$$

---

[10] See (1.2.16) below for the scaling condition on $\mathbb{R}^d$.

[11] One can choose $C_\varepsilon = \big(\frac{1}{c} \log \frac{C}{\varepsilon}\big)^{\frac{3}{2}}$.

Recalling that $\ell^1 \subset \ell^2$, we see that there is a significant improvement in (1.2.12) under randomization of the sequences, which was a key in establishing crucial nonlinear estimates in a probabilistic manner in [10, 27].

1.2.3 Random Initial Data on $\mathbb{R}^d$

We conclude this section by briefly going over the randomization of a function on $\mathbb{R}^d$ analogous to (1.2.9) on $\mathbb{T}^d$. See [5, 52, 92]. On compact domains, there is a countable basis of eigenfunctions of the Laplacian and thus there is a natural way to introduce a randomization via (1.2.9). On the other hand, on $\mathbb{R}^d$, there is no countable basis of $L^2(\mathbb{R}^d)$ consisting of eigenfunctions of the Laplacian and hence there is no "natural" way to introduce a randomization. In the following, we discuss a randomization adapted to the so-called Wiener decomposition [89] of the frequency space: $\mathbb{R}^d = \bigcup_{n\in\mathbb{Z}^d} Q_n$, where Q_n is the unit cube centered at $n \in \mathbb{Z}^d$.

Let $\psi \in \mathscr{S}(\mathbb{R}^d)$ such that

$$\operatorname{supp}\psi \subset [-1, 1]^d \quad \text{and} \quad \sum_{n\in\mathbb{Z}^d} \psi(\xi - n) \equiv 1 \quad \text{for any } \xi \in \mathbb{R}^d.$$

Then, given a function u_0 on $\mathbb{R}^d$, we have

$$u_0 = \sum_{n\in\mathbb{Z}^d} \psi(D - n)u_0,$$

where $\psi(D-n)$ is defined by $\psi(D-n)u_0(x) = \int_{\mathbb{R}^d} \psi(\xi - n)\widehat{u}_0(\xi)e^{ix\cdot\xi}d\xi$, namely the Fourier multiplier operator with symbol $\mathbb{N}_{Q_n}$ conveniently smoothed. This decomposition leads to the following randomization of u_0 adapted to the Wiener decomposition. Let $\{g_n\}_{n\in\mathbb{Z}^d}$ be a sequence of independent mean-zero complex-valued random variables as in (1.2.9). Then, we can define the Wiener randomization[12] of u_0 by

$$u_0^\omega := \sum_{n\in\mathbb{Z}^d} g_n(\omega)\psi(D - n)u_0. \tag{1.2.13}$$

Compare this with the randomization (1.2.9) on $\mathbb{T}^d$. Under the assumption (1.2.10), Lemmas 1.2 and 1.3 also hold for the Wiener randomization (1.2.13) of a given function on $\mathbb{R}^d$. The proofs remain essentially the same with an extra ingredient of the following version of Bernstein's inequality:

$$\|\psi(D-n)u_0\|_{L^q(\mathbb{R}^d)} \lesssim \|\psi(D-n)u_0\|_{L^p(\mathbb{R}^d)}, \qquad 1 \le p \le q \le \infty, \tag{1.2.14}$$

[12] It is also called the unit-scale randomization in [33].

for all $n \in \mathbb{Z}^d$. The point of (1.2.14) is that once a function is (roughly) restricted to a unit cube, we do not need to lose any derivative to go from the L^q-norm to the L^p-norm, $q \geq p$. See [5] for the proofs of the analogues of Lemmas 1.2 and 1.3.

Note that the probabilistic Strichartz estimates in Lemma 1.3 hold only locally in time. On $\mathbb{T}^d$, this does not cause any loss since the usual deterministic Strichartz estimates also hold only locally in time. On the other hand, the Strichartz estimates on $\mathbb{R}^d$ hold globally in time:

$$\|e^{-it\Delta}u_0\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L_x^2(\mathbb{R}^d)} \tag{1.2.15}$$

for any Schrödinger admissible pair (q, r), satisfying

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \tag{1.2.16}$$

with $2 \leq q, r \leq \infty$ and $(q, r, d) \neq (2, \infty, 2)$. By incorporating the global-in-time estimate (1.2.15), one can also obtain the following global-in-time probabilistic Strichartz estimates.

Lemma 1.6 *Given $u_0 \in L^2(\mathbb{R}^d)$, let u_0^ω be its Wiener randomization defined in* (1.2.13)*, satisfying* (1.2.10)*. Given a Schrödinger admissible pair (q, r) with $q, r < \infty$, let $\widetilde{r} \geq r$. Then, there exist $C, c > 0$ such that*

$$P\Big(\|e^{-it\Delta}u_0^\omega\|_{L_t^q L_x^{\widetilde{r}}(\mathbb{R}\times\mathbb{R}^d)} > \lambda\Big) \leq Ce^{-c\lambda^2\|u_0\|_{L^2}^{-2}}$$

for all $\lambda > 0$.

As in the periodic setting, similar global-in-time probabilistic Strichartz estimates also hold for the wave equation. See [52, 65, 79].

Remark 1.1 (i) As we point out below, the Wiener randomization is special among other possible randomizations stemming from functions spaces in time–frequency analysis. Recall the following definition of the modulation spaces in time–frequency analysis [35–37]. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$; $M_s^{p,q}$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ for which the (quasi) norm

$$\|u\|_{M_s^{p,q}(\mathbb{R}^d)} := \Big\| \langle n\rangle^s \|\psi(D-n)u\|_{L_x^p(\mathbb{R}^d)} \Big\|_{\ell_n^q(\mathbb{Z}^d)}$$

is finite, where $\psi(D-n)$ is as above. In particular, we see that the Wiener randomization (1.2.13) based on the unit cube decomposition of the frequency space is very natural from the perspective of time–frequency analysis associated with the modulation spaces.

(ii) Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\operatorname{supp}\varphi_0 \subset \{|\xi| \leq 2\}$, $\operatorname{supp}\varphi \subset \{\frac{1}{2} \leq |\xi| \leq 2\}$, and $\varphi_0(\xi) + \sum_{j=1}^\infty \varphi(2^{-j}\xi) \equiv 1$. With $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, one may consider the following decomposition of a function:

$$u_0 = \sum_{j=0}^{\infty} \varphi_j(D)u_0 \tag{1.2.17}$$

and introduce the following randomization:

$$u_0^\omega := \sum_{j=0}^{\infty} g_n(\omega)\varphi_j(D)u_0.$$

Note that (1.2.17) can be viewed as a decomposition associated with the Besov spaces. In view of the Littlewood–Paley theory, such a randomization does not yield any improvement on differentiability or integrability and thus it is of no interest.

(iii) Consider the following wavelet series of a function:

$$u_0 = \sum_{\lambda \in \Lambda} \langle u_0, \psi_\lambda \rangle \psi_\lambda, \tag{1.2.18}$$

where $\{\psi_\lambda\}_{\lambda\in\Lambda}$ is a wavelet basis of $L^2(\mathbb{R}^d)$. One may also fancy the following randomization based on the wavelet expansion (1.2.18):

$$u_0^\omega := \sum_{\lambda \in \Lambda} g_\lambda(\omega) \langle u_0, \psi_\lambda \rangle \psi_\lambda. \tag{1.2.19}$$

Under some regularity assumption on ψ_λ, we have the following characterization of the L^p-norm [55, Chapter 6]:

$$\|u_0^\omega\|_{L^p(\mathbb{R}^d)} \sim \bigg\| \bigg(\sum_{\lambda\in\Lambda} |g_\lambda(\omega)|^2 |\langle u_0, \psi_\lambda\rangle|^2 |\psi_\lambda(x)|^2 \bigg)^{\frac{1}{2}} \bigg\|_{L^p(\mathbb{R}^d)} \tag{1.2.20}$$

for $1 < p < \infty$. For example, if $\{g_\lambda\}_{\lambda\in\Lambda}$ is a sequence of independent Bernoulli random variables, then it follows from (1.2.20) that $\|u_0^\omega\|_{L^p} \sim \|u_0\|_{L^p}$ and hence we see no improvement on integrability under the randomization (1.2.19).

1.3 Probabilistic Well-Posedness of NLW and NLS

In this section, we go over some aspects of probabilistic well-posedness of nonlinear dispersive PDEs. In recent years, there has been an increasing number of probabilistic well-posedness results for these equations. See [6, 63, 72, 79] and the references therein.

1.3.1 Basic Almost Sure Local Well-Posedness Argument

In the following, we consider the defocusing[13] cubic NLW on $\mathbb{T}^3$:

$$\begin{cases} \partial_t^2 u = \Delta u - u^3 \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3. \tag{1.3.1}$$

We say that u is a solution to (1.3.1) if u satisfies the following Duhamel formulation:

$$u(t) = S(t)(u_0, u_1) - \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} u^3(t')dt', \tag{1.3.2}$$

where $S(t)$ denotes the linear wave operator:

$$S(t)\,(u_0, u_1) := \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1.$$

In view of (1.1.4), we see that the scaling-critical regularity for (1.3.1) is $s_{\text{crit}} = \frac{1}{2}$. When $s \ge \frac{1}{2}$, local well-posedness of (1.3.1) in $\mathcal{H}^s(\mathbb{T}^3)$ follows from a standard fixed point argument with the (deterministic) Strichartz estimates. Moreover, the equation (1.3.1) is known to be ill-posed in $\mathcal{H}^s(\mathbb{T}^3)$ for $s < \frac{1}{2}$ [21, 26, 90]. In the following, we take initial data (u_0, u_1) to be in $\mathcal{H}^s(\mathbb{T}^3) \setminus \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$ for some appropriate $s < \frac{1}{2}$.

Fix $(u_0, u_1) \in \mathcal{H}^s(\mathbb{T}^3) \setminus \mathcal{H}^{\frac{1}{2}}(\mathbb{T}^3)$. We apply the randomization defined in (1.2.9) to (u_0, u_1). More precisely, we set

$$(u_0^\omega, u_1^\omega)(x) := \bigg(\sum_{n \in \mathbb{Z}^3} g_{0,n}(\omega)\widehat{u}_0(n)e^{in\cdot x}, \sum_{n \in \mathbb{Z}^3} g_{1,n}(\omega)\widehat{u}_1(n)e^{in\cdot x} \bigg), \tag{1.3.3}$$

where $\{g_{j,n}\}_{j=0,1,n\in\mathbb{Z}^3}$ is a sequence of independent mean-zero complex-valued random variables conditioned that $g_{j,-n} = \overline{g_{j,n}}$, $j = 0, 1, n \in \mathbb{Z}^3$. Moreover, we assume the exponential moment bound of type (1.2.10).

Theorem 1.1 *Let $s \ge 0$. Then, the cubic NLW (1.3.1) on $\mathbb{T}^3$ is almost surely locally well-posed with respect to the randomization (1.3.3). Moreover, the solution u lies in the class:*

$$S(t)(u_0^\omega, u_1^\omega) + C([0, T_\omega]; H^1(\mathbb{T}^3)) \subset C([0, T_\omega]; L^2(\mathbb{T}^3))$$

for $T_\omega = T_\omega(u_0, u_1) > 0$ almost surely.

This theorem is implicitly included in [22]. See also [79]. We point out, however, that the main goal of the paper [22] is to establish almost sure global well-posedness

[13] For the local-in-time argument, the defocusing/focusing nature of the equation does not play any role.

(see the next subsection) and to introduce the notion of probabilistic continuous dependence. See [22] for details.

In view of the Duhamel formulation (1.3.2), we write the solution u as

$$u = z + v, \tag{1.3.4}$$

where $z = z^\omega = S(t)(u_0^\omega, u_1^\omega)$ denotes the random linear solution. Then, instead of studying the original Eq. (1.3.1), we study the equation satisfied by the nonlinear part v:

$$\begin{cases} \partial_t^2 v = \Delta v - (v + z^\omega)^3 \\ (v, \partial_t v)|_{t=0} = (0, 0) \end{cases} \tag{1.3.5}$$

by viewing the random linear solution z^ω as an explicit external source term. Given $\omega \in \Omega$, define Γ^ω by

$$\Gamma^\omega(v)(t) = -\int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}(v + z^\omega)^3(t')dt'. \tag{1.3.6}$$

Our main goal is to show that

$$v = \Gamma^\omega(v) \tag{1.3.7}$$

on some random time interval $[0, T_\omega]$ with $T_\omega > 0$ almost surely.[14] Then, the solution u to (1.3.1) with the randomized initial data (u_0^ω, u_1^ω) in (1.3.3) is given by (1.3.4).

Given $T > 0$, we use the following shorthand notations: $C_T B = C([0, T]; B)$ and $L_T^q B = L^q([0, T]; B)$. We also denote by $B_1 \subset C_T \dot{H}^1$ the unit ball in $C_T \dot{H}^1$ centered at the origin. Suppose that $(u_0, u_1) \in \mathcal{H}^0(\mathbb{T}^3)$. Then, by Sobolev's inequality, we have

$$\begin{aligned} \|\Gamma^\omega(v)\|_{C_T \dot{H}^1} \le \|v + z^\omega\|^3_{L_T^3 L_x^6} &\le C_1 T \|v\|^3_{C_T \dot{H}^1} + C_2 \|z^\omega\|^3_{L_T^3 L_x^6} \\ &\le C_1 T \|v\|^3_{C_T \dot{H}^1} + \tfrac{1}{2}, \end{aligned}$$

where the last inequality holds on a set Ω_T thanks to the probabilistic Strichartz estimate (i.e., an analogue to Lemma 1.3 for the linear wave equation). Moreover, we have

$$P(\Omega_T^c) \to 0 \text{ as } T \to 0. \tag{1.3.8}$$

By taking $T > 0$ sufficiently small, we conclude that

$$\|\Gamma^\omega(v)\|_{C_T \dot{H}^1} \le 1 \tag{1.3.9}$$

---

[14] Needless to say, the solution v is random since it depends on the random linear solution z^ω. For simplicity, however, we suppress the superscript ω.

for any $v \in B_1$ and $\omega \in \Omega_T$.

Similarly, by taking $T > 0$ sufficiently small, the following difference estimate holds:

$$\begin{aligned}\|\Gamma^\omega(v_1) - \Gamma^\omega(v_2)\|_{C_T \dot{H}^1} &\le C T^{\frac{1}{3}} \Big(\sum_{j=1}^{2} T^{\frac{2}{3}} \|v_j\|^2_{C_T \dot{H}^1} + \|z^\omega\|^2_{L^3_T L^6_x} \Big) \|v_1 - v_2\|_{C_T \dot{H}^1} \\ &\le \tfrac{1}{2} \|v_1 - v_2\|_{C_T \dot{H}^1} \end{aligned} \tag{1.3.10}$$

for any $v_1, v_2 \in B_1$ and $\omega \in \Omega_T$. Therefore, from (1.3.9) and (1.3.10), we conclude that, given $T > 0$ sufficiently small, Γ^ω is a contraction on B_1 for any $\omega \in \Omega_T$. By the fundamental theorem of calculus: $v(t) = \int_0^t \partial_t v(t')dt'$ and Minkowski's inequality, we also conclude that $v \in C([0, T]; H^1(\mathbb{T}^3))$.

Now, set $\Sigma = \bigcup_{0<T\ll 1} \Omega_T$. Then, for each $\omega \in \Sigma$, there exists a unique solution v to (1.3.7) on $[0, T_\omega]$ with $T_\omega > 0$. Moreover, it follows from (1.3.8) that $P(\Sigma) = 1$. This proves Theorem 1.1.

The main point of the argument above is (i) the decomposition (1.3.4) and (ii) the gain of integrability for the random linear solution z thanks to the probabilistic Strichartz estimates. Then, the gain of one derivative in the Duhamel integral operator in (1.3.6) allows us to conclude the desired result in a straightforward manner. Lastly, note that the same almost sure local well-posedness result holds for the cubic NLW posed on $\mathbb{R}^3$ with the verbatim proof.

Remark 1.2 The decomposition (1.3.4) allows us to separate the unknown part v from the explicitly known random part z and exploit the gain of integrability on z. This idea goes back to the work of McKean [54] and Bourgain [10]. See also Burq–Tzvetkov [21]. In the field of stochastic parabolic PDEs, this decomposition is usually referred to as the Da Prato–Debussche trick [28].

Next, let us briefly discuss the situation for the cubic NLS on $\mathbb{R}^3$:

$$\begin{cases} i\partial_t u = \Delta u - |u|^2 u \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^3), \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \tag{1.3.11}$$

As before, the scaling-critical regularity for (1.3.11) is $s_{\text{crit}} = \frac{1}{2}$ and local well-posedness of (1.3.11) in $H^s(\mathbb{R}^3)$, $s \ge \frac{1}{2}$, follows from a standard fixed point argument with the Strichartz estimates. Moreover, the Eq. (1.3.11) is known to be ill-posed in $H^s(\mathbb{R}^3)$ for $s < \frac{1}{2}$. Nonetheless, we have the following almost sure local well-posedness.

Theorem 1.2 *Let $\frac{1}{4} < s < \frac{1}{2}$. Given $u_0 \in H^s(\mathbb{R}^3)$, let u_0^ω be its Wiener randomization defined in (1.2.13), satisfying (1.2.10). Then, the cubic NLS (1.3.11) on $\mathbb{R}^3$ is almost surely locally well-posed with respect to the random initial data u_0^ω. Moreover, the solution u lies in the class:*

$$e^{-it\Delta}u_0^\omega + C([0, T]; H^{\frac{1}{2}}(\mathbb{R}^3)) \subset C([0, T]; H^s(\mathbb{R}^3))$$

for $T_\omega = T_\omega(u_0) > 0$ *almost surely.*

As in the case of the cubic NLW, write the solution u as

$$u = z + v, \tag{1.3.12}$$

where z denotes the random linear solution:

$$z = z^\omega = e^{-it\Delta}u_0^\omega. \tag{1.3.13}$$

Then, the residual part v satisfies the following perturbed NLS:

$$\begin{cases} i\partial_t v = \Delta v - |v + z|^2(v + z) \\ v|_{t=0} = 0. \end{cases} \tag{1.3.14}$$

In terms of the Duhamel formulation, we have

$$v(t) = i\int_0^t e^{-i(t-t')\Delta}|v + z|^2(v + z)(t')dt'. \tag{1.3.15}$$

A key difference from (1.3.6) is that there is no explicit smoothing on the Duhamel integral operator in (1.3.15). Hence, we need another mechanism to gain derivatives. For this purpose, we employ the Fourier restriction norm method introduced by Bourgain [8]. The basic strategy for proving Theorem 1.2 is to expand the product $|v + z|^2(v + z)$ in (1.3.15) and carry out case-by-case analysis on terms of the form:

$$v\overline{v}v, \quad v\overline{v}z, \quad v\overline{z}z, \quad \dots, \quad z\overline{z}z. \tag{1.3.16}$$

In the following, we describe the main idea of the argument. See [6] for the full details. By the duality, it suffices to estimate

$$\left| \int_0^T \int_{\mathbb{R}^3} \langle\nabla\rangle^{\frac{1}{2}}(v_1\overline{v}_2v_3)\overline{v}_4 dxdt \right|, \tag{1.3.17}$$

where $v_j = v$ or z, $j = 1, 2, 3$, and v_4 denotes the duality variable at the spatial regularity 0. By applying the dyadic decompositions to each function in (1.3.17), we separate the argument into several cases. For the sake of the argument, let us denote by N_j the dyadic size of the spatial frequencies of v_j after the dyadic decomposition and assume $N_1 \sim N_4 \geq N_2 \geq N_3$.[15] (i) When $N_2, N_3 \gtrsim N_1^\theta$ for some suitable $\theta = \theta(s) \in (0, 1)$, we can move the derivatives from v_1 to v_2 and v_3. (ii) When $N_2, N_3 \ll N_1^\theta$,

[15] Note that, due to the spatial integration in (1.3.17), the largest two frequencies of the dyadic pieces must be comparable.

we group $v_1\overline{v}_2$ and $v_3\overline{v}_4$ and apply the bilinear refinement of the Strichartz estimate [12, 77], which allows us to gain some derivative. (iii) When $N_2 \gtrsim N_1^\theta \geq N_3$, we combine (i) and (ii). This allows us to prove Theorem 1.2.

Remark 1.3 (i) Note that we did not need to perform any refined case-by-case analysis in proving Theorem 1.1 for the cubic NLW (1.3.1). This is thanks to the explicit gain of one derivative in the Duhamel integral operator (1.3.6).

(ii) One may also study almost sure local well-posedness of the cubic NLS posed on $\mathbb{T}^3$ below the scaling-critical regularity. In this case, the argument becomes more involved due to the lack of the bilinear refinement of the Strichartz estimate, which was the main tool for gaining derivatives in the problem on $\mathbb{R}^3$. See, for example, [10, 27, 58]. We point out that, in these works, the random initial data was taken to be of the specific form (1.2.6). Then, the main ingredient is the probabilistic improvement of Young's inequality (Lemma 1.5), which allows us to save some summations. Note that with the random initial data of the form (1.2.6), this probabilistic improvement on summability allows us to gain derivatives thanks to the reciprocal powers of the spatial frequencies in (1.2.6). At this point, however, for the general randomized initial data (1.2.9) on $\mathbb{T}^d$, it is not clear to us how to prove almost sure local well-posedness below the critical regularity.

Remark 1.4 (i) The solution map: $(u_0, u_1) \mapsto u$ for (1.3.1) is classically ill-posed when $s < s_{\text{crit}} = \frac{1}{2}$. The decomposition (1.3.4) tells us that we can decompose the ill-posed solution map as

$$(u_0^\omega, u_1^\omega) \overset{①}{\longmapsto} z^\omega \overset{②}{\longmapsto} v \longmapsto u = z^\omega + v, \tag{1.3.18}$$

where the first step ① involves stochastic analysis (i.e., the probabilistic Strichartz estimates) and the second step ② is entirely deterministic. Moreover, there is continuous dependence of the nonlinear part v on the random linear part z in the second step. See Remark 1.5 for more on this issue.

(ii) By a similar argument, one can prove almost sure local well-posedness with initial data of the form: "a smooth deterministic function + a rough random perturbation." For example, given deterministic $(v_0, v_1) \in \mathcal{H}^1(\mathbb{T}^3)$, consider the random initial data $(v_0, v_1) + (u_0^\omega, u_1^\omega)$ for (1.3.1), where (u_0^ω, u_1^ω) is as in (1.3.3). Then, the only modification in the proof of Theorem 1.1 appears in that the initial data for the perturbed equation (1.3.5) is now given by (v_0, v_1). In this case, we have the following decomposition of the ill-posed solution map:

$$(v_0 + u_0^\omega, v_1 + u_1^\omega) \longmapsto (v_0, v_1, z^\omega) \longmapsto v \longmapsto u = z^\omega + v.$$

See [69] for a further discussion on the random initial data of this type.

(iii) In Theorems 1.1 and 1.2, we used the term "almost sure local well-posedness" in a loose manner, following Bourgain. In fact, what is claimed in Theorems 1.1 and 1.2 is simply almost sure local existence of unique solutions. We point out, however, that the decompositions (1.3.4) and (1.3.12) provide continuous dependence of the nonlinear part v (in a higher regularity) on the random linear solution z as mentioned above.

(iv) In [22], Burq–Tzvetkov introduced the notion of probabilistic continuous dependence of u on the random initial data, thus providing a more complete notion of probabilistic well-posedness.

(v) Unlike the usual deterministic theory, the approximation property of the random solution u^ω constructed in Theorem 1.1 by smooth solutions crucially depends on a method of approximation. On the one hand, Xia [90] showed that the solution map: $(u_0, u_1) \mapsto u$ for (1.3.1) is discontinuous everywhere in $\mathcal{H}^s(\mathbb{T}^3)$ when $s < \frac{1}{2}$. This in particular shows that the solution map for (1.3.1), a priori defined on smooth functions, does not extend continuously to rough functions, including the case of the random initial data (u_0^ω, u_1^ω) considered in Theorem 1.1.[16] On the other hand, by considering mollified initial data $(\eta_\varepsilon * u_0^\omega, \eta_\varepsilon * u_1^\omega)$, one can show that the corresponding smooth solution u_ε converges almost surely to the solution u^ω constructed in Theorem 1.1. Moreover, the limit is independent of the mollification kernel η. See [64, 88, 90].

(vi) In [63], Oh–Okamoto–Pocovnicu proved almost sure local well-posedness of the following NLS without gauge invariance:

$$i\partial_t u = \Delta u - |u|^p$$

in the regime where nonexistence of solutions is known. This shows an example of a probabilistic argument overcoming a stronger form of ill-posedness than discontinuity of a solution map (which is the case for NLW (1.3.1) and NLS (1.3.11) discussed above). In the same paper, the authors also discussed a probabilistic construction of finite time blow-up solutions below the scaling-critical regularity.

(vii) In this section, we discussed almost sure local well-posedness based on a simple Banach fixed point argument. There are also probabilistic constructions of local-in-time solutions which are not based on a contraction argument. See [61, 75, 82]. See also Sect. 1.3.3.

1.3.2 On Almost Sure Global Well-Posedness

Before going over some of the almost sure global well-posedness results in the literature, we point out that, in the stochastic setting, it suffices to prove the following statement to conclude almost sure global well-posedness.

Lemma 1.7 ("Almost" almost sure global well-posedness) *Given $T > 0$ and $\varepsilon > 0$, there exists $\Omega_{T,\varepsilon} \subset \Omega$ with $P(\Omega_{T,\varepsilon}^c) < \varepsilon$ such that a solution u^ω exists on $[-T, T]$ for any $\omega \in \Omega_{T,\varepsilon}$.*

[16]In fact, almost sure norm inflation at (u_0^ω, u_1^ω) holds.

Then, almost sure global well-posedness follows from Lemma 1.7 and Borel–Cantelli lemma.

In [9], Bourgain studied the invariance property of the Gibbs measure ρ in (1.2.7) for NLS (1.1.1) on $\mathbb{T}$. The main difficulty of this problem is the construction of global-in-time dynamics in the support of the Gibbs measure, i.e., in $H^\sigma(\mathbb{T})$, $\sigma < \frac{1}{2}$. By introducing the Fourier restriction norm method, Bourgain [8] proved local well-posedness of NLS below $H^{\frac{1}{2}}(\mathbb{T})$. Global well-posedness, however, was obtained only for the cubic case ($p = 3$). In [9], Bourgain combined PDE analysis with ideas from probability and dynamical systems and proved global well-posedness of NLS almost surely with respect to the Gibbs measure ρ. The main idea is to use the (formal) invariance of the Gibbs measure ρ as a replacement of a conservation law, providing a control on the growth of the relevant norm of solutions in a probabilistic manner. More precisely, he exploited the invariance of the truncated Gibbs measure ρ_N associated with the truncated NLS:

$$i\partial_t u_N = \Delta u_N - \mathbf{P}_N(|\mathbf{P}_N u_N|^{p-1}\mathbf{P}_N u_N), \tag{1.3.19}$$

where $\mathbf{P}_N$ denotes the Dirichlet projection onto the frequencies $\{|n| \le N\}$, and proved the following growth bound; given $N \in \mathbb{N}$, $T > 0$, and $\varepsilon > 0$, there exists $\Omega_{N,T,\varepsilon} \subset \Omega$ with $P(\Omega_{N,T,\varepsilon}^c) < \varepsilon$ such that for $\omega \in \Omega_{N,T,\varepsilon}$, the solution u_N^ω to (1.3.19) satisfies

$$\|u_N(t)\|_{H^\sigma} \le C\Big(\log\frac{T}{\varepsilon}\Big)^{\frac{1}{2}} \tag{1.3.20}$$

for any $t \in [-T, T]$, where C is independent of N. Combining (1.3.20) with a standard PDE analysis, the same estimate also holds for the solution u to (1.1.1) on a set $\Omega_{T,\varepsilon}$ with $P(\Omega_{T,\varepsilon}^c) < \varepsilon$, yielding Lemma 1.7. We point out that invariance of the Gibbs measure ρ follows easily once we have well-defined global-in-time dynamics. This argument is now known as *Bourgain's invariant measure argument* and is widely applied[17] in situations where there is a formally invariant measure.

Note that while Bourgain's invariant measure argument is very useful, its use is restricted to the situation where there is a formally invariant measure. Namely, it cannot be used to study the global-in-time behavior of solutions to an evolution equation with general random initial data. In the following, we list various methods in establishing almost sure global well-posedness of nonlinear dispersive PDEs with general random initial data.

• *Bourgain's high-low method in the probabilistic context*: In [12], Bourgain introduced an argument to prove global well-posedness of NLS below the energy space. The main idea is to divide the dynamics into the low- and high-frequency parts, where the low-frequency part lies in H^1 and hence the energy conservation is available. The main ingredient in this argument is the nonlinear smoothing property of the

[17] This argument is not limited to nonlinear dispersive PDEs. For instance, see [45] for an application of this argument in studying a stochastic parabolic PDE.

high-frequency part. By exhibiting nonlinear smoothing in a probabilistic manner,[18] Colliander–Oh [27] implemented this argument in the probabilistic setting to prove almost sure global well-posedness of the (renormalized) cubic NLS on $\mathbb{T}$ in negative Sobolev spaces. See also [52, 81].

• *Probabilistic a priori energy bound*: The most basic way to prove global well-posedness is to iterate a local well-posedness argument. This can be implemented in a situation, where one has (deterministic) local well-posedness in the subcritical sense[19] and the relevant norm of a solution is controlled by a conservation law. In the probabilistic setting, one can implement a similar idea. Burq–Tzvetkov [22] proved almost sure global well-posedness of the defocusing cubic NLW on $\mathbb{T}^3$ by estimating the growth of the (non-conserved) energy:

$$H(v) = \frac{1}{2}\int_{\mathbb{T}^3} |\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{T}^3} (\partial_t v)^2 dx + \frac{1}{4}\int_{\mathbb{T}^3} v^4 dx$$

of the nonlinear part $v = u - z$, solving the perturbed NLW (1.3.5). The argument is based on Gronwall's inequality along with Cauchy–Schwarz' inequality and the probabilistic Strichartz estimates. There is a slight loss of derivative in controlling the L^∞_x-norm of the random linear solution, and thus, this argument works only for $s > 0$. In the endpoint case ($s = 0$), Burq–Tzvetkov adapted Yudovich's argument [91] to control the energy growth of the nonlinear part v.

We point out that this argument based on Cauchy–Schwarz' and Gronwall's inequalities works only for the cubic case. In [65], Oh–Pocovnicu proved almost sure global well-posedness of the energy-critical defocusing quintic NLW on $\mathbb{R}^3$ (see the next item). The main new ingredient is a new probabilistic a priori energy bound on the nonlinear part $v = u - z$. See [53, 84] for almost sure global well-posedness of the three-dimensional NLW $3 < p < 5$, following the idea in [65].

• *Almost sure global existence by a compactness argument*: In [19], Burq–Thomann–Tzvetkov studied the defocusing cubic NLW (1.3.1) on $\mathbb{T}^d$, $d \geq 4$. By establishing a probabilistic energy bound on the nonlinear part v_N of the solution to the truncated NLW, which is uniform in the truncation parameter $N \in \mathbb{N}$, they established a compactness property of $\{v_N\}_{N\in\mathbb{N}}$, which allowed them to prove almost sure global existence. See Nahmod–Pavlović–Staffilani [57] for a precursor of this argument in the context of the Navier–Stokes equations.

In [20], Burq–Thomann–Tzvetkov adapted a different kind of compactness argument in fluids by Albeverio–Cruzeiro [1] to the dispersive setting and constructed almost sure global-in-time dynamics with respect to Gibbs measures ρ for various equations. By exploiting the invariance of the truncated Gibbs measure ρ_N for the truncated dynamics, they showed a compactness property of the measures $\{\nu_N\}_{N\in\mathbb{N}}$ on space-time functions, where $\nu_N = \rho_N \circ \Phi_N^{-1}$ denotes the pushforward of the truncated Gibbs measure ρ_N under the global-in-time solution map Φ_N of the truncated

[18] See Theorems 1.1 and 1.2 for such nonlinear smoothing in the probabilistic setting.

[19] Namely, the local existence time depends only on the norm of initial data.

dynamics. Skorokhod's theorem then allowed them to construct a function u as an almost sure limit of the solutions u_N to the truncated dynamics (distributed according to ν_N), yielding almost sure global existence. See [32, 70] for more on this method. This construction of global-in-time solutions is closely related to the notion of martingale solutions in the field of stochastic PDEs. See [29].

Due to the use of compactness, the global-in-time solutions constructed above are not unique (except for the two-dimensional case in [57]). In the four-dimensional case, the solutions to the defocusing cubic NLW constructed in [19] were shown to be unique in [66], relying on the result [79], which we discuss in the next item.

• *Almost sure global well-posedness in the energy-critical case via perturbation/stability results*: In the deterministic setting, the energy conservation allows us to prove global well-posedness in H^1 of energy-subcritical defocusing NLW and NLS. In the energy-critical setting, however, the situation is more complicated and the energy conservation is not enough. Over the last several decades, substantial effort was made in understanding global-in-time behavior of solutions to the energy-critical NLW and NLS. See [6, 63, 79] for the references therein.

In an analogous manner, a probabilistic a priori energy bound discussed above does not suffice to prove almost sure global well-posedness of the energy-critical defocusing NLW and NLS. In [6], we introduced a new argument, using perturbation/stability results for NLS to approximate the dynamics of the perturbed NLS (1.3.14) by unperturbed NLS dynamics on short time intervals, and proved conditional almost sure global well-posedness of the energy-critical defocusing cubic NLS on $\mathbb{R}^4$, provided that the energy of the nonlinear part $v = u - z$ remains bounded almost surely for each finite time. Subsequently, by establishing probabilistic a priori energy bounds, Pocovnicu [79] and Oh–Pocovnicu [65, 66] applied this argument and proved almost sure global well-posedness of the energy-critical defocusing NLW on $\mathbb{R}^d$ and $\mathbb{T}^d$, $d = 3, 4, 5$. More recently, Oh–Okamoto–Pocovnicu [63] established probabilistic a priori energy bounds for the energy-critical defocusing NLS on $\mathbb{R}^d$, $d = 5, 6$ and proved almost sure global well-posedness for these equations.

• *Almost sure scattering results via a double bootstrap argument:* The almost sure global well-posedness results mentioned above do not give us any information on the qualitative behavior of global-in-time solutions such as scattering. This is due to the lack of global-in-time space-time bounds in the argument mentioned above. In a recent paper, Dodson–Lührmann–Mendelson [33] studied the energy-critical defocusing NLW on $\mathbb{R}^4$ in the radial setting and proved almost sure scattering in this setting. The argument is once again based on applying perturbation/stability results as in [6, 65, 79]. The main new ingredient is the Morawetz estimate for the perturbed NLW (1.3.5) satisfied by the nonlinear part $v = u - z$. More precisely, they implemented a double bootstrap argument, controlling the energy and the Morawetz quantity for the nonlinear part v in an intertwining manner. Here, the radial assumption plays a crucial role in applying the radial improvement of the Strichartz estimates. We also point out that even if the original initial data is radial, its randomization is no longer radial and some care must be taken in order to make use of the radial assumption. In [47], Killip–Murphy–Vişan proved an analogous almost sure scattering result for the energy-critical defocusing NLS on $\mathbb{R}^4$ in the radial setting.[20] It would be of interest to remove the radial assumption imposed in the aforementioned work [33, 47].

We also mention the work [18, 31] on the almost sure scattering results based on (a variant of) Bourgain's invariant measure argument combined with the equation-specific transforms.

In this subsection, we went over various globalization arguments. We point out, however, that, except for Bourgain's invariant measure argument (which is restricted to a very particular setting), all the almost sure globalization arguments are based on known deterministic globalization arguments. Namely, there is *no* probabilistic argument at this point that is not based on a known deterministic argument for generating global-in-time solutions.

1.3.3 Further Discussions

We conclude this section with a further discussion on probabilistic construction of solutions to nonlinear dispersive PDEs.

• *Higher-order expansions*: In Sect. 1.3.1, we described the basic strategy for proving almost sure local well-posedness of NLW and NLS. In the following, we describe the main idea on how to improve the regularity thresholds stated in Theorems 1.1 and 1.2.

We first consider the cubic NLS on $\mathbb{R}^3$. The almost sure local well-posedness stated in Theorem 1.2 follows from the case-by-case analysis in (1.3.16) within the framework of the Fourier restriction norm method. See [6] for the details. By examining the case-by-case analysis in [6], we see that the regularity restriction $s > \frac{1}{4}$ in Theorem 1.2 comes from the cubic interaction of the random linear solution:

$$z_3(t) := i \int_0^t S(t - t')|z_1|^2 z_1(t')dt', \tag{1.3.21}$$

where $z_1 := z^\omega = e^{-it\Delta}u_0^\omega$ defined in (1.3.13). On the one hand, given $u_0 \in H^s(\mathbb{R}^3)$, $0 \le s < 1$, we can prove that z_3 in (1.3.21) has spatial regularity $2s - \varepsilon$ for any $\varepsilon > 0$. On the other hand, we need $2s - \varepsilon > s_{\text{crit}} = \frac{1}{2}$ in order to close the argument. This yields the regularity restriction $s > \frac{1}{4}$ stated in Theorem 1.2.

By noting that all the other interactions in (1.3.16) behave better than $z_1\overline{z_1}z_1$, we introduce the following second-order expansion:

$$u = z_1 + z_3 + v$$

[20] We also mention a recent work by Dodson–Lührmann–Mendelson [34] that appeared after the completion of this note. The main new idea in [34] is to adapt the functional framework for the derivative NLS and Schrödinger maps to study the perturbed NLS (1.3.14).

to remove the worst interaction $z_1\overline{z_1}z_1$. In this case, the residual term $v := u - z_1 - z_3$ satisfies the following equation:

$$\begin{cases} i\partial_t v = \Delta v - \mathcal{N}(v + z_1 + z_3) + \mathcal{N}(z_1) \\ v|_{t=0} = 0, \end{cases}$$

where $\mathcal{N}(u) = |u|^2u$. In terms of the Duhamel formulation, we have

$$v(t) = i\int_0^t e^{-i(t-t')\Delta}\big\{\mathcal{N}(v + z_1 + z_3) - \mathcal{N}(z_1)\big\}(t')dt' \tag{1.3.22}$$

Note that we have removed the worst interaction $z_1\overline{z_1}z_1$ appearing in the case-by-case analysis (1.3.16). There is, however, a price to pay; we need to carry out the following case-by-case analysis

$$v_1\overline{v_2}v_3, \quad \text{for } v_i = v,\ z_1,\ \text{or } z_3,\ i = 1, 2, 3,\ \text{but not all } v_i \text{ equal to } z_1, \tag{1.3.23}$$

containing more terms than the previous case-by-case analysis (1.3.16). Nonetheless, by studying the fixed point problem (1.3.22) for v, we can lower the regularity threshold for almost sure local well-posedness. Note that the solution u thus constructed lies in the class:

$$z_1 + z_3 + C([0, T]; H^{\frac{1}{2}}(\mathbb{R}^3)) \subset C([0, T]; H^s(\mathbb{R}^3))$$

for some appropriate $s < \frac{1}{2}$.

By examining the case-by-case analysis (1.3.23), we see that the worst interaction appears in

$$z_{j_1}\overline{z_{j_2}}z_{j_3} \quad \text{with } (j_1, j_2, j_3) = (1, 1, 3) \text{ up to permutations}, \tag{1.3.24}$$

giving rise to the following third-order term:

$$z_5(t) := i\sum_{\substack{j_1+j_2+j_3=5 \\ j_1, j_2, j_3\in\{1,3\}}}\int_0^t S(t - t')z_{j_1}\overline{z_{j_2}}z_{j_3}(t')dt'. \tag{1.3.25}$$

A natural next step is to remove this non-desirable interaction in (1.3.24) in the case-by-case analysis in (1.3.23) by considering the following third-order expansion:

$$u = z_1 + z_3 + z_5 + v.$$

In this case, the residual term $v := u - z_1 - z_3 - z_5$ satisfies the following equation:

$$\begin{cases} i\partial_t v = \Delta v - \mathcal{N}(v + z_1 + z_3 + z_5) + \displaystyle\sum_{\substack{j_1+j_2+j_3\in\{3,5\}\\ j_1,j_2,j_3\in\{1,3\}}} z_{j_1}\overline{z_{j_2}}z_{j_3} \\ v|_{t=0} = 0 \end{cases}$$

and thus we need to carry out the following case-by-case analysis:

$$v_1\overline{v_2}v_3 \quad \text{for } v_i = v, z_1, z_3, \text{ or } z_5, \, i = 1, 2, 3, \text{ such that}$$
$$\text{it is not of the form } z_{j_1}\overline{z_{j_2}}z_{j_3} \text{ with } j_1 + j_2 + j_3 \in \{3, 5\}.$$

Note the increasing number of combinations. While it is theoretically possible to repeat this procedure based on partial power series expansions, it seems virtually impossible to keep track of all the terms as the order of the expansion grows. In [7], we introduced modified higher-order expansions to avoid this combinatorial nightmare and established improved almost sure local well-posedness based on the modified higher-order expansions of arbitrary length.

Next, we briefly discuss the cubic NLW (1.3.1) on $\mathbb{T}^3$. Theorem 1.1 already provides almost sure local well-posedness for $s \geq 0$, and hence, we now need to take $s < 0$. In this case, the cubic product z^3 appearing in the perturbed NLW (1.3.5) does not make sense and we need to *renormalize* the nonlinearity. We assume that the randomized initial data (u_0^ω, u_1^ω) is of the form (1.3.3) with $\widehat{u}_j(n) = \langle n\rangle^{-\alpha+j}$, $j = 0, 1$, and that $\{g_{j,n}\}_{j=0,1,n\in\mathbb{Z}^3}$ in (1.3.3) is a sequence of independent standard complex-valued Gaussian random variables conditioned that $g_{j,-n} = \overline{g_{j,n}}$, $j = 0, 1$, $n \in \mathbb{Z}^3$. Comparing this with (1.2.6), we see that (u_0^ω, u_1^ω) is distributed according to the Gaussian measure $\mu_\alpha \otimes \mu_{\alpha-1}$ supported on $\mathcal{H}^s(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$. In this case, we can apply the Wick renormalization (see [41, 71]) and obtain the following renormalized equation for $v = u - z$:

$$\begin{aligned} \partial_t^2 v &= \Delta v - :(v+z)^3: \\ &= \Delta v \; - v^3 - 3v^2 z - 3v :z^2: - :z^3:, \end{aligned} \tag{1.3.26}$$

where $:z^\ell:$ is the standard Wick power of z, having the spatial regularity $\ell(\alpha - \frac{3}{2}) - \varepsilon$, $\ell = 2, 3$. By studying the equation (1.3.26), it is easy to prove almost sure local well-posedness for $\alpha > \frac{4}{3}$, corresponding to $s > -\frac{1}{6}$. Note that the worst term in (1.3.26) is given by $:z^3:$ with the spatial regularity $3\alpha - \frac{9}{2} - \varepsilon$.

As in the case of the cubic NLS, we shall consider the second-order expansion:

$$u = z_1 + z_3 + v, \tag{1.3.27}$$

where $z_1 = z$ and

$$z_3(t) := -\int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} :z_1^3(t'):\, dt'.$$

Then, the residual term $v = u - z_1 - z_3$ satisfies

$$\begin{aligned}\partial_t^2 v &= \Delta v - :(v + z_1 + z_3)^3: + :z_1^3: \\ &= \Delta v \ - (v + z_3)^3 - 3(v + z_3)^2 z_1 - 3(v + z_3) :z_1^2: .\end{aligned} \tag{1.3.28}$$

Note that the worst term $:z_1^3:$ in (1.3.26) is now eliminated. In (1.3.28), the worst contribution is given by $3(v + z_3) :z_1^2:$ with regularity $2\alpha - 3 - \varepsilon$.[21] By solving the fixed point problem (1.3.28) for v, we can lower the regularity threshold. In this formulation, the regularity restriction arises in making sense of the product $z_3 \cdot :z_1^2:$ as distributions of regularities $3\alpha - \frac{7}{2} - \varepsilon$ and $2\alpha - 3 - \varepsilon$.

Remark 1.5 The second-order formulation (1.3.27) yields the following decomposition of the ill-posed solution map:

$$(u_0^\omega, u_1^\omega) \longmapsto (z_1, :z_1^2:, z_3) \longmapsto v \longmapsto u = z_1 + z_3 + v.$$

Compare this with (1.3.18). As before, the first step involves stochastic analysis, while the second step is entirely deterministic. One can establish a further improvement to the argument sketched above, providing a meaning to $z_3 \cdot :z_1^2:$ in a probabilistic manner:[22]

$$(u_0^\omega, u_1^\omega) \longmapsto (z_1, :z_1^2:, z_3, z_3 \circleddash :z_1^2:) \longmapsto v \longmapsto u = z_1 + z_3 + v.$$

See [67] for details.

Remark 1.6 In a recent paper [80], Pocovnicu–Wang provided a simple argument for constructing unique solutions to NLS with random initial data by exploiting the dispersive estimate. In the context of the cubic NLS (1.3.11) on $\mathbb{R}^3$, the random initial data can be taken to be only in $L^2(\mathbb{R}^3)$. Compare this with Theorem 1.2. Their construction, however, places a solution u only in the class:

$$e^{-it\Delta} u_0^\omega + C([0, T]; L^4(\mathbb{R}^3)),$$

which does not embed in $C([0, T]; H^s(\mathbb{R}^3))$. See also [68] for a related result in the context of the stochastic NLS on $\mathbb{R}^d$.

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[21] Here, we assume that z_3 has positive regularity. For example, we know that z_3 has spatial regularity at least $3\alpha - \frac{9}{2} + 1 - \varepsilon$ and hence $\alpha > \frac{7}{6}$ suffices.

[22] Recall the following paraproduct decomposition of the product fg of two functions f and g:

$$\begin{aligned} fg &= f \circledless g + f \circleddash g + f \circledgtr g \\ &:= \sum_{j<k-1} \varphi_j(D) f \, \varphi_k(D) g + \sum_{|j-k|\le 1} \varphi_j(D) f \, \varphi_k(D) g + \sum_{k<j-1} \varphi_j(D) f \, \varphi_k(D) g. \end{aligned}$$

Since the paraproducts $z_3 \circledless :z_1^2:$ and $z_3 \circledgtr :z_1^2:$ always make sense as distributions, it suffices to give a meaning to the resonant product $z_3 \circleddash :z_1^2:$ in a probabilistic manner.

• *Bourgain–Bulut's argument*: In Sect. 1.3.2, we described a globalization argument when there is a formally invariant measure (Bourgain's invariant measure argument). We point out that this globalization argument requires a separate local well-posedness argument (whether deterministic or probabilistic). In [13–15], Bourgain–Bulut presented a new argument, where they exploited formal invariance already in the construction of local-in-time solutions. In the following, we briefly sketch the essential idea in [13–15] by taking NLS (1.1.1) on $\mathbb{T}$ as an example.

The main goal is to show that the solution u_N to the truncated NLS (1.3.19) converges to some space-time distribution u, which turns out to satisfy the original NLS (1.1.1). This is done by exploiting the invariance of the truncated Gibbs measure ρ_N:

$$d\rho_N = Z^{-1} e^{-\frac{1}{p+1}\int |\mathbf{P}_N u|^{p+1}} d\mu_1,$$

for the truncated equation (1.3.19). Let $T \le 1$. Then, by the probabilistic Strichartz estimates (Lemma 1.3), we have

$$\mu_1\Big(u_0 : \|e^{-it\Delta}u_0\|_{L^q_t W^{\sigma,r}_x([0,T]\times\mathbb{T})} > \lambda\Big) \le C\exp(-c\lambda^2)$$

for any $\sigma < \frac{1}{2}$, finite $q \ge 2$, and $2 \le r \le \infty$. Then, by using the mutual absolute continuity between μ_1 and ρ_N and exploiting the invariance of ρ_N, we can upgrade this estimate to

$$\rho_N\Big(u_0 : \|u_N\|_{L^q_t W^{\sigma,r}_x([0,T]\times\mathbb{T})} > \lambda\Big) \le C\exp(-c\lambda^{c'}), \tag{1.3.29}$$

where u_N is the solution to the truncated NLS (1.3.19) with $u_N|_{t=0} = u_0$. We stress that the constants in (1.3.29) are independent of $N \in \mathbb{N}$.

Given $M \ge N \ge 1$, let u_M and u_N be the solutions to (1.3.19) with the truncation size M and N, respectively. Then, on a time interval $I_j = [t_j, t_{j+1}] \subset [0, T]$, we have

$$\begin{aligned} u_M(t) - u_N(t) &= e^{-i(t-t_j)\partial_x^2}(u_M(t_j) - u_N(t_j)) \\ &\quad + i\int_{t_j}^{t} e^{-i(t-t')\partial_x^2}(\mathbf{P}_M - \mathbf{P}_N)|u_M|^{p-1}u_M(t')dt' \\ &\quad + i\int_{t_j}^{t} e^{-i(t-t')\partial_x^2}\mathbf{P}_N(|u_M|^{p-1}u_M - |u_N|^{p-1}u_N)(t')dt'. \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned} \tag{1.3.30}$$

Fix $s < \frac{1}{2}$ sufficiently close to $\frac{1}{2}$. We estimate each term on the right-hand side of (1.3.30) in the $X^{s,b}$-norm with $b = \frac{1}{2}+$. See [85] for the basic properties of the $X^{s,b}$-spaces. The first term I is trivially bounded by $\|u_M(t_j) - u_N(t_j)\|_{H^s}$. Noting that the second term II is supported on high frequencies $\{|n| > N\}$, we can show that it tends to 0 as $N \to \infty$. Here, we used (1.3.29) with $s < \sigma < \frac{1}{2}$, giving a decay of $N^{s-\sigma}$ for $\|\mathrm{II}\|_{X^{s,b}([t_j,t_{j+1}])}$. As for the last term III, by the fractional Leibniz rule, (1.3.29), and the periodic Strichartz estimate, we can estimate it by $\|u_M - u_N\|_{X^{s,b}([t_j,t_{j+1}])}$.[23] This allows us to iterate the argument on intervals I_j to cover the entire interval $[0, T]$ and show that $\{u_N\}_{N\in\mathbb{N}}$ is Cauchy in $X^{s,b}([0, T])$.

The almost sure local well-posedness argument presented in Sect. 1.3.1 exploited the gain of integrability only at the level of the random linear solution. The main novelty of the argument presented above is the use of invariance in constructing local-in-time solutions, which yields the gain of integrability for the truncated solutions u_N, uniformly in $N \in \mathbb{N}$.

1.4 Remarks and Comments

(i) In (1.1.4), we introduced the scaling-critical Sobolev regularity s_{crit}. Note that this regularity is based on the (homogeneous) L^2-based Sobolev spaces $\dot{H}^s = \dot{W}^{s,2}$. Given $1 \leq r \leq \infty$, we can also consider the scaling-critical Sobolev regularity adapted to the L^r-based Sobolev spaces $\dot{W}^{s,r}$:

$$s_{\text{crit}}(r) = \frac{d}{r} - \frac{2}{p-1}$$

such that the homogeneous $\dot{W}^{s_{\text{crit}}(r),r}$-norm is invariant under the dilation symmetry (1.1.3). Heuristically speaking, the gain of integrability depicted in Lemmas 1.2 and 1.3 allows us to lower the critical regularity from $s_{\text{crit}} = s_{\text{crit}}(2) = \frac{d}{2} - \frac{2}{p-1}$ to $s_{\text{crit}}(r) = -\frac{2}{p-1} + \varepsilon$ for $r \gg 1$.[24] For example, in the cubic case ($p = 3$), we have $s_{\text{crit}}(r) = \frac{d}{r} - 1 \to -1$ as $r \to \infty$, which makes the problem considered in Sect. 1.3.1 *subcritical* in some appropriate sense.

(ii) In recent years, there has been a significant development in the well-posedness theory of singular stochastic parabolic PDEs. For example, the theory of regular-

[23]Here, the implicit constant depends on the choice of λ in (1.3.29), which needs to be chosen in terms of N. See [14] for more on this issue.

[24]Things are not as simple as stated here due to the unboundedness of the linear solution operator on L^r, $r \neq 2$, for dispersive equations. In the case of the nonlinear heat equation, however, this heuristics can be seen more clearly. Consider the following nonlinear heat equation on $\mathbb{R}^d$:

$$\partial_t u = \Delta u - |u|^{p-1} u \tag{1.4.1}$$

with initial data $u_0 \in L^2(\mathbb{R}^d)$. In general, (when $4 < d(p-1)$ for example), we do not know how to construct a solution with initial data in $L^2(\mathbb{R}^d)$. By randomizing the initial data u_0 as in (1.2.13), we see that the randomized initial data u_0^ω lies almost surely in $L^r(\mathbb{R}^d)$ for any finite $r \geq 2$. Then, by taking $r > \frac{d(p-1)}{2}$, we can apply the deterministic subcritical local well-posedness result in [17] to conclude (rather trivial) almost sure local well-posedness of (1.4.1) with respect to the Wiener randomization u_0^ω. This is an instance of "making the problem subcritical" by randomization.

ity structures by Hairer [44] and the paracontrolled distributions introduced by Gubinelli–Imkeller–Perkowski [23, 40] allow us to make sense of the following stochastic quantization equation (SQE, dynamical Φ_3^4 model) on $\mathbb{T}^3$:

$$\partial_t u = \Delta u - u^3 + \infty \cdot u + \xi, \tag{1.4.2}$$

where ξ denotes the space-time white noise. See also [50]. Moreover, it has been shown that the Gibbs measure ρ in (1.2.7) (in a renormalized form) is invariant under the dynamics of (1.4.2) [2, 45]. It would be of great interest to study a similar problem for the defocusing cubic NLS and NLW on $\mathbb{T}^3$ with the Gibbs measure ρ as initial data. While the NLS problem seems to be out of reach (see (iii) below), one may approach the NLW problem by adapting the paracontrolled calculus[25] to the wave case.

(iii) There has also been some development in the solution theory for singular stochastic nonlinear dispersive PDEs [41, 60]. The problem of particular importance is the following (renormalized) stochastic cubic nonlinear Schrödinger equation (SNLS) on $\mathbb{T}$ with additive space-time white noise forcing:

$$i\partial_t u = \Delta u - |u|^2 u + 2\infty \cdot u + \xi. \tag{1.4.3}$$

Since SNLS (1.4.3) on $\mathbb{T}^d$ scales like (1.4.2) on $\mathbb{T}^d$, one may be tempted to think that they are of equal difficulty. This, however, is completely false; while SQE on $\mathbb{T}^d$, $d = 1, 2, 3$ is subcritical, SNLS on the one-dimensional torus $\mathbb{T}$ is *critical* in the following sense.

The linear heat semigroup $e^{t\Delta}$ is bounded on L^∞, and thus, the scaling-critical regularity for the cubic heat equation is given by $s_{\text{crit}}(\infty) = -1$ for any dimension. The space-time white noise under the Duhamel integral operator:[26] $\int_0^t e^{(t-t')\Delta}\xi(dt')$ has (spatial) regularity $-\frac{d}{2} + 1 - \varepsilon$. Comparing these two regularities, we see that SQE (1.4.2) is critical when $d = 4$ and is subcritical when $d = 1, 2, 3$. We point out that both the theory of regularity structures and the theory of paracontrolled distributions are subcritical theories and cannot handle (1.4.2) when $d = 4$.

Similarly, by recalling that the linear Schrödinger group $e^{-it\Delta}$ is bounded on L^2 and is unbounded on any L^r, $r \neq 2$, it seems reasonable to use $r = 2$ to compute the scaling-critical regularity for the cubic NLS, thus giving $s_{\text{crit}}(2) = \frac{d}{2} - 1$. On the other hand, the stochastic convolution $\int_0^t e^{-i(t-t')\Delta}\xi(dt')$ in this case does not experience any smoothing and thus has (spatial) regularity $-\frac{d}{2} - \varepsilon$. Comparing these two regularities, we see that SNLS (1.4.3) is critical already when $d = 1$. We point out that the (deterministic) cubic NLS on $\mathbb{T}$ with the spatial white noise as initial data, i.e., (1.2.6) with $s = 0$, basically has the same difficulty. This criticality is also manifested in the fact that the higher-order iterates such as z_3 in (1.3.21) and z_5 in

[25] At this point, we do not know how to apply the theory of regularity structures to study dispersive PDEs, partly because we do not know how to lift the Duhamel integral operator for dispersive PDEs to regularity structures.

[26] This is the so-called stochastic convolution.

(1.3.25) do not experience any smoothing.[27] Lastly, we mention a recent work [38], where the second author (with Forlano and Y. Wang) established local well-posedness of (1.4.3) with a slightly smooth noise $\langle \partial_x \rangle^{-\varepsilon}\xi$, $\varepsilon > 0$.

(iv) Various methods and ideas in the random data Cauchy theory for nonlinear dispersive PDEs are applicable to study stochastic nonlinear dispersive PDEs thanks to the gain of integrability on a stochastic forcing term.[28] For example, the almost sure local well-posedness result for the (renormalized) cubic NLS in $H^s(\mathbb{T})$, $s > -\frac{1}{3}$, by Colliander–Oh [27] essentially implies local well-posedness of SNLS (1.4.3) with a smoothed noise $\langle \partial_x \rangle^{-\varepsilon}\xi$, $\varepsilon > \frac{1}{6}$.

(v) Thanks to Bourgain's invariant measure argument, we now have a good understanding of how to build an invariant measure of Gibbs type based on a conservation law. Note, however, that these measures are supported on rough functions (except for completely integrable equations) and we do not know how to construct invariant measures supported on smooth functions. Bourgain [11] wrote "the most important challenge is perhaps the question if we may produce an invariant measure which is supported by smooth functions."

(vi) So far, we discussed how to construct solutions in a probabilistic manner. It would be of interest to develop a probabilistic argument to get more qualitative information of solutions. For example, the random initial data (1.2.6) lies almost surely in $W^{\sigma,\infty}(\mathbb{T}^d)$, $\sigma < s - \frac{d}{2}$. On the one hand, the deterministic well-posedness theory propagates only the H^σ-regularity of solutions. On the other hand, quasi-invariance of μ_s [72–74, 87] implies that the $W^{\sigma,\infty}$-regularity is also propagated in an almost sure manner. An interesting problem may be to use probabilistic tools to study the growth of high Sobolev norms of solutions. For example, the argument in [72, 87] provides a probabilistic proof of a polynomial upper bound.

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[27] For example, for the subcritical SQE on $\mathbb{T}^3$, the second-order iterate (an analogue of z_3 in (1.3.21)) gains one derivative as compared to the stochastic convolution.

[28] At least on $\mathbb{T}^d$. On $\mathbb{R}^d$, there is a limitation on the gain of integrability. See [30, 68].

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Chapter 2
Endpoint Results for Fourier Integral Operators on Noncompact Symmetric Spaces

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Abstract Let $\mathbb{X}$ be a noncompact symmetric space of rank one, and let $\mathfrak{h}^1(\mathbb{X})$ be a local atomic Hardy space. We prove the boundedness from $\mathfrak{h}^1(\mathbb{X})$ to $L^1(\mathbb{X})$ and on $\mathfrak{h}^1(\mathbb{X})$ of some classes of Fourier integral operators related to the wave equation associated with the Laplacian on $\mathbb{X}$, and we estimate the growth of their norms depending on time.

2.1 Introduction

Given a second-order differential operator $\mathscr{L}$ on a manifold $\mathbb{M}$, consider the Cauchy problem for the associated wave equation

$$\begin{cases} \partial_t^2 u(t,x) + \mathscr{L}u(t,x) = 0, \\ u(0,x) = f(x), \\ \partial_t u(0,x) = g(x) \qquad t \in \mathbb{R},\ x \in \mathbb{M}. \end{cases} \tag{2.1.1}$$

An interesting problem is to find L^p-bounds of the solution u at a certain time in terms of Sobolev norms of the initial data f and g. This problem is well understood for the standard Laplacian in $\mathbb{R}^n$ [15, 18]. It was also studied for the Laplace–Beltrami operator on compact manifolds [19], for the sub-Laplacian on groups of Heisenberg type [16, 17] and for the Laplacian on compact Lie groups [5]. Ionescu [12] investigated the same problem on noncompact symmetric spaces of rank one. More precisely, let $\mathbb{X}$ be a noncompact symmetric space of rank one and dimension n and

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denote with d the number $(n-1)/2$. Let Δ denote the Laplace–Beltrami operator on $\mathbb{X}$, whose L^2-spectrum is the half-line $[\rho^2, \infty)$, and set $\mathscr{L} = \Delta - \rho^2$ (see Sect. 2.2 for the definition of ρ). The wave equation associated with $\mathscr{L}$ was considered in [2–4, 11, 12, 21]. By the spectral theorem, the solution of the Cauchy problem (2.1.1) associated with $\mathscr{L}$ is given by

$$u(t,\cdot) = \cos(t\sqrt{\mathscr{L}})\, f + \frac{\sin(t\sqrt{\mathscr{L}})}{\sqrt{\mathscr{L}}}\, g.$$

Finding L^p-bounds for u amounts to prove the boundedness on $L^p(\mathbb{X})$ of the operators

$$\mathscr{T}_t = m(\sqrt{\mathscr{L}})\cos(t\sqrt{\mathscr{L}}) \qquad \text{and} \qquad \mathscr{S}_t = m(\sqrt{\mathscr{L}})\frac{\sin(t\sqrt{\mathscr{L}})}{\sqrt{\mathscr{L}}},$$

for suitable symbols m, and estimates the growth of their norm on $L^p(\mathbb{X})$ depending on t.

In this paper, we prove endpoint results at $p = 1$ for $\mathscr{T}_t$. To state our result, we need some notation. For every $a \geq 0$ and $b \in \mathbb{R}$, let S_a^b be the set of continuous functions m on the complex tube $\{\lambda \in \mathbb{C} \colon |\mathrm{Im}\,\lambda| \leq a\}$, analytic in the interior of the tube, infinitely differentiable on the two lines $|\mathrm{Im}\,\lambda| = a$, which satisfy the symbol inequalities

$$|\partial_\lambda^\alpha m(\lambda)| \leq C\,(1 + |\mathrm{Re}\,\lambda|)^{b-\alpha} \qquad \forall\, \alpha \in \mathbb{N}, \quad |\mathrm{Im}\,\lambda| \leq a.$$

If $m \in S_a^b$, the real number b is called the *order* of m.

In [12], Ionescu proved an endpoint result for $\mathscr{T}_t$ at $p = \infty$. Indeed, he showed that if $m \in S_\rho^{-d}$ is an even symbol, then the operator $\mathscr{T}_t$ is bounded from $L^\infty(\mathbb{X})$ to a suitable $BMO(\mathbb{X})$ space. From this, he deduced the boundedness of $\mathscr{T}_t$ on $L^p(\mathbb{X})$ for every $p \in (1, \infty)$. Let us also mention that previously Giulini and Meda [8] proved L^p-estimates, $p \in (1, \infty)$, for oscillating multipliers of the form $\Delta^{-\beta/2} e^{i\Delta^{\alpha/2}}$, $\alpha > 0$, $\mathrm{Re}\,\beta \geq 0$. When $\alpha = 1$ and $\beta = d$, these operators are related to $\mathscr{T}_1$. Note, however, that on a noncompact symmetric space, the growth in t of the norm of $\mathscr{T}_t$ cannot be deduced from its norm at $t = 1$, as one can do in other contexts equipped with a dilation structure (e.g. Euclidean spaces and stratified nilpotent groups).

Let $\mathfrak{h}^1(\mathbb{X})$ be the local atomic Hardy space of Goldberg type defined by Taylor [22] and Meda and Volpi [14] (see Definition 2.2 below). The main result of this paper is the following.

Theorem 2.1 *Let $t > 0$. Then the following hold:*

(i) if $m \in S_\rho^{-d}$ is an even symbol, then the operator $\mathscr{T}_t$ is bounded from $\mathfrak{h}^1(\mathbb{X})$ to $L^1(\mathbb{X})$ and $\|\mathscr{T}_t f\|_{\mathfrak{h}^1 \to L^1} \leq C\, e^{\rho t}$;

(ii) if $m \in S_\rho^b$ is an even symbol and $b < -d$, then the operator $\mathscr{T}_t$ is bounded on $\mathfrak{h}^1(\mathbb{X})$ and $\|\mathscr{T}_t f\|_{\mathfrak{h}^1 \to \mathfrak{h}^1} \leq C\, e^{\rho t}$.

The results of Theorem 2.1 are endpoint results for $\mathscr{T}_t$ at $p = 1$. The $\mathfrak{h}^1 \to L^1$ boundedness may be considered as the counterpart at $p = 1$ of Ionescu's result; observe, however, that it does not descend from this by duality, for $\mathfrak{h}^1(\mathbb{X})$ is not the dual of $BMO(\mathbb{X})$. Nevertheless, the proof of part (i) is strongly related to Ionescu's proof. Part (ii), instead, gives a more precise endpoint result but requires higher regularity of the multiplier m. It would be interesting to know whether this regularity condition is really necessary, or whether it can be weakened up to the value $b = -d$, which as part (i) shows is enough for the $\mathfrak{h}^1 \to L^1$ boundedness. The proof of part (ii) goes through a pointwise decomposition of the convolution kernel k_t of $\mathscr{T}_t$ as a sum of compactly supported functions in certain annuli, whose $\mathfrak{h}^1$-norm we estimate separately. We do this by means of precise estimates of both k_t and its derivative. In applying this procedure, the condition $b < -d$ turns out to be fundamental.

We finally observe that, by analytic interpolation with $L^2(\mathbb{X})$ and by duality, one can re-obtain Ionescu's result of $L^p(\mathbb{X})$ boundedness of $\mathscr{T}_t$, $p \in (1, \infty)$.

The paper is organized as follows. In Sect. 2.2, we summarize the notation for noncompact symmetric spaces of rank one and the spherical analysis on them. In Sect. 2.3, we recall the definition of the local Hardy space $\mathfrak{h}^1(\mathbb{X})$ and we prove some technical lemmata which will be of use later on. In Sect. 2.4, we prove Theorem 2.1 (i), while Section 2.5 is devoted to the proof of Theorem 2.1 (ii).

2.2 Notation

We shall use the same notation as in [12] and refer the reader to [1, 7, 10] for more details on noncompact symmetric spaces and spherical analysis on them.

Let G be a connected noncompact semisimple Lie group with finite centre, $\mathfrak{g}$ its Lie algebra, θ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition. Let K be a maximal compact subgroup of G and $\mathbb{X} = G/K$ be the associated symmetric space of dimension n. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. We will assume that the dimension of $\mathfrak{a}$ is one, i.e. that the rank of $\mathbb{X}$ is one. The Killing form on $\mathfrak{g}$ induces a G-invariant distance on $\mathbb{X}$, which we shall denote by $d(\cdot, \cdot)$. For every $x \in \mathbb{X}$, we denote by $|x|$ the distance $d(x, o)$, where $o = eK$ and e is the identity of G. Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and for $\alpha \in \mathfrak{a}^*$ let $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \quad \forall H \in \mathfrak{a}\}$. Let $\Sigma = \{\alpha \in \mathfrak{a}^* \setminus \{0\} : \dim \mathfrak{g}_\alpha \neq 0\}$ be the set of nonzero roots. It is well known that either $\Sigma = \{-\alpha, \alpha\}$ or $\Sigma = \{-2\alpha, -\alpha, \alpha, 2\alpha\}$. Let $m_1 = \dim \mathfrak{g}_\alpha$, $m_2 = \dim \mathfrak{g}_{2\alpha}$ and $\rho = (m_1 + 2m_2)/2$. Set $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ and $N = \exp \mathfrak{n}$.

In the sequel, we shall identify $A = \exp \mathfrak{a}$ with $\mathbb{R}$ by choosing the unique element H_0 of $\mathfrak{a}$ such that $\alpha(H_0) = 1$ and considering the diffeomeorphism $a : \mathbb{R} \to A$ defined by $a(s) = \exp(sH_0)$. It is well known that G admits the Cartan decomposition $G = KA^+K$, where $A^+ = \{a(s) : s \geq 0\}$ and the Iwasawa decomposition $G = NAK$. For every $g \in G$ we denote by $H(g)$ the unique element in $\mathbb{R}$ such that $g = n \exp(H(g)H_0)k$, for some $n \in N$ and $k \in K$.

For every $r > 0$ and $x \in \mathbb{X}$, we denote by $B(x, r)$ the closed ball centred at the point x of radius r. For every $0 < r < R$, we denote by A_r^R the annulus $A_r^R = \{x \in \mathbb{X} \colon r \leq |x| \leq R\}$. As a convention, A_r^R when $r \leq 0$ shall be intended as the ball $B(o, R)$.

For every integrable function f on G, we have

$$\int_G f(g)\, dg = C \int_K \int_{\mathbb{R}^+} \int_K f(k_1 a(s) k_2)\, \delta(s)\, dk_1\, ds\, dk_2,$$

where dg is the Haar measure of G, dk is the Haar measure of K normalized in such a way that $\int_K dk = 1$ and

$$\delta(s) = C(\sinh s)^{m_1}(\sinh 2s)^{m_2} \asymp \begin{cases} s^{n-1} & s \leq 1 \\ e^{2\rho s} & s > 1. \end{cases}$$

We identify right K-invariant functions on G with functions on $\mathbb{X}$, and K-biinvariant functions on G with K-invariant functions on $\mathbb{X}$ which can also be identified with functions depending only on the coordinate $s \in \mathbb{R}^+$. More precisely, if f is a K-biinvariant function on G, we shall denote by $F : \mathbb{R}^+ \to \mathbb{C}$ the function such that $f(k_1 a(s) k_2) = F(s)$ for every $s \in \mathbb{R}^+, k_1, k_2 \in K$. We define the convolution of two functions f_1, f_2 on $\mathbb{X}$, when it exists, as

$$f_1 * f_2(x) = \int_G f_1(gh) f_2(h^{-1})\, dh \qquad \forall\, x = gK \in \mathbb{X}.$$

We denote by μ the Riemannian measure on $\mathbb{X}$, and for every $p \in [1, \infty)$, let $L^p(\mathbb{X})$ be the space of measurable functions f such that $\|f\|_{L^p}^p = \int_{\mathbb{X}} |f|^p d\mu < \infty$. For every K-invariant function f on $\mathbb{X}$

$$\int_{\mathbb{X}} f(x)\, d\mu(x) = \int_{\mathbb{R}^+} F(s)\, \delta(s)\, ds,$$

where F is defined above. By this and the left invariance of the metric

$$\mu(B(x, r)) = \mu(B(o, r)) \asymp \begin{cases} r^n & r \leq 1 \\ e^{2\rho r} & r > 1 \end{cases} \qquad \forall\, r > 0,\ x \in \mathbb{X}. \tag{2.2.1}$$

Observe moreover that

$$\mu(A_{R-r}^{R+r}) \asymp e^{2\rho R} r, \qquad \forall\, R > 1,\ r < 1. \tag{2.2.2}$$

We recall that a spherical Fourier transform on the symmetric space is defined. It associates to each left K-invariant function f on $\mathbb{X}$, i.e. to each radial function, its spherical Fourier transform $\widetilde{f}$, defined by

$$\widetilde{f}(\lambda) = \int_G f(g)\, \phi_\lambda(g)\, dg \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*,$$

where the spherical functions are defined by

$$\phi_\lambda(g) = \int_K \exp[(i\lambda + \rho)H(kg)]\, dk \qquad g \in G,\ \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

It is well known that for every radial function in $L^2(\mathbb{X})$

$$\|f\|_{L^2}^2 = C \int_0^\infty |\widetilde{f}(\lambda)|^2\, |\mathbf{c}(\lambda)|^{-2} d\lambda, \tag{2.2.3}$$

and

$$f(x) = C \int_0^\infty \widetilde{f}(\lambda)\, \phi_\lambda(x)\, |\mathbf{c}(\lambda)|^{-2} d\lambda, \tag{2.2.4}$$

where $\mathbf{c}$ is the Harish-Chandra function. In particular, by the Plancherel and the inversion formulae above, any bounded function $m : \mathbb{R}^+ \to \mathbb{C}$ defines a bounded operator on $L^2(\mathbb{X})$ given by $\widetilde{\mathcal{U}_m f}(\lambda) = m(\lambda)\, \widetilde{f}(\lambda)$.

All throughout the paper, we shall write $A \lesssim B$ when there exists a positive constant C such that $A \leq C\, B$, whose value may change from line to line. If $A \lesssim B$ and $B \lesssim A$, we write $A \asymp B$.

2.3 The Local Hardy Space $\mathfrak{h}^1(\mathbb{X})$

We recall here the definition of the local atomic Hardy space $\mathfrak{h}^1(\mathbb{X})$, which can be thought as the analogue in the context of noncompact symmetric space of the local Hardy space introduced by Goldberg in the Euclidean setting [9]. The space $\mathfrak{h}^1(\mathbb{X})$ was introduced and studied by Meda and Volpi [14] and Taylor [22] in more general contexts. It is easy to see that noncompact symmetric spaces satisfy the geometric assumptions of [14, 22], so that the theory developed in those papers can be applied in our setting.

Definition 2.1 A **standard $\mathfrak{h}^1$-atom** is a function a in $L^1(\mathbb{X})$ supported in a ball B of radius ≤ 1 such that

(i) $\|a\|_{L^2} \leq \mu(B)^{-1/2}$ (size condition);
(ii) $\int a\, d\mu = 0$ (cancellation condition).

A **global $\mathfrak{h}^1$-atom** is a function a in $L^1(\mathbb{X})$ supported in a ball B of radius 1 such that $\|a\|_{L^2} \leq \mu(B)^{-1/2}$. Standard and global $\mathfrak{h}^1$-atoms will be referred to as **admissible atoms**.

Definition 2.2 The Hardy space $\mathfrak{h}^1(\mathbb{X})$ is the space of functions f in $L^1(\mathbb{X})$ such that $f = \sum_j c_j a_j$, where $\sum_j |c_j| < \infty$ and a_j are admissible atoms. The norm $\|f\|_{\mathfrak{h}^1}$ is defined as the infimum of $\sum_j |c_j| < \infty$ over all atomic decompositions of f.

By means of the atomic structure of $\mathfrak{h}^1(\mathbb{X})$ and of the following result, the boundedness from $\mathfrak{h}^1(\mathbb{X})$ of an operator bounded on $L^2(\mathbb{X})$ may be tested only on atoms. Its proof is an easy adaptation of the proof of [14, Theorem 4 and Proposition 4] and is omitted.

Proposition 2.1 *Let Y be either $L^1(\mathbb{X})$ or $\mathfrak{h}^1(\mathbb{X})$. Suppose that $\mathscr{U}$ is a Y-valued linear operator defined on finite linear combination of admissible atoms such that*

$$A := \sup\{\|\mathscr{U} a\|_Y : a \ \mathfrak{h}^1\text{-atom}\} < \infty.$$

Then there exists a unique bounded operator $\mathscr{U}'$ from $\mathfrak{h}^1(\mathbb{X})$ to Y which extends $\mathscr{U}$ with norm $\|\mathscr{U}'\|_{\mathfrak{h}^1 \to Y} \lesssim A$. If $\mathscr{U}$ is bounded on $L^2(\mathbb{X})$, then $\mathscr{U}'$ and $\mathscr{U}$ coincide on $Y \cap L^2(\mathbb{X})$.

We now collect some technical lemmata where we estimate the $\mathfrak{h}^1$-norm of L^2-functions supported either in a ball or in an annulus, which will be useful later on. We shall repeatedly use the notion of discretization of the space $\mathbb{X}$, which we now recall.

For every $r \in (0, 1]$, we call $r/3$-discretization Σ of $\mathbb{X}$ a set of points which is maximal with respect to the properties

$$\min\{d(z, w) : z, w \in \Sigma, z \neq w\} > \frac{r}{3}, \qquad d(x, \Sigma) \leq \frac{r}{3} \quad \forall x \in \mathbb{X}.$$

Let Σ be a $r/3$-discretization of $\mathbb{X}$, for some $r \in (0, 1]$. Then the family of balls $\mathscr{B} = \{B(z, r) : z \in \Sigma\}$ is a uniformly locally finite covering of $\mathbb{X}$. More precisely, there exists a constant M, independent of r, such that

$$1 \leq \sum_{B \in \mathscr{B}} \chi_B(x) \leq M \qquad \forall x \in \mathbb{X}. \tag{2.3.1}$$

Indeed, given any point $x \in \mathbb{X}$, if $x \in B(z, r)$, then $z \in B(x, r)$. Thus $\sum_{B \in \mathscr{B}} \chi_B(x) = M(x) = |\Sigma \cap B(x, r)|$. Let $\{w_1, \ldots, w_{M(x)}\} = \Sigma \cap B(x, r)$. If $w_i, w_j \in \Sigma \cap B(x, r)$, with $w_i \neq w_j$, then $B(w_i, \frac{r}{6}) \cap B(w_j, \frac{r}{6}) = \emptyset$. Thus, $\bigcup_{i=1}^{M(x)} B(w_i, \frac{r}{6}) \subseteq B(x, r + \frac{r}{6})$ and by (2.2.1)

$$C\, M(x) r^n \leq \mu\left(\bigcup_{i=1}^{M(x)} B(w_i, \tfrac{r}{6})\right) \leq \mu\left(B(x, r + \tfrac{r}{6})\right) \leq C\, r^n.$$

Thus, there exists a constant M independent of x and r such that $M(x) \leq M$, which proves (2.3.1).

Lemma 2.1 *Let f be a function in $L^2(\mathbb{X})$ supported in a ball $B = B(o, R)$. If*

- *either $R \leq 1$ and f has vanishing integral,*
- *or $R \geq 1$,*

then $\|f\|_{\mathfrak{h}^1} \lesssim \mu(B)^{1/2}\, \|f\|_{L^2}$.

Proof If $R \leq 1$ and f has vanishing integral, it suffices to notice that $\frac{f}{\mu(B)^{1/2}\|f\|_{L^2}}$ is a standard atom.

If $R \geq 1$, we follow the line of [14, Lemma 3.3] with slight modifications. Let Σ be a 1/3-discretization of $\mathbb{X}$. Denote by $z_1, \ldots, z_N$ the points in Σ such that $B(z_j, 1) \cap B \neq \emptyset$. Note that $N \leq C\, \mu(B)$. Denote by B_j the ball $B(z_j, 1)$ and define

$$\psi_j = \frac{\chi_{B_j}}{\sum_{k=1}^N \chi_{B_k}}.$$

We have $f = \sum_{j=1}^N f_j$, where $f_j = f\, \psi_j$. Since $\frac{f_j}{\mu(B_j)^{1/2}\|f_j\|_{L^2}}$ is a global atom, then

$$\|f\|_{\mathfrak{h}^1} \leq \sum_{j=1}^N \mu(B_j)^{\frac{1}{2}} \|f_j\|_{L^2} \lesssim \sum_{j=1}^N \|f_j\|_{L^2} \lesssim N^{\frac{1}{2}} \left(\sum_{j=1}^N \|f_j\|_{L^2}^2 \right)^{1/2} \lesssim \mu(B)^{\frac{1}{2}} \|f\|_{L^2},$$

where we used Schwarz's inequality and the fact that $N \lesssim \mu(B)$. ■

Lemma 2.2 *Let f be a function in $L^2(\mathbb{X})$ with vanishing integral supported in an annulus A_{R-r}^{R+r}, $r \in (0, 1]$, $R > r$. Then f is in $\mathfrak{h}^1(\mathbb{X})$ and*

$$\|f\|_{\mathfrak{h}^1} \lesssim \log(1/r)\, e^{\rho R} r^{1/2}\, \|f\|_{L^2}.$$

Proof We take a $r/3$-discretization Σ of $\mathbb{X}$. The set $A_{R-r}^{R+r} \cap \Sigma$ has at most N elements $z_1, \ldots, z_N$. Then $A_{R-r}^{R+r} \subseteq \cup_{j=1}^N B_j \subseteq A_{R-2r}^{R+2r}$, so that

$$N \leq C\, r^{-n} \mu(A_{R-2r}^{R+2r}) \lesssim r^{-n+1} e^{2\rho R}, \tag{2.3.2}$$

the second inequality by (2.2.2). Let K be the lowest integer such that $2^K r > 1$, and for every $k = 0, \ldots, K$ and $j = 1, \ldots, N$, denote by B_j^k the ball $B(z_j, 2^k r)$ and define

$$\psi_j = \frac{\chi_{B_j^0}}{\sum_{i=1}^N \chi_{B_i^0}}, \qquad \phi_j^k = \frac{\chi_{B_j^k}}{\mu(B_j^k)}.$$

Clearly $\int \phi_j^k \, d\mu = 1$ and $\|\phi_j^k\|_{L^2} = \mu(B_j^k)^{-1/2}$. Set $f_j^0 = f\,\psi_j$, so that $f = \sum_{j=1}^N f_j^0$. Next, define

$$a_j^0 = f_j^0 - \phi_j^0 \int f_j^0 \, d\mu,$$
$$a_j^k = (\phi_j^{k-1} - \phi_j^k) \int f_j^0 \, d\mu \qquad k = 1, \ldots, K-1,$$
$$a_j^K = \phi_j^{K-1} \int f_j^0 \, d\mu.$$

Then, the support of a_j^0 is contained in B_j^0, the integral of a_j^0 vanishes, and

$$\|a_j^0\|_{L^2} \le \|f_j^0\|_{L^2} + \mu(B_j^0)^{-1/2} \|f_j^0\|_{L^2} \mu(B_j^0)^{1/2} = 2 \|f_j^0\|_{L^2} .$$

Hence, by Lemma 2.1

$$\|a_j^0\|_{\mathfrak{h}^1} \le 2 \|f_j^0\|_{L^2} \mu(B_j^0)^{1/2} .$$

The function a_j^k is supported in B_j^k, the integral of a_j^k vanishes, and

$$\|a_j^k\|_{L^2} \le \|f_j^0\|_{L^2} \, \mu(B_j^0)^{1/2} (\mu(B_j^{k-1})^{-1/2} + \mu(B_j^k)^{-1/2}) .$$

Then, again by Lemma 2.1

$$\begin{aligned} \|a_j^k\|_{\mathfrak{h}^1} &\le \|f_j^0\|_{L^2} \, \mu(B_j^0)^{1/2} \, \mu(B_j^k)^{1/2} (\mu(B_j^{k-1})^{-1/2} + \mu(B_j^k)^{-1/2}) \\ &= \|f_j^0\|_{L^2} \, \mu(B_j^0)^{1/2} \, \frac{\mu(B_j^{k-1})^{1/2} + \mu(B_j^k)^{1/2}}{\mu(B_j^{k-1})^{1/2}} \lesssim \|f_j^0\|_{L^2} \, \mu(B_j^0)^{1/2}. \end{aligned}$$

Finally, the function a_j^K is supported in B_j^K, whose radius is bigger than 1 but smaller than 2, so that by Lemma 2.1

$$\|a_j^K\|_{\mathfrak{h}^1} \lesssim \|a_j^K\|_{L^2} \lesssim \|f_j^0\|_{L^2} \, \mu(B_j^0)^{1/2}.$$

It follows that $f = \sum_{j=1}^N f_j^0 = \sum_{j=1}^N \sum_{k=0}^K a_j^k$ and

$$\begin{aligned} \|f\|_{\mathfrak{h}^1} &\lesssim \sum_{k=0}^K \sum_{j=1}^N \|f_j^0\|_{L^2} \, \mu(B_j^0)^{1/2} \\ &\le K \, N^{1/2} \left(\sum_{j=1}^N \|f_j^0\|_{L^2}^2 \right)^{1/2} r^{n/2} \lesssim \log(1/r) \, e^{\rho R} r^{1/2} \|f\|_{L^2}, \end{aligned}$$

the last inequality by (2.3.2) and since $\sum_{j=1}^N \|f_j^0\|_{L^2}^2 \le M \|f\|_{L^2}^2$, where M is the constant in (2.3.1). This completes the proof of the lemma. ∎

Lemma 2.3 *Let γ be a radial function supported in $B(o, \beta)$.*

(i) If a is a global atom at scale 1 supported in $B(o, 1)$, then

$$\|a * \gamma\|_{\mathfrak{h}^1} \le C\, \mu(B(o, 1+\beta))^{1/2}\, \|\gamma\|_{L^2};$$

(ii) if a is a standard atom supported in $B(o, r)$, $r \in (0, 1]$, then

$$\|a * \gamma\|_{\mathfrak{h}^1} \le C\, \mu(B(o, r+\beta))^{1/2}\, \min(\|\gamma\|_{L^2}, r\, \|\nabla\gamma\|_{L^2}),$$

where ∇ is the Riemannian gradient.

Proof To prove (i), if a is a global atom supported in $B(o, 1)$, then $a * \gamma$ is supported in $B(o, 1+\beta)$ and

$$\|a * \gamma\|_2 \le \|a\|_{L^1}\, \|\gamma\|_{L^2} \le \|\gamma\|_{L^2}.$$

Thus, (i) follows from Lemma 2.1.

To prove (ii), if a is a standard atom supported in $B(o, r)$, $r \le 1$, then $a * \gamma$ is supported in $B(o, r+\beta)$ and again

$$\|a * \gamma\|_2 \le \|a\|_{L^1}\, \|\gamma\|_{L^2} \le \|\gamma\|_{L^2}.$$

By arguing as in [13, Lemma 2.7] and using the cancellation of the atom, we obtain that

$$\|a * \gamma\|_2 \le r\, \|\nabla\gamma\|_{L^2}. \tag{2.3.3}$$

Thus, (ii) follows from Lemma 2.1. ∎

Lemma 2.4 *Let m be an even symbol in S_0^b and $\mathscr{U}_m$ be the operator defined by the Fourier multiplier m. The following hold:*

(i) if $2 \le q < \infty$ and $\frac{1}{q} = \frac{1}{2} + \frac{b}{n}$, then $\mathscr{U}_m$ is bounded from $L^2(\mathbb{X})$ to $L^q(\mathbb{X})$;
(ii) if $1 < s \le 2$ and $\frac{1}{s} = \frac{1}{2} - \frac{b}{n}$, then $\mathscr{U}_m$ is bounded from $L^s(\mathbb{X})$ to $L^2(\mathbb{X})$.

Proof Part (i) is proved in [12, Lemma 3].

Part (ii) follows by a duality argument. Indeed, the adjoint of $\mathscr{U}_m$ is the operator $\mathscr{U}_{\overline{m}}$. Since $m \in S_0^b$ also $\overline{m} \in S_0^b$. By (i) the operator $\mathscr{U}_{\overline{m}}$ is bounded from $L^2(\mathbb{X})$ to $L^q(\mathbb{X})$, with $2 \le q < \infty$ and $\frac{1}{q} = \frac{1}{2} + \frac{b}{n}$. Then $\mathscr{U}_m$ is bounded from $L^{q'}(\mathbb{X})$ to $L^2(\mathbb{X})$. Let $s = q'$. Then $1 < s \le 2$ and $\frac{1}{s} = 1 - \frac{1}{q} = 1 - \frac{1}{2} - \frac{b}{n} = \frac{1}{2} - \frac{b}{n}$, as required. ∎

2.4 Boundedness of $\mathscr{T}_t$ from $\mathfrak{h}^1(\mathbb{X})$ to $L^1(\mathbb{X})$

In this section, we prove part (i) of Theorem 2.1. The proof is inspired to that of [12, Proposition 4].

Proof (*of Theorem* 2.1 (*i*)) By Proposition 2.1 and since $\mathscr{T}_t$ is left invariant, it is enough to prove that

$$\sup\{\|\mathscr{T}_t a\|_{L^1} : a\ \mathfrak{h}^1\text{-atom supported in } B(o,r), r \le 1\} \lesssim e^{\rho t}.$$

Let a be an atom supported in $B(o,r)$, $r \le 1$. We separate two different cases, according to the values of t.

Case I: $t \ge 1/2$. We define the set

$$B^* := \{x \in \mathbb{X}\colon ||x| - t| < 10r\},$$

whose measure is $\mu(B^*) \lesssim re^{2\rho t}$, and split

$$\|\mathscr{T}_t a\|_{L^1} = \|\mathscr{T}_t a\|_{L^1(B^*)} + \|\mathscr{T}_t a\|_{L^1((B^*)^c)}.$$

We observe that by Hölder inequality

$$\|\mathscr{T}_t a\|_{L^1(B^*)} \le \mu(B^*)^{1/2}\|\mathscr{T}_t a\|_{L^2} \lesssim e^{\rho t} r^{1/2}\|\mathscr{T}_t a\|_{L^2}.$$

Moreover, by Lemma 2.4 (ii) with $\frac{1}{s} = \frac{1}{2} - \left(-\frac{d}{n}\right) = \frac{1}{2} + \frac{n-1}{2n} = 1 - \frac{1}{2n}$, Hölder inequality and the size condition of the atom

$$\|\mathscr{T}_t a\|_{L^2} \lesssim \|a\|_{L^s} \lesssim \mu(B)^{-1+1/s} \lesssim r^{-1/2}. \tag{2.4.1}$$

Thus, $\|\mathscr{T}_t a\|_{L^1(B^*)} \lesssim e^{\rho t}$.

Let now k_t be the radial kernel of the operator $\mathscr{T}_t$, and let K_t be the function on $[0,\infty)$ such that $k_t(x) = K_t(|x|)$. It remains to estimate the L^1-norm of $a * k_t$ on $(B^*)^c$. In order to do this, we take a function

$$\psi_t \in C_c^\infty(\mathbb{X}), \quad \psi_t(x) = 1 \text{ if } ||x| - t| < \tfrac{1}{10}, \quad \psi_t(x) = 0 \text{ if } ||x| - t| \ge \tfrac{2}{10},$$

with values in $[0, 1]$, define $\Psi_t(|x|) = \psi_t(x)$, and split the kernel k_t in its singular part s_t and its good part g_t as

$$k_t = k_t\psi_t + k_t(1-\psi_t) =: s_t + g_t.$$

Observe that this induces a splitting $K_t = K_t\Psi_t + K_t(1-\Psi_t) =: S_t + G_t$ of functions defined on $\mathbb{R}^+$. It is proved in [12, p. 287] that

$$|G_t(s)| \lesssim \begin{cases} s^{-d-1} & \text{if } s \le \frac{1}{10} \\ e^{-\rho s}|t-s|^{-2} & \text{if } \frac{1}{10} \le s \le t - \frac{1}{10} \\ e^{\rho t}e^{-2\rho s}|t-s|^{-2} & \text{if } s \ge t + \frac{1}{10} \end{cases} \tag{2.4.2}$$

from which $\|g_t\|_{L^1} \le e^{\rho t}$. Thus,

$$\|a * g_t\|_{L^1((B^*)^c)} \le \|a * g_t\|_{L^1} \le \|a\|_{L^1}\|g_t\|_{L^1} \le e^{\rho t}.$$

As for the convolution with s_t, we first consider the case when a is a global atom. Since ψ_t is supported in the annulus $A_{t-2/10}^{t+2/10}$, the convolution $a * s_t$ is supported in the annulus $A_{t-6/5}^{t+6/5}$. Then by Hölder inequality

$$\|a * s_t\|_{L^1((B^*)^c)} \le \|a * s_t\|_{L^1} \lesssim \mu(A_{t-6/5}^{t+6/5})^{1/2}\|a * s_t\|_{L^2} \lesssim e^{\rho t}\|a * s_t\|_{L^2}$$

where

$$\|a * s_t\|_{L^2} \lesssim \|a * k_t\|_{L^2} + \|a * g_t\|_{L^2} \lesssim \|\mathscr{T}_t\|_{L^2\to L^2}\|a\|_{L^2} + \|g_t\|_{L^2}\|a\|_{L^1} \lesssim 1,$$

since $\|g_t\|_{L^2} \lesssim 1$ by (2.4.2). Thus, $\|a * s_t\|_{L^1((B^*)^c)} \lesssim e^{\rho t}$. If instead a is a standard atom, by its cancellation condition it is easy to see that

$$a * s_t(x) = \int_G a(z)\left[s_t(z^{-1}x) - s_t(x)\right]\,dz = \int_B a(z)\left[S_t(|z^{-1}x|) - S_t(|x|)\right]\,dz$$

for every $x \in \mathbb{X}$, so that

$$\|a * s_t\|_{L^1((B^*)^c)} \le \int_B |a(z)| \int_{(B^*)^c} |S_t(|z^{-1}x|) - S_t(|x|)|\,dx\,dz.$$

It remains to observe that, since $|\partial_s S_t(s)| \lesssim e^{-\rho t}|t-s|^{-2}$ as shown in [12, p. 287],

$$\begin{aligned}\sup_{z\in B}\int_{(B^*)^c} |S_t(|z^{-1}x|) - S_t(|x|)|\,dx &\lesssim \sup_{z\in B}|z|\int_{10r\le||x|-t|\le r+2/10} |\partial_s S_t(|x|)|\,dx\\ &\lesssim re^{-\rho t}\int_{10r\le||x|-t|\le r+2/10} ||x|-t|^{-2}\,dx \lesssim e^{\rho t},\end{aligned}$$

which concludes the proof of the Case I.

Case II: $t < 1/2$. After defining the set

$$B^* := \{x \in \mathbb{X}\colon ||x| - t| < 10r\} \cup B(0, 10r),$$

we proceed as in the previous case. Since $\mu(B^*) \lesssim r$, we get $\|\mathscr{T}_t a\|_{L^1(B^*)} \le C$ again by (2.4.1). In order to estimate $\|\mathscr{T}_t a\|_{L^1((B^*)^c)}$, we pick a function

$$\psi_0 \in C_c^\infty(\mathbb{X}), \quad \psi_0(x) = 1 \ \text{ if } \ |x| \le \tfrac{3}{4}, \quad \psi_0(x) = 0 \ \text{ if } \ |x| \ge 1,$$

and split again the kernel k_t as

$$k_t = k_t\psi_0 + k_t(1-\psi_0) = s_t + g_t.$$

We let Ψ_0, S_t and G_t be the associated functions on $\mathbb{R}^+$. It is proved in [12, p. 288] that

$$|G_t(s)| \lesssim e^{-2\rho s}|t-s|^{-2} \qquad \forall\, s \geq \tfrac{3}{4},$$

so that $\|g_t\|_{L^1} \leq C$; hence, $\|a * g_t\|_{L^1((B^*)^c)} \leq C$. As for the convolution with s_t, if a is a global atom then as before

$$\|a * s_t\|_{L^1} \lesssim \mu(A_{t-1}^{t+1})^{1/2}\|a * s_t\|_{L^2} \lesssim \|\mathscr{T}_t\|_{L^2\to L^2}\|a\|_{L^2} + \|g_t\|_{L^2}\|a\|_{L^1} \lesssim 1,$$

while if a is a standard atom, by its cancellation condition we obtain again

$$\|a * s_t\|_{L^1((B^*)^c)} \leq \int_B |a(z)| \int_{(B^*)^c} |S_t(|z^{-1}x|) - S_t(|x|)|\, dx\, dz.$$

Proceeding as in [12, p. 288], S_t may be written as the sum of two functions $S_{1,t} + S_{2,t}$ such that $S_{1,t}(s) \leq s^{-d-1}$ (hence $s_{1,t} \in L^1(\mathbb{X})$) while

$$\left|\partial_s S_{2,t}(s)\right| \lesssim s^{-d}\left(|t-s|^{-2} + s|t-s|^{-1}\right).$$

The proof may be completed as before. ∎

2.5 Boundedness of $\mathscr{T}_t$ on $\mathfrak{h}^1(\mathbb{X})$

In this section, we prove part (ii) of Theorem 2.1, but first we need some preliminary results. We recall the behaviour of the Harish-Chandra function and of spherical functions on noncompact symmetric spaces of rank one. It follows from [12, Propositions A.1, A.2] and is based on various results in [20]. We denote by ρ' the number $\rho + \frac{1}{10}$.

Lemma 2.5 *The Harish-Chandra function* $\mathbf{c}$ *satisfies the following:*

(i) for all $\lambda \in \mathbb{R}$

$$|\mathbf{c}(\lambda)|^{-2} = \mathbf{c}(\lambda)^{-1}\,\mathbf{c}(-\lambda)^{-1};$$

(ii) the function $\lambda \mapsto \lambda^{-1}\,\mathbf{c}(-\lambda)^{-1}$ *is analytic inside the region* $\operatorname{Im}\lambda \geq 0$, *and for all* $\alpha \geq 0$, *there exists a positive constant* C_α *such that*

$$\left|\partial_\lambda^\alpha(\lambda^{-1}\,\mathbf{c}(-\lambda)^{-1})\right| \leq C_\alpha\,(1+|\operatorname{Re}\lambda|)^{d-1-\alpha} \qquad \forall\, 0 \leq \operatorname{Im}\lambda \leq \rho';$$

(iii) the function $\lambda \mapsto \lambda\,\mathbf{c}(\lambda)$ *is analytic in a neighbourhood of the real axis, and for all* $\alpha \geq 0$, *there exists a positive constant* C_α *such that*

$$\left|\partial_\lambda^\alpha(\lambda\,\mathbf{c}(\lambda))\right| \leq C_\alpha\,(1+|\operatorname{Re}\lambda|)^{1-d-\alpha} \qquad \forall\, \lambda \in \mathbb{R}.$$

The spherical functions ϕ_λ satisfy the following properties:

(a) $|\partial_s^\ell \phi_\lambda(s)| \le C\, e^{-\rho s}(1+s)\,(1+|\lambda|)^\ell \qquad \forall\, \lambda, s \in \mathbb{R},\ \ell \in \mathbb{N}.$

(b) If $s \le 1$, $\lambda \in \mathbb{R}$ and $s|\lambda| \ge 1$, for every $N \in \mathbb{N}$, ϕ_λ can be written as

$$\phi_\lambda(s) = e^{i\lambda s} a_1(\lambda, s) + e^{-i\lambda s} a_1(-\lambda, s) + O(\lambda, s),$$

where the functions $a_1, O : \{(s,\lambda) \in \mathbb{R} \times [0,1] \colon s|\lambda| \ge 1\} \to \mathbb{C}$ satisfy

$$\left|\partial_\lambda^\alpha \partial_s^\ell a_1(\lambda, s)\right| \le C\,[s(1+|\lambda|)]^{-d}\, s^{-\ell}\,(1+|\lambda|)^{-\alpha} \qquad \ell \in \{0,1\},\ \alpha \in [0,N]$$

and

$$|\partial_s^\ell O(\lambda, s)| \le C\,[s(1+|\lambda|)]^{-d-N-1-\ell}.$$

(c) If $s \ge 1/10$, then

$$\phi_\lambda(s) = e^{-\rho s}\left(e^{i\lambda s}\mathbf{c}(\lambda) a_2(\lambda, s) + e^{-i\lambda s}\mathbf{c}(-\lambda) a_2(-\lambda, s)\right),$$

where the function a_2 is such that for all $\alpha \ge 0$, there exist positive constants C_α such that

$$\left|\partial_\lambda^\alpha \partial_s^\ell a_2(\lambda, s)\right| \le C_\alpha\,(1+|\mathrm{Re}\,\lambda|)^{-\alpha} \qquad \forall\, \ell \in \{0,1\},\ s \ge \tfrac{1}{10},\ 0 \le \mathrm{Im}\,\lambda \le \rho'.$$

Proof The properties of the Harish–Chandra function were given in [12]. See also [1, Formula (2.2.5)].

Formula (a) follows from [7, Formula 5.1.18].

The proof of (b) follows the same outline of the proof of [12, Proposition A.2 (b)]. The only difference is that following the same arguments, it is possible to estimate the derivatives of the term $O(\lambda, s)$ which were not estimated in [12].

The proof of (c) is given in [12, Proposition A.2 (c)]. ■

In the following proposition, we shall prove pointwise estimates of the kernel of the operator $\mathcal{T}_t$ and of its derivative. We will distinguish the cases when t is either large or small. Let us mention that Ionescu [12] estimated the kernels of the operator $\mathcal{T}_t$ (but not their derivatives) far from the sphere of radius t, while he gave estimates of the derivatives of the kernels (but not of the kernels) near the sphere of radius t.

Proposition 2.2 *Let $\varepsilon > 0$ and $m \in S_\rho^{-d-\varepsilon}$ be an even symbol. Let k_t be the radial kernel of the operator $\mathcal{T}_t$ and K_t be the function on $[0,\infty)$ such that $k_t(x) = K_t(|x|)$. If $t \ge \frac{1}{2}$, then*

$$|K_t(s)| \lesssim \begin{cases} s^{-d-1+\varepsilon} & s \le \frac{1}{10} \\ e^{-\rho s}|t-s|^{-2+[\varepsilon]} & \frac{1}{10} \le s \le t - \frac{2}{10} \\ e^{-\rho t}|t-s|^{-1+\varepsilon} & t - \frac{2}{10} \le s \le t + \frac{2}{10} \\ e^{\rho t}\, e^{-2\rho s}\,|t-s|^{-2+[\varepsilon]} & s \ge t + \frac{2}{10}; \end{cases} \tag{2.5.1}$$

$$|K_t'(s)| \lesssim \begin{cases} s^{-d-2+\varepsilon} & s \leq \frac{1}{10} \\ e^{-\rho s}|t-s|^{-2+[\varepsilon]} & \frac{1}{10} \leq s \leq t - \frac{2}{10} \\ e^{-\rho t}|t-s|^{-2+\varepsilon} & t - \frac{2}{10} \leq s \leq t + \frac{2}{10} \\ e^{\rho t}\, e^{-2\rho s}\, |t-s|^{-2+[\varepsilon]} & s \geq t + \frac{2}{10}. \end{cases} \tag{2.5.2}$$

If $t < \frac{1}{2}$, *then*

$$|K_t(s)| \lesssim \begin{cases} e^{-2\rho s}\, |t-s|^{-2+[\varepsilon]} & s \geq 1 \\ s^{-d-1+\varepsilon} + s^{-d}|t-s|^{-1+\varepsilon} & s \leq 1\,; \end{cases} \tag{2.5.3}$$

$$|K_t'(s)| \lesssim \begin{cases} e^{-2\rho s}\, |t-s|^{-2+[\varepsilon]} & s \geq 1 \\ s^{-d-2+\varepsilon} + s^{-d}|t-s|^{-2+\varepsilon} + s^{-d-1}|t-s|^{-1+\varepsilon} & s \leq 1. \end{cases} \tag{2.5.4}$$

Proof Since the operator $\mathscr{T}_t$ corresponds to the spherical Fourier multiplier $\lambda \mapsto m(\lambda)\cos(t\lambda)$, by the inversion formula for the spherical transform (2.2.4) we get

$$K_t(s) = C \int_{\mathbb{R}} m(\lambda)\, \cos(t\lambda)\, \phi_\lambda(s)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda. \tag{2.5.5}$$

We distinguish the cases when t is either large or small.

Case I: $t \geq 1/2$. Let Ψ_t be a smooth cutoff function such that

$$\Psi_t(s) = 1 \;\text{ if }\; |s-t| \leq \tfrac{1}{10}, \quad \Psi_t(s) = 0 \;\text{ if }\; |s-t| \geq \tfrac{2}{10}.$$

Let $S_t := \Psi_t\, K_t$ and $G_t := (1-\Psi_t)\, K_t$. To prove (2.5.1) and (2.5.2), it is enough to estimate S_t and G_t and their derivatives. We shall repeatedly use, without further mention, [3, Lemma A.2] to estimate the Fourier transform of a symbol of some given order.

We first consider S_t. Observe that $S_t(s) = 0$ unless $|t-s| \leq \frac{2}{10}$, i.e. $t - \frac{2}{10} \leq s \leq t + \frac{2}{10}$. From (2.5.5) and Lemma 2.5, we deduce that

$$S_t(s) = C\, \Psi_t(s)\, e^{-\rho s} \int_{\mathbb{R}} m(\lambda)\, \cos(t\lambda)\, e^{i\lambda s}\, a_2(\lambda, s)\, \mathbf{c}(-\lambda)^{-1}\, d\lambda.$$

Since by Lemma 2.5 (c), the function $\lambda \mapsto m(\lambda)\; a_2(\lambda, s)\, \mathbf{c}(-\lambda)^{-1}$ is a symbol on the real line of order $-\varepsilon$

$$|S_t(s)| \lesssim e^{-\rho t}\, |t-s|^{-1+\varepsilon}.$$

Similarly, one can see that $|S_t'(s)| \lesssim e^{-\rho t}\, |t-s|^{-2+\varepsilon}$.

To estimate G_t and its derivative, we observe that $G_t(s) = 0$ unless $|t-s| \geq \frac{1}{10}$. The function G_t can be estimated as in [12, Formula (3.9)] (see also (2.4.2)). To estimate the derivative of G_t, we distinguish different cases.

We first consider the case when $s \le \frac{1}{10}$. We choose a smooth cutoff function η such that

$$\eta(v) = 1 \text{ if } |v| \le 1, \quad \eta(v) = 0 \text{ if } |v| \ge 2.$$

By Lemma 2.5 (b), we write

$$\begin{aligned} G_t(s) &= C\ (1 - \Psi_t(s)) \int_{\mathbb{R}} \eta(\lambda\, s)\, \phi_\lambda(s)\, m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &+ C\ (1 - \Psi_t(s)) \int_{\mathbb{R}} (1 - \eta(\lambda s))\ O(\lambda, s)\, m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &+ C\ (1 - \Psi_t(s)) \int_{\mathbb{R}} (1 - \eta(\lambda s))\, e^{i\lambda s}\, a_1(\lambda, s)\, m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda. \end{aligned}$$

Then

$$\begin{aligned} G_t'(s) = C &\int_{\mathbb{R}} \Big[- \Psi_t'(s)\eta(\lambda\, s)\, \phi_\lambda(s)\ + (1 - \Psi_t(s))\, \lambda\eta'(\lambda s)\, \phi_\lambda(s) \\ &+ (1 - \Psi_t(s))\, \eta(\lambda s)\, \partial_s \phi_\lambda(s) \Big] m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &+ C \int_{\mathbb{R}} \Big[- \Psi_t'(s)\, (1 - \eta(\lambda\, s))\ O(\lambda, s)\ - (1 - \Psi_t(s))\, \lambda\eta'(\lambda s)\, O(\lambda, s) \\ &+ (1 - \Psi_t(s))\, (1 - \eta(\lambda s))\ \partial_s O(\lambda, s) \Big] m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &+ C \int_{\mathbb{R}} \Big[- \Psi_t'(s)\, (1 - \eta(\lambda\, s))\ a_1(\lambda, s)\ - (1 - \Psi_t(s))\, \lambda\eta'(\lambda s)\, a_1(\lambda, s) \\ &+ (1 - \Psi_t(s))\, (1 - \eta(\lambda s))\ i\lambda\, a_1(\lambda, s)\ + (1 - \Psi_t(s))\, (1 - \eta(\lambda s))\ \partial_s a_1(\lambda, s) \Big] \\ &\times e^{i\lambda s}\, m(\lambda)\ \cos(t\lambda)\ |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ = &\ G_{1,t}'(s) + G_{2,t}'(s) + G_{3,t}'(s). \end{aligned} \tag{2.5.6}$$

By Lemma 2.5 (a)

$$|G_{1,t}'(s)| \lesssim \int_0^{2/s} (1 + \lambda)^{d-\varepsilon+1}\, d\lambda \lesssim s^{-d-2+\varepsilon}.$$

Similarly, by Lemma 2.5 (b), (with $N = 0$)

$$|G_{2,t}'(s)| \lesssim \int_{1/s}^{\infty} s^{-d-1}\, \lambda^{-\varepsilon-1}\, d\lambda \lesssim s^{-d-1+\varepsilon}.$$

To estimate $G_{3,t}'$, we write $\cos(t\lambda) = (e^{it\lambda} + e^{-it\lambda})/2$ and integrate by parts twice:

$$
\begin{aligned}
|G_{3,t}'(s)| \lesssim \frac{1}{|t-s|^2} \int_{\mathbb{R}} \Big| \partial_\lambda^2 \Big[-\Psi_t'(s)\,(1-\eta(\lambda\, s))\; a_1(\lambda, s)\; - (1-\Psi_t(s))\,\lambda\eta'(\lambda s)\, a_1(\lambda, s) \\
+ (1-\Psi_t(s))\,(1-\eta(\lambda s))\; i\lambda\, a_1(\lambda, s)\; + (1-\Psi_t(s))\,(1-\eta(\lambda s))\; \partial_s a_1(\lambda, s) \Big] \\
\times\; m(\lambda)\; |\mathbf{c}(\lambda)|^{-2} \Big| \, d\lambda.
\end{aligned}
$$

By applying Lemma 2.5 (b), we can easily show that $|G_{3,t}'(s)| \lesssim s^{-d-2+\varepsilon}$.

Thus, from (2.5.6) and the estimates above, we deduce that for every $s \le \frac{1}{10}$, $|G_t'(s)| \lesssim s^{-d-2+\varepsilon}$.

We now consider the case $s \ge \frac{1}{10}$. By Lemma 2.5 (c), we have

$$
G_t(s) = C\,(1-\Psi_t(s))\, e^{-\rho s} \int_{\mathbb{R}} m(\lambda)\, a_2(\lambda, s)\, \mathbf{c}(-\lambda)^{-1}\, e^{i\lambda s}\; \cos(t\lambda)\, d\lambda, \quad (2.5.7)
$$

so that

$$
\begin{aligned}
G_t'(s) = -C\Psi_t'(s)\; e^{-\rho s} \int_{\mathbb{R}} m(\lambda)\, a_2(\lambda, s)\, \mathbf{c}(-\lambda)^{-1}\, e^{i\lambda s}\; \cos(t\lambda)\, d\lambda \\
+ C\,(1-\Psi_t(s))\, e^{-\rho s} \int_{\mathbb{R}} m(\lambda)\; (\partial_s a_2(\lambda, s) + (-\rho + i\lambda)\, a_2(\lambda, s)) \\
\times\, \mathbf{c}(-\lambda)^{-1}\, e^{i\lambda s}\; \cos(t\lambda)\, d\lambda.
\end{aligned}
$$

Since $\lambda \mapsto m(\lambda)\; (a_2(\lambda, s) + \partial_s a_2(\lambda, s))\; \mathbf{c}(-\lambda)^{-1}$ is a symbol of order $-\varepsilon$, and $\lambda \mapsto m(\lambda)\, i\lambda\, a_2(\lambda, s)\, \mathbf{c}(-\lambda)^{-1}$ is a symbol of order $1-\varepsilon$, we obtain that

$$
|G_t'(s)| \lesssim e^{-\rho s}|t-s|^{-2+[\varepsilon]} \qquad \text{if } \tfrac{1}{10} \le s \le t - \tfrac{1}{10}.
$$

It remains to consider the case when $s \ge t + \frac{1}{10}$. In order to do this, we move the contour of integration in formula (2.5.7) to the line $\mathbb{R} + i\rho$ and obtain

$$
\begin{aligned}
G_t(s) =\; C\,(1-\Psi_t(s))\, e^{-2\rho s} \\
\times \int_{\mathbb{R}} m(\lambda + i\rho)\, a_2(\lambda + i\rho, s)\, \mathbf{c}(-\lambda - i\rho)^{-1}\, e^{i\lambda s} \cos(t(\lambda + i\rho))\, d\lambda.
\end{aligned}
$$

By taking the derivative, we get

$$
\begin{aligned}
G_t'(s) = e^{-2\rho s} \int_{\mathbb{R}} m(\lambda + i\rho)\mathbf{c}(-\lambda - i\rho)^{-1} \cos(t(\lambda + i\rho)) e^{i\lambda s} \Big\{ -\Psi_t'(s) a_2(\lambda + i\rho, s) \\
+ (1-\Psi_t(s))\,[a_2(\lambda + i\rho, s)(-2\rho + i\lambda) + \partial_s a_2(\lambda + i\rho, s)] \Big\}\, d\lambda.
\end{aligned}
$$

The estimates of the derivatives of a_2 and $\mathbf{c}^{-1}$ contained in Lemma 2.5 imply that

$$
|G_t'(s)| \lesssim e^{-2\rho s}\, e^{\rho t}\, |t-s|^{-2+[\varepsilon]}.
$$

By combining the estimates of G_t, G_t', S_t and S_t', one deduces the required estimates of K_t and its first derivative for t large.

Case II: $t < 1/2$. Let Ψ_0 be a smooth cutoff function such that

$$\Psi_0(s) = 1 \text{ if } s \leq \tfrac{3}{4}, \qquad \Psi_0(s) = 0 \text{ if } s \geq 1.$$

Let $S_t = \Psi_0\, K_t$ and $G_t = (1 - \Psi_0)\, K_t$.

We first analyse G_t and notice that $G_t(s) = 0$ if $s \leq 3/4$. If $s > 3/4$, then by Lemma 2.5 (c)

$$G_t(s) = C\ (1 - \Psi_0(s)) \int_{\mathbb{R}} m(\lambda)\ \cos(t\lambda) e^{-\rho s}\, e^{i\lambda s}\, a_2(\lambda, s)\, \mathbf{c}(-\lambda)^{-1} d\lambda,$$

which by moving the contour of integration from the real line to $\mathbb{R} + i\rho$ becomes

$$\begin{aligned} G_t(s) = {} & C\ (1 - \Psi_0(s))\ e^{-2\rho s} \\ & \times \int_{\mathbb{R}} m(\lambda + i\rho)\, e^{i\lambda s}\, a_2(\lambda + i\rho, s)\ \cos(t(\lambda + i\rho))\, \mathbf{c}(-\lambda - i\rho)^{-1} d\lambda. \end{aligned}$$

The function G_t can be estimated as in [12, p. 289]. Since G_t is the Fourier transform at $s \pm t$ of a symbol of order $-\varepsilon$, $s > 3/4$ and $t < 1/2$,

$$|G_t'(s)| \lesssim e^{-2\rho s}\, |t - s|^{-2+[\varepsilon]}.$$

It remains to consider S_t. Observe that $S_t(s) = 0$ unless $s \leq 1$; hence, we use Lemma 2.5 (c) (with $N = 0$) to write

$$\begin{aligned} S_t(s) &= \Psi_0(s) \int \eta(\lambda s)\, \phi_\lambda(s)\, m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &\quad + \Psi_0(s) \int (1 - \eta(\lambda s))\ O(\lambda, s)\, m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &\quad + \Psi_0(s) \int (1 - \eta(\lambda s)) e^{i\lambda s}\, a_1(\lambda, s)\, m(\lambda)\ \cos(t\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \\ &= S_{1,t}(s) + S_{2,t}(s) + S_{3,t}(s), \end{aligned}$$

where η is a smooth cutoff function such that $\eta(v) = 1$ if $|v| \leq 1$ and $\eta(v) = 0$ if $|v| \geq 2$. For every $s \leq 1$, we have

$$|S_{1,t}(s)| \lesssim \int_0^{2/s} \lambda^{d-\varepsilon} d\lambda \lesssim s^{-d-1+\varepsilon}$$

and

$$|S_{2,t}(s)| \lesssim \int_{1/s}^{\infty} (s\lambda)^{-d-1} \lambda^{d-\varepsilon} d\lambda \lesssim s^{-d-1+\varepsilon}.$$

Finally, $S_{3,t}$ is the inverse Fourier transform computed at $s \pm t$ of the symbol $\lambda \mapsto (1-\eta(\lambda s))\, a_1(\lambda, s)\, m(\lambda)\, |\mathbf{c}(\lambda)|^{-2}$ of order $-\varepsilon$. Then,

$$|S_{3,t}(s)| \lesssim s^{-d}\,|t-s|^{-1+\varepsilon}.$$

It then follows that for every $s \leq 1$,

$$|S_t(s)| \lesssim s^{-d-1+\varepsilon} + s^{-d}|t-s|^{-1+\varepsilon}.$$

In a similar way, one can prove that for every $s \leq 1$,

$$|S_t'(s)| \lesssim s^{-d-2+\varepsilon} + s^{-d}|t-s|^{-2+\varepsilon} + s^{-d-1}|t-s|^{-1+\varepsilon}.$$

By combining the estimates of G_t, G_t', S_t and S_t', one deduces the required estimates of K_t and its first derivative for t small. ■

Remark 2.1 Let us notice that the kernel K_t and its derivative behave in the same way far from the singularities, i.e. far from the point o and the sphere of radius t, while they have a different behaviour near o and near the sphere of radius t. Observe moreover that when $t \geq \frac{1}{2}$ and either $\frac{1}{10} \leq s \leq t - \frac{2}{10}$ or $s \geq t + \frac{2}{10}$, or when $t < \frac{1}{2}$ and $s \geq 1$, the power $|t-s|^{-2+[\varepsilon]}$ in the estimates of $K_t(s)$ and $K_t'(s)$ may be replaced with $|t-s|^{-M}$ for any integer $M \geq -2+[\varepsilon]$, provided the constant (which might depend on M) is properly chosen. This is a consequence of [3, Lemma A.2].

We are now in the position to prove the part (ii) of Theorem 2.1. The strategy we shall adopt consists in decomposing the kernel k_t of $\mathscr{T}_t$ into a sum of compactly supported functions which we shall consider separately. We also treat separately the cases when t is either large or small. The proof turns out to be more delicate when a is a standard atom supported in a ball of small radius, and in this case, the cancellation condition of the atom is crucial together with the estimates of the derivative of the kernel. When the atom is either a global atom or a standard atom supported in a ball of radius not too small when compared with t and 1, instead, the cancellation of the atom plays no role and only the estimates of the kernel are involved.

In order to do this, we shall repeatedly use smooth cutoff radial functions, which are introduced below. We fix $r \in (0,1]$ and $t > 0$.

Take a function $\phi \in C_c^\infty(\mathbb{R})$ supported in $[1/2, 2]$ such that $0 \leq \phi \leq 1$, $\phi = 1$ in $[1, 3/2]$, $\phi(s) = 1 - \phi(s/2)$ for every $s \in (1,2)$ and $|\phi'| \leq C$. For every $i \in \mathbb{N}$ and every $x \in \mathbb{X}$, define

$$\phi_i(x) = \phi\left(\frac{|x|}{2^i r}\right). \tag{2.5.8}$$

Observe that ϕ_i is supported in the annulus $A_{2^{i-1}r}^{2^{i+1}r}$, $0 \leq \phi_i \leq 1$ and $|\nabla\phi_i| \leq C\,(2^i r)^{-1}$.

For every $h \in \mathbb{N}$ and $x \in \mathbb{X}$, define

$$\eta_h(x) = \phi\left(\frac{t-|x|}{2^h r}\right), \qquad \omega_h(x) = \phi\left(\frac{|x|-t}{2^h r}\right). \tag{2.5.9}$$

The function η_h is supported in $A_{t-2^{h+1}r}^{t-2^{h-1}r}$, $0 \leq \eta_h \leq 1$ and $|\nabla \eta_h| \leq C\,(2^h r)^{-1}$. Similarly, ω_h is supported in $A_{t+2^{h-1}r}^{t+2^{h+1}r}$, $0 \leq \omega_h \leq 1$ and $|\nabla \omega_h| \leq C\,(2^h r)^{-1}$.

Finally, take a function $\psi \in C_c^\infty(\mathbb{R})$ supported in $[0, 2]$ such that $0 \leq \psi \leq 1$, $\psi = 1$ in $[2/3, 4/3]$ and $\psi(s+1) = 1 - \psi(s)$ for every $s \in (0, 1)$. For every $j \geq 2$ and $x \in \mathbb{X}$, define

$$\psi_j(x) = \psi\left(|x| - j + 1\right). \tag{2.5.10}$$

The function ψ_j is supported in A_{j-1}^{j+1} and $0 \leq \psi_j \leq 1$.

Proof (*of Theorem* 2.1 (*ii*)) By Proposition 2.1 and the left invariance of $\mathscr{T}_t$, it is enough to prove that

$$\sup\{\|\mathscr{T}_t a\|_{\mathfrak{h}^1} : a\ \mathfrak{h}^1\text{-atom supported in } B(o, r), r \leq 1\} \lesssim e^{\rho t}.$$

All throughout the proof, we let $\varepsilon := -b - d > 0$, so that $m \in S_\rho^{-d-\varepsilon}$. It will be crucial for the following to notice that by Lemma 2.4 (ii) with $\frac{1}{s} = \frac{1}{2} - \left(-\frac{d+\varepsilon}{n}\right) = 1 - \frac{1}{2n} + \frac{\varepsilon}{n}$, Hölder inequality and the size condition of the atom, we get

$$\|\mathscr{T}_t a\|_{L^2} \lesssim \|a\|_{L^s} \lesssim \mu(B)^{-1+1/s} \lesssim r^{-\frac{1}{2}+\varepsilon}. \tag{2.5.11}$$

Case I: $t \geq 1/2$.

Choose J such that $J - 2 \leq t + \frac{2}{10} \leq J - 1$. Then for every $j \geq J$, the function $a * (\psi_j k_t)$ is supported in $B(o, j + 1 + r)$. By Lemma 2.3 and estimate (2.5.1), we obtain

$$\begin{aligned}\|a * (\psi_j k_t)\|_{\mathfrak{h}^1} &\lesssim (\mu(B(o, j + r + 1))^{1/2}\, \|\psi_j k_t\|_{L^2} \\ &\lesssim e^{\rho j} \left(\int_{j-1}^{j+1} e^{2\rho t} e^{-4\rho s}\, |t - s|^{-4}\, e^{2\rho s}\, ds\right)^{1/2} \lesssim e^{\rho t}\, |t - j|^{-2}.\end{aligned}$$

Thus,

$$\sum_{j=J}^{\infty} \|a * (\psi_j k_t)\|_{\mathfrak{h}^1} \lesssim e^{\rho t} \sum_{j=J}^{\infty} (j - t)^{-2} \lesssim e^{\rho t} \int_J^\infty \frac{du}{(u - t)^2} \lesssim e^{\rho t}, \tag{2.5.12}$$

where we have used the fact that $J - 2 \leq t + \frac{2}{10} \leq J - 1$.

Subcase IA: $r \leq \frac{1}{10}$.

Let ϕ_0 be a smooth function taking values in $[0, 1]$ supported in $B(o, 3r)$ such that

$$\phi_0 + \sum_{i=1}^{I_1} \phi_i + \sum_{i=I_1+1}^{I_2} \phi_i + \sum_{h=3}^{H_1} \eta_h + \sum_{h=3}^{H_2} \omega_h + \sum_{j=J}^{\infty} \psi_j = 1 \qquad \text{in} \quad \mathbb{X} \setminus A_{t-10r}^{t+10r},$$

where $\phi_i, \eta_h, \omega_h, \psi_j$ are defined by formulae (2.5.8), (2.5.9), (2.5.10) and

$$\begin{aligned} 2^{I_1-1}r &\le \tfrac{1}{10} \le 2^{I_1+1}r, \\ 2^{I_2-1}r &\le t - \tfrac{2}{10} \le 2^{I_2+1}r, \\ t - 2^{H_1+1}r &\le t - \tfrac{2}{10} \le t - 2^{H_1-1}r, \\ t + 2^{H_2-1}r &\le t + \tfrac{2}{10} \le t + 2^{H_2+1}r. \end{aligned}$$

Define

$$\sigma_t = \left[1 - \phi_0 + \sum_{i=1}^{I_1} \phi_i + \sum_{i=I_1+1}^{I_2} \phi_i + \sum_{h=3}^{H_1} \eta_h + \sum_{h=3}^{H_2} \omega_h + \sum_{j=J}^{\infty} \psi_j \right] k_t,$$

so that

$$\mathscr{T}_t a = a * (\phi_0 k_t) + \sum_{i=1}^{I_2} a * (\phi_i k_t) + \sum_{h=3}^{H_1} a * (\eta_h k_t) + \sum_{h=3}^{H_2} a * (\omega_h k_t) + a * \sigma_t + \sum_{j=J}^{\infty} a * (\psi_j k_t).$$

The $\mathfrak{h}^1$-norm of the last term of the sum has been already estimated in (2.5.12). We now concentrate on the remaining terms.

The function $a * (\phi_0 k_t)$ is supported in $B(o, 4r)$ and by Lemma 2.1

$$\|a * (\phi_0 k_t)\|_{\mathfrak{h}^1} \le \mu(B(o, 4r))^{1/2} \|a * (\phi_0 k_t)\|_{L^2} \lesssim r^{n/2} \|\mathscr{T}_t\|_{L^2 \to L^2} \|a\|_{L^2} \lesssim 1, \tag{2.5.13}$$

where we have used the size condition of the atom and the fact that the norm of the operator $f \mapsto f * (\phi_0 k_t)$ on $L^2(\mathbb{X})$ is bounded by the norm of $\mathscr{T}_t$ on $L^2(\mathbb{X})$ (see e.g. [13, proof of Theorem 3.1]).

Consider now the cases $i = 1, \dots, I_1$. The function $a * (\phi_i k_t)$ is supported in $B(o, (2^{i+1} + 1)r)$. By Lemma 2.3 and by estimates (2.5.1) and (2.5.2), we obtain that

$$\begin{aligned} \|a * (\phi_i k_t)\|_{\mathfrak{h}^1} &\lesssim (\mu(B(o, (2^{i+1} + 1)r))^{1/2}\, r\, \|\nabla(\phi_i k_t)\|_{L^2} \\ &\lesssim (2^i r)^{n/2}\, r \left(\int_{2^{i-1}r}^{2^{i+1}r} [(2^i r)^{-2} s^{-2d-2+2\varepsilon} + s^{-2d-4+2\varepsilon}] s^{n-1}\, ds \right)^{1/2} \\ &\lesssim (2^i)^{\varepsilon+(n-3)/2} r^{\varepsilon+(n-1)/2}. \end{aligned}$$

Thus, since $I_1 \asymp \log_2\left(\frac{1}{10r}\right)$, we get

$$\sum_{i=1}^{I_1} \|a * (\phi_i k_t)\|_{\mathfrak{h}^1} \lesssim r^{\varepsilon+(n-1)/2} \int_1^{\log_2\left(\frac{1}{10r}\right)} (2^u)^{\varepsilon+(n-3)/2}\, du \lesssim r.$$

Consider now the cases when $i = I_1 + 1, \dots, I_2$. The function $a * (\phi_i k_t)$ is supported in $B(o, (2^{i+1} + 1)r)$. By Lemma 2.3 and by estimates (2.5.1) and (2.5.2), we obtain that

$$\begin{aligned}\|a*(\phi_i k_t)\|_{\mathfrak{h}^1} &\lesssim (\mu(B(o,(2^{i+1}+1)r))^{1/2}\, r\,\|\nabla(\phi_i k_t)\|_{L^2}\\ &\lesssim e^{\rho 2^i r} r\left(\int_{2^{i-1}r}^{2^i r}[(2^i r)^{-2}e^{-2\rho s}+e^{-2\rho s}]e^{2\rho s}\,ds\right)^{1/2}\lesssim e^{\rho 2^i r}\, r^{3/2}\, 2^{i/2}.\end{aligned}$$

Thus, since $2^{I_2}r \asymp t-\frac{1}{10}$, we get

$$\begin{aligned}\sum_{i=I_1+1}^{I_2}\|a*(\phi_i k_t)\|_{\mathfrak{h}^1} &\lesssim r^{3/2}\sum_{i=I_1+1}^{I_2} e^{\rho 2^i r}\, 2^{i/2}\lesssim r^{3/2}\int_{I_1+1}^{I_2} e^{\rho r 2^u}2^{u/2}\,du\\ &\lesssim r^{3/2}\int_{2^{I_1+1}}^{2^{I_2}} v^{-1/2}e^{\rho v r}\,dv\lesssim r^{1/2}e^{\rho t}.\end{aligned}$$

Consider now $3\le h\le H_1$. By the triangular inequality, the function $a*(\eta_h k_t)$ is supported in $A^{t-(2^{h-1}-1)r}_{t-(2^{h+1}+1)r}$ and has vanishing integral and by Lemma 2.2, by (2.3.3) and estimates (2.5.1) and (2.5.2)

$$\begin{aligned}\|a*(\eta_h k_t)\|_{\mathfrak{h}^1} &\lesssim \log(1/(2^h r))(2^h r)^{1/2}\, e^{\rho t}\, r\,\|\nabla(\eta_h k_t)\|_{L^2}\\ &\lesssim e^{\rho t} r\,(2^h r)^{-1+\varepsilon}\log(1/(2^h r))\end{aligned}$$

since $\|\nabla(\eta_h k_t)\|_{L^2}\lesssim (2^h r)^{-3/2+\varepsilon}$. Using the fact that $2^{H_1}r\asymp\frac{2}{10}$ and then changing variables $2^h r=v$, we obtain

$$\begin{aligned}\sum_{h=3}^{H_1}\|a*(\eta_h k_t)\|_{\mathfrak{h}^1} &\lesssim e^{\rho t} r\sum_{h=3}^{H_1}(2^h r)^{-1+\varepsilon}\log(1/(2^h r))\\ &\lesssim e^{\rho t} r\int_3^{H_1}(2^h r)^{-1+\varepsilon}\log(1/(2^h r))\,dh\\ &\lesssim e^{\rho t} r\int_{8r}^{2/10} v^{-2+\varepsilon}\log(1/v)\,dv\lesssim e^{\rho t} r^{\varepsilon}\log(1/r)\lesssim e^{\rho t}.\end{aligned}$$

Similar computations can be done for $a*(\omega_h k_t)$, proving that

$$\sum_{h=3}^{H_2}\|a*(\omega_h k_t)\|_{\mathfrak{h}^1}\lesssim e^{\rho t}.$$

It remains to consider $a*\sigma_t$, where σ_t is the singular part of the kernel supported in A^{t+10r}_{t-10r}. By the triangular inequality, the function $a*\sigma_t$ is supported in A^{t+11r}_{t-11r}. For every $x\in A^{t+11r}_{t-11r}$, we have

$$\mathscr{T}_t a(x)=a*\sigma_t(x)+a*(\eta_3 k_t)(x)+a*(\omega_3 k_t)(x),$$

so that

$$\begin{aligned}\|a*\sigma_t\|_{L^2} &\le \|\mathscr{T}_t a\|_{L^2} + \|a*(\eta_3 k_t)\|_{L^2} + \|a*(\omega_3 k_t)\|_{L^2}\\ &\lesssim r^{-1/2+\varepsilon} + r\|\nabla(\eta_3 k_t)\|_{L^2} + r\|\nabla(\eta_3 k_t)\|_{L^2} \lesssim r^{-1/2+\varepsilon}.\end{aligned} \tag{2.5.14}$$

The second inequality follows from (2.5.11) and (2.3.3), while the third follows from the computations we made before for $\nabla(\eta_h k_t)$ and a similar computation for $\nabla(\omega_h k_t)$.

We deduce from Lemma 2.2 and (2.5.14) that

$$\|a*\sigma_t\|_{\mathfrak{h}^1} \lesssim \log(1/r) e^{\rho t} r^{1/2} r^{-1/2+\varepsilon} \lesssim e^{\rho t}.$$

Subcase IB: $\frac{1}{10} < r \le 1$.

Choose two smooth cutoff functions ϕ_0 and ϕ_t with values in [0, 1] such that

$$\operatorname{supp}(\phi_0) \subseteq B(o,3), \qquad \operatorname{supp}(\phi_t) \subseteq A_2^{t-\frac{1}{10}}$$
$$\phi_0 + \phi_t + \sum_{j=J}^{\infty} \psi_j = 1 \qquad \text{in} \quad \mathbb{X} \setminus A_{t-\frac{1}{10}}^{t+\frac{1}{10}},$$

(if $t - 1/10 < 2$, then just $\phi_t \equiv 0$) and define

$$\sigma_t = \left[1 - \phi_0 - \phi_t - \sum_{j=J}^{\infty} \psi_j\right] k_t.$$

The convolution of a with the sum of the ψ_j's has been already estimated in (2.5.12). The function $a*(\phi_0 k_t)$ is supported in $B(o, 3+r)$ and by Lemma 2.1

$$\|a*(\phi_0 k_t)\|_{\mathfrak{h}^1} \lesssim \mu(B(o,4))^{1/2}\, \|\mathscr{T}_t\|_{L^2\to L^2}\, \|a\|_{L^2} \lesssim 1, \tag{2.5.15}$$

where we argued as in (2.5.13). By Lemma 2.3 and estimates (2.5.1), we get

$$\begin{aligned}\|a*(\phi_t k_t)\|_{\mathfrak{h}^1} &\lesssim \mu(B(o, t-\tfrac{1}{5}+r))^{1/2}\, \|\phi_t k_t\|_{L^2}\\ &\lesssim e^{\rho t} \left(\int_2^{t-\frac{1}{5}} e^{-2\rho s} |t-s|^{-4} e^{2\rho s}\, ds\right)^{1/2} \lesssim e^{\rho t}.\end{aligned} \tag{2.5.16}$$

It remains to estimate the $\mathfrak{h}^1$-norm of $a*\sigma_t$, which is supported in $A_{t-r-1/10}^{t+r+1/10}$. Since

$$\mathscr{T}_t a(x) = a*\sigma_t(x) + a*(\phi_t k_t)(x) + a*(\psi_J k_t)(x) \qquad \forall\, x \in A_{t-r-1/10}^{t+r+1/10},$$

then

$$\begin{aligned}\|a*\sigma_t\|_{L^2} &\le \|\mathscr{T}_t a\|_{L^2} + \|a*(\phi_t k_t)\|_{L^2} + \|a*(\psi_J k_t)\|_{L^2}\\ &\le \|\mathscr{T}_t\|_{L^2\to L^2}\, \|a\|_{L^2} + \|\phi_t k_t\|_{L^2} + \|\psi_J k_t\|_{L^2} \lesssim 1,\end{aligned}$$

which follows from (2.5.11) and the computations we made in (2.5.16) and (2.5.12). Thus,

$$\|a * \sigma_t\|_{\mathfrak{h}^1} \lesssim \mu\left(B(o, t + \tfrac{1}{10} + r)\right)^{1/2} \|a * \sigma_t\|_{L^2} \lesssim e^{\rho t}.$$

The proof in the case $t \geq 1/2$ is then complete.

Case II: $t < 1/2$.

For every $j \geq 2$ by Lemma 2.3 and estimates (2.5.3), we get

$$\begin{aligned}\|a * (\psi_j k_t)\|_{\mathfrak{h}^1} &\lesssim \mu(B(o, j+2))^{1/2} \|\psi_j k_t\|_{L^2} \\ &\lesssim e^{\rho j} \left(\int_{j-1}^{j+1} e^{-4\rho s}(1 + |t-s|^{-2})^2 e^{2\rho s}\, ds\right)^{1/2} \lesssim (j-t)^{-2} \lesssim j^{-2},\end{aligned}$$

where the functions ψ_j are defined in (2.5.10). Thus,

$$\sum_{j=2}^{\infty} \|a * (\psi_j k_t)\|_{\mathfrak{h}^1} \lesssim \sum_{j=2}^{\infty} j^{-2} \lesssim 1. \tag{2.5.17}$$

Subcase IIA: $r \leq \frac{t}{20}$.

Let ϕ_0 be a cutoff function supported in $B(o, 3r)$ taking values in [0, 1] such that

$$\phi_0 + \sum_{i=2}^{I} \phi_i + \sum_{i=I_1}^{I_2} \phi_i + \sum_{j=2}^{\infty} \psi_j = 1 \qquad \text{in} \quad \mathbb{X} \setminus A_{t-10r}^{t+10r},$$

where the ϕ_i's are defined by (2.5.8) and

$$\begin{aligned} 2^{I-1}r &< t - 10r < 2^{I+1}r, \\ 2^{I_1-1}r &< t + 10r < 2^{I_1+1}r, \\ 2^{I_2-1}r &< 1 < 2^{I_2+1}r. \end{aligned}$$

Define

$$\sigma_t = \left[1 - \phi_0 - \sum_{i=3}^{I} \phi_i - \sum_{i=I_1}^{I_2} \phi_i - \sum_{j=1}^{\infty} \psi_j\right] k_t.$$

The $\mathfrak{h}^1$-norm of the convolution with the ψ_j's has been already estimated in (2.5.17). Since $a * (\phi_0 k_t)$ is supported in $B(o, 4r)$

$$\|a * (\phi_0 k_t)\|_{\mathfrak{h}^1} \lesssim \mu((B(o, 4r)))^{1/2} \|a * (\phi_0 k_t)\|_{L^2} \lesssim r^{n/2} \|a\|_{L^2} \, \|\mathscr{T}_t\|_{L^2 \to L^2} \lesssim 1,$$

where we argued as in (2.5.13). For every $i \in \{2, \ldots, I\} \cup \{I_1, \ldots, I_2\}$, the function $a * (\phi_i k_t)$ is supported in $B(o, 2^{i+1}r + r)$ and by Lemma 2.3 and estimates (2.5.3) and (2.5.4)

$$\begin{aligned}\|a*(\phi_i k_t)\|_{\mathfrak{h}^1} &\lesssim \mu(B(o,2^{i+1}r+r))^{1/2}\, r\, \|\nabla(\phi_i k_t)\|_{L^2}\\ &\lesssim (2^i r)^{n/2}\, r \left(\int_{2^{i-1}r}^{2^{i+1}r} [(2^i r)^{-2} s^{-2-2d+2\varepsilon} + (2^i r)^{-2} s^{-2d} |t-s|^{-2+2\varepsilon} \right.\\ &\qquad \left. + s^{-2d-4+2\varepsilon} + s^{-2d}|t-s|^{-4+2\varepsilon} + s^{-2d-2}|t-s|^{-2+2\varepsilon}] s^{n-1}\, ds \right)^{1/2}\\ &\lesssim r \left[(2^i r)^{\frac{n-3}{2}+\varepsilon} + (2^i r)^{\frac{n-1}{2}} |t-2^i r|^{-1+\varepsilon} + (2^i r)^{\frac{n+1}{2}} |t-2^i r|^{-2+\varepsilon} \right].\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{i=2}^{I} \|a*(\phi_i k_t)\|_{\mathfrak{h}^1} &\lesssim r^{\frac{n-1}{2}+\varepsilon} \int_2^I (2^u)^{\frac{n-3}{2}+\varepsilon}\, du + r^{\frac{n+1}{2}} \int_2^I (2^u)^{\frac{n-1}{2}} |t-2^u r|^{-1+\varepsilon}\, du\\ &\qquad + r^{\frac{n+3}{2}} \int_2^I (2^u)^{\frac{n+1}{2}} |t-2^u r|^{-2+\varepsilon}\, du.\end{aligned}$$

By the change of variables $2^u r = w$ and recalling that $2^I r \asymp t-10r < t < 1/2$,

$$\begin{aligned}\sum_{i=3}^{I} \|a*(\phi_i k_t)\|_{\mathfrak{h}^1} &\lesssim r + r\int_{4r}^{2^I r} w^{\frac{n-3}{2}} |t-w|^{-1+\varepsilon}\, dw + r\int_{4r}^{2^I r} w^{\frac{n-1}{2}} |t-w|^{-2+\varepsilon}\, dw\\ &\lesssim r^{\varepsilon} \lesssim 1\end{aligned}$$

since $|t-w| \geq |t-4r| \gtrsim r$. Arguing as before, we can also prove that

$$\sum_{i=I_1}^{I_2} \|a*(\phi_i k_t)\|_{\mathfrak{h}^1} \lesssim 1.$$

It remains to consider $a*\sigma_t$, where σ_t is the singular part of the kernel supported in A_{t-10r}^{t+10r}. By the triangular inequality, $a*\sigma_t$ is supported in A_{t-11r}^{t+11r}. For every $x \in A_{t-11r}^{t+11r}$, we have

$$\mathscr{T}_t a(x) = a*\sigma_t(x) + a*(\phi_I k_t)(x) + a*(\phi_{I_1} k_t)(x),$$

so that

$$\begin{aligned}\|a*\sigma_t\|_{L^2} &\leq \|\mathscr{T}_t a\|_{L^2} + \|a*(\phi_I k_t)\|_{L^2} + \|a*(\phi_{I_1} k_t)\|_{L^2}\\ &\lesssim r^{-1/2+\varepsilon} + r\|\nabla(\phi_I k_t)\|_{L^2} + r\|\nabla(\phi_{I_1} k_t)\|_{L^2} \lesssim r^{-1/2+\varepsilon},\end{aligned}$$

where we have applied (2.5.11) and the computations we made above. Then by Lemma 2.2,

$$\|a*\sigma_t\|_{\mathfrak{h}^1} \lesssim \log(1/r)\mu(A_{t-11r}^{t+11r})^{1/2} \|a*\sigma_t\|_{L^2} \lesssim \log(1/r) r^{\varepsilon} \lesssim 1.$$

Subcase IIB: $\frac{t}{20} < r \leq 1$.

Notice that $t + 10r < 30r$. We choose a smooth cutoff function ϕ_0 supported in $B(o, 30r)$ taking values in [0, 1] such that

$$\phi_0 + \sum_{i=5}^{I} \phi_i + \sum_{j=2}^{\infty} \psi_j = 1$$

in $\mathbb{X}$, where I is such that $2^{I-1}r < 1 < 2^{I+1}r$. We split the kernel k_t accordingly as we did before. Then $a * (\phi_0 k_t)$ is supported in $B(o, 31r)$ and

$$\|a * (\phi_0 k_t)\|_{\mathfrak{h}^1} \lesssim \mu(B(o, 31r))^{1/2} \|a\|_{L^2} \, \|\mathscr{T}_t\|_{L^2 \to L^2} \lesssim 1,$$

where we argued as in (2.5.13). For every $i = 5, \ldots, I$ by Lemma 2.3 and estimate (2.5.3), one can see that

$$\|a * (\phi_i k_t)\|_{\mathfrak{h}^1} \lesssim (2^i r)^{n/2} \|\phi_i k_t\|_{L^2} \lesssim (2^i r)^{\frac{n-1}{2}+\varepsilon} + (2^i r)^{\frac{n+1}{2}} |t - 2^i r|^{-1+\varepsilon},$$

which yields

$$\sum_{i=5}^{I} \|a * (\phi_i k_t)\|_{\mathfrak{h}^1} \lesssim r^{\frac{n-1}{2}+\varepsilon} \int_{2^5}^{2^I} v^{\frac{n-3}{2}+\varepsilon} \, dv + \int_{32r}^{1} v^{\frac{n-1}{2}} |t - v|^{-1+\varepsilon} \, dv \lesssim 1$$

where we used the fact that $2^{I-1}r < 1 < 2^{I+1}r$. This concludes the proof of the case $t < 1/2$ and of the theorem. ■

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Chapter 3
Weak-Type Estimates for the Metaplectic Representation Restricted to the Shearing and Dilation Subgroup of $SL(2, \mathbb{R})$

Alessandra Cauli

Abstract We consider the subgroup G of $SL(2, \mathbb{R})$ consisting of shearing and dilations, and we study the decay at infinity of the matrix coefficients of the metaplectic representation restricted to G. We prove weak-type estimates for such coefficients, which are uniform for functions in the modulation space M^1. This work represents a continuation of a project aiming at studying weak-type and Strichartz estimates for unitary representations of non-compact Lie groups.

3.1 Introduction

In [1], we started a project aiming at studying weak-type and Strichartz estimates for unitary representations of non-compact Lie groups. There, we consider the case of the metaplectic representation, which is a faithful representation of the metaplectic group $Mp(n, \mathbb{R})$ in $L^2(\mathbb{R}^n)$, being the double covering of the symplectic group $Sp(n, \mathbb{R})$. It can therefore be identified with a subgroup of unitary operators $\hat{S}$ on $L^2(\mathbb{R}^n)$. We denote by $\hat{S} \mapsto S \in Sp(n, \mathbb{R})$ the projection. Among other things, we proved the *uniform* and sharp weak-type estimate

$$\left\| \left\langle \hat{S}\varphi_1, \varphi_2 \right\rangle \right\|_{L^{4,\infty}(Mp(n,\mathbb{R}))} \lesssim \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1} ,$$

where $Mp(n, \mathbb{R})$ is endowed with its Haar measure. In this formula, M^1 denotes a modulation space—a Banach space well known in time–frequency analysis [16]—whose definition will be recalled in the next section. Here it is enough to observe

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that it is dense in $L^2(\mathbb{R}^n)$ and contains the Schwartz space. From the above estimate, we can deduce Strichartz-type estimates for the metaplectic representation. In this paper, we consider a similar problem, but for the metapletic representation restricted to the shearing and dilation subgroup G of $SL(2,\mathbb{R})$, constituted by the matrices of the form $l_t d_{s^{1/2}}$, with $t \in \mathbb{R}$, $s > 0$, where

$$d_s = \begin{pmatrix} s^{-1} & 0 \\ 0 & s \end{pmatrix}, s > 0, l_t = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, t \in \mathbb{R}$$

hence

$$l_t d_{s^{1/2}} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} s^{-1/2} & 0 \\ 0 & s^{1/2} \end{pmatrix} = \begin{pmatrix} s^{-1/2} & 0 \\ ts^{-1/2} & s^{1/2} \end{pmatrix}.$$

The corresponding Haar measure is $\frac{dtds}{s^2}$. This group is particularly important in time–frequency analysis because it is reproducing, see [13, Lemma 2.1] as well as [2–5], where it is shown that the metaplectic representation plays a key role in the classification of the reproducing subgroups of the affine symplectic group.

Now, for any $\varphi_1, \varphi_2 \in M^1$ we consider again the matrix coefficient $G \to \mathbb{R}$, $S \mapsto \left|\left\langle \hat{S}\varphi_1, \varphi_2 \right\rangle\right|$. Our main result reads as follows.

Theorem 3.1 *Given G as above, we have:*

$$\left\|\left\langle \hat{S}\varphi_1, \varphi_2 \right\rangle\right\|_{L^{4,\infty}} \lesssim \|\varphi_1\|_{M^1} \|\varphi_2\|_{M^1}.$$

From this result, it is easy to obtain corresponding Strichartz-type estimates, following the pattern in [1]. The results on Strichartz estimates are often applied to wellposedness and scattering of nonlinear PDEs, see [17] for further details. We also refer to [10–12] for general results concerning decay estimates for matrix coefficients of unitary representation and to [6–9] for the role of the metaplectic representation in PDEs from a time–frequency analysis perspective.

The paper is organized as follows. Section 3.2 contains some preliminary results, Sect. 3.3 is devoted to a survey of the results on reproducing groups, with the aim of describing the role of the above group G in time–frequency analysis. In Sect. 3.4, we recall some results from [1], while in Sect. 3.5, we prove our main result.

3.2 Preliminaries

3.2.1 Integration on the Symplectic Group

The symplectic group $Sp(n,\mathbb{R})$ is the group of $2n \times 2n$ real matrices S such that $SJ = JS$, where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is the standard symplectic form

$$\omega(x, y) = x^T J y, \quad x, y \in \mathbb{R}^{2n}. \tag{3.2.1}$$

$Sp(n, \mathbb{R})$ turns out to be a unimodular Lie group. The following integration formula for $U(2n, \mathbb{R})$-bi-invariant functions on $Sp(n, \mathbb{R})$ will be crucial in the following.

Recall that $f : Sp(n, \mathbb{R}) \to \mathbb{C}$ is called $U(2n, \mathbb{R})$-bi-invariant if $f(U_1 S U_2) = f(S)$ for every $S \in Sp(n, \mathbb{R})$, $U_1, U_2 \in U(2n, \mathbb{R})$.

Consider the abelian subgroup $A = \{a_t\}$ of $Sp(n, \mathbb{R})$ given by

$$a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}, \quad t = \mathrm{diag}(t_1, \ldots, t_n), \ (t_1, \ldots, t_n) \in \mathbb{R}^n.$$

If f is a $U(2n, \mathbb{R})$-bi-invariant function on $Sp(n, \mathbb{R})$, its integral with respect to the Haar measure is given by

$$\int_{Sp(n,\mathbb{R})} f(S) dS = C \int_{t_1 \geq \ldots \geq t_n \geq 0} f(a_t) \prod_{i<j} \sinh \frac{t_i - t_j}{2} \prod_{i \leq j} \sinh \frac{t_i + t_j}{2} \, dt_1 \ldots dt_n \tag{3.2.2}$$

for some constant $C > 0$.

3.2.2 *The Metaplectic Representation*

The metaplectic representation μ links the standard Schrödinger representation ρ of the Heisenberg group $\mathbb{H}^n$ to the representation obtained from it by composing ρ with the action of $Sp(n, \mathbb{R})$ by automorphisms on $\mathbb{H}^n$. We recall here its construction. The product in the Heisenberg group $\mathbb{H}^n$ is defined by

$$(z, t) \cdot (z', t') = \left(z + z', t + t' - \frac{1}{2}\omega(z, z') \right)$$

on $\mathbb{R}^{2n+1}$, where ω is the standard symplectic form in $\mathbb{R}^{2n}$ given by (3.2.1). We denote the translation and modulation operators on $L^2(\mathbb{R}^n)$ by

$$T_x f(t) = f(t - x) \quad and \quad M_\xi f(t) = e^{2\pi i \langle \xi, t \rangle} f(t).$$

The Schrödinger representation of the group $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$ is then defined by

$$\rho(x, \xi, t) f(y) = e^{2\pi i t} e^{\pi i \langle x, \xi \rangle} e^{2\pi i \langle \xi, y - x \rangle} f(y - x) = e^{2\pi i t} e^{\pi i \langle x, \xi \rangle} T_x M_\xi f(y),$$

where we write $z = (x, \xi)$ when we separate space components from frequency components in a point z in the phase space $\mathbb{R}^{2n}$. The symplectic group acts on $\mathbb{H}^n$ via automorphisms that leave the center $\{(0, t) : t \in \mathbb{R}\} \simeq \mathbb{R}$ of $\mathbb{H}^n$ point-wise fixed:

$$A \cdot (z, t) = (Az, t).$$

Moreover, for any fixed $A \in Sp(n, \mathbb{R})$ there is a representation

$$\rho_A : \mathbb{H}^n \longrightarrow \mathscr{U}(L^2(\mathbb{R}^n)), \quad (z, t) \mapsto \rho(A \cdot (z, t))$$

whose restriction to the center is a multiple of the identity. By the Stone–von Neumann theorem, ρ_A is equivalent to ρ. This means that there exists an unitary operator $\mu(A) \in \mathscr{U}(L^2(\mathbb{R}^n))$ such that $\rho_A(z, t) = \mu(A) \circ \rho(z, t) \circ \mu(A)^{-1}$, for all $(z, t) \in \mathbb{H}^n$. By Schur's lemma, μ is determined up to a phase factor e^{is}, $s \in \mathbb{R}$. Actually, the phase ambiguity is really a sign and μ lifts to a representation of the double cover of the symplectic group. It is exactly the famous metaplectic or Shale–Weil representation.

The representations ρ and μ can be combined and produced the extended metaplectic representation of the group $G = \mathbb{H}^n \rtimes Sp(n, \mathbb{R})$. The group law on G is

$$((z, t), A) \cdot ((z', t'), A') = ((z, t) \cdot (Az', t'), AA')$$

and the extended metaplectic representation μ_e of G is

$$\mu_e((z, t), A) = \rho(z, t) \circ \mu(A).$$

The role of the center of the Heisenberg group is irrelevant, and the true group under consideration is $\mathbb{R}^{2n} \rtimes Sp(n, \mathbb{R})$, which we denote again by G. G acts naturally by affine transformations on the phase space, namely

$$g \cdot (x, \xi) = ((q, p), A) \cdot (x, \xi) = A^T (x, \xi) + (q, p)^T.$$

For elements of $Sp(n, \mathbb{R})$ in special form and for $f \in L^2(\mathbb{R}^n)$, the metaplectic representation can be computed explicitly in a simple way, so we have:

$$\mu\left(\begin{bmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{bmatrix}\right) f(x) = (det A)^{-1/2} f(A^{-1}x) \tag{3.2.3}$$

$$\mu\left(\begin{bmatrix} I & 0 \\ C & I \end{bmatrix}\right) f(x) = \pm e^{i\pi\langle Cx, x\rangle} f(x) \tag{3.2.4}$$

$$\mu(J) = (-i)^{d/2} \mathscr{F} \tag{3.2.5}$$

where $\mathscr{F}$ denotes the Fourier transform

$$\mathscr{F} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle x,\xi\rangle} dx, \quad f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n).$$

3.2.3 Modulation Spaces

Fix a window function $\varphi \in \mathscr{S}(\mathbb{R}^n) \setminus \{0\}$. The short-time Fourier transform (STFT) of a function–temperate distribution $\psi \in \mathscr{S}'(\mathbb{R}^n)$ with respect to φ is defined by

$$V_\varphi \psi(x, \xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot y} \psi(y) \overline{\varphi(y - x)} dy, \quad x, \xi \in \mathbb{R}^n.$$

For $1 \le p, q \le \infty$ and a Schwartz function $\varphi \in \mathscr{S}(\mathbb{R}^n) \setminus \{0\}$, the modulation space $M^{p,q}(\mathbb{R}^n)$ is defined as the space of $\psi \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$\|\psi\|_{M^{p,q}} := \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |V_\varphi \psi(x, \xi)|^p dx \Big)^{q/p} d\xi \Big)^{1/q} < \infty,$$

with obvious changes if $p = \infty$ or $q = \infty$.

If $p = q$, then we write M^p instead of $M^{p,p}$.

We will also need a variant, sometimes called Wiener amalgam space in the literature, whose norm is

$$\|\psi\|_{W(FL^p,L^q)} := \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} |V_\varphi \psi(x, \xi)|^p d\xi \Big)^{q/p} dx \Big)^{1/q};$$

here the Lebesgue norms appear in the inverse order. Both these norms provide a measure of the time–frequency concentration of a function and are widely used in time–frequency analysis [14, 16].

We have $M^{p_1,q_1} \subseteq M^{p_2,q_2}$ if and only if $p_1 \le p_2$ and $q_1 \le q_2$. Similarly, $W(FL^{p_1}, L^{q_1}) \subseteq W(FL^{p_2}, L^{q_2})$ if and only if $p_1 \le p_2$ and $q_1 \le q_2$.

The duality goes as expected:

$$(M^{p,q})' = M^{p',q'}, \quad 1 \le p, q < \infty,$$

and in particular

$$|\langle f, g\rangle| \lesssim \|f\|_{M^p} \|g\|_{M^{p'}}. \tag{3.2.6}$$

In the dispersive estimates, we meet, in particular, the Gelfand triple

$$M^1 \subset L^2(\mathbb{R}^n) \subset M^\infty.$$

We observe that

$$\mathscr{S}(\mathbb{R}^n) \subset M^1 \subset L^2(\mathbb{R}^n)$$

with dense and strict inclusions. For atomic characterizations of the space M^1, we refer to [14, 16].

We will also use the complex interpolation theory for modulation spaces, which reads as follows: for $1 \leq p, q, p_i, q_i \leq \infty$, $i = 0, 1$, $0 \leq \theta \leq 1$,

$$\frac{1}{p} = \frac{1-\vartheta}{p_0} + \frac{\vartheta}{p_1}, \quad \frac{1}{q} = \frac{1-\vartheta}{q_0} + \frac{\vartheta}{q_1},$$

we have

$$(M^{p_0,q_0}, M^{p_1,q_1})_\vartheta = M^{p,q}.$$

3.3 The Role of the Metaplectic Representation in Time–Frequency Analysis

The metaplectic representation became an essential tool in time–frequency analysis as it is very important in Weyl pseudo-differential calculus, and it is strictly related to a quadratic time–frequency representation called Wigner distribution, which is regarded as one of the best mathematical descriptions of the time–frequency behavior of signals. The metaplectic representation is also important in order to understand when a subgroup is reproducing as we will see in the following.

3.3.1 The Wigner Distribution and Some of its Properties

In this paragraph, we recall some basic definition and properties; we refer to [14, 15] for more details.

Definition 3.1 The Wigner distribution $W(\varphi)$ of a function $\varphi \in L^2(\mathbb{R}^n)$ is defined to be

$$W\varphi(x,\xi) = \int_{\mathbb{R}^n} \varphi\left(x + \frac{t}{2}\right) \overline{\varphi\left(x - \frac{t}{2}\right)} e^{-2\pi i \xi \cdot t} dt.$$

By polarizing the quadratic expression, one obtains the cross-Wigner distribution of $\varphi, \psi \in L^2(\mathbb{R}^n)$:

$$W(\varphi,\psi)(x,\xi) = \int_{\mathbb{R}^n} \varphi\left(x + \frac{t}{2}\right) \overline{\psi\left(x - \frac{t}{2}\right)} e^{-2\pi i \xi \cdot t} dt.$$

We also set $W(\varphi) = W(\varphi, \varphi)$.

Proposition 3.1 *The cross-Wigner distribution of $\varphi, \psi \in L^2(\mathbb{R}^n)$ satisfies:*

1. *$W(\varphi, \psi)$ is uniformly continuous on $\mathbb{R}^{2n}$ and $\|W(\varphi,\psi)\|_\infty \leq 2^n \|\varphi\|_2 \|\psi\|_2$.*

2. $W(\varphi, \psi) = \overline{W(\psi, \varphi)}$; *in particular,* $W(\varphi)$ *is real valued.*
3. *For* $u, v, \eta, \gamma \in \mathbb{R}^n$, *we have*

$$W(T_u M_\eta \varphi, T_v M_\gamma \psi)(x, \xi) =$$

$$= e^{\pi i(u+v)\cdot(\gamma-\eta)} e^{2\pi i x\cdot(\eta-\gamma)} e^{-2\pi i \xi\cdot(u-v)} \cdot W(\varphi, \psi)\left(x - \frac{u+v}{2}, \xi - \frac{\eta+\gamma}{2}\right).$$

In particular, $W\varphi$ *is covariant, that is,*

$$W(T_u M_\eta \varphi)(x, \xi) = W\varphi(x - u, \xi - \eta).$$

4. $W(\hat{\varphi}, \hat{\psi})(x, \xi) = W(\varphi, \psi)(-\xi, x)$.
5. *Moyal's formula: for* $\varphi_1, \varphi_2, \psi_1, \psi_2 \in L^2(\mathbb{R}^n)$,

$$\langle W(\varphi_1, \psi_1), W(\varphi_2, \psi_2)\rangle_{L^2(\mathbb{R}^{2n})} = \langle \varphi_1, \varphi_2\rangle \overline{\langle \psi_1, \psi_2\rangle}. \tag{3.3.1}$$

6. *Marginal densities: If* $\varphi, \hat{\varphi} \in L^1 \cap L^2(\mathbb{R}^n)$, *then*

$$\int_{\mathbb{R}^n} W\varphi(x, \xi) d\xi = |\varphi(x)|^2, \int_{\mathbb{R}^n} W\hat{\varphi}(x, \xi) d\xi = \left|\hat{\varphi}(x)\right|^2. \tag{3.3.2}$$

In particular,

$$\int\int_{\mathbb{R}^{2n}} W\varphi(x, \xi) dx d\xi = \|\varphi\|_2^2. \tag{3.3.3}$$

3.3.2 Reproducing Groups

We now come to the notion of reproducing groups, and we show how the metaplectic representation is essential in their characterization. We consider a Lie group H with left Haar measure dh, and $\phi \in L^2(\mathbb{R}^n)$ is a unitary representation of H in $L^2(\mathbb{R}^n)$, and we are interested in reproducing formulae of the type:

$$f = \int_H \langle f, \phi_h\rangle \phi_h dh, \quad f \in \mathscr{H}, \tag{3.3.4}$$

In particular, we consider the case when the Lie group H in (3.3.4) is a subgroup of the semi-direct product $G = \mathbb{H}^n \rtimes Sp(n, \mathbb{R})$ of the Heisenberg group and the symplectic group, while the representation $h \mapsto \phi_h$ arises from the restriction to H of the reducible (extended) metaplectic representation μ_e of G as applied to a fixed and suitable window function $\phi \in L^2(\mathbb{R}^n)$. A group H for which there exists a window ϕ such that (3.3.4) holds is said to be *reproducing*.

Definition 3.2 We say that a connected Lie subgroup H of $G = \mathbb{R}^{2n} \rtimes Sp(n, \mathbb{R})$ is a reproducing group for μ_e if there exists a function $\phi \in L^2(\mathbb{R}^n)$ such that

$$f = \int_H \langle f, \mu_e(h)\phi \rangle \, \mu_e(h)\phi dh, \quad for \quad all \quad f \in L^2(\mathbb{R}^n). \tag{3.3.5}$$

Any $\phi \in L^2(\mathbb{R}^n)$ for which (3.3.5) holds is called a reproducing function.

Remark 3.1 We notice that we do require Formula (3.3.5) to hold for all functions in $L^2(\mathbb{R}^n)$ for the same window ϕ, but we do not require the restriction of μ_e to H to be irreducible.

One of the most important features of μ_e is that it may be realized by affine actions on $\mathbb{R}^{2n}$ by means of the Wigner distribution. Since the reproducing formula is insensitive to phase factors, i.e., to the action of the center of $\mathbb{H}^n$, the group G is truly $\mathbb{R}^{2n} \rtimes Sp(n, \mathbb{R})$. The following notion of *admissible* subgroup H of $G = \mathbb{H}^n \rtimes Sp(n, \mathbb{R})$ relative to the extended metaplectic representation μ_e via the Wigner distribution is important to establish when a subgroup is reproducing, together with some additional integrability and boundedness properties of $W(\psi)(h^{-1} \cdot (x, \xi))$.

Definition 3.3 We say that a connected Lie subgroup H of $G = \mathbb{R}^{2n} \rtimes Sp(n, \mathbb{R})$ is an admissible group for μ_e if there exists a function $\phi \in L^2\mathbb{R}^n$ such that

$$\int_H W(\phi)(h^{-1} \cdot (x, \xi))dh = 1 \quad for \quad a.e. \quad (x, \xi) \in \mathbb{R}^{2n}. \tag{3.3.6}$$

Any $\phi \in L^2(\mathbb{R}^n)$ for which (3.3.6) holds is called an admissible function.

Theorem 3.2 *Suppose that $\phi \in L^2(\mathbb{R}^n)$ is such that the mapping*

$$h \mapsto W(\mu_e(h)\phi)(x, \xi) = W(\phi)(h^{-1} \cdot (x, \xi)) \tag{3.3.7}$$

is in $L^1(H)$ for a.e. $(x, \xi) \in \mathbb{R}^{2n}$ and

$$\int_H \left| W(\phi)(h^{-1} \cdot (x, \xi)) \right| dh \leq M, \quad for \quad a.e. \quad (x, \xi) \in \mathbb{R}^{2n}. \tag{3.3.8}$$

The condition (3.3.5) holds for all $f \in L^2(\mathbb{R}^n)$ if and only if the following admissibility condition is satisfied:

$$\int_H W(\phi)(h^{-1} \cdot (x, \xi))dh = 1 \quad for \quad a.e. \quad (x, \xi) \in \mathbb{R}^{2n}. \tag{3.3.9}$$

We now dispose of two different tools for checking whether a subgroup H of $G = \mathbb{R}^{2n} \rtimes Sp(n, \mathbb{R})$ is reproducing or not. Either we find a window function ϕ for which (3.3.5) holds or we check the admissibility of the subgroup H and use Theorem 3.2.

3.4 On Dispersive and Strichartz Estimates for the Metaplectic Representation

We recall here a number of definitions and results from [1] that we will use in the following. As anticipated in the introduction, in [1] we proved the following dispersive-type estimate.

Theorem 3.3 (Dispersive estimate) *The following estimate holds:*

$$\|\widehat{S}\psi\|_{M^\infty} \lesssim (\lambda_1(S)\dots\lambda_n(S))^{-1/2}\|\psi\|_{M^1} \tag{3.4.1}$$

for $\widehat{S} \in Mp(n,\mathbb{R})$, $\psi \in \mathscr{S}(\mathbb{R}^n)$*, where* $\lambda_1(S),\dots,\lambda_n(S)$ *are the singular values* ≥ 1 *of* $S = \pi(\widehat{S}) \in Sp(n,\mathbb{R})$.

By duality, this is in fact equivalent to:

$$|\langle\widehat{S}\psi,\varphi\rangle| \lesssim (\lambda_1(S)\dots\lambda_n(S))^{-1/2}\|\psi\|_{M^1}\|\varphi\|_{M^1}, \forall\widehat{S} \in Mp(n,\mathbb{R}). \tag{3.4.2}$$

Corollary 3.1 (Uniform weak-type estimate for matrix coefficients) *Let* $G = Mp(n,\mathbb{R})$ *with the Haar measure. The following estimate holds:*

$$\|\langle\widehat{S}\varphi_1,\varphi_2\rangle\|_{L^{4n,\infty}(G)} \lesssim \|\varphi_1\|_{M^1}\|\varphi_2\|_{M^1}, \tag{3.4.3}$$

for $\varphi_1,\varphi_2 \in \mathscr{S}(\mathbb{R}^n)$.

The proof of the above corollary is in fact based on Theorem 3.3 and the following result:

Proposition 3.2 *Let* $\alpha > 0$*,* $\beta > 0$*. Consider the function*

$$h(S) = (\lambda_1(S)\dots\lambda_n(S))^{-\alpha}$$

on $Sp(n,\mathbb{R})$*, where* $\lambda_1(S)\dots\lambda_n(S)$ *are the singular values* ≥ 1 *of the symplectic matrix* S.

We have $h \in L^{\beta,\infty}$ *on* $Sp(n,\mathbb{R})$*, with respect to the Haar measure, if*

$$\alpha\beta \geq 2n.$$

Proposition 3.2 is in turn proved by using the explicit formula for the integration of functions on $Mp(n,\mathbb{R})$ (i.e., $Sp(n,\mathbb{R})$), that we recalled in the previous section.

Finally, as a consequence of the dispersive estimates we therefore obtained the following Strichartz-type estimates.

Theorem 3.4 (Strichartz estimates) *Let* $G = Mp(n,\mathbb{R})$ *with the Haar measure. The following estimates hold:*

$$\|\widehat{S}\psi\|_{L^q(G;M^r)} \lesssim \|\psi\|_{L^2},$$

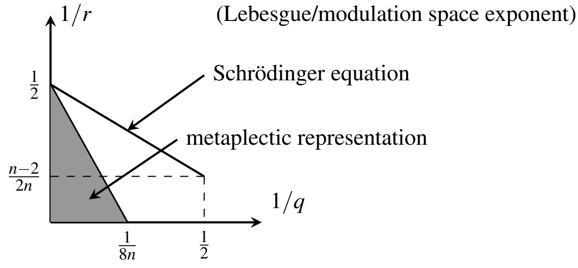

Fig. 3.1 Admissible pairs for Strichartz estimates

for

$$\frac{4n}{q} + \frac{1}{r} \leq \frac{1}{2}, \quad 2 \leq q, r \leq \infty.$$

The range of admissible pairs (q, r) in Theorem 3.4 is represented in Fig. 3.1.

For the sake of completeness, we now report a sketch of the proof of the previous theorem.

Proof (*Proof of Theorem* 3.4) We know that

$$\|\widehat{S}\psi\|_{L^2} = \|\psi\|_{L^2} \tag{3.4.4}$$

for $\psi \in L^2(\mathbb{R}^n)$, which gives the desired Strichartz estimate for $q = \infty$, $r = 2$, because $M^2 = L^2$, and also for $q = \infty$, $2 \leq r \leq \infty$, because $L^2 \hookrightarrow M^r$ for $r \geq 2$. Hence from now on, we can suppose $q < \infty$.

Now by Theorem 3.3,

$$\|\widehat{S}\psi\|_{M^\infty} \lesssim (\lambda_1(S) \ldots \lambda_n(S))^{-1/2} \|\psi\|_{M^1}.$$

By interpolation with (3.4.4), we obtain, for every $2 \leq r \leq \infty$,

$$\|\widehat{S}\psi\|_{M^r} \lesssim (\lambda_1(S) \ldots \lambda_n(S))^{-(\frac{1}{2} - \frac{1}{r})} \|\psi\|_{M^{r'}}. \tag{3.4.5}$$

Let now $G = Mp(n, \mathbb{R})$, as in the statement. Then one concludes by applying the usual TT^* method (see [17, page 75]) to the operator

$$T : L^2 \to L^q(G; M^r), T\psi = \widehat{S}\psi.$$

3.5 Proof of the Main Results

Since $G \subset SL(2, \mathbb{R}) = Sp(1, \mathbb{R})$, the estimate (3.4.2) holds with $n = 1$ for $S \in G$. Let now

$$S = l_t d_{s^{1/2}} = \begin{pmatrix} s^{-1/2} & 0 \\ ts^{-1/2} & s^{1/2} \end{pmatrix}$$

be a generic element of G, so that

$$S^* = \begin{pmatrix} s^{-1/2} & ts^{-1/2} \\ 0 & s^{1/2} \end{pmatrix}$$

and

$$S^*S = \begin{pmatrix} s^{-1/2} & ts^{-1/2} \\ 0 & s^{1/2} \end{pmatrix} \begin{pmatrix} s^{-1/2} & 0 \\ ts^{-1/2} & s^{1/2} \end{pmatrix} = \begin{pmatrix} s^{-1} + t^2 s^{-1} & t \\ t & s \end{pmatrix}.$$

We determine the singular values of S, which are the square roots of the eigenvalues of S^*S. Now,

$$det(S^*S - \lambda I) = \lambda^2 - (s + s^{-1} + t^2 s^{-1})\lambda + 1$$

from which

$$\lambda_{1,2} = \frac{s + s^{-1} + t^2 s^{-1} \pm \sqrt{(s - s^{-1})^2 + t^2(t^2 s^{-2} + 2s^{-2} + 2)}}{2}.$$

We assume $\lambda_2 \leq \lambda_1$, and we have $\lambda_2 = \lambda_1^{-1}$, so that $\lambda_1 \geq 1$. We get:

$$\lambda_1^{-1/2} = \left(\frac{1}{2}\right)^{-1/2} (s + s^{-1}(1 + t^2) + \sqrt{(s - s^{-1})^2 + t^2 s^{-2}(t^2 + 2) + 2t^2})^{-1/2}.$$

Let us observe that

$$\sqrt{(s - s^{-1})^2 + t^2 s^{-2}(t^2 + 2) + 2t^2} \asymp |s - s^{-1}| + |t| s^{-1} \sqrt{t^2 + 1} + |t|.$$

We have to estimate the measure of the set

$$D_\lambda = \left\{(t, s) \in \mathbb{R} \times \mathbb{R}_+ : \lambda_1(t, s)^{-1/4} \geq \lambda\right\},$$

$\lambda > 0$ with respect to the Haar measure $\frac{dtds}{s^2}$ i.e.,

$$\int_{\mathbb{R} \times \mathbb{R}_+} \chi_{D_\lambda} \frac{dtds}{s^2},$$

where χ_{D_λ} is the indicator function of D_λ. We split the domain of the indicator function into several regions. As $D_\lambda = \emptyset$ if $\lambda > 1$, we will consider $\lambda \leq 1$ in the following.

Let us consider $D_1 : |t| \leq 2, s \geq 2$. Then we have:

$$s + s^{-1}(1 + t^2) \asymp s + s^{-1} \asymp s$$

and

$$|s - s^{-1}| + |t|s^{-1}\sqrt{t^2 + 1} + |t| \asymp s + |t|s^{-1} + |t| = s + |t|(s^{-1} + 1)$$
$$\asymp s + |t| \asymp s.$$

Hence,

$$\lambda_1(t, s) \asymp s.$$

Since

$$s^{-1/4} \geq \lambda \Leftrightarrow s \leq \lambda^{-4},$$

we have

$$\int_{D_1} \chi_{D_1} \frac{dtds}{s^2} \leq \int_{|t| \leq 2, 2 \leq s \lesssim \lambda^{-4}} \frac{dtds}{s^2} = \mathcal{O}(1)$$

as $\lambda \to 0^+$.

Then consider $D_2 : |t| \geq 2, s > 2$. We have:

$$s + s^{-1}(1 + t^2) \asymp s + s^{-1}t^2$$

and

$$|s - s^{-1}| + |t|s^{-1}\sqrt{t^2 + 1} + |t| \asymp s + |t|(1 + |t|s^{-1}).$$

Hence,

$$\lambda_1(t, s) \asymp s + |t|(1 + s^{-1}|t|).$$

We consider two cases:

- If $s^{-1}|t| \geq 1$, $\lambda_1(t, s) \asymp s + s^{-1}t^2$.
  Clearly,

  $$s + s^{-1}t^2 \geq s^{-1}t^2$$

  and

  $$(s^{-1}t^2) \geq \lambda^2$$

  if

  $$|t| \leq \lambda^{-1}s^{-1/2}.$$

  Then,

$$\int_{D_2} \chi_{D_2} \frac{dtds}{s^2} \leq \int_2^{+\infty} \frac{1}{s^2} \int_{|t| \lesssim \lambda^{-1} s^{-1/2}} dtds$$
$$\asymp \lambda^{-1} \int_2^{+\infty} \frac{1}{s^{3/2}} ds = \mathscr{O}(\lambda^{-1}).$$

- If $s^{-1}|t| \leq 1$, $\lambda_1(t, s) \asymp s + |t|$.

Since
$$s + |t| \leq \lambda^{-4} \Rightarrow |t| \leq \lambda^{-4},$$

$$\int_{D_2} \chi_{D_2} \frac{dtds}{s^2} \leq \int_{|t| \lesssim \lambda^{-4}, s \geq 2} \frac{dtds}{s^2}$$
$$\lesssim \lambda^{-4} \int_2^{+\infty} \frac{ds}{s^2} = \mathscr{O}(\lambda^{-4}).$$

So we have
$$\int_{D_2} \chi_{D_2} \frac{dtds}{s^2} \lesssim \lambda^{-4}.$$

Then consider $D_3 : |t| \geq 2, 0 < s < \frac{1}{2}$. Then we have:

$$s + s^{-1}(1 + t^2) \asymp s + s^{-1}t^2 \asymp s + s^{-1}t^2$$

and

$$|s - s^{-1}| + |t|s^{-1}\sqrt{t^2 + 1} + |t| \asymp s^{-1} + t^2 s^{-1} + |t| = s^{-1}(1 + t^2) + |t|$$
$$\asymp s^{-1}t^2 + |t| = |t|(s^{-1}|t| + 1) \asymp s^{-1}t^2.$$

So,
$$\lambda_1(t, s) \asymp s + s^{-1}t^2$$

and
$$s + s^{-1}t^2 \leq \lambda^{-4}$$

implies
$$|t| \leq s^{1/2}\lambda^{-2}.$$

Then,
$$\int_{D_3} \chi_{D_3} \frac{dtds}{s^2} \leq \int_{\lambda^2 \lesssim s \leq 1/2} \frac{1}{s^2} \int_{2 \leq |t| \lesssim s^{1/2}\lambda^{-2}} dtds \asymp \lambda^{-2}.$$

Consider $D_4 : |t| \geq 2$, $\frac{1}{2} < s < 2$. So we have:

$$s+s^{-1}(1+t^2)\asymp t^2$$

and

$$|s-s^{-1}|+|t|s^{-1}\sqrt{t^2+1}+|t|\asymp|s-1|+t^2\asymp t^2.$$

On the other hand,

$$t^2\le\lambda^{-4}\Rightarrow|t|\le\lambda^{-2}$$

and therefore,

$$\int_{D_4}\chi_{D_4}\frac{dtds}{s^2}\lesssim\int_{2\le|t|\lesssim\lambda^{-2},\,\frac{1}{2}<s<2}1\frac{dtds}{s^2}\asymp\lambda^{-2}.$$

Consider $D_5 : |t|\le 2,\ 0<s<\frac{1}{2}$. Then we have:

$$s+s^{-1}(1+t^2)\asymp s+s^{-1}\asymp s^{-1}$$

and

$$\begin{aligned}|s-s^{-1}|+|t|s^{-1}\sqrt{t^2+1}+|t|&\asymp|s-s^{-1}|+|t|s^{-1}\sqrt{t^2+1}+|t|\\&\asymp s^{-1}+|t|s^{-1}+|t|=s^{-1}(|t|+1)+|t|\asymp s^{-1}+|t|.\end{aligned}$$

Hence,

$$\lambda_1(t,s)\asymp s^{-1}+|t|$$

and

$$s^{-1}+|t|\le\lambda^{-4}\Rightarrow s^{-1}\le\lambda^{-4}\Rightarrow s\ge\lambda^4$$

so that

$$\int_{D_5}\chi_{D_5}\frac{dtds}{s^2}\le\int_{|t|\le 2,\lambda^4\lesssim s<\frac{1}{2}}\frac{dtds}{s^2}\lesssim\lambda^{-4}.$$

Consider $D_6 : |t|\le 2,\ \frac{1}{2}<s<2$. Then we have:

$$\int_{D_6}\chi_{D_6}\frac{dtds}{s^2}\le\int_{|t|\le 2,\,\frac{1}{2}<s<2}\frac{dtds}{s^2}=\mathcal{O}(1)$$

for $\lambda\to 0^+$. Considering the contribution of each integral, we have just calculated and observing that

$$\lambda^{-4}\lesssim\lambda^{-\beta}$$

for $0<\lambda\le 1$, if $\beta\ge 4$ we conclude the proof.

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Chapter 4
On the Atomic Decomposition of Coorbit Spaces with Non-integrable Kernel

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Abstract This chapter is concerned with recent progress in the context of coorbit space theory. Based on a square-integrable group representation, the coorbit theory provides new families of associated smoothness spaces, where the smoothness of a function is measured by the decay of the associated voice transform. Moreover, by discretizing the representation, atomic decompositions and Banach frames can be constructed. Usually, the whole machinery works well if the associated reproducing kernel is *integrable* with respect to a weighted Haar measure on the group. In recent studies, it has turned out that to some extent coorbit spaces can still be

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established if this condition is violated. In this chapter, we clarify in which sense atomic decompositions and Banach frames for these generalized coorbit spaces can be obtained.

4.1 Introduction

This chapter is concerned with specific problems arising in the context of signal analysis. The overall goal in signal analysis is the efficient extraction of the relevant information one is interested in. For this, the signal—usually modeled as an element in a suitable function space—has to be processed, denoised, compressed, etc. The first step is always to decompose the signal into appropriate building blocks. This is performed by an associated transform, such as the wavelet transform, the Gabor transform, or the shearlet transform, just to name a few. Which transform to choose clearly depends on the type of information one wants to extract from the signal. In recent years, it has turned out that group theory—in particular representation theory—acts as a common thread behind many transforms. Indeed, many transforms are related with square-integrable representations of certain locally compact groups. For instance, the wavelet transform is associated with the affine group whereas the Gabor transform stems from the Weyl-Heisenberg group. We refer e.g., to [12, 13] for details.

This connection with group theory paves the way to the application of another very important concept, namely coorbit theory. This theory has been developed by Feichtinger and Gröchenig already in the late 1980s, see [12–14, 21]. In recent years, coorbit theory has experienced a real renaissance. Among other things, the connections to the various shearlet transforms [6] and to the concept of decomposition spaces [18, 28] have been investigated.

Based on a square-integrable group representation, by means of coorbit space theory, it is possible to construct canonical smoothness spaces, the coorbit spaces, by collecting all functions for which the associated voice transform has a certain decay. Moreover, by discretizing the underlying representation, it is possible to obtain atomic decompositions for the coorbit spaces. Moreover, also Banach frames can be constructed.

The coorbit space theory is based on certain assumptions. In particular, it is not enough that the representation is square-integrable, it must also be *integrable*, i.e., the reproducing kernel must be contained in a weighted L_1-space on the group. Unfortunately, this condition is restrictive, and even in very simple settings such as for the case of band-limited functions, it is not satisfied. Nevertheless, in [5], it has been shown that there is a way out. Instead of using a classical L_1-space as the space of generalized test functions, one can work with the weaker concept of Fréchet spaces. Then, more or less all the basic steps to establish the associated coorbit spaces can be performed. We refer to Sect. 4.2 for brief discussion of this approach.

However, in [5] one issue remained open, namely the construction of atomic decompositions for the resulting coorbit spaces. This is exactly the problem we are concerned with here. As a surprise, it turns out that this part of the coorbit space theory does not directly carry over to the Fréchet setting. There are two essential differences: First of all, a synthesis map can be constructed, but only at the price that the integrability parameters of the discrete norms on the coefficient spaces and of the coorbit norms are different. At first sight, this might look strange, but in the setting of non-integrable kernels this is in a certain sense not too surprising. Indeed, in the context of coorbit space theory, sooner or later convolution estimates of Young type have to be employed, which yield bounded mappings between L_p-spaces with *different* integrability exponents for domain and codomain if the convolution kernel is not in L_1. Concerning the atomic decomposition part, the situation is even more involved. It turns out that for any element in the coorbit space a suitable approximation by linear combinations of the atoms can be derived, but at the price that the weighted sequence norms of the expansion coefficients cannot be uniformly bounded by the coorbit norm. These results will be stated and proved in Sect. 4.3, see in particular Theorem 4.3.

Looking at these results, the inclined reader might have the impression that the authors were simply unable to prove sharper results, whereas such results might still be true, and provable with a more refined analysis. This might be true, but only partially. Indeed, in Sect. 4.4, we prove an additional result which shows that, under some very natural conditions, *uniform* bounds can only be obtained if the kernel operator acts as a bounded operator on the weighted L_p-spaces, that is, this additional assumption is necessary for obtaining uniform bounds. These facts strongly indicate that with the decomposition results stated in Sect. 4.3, we have almost reached the ceiling. However, there is still a little bit of flexibility which we can use to improve our results. Indeed, in Sect. 4.5, we prove that if there exists a second kernel W that satisfies additional smoothness assumptions and acts as the identity by left and right convolution on the reproducing kernel of the representation, then *uniform* bounds for both, the synthesis and the analysis part, can be obtained. Fortunately, in one important practical application given by the Paley–Wiener spaces such a kernel can be found.

This chapter is organized as follows. First of all, in Sect. 4.2, we recall the construction of coorbit spaces based on non-integrable kernels. We keep the explanation as short as possible and refer to [5] for further details. Then, in Sect. 4.3, we provide first discretization results for the associated coorbit spaces; the main result is Theorem 4.3. Then, in Sect. 4.4, we are concerned with "negative" results. Indeed, in Theorem 4.4 we show that stable decompositions can only be obtained if the right convolution by the reproducing kernel is bounded on the underlying L_p-spaces. Finally, in Sect. 4.5 we present satisfactory discretization results with the aid of an additional kernel W. Indeed, in the Theorems 4.5 and 4.6, respectively, we show that atomic decompositions and Banach frames with uniform bounds can be constructed, just as in the context of the classical coorbit theory.

4.2 An Overview

Throughout this paper, G denotes a fixed locally compact second countable group with left *Haar measure* β and *modular function* Δ. For a definition of these terms, we refer to [15]. We simply write $\int_G f(x)\,dx$ instead of $\int_G f(x)\,d\beta(x)$ and we denote by $L_0(G)$ the space of Borel measurable functions. Given $f \in L_0(G)$ the functions $\breve{f}$ and $\overline{f}$ are

$$\breve{f}(x) = f(x^{-1}), \qquad \overline{f}(x) = \overline{f(x)},$$

and for all $x \in G$ the left and right regular representations λ and ρ act on f as

$$\lambda(x)f(y) = f(x^{-1}y) \qquad \text{a.e } y \in G,$$
$$\rho(x)f(y) = f(yx) \qquad \text{a.e } y \in G.$$

Finally, the convolution $f * g$ between $f, g \in L_0(G)$ is the function

$$f * g(x) = \int_G f(y)g(y^{-1}x)\,dy = \int_G f(y)\cdot(\lambda(x)\breve{g})(y)\,dy \qquad \text{a.e. } x \in G,$$

provided that, for almost all $x \in G$, the function $y \mapsto f(y)\cdot(\lambda(x)\breve{g})(y)$ is integrable.

Furthermore, given two functions $f, g \in L_0(G)$, with slight abuse of notations, we write

$$\langle f, g\rangle_{L_2} = \int_G f(x)\overline{g(x)}\,dx,$$

provided that the function $f g$ is integrable.

We fix a continuous weight $w : G \to (0, \infty)$ satisfying

$$w(xy) \le w(x)w(y), \tag{4.2.1a}$$
$$w(x) = w(x^{-1}) \tag{4.2.1b}$$

for all $x, y \in G$. As a consequence, it also holds that

$$\inf_{x\in G} w(x) \ge 1. \tag{4.2.1c}$$

The symmetry (4.2.1b) can always be satisfied by replacing w with $w + \breve{w}$, where the latter weight is easily seen to still satisfy the submultiplicativity condition (4.2.1a).

For all $p \in [1, \infty)$, define the separable Banach space

$$L_{p,w}(G) = \left\{ f \in L_0(G) \;\middle|\; \int_G |w(x)f(x)|^p\,dx < \infty \right\}$$

with norm

$$\|f\|_{L_{p,w}}^p = \int_G |w(x)f(x)|^p \, dx,$$

and the obvious modifications for $L_\infty(G)$, which however is not separable. When $w \equiv 1$, we simply write $L_p(G)$.

With terminology as in [5], we choose, as a *target space* for the coorbit space theory, the following space

$$\mathscr{T}_w = \bigcap_{1<p<\infty} L_{p,w}(G).$$

We recall some basic properties of $\mathscr{T}_w$; for proofs, we refer to Theorem 4.3 of [5], which is based on results in [7]. We endow $\mathscr{T}_w$ with the (unique) topology such that a sequence $(f_n)_{\in\mathbb{N}}$ in $\mathscr{T}_w$ converges to 0 if and only if $\lim_{n\to+\infty}\|f_n\|_{L_{p,w}} = 0$ for all $1 < p < \infty$. With this topology, $\mathscr{T}_w$ becomes a reflexive Frechét space. The (anti-)linear dual space of $\mathscr{T}_w$ can be identified with

$$\mathscr{U}_w = \operatorname{span} \bigcup_{1<q<\infty} L_{q,w^{-1}}(G)$$

under the pairing

$$\int_G \Phi(x)\overline{f(x)}\, dx = \langle \Phi, f\rangle_w, \qquad \Phi \in \mathscr{U}_w,\ f \in \mathscr{T}_w. \tag{4.2.2}$$

Remark 4.1 The space $\mathscr{U}_w$ is endowed with one of the following equivalent topologies, both compatible with the pairing (4.2.2).

(i) The finest topology making the inclusions $L_{q,w^{-1}}(G) \hookrightarrow \mathscr{U}_w$ continuous for all $1 < q < \infty$.
(ii) The topology induced by the family of semi-norms $\left(\|\cdot\|_{p,r}\right)_{1<p<r<\infty}$, where

$$\|\Phi\|_{p,r} = \sup\left\{|\langle\Phi, f\rangle_w| \;\middle|\; f \in \mathscr{T}_w \text{ and } \max\left\{\|f\|_{L_{p,w}}, \|f\|_{L_{r,w}}\right\} \le 1\right\},$$

for $\Phi \in \mathscr{U}_w$.

The representation λ leaves invariant both $\mathscr{T}_w$ and $\mathscr{U}_w$, it acts continuously on $\mathscr{T}_w$, and the contragradient representation ${}^t\lambda$ of $\lambda_{|\mathscr{T}_w}$, given by

$$\langle {}^t\lambda_g\Phi, f\rangle_w = \langle\Phi, \lambda_{g^{-1}} f\rangle_w \quad \text{for} \quad \Phi \in \mathscr{U}_w \quad \text{and} \quad f \in \mathscr{T}_w,$$

is simply ${}^t\lambda = \lambda_{|\mathscr{U}_w}$.

Take $g \in \mathscr{T}_w$ with $\check{g} \in \mathscr{T}_w$. For all $f \in \mathscr{T}_w$, the convolution $f * g$ is in $\mathscr{T}_w$ and the map

$$f \mapsto f * g$$

is continuous from $\mathscr{T}_w$ into $\mathscr{T}_w$. Furthermore, for all $\Phi \in \mathscr{U}_w$, the convolution $\Phi * g$ is in $\mathscr{U}_w$ and the map

$$\Phi \mapsto \Phi * g$$

is continuous from $\mathscr{U}_w$ into $\mathscr{U}_w$.

Take now a (strongly continuous) unitary representation π of G acting on a separable complex Hilbert space $\mathscr{H}$ with scalar product $\langle \cdot, \cdot \rangle_{\mathscr{H}}$ linear in the first entry. We assume that π is reproducing, namely there exists a vector $u \in \mathscr{H}$ such that the corresponding voice transform

$$Vv(x) = \langle v, \pi(x)u \rangle_{\mathscr{H}}, \qquad v \in \mathscr{H},\ x \in G,$$

is an isometry from $\mathscr{H}$ into $L_2(G)$. We observe that this implies that V is injective, whence span $\{\pi(x)u\}_{x\in G}$ is dense in $\mathscr{H}$.

We denote by K the reproducing kernel

$$K(x) = Vu(x) = \langle u, \pi(x)u \rangle_{\mathscr{H}}, \qquad x \in G, v \in \mathscr{H}, \tag{4.2.3}$$

which is a bounded continuous function and enjoys the following basic properties

$$\overline{K} = \breve{K}, \tag{4.2.4a}$$

$$\sum_{i,j=1}^{n} c_i \overline{c_j} K(x_i^{-1} x_j) \geq 0, \qquad c_1, \ldots, c_n \in \mathbb{C},\ x_1, \ldots, x_n \in G, \tag{4.2.4b}$$

$$K * K = K \in L_2(G). \tag{4.2.4c}$$

In general, π is not assumed to be irreducible, but the reproducing assumption implies that u is a cyclic vector. Properties (4.2.4a) and (4.2.4b) uniquely define the representation π up to an unitary equivalence, see Theorem 3.20 and Proposition 3.35 of [15]. Equation (4.2.4c) states that π is equivalent to the subrepresentation of the left-regular representation (on $L_2(G)$) having K as a cyclic vector. Conversely, if a bounded continuous function K satisfies (4.2.4a), (4.2.4b), and (4.2.4c), then there exists a unique (up to an unitary equivalence) reproducing representation π whose reproducing kernel is K.

For the remainder of the paper, we will always impose the following basic assumption:

Assumption 4.1 We assume $K \in \mathscr{T}_w$, *i.e*,

$$K \in L_{p,w}(G) \text{ for all } 1 < p < \infty. \tag{4.2.5}$$

We add some remarks.

Remark 4.2 (i) Since $w(x) \geq 1$, Assumption (4.2.5) implies that $K \in L_p(G)$ for all $p > 1$. If π is irreducible, this last fact gives that V is an isometry up to a constant, so that π is always a reproducing representation. If π is reducible, condition (4.2.5) is not sufficient to ensure that π is reproducing; however, if $K * K = K$, then π is always reproducing.

(ii) If w^{-1} belongs to $L_q(G)$ for some $1 < q < \infty$, then Hölder's inequality shows $K \in L_1(G)$, but in general $K \notin L_{1,w}(G)$. However in many interesting examples, w is independent of one or more variables, so that $w^{-1} \notin L_q(G)$ for all $1 < q < \infty$.

We now define the *test space* $\mathscr{S}_w$ as

$$\mathscr{S}_w = \left\{ v \in \mathscr{H} \;\middle|\; Vv \in L_{p,w}(G) \text{ for all } 1 < p < \infty \right\}, \tag{4.2.6}$$

which becomes a locally convex topological vector space under the family of seminorms

$$\|v\|_{p,\mathscr{S}_w} = \|Vv\|_{L_{p,w}}. \tag{4.2.7}$$

We recall the main properties of $\mathscr{S}_w$.

Theorem 4.1 (Theorem 4.4 of [5]) *Under Assumption* (4.2.5)*, the following hold:*

(i) the space $\mathscr{S}_w$ is a reflexive Fréchet space, continuously and densely embedded in $\mathscr{H}$;

(ii) the representation π leaves $\mathscr{S}_w$ invariant and its restriction to $\mathscr{S}_w$ is a continuous representation;

(iii) the space $\mathscr{H}$ is continuously and densely embedded into the (anti-)linear dual $\mathscr{S}'_w$, where both spaces are endowed with the weak topology;

(iv) the restriction of the voice transform $V : \mathscr{S}_w \to \mathscr{T}_w$ is a topological isomorphism from $\mathscr{S}_w$ onto the closed subspace $\mathscr{M}^{\mathscr{T}_w}$ of $\mathscr{T}_w$, given by

$$\mathscr{M}^{\mathscr{T}_w} = \left\{ f \in \mathscr{T}_w \;\middle|\; f * K = f \right\},$$

and it intertwines π and λ;

(v) for every $f \in \mathscr{T}_w$, there exists a unique element $\pi(f)u \in \mathscr{S}_w$ such that

$$\langle \pi(f)u, v\rangle_{\mathscr{H}} = \int_G f(x)\langle \pi(x)u, v\rangle_{\mathscr{H}}\, dx = \int_G f(x)\overline{Vv(x)}\, dx, \qquad v \in \mathscr{H}.$$

Furthermore, it holds that

$$V\pi(f)u = f * K,$$

and the map

$$\mathscr{T}_w \ni f \mapsto \pi(f)u \in \mathscr{S}_w$$

is continuous and its restriction to $\mathscr{M}^{\mathscr{T}_w}$ is the inverse of V.

Here and in the following, the notation $\pi(f)u$ is motivated by the following fact.

Remark 4.3 In the framework of abstract harmonic analysis, any function $f \in L_1(G)$ defines a bounded operator $\pi(f)$ on $\mathscr{H}$, which is weakly given by

$$\langle \pi(f)v, v' \rangle_{\mathscr{H}} = \int_G f(x) \langle \pi(x)v, v' \rangle_{\mathscr{H}}\, dx, \qquad v, v' \in \mathscr{H},$$

see for example Sect. 3.2 of [15]. However, if $f \notin L_1(G)$, then in general $\pi(f)v$ is well-defined only if $v = u$, where u is an admissible vector for the representation π.

Recalling that the (anti-)dual of $\mathscr{T}_w$ is $\mathscr{U}_w$ under the pairing (4.2.2), we denote by ${}^tV$ the contragradient map ${}^tV : \mathscr{U}_w \to \mathscr{S}'_w$ given by

$$\langle {}^tV\Phi, v \rangle_{\mathscr{S}_w} = \langle \Phi, Vv \rangle_w, \qquad \Phi \in \mathscr{U}_w,\ v \in \mathscr{S}_w.$$

As usual, we extend the voice transform from $\mathscr{H}$ to the (anti-)dual $\mathscr{S}'_w$ of $\mathscr{S}_w$, where $\mathscr{S}'_w$ plays the role of the space of distributions. For all $T \in \mathscr{S}'_w$, we set

$$V_e T(x) = \langle T, \pi(x)u \rangle_{\mathscr{S}_w}, \qquad x \in G, \tag{4.2.8}$$

which is a continuous function on G by item (ii) of the previous theorem and $\langle \cdot, \cdot \rangle_{\mathscr{S}_w}$ denotes the pairing between $\mathscr{S}_w$ and $\mathscr{S}'_w$, whereas $\langle \cdot, \cdot \rangle_w$ is the pairing between $\mathscr{T}_w$ and $\mathscr{U}_w$.

We summarize the main properties of the extended voice transform in the following theorem.

Theorem 4.2 (Theorem 4.4 of [5]) *Under assumption (4.2.5), the following hold:*

(i) for every $\Phi \in \mathscr{U}_w$ there exists a unique element $\pi(\Phi)u \in \mathscr{S}'_w$ such that

$$\langle \pi(\Phi)u, v \rangle_{\mathscr{S}_w} = \int_G \Phi(x) \langle \pi(x)u, v \rangle_{\mathscr{H}}\, dx = \int_G \Phi(x) \overline{Vv(x)}\, dx, \qquad v \in \mathscr{S}_w.$$

Furthermore, it holds that

$$V_e \pi(\Phi)u = \Phi * K;$$

(ii) for all $T \in \mathscr{S}'_w$ the voice transform $V_e T$ is in $\mathscr{U}_w$ and satisfies

$$V_e T = V_e T * K, \tag{4.2.9}$$
$$\langle T, v \rangle_{\mathscr{S}_w} = \langle V_e T, Vv \rangle_w, \qquad v \in \mathscr{S}_w; \tag{4.2.10}$$

(iii) the extended voice transform V_e is injective, continuous from $\mathscr{S}'_w$ into $\mathscr{U}_w$ (when both spaces are endowed with the strong topology), its range is the closed subspace

$$\mathscr{M}^{\mathscr{U}_w} = \left\{\Phi \in \mathscr{U}_w \;\middle|\; \Phi * K = \Phi\right\} = \text{span} \bigcup_{p \in (1,\infty)} \mathscr{M}^{L_{p,w}(G)} \subset L_{\infty,w^{-1}}(G) \tag{4.2.11}$$

and it intertwines the contragradient representation of $\pi_{|\mathscr{S}_w}$ *and* $\lambda_{|\mathscr{U}_w}$*;*

(iv) the map

$$\mathscr{M}^{\mathscr{U}_w} \ni \Phi \mapsto \pi(\Phi)u \in \mathscr{S}'_w$$

is the left inverse of V_e *and coincides with the restriction of the map* ${}^tV$ *to* $\mathscr{M}^{\mathscr{U}_w}$*, namely*

$$V_e({}^tV\Phi) = V_e\pi(\Phi)u = \Phi, \qquad \Phi \in \mathscr{M}^{\mathscr{U}_w}; \tag{4.2.12}$$

(v) regarding $\mathscr{S}_w \hookrightarrow \mathscr{H} \hookrightarrow \mathscr{S}'_w$*, it holds*

$$\mathscr{S}_w = \left\{T \in \mathscr{S}'_w \;\middle|\; V_eT \in \mathscr{T}_w\right\} = \left\{\pi(f)u \;\middle|\; f \in \mathscr{M}^{\mathscr{T}_w}\right\}.$$

Item (ii) of the previous theorem states that the voice transform of any distribution $T \in \mathscr{S}'_w$ satisfies the reproducing formula (4.2.9) and uniquely defines the distribution T by means of the reconstruction formula (4.2.10), i.e.,

$$T = \int_G \langle T, \pi(x)u\rangle_{\mathscr{S}_w} \pi(x)u \, dx,$$

where the integral is a Dunford-Pettis integral with respect to the duality between $\mathscr{S}_w$ and $\mathscr{S}'_w$, see, for example, Appendix 3 of [15].

We now fix an exponent $r \in [1, \infty)$, and a w-moderate weight m, i.e., a continuous function $m : G \to (0, \infty)$ such that

$$m(xy) \leq w(x) \cdot m(y) \quad \text{and} \quad m(xy) \leq m(x) \cdot w(y) \quad \text{for all } x, y \in G\,. \tag{4.2.13}$$

Remark 4.4 The definition (4.2.13) of a w-moderate weight m is equivalent to the condition

$$m(xyz) \leq w(x) \cdot m(y) \cdot w(z) \quad \text{for all } x, y, z \in G$$

up to the constant $w(e)$.

The result of the following lemma is used multiple times in this chapter.

Lemma 4.1 *If m is a w-moderate weight on G, then so is m^{-1}.*

Proof To prove the estimates in (4.2.13) for m^{-1}, we fix $x, y \in G$, then by the w-moderateness of m it holds

$$m(y) = m(x^{-1}xy) \leq w(x^{-1}) \cdot m(xy) = w(x) \cdot m(xy),$$

which implies $m(xy)^{-1} \leq w(x) \cdot m(y)^{-1}$. Similarly, we observe that

$$m(x) = m(xyy^{-1}) \leq m(xy) \cdot w(y^{-1}) = m(xy) \cdot w(y),$$

which in turn implies $m(xy)^{-1} \leq m(x)^{-1} \cdot w(y)$. □

With terminology as in [5], we choose as a *model space* for the coorbit space theory, the Banach space $Y = L_{r,m}(G)$ with $r \in (1, \infty)$. The corresponding coorbit space is defined as

$$\mathrm{Co}(Y) = \left\{ T \in \mathscr{S}_w' \,\middle|\, V_e T \in Y \right\} \tag{4.2.14}$$

endowed with the norm

$$\|T\|_{\mathrm{Co}(Y)} = \|V_e T\|_Y. \tag{4.2.15}$$

We summarize the main properties of $\mathrm{Co}(Y)$ in the following proposition.

Proposition 4.1 *The space* $\mathrm{Co}(Y)$ *is a Banach space invariant under the action of the contragradient representation of* $\pi_{|\mathscr{S}_w}$. *The extended voice transform is an isometry from* $\mathrm{Co}(Y)$ *onto the* λ*-invariant closed subspace*

$$\mathscr{M}^Y = \left\{ F \in Y \,\middle|\, F * K = F \right\} \subset \mathscr{U}_w \,,$$

and we have

$$\mathrm{Co}(Y) = \left\{ \pi(F)u \,\middle|\, F \in \mathscr{M}^Y \right\}.$$

Furthermore

$$V_e \pi(F)u = F, \qquad F \in \mathscr{M}^Y, \tag{4.2.16}$$

$$\pi(V_e T)u = T, \qquad T \in \mathrm{Co}(Y). \tag{4.2.17}$$

Proof The proof is essentially an application of Theorem 3.5 in [5]. We first note that convergence with respect to $\|\cdot\|_Y = \|\cdot\|_{L_{r,m}}$ implies convergence in measure. Furthermore, since m is w-moderate, it is not hard to see that $Y = L_{r,m}(G)$ is λ-invariant, and that the restriction of λ to Y is a continuous representation of G. Therefore, we only need to prove that Assumptions 5 and 6 in [5] are satisfied.

We first show that $Y \subset \mathscr{U}_w$. By (4.2.13) and (4.2.1b), we get for any $x \in G$ that

$$m(e) = m(xx^{-1}) \leq m(x) \cdot w(x^{-1}) = m(x) \cdot w(x)\,, \tag{4.2.18}$$

and hence $[w(x)]^{-1} \cdot m(e) \leq m(x)$, whence $Y = L_{r,m}(G) \hookrightarrow L_{r,w^{-1}}(G) \subset \mathscr{U}_w$ since $r > 1$.

Since $\mathscr{U}_w = \mathscr{T}_w'$ under the pairing (4.2.2), for all $F \in Y$ and $f \in \mathscr{T}_w$ it holds that $Ff \in L_1(G)$. In particular, by assumption (4.2.5), $FK \in L_1(G)$ for all $F \in Y$ and, by construction, $F\,Vv \in L_1(G)$ for all $v \in \mathscr{S}_w$ and $F \in \mathscr{M}^Y$, so that Assumption 5 and Assumption 6 in [5] hold true. □

4.3 Discretization

The aim of this section is to establish certain atomic decompositions for the coorbit spaces described in Sect. 4.2. In particular, we recall that

$$\mathscr{T}_w = \bigcap_{p\in(1,\infty)} L_{p,w}(G), \qquad \mathscr{T}_w' = \mathscr{U}_w = \operatorname{span} \bigcup_{q\in(1,\infty)} L_{q,w^{-1}}(G) \tag{4.3.1}$$

and for some $1 < r < \infty$,

$$Y = L_{r,m}(G). \tag{4.3.2}$$

Proposition 4.1 shows that the correspondence principle holds, i.e., the extended voice transform V_e is an isometry from the associated coorbit space

$$\mathrm{Co}(L_{r,m}) := \left\{T \in \mathscr{S}_w' \;\middle|\; V_e(T) \in L_{r,m}(G)\right\} \tag{4.3.3}$$

onto the corresponding reproducing kernel Banach space

$$\mathscr{M}_{r,m} = \mathscr{M}^{L_{r,m}(G)} = \left\{f \in L_{r,m}(G) \;\middle|\; f * K = f\right\}. \tag{4.3.4}$$

Remark 4.5 Assumption (4.2.5) on the kernel K and the fact that m is w-moderate imply that for all $f \in L_{r,m}(G)$ the convolution $f * K$ is well-defined; see Proposition 4.13.

In this setting, we can characterize the antidual $\mathscr{M}_{r,m}'$ of the reproducing kernel space.

Lemma 4.2 *The antidual $\mathscr{M}_{r,m}'$ of $\mathscr{M}_{r,m}$ is canonically isomorphic to $L_{r',m^{-1}}(G)/\mathscr{M}_{r,m}^{\perp}$, where*

$$\mathscr{M}_{r,m}^{\perp} = \left\{\widetilde{F} \in L_{r',m^{-1}}(G) \;\middle|\; \langle \widetilde{F}, F\rangle_{L_2} = 0 \textit{ for all } F \in \mathscr{M}_{r,m}\right\} \tag{4.3.5}$$

and $1/r + 1/r' = 1$. Hence, for every $\Gamma \in \mathscr{M}_{r,m}'$, there is a $\widetilde{F} \in L_{r',m^{-1}}(G)$ such that $\Gamma(F) = \langle \widetilde{F}, F\rangle_{L_2}$ for all $F \in \mathscr{M}_{r,m}$.

Proof Since $\mathscr{M}_{r,m}$ is a closed subspace of $L_{r,m}(G)$, [25, Proposition 1.4] yields that $\mathscr{M}_{r,m}'$ is canonically isomorphic to $L_{r,m}'(G)/\mathscr{M}_{r,m}^{\perp}$. The claim follows because $L_{r,m}'(G)$ is canonically isomorphic to $L_{r',m^{-1}}(G)$. □

Some more preparations are necessary. Given a compact neighborhood $Q \subset G$ of e with $Q = \overline{\operatorname{int} Q}$, the *local maximal function (with respect to the right regular representation)* $M_Q^{\rho} f$ of $f \in L_0(G)$ is defined by

$$M_Q^{\rho} f(x) := \|f \cdot \rho(x)\chi_Q\|_{L_\infty}, \quad \text{whence} \quad \widetilde{M}_Q^{\rho} f(x) := M_Q^{\rho} f(x^{-1}) = \|f\|_{L_\infty(Qx)}. \tag{4.3.6}$$

Then, for a function space Y on G, we define

$$\mathscr{M}_Q^\rho(Y) := \left\{ f \in L_0(G) \;\middle|\; \breve{M}_Q^\rho f \in Y \right\}. \tag{4.3.7}$$

Now we define the *Q-oscillation* of a function f with respect to Q as

$$\operatorname{osc}_Q f(x) := \sup_{u \in Q} |f(ux) - f(x)|. \tag{4.3.8}$$

The decay properties of the Q-oscillation play an important role in view of the discretization of coorbit spaces. To this end, the following lemma is useful. Since the proof is a simple generalization of the proof of [21, Lemma 4.6], it is deferred to the appendix.

Lemma 4.3 *Let w be a weight on G, let $p \in (1, \infty)$, and assume that $f : G \to \mathbb{C}$ is continuous and that $f \in \mathscr{M}_{Q_0}^\rho(L_{p,w})$ for some compact unit neighborhood Q_0 with $Q_0 = \overline{\operatorname{int} Q_0}$. Then the following hold:*

(i) $\|\operatorname{osc}_{Q_0} f\|_{L_{p,w}} < \infty$.
(ii) *For arbitrary $\varepsilon > 0$, there is a unit neighborhood $Q_\varepsilon \subset Q_0$ such that for each unit neighborhood $Q \subset Q_\varepsilon$, we have $\|\operatorname{osc}_Q f\|_{L_{p,w}} < \varepsilon$. Put briefly,*

$$\lim_{Q \to \{e\}} \|\operatorname{osc}_Q f\|_{L_{p,w}} = 0.$$

4.3.1 An Assumption on the Kernel

From now on we make the following assumption on the reproducing kernel space.

Assumption 4.2 Assumption 4.1 is satisfied, and $\operatorname{span}\{\lambda(x)K\}_{x\in G}$ is dense in $\mathscr{M}_{r,m}$.

This assumption is similar to the density of span $\{\pi(x)K\}_{x\in G}$ in $\mathscr{H}$—which is equivalent to K being a cyclic vector for the representation π on $\mathscr{H}$—and in $\mathscr{M}^{\mathscr{T}_w}$, which is Assumption 3 of [5] and fulfilled in our setting, as can be seen by combining Theorems 4.1 and 4.2.

In the following, we will denote with RC_K the right convolution operator $RC_K f := f * K$, where the space on which RC_K acts may vary depending on the context.

Before we provide a sufficient condition under which Assumption 4.2 is fulfilled (see Lemma 4.4), we need a couple of auxiliary results.

Proposition 4.2 *Assume that for all $f \in L_{r,m}(G)$, f and K are convolvable (in the sense that $f \cdot \lambda(x)\breve{K} \in L_1(G)$ for almost all $x \in G$) and $f * K \in L_{r,m}(G)$, then the right convolution operator*

$$RC_K : L_{r,m}(G) \to L_{r,m}(G), \qquad RC_K f = f * K,$$

is bounded.

Proof For $r = 2$, the result is stated in [22, Proposition 3.10], whose proof holds true for any p. Indeed, by the closed graph theorem, it is enough to show that RC_K is a closed operator. Take a sequence $(f_n)_{n\in\mathbb{N}}$ converging to $f \in L_{r,m}(G)$ such that $(RC_K f_n)_{n\in\mathbb{N}}$ converges to $g \in L_{r,m}(G)$. By a sharp version of the Riesz–Fischer theorem, see [1, Theorem 13.6], there exists a positive function $g \in L_{r,m}(G)$ such that, possibly passing twice to a subsequence, there exist two null sets E, F such that for all $y \in G \setminus E$ and $x \in G\setminus F$

$$\begin{aligned} &|f_n(y)| \le g(y), \\ &\lim_{n\to\infty} f_n(y) = f(y), \\ &\lim_{n\to\infty} RC_K f_n(x) = g(x). \end{aligned}$$

Furthermore, by definition of convolution and possibly redefining the null set F, we get that for all $x \in G\setminus F$ and all $n \in \mathbb{N}$ the mappings

$$y \mapsto f_n(y)K(y^{-1}x), \qquad y \mapsto g(y)K(y^{-1}x)$$

are integrable. Then, given $x \in G\setminus F$, for all $y \in G\setminus E$

$$|f_n(y)K(y^{-1}x)| \le |g(y)K(y^{-1}x)|, \quad \lim_{n\to\infty} f_n(y)K(y^{-1}x) = f(y)K(y^{-1}x).$$

For $x \in G\setminus F$, the function $y \mapsto g(y)K(y^{-1}x)$ is integrable, so that by dominated convergence we obtain

$$g(x) = \lim_{n\to\infty} \int_G f_n(y)K(y^{-1}x)\,dy = \int_G f(y)K(y^{-1}x)\,dy = f * K(x),$$

so RC_K is indeed closed. □

Proposition 4.3 *Denote by r' the dual exponent $1/r + 1/r' = 1$. Assume that the right convolution operator*

$$RC_K : L_{r,m}(G) \to L_{r,m}(G), \qquad RC_K f = f * K$$

is bounded, then

(i) the right convolution operator is bounded on $L_{r',m^{-1}}(G)$ and it coincides with the adjoint of RC_K;
(ii) the operator RC_K is a projection from $L_{r,m}(G)$ onto the reproducing kernel Banach space $\mathscr{M}_{r,m}$.

Here and in the following, the duality pairing is the sesqui-linear form

$$\langle f, g\rangle_{L_2} = \int_G f(x)\overline{g(x)}\,dx, \qquad f \in L_{r,m}(G),\ g \in L_{r',m^{-1}}(G).$$

Proof Since RC_K is a bounded operator on $L_{rm}(G)$, the adjoint is a bounded operator on $L_{r,m}(G)'$. Take $g \in L_{r',m^{-1}}(G)$ and $f \in C_c(G) \subset L_{r,m}(G)$, then

$$\begin{aligned}\langle RC_K^* g, f\rangle_{L_2} = \langle g, RC_K f\rangle_{L_2} &= \int_G g(x)\left(\int_G \overline{f(y)K(y^{-1}x)}\,dy\right)dx\\ &= \int_G\left(\int_G g(x)K(x^{-1}y)\,dx\right)\overline{f(y)}\,dy\\ &= \langle g * K, f\rangle_{L_2},\end{aligned}$$

where $\overline{K(y^{-1}x)} = K(x^{-1}y)$. Note that we can interchange the integral by Fubini's theorem since

$$\int_G |g(x)|\left(\int_G |f(y)K(y^{-1}x)|\,dy\right)dx \le \|g\|_{L_{r',m^{-1}}} \cdot \||f| * |K|\|_{L_{r,m}}$$

and $|f| * |K| \in L_{r,m}(G)$ by Young's inequality (4.6.5) with $q = 1$ and $p = r$, $f \in L_{1,m}(G)$ and $g = K \in L_{r,w}(G)$. Note that Fubini's theorem shows that

$$\int |f(y)| \cdot (|g| * |K|)(y)\,dy < \infty\,.$$

Since this holds for any $f \in C_c(G)$, we see $|g| * |K| < \infty$ almost everywhere, so that g and K are convolvable. By density of $C_c(G)$ in $L_{r,m}(G)$, we get that $RC_K^* g = g * K$, so that $g * K \in L_{r',m^{-1}}(G)$. Hence the convolution operator acts continuously on $L_{r',m^{-1}}(G)$, and it coincides with RC_K^*.

To show the second claim, observe first that for any $f \in C_c(G) \subset \mathscr{T}_w \subset L_{r',m^{-1}}G)$, since $K \in \mathscr{T}_w$, both $|f| * |K|$ and $(|f| * |K|) * |K|$ exist, so that by (77d) of [5] the convolution is associative and

$$RC_K^2 f = (f * K) * K = f * (K * K) = f * K = RC_K f.$$

By density, and since RC_K is bounded on $L_{r,m}(G)$ by assumption, we get that $RC_K^2 = RC_K$ and hence Ran $RC_K \subset \mathscr{M}_{r,m}$. The other inclusion is trivial. □

As a consequence of the above result, we get the following corollary.

Corollary 4.1 *Denote by r' the dual exponent $1/r + 1/r' = 1$ and assume that the right convolution operator RC_K is bounded on $L_{r,m}(G)$. The sesqui-linear pairing on* $\mathrm{Co}(L_{r,m}) \times \mathrm{Co}(L_{r',m^{-1}})$ *given by*

$$\langle T, T' \rangle_{\mathrm{Co}(L_{r,m})} = \langle V_e T, V_e T' \rangle_{L_2}$$

is such that the linear map

$$T' \mapsto \left(T \mapsto \overline{\langle T, T' \rangle_{\mathrm{Co}(L_{r,m})}} \right)$$

is an isomorphism of $\mathrm{Co}(L_{r',m^{-1}})$ *onto the antilinear dual of* $\mathrm{Co}(L_{r,m})$.

Proof We identify $\mathrm{Co}(L_{r,m})$ with $\mathscr{M}_{r,m}$ by the extended voice transform V_e, so that the pairing becomes

$$\langle f, g \rangle_{L_2} = \int_G f(x) \overline{g(x)}\, dx, \qquad f \in \mathscr{M}_{r,m},\ g \in \mathscr{M}_{r',m^{-1}}.$$

Since $g \in L_{r',m^{-1}}(G)$, clearly $f \mapsto \overline{\langle f, g \rangle_{L_2}}$ is a continuous antilinear map, which we denote by Γ_g, on $\mathscr{M}_{r,m}$ whose norm is

$$\begin{aligned} \|\Gamma_g\| &= \sup \left\{ |\langle f, g \rangle_{L_2}| \mid f \in \mathscr{M}_{r,m},\ \|f\|_{L_{r,m}} \le 1 \right\} \\ &\le \sup \left\{ |\langle h, g \rangle_{L_2}| \mid h \in L_{r,m}(G),\ \|h\|_{L_{r,m}} \le 1 \right\} = \|g\|_{L_{r',m^{-1}}}. \end{aligned}$$

Next, since $L_{r,m}(G)$ is the dual of $L_{r',m^{-1}}(G)$, there is $h \in L_{r,m}(G)$ with $\|h\|_{L_{r,m}} \le 1$ such that $\|g\|_{L_{r',m^{-1}}} = \langle h, g \rangle_{L_2}$. Now, setting $c := \|RC_K\|_{L_{r,m} \to L_{r,m}}$ and $f = c^{-1} \cdot RC_K h$, we have $\|f\|_{L_{r,m}} \le 1$ and

$$\langle f, g \rangle_{L_2} = c^{-1} \langle RC_K h, g \rangle_{L_2} = c^{-1} \langle h, RC_K g \rangle_{L_2} = c^{-1} \langle h, g \rangle_{L_2} = c^{-1} \|g\|_{L_{r',m^{-1}}}.$$

Hence, $c^{-1} \cdot \|g\|_{L_{r',m^{-1}}} \le \|\Gamma_g\| \le \|g\|_{L_{r',m^{-1}}}$.

We now prove that the map $g \mapsto \Gamma_g$ is surjective. Take Γ in the antilinear dual of $\mathscr{M}_{r,m}$. Since $\mathscr{M}_{r,m}$ is a subspace of $L_{r,m}(G)$ there exists $g' \in L_{r',m^{-1}}(G)$ such that $\Gamma(f) = \overline{\langle f, g' \rangle_{L_2}}$ for all $f \in \mathscr{M}_{r,m}$. By setting $g = RC_K g' \in \mathscr{M}_{r',m^{-1}}$, as above

$$\Gamma(f) = \overline{\langle f, g' \rangle_{L_2}} = \overline{\langle RC_K f, g \rangle_{L_2}} = \overline{\langle f, g \rangle_{L_2}} = \Gamma_g(f) \quad \text{for all } f \in \mathscr{M}_{r,m},$$

thus $\Gamma = \Gamma_g$. □

Now we can prove that in the following setting Assumption 4.2 is fulfilled.

Lemma 4.4 *Fix $r \in (1, \infty)$ and assume that the right convolution operator RC_K is bounded on $L_{r,m}(G)$. Then the sets* $\mathrm{span}\{\pi(x)u\}_{x \in G}$ *and* $\mathrm{span}\{\lambda(x)K\}_{x \in G}$ *are dense in* $\mathrm{Co}(L_{r,m})$ *and $\mathscr{M}_{r,m}$, respectively. Thus, Assumption 4.2 is fulfilled.*

Proof By the correspondence principle, it is enough to show the second claim. Let $\Gamma \in \mathscr{M}'_{r,m}$ be such that for all $x \in G$,

$$\Gamma(\lambda(x)K) = 0.$$

By the above corollary, there exists $g \in \mathscr{M}_{r',m^{-1}}$ such that $\Gamma(f) = \langle g, f\rangle_{L_2}$ for all $f \in \mathscr{M}_{r,m}$. In particular,

$$0 = \Gamma\big(\lambda(x)K\big) = \langle g, \lambda(x)K\rangle_{L_2} = g * K(x) = RC_K g(x)$$

for all $x \in G$, that is, $RC_K g = 0$. Since $g \in \mathscr{M}_{r',m^{-1}}$, this implies $g = 0$ and then $\Gamma = 0$. Since this holds for any $\Gamma \in \mathscr{M}'_{r,m}$ such that $\Gamma(\lambda(x)K) = 0$ for all $x \in G$, we see that $\mathrm{span}\{\lambda(x)K\}_{x\in G}$ is dense in $\mathscr{M}_{r,m}$. □

By Young's inequality, we know that the $L_1(G)$-integrability of $K \cdot w$ implies that the (right) convolution operator RC_K is a bounded operator acting on $L_{p,m}(G)$ for all $1 < p < \infty$. But for general $K \in \mathscr{T}_w$, this question is unclear. As we will show in Sect. 4.3.3, there are kernels that act boundedly on all $L_p(G)$ without being integrable. But in Sect. 4.4, we also show that there exist kernels for a very similar setting that are contained in $\mathscr{T}_w$ but that do *not* give rise to bounded operators on $L_{p,m}(G)$.

4.3.2 Atomic Decompositions

This section is dedicated to finding possible atomic decompositions of coorbit spaces, provided that Assumption 4.2 is fulfilled. The main results of this section will be stated in Theorem 4.3.

But before that we need to introduce some notation. First, for each $n \in \mathbb{N}$, we choose a countable subset $Y_n = \{x_{j,n}\}_{j\in\mathscr{J}_n} \subset G$ such that

$$Y_n \subset Y_{n+1}, \tag{4.3.9}$$

$$\overline{\bigcup_{n\in\mathbb{N}} Y_n} = G. \tag{4.3.10}$$

Moreover, for every $n \in \mathbb{N}$, we assume that there exists a compact neighborhood Q_n of the identity $e \in G$, such that Y_n is Q_n-*dense* in G, i.e.,

$$G = \bigcup_{j\in\mathscr{J}_n} x_{j,n} Q_n. \tag{4.3.11}$$

Additionally, we assume each Y_n to be uniformly relatively Q_n-separated, i.e., there exists an integer $\mathscr{I}$, independent of n, and subsets $Z_{n,i} \subset Y_n$, $1 \le i \le \mathscr{I}$, such that

$$Y_n = \bigcup_{i=1}^{\mathscr{I}} Z_{n,i} \tag{4.3.12}$$

and for all $x, y \in Z_{n,i}$, $1 \leq i \leq \mathscr{I}$, it holds $xQ_n \cap yQ_n \neq \emptyset$ if and only if $x = y$.

By $\Psi_n = \{\psi_{n,x}\}_{x \in Y_n}$, we denote a *partition of unity* subordinate to the Q_n-dense set Y_n, i.e.,

$$0 \leq \psi_{n,x} \leq 1, \tag{4.3.13}$$

$$\sum_{x \in Y_n} \psi_{n,x} \equiv 1, \tag{4.3.14}$$

$$\operatorname{supp}(\psi_{n,x}) \subset xQ_n. \tag{4.3.15}$$

We also assume that the family $\Psi_n = \{\psi_{n,x}\}_{x \in Y_n}$ is linearly independent as a.e. defined functions, i.e., for any finite subset $X \subset Y_n$ and $(\alpha_x)_{x \in X} \in \mathbb{C}^X$, the condition

$$\sum_{x \in X} \alpha_x \psi_{n,x}(y) = 0$$

for almost all $y \in G$ implies that $\alpha_x = 0$ for all $x \in X$.

We now denote with X_n a *finite* subset of Y_n, such that

$$X_n \subset X_{n+1}, \tag{4.3.16}$$

$$\overline{\bigcup_{n \in \mathbb{N}} X_n} = G. \tag{4.3.17}$$

Therefore, for every $n \in \mathbb{N}$, the finite set of functions $\{\psi_{n,x}\}_{x \in X_n}$ is similar to a partition of unity subordinate to the family $(xQ_n)_{x \in X_n}$.

For each $n \in \mathbb{N}$ and $1 < r < \infty$, set

$$T_n : L_{r,m}(G) \to \mathscr{M}_{r,m}, \qquad T_n F := \sum_{x \in X_n} \langle F, \psi_{n,x} \rangle_{L_2} \lambda(x) K. \tag{4.3.18}$$

We observe that this operator is well-defined. Since the sum is finite, we only have to verify that each term of the sum is a well-defined element of $\mathscr{M}_{r,m}$. It is easy to verify that the reproducing identity holds for $\lambda(x)K$, since it holds for K. Moreover, we have $\lambda(x)K \in L_{r,m}(G)$ by Assumption 4.1 and by translation invariance of the spaces $L_{r,m}(G)$; thus, $\lambda(x)K \in \mathscr{M}_{r,m}$. Finally, the pairing

$$\langle F, \psi_{n,x} \rangle_{L_2} = \int_G F(y) \psi_{n,x}(y)\, dy$$

is well-defined for all $x \in X_n$, since $\psi_{n,x}$ is bounded with compact support, so that $\psi_{n,x} \in L_{r',m^{-1}}(G)$.

Now we define $V_n = \operatorname{Ran} T_n$, which is a finite dimensional subspace of $\mathscr{M}_{r,m}$, as well as $\widetilde{V}_n = V_e^{-1}(V_n)$, which is a finite dimensional subspace of $\operatorname{Co}(L_{r,m})$ by the correspondence principle. We show the following result concerning the structure of the spaces V_n:

Lemma 4.5 *The following holds for all $n \in \mathbb{N}$:*

$$V_n = \operatorname{span}\{\lambda(x)K\}_{x\in X_n}, \tag{4.3.19}$$

$$V_n \subset V_{n+1}, \tag{4.3.20}$$

$$\overline{\bigcup_{n\geq 1} V_n} = \mathscr{M}_{r,m}. \tag{4.3.21}$$

Proof We start by showing (4.3.19). By the construction, we made above, $V_n \subseteq \operatorname{span}\{\lambda(x)K\}_{x\in X_n}$.

We first observe that the map

$$F \mapsto \big(\langle F, \psi_{n,x}\rangle_{L_2(G)}\big)_{x\in X_n}$$

is surjective from $L_{r,m}(G)$ to $\mathbb{C}^{X_n}$. Indeed, if this was not true, there would be a non-zero family $(\alpha_x)_{x\in X_n} \in \mathbb{C}^{X_n}$ satisfying $\sum_{x\in X_n} \alpha_x \langle F, \psi_{n,x}\rangle_{L_2} = 0$ for all $F \in L_{r,m}(G)$, then $\sum_{x\in X_n} \alpha_x \psi_{n,x} = 0$ in $L_{r',m^{-1}}(G)$ and, hence, almost everywhere, then by assumption $\alpha_x = 0$ for all $x \in X_n$, a contradiction. It follows that $\operatorname{span}\{\lambda(x)K\}_{x\in X_n} \subseteq V_n$ and (4.3.19) holds true.

Equation (4.3.20) is an easy consequence of (4.3.16) and (4.3.19).

It remains to show (4.3.21). Since the sequence $(V_n)_{n\in\mathbb{N}}$ is an increasing family of subspaces, and since $V_n \subset \mathscr{M}_{r,m}$ for all $n \in \mathbb{N}$, the set $\overline{\bigcup_{n\geq 1} V_n}$ is a subspace of the closed space $\mathscr{M}_{r,m}$. Hence, by the Hahn–Banach theorem, condition (4.3.21) is equivalent to the following condition: If $\Gamma \in \mathscr{M}'_{r,m}$ satisfies

$$\langle \Gamma, F\rangle_{\mathscr{M}'_{r,m}\times\mathscr{M}_{r,m}} = 0, \quad \text{for all } F \in V_n,\ n \in \mathbb{N},$$

then $\Gamma = 0$ in $\mathscr{M}'_{r,m}$. By Lemma 4.2, we can write $\langle \Gamma, F\rangle_{\mathscr{M}'_{r,m}\times\mathscr{M}_{r,m}} = \langle g, F\rangle_{L_2}$ for all $F \in \mathscr{M}_{r,m}$, for a suitable $g \in L_{r',m^{-1}}(G)$. Since $\lambda(x)K \in \mathscr{M}_{r,m}$, $x \in G$, for every $f \in L_{r',m^{-1}}(G)$ with $f - g \in \mathscr{M}^{\perp}_{r,m}$, it holds for all $x \in G$,

$$(g * K)(x) = \langle g, \lambda(x)\, K\rangle_{L_2} = \langle \Gamma, \lambda(x)\, K\rangle_{\mathscr{M}'_{r,m}\times\mathscr{M}_{r,m}}.$$

Now, with $F = T_n f$ for some $f \in L_{r,m}(G)$, we obtain

$$\begin{aligned} 0 = \langle \Gamma, T_n f\rangle_{\mathscr{M}'_{r,m}\times\mathscr{M}_{r,m}} &= \sum_{x\in X_n} \langle \psi_{n,x}, f\rangle_{L_2} \cdot \langle \Gamma, \lambda(x)K\rangle_{\mathscr{M}'_{r,m}\times\mathscr{M}_{r,m}} \\ &= \sum_{x\in X_n} \langle \psi_{n,x}, f\rangle_{L_2} \cdot (g * K)(x) = \langle \sum_{x\in X_n} (g * K)(x)\psi_{n,x}, f\rangle_{L_2}. \end{aligned}$$

Since this holds for any $f \in L_{r,m}(G)$, we get $\sum_{x \in X_n} (g * K)(x)\psi_{n,x} = 0$ in $L_{r',m^{-1}}(G)$ for all $n \in \mathbb{N}$. Because the finite family $\{\psi_{n,x}\}_{x \in X_n}$ is linearly independent as elements of $L_{r',m^{-1}}(G)$, we have $(g * K)(x) = 0$ for all $x \in X_n$ and $n \in \mathbb{N}$. Therefore, by (4.3.17), the function $g * K$ vanishes on a dense subset of G. But since we have $g * K(x) = \langle g, \lambda(x)K \rangle_{L_2}$ with $g \in L_{r',m^{-1}}(G)$, and since the map $G \to L_{r,m}(G)$, $x \mapsto \lambda(x)K$ is continuous, we see that $g * K : G \to \mathbb{C}$ is a continuous functions, so that we get $g * K \equiv 0$, i.e., $\langle \Gamma, \lambda(x)K \rangle_{\mathscr{M}'_{r,m} \times \mathscr{M}_{r,m}} = 0$ for all $x \in G$.

By Assumption 4.2, this implies $\Gamma = 0$ as an element of $\mathscr{M}'_{r,m}$, which proves (4.3.21). □

Remark 4.6 By the correspondence principle, analogous results to (4.3.19), (4.3.20), and (4.3.21) hold true for $\widetilde{V}_n$. This can be seen as follows: Since it holds $V_e \pi(x)u = \lambda(x)K$ for all $x \in X_n$, by (4.3.19), we obtain

$$\widetilde{V}_n = \operatorname{span} \{\pi(x)u\}_{x \in X_n} . \tag{4.3.22}$$

Hence, the nesting property $\widetilde{V}_n \subset \widetilde{V}_{n+1}$ analogous to (4.3.20) is straightforward. By the correspondence principle, it follows from (4.3.20) that

$$\overline{\bigcup_{n \in \mathbb{N}} \widetilde{V}_n} = \operatorname{Co}(L_{r,m}). \tag{4.3.23}$$

With the spaces V_n at hand, in the following, we will turn to projections from $\mathscr{M}_{r,m}$ onto V_n and their properties. To this end, let $\pi_n : \mathscr{M}_{r,m} \to V_n$ be the metric projection defined by

$$\pi_n(F) = \operatorname{argmin}_{g \in V_n} \|F - g\|_{\mathscr{M}_{r,m}}. \tag{4.3.24}$$

Since $\mathscr{M}_{r,m}$ is a closed subspace of $L_{r,m}(G)$ with $1 < r < \infty$, the space $\mathscr{M}_{r,m}$ is a uniformly convex Banach space and every V_n is convex and closed; therefore π_n is a well-defined and unique function, see [19, Proposition 3.1]. Similarly, we define the projection $\widetilde{\pi}_n : \operatorname{Co}(L_{r,m}) \to \widetilde{V}_n$ by setting $\widetilde{\pi}_n = V_e^{-1} \pi_n V_e$.

The following lemma gives us a first norm estimate for this metric projection.

Lemma 4.6 *Given $\varepsilon > 0$ and $F \in \mathscr{M}_{r,m}$, there exists $n^* = n^*_{F,\varepsilon} \in \mathbb{N}$ such that for all $n \geq n^*$ it holds*

$$\|F - \pi_n(F)\|_{\mathscr{M}_{r,m}} \leq \varepsilon, \tag{4.3.25}$$

$$\|\pi_n(F)\|_{\mathscr{M}_{r,m}} \leq (1 + \varepsilon)\|F\|_{\mathscr{M}_{r,m}}. \tag{4.3.26}$$

Proof If $F = 0$, the claim is clear since $0 \in V_n$ so that $\pi_n(F) = 0$. Hence, we can assume that $F \neq 0$. Let $\delta := \min\left\{1, \|F\|_{\mathscr{M}_{r,m}}\right\} \cdot \varepsilon > 0$. By (4.3.21), there exists $n^* \geq 1$ and $g \in V_{n^*}$ such that $\|F - g\|_{\mathscr{M}_{r,m}} \leq \delta$. For all $n \geq n^*$, by (4.3.20), we have $g \in V_n$ and by definition of the metric projection,

$$\|F - \pi_n(F)\|_{\mathscr{M}_{r,m}} \leq \|F - g\|_{\mathscr{M}_{r,m}} \leq \delta \leq \varepsilon.$$

The triangle inequality gives

$$\|\pi_n(F)\|_{\mathscr{M}_{r,m}} \leq \|F - \pi_n(F)\|_{\mathscr{M}_{r,m}} + \|F\|_{\mathscr{M}_{r,m}} \leq \delta + \|F\|_{\mathscr{M}_{r,m}} \leq (1+\varepsilon)\|F\|_{\mathscr{M}_{r,m}},$$

which concludes the proof. □

The following auxiliary result establishes a first upper bound for certain coefficients related to functions $F \in \mathscr{M}_{r,m}$. This will be used for the atomic decomposition afterward.

Proposition 4.4 *For any $F \in L_{r,m}(G)$ and $n \in \mathbb{N}$, let the coefficients $c_{n,x} \in \mathbb{C}$, $x \in X_n$, be defined via*

$$c_{n,x} := \int_G F(y)\psi_{n,x}(y)\,dy.$$

Then the inequality

$$\left(\sum_{x \in X_n} |c_{n,x}|^r m(x)^r\right)^{1/r} \leq |Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q) \cdot \|F\|_{L_{r,m}} \tag{4.3.27}$$

holds, where $|Q_n|$ denotes the Haar measure of the set Q_n and r' denotes the dual exponent of r.

Proof We first note that, since $\psi_{n,x}$ is compactly supported and bounded, the coefficient $c_{n,x}$ is well-defined.

Next, we observe that if $\psi_{n,x}(y) \neq 0$, then $y = xq_n$ for some $q_n \in Q_n$, and hence $m(x) = m(xq_nq_n^{-1}) \leq m(xq_n) \cdot w(q_n^{-1}) \leq m(y) \cdot \sup_{q \in Q_n} w(q)$. This shows

$$\begin{aligned} m(x) \cdot |c_{n,x}| &\leq m(x) \cdot \int_G |F(y)| \cdot \psi_{n,x}(y)\,dy \\ &\leq \sup_{q \in Q_n} w(q) \cdot \int_G |(mF)(y)| \cdot \psi_{n,x}(y)\,dy\,. \end{aligned} \tag{4.3.28}$$

We will now further estimate the integral on the right-hand side, setting $F_0 := m \cdot F$ for brevity.

To this end, we define the measure $d\mu_x$ on G (for $x \in X_n$) by setting

$$d\mu_x(y) = \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}}\,dy$$

and readily observe that $\int_G 1 \;\; d\mu_x = 1$. Thus, by Jensen's inequality, see [8, Theorem 10.2.6], we obtain

$$\left(\int_G |F_0(y)| \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}}\, dy\right)^r = \left(\int_G |F_0(y)|\, d\mu_x(y)\right)^r$$
$$\leq \int_G |F_0(y)|^r\, d\mu_x(y)$$
$$= \int_G |F_0(y)|^r \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}}\, dy.$$

By the properties of Ψ_n, see (4.3.13), (4.3.14) and (4.3.15), it holds

$$\|\psi_{n,x}\|_{L_1} = \int_G \psi_{n,x}(y)\, dy \leq \int_{xQ_n} 1\, dy = \int_{Q_n} 1\, dy = |Q_n|.$$

Recalling (4.3.28), we thus see

$$\sum_{x\in X_n} (m(x)\cdot |c_{n,x}|)^r$$
$$\leq \sup_{q\in Q_n} w(q)^r \cdot \sum_{x\in X_n} \|\psi_{n,x}\|_{L_1}^r \left(\int_G |F_0(y)| \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}}\, dy\right)^r$$
$$\leq \sup_{q\in Q_n} w(q)^r \cdot \sum_{x\in X_n} \|\psi_{n,x}\|_{L_1}^r \int_G |F_0(y)|^r \frac{\psi_{n,x}(y)}{\|\psi_{n,x}\|_{L_1}}\, dy$$
$$\leq \sup_{q\in Q_n} w(q)^r \cdot \sup_{x\in X_n} \|\psi_{n,x}\|_{L_1}^{r-1} \sum_{x\in X_n} \int_G |F_0(y)|^r \psi_{n,x}(y)\, dy$$
$$\leq \sup_{q\in Q_n} w(q)^r \cdot |Q_n|^{r-1} \cdot \|F\|_{L_{r,m}}^r,$$

which concludes the proof. □

With this at hand, we are able to give a first atomic decomposition for functions $F \in V_n$, $n \in \mathbb{N}$, as well as an estimate for the norm of the coefficients involved.

Lemma 4.7 *Given $n \in \mathbb{N}$, for all $F \in V_n$ the following atomic decomposition holds true:*

$$F = \sum_{x\in X_n} c(F)_{n,x}\, \lambda(x)K, \tag{4.3.29}$$

where the coefficients $c(F)_{n,x}$ are of the form

$$c(F)_{n,x} = \langle S_n F, \psi_{n,x}\rangle_{L_2}, \tag{4.3.30}$$

where S_n denotes any linear right inverse of $T_n : L_{r,m}(G) \to V_n$. In particular, the coefficients depend linearly on F and they satisfy

$$\left(\sum_{x \in X_n} |c(F)_{n,x}|^r m(x)^r \right)^{1/r} \leq C_n \|F\|_{\mathscr{M}_{r,m}}, \tag{4.3.31}$$

with $C_n = \|S_n\| \cdot |Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q)$.

Proof We first observe that the operator T_n admits a bounded right inverse $S_n : V_n \to L_{r,m}(G)$. Indeed, by [4, Theorem 2.12], the existence of a bounded right inverse is equivalent to the existence of a topological supplement of the kernel of T_n. However, since the spaces V_n are finite dimensional, such a topological supplement exists, see [4, Example 2.4.2].

In the remainder of the proof, we denote by S_n an arbitrary linear right inverse of T_n. Thus, for all $F \in V_n$ we have the decomposition

$$F = T_n S_n F = \sum_{x \in X_n} \langle S_n F, \psi_{n,x} \rangle_{L_2} \lambda(x) K,$$

so that (4.3.29) holds true if we define the coefficients $c(F)_{n,x}$ as in (4.3.30). With this notation the coefficients depend linearly on F. By applying (4.3.27), we obtain the estimate

$$\left(\sum_{x \in X_n} |c(F)_{n,x}|^r m(x)^r \right)^{1/r} \leq |Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q) \cdot \|S_n F\|_{\mathscr{M}_{r,m}} \leq C_n \|F\|_{\mathscr{M}_{r,m}},$$

where C_n is as in the statement of the lemma, and where $\|S_n\|$ is the operator norm of S_n as an operator from V_n into $L_{r,m}(G)$. This proves (4.3.31). □

Remark 4.7 Note that if the sequence $(|Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q) \cdot \|S_n\|)_{n \in \mathbb{N}}$ is bounded, then the constant C_n in (4.3.31) can be bounded independently of n. Naturally, the question arises under which conditions this really is the case. To answer this question, it is necessary to determine the asymptotic behavior of the operator norm of S_n. As we will show in Sect. 4.3.3, this task is already non-trivial for a very simple setting. Still, in Lemma 4.20, we give a partial answer, as we present a technique to characterize the operator norm in a different manner.

The proof of the following technical lemma can be found in the appendix. We recall that the integer $\mathscr{I}$ is defined through assumption (4.3.12).

Lemma 4.8 *Let* $1 \leq p \leq \infty$ *and* $(d_x)_{x \in Y_n} \in \ell_{p,m}(Y_n)$ *for some* $n \in \mathbb{N}$, *then*

$$\left\| \sum_{x \in Y_n} |d_x| \chi_{x Q_n} \right\|_{L_{p,m}} \leq \mathscr{I}^{1-\frac{1}{p}} \cdot \sup_{q \in Q_n} w(q) \cdot |Q_n|^{\frac{1}{p}} \cdot \|(d_x)_{x \in Y_n}\|_{\ell_{p,m}}$$

with the convention $\frac{1}{\infty} := 0$.

With the auxiliary results above, we are in the position to state and prove our main result.

Theorem 4.3 *We assume that K satisfies (4.2.5) and that there exists $p < r$ such that*

$$\begin{aligned} &K \in L_{p,w\Delta^{-1/p}}(G), \\ &\operatorname{osc}_{Q_n}(K) \in L_{p,w}(G) \cap L_{p,w\Delta^{-1/p}}(G), \end{aligned} \tag{4.3.32}$$

for all $n \in \mathbb{N}$.

(i) Fix $\varepsilon > 0$; then for any $T \in \mathrm{Co}(L_{r,m})$ there exists $n^* = n^*_{T,\varepsilon} \in \mathbb{N}$ such that for all $n \geq n^*$

$$\Big\| T - \sum_{x \in X_n} c(T)_{n,x} \pi(x) u \Big\|_{\mathrm{Co}(L_{r,m})} \leq \varepsilon,$$

where the family $(c(T)_{n,x})_{x \in X_n}$ satisfies

$$\|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{r,m}} \leq C_n(1+\varepsilon)\|T\|_{\mathrm{Co}(L_{r,m})},$$

with $C_n = |Q_n|^{1/r'} \cdot \sup_{q \in Q_n} w(q) \cdot \|S_n\|$, where S_n denotes any linear right inverse to the operator $T_n : L_{r,m}(G) \to V_n$ defined in (4.3.18).

(ii) Let $n \in \mathbb{N}$, and let $d = (d_x)_{x \in Y_n} \in \ell_{q,m}(Y_n)$. Then $T = \sum_{x \in Y_n} d_x \pi(x) u$ is in $\mathrm{Co}(L_{r,m})$. Furthermore the estimate

$$\|T\|_{\mathrm{Co}(L_{r,m})} \leq D_n \|(d_x)_{x \in Y_n}\|_{\ell_{q,m}}$$

holds, where $1/q + 1/p = 1 + 1/r$, and

$$D_n := |Q_n|^{\frac{1}{q}-1} \cdot \mathscr{I}^{1-\frac{1}{q}} \cdot \sup_{q \in Q_n} w(q) \cdot \theta_n \tag{4.3.33}$$

with $\theta_n := \max \Big\{ \| \operatorname{osc}_{Q_n}(K) + |K| \, \|_{L_{p,w}}, \ \| \operatorname{osc}_{Q_n}(K) + |K| \, \|_{L_{p,w\Delta^{-1/p}}} \Big\}$.

Proof To prove (i), choose $n^* = n^*_{F,\varepsilon}$ as in Lemma 4.6 with $F = V_e T \in \mathscr{M}_{r,m}$. By applying (4.3.29) and (4.3.30) to $\pi_n(F) \in V_n$, we obtain the atomic decomposition

$$\begin{aligned}
\widetilde{\pi}_n(T) &= V_e^{-1}\pi_n(F)\\
&= V_e^{-1}\left(\sum_{x\in X_n}\langle S_n\pi_n(F),\psi_{n,x}\rangle_{L_2}\cdot\lambda(x)K\right)\\
&= \sum_{x\in X_n}\langle S_n\pi_n V_e(T),\psi_{n,x}\rangle_{L_2}\cdot V_e^{-1}\lambda(x)K\\
&= \sum_{x\in X_n} c(T)_{n,x}\pi(x)u,
\end{aligned}$$

where $c(T)_{n,x} = \langle S_n\pi_n V_e(T),\psi_{n,x}\rangle_{L_2}$. Using (4.3.25) and the correspondence principle, we derive

$$\begin{aligned}
\Big\|T - \sum_{x\in X_n} c(T)_{n,x}\pi(x)u\Big\|_{\mathrm{Co}(L_{r,m})} &= \big\|T-\widetilde{\pi}_n(T)\big\|_{\mathrm{Co}(L_{r,m})}\\
&= \big\|V_eT - V_e\widetilde{\pi}_n(T)\big\|_{\mathscr{M}_{r,m}}\\
&= \big\|F-\pi_n(F)\big\|_{\mathscr{M}_{r,m}} \le \varepsilon\,.
\end{aligned}$$

Now (4.3.31) and (4.3.26) yield the estimate

$$\begin{aligned}
\left(\sum_{x\in X_n}|c(T)_{n,x}|^r m(x)^r\right)^{\frac{1}{r}} &\le C_n\|\pi_n(V_eT)\|_{\mathscr{M}_{r,m}} \le C_n(1+\varepsilon)\|V_eT\|_{\mathscr{M}_{r,m}}\\
&= C_n(1+\varepsilon)\|T\|_{\mathrm{Co}(L_{r,m})}
\end{aligned}$$

for any $n \ge n^*$.

It remains to prove (ii). In [6, Chap. 3, p. 100], the following pointwise estimate for $y \in G$ has been established:

$$|\sum_{x\in Y_n} d_x\lambda(x)K(y)| \le \left(\sum_{x\in Y_n}|d_x|\frac{\chi_{xQ_n}}{|Q_n|}\right) * \big(\mathrm{osc}_{Q_n}(K)+|K|\big)\,(y).$$

Let now $q > 1$ such that $1/q + 1/p = 1 + 1/r$. By using Young's inequality, see Proposition 4.13, and Lemma 4.8, we obtain

$$\begin{aligned}
&\Big\|\sum_{x\in Y_n} d_x\pi(x)u\Big\|_{\mathrm{Co}(L_{r,m})} = \Big\|\sum_{x\in Y_n} d_x\lambda(x)K\Big\|_{L_{r,m}}\\
&\le \Big\|\Big(\sum_{x\in Y_n}|d_x|\chi_{xQ_n}\Big) * \big(\mathrm{osc}_{Q_n}(K)+|K|\big)\Big\|_{L_{r,m}}\cdot|Q_n|^{-1}
\end{aligned}$$

$$\leq \Big\| \sum_{x \in Y_n} |d_x| \chi_{xQ_n} \Big\|_{L_{q,m}}$$
$$\cdot \max \left\{ \|\mathrm{osc}_{Q_n}(K) + |K|\|_{L_{p,w}}, \|\mathrm{osc}_{Q_n}(K) + |K|\|_{L_{p,w\Delta^{-1/p}}} \right\} \cdot |Q_n|^{-1}$$
$$\leq |Q_n|^{\frac{1}{q}-1} \cdot \mathscr{I}^{1-\frac{1}{q}} \cdot \sup_{q \in Q_n} w(q) \cdot \|(d_x)\|_{\ell_{q,m}}$$
$$\cdot \max \left\{ \|\mathrm{osc}_{Q_n}(K) + |K|\|_{L_{p,w}}, \|\mathrm{osc}_{Q_n}(K) + |K|\|_{L_{p,w\Delta^{-1/p}}} \right\}.$$

By the assumption (4.3.32), the expression on the right-hand side is finite. □

Remark 4.8 The coefficients $c(T)_{n,x}$, $x \in X_n$, in Theorem 4.3(i), depend linearly on T if and only if the projection π_n from (4.3.24) is linear.

The following proposition presents a slight variation of Theorem 4.3.

Proposition 4.5 *Under the same assumptions as in Theorem 4.3, the following holds: Fix $\varepsilon > 0$ and $T \in \mathrm{Co}(L_{r,m})$; then there exists $n^* = n^*_{T,\varepsilon} \in \mathbb{N}$ such that for all $n \geq n^*$*

$$\frac{1}{\tau_n(1+\varepsilon)} \cdot \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{q,m}} \leq \|T\|_{\mathrm{Co}(L_{r,m})} \tag{4.3.34}$$

and

$$\|T\|_{\mathrm{Co}(L_{r,m})} \leq \varepsilon + D_n \cdot \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{q,m}}, \tag{4.3.35}$$

where D_n as in (4.3.33) and $\tau_n := C_n \cdot |X_n|^{\frac{1}{q}-\frac{1}{r}}$, $|X_n|$ is the cardinality of X_n and $1/q + 1/p = 1 + 1/r$.

Proof Throughout this proof, we use the same notations as in Theorem 4.3. We first note that for any finite sequence $(d_x)_{x \in X_n}$, $n \in \mathbb{N}$, by Hölder's inequality, it holds that

$$\|(d_x)_{x \in X_n}\|_{\ell_{q,m}} \leq \|(d_x)_{x \in X_n}\|_{\ell_{r,m}} \cdot \|1_{X_n}\|_{\ell_{\frac{rq}{r-q}}},$$

where 1_{X_n} is a sequence of ones only. Furthermore, it holds

$$\|1_{X_n}\|_{\ell_{\frac{rq}{r-q}}} = |X_n|^{\frac{1}{q}-\frac{1}{r}}.$$

With $\tau_n := C_n \cdot |X_n|^{\frac{1}{q}-\frac{1}{r}}$, we then obtain from Theorem 4.3(i) the estimate

$$\frac{1}{\tau_n(1+\varepsilon)} \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{q,m}} \leq \frac{1}{C_n(1+\varepsilon)} \|(c(T)_{n,x})_{x \in X_n}\|_{\ell_{r,m}} \leq \|T\|_{\mathrm{Co}(L_{r,m})},$$

which proves (4.3.34).

It remains to show the second inequality (4.3.35). For this we note that the sequence $(c(T)_{n,x})_{x\in X_n}$ can be understood as a sequence over the index set Y_n with only finitely many non-zero entries. Therefore, by (4.3.25) and Theorem 4.3(ii), this yields

$$\begin{aligned}\|T\|_{\mathrm{Co}(L_{r,m})} &\leq \varepsilon + \Big\| \sum_{x\in X_n} c(T)_{n,x}\pi(x)u \Big\|_{\mathrm{Co}(L_{r,m})} \\ &\leq \varepsilon + |Q_n|^{\frac{1}{q}-1} \cdot \theta_n \cdot \|(c(T)_{n,x})_{x\in X_n}\|_{\ell_{q,m}},\end{aligned}$$

which concludes the proof. □

4.3.3 An Example: Coorbit Theory for Paley–Wiener Spaces

As an example, we will discuss the case of band-limited functions on the real line. This case cannot be handled with the classical coorbit theory, since the reproducing kernel that arises is the sinc function, which is not integrable. Thus, the band-limited functions are a suitable example for our setting.

We will briefly recall the setting following the lines of Sect. 4.2 in [5]. Let G denote the additive group $\mathbb{R}$ whose Haar measure is the Lebesgue measure dx. Since the group is abelian, $\mathbb{R}$ is unimodular. We denote by $\mathbb{S}(\mathbb{R})$ the Schwartz space of smooth, rapidly decaying functions and by $\mathbb{S}'(\mathbb{R})$ the space of tempered distributions. The Fourier transform on $\mathbb{S}(\mathbb{R})$ and $\mathbb{S}'(\mathbb{R})$—defined for $f \in L^1(\mathbb{R})$ as $\mathscr{F} f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx$—is denoted by $\mathscr{F}$. If $v \in \mathbb{S}'(\mathbb{R})$, we also set $\widehat{v} = \mathscr{F} v$.

The Hilbert space $\mathscr{H}$ we are interested in is the Paley–Wiener space of functions with band in the fixed set $\Omega \subset \mathbb{R}$, namely

$$\mathscr{H} = B^2_\Omega = \left\{ v \in L_2(\mathbb{R}) \;\middle|\; \operatorname{supp}(\widehat{v}) \subseteq \Omega \right\}$$

equipped with the $L_2(\mathbb{R})$ scalar product. Then, by defining π for $b \in \mathbb{R}$ as

$$\pi(b)v(x) = v(x-b), \quad v \in B^2_\Omega,$$

π becomes an unitary representation of the group $\mathbb{R}$ acting on B^2_Ω. With this definition of π, on the frequency side $\widehat{\pi} = \mathscr{F}\pi\mathscr{F}^{-1}$ acts on $\mathscr{F}\mathscr{H} = L_2(\Omega)$ by modulations:

$$\widehat{\pi}(b)\widehat{v}(\xi) = e^{2\pi i b\xi}\widehat{v}(\xi), \quad v \in B^2_\Omega.$$

From now on we set Ω to be a symmetrical interval, $\Omega = [-\omega, \omega]$. Proposition 4.6 in [5] then shows that by choosing as admissible vector the function $u = \mathscr{F}^{-1}\chi_\Omega \in B^2_\Omega$, the resulting kernel K as defined in (4.2.3) is the sinc function

$$K(b) = \mathscr{F}^{-1}\chi_{\Omega}(b) = 2\omega\,\mathrm{sinc}(2\omega\pi b) = \frac{\sin(2\omega\pi b)}{\pi b}, \tag{4.3.36}$$

where $\mathrm{sinc}\, x = \sin\, x/x$. Clearly, K is not in $L_1(\mathbb{R})$, but it belongs to $L_p(\mathbb{R})$ for every $p > 1$. Therefore, we choose the weight $w = 1$ and take

$$\mathscr{T} = \bigcap_{1<p<\infty} L_p(\mathbb{R})$$

as a target space to construct coorbits, see (4.3.1). As above, the (anti-)dual of $\mathscr{T}$ can be identified with

$$\mathscr{T}' = \mathscr{U} = \mathrm{span} \bigcup_{1<q<\infty} L_q(\mathbb{R}).$$

For $p \in [1, \infty)$, we define the Paley–Wiener p-spaces

$$B^p_\Omega := \left\{ f \in L_p(\mathbb{R}) \;\middle|\; \mathrm{supp}(\mathscr{F} f) \subseteq \Omega \right\}.$$

Recall that the Fourier transform maps $L_p(\mathbb{R})$ to $L_{p'}(\mathbb{R})$ for $p \le 2$, which follows from the Hausdorff–Young inequality. In contrast, for $p > 2$, the space $\mathscr{F} L_p(\mathbb{R})$ contains distributions that in general are not functions, see [23, Theorem 7.6.6].

The spaces B^p_Ω are sometimes defined in the literature as the spaces of the entire functions of fixed exponential type whose restriction to the real line is in $L_p(\mathbb{R})$. This definition is equivalent to ours since a Paley–Wiener theorem holds for all $p \in [1, \infty)$. In particular, all these functions are infinitely differentiable on $\mathbb{R}$. Moreover, if $f \in B^p_\Omega$ with $p < \infty$, then $f(x) \to 0$ as $x \to \pm\infty$, and hence

$$B^p_\Omega \subset \mathscr{C}^\infty_0(\mathbb{R}) = \left\{ f \in \mathscr{C}^\infty(\mathbb{R}) \;\middle|\; f(x) \to 0 \text{ as } x \to \pm\infty \right\}, \quad 1 \le p < \infty.$$

Consequently, the Paley–Wiener spaces are nested and increase with p:

$$B^p_\Omega \subseteq B^q_\Omega, \quad 1 \le p \le q < \infty.$$

Proposition 4.6 (Proposition 4.8 of [5]) *Let $\Omega = [-\omega, \omega]$ and define $u := K := \mathscr{F}^{-1}\chi_\Omega$. The "test space" (as defined in 4.2.6) is*

$$\mathscr{S} = \bigcap_{p\in(1,\infty)} B^p_\Omega$$

and its dual space is

$$\mathscr{S}' = \bigcup_{p\in(1,\infty)} B^p_\Omega.$$

The extended voice transform is the inclusion

$$V_e : \mathcal{S}' \hookrightarrow \mathcal{U}$$

and the following identification holds:

$$\mathrm{Co}(L_p(\mathbb{R})) = \mathcal{M}^p = B^p_\Omega.$$

To obtain a discretization as laid out in Sect. 4.3, we first need to show that Assumption 4.2 is fulfilled. By Lemma 4.4, it suffices to show that the convolution operator associated to K is a bounded operator on $L_p(\mathbb{R})$.

Corollary 4.2 *Let* $1 < p < \infty$, *then* RC_K *is a bounded operator on* $L_p(\mathbb{R})$.

Proof Since $K = \mathcal{F}^{-1}\chi_\Omega$, the convolution with K is a bounded operator on $L_p(\mathbb{R})$ if and only if χ_Ω is a Fourier multiplier on $L_p(\mathbb{R})$. By [20, Example 2.5.15], this is true if and only if $\chi_{[0,1]}$ is a Fourier multiplier on $L_p(\mathbb{R})$. However, it is well known that this is true because the Hilbert transform is bounded as an operator acting on $L_p(\mathbb{R})$, see [20, Theorem 5.1.7]. □

We will now apply the analysis outlined in Sect. 4.3.2 to obtain a discretization for these spaces. To this end, for $n \in \mathbb{N}$, let

$$Y_n := \left\{2^{-n}k\right\}_{k\in\mathbb{Z}} \subset \mathbb{R}. \tag{4.3.37}$$

Furthermore, we fix

$$Q_n = \left[-2^{-n-1}, 2^{-n-1}\right], \tag{4.3.38}$$

which is a compact neighborhood of zero, and we set

$$\psi_{n,k} := \chi_{[-2^{-n-1},2^{-n-1})}(\cdot - 2^{-n}k) \tag{4.3.39}$$

for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, where χ denotes the characteristic function. Then, the verification of (4.3.13)–(4.3.15) is straightforward. For later use, we note that $|Q_n| = 2^{-n}$. Furthermore, the system $\{\psi_{n,k}\}_{k\in\mathbb{Z}}$, $n \in \mathbb{N}$ fixed, is orthogonal with $\|\psi_{n,k}\|^2_{L_2(\mathbb{R})} = |Q_n|$. As a finite subset of Y_n, $n \in \mathbb{N}$, we set

$$X_n := \left\{2^{-n}k \;\middle|\; -N(n) \le k \le N(n)\right\}, \tag{4.3.40}$$

where $N(n) \in \mathbb{N}$ is chosen such that (4.3.16) and (4.3.17) are fulfilled. A possible choice is $N = N(n) = n \cdot 2^n$.

According to (4.3.18), the operator $T_n : B^p_\Omega \to V_n \subset B^p_\Omega$ is defined via

$$T_n f(x) = \sum_{k=-N(n)}^{N(n)} \langle f, \psi_{n,k} \rangle_{L_2} K(x - 2^{-n}k),$$

for $f \in B^p_\Omega$, where

$$\langle f, \psi_{n,k} \rangle_{L_2} = \int_{2^{-n}(k-1/2)}^{2^{-n}(k+1/2)} f(y)\, dy.$$

By (4.3.19), this means

$$V_n = \operatorname{span}\left\{ \operatorname{sinc}(2\pi\omega(\cdot - 2^{-n}k)) \;\middle|\; -N(n) \leq k \leq N(n) \right\}.$$

In order to apply Theorem 4.3, we first need to show the following:

Lemma 4.9 *It holds $K \in \mathscr{M}^\rho_{Q_n}(L_r)$, and therefore $\operatorname{osc}_{Q_n} f \in L_r(\mathbb{R})$ for all $n \in \mathbb{N}$.*

Proof We have $|K(y)| \leq 2\omega$ for all $y \in \mathbb{R}$, and $|K(b)| \leq 1/(\pi|b|)$ for all $b \neq 0$. This implies

$$|K(y)| \leq \frac{1+4\omega}{1+|y|}.$$

Indeed, for $|y| \leq 1$, we have $|K(y)| \leq 2\omega \leq \frac{4\omega}{1+|y|}$, while for $|y| \geq 1$, we have $1/|y| \leq 2/(1+|y|)$, and thus $|K(y)| \leq \frac{2}{\pi}\frac{1}{|y|} \leq \frac{1+4\omega}{1+|y|}$.

Now, for $y \in x + Q_n \subset x + [-1, 1]$ we have $1 + |x| \leq 2 + |y| \leq 2(1 + |y|)$, so that $|K(y)| \leq \frac{1+4\omega}{1+|y|} \leq \frac{2+8\omega}{1+|x|}$. Hence,

$$\sup_{y \in x+Q_n} |K(y)|^r \leq \left(\frac{2+8\omega}{1+|x|} \right)^r,$$

and thus

$$\int_{\mathbb{R}} \sup_{y \in x+Q_n} |K(y)|^r \, dx < \infty,$$

concluding the proof. □

With this at hand, we can discretize the Paley–Wiener p-spaces according to Theorem 4.3.

Proposition 4.7 *Let $1 < p < \infty$.*

(i) Fix $\varepsilon > 0$; then for any $f \in B^p_\Omega$ there exists an integer $n^* = n^*_{f,\varepsilon} \in \mathbb{N}$, such that for all $n \geq n^*$

$$\left\| f - \sum_{k=-N(n)}^{N(n)} c(f)_{n,k} K(\cdot - 2^{-n}k) \right\|_{L_p} \leq \varepsilon,$$

where the family of coefficients $(c(f)_{n,k})_{-N(n)\le k\le N(n)}$ *satisfies*

$$\|(c(f)_{n,k})_{-N(n)\le k\le N(n)}\|_{\ell_p} \le 2^{-n/p'}(1+\varepsilon)\|S_n\| \cdot \|f\|_{L_p}.$$

Here, as usual, S_n is a linear right inverse for the operator T_n defined in (4.3.18).

(ii) For any sequence $(d_x)_{x\in Y_n} \in \ell_q(Y_n)$, $n \in \mathbb{N}$, the function f defined by $f = \sum_{k\in\mathbb{Z}} d_{2^{-n}k} K(\cdot - 2^{-n}k)$ is in B^p_Ω with

$$\|f\|_{L_p(\mathbb{R})} \le C \cdot 2^{n(1-1/q)}\|(d_x)_{x\in Y_n}\|_{\ell_q},$$

where $C = C(p,q) > 0$ is a constant and $q < p$.

Proof (i) is an application of Theorem 4.3(i), with $|Q_n| = 2^{-n}$.

It remains to prove (ii). Again, we can apply Theorem 4.3(ii) and note that, by Lemma 4.9, the assumption (4.3.32) is fulfilled. Moreover, Lemma 4.9 shows that $\|\mathrm{osc}_{Q_n}(K) + |K|\|_{L_r}$ can be estimated from above by a constant $C > 0$ independent of $n \in \mathbb{N}$. □

As stated in Remark 4.7, the asymptotic behavior of the operator norm of S_n is crucial. In the following, we apply Lemma 4.20 to obtain a useful characterization of $\|S_n\|$.

For this, we restrict ourselves to the case $p = 2$ and obtain with the notation of Lemma 4.20

$$\begin{aligned}
\|S_n\|^{-1} = \varepsilon &= \inf\left\{ \frac{\|T_n f\|_{L_2}}{\|f\|_{L_2}} \,\middle|\, f \in (\mathrm{Ker}\, T_n)^\perp \right\} \\
&= \inf\left\{ \frac{\langle T_n^* T_n f, f\rangle_{L_2}}{\langle f, f\rangle_{L_2}} \,\middle|\, f \in (\mathrm{Ker}\, T_n)^\perp \right\}^{1/2} \\
&= \lambda_{\min}(U_n)^{1/2},
\end{aligned}$$

where $\lambda_{\min}(U_n)$ denotes the smallest eigenvalue of the operator

$$U_n := T_n^* T_n : (\mathrm{Ker}\, T_n)^\perp \to (\mathrm{Ker}\, T_n)^\perp .$$

Here, we used the well-known inclusion $\mathrm{Ran}\, A^* \subset (\mathrm{Ker}\, A)^\perp$ which guarantees that U_n is well-defined.

We have thus shown that the asymptotic behavior of the smallest eigenvalue of U_n is equivalent to the asymptotic behavior of $\|S_n\|$.

By using

$$T_n f = \sum_{j=-N(n)}^{N(n)} \langle f, \psi_{n,j}\rangle_{L_2} K(\cdot - 2^{-n}j), \quad T_n^* g = \sum_{k=-N(n)}^{N(n)} \langle g, K(\cdot - 2^{-n}k)\rangle_{L_2} \psi_{n,k},$$

we can rewrite U_n as

$$
\begin{aligned}
U_n f &= \sum_{j,k=-N(n)}^{N(n)} \langle f, \psi_{n,j}\rangle_{L_2(\mathbb{R})} \langle K(\cdot - 2^{-n} j), K(\cdot - 2^{-n}k)\rangle_{L_2(\mathbb{R})} \psi_{n,k} \\
&= \sum_{j,k=-N(n)}^{N(n)} \langle f, \psi_{n,j}\rangle_{L_2(\mathbb{R})} K(2^{-n}(k-j))\psi_{n,k}
\end{aligned}
$$

for $f \in (\operatorname{Ker} T_n)^{\perp}$.

We set $W_n := \operatorname{span}\left\{\psi_{n,k} \;\middle|\; -N(n) \le k \le N(n)\right\}$ and obtain the relation $W_n^{\perp} \subset \operatorname{Ker} T_n$; thus $(\operatorname{Ker} T_n)^{\perp} \subset W_n$. Next, we note that the family $\{\lambda(x)K\}_{x\in\mathbb{R}}$ is linearly independent; indeed, we have $\mathscr{F}(\lambda(x)K) = e^{-2\pi i x \cdot}\chi_{[-\omega,\omega]}$, and by analyticity these functions are linearly independent if and only if the functions $(\mathbb{R} \to \mathbb{C}, \xi \mapsto e^{-2\pi i x\xi})$ are. But each of these functions is an eigenvector of the differential operator $d/d\xi$ with pairwise distinct eigenvalues $2\pi i x$, $x \in \mathbb{R}$, which yields the linear independence. From this and from Lemma 4.5, we see that $\operatorname{Ran} T_n = V_n = \operatorname{span}\{\lambda(x)K\}_{x\in X_n}$ satisfies $\dim \operatorname{Ran} T_n = |X_n| = 1 + 2N(n)$. But since $T_n : (\operatorname{Ker} T_n)^{\perp} \to V_n$ is an isomorphism, we see $\dim(\operatorname{Ker} T_n)^{\perp} = 1 + 2N(n)$ as well, so that we finally see $W_n = (\operatorname{Ker} T_n)^{\perp}$ by comparing dimensions. Hence, $U_n : W_n \to W_n$.

Moreover, by the orthogonality of the family $\{\psi_{n,k}\}$, we see that

$$
U_n\, \psi_{n,k} = \|\psi_{n,k}\|_{L_2}^2 \sum_{\ell=-N(n)}^{N(n)} K(2^{-n}(\ell - k))\, \psi_{n,\ell} \tag{4.3.41}
$$

for any $-N(n) \le k \le N(n)$. Since $\dim W_n = 2N(n) + 1 < \infty$, we may define an isomorphism

$$
P_n : W_n \to \mathbb{R}^{2N(n)+1}, \quad P_n(\psi_{n,k}) = \|\psi_{n,k}\|_{L_2} e_k, \tag{4.3.42}
$$

where e_k denotes the k-th canonical unit vector of $\mathbb{R}^{2N(n)+1}$. Note that P_n maps the orthonormal basis $(\psi_{n,k}/\|\psi_{n,k}\|_{L_2(\mathbb{R})})$ to the orthonormal basis $(e_k)_{k\in\mathbb{N}}$, so that P_n is unitary.

The linear map $P_n U_n P_n^{-1} : \mathbb{R}^{2N(n)+1} \to \mathbb{R}^{2N(n)+1}$ is represented by a matrix M_n, whose entries are given via

$$
\begin{aligned}
(M_n)_{j,k} &= \langle P_n U_n P_n^{-1} e_k, e_j\rangle_{\mathbb{R}^{2N(n)+1}} = \langle U_n \frac{\psi_{n,k}}{\|\psi_{n,k}\|_{L_2}}, \frac{\psi_{n,j}}{\|\psi_{n,j}\|_{L_2(\mathbb{R})}}\rangle_{L_2} \\
&= \frac{\|\psi_{n,k}\|_{L_2}}{\|\psi_{n,j}\|_{L_2}} \sum_{\ell=-N(n)}^{N(n)} K(2^{-n}(\ell-k))\langle \psi_{n,\ell}, \psi_{n,j}\rangle_{L_2} \\
&= \|\psi_{n,k}\|_{L_2}\|\psi_{n,j}\|_{L_2} K(2^{-n}(j-k)) \\
&= 2^{-n} K(2^{-n}(j-k)),
\end{aligned}
$$

$1 \leq j, k \leq 2N(n) + 1$. Since K is real, the matrix M_n is a *symmetric Toeplitz matrix*, which means that the entries of M_n only depend on the quantity $|k - j|$, thus yielding a band structure. Since the eigenvalues of M_n coincide with those of the map U_n, finding the smallest eigenvalue of U_n is equivalent to finding the smallest eigenvalue of the Toeplitz matrix M_n.

Unfortunately, this task is very difficult. To the best knowledge of the authors, it is not possible to properly characterize the asymptotic behavior of the smallest eigenvalue of such a Toeplitz matrix. We further refer to [3], where the authors were told that leading experts on the field of Toeplitz matrices are unaware of these asymptotics.

Since there are already big obstacles in understanding the asymptotic behavior of $\|S_n\|$ in this rather simple setting, one cannot hope that easy answers are available when turning to more complex groups and their associated coorbit spaces.

4.4 Obstructions to Discretization for Non-integrable Kernels

In classical coorbit theory, the kernel $K(x) = Vu(x) = \langle u, \pi(x)\, u\rangle_{\mathscr{H}}$ is assumed to be integrable; in other words, it has to satisfy $K \in L_{1,w}(G)$ for a suitable weight $w \geq 1$ on G. This assumption is introduced in order to guarantee two independent properties: First, it ensures that one can construct a suitable reservoir of "distributions," and thus obtains well-defined coorbit spaces. Second, it ensures that the right convolution operator $f \mapsto f * K$ acts boundedly on the function space Y which is used to define the coorbit space $\mathrm{Co}(Y)$. For instance, this is the case if $Y = L_{r,m}(G)$ with a w-moderate weight m.

Replacing the integrability condition $K \in L_{1,w}(G)$ by the weaker assumption $K \in \bigcap_{1<p<\infty} L_{p,w}(G)$, one can still define a suitable reservoir and obtains well-defined decomposition spaces, as we saw in Sect. 4.2. However, we will see in the present section—precisely, in Proposition 4.8—that the modified assumption $K \in \bigcap_{1<p<\infty} L_{p,w}(G)$ is in general too weak to ensure that right convolution with K defines a bounded operator on $L_{r,m}(G)$. In other words, a given kernel K satisfying the weak integrability assumption might or might not act boundedly on $L_{r,m}(G)$ by right convolution.

For such "bad" kernels that do not act boundedly, no discretization results similar to those from classical coorbit theory can hold, as we will prove in the present section. Therefore, if such discretization results for the coorbit space $\mathrm{Co}(L_{r,m})$ are desired, one needs to assume that $K \in \bigcap_{1<p<\infty} L_{p,w}(G)$ and additionally that $f \mapsto f * K$ defines a bounded operator on $L_{r,m}(G)$. This second condition is highly non-trivial to verify in many cases where the kernel K is not integrable. However, it is possible in the setting of the group $(\mathbb{R}, +)$ as discussed in Sect. 4.3.3.

Since we aim to show that no discretization as for classical coorbit theory is possible, we briefly recall these results: Assuming the kernel K to be well behaved, a combination of Lemma 3.5(v) and Theorem 6.1 in [13] shows that the *synthesis operator*

$$\mathrm{Synth}_X : \ell_{r,m_X}(I) \to \mathrm{Co}(L_{r,m}),\ (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \cdot \pi(x_i)\, u \quad \text{with} \quad (m_X)_i = m(x_i)$$

is well-defined and bounded, for each $r \in (1,\infty)$, each w-moderate weight m, and each family $X = (x_i)_{i\in I}$ in G that is sufficiently *separated*—similar to $\delta\mathbb{Z}^d$ in $G = \mathbb{R}^d$. The operator Synth_X even has a bounded linear right inverse, provided that the family X is sufficiently dense in G, where the required density only depends on w, u. If Synth_X indeed has a bounded linear right inverse, the family $(\pi(x_i)\,u)_{i\in I}$ is called a family of atoms for $\mathrm{Co}(L_{r,m})$ with coefficient space $\ell_{r,m_X}(I)$.

Dual to the concept of atomic decompositions is the notion of *Banach frames*, which was introduced in [21]. By definition, the family $(\pi(x_i)\,u)_{i\in I}$ is a Banach frame for $\mathrm{Co}(L_{r,m})$ with coefficient space $\ell_{r,m_X}(I)$ if the *analysis operator*

$$A_X : \mathrm{Co}(L_{r,m}) \to \ell_{r,m_X}(I),\ f \mapsto \big(\langle f, \pi(x_i)\,u\rangle_{\mathscr{S}_w}\big)_{i\in I}$$

is well-defined and bounded and has a bounded linear left inverse. As shown in [21, Theorem 5.3], this is satisfied if the sampling points $X = (x_i)_{i\in I}$ satisfy the same properties as above: they should be sufficiently separated and dense enough in G, where these conditions only depend on w and u, but not on the integrability exponent r or the w-moderate weight m. Provided that $(\pi(x_i)\,u)_{i\in I}$ is a Banach frame for $\mathrm{Co}(L_{r,m})$, we have in particular $\|A_X f\|_{\ell_{r,m_X}} \asymp \|f\|_{\mathrm{Co}(L_{r,m})}$ for all $f \in \mathrm{Co}(L_{r,m})$; but in general, this latter property is weaker than the Banach frame property.

The preceding statements hold for all w-moderate weights m and for all exponents $r \in (1,\infty)$. Since the reciprocal m^{-1} of a w-moderate weight m is again w-moderate, see Lemma 4.1, it follows that if the above properties hold for $L_{r,m}(G)$, then they also hold for $L_{r',m^{-1}}(G)$. Therefore, classical coorbit theory provides discretization results that are stronger than the assumptions of the following theorem. The following theorem thus shows that discretization results as in classical coorbit theory can only hold if the kernel K acts boundedly on $L_{r,m}(G)$ via right convolutions.

Theorem 4.4 *Let $r \in (1,\infty)$ be arbitrary. Assume that Assumption 4.1 is satisfied, and let $m : G \to (0,\infty)$ be a w-moderate weight. Furthermore, assume that for some family $(x_i)_{i\in I}$ in G and for some weight $\theta = (\theta_i)_{i\in I}$ on the index set I, the following hold:*

(i) "Weak Banach frame condition for $\mathrm{Co}(L_{r,m})$": The analysis map

$$A : \mathrm{Co}(L_{r,m}) \to \ell_{r,\theta}(I),\ \varphi \mapsto \big(\langle \varphi\,,\ \pi(x_i)\,u\rangle_{\mathscr{S}_w}\big)_{i\in I}$$

is well-defined and bounded, with

$$\|A\,\varphi\|_{\ell_{r,\theta}} \asymp \|\varphi\|_{\mathrm{Co}(L_{r,m})} \qquad \text{for all } \varphi \in \mathrm{Co}(L_{r,m})\,. \tag{4.4.1}$$

(ii) "Weak atomic decomposition condition for $\mathrm{Co}(L_{r',m^{-1}})$*": The synthesis map*

$$S : \ell_{r',\theta^{-1}}(I) \to \mathrm{Co}(L_{r',m^{-1}}), \, (c_i)_{i\in I} \mapsto \sum_{i\in I} [c_i \cdot \pi(x_i)\, u]$$

is well-defined and bounded.

Then the right convolution operator $RC_K : f \mapsto f * K$ *defines a bounded linear operator on* $L_{r,m}(G)$.

For the proof of this theorem, we will need several technical lemmata. Having shown in Sect. 4.2 that the voice transform can be extended from $\mathscr{H}$ to the reservoir $\mathscr{S}'_w$ (and thus to the coorbit spaces $\mathrm{Co}(L_{r,m})$), our first lemma shows that one can also define a version of the voice transform on the (anti-)dual space $\left[\mathrm{Co}(L_{r,m})\right]'$.

Lemma 4.10 *If Assumption 4.1 is satisfied for* $r \in (1, \infty)$*, and if* $m : G \to (0, \infty)$ *is* w*-moderate, then there is a constant* $C = C(m, r, w, K) > 0$ *such that*

$$\text{for all } x \in G : \quad \pi(x)\, u \in \mathrm{Co}(L_{r,m}) \quad \text{and} \quad \|\pi(x)\, u\|_{\mathrm{Co}(L_{r,m})} \le C \cdot w(x)\,.$$

Therefore, for any (antilinear) continuous functional $\varphi \in \left[\mathrm{Co}(L_{r,m})\right]'$*, the* special voice transform

$$V_{\mathrm{sp}}\, \varphi : G \to \mathbb{C}, x \mapsto \varphi(\pi(x)\, u) = \langle \varphi\,,\, \pi(x)u\rangle_{[\mathrm{Co}(L_{r,m})]' \times \mathrm{Co}(L_{r,m})}$$

is a well-defined function.

Proof First, let us set $C_1 := m(e)$, where e is the unit element of G. Since m is w-moderate (see 4.2.13), we have

$$m(x) = m(x \cdot e) \le w(x) \cdot m(e) \le C_1 \cdot w(x) \qquad \text{for all } x \in G\,.$$

Furthermore,

$$w(y) = w(xx^{-1}y) \le w(x) \cdot w(x^{-1}y) = w(x) \cdot (\lambda(x)w)(y)\,.$$

Now, recall from Sect. 4.2, the embedding $\mathscr{H} \hookrightarrow \mathscr{S}'_w$, and that the extended voice transform V_e coincides with the usual voice transform on $\mathscr{H}$. Therefore, since $\pi(x)\, u \in \mathscr{H}$, and since $K = Vu$, we get

$$\begin{aligned}
\|V_e\,[\pi(x)u]\|_{L_{r,m}} &= \|V\,[\pi(x)u]\|_{L_{r,m}} \le C_1 \cdot \|V\,[\pi(x)u]\|_{L_{r,w}} \\
&= C_1 \cdot \|w \cdot \lambda(x)\,[Vu]\|_{L_r} \le C_1 \cdot w(x) \cdot \|\lambda(x)\,[w \cdot Vu]\|_{L_r} \\
&= C_1 \cdot w(x) \cdot \|w \cdot Vu\|_{L_r} = C_1 \cdot w(x) \cdot \|K\|_{L_{r,w}} = C \cdot w(x)\,,
\end{aligned}$$

where $C := C_1 \cdot \|K\|_{L_{r,w}}$ is finite thanks to Assumption 4.1. This proves the first part of the lemma, which then trivially implies that $V_{\mathrm{sp}}\,\varphi$ is a well-defined function, for any $\varphi \in [\mathrm{Co}(L_{r,m})]'$. □

Our next lemma shows that if the right convolution with K does *not* act boundedly on $L_{r',m^{-1}}(G)$, then there exist certain pathological functionals on $\mathrm{Co}(L_{r,m})$.

Lemma 4.11 *Assume that Assumption 4.1 is satisfied, and let $r \in (1,\infty)$. If the right convolution operator $RC_K : f \mapsto f * K$ does* not *yield a well-defined bounded linear operator on $L_{r',m^{-1}}(G)$, then there is an (antilinear) continuous functional $\varphi \in \left[\mathrm{Co}(L_{r,m})\right]'$ satisfying $V_{\mathrm{sp}}\,\varphi \notin L_{r',m^{-1}}(G)$.*

Proof We first claim that there is some $\Phi \in L_{r',m^{-1}}(G)$ with $\Phi * K \notin L_{r',m^{-1}}(G)$; that is, we claim that $RC_K : L_{r',m^{-1}}(G) \to L_{r',m^{-1}}(G)$ is not well-defined.

To see this, recall from Assumption 4.1 that $K \in \bigcap_{1<p<\infty} L_{p,w}(G)$. Thus, since m^{-1} is w-moderate (see Lemma 4.1), Young's inequality (see Proposition 4.13) shows that the right convolution operator RC_K is bounded as a map $RC_K : L_{r',m^{-1}}(G) \to L_{q,m^{-1}}(G)$ for any $q \in (r',\infty)$. Therefore, if $RC_K : L_{r',m^{-1}}(G) \to L_{r',m^{-1}}(G)$ *was* well-defined, then the closed graph theorem would imply that $RC_K : L_{r',m^{-1}}(G) \to L_{r',m^{-1}}(G)$ is bounded, contradicting our assumptions. Hence, there is a function Φ as desired.

Now, define the antilinear functional

$$\varphi : \mathrm{Co}(L_{r,m}) \to \mathbb{C}, \quad f \mapsto \int_G \Phi(y) \cdot \overline{V_e f(y)}\, dy.$$

It is easy to see that φ is well-defined and bounded; in fact,

$$|\varphi(f)| \le \|\Phi\|_{L_{r',m^{-1}}} \cdot \|V_e f\|_{L_{r,m}} = \|\Phi\|_{L_{r',m^{-1}}} \cdot \|f\|_{\mathrm{Co}(L_{r,m})}\,.$$

Finally, note for all $x \in G$ that

$$\begin{aligned} V_{\mathrm{sp}}\,\varphi(x) &= \langle \varphi\,,\, \pi(x)\,u\rangle_{[\mathrm{Co}(L_{r,m})]' \times \mathrm{Co}(L_{r,m})} = \int_G \Phi(y) \cdot \overline{V_e\,[\pi(x)\,u]\,(y)}\, dy \\ &= \int_G \Phi(y) \cdot \overline{\langle \pi(x) u\,,\, \pi(y)\,u\rangle_{\mathscr{H}}}\, dy = \int_G \Phi(y) \cdot \langle u\,,\, \pi(y^{-1}x)\,u\rangle_{\mathscr{H}}\, dy \\ &= \int_G \Phi(y) \cdot K(y^{-1}x)\, dy = (\Phi * K)(x) \end{aligned}$$

with $\Phi * K \in L_{q,m^{-1}}(G)$ for all $q \in (r',\infty)$. But by our choice of Φ, we have $V_{\mathrm{sp}}\,\varphi = \Phi * K \notin L_{r',m^{-1}}(G)$, as desired. □

Our next lemma shows that the assumptions of Theorem 4.4 exclude the existence of pathological functionals as in the preceding lemma.

Lemma 4.12 *Under the assumptions of Theorem 4.4 and with notation as in Lemma 4.10, every antilinear continuous functional* $\varphi \in \left[\mathrm{Co}(L_{r,m})\right]'$ *satisfies* $V_{\mathrm{sp}}\,\varphi \in L_{r',m^{-1}}(G)$.

Proof Let $\varphi \in \left[\mathrm{Co}(L_{r,m})\right]'$ be arbitrary, and let the analysis operator A be as in the assumptions of Theorem 4.4. Using this operator, we define the (antilinear) functional

$$\Lambda_0 : A\left(\mathrm{Co}(L_{r,m})\right) \to \mathbb{C},\, Af \mapsto \varphi(f)\,.$$

Note that this is well-defined, since (4.4.1) ensures that A is injective. Furthermore, with $A\left(\mathrm{Co}(L_{r,m})\right)$ considered as a subspace of $\ell_{r,\theta}(I)$, the functional Λ_0 is bounded, since (4.4.1) yields a constant $C > 0$ such that each $c = Af \in A\left(\mathrm{Co}(L_{r,m})\right)$ satisfies

$$\begin{aligned}|\Lambda_0(c)| = |\varphi(f)| &\le \|\varphi\|_{[\mathrm{Co}(L_{r,m})]'} \cdot \|f\|_{\mathrm{Co}(L_{r,m})}\\ &\le C\|\varphi\|_{[\mathrm{Co}(L_{r,m})]'} \cdot \|Af\|_{\ell_{r,\theta}} = C\|\varphi\|_{[\mathrm{Co}(L_{r,m})]'} \cdot \|c\|_{\ell_{r,\theta}}\,.\end{aligned}$$

With Λ_0 being bounded, an antilinear version of the Hahn–Banach theorem yields a bounded (antilinear) extension $\Lambda : \ell_{r,\theta}(I) \to \mathbb{C}$ of Λ_0. Therefore, an antilinear version of the Riesz representation theorem for the dual of $\ell_{r,\theta}(I)$ ensures the existence of $\varrho = (\varrho_i)_{i\in I} \in \ell_{r',\theta^{-1}}(I)$ satisfying $\Lambda(c) = \langle \varrho\,,\, c\rangle_{\ell_{r',\theta^{-1}} \times \ell_{r,\theta}}$ for all $c \in \ell_{r,\theta}(I)$. Here, the pairing between $\ell_{r',\theta^{-1}}(I)$ and $\ell_{r,\theta}(I)$ is given by $\langle (c_i)_{i\in I}, (e_i)_{i\in I}\rangle_{\ell_{r',\theta^{-1}} \times \ell_{r,\theta}} = \sum_{i\in I} c_i \cdot \overline{e_i}$.

Having constructed the sequence $\varrho \in \ell_{r',\theta^{-1}}(I)$, we can now apply the second assumption of Theorem 4.4—the boundedness of the synthesis operator S—to define $g := S\varrho \in \mathrm{Co}(L_{r',m^{-1}})$. Furthermore, for arbitrary $x \in G$, we recall from Lemma 4.10 that $\pi(x)\,u \in \mathrm{Co}(L_{r,m})$, so that

$$c^{(x)} = \left(c_i^{(x)}\right)_{i\in I} := A\left(\pi(x)u\right) = \left(\langle \pi(x)\,u\,,\,\pi(x_i)\,u\rangle_{\mathscr{H}}\right)_{i\in I} \in \ell_{r,\theta}(I)$$

is well-defined. Combining our preceding observations, we see

$$\begin{aligned}V_{\mathrm{sp}}\,\varphi(x) &= \varphi(\pi(x)\,u) = \Lambda_0\big(A(\pi(x)\,u)\big) = \Lambda(c^{(x)}) = \langle \varrho, c^{(x)}\rangle_{\ell_{r',\theta^{-1}}, \times \ell_{r,\theta}}\\ &= \sum_{i\in I}\left[\varrho_i \cdot \overline{\langle \pi(x)\,u\,,\,\pi(x_i)\,u\rangle_{\mathscr{H}}}\right] = \sum_{i\in I}\left[\varrho_i \cdot \langle \pi(x_i)\,u\,,\,\pi(x)\,u\rangle_{\mathscr{S}_w}\right]\\ &\overset{(*)}{=} \left\langle \sum_{i\in I}\left(\varrho_i \cdot \pi(x_i)\,u\right)\,,\,\pi(x)\,u\right\rangle_{\mathscr{S}_w}\\ &= \langle S\varrho\,,\,\pi(x)\,u\rangle_{\mathscr{S}_w} = [V_e\,g]\,(x).\end{aligned} \tag{4.4.2}$$

This identity—which will be fully justified below—completes the proof, since we have $g = S\varrho \in \mathrm{Co}(L_{r',m^{-1}})$, that is $V_e\,g \in L_{r',m^{-1}}(G)$. Therefore, (4.4.2) implies $V_{\mathrm{sp}}\,\varphi = V_e\,g \in L_{r',m^{-1}}(G)$, as claimed.

It remains to justify the step marked with $(*)$ in (4.4.2).

At that step, we used on the one hand that $S\varrho = \sum_{i\in I}\left[\varrho_i \cdot \pi(x_i)\,u\right]$ with unconditional convergence in $\mathrm{Co}(L_{r',m^{-1}})$. To see that this indeed holds, recall that $r' < \infty$, so that $\varrho = \sum_{i \in I} \varrho_i \, \delta_i$, with unconditional convergence in $\ell_{r',\theta^{-1}}(I)$; by the boundedness of S, this implies the claimed identity. On the other hand, we also used at $(*)$ that $\mathrm{Co}(L_{r',m^{-1}}) \to \mathbb{C}$, $f \mapsto \langle f \,,\, \pi(x)\, u\rangle_{\mathscr{S}_w}$ is a bounded linear functional. Indeed, (4.2.10) and Lemma 4.10 imply

$$
\begin{aligned}
\left|\langle f \,,\, \pi(x)\, u\rangle_{\mathscr{S}_w}\right| &= \left|\langle V_e f \,,\, V\,[\pi(x)\, u]\rangle_{L_2}\right| \le \|V_e f\|_{L_{r',m^{-1}}} \cdot \|V\,[\pi(x)\, u]\|_{L_{r,m}} \\
&= \|f\|_{\mathrm{Co}(L_{r',m^{-1}})} \cdot \|V\,[\pi(x)\, u]\|_{L_{r,m}} \\
&\le C \cdot \|f\|_{\mathrm{Co}(L_{r',m^{-1}})} \cdot w(x)\,,
\end{aligned}
$$

with $C = C(m, w, u, r)$. □

We can now finally prove Theorem 4.4.

Proof (*of Theorem* 4.4) Assume toward a contradiction that the right convolution operator $RC_K : L_{r,m}(G) \to L_{r,m}(G)$ is not bounded. By Proposition 4.3, and since the prerequisites of Theorem 4.4 include Assumption 4.1, this implies that $RC_K : L_{r',m^{-1}}(G) \to L_{r',m^{-1}}(G)$ is also not bounded. Therefore, Lemma 4.11 yields an antilinear continuous functional $\varphi \in [\mathrm{Co}(L_{r,m})]'$ with $V_{\mathrm{sp}}\varphi \notin L_{r',m^{-1}}(G)$. In view of Lemma 4.12, this yields the desired contradiction. □

Before closing this section, we show that the “weak integrability assumption” $K \in \bigcap_{1<p<\infty} L_{p,w}(G)$ does *not* imply in general that the right convolution operator $RC_K : f \mapsto f * K$ acts boundedly on any L_p-space with $p \neq 2$.

To this end, we consider as in Sect. 4.3.3 the Paley–Wiener space

$$
\mathscr{H} = B^2_\Omega = \{f \in L_2(\mathbb{R}) \;:\; \widehat{f} \equiv 0 \text{ almost everywhere on } \mathbb{R}\setminus\Omega\} \tag{4.4.3}
$$

for a fixed measurable subset $\Omega \subset \mathbb{R}$ of finite measure. As seen in Sect. 4.3.3, the group $G = \mathbb{R}$ acts on this space by translations; that is, if we set $\pi(x) f = \lambda(x) f$ for $f \in B^2_\Omega$, then π is an unitary representation of $\mathbb{R}$. Setting $u := \mathscr{F}^{-1}\chi_\Omega \in B^2_\Omega$, using Plancherel’s theorem, and noting $\widehat{f} = \widehat{f} \cdot \chi_\Omega = \widehat{f} \cdot \overline{\widehat{u}}$ for $f \in B^2_\Omega$, we see that the associated voice transform is given by

$$
\begin{aligned}
Vf(x) &= \langle f \,,\, \pi(x) u\rangle_{L_2} = \langle \widehat{f} \,,\, e^{-2\pi i x \cdot}\, \widehat{u}\,\rangle_{L_2} = \int_{\mathbb{R}} \widehat{f}(\xi) \cdot e^{2\pi i x \xi}\, d\xi \\
&= (\mathscr{F}^{-1}\widehat{f}\,)(x) = f(x)\,.
\end{aligned}
$$

Thus, $V : B^2_\Omega \to L_2(\mathbb{R})$ is an isometry and the reproducing kernel K is simply given by $K(x) = Vu(x) = u(x)$ for $x \in \mathbb{R}$. In view of these remarks, the following proposition shows that there is a reproducing kernel that satisfies the weak integrability assumption, but for which the associated right convolution operator does *not* act boundedly on $L_p(\mathbb{R})$ for any $p \neq 2$.

Proposition 4.8 *There is a compact set $C \subset [0,1]$ with the following properties:*

(i) $\mathscr{F}^{-1}\chi_C \in \bigcap_{1<p\leq\infty} L_p(\mathbb{R})$.

(ii) *For any $p \in (1,\infty)\setminus\{2\}$, the convolution operator $f \mapsto f * \mathscr{F}^{-1}\chi_C$ is* not *bounded, and by Proposition 4.2 not well-defined, as an operator on $L_p(\mathbb{R})$.*

Since the construction of the set C is quite technical, we refer the proof to the appendix.

4.5 Improved Discretization Results Under Additional Assumptions

In the preceding section, we have seen that there are limitations to the possible discretization theory for coorbit spaces with "bad" kernels, that is, for kernels K for which the right convolution with K does not act boundedly on $L_{r,m}(G)$.

But even if this right convolution operator *does* act boundedly, the results in the preceding sections only yield discretization results that are weaker than those that one would expect to hold when coming from classical coorbit theory. In the present section, we will see that a "proper" discretization theory is possible even for relatively bad (i.e., non-integrable) kernels, as long as the kernel in question acts boundedly on $L_{r,m}(G)$ and is compatible with another "well-behaved" kernel $W : G \to \mathbb{C}$, in the sense that it satisfies $K * W = K$ for the construction of Banach frames, or $W * K = K$ for the construction of atomic decompositions.

We emphasize that we do *not* assume that the kernel W satisfies $W * W = W$, thereby allowing a larger freedom in the choice of W. To see that the property $K * K = K$ is indeed quite restrictive, let us consider the case when $G = \mathbb{R}$ is the real line. Then $K * K = K$ implies that $\widehat{K} = \widehat{K} \cdot \widehat{K}$, so that $\widehat{K} = \chi_\Omega$ must be the indicator function of a (measurable) set, see also [5]. In particular, $K \notin L_1(\mathbb{R})$ (unless $K \equiv 0$), since otherwise $\widehat{K}$ would be continuous. In stark contrast, at least if the set Ω is bounded, one can choose a Schwartz function ψ with $\psi \equiv 1$ on Ω, so that $W := \mathscr{F}^{-1}\psi \in \mathbb{S}(\mathbb{R})$ satisfies $\widehat{W * K} = \widehat{W} \cdot \chi_\Omega = \chi_\Omega = \widehat{K}$, and thus $K * W = W * K = K$. It is worth noting that a related approach has been established in [17].

The section is structured as follows: In the first subsection, we recall some basic notions from classical coorbit theory: Relatively separated sets, BUPUs, etc. Then, in Sect. 4.5.2, we discuss conditions on the well-behaved kernel W which guarantee the existence of Banach frames for the coorbit spaces. The existence of atomic decompositions, under similar but different conditions on W, is discussed in Sect. 4.5.3. In the last subsection, we apply the abstract results to the setting of Paley–Wiener spaces.

Finally, we should mention that most of the proofs in this section are heavily inspired by the original coorbit papers [12–14, 21]. The main novel ingredient here is the observation that instead of the idempotent reproducing formula $K * K = K$, it suffices to have $K * W = K$ or $W * K = K$ for potentially different kernels K, W.

Remark 4.9 Most of the results in this section can also be obtained for coorbit spaces $\mathrm{Co}(Y)$ where Y is a solid Banach space continuously embedded into $L_0(G)$. For simplicity, we restrict our attention to the case $Y = L_{r,m}(G)$ as in the rest of the paper.

4.5.1 Required Notions from Classical Coorbit Theory

We would like to sample the continuous frame $\big(\pi(x)u\big)_{x\in G}$ to obtain a discrete (Banach) frame $\big(\pi(x_i)u\big)_{i\in I}$. In order for this to succeed, the family of sampling points $(x_i)_{i\in I}$ needs to be sufficiently well distributed in G. This intuition is made precise in the following definition. The reader might compare this to the definitions in the beginning of Sect. 4.3.2.

Definition 4.1 (*cf.* [13, Definition 3.2]) Let $X = (x_i)_{i\in I}$ be a family in G.

(i) X is V-*dense* in G, for a unit neighborhood $V \subset G$, if $G = \bigcup_{i\in I} x_i V$.
(ii) X is V-*separated*, for a unit neighborhood $V \subset G$, if the family $(x_i V)_{i\in I}$ is pairwise disjoint.
(iii) X is *relatively separated* if for every compact unit neighborhood $Q \subset G$, there is a constant $N = N(X, Q) \in \mathbb{N}$ with

$$\sum_{i\in I} \chi_{x_i Q}(x) \le N \qquad \text{for all } x \in G\,.$$

(iv) X is V-*well-spread* for a unit neighborhood $V \subset G$ if X is relatively separated and V-dense.

Remark 4.10 (i) Since we always assume the underlying group G to be second countable, G is in particular σ-compact. Therefore, [28, Lemma 2.3.10] shows that (the index set of) every relatively separated family in G is countable.
(ii) Usually, X is called relatively separated if X is a finite union of V-separated sets, for some compact unit neighborhood V. The two definitions are shown to be equivalent in in [11, Lemma 2.9] and [28, Lemma 2.3.11].

Given a V-well-spread family $X = (x_i)_{i\in I}$, one often wants to decompose a given function f into building blocks f_i which are supported in the sets $(x_i V)_{i\in I}$. This can be done using suitable partitions of unity; again the reader might compare this to Sect. 4.3.2.

Definition 4.2 (*cf.* [13, Definition 3.6]) Let $V \subset G$ be a compact unit neighborhood. A family $\Psi = (\psi_i)_{i\in I}$ is called a V-*BUPU (bounded uniform partition of unity) with localizing family* $X = (x_i)_{i\in I}$ if the following holds:

(i) Each $\psi_i : G \to [0,1]$ is a measurable function.
(ii) X is relatively separated and $\psi_i \equiv 0$ on $G \backslash x_i V$ for all $i \in I$.
(iii) We have $\sum_{i \in I} \psi_i \equiv 1$ on G.

One can find a V-BUPU for any compact unit neighborhood V:

Lemma 4.13 (cf. [10, Theorem 2] and [28, Lemma 2.3.12]) *Let $V \subset G$ be an arbitrary compact unit neighborhood. Then there exists a V-BUPU $\Psi = (\psi_i)_{i \in I}$ with $\psi_i \in C_c(G)$ for all $i \in I$.*

The following lemma points out an important property of relatively separated families that we will use time and again:

Lemma 4.14 *Let $X = (x_i)_{i \in I}$ be a relatively separated family and let $r \in [1, \infty)$. Let further $m : G \to (0, \infty)$ be a w-moderate weight. Define the weight m_X on the index set I by $(m_X)_i := m(x_i)$ for $i \in I$.*

Then for every compact unit neighborhood $U \subset G$, the synthesis operator

$$\mathrm{Synth}_{X,U} : \ell_{r,m_X}(I) \to L_{r,m}(G),\ (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i\, \chi_{x_i U}$$

is well-defined and bounded, with pointwise absolute convergence of the defining series.

Furthermore, if $\Psi = (\psi_i)_{i \in I}$ is a U-BUPU with localizing family X, then the synthesis operator

$$\mathrm{Synth}_{X,\Psi} : \ell_{r,m_X}(I) \to L_{r,m}(G),\ (c_i)_{i \in I} \mapsto \sum_{i \in I} c_i\, \psi_i$$

is well-defined and bounded, with pointwise absolute convergence of the defining series.

Proof The second part of the lemma is a consequence of the first one: Since $0 \leq \psi_i \leq 1$, and since ψ_i vanishes outside of $x_i U$, we have

$$|(\mathrm{Synth}_{X,\Psi}\, c)(x)| \leq \sum_{i \in I} |c_i|\, \psi_i(x) \leq \sum_{i \in I} |c_i|\, \chi_{x_i U}(x) = (\mathrm{Synth}_{X,U}\, |c|)(x) < \infty$$

for all $x \in G$ and all $c = (c_i)_{i \in I} \in \ell_{r,m_X}(I)$, where $|c| = (|c_i|)_{i \in I} \in \ell_{r,m_X}(I)$ with $\|\, |c|\, \|_{\ell_{r,m_X}} = \|c\|_{\ell_{r,m_X}}$, so that

$$\|\,\mathrm{Synth}_{X,\Psi}\, c\|_{L_{r,m}} \leq \|\,\mathrm{Synth}_{X,U}\, |c|\,\|_{L_{r,m}} \lesssim \|\, |c|\,\|_{\ell_{r,m_X}} = \|c\|_{\ell_{r,m_X}}\,.$$

Thus, it remains to prove the first part of the lemma.

By definition of a relatively separated family, there is $N = N(X, U) > 0$ with $\sum_{i \in I} \chi_{x_i U} \leq N$. On the one hand, this shows that for each $x \in G$ only finitely many terms of the series defining $(\mathrm{Synth}_{X,U}\, c)(x)$ do no vanish; in particular, the defining series is pointwise absolutely convergent. On the other hand, we see

$$\begin{aligned}
\left| \left(\mathrm{Synth}_{X,U}\, c\right)(x) \right|^r &\leq \left(\sum_{i\in I} |c_i|\, \chi_{x_i U}(x)\, \chi_{x_i U}(x) \right)^r \\
&\leq \left(\sup_{j\in I} |c_j|\, \chi_{x_j U}(x) \cdot \sum_{i\in I} \chi_{x_i U}(x) \right)^r \\
&\leq N^r \cdot \sup_{j\in I} |c_j|^r\, \chi_{x_j U}(x) \leq N^r \cdot \sum_{i\in I} |c_i|^r\, \chi_{x_i U}(x)\,.
\end{aligned}$$

Thus,

$$\begin{aligned}
\| \mathrm{Synth}_{X,U}\, c \|_{L_{r,m}}^r &\leq N^r \cdot \int_G (m(x))^r \cdot \sum_{i\in I} |c_i|^r\, \chi_{x_i U}(x)\, dx \\
&\leq N^r \cdot \sum_{i\in I} \left(|c_i|^r \cdot \int_{x_i U} (m(x))^r\, dx \right).
\end{aligned}$$

But for $x = x_i u \in x_i U$, we have $m(x) = m(x_i u) \leq m(x_i) \cdot w(u) \leq C \cdot m(x_i)$ for $C := \sup_{u\in U} w(u)$, which is finite since U is compact and w is continuous. Overall, since $|x_i U| = |U|$ for all $i \in I$, where $|U|$ is the Haar measure of U, we see

$$\| \mathrm{Synth}_{X,U}\, c \|_{L_{r,m}}^r \leq N^r \cdot C^r \cdot |U| \cdot \sum_{i\in I} \left(m(x_i) \cdot |c_i| \right)^r,$$

which easily yields the boundedness of $\mathrm{Synth}_{X,U}$. □

4.5.2 *Banach Frames*

In this subsection, we will assume the following:

Assumption 4.3 We fix some $r \in (1, \infty)$ and a w-moderate weight $m : G \to (0, \infty)$ and assume that the kernel K from (4.2.3) satisfies the following:

- (i) Assumption 4.1 is satisfied, that is, $K \in L_{p,w}(G)$ for all $p \in (1, \infty)$.
- (ii) The right convolution operator $RC_K : f \mapsto f * K$ is well-defined, and by Proposition 4.2 bounded, as an operator on $L_{r,m}(G)$.
- (iii) There is some unit neighborhood $U_0 \subset G$ such that for each unit neighborhood $U \subset U_0$ there is a constant $C_U > 0$ with

$$\text{for all } f \in \mathscr{M}_{r,m} : \quad \| \mathrm{osc}_U^{\rho} f \|_{L_{r,m}} \leq C_U \cdot \| f \|_{L_{r,m}}\,. \tag{4.5.1}$$

Here, $\mathcal{M}_{r,m}$ is the reproducing kernel space from (4.3.4), and

$$\operatorname{osc}_U^{\rho} f(x) := \sup_{u \in U} |f(xu) - f(x)| \tag{4.5.2}$$

similar to (4.3.8).

(iv) The constants C_U from the preceding point satisfy $C_U \to 0$ as $U \to \{e\}$. More precisely, for every $\varepsilon > 0$ there is a unit neighborhood $U_\varepsilon \subset U_0$ with $C_U \leq \varepsilon$ for all unit neighborhoods $U \subset U_\varepsilon$.

At a first glance, it seems that the preceding assumptions have nothing to do with the existence of a "well-behaved" kernel W which is compatible with the kernel K. But it turns out that the existence of such a kernel provides an easy way of verifying the preceding assumptions:

Lemma 4.15 *Assume that $K \in L_{p,w}(G)$ for all $p \in (1, \infty)$ and that the operator $RC_K : L_{r,m}(G) \to L_{r,m}(G)$, $f \mapsto f * K$ is well-defined and bounded.*

Furthermore assume that there is a kernel $W : G \to \mathbb{C}$ with the following properties:

(i) W is continuous.

(ii) $K * W = K$.

(iii) $M_{U_0}^{\lambda} W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ for some compact unit neighborhood $U_0 \subset G$. Here

$$M_{U_0}^{\lambda} W(x) := \|W\|_{L_\infty(xU_0)}, \qquad x \in G \tag{4.5.3}$$

is the local maximal function (with respect to left-regular representation)*, similar to (4.3.6).*

Then Assumption 4.3 is satisfied.

Proof We first note that our assumptions imply $M_U^{\lambda} W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ for *every* compact unit neighborhood $U \subset G$. Indeed, by compactness, and since $U \subset \bigcup_{x \in G} x \operatorname{int}(U_0)$, there is a finite family $(x_i)_{i=1,\ldots,n}$ with $U \subset \bigcup_{i=1}^n x_i U_0$. Therefore, $xU \subset \bigcup_{i=1}^n x x_i U_0$, whence

$$\begin{aligned} M_U^{\lambda} W(x) = \|W\|_{L_\infty(xU)} &\leq \sum_{i=1}^{n} \|W\|_{L_\infty(xx_iU_0)} \\ &= \sum_{i=1}^{n} M_{U_0}^{\lambda} W(xx_i) = \sum_{i=1}^{n} \left[\rho(x_i)(M_{U_0}^{\lambda} W)\right](x) . \end{aligned}$$

But since w and $w\Delta^{-1}$ are submultiplicative, both $L_{1,w}(G)$ and $L_{1,w\Delta^{-1}}(G)$ are invariant under right translations, and hence $M_U^{\lambda} W \in L_{1,w}(G) \cap L_{1,\underline{w\Delta_{-1}}}(G)$.

Next, if V is an *open* precompact unit neighborhood and $U := \overline{V}$, then by continuity of W, we have $|W(x)| \leq \sup_{v \in V} |W(xv)| = \|W\|_{L_\infty(xV)} \leq M_U^{\lambda} W(x)$ for all $x \in G$. Therefore, $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$.

Since by assumption, the right convolution operator RC_K acts boundedly on $L_{r,m}(G)$, Lemma 4.4 shows that the set $X_0 := \operatorname{span}\{\lambda(x)K\}_{x\in G}$ is dense in the reproducing kernel space $\mathscr{M}_{r,m}$. Furthermore, the assumption $K * W = K$ yields $(\lambda(x)K) * W = \lambda(x)(K * W) = \lambda(x)K$ for all $x \in G$, and thus $f * W = f$ for all $f \in X_0$. By density of X_0 in $\mathscr{M}_{r,m}$, and since the right convolution operator $f \mapsto f * W$ is continuous on $L_{r,m}(G)$ thanks to $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ and Young's inequality (Proposition 4.13), we see

$$f * W = f \quad \text{for all} \quad f \in \mathscr{M}_{r,m} . \tag{4.5.4}$$

We now use (4.5.4) to prove (4.5.1). To this end, let $U \subset G$ be an arbitrary compact unit neighborhood. Let $f \in \mathscr{M}_{r,m}$, $x \in G$ and $u \in U$ be arbitrary. Then

$$\begin{aligned} |f(xu) - f(x)| &= |(f * W)(xu) - (f * W)(x)| \\ &\leq \int_G |f(y)| \cdot |W(y^{-1}xu) - W(y^{-1}x)| \, dy \\ &\leq \int_G |f(y)| \cdot (\operatorname{osc}_U^\rho W)(y^{-1}x) \, dy = \big(|f| * (\operatorname{osc}_U^\rho W)\big)(x) . \end{aligned}$$

Since this holds for every $u \in U$, we get $\operatorname{osc}_U^\rho f(x) \leq |f| * (\operatorname{osc}_U^\rho W)(x)$ for all $x \in G$. By solidity of $L_{r,m}(G)$ and in view of Young's inequality (Proposition 4.13), this implies

$$\| \operatorname{osc}_U^\rho f \|_{L_{r,m}} \leq \| f \|_{L_{r,m}} \cdot \max\{\| \operatorname{osc}_U^\rho W \|_{L_{1,w}}, \| \operatorname{osc}_U^\rho W \|_{L_{1,w\Delta^{-1}}}\} .$$

But an easy generalization of Lemma 4.3 shows that $\| \operatorname{osc}_U^\rho W \|_{L_{1,v}} \to 0$ as $U \to \{e\}$, for $v = w$ as well as for $v = w\Delta^{-1}$. From this, it is not hard to see that the two remaining properties from Assumption 4.3 are satisfied. □

We now prove that Assumption 4.3 ensures that a sufficiently fine sampling of the continuous frame $(\pi(x)\, u)_{x\in G}$ provides a Banach frame for the coorbit space $\operatorname{Co}(L_{r,m})$. For this, we will first show that we can sample the point evaluation functionals to obtain a Banach frame for the reproducing kernel space $\mathscr{M}_{r,m}$. In the end, we will then use the correspondence principle to transfer the result from the reproducing kernel space to the coorbit space.

We begin by showing that a certain sampling operator is bounded:

Lemma 4.16 *Let Assumption 4.3 be satisfied, and let $X = (x_i)_{i\in I}$ be a relatively separated family in G.*

Then, with the weight m_X as in Lemma 4.14, the sampling operator

$$\operatorname{Samp}_X : \mathscr{M}_{r,m} \to \ell_{r,m_X}(I),\ f \mapsto (f(x_i))_{i\in I} = \big(\langle f, \lambda_{x_i} K\rangle_{L_2}\big)_{i\in I}$$

is well-defined and bounded.

Proof We first recall that each $f \in \mathscr{M}_{r,m}$ satisfies $f = f * K$, and hence

$$f(x) = f * K(x) = \int_G f(y) \cdot K(y^{-1}x)\, dy = \int_G f(y) \cdot \overline{K(x^{-1}y)}\, dy = \langle f, \lambda(x)K \rangle_{L_2} .$$

But $K \in L_{r',w}(G)$, and thus also $\lambda(x)K \in L_{r',w}(G)$, since w is submultiplicative. Furthermore, since m is w-moderate, we have $m(e) = m(xx^{-1}) \le m(x)w(x^{-1})$, and thus $[m(x)]^{-1} \le w(x^{-1})/m(e) = w(x)/m(e)$, whence $L_{r',w}(G) \hookrightarrow L_{r',m^{-1}}(G)$. Thus, the dual pairing $\langle f, \lambda(x)K \rangle_{L_2} \in \mathbb{C}$ is well-defined for every $x \in \mathbb{R}$. Therefore, each entry $f(x_i)$ of the sequence $\mathrm{Samp}_X\ f = (f(x_i))_{i \in I}$ makes sense.

Now, let U be a compact unit neighborhood with $\| \operatorname{osc}_U^\rho f \|_{L_{r,m}} \le C \cdot \|f\|_{L_{r,m}}$ for all $f \in \mathscr{M}_{r,m}$. Such a neighborhood exists by virtue of Assumption 4.3. Note that U^{-1} is also a compact unit neighborhood, so that by definition of a relatively separated family there is a constant $N > 0$ with $\sum_{i \in I} \chi_{x_i U^{-1}}(x) \le N$ for all $x \in G$.

Next, fix any $i \in I$ and note that $\chi_{x_i U^{-1}}(x) \neq 0$ can only hold if $x = x_i u^{-1}$ and thus $x_i = xu$ for some $u \in U$. But in this case, we see by definition of the oscillation $\operatorname{osc}_U^\rho f$ that

$$|f(x_i)| \le |f(x)| + |f(x_i) - f(x)| \le |f(x)| + (\operatorname{osc}_U^\rho f)(x) =: F(x) .$$

We have thus shown $|f(x_i)| \cdot \chi_{x_i U^{-1}}(x) \le F(x) \cdot \chi_{x_i U^{-1}}(x)$ for all $x \in G$. Summing over $i \in I$, we see

$$\Theta_f(x) := \sum_{i \in I} |f(x_i)| \cdot \chi_{x_i U^{-1}}(x) \le \left(\sum_{i \in I} \chi_{x_i U^{-1}}(x) \right) \cdot F(x) \le N \cdot F(x)$$

for all $x \in G$.

Because of $r \ge 1$, we have $\ell_1(I) \hookrightarrow \ell_r(I)$, which implies $\sum_{i \in I} c_i^r \le (\sum_{i \in I} c_i)^r$ for arbitrary $c_i \ge 0$. Therefore,

$$\begin{aligned} \int_G (m(x))^r \cdot \sum_{i \in I} |f(x_i)|^r \cdot \chi_{x_i U^{-1}}(x)\, dx &\le \|\Theta_f\|_{L_{r,m}}^r \le N^r \cdot \|F\|_{L_{r,m}}^r \\ &\le N^r \cdot \big(\|f\|_{L_{r,m}} + \| \operatorname{osc}_U^\rho f \|_{L_{r,m}} \big)^r \\ &\le N^r \cdot (1 + C)^r \cdot \|f\|_{L_{r,m}}^r . \end{aligned}$$

Finally, if $\chi_{x_i U^{-1}}(x) \neq 0$, then $x = x_i u^{-1}$ for some $u \in U$, and therefore $m(x_i) = m(xu) \le m(x) \cdot w(u) \le C' \cdot m(x)$ for $C' := \sup_{u \in U} w(u)$, which is finite since w is continuous and U is compact.

Overall, we have thus shown

$$\begin{aligned} (C')^{-r} \cdot \sum_{i \in I} (m(x_i))^r \cdot |f(x_i)|^r \cdot |x_i U^{-1}| &\le \int_G (m(x))^r \cdot \sum_{i \in I} |f(x_i)|^r \cdot \chi_{x_i U^{-1}}(x)\, dx \\ &\le N^r \cdot (1 + C)^r \cdot \|f\|_{L_{r,m}}^r , \end{aligned}$$

which—because of $|x_i U^{-1}| = |U^{-1}|$—shows

$$\| \operatorname{Samp}_X f\|_{\ell_{r,m_X}} \leq C' N(1+C) \cdot |U^{-1}|^{-1/r} \cdot \|f\|_{L_{r,m}} \quad \text{for all } f \in \mathscr{M}_{r,m}\,,$$

which finally proves that Samp_X is well-defined and bounded. □

Now we can prove that a sufficiently fine sampling of the point evaluations yields a Banach frame for the reproducing kernel space $\mathscr{M}_{r,m}$.

Proposition 4.9 *Let Assumption 4.3 be satisfied, and let $U \subset U_0^{-1}$ be a compact unit neighborhood such that the constant $C_{U^{-1}}$ from (4.5.1) satisfies*

$$\|RC_K\|_{L_{r,m} \to L_{r,m}} \cdot C_{U^{-1}} < 1\,. \tag{4.5.5}$$

Let $X = (x_i)_{i\in I}$ be any relatively separated family in G for which there exists a U-BUPU $\Psi = (\psi_i)_{i\in I}$ with localizing family X, and let the weight m_X be defined as in Lemma 4.14.

Then there is a bounded linear reconstruction map $R : \ell_{r,m_X}(I) \to \mathscr{M}_{r,m}$ *which satisfies $R \circ \operatorname{Samp}_X = \operatorname{id}_{\mathscr{M}_{r,m}}$ for the sampling map* Samp_X *from Lemma 4.16.*

In other words, the family $(\delta_{x_i})_{i\in I}$ of point evaluations forms a Banach frame *for $\mathscr{M}_{r,m}$ with coefficient space $\ell_{r,m_X}(I)$.*

Remark 4.11 The proof shows that the action of the reconstruction operator is *independent* of the choice of r, m.

In other words, if (4.5.5) is satisfied for $L_{r_1,m_1}(G)$ and $L_{r_2,m_2}(G)$ and if $R_1 : \ell_{r_1,m_{1,X}}(I) \to \mathscr{M}_{r_1,m_1}$ and $R_2 : \ell_{r_2,m_{2,X}}(I) \to \mathscr{M}_{r_2,m_2}$ denote the respective reconstruction operators, then $R_1 c = R_2 c$ for all $c \in \ell_{r_1,m_{1,X}}(I) \cap \ell_{r_2,m_{2,X}}(I)$.

Proof With the synthesis operator $\operatorname{Synth}_{X,\Psi}$ from Lemma 4.14, we define

$$B := \operatorname{Synth}_{X,\Psi} \circ \operatorname{Samp}_X : \mathscr{M}_{r,m} \to L_{r,m}(G)\,.$$

Because of $f(x) = \sum_{i\in I} \psi_i(x) f(x)$, we have

$$|f(x) - Bf(x)| \leq \sum_{i\in I} \psi_i(x) \cdot |f(x) - f(x_i)|\,.$$

But if $\psi_i(x) \neq 0$, then $x = x_i u \in x_i U$, so that $x_i = xu^{-1} \in xU^{-1}$, and hence $|f(x) - f(x_i)| = |f(x) - f(xu^{-1})| \leq \operatorname{osc}^{\rho}_{U^{-1}} f(x)$. Therefore,

$$|f(x) - Bf(x)| \leq \sum_{i\in I} \psi_i(x) \operatorname{osc}^{\rho}_{U^{-1}} f(x) = \operatorname{osc}^{\rho}_{U^{-1}} f(x)\,.$$

By Proposition 4.3 the operator RC_K is a projection onto $\mathscr{M}_{r,m}$, therefore $RC_K f = f$ for $f \in \mathscr{M}_{r,m}$. Thus, the operator $A := RC_K \circ B : \mathscr{M}_{r,m} \to \mathscr{M}_{r,m}$ is well-defined and bounded, and we have

$$\begin{aligned}\|f - Af\|_{L_{r,m}} = \|RC_K(f - Bf)\|_{L_{r,m}} &\le \|RC_K\|_{L_{r,m}\to L_{r,m}} \cdot \|f - Bf\|_{L_{r,m}} \\ &\le \|RC_K\|_{L_{r,m}\to L_{r,m}} \cdot \|\operatorname{osc}^{\rho}_{U^{-1}} f\|_{L_{r,m}} \\ &\le \|RC_K\|_{L_{r,m}\to L_{r,m}} \cdot C_{U^{-1}} \cdot \|f\|_{L_{r,m}}\end{aligned}$$

for all $f \in \mathscr{M}_{r,m}$.

In view of (4.5.5), a Neumann series argument (see [2, Sect. 5.7]) shows that the bounded linear operator $R_0 := \sum_{n=0}^{\infty}(\mathrm{id}_{\mathscr{M}_{r,m}} - A)^n : \mathscr{M}_{r,m} \to \mathscr{M}_{r,m}$ satisfies

$$(R_0 \circ RC_K \circ \mathrm{Synth}_{X,\Psi}) \circ \mathrm{Samp}_X = R_0 \circ A = \mathrm{id}_{\mathscr{M}_{r,m}} .$$

Thus, $R := R_0 \circ RC_K \circ \mathrm{Synth}_{X,\Psi} : \ell_{r,m_X}(I) \to \mathscr{M}_{r,m}$ is the desired reconstruction operator. Note that the action of this operator on a given sequence is independent of the choice of r, m, since the action of the operators RC_K, $\mathrm{Synth}_{X,\Psi}$ and $A = RC_K \circ \mathrm{Synth}_{X,\Psi} \circ \mathrm{Samp}_X$ is independent of r, m, so that the same holds for $R_0 = \sum_{n=0}^{\infty}(\mathrm{id} - A)^n$. □

Using the correspondence principle, we can finally lift the result from the reproducing kernel space $\mathscr{M}_{r,m}$ to the coorbit space $\mathrm{Co}(L_{r,m})$.

Theorem 4.5 *Under the assumptions of Proposition 4.9, the sampled family* $(\pi(x_i)u)_{i\in I} \subset (\mathrm{Co}(L_{r,m}))'$ *forms a Banach frame for* $\mathrm{Co}(L_{r,m})$ *with coefficient space* $\ell_{r,m_X}(I)$.

More precisely, the sampling operator

$$\mathrm{Samp}_{X,\mathrm{Co}} : \mathrm{Co}(L_{r,m}) \to \ell_{r,m_X}(I),\ f \mapsto \big(V_e f(x_i)\big)_{i\in I} = \big(\langle f, \pi(x_i)u\rangle_{\mathscr{S}_w}\big)_{i\in I}$$

is well-defined and bounded, and there is a bounded linear reconstruction operator $R_{\mathrm{Co}} : \ell_{r,m_X}(I) \to \mathrm{Co}(L_{r,m})$ *satisfying* $R_{\mathrm{Co}} \circ \mathrm{Samp}_{X,\mathrm{Co}} = \mathrm{id}_{\mathrm{Co}(L_{r,m})}$.

Finally, the action of the reconstruction operator R_{Co} *is independent of the choice of* r, m*, that is, if the assumptions of the current theorem are satisfied for* $L_{r_1,m_1}(G)$ *and for* $L_{r_2,m_2}(G)$ *and if* R_1, R_2 *denote the corresponding reconstruction operators, then* $R_1 c = R_2 c$ *for all* $c \in \ell_{r_1,m_{1,X}}(I) \cap \ell_{r_2,m_{2,X}}(I)$.

Proof The correspondence principle (Proposition 4.1) states that the extended voice transform $V_e : \mathrm{Co}(L_{r,m}) \to \mathscr{M}_{r,m}$ is an isometric isomorphism. Now, with the sampling map Samp_X from Proposition 4.9, we have

$$(\mathrm{Samp}_X \circ V_e) f = \big(V_e f(x_i)\big)_{i\in I} = \big(\langle f, \pi(x_i)u\rangle_{\mathscr{S}_w}\big)_{i\in I} = \mathrm{Samp}_{X,\mathrm{Co}} f .$$

Thus, the sampling operator $\mathrm{Samp}_{X,\mathrm{Co}} = \mathrm{Samp}_X \circ V_e : \mathrm{Co}(L_{r,m}) \to \ell_{r,m_X}(I)$ is indeed well-defined and bounded.

Now, with the reconstruction operator $R : \ell_{r,m_X}(I) \to \mathscr{M}_{r,m}$ from Proposition 4.9, define $R_{\mathrm{Co}} := V_e^{-1} \circ R : \ell_{r,m_X}(I) \to \mathrm{Co}(L_{r,m})$. Then

$$R_{\mathrm{Co}} \circ \mathrm{Samp}_{X,\mathrm{Co}} = V_e^{-1} \circ R \circ \mathrm{Samp}_X \circ V_e = V_e^{-1} \circ V_e = \mathrm{id}_{\mathrm{Co}(L_{r,m})},$$

as desired. Since the action of R is independent of the choice of r, m, so is the action of R_{Co}. □

4.5.3 Atomic Decompositions

For the case of atomic decompositions, we will impose slightly different conditions compared to the case of Banach frames. In this case, our assumptions immediately refer to a "well-behaved" kernel W.

Assumption 4.4 We fix some $r \in (1, \infty)$ and some w-moderate weight $m : G \to (0, \infty)$, and we assume that the kernel K from (4.2.3) satisfies the following:

(i) Assumption 4.1 is satisfied, that is, $K \in L_{p,w}(G)$ for all $p \in (1, \infty)$.
(ii) The right convolution operator $RC_K : f \mapsto f * K$ is well-defined and bounded as an operator on $L_{r,m}(G)$.
(iii) There is a continuous kernel $W : G \to \mathbb{C}$ with the following properties:
   (a) $W * K = K$.
   (b) $\widetilde{M}^{\rho}_{Q} W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ for some compact unit neighborhood $Q \subset G$. Here, the maximal function $\widetilde{M}^{\rho}_{Q} W$ is defined as in (4.3.6).

Remark 4.12 We will use below that $\widetilde{M}^{\rho}_{U} W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ for *every* compact unit neighborhood $U \subset G$ if we assume $\widetilde{M}^{\rho}_{Q} W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ for *some* unit neighborhood $Q \subset G$.

Indeed, by compactness of U, and since $U \subset \bigcup_{x \in G} (\operatorname{int} Q)x$, there is a finite family $(x_i)_{i=1,\ldots,n}$ in G with $U \subset \bigcup_{i=1}^{n} Q x_i$. Therefore, $Ux \subset \bigcup_{i=1}^{n} Q x_i x$, whence

$$\widetilde{M}^{\rho}_{U} W(x) = \|W\|_{L_\infty(Ux)} \leq \sum_{i=1}^{n} \|W\|_{L_\infty(Qx_i x)} = \sum_{i=1}^{n} (\widetilde{M}^{\rho}_{Q} W)(x_i x)\,.$$

By solidity and (left) translation invariance of $L_{1,v}(G)$ for $v = w$ or $v = w\Delta^{-1}$, this implies

$$\|\widetilde{M}^{\rho}_{U} W\|_{L_{1,v}} \leq \sum_{i=1}^{n} \|\lambda(x_i^{-1})(\widetilde{M}^{\rho}_{Q} W)\|_{L_{1,v}} < \infty\,.$$

Here, the left-translation invariance of $L_{1,v}(G)$ is a consequence of the submultiplicativity of v.

As in the preceding subsection, our first goal is to show that certain synthesis and analysis operators are bounded.

Lemma 4.17 *Let Assumption 4.4 be satisfied, and let $X = (x_i)_{i \in I}$ be any relatively separated family in G. Let the weight m_X be as in Lemma 4.14. Then the following hold:*

(i) *If $\Psi = (\psi_i)_{i\in I}$ is a U-BUPU with localizing family X, then the analysis operator*

$$\mathrm{Ana}_{X,\Psi} : L_{r,m}(G) \to \ell_{r,m_X}(I),\ f \mapsto \big(\langle f, \psi_i\rangle_{L_2}\big)_{i\in I}$$

is a well-defined bounded linear map.

(ii) *The synthesis map*

$$\mathrm{Synth}_{X,W} : \ell_{r,m_X}(I) \to L_{r,m}(G),\ (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \cdot \lambda(x_i)W$$

is a well-defined bounded linear map, where the defining series is almost everywhere absolutely convergent.

Proof For $x = x_i u \in x_i U$, we have $m(x_i) = m(xu^{-1}) \le m(x)w(u^{-1}) \le C \cdot m(x)$, where $C := \sup_{u\in U} w(u^{-1})$ is finite by continuity of w and compactness of U. Since we also have $\psi_i \equiv 0$ on $G\backslash x_i U$, then we see by following the lines of the proof of Proposition 4.4 and using Jensen's inequality, see [8, Theorem 10.2.6]:

$$\begin{aligned}
(m(x_i))^r \cdot |\langle f, \psi_i\rangle_{L_2}|^r &\le |x_i U|^r \cdot \left(\int_{x_i U} |f(x)| m(x_i)\psi_i(x)\frac{dx}{|x_i U|}\right)^r \\
&\le |x_i U|^r \cdot \int_{x_i U} (|f(x)| \cdot C \cdot m(x)\psi_i(x))^r \frac{dx}{|x_i U|} \\
&\le |U|^{r-1} \cdot C^r \cdot \int_G |(m \cdot f)(x)|^r \cdot \psi_i(x)\, dx\,,
\end{aligned}$$

where the last step used the left invariance of the Haar measure and the estimate $(\psi_i(x))^r \le \psi_i(x)$ which holds since $\psi_i(x) \in [0, 1]$ and $r \ge 1$.

Summing over $i \in I$ and applying the monotone convergence theorem, we thus get

$$\begin{aligned}
\|\mathrm{Ana}_{X,\Psi} f\|^r_{\ell_{r,m_X}} &= \sum_{i\in I} \big(m(x_i) \cdot |\langle f, \psi_i\rangle_{L_2}|\big)^r \\
&\le |U|^{r-1} \cdot C^r \cdot \int_G |(m \cdot f)(x)|^r \cdot \sum_{i\in I} \psi_i(x)\, dx \\
&= |U|^{r-1} \cdot C^r \cdot \|f\|^r_{L_{r,m}} < \infty\,,
\end{aligned}$$

thereby proving the boundedness and well-definedness of $\mathrm{Ana}_{X,\Psi}$.

We now consider the synthesis map $\mathrm{Synth}_{X,W}$. Let $V \subset G$ be any compact unit neighborhood, and set $Q := \mathrm{int}\, V$, so that $U := \overline{Q} \subset V$ is a compact unit neighborhood that satisfies $\overline{\mathrm{int}\, U} \supset \overline{Q} = U$ and hence $U = \overline{\mathrm{int}\, U}$. As a consequence, as seen in the proof of Lemma 4.3 (see p. 138), we have $\|W\|_{L_\infty(Ux)} = \sup_{y\in Ux} |W(y)|$ for all $x \in G$. Here we used that W is continuous.

Now, let $x \in G$ and $i \in I$ be arbitrary. For any $y = x_i u \in x_i U$, we then have $x_i^{-1} x = (yu^{-1})^{-1} x = u y^{-1} x \in U y^{-1} x$, and thus

$$|W(x_i^{-1} x)| \leq \|W\|_{L_\infty(U y^{-1} x)} = (\breve{M}_U^\rho W)(y^{-1} x) \quad \text{for all } x \in G \text{ and } y \in x_i U\,.$$

Writing $\Theta := \breve{M}_U^\rho W$, and averaging this estimate over $y \in x_i U$, we get

$$|\lambda_{x_i} W(x)| \leq \frac{1}{|U|} \int_G \chi_{x_i U}(y) \cdot \Theta(y^{-1} x)\, dy \quad \text{for all } x \in G,\ i \in I\,. \tag{4.5.6}$$

Now, let $c = (c_i)_{i \in I} \in \ell_{r, m_X}(I)$ be arbitrary, and set $\Upsilon := \sum_{i \in I} |c_i| \cdot \chi_{x_i U}$. With the notation introduced in Lemma 4.14, we get $\Upsilon = \mathrm{Synth}_{X,U} |c|$ with $|c| = (|c_i|)_{i \in I}$. This easily implies $\Upsilon \in L_{r,m}(G)$ with

$$\|\Upsilon\|_{L_{r,m}} \leq C \cdot \|c\|_{\ell_{r, m_X}} \tag{4.5.7}$$

for a constant $C = C(m, X, U, r)$ independent of c.

By weighting estimate (4.5.6) with $|c_i|$ and summing over $i \in I$, and by invoking the monotone convergence theorem, we see for all $x \in G$ that

$$\sum_{i \in I} |c_i| \cdot |(\lambda(x_i) W)(x)| \leq \frac{1}{|U|} \cdot \int_G \Upsilon(y) \cdot \Theta(y^{-1} x)\, dy = \frac{1}{|U|} \cdot (\Upsilon * \Theta)(x)\,.$$

But since $\Theta = \breve{M}_U^\rho W \in L_{1,w}(G) \cap L_{1, w\Delta^{-1}}(G)$ (see Assumption 4.4 and Remark 4.12) and since $\Upsilon \in L_{r,m}(G)$, Young's inequality (Proposition 4.13) shows $\Upsilon * \Theta \in L_{r,m}(G)$. In particular, $\Upsilon * \Theta(x) < \infty$ almost everywhere. Therefore, we already see that the series defining $\mathrm{Synth}_{X,W}\, c$ is almost everywhere absolutely convergent. Finally, we also see

$$\begin{aligned} \|\mathrm{Synth}_{X,W}\, c\|_{L_{r,m}} &\leq \Big\| \sum_{i \in I} |c_i| \cdot |\lambda(x_i) W| \Big\|_{L_{r,m}} \leq \frac{1}{|U|} \cdot \|\Upsilon * \Theta\|_{L_{r,m}} \\ &\leq \frac{1}{|U|} \cdot \max\{\|\Theta\|_{L_{1,w}}, \|\Theta\|_{L_{1,w\Delta^{-1}}}\} \cdot \|\Upsilon\|_{L_{r,m}}\,. \end{aligned}$$

In view of (4.5.7), this proves the boundedness and well-definedness of $\mathrm{Synth}_{X,W}$. □

Now we can prove the desired atomic decomposition result:

Proposition 4.10 *Let Assumption 4.4 be satisfied. For each compact unit neighborhood $U \subset G$ write*

$$C_U := \max\{\|\operatorname{osc}_U W\|_{L_{1,w}}, \|\operatorname{osc}_U W\|_{L_{1,w\Delta^{-1}}}\}. \tag{4.5.8}$$

Assume that

$$C_U \cdot \|RC_K\|_{L_{r,m}\to L_{r,m}} < 1\,. \tag{4.5.9}$$

Finally, let $X = (x_i)_{i\in I}$ *be a relatively separated family for which there exists a* U*-BUPU* $\Psi = (\psi_i)_{i\in I}$ *with localizing family* X*, and let the weight* m_X *be as defined in Lemma 4.14.*

Then the family $(\lambda(x_i)K)_{i\in I}$ *forms a family of atoms for* $\mathscr{M}_{r,m}$ *with associated sequence space* $\ell_{r,m_X}(I)$*. This means:*

(i) The synthesis operator

$$\mathrm{Synth}_{X,K} : \ell_{r,m_X}(I) \to \mathscr{M}_{r,m},\ (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \cdot \lambda(x_i)K$$

is well-defined and bounded, with unconditional convergence of the defining series. This even holds without assuming (4.5.9).

(ii) There is a bounded coefficient operator

$$C : \mathscr{M}_{r,m} \to \ell_{r,m_X}(I) \quad \textit{with} \quad \mathrm{Synth}_{X,K}\circ C = \mathrm{id}_{\mathscr{M}_{r,m}}\,.$$

Remark 4.13 (i) We note that condition (4.5.9) is always satisfied for U small enough, thanks to Lemma 4.3 and Assumption 4.4.

(ii) As in Proposition 4.9, the action of the coefficient operator C is independent of the choice of r, m, that is, if condition (4.5.9) is satisfied for $L_{r_1,m_1}(G)$ and $L_{r_2,m_2}(G)$ and if $C_1 : \mathscr{M}_{r_1,m_1} \to \ell_{r_1,m_{1,X}}(I)$ and $C_2 : \mathscr{M}_{r_2,m_2} \to \ell_{r_2,m_{2,X}}(I)$ are the respective coefficient operators, then $C_1 f = C_2 f$ for all $f \in \mathscr{M}_{r_1,m_1} \cap \mathscr{M}_{r_2,m_2}$.

Proof **Step 1** (**Boundedness of the synthesis operator**): For this step, we will *not* use condition (4.5.9). By Assumption 4.4, $RC_K : L_{r,m}(G) \to L_{r,m}(G)$ is bounded, and we have $W * K = K$, which implies $(\lambda(x)W) * K = \lambda(x)(W * K) = \lambda(x)K$ for all $x \in G$.

Furthermore, Lemma 4.17 shows that the map

$$\mathrm{Synth}_{X,W} : \ell_{r,m_X}(I) \to L_{r,m}(G),\ (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \cdot \lambda(x_i)W$$

is well-defined and bounded. Because of $r < \infty$, each $c = (c_i)_{i\in I} \in \ell_{r,m_X}(I)$ satisfies $c = \sum_{i\in I} c_i\delta_i$ with unconditional convergence in $\ell_{r,m_X}(I)$, where $(\delta_i)_{i\in I}$ denotes the standard basis of $\ell_{r,m_X}(I)$. This implies that the series defining $\mathrm{Synth}_{X,W}\, c$ converges unconditionally in $L_{r,m}(G)$. Since bounded operators preserve unconditional convergence, we see

$$RC_K\big(\mathrm{Synth}_{X,W}\, c\big) = RC_K\left(\sum_{i\in I} c_i\, \lambda(x_i) W\right)$$
$$= \sum_{i\in I} c_i\, [(\lambda(x_i)W) * K] = \sum_{i\in I} c_i\, \lambda(x_i) K$$

with unconditional convergence of the series. We have thus shown that $\mathrm{Synth}_{X,K} = RC_K \circ \mathrm{Synth}_{X,W} : \ell_{r,m_X}(I) \to L_{r,m}(G)$ is well-defined and bounded, with unconditional convergence of the defining series.

Since $\lambda(x_i)K \in \mathscr{M}_{r,m}$ for all $i \in I$, we also see that the range of $\mathrm{Synth}_{X,K}$ is contained in the closed subspace $\mathscr{M}_{r,m} \subset L_{r,m}(G)$.

Step 2 (**An alternative reproducing formula for** $\mathscr{M}_{r,m}$): In this step, we will prove

$$f = (f * W) * K \qquad \text{for all } f \in \mathscr{M}_{r,m}\,. \tag{4.5.10}$$

This is almost trivial: For $f \in \mathscr{M}_{r,m}$, we have $f = f * K$ by definition of $\mathscr{M}_{r,m}$, and we have $K = W * K$ by Assumption 4.4. By combining these facts, we get $f = f * K = f * (W * K)$. Thus, all we need to verify is that the convolution is associative in the setting that we consider here.

In light of [5, Lemma 6.3] to prove this, it remains to show $((|f| * |W|) * |K|)(x) < \infty$ for almost all $x \in G$. To this end, we first show $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$. In order to see this, let $V \subset G$ be any compact unit neighborhood, and set $Q := \operatorname{int} V$, so that $U := \overline{Q} \subset V$ is a compact unit neighborhood that satisfies $\operatorname{int} U \supset \overline{Q} = U$ and hence $U = \overline{\operatorname{int} U}$. As a consequence of this and of the continuity of W, as seen in the proof of Lemma 4.3 (see p. 138), we have

$$\widetilde{M}_U^{\rho} W(x) = \|W\|_{L_\infty(Ux)} = \sup_{y\in Ux} |W(y)| \geq |W(x)| \quad \text{for all } x \in G\,.$$

Since $\widetilde{M}_U^{\rho} W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$ (see Assumption 4.4 and Remark 4.12), we see $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$.

Now, fix some $s \in (r, \infty)$ and let $f \in \mathscr{M}_{r,m}$. Because of $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$, Proposition 4.13 shows $|f| * |W| \in L_{r,m}(G)$. Therefore, by the second part of Proposition 4.13, we see $(|f| * |W|) * |K| \in L_{s,m}(G)$, since $|K(x^{-1})| = |K(x)|$ and since $K \in L_{p,w}(G)$ for all $p \in (1, \infty)$. In particular, $((|f| * |W|) * |K|)(x) < \infty$ for almost all $x \in G$. By the considerations from above, we thus see that (4.5.10) holds.

Step 3 (**Approximating** $f \mapsto f * W$): Let $\mathrm{Ana}_{X,\Psi} : L_{r,m}(G) \to \ell_{r,m_X}(I)$ and $\mathrm{Synth}_{X,W} : \ell_{r,m_X}(I) \to L_{r,m}(G)$ be as defined in Lemma 4.17, and define $A := \mathrm{Synth}_{X,W} \circ \mathrm{Ana}_{X,\Psi} : L_{r,m}(G) \to L_{r,m}(G)$. In this step, we will show

$$\|f * W - Af\|_{L_{r,m}} \leq C_U \cdot \|f\|_{L_{r,m}} \qquad \text{for all } f \in L_{r,m}(G)\,, \tag{4.5.11}$$

with C_U as in (4.5.8).

To this end, recall from the previous step that $W \in L_{1,w}(G) \cap L_{1,w\Delta^{-1}}(G)$, so that Young's inequality (Proposition 4.13) shows $|f| * |W| \in L_{r,m}(G)$ for $f \in L_{r,m}(G)$.

In particular, this implies $|f| * |W|(x) < \infty$ for almost all $x \in G$. For each such $x \in G$, the dominated convergence theorem and the definition of the operators $\mathrm{Ana}_{X,\Psi}$, $\mathrm{Synth}_{X,W}$ and A shows

$$\begin{aligned} |f * W(x) - Af(x)| &= \left| \sum_{i\in I} \int_G f(y)\psi_i(y) W(y^{-1}x)\, dy - \sum_{i\in I} \langle f, \psi_i\rangle_{L_2} (\lambda(x_i)W)(x) \right| \\ &\leq \sum_{i\in I} \int_G \psi_i(y) \cdot |f(y)| \cdot |W(y^{-1}x) - W(x_i^{-1}x)|\, dy\,. \end{aligned}$$

Fix $i \in I$ for the moment. For $y \in G$ with $\psi_i(y) \neq 0$, we have $y = x_i u \in x_i U$, and thus $x_i^{-1}x = uy^{-1}x \in Uy^{-1}x$. Therefore, $|W(y^{-1}x) - W(x_i^{-1}x)| \leq (\mathrm{osc}_U\, W)\,(y^{-1}x)$, by definition of the oscillation $\mathrm{osc}_U\, W$ (see 4.3.8).

If we combine this with the estimate from above and with the monotone convergence theorem, we get

$$\begin{aligned} |f * W(x) - Af(x)| &\leq \sum_{i\in I} \int_G \psi_i(y) \cdot |f(y)| \cdot (\mathrm{osc}_U\, W)(y^{-1}x)\, dy \\ &= \int_G |f(y)| \cdot (\mathrm{osc}_U\, W)(y^{-1}x)\, dy = (|f| * \mathrm{osc}_U\, W)(x)\,. \end{aligned}$$

In view of Young's inequality (Proposition 4.13) and the definition of C_U (see 4.5.8), we see that (4.5.11) holds true.

Step 4 (**Completing the proof**): Recall that $RC_K : L_{r,m}(G) \to \mathscr{M}_{r,m}$ is bounded by Assumption 4.4. Thus, $B := RC_K \circ A|_{\mathscr{M}_{r,m}} : \mathscr{M}_{r,m} \to \mathscr{M}_{r,m}$ is well-defined and bounded, with A as in the preceding step. Now, for arbitrary $f \in \mathscr{M}_{r,m}$ our results from Steps 2 and 3 show

$$\|f - Bf\|_{L_{r,m}} = \|RC_K(f * W - Af)\|_{L_{r,m}} \leq \|RC_K\|_{L_{r,m}\to L_{r,m}} \cdot C_U \cdot \|f\|_{L_{r,m}}\,.$$

In view of our assumption (4.5.9), a Neumann series argument (see [2, Sect. 5.7]) shows that $C_0 := \sum_{n=0}^{\infty} (\mathrm{id}_{\mathscr{M}_{r,m}} - B)^n$ defines a bounded linear operator $C_0 : \mathscr{M}_{r,m} \to \mathscr{M}_{r,m}$ with $B \circ C_0 = \mathrm{id}_{\mathscr{M}_{r,m}}$.

But we saw in Step 1 that $\mathrm{Synth}_{X,K} = RC_K \circ \mathrm{Synth}_{X,W}$, so that

$$\begin{aligned} B = RC_K \circ A|_{\mathscr{M}_{r,m}} &= RC_K \circ \mathrm{Synth}_{X,W} \circ \mathrm{Ana}_{X,\Psi}\,|_{\mathscr{M}_{r,m}} \\ &= \mathrm{Synth}_{X,K} \circ \mathrm{Ana}_{X,\Psi}\,|_{\mathscr{M}_{r,m}}\,. \end{aligned}$$

Thus, the operator $C := \mathrm{Ana}_{X,\Psi}\,|_{\mathscr{M}_{r,m}} \circ C_0 : \mathscr{M}_{r,m} \to \ell_{r,m_X}(I)$ satisfies

$$\mathrm{Synth}_{X,K} \circ C = B \circ C_0 = \mathrm{id}_{\mathscr{M}_{r,m}}\,.$$

It is not hard to see that the action of the coefficient operator C is independent of the choice of r, m. For more details see the end of the proof of Proposition 4.9, where a similar claim is shown. □

Finally, as in the preceding section, we apply the correspondence principle to obtain atomic decomposition results for the coorbit spaces from those for the reproducing kernel spaces.

Theorem 4.6 *Under the assumptions of Proposition 4.10, the sampled family* $(\pi(x_i)u)_{i\in I} \subset \mathrm{Co}(L_{r,m})$ *forms a family of atoms for* $\mathrm{Co}(L_{r,m})$ *with coefficient space* $\ell_{r,m_X}(I)$.

More precisely, the synthesis operator

$$\mathrm{Synth}_{X,\mathrm{Co}} : \ell_{r,m_X}(I) \to \mathrm{Co}(L_{r,m}),\ (c_i)_{i\in I} \mapsto \sum_{i\in I} c_i \cdot \pi(x_i)u$$

is well-defined and bounded, and there is a bounded linear coefficient operator $C_{\mathrm{Co}} : \mathrm{Co}(L_{r,m}) \to \ell_{r,m_X}(I)$ *satisfying* $\mathrm{Synth}_{X,\mathrm{Co}} \circ C_{\mathrm{Co}} = \mathrm{id}_{\mathrm{Co}(L_{r,m})}$.

Finally, the action of the coefficient operator C_{Co} *is independent of the choice of* r, m. *In other words, if the assumptions of the current theorem are satisfied for* L_{r_1,m_1} *and for* L_{r_2,m_2} *and if* C_1, C_2 *denote the corresponding coefficient operators, then* $C_1 f = C_2 f$ *for all* $f \in \mathrm{Co}(L_{r_1,m_1}) \cap \mathrm{Co}(L_{r_2,m_2})$.

Proof The correspondence principle (Proposition 4.1) states that the extended voice transform $V_e : \mathrm{Co}(L_{r,m}) \to \mathscr{M}_{r,m}$ is an isometric isomorphism. Furthermore,

$$\begin{aligned}(V_e\,\pi(x)\,u)(y) &= \langle \pi(x)\,u, \pi(y)\,u\rangle_{\mathscr{S}_w} = \langle \pi(x)\,u, \pi(y)\,u\rangle_{\mathscr{H}} \\ &= K(x^{-1}y) = (\lambda(x)K)(y)\end{aligned}$$

for all $x, y \in G$. In other words, $V_e\,\pi(x)\,u = \lambda(x)K$ for all $x \in G$.

Now, since the bounded linear operator $V_e^{-1} : \mathscr{M}_{r,m} \to \mathrm{Co}(L_{r,m})$ preserves unconditional convergence of series, the synthesis operator $\mathrm{Synth}_{X,K}$ from Proposition 4.10 satisfies

$$\begin{aligned}(V_e^{-1} \circ \mathrm{Synth}_{X,K})\ (c_i)_{i\in I} &= V_e^{-1}\left(\sum_{i\in I} c_i \cdot \lambda(x_i)K\right) = \sum_{i\in I} c_i \cdot V_e^{-1}(\lambda(x_i)K) \\ &= \sum_{i\in I} c_i \cdot \pi(x_i)u = \mathrm{Synth}_{X,\mathrm{Co}}\ (c_i)_{i\in I}\,,\end{aligned}$$

for arbitrary $(c_i)_{i\in I} \in \ell_{r,m_X}(I)$, with unconditional convergence of all involved series. Thus, the operator $\mathrm{Synth}_{X,\mathrm{Co}} = V_e^{-1} \circ \mathrm{Synth}_{X,K} : \ell_{r,m_X}(I) \to \mathrm{Co}(L_{r,m})$ is indeed well-defined and bounded.

Now, with the coefficient operator $C : \mathscr{M}_{r,m} \to \ell_{r,m_X}(I)$ from Proposition 4.10, define $C_{\mathrm{Co}} := C \circ V_e : \mathrm{Co}(L_{r,m}) \to \ell_{r,m_X}(I)$. Then

$$\mathrm{Synth}_{X,\mathrm{Co}} \circ C_{\mathrm{Co}} = V_e^{-1} \circ \mathrm{Synth}_{X,K} \circ C \circ V_e = V_e^{-1} \circ V_e = \mathrm{id}_{\mathrm{Co}(L_{r,m})} ,$$

as desired. Since the action of C is independent of the choice of r, m, so is the action of C_{Co}. □

4.5.4 An Application: Discretization Results for General Paley–Wiener Spaces

In this subsection, we will apply the abstract results from this section to the Paley–Wiener spaces B^p_Ω, thereby improving on the discretization results derived in Sect. 4.3.3. Furthermore, our proofs clearly point out those properties that the set $\Omega \subset \mathbb{R}$ has to satisfy if one wants discretization results to hold for the associated Paley–Wiener spaces:

Assumption 4.5 Let $\Omega \subset \mathbb{R}$ be measurable, and let $r \in (1, \infty)\setminus\{2\}$. Assume that the following properties hold:

(i) Ω is bounded;
(ii) the kernel $K := \mathscr{F}^{-1}\chi_\Omega$ satisfies $K \in \bigcap_{1<p<\infty} L_p(\mathbb{R})$;
(iii) the convolution operator RC_K is well-defined on $L_r(\mathbb{R})$.

Remark 4.14 The last property means that the indicator function χ_Ω is an $L_r(\mathbb{R})$-Fourier multiplier, which implies that $\chi_{\Omega^c} = 1 - \chi_\Omega$ is an $L_r(\mathbb{R})$-Fourier multiplier as well. Therefore, [24, Theorem 1] shows that there is an open set $U \subset \mathbb{R}$ with $\chi_{\Omega^c} = \chi_U$ almost everywhere, and thus $\chi_{U^c} = \chi_\Omega$ almost everywhere. But since Ω is bounded, we have $\Omega \subset [-R, R]$ for some $R > 0$, and then $\chi_\Omega = \chi_\Omega \chi_{[-R,R]} = \chi_{U^c}\chi_{[-R,R]} = \chi_{[-R,R]\setminus U}$ almost everywhere. Thus, by modifying Ω on a null set, we can (and will) always assume that Ω is compact. This neither changes the kernel K, nor the underlying Hilbert space

$$\mathscr{H} := B^2_\Omega := \{f \in L_2(\mathbb{R}) \,:\, \widehat{f} \equiv 0 \text{ a.e. on } \mathbb{R}\setminus\Omega\}\,.$$

As seen in the discussion before Proposition 4.8, if we set $u := K = \mathscr{F}^{-1}\chi_\Omega$, then all standing assumptions from Sect. 4.2 are satisfied for $m = w \equiv 1$, so that the coorbit spaces $\mathrm{Co}(L_p)$ are well-defined for $1 < p < \infty$. Furthermore, we saw before Proposition 4.8 that the associated voice transform satisfies $Vf = f$ for all $f \in \mathscr{H} = B^2_\Omega \subset L_2(\mathbb{R})$. Using this identity, we can now identify the abstractly defined coorbit spaces with more concrete reproducing kernel or Paley–Wiener spaces.

Lemma 4.18 *Setting $\mathscr{T} := \bigcap_{1<p<\infty} L_p(\mathbb{R})$, the space $\mathscr{S}$ from (4.2.6) satisfies*

$$\mathscr{S} = \left\{ f \in \mathscr{T} \;\middle|\; f * K = f \right\} ,$$

with topology generated by the norms $(\|\cdot\|_{L_p})_{1<p<\infty}$.

Furthermore, with $\mathscr{M}_p = \{f \in L_p(\mathbb{R}) \mid f * K = f\}$ *and* $\mathscr{M} := \bigcup_{1<p<\infty} \mathscr{M}_p$, *the map*

$$\iota : \mathscr{M} \to \mathscr{S}', f \mapsto \Phi_f \quad \text{with} \quad \langle \Phi_f \,,\, g \rangle_{\mathscr{S}} := \langle f, g \rangle_{L_2}$$

is a bijection. If we identify each $\varphi \in \mathscr{S}'$ *with its inverse image* $\iota^{-1}\varphi \in \mathscr{M}$ *under this map, then the extended voice transform is the identity map, that is* $V_e\,\varphi = \iota^{-1}\varphi$.

According to the general result, the coorbit spaces $\mathrm{Co}(L_p)$ *are given by*

$$\mathrm{Co}(L_p) = \iota(\mathscr{M}_p) \quad \text{for all } \ p \in (1, \infty)\,, \tag{4.5.12}$$

which means that if we identify φ *with* $\iota^{-1}\varphi$, *then* $\mathrm{Co}(L_p) = \mathscr{M}_p$.

Finally for $p \in (1, 2]$ *we have*

$$\mathscr{M}_p = B_\Omega^p := \left\{ f \in L_p(\mathbb{R}) \,\middle|\, \widehat{f} \equiv 0 \text{ a.e. on } \mathbb{R}\backslash\Omega \right\}.$$

Therefore, up to canonical identifications, the coorbit spaces $\mathrm{Co}(L_p)$ *coincide with the Paley–Wiener spaces* B_Ω^p, *at least for* $p \in (1, 2]$.

Remark 4.15 We do not know if in general the identity $\mathscr{M}_p = B_\Omega^p$ with

$$B_\Omega^p = \left\{ f \in L_p(\mathbb{R}) \,\middle|\, \text{the tempered dist. } \widehat{f} \text{ has } \operatorname{supp}(\widehat{f}) \subset \Omega \right\}$$

also holds for $p \in (2, \infty)$. In case of $\Omega = [-\omega, \omega]$, it was shown in [5, Proposition 4.8] that this is true. Using this, one can show $\mathscr{M}_p = B_\Omega^p$ even if Ω is a finite disjoint union of compact intervals. For more general sets Ω, however, we do not know whether $\mathscr{M}_p = B_\Omega^p$ for $p \in (2, \infty)$.

Proof (*of Lemma* 4.18) The following proof is similar to the proof of [5, Proposition 4.8] with some significant improvements and generalizations.

The first property is an immediate consequence of the definitions, combined with $Vf = f$ for $f \in \mathscr{H}$.

The map ι is indeed well-defined, since if $f \in \mathscr{M}_p$ for some $p \in (1, \infty)$, then $|\langle f, g \rangle_{L_2}| \leq \|f\|_{L_p} \cdot \|g\|_{L_{p'}}$, where $\|\cdot\|_{L_{p'}}$ is a continuous norm on $\mathscr{S}$.

To prove the surjectivity of ι, we first show that $\mathscr{M}$ is a (complex) vector space. Since each $\mathscr{M}_p$ is closed under multiplication with complex numbers, we only need to show that $\mathscr{M}$ is closed under addition. To this end, note for $f \in \mathscr{M}_p$ because of $K \in L_{p'}(\mathbb{R})$ that

$$|f(x)| = |(f * K)(x)| = |\langle f, \lambda(x)K \rangle_{L_2}| \leq \|f\|_{L_p} \cdot \|\lambda(x)K\|_{L_{p'}} \leq C_p \cdot \|f\|_{L_p}$$

for all $x \in \mathbb{R}$, and thus $\mathscr{M}_p \hookrightarrow L_\infty$. This embedding implies $\mathscr{M}_p \subset \mathscr{M}_q$ for $p \leq q$, and thus $\mathscr{M}_p + \mathscr{M}_q \subset \mathscr{M}_q + \mathscr{M}_q = \mathscr{M}_q \subset \mathscr{M}$. From this, we easily see that $\mathscr{M}$ is a vector space.

With $\mathscr{M}$ being a vector space, we see $\mathscr{M} = \operatorname{span} \bigcup_{1<p<\infty} \mathscr{M}_p$. With notation as in (4.2.11), this means $\mathscr{M} = \mathscr{M}^{\mathscr{U}}$. Hence, Theorem 4.2 shows for arbitrary $\varphi \in \mathscr{S}'$ that $f := V_e\varphi \in \mathscr{M}^{\mathscr{U}} = \mathscr{M}$, and (4.2.10) shows because of $Vg = g$ for $g \in \mathscr{S} \subset \mathscr{H}$ that

$$\langle \Phi_f\,,\, g\rangle_{\mathscr{S}} = \langle f, g\rangle_{L_2} = \langle V_e\varphi\,,\, Vg\rangle_{L_2} = \langle \varphi, g\rangle_{\mathscr{S}}\,.$$

Hence, $\varphi = \Phi_f = \iota f = \iota V_e\varphi$. On the one hand, this shows that ι is surjective, and on the other hand—once we know that ι is bijective—it proves that the inverse of ι is given by $\iota^{-1} = V_e : \mathscr{S}' \to \mathscr{M}$.

In order to prove that ι is injective, note $\lambda(x)K \in \mathscr{S}$ for all $x \in \mathbb{R}$ and recall $\overline{K(x)} = K(-x)$. Hence,

$$\langle \Phi_f\,,\, \lambda(x)K\rangle_{\mathscr{S}} = \langle f, \lambda(x)K\rangle_{L_2} = (f * K)(x) = f(x) \quad \text{for} \quad f \in \mathscr{M}\,.$$

Therefore, if $\Phi_f = 0$, then $f = 0$ as well. Since the domain $\mathscr{M}$ of ι is a vector space, this shows that ι is injective.

Equation (4.5.12) is seen to be true by combining the identity $V_e = \iota^{-1}$ with the correspondence principle (see Proposition 4.1), which states that $V_e : \operatorname{Co}(L_p) \to \big\{f \in L_p(\mathbb{R}) \bigm| f * K = f\big\} = \mathscr{M}_p$ is an isomorphism.

To prove $\mathscr{M}_p = B^p_\Omega$ for $p \in (1, 2]$, first note $\mathscr{F}(f * g) = \widehat{f} \cdot \widehat{g}$ for arbitrary $f, g \in L_2$, see e.g. [26, p. 270]. Therefore, for $f \in \mathscr{M}_p \hookrightarrow \mathscr{M}_2$ (here we used that $p \le 2$) we see that $\widehat{f} = \widehat{f * K} = \widehat{f} \cdot \widehat{K} = \widehat{f} \cdot \chi_\Omega$, where the equality holds in the sense of tempered distributions. But since both sides are $L_2(\mathbb{R})$ functions, this implies $\widehat{f} = \widehat{f} \cdot \chi_\Omega$ almost everywhere, and thus $f \in B^p_\Omega$.

Conversely, let $f \in B^p_\Omega$ be arbitrary. Because of $p \le 2$, [27, Theorem in Sect. 1.4.1] shows $f \in L_2(\mathbb{R})$. Furthermore, since $\widehat{f} \equiv 0$ almost everywhere on $\mathbb{R}\setminus\Omega$, we have $\mathscr{F}(f * K) = \widehat{f} \cdot \widehat{K} = \widehat{f} \cdot \chi_\Omega = \widehat{f}$, and thus $f = f * K$, so that $f \in \mathscr{M}_p$. □

With Lemma 4.18 showing that the coorbit spaces $\operatorname{Co}(L_p)$ coincide with the reproducing kernel spaces $\mathscr{M}_p$, we will in the following concentrate on the latter spaces for proving discretization results.

In Sects. 4.5.2 and 4.5.3, we showed that the sampled frame $(\pi(x_i)\,u)_{i\in I}$ forms a Banach frame or an atomic decomposition for the coorbit space $\operatorname{Co}(L_{r,m})$ *if the family of sampling points* $(x_i)_{i\in I}$ *is sufficiently dense in* G. For the case of the Paley–Wiener spaces, one can state quite precisely how dense the sampling points need to be:

Proposition 4.11 *Suppose that Assumption 4.5 is satisfied, and choose* $R > 0$ *and* $\xi_0 \in \mathbb{R}$ *with* $\Omega \subset \xi_0 + [-R, R]$.

Then the family $(\lambda(k/(2R))K)_{k\in\mathbb{Z}}$ *forms a Banach frame and an atomic decomposition for the reproducing kernel space* $\mathscr{M}_r$ *with coefficient space* $\ell_r(\mathbb{Z})$. *More precisely, the operators*

$$\operatorname{Samp} : \mathscr{M}_r \to \ell_r(\mathbb{Z}),\ f \mapsto \big(f(k/(2R))\big)_{k\in\mathbb{Z}} = \big(\langle f, \lambda(k/(2R))K\rangle_{L_2}\big)_{k\in\mathbb{Z}}$$

and

$$\mathrm{Synth} : \ell_r(\mathbb{Z}) \to \mathscr{M}_r, \, (c_k)_{k\in\mathbb{Z}} \mapsto \sum_{k\in\mathbb{Z}} c_k \cdot \lambda(k/(2R))K$$

are well-defined and bounded with $\mathrm{Synth} \circ \mathrm{Samp} = (2R)^{-1} \cdot \mathrm{id}_{\mathscr{M}_r}$.

Proof Since $\Omega \subset \mathbb{R}$ is bounded, there is a Schwartz function $\psi \in \mathbb{S}(\mathbb{R})$ with $\psi \equiv 1$ on Ω. We then have $W := \mathscr{F}^{-1}\psi \in \mathbb{S}(\mathbb{R})$, so that W is continuous. Furthermore,

$$\widehat{W * K} = \widehat{K * W} = \widehat{K} \cdot \widehat{W} = \chi_\Omega \cdot \psi = \chi_\Omega = \widehat{K}\,,$$

and hence $W * K = K * W = K$. Since W is a Schwartz function, there is some $C > 0$ with $|W(x)| \le C \cdot (1 + |x|)^{-2}$ for all $x \in \mathbb{R}$. Because of

$$1 + |x| \le 2 + |x - y| \le 2 \cdot (1 + |x - y|)$$

for any $y \in Q := U_0 := [-1, 1]$, this shows $|W(x + y)| \le 4C \cdot (1 + |x|)^{-2}$, and hence $\widetilde{M}_Q^{\rho} W \in L_1(\mathbb{R})$ and $M_{U_0}^{\lambda} W \in L_1(\mathbb{R})$. Overall, noting that the modular function Δ of the abelian group $G = \mathbb{R}$ satisfies $\Delta \equiv 1$, we see using Lemma 4.15 that Assumptions 4.3 and 4.4 are both satisfied for $w \equiv m \equiv 1$.

Now, define $I := \mathbb{Z}$ and $x_k := k/(2R)$ for $k \in \mathbb{Z}$. It is not hard to see that the family $(x_k)_{k\in\mathbb{Z}}$ is relatively separated in $G = \mathbb{R}$. Therefore, Lemma 4.16 and Proposition 4.10 show that the two operators from the statement of the current proposition are well-defined and bounded. It remains to show $\mathrm{Synth} \circ \mathrm{Samp} = (2R)^{-1}\,\mathrm{id}_{\mathscr{M}_r}$.

For this, it suffices to show $\mathrm{Synth}(\mathrm{Samp}\, f) = (2R)^{-1} \cdot f$ for $f \in \mathscr{M}_r \cap L_2(\mathbb{R})$, since Lemma 4.4 shows that $\mathrm{span}\{\lambda(x)K\}_{x\in\mathbb{R}} \subset \mathscr{M}_r \cap L_2(\mathbb{R})$ is dense in $\mathscr{M}_r$. But it is well-known that the family $(e_k)_{k\in\mathbb{Z}} = \big((2R)^{-1/2} \cdot e^{2\pi i \frac{k}{2R}\cdot}\big)_{k\in\mathbb{Z}}$ forms an orthonormal basis of $L_2(\Omega_0)$ where $\Omega_0 := \xi_0 + [-R, R]$. To make use of this orthonormal basis, first note for $f \in \mathscr{M}_r \cap L_2(\mathbb{R})$ that $\widehat{f} = \widehat{f * K} = \widehat{f} \cdot \widehat{K} = \chi_\Omega \cdot \widehat{f}$. Because of $\widehat{f} = \widehat{f} \cdot \chi_\Omega$, we get $\widehat{f} \equiv 0$ almost everywhere on $\mathbb{R}\setminus\Omega \supset \mathbb{R}\setminus\Omega_0$.

Overall, since $\mathscr{F}(\lambda(k/(2R))K) = e^{-2\pi i \frac{k}{2R}\cdot}\,\chi_\Omega = (2R)^{1/2} \cdot e_{-k} \cdot \chi_\Omega$, we see

$$\begin{aligned}
\widehat{f} = \chi_\Omega \cdot \widehat{f} &= \chi_\Omega \cdot \sum_{k\in\mathbb{Z}} \langle \widehat{f}, e_k\rangle_{L_2}\, e_k \\
&= (2R)^{-1} \cdot \sum_{k\in\mathbb{Z}} \Big\langle \widehat{f}, \mathscr{F}\big(\lambda(-k/(2R))K\big)\Big\rangle_{L_2} \cdot \mathscr{F}(\lambda(-k/(2R))\,K) \\
&= (2R)^{-1} \cdot \mathscr{F}\left(\sum_{\ell\in\mathbb{Z}} \langle f, \lambda(\ell/(2R))K\rangle_{L_2} \cdot \lambda(\ell/(2R))\,K\right) \\
&= (2R)^{-1} \cdot \mathscr{F}\left(\mathrm{Synth}(\mathrm{Samp}\, f)\right)\,,
\end{aligned}$$

which implies $f = (2R)^{-1} \cdot (\mathrm{Synth} \circ \mathrm{Samp}) f$ for all $f \in L_2(\mathbb{R}) \cap \mathscr{M}_r$, as desired. □

To close this section, we show that the existence of a "well-behaved" kernel W with $K * W = K$ is *independent* of the property that K acts boundedly on $L_r(\mathbb{R})$ via right convolutions, even when we restrict to the class of reproducing kernels K which satisfy the weak integrability property $K \in \bigcap_{1<p<\infty} L_p(\mathbb{R})$. In the proof of Proposition 4.11, we saw that for every *bounded* set $\Omega \subset \mathbb{R}$, there is a such a well-behaved kernel W associated to the reproducing kernel $K = \mathcal{F}^{-1}\chi_\Omega$. But the set $C \subset [0, 1]$ that we constructed in Proposition 4.8 is bounded and the associated kernel $K = \mathcal{F}^{-1}\chi_C$ satisfies the weak integrability property. Still, K does not act boundedly via right convolution on *any* $L_p(\mathbb{R})$ space with $p \neq 2$. Conversely, the following proposition shows the existence of a kernel K that acts boundedly via right convolution on all L_p spaces for $1 < p < \infty$, but for which no well-behaved kernel W with $K = W * K$ exists.

Proposition 4.12 *The set* $\Omega := \bigcup_{j=1}^{\infty}[3 \cdot 2^{j-2} + (0, 2^{-2j})]$ *with the associated kernel* $K := \mathcal{F}^{-1}\chi_\Omega$ *has the following properties:*

(i) $K \in \bigcap_{1<p<\infty} L_p(\mathbb{R})$.
(ii) There is no $W \in L_1(\mathbb{R})$ *with* $K = K * W$.
(iii) The operator RC_K *is bounded on* $L_p(\mathbb{R})$ *for every* $p \in (1, \infty)$.

Proof We first verify that the union defining Ω is indeed disjoint. To this end, set $I_j := 3 \cdot 2^{j-2} + (0, 2^{-2j})$ for $j \in \mathbb{N}$, and note $3j - 2 > 0$, so that $2^{3j-2} > 1$, and hence $2^{-2j} < 2^{j-2}$. This implies

$$2^{j-1} = 2 \cdot 2^{j-2} < 3 \cdot 2^{j-2} < 3 \cdot 2^{j-2} + 2^{-2j} < 3 \cdot 2^{j-2} + 2^{j-2} = 2^j . \quad (4.5.13)$$

Therefore, $I_j \subset (2^{j-1}, 2^j)$, which easily yields the desired disjointness. Next, we verify the three claimed properties.

First property: A direct computation shows that $F := \mathcal{F}^{-1}\chi_{(0,1)}$ satisfies $F(x) = \frac{e^{2\pi ix}-1}{2\pi ix}$ for $x \neq 0$, and hence $F \in \bigcap_{1<p<\infty} L_p(\mathbb{R})$. Since $\chi_{I_j} = \lambda(3 \cdot 2^{j-2})\big(\chi_{(0,1)}(2^{2j}\cdot)\big)$, we thus see by elementary properties of the Fourier transform that $\mathcal{F}^{-1}\chi_{I_j}(x) = 2^{-2j} \cdot e^{6\pi i 2^{j-2}x} \cdot F(2^{-2j}x)$. Therefore,

$$\begin{aligned}\|K\|_{L_p} &= \Big\|\sum_{j=1}^{\infty}\mathcal{F}^{-1}\chi_{I_j}\Big\|_{L_p} \le \sum_{j=1}^{\infty} 2^{-2j} \cdot \|F(2^{-2j}\cdot)\|_{L_p} \\ &= \|F\|_{L_p} \cdot \sum_{j=1}^{\infty} 2^{-2j(1-p^{-1})} < \infty\end{aligned}$$

for arbitrary $p \in (1, \infty)$.

Second property: Assume toward a contradiction that $K = K * W$ for some $W \in L_1(\mathbb{R})$. This implies $\chi_\Omega = \widehat{K} = \widehat{K} \cdot \widehat{W} = \chi_\Omega \cdot \widehat{W}$ almost everywhere. In particular, there is a null set $N \subset \mathbb{R}$ with $\widehat{W}(\xi) = 1$ for all $\xi \in \Omega \setminus N$. But the Riemann-Lebesgue lemma shows $\lim_{\xi\to\infty} \widehat{W}(\xi) = 0$, so that $|\widehat{W}(\xi)| \le 1/2$ for all $\xi \in \mathbb{R}$ with

$|\xi| \geq 2^{j_0-2}$, for a suitable $j_0 \in \mathbb{N}$. Hence, for any ξ belonging to the positive measure set $I_{j_0} \setminus N = (3 \cdot 2^{j_0-2}, 3 \cdot 2^{j_0-2} + 2^{-2j_0}) \setminus N \subset \Omega \setminus N$, we have $1 = |\widehat{W}(\xi)| \leq 1/2$, a contradiction.

Third property: Here, we will use the *strong Marcinkiewicz multiplier theorem* which states the following:

Strong Marcinkiewicz multiplier theorem (see [9, Theorem 8.3.1]) *Let* $(\Delta_j)_{j\in\mathbb{Z}}$ *denote the usual dyadic decomposition of* $\mathbb{R}$*,*

$$\Delta_j := \begin{cases} [2^{j-1}, 2^j), & \text{if } j > 0\,, \\ (-1, 1), & \text{if } j = 0\,, \\ (-2^{|j|}, -2^{|j|-1}], & \text{if } j < 0\,. \end{cases}$$

Assume that $\phi : \mathbb{R} \to \mathbb{C}$ *is measurable and satisfies*

$$\sup_{\xi\in\mathbb{R}} |\phi(\xi)| < \infty \quad \text{and} \quad \sup_{j\in\mathbb{Z}} \operatorname{Var}_{\Delta_j} \phi < \infty\,,$$

where $\operatorname{Var}_I \phi$ *denotes the total variation of the function* ϕ *when restricted to the interval* I*.*

Then ϕ *is an* $L_p(\mathbb{R})$*-Fourier multiplier for all* $p \in (1, \infty)$*. In other words, the map* $\mathbb{S}(\mathbb{R}) \to \mathbb{S}'(\mathbb{R})$, $f \mapsto \mathscr{F}^{-1}(\widehat{f} \cdot \phi)$ *extends to a bounded linear operator on* $L_p(\mathbb{R})$*, for any* $p \in (1, \infty)$*.*

We want to apply this theorem for $\phi := \chi_\Omega$. To this end, first note $\sup_{\xi\in\mathbb{R}} |\phi(\xi)| = 1 < \infty$. Second, (4.5.13) shows for $j \in \mathbb{Z}$ with $j \leq 0$ that $\phi|_{\Delta_j} \equiv 0$, and for $j \in \mathbb{N}$ that $\phi|_{\Delta_j} = \chi_{I_j}$ is the indicator function of an interval. In both cases, $\operatorname{Var}_{\Delta_j}\phi \leq 2$. All in all, the strong Marcinkiewicz multiplier theorem shows that the map $\mathbb{S}(\mathbb{R}) \to \mathbb{S}'(\mathbb{R})$, $f \mapsto \mathscr{F}^{-1}(\phi \cdot \widehat{f}) = f * K$ extends to a bounded linear operator on $L_p(\mathbb{R})$ for any $p \in (1, \infty)$. Finally, since $K \in \bigcap_{1<p<\infty} L_p(\mathbb{R})$, Young's inequality (Proposition 4.13) shows that $L_p(\mathbb{R}) \to L_q(\mathbb{R})$, $f \mapsto f * K$ is well-defined and bounded for any $q \in (p, \infty)$. Therefore, the extended map is still given by $L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $f \mapsto f * K$. □

4.6 Appendix

In this appendix, we provide proofs for several technical auxiliary results that we used above. We first present some weighted versions of well-known facts for the reader's convenience.

The first lemma is a weighted version of Schur's test.

Lemma 4.19 (Schur's test) *Let* $K : G \times G \to \mathbb{C}$ *be measurable, let* $w > 0$ *denote a weight on* G*, and let* $p, q, r \in [1, \infty]$ *with* $1 + 1/p = 1/q + 1/r$*. Assume that there is a constant* $C_K > 0$ *such that*

$$\left\| K(x,\cdot) \cdot \frac{w(x)}{w} \right\|_{L_r} \leq C_K \quad \textit{for a.e. } x \in G, \tag{4.6.1}$$

$$\left\| K(\cdot,y) \cdot \frac{w}{w(y)} \right\|_{L_r} \leq C_K \quad \textit{for a.e. } y \in G. \tag{4.6.2}$$

If $f \in L_{q,w}(G)$*, then the integral*

$$I_f(x) = \int_G K(x,y) f(y)\, dy$$

converges for a.e. $x \in G$*. The function* I_f *is in* $L_{p,w}(G)$ *and fulfills*

$$\|I_f\|_{L_{p,w}} \leq C_K \|f\|_{L_{q,w}}.$$

Proof It suffices to assume $f \geq 0$ and $K \geq 0$. Indeed, temporarily writing $I_{K,f}$ instead of I_f to emphasize the role of the kernel K, we have $|I_{K,f}| \leq I_{|K|,|f|}$; furthermore, if (4.6.1) and (4.6.2) hold for K, then they also hold for $|K|$, with the same constants, and we have $\|f\|_{L_{q,w}} = \|\,|f|\,\|_{L_{q,w}}$. Hence, if the claim holds for $K, f \geq 0$, then also

$$\|I_{K,f}\|_{L_{p,w}} \leq \|I_{|K|,|f|}\|_{L_{p,w}} \leq C_{|K|} \cdot \|\,|f|\,\|_{L_{q,w}} = C_K \cdot \|f\|_{L_{q,w}}.$$

Thus, we will assume in the following that $K, f \geq 0$. Hence, also $I_f \geq 0$, so that [16, Theorem 6.14] shows

$$\|I_f\|_{L_{p,w}} = \sup_{0 \leq h \in L_{p',w^{-1}}(G) \setminus \{0\}} \frac{\langle I_f, h \rangle_{L_2}}{\|h\|_{L_{p',w^{-1}}}}. \tag{4.6.3}$$

We denote with $d(x, y)$ the product measure on $G \times G$. Furthermore, for brevity, we set $M_x(y) := \frac{w(x)}{w(y)} \cdot K(x, y)$ and observe $\|M_x\|_{L_r} \leq C_K$ for almost all $x \in G$, thanks to (4.6.1). Likewise, (4.6.2) shows $\|M^{(y)}\|_{L_r} \leq C_K$ for almost all $y \in G$, where $M^{(y)}(x) := \frac{w(x)}{w(y)} \cdot K(x, y)$.

We first consider a number of special cases, so that we can then concentrate on the case where $p, q, r \in (1, \infty)$.

Case 1: At least one of p, q, r is infinite. In case of $p < \infty$, we have $1 < 1 + p^{-1} = q^{-1} + r^{-1}$. But if $q = \infty$, then the right-hand side of this inequality is $r^{-1} \leq 1$, which leads to a contradiction. Similarly, we see that $r = \infty$ leads to a contradiction. Therefore, we necessarily have $p = \infty$ in the present case.

Because of $1 = 1 + p^{-1} = q^{-1} + r^{-1}$, this implies $q = r'$, and hence

$$w(x) \cdot I_f(x) = \int_G M_x(y) \cdot w(y) \cdot f(y)\, dy \leq \|M_x\|_{L_r} \cdot \|f\|_{L_{r',w}} \leq C_K \cdot \|f\|_{L_{q,w}} < \infty$$

for almost all $x \in G$, proving the claim in Case 1, since $p = \infty$.

Case 2: We have $p, q, r < \infty$, but at least one of p, q, r is equal to one. This leaves three subcases:

Case 2-A: We have $p = 1$, and hence $2 = 1 + p^{-1} = q^{-1} + r^{-1} \le 2$, which implies $q = r = 1$. Hence, by Fubini's theorem,

$$\begin{aligned}\|I_f\|_{L_{p,w}} &= \int_G w(x) \int_G K(x,y) \cdot f(y)\, dy\, dx = \int_G w(y) \cdot f(y) \cdot \int_G M_x(y)\, dx\, dy \\ &\le C_K \cdot \|f\|_{L_{1,w}} = C_K \cdot \|f\|_{L_{q,w}}\,,\end{aligned}$$

which proves the claim in Case 2-A.

Case 2-B: We have $p \in (1, \infty)$, but $r = 1$. Since $1 + p^{-1} = q^{-1} + r^{-1} = 1 + q^{-1}$, this implies $p = q \in (1, \infty)$. Hence, for each nonnegative $h \in L_{p',w^{-1}}(G)\setminus\{0\}$, Fubini's theorem and Hölder's inequality show

$$\begin{aligned}\langle I_f, h\rangle_2 &= \int_G h(x) \int_G K(x,y)\, f(y)\, dy\, dx \\ &= \int_{G\times G} \frac{h(x)}{w(x)} \cdot [M^{(y)}(x)]^{\frac{1}{p'}}\, [M^{(y)}(x)]^{\frac{1}{p}} \cdot w(y)\, f(y)\, d(x,y) \\ &\le \left(\int_G \left(\frac{h(x)}{w(x)}\right)^{p'} \int_G M^{(y)}(x)\, dy\, dx\right)^{1/p'} \\ &\quad \cdot \left(\int_G (w(y)\, f(y))^p \int_G M^{(y)}(x)\, dx\, dy\right)^{1/p} \\ &\le C_K \cdot \|h\|_{L_{p',w^{-1}}} \cdot \|f\|_{L_{p,w}}\,.\end{aligned}$$

In view of (4.6.3) and because of $p = q$, this proves the claim in Case 2-B.

Case 2-C: We have $p, r \in (1, \infty)$, but $q = 1$. This implies $p = r \in (1, \infty)$, since $1 + p^{-1} = q^{-1} + r^{-1} = 1 + r^{-1}$, For nonnegative $h \in L_{p',w^{-1}}(G) = L_{r',w^{-1}}(G)$, we thus have

$$\begin{aligned}\langle I_f, h\rangle_{L_2} &= \int_G w(y) \cdot f(y) \int_G M_x(y) \cdot \frac{h(x)}{w(x)}\, dx\, dy \\ &\le \int_G w(y) \cdot f(y) \cdot \|M_x\|_{L_p} \cdot \|h\|_{L_{p',w^{-1}}}\, dy \\ &\le C_K \cdot \|h\|_{L_{p',w^{-1}}} \cdot \|f\|_{L_{1,w}} = C_K \cdot \|h\|_{L_{p',w^{-1}}} \cdot \|f\|_{L_{q,w}}\,.\end{aligned}$$

In view of (4.6.3), this proves the claim in Case 2-C.

Finally, we handle the case $p, q, r \in (1, \infty)$. By elementary calculations, one can show $r/p + r/q' = q/p + q/r' = p'/q' + p'/r' = 1$, where all occurring numbers $\frac{r}{p}, \frac{r}{q'}$ and so on are elements of the interval $(0, 1)$. Thus, for any $0 \le h \in L_{p',w^{-1}}(G)$, it follows from Hölder's inequality and Fubini's theorem that

$$\begin{aligned}
\langle I_f, h\rangle_{L_2} &= \int_{G\times G} K(x,y)\frac{w(x)}{w(y)} \cdot f(y)w(y) \cdot h(x)w(x)^{-1}\, d(x,y)\\
&= \int_{G\times G} \big(M^{(y)}(x)\big)^{r/p} \cdot \big(f(y)w(y)\big)^{q/p} \cdot \big(M_x(y)\big)^{r/q'}\\
&\qquad \cdot \big(h(x)w(x)^{-1}\big)^{p'/q'} \cdot \big(f(y)w(y)\big)^{q/r'} \cdot \big(h(x)w(x)^{-1}\big)^{p'/r'}\, d(x,y)\\
&\overset{(*)}{\leq} \left(\int_G |f(y)w(y)|^q \int_G \big(M^{(y)}(x)\big)^r\, dx\, dy\right)^{1/p}\\
&\qquad \cdot \left(\int_G |h(x)w(x)^{-1}|^{p'} \int_G \big(M_x(y)\big)^r\, dy\, dx\right)^{1/q'}\\
&\qquad \cdot \left(\int_{G\times G} |f(y)w(y)|^q |h(x)w(x)^{-1}|^{p'}\, d(x,y)\right)^{1/r'}\\
&\leq C_K \cdot \|f\|_{L_{q,w}} \cdot \|h\|_{L_{p',w^{-1}}} < \infty,
\end{aligned}$$

where the step marked with $(*)$ used $\frac{1}{p} + \frac{1}{q'} + \frac{1}{r'} = \frac{1}{p} + 1 - \frac{1}{q} + 1 - \frac{1}{r} = 1$. In view of (4.6.3), this proves the claim for the case $p, q, r \in (1, \infty)$. □

Next we present a weighted version of the classical Young's inequality.

Proposition 4.13 (Young's inequality) *Let m be a w-moderate weight on G, see (4.2.13), and let $p, q, r \in [1, \infty]$ such that $1 + 1/p = 1/q + 1/r$. Then it follows for $f \in L_{q,m}(G)$ and $g \in L_{r,w}(G) \cap L_{r,w\Delta^{-1/r}}(G)$ that $f * g \in L_{p,m}(G)$ and*

$$\|f * g\|_{L_{p,m}} \leq \max\{\|g\|_{L_{r,w}}, \|g\|_{L_{r,w\Delta^{-1/r}}}\} \cdot \|f\|_{L_{q,m}}. \tag{4.6.4}$$

If, instead of $g \in L_{r,w}(G) \cap L_{r,w\Delta^{-1/r}}(G)$, it holds $g \in L_{r,w}(G)$ and $|g(x)| = |g(x^{-1})|$ as well as $w(x) = w(x^{-1})$ for all $x \in G$, or if $g \in L_{r,w}(G)$ and G is unimodular, then

$$\|f * g\|_{L_{p,m}} \leq \|g\|_{L_{r,w}} \cdot \|f\|_{L_{q,m}}. \tag{4.6.5}$$

Proof We apply Lemma 4.19 for the case $K(x,y) = g(y^{-1}x)$ and the weight m. It suffices to show that there exists a constant C_K that fulfills (4.6.1) and (4.6.2). We first consider the case $r < \infty$ and use (4.2.13) and the left invariance of the Haar measure to conclude

$$\begin{aligned}
\int_G |g(y^{-1}x)|^r \cdot \frac{m(x)^r}{m(y)^r}\, dx &= \int_G |g(z)|^r \cdot \frac{m(yz)^r}{m(y)^r}\, dz\\
&\leq \int_G |g(z)|^r \cdot \frac{m(y)^r w(z)^r}{m(y)^r}\, dz\\
&= \int_G |g(z)|^r \cdot w(z)^r\, dz = \|g\|_{L_{r,w}}^r
\end{aligned}$$

for almost all $y \in G$. Now, using the change of variables $z = x^{-1}y$, and recalling the formula $d\varrho(x) = \Delta(x^{-1})dx$ (see [15, Proposition 2.31]) for the right Haar measure ϱ given by $\varrho(M) = \beta(M^{-1})$, we see

$$\begin{aligned}\int_G |g(y^{-1}x)|^r \cdot \frac{m(x)^r}{m(y)^r}\, dy &= \int_G |g(z^{-1})|^r \cdot \frac{m(x)^r}{m(xz)^r}\, dz \\ &\le \int_G |g(z^{-1})|^r \cdot [w(z^{-1})]^r\, dz \\ &= \int_G |g(y)|^r \cdot [w(y)]^r \cdot \Delta(y)^{-1}\, dy = \|g\|_{L_{r,w\Delta^{-1/r}}}^r\end{aligned}$$

for almost all $x \in G$. By setting $C_K = \max\{\|g\|_{L_{r,w}}, \|g\|_{L_{r,w\Delta^{-1/r}}}\} < \infty$, Lemma 4.19 yields

$$\|f * g\|_{L_{p,m}} \le C_K \cdot \|f\|_{L_{q,m}} \qquad \text{for all } f \in L_{q,m}(G)\,,$$

which proves (4.6.4).

Finally, for the case $r = \infty$, observe $m(x) = m(yy^{-1}x) \le m(y) \cdot w(y^{-1}x)$, so that we get

$$|g(y^{-1}x)| \cdot \frac{m(x)}{m(y)} \le |g(y^{-1}x)| \cdot w(y^{-1}x) \le \|g\|_{L_{\infty,w}}$$

for almost every $x \in G$ and almost every $y \in G$, which establishes (4.6.1) and (4.6.2).

It remains to prove (4.6.5). If we assume $|g(x)| = |g(x^{-1})|$ and $w(x) = w(x^{-1})$, the formula $d\varrho(x) = \Delta(x^{-1})dx$ from above yields for $r < \infty$ that

$$\begin{aligned}\|g\|_{L_{r,w\Delta^{-1/r}}}^r &= \int_G |g(y)|^r \cdot [w(y)]^r \cdot \Delta(y^{-1})\, dy = \int_G |g(z^{-1})|^r \cdot [w(z^{-1})]^r\, dz \\ &= \int_G |g(z)|^r \cdot [w(z)]^r\, dz = \|g\|_{L_{r,w}}^r\,.\end{aligned}$$

This identity trivially holds if G is unimodular, so that $\Delta \equiv 1$. For $r = \infty$, we always have $\|g\|_{L_{r,w\Delta^{-1/r}}} = \|g\|_{L_{r,w}}$. In all of these cases, (4.6.5) is a direct consequence of (4.6.4). □

Lemma 4.20 *Let A be a bounded and surjective linear operator that maps a Banach space W onto a Banach space V. Suppose that the kernel of A admits a complement L in W. Set*

$$\varepsilon := \inf\left\{\sup\left\{|\langle Ax, y\rangle_{V\times V^*}| \mid y \in V^*, \|y\|_{V^*} = 1\right\} \,\middle|\, x \in L, \|x\|_W = 1\right\}.$$

Then the map $S := (A|_L)^{-1} : V \to L \subset W$ is a linear right inverse of A with

$$\|S\| = \varepsilon^{-1}.$$

Proof It is straightforward that $A|_L : L \to V$ is a bijection. Therefore S is indeed a linear right inverse of A, and we have

$$
\begin{aligned}
&\inf \left\{\sup\left\{|\langle Ax, y\rangle_{V\times V^*}| \mid y \in V^*, \|y\|_{V^*} = 1\right\} \mid x \in L, \|x\|_W = 1\right\} \\
&\quad = \inf\{\|Ax\|_V \mid x \in L, \|x\|_W = 1\} \\
&\quad = \inf\left\{\frac{\|Ax\|_V}{\|x\|_W} \,\middle|\, x \in L\setminus\{0\}\right\} = \inf\left\{\frac{\|ASv\|_V}{\|Sv\|_W} \,\middle|\, v \in V\setminus\{0\}\right\} \\
&\quad = \left(\sup\left\{\frac{\|Sv\|_W}{\|ASv\|_V} \,\middle|\, v \in V\setminus\{0\}\right\}\right)^{-1} = \left(\sup\left\{\frac{\|Sv\|_W}{\|v\|_V} \,\middle|\, v \in V\setminus\{0\}\right\}\right)^{-1} \\
&\quad = \|S\|^{-1},
\end{aligned}
$$

which proves the claim. □

Lemmas 4.3 and 4.8 as well as Proposition 4.8 were left unproven. The proofs are presented here.

Proof of Lemma 4.3. We start with an auxiliary observation: We claim that $\|g\|_{L_\infty(Qx)} = \sup_{y\in Qx} |g(y)|$ if $g : G \to \mathbb{C}$ is continuous and if $Q \subset G$ is a compact unit neighborhood with $Q = \overline{\text{int } Q}$.

Indeed, the inequality "$\leq$" is trivial. Conversely, if we set $\alpha := \|g\|_{L_\infty(Qx)}$, then the set $M := \{y \in G \mid |g(y)| > \alpha\}$ is open, and $M \cap Qx$ is a null set. Hence, $M \cap (\text{int } Q)x = \emptyset$, since this is an open null set. In other words, $|g(y)| \leq \alpha$ for all $y \in (\text{int } Q)x$. By continuity of g and since $Q \subset \overline{\text{int } Q}$, we see $|g(y)| \leq \alpha$ for all $y \in Qx$.

In particular, this implies $\breve{M}^{\rho}_{Q} g(x) = \sup_{q\in Q} |g(qx)|$, and thus (because of $e \in Q$) $\breve{M}^{\rho}_{Q} g \geq |g|$.

To prove (i) we note that $\breve{M}^{\rho}_{Q_0} f \in L_{p,w}(G)$ which implies $f \in L_{p,w}(G)$, since we just saw that $\breve{M}^{\rho}_{Q_0} f \geq |f|$. We intend to show $\|\text{osc}_{Q_0} f\|_{L_{p,w}} < \infty$. But we have

$$
\text{osc}_{Q_0} f(x) = \sup_{q\in Q_0} |f(qx) - f(x)| \leq \sup_{q\in Q_0} |f(qx)| + |f(x)| \leq |f(x)| + \breve{M}^{\rho}_{Q_0} f(x)\,.
$$

Therefore,

$$
\|\text{osc}_{Q_0} f\|_{L_{p,w}} \leq \|\breve{M}^{\rho}_{Q_0} f\|_{L_{p,w}} + \|f\|_{L_{p,w}} < \infty\,. \tag{4.6.6}
$$

It remains to prove (ii). For this, we first note that $\text{osc}_Q f \leq \text{osc}_{Q_0} f$ if $Q \subset Q_0$. Furthermore, by part (i) we have $\text{osc}_{Q_0} f \in L_{p,w}(G)$. Hence, since G is σ-compact, for any $\varepsilon > 0$, there exists a compact set $K \subset G$ of positive measure such that

$$
\int_{G\setminus K} |\text{osc}_Q f(x) w(x)|^p \, dx \leq \int_{G\setminus K} |\text{osc}_{Q_0} f(x) w(x)|^p \, dx < \frac{\varepsilon}{2} \tag{4.6.7}
$$

for all unit neighborhoods $Q \subset Q_0$.

Next, we observe that since f is continuous, it is uniformly continuous on K in the following sense: For every $\delta > 0$, there is a unit neighborhood $U_\delta \subset G$ with $|f(x) - f(ux)| < \delta$ for all $x \in K$ and $u \in U_\delta$.

The uniform continuity described above simply means $\operatorname{osc}_{U_\delta} f(x) \le \delta$ for all $x \in K$. Choosing $\delta := \varepsilon^{1/p}/([2 \cdot |K|]^{1/p} \sup_{y\in K} w(y))$, we see for every unit neighborhood $Q \subset Q_0 \cap U_\delta$ that

$$\int_K |\operatorname{osc}_Q f(x) w(x)|^p \, dx \le \int_K \frac{\varepsilon}{2|K|} \cdot \frac{w(x)^p}{\sup_{y\in K} w(y)^p} \, dx \le \int_K \frac{\varepsilon}{2|K|} \, dx = \frac{\varepsilon}{2}. \tag{4.6.8}$$

Equations (4.6.7) and (4.6.8) yield $\|\operatorname{osc}_Q f\|_{L_{p,w}}^p < \varepsilon$, which concludes the proof. □

Proof of Lemma 4.8. Let $1 \le p < \infty$ and $(d_x)_{x\in Y_n} \in \ell_{p,m}(Y_n)$, then we first note that for all $x \in G$ it holds

$$\begin{aligned}\int_{xQ_n} m(y)^p \, dy = \int_{Q_n} m(xy)^p \, dy &\le m(x)^p \int_{Q_n} w(y)^p \, dy \\ &\le \sup_{q\in Q_n} w(q)^p \cdot |Q_n| \cdot m(x)^p.\end{aligned}$$

With this at hand and since Y_n is relatively Q_n-separated, as stated in (4.3.12), we derive

$$\begin{aligned}\Big\| \sum_{x\in Y_n} |d_x| \chi_{xQ_n} \Big\|_{L_{p,m}} &\le \sum_{i=1}^{\mathscr{I}} \Big\| \sum_{x\in Z_{n,i}} |d_x| \chi_{xQ_n} \Big\|_{L_{p,m}} \\ &= \sum_{i=1}^{\mathscr{I}} \left(\sum_{x\in Z_{n,i}} |d_x|^p \int_{xQ_n} m(y)^p \, dy \right)^{\frac{1}{p}} \\ &\le \sum_{i=1}^{\mathscr{I}} \left(\sum_{x\in Z_{n,i}} |d_x|^p m(x)^p \right)^{\frac{1}{p}} \cdot \sup_{q\in Q_n} w(q) \cdot |Q_n|^{\frac{1}{p}} \\ &\le \mathscr{I}^{1-\frac{1}{p}} \cdot \sup_{q\in Q_n} w(q) \cdot |Q_n|^{\frac{1}{p}} \cdot \|(d_x)_{x\in Y_n}\|_{\ell_{p,m}}.\end{aligned}$$

It remains to prove the case $p = \infty$. Similarly as above, we see that

$$\begin{aligned}\Big\| \sum_{x\in Y_n} |d_x| \chi_{xQ_n} \Big\|_{L_{\infty,m}} &\leq \sum_{i=1}^{\mathscr{I}} \Big\| \sum_{x\in Z_{n,i}} |d_x| \chi_{xQ_n} \Big\|_{L_{\infty,m}} \\ &= \sum_{i=1}^{\mathscr{I}} \sup_{x\in Z_{n,i}} \left(|d_x| \cdot \sup_{y\in xQ_n} m(y) \right) \\ &\leq \mathscr{I} \cdot \left(\sup_{x\in Y_n} |d_x| \cdot m(x) \right) \cdot \sup_{y\in Q_n} w(y) \\ &= \mathscr{I} \cdot \sup_{q\in Q_n} w(q) \cdot \|(d_x)_{x\in Y_n}\|_{\ell_{\infty,m}}. \end{aligned}$$ □

Proof of Proposition 4.8. We will construct $C \subset [0, 1]$ as a certain "fat Cantor set". In particular, we will show below that C has positive measure and fulfills the following two additional properties:

$$|C \cap B| < |B| \qquad \text{for all open intervals } \emptyset \neq B \subset \mathbb{R}, \tag{4.6.9}$$

and

$$C^c = \bigcup_{n=0}^{\infty} \bigcup_{j=0}^{2^n-1} B_j^n \quad \text{with} \quad B_j^n := \frac{a_j^{(n)} + b_j^{(n)}}{2} + \left(-\frac{\mu_{n+1}}{2}, \frac{\mu_{n+1}}{2}\right), \tag{4.6.10}$$

where the complement C^c is taken relative to $[0, 1]$, and where $a_j^{(n)}, b_j^{(n)} \in \mathbb{R}$ are suitable, while $\mu_n := \min\{4^{-n}, n^{-n}\}$ for $n \in \mathbb{N}$.

Before we provide the precise construction of such a set C, let us see how the properties (4.6.9) and (4.6.10) imply the properties of C that are stated in the proposition.

First, [24, Theorem 1] shows that if the operator $f \mapsto f * \mathscr{F}^{-1}\chi_C$ is bounded on $L_p(\mathbb{R})$ for some $p \in (1, \infty)\setminus\{2\}$, that is, if χ_C is an $L_p(\mathbb{R})$-Fourier multiplier, then C would be equivalent to an open set. In other words, there would be an open set $U \subset \mathbb{R}$ with $\chi_C = \chi_U$ Lebesgue almost everywhere. But since C has positive measure, this is only possible if U is a *nonempty* open set. Therefore, U contains a nonempty open interval $B \subset U$. Since $\chi_C = \chi_U$ almost everywhere, this implies $|B \cap C| = |B \cap U| = |B|$, in contradiction to (4.6.9). In summary, we have thus shown that the convolution operator $f \mapsto f * \mathscr{F}^{-1}\chi_C$ is *not* bounded on any $L_p(\mathbb{R})$ space for $p \in (1, \infty)\setminus\{2\}$. But this even implies that $L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $f \mapsto f * \mathscr{F}^{-1}\chi_C$ is not well-defined, by Proposition 4.2.

Second, we will see that (4.6.10) ensures $\mathscr{F}^{-1}\chi_{C^c} \in \bigcap_{1<p\leq\infty} L_p(\mathbb{R})$, which then implies $\mathscr{F}^{-1}\chi_C = \mathscr{F}^{-1}\chi_{(0,1)} - \mathscr{F}^{-1}\chi_{C^c} \in \bigcap_{1<p\leq\infty} L_p(\mathbb{R})$. Here, we used that $F := \mathscr{F}^{-1}\chi_{(0,1)} \in \bigcap_{1<p\leq\infty} L_p(\mathbb{R})$, since a direct computation shows $F(x) = \frac{e^{2\pi ix}-1}{2\pi ix}$ for $x \neq 0$, which implies $|F(x)| \lesssim (1 + |x|)^{-1}$. It remains to show $\mathscr{F}^{-1}\chi_{C^c} \in \bigcap_{1<p\leq\infty} L_p(\mathbb{R})$. To this end, we set $\xi_j^{(n)} := \frac{a_j^{(n)}+b_j^{(n)}}{2} - \frac{\mu_{n+1}}{2}$, recall the definition of

the intervals $B_j^n = \xi_j^{(n)} + \mu_{n+1} \cdot (0,1)$ from (4.6.10), and use standard properties of the Fourier transform to compute

$$\mathscr{F}^{-1}\chi_{B_j^n} = \mu_{n+1} \cdot M_{\xi_j^{(n)}}\big[(\mathscr{F}^{-1}\chi_{(0,1)})(\mu_{n+1}\cdot)\big] = \mu_{n+1} \cdot M_{\xi_j^{(n)}}\big(F(\mu_{n+1}\cdot)\big)\,,$$

where $(M_\xi f)(x) = e^{2\pi i x \xi} f(x)$ denotes the *modulation* with frequency ξ of a function f. Next, (4.6.10) shows

$$\mathscr{F}^{-1}\chi_{C^c} = \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1} \mathscr{F}^{-1}\chi_{B_j^n}\,.$$

Combining this with the triangle inequality for L_p and with the elementary identities $\|M_\xi f\|_{L_p} = \|f\|_{L_p}$ and $\|f(a\cdot)\|_{L_p(\mathbb{R})} = a^{-1/p}\|f\|_{L_p(\mathbb{R})}$ for $a>0$ and $f \in L_p(\mathbb{R})$, we see because of $\mu_n \le n^{-n}$ and $1 - p^{-1} > 0$ for each fixed $p \in (1,\infty]$ that

$$\begin{aligned}\|\mathscr{F}^{-1}\chi_{C^c}\|_{L_p} &\le \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1} \mu_{n+1}\cdot \|M_{\xi_j^{(n)}}\big(F(\mu_{n+1}\cdot)\big)\|_{L_p} \\ &\le \|F\|_{L_p}\cdot \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}\mu_{n+1}^{1-p^{-1}} \le \|F\|_{L_p}\cdot\sum_{\ell=1}^{\infty} 2^{\ell-1}\cdot \ell^{-\ell(1-p^{-1})}\,.\end{aligned} \tag{4.6.11}$$

But for $\ell \ge \ell_0 = \ell_0(p)$, we have $(1-p^{-1})\cdot \log_2(\ell) \ge 2$, and thus

$$2^{\ell-1}\cdot \ell^{-\ell(1-p^{-1})} = \frac{1}{2}\cdot 2^{\ell}\cdot 2^{-\ell(1-p^{-1})\cdot\log_2(\ell)} \le 2^{\ell\left(1-(1-p^{-1})\cdot\log_2(\ell)\right)} \le 2^{-\ell}\,,$$

so that the series on the right-hand side of (4.6.11) converges. Hence, $\mathscr{F}^{-1}\chi_{C^c} \in L_p(\mathbb{R})$ for every $p \in (1,\infty]$.

Finally, we note because of $\mu_n \le 4^{-n}$ that property (4.6.10) also implies

$$|C^c| = \sum_{n=0}^{\infty}\sum_{j=0}^{2^n-1}|B_j^n| = \sum_{n=0}^{\infty} 2^n \mu_{n+1} \le \sum_{n=0}^{\infty} 2^n\cdot 4^{-(n+1)} \le \frac{1}{4}\cdot\sum_{n=0}^{\infty}2^{-n} = \frac{1}{2} < 1\,,$$

so that $C \subset [0,1]$ necessarily has positive measure if it satisfies properties (4.6.9) and (4.6.10). It remains to show that one can indeed construct a compact set $C \subset [0,1]$ that satisfies properties (4.6.9) and (4.6.10).

To this end, as for the construction of the classical Cantor set, we will set $C := \bigcap_{n=0}^{\infty} C^n$ where the sets $C^n := \bigcup_{j=0}^{2^n-1} C_j^n$ will be defined inductively.

For the start of the induction set $C_0^0 := [a_1^{(0)}, b_1^{(0)}] := [0,1]$.

For the induction step, assume for some $n \in \mathbb{N}_0$ that we have constructed closed intervals $C_\ell^n = [a_\ell^{(n)}, b_\ell^{(n)}] \subset [0,1]$, for $\ell = 0,\ldots,2^n-1$, with

$$4^{-n} \leq b_\ell^{(n)} - a_\ell^{(n)} \leq 2^{-n} \quad \text{for all} \quad 0 \leq \ell < 2^n \tag{4.6.12}$$

and

$$b_\ell^{(n)} < a_{\ell+1}^{(n)} \qquad \text{for} \qquad 0 \leq \ell < 2^n - 1\,. \tag{4.6.13}$$

Now, for $0 \leq j < 2^{n+1}$ we can write $j = 2\ell + k$ with uniquely determined $k \in \{0, 1\}$ and $0 \leq \ell < 2^n$. We then recall from after (4.6.10) that $\mu_{n+1} = \min\{4^{-(n+1)}, (n+1)^{-(n+1)}\}$, and define

$$\begin{aligned} C_j^{n+1} &:= [a_j^{(n+1)}, b_j^{(n+1)}] \\ &:= \begin{cases} \left[a_\ell^{(n)}, \frac{a_\ell^{(n)}+b_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2}\right] \subset [a_\ell^{(n)}, b_\ell^{(n)}] = C_\ell^n & \text{if } k = 0\,, \\ \left[\frac{a_\ell^{(n)}+b_\ell^{(n)}}{2} + \frac{\mu_{n+1}}{2}, b_\ell^{(n)}\right] \subset [a_\ell^{(n)}, b_\ell^{(n)}] = C_\ell^n & \text{if } k = 1\,. \end{cases} \end{aligned} \tag{4.6.14}$$

With this choice, we see from (4.6.12) and because of $\mu_{n+1} \leq 4^{-(n+1)}$ that

$$b_j^{(n+1)} - a_j^{(n+1)} = \frac{b_\ell^{(n)} - a_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2} \geq \frac{1}{2} \cdot \left(4^{-n} - 4^{-(n+1)}\right) = \frac{3}{8} \cdot 4^{-n} \geq 4^{-(n+1)}$$

and

$$b_j^{(n+1)} - a_j^{(n+1)} = \frac{b_\ell^{(n)} - a_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2} \leq \frac{1}{2}(b_\ell^{(n)} - a_\ell^{(n)}) \leq 2^{-(n+1)}\,,$$

thereby proving (4.6.12) for $n + 1$ instead of n.

For the proof of (4.6.13) for $0 \leq j < 2^{n+1} - 1$ with $j = 2\ell + k$ and $k \in \{0, 1\}$, we distinguish two cases:

Case 1: $k = 0$. In this case, $j + 1 = 2\ell + 1$, and hence

$$b_j^{(n+1)} = \frac{a_\ell^{(n)} + b_\ell^{(n)}}{2} - \frac{\mu_{n+1}}{2} < \frac{a_\ell^{(n)} + b_\ell^{(n)}}{2} + \frac{\mu_{n+1}}{2} = a_{j+1}^{(n+1)}\,.$$

Case 2: $k = 1$. In this case, $2(\ell + 1) + 0 = j + 1 < 2^{n+1}$, so that $1 \leq \ell + 1 < 2^n$. Therefore, (4.6.13) shows $b_j^{(n+1)} = b_\ell^{(n)} < a_{\ell+1}^{(n)} = a_{j+1}^{(n+1)}$.

We have thus verified (4.6.13) for $n + 1$ instead of n.

As indicated above, we define $C^n := \bigcup_{j=0}^{2^n-1} C_j^n$ and observe as a consequence of (4.6.14) that each C^n is closed with $C^{n+1} \subset C^n$ for all $n \in \mathbb{N}_0$. Hence, $C := \bigcap_{n=0}^\infty C^n \subset C^0 = [0, 1]$ is compact.

Having defined the set C, our first goal is to prove property (4.6.9). Let $B \subset \mathbb{R}$ be a nonempty open interval. If $C \cap B$ is a finite set, the inequality in (4.6.9) is trivially satisfied. Hence, we can assume that $C \cap B$ is infinite, so that there are $x, y \in C \cap B$ with $x < y$. Choose $n \in \mathbb{N}_0$ with $2^{-n} < y - x$ and note because of $x, y \in C \subset C^n = \bigcup_{j=0}^{2^n-1} C_j^n$ that there are $j_x, j_y \in \{0, \ldots, 2^n - 1\}$ with $x \in C_{j_x}^n$ and $y \in C_{j_y}^n$. In case of $j_y \leq j_x$, we would get because of $a_\ell^{(n)} \leq b_\ell^{(n)} \leq a_{\ell+1}^{(n)}$ for $0 \leq \ell < 2^n - 1$ and because of $b_\ell^{(n)} - a_\ell^{(n)} \leq 2^{-n}$ for $0 \leq \ell < 2^n$ (see 4.6.12, 4.6.13) that

$$2^{-n} < y - x \le b_{j_y}^{(n)} - a_{j_x}^{(n)} \le b_{j_y}^{(n)} - a_{j_y}^{(n)} \le 2^{-n},$$

a contradiction. Hence, $j_y > j_x$, so that (4.6.12) and (4.6.13) show

$$B \ni x \le b_{j_x}^{(n)} \le b_{j_y-1}^{(n)} < a_{j_y}^{(n)} \le y \in B\,,$$

and thus $(b_{j_y-1}^{(n)}, a_{j_y}^{(n)}) \subset B \setminus C^n \subset B \setminus C$. But since this interval has positive measure, we see $|B| = |B \setminus C| + |B \cap C| > |B \cap C|$, thereby proving (4.6.9).

Finally, we prove the formula (4.6.10) for the complement C^c of C, with the complement taken relative to $[0, 1]$. To see this, note $C^c = \bigcup_{n=0}^{\infty} (C^n)^c$. By disjointization, and since $(C^0)^c = \emptyset$ and $(C^n)^c \subset (C^{n+1})^c$, this yields

$$C^c = \bigcup_{n=1}^{\infty} (C^n)^c \setminus (C^{n-1})^c = \bigcup_{n=1}^{\infty} C^{n-1} \setminus C^n = \bigcup_{n=0}^{\infty} C^n \setminus C^{n+1}\,.$$

Next, recall $C^n = \bigcup_{j=0}^{2^n-1} C_j^n$ and also recall from (4.6.14) that $C_{2\ell+k}^{n+1} \subset C_\ell^n$ for $0 \le \ell < 2^n$ and $k \in \{0, 1\}$. Therefore, by (4.6.14) and the definition of B_j^n in (4.6.10) it holds

$$C_j^n \setminus C^{n+1} = \bigcap_{\ell=0}^{2^n-1} \bigcap_{k=0}^{1} C_j^n \setminus C_{2\ell+k}^{n+1} = \bigcap_{k=0}^{1} C_j^n \setminus C_{2j+k}^{n+1} = B_j^n\,.$$

Putting everything together, we see that (4.6.10) holds. □

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Chapter 5
On the Purity and Entropy of Mixed Gaussian States

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Abstract The notions of purity and entropy play a fundamental role in the theory of density operators. These are nonnegative trace class operators with unit trace. We review and complement some results from a rigorous point of view.

5.1 Introduction

The number of papers devoted to probability distributions and their quantum counterparts defies the imagination. (For a recent survey see Adesso et al. [1]). One of the aims of this study is to derive and justify rigorously some formulas which are often found in the physical literature. We will use notation slightly different from that which is customary in the literature on pseudodifferential operators, or in time frequency analysis, we will make explicit the dependence on a parameter $\hbar$ of the functions we work with (in physics, this parameter is identified with Planck's constant h divided by 2π). In fact, $\hbar$ is usually taken equal to one in pseudodifferential theory and equal to $1/2\pi$ in time-frequency analysis, and this often makes the passage from formulas obtained in these theories to quantum mechanics become rather acrobatic (a good example of this situation is Folland's book [10]). The advantage of introducing an undetermined positive parameter $\hbar$ is that everyone is free to fix it in his guise, following his needs or interests.

We will mainly focus on the mixed states from quantum mechanics which are represented by a Gaussian phase space distribution of the type

$$\rho_\Sigma(z) = (2\pi)^{-n}\sqrt{\det \Sigma^{-1}}e^{-\frac{1}{2}\Sigma^{-1}(z-z_0)^2}. \tag{5.1.1}$$

Here Σ is a positive-definite symmetric (real) $2n \times 2n$ matrix, and $z_0 = (x_0, p_0)$ a fixed point in phase space $\mathbb{R}^{2n}_z \equiv \mathbb{R}^n_x \times \mathbb{R}^n_p$. Since $\rho \geq 0$ and

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$$\int \rho_\Sigma(z)d^{2n}z = 1$$

such a function ρ can always be viewed as a classical probability distribution whose covariance matrix is Σ and centered at z_0. Consider the operator $\widehat{\rho}_\Sigma$ defined, for $\psi \in L^2(\mathbb{R}^n)$, by

$$\widehat{\rho}_\Sigma\psi(x) = \iint e^{\frac{i}{\hbar}\langle p,x-y\rangle}\rho_\Sigma(z)(\tfrac{1}{2}(x+y), p)\psi(y)d^n y d^n p; \tag{5.1.2}$$

this operator will be the density operator of some quantum state if the three following conditions are fulfilled:

- $\widehat{\rho}_\Sigma$ *is self-adjoint:* $\widehat{\rho}_\Sigma = \widehat{\rho}_\Sigma^*$;
- $\widehat{\rho}_\Sigma$ *is of trace class and* $Tr\widehat{\rho}_\Sigma = 1$;
- $\widehat{\rho}_\Sigma$ *is positive semidefinite:* $\widehat{\rho}_\Sigma \geq 0$.

It is the third of these conditions that poses a problem. While it is clear that $\widehat{\rho}_\Sigma = \widehat{\rho}_\Sigma^*$ (because $\widehat{\rho}_\Sigma$ is, up to a factor, the Weyl operator with the real symbol ρ_Σ) and that $\widehat{\rho}_\Sigma$ is of trace class (because $\rho_\Sigma \in L^2(\mathbb{R}^n)$) and has trace

$$Tr(\widehat{\rho}_\Sigma) = \int \rho_\Sigma(z)d^{2n}z = 1$$

it is not at all clear that $\widehat{\rho}_\Sigma \geq 0$, that is $(\widehat{\rho}_\Sigma\psi|\psi)_{L^2} \geq 0$ for all $\psi \in L^2(\mathbb{R}^n)$. It turns out that this is the case if and only if the covariance matrix Σ satisfies the positivity condition

$$\Sigma + \frac{i\hbar}{2}J \geq 0 \tag{5.1.3}$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ is the standard $2n \times 2n$ symplectic matrix. This condition means that all the eigenvalues of the complex self-adjoint matrix $\Sigma + (i\hbar/2)J$ are ≥ 0. This criterion can be proven in several ways [3, 9] , but none of the available proofs is really elementary. We now ask: What happens if we change Planck's constant in such a way that $\hbar$ becomes a number η? In this case, ρ_Σ will still be the Wigner distribution of a quantum state provided that the covariance matrix satisfies the new condition

$$\Sigma + \frac{i\eta}{2}J \geq 0. \tag{5.1.4}$$

Setting $\eta = r\hbar$ we have

$$\Sigma + \frac{i\eta}{2}J = (1-r)\Sigma + r\left(\Sigma + \frac{i\hbar}{2}J\right);$$

since $\Sigma > 0$ and $\Sigma + (i\hbar/2)J \geq 0$ condition (5.1.4) will hold for $0 \leq r \leq 1$; discarding the case $r = 0$ which corresponds to the case of a classical probability distribution, the operator $\widehat{\rho}_\Sigma$ will be a density operator for all $\eta \leq \hbar$.

5.2 Interpretations of the Condition $\Sigma + \frac{i\eta}{2} J \geq 0$

Condition (5.1.4) can be restated in terms of the symplectic spectrum of the covariance matrix Σ. Observing that the product $J\Sigma$ has the same eigenvalues as the antisymmetric matrix $\Sigma^{1/2} J \Sigma^{1/2}$ (because they are conjugate), its eigenvalues are pure imaginary numbers $\pm i\lambda_1^\sigma, \pm i\lambda_2^\sigma, ..., \pm i\lambda_n^\sigma$ where $\lambda_j^\sigma > 0$ for $j = 1, 2, ..., n$. The set $\{\lambda_1^\sigma, \lambda_2^\sigma, ..., \lambda_n^\sigma\}$ is called the *symplectic spectrum* of Σ. Now, there exists a symplectic matrix S (*i.e.,* a matrix such that $S^T J S = J$) diagonalizing Σ as follows:

$$\Sigma = S^T D S\,,\; D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \tag{5.2.1}$$

where Λ is the diagonal matrix with nonzero entries the positive numbers $\lambda_1^\sigma, \lambda_2^\sigma, ...,$ λ_n^σ (this is called a symplectic, or Williamson, diagonalization of Σ [3, 5, 9]). Since $S^T J S = J$ we have

$$\Sigma + \frac{i\eta}{2} J = S^T D S + \frac{i\eta}{2} J = S^T (D + \frac{i\eta}{2} J) S$$

hence, the condition $\Sigma + \frac{i\eta}{2} J \geq 0$ is equivalent to $D + \frac{i\eta}{2} J \geq 0$. Now, the characteristic polynomial of the matrix $D + \frac{i\eta}{2} J$ is the product $P_1(\lambda) \cdots P_n(\lambda)$ where the P_j are the second-degree polynomials $P_j(\lambda) = (\lambda_j^\sigma - \lambda)^2 - \frac{\eta^2}{4}$; hence, the eigenvalues λ of $D + \frac{i\eta}{2} J$ are the numbers $\lambda = \lambda_j^\sigma \pm \frac{1}{2}\eta$; the condition $D + \frac{i\eta}{2} J \geq 0$ implies that all these eigenvalues λ_j must be ≥ 0, and hence, $\lambda_j^\sigma \geq \sup\{\pm \frac{1}{2}\eta\} = \frac{1}{2}|\eta|$ for all j. We have thus proven the equivalence

$$\Sigma + \frac{i\eta}{2} J \geq 0 \Longleftrightarrow |\eta| \leq 2\lambda_{\min}^\sigma \tag{5.2.2}$$

where $\lambda_{\min}^\sigma$ is the smallest symplectic eigenvalue of the covariance matrix Σ. It follows in particular that we have

$$\det \Sigma \geq \left(\frac{\eta}{2}\right)^n . \tag{5.2.3}$$

It turns out that the reformulation (5.2.2) opens the gate to an attractive geometric interpretation in terms of the "quantum blobs" we have introduced elsewhere [3, 4, 7]. A quantum blob is the image by a linear (or affine) symplectic transformation of the phase space ball $B^{2n}(\sqrt{\hbar})$. It is thus an ellipsoid of a particular type, which can be viewed quantum mechanically as a "minimum uncertainty ellipsoid." Consider now the "covariance ellipsoid"

$$\Omega_\Sigma = \{z \in \mathbb{R}^{2n} : \tfrac{1}{2}\Sigma^{-1}z \cdot z \leq 1\}.$$

One proves [3, 4, 8] that the symplectic capacity of this ellipsoid satisfies

$$c(\Omega_\Sigma) \geq \pi\hbar$$

so that it must contain the image by a symplectic transformation of $B^{2n}(\sqrt{\hbar})$, that is a quantum blob. This is a geometric version of the uncertainty principle: The only quantum mechanically admissible covariance matrices are those for which the associated covariance ellipsoids contain a minimum uncertainty ellipsoid.

5.3 The Purity of a Gaussian State

Let $\widehat{\rho}$ be a density operator; then, $\widehat{\rho}^2$ is a trace class operator, and one has $\mathrm{Tr}(\widehat{\rho}^2) \leq 1$ with equality if and only if $\widehat{\rho}$ represents a pure state, that is, if $\widehat{\rho}\psi = (\psi|\phi)\phi$ for some $\phi \in L^2(\mathbb{R}^n)$. The number $\mu(\widehat{\rho}) = \mathrm{Tr}(\widehat{\rho}^2)$ is called the *purity* of $\widehat{\rho}$; it will, in general, depend on the value of Planck's constant, that is, on the parameter η. The purity is multiplicative under tensor products: If $\widehat{\rho}_{(1)}$ and $\widehat{\rho}_{(2)}$ are density operators on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, then $\widehat{\rho}_{(1)} \otimes \widehat{\rho}_{(2)}$ is a density operator on $\mathbb{R}^n$, $n = n_1 + n_2$ and we have

$$\mu(\widehat{\rho}_{(1)} \otimes \widehat{\rho}_{(2)}) = \mu(\widehat{\rho}_{(1)})\mu(\widehat{\rho}_{(2)}). \tag{5.3.1}$$

In the Gaussian case, we have an explicit formula for the purity of the state. To make things rigorous we need the following lemma.

Lemma 5.1 *Let $\widehat{A}$ and $\widehat{B}$ be trace class operators with η-Weyl symbols a and b, respectively. Then $\widehat{A}\widehat{B}$ is of trace class and*

$$Tr(\widehat{A}\widehat{B}) = \left(\tfrac{1}{2\pi\eta}\right)^n \int a(z)b(z)d^{2n}z. \tag{5.3.2}$$

For a proof see [6]; note that the result holds if one only supposes that $\widehat{A}$ and $\widehat{B}$ are Hilbert–Schmidt operators.

In particular, if $\widehat{\rho}'$ and $\widehat{\rho}''$ have Wigner distributions ρ' and ρ'' on $\mathbb{R}^{2n}$ then

$$\mathrm{Tr}(\widehat{\rho}'\widehat{\rho}'') = (2\pi\eta)^n \int \rho'(z)\rho''(z)d^{2n}z \tag{5.3.3}$$

since the Weyl symbols of $\widehat{\rho}$ and $\widehat{\rho}'$ are $(2\pi\eta)^n\rho$ and $(2\pi\eta)^n\rho'$, respectively. Notice that, however, if $\widehat{\rho}'$ and $\widehat{\rho}''$ have Wigner distributions ρ' and ρ'' defined on $\mathbb{R}^{2n'}$ and $\mathbb{R}^{2n''}$, respectively, then

$$\operatorname{Tr}(\widehat{\rho}' \otimes \widehat{\rho}'') = \operatorname{Tr}(\widehat{\rho}')\operatorname{Tr}(\widehat{\rho}''). \tag{5.3.4}$$

Proposition 5.1 *Let $\widehat{\rho_\Sigma}$ be the η-density operator corresponding to the Gaussian ρ_Σ where $\Sigma + \frac{i\eta}{2}J \geq 0$. The purity of $\widehat{\rho_\Sigma}$ is*

$$\mu(\widehat{\rho_\Sigma}) = \left(\frac{\eta}{2}\right)^n \det(\Sigma^{-1/2}). \tag{5.3.5}$$

Proof (Cf. [5], p. 302). The Weyl symbol of $\widehat{\rho_\Sigma}$ is $(2\pi\eta)^n\rho_\Sigma$; hence, using formula (5.3.2),

$$\operatorname{Tr}(\widehat{\rho_\Sigma}^2) = (2\pi\eta)^n \int \rho_\Sigma^2(z)d^{2n}z$$

Now

$$\int \rho_\Sigma^2(z)d^{2n}z = \left(\tfrac{1}{2\pi}\right)^{2n} (\det \Sigma)^{-1} \int e^{-\Sigma^{-1}z^2} d^{2n}z$$

hence, using the formula

$$\int e^{-Mz^2} d^{2n}z = \pi^n (\det M)^{-1/2}$$

which is valid for every positive-definite symmetric matrix M, we get

$$\int \rho_\Sigma^2(z)d^{2n}z = \left(\tfrac{1}{4\pi}\right)^n (\det \Sigma)^{1/2};$$

formula (5.3.5) follows.

Notice that since $\det \Sigma \geq (\frac{1}{2}\eta)^n$, we indeed have $\operatorname{Tr}(\widehat{\rho_{\Sigma}}_\eta^2) \leq 1$. The purity of the corresponding η-density matrix is $\mu(\widehat{\rho_{\Sigma}}_\eta) = 1$ if and only if $\det(\Sigma) = (\eta/2)^n$. Since $\det(\Sigma) = \det(J\Sigma) = (\lambda_1^\sigma)^2 \cdots (\lambda_n^\sigma)^2$, this requires that $\lambda_j^\sigma = 1$ for all $j = 1, 2, ..., n$ in view (5.2.2). In this case, the matrix D in (5.2.1) is the identity and $\Sigma = S^T S$ which is a positive-definite symplectic matrix and the corresponding state is then a squeezed coherent state . It is in fact the image of the fiducial coherent state $\phi_0(x) = (\pi\hbar)^{-n}e^{-|x|^2/2\hbar}$ by any of the two metaplectic operators $\pm\widehat{S}$ covering the symplectic matrix S [3, 5, 6].

To summarize, we have the following situation (we assume here for simplicity that $\eta > 0$): Suppose that (5.1.4) holds for $\eta = \hbar$. Then the system is a mixed quantum state for all $\eta \leq \hbar$; when $\hbar \leq \eta \leq 2\lambda_{\min}^\sigma$, it is still a mixed state unless $\eta = \lambda_1^\sigma = \cdots = \lambda_n^\sigma$ in which case it becomes a coherent state; when $\eta > 2\lambda_{\min}^\sigma$...

5.4 The Notion of η-Weyl Operator

Let a be some symbol, belonging to an adequate symbol space, which we need not define precisely for the moment; we may assume without restricting the generality of our arguments that $a \in \mathcal{S}(\mathbb{R}^{2n})$ the Schwartz space of functions decreasing at infinity, together with their derivatives, faster that the inverse of any power of $|x|$. In the theory of pseudodifferential operators, it is customary to define the Weyl operator with symbol a by the formula

$$\mathrm{Op}^{\mathrm{W}}(a)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi} a(\tfrac{1}{2}(x+y), \xi)u(y)d^n p d^n y$$

while in quantum mechanics, one uses a slightly different definition, namely

$$\widehat{A}\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint e^{\frac{i}{\hbar}(x-y)\xi} a(\tfrac{1}{2}(x+y), p)\psi(y)d^n p d^n y.$$

Both definitions coincide when $\hbar = 1$; the choice $\hbar = 1/2\pi$ leads to the definition of Weyl operators mostly used in harmonic analysis:

$$Af(x) = \iint e^{2\pi i(x-y)\omega} a(\tfrac{1}{2}(x+y), \omega) f(y)d^n p d^n y.$$

We are going to prove an elementary but useful result which allows to toggle painlessly between these various definitions. Let us introduce the following definition: For any real number $\eta > 0$, we define the η-Weyl operator $\widehat{A}_\eta = \mathrm{Op}_\eta^{\mathrm{W}}(a)$ by the formula

$$\widehat{A}_\eta\psi(x) = \left(\frac{1}{2\pi\eta}\right)^n \iint e^{\frac{i}{\eta}p(x-y)} a(\tfrac{1}{2}(x+y), p)\psi(y)d^n p d^n y.$$

The three operators above correspond to the choices $\eta = 1$, $\eta = \hbar$, and $\eta = 1/2\pi$, respectively. Notice that the normalization constant $(2\pi\eta)^{-n}$ is chosen so that to $a = 1$ corresponds the identity operator: $\mathrm{Op}_\eta^{\mathrm{W}}(1) = I_{\mathrm{d}}$.

Lemma 5.2 *Let η and η' be two positive real numbers. We have*

$$Op_\eta^{\mathrm{W}}(a) = M_{\sqrt{\eta'/\eta}} Op_{\eta'}^{\mathrm{W}}(a \circ \sqrt{\eta/\eta'}) M_{\sqrt{\eta/\eta'}} \tag{5.4.1}$$

where for $r > 0$, $a \circ r(x, p) = a(rx, rp)$ and $M_r\psi(x) = \psi(rx)$.

For any number $r > 0$, we have

$$\mathrm{Op}_{\eta'}^{\mathrm{W}}(a \circ r)\psi(x) = \left(\frac{1}{2\pi\eta'}\right)^n \iint e^{\frac{i}{\eta'}p(x-y)} a(\tfrac{1}{2}(rx+ry), rp)\psi(y)d^n p d^n y$$

and hence, making the change of variables $(y, p) \longmapsto (y/r, p/r)$ and replacing x with x/r,

$$\mathrm{Op}^{\mathrm{W}}_{\eta'}(a \circ r)\psi(x/r) = \left(\tfrac{1}{2\pi r^2\eta'}\right)^n \iint e^{\frac{i}{r^2\eta'}p(x-y)} a(\tfrac{1}{2}(x+y), p)\psi(y/r)d^n p d^n y$$

that is

$$M_{1/r}\mathrm{Op}^{\mathrm{W}}_{\eta'}(a \circ r)\psi(x) = \left(\tfrac{1}{2\pi r^2\eta'}\right)^n \iint e^{\frac{i}{r^2\eta'}p(x-y)} a(\tfrac{1}{2}(x+y), p)M_{1/r}\psi(y)d^n p d^n y$$

Choosing $r = \sqrt{\eta/\eta'}$, we get formula (5.4.1).

It is often advantageous to express Weyl operators in terms of the displacement and reflection operators $\widehat{T}(z_0)$ and $\widehat{\Pi}(z_0)$. These are defined for $z_0 = (x_0, p_0)$ by

$$\widehat{T}(z_0)\psi(x) = e^{\frac{2i}{\eta}(p_0 x - \frac{1}{2}p_0 x_0)}\psi(x - x_0)$$
$$\widehat{\Pi}(z_0)\psi(x) = e^{\frac{i}{\eta}p_0(x-x_0)}\psi(2x_0 - x);$$

one verifies that $\widehat{\Pi}(z_0) = \widehat{T}(z_0)R\widehat{T}(-z_0)$ where $R\psi(x) = \psi(-x)$ and one proves that

$$\mathrm{Op}^{\mathrm{W}}_{\eta}(a) = \left(\tfrac{1}{2\pi\eta}\right)^n \int a_\sigma(z_0)\widehat{T}(z_0)d^{2n}z_0 \tag{5.4.2}$$

$$\mathrm{Op}^{\mathrm{W}}_{\eta}(a) = \left(\tfrac{1}{\pi\eta}\right)^n \int a(z_0)\widehat{\Pi}(z_0)d^{2n}z_0 \tag{5.4.3}$$

where in the first formula a_σ is the symplectic Fourier transform of the symbol a:

$$a_\sigma(z_0) = \left(\tfrac{1}{2\pi\eta}\right)^n \int e^{-\frac{i}{\eta}\sigma(z_0,z)}a(z)d^{2n}z.$$

A straightforward consequence of this rescaling result is that it allows to immediately obtain a quantization result which is usually stated only in the case $\eta = 1/2\pi$ in the literature.

Lemma 5.3 *Let $n = 1$ and set, for $0 < s < 1$,*

$$a(x, p) = \exp\left(-\frac{s}{\eta}(x^2 + p^2)\right).$$

We have $Op^{\mathrm{W}}_{\eta}(a) > 0$ and

$$Op^{\mathrm{W}}_{\eta}(a) = (1 - s^2)^{-1/2} \exp\left[\frac{1}{2\eta}\ln\left(\frac{1-s}{1+s}\right)(\widehat{x}^2 + \widehat{p}^2)\right] \tag{5.4.4}$$

where $\widehat{x}\psi = x\psi$ *and* $\widehat{p}\psi = -i\eta\partial_x$

Proof Set $a_\zeta(x,p) = e^{-2\pi\zeta(x^2+p^2)}$, $\zeta \neq 1$. In view of Corollary (5.29) in Folland [10], we have

$$\mathrm{Op}^{\mathrm{W}}_{1/2\pi}(a_\zeta) = (1-\zeta^2)^{-1/2}\exp\left[\pi\ln\left(\frac{1-\zeta}{1+\zeta}\right)(x^2 + \left(\tfrac{1}{2\pi i}\partial_x\right)^2)\right],$$

choosing $\eta' = 1/2\pi$ in formula (5.4.1) of Lemma 5.2 above yields formula (5.4.4).

In our study of entropy, we will need the following result.

Lemma 5.4 *Let* $Op^{\mathrm{W}}_\eta(a)$ *be defined as above with* $0 < s < 1$. *The logarithm of* $Op^{\mathrm{W}}_\eta(a)$ *exists and is given by*

$$\ln Op^{\mathrm{W}}_\eta(a) = -\frac{1}{2}\ln(1-s^2) + \frac{1}{2\eta}\ln\left(\frac{1-s}{1+s}\right)(\widehat{x}^2 + \widehat{p}^2). \tag{5.4.5}$$

Proof The result immediately follows from Lemma 5.3.

5.5 Metaplectic Covariance

Viewed abstractly, the metaplectic group $\mathrm{Mp}(n)$ is a unitary representation of the double cover $\mathrm{Sp}_2(n)$of the symplectic group $\mathrm{Sp}(n)$. The simplest (but not necessarily the most useful) way of describing $\mathrm{Mp}(n)$ is to use its elementary generators $\widehat{J}$, $\widehat{V}_{-P}$, and $\widehat{M}_{L,m}$; denoting by π^{Mp} the covering projection $\mathrm{Mp}(n) \longrightarrow \mathrm{Sp}(n)$ these operators and their projections are given by

$$\widehat{J}\psi(x) = e^{-in\pi/4}F\psi(x)\,,\ \pi^{\mathrm{Mp}}(\widehat{J}) = J$$
$$\widehat{V}_{-P}\psi(x) = e^{\frac{i}{2\eta}Px^2}\psi(x)\,,\ \pi^{\mathrm{Mp}}(\widehat{V}_{-P}) = V_{-P}$$
$$\widehat{M}_{L,m}\psi(x) = i^m\sqrt{|\det L|}\psi(Lx)\,,\ \pi^{\mathrm{Mp}}(\widehat{M}_{L,m}) = M_{L,m}.$$

here F is the Fourier transform

$$F\psi(x) = \left(\tfrac{1}{2\pi\eta}\right)^n \int e^{-\frac{i}{\eta}xx'}\psi(x')d^nx'$$

and V_{-P} $(P = P^T)$, $M_{L,m}$ $(\det L \neq 0)$ are the symplectic matrices

$$V_{-P} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix},\ M_{L,m} = \begin{pmatrix} L^{-1} & 0 \\ 0 & L^T \end{pmatrix}.$$

The index m in $\widehat{M}_{L,m}$ is an integer corresponding to a choice of $\arg\det L$: m and is even if $\det L > 0$ and odd if $\det L < 0$.

Using these generators, it is a simple exercise to show that the translation and reflection operators satisfy the symplectic covariance relations

$$\widehat{T}(Sz_0) = \widehat{S}\widehat{T}(z_0)\widehat{S}^{-1}\ ,\ \widehat{\Pi}(Sz_0) = \widehat{S}\widehat{\Pi}(z_0)\widehat{S}^{-1}$$

for every $\widehat{S} \in \mathrm{Mp}(n)$, $S = \pi^{\mathrm{Mp}}(\widehat{S})$. It follows from formula (5.4.2) or (5.4.3) that Weyl operators satisfy the similar formula

$$\mathrm{Op}_\eta^{\mathrm{W}}(a \circ S^{-1}) = \widehat{S}\mathrm{Op}_\eta^{\mathrm{W}}(a)\widehat{S}^{-1}.$$

Notice that a density operator $\widehat{\rho}$ remains a density operator under metaplectic conjugation: we have $\widehat{S}\widehat{\rho}\widehat{S}^{-1} \geq 0$ and $\mathrm{Tr}(\widehat{S}\widehat{\rho}\widehat{S}^{-1}) = \mathrm{Tr}(\widehat{\rho}) = 1$. In fact, conjugation does not affect the purity of the state since we have likewise

$$\mathrm{Tr}[(\widehat{S}\widehat{\rho}\widehat{S}^{-1})^2] = \mathrm{Tr}(\widehat{S}\widehat{\rho}^2\widehat{S}^{-1}) = \mathrm{Tr}(\widehat{\rho}^2).$$

5.6 Gaussian Density Operators

Let us return to the probability distribution ρ_Σ; we assume for simplicity that $z_0 = 0$ so

$$\rho_\Sigma(z) = (2\pi)^{-n}\sqrt{\det \Sigma^{-1}}e^{-\frac{1}{2}\Sigma^{-1}z^2}. \tag{5.6.1}$$

In view of Williamson's symplectic diagonalization theorem [5], there exists $S \in \mathrm{Sp}(n)$ such that $S\Sigma S^T = D$ where

$$D = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}\ ,\ \Lambda = \mathrm{diag}(\lambda_1^\sigma, \lambda_2^\sigma, ..., \lambda_n^\sigma)$$

the λ_j^σ being the symplectic eigenvalues of the covariance matrix Σ. We thus have

$$\rho_\Sigma(S^{-1}z) = (2\pi)^{-n}\sqrt{\det D^{-1}}e^{-\frac{1}{2}D^{-1}z^2}$$

that is

$$\rho_\Sigma \circ S^{-1} = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n \tag{5.6.2}$$

with

$$\rho_j(x_j, p_j) = \frac{1}{2\pi\lambda_j^\sigma}\exp\left(-\frac{1}{2\lambda_j^\sigma}(x_j^2 + p_j^2)\right). \tag{5.6.3}$$

Observing that $\mathrm{Op}_\eta^{\mathrm{W}}(\rho_\Sigma \circ S^{-1}) = \widehat{S}\mathrm{Op}_\eta^{\mathrm{W}}(\rho_\Sigma)\widehat{S}^{-1}$ where $\widehat{S} \in \mathrm{Mp}(n)$ covers S, we have

$$\widehat{S}\mathrm{Op}_\eta^{\mathrm{W}}(\rho_\Sigma)\widehat{S}^{-1} = \widehat{\rho}_1 \otimes \widehat{\rho}_2 \otimes \cdots \otimes \widehat{\rho}_n.$$

We have $\widehat{\rho}_j = (2\pi\eta)\mathrm{Op}_\eta^{\mathrm{W}}(\rho_j)$; hence, $\widehat{\rho}_j$ is the Weyl operator with symbol

$$a_j(x_j, p_j) = \frac{\eta}{\lambda_j^\sigma} \exp\left(-\frac{1}{2\lambda_j^\sigma}(x_j^2 + p_j^2)\right)$$

setting $s = \eta/2\lambda_j^\sigma$ in formula (5.4.4) in Lemma 5.3 we get

$$\widehat{\rho}_j = \frac{\eta}{\lambda_j^\sigma}(1 - (\eta/2\lambda_j^\sigma)^2)^{-1/2} \exp\left[\frac{1}{2\eta} \ln\left(\frac{1 - \eta/2\lambda_j^\sigma}{1 + \eta/2\lambda_j^\sigma}\right)(\widehat{x_j}^2 + \widehat{p_j}^2)\right]. \qquad (5.6.4)$$

The operator $\widehat{S}\mathrm{Op}_\eta^{\mathrm{W}}(\rho_\Sigma)\widehat{S}^{-1}$ and hence $\mathrm{Op}_\eta^{\mathrm{W}}(\rho_\Sigma)$ are positive-definite if and only if each $\widehat{\rho}_j$ is, that is if $s = \eta/2\lambda_j^\sigma < 1$. But this is the case in view of the quantization condition (5.2.2).

It follows from Proposition 5.4 that $\ln \widehat{\rho}_j = \ln((2\pi\eta)\mathrm{Op}_\eta^{\mathrm{W}}(a))$ is given by

$$\ln \widehat{\rho}_j = \ln\left(\eta/2\lambda_j^\sigma\right) - \frac{1}{2}\ln(1 - (\eta/2\lambda_j^\sigma)^2) + \frac{1}{2\eta} \ln\left(\frac{1 - \eta/2\lambda_j^\sigma}{1 + \eta/2\lambda_j^\sigma}\right)(\widehat{x_j}^2 + \widehat{p_j}^2). \qquad (5.6.5)$$

5.7 The Entropy of a Gaussian State

By definition, the (von Neumann) entropy of a density operator $\widehat{\rho}$ is the nonnegative number

$$\boldsymbol{S}(\widehat{\rho}) = -\mathrm{Tr}(\widehat{\rho}\ln\widehat{\rho}) \qquad (5.7.1)$$

where the logarithm $\ln\widehat{\rho}$ is defined as follows: Suppose that $\widehat{\rho}$ has the spectral decomposition

$$\widehat{\rho} = \sum_j \lambda_j \widehat{\rho}_j\,, \quad \sum_j \lambda_j = 1 \qquad (5.7.2)$$

where the λ_j are > 0 and $\widehat{\rho}_j$ are rank-one orthogonal projections in $L^2(\mathbb{R}^n)$. Then

$$\ln\widehat{\rho} = \sum_j (\ln\lambda_j)\widehat{\rho}_j. \qquad (5.7.3)$$

It follows from this definition that $\ln\widehat{\rho}$ is also a trace class operator. The Von Neumann entropy is the quantum variant of the Gibbs–Boltzmann entropy, defined for an usual probability distribution by

$$\boldsymbol{S}_{\mathrm{B}}(\rho) = -k_{\mathrm{B}} \int \rho(z)\ln\rho(z) d^{2n}z$$

where k_{B} is Boltzmann's constant; it is closely related to the Shannon entropy.

The following lemma summarizes the properties of entropy we will need.

Lemma 5.5 *Assume that $\widehat{\rho}$ has the spectral decomposition (5.7.2). (i) Then,*

$$\boldsymbol{S}(\widehat{\rho}) = -\sum_j \lambda_j \ln \lambda_j. \tag{5.7.4}$$

(ii) For every $\widehat{S} \in Mp(n)$, the density operators $\widehat{\rho}$ and $\widehat{S}\widehat{\rho}\widehat{S}^{-1}$ have same entropy:

$$\boldsymbol{S}(\widehat{S}\widehat{\rho}\widehat{S}^{-1}) = \boldsymbol{S}(\widehat{\rho}). \tag{5.7.5}$$

(iii) Let $\widehat{\rho}_{(1)}$ and $\widehat{\rho}_{(2)}$ be density operators on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. Then $\widehat{\rho}_{(1)} \otimes \widehat{\rho}_{(2)}$ is a density operator on $\mathbb{R}^n$, $n = n_1 + n_2$ and we have the additivity property

$$\boldsymbol{S}(\widehat{\rho}_{(1)} \otimes \widehat{\rho}_{(2)}) = \boldsymbol{S}(\widehat{\rho}_{(1)}) + \boldsymbol{S}(\widehat{\rho}_{(2)}. \tag{5.7.6}$$

Proof (i) The operator $\widehat{\rho} \ln \widehat{\rho}$ is the product of two trace class operators and is hence of trace class, and we have

$$\widehat{\rho} \ln \widehat{\rho} = \left(\sum_j \lambda_j \widehat{\rho}_j\right) \left(\sum_j \ln \lambda_j \widehat{\rho}_j\right) = \sum_j (\lambda_j \ln \lambda_j) \widehat{\rho}_j$$

the second equality because $\widehat{\rho}_j \widehat{\rho}_k = 0$ for $j \neq k$ and $\widehat{\rho}_j^2 = \widehat{\rho}_j$. It follows that the Weyl symbol of $\widehat{\rho} \ln \widehat{\rho}$ is the function

$$(2\pi\eta)^n \sum_j (\lambda_j \ln \lambda_j) \rho_j.$$

Let now $\rho = \sum_j \lambda_j \rho_j$ and $\sum_j \ln \lambda_j \rho_j$ be the Wigner distributions of $\widehat{\rho}$ and $\ln \widehat{\rho}$, respectively. In view of formula (5.3.2) in Lemma 5.1, we have

$$\begin{aligned}
\operatorname{Tr}(\widehat{\rho} \ln \widehat{\rho}) &= (2\pi\eta)^n \int \left(\sum_j \lambda_j \rho_j(z)\right) \left(\sum_j \ln \lambda_j \rho_j(z)\right) d^{2n}z \\
&= (2\pi\eta)^n \int \left(\sum_{j,k} \lambda_j \ln \lambda_k \rho_j(z) \rho_k(z)\right) d^{2n}z \\
&= (2\pi\eta)^n \sum_{j,k} \lambda_j \ln \lambda_k \int \rho_j(z) \rho_k(z) d^{2n}z.
\end{aligned}$$

Writing

$$\rho_j(z) = W\psi_j(z) = \left(\tfrac{1}{2\pi\eta}\right)^n \int e^{-\frac{i}{\eta}py} \psi_j(x + \tfrac{1}{2}y) \psi_j^*(x - \tfrac{1}{2}y) d^n y$$

Moyal's identity implies that

$$\int \rho_j(z) \rho_k(z) d^{2n}z = 0$$

for $j \neq k$ hence,

$$\mathrm{Tr}(\widehat{\rho} \ln \widehat{\rho}) = (2\pi\eta)^n \sum_j \lambda_j \ln \lambda_j \int \rho_j(z)^2 d^{2n}z.$$

Using once again the Moyal identity we have

$$\int \rho_j(z)^2 d^{2n}z = \left(\tfrac{1}{2\pi\eta}\right)^n ||\psi_j||^4 = \left(\tfrac{1}{2\pi\eta}\right)^n$$

and hence

$$\mathrm{Tr}(\widehat{\rho} \ln \widehat{\rho}) = \sum_j \lambda_j \ln \lambda_j$$

which proves (5.7.4). (ii) Formula (5.7.5) I is straightforward since $\widehat{S}\widehat{\rho}\widehat{S}^{-1}$ and $\widehat{\rho}$ have spectral decompositions with the same coefficients. (iii) Writing $\widehat{\rho}_{(1)} = \sum_j \lambda_j \widehat{\rho}_j$ and $\widehat{\rho}_{(2)} = \sum_k \mu_k \widehat{\rho}'_k$, we have

$$\widehat{\rho}_{(1)} \otimes \widehat{\rho}_{(2)} = \sum_{j,k} \lambda_j \mu_k \widehat{\rho}_j \otimes \widehat{\rho}'_k$$

hence,

$$\boldsymbol{S}(\widehat{\rho}_{(1)} \otimes \widehat{\rho}_{(2)}) = \sum_{j,k} \lambda_j \mu_k \ln \lambda_j \mu_k = \boldsymbol{S}(\widehat{\rho}_{(1)}) + \boldsymbol{S}(\widehat{\rho}_{(2)})$$

since $\sum_j \lambda = \sum_k \mu_k = 1$.

We now set out to calculate explicitly the entropy of the operator $\widehat{\rho_\Sigma}$ determined by the centered Gaussian (5.6.1). We begin by studying the case $n = 1$, which leads to a rigorous proof of a formula used in the physical literature and originally due to Agarwal [2].

Proposition 5.2 *Let $\widehat{\rho}_j$ be the Gaussian density operator with Wigner distribution (5.6.3). The entropy $\boldsymbol{S}(\widehat{\rho}_j) = -Tr(\widehat{\rho}_j \ln \widehat{\rho}_j)$ of $\widehat{\rho}_j$ is given by*

$$\boldsymbol{S}(\widehat{\rho}_j) = -\ln 2\mu_j + \frac{1}{2}\ln(1-\mu_j^2) - \frac{1}{2\mu_j}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right) \tag{5.7.7}$$

where $\mu_j = \eta/2\lambda_j^\sigma$ is the purity of $\widehat{\rho}_j$.

Proof Using formula (5.3.2), we have

$$\mathrm{Tr}(\widehat{\rho}_j \ln \widehat{\rho}_j) = \frac{1}{2\pi\eta}\int a(z)b(z)dz$$

where a and b are the symbols of

$$\widehat{\rho}_j = 2\mu_j(1-\mu_j^2)^{-1/2} \exp\left[\frac{1}{2\eta}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)(\widehat{x_j}^2 + \widehat{p_j}^2)\right]$$
$$\ln\widehat{\rho}_j = \ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2) + \frac{1}{2\eta}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)(\widehat{x_j}^2 + \widehat{p_j}^2)$$

respectively. We have

$$a(z) = (2\pi\eta)\frac{1}{2\pi\lambda_j^\sigma}e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)}$$
$$b(z) = \ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2) + \frac{1}{2\eta}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)(x_j^2 + p_j^2)$$

and hence,

$$\begin{aligned}\tfrac{1}{2\pi\eta}a(z)b(z) &= \frac{1}{2\pi\lambda_j^\sigma}e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)} \\ &\quad\times\left[\ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2) + \frac{1}{2\eta}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)(x_j^2+p_j^2)\right] \\ &= \frac{1}{2\pi\lambda_j^\sigma}e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)}\left[\ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2)\right] \\ &\quad+ \frac{1}{4\pi\eta\lambda_j^\sigma}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)}(x_j^2+p_j^2).\end{aligned}$$

Integrating, we get

$$\begin{aligned}\mathrm{Tr}(\widehat{\rho}_j\ln\widehat{\rho}_j) &= \left(\ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2)\right)\int\frac{1}{2\pi\lambda_j^\sigma}e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)}dp_j dx_j \\ &\quad+ \frac{1}{4\pi\eta\lambda_j^\sigma}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)\int e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)}(x_j^2+p_j^2)dp_j dx_j.\end{aligned}$$

Using the relations

$$\int\frac{1}{2\pi\lambda_j^\sigma}e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)}dp_j dx_j = 1$$

and

$$\int e^{-\frac{1}{2\lambda_j^\sigma}(x_j^2+p_j^2)}(x_j^2+p_j^2)dp_j dx_j = 4\pi(\lambda_j^\sigma)^2$$

we get, after a few simplifications,

$$\begin{aligned}\mathrm{Tr}(\widehat{\rho}_j \ln \widehat{\rho}_j) &= \ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2) + \frac{1}{4\pi\eta\lambda_j^\sigma}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)4\pi(\lambda_j^\sigma)^2 \\ &= \ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2) + \frac{\lambda_j^\sigma}{\eta}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right) \\ &= \ln 2\mu_j - \frac{1}{2}\ln(1-\mu_j^2) + \frac{1}{2\mu_j}\ln\left(\frac{1-\mu_j}{1+\mu_j}\right)\end{aligned}$$

since $\mu_j = \eta/2\lambda_j^\sigma \iff 2\mu_j = \eta/\lambda_j^\sigma \iff \lambda_j^\sigma/\eta = 1/2\mu_j$. Formula (5.7.7) follows since $\boldsymbol{S}(\widehat{\rho}_j) = -\mathrm{Tr}(\widehat{\rho}_j \ln \widehat{\rho}_j)$.

The case or arbitrary n immediately follows.

Corollary 5.1 *The entropy of the Gaussian mixed state $\widehat{\rho}_\Sigma$ is given by*

$$\boldsymbol{S}(\widehat{\rho}_\Sigma) = \sum_{j=1}^{n} \boldsymbol{S}(\widehat{\rho}_j) \tag{5.7.8}$$

where $\boldsymbol{S}(\widehat{\rho}_j)$ is given by formula (5.7.7).

Proof The trace, the purity, and the entropy are invariant under conjugation with unitary operators; hence, it is sufficient to assume that $\widehat{\rho}_\Sigma = \widehat{\rho}_1 \otimes \cdots \otimes \widehat{\rho}_n$. Formula (5.7.8) follows from the additivity property (5.7.6) of the entropy.

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Chapter 6
On the Continuity of τ-Wigner Pseudodifferential Operators

Lorenza D'Elia

Abstract In this survey, we recollect the latest results about the continuity of τ-pseudodifferential operators. We obtain boundedness results for these operators with symbols in Wiener amalgam spaces for $\tau \in (0, 1)$, exhibiting a function of real parameter τ which is an upper bound for the operator norm. In general, for $\tau = 0$ and $\tau = 1$ the corresponding operators are unbounded. For the well-known continuity properties of τ-pseudodifferential operators with symbols in modulation spaces, we find an upper bound for the operator norm which does not depend on τ.

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The purpose of this survey is to shed light on recent development on continuity properties of τ-pseudodifferential operators with symbols in modulation and Wiener amalgam spaces.

The theory of pseudodifferential operators is relatively young; in its modern form, it was developed about mid-sixties. One of the predecessors of this theory is surely Calderón [5], who employed the Fourier transform in order to turn a study of linear partial differential equations into an algebraic analysis of characteristic polynomials (or symbols) of differential equations. This revolutionary viewpoint has been extended and investigated by the works of Kohn and Nirenberg [42], Hörmander [40] who coined the modern form of pseudodifferential operators:

$$\mathrm{Op}_0(a)f(x) = \int_{\mathbb{R}^d} a(x,\xi)\hat{f}(\xi)e^{2\pi i x\xi}\mathrm{d}\xi, \qquad f \in S(\mathbb{R}^d), \tag{6.0.1}$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x)\mathrm{d}x$ is the Fourier transform and $a(x,\xi)$ is the so-called symbol. Formula (6.0.1) is known as Kohn–Nirenberg operator. It is broadly used in the context of partial differential equations. The theory of pseudodifferential operators is also employed in another field: quantum mechanics. Indeed, it provides

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good mathematical tools in order to solve the quantization problem. Roughly speaking, a quantization is a rule that assigns an operator to a function, called symbol, on the phase space $\mathbb{R}^{2d}$. Weyl proposed in [56] the first quantization procedure which is widely used nowadays. The Weyl operator or pseudodifferential operator in Weyl form is defined as

$$\mathrm{Op}_{1/2}(a)f(x) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a\left(\frac{x+y}{2}, \xi\right) f(y) e^{2\pi i(x-y)\xi} \mathrm{d}y\mathrm{d}\xi, \qquad f \in S(\mathbb{R}^d). \tag{6.0.2}$$

A comprehensive study of this operator can be found in Wong's book [58].

Since the end of 90s, the tools of time–frequency analysis turned out to be appropriate in order to extend the investigation of pseudodifferential operators. Here, their weak definition is used: they are defined by means of duality pairing between the time–frequency representations and the symbols. From a mathematical point of view, time–frequency representations, which allow the time and the frequency variable to jointly coexist, are described by quadratic forms, originated from their associated sesquilinear forms. The advantage of using the weak definition is due to the existence of a one-to-one correspondence between operators and sesquilinear forms, which shows that the boundedness of operators defined on suitable function spaces is equivalent to that of their associated forms (cf. [2]). With this in mind, (6.0.1) may be written as

$$\langle \mathrm{Op}_0(a)f, g\rangle = \langle a, R(g, f)\rangle, \qquad f, g \in S(\mathbb{R}^d),\ a \in S'(\mathbb{R}^{2d}),$$

where $R(g,\ f) = e^{-2\pi i x\xi} g(x)\overline{\hat{f}(\xi)}$ is the Rihaczek distribution. Likewise, formula (6.0.2) is rewritten as

$$\langle \mathrm{Op}_{1/2}(a)f, g\rangle = \langle a, W(g, f)\rangle, \qquad f, g \in S(\mathbb{R}^d),\ a \in S'(\mathbb{R}^{2d}).$$

In this case, the time–frequency representation associated to Weyl operator is one of the most popular ones, the well-known (cross-)Wigner distribution [55, 57]

$$W(f, g)(x, \xi) = \int_{\mathbb{R}^d} f\left(x + \frac{t}{2}\right)\overline{g\left(x - \frac{t}{2}\right)} e^{-2\pi i t\xi} \mathrm{d}t, \qquad f, g \in S(\mathbb{R}^d). \tag{6.0.3}$$

For $f = g$, $W(f, f)$ is simply called the Wigner distribution. For a collection of its properties, we refer to Gröchenig's book [36].

In this survey, we are interested in studying a particular class of pseudodifferential operators depending on a real parameter τ, called τ-operators or Shubin operators [46]. From a time–frequency analysis perspective, for $\tau \in [0, 1]$, they are defined as

$$\langle \mathrm{Op}_\tau(a)f, g\rangle = \langle a, W_\tau(g, f)\rangle, \qquad f, g \in S(\mathbb{R}^d),\ a \in S'(\mathbb{R}^{2d}), \tag{6.0.4}$$

where τ-Wigner distribution $W_\tau(f, g)$ is defined as

$$W_\tau(f,g)(x,\xi) = \int_{\mathbb{R}^d} f(x+\tau t)\overline{g(x-(1-\tau)t)}e^{-2\pi i\xi t}\mathrm{d}t, \qquad f,g \in S(\mathbb{R}^d).$$

Varying the parameter τ, $W_\tau(f,g)$ is a family of time–frequency representations, where in the left endpoint $\tau = 0$ $W_0(f,g)$ coincides with the Rihaczek distribution $R(f,g)$, in the right endpoint $\tau = 1$ $W_1(f,g)$ becomes the conjugate Rihaczek distribution

$$W_1(f,g)(x,\xi) = \overline{R}(f,g)(x,\xi) = e^{2\pi i x\xi}\overline{g}(x)\hat{f}(\xi),$$

and in the middle point $\tau = 1/2$, we recover the Wigner distribution. We may think of W_τ as a path joining the Rihaczek distribution and its conjugate and having as middle point the Wigner distribution. Likewise, Formula (6.0.4) amounts to a collection of pseudodifferential operators: Op_0 and Op_1 are, respectively, the Kohn–Nirenberg and its adjoint (also called anti-Kohn–Nirenberg) operators, whereas in the middle point $\mathrm{Op}_{1/2}$ becomes the classical Weyl operator (6.0.2).

The continuity of pseudodifferential operators has been studied by plenty of authors. In the framework of Kohn–Nirenberg operator, a deep analysis has been carried on for the so-called Hörmander class [40] $\mathscr{S}^m_{\rho,\delta}$, $m \in \mathbb{R}$, $0 \le \delta \le \rho \le 1$. This class of symbols consists of all functions $a(x,\xi) \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m+\delta|\alpha|-\rho|\beta|},$$

for any multi-index $\alpha, \beta \in \mathbb{Z}^n_+$. Calderón and Vaillancourt [6] first showed that any pseudodifferential operators with symbols in the class $\mathscr{S}^0_{0,0}$ are continuous on $L^2(\mathbb{R}^d)$. The request that the symbol belongs to $\mathscr{S}^0_{0,0}$ means all of its derivatives are bounded. Many efforts have been done in order to seek the minimal assumptions on the regularity of the symbols: we mention Coifamn and Meyer [7], Cordes [30], Kato [41], Nagase [45] who proved that to obtain the continuity on $L^2(\mathbb{R}^d)$ is sufficient the boundedness of the symbol's derivatives up to a certain order. The investigation of continuity properties on $L^2(\mathbb{R}^d)$ leads to a class of symbols, larger than $\mathscr{S}^0_{0,0}$, introduced by Sjöstrand [47] and then recognized to be the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$. These spaces was introduced in the context of time–frequency analysis by Feichtinger [33–35]: they measure the time–frequency decay of a function/distribution in the phase space. Many authors investigated the continuity of pseudodifferential operators with symbols in classical modulation spaces. The earliest paper was that of Gröchenig and Heil [38] in which they showed that if the symbol $a \in M^{\infty,1}(\mathbb{R}^{2d})$, the corresponding operator is bounded on $L^2(\mathbb{R}^d)$ and on the modulation spaces $M^p(\mathbb{R}^d)$, $1 \le p \le \infty$. Sugimoto and Tomita [48] investigated the boundedness of pseudodifferential operators with symbols in the Hörmander class $S^m_{\rho,\delta}$ on the modulation spaces $M^{p,q}(\mathbb{R}^d)$. In [15], Cordero and Nicola give a complete characterization of the continuity of pseudodifferential operators with symbols in modulation spaces $M^{p,q}(\mathbb{R}^{2d})$ acting on Lebesgue spaces $L^r(\mathbb{R}^d)$ and on Wiener amalgam spaces $W(L^r, L^s)(\mathbb{R}^d)$. These spaces arise as the Fourier transform of the modulation spaces $M^{p,q}(\mathbb{R}^d)$ and their inventor Feichtinger [35] suggests to call them simply modulation spaces. In [32], we handle the boundedness properties of Weyl and Kohn–Nirenberg operators acting on weighted modulation spaces $M^{r_1,r_2}(\mathbb{R}^d)$ with symbol in Wiener amalgam spaces $W(\mathscr{F}L^p, L^q)(\mathbb{R}^{2d})$. We mention also other important contributions in this framework [1, 8, 9, 18, 29, 37, 43, 51–53].

Pseudodifferential operators are a particular case of Fourier integral operators (FIOs), which are defined as

$$\int_{\mathbb{R}^d} e^{2\pi i\Phi(x,\xi)} a(x,\xi)\hat{f}(\xi)\mathrm{d}\xi.$$

The function a is called the symbol and Φ is the phase function. We recognize at once that pseudodifferential operators are FIOs with phase function $\Phi(x,\xi) = 2\pi i x\xi$. FIOs are extensively used in the analysis of the behaviour of the solution of Schrödinger equations. We bring to mind some of the works focused on the continuity properties of FIOs: Cordero and Nicola [14] provide assumptions on the symbol and on the phase function which guarantee the boundedness of FIOs acting on the modulation spaces $M^{p,q}(\mathbb{R}^d)$, $1 \le q < p \le \infty$. In [28], Cordero, Tabacco, Wahlberg give the optimal boundedness result for FIOs with symbol in modulation spaces and acting on the same spaces. We recall also the paper of Cordero, Gröchenig, Nicola and Rodino [11], Cordero and Nicola [13, 14, 17], Cordero, Nicola and Rodino [19–26], in which they continue the investigation of continuity of FIOs on the modulation spaces. Moreover, we mention Trèves' book [54] where the author introduces the classical theory of FIOs.

Localization operators, first introduced in time–frequency analysis by Daubechies [31], can be regarded as pseudodifferential operators. More precisely, the localization operator $A_a^{\varphi_1,\varphi_2}$ with symbol $a \in S'(\mathbb{R}^{2d})$ and windows $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$ is the Weyl operator $\mathrm{Op}_{1/2}(a * W(\varphi_1,\varphi_2))$, where the symbol is the convolution of a with the Wigner distribution of the windows $W(\varphi_1,\varphi_2)$. Namely, for any $f, g \in S(\mathbb{R}^d)$, $a \in S'(\mathbb{R}^{2d})$

$$\langle A_a^{\varphi_1,\varphi_2} f, g\rangle = \langle \mathrm{Op}_{1/2}(a * W(\varphi_1,\varphi_2)) f, g\rangle = \langle a * W(\varphi_1,\varphi_2), W(g, f)\rangle.$$

The above formula permits us to study these operators in the realm of modulation spaces. In [10], the authors showed that if the symbol a of the localization operator satisfies some weak time–frequency concentration assumptions, i.e. $a \in M^\infty(\mathbb{R}^{2d})$ and the windows $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, then $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$. Moreover, if the localization operator $A_a^{\varphi_1,\varphi_2}$ is bounded on $L^2(\mathbb{R}^d)$ uniformly with respect to all windows $\varphi_1, \varphi_2 \in M^1(\mathbb{R}^d)$, then the symbol a necessarily belongs to the modulation space $M^\infty(\mathbb{R}^{2d})$. Cordero and Nicola [16] give sufficient and necessary conditions for the localization operators in order to be a continuous map on the Lebegues space $L^p(\mathbb{R}^d)$ and Wiener amalgam spaces $W(L^p, L^q)(\mathbb{R}^d)$ when the symbol a belongs to the same spaces. Their approach differs from [10] since they write $A_a^{\varphi_1,\varphi_2}$ as an integral operator and not as Weyl operator. The paper [12] recollects some of the most important boundedness result regarding localization operators. Other continuity properties can be found in [27, 39, 49–51].

In this paper, we persist in analysing the continuity of τ-operators with symbols in the Wiener amalgam spaces. We prove the continuity for $\tau \in (0, 1)$, while in the endpoints $\tau = 0$ and $\tau = 1$ the Kohn–Nirenberg and its adjoint fail to be bounded. This claim is proved by a counterexample, which generalizes one-dimensional one exhibited by Boulkhermair in [4]. Furthermore, in the case of bounded τ-operators, i.e. for $\tau \in (0, 1)$, we produce a function of real paramater τ, which is an upper bound for the operator norm: for $1 \leq r_1, r_2 \leq \infty$, if we set $\gamma = 1/r_1 - 1/r_2$ ones has:

$$\alpha_{(r_1,r_2)}(\tau) = \frac{1}{\tau^{d(1-\gamma)}(1-\tau)^{d(1+\gamma)}}. \tag{6.0.5}$$

Then we discuss the continuity properties of τ-operators with symbols in modulation spaces. It is well known that the results for Weyl operators with symbols in modulation spaces are still true for any τ-operators (cf. [51, Proposition 1.2 (5)]). We find an upper bound for the operator norm which actually does not depend on parameter τ, as expected.

The paper is organized as follows: in Sect. 6.2, we recall the necessary definitions and notations that we will use. In Sect. 6.3, we present the existing boundedness properties of τ-operators in Lebesgue spaces. Section 6.3 is addressed to introduce the new results of continuity of τ-operators with symbols in Wiener amalgam and modulation spaces.

6.1 Preliminaries

In this section, we recollect some definitions and notations concerning our setting.

A weight function on $\mathbb{R}^{2d}$ is a positive and locally integrable function. A submultiplicative weight v on $\mathbb{R}^{2d}$ is such that

$$v(z_1 + z_2) \leq v(z_1)v(z_2), \qquad z_1, z_2 \in \mathbb{R}^{2d}.$$

Given a submultiplicative weight v, a positive function m on $\mathbb{R}^{2d}$ is a v-moderate weight if there exists a constant $C \geq 0$ such that

$$m(z_1 + z_2) \leq Cv(z_1)m(z_2), \qquad z_1, z_2 \in \mathbb{R}^{2d}.$$

In what follows, we denote with $\mathscr{M}_v(\mathbb{R}^{2d})$ the set of all v-moderate weights. The Schwartz class is denoted by $S(\mathbb{R}^d)$ and its dual with $S'(\mathbb{R}^d)$. The inner product on $\mathbb{R}^d$ is simply written as $xy = x \cdot y$.

Among time–frequency representations, we recall the definition of the short-time frequency representation, which is the key ingredient in order to define our Banach function spaces. Given a nonzero window $g \in S(\mathbb{R}^d)$, the short-time Fourier transform (STFT) is defined as

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)}e^{-2\pi i t\xi}\mathrm{d}t, \qquad f \in S(\mathbb{R}^d).$$

We refer the readers to [36] for its properties. Fixed a nonzero window $g \in S(\mathbb{R}^d)$ and $m \in \mathscr{M}_v(\mathbb{R}^{2d})$, the modulation space $M_m^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, consists of all tempered distribution $f \in S'(\mathbb{R}^d)$ such that

$$\|f\|_{M_m^{p,q}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p m^p(x,\xi)\mathrm{d}x\right)^{q/p}\mathrm{d}\xi\right)^{1/q} < \infty,$$

with obvious modifications for the case $p = \infty$ or $q = \infty$. Given even weights u, v on $\mathbb{R}^d$, the Wiener amalgam space $W(\mathscr{F}L_u^p, L_w^q)(\mathbb{R}^d)$ is the set of all $f \in S'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathscr{F}L_u^p, L_w^q)} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_g f(x,\xi)|^p u^p(\xi)\mathrm{d}\xi\right)^{q/p} w^q(x)\mathrm{d}x\right)^{1/q} < \infty.$$

It is easy to check that Wiener amalgam spaces are the image under Fourier transform of modulation spaces, i.e. $\mathscr{F}(M_{u\otimes v}^{p,q}) = W(\mathscr{F}L_u^p, L_w^q)$. Both modulation and Wiener amalgam spaces are Banach spaces independent of the choice of window function, in the sense that different window g yield equivalent norm. For their basic properties, we refer to [36].

In the sequel, we denote by J the canonical symplectic matrix in $\mathbb{R}^{2d}$

$$J = \begin{pmatrix} 0_{d\times d} & Id_{d\times d} \\ -Id_{d\times d} & 0_{d\times d} \end{pmatrix} \in Sp(2d,\mathbb{R}),$$

where the symplectic group $Sp(2d,\mathbb{R})$ is defined as

$$Sp(2d,\mathbb{R}) = \{M \in GL(2d,\mathbb{R}) : M^T J M = J\}.$$

Fixed a submultiplicative weight v, we set $v_J(x) = v(Jx)$. The conjugate exponent p' of p, $1 \le p \le \infty$, is defined as $1/p + 1/p' = 1$.

6.2 Boundedness Properties in Lebesgue Spaces

The boundedness of τ-operators $\mathrm{Op}_\tau(a)$ with symbols in classical Lebegue spaces $L^p(\mathbb{R}^{2d})$ was extensively studied. In [58], Wong shows that if the symbol $a \in L^p(\mathbb{R}^{2d})$, $1 \le p \le 2$, then the corresponding τ-operator is continuous map on the Hilbert space $L^2(\mathbb{R}^d)$. An extended version of this result is been provided in [59] by the same author. Here, he proved that if the symbol $a \in L^1(\mathbb{R}^{2d})$, then $\mathrm{Op}_\tau(a)$ is a bounded operator on $L^p(\mathbb{R}^d)$. A deep analysis of pseudodifferential operators is carried on [60]. Let us note that the above results hold for every $\tau \in [0, 1]$.

We mention the results of Boggiatto et al. [3] in which they completely characterize the continuity properties of τ-operators in the Lebesgue spaces.

Theorem 6.1 ([3] Theorem 6.6)

(i) *If* $\tau \in (0, 1)$, *the map*

$$a \in L^q(\mathbb{R}^{2d}) \mapsto Op_\tau(a) \in B(L^p(\mathbb{R}^d))$$

is continuous if and only if $q \leq 2$ *and* $q \leq p \leq q'$, *with corresponding norm estimate*

$$\|Op_\tau(a)\|_{B(L^p)} \leq C\|a\|_{L^q}, \qquad C > 0.$$

(ii) *If* $\tau = 0$, *the Kohn–Nirenberg correspondence*

$$a \in L^q(\mathbb{R}^{2d}) \mapsto Op_0(a) \in B(L^p(\mathbb{R}^d))$$

is continuous if and only if $p = q$ *and* $q \leq 2$. *The norm estimate is*

$$\|Op_0(a)\|_{B(L^p)} \leq C\|a\|_{L^q}, \qquad C > 0.$$

(iii) *If* $\tau = 1$, *the map*

$$a \in L^q(\mathbb{R}^{2d}) \mapsto Op_1(a) \in B(L^p(\mathbb{R}^d))$$

is continuous if and only if $p = q'$ *and* $q \leq 2$. *Moreover,*

$$\|Op_1(a)\|_{B(L^p)} \leq C\|a\|_{L^q}, \qquad C > 0.$$

6.3 Continuity of τ-Operators with Symbols in Wiener and Modulation Spaces

This section concerns the discussion of the new boundedness results of τ-operators with symbols in Wiener amalgam and modulation spaces.

The key tools in order to obtain the boundedness of τ-operators are the norm estimates of τ-Wigner distributions. Firstly, we compute their STFTs, since they allow us to compute the Wiener amalgam and modulation norms.

Lemma 6.1 *(i) Consider $\tau \in (0,1)$. Let $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$, $f, g \in S(\mathbb{R}^d)$ and set $\Phi_\tau = W_\tau(\varphi_1, \varphi_2)$. Then,*

$$V_{\Phi_\tau} W_\tau(g, f)(z, \zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1 - \tau\zeta_2, z_2 + (1-\tau)\zeta_1) \times \overline{V_{\varphi_2} f(z_1 + (1-\tau)\zeta_2, z_2 - \tau\zeta_1)}$$

where $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$. Equivalently,

$$V_{\Phi_\tau} W_\tau(g, f)(z, \zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z + \sqrt{\tau(1-\tau)} A_\tau^T \zeta) \overline{V_{\varphi_2} f(z + \sqrt{\tau(1-\tau)} A_\tau \zeta)},$$

where A_τ is symplectic matrix defined as

$$A_\tau = \begin{pmatrix} 0_{d\times d} & (\frac{1-\tau}{\tau})^{1/2} I_{d\times d} \\ -(\frac{\tau}{1-\tau})^{1/2} I_{d\times d} & 0_{d\times d} \end{pmatrix}, \quad \tau \in (0,1).$$

(ii) Let $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$, $f, g \in S(\mathbb{R}^d)$ and set $\Phi_0 = W_0(\varphi_1, \varphi_2)$. Then,

$$V_{\Phi_0} W_0(g, f)(z, \zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1, z_2 + \zeta_1) \overline{V_{\varphi_2} f(z_1 + \zeta_2, z_2)},$$

where $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

(iii) Let $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$, $f, g \in S(\mathbb{R}^d)$ and set $\Phi_1 = W_1(\varphi_1, \varphi_2)$. Then,

$$V_{\Phi_1} W_1(g, f)(z, \zeta) = e^{-2\pi i z_2 \zeta_2} V_{\varphi_1} g(z_1 - \zeta_2, z_2) \overline{V_{\varphi_2} f(z_1, z_2 - \zeta_1)},$$

where $z = (z_1, z_2)$, $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^{2d}$.

Let us emphasize that the computation of STFT is been made with respect to a window function which depends on the real parameter τ.

Lemma 6.2 *Consider $\Phi(x, \xi) = e^{-\pi(x^2+\xi^2)}$, $(x, \xi) \in \mathbb{R}^{2d}$, and $\Phi_\tau = W_\tau(\varphi, \varphi)$, where $\varphi(t) = e^{-\pi t^2}$, $t \in \mathbb{R}^d$. Then there exists a constant $C > 0$ such that*

$$\|V_\Phi \Phi_\tau\|_{L^1_{1\otimes v_J}} \le C, \quad \forall \tau \in [0,1].$$

Consequently,

$$\|\Phi_\tau\|_{M^1_{1\otimes v_J}} \le C, \quad \forall \tau \in [0,1].$$

Lemma 6.1 along with Lemma 6.2 permits us to release the dependence of τ from $W(\mathscr{F}L^1_{1/v_J}, L^\infty)(\mathbb{R}^d)$ and $W(\mathscr{F}L^2_{1/v_J}, L^2)(\mathbb{R}^d)$ norm of τ-Wigner distribution, obtaning uniform estimates.

Proposition 6.1 *(i) Assume that $m \in \mathscr{M}_v(\mathbb{R}^{2d})$, $1 \le p_1, p_2 \le \infty$, $f \in M^{p_1,p_2}_m(\mathbb{R}^d)$, $g \in M^{p_1',p_2'}_{1/m}(\mathbb{R}^d)$. Then for every $\tau \in (0,1)$, the τ-Wigner distribution $W_\tau(g, f)$ is in $W(\mathscr{F}L^1_{1/v_J}, L^\infty)(\mathbb{R}^{2d})$, with*

$$\|W_\tau(g,f)\|_{W(\mathscr{F}L^1_{1/v_J},L^\infty)} \leq C\alpha_{(p_1,p_2)}(\tau)\|f\|_{M^{p_1,p_2}_m}\|g\|_{M^{p_1',p_2'}_{1/m}}, \tag{6.3.1}$$

where the function $\alpha_{(p_1,p_2)}(\tau)$ is defined in (6.0.5) and $C>0$ is independent of τ.

(ii) Let $m \in \mathscr{M}_v(\mathbb{R}^{2d})$, $f \in M^2_m(\mathbb{R}^d)$ and $g \in M^2_{1/m}(\mathbb{R}^d)$. For $\tau \in (0,1)$, the $W_\tau(g,f) \in W(\mathscr{F}L^2_{1/v_J}, L^2)(\mathbb{R}^{2d})$, with the uniform estimate

$$\|W_\tau(g,f)\|_{W(\mathscr{F}\ L^2_{1/v_J},L^2)} \leq C\|f\|_{M^2_m}\|g\|_{M^2_{1/m}}, \tag{6.3.2}$$

where the positive constant C is independent of τ.

These estimates for the τ-Wigner distribution are translated in ones for the corresponding τ-operators, which are assumed to be defined in th weak sense.

Proposition 6.2 *Consider $m \in \mathscr{M}_v(\mathbb{R}^{2d})$ and a symbol $a \in W(\mathscr{F}L^\infty_{v_J}, L^1)(\mathbb{R}^{2d})$. Then for every $\tau \in (0,1)$, the τ-pseudodifferential operator $Op_\tau(a)$ is bounded on $M^{p_1,p_2}_m(\mathbb{R}^d)$, for every $1 \leq p_1, p_2 \leq \infty$, with*

$$\|Op_\tau(a)f\|_{M^{p_1,p_2}_m} \leq C\alpha_{(p_1,p_2)}(\tau)\|a\|_{W(\mathscr{F}L^\infty_{v_J},L^1)}\|f\|_{M^{p_1,p_2}_m}, \tag{6.3.3}$$

where the function $\alpha_{(p_1,p_2)}(\tau)$ is defined by (6.0.5) and the constant $C>0$ does not depend on τ.

Proposition 6.3 *Given $m \in \mathscr{M}_v(\mathbb{R}^{2d})$, $a \in W(\mathscr{F}L^2_{v_J}, L^2)(\mathbb{R}^{2d})$ and $\tau \in (0,1)$. Then the operator $Op_\tau(a)$ is bounded on $M^2_m(\mathbb{R}^d)$ with*

$$\|Op_\tau(a)f\|_{M^2_m} \leq C\|a\|_{W(\mathscr{F}L^2_{v_J},L^2)}\|f\|_{M^2_m}, \tag{6.3.4}$$

and $C>0$ is independent of τ.

By exploiting the complex interpolation, we show a more general continuity property of Op_τ, which provides an upper bound for the operator norm which is a function of the real parameter τ.

Theorem 6.2 *Let $1 \leq p,q,r_1,r_2 \leq \infty$ be such that*

$$q \leq p', \qquad \text{and} \qquad \max\{r_1,r_2,r_1',r_2'\} \leq p. \tag{6.3.5}$$

Let $m \in \mathscr{M}_v(\mathbb{R}^{2d})$ and $a \in W(\mathscr{F}L^p_{v_J}, L^q)(\mathbb{R}^{2d})$. Every τ-pseudodifferential operator $Op_\tau(a)$, $\tau \in (0,1)$, is a bounded on $M^{r_1,r_2}_m(\mathbb{R}^{2d})$. Moreover, there exists a constant $C>0$ independent of τ such that

$$\|Op_\tau(a)f\|_{M^{r_1,r_2}_m} \leq C\alpha_{(r_1,r_2)}(\tau)\|a\|_{W(\mathscr{F}L^p_{v_J},L^q)}\|f\|_{M^{r_1,r_2}_m}, \quad \tau \in (0,1). \tag{6.3.6}$$

Proof By regarding Op_τ as the bilinear map $(a,f) \mapsto \mathrm{Op}_\tau(a)f$, Propositions 6.2 and 6.3 give the continuity of the $\mathrm{Op}_\tau(a)$ on the following function spaces

$$W(\mathscr{F}L^\infty_{v_J}, L^1)(\mathbb{R}^{2d}) \times M^{p_1,p_2}_m(\mathbb{R}^d) \to M^{p_1,p_2}_m(\mathbb{R}^d),$$
$$W(\mathscr{F}L^2_{v_J}, L^2)(\mathbb{R}^{2d}) \times M^2_m(\mathbb{R}^d) \to M^2_m(\mathbb{R}^d),$$

for $1 \le p_1, p_2 \le \infty$. Using the complex interpolation between Wiener amalgam and modulation spaces, for $\theta \in [0,1]$, we have

$$[W(\mathscr{F}L^\infty_{v_J}, L^1), W(\mathscr{F}L^2_{v_J}, L^2)]_\theta = W(\mathscr{F}L^p_{v_J}, L^{p'}),$$

with $2 \le p \le \infty$, and $[M^{p_1,p_2}_m, M^2_m]_\theta = M^{r_1,r_2}_m$, with

$$\frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{2} = \frac{1-\theta}{p_1} + \frac{1}{p}$$

and

$$\frac{1}{r_2} = \frac{1-\theta}{p_2} + \frac{\theta}{2} = \frac{1-\theta}{p_2} + \frac{1}{p}$$

so that $r_1, r_2 \le p$. Similarly, we obtain $r_1', r_2' \le p$, and thus the second relation of (6.3.5). Due to inclusion relations for Wiener amalgam spaces, we relax the assumptions on symbols, so that the symbol a may belong to $W(\mathscr{F}L^p_{v_J}, L^q)(\mathbb{R}^{2d})$, with $q \le p'$, which gives the first relation of (6.3.5). Finally, the norm is provided by

$$\begin{aligned}
\|\mathrm{Op}_\tau\|_{B(W(\mathscr{F}L^p_{v_J},L^q)\times M^{r_1,r_2}_m, M^{r_1,r_2}_m)} &\le \|\mathrm{Op}_\tau\|^{1-\theta}_{B(W(\mathscr{F}L^\infty_{v_J},L^1)\times M^{p_1,p_2}_m, M^{p_1,p_2}_m)} \\
&\qquad \times \|\mathrm{Op}_\tau\|^{\theta}_{B(W(\mathscr{F}L^2_{v_J},L^2)\times M^2_m, M^2_m)} \\
&\le C \frac{1}{\tau^{d(1-\theta)\left(1-\frac{1}{p_1}+\frac{1}{p_2}\right)}(1-\tau)^{d(1-\theta)\left(1+\frac{1}{p_1}-\frac{1}{p_2}\right)}} \\
&\le C \frac{1}{\tau^{d\left(1-\frac{1}{p_1}+\frac{1}{p_2}\right)}(1-\tau)^{d\left(1+\frac{1}{p_1}-\frac{1}{p_2}\right)}},
\end{aligned}$$

since $1-\theta \le 1$. This concludes the proof. □

This result does not hold in the endpoints $\tau = 0$ and $\tau = 1$. A first suggestion comes from the fact the function $\alpha_{(r_1,r_2)}(\tau)$ is unbounded on $(0,1)$: indeed for $(r_1, r_2) \ne \{(1,+\infty), (+\infty, 1)\}$,

$$\lim_{\tau\to 0^+} \alpha_{(r_1,r_2)}(\tau) = \lim_{\tau \to 1^-} \alpha_{(r_1,r_2)}(\tau) = +\infty,$$

and for $(r_1, r_2) = (1, +\infty)$, $\lim_{\tau\to 1^-} \alpha_{(1,+\infty)}(\tau) = +\infty$ and for $(r_1, r_2) = (+\infty, 1)$, $\lim_{\tau\to 0^+} \alpha_{(+\infty,1)}(\tau) = +\infty$. Inspired by [4], we have found a counterexample that permits us to shows that Kohn–Nirenberg and its adjoint operator are unbounded on $L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d)$. We define the symbol

$$a(x_1, \ldots, x_d, \xi_1, \ldots, \xi_d) = x_1^{-1/2} \cdots x_d^{-1/2} \chi_{(0,1]}(x_1) \cdots \chi_{(0,1]}(x_d) e^{\pi \xi^2},$$

then it is easy to compute that $\mathrm{Op}_0(a)f \notin L^2(\mathbb{R}^d)$, when f is the Gaussian function $f(t) = e^{\pi t} \in L^2(\mathbb{R}^d)$. As a consequence, the anti-Kohn–Nirenberg operator $\mathrm{Op}_1(a)f$ is also unbounded, since it is the adjoint Kohn–Nirenberg one.

As far as the symbols in modulation spaces concerned, we obtain the following result. As expected, the norm estimate does not depend of τ.

Theorem 6.3 *Let $p_1, p_2, q_1, q_2, p, q \in [0, 1]$ be such that*

$$p_1, p_2', q_1, q_2' \leq q', \qquad \frac{1}{p_1} + \frac{1}{p_2'} \geq \frac{1}{p'} + \frac{1}{q'}, \qquad \frac{1}{q_1} + \frac{1}{q_2'} \geq \frac{1}{p'} + \frac{1}{q'}.$$

Let $m \in \mathscr{M}_v(\mathbb{R}^{2d})$. For any $\tau \in [0, 1]$, $Op_\tau(a)$ with symbol $a \in M^{p,q}_{1\otimes v_J}(\mathbb{R}^{2d})$, is a bounded operator from $M^{p_1,q_1}_m(\mathbb{R}^d)$ to $M^{p_1,q_1}_m(\mathbb{R}^d)$. Furthermore,

$$\|Op_\tau(a)f\|_{M^{p_2,q_2}_m} \leq C\|a\|_{M^{p,q}_{1\otimes v_J}} \|f\|_{M^{p_1,q_1}_m},$$

with $C > 0$ independent of parameter τ.

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Chapter 7
Gabor Expansions of Signals: Computational Aspects and Open Questions

Hans G. Feichtinger

Abstract In the last 40 years the foundations of Gabor analysis, even in the context of locally compact Abelian (LCA) groups, have been widely developed. We know a lot about function spaces, in particular *modulation spaces*, characterization of these spaces via Gabor expansions, or mapping properties of operators between such spaces, even the description of solutions for PDEs can nowadays be given in this context. In contrast, the applied literature gives the impression that the computation of dual Gabor windows in the standard situation, i.e. for the Hilbert space $\boldsymbol{L}^2(\mathbb{R})$, and a time–frequency lattice of the form $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ is still the most important problem in (numerical) Gabor analysis. The emphasis of this note is on the *value of numerical work, which is much more than just numerical realization of theoretical concepts*. It has been in many cases the inspiration for the derivation of theoretical results, based on sometimes surprising observations or systematic numerical simulations. According to our experience, numerical Gabor analysis provides a lot of additional insight about the concrete situation; it may suggest new directions and ask for new theory, but of course efficient algorithms often make use of underlying theory. Overall, we observe that there is an urgent need for a *stronger link between computational and theoretical Gabor analysis*. The note also contains a number of suggestions and even conjectures which are likely to encourage research in the direction indicated above.

7.1 Introduction

Gabor analysis is a particular branch of *time–frequency* analysis. The traditional approach to Harmonic Analysis is to look at a function (or tempered distribution) on the "time side" or on the "frequency side", i.e. one tries to understand properties of

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a "signal", meaning a function or distribution, by either looking at its representation as a function of time or by studying the behaviour of its Fourier transform. The first representation makes it easy to find those segments of a function which contain high energy, while the second tells us which frequencies (on average) are predominant in the signal.

It is also well known that decay on the Fourier transform side corresponds qualitatively to smoothness in the time domain, and so on. Due to Plancherel's theorem (showing that the Fourier transform $\mathscr{F}$ preserves the $\boldsymbol{L}^2$-norm) we can think of $|f|^2$ or $|\widehat{f}|^2$ as energy distributions. In contrast, *time–frequency analysis* (TFA) looks at the spectrogram, i.e. the absolute value of the STFT, or *Sliding Window Fourier transform*) resp. its square and thus presents a (smooth, non-negative) energy distribution within *phase space*. The *energy preservation principle* appears in the form of *Moyal's identity*; see [66]. The STFT describes the energy distribution within *phase space*, well comparable with a musical score describing the melodies within a piece of music. Since this continuous version is highly redundant it is natural to replace it by a discretized version, i.e. to sample the STFT with respect to a lattice. This is where *Gabor analysis* takes off. Here we write $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ for the so-called *time–frequency plane* (or phase space), and in the case of $d = 1$, the most important case which is also relevant for effective audio processing, we thus consider $\mathbb{R} \times \widehat{\mathbb{R}}$ sometimes also as the complex plane.

It is often argued that one of the drawbacks of the STFT is the fact that one has to choose a window. Different windows may provide different information about the signal to be analysed. The rule of thumb is that long windows allow for a better frequency resolution at the cost of poor time resolution. In contrast, short windows allow for good time resolution but provide bad resolution in the frequency direction. The *Heisenberg uncertainty relation* is known to prohibit high precision in both ways, and this is why Gabor was suggesting to use the Gauss function $g_0(t) = exp(-\pi|t|^2)$, because it is a minimizer to the Heisenberg inequality. But one has to be aware that there might be other choices more appropriate for a given concrete task.

Once the choice of the window has been made, one can then look for possible discretizations. It is well known that it is convenient to choose some *lattice*, i.e. a discrete subgroup Λ of $\mathbb{R}^{2d}$ (generated by a suitable $2d \times 2d$ non-singular matrix $\boldsymbol{A}$, or $\Lambda = \boldsymbol{A} * \mathbb{Z}^{2d}$). We think that too much attention is given to the simple special case where $\boldsymbol{A}$ is a simple diagonal matrix, or just even $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, for some pair (a, b) of positive lattice constants, the *time gap* a and the *frequency gap* b.

Gabor gave arguments why one should use the so-called *Neumann lattice*, or simply choose $a = 1 = b$. There are in fact strong arguments against the case $a \cdot b > 1$ (lack of density of the resulting Gabor sums) and $a \cdot b < 1$ (lack of uniqueness), but as it turns out one has to live with this redundancy because otherwise not every $\boldsymbol{L}^2(\mathbb{R}^d)$-function has a representation as a Gabor series (double sum involving TF-shifted copies of the *Gabor atom*); see [77].

We will rely mostly on the standard notations as given in the books [66] or [49, 50]. In particular we denote time–frequency shift operators $M_\omega T_t$ by $\pi(\lambda)$ with $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and the STFT of a *signal* f for the *Gabor window* g by

$$V_g(f)(\lambda) = V_g(f)(t, \omega) = \langle f, \pi(\lambda)g \rangle, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{7.1.1}$$

This continuous function is first defined for pairs $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$, but for $g \in \mathscr{S}(\mathbb{R}^d)$ (e.g. the Gauss function) this function is also well defined for general tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ and gives a continuous function of at most polynomial growth. The Schwartz space $\mathscr{S}(\mathbb{R}^d)$ can be characterized as the subspace of $\mathscr{S}'(\mathbb{R}^d)$s whose elements have an STFT decaying faster than any polynomial (see [63]).

7.2 Goals for Numerical Gabor Analysis

Let us jump right away into a description of what could be seen as the main goals for Gabor analysis, first at a theoretical level, but then more precisely at a numerical or computational level. Of course such goals have a subjective component, but we hope to formulate topics which are useful for the whole community.

One may formulate as the **first goal of Gabor analysis** to obtain an *expansion of a given function* f or (tempered) distribution as a *Gabor series*, which requires to *compute the (minimal norm) coefficients* properly, or equivalently (as theory tells us) to *recover the signal exactly or approximately from the sampled spectrogram* in a linear way.

In doing so we should have the theoretical backup which guarantees that the representation is *local in both the time and the Fourier domain*, but also that imprecise knowledge about the window or the lattice does not matter too much. The *theory of Gabor frames* provides useful information: instead of inverting the *Gabor frame operator* it is enough to compute (precise or approximately) the *dual Gabor atom* for the given situation, and this can be done in a constructive way (see [78]).

As a **second goal** we see the characterization of smoothness and decay of a function directly from the Gabor coefficients, without resynthesis. The theory of Gabor expansions is closely related to the theory of *modulation spaces* (see [56, 66]), which in the sense of *coorbit spaces* (see [39]) are those Banach spaces of (tempered or even ultra-)distributions whose STFT shows a certain behaviour over phase space, typically expressed through decay and summability properties; see [57]. In particular, we say that a distribution f belongs to the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ (also known as modulation space $M^1(\mathbb{R}^d)$) if its STFT $V_g(f)$ is in $L^1(\mathbb{R}^{2d})$. Its definition does not depend on the choice of the window function $g \in \mathscr{S}(\mathbb{R}^d)$. Moreover $\|f\|_{\boldsymbol{S}_0(\mathbb{R}^d)} = \|V_g(f)\|_{L^1(\mathbb{R}^{2d})}$.

Aside from discretization over phase space (which is theoretically well justified) it should also be OK to work with a sampled version of the input function g in order to find out whether a given function belongs to a certain modulation space or not, at least under mild and practically applicable extra conditions and after appropriate "high-quality" sampling.

While discrete, finite dimensional versions of Gabor analysis (e.g. computation of the canonical dual) can be *realized computationally* (we have called it a *constructive realizable approach* in [48]) the transition to the continuous, non-compact limit has not been analysed properly so far. In some sense we claim that *Gabor analysis over finite groups should be seen as a good approximation to Gabor analysis over a continuous and non-periodic domain* such as $\mathbb{R}^d$. Here we will need more insight through a combination of computational exploration, good algorithms, but also theoretical justification for the transition between the two domains. Although this is not the main subject of this note several observations made here will be relevant for future publications. We can still regard this as the **third goal** for future research in Gabor analysis.

Finally let us mention what we see as the **fourth goal** (and in the long run probably the most important one), which we see as an emerging branch of time–frequency analysis: make use of Gabor expansions and time–frequency descriptions of *operators* in order to treat in a numerical way *pseudo-differential operators* or even *Fourier integral operators*. There are already various interesting papers available, still mostly at the theoretical level, but there should be much more computational work and more intensive collaborations with applied scientists in order to demonstrate the practical relevance of these investigations. The kernel theorem [75] gives many hints in this direction.

Examples of work of the combined efforts of colleagues from NuHAG and the time–frequency community in Torino are [14, 20–22], just to give a few examples.

7.3 The Standard Literature

Let us briefly give a summary of the established literature on Gabor analysis. We will do it in various sections, emphasizing different aspects of Gabor analysis.

The classical and still traditional question showing up in early papers on Gabor analysis is to assume that we have a problem concerning frames $\mathcal{G}(g, a, b)$ arising in a specific way, using TF-shifts along a lattice $a\mathbb{Z} \times b\mathbb{Z}$, in the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\big)$ or $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$.

A good summary describing the coarse structure of Gabor analysis for this case is given in the recent paper [60], but one finds valuable information also in the books [66], or [49, 50].

The typical questions are of the following form:

1. Given $0 \neq g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and lattice constants $a, b > 0$, can one show that the Gabor system $\mathcal{G}(g, a, b)$ forms a frame (or a Riesz basic sequence), perhaps with some estimates for the frame bounds;
2. More generally, what are sufficient conditions for such systems to be frames, conditions which can be applied to a larger family *Gabor atoms* g, and families of lattices (including non-separable ones);
3. The only functions for which we know all possible sampling patterns (even irregular patterns) which give a frame are the one-dimensional Gaussians due to the work of Lyubarskii [85], Seip [102] and Seip and Wallsten [101] proving a con-

jecture of Daubechies et al. [26]. For a Gaussian we always get a frame if the (lower Beurling) density of the lattice (pattern) is greater than 1;

4. Quite recently it has been shown by Gröchenig and Stöckler (see [62], but also [4, 79, 80]) that a quite large family of *totally positive* functions have the maximal *frame-set*, i.e. they form Gabor frames for any pair with $ab < 1$;
5. Fixing $g \neq 0$ one can ask what the set of all pairs (a, b) (with necessarily $ab < 1$) is such that $\mathscr{G}(g, a, b)$ is a Gabor frame; this set is known to be open for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ [43]; but even for Hermite functions this set can have unexpected exclusions areas (see [84]), while for rectangular function the so-called *abc* problem is surprisingly difficult to handle (cf. [24, 25, 68]);
6. Negative results, as for example Balian–Low-type results, which exclude the existence of Gabor frames at critical density with reasonable good windows (and duals), combined with observations how the frame bounds deteriorate as the density approaches the critical density, i.e. for ab close to 1;
7. Questions concerning perturbation or stability of Gabor frames with respect to a change of the lattice constants [43].

Although Gabor's seminal paper appeared already in 1946 [59] its possible relevance was only recognized in the early 80s, mostly through the investigations of A.J.E.M. Janssen, who was checking whether the claims of Gabor are valid if one allows distributional convergence; see [77]. One can say that a relatively small number of papers which appeared in this period have led the basis for the development of the subject later (see the book [50] for a first summary, and [49] for some advances).

First of all we have to mention the "painless" paper [27] which describes the situation of windows of finite length, so that for sufficiently small lattice parameters b the Gabor frame operator is in fact multiplication operator (hence inversion is "painless"). This case is also convenient for discrete case and one could even argue that an important step of the MP3 coding algorithm (invented by the Fraunhofer research group around K. Brandenburg, in Erlangen) is making use of this fact. It can also be seen as the kickoff for the *frame movement* and a revival of the work done by Duffin and Schaeffer [32] on what is known as *frames in Hilbert spaces*.

Around that time also interest in the connection between the Gabor approach to signal analysis and attempts to understand the *human visual system* received great interest, through the work of Porat and Zeevi [87] and Daugman ([31], with a huge number of citations nowadays).

One has to mention the work of Walnut [111], where the so-called *Walnut representation* of the frame operator was introduced, and the investigations by Ron and Shen [98, 99], but also the two important papers which appeared in a single issue of the J. Fourier Anal. Appl. 1995, namely [28, 76]. All of the last mentioned papers discuss in one way or another that a Gabor family arising from a fixed window $g \neq 0$ with parameters (a, b) and describe that it is a frame if and only if the corresponding *adjoint family* with parameters $(1/b, 1/a)$ is a Riesz basic sequence, i.e. a Riesz basis for its closed linear span. The first and in fact numerically motivated result in this direction is the biorthogonality characterization of Gabor frames described by Wexler and Raz in [112]. From now on Λ° is the *adjoint group* for the given lattice

Λ, resp. the group of all TF-shift parameters commuting with the original family of TF-shifts $\{\pi(\lambda), \lambda \in \Lambda\}$ (see [44]). It then took a while until the role of this lattice was understood, especially in the non-separable case (see [44]), so that nowadays it is clear that the adjoint lattice Λ° or *Weyl–Heisenberg commutator* of the original lattice plays an important role, and that the (weak) duality condition of Gabor windows (cf. [51]) is equivalent to the biorthogonality over Λ° and can be expressed by $V_g(\gamma)(\lambda^\circ) = \delta_{0,\lambda^\circ}, \lambda^0 \in \Lambda^0$.

7.4 The Classical Frame Algorithm

Reading standard texts on frame theory (see [11, 29]) it seems to be very important to determine the lower and upper frame bounds A and B, among others in order to check then the closeness of the optimally scaled version of the frame operator S to the identity operator on the given Hilbert space, i.e. to verify the validity of an estimate of the form

$$\|\gamma^{-1} S - Id_{\mathscr{H}}\|_{\mathscr{H}} < 1, \text{ with } \gamma = (A+B)/2. \tag{7.4.1}$$

Although it is good to have reasonable estimates for these two frame bounds A and B, this information is by far *not so really relevant for application,* for a variety of reasons: first of all a fine estimate or exact determination of these two parameters is not crucial for the convergence of the usual frame iteration method. It does not put the convergence of the algorithm at risk, a somewhat inaccurate estimate just results in a slower convergence rate, but does not cause divergence. As a rule of thumb it makes sense to just choose as scaling factor γ the inverse of the redundancy factor, or a value which is slightly below, if one definitely wants to avoid divergence of the naive iterative (frame) algorithm which follows directly from the Neumann representation of the inverse operator

$$(\gamma S)^{-1} = \sum_{k=0}^{\infty} (Id - \gamma S)^k. \tag{7.4.2}$$

Since we are only interested to solve the equation $S(\widetilde{g}) = g$ we are not so much interested in the inverse operator, but only what the partial sums are doing when acting on the Gabor atom g, resp. we can describe the approximations g_n to $\widetilde{g}$ via

$$g_n = \sum_{k=0}^{n} (Id - \gamma S)^k g, \quad n \to \infty, \tag{7.4.3}$$

which can be described recursively with $g_1 = \gamma g$ and

$$g_{n+1} = g_n - \gamma(S(g_n) - g), \quad n \geq 1. \tag{7.4.4}$$

By measuring the change between g_{n+1} and g_n one can even find out which parameter γ is more suitable and can expect to find a good suboptimal value for this scaling parameter during the iterations for the case that these iterations are working not fast enough. It has also often been observed (but not proved theoretically, to our knowledge) that the optimal parameter $2/(A+B)$ is rather close to $1/\mathrm{red}(\Lambda)$, at least for well-balanced lattices and Gauss-like Gabor atoms. But clearly, computing first the eigenvalues of $S_{g,\Lambda}$ in order to faster find the dual windows is not a very efficient strategy.

Looking at the problem of the (canonical) dual Gabor window it is much better to make use of the fact that the frame operator is always a symmetric (Hermitian) operator, and correspondingly the equation $S(\widetilde{g}) = g$ should be solved by a much more efficient method, such as the *conjugate gradients method*, ideally combined making use of sparsity considerations, which again stem from the invariance properties of the Gabor frame operator. Early examples of such strategies are given in [90, 91]. The block structure there reduced the cost of matrix–matrix and matrix–vector multiplication using the properties of the Walnut representation (namely at most b nonzero (cyclic) side-diagonals, each of which is a-periodic). This strategy had been already quite successful in connection with the irregular sampling problem for band-limited functions, which also was understood at that time as a problem concerning frames (see, e.g. [64] for an analysis of the situation there).

There are various elements of abstract Gabor analysis which have been quite useful and in some sense observed independently when we started to carry out numerical work on Gabor analysis in the early 90s. In particular, during the Ph.D. work of Qiu [95] many MATLAB programs have been developed which up to now are used for recurrent experiments on this topic.

First of all, as soon as one forms the *matrix describing the Gabor frame operator*, with Gabor atom being a signal of length n, and time shifts which are multiples of a, a divisor of n, and sampling the Fourier transform on a sublattice, with lattice constant b, also a divisor of n, it is easy to see that there are typically (independent of the atom) b side-diagonals (in the sense of a cyclic matrix) at a distance of n/b, which is in fact another way to express the existence of a Walnut representation for the Gabor frame operator. In addition, these side-diagonals are a-periodic functions.

This observation led immediately to the idea to store the information about the Gabor frame matrix in the format of a small block matrix (the block nonzero matrix, as it was called) of size $a \times b$, where each of those b (row) vectors contains the information about the basic period of each of those b side-diagonals. The first entry describes the main diagonal, which contains the a-periodic version of $|g|^2$ (which has to be nonzero whenever the given Gabor family is a frame).

It was also plausible, as an alternative, to describe the operator by taking the Fourier coefficients (again there are a such Fourier coefficients for each a-periodic side-diagonal). As it turned out this is more or less equivalent to the Janssen representation of the Gabor frame operator, but we will not pursue this aspect here. We just wanted to mention that one can be lead to the idea of a Janssen representation directly by studying the properties of Gabor systems in a numerical fashion and inspecting the particular properties of the Gabor frame matrix.

It can also be observed that for b small and windows g with small support there are only few side-diagonals and if the length of such a window g is shorter than n/b then one finds that the Gabor frame operator is just a diagonal matrix, which is easy to invert. In the case of strong decay the side-diagonals are not very important and one can view the main diagonal as the dominant part of the Gabor frame operator. The case that this matrix is diagonal dominant implies of course that the matrix is invertible, or that the given Gabor family forms a frame (see [9], or more recently [10]). The condition given in early papers by I. Daubechies implies that this condition is satisfied. Translated to the finite discrete case it reads: assume that the main diagonal of the frame matrix is non-vanishing, then for sufficiently small b the Gabor family will form a Gabor frame.

Given this concrete *sparse structure* of the Gabor frame matrix it is quite natural (as it has been carried out in a series of papers by S. Qiu, during his time at NuHAG) to describe the action of the Gabor frame operator on a vector by a "compressed" matrix–vector multiplication, resp. the composition of two matrices of this type by a fast block/block matrix multiplication. Given the fact that the Gabor frame matrix is positive definite and that the determination of the dual Gabor family can be reduced to the determination of the (canonical) dual Gabor window $\widetilde{g} = S_{g,\Lambda}^{-1}(g)$ one finds out easily that the *Conjugate Gradients* strategy for solving the linear equation $S_{g,\Lambda}(\widetilde{g}) = g$ for $\widetilde{g}$ is the way to go, using during the iterations the sparse structure as described above. The details of this approach can be found in the papers [90–94, 96, 97] and others.

The numerical work in Gabor analysis is to a large extent concentrating on the very classical case where one or several Gabor atoms are given, and then the question is formulated: for which lattice constants (in the finite discrete case this amounts to choosing lattice constants a, b which are divisors of the signal length) is a given Gabor family a Gabor frame, and if this is the case, how and when can one determine the *canonical dual Gabor atom*, or sometimes *some dual Gabor atom* or at least a good *approximate dual Gabor atom*.

Since Gabor analysis can be done over any finite Abelian group (see [52]) is in principle quite possible to treat Gabor families arising from an arbitrary lattice $\Lambda \lhd G \times \widehat{G}$. Many of the available numerical algorithms for the determination of dual Gabor atoms allow to work for general lattices (including the non-separable ones) for $1D$-signals. Although there exist papers and numerical work in that direction, especially with applications in image analysis, we think that this direction is still a bit underdeveloped. New problems arise from the size and memory restrictions during actual computation. For a reasonable image format it is simply not possible to store an irregular, dual Gabor frame, because that would require to store somehow more images of the given format than pixels. But we cannot discuss this problem here.

Let us also mention that despite the well-known fact that every finite Abelian group is isomorphic to a product of cyclic groups this representation is only unique up to isomorphism, and isomorphic groups have an "isomorphic" Gabor theory, and the same subgroup structure "in abstracto". For example, groups of the form $\mathbb{Z}_N \times \mathbb{Z}_M$ can be treated using the $1D$ code if M and N are relatively prime, because then they are naturally isomorphic to $\mathbb{Z}_{M \cdot N}$ (see [41]).

For the discussion of *non-separable cases* and a subsequent *exhaustive analysis of performance parameters* (like the condition number of the frame operator or the numerical $\boldsymbol{S}_0$-norm of the dual window) recent results providing an explicit and constructive catalogue of subgroups (of all orders) are giving us important information; see [67, 72] and more recently [106]. They have been important for the UnlocX project.

7.5 The Idea of Double Preconditioning

The idea of double preconditioning of Gabor frame operators has been introduced in [3], based on numerical experience. Despite its algorithmic efficiency it still waits for theoretical justification. The idea is the following: if we take the normalized Gabor frame operator $S_{g,\Lambda}/\mathrm{red}(\Lambda)$ then one can show (see [51]) at least for nonzero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ that these operators converge to identity operator for $(a, b) \to (0, 0)$, but despite the fact that it was known that for the Gauss function $g_0(t) = e^{-\pi t^2}$ determines a Gabor frame for any pair (a, b) with $0 < ab < 1$ the corresponding frame matrices may not be very close to the identity, or even if they satisfy (7.4.1) the estimate may still not guarantee a good rate of convergence, mostly because the diagonal part of the frame matrix is highly oscillating. This will happen typically if a is relatively large compared to b, in other words of the shift parameter a is larger than a value that would allow to approximate relatively well the constant function 1 by shifted copies of the absolute square $|g|^2$ of the Gabor atom, by making it a-periodic.

But inverting the diagonal part of the matrix is very easy and thus it appears as a natural idea to use it as a *preconditioner* in order to bring the frame operator closer to the identity.

In some cases this simple trick already produces a perfect solution: in the *painless case* (see [27]), namely in the case that the length of the support of g is smaller than $1/b$. It is also plausible and provable that the main diagonal will play a crucial part of the frame operator whenever the Gabor atom is well concentrated. Fine estimates of the effect of tails are given, e.g. in [10].

In the alternative case of relatively large values for b compared to a we will see that the Fourier version of the Gabor frame operator has a prominent diagonal part. In fact, the question whether the Fourier transform of g generates a Gabor frame for the lattice constants (b, a) is unitarily equivalent to the frame question for the triple (g, a, b). There is also a direct way to see the possible effect on the "time side". One often has Gabor frame operators with significant side-diagonals, but not too much oscillations. Clearly the best approximation to such an operator (in the Hilbert–Schmidt or Frobenius norm) is to replace the side-diagonals by their respective mean values. The inversion of such a convolution operator is of course best done by a small FFT, so we have another cheap preconditioner (all with their spreading support inside the adjoint lattice Λ°!).

The combination of the two preconditioner methods is based on the numerical evidence that one has, at least for decent windows, such as Gaussian or similar, the following effect.

- Typically one of the two steps is helpful, bringing the frame operator closer to identity;
- In some interesting (balanced) cases (not far from critical sampling) one can profit from both methods. In other words, the Gabor atom which is obtained by simply applying the inverse of the diagonal part and then the inverse of the convolutional part of the Gabor frame operator is already quite close to the true canonical dual window (even in the $\boldsymbol{S}_0$-norm);
- Whenever only one of these two methods is less relevant, either the first or the second one, it does not matter to blindly apply both of them, without checking which one is the good one, because the other one typically does not deteriorate the outcome, at least not in a practical sense;
- Asking about the order of the two preconditioners one can say that it turns out to matter not very much. So we do not have any specific recommendation for this. But clearly the amount of oscillation of the diagonal part (on the time resp. Fourier side) is a strong indicator for the expected performance;
- Experimentally also multiple preconditioning has been tried out. This would be a good strategy for non-separable Gabor lattices, in the extreme case for something similar to a hexagonal lattice, which has three directions of equal importance. In any case the preconditioners will arise within the Banach algebra of linear combinations of TF-shift operators from the adjoint lattice!

As indicated already earlier, all these observations *wait for a theoretical underpinning and quantitative guarantees*. Under which conditions can we expect a certain closeness of the preconditioned frame operator to the identity operator (acting on $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$, but also on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$)? Observe that both preconditioners and hence all the operators arising by the preconditioning procedure still belong to the Banach algebra of matrices commuting with all the TF-shifts $\pi\lambda \in \Lambda$, and thus the explicit sparsity can still be used. On the other hand, the estimate that we can give using the Janssen representation (cf. below) allows then to show that we have sufficient conditions for the inverse frame operator also being invertible on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, without using the deep results obtained in [40, 61].

7.6 The Janssen Representation

It is our understanding that the *Janssen representation* is the most important structural fact (see [44, 52], or [45] for technical details) concerning the *Gabor frame operator*. Only for the case of separable lattices it is equivalent to the *Walnut representation* (see [66], where this representation is playing the dominant role), while the Janssen representation also makes sense for non-separable lattices $\Lambda \lhd G \times \widehat{G}$. As we will point out it is also a good basis for the derivation of efficient algorithms.

The frame operator for a Gabor system $G(g, \Lambda)$ is given by

$$S_{g,\Lambda}(f) = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g \tag{7.6.1}$$

Then *Janssen's representation* of $S_{g,\gamma}$ (see [66], Sect. 7.2), for $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, is given by

$$S_{g,\gamma} = (ab)^{-d} \sum\sum_{l,n\in\mathbb{Z}^d} \big\langle \gamma, M_{l/a}T_{n/b}g\big\rangle M_{l/a}T_{n/b}.$$

The Janssen representation of $S_{g,\Lambda}$ follows from the commutation relation

$$S_{g,\Lambda} \circ \pi(\lambda) = \pi(\lambda) \circ S_{g,\Lambda} \quad \forall \lambda\in\Lambda, \tag{7.6.2}$$

and also to the *Wexler–Raz biorthogonality property*, characterizing (weakly) *dual Gabor atoms* γ (see [51]) via the following property of its STFT:

$$V_g(\gamma)(\lambda^\circ) = \delta_{0,\lambda^\circ} \quad \text{the Kronecker symbol over } \Lambda^\circ. \tag{7.6.3}$$

Recall that Λ° is the adjoint lattice for Λ. Given the general theory of *spreading distributions* for operators one can link the commutation relation to the fact that the *spreading support* is inside Λ°. On the other hand, viewing $S_{g,\Lambda}$ as TF-periodization of the rank one projection $P_g : f \mapsto \langle f, g\rangle g$ (assuming $\|g\|_2 = 1$!) one also finds that the *spreading coefficients* can be described as the sampling values of $V_g(g)$ over Λ°, i.e. we have altogether

$$S_{g,\Lambda} = \mathrm{red}(\Lambda) \sum_{\lambda^\circ\in\Lambda^\circ} V_g(g)(\lambda^\circ)\pi(\lambda^\circ). \tag{7.6.4}$$

Here $\mathrm{red}(\Lambda)$ is inverse of the volume of the fundamental domain for Λ, or equivalently the area of the fundamental domain of Λ°, which can be computed as the absolute value of the determinant of any matrix $\boldsymbol{A}$ with $\Lambda = \boldsymbol{A}(\mathbb{Z}^{2d})$. Let us denote the *reduced frame operator* $\mathrm{red}(\Lambda)^{-1}S_{g,\Lambda}$ by $R_{g,\Lambda}$. Then we have under the usual assumption $\|g\|_2 = 1$

$$\|R_{g,\Lambda} - \mathrm{Id}_{\boldsymbol{L}^2}\| \leq \sum_{\lambda^\circ\in\Lambda^\circ\setminus\{0\}} |V_g(g)(\lambda^\circ)|. \tag{7.6.5}$$

If $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ then we also have $V_g g \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ and hence the sum on the right hand side is certainly absolutely convergent (see [45]). Moreover, the more dense the lattice Λ is in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ the more sparse Λ° will be. In fact, if Λ is generated from $\mathbb{Z}^{2d}$ by the matrix $\boldsymbol{A}$ then Λ° is generated by the inverse transposed matrix $\boldsymbol{B} = (\boldsymbol{A}^t)^{-1}$. Hence one can use this situation in order to derive not only that for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one can guarantee that the frame operator is invertible on $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$, but even on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ and that consequently the *dual Gabor atom* $\widetilde{g}$, which can be described as $\widetilde{g} = S_{g,\Lambda}^{-1}(g)$, also belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ (and in fact depends continuously on the matrix $\boldsymbol{A}$; see [43]). This is also the basis for the proof (see [51]) that the (normalized) dual windows converge to the original Gabor atom (even in the $\boldsymbol{S}_0$-norm) as the density of Λ tends to infinity (in the sense the $(a_n, b_n) \to 0$).

Another consequence of the Janssen representation of $S_{g,\Lambda}$ is the fact that it brings in the structure of a Banach algebra, namely a *twisted convolution algebra* of operators of the form $\sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \pi(\lambda^\circ)$, with the coefficient belonging to some (solid) Banach space of sequences over Λ°. The twist arises from the phase factors arising when one composes two TF-shifts.

An instructive MATLAB experiment exposing the unique role of the ZAK transform and also justifying at least numerically its form works as follows (using NuHAG M-files): clearly for a given signal length n vectors $\mathbf{x} \in \mathbb{C}^n$ can be considered as functions on $\mathbb{Z}_n$ (with cyclic shift). Given any divisor a of n (e.g. $n = 480$ and $a = 20$) the lattice constant $b = n/a = 24$ determines a critical lattice. The adjoint lattice is then of the form $a_0 = n/b$ and $b_0 = n/a$, in other words the TF-shifts from Λ commute. We create a random matrix with spreading support in $\Lambda^\circ = \Lambda$ (in this case), e.g. by the commands

```
RM=spr2mat(rand(a,n/a)); SRM=RM'*RM; [VZ,EZ]=eig(SRM);
```

and then inspecting the columns of the matrix VZ of eigenvectors of such an operator. One will find that these are all modulated and shifted version of a Dirac comb, with lattice constant a! In other words, such an experiment would suggest (if it wasn't well known already) to define something like the ZAK transform by first defining for decent signals $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ as the STFT with respect to a Dirac comb $\sum_{n\in\mathbb{Z}} \delta_{an}$ and then restrict it to an appropriate rectangle in the phase space domain. This turns out to be just the *Zak transform*!

At the critical density we can also expect problems with the Gabor matrix (now an $n \times n$-matrix): it does not have full rank (one dimension missing) and has a one-dimensional null-space.

The *Janssen representation* of the Gabor frame operator $S_{g,\Lambda}$ is not only an important theoretical concept but also very well suited for its inversion.

First of all it can be used to demonstrate that for $0 \neq g \in \boldsymbol{S}_0(\mathbb{R}^d)$ it is guaranteed that for any (separable or non-separable) lattice which is fine enough (in the sense of allowing a sufficiently small fundamental domain, contained in a ball of radius δ, depending only on g) will form a Gabor frame.

In fact, one can always assume without loss of generality that $\|g\|_2 = 1$ or equivalently that the identity operator $\mathrm{Id}_{\boldsymbol{L}^2} = \pi(0,0) = M_0 T_0$ appears with coefficient $1 = V_g(g)(0,0) = \langle g, g \rangle$ in the Janssen representation of reduced frame operator $S_{g,\Lambda}$.

The assumption $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ implies in fact that $V_g(g) \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ and hence it is in Wiener's algebra $\boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^{2d})$. Saying that Λ has a small fundamental domain is equivalent to say that the adjoint lattice Λ° is sufficiently coarse, so that for any given function $F \in \boldsymbol{W}(\boldsymbol{C}_0, \boldsymbol{\ell}^1)(\mathbb{R}^{2d})$, in our case $F = V_g(g)$ one can say that

$$\sum_{\lambda^\circ \in \Lambda^\circ \setminus \{0\}} |F(\lambda^\circ)| \leq \gamma < 1,$$

uniformly over all these lattices Λ°. This in turn implies naturally that the frame operator, taken as an operator on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$, or on $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ or even on any of the spaces $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$, with $1 \leq p \leq \infty$ is invertible (with uniform control of the inverse operator on all these spaces).

The (equivalent) condition, whether

$$\sum_{\lambda^\circ \in \Lambda^\circ} |V_g(g)(\lambda^\circ)| < 2\|g\|_2^2 \tag{7.6.6}$$

is very easily verified and requires only to compute the $\boldsymbol{\ell}^1$-norm if a sampled STFT (resp. ambiguity function) over Λ°.

It has been first suggested as a "criterion" (in the sense of an easy, sufficient condition of invertibility) in the master thesis of Tschurtschenthaler [107] from 2000 (see Chap. 4: Preconditioning the Gabor frame operator). It applies—according to our experience—to a large variety of examples where one has reasonable redundancy (even close to 1) and modest excentricity (i.e. in the separable case the situations where a and b are not too different).

7.7 Gabor Multipliers

Gabor multipliers and related operators, the so-called *underspread operators*, described as *slowly varying linear systems* in communication theory (see [69, 70, 81, 83] and in particular [82]) are exactly the class of linear operators which are well described via Gabor systems. There are different names for these operators, e.g. *STFT multipliers* (because the essential step in their description is a pointwise multiplication of the STFT of the input signal) or *Anti-Wick operators* (see [6, 7, 17]).

At the phase space level these operators are of the form $F \mapsto (M \cdot F) * G$ (with $F = V_g(f)$, $G = V_g(g)$), where "$*$" denotes (twisted) convolution over phase space, or ordinary convolution on the corresponding *reduced Heisenberg group*, and $H \mapsto H * G$ describes the projection from $(\boldsymbol{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2)$ onto the (closed) range of $\boldsymbol{L}^2(\mathbb{R}^d)$ under the STFT with fixed window g, given by $f \mapsto V_g(f)$. Hence these operators are also viewed as Toeplitz operators on a Hilbert space. There is a rich body of literature on this direction (e.g. Englis: [33], or Hutnik [74]).

The paper [7] treats kernels in distributional Sobolev spaces, [6] with symbols in $\boldsymbol{L}^p(\mathbb{R}^{2d})$. For symbols which are indicator functions of sets in phase space or which are non-negative and well decaying towards infinity the resulting operators also known as *localization operators*, going back to the meanwhile classical paper by Daubechies [30]. Further relevant results about localization operators are given in

[13–15, 18, 23], or [16] and provide a potentially useful theoretical foundation for corresponding numerical work. Related work has been done in [12, 47], or [105].

In the area of *Gabor multipliers* a lot of insight can be gained by numerical experiments, and in fact many abstract results have been inspired by comprehensive simulations by the author.

One can think of such operators as linear operators which behave locally like convolution operators, but with some extra freedom to *change the convolution kernel* slowly over time (or location, e.g. a space-variant blurring in image processing). The main tools for the analysis of such systems are the spreading function (resp. distribution), the Gabor matrix representation of the operator and in some cases the Kohn–Nirenberg representation known from the theory of partial differential equations with variable coefficients.

7.8 Numerical Illustrations

MATLAB experiments can also be used quite well in order to illustrate (well-known) theoretical results by exhibiting special cases or some valid approximations thereof. Sometimes it is also instructive to have quantitative versions rather than just asymptotic estimates.

One of the most well-known principles in Gabor analysis is the so-called Balian–Low principle [66]. According to this principle it is impossible to have a Gaborian Riesz basis for the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}^d), \, \|\cdot\|_2\big)$ (even for $d = 1$), except for "bad" Gabor atoms, such as the indicator function $\mathbf{1}_{[0,1)}$, which of course generated an ONB for $\big(\boldsymbol{L}^2(\mathbb{R}), \, \|\cdot\|_2\big)$ with respect to the integer (von Neumann) lattice $\mathbb{Z} \times \mathbb{Z}$ [110].

So in particular, the Gabor system at the critical density with the (Fourier invariant) Gauss function $g_0(t) = e^{-\pi t^2}$ is a total family in $\big(\boldsymbol{L}^2(\mathbb{R}), \, \|\cdot\|_2\big)$ (i.e. finite linear combinations are dense on $\boldsymbol{L}^2(\mathbb{R}^d)$), but it is not possible to represent every element as a series with coefficients in $\boldsymbol{\ell}^2(\mathbb{Z}^2)$.

The usual argument of proof makes use of the so-called *Zak transform* which is (the natural) way of diagonalizing the collection of TF-shifts along the integer lattice $\mathbb{Z} \times \mathbb{Z}$. The characteristic property of such a family is of course its *commutativity* (due to the fact that phase factors appearing usually in the composition law for TF-shifts are trivial in this particular case), combined with the periodicity properties (on the time direction) and the quasi-periodicity property in the frequency direction.

Usually a topological argument is then used in order to derive that whenever the Zak transform of $Z(g)$ of g, for example, whenever g belongs to the modulation space $\boldsymbol{M}^1(\mathbb{R}^d)$ (resp. the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$), it *must have some zero* in its domain, which is the unit square (fundamental domain of $\mathbb{R} \times \widehat{\mathbb{R}}$ over the lattice $\mathbb{Z}^2$).

For the concrete case of the Gauss function this zero is located exactly in the mid-point of its domain, so $Z(g_0)(1/2, 1/2) = 0$.

In order to check that there is a problem at the so-called critical density one can do the following MATLAB experiment, recalling that in the context of linear algebra the "critical" lattices in $\mathbb{Z}_n \times \mathbb{Z}_n$ are exactly those with n elements, because they are producing Gabor families with the chance of providing a basis to $\mathbb{C}^n$. Any lattice with more than n elements clearly will have linear dependencies built in, while any lattice with less than n elements will produce families which cannot span all of $\mathbb{C}^n$.

So let us consider a finite dimensional analogue: we start with the Gauss function and for any natural number k (e.g. for $k = 16$) produce a discrete version of the Gauss function of length k^2 (i.e. $n = 256 = 16^2$).

Formally (but numerically irrelevant) this is obtained by periodizing g_0 to the period $p = 16$ and then sampling it (or vice versa, which is the same!) at the rate $h = 1/16$. This gives a sequence of length 256, with the sampling starting at zero up to $16 - 1/16$ (the value at $16 = 256 * h$ equals of course the value at zero, due to the p-periodicity).

It is not difficult to check that up to the normalizing factor 1/4, where $4 = \sqrt{16}$, this finite sequence is invariant under the `fft`, or up to numerical precision we have equality of `fft(g)` and $4 * g$. In fact, the parameters p (length of the period) and h sampling rate change their role; i.e. we get a sampled and periodized version of the function, i.e. the new sampling is at the lattice $\mathbb{Z}/p$ and the new period is $1/p$, but due to the choice of the function g_0, which is Fourier invariant, and the fact that we have $h = 1/p$ we get (theoretically and numerically) the `fft`-invariance of our vector $g \in \mathbb{R}^{256}$.

After forming the matrix of size 256×256 containing (e.g. as columns) all the TF-shifted version along the lattice we can check for the rank of that matrix and will find that it is only 255. This is due to the fact that we have taken the correct version of the Gauss function with the correct form of symmetry (not under the simple flip operator, but under the operator which flips only the coordinates, but the one related properly to the involution for functions, of the form $f^{\checkmark}(x) = f(-x)$). Otherwise we would maybe believe that the span is still everything, but with a rather bad condition number (see Sect. 7.12).

Practically all the basic properties and formulas valid for regular (meaning the lattice case) or irregular Gabor families, separable or not, can in principle be verified experimentally or in an explorative way based on the set of MATLAB files found within LTFAT resp. the NuHAG repository.

One can virtually *take a look* at the shape of the Gabor frame matrix, seeing its pattern (Walnut representation) by simple inspection (as was done during the work with Sigang Qiu), but also verify easily that there is a simple connection between matrices describing linear mappings on $\mathbb{C}^n$ and the corresponding spreading representation (as a matrix indexed by the discrete phase space, so in terms of format still $n \times n$-matrices), more or less the content of [44] in a discrete form. The routines mapping the matrix kernel to its spreading representation or its Kohn–Nirenberg symbol are easily shown to have an inverse, and in this way one can demonstrate with a few lines of code the concentration of the spreading function of a given matrix on the adjoint lattice, which is connected with the original lattice (described by its indicator function, which is more or less the Dirac comb for the lattice Λ, or a sum of all the unit vectors which are labelled by the elements of the lattice) by the symplectic Fourier transform.

These routines do not only implement the various isomorphisms relevant for a good understanding of Gabor analysis in a numerically perfect way, they also suggest how to go ahead with similar ideas in the continuous setting, making use of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

First one can replace the discrete variables by continuous variables, and vectors in $\mathbb{C}^n$ (recall, they are essentially functions on $\mathbb{Z}_n$) by test functions in $\boldsymbol{S}_0(\mathbb{R}^d)$. For those decent (continuous and Riemann integrable) functions it is no problem to replace the continuous summations or e.g. (partial) Fourier transform by the corresponding integrals. Just to give an idea: the scalar product of $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$ is of course just $\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx$, an the FFT is replaced by the usual continuous FT given by (Riemann) integrals, both for the forward and inverse direction. In the next step one shows that most of these operations are compatible with the scalar product (usually taken as a starting point for the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$), and then (usually by means of duality considerations) one extends the transformations of interest to outer level of those "rigged Hilbert spaces", in our case to operators having a *kernel* (or equivalently a spreading symbol or equivalently a KNS symbol) in $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ resp. describe operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$. We do not go into further details about this more functional analytic part of application-oriented Harmonic Analysis, but maybe some readers find the studies related to the kernel theorem (see [75]) interesting for further studies.

7.9 Computations Benefitting from Theory

There are many ways in which the computation of objects relevant for the study of Gabor analysis can benefit greatly from equivalent descriptions, allowing efficient implementations. In addition, one may observe that the user's view on formulas which make such equivalent characterizations possible may change his appreciation for one or the other of these formulas. He may find that some of those characterizations are very important and useful, while others are only theoretically relevant but do not help in practice. This does not say anything about the "value" of a given result in general, just for its usefulness with respect to applications, but experience indicates that on average there is a strong (positive) correlation between (a kind of overall) importance of a mathematical result and its usefulness. The list of possible examples is very long, so we will restrict our attention to a few examples.

1. The synthesis of the Gabor frame matrix or of Gabor multipliers;
   Here the issue is to build the matrix describing the frame operator S which maps f to $\sum_{\lambda\in\Lambda}\langle f, \pi(\lambda)g\rangle\pi(\lambda)g$. Since the individual term $f \mapsto \langle f, \pi(\lambda)g\rangle\pi(\lambda)g = P_\lambda(f)$ is described by the outer product of the Gabor building block $g_\lambda = \pi(\lambda)g$ by itself the *naive way* of computing that matrix is to sum up over $\#\Lambda$ many of such projection operators.
   In the same way one could also create the matrices which generate Gabor multipliers, because Gabor multipliers with respect to a given Gabor frame $(g_\lambda)_{\lambda\in\Lambda}$,

with symbols $(m_\lambda)_{\lambda\in\Lambda}$ are nothing else but a weighted sum of such projection operators; i.e. it is of the form

$$G_{g,\Lambda,\mathbf{m}} = \sum_{\lambda\in\Lambda} m_\lambda P_\lambda.$$

Such a description is fine in order to understand the dependence of the Gabor multiplier on its ingredients, specifically the question when and whether the symbols uniquely determine the Gabor multipliers, or when and how (stable or not) it is possible to recover that (upper, Berezin) symbol $\mathbf{m} = (m_\lambda)_{\lambda\in\Lambda}$ from the diagonal part of the Gabor frame matrix, which is the so-called *lower symbol* of an operator, namely

$$\lambda \mapsto \langle Tg_\lambda, g_\lambda\rangle, \quad \lambda\in\Lambda.$$

Unfortunately the formation of such operators is getting more and more cumbersome the larger the signal size n is and also the more lattice points are involved. In this sense, a simple replacement of the lattice constants a, b (say even) by $a/2, b/2$ would increase the time needed to build the corresponding Gabor multiplier by the factor 4. A study of STFT multipliers (Gabor multipliers with full lattice, or $a = 1 = b$, thus n^2 projection operators) would be virtually impossible. Here it is very helpful to observe that the projection operator P_λ can be obtained by conjugation of $P_0 : f \mapsto \langle f, g_0\rangle g_0$ with the TF-shift $\pi(\lambda)$, since $P_\lambda = \pi(\lambda) \circ P_0 \circ \pi(\lambda)'$ (transpose conjugate, equals inverse TF-shift; see [44]).
Now, the mapping $\pi \otimes \pi^*$, conjugating every Hilbert Schmidt operator (here we just have $n \times n$-matrices with the Frobenius norm resp. with the scalar product inherited by the Euclidean structure of $\mathbb{C}^{n^2}$) with $\pi(\lambda)$ is in fact a *unitary representation of the cyclic group* $\mathbb{Z}_n$ *of order n on the space of matrices*, satisfying

$$[\pi(\lambda_1) \otimes \pi(\lambda_1)^*] \circ [\pi(\lambda_2) \otimes \pi(\lambda_2)^*] = \pi(\lambda_1 + \lambda_2) \otimes \pi(\lambda_1 + \lambda_2)^*, \quad (7.9.1)$$

and it is the so-called *Kohn–Nirenberg* description of operators (we refer again only to [44] for details) which intertwines this action with ordinary translation (on phase space). In other words, if $\kappa(T)$ is the KNS of some matrix (resp. operator T acting as a matrix through vector–matrix multiplication, i.e. $\mathbf{x} \mapsto \boldsymbol{A} * \mathbf{x}$) then we have for the new matrix $\boldsymbol{A}_\lambda := \pi(\lambda) * \boldsymbol{A} * \pi(\lambda)'$ the KNS

$$\kappa(\boldsymbol{A}_\lambda) = T_\lambda(\kappa(\boldsymbol{A})), \quad \lambda \in \mathbb{Z}_n \times \mathbb{Z}_n, \quad (7.9.2)$$

or in the case of Gabor multipliers

$$\kappa(G_{g,\Lambda,\mathbf{m}}) = \sum_{\lambda\in\Lambda} m_\lambda \kappa(P_\lambda) = \sum_{\lambda\in\Lambda} m_\lambda T_\lambda \kappa(P_0) = (\sum_{\lambda\in\Lambda} m_\lambda \delta_\lambda) * *\kappa(P_0), \quad (7.9.3)$$

where $**$ denotes $2D$-convolution over phase space. Clearly such a convolution can be performed easily by means of the `fft2` command.

The benefit of this consideration is clearly that the influence of the size of n is weakened very much, but above all that such a routine is—concerning computational costs—independent of the number of points in the lattice Λ (it would be even possible to use an irregular Gabor multiplier, with a more general discrete set of points Λ, which could be interesting to study the effect of removing certain elements from a given Gabor frame and check for the frame properties of the remaining Gabor family).

In addition one can use the fact that the KNS symbol of any operator is just the *symplectic Fourier transform* of its spreading function, and that the spreading function of a rank one operator[1]

$$f \mapsto \langle f, h\rangle g = (g * h') * f$$

is essentially the STFT $V_g(h)$, so one can use the STFT routine in order to form the KNS symbol of a Gabor multiplier in a cheap way. Of course, one also has to have (which is clearly available) an efficient FFT-based routine in order to come back from the KNS symbol of the operator to the operator resp. matrix itself, because this is what we are going for.

Having now an easy way to build Gabor multipliers it is meaningful to study various properties of such operators, e.g. how the choice of the symbol influence the behaviour of eigenvalues, when a Gabor multiplier is compact, or how a "fine Gabor multiplier", i.e. an STFT multiplier (imitating an Anti-Wick operator in the discrete setting) can be well approximated to be a rough one (for a low-redundant lattice Λ), and so on.

Let us only mention that such experiments give on the one hand plenty of insight into the general situation, because naive or smart questions can be answered usually quite quickly by a small experiment, but at the same time it raises the question to which extent the findings of such experiment conform with theoretical knowledge, especially for the related, continuous setting. Let us just reinforce the need of considerations on this point here, without going into details.

Finally let us mention (on the practical side) that to form the matrix of a Gabor frame operator one just has to view it as a Gabor multiplier with constant symbol $m(\lambda) \equiv 1$. Of course, for the case of separable lattice one can even further speed up the procedure by making use of another theoretical fact, the so-called *Walnut representation*, which—in terms of matrix representations—means that the matrix of the Gabor frame operator $S = S_{g,\Lambda}$ has only b side-diagonals which are all a-periodic (see [66, 111]).

2. As a second point where theory appears to help enormously let us mention the following relatively recent observation: in order to judge the quality of Gabor frames there are many possible measures of quality. We can say that a Gabor frame is not so good if the norm of the Gabor frame operator is large. There is *evidence from many numerical examples* (but to my knowledge no formal

[1] Recall that at the MATLAB command line the "$*$" represents matrix multiplication resp. matrix–vector multiplication!

connection at a theoretical level) that at least two distinct quantities are highly correlated:

a. The condition number of the frame operator (on $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ resp. perhaps better on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$);
b. The $\boldsymbol{S}_0$-norm of the canonical dual window $\widetilde{g}$;
c. The covering properties of the contour lines (e.g. for level $= 0.5$) of the function $|V_g(g)| = |STFT(g,g)|$.

Let us shortly explain these terms and provide background information. The first condition tells us about the numerical stability of the Gabor frame expansions. In other words, it indicates how much one can trust the closeness of Gabor coefficients (sampled versions of $V_g(f)$ over the lattice Λ) in order to conclude that also the corresponding signals are close.
Since for all practical purposes it makes sense to assume that the chosen window g is in $\boldsymbol{S}_0(\mathbb{R}^d)$ we know from theory that also the canonical dual window is in $\boldsymbol{S}_0(\mathbb{R}^d)$ (see [61]) and consequently that the mapping $f \mapsto (V_g(f)(\lambda)_{\lambda\in\Lambda}$ from signals to sampled STFTs establishes an Banach Gelfand triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(\Lambda)$ (see [5, 19, 54]); one can also ask for the condition number of this Banach Gelfand triple isomorphism, which is more or less the same as looking at the condition number of this isomorphism as a mapping from $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ to $\big(\boldsymbol{\ell}^1(\Lambda), \|\cdot\|_1\big)$.
Again theory tells us that this term is closely related to the $\boldsymbol{S}_0$-norm of $\widetilde{g}$, because on the one hand it is known that we have $\widetilde{g} = S_{g,\Lambda}^{-1}(g)$, hence knowing the properties of the inverse frame operator on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ implies that we have control over $\|\widetilde{g}\|_{\boldsymbol{S}_0}$ by controlling the $\boldsymbol{S}_0(\mathbb{R}^d)$-norm of g (which is just $\|g_0\|_{\boldsymbol{S}_0} = 2$ for the Gaussian). On the other hand, function space estimates allow us to control both the coefficient mapping $f \mapsto (V_g(f)(\lambda)$ and the synthesis mapping $c \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$ by the $\boldsymbol{S}_0$-norm of $\widetilde{g}$, using again the fact that the dual window induces the inverse of the frame operator, or $S_{g,\Lambda}^{-1} = S_{\widetilde{g},\Lambda}$.
The connection to the covering numbers (for the case of the Gauss function $g = g_0$ these are just circles, for generalized Gaussians one has ellipses) is giving a nice criterion for "geometric fit" between windows (better the shape or concentration of $|V_g(g)|$) and the lattice properties, but this is still under investigation and a hard task, even for special cases (see some recent work of Faulhuber [34, 35, 37, 38])
Coming back to the issue of numerical investigations, recall that one of our goals is to find theoretical justifications for the connection between quantities described in (i), (ii) and (iii), ideally at a quantitative level. Because it turns out that for large n the actual accurate computation of the condition number of the frame operator (which is actually only possible in the Hilbert space setting, i.e. between Euclidean norms) is computationally expensive, whereas it is easier to determine covering numbers making use of the lattice description by their generating matrices (independent on n).

3. But it is not always necessary to avoid large signal sizes. We have meanwhile a collection of good algorithms allowing us to compute the dual Gabor atom for significant signal dimensions (even beyond $n = 15000$; see the LTFAT toolbox for details), as well as discrete version of computing the $\boldsymbol{S}_0$ of a dual Gabor atom. This is computed as the $\boldsymbol{\ell}^1$-norm of the numerical computed STFT suitably normalized, so that for any sufficiently large n, e.g. $n \geq 100$, one gets as a numerical value for the discrete Gauss sequence of length n a value very close to 2, as is the analytically computed value for the continuous case. Thus overall it should be computationally feasible to compute the $\boldsymbol{S}_0$-norm of dual windows for different configurations (atoms, lattices, signal dimensions) and study in this way the influence of the Gabor atom, the lattice, and their (mostly geometric) match for the quality of the resulting Gabor frame.
   But *trying to realize this program* one observes that this is not a computationally cheap task either. Already for $n \leq 5000$ this approach gets slow, and thus simulating the situation of more and more critical lattices, requiring to take larger and larger n in order to allow to have some redundancy of the lattices which is closer and closer to one is not really feasible in this way.
   So in fact we have to go to a coarser version of the STFT, which is still reasonably fast and thus allows to approximate the $\boldsymbol{S}_0$-norm by a kind of discrete sum. Trying to understand the behaviour of $\boldsymbol{S}_0$-norms of dual windows thus requires the *ability to carry out the computation* of the *dual Gabor atom* first and then to compute a reliable approximation of $\|\widetilde{g}\|_{\boldsymbol{S}_0}$. This is all possible by now and seems to indicate the expected behaviour (some results in this direction have been obtained in [8]). What is missing at the moment (at least in an explicit form) is a justification, which would require to have equivalence bounds valid uniformly over a range of signal size parameters ($n \geq n_0$, for some $n_0 \in \mathbb{N}$) combined with estimates valid for all lattices which are "not too rough", i.e. with a range of redundancy larger than, say 3.5 and maybe restrictions on the excentricity of lattices allowed in the approximation.
   So we have another example, where numerical computations ask for a simplification by taking some approximation, which in turn raises the need of more general and uniform estimates than those found in the literature. Of course, while writing this, I can report that a couple of numerical experiments carried out in that direction provide good indications that there is a fairly realistic chance to be able to proceed by this combination of methods. But this in itself is only a *plausibility argument and not a proof.*

7.10 What are Good Gabor Systems?

During the UnlocX project, entitled *Uncertainty principles versus localization properties, function systems for efficient coding schemes* (2010–2013) one of the key issues for the NuHAG team was to identify "good Gabor systems". Although this is a priori a vague question the numerical simulations carried out during that project brought up a quite clear picture. We can give here only a short outline and suggest the concept of a *compound condition number* for a (tight) Gabor system.

Intuitively Gabor systems that are of interest have highly localized (in the TF-sense) atoms, but they also have very localized dual windows, because otherwise the representation is not at all local, and the coefficients obtained with the help of "rough" dual windows are very sensitive to noise and also very non-local. So in short: the Balian–Low theorem prohibits the existence of *well-localized Gaborian Riesz basis* for $\left(\boldsymbol{L}^2(\mathbb{R}^d),\ \|\cdot\|_2\right)$ and one has to expect (correctly) that systems close to the critical density will not have a good joint TF-localization either. For Gabor multipliers it is advisable to start from tight Gabor families, and also these Gabor families do not have good TF-localization near the critical density.

Consequently, going for well-localized Gabor systems one has to allow a certain amount of redundancy for *Gabor frames*, or equivalently not too high density for *Gaborian Riesz basis sequences* for applications in mobile communication. Interestingly enough the Daubechies–Landau–Landau/Janssen/Ron–Shen duality principle implies that these two properties are equivalent (thanks to a smart use of Poisson's formula for the symplectic Fourier transform; see [45]). The condition number of a Gabor frame with lattice Λ equals the condition number of the Gaborian Riesz sequence arising from the adjoint TF-lattice Λ°.

If one takes the *condition number of the Gabor frame operator* (or the condition number of the Gram matrix for a Gaborian Riesz sequence) as the only measure of quality it is plausible (and evident from the numerical experiments) that high redundancy gives very good frames. Equivalently very sparse lattices Λ° give rise to almost orthonormal systems (but at very low density, so not so interesting for applications, because this means that one has channels of low capacity).

So essentially the following is a reasonable question: **Given the Gabor atom and the redundancy**, one can search for the "best lattice" Λ_0, i.e. for the lattice (within a given family of lattices) which minimizes the condition number among all frame operators of the form $S_{g,\Lambda}$, for that given redundancy. A typical value would be a *redundancy factor* of 1.5, which means in the finite case that one has 50% more Gabor atoms compared to the dimension of the signal space. For signals of length n this means that one has to choose a and b such that both divide n and $1.5ab = n$. We can take $n = 480$, $a = 20$, $b = 16$ as a concrete example, resulting in a Gabor family of 720 elements. But there are of course many more non-separable lattices in this case, in fact altogether 186 of them which all have the same number of elements.

During the UnlocX project we have systematically evaluated various alternative measures of quality, and quite a few of them showed strong correlation with the condition number of the frame operator. This even suggests that under certain assumptions on the TF-concentration of the Gabor atom one might be able to estimate the operator norm of the inverse frame operator on $\left(\boldsymbol{S}_0(\mathbb{R}^d),\ \|\cdot\|_{\boldsymbol{S}_0}\right)$ by the operator norm on $\left(\boldsymbol{L}^2(\mathbb{R}^d),\ \|\cdot\|_2\right)$, but to our knowledge such estimates are not theoretically available up to now, nor do we have counterexamples.

Taking a closer look at those lattices which show optimal performance for a discrete Gaussian window clearly indicates that in all the cases analysed so far experimentally it was always the most hexagonal lattice which was showing the best performance.

For the continuous case considerations in that direction have been undertaken by Markus Faulhuber (in his Ph.D. thesis [35] and later on in [34]). His works indicate that for the Gaussian $g_0(t) = e^{-\pi|t|^2}$, the best case among all lattices of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$ for a given redundancy $red = 1/(ab)$ are the balanced ones, i.e. those with $a = b$.

For more general lattice one expects that for $d = 1$, i.e. when the phase space which $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ can be identified with the complex plane, a hexagonal lattice with the given size of the fundamental domain (< 1!) is the best possible choice. A proof of this claim would solve the *Strohmer–Beaver conjecture* positively (see [104]).

Finally we would like to mention that also an alternative view on Gabor frames generated by pairs (g, Λ) is possible. One may ask: Given the lattice Λ (with $\text{red}(\Lambda) > 1$), what is the best Gabor atom for the given frame, e.g. within the family of *generalized Gaussian functions* of the form $g(t) = e^{Q(x)}$ where $Q(x)$ is a complex-valued quadratic form? Again experimental insight was that the optimal lattices are those which are distorted hexagonal lattices, with a symplectic distortion related to the quadratic form Q in a natural fashion. However, the results in [36] show that given, e.g. the hexagonal lattice, we can choose a very ill-localized Gaussian window (very long and possibly rotated) and still get the same optimal frame bounds as for the round standard Gaussian. The reason is that the lattice is invariant under the action of the modular group $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\pm I$. In particular, we can choose from countably many possible bases for a lattice and to each choice of a basis we have a different Gaussian window which can be transformed into the standard Gaussian by choosing the canonical basis for the hexagonal window.

But aside from the decision to fix the redundancy, typically in the range $[1.2, 1.6]$ or $[1.1, 2]$ we may ask: *How can one determine, maybe first in some abstract way, the right redundancy for Gabor systems?*

The suggestion given below (named *compound condition number*) relies on some observations which we have to explain first.

We start with the following question: In which cases are Gabor expansions useful? As a first answer one can say, for the analysis of operators T which have a strong diagonal concentration in the Gabor frame matrix representation, which can be described as the infinite matrix where each column contains the Gabor coefficients of $T(\pi(\lambda)g)$, the image of some Gabor atom under the operator. Such operators, often called *underspread operators*, because their (essential) spreading support is relatively small, can be well approximated by Gabor multipliers, i.e. by a weighted sum of projection operators on the elements of a Gabor frame.

Although one can do some fine analysis on the dependency of the class of operators which are well approximated by Gabor multipliers (e.g. Hilbert Schmidt operators, which are well approximated in the HS-sense), as a function of their spreading support and the corresponding Gabor atom we will try to explain the situation for the case of a Gaussian Gabor atom, resp. a tight Gabor frame derived from a given Gabor frame using the square root inverse of the frame operator.

In most of the cases of interest it turned out that the family of (rank one) projection operators $(P_\lambda)_{\lambda\in\Lambda}$ forms a Riesz basic sequence within the Hilbert space $\mathscr{HS}$ of Hilbert Schmidt operators (see [46]).

In fact, they form even a *Riesz projection basis* for the Banach Gelfand Triple of Gabor multipliers in $(\mathscr{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathscr{HS}, \mathscr{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ (see [19]).

A natural quality criterion for such a Riesz basic sequence is its *Riesz condition number*. This condition number can be easily computed from the theory of spline-type space and the connection to the KNS representation of operators (which is in fact a unitary Banach Gelfand triple morphism between these operators and their Kohn–Nirenberg symbols, in $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ (see [58]).

Doing a few numerical experiments one finds (what is at the end quite easily to verify theoretically): Whenever the lattice is getting more and more dense the condition number of this family is increasing, and one certainly can give estimates how much it grows with increased density of the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$.

In practical terms this means, that the stability of approximation of operators is quickly degrading if the lattice Λ is too dense. And there is in fact not much to gain. It is the disadvantageous situation that one has to pay a high price of numerical instability for a minimal improvement of approximation quality. Consequently we think that it is a good idea to keep that condition number under control as well.

As a consequence we suggest to define the **compound condition number** as a quality criterion, which has the effect that it leads (at least according to our experiments) to a well-localized Gabor frame with a relatively stable procedure allowing us to approximate underspread operators, typically by Gabor multipliers with the corresponding tight Gabor family.

Definition 7.1

$$\boldsymbol{\kappa\kappa}(\mathbf{g}, \mathbf{a}, \mathbf{b}) := \sqrt{\mathrm{cond}(S_{g,a,b}) \cdot \mathrm{cond}(P_g)},$$

where P_g is the condition number of the Riesz basic sequence of the family of projections onto the Gabor atoms, i.e. the quotient of the two Riesz bounds for this family of operators within the Hilbert space of Hilbert–Schmidt operators.

This figure of merit certainly guarantees that for those lattices (for a given Gabor atom g) which show optimal performance one has a well-conditioned Gabor frame, but at the same time a stable and numerical realizable procedure to approximate a given operator by Gabor multipliers, preferably with tight Gabor atoms.

Since the interesting parameters describing the lattice of interest (either the separable ones, or the ones given by general matrices) form a compact domain one can expect (due to the continuous dependence of the individual condition numbers on the lattice parameters) that the optimal value is achieved, both within the family of separable lattices as well as in the family of all (reasonable) lattices.

7.11 The Role of the Zak Transform

It is clear that *painless Gabor expansions* are possible if the Gabor frame operator is just a multiplication operator. In this case, after realizing the Gabor frame operator it is enough to multiply the result of the Gabor frame operator by the *inverse of its diagonal term*, i.e. with the pointwise inverse of the a-periodic version of $|g|^2$ according to the Walnut representation. This fact is equivalent to the claim that the dual Gabor atom is obtained by a simple multiplication and thus the atom g and its canonical dual $\widetilde{g}$ have the same compact support, which is also very convenient in practice. The same is true for the canonical tight Gabor frame, obtained by dividing by the square root of the diagonal term.

Alternatively it is easy to obtain the correct set of (canonical) coefficients for the Gabor expansion of a given signal f with respect to such a Gabor atom by premultiplying f and then taking the sampled STFT with respect to the given atom g.

The situation can also be described equivalently in the Janssen picture: In this case the support of the spreading function of $S_{g,\Lambda}$ is part if the (commutative) subgroup $\{0\} \times (n/a)\mathbb{Z} \subset \mathbb{R} \times \widehat{\mathbb{R}}$ of all modulation operators. In other words, the frame operator (and its inverse and square root inverse) all belong to a simple commutative algebra of TF-shifts.

For example, one can have the same situation on the Fourier transform side. Assume that one wants to use a band-limited Gabor atom g with the length of the support of $\hat{g}$ being smaller than $1/a$ (or in other words, so that a is below the Nyquist rate for the band-limited atom g) we have the same situation on the Fourier transform side.

But this is not the only situation, where despite the typical non-orthogonality of the Gabor family one has a cheap way of inverting the Gabor frame operator. *Any commutative group of TF-shifts* allows the diagonalization of the set of *all* Gabor frame operators with respect to such a lattice.

In the terminology of general TF-lattice we have the following situation: We are interested in the situation where the *adjoint lattice* Λ° is commutative, or in other words where $\Lambda^\circ \subset \Lambda^{\circ\circ} = \Lambda$.

For the classical case of finite Gabor analysis for signals of length n with TF-lattice parameters (a, b) (both dividing n) the commutative situation is equivalent with integer redundancy, or in other words with the fact that ab divides n. As we know that we cannot have "nice" (resp. well TF-concentrated Gabor atoms and duals of the same form) Gabor families at *critical density*, resp. for the case $ab = n$, we talk about the cases $n = 2 \cdot ab$ or $n = k \cdot ab$, with $k \in \mathbb{N}$, which at least for $k \geq 4$ are considered already high redundancy case.

It is in this situation that one can compute dual Gabor atoms easily and with very fast algorithms, based on the use of the Zak transform in the classical, separable case. In fact the rational case, which is often distinguished in discussions about Gabor families from the irrational case (see [40] versus [61]) can and should be viewed in many cases as the case of a multi-window Gabor family with respect to some commutative lattice. This fact points to a possible explanation, by partial Zak transform methods appear in a number of algorithms, starting with the often cited paper by Zeevi and Zibulski [114, 115].

7.12 Transfer from $\mathbb{R}$ to $\mathbb{Z}_n$

At the very end let us address an important but *often neglected* issue: the transfer of information about continuous functions to the finite discrete (or discrete periodic) case and vice versa. We cannot discuss the details here, but we want at least to raise awareness for this problem and indicate how to avoid problems with it.

Typically starting from the argument that the computer allows only to work with finite vectors one easily agrees that one has to sample a continuous function on the real line, of course equidistantly and fine enough on a sufficiently long interval in order to capture the rough shape of the (smooth) function under consideration. Of course one may expect then to replace the Fourier transform (given as an integral transform) by the FFT (the fast implementation of the DFT, the Discrete Fourier transform), resp. "the" Fourier transform over the group $\mathbb{Z}_n$, but it is also clear that this *natural analogue* needs some justification, which is rarely given.

Unfortunately this transfer and the transfer back to the continuous domain, e.g. by piecewise linear interpolation is viewed only as a marginal question, although neglecting this harmless looking but delicate issue results in a number of complications in the use if the FFT (interesting examples can be found [1, 109]), explained somehow as bad properties of the discrete transform or motivation to teach additional "tricks" to the user. Among others it seems that the MATLAB command `fftshift` has been introduced to tackle this problem, but it needs to used properly in order to provide good results.

Summarizing the set of problems we can list the following issues in this context:

- How should one insert the information on the given continuous functions via (regular, i.e. equidistant) sampling into a finite vector, representing a discrete and periodic signal?
- How can one plot the (input or output) data properly so that they represent the continuous curve at least approximately, or better, as good as possible?
- Which function spaces allow to guarantee preservation of more and more information about the underlying continuous function as the number of samples is getting larger?
- How can one obtain quantitative version for the mutual approximation?
- Which theoretical justifications are most suitable to better understand the connection and avoid pitfalls?

We believe, that distribution theory (either using the Schwartz theory of tempered distributions, or the much simpler theory of Banach Gelfand Triples, additional reference: (see [19])) provides the clue to a correct understanding.

Starting from a test function in the relatively large space $S_0(\mathbb{R})$ (similar for $S_0(\mathbb{R}^d)$) one can sample it by pointwise multiplication with a *Dirac comb*, typically $⊔⊔_\alpha = \sum_{k\in\mathbb{Z}} \delta_{\alpha k}$ and then periodize it, with a period p which should be a multiple of α, via convolution with the Dirac comb $⊔⊔_p = \sum_{k\in\mathbb{Z}} \delta_{pk}$. The result is a discrete and periodic (hence unbounded) measure, or in engineering terminology a periodic and discrete signal. Now it is our choice to assign the amplitudes arising in this sum of the form $\sum_{k\in\mathbb{Z}} c_k\delta_{\alpha k}$ to a vector in $\mathbb{C}^n$, with $n = p/\alpha$, i.e. to turn that p-periodic sequence into a function on $\mathbb{Z}_n$. Clearly we will assign the value at zero, i.e. the coefficient c_0 to the zero element of the multiplicative group of unit roots, i.e. to $\omega_n^0 = 1$. Starting from there it is plausible to follow the unit roots in the mathematical positive sense, and assign the amplitudes c_k to the element ω_n^{k-1}.

In this way the generating sequence $\mathbf{c} = [c_0, c_1, ...c_{n-1}] \in \mathbb{C}^n$ arises in a correct way, *but* MATLAB does not allow indices zero, so one has to attach (formally) to the "abstract vector" $\mathbf{c}$ the "concrete vector" $\mathbf{d} = [d_1, ...d_n]$ with $d_k = c_{k-1}$.

It is also clear that the negative coordinates are the ones coming at the end of the sequence, because due to the periodicity we have $c_{n-1} = c_{-1}, c_{n-2} = c_{n-2}$ etc.. This procedure is also compatible with the flip operator $f \mapsto f^{\checkmark}$, given by $f^{\checkmark}(x) = f(-x)$. It corresponds naturally to the (correct!) flip operator on the torus, meaning that it gives us the sequence $\mathbf{c}^{\checkmark} = [c_0, c_{n-1}, ...c_2, c_1]$, or expressed in MATLAB coordinates $[d_1, d_n, d_{n-1}, ...d_3, d_2]$.

Now the key point if this interpretation is the following statement, which is making a *theoretical connection* between the (generalized) Fourier transform for distributions (in our case discrete and periodic distributions) and the numerical DFT (or FFT) algorithm.

Simply starting from the basic rules for the (generalized) Fourier transform, according to which the Fourier transform of a shifted version of a function or distribution equals a modulated version of its Fourier transform, combined with the fact that (according to Poisson's formula, interpreted in a distributional setting) one has $\widehat{⊔⊔} = ⊔⊔$, and finally the dilation theorem applied to this formula, which tells us that (up to normalization) $\widehat{⊔⊔_\alpha} = C_\alpha ⊔⊔_{1/\alpha}$ one can end with the conclusions:

Up to suitable normalization the (abstract) Fourier transform of a p-periodic sum of Dirac measures concentrated on the lattice $\alpha\mathbb{Z}$ is the $1/\alpha$-periodic sum of Dirac measures concentrated on the lattice $1/p \cdot \mathbb{Z}$, whose generating sequence $\mathbf{f} = [f_0, f_1, ...f_{n-1}]$ *is just the DFT of the original sequence* $\mathbf{c}$.

It is plausible that for a "decent function" with some smoothness and good decay at infinity the periodization will not play an important role (resp. will cause only a small error compared to just sampling its non-periodized version essentially over the interval $[-p/2, p/2]$) and thus one can hope to almost recover f from its sampled and periodized version (and the same applies on the Fourier transform side).

To make that argument valid it is of course (again) important to choose the period as a multiple of the sampling rate, because then the order of *sampling and periodization* can be *interchanged without affecting the result!*. In some sense it is most natural to choose p as some (even) natural number and $\alpha = 1/p$. In this case we obtain signals of length $n = p^2$. The symmetry of the situation then implies that a Fourier invariant function (like the Gauss function, or any of the Hermite functions $h_j, j = 4 * l, l =$

0, 1, ... and any linear combination thereof) results in a discrete version which is in a numerical precise sense and up to the usual factor $\sqrt{n}$ invariant under the FFT, thus perfectly imitating the continuous situation. In addition we can be sure that a real-valued function will have a symmetric Fourier transform and a symmetric (in the sense of the discrete flip operator) function will have a real-valued FFT, as we would expect it.

As with respect to the convergence property and approximate recovery of f from a (fine) sampled and (coarsely) periodized version via piecewise linear interpolation, or more generally suitable quasi-interpolation operator, a first general approach is provided by the paper [42], showing that the test function space $\boldsymbol{S}_0(\mathbb{R}^d)$ provides a very natural setting. Due to the fact that $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ is continuously embedded into $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ and $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, and consequently into all of the $\boldsymbol{L}^p$-spaces, in particular $\big(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$, one can derive convergence results in all those norms whenever the input is in $\boldsymbol{S}_0(\mathbb{R}^d)$.

Essentially, this approach indicates that there is a sequence of recovery operators R_p which can be applied to the sampled and discretized versions of f, allowing to almost recover f. By writing PS_p (for the case $\alpha = 1/p$, hence $n = p^2$) we have the boundedness from $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ into $\mathbb{C}^n$ and R_p back from $\mathbb{C}^n$, so that the combined operator sequence

$$R_p \circ PS_p$$

is uniformly bounded on $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ and pointwise convergent to the identity in the strong operator sense, i.e.

$$\|f - R_p(PS_p(f))\|_{\boldsymbol{S}_0} \to 0 \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{7.12.1}$$

according to [42].

It is an easy exercise to estimate in such a situation that one has uniform convergence over relatively compact subsets of $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ (see [53, 55]), typically over the unit balls of Banach spaces compactly embedded into $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$, such as the Shubin classes $\boldsymbol{Q}_s(\mathbb{R}^d)$ (for $s > d$), or the spaces described in [65] providing sufficient conditions for $\boldsymbol{S}_0(\mathbb{R}^d)$ via weighted norms on both the time and the frequency side. For those examples one can expect nice estimates for the convergence rates, in dependence of α, p and the function space parameters.

As a plausibility argument for the use of the “simple-minded approach”, which consists in taking samples as a sequence, starting with the first sample at the left end of the sampling interval, and progressing until the right end, one can take the desire that it appears to be the most natural way, sampling at a regular distance from left to right, also for convenient plotting. The MATLAB command `bas = linspace(a,b,n)` provides exactly such a list. For example one gets from `bas = linspace(-1,1,4)` the sampling positions

$$[-1, -.3333, .3333, 1]$$

which is clearly missing the central value 0, as any even number n for the case of a symmetric interval $[-p/2, p/2]$.

In fact, in order to form a proper sampling sequence (less convenient to plot, but correct with respect to the transfer setting) one produces a vector which starts with the value of the function at 0 and ends at $p/2$ with $1 + p/2$ entries (for even n) or $(1 + p)/2$for p odd, and a complementary series describing the remaining sampling positions from α, until the length n of values is reached. It turns out that for even n the last value of the first segment corresponds just to the amplitude taken at $-1 = \omega_n^{n/2}$, which is another flip-invariant coordinate. In contrast, the situation of odd signal length n appears more natural in this setting, especially on the Fourier transform side, because the first coordinate corresponds then to the zero-frequency (DC) part of the signal, while the number of positive and negative frequencies is equal.

So, despite the fact that the plotting is more convenient in the "naive setting" one can of course make use of reindexing in order to plot functions properly, even with the right labels, including zero (0) an positive as well as negative values. The corresponding NuHAG routine is called `plotc.m` (`plot` a complex-valued signal in a **c**entred way). A corresponding version for images or spectrograms is denoted by `imgc.m`. Here only the absolute value of the complex signal is used, but with the same idea.

For a discrete Gauss function `g = gaussnk(n)` the command `plotc(g);` displays a discrete (and FFT invariant) Gaussian signal in the expected (nice) way, while one can visualize the corresponding $2D$ Gaussian given by `gg = g(:)*g(:).'` by the command `imgc(gg);`.

7.13 Acknowledgement

The author would like to thank his many students and internship candidates over the years who have produced large amounts of Gabor related work, producing valuable MATLAB code and helping in creating insight through systematic simulations. Many of the resulting routines (M-files) can be retrieved from the NuHAG server or by request to the author, much of it still in a rather experimental format.

Among the young people who have contributed to the explorative work over more than a decade I have to mention teams of master students from Jacobs University in Bremen (sent by Götz Pfander), in particular Vlad Nicol, Radu Frunza and Ion Victor Gosea. In addition Missbauer [86], Julio Estevez and many others have done extensive experimental work. The Ph.D. theses of Qiu [95], Holighaus [73], Wiesmeyer [113], Velasco [108] and Faulhuber [35] have been heavily, and I think positively influenced by a strong numerical component. Some of the more recent contributions will be part of the Ph.D. thesis of Anna Grybos.

In particular we would like to indicate our appreciation of Peter Balazs and his group at ARI (Acoustic Research Institute of the Austrian Academy of Sciences), who keep the LTFAT toolbox, originally created by Peter Soendergaard, under devel-

opment, which is a tremendous service to the community (see [88, 100, 103]). In the last years Zdenek Pruza has taken care of this toolbox (and much more, e.g.[89]).

Work on Gabor-like adaptive TF-frames published in [2, 71] has been the basis for an interesting webpage called the *GABORATOR*, at www.gaborator.com, which went online in November 2017. It can be seen as a convincing example of how theory can support applications, in this case audio freaks, who could not do signal manipulations on the basis of spectrograms adapted to the human hearing system until a perfect reconstruction filter bank had been made available, described both theoretically and demonstrated on the basis of MATLAB.

Results concerning the *quality of Gabor families*, i.e. computational exploration of the fit between windows (typically generalized Gaussian, with elliptic contour lines for their ambiguity functions) have been carried out in the framework of the European Project UnlocX, Proj. Number 255931, entitled *Uncertainty principles versus localization properties, function systems for efficient coding schemes*.

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Contact and Cooperation

This manuscript contains a number of suggestions, conjectures and vague statements. Some of these statements might encourage colleagues to follow up on them and try to go deeper, and/or to run more systematic experiments or simulations based on the ideas of this paper.

In order to avoid that several people work on the same subject without knowing about each other's work, with the additional trouble of then having perhaps problems to publish it, I suggest that those who are interested in a concrete subject should contact the author (hgfei) and coordinate their efforts, so that the gain for the overall community interested in questions related to Gabor analysis is maximized. This suggested procedure would also be beneficial for individuals, avoiding to some extent a long-lasting experimental phase due to potentially harmless problems, which are however sometimes hard to overcome if one works in isolation. Maybe later on it will be possible to even establish a research platform which helps to openly exchange ideas, while at the same time ensuring recognition by the academic community. In this sense a cooperative research enterprise might be initiated if there is sufficiently strong response.

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Chapter 8
L^p Continuity and Microlocal Properties for Pseudodifferential Operators

Gianluca Garello and Alessandro Morando

Abstract The aim of this paper is to give a brief survey about L^p continuity and microlocal regularity for classical pseudodifferential operators, with $p \neq 2$. In particular, we focus on some classes of operators with smooth symbol satisfying decay properties of quasi-homogeneous or completely non-homogeneous type.

8.1 Introduction

Let us consider the pseudodifferential operators with classical Kohn–Nirenberg quantization:

$$a(x, D)u(x) = (2\pi)^{-n} \int e^{ix\cdot\xi} a(x, \xi)\hat{u}(\xi)\, d\xi, \quad u \in \mathscr{S}(\mathbb{R}^n), \tag{8.1.1}$$

where $\hat{u}(\xi)$ is the Fourier transform of u. The symbol $a(x, \xi)$ is considered in the Hörmander classes $S^m_{\rho,\delta}$, characterized, for $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$ by the estimates:

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq c_{\alpha,\beta} \,(1 + |\xi|)^{m-\rho|\alpha|+\delta|\beta|}, \quad x, \xi \in \mathbb{R}^n. \tag{8.1.2}$$

It is well known, see e.g. Hörmander [19], that for $0 \leq \delta < \rho \leq 1$ the pseudodifferential operators of order $m = 0$ are L^2 bounded and the same is true when $\delta = \rho \neq 1$. In the case $\delta = \rho = 1$, a counterexample of Ching, [6] 1972, showed that the L^2 continuity is not in general assured. Restrictive conditions which assure the L^2 boundedness of pseudodifferential operators in Op $S^m_{1,1}$ are given in the theory of paradifferential operators, introduced in Bony [5] and well summarized in [24].

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A result of microlocal continuity in the Sobolev (Bessel potential) spaces $H^{s,2}$ is given in [10].

In this paper, the aim is to give a brief survey about L^p continuity and microlocal regularity for classical pseudodifferential operators, with $p \neq 2$.

The problem of L^p boundedness, $1 < p < \infty$, when $\rho = 1$ is completely solved since the early years of development of the pseudodifferential theory; see, e.g., [29, 32]. The same is true for the study of the regularity of solutions to elliptic pseudodifferential equations. Here and in the following, the L^p and the related Sobolev spaces are intended on $\mathbb{R}^n$, in any occurrence where, for short, this is omitted.

In Sect. 8.2, we summarize some informations about the literature on the $\rho \neq 1$ case, starting from the very basic results of Hörmander [19], which clarify that the pseudodifferential operators of order zero are not in general L^p bounded, but one needs operators of suitable negative order which strictly depends on ρ itself.

The issue of the negative order for obtaining the L^p boundedness was completely solved by a sharp result of Fefferman [7], as specified in next Theorem 8.3.

Let us notice that also the generalized weighted symbol classes, introduced by R. Beals [1–3], require $\delta < \rho = 1$ for obtaining the L^p continuity for pseudodifferential operators of zero order.

In the last part of Sect. 8.2, we describe the layout of Taylor [29], by considering pseudodifferential operators of order zero which satisfy an additional property, that directly assures their boundedness on L^p, $1 < p < \infty$. The difference with respect to the preceding cases is that now we can apply the Marcinkiewicz lemma on the L^p continuity of Fourier multipliers.

The pseudodifferential operators of *quasi-homogeneous type*, introduced at first by Lascar [22], fit in a natural way the Taylor model. We describe this case in Sects. 8.3 and 8.4, where we construct also a quasi-homogeneous version of the tools needed for studying the microlocal regularity in the generalized Sobolev spaces $H^{s,p}_M$. We refer here to the results obtained in [14]. By means of the quasi-homogeneous weights, we can plainly study the hypoellipticity of the heat operator and microlocal propagation of singularities for Schrödinger operator.

For the study of continuity and microlocal properties of quasi-homogeneous pseudodifferential operators in the case $\delta = \rho = 1$ and Sobolev space with summability exponent $p = 2$, we address [11].

In Sect. 8.5, following the approach of Rodino [25], we introduce a class of local vector weighted symbols $S_{m,\Lambda}(\Omega)$, where $m(\xi)$ and $\Lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))$ are positive continuous weight and weight vector. The corresponding pseudodifferential operators could be considered in the frame of general pseudodifferential calculus of Beals [1], Hörmander [20]. In this framework, we study the microlocal regularity on a scale of Sobolev spaces H^p_m. From this point of view, the main problem consists in the complete lack of any homogeneity property of the weights $m(\xi)$ and $\Lambda(\xi)$. Thus for describing the microlocal regularity of a distribution $u \in \mathscr{D}'(\Omega)$, we cannot use the conic neighborhoods as done in the classical definition of the Hörmander wave front set; see [21, I]. Also the quasi-homogeneous wave front set introduced in Sect. 8.3 is not useful in this case. For this reason, following the arguments in [9, 25], we introduce the concepts of characteristic filter of a pseudodifferential operator and of filter of weighted Sobolev regularity of $u \in \mathscr{D}'(\Omega)$. We can then give a result of microlocal propagation of singularities for solutions to (pseudo)differential equations. The results are all proved in [16].

8.2 Classical Estimates

Let us consider the problem of L^p continuity for pseudodifferential operators in $\mathrm{Op}S^m_{\rho,\delta}$, starting from the following result of Hörmander [18, Theorem 1.1].

Theorem 8.1 *If A is a bounded translation invariant operator from L^q to L^p and $q > p$, we have $A = 0$ if $q < \infty$, and if $q = \infty$, the restriction of A to L^∞_0 is* 0.

Here, L^∞_0 is the set of L^∞ functions which tend to 0 at infinitive.

We then obtain the necessary condition $q \leq p$ for a not trivial pseudodifferential operator to satisfy, for $1 \leq p, q \leq \infty$ and some positive C, the inequality

$$\|a(x, D)u\|_{L^p} \leq C\|u\|_{L^q}, \quad u \in \mathscr{S}(\mathbb{R}^n). \tag{8.2.1}$$

The following result is proved in [19] as a quite easy consequence of the L^2 continuity of the pseudodifferential operators in $\mathrm{Op}S^m_{\rho,\delta}$, $0 \leq \delta < \rho \leq 1$.

Theorem 8.2 *If $q \leq 2 \leq p$, $0 \leq \delta < \rho \leq 1$, the estimate (8.2.1) is valid for any $a(x, \xi) \in S^{-m}_{\rho,\delta}$ if and only if*

$$m \geq n\left(\frac{1}{q} - \frac{1}{p}\right), \tag{8.2.2}$$

with strict inequality if $q = 1$ or $p = \infty$.

In the cases $q \leq p \leq 2$ or $2 \leq q \leq p$, the result depends on the choice of $0 < \rho \leq 1$. In order to obtain (8.2.1) for any $a(x, \xi) \in S^{-m}_{\rho,\delta}$, it is essentially proved in Wainger [31], 1965, that we must have:

$$m \geq n\left\{\frac{1}{q} - \frac{1}{p} + (1-\rho)\max\left(\frac{1}{2} - \frac{1}{q}, \frac{1}{p} - \frac{1}{2}, 0\right)\right\}, \tag{8.2.3}$$

with strict inequality if $q = 1$ or $p = \infty$. The inequality (8.2.3) clearly agrees with (8.2.2) when $q \leq 2 \leq p$.

When $p = \infty$ and $q > 2$, the sufficiency of the strict inequality (8.2.3) is also essentially contained in the results of Wainger [31].

By an interpolation theorem of Stein [27], the sufficiency of the strict inequality (8.2.3) follows in general since Theorem 8.2 has already shown it for $q \leq 2 \leq p$.

At the end of Hörmander [19], it is left as an open problem whether (8.2.1) is satisfied by all $a(x, D) \in \mathrm{Op}S^{-m}_{\rho,\delta}$, $0 \leq \delta < \rho < 1$, when there is equality in (8.2.3), in which case Theorem 8.2 is not applicable.

If we consider the case $1 \le p = q \le \infty$, $p \ne 2$, then (8.2.3) reduces to

$$m \ge n(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{8.2.4}$$

In this case, the problem is completely solved by

Theorem 8.3 (Feffermann [7], 1973)

(a) Let $a(x,\xi)$ be a symbol in $S^{-m}_{\rho,\delta}$ with $0 \le \delta < \rho < 1$ and $m < \frac{n}{2}(1-\rho)$. Then the operator $a(x,D)$ is bounded on L^p, $1 < p < \infty$, provided that:

$$m \ge n(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{8.2.5}$$

(b) If $m < n(1-\rho)\left|\frac{1}{p} - \frac{1}{2}\right|$, then the symbol

$$a_{\rho,m}(\xi) = \frac{e^{i|\xi|^{1-\rho}}}{1+|\xi|^m} \tag{8.2.6}$$

belongs to the class $S^{-m}_{\rho,0}$ and provides an operator $a_{\rho,m}(D)$ unbounded on L^p.

(c) Let $a(x,\xi)$ belong to $S^{-\frac{n}{2}(1-\rho)}_{\rho,\delta}$ so that the critical L^p space is L^1. Although $a(x,\xi)$ is unbounded on L^1, it is bounded on the Hardy space H^1.

The part (b) of the previous theorem is exactly the counterexample of Wainger [31], and the part (a) must be proved in the critical case when the inequality is not strict. Feffermann observes that (a) may be obtained from (c) by a not trivial interpolation. Moreover, the proof of (a) and (c) requires a technique of Feffermann and Stein [8], based on the class of functions with bounded mean oscillation (BMO).

A different point of view was introduced by Taylor, who realized that the proof of the L^p continuity of pseudodifferential operators in $\mathrm{Op}S^0_{1,0}$ may be adapted to show the L^p boundedness of some suitable subclasses of the zero-order operators in $\mathrm{Op}\, S^0_{\rho,0}$, $\rho \in]0,1[$. This is obtained by replacing the Mikhlin–Hörmander lemma about Fourier multipliers with the following analogous result due to Marcinkiewicz and Lizorkin (see [23, 28]).

Lemma 8.1 *Set $\mathbb{K} := \{0,1\}^n$. Let the function $a(\xi)$ be continuous on $\mathbb{R}^n$ together with all its derivatives of the form $\partial^\gamma_\xi a(\xi)$, $\gamma \in \mathbb{K}$. If a constant $B > 0$ exists such that*

$$|\xi^\gamma \partial^\gamma_\xi a(\xi)| \le B, \qquad \xi \in \mathbb{R}^n, \quad \gamma \in \mathbb{K}, \tag{8.2.7}$$

then for every $p \in]1,+\infty[$, we can find a constant $A_p > 0$, depending only on p, B and the dimension n, such that

$$\|a(D)u\|_p \le A_p \|u\|_p, \qquad u \in \mathscr{S}(\mathbb{R}^n).$$

According to Taylor [29], we give the following

Definition 8.1 For $m \in \mathbb{R}$ and $\rho \in]0,1]$, M_ρ^m is the class of all functions $a \in C^\infty(\mathbb{R}^{2n})$ such that for every $\gamma \in \mathbb{K}$

$$\xi^\gamma \partial_\xi^\gamma a(x,\xi) \in S^m_{\rho,0}. \tag{8.2.8}$$

Notice that the requirement (8.2.8) is equivalent to assuming that for any multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ and some $C_{\alpha,\beta} > 0$,

$$|\xi^\gamma \partial_\xi^{\gamma+\alpha} \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|}, \quad (x,\xi) \in \mathbb{R}^{2n},\ \gamma \in \mathbb{K} \tag{8.2.9}$$

(cf. [12], Proposition 3.1). The usual symbolic calculus of pseudodifferential operators holds true in Op M_ρ^m. For zero-order operators, the following result holds (cf. [29]).

Theorem 8.4 *Let $a(x,\xi)$ belong to M_ρ^0. Then for every $p \in]1,+\infty[$ the operator $a(x,D)$ is L^p bounded.*

The L^p continuity result stated by Theorem 8.4 can be easily extended to pseudodifferential operators in Op M_ρ^m with arbitrary order $m \in \mathbb{R}$. After introducing the Sobolev (Bessel potential) spaces $H^{s,p} := \left\{u \in \mathscr{S}'(\mathbb{R}^n) :\ \langle D \rangle^s u \in L^p(\mathbb{R}^n)\right\}$, $p \in]1,+\infty[$ and $s \in \mathbb{R}$, where $\langle D \rangle^s u = \mathscr{F}^{-1}\left(\langle \cdot \rangle^s \widehat{u}\right)$, we obtain

Proposition 8.1 *Let $a(x,\xi)$ belong to M_ρ^m with arbitrary m. Then, for every real s, the following operator*

$$a(x,D) : H^{s+m,p} \to H^{s,p}$$

is linear and continuous, whenever p belongs to $]1,+\infty[$.

8.3 Quasi-Homogeneous Pseudodifferential Operators

In order to provide regularity results to partial differential equations of generalized elliptic type, as for example the heat equation, in this section we are concerned with continuity of pseudodifferential operators of quasi-homogeneous type, as originally introduced by Lascar [22]; see also Segàla [26], Yamazaki [33], Garello [11]. All the results in the next two sections are proved in [14], or in the references given there.

Consider a vector $M = (m_1, \ldots, m_n)$ with positive integer components such that $\min\limits_{1\le j\le n} m_j = 1$. The *quasi-homogeneous weight* in $\mathbb{R}^n$ is defined by

$$\langle \xi \rangle_M^2 = 1 + |\xi|_M^2, \quad \text{where} \quad |\xi|_M := \left(\sum_{j=1}^n \xi_j^{2m_j}\right)^{\frac{1}{2}}, \quad \xi \in \mathbb{R}^n. \tag{8.3.1}$$

We set $1/M := (1/m_1, \ldots, 1/m_n)$, $\alpha \cdot 1/M := \sum_{j=1}^{n} \alpha_j/m_j$, $m^* := \max_{1 \le j \le n} m_j$. The usual Euclidean norm $|\xi|$ corresponds to the case $M = (1, \ldots, 1)$.

Proposition 8.2 *For any vector $M = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ satisfying the previous assumptions, there exists a suitable positive constant C such that*

(i) $\frac{1}{C}(1 + |\xi|) \le \langle\xi\rangle_M \le C(1 + |\xi|)^{m^*}, \quad \forall \xi \in \mathbb{R}^n$ *;*
(ii) $|\xi + \eta|_M \le C(|\xi|_M + |\eta|_M), \quad \forall \xi, \eta \in \mathbb{R}^n$ *;*
(iii) (quasi-homogeneity) $|t^{1/M}\xi|_M = t|\xi|_M, \quad \forall t > 0, \forall \xi \in \mathbb{R}^n$*,*
where $t^{1/M}\xi = (t^{1/m_1}\xi_1, \ldots, t^{1/m_n}\xi_n)$*;*
(iv) $\xi^\gamma \partial^{\alpha+\gamma}|\xi|_M \le C_{\alpha,\gamma}\langle\xi\rangle_M^{1-\alpha\cdot 1/M}, \ \forall \alpha, \gamma \in \mathbb{Z}_+^n, \forall \xi \ne 0.$

Definition 8.2 Given $m \in \mathbb{R}$ and $\delta \in [0, 1]$, $S^m_{M,\delta}$ will be the class of functions $a(x, \xi) \in C^\infty(\mathbb{R}^{2n})$ such that for all multi-indices $\alpha, \beta \in \mathbb{Z}_+^n$ there exists $C_{\alpha,\beta} > 0$ such that:

$$|\partial_x^\beta \partial_\xi^\alpha a(x, \xi)| \le C_{\alpha,\beta}\langle\xi\rangle_M^{m-\alpha\cdot 1/M+\delta\beta\cdot 1/M}, \quad (x, \xi) \in \mathbb{R}^{2n}. \tag{8.3.2}$$

The estimates in Proposition 8.2.*i* yield the inclusion

$$S^m_{M,\delta} \subset S^{\max\{mm^*,m\}}_{1/m^*,\delta m^*}, \tag{8.3.3}$$

which establishes a suitable relation between the *quasi-homogeneous classes* $S^m_{M,\delta}$ and the Hörmander symbol classes $S^m_{\rho,\delta}$, where $0 \le \rho < 1$, excluding the trivial case $M = (1, \ldots, 1)$.

For the adjoint and the product of pseudodifferential operators in $\mathrm{Op}\, S^m_{M,\delta}$, a suitable symbolic calculus is developed in [13] with some restrictions on δ; we quote here the result.

Proposition 8.3 (Symbolic calculus)

1. *If $a(x, D) \in \mathrm{Op}S^m_{M,\delta}$ with $0 \le \delta < 1/m^*$, the adjoint operator $a(x, D)^*$ still belongs to $\mathrm{Op}S^m_{M,\delta}$ and its symbol $a^*(x, \xi)$ satisfies for any integer $k > 0$*

$$a^*(x, \xi) - \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha \partial_x^\alpha \overline{a(x, \xi)} \in S^{m-(1/m^*-\delta)k}_{M,\delta};$$

we write: $a^* \sim \sum_{\alpha \ge 0} \frac{(-i)^{|\alpha|}}{\alpha!}\partial_\xi^\alpha \partial_x^\alpha \overline{a}$.

2. *If $a(x, D) \in \mathrm{Op}S^{m_1}_{M,\delta_1}$, $b(x, D) \in \mathrm{Op}S^{m_2}_{M,\delta_2}$ with $0 \le \delta_2 < 1/m^*$, then*

$$a(x, D)b(x, D) \in \mathrm{Op}S^{m_1+m_2}_{M,\delta},$$

with $\delta := \max\{\delta_1, \delta_2\}$, and the symbol $a\sharp b$ of the product satisfies for any integer $k > 0$

$$a\sharp b(x,\xi) - \sum_{|\alpha|<k} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi) \in S_{M,\delta}^{m_1+m_2-(1/m^*-\delta_2)k}; \tag{8.3.4}$$

we write: $a\sharp b \sim \sum_{\alpha\geq 0} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b.$

We observe that symbols in the class $S^0_{M,\delta}$ plainly satisfy the Lizorkin–Marcinkiewicz condition (8.2.7), which allows to develop the L^p-theory of the pseudodiffererential operators in Op $S^m_{M,\delta}$ for $1 < p < \infty$.

The suitable functional setting is provided by a quasi-homogeneous counterpart of the Sobolev spaces, namely for $s \in \mathbb{R}$ and $p \in]1, +\infty[$, we say that a distribution $u \in \mathscr{S}'(\mathbb{R}^n)$ belongs to the *quasi-homogeneous Sobolev space* $H^{s,p}_M$, if

$$\langle D\rangle^s_M u := \mathscr{F}^{-1}(\langle\cdot\rangle^s_M \hat{u}) \in L^p(\mathbb{R}^n)\,. \tag{8.3.5}$$

$H^{s,p}_M$ becomes a Banach space, when provided with the norm $||u||_{H^{s,p}_M} := ||\langle D\rangle^s_M u||_{L^p}$.

The case $p = 1$ is not considered since, in order to prove the continuity result in Theorem 8.5, $H^s_{M,p}$ must be stated in terms of dyadic decompositions, which characterize $H^{s,p}_M$ only for $1 < p < \infty$; see [13, 30, Sect. 2.3.5].
The following continuous embeddings can be easily established

$$\mathscr{S}(\mathbb{R}^n) \subset H^{s,p}_M \subset H^{r,p}_M\,, \quad r < s \quad \text{and} \quad p \in]1, +\infty[. \tag{8.3.6}$$

Theorem 8.5 *If* $a(x,\xi) \in S^m_{M,\delta}$, $m \in \mathbb{R}$, $\delta \in [0, 1[$ *then*

$$a(x, D) : H^{s+m,p}_M \to H^{s,p}_M \quad \textit{for any} \quad s \in \mathbb{R} \tag{8.3.7}$$

is a linear continuous operator.

If $\delta = 1$*, then the mapping property* (8.3.7) *is still true for* $s > 0$.

Recall that pseudodifferential operators with symbol $a \in S^{-\infty} := \bigcap_{m\in\mathbb{R}} S^m_{1,0}$ are *regularizing* operators; namely they can be extended to linear bounded operators from $\mathscr{S}'(\mathbb{R}^n)$ into the space of *polynomially bounded* C^∞ functions in $\mathbb{R}^n$, with polynomially bounded derivatives, and from the space $\mathscr{E}'(\mathbb{R}^n)$ of the compactly supported distributions in $\mathbb{R}^n$ into the Schwartz space $\mathscr{S}(\mathbb{R}^n)$.

8.4 Microlocal Sobolev Regularity

In this section, we introduce a quasi-homogeneous Sobolev version of Hörmander wave front set; see [21, I].
Let M be a vector as defined at the beginning of Sect. 8.3, we say that a set $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ is $M-$conic, if

$$\xi \in \Gamma_M \quad \Rightarrow \quad t^{1/M}\xi \in \Gamma_M \;,\; \forall\, t > 0\,.$$

Moreover, $a \in \mathscr{S}'(\mathbb{R}^{2n})$ is *microlocally regularizing* in $(x_0, \xi^0) \in T^\circ\mathbb{R}^n := \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ if there exist U open neighborhood of x_0 and Γ_M open $M-$conic neighborhood of ξ^0 such that $a_{|\, U\times\Gamma_M} \in C^\infty(U \times \Gamma_M)$ and for every $m > 0$ and all $\alpha, \beta \in \mathbb{Z}^n_+$ a positive constant $C_{m,\alpha,\beta} > 0$ is found in such a way that:

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \le C_{m,\alpha,\beta}(1 + |\xi|)^{-m}\,, \quad (x, \xi) \in U \times \Gamma_M\,. \tag{8.4.1}$$

We write $a(x, \xi) \in mcl\, S^{-\infty}(U \times \Gamma_M)$.

Definition 8.3 We say that a symbol $a \in S^m_{M,\delta}$ is *microlocally $M-$elliptic* at $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$ if there exist an open neighborhood U of x_0 and an $M-$conic open neighborhood Γ_M of ξ^0 such that for $c_0 > 0$, $\rho_0 > 0$:

$$|a(x, \xi)| \ge c_0\langle\xi\rangle_M^m\,, \quad (x, \xi) \in U \times \Gamma_M\,, \quad |\xi|_M > \rho_0\,. \tag{8.4.2}$$

Moreover the *M-characteristic set* of $a \in S^m_{M,\delta}$ (or $a(x, D)$) is $\mathrm{Char_M}(a) \subset T^\circ\mathbb{R}^n$ defined by

$$(x_0, \xi^0) \in T^\circ\mathbb{R}^n \setminus \mathrm{Char_M}(a) \;\; \Leftrightarrow \;\; a \text{ is microlocally M-elliptic at } (x_0, \xi^0)\,. \tag{8.4.3}$$

Proposition 8.4 Microlocal parametrix. *Assume that* $0 \le \delta < 1/m^*$. *Then* $a \in S^m_{M,\delta}$ *is microlocally $M-$elliptic at* $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$ *if and only if there exist symbols* $b, c \in S^{-m}_{M,\delta}$ *such that*

$$a \sharp b - 1 \quad \textit{and} \quad c \sharp a - 1 \tag{8.4.4}$$

are microlocally regularizing at (x_0, ξ^0).

For $s \in \mathbb{R}$, $1 < p < \infty$, we define the $H^{s,p}_M$- *wave front set* of $u \in \mathscr{S}'(\mathbb{R}^n)$, denoted $WF_{H^{s,p}_M}(u)$, as follows

Definition 8.4 Consider $(x_0, \xi^0) \in T^\circ\mathbb{R}^n$ then

$$(x_0, \xi^0) \notin WF_{H^{s,p}_M}(u) \tag{8.4.5}$$

if there exist $\phi \in C_0^\infty(\mathbb{R}^n)$ identically one in a neighborhood of x_0, $\psi(\xi) \in S^0_M$ identically one on $\Gamma_M \cap \{|\xi|_M > \varepsilon_0\}$, for $\Gamma_M \subset \mathbb{R}^n \setminus \{0\}$ $M-$conic neighborhood of ξ^0 and $0 < \varepsilon_0 < |\xi^0|_M$, satisfying

$$\psi(D)(\phi u) \in H^{s,p}_M\,. \tag{8.4.6}$$

If (8.4.6) is satified, we also say that $u \in \mathscr{S}'(\mathbb{R}^n)$ is *microlocally in* $H^{s,p}_M$ at (x_0, ξ^0). We say that $x_0 \notin H^{s,p}_M - \text{singsupp}\,(u)$ if and only if there exists a function $\phi \in C_0^\infty(\mathbb{R}^n)$, $\phi \equiv 1$ in some open neighborhood of x_0, such that $\phi u \in H^{s,p}_M$.

$H^{s,p}_M$ − singsupp (u) and $WF_{H^{s,p}_M}(u)$ are closed subsets, respectively, of $\mathbb{R}^n$ and $T^\circ\mathbb{R}^n$; moreover, $WF_{H^{s,p}_M}(u)$ is $M - conic$ in the ξ variable.

Proposition 8.5 *For every $u \in \mathscr{S}'(\mathbb{R}^n)$ and $s \in \mathbb{R}$, we have:*

$$H^{s,p}_M - \text{singsupp}(u) = \pi_1(WF_{H^{s,p}_M}(u)),$$

where π_1 is the canonical projection of $T^\circ\mathbb{R}^n$ onto $\mathbb{R}^n$.

We then state the main result of microlocal Sobolev continuity and regularity for pseudodifferential operators in $\mathrm{Op}S^m_{M,\delta}$ and give few examples.

Theorem 8.6 *Consider $m \in \mathbb{R}$, $\delta \in [0, 1/m^*[$ and $a \in S^m_{M,\delta}$, then for every $s \in \mathbb{R}$, $1 < p < \infty$ and $u \in \mathscr{S}'(\mathbb{R}^n)$*

$$WF_{H^{s,p}_M}(a(x,D)u) \subset WF_{H^{s+m,p}_M}(u) \subset WF_{H^{s,p}_M}(a(x,D)u) \cup \mathrm{Char_M}(a). \quad (8.4.7)$$

Examples

4.1 Consider $M = (1, 2)$. Both the heat operator: $H(\partial) = \partial_{x_1} - \partial^2_{x_2}$ and the Schrödinger operator: $S(\partial) = i\partial_{x_1} - \partial^2_{x_2}$ may be considered as operators in $\mathrm{Op}S^1_M$. The first one is clearly M-elliptic, while the M-characteristic set of the second one is the set $\{(x_1, x_2, \eta^2, \eta),\ x_1, x_2 \in \mathbb{R},\ \eta \in \mathbb{R} \setminus \{0\}\}$.

4.2 Consider the operators

$$Q(x,\partial) = x_1\partial_{x_1} + i\partial_{x_1} - \partial^2_{x_2} \text{ with symbol } q(x,\xi) = ix_1\xi_1 - \xi_1 + \xi_2^2; \quad (8.4.8)$$

$$P(x,\partial) = x_1\partial_{x_1} - \partial^2_{x_2} \text{ with symbol } p(x,\xi) = ix_1\xi_1 + \xi_2^2. \quad (8.4.9)$$

Both of them can be considered as operators in $\mathrm{Op}S^1_M$. We have

$$\mathrm{Char}_M(q) = \left\{(0, x_2, \xi_1, \xi_2);\ \ x_2 \in \mathbb{R},\ \xi_1 = \xi_2^2,\ \xi_2 \neq 0\right\}.$$

About the operator $P(x,\partial)$, we can notice that for $x^0 = (0, x^0_2)$, with an arbitrary $x^0_2 \in \mathbb{R}$, $p(x^0, \xi) = 0$ if and only if $\xi_2 = 0$; thus, the M-characteristic set $\mathrm{Char}_M(p) = \{(0, x_2, \xi_1, 0);\ \ \xi_1 \neq 0\}$ coincides with the classical (conic) characteristic set $\mathrm{Char}(p)$.

8.5 m-Pseudodifferential Operators on L^p

In the following, the Sobolev microlocal regularity results, previously obtained for quasi-homogeneous pseudodifferential operators, are extended to a more general framework of pseudodifferential operators, where the decay of symbols and their derivatives are estimated by general non-homogeneous weights.

Notations. For $\chi(\xi)$, $\kappa(\xi)$ positive continuous functions of $\xi \in \mathbb{R}^n$ and C, c positive constants, we set:

- $\chi(\xi) \asymp \kappa(\xi)$, if $c \leq \frac{\chi(\xi)}{\kappa(\xi)} \leq C$, for any $\xi \in \mathbb{R}^n$;
- $\chi(\xi) \approx \chi(\eta)$ in a domain D if $c \leq \frac{\chi(\eta)}{\chi(\xi)} \leq C$, for any $\xi, \eta \in D$.

Definition 8.5 A vector-valued function $\Lambda(\xi) = (\lambda_1(\xi), \dots, \lambda_n(\xi))$, $\xi \in \mathbb{R}^n$, with positive continuous components, is a *weight vector* if there exist positive constants C, c such that for any $j = 1, \dots, n$:

$$c\langle\xi\rangle^c \leq \lambda_j(\xi) \leq C\langle\xi\rangle^C \text{ (polynomial growth);} \tag{8.5.1}$$

$$\lambda_j(\xi) \geq c|\xi_j| \text{ (M-condition);} \tag{8.5.2}$$

$$\lambda_j(\eta) \approx \lambda_j(\xi) \text{ when } \sum_{k=1}^{n} |\xi_k - \eta_k|\lambda_k(\eta)^{-1} \leq c \text{ (slowly varying condition)} \tag{8.5.3}$$

A positive real continuous function $m(\xi)$ is an *admissible weight*, associated with $\Lambda(\xi)$, if for some positive constants N, C, c

$$m(\eta) \leq C\, m(\xi)\, (1 + |\eta - \xi|)^N \text{ (temperance);} \tag{8.5.4}$$

$$m(\eta) \approx m(\xi) \quad \text{when} \quad \sum_{k=1}^{n} |\xi_k - \eta_k|\lambda_k(\eta)^{-1} \leq c. \tag{8.5.5}$$

From (8.5.4), it follows that $c\langle\xi\rangle^{-N} \leq m(\xi) \leq C\langle\xi\rangle^N$.

It is trivial that any positive constant function on $\mathbb{R}^n$ is an admissible weight associated with any weight vector $\Lambda(\xi)$.

Consider $\tilde{\Lambda}(\xi) \asymp \Lambda(\xi)$, that is, $\lambda_j(\xi) \asymp \tilde{\lambda}_j(\xi)$, $j = 1, \dots, n$; then $\tilde{\Lambda}(\xi)$ is again a weight vector. Similarly, $\tilde{m}(\xi) \asymp m(\xi)$ is an admissible weight.

Examples

5.1. For $\langle\xi\rangle_M$ defined in (8.3.1), $\Lambda_M(\xi) = \left(\langle\xi\rangle_M^{1/m_1}, \dots, \langle\xi\rangle_M^{1/m_n}\right)$ is a weight vector. For $s \in \mathbb{R}$, $\langle\xi\rangle_M^s$ is an admissible weight.

5.2. Consider a continuous function $\lambda(\xi)$ satisfying (8.5.1) and the *strong* slowly varying condition:

$$\lambda(\eta) \approx \lambda(\xi) \text{ if for some } c, \mu > 0 \ \sum_{j=1}^{n} |\eta_j - \xi_j| \left(\lambda(\eta)^{\frac{1}{\mu}} + |\eta_j|\right)^{-1} \leq c. \tag{8.5.6}$$

Then the vector $\Lambda(\xi) := \left(\lambda(\xi)^{\frac{1}{\mu}} + |\xi_1|, \dots, \lambda(\xi)^{\frac{1}{\mu}} + |\xi_n|\right)$ is a weight vector; see [15, Proposition 1] for the proof. For $s \in \mathbb{R}$, $\lambda(\xi)^s$ is an admissible weight. In such frame, emphasis is given to the *multi-quasi-homogeneous* weights $\lambda_{\mathscr{P}}(\xi) = \left(\sum_{\alpha \in V(\mathscr{P})} \xi^{2\alpha}\right)^{1/2}$, where $V(\mathscr{P})$ is the set of the vertices of a *complete Newton polyhedron* $\mathscr{P}$ as introduced in [17]; see also [4]; in this case, the value μ in (8.5.6) is called *formal order* of $\mathscr{P}$.

For the sake of generality, we introduce the symbol classes in local form. All the results obtained for quasi-homogeneous pseudodifferential operators in the preceding Sects. 8.3 and 8.4 may be translated into such local point of view.

For Ω open subset of $\mathbb{R}^n$, we set $K \subset\subset \Omega$, when K is a compact subset of Ω.

Definition 8.6 For $\Lambda(\xi)$ weight vector and $m(\xi)$ admissible weight, the symbol class $S_{m,\Lambda}(\Omega)$ is given by all the functions $a(x,\xi) \in C^\infty(\Omega \times \mathbb{R}^n)$, such that for any $K \subset\subset \Omega$, $\alpha, \beta \in \mathbb{Z}_+^n$ and suitable $c_{\alpha,\beta,K} > 0$:

$$\sup_{x\in K} |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le c_{\alpha,\beta,K}\, m(\xi)\Lambda(\xi)^{-\alpha}, \quad \xi \in \mathbb{R}^n \tag{8.5.7}$$

where, with standard vectorial notation, $\Lambda(\xi)^\gamma = \prod_{k=1}^n \lambda_k(\xi)^{\gamma_k}$.

$S_{m,\Lambda}(\Omega)$ turns out to be a Fréchet space, with respect to the family of natural semi-norms defined as the best constants $c_{\alpha,\beta,K}$ involved in the estimates (8.5.7).

Henceforth, $\Lambda(\xi)$ will always be a weight vector and all the admissible weights $m(\xi)$ will be referred to it.

Remark 8.1 1. Considering the constants C, c in (8.5.1) and N in (8.5.4), the following relation with the usual Hörmander [21] symbol classes $S^m_{\rho,\delta}(\Omega)$, $0 \le \delta < \rho \le 1$, is trivial:

$$S_{m,\Lambda}(\Omega) \subset S^N_{c,0}(\Omega)\,. \tag{8.5.8}$$

Then for any weight vector $\Lambda(\xi)$, $m(\xi)$ admissible weight and $a(x,\xi) \in S_{m,\Lambda}(\Omega)$, the $m-$pseudodifferential operator $a(x,D)$ is defined by (8.1.1).

The operator $a(x,D)$ maps continuously $C_0^\infty(\Omega)$ to $C^\infty(\Omega)$ and it extends to a bounded linear operator from $\mathscr{E}'(\Omega)$ to $\mathscr{D}'(\Omega)$.

2. If m_1, m_2 are admissible weights such that $m_1 \le Cm_2$, then $S_{m_1,\Lambda}(\Omega) \subset S_{m_2,\Lambda}(\Omega)$, with continuous embedding. In particular $S_{m_1,\Lambda}(\Omega) = S_{m_2,\Lambda}(\Omega)$, as long as $m_1 \asymp m_2$.

When the admissible weight m is an arbitrary positive constant function, $S_{m,\Lambda}(\Omega)$ will be just denoted by $S_\Lambda(\Omega)$ and $a(x,\xi) \in S_\Lambda(\Omega)$ will be called a *zero-order symbol*.

The proofs of all the results in the following are given in [16], and the references given there.

Proposition 8.6 *For $\Lambda(\xi) = (\lambda_1(\xi), \ldots, \lambda_n(\xi))$ weight vector, the following properties are satisfied:*

(i) the function

$$\pi(\xi) = \min_{1\le j\le n} \lambda_j(\xi), \quad \xi \in \mathbb{R}^n \tag{8.5.9}$$

is an admissible weight associated with $\Lambda(\xi)$ and it moreover satisfies (8.5.6), with $\mu = 1$;

(ii) *If m, m' are admissible weights associated with the weight vector $\Lambda(\xi)$, then the same property is fulfilled by mm' and $1/m$.*

For any $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ and any sequence $a_k(x,\xi) \in S_{m\pi^{-k},\Lambda}(\Omega)$, $k \in \mathbb{Z}_+$, we write

$$a(x,\xi) \sim \sum_{k=0}^{\infty} a_k(x,\xi), \tag{8.5.10}$$

if for every integer $N \geq 1$:

$$a(x,\xi) - \sum_{k<N} a_k(x,\xi) \in S_{m\pi^{-N},\Lambda}(\Omega)\,. \tag{8.5.11}$$

By $\widetilde{\mathrm{Op}}S_{m,\Lambda}(\Omega)$, we denote the class of *properly supported* $m-$pseudodifferential operators which map $C_0^\infty(\Omega)$ to $C_0^\infty(\Omega)$, $C^\infty(\Omega)$ to $C^\infty(\Omega)$ and extend to bounded linear operators on $\mathscr{E}'(\Omega)$ and $\mathscr{D}'(\Omega)$.

Proposition 8.7 (Symbolic calculus) *Consider $a(x,D) \in Op\,S_{m,\Lambda}(\Omega)$ and $b(x,D) \in \widetilde{Op}\,S_{m',\Lambda}(\Omega)$, where $m(\xi)$, $m'(\xi)$ are admissible weights, both associated with the same weight vector $\Lambda(\xi)$. Then, we have:*

(i) *$a(x,D)^* \in Op\,S_{m,\Lambda}(\Omega)$ and $a(x,D)^* = a^*(x,D)$, where $a^*(x,\xi) \in S_{m,\Lambda}(\Omega)$ satisfies the following aymptotic expansion:*

$$a^*(x,\xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \partial_x^\alpha \overline{a(x,\xi)}\,. \tag{8.5.12}$$

Moreover $a(x,D)^* \in \widetilde{Op}\,S_{m,\Lambda}(\Omega)$ if $a(x,D)$ is assumed properly supported.

(ii) *$b(x,D)a(x,D) \in Op\,S_{mm',\Lambda}(\Omega)$ and its symbol $b\sharp a$ satisfies*

$$b\sharp a(x.\xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha b(x,\xi) \partial_x^\alpha a(x,\xi). \tag{8.5.13}$$

Considering $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ and using (8.5.7), (8.5.4), (8.5.1), (8.5.2), it immediately follows that for any $\alpha, \gamma \in \mathbb{Z}_+^n$, $K \subset\subset \Omega$,

$$\sup_{x\in K} |\xi^\gamma \partial_\xi^{\alpha+\gamma} a(x,\xi)| \leq M_{\alpha,\gamma,K} \langle\xi\rangle^{N-c|\alpha|}\,, \quad \xi \in \mathbb{R}^n\,,$$

with some positive constants $M_{\alpha,\gamma,K}$, N, c. Then $S_{m,\Lambda}(\Omega) \subset M_c^N(\Omega)$. Here $M_\rho^r(\Omega)$, $0 < \rho \leq 1$, are the local version of the symbol classes defined in Taylor [29]; see also Definition 8.1. Then, the next result immediately follows from Theorem 8.4.

Theorem 8.7 *If $a(x,\xi) \in S_\Lambda(\Omega)$, then for any $1 < p < \infty$:*

$$a(x,D) : L^p_{\mathrm{comp}}(\Omega) \mapsto L^p_{\mathrm{loc}}(\Omega) \quad \textit{continuously.}$$

If $a(x, D)$ is assumed to be properly supported, then it is bounded both as operator on $L^p_{\mathrm{comp}}(\Omega)$ and on $L^p_{\mathrm{loc}}(\Omega)$.

It may be proved by means of technical arguments that for any admissible weight $m(\xi)$, there exists a smooth equivalent weight $\tilde{m}(\xi)$ whose derivatives satisfy the estimates

$$|\partial^\alpha \tilde{m}(\xi)| \le c_\alpha m(\xi)\Lambda(\xi)^{-\alpha}, \quad \text{for some} \quad c_\alpha > 0. \tag{8.5.14}$$

Identifying now $m(\xi)$ and $\tilde{m}(\xi)$, we can define for $1 < p < \infty$ the weighted Sobolev space:

$$H^p_m := \left\{u \in \mathscr{S}'(\mathbb{R}^n), \text{ such that } m(D)u \in L^p(\mathbb{R}^n)\right\}. \tag{8.5.15}$$

H^p_m is a Banach space when equipped with the norm $\|u\|_{p,m} := \|m(D)u\|_{L^p}$ (Hilbert space when $p = 2$).

For any open subset $\Omega \subset \mathbb{R}^n$, the following local spaces may be introduced:

$$H^p_{m,\mathrm{loc}}(\Omega) = \left\{u \in \mathscr{D}'(\Omega) \text{ such that, for any } \varphi \in C_0^\infty(\Omega),\ \varphi u \in H^p_m\right\}; \tag{8.5.16}$$

$$H^p_{m,\mathrm{comp}}(\Omega) = \bigcup_{K \subset\subset \Omega} H^p_m(K), \tag{8.5.17}$$

where $H^p_m(K)$ is the closed subspace of H^p_m, consisting of the distributions supported in the compact set K. The notations in Theorem 8.7 are now completely clarified.

$H^p_{m,\mathrm{loc}}(\Omega)$ equipped with the semi-norms $p_\psi(\cdot) := \|\psi \cdot\|_{m,p} = \|m(D)\psi \cdot\|_{L^p}$, $\psi \in C_0^\infty(\Omega)$ arbitrary is a Fréchet space. $H^p_{m,\mathrm{comp}}(\Omega)$ is provided with the inductive limit topology of the spaces $H^p_m(K)$, for K ranging on the collection of all compact subsets of Ω.

Proposition 8.8 *Assume m, m' admissible weights, $a(x, \xi) \in S_{m',\Lambda}(\Omega)$ and $p \in]1, \infty[$. Then, $a(x, D)$ extends to a bounded linear operator:*

$$a(x, D) : H^p_{m,\mathrm{comp}}(\Omega) \mapsto H^p_{m/m',\mathrm{loc}}(\Omega). \tag{8.5.18}$$

If moreover $a(x, D)$ is a properly supported operator, then the following maps are continuous:

$$a(x, D) : H^p_{m,\mathrm{comp}}(\Omega) \mapsto H^p_{m/m',\mathrm{comp}}(\Omega); \tag{8.5.19}$$

$$a(x, D) : H^p_{m,\mathrm{loc}}(\Omega) \mapsto H^p_{m/m',\mathrm{loc}}(\Omega). \tag{8.5.20}$$

The complete lack of any homogeneity property of $\Lambda(\xi)$ and $m(\xi)$ prevents us from defining the characteristic set of $a(x, D)$ in terms of conic or $M-$conic neighborhoods in $\mathbb{R}^n_\xi$. We try then an alternative way to solve the problem by the introduction of the following tools.

Following Rodino [25] and Garello [9], the Λ-neighborhood of a set $X \subset \mathbb{R}^n$ with length $\varepsilon > 0$, is defined as the open set:

$$X_{\varepsilon\Lambda} := \bigcup_{\xi^0\in X}\left\{|\xi_j-\xi_j^0|<\varepsilon\lambda_j(\xi^0), \quad \text{for} \quad j=1,\dots,n\right\}. \tag{8.5.21}$$

Moreover for $x_0 \in \Omega$, we set:

$$X(x_0) := \{x_0\}\times X \quad , \quad X_{\varepsilon\Lambda}(x_0) := B_\varepsilon(x_0)\times X_{\varepsilon\Lambda}, \tag{8.5.22}$$

where $B_\varepsilon(x_0)$ is the open ball in Ω centered at x_0 with radius ε. For every $\varepsilon > 0$, a suitable $0 < \varepsilon^* < \varepsilon$, depending only on ε and Λ, can be found in such a way that for every $X \subset \mathbb{R}^n$:

$$(X_{\varepsilon^*\Lambda})_{\varepsilon^*\Lambda} \subset X_{\varepsilon\Lambda}; \tag{8.5.23}$$
$$\left(\mathbb{R}^n\setminus X_{\varepsilon\Lambda}\right)_{\varepsilon^*\Lambda} \subset \mathbb{R}^n\setminus X_{\varepsilon^*\Lambda}. \tag{8.5.24}$$

Definition 8.7 Consider $x_0 \in \Omega$, $X \subset \mathbb{R}^n_\xi$, m admissible weight. We say that $a(x,\xi) \in S_{m,\Lambda}(\Omega)$ is *m−microlocally elliptic* in X at the point x_0 if, for some positive c_0, R_0,

$$|a(x_0,\xi)| \ge c_0\, m(\xi), \quad \text{when} \quad \xi\in X, \quad |\xi|>R_0\,. \tag{8.5.25}$$

We write $a(x,\xi) \in \text{mce}_{m,\Lambda}X(x_0)$.

We denote with $\text{mce}_\Lambda X(x_0)$ the class of *zero-order* microlocal symbols in X at the point x_0.

Definition 8.8 For $X \subset \mathbb{R}^n$, $x_0 \in \Omega$, and $p \in]1,\infty[$, we say that $u \in \mathscr{D}'(\Omega)$ is microlocally H^p_m−regular in X at the point $x_0 \in \Omega$, and write $u \in \text{mcl}H^p_m X(x_0)$, if there exists a properly supported operator $a(x,D) \in \text{Op}\,\text{mce}_\Lambda X(x_0)$, such that $a(x,D)u \in H^p_{m,\text{loc}}(\Omega)$.

We say that a set family $\Xi_\Lambda \subset \mathbb{R}^n$ is a *Λ-filter* if it is closed with respect to the intersection of any finite number of its elements and moreover:

$$X\in\Xi_\Lambda \text{ and } X\subset Y, \text{ then } Y\in\Xi_\Lambda; \tag{8.5.26}$$
$$\text{for any } X\in\Xi_\Lambda, \text{ there exists } Y\in\Xi_\Lambda \text{ and } \varepsilon>0 \text{ such that } Y_{\varepsilon\Lambda}\subset X. \tag{8.5.27}$$

Proposition 8.9 *For $u \in \mathscr{D}'(\Omega)$ and $a(x,\xi) \in S_{m,\Lambda}(\Omega)$, the following families of subsets of $\mathbb{R}^n$:*

$$\mathscr{W}^p_{m,x_0}u := \left\{X\subset\mathbb{R}^n\,;\; u\in mclH^p_m(\mathbb{R}^n\setminus X)(x_0)\right\}, \quad 1<p<\infty; \tag{8.5.28}$$
$$\Sigma_{m,x_0}a := \left\{X\subset\mathbb{R}^n\,,\; a(x,\xi)\in mce_{m,\Lambda}(\mathbb{R}^n\setminus X)(x_0)\right\}, \tag{8.5.29}$$

are both Λ-filters.

We refer to the Λ-filters in the previous proposition, respectively, as *filter of Sobolev singularities of $u \in \mathscr{D}'(\Omega)$* and *characteristic filter of $a(x,\xi) \in S_{m,\Lambda}(\Omega)$*.

Concerning the filter of Sobolev singularities of a distribution, the next result holds.

Proposition 8.10 *The following conditions are equivalent:*

a. $\emptyset \in \mathscr{W}^p_{m,x_0}u$;
b. *There exists* $\phi \in C_0^\infty(\Omega)$, *with* $\phi(x_0) \neq 0$, *such that* $\phi u \in H^p_m$;
c. *There exist* $X_1, \ldots, X_H \subset \mathbb{R}^n$, *with* $\bigcup_{h=1}^{H} X_h = \mathbb{R}^n$, *such that* $u \in mclH^p_m X_h(x_0)$ *for* $h = 1, \ldots, H$.

We are now able to generalize the result of microlocal regularity stated in Theorem 8.6, in terms of Λ - filters.

Theorem 8.8 *For* m, m' *arbitrary admissible weights, associated with the same weight vector* Λ, *consider* $a(x, D) \in \widetilde{Op}\, S_{m,\Lambda}(\Omega)$, $x_0 \in \Omega$, $p \in]1, \infty[$. *Then for any* $u \in \mathscr{D}'(\Omega)$, *we have:*

$$\mathscr{W}^p_{m'/m\,,\,x_0} a(x, D)u \cap \Sigma_{m,\,x_0} a \subset \mathscr{W}^p_{m'\,,\,x_0} u \subset \mathscr{W}^p_{m'/m\,,\,x_0} a(x, D)u. \tag{8.5.30}$$

The statement may be expressed in more explicit form by the following

Proposition 8.11 *Consider* $x_0 \in \Omega$, $X \subset \mathbb{R}^n$, $p \in]1, \infty[$, m *admissible weight,* $a(x, D) \in \widetilde{Op} S_{m,\Lambda}(\Omega)$, $u \in \mathscr{D}'(\Omega)$. *Then we have:*

(i) $u \in mclH^p_{m'} X(x_0) \Rightarrow a(x, D)u \in mclH^p_{m'/m} X(x_0)$;
(ii) $a(x, \xi) \in mce_{m,\Lambda}(X(x_0))$ *and* $a(x, D)u \in mclH^p_{m'/m} X(x_0) \Rightarrow u \in mclH^p_{m'} X(x_0)$.

Example

5.3. Let us define now the positive function in $\mathbb{R}^2$

$$\lambda(\xi) := (1 + \xi_1^6 + \xi_1^4\xi_2^4 + \xi_2^6)^{1/2}, \tag{8.5.31}$$

which may be considered as *multi-quasi-homogeneous* weight in Example 5.2; precisely, here the set of vertices of the complete Newton polyhedron is

$$V(\mathscr{P}) = \{(0, 0), (3, 0), (2, 2), (0, 3)\}\ .$$

The formal order of $\lambda(\xi)$ is $\mu = 6$. Then $\Lambda(\xi) = \left(\lambda(\xi)^{1/6} + |\xi_1|, \lambda(\xi)^{1/6} + |\xi_2|\right)$ is a weight vector and, for any $r \in \mathbb{R}$, $\lambda(\xi)^r$ is an admissible weight. Consider the linear partial differential operators in Op $S_{\lambda,\Lambda}(\mathbb{R}^2)$:

$$A(x, \partial) = (x_1\partial_{x_1} - \partial^2_{x_2})(x_2\partial_{x_2} - \partial^2_{x_1}), \tag{8.5.32}$$

$$B(x, \partial) = (x_1\partial_{x_1} + i\partial_{x_1} - \partial^2_{x_2})(x_2\partial_{x_2} + i\partial_{x_2} - \partial^2_{x_1}). \tag{8.5.33}$$

According to the rules of the symbolic calculus (see Proposition 8.7), their symbols $a(x,\xi)$ and $b(x,\xi)$ can be written in the form

$$a(x,\xi) = (ix_1\xi_1 + \xi_2^2)(ix_2\xi_2 + \xi_1^2) + 2\xi_2^2, \tag{8.5.34}$$

$$b(x,\xi) = (ix_1\xi_1 - \xi_1 + \xi_2^2)(ix_2\xi_2 - \xi_2 + \xi_1^2) + 2\xi_2^2. \tag{8.5.35}$$

The term $2\xi_2^2$ that appears in the right-hand side of both formulas (8.5.34), (8.5.35) behaves as a lower-order symbol with respect to the weight function (8.5.31). Observing that $\lambda(\xi) \asymp (1+\xi_1^2+\xi_2^4)^{\frac{1}{2}}(1+\xi_1^4+\xi_2^2)^{\frac{1}{2}}$, it rightly follows that $A(x,\partial)$ and $B(x,\partial)$ are λ-elliptic operators in $\mathrm{Op}S_{\lambda,\Lambda}(\Omega)$, where $\Omega := \mathbb{R}^2 \setminus \bigcup_{j=1,2}\{x_j = 0\}$.

As operators in $\mathrm{Op}S_{\lambda,\Lambda}(\mathbb{R}^2)$, $A(x,\partial)$ and $B(x,\partial)$ fail to be $\lambda-$elliptic at the points of the coordinate axes. The behavior of $B(x,\partial)$ along the coordinate axes can be summarized as follows.

i. At any point $x^0 = (0, x_2^0)$, $x_2^0 \neq 0$, the symbol of $B(x,\partial)$ reduces to $b(x^0,\xi) = (-\xi_1 + \xi_2^2)(ix_2^0\xi_2 - \xi_2 + \xi_1^2) + 2\xi_2^2$; hence, it is $\lambda-$microlocally elliptic in all sets X_k of the type

$$\left\{(\xi_1,\xi_2) \in \mathbb{R}^2;\, \xi_1 < (1-k)\xi_2^2 \quad \text{or} \quad \xi_1 > \frac{1}{1-k}\xi_2^2\right\}, \quad 0 < k < 1\,. \tag{8.5.36}$$

This means that a base of the characteristic filter $\Sigma_{\lambda,x^0}b$ is given by the family of sets:

$$\left\{\xi \in \mathbb{R}^2 \quad ; \quad (1-k)\xi_2^2 \le \xi_1 \le \frac{1}{1-k}\xi_2^2\right\}_{0<k<1}.$$

Arguing similarly, we obtain that:

ii. At any point $y^0 = (y_1^0, 0)$ (with an arbitrary $y_1^0 \neq 0$), the characteristic filter $\Sigma_{\lambda,y^0}b$ admits as base the family of sets:

$$\left\{\xi \in \mathbb{R}^2 \quad ; \quad (1-k)\xi_1^2 \le \xi_2 \le \frac{1}{1-k}\xi_1^2\right\}_{0<k<1};$$

iii. At the origin $\mathbf{0} = (0,0)$, the characteristic filter $\Sigma_{\lambda,\mathbf{0}}b$ admits as base the family of sets:

$$\left\{\xi \in \mathbb{R}^2 \quad ; \quad \begin{array}{c} (1-k)\xi_2^2 \le \xi_1 \le \frac{1}{1-k}\xi_2^2 \\ \text{or} \\ (1-k')\xi_1^2 \le \xi_2 \le \frac{1}{1-k'}\xi_1^2 \end{array}\right\}_{0<k,k'<1}.$$

Applying Theorem 8.8, we obtain that for any solution u of the equation $B(x, \partial)u = f \in H^p_{\lambda^s,\mathrm{loc}}(\mathbb{R}^2)$, the filter of Sobolev singularities $\mathscr{W}^p_{\lambda^{s+1},x^0}u$ at every point $x^0 = (x^0_1, x^0_2)$ belonging to the coordinate axes (that is such that $x^0_1 x^0_2 = 0$) contains the characteristic filter $\Sigma_{\lambda,x^0}b$.

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Chapter 9
Hyperbolic Wavelet Frames and Multiresolution in the Weighted Bergman Spaces

Margit Pap

Abstract In this paper, we construct so-called hyperbolic wavelet frames in weighted Bergman spaces and a multiresolution analysis (MRA) generated by them. The construction is based on a new example of sampling set for the weighted Bergman space, which is related to the Blaschke group operation. The introduced MRA is an analog of the MRA generated by the affine wavelets in the space of the square integrable functions on the real line, and in fact is the discretization of the continuous voice transform generated by a representation of the Blaschke group over the weighted Bergman space. The projection to the resolution levels is an interpolation operator. This projection operator gives opportunity of practical realization of the hyperbolic wavelet representation of a function belonging to the weighted Bergman space, if we can measure the values of the function on a given set of points inside the unit disc. Convergence properties of the hyperbolic wavelet representation are studied.

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9.1 Intoduction

Grossman, Morlet and Paul showed that the properties of affine wavelet transform and the Gábor transform are related to square integrable group representations of certain groups, namely the affine group and Heisenberg group, respectively (see [13]). The common generalization of these transforms is the voice transform. We consider a locally compact topological group $(G, \cdot)$. A unitary representation of the group $(G, \cdot)$ on a Hilbert space $(H, \langle \cdot, \cdot \rangle)$.

The voice transform of $f \in H$ generated by the representation U and by the parameter $\rho \in H$ is the (complex-valued) function on G defined by

$$(V_\rho f)(x) := \langle f, U_x \rho \rangle \quad (x \in G,\ f, \rho \in H).$$

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For analogy, we consider the affine wavelet transform (the simplest version) which is the voice transform of the affine group (see [9, 17, 33, 35]). Indeed let consider $(G, \circ)$ equal to the the affine group, where

$$G = \{\ell_{(a,b)} : \mathbb{R} \to \mathbb{R} : (a, b) \in \mathbb{R}^* \times \mathbb{R}\},$$

$$\ell_{(a,b)}(x) = ax + b, \ \mathbb{R}^* := \mathbb{R} \setminus \{0\}, \ \ell_1 \circ \ell_2(x) = \ell_1(\ell_2(x)) = a_1a_2x + a_1b_2 + b_1.$$

The representation of the affine group G on $L^2(\mathbb{R})$ is given by

$$U_{(a,b)} f(x) = |a|^{-1/2} f(a^{-1}x - b),$$

where a is the dilatation parameter and b the translation parameter.

The continuous affine wavelet transform is the voice transform of the affine group generated by this representation, i.e.,

$$W_\psi f(a, b) = |a|^{-1/2} \int_{\mathbb{R}} f(t)\overline{\psi(a^{-1}t - b)}dt = \langle f, U_{(a,b)}\psi\rangle, \quad f, \psi \in L^2(\mathbb{R}).$$

There is a rich bibliography of the affine wavelet theory (see for example [3–5, 17, 19, 20]). One important question is the construction of the discrete version, i.e., to find ϕ so that the discrete translates and dilates

$$\phi_{n,k} = 2^{-n/2}\phi(2^{-n}x - k)$$

form an orthonormal, or frame basis, and generate a multiresolution analysis in $L^2(\mathbb{R})$ (see [4, 5, 20]). Roughly speaking we want to approximate the function f, if we know the coefficients corresponding to the bases $\{\phi_{n,k}\}$, which are the values of the affine wavelet transform on a special discrete lattice:

$$\langle f, \phi_{n,k}\rangle = W_\phi(2^{-n}, k).$$

The discrete lattice in the affine case is determined by the following discrete subset of the affine group:

$$G_{n,k} = \{\ell_{(2^{-n}, -k)} : \mathbb{R} \to \mathbb{R} : n \in \mathbb{Z}, \ k \in \mathbb{Z}\}.$$

On abstract level, the discretization of the voice transform can be achieved using the unified approach of the atomic decomposition elaborated by Feichtinger and Gröchenig [9, 10]. This general description can be applied, when the representation which induces the voice transform satisfies both square integrability and the integrability conditions. In the affine wavelet case, the integrability condition is not satisfied, but it can be constructed multiresolution analysis in order to discretize it. The definition of the affine wavelet multiresolution analysis (MRA) in $L^2(\mathbb{R})$ is the following.

Definition 9.1.1 Let $V_j,\ \ j \in \mathbb{Z}$ be a sequence of subspaces of $L^2(\mathbb{R})$. The collections of spaces $\{V_j,\ \ j \in \mathbb{Z}\}$ is called a multiresolution analysis with scaling function ϕ if the following conditions hold:

1. (nested) $V_j \subset V_{j+1}$
2. (density) $\overline{\cup V_j} = L^2(R)$
3. (separation) $\cap V_j = \{0\}$
4. (basis) The function ϕ belongs to V_0 and the set $\{2^{n/2}\phi(2^n x - k),\ \ k \in \mathbb{Z}\}$ is a (orthonormal or frame) bases in V_n.

In the construction of affine wavelet multiresolution, the dilatation is used to obtain a higher resolution level, i.e., if $(f(x) \in V_n$ then $f(2x) \in V_{n+1})$ and applying the translation we remain on the same level of resolution. This field has a rich bibliography (see for example [3–5, 17, 20]).

Starting from 1980 Meyer and Daubechies, among others, constructed smooth orthonormal wavelet systems of the form $\phi_{n,k}(x) = 2^{n/2}\phi(2^n x - k)$, using dilation and translation of a single function ϕ (mother wavelet), and related MRA-s in different function spaces. Excepting the Haar wavelet system, the construction of such systems is a hard task. Despite the fact that in general ϕ cannot be given in an explicit form, the wavelet Fourier series enjoy nice convergence and approximation properties. The kernel functions of the partial sums can be well estimated and the wavelet Fourier coefficients can be calculated by a fast algorithm. Since 1980 the theory of wavelets has been growing fast, and it turned out that the wavelet theory has many practical applications.

In the last years, it turned out that affine wavelet frames with a multiresolution structure are also very important in applications, since this guarantees the existence of the fast decomposition and reconstruction algorithms. Recently, tight affine wavelet frames derived by the multiresolution analysis are used to open a few new areas of applications of frames. The application of tight wavelet frames in image restorations is one of them that includes image inpainting, image denoising, image deblurring and blind deburring, and image decompositions. [1, 6, 31].

An up-to-date monograph in this domain is [18], where are collected the most important ones and multivariate results connected to affine wavelet frames (framelets) and the related MRA-s, and their application in the image recovery from incomplete observed data, including the tasks of inpainting and image/video enhancement. In the recovery of missing data from incomplete and/or damaged and noisy samples, application of wavelet methods based on frames is more advanced due to the redundancy of frame systems.

Y. Meyer (Abel Prize 2017) formulated the following question: Is there any "regular" (smooth or analytic and with decay condition) affine wavelet orthonormal basis and any multiresolution analysis (MRA) generated by this basis? Auscher gave in 1995 a negative answer to this question (see [2]). Applying dilation and translation to a single function, it is not possible to construct analytic wavelets satisfying some "regularity" conditions.

Question: Is there any other way to construct analytic (very regular) wavelets or wavelet frames, and to generate multiresolution analysis in analytic function spaces, like in Hardy spaces of the unit disc, or upper half plane, or in weighted Bergman spaces?

In order to construct analytic wavelets, joint with Schipp, we have considered, instead of the affine group, the group generated by the composition of the special linear fractional transformations which preserve the unit circle and unit disc, the so-called Blaschke functions. The generated group is called the Blaschke group. These functions are closely related to hyperbolic geometry of the unit disc and to the theory of analytic function spaces. The congruences in the Poincare model of the hyperbolic geometry can be described by using Blaschke functions. We have considered representations of the Blaschke group on the Hardy space of the unit disc, on the weighted Bergman spaces and the voice transforms induced by these representations.

Results connected to these voice transforms of the Blaschke group, the so-called hyperbolic wavelet transforms, were published in [11, 12, 22–29, 32]. It turned out that the general theory of atomic decompositions developed by Feichtinger and Gröchenig (see [9, 10]) can be applied only for some weighted Bergman spaces. In this way, in those cases, new atomic decomposition results can be derived (see [12, 27]). If the representation which induces the hyperbolic wavelet transform is not integrable, then it is showed that an analog of MRA can be constructed. In the case of the Hardy spaces, we constructed hyperbolic analytic wavelets given by explicit formulas and an analog of MRA (see [12, 26, 29]). Recently was published a survey paper by Nowak and Pap summarizing this new method of construction of analytic wavelets (see [21]), where it was formulated the problem of extension of the construction for weighted Bergman spaces.

Our goal is to construct an analog of the MRA in the weighted Bergman space. But it turns out, that in this case the construction is more complicated, then in the Hardy spaces. The construction is based on a new example of sampling set for the weighted Bergman space, which is related to the Blaschke group operation. The constructed discretization scheme, the projection to the resolution levels, is an interpolation operator. This projection operator gives opportunity of practical realization of the hyperbolic wavelet representation of a function belonging to the weighted Bergman space, if we can measure the values of the function on a given set of points inside the unit disc. Convergence properties of the hyperbolic wavelet representation are studied.

The plan of this paper is as follows. First we present some basic results connected to the weighted Bergman spaces, we give the definition of the Blascke group, and basic properties of the voice transform generated by a representation of the Blaschke group on the weighted Bergman space.

In the second section, we introduce a discrete subset of the Blaschke group, which is sampling set for the weighted Bergman space. Using this special sampling set, we consider hyperbolic wavelet frames and we construct an analog of MRA decomposition in the weighted Bergman space. First the different resolution spaces will be defined using the introduced non-orthogonal hyperbolic wavelet frames. Applying the Gram–Schmidt orthogonalization we consider the rational orthogonal basis on the n-th multiresolution level V_n. This system is the analog of the Malmquist-

Takenaka system in the Hardy spaces possesses similar properties and is connected to the contractive zero divisors of a finite set in Bergman space. We prove that the projection operator $P_n f(z)$ on the resolution level V_n is convergent in A^2_α norm to f, is interpolation operator on the set the $\bigcup_{k=0}^n \mathscr{A}_k$, where $\mathscr{A}_k$ is defined by (9.2.7) with minimal norm and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc.

9.1.1 The Hyperbolic Wavelet Transform

The hyperbolic wavelet transform is the voice transform generated by a representation of the Blaschke group. In this paper, we consider a representation on the weighted Bergman space.

9.1.1.1 The Weighted Bergman Spaces A^p_α

In this section, we summarize the basic results connected to the weighted Bergman spaces (see [7, 16, 37]). Let us consider the unit disc and the open unit circle in the complex plane denoted by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, respectively. Let us denote by $\mathscr{A}$ the set of functions $f : \mathbb{D} \to \mathbb{C}$ which are analytic in $\mathbb{D}$. Denote the weighted area measure on $\mathbb{D}$ by

$$dA_\alpha(z) := \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha \, dxdy \ , z = x + iy.$$

For all $\alpha > -1$ the weighted Bergman spaces A^p_α are subsets of analytic functions with the following property

$$A^p_\alpha := \{f \in \mathscr{A} : \int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) < \infty\}.$$

For $p = 2$, the set A^2_α is a Hilbert space, with the following scalar product

$$\langle f, g\rangle_\alpha := \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z).$$

For $\alpha = 0$ we get back the unweighted case, $A^2 = A^2_0$, which is called the Bergman space (see [7, 16]). For $0 < p < \infty$, and $-1 < \alpha < \infty$ the weighted Bergman space A^p_α is a closed subspace of $L^p(\mathbb{D}, dA_\alpha) = L^p$. For any function $f \in A^p_\alpha$, and for any compact subset K of $\mathbb{D}$, there exists a positive constant $C = C(n, K, p, \alpha)$, such that

$$\sup\{|f^{(n)}(z)| : z \in K\} \leqq C\|f\|_{A^p_\alpha}.$$

This inequality implies, that the point-evaluation map is a bounded linear functional on A^p_α, and the norm convergence in A^p_α implies the locally uniform convergence on $\mathbb{D}$.

The weighted Bergman space A^2_α is a reproducing kernel Hilbert space, and the reproducing kernel, the weighted Bergman kernel, is given by the following formula

$$K_\alpha(\xi, z) = \frac{1}{(1 - \overline{z}\xi)^{\alpha+2}}\,.$$

For $-1 < \alpha < +\infty$, the weighted Bergman projection defined by

$$P_\alpha : L^2(\mathbb{D}, dA_\alpha) \to A^2_\alpha, \quad P_\alpha f(z) = \int_{\mathbb{D}} f(\xi)\frac{1}{(1 - \overline{\xi}z)^{\alpha+2}} dA_\alpha(\xi),$$

is an orthogonal projection operator, which satisfies $P_\alpha f = f$ for every $f \in A^2_\alpha$. The projection operator can be extended to $L^1(\mathbb{D}, dA_\alpha)$ by mapping each $f \in L^1(\mathbb{D}, dA_\alpha)$ to an analytic function, and $P_\alpha f = f$, for every $f \in A^1_\alpha$ (see [16] p. 6).

For $0 < p < \infty$, a sequence of points $\Gamma = \{z_k : k \in \mathbb{N}\}$ in the unit disc is sampling sequence for A^p_α, if there exist positive constants A and B such that

$$A||f||^p \le \sum_{k=1}^{\infty} |f(z_k)|^p(1 - |z_k|^2)^{2+\alpha} \le B||f||^p, \quad f \in A^p_\alpha.$$

For $p = 2$, this inequality can be expressed in equivalent form, using the localized weighted Bergman kernels in z_k. If $\varphi_k(z) = K(z, z_k)/\|K(z, z_k)\|$, is the localized and normalized weighted Bergman kernel, then the previous inequality is equivalent with the following

$$A||f||^2 \le \sum_{k=1}^{\infty} |\langle f, \varphi_k\rangle|^2 \le B||f||^2, \quad f \in A^2.$$

However, this last inequality shows that $\{\varphi_k(z), k \in \mathbb{N}\}$ will constitute a frame for A^2_α, if and only if $\Gamma = \{z_k : k \in \mathbb{N}\}$ is a sampling set for A^2_α. The Bergman spaces A^p_α do have sampling sequences, but their construction is a difficult task. Some explicit examples are due to Seip, Duren, Schuster, Horowitz, Luecking (see for ex in [7]). An A^p_α sampling sequence is never an A^p_α zero set. A total characterization of sampling sequences can be given with the uniformly discrete property and upper and lower Seip density of the set (see [7]). But the computation of the upper and lower density of a set in general is difficult. Duren, Schuster and Vukotic in [8] gave sufficient conditions based on the pseudohyperbolic metric. Using this sufficient condition it is easier to verify, if a set of points from the unit disc is sampling set.

The pseudohyperbolic metric in the unit disc is defined by the following formula

$$\rho(z, y) = \left| \frac{y - z}{1 - \overline{y}z} \right| \quad (y, z \in \mathbb{D}).$$

A sequence of points $\Gamma = \{z_k\}$ of points in the unit disc is uniformly discrete (separated), if

$$\delta(\Gamma) = \inf_{j \neq k} \rho(z_j, z_k) = \delta > 0.$$

For $0 < \varepsilon < 1$, a sequence of points $\Gamma = \{z_k : k \in \mathbb{N}\}$ of points in the unit disc is said to be ε-net, if each point $z \in \mathbb{D}$ has the property $\rho(z, z_k) < \varepsilon$ for some z_k in Γ. An equivalent statement is, that $\mathbb{D} = \bigcup_{k=1}^{\infty} \Delta(z_k, \varepsilon)$, where $\Delta(z_k, \varepsilon)$ denotes a pseudohyperbolic disc.

In [8] it is shown that, if Γ is ε-net, then its lower density satisfies the following inequality

$$D^{-}(\Gamma) \geq \frac{(1 - \varepsilon)^2}{2\varepsilon^2}.$$

If Γ is separated (uniformly discrete), and $D^{-}(\Gamma) > (\alpha + 1)/p$, then is a sampling set for A_α^p (Theorem 5.23 of [16]). We will use this last sufficient condition in order to construct a sampling sequence in A_α^p.

9.1.1.2 The Blaschke Group

The pseudohyperbolic metric can be expressed using the Blaschke functions, i.e., $\rho(z, a) = |B_a(z)|$, where

$$B_a(z) := \varepsilon \frac{z - b}{1 - \bar{b}z} \quad (z \in \mathbb{C}, a = (b, \varepsilon) \in \mathbb{B} := \mathbb{D} \times \mathbb{T}, \bar{b}z \neq 1).$$

These functions can be used to represent the congruences in the Poincaré model of the hyperbolic Bolyai-Lobachevsky geometry.

If $a \in \mathbb{B}$, then B_a is an $1 - 1$ map on $\mathbb{T}$ and $\mathbb{D}$, respectively. The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbb{B} := \mathbb{D} \times \mathbb{T}$, let us define the operation induced by the function composition in the following way $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The set of the parameters with the induced operation $(\mathbb{B}, \circ)$ is called the Blaschke group. If we use the notations $a_j := (b_j, \varepsilon_j),\ j \in \{1, 2\}$ and $a := (b, \varepsilon) =: a_1 \circ a_2$, then we have

$$b = \frac{b_1 \overline{\varepsilon}_2 + b_2}{1 + b_1 \overline{b}_2 \overline{\varepsilon}_2} = B_{(-b_2 \varepsilon_2, \overline{\varepsilon_2})}(b_1), \quad \varepsilon = \varepsilon_1 \frac{\varepsilon_2 + b_1 \overline{b}_2}{1 + \varepsilon_2 \overline{b}_1 b_2} = B_{(-b_1 \overline{b}_2, \varepsilon_1)}(\varepsilon_2).$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \varepsilon) \in \mathbb{B}$ is $a^{-1} = (-b\varepsilon, \overline{\varepsilon})$.

This group and the voice transforms on this group, the so-called hyperbolic wavelet transforms, were introduced and studied by Pap and Schipp in [22–29].

9.1.1.3 The Representation of Blaschke Group on the Weighted Bergman Space A^2_α

The representation of the Blaschke group on the weighted Bergman space is given by the following formula (see [24, 25])

$$(U^\alpha_{a^{-1}} f)(z) := e^{i\frac{\alpha+2}{2}\psi} \frac{(1-|b|^2)^{\frac{\alpha+2}{2}}}{(1-\overline{b}z)^{\alpha+2}} f\left(e^{i\psi}\frac{z-b}{1-\overline{b}z}\right) \quad (a=(b,e^{i\psi})\in\mathbb{B}).$$

It can be proved proved that for all $\alpha \geq 0,\quad U^\alpha_a(a\in\mathbb{B})$ is a unitary irreducible representation of the group $\mathbb{B}$ on the Hilbert space A^2_α. The unitarity means that

$$\langle f,g\rangle = \langle f,g\rangle_\alpha := \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA_\alpha(z) = \langle U^\alpha_a f, U^\alpha_a g\rangle_\alpha .$$

We consider the hyperbolic wavelet transform induced by this representation

$$(V_g f)(a^{-1}) = (V_g f)(-b\varepsilon,\overline{\varepsilon}) := \langle f, U^\alpha_{a^{-1}} g\rangle_\alpha \quad (a=(b,e^{i\psi})\in\mathbb{B},\ f,\rho\in A^2_\alpha).$$

This transform is in same relation with the Blaschke group and the weighted Bergman space, as the affine wavelet transform with the affine group and the $L^2(\mathbb{R})$ (see [9, 17, 33]).

From the general theory (see [17, 35]), it follows that the voice transform generated by representation $U^\alpha_a\ (a\in\mathbb{B})$ is one to one. The function $V_g f$ is continuous and bounded on $\mathbb{B}$. It can be shown that every element from A^2_α is admissible. Taking in consideration that the Blaschke group is unimodular, from the general theory of voice transform follows, that there exists a constant C such that for $f, g\in A^2_\alpha, g\neq 0$ and $\|Cg\| = 1$, the following reproducing formula is valid:

$$V_g f = V_g f * V_g g, \quad \text{i.e.,}\quad V_g f(y^{-1}) = \int_{\mathbb{B}} V_g f(x^{-1}) V_g g(x\circ y^{-1}) dm(x).$$

It was shown that, in the case of the weighted Bergman spaces, where the weight is generated by a positive α, under some other restrictions to the weight, both the integrability and square integrability conditions of the voice transform are satisfied. Consequently, the general theory of atomic decomposition can be applied, and in this way, new atomic decomposition results can be obtained for some weighted Bergman spaces (see [27]). For the Bergman space, unweighted case $\alpha = 0$, the integrability condition of the representation is not satisfied. For this case in [28] it is showed that, it is possible to construct a multiresolution analysis, using localized Bergman kernels in special sampling points.

In this paper, we will extend these results for the weighted Bergman spaces. This result is interesting, especially for those weighted Bergman spaces, where the integrability condition of the representation is not satisfied. Our goal is to answer the following question:

Question: Is it possible to find a discrete subset $\{a_{k\ell} = (z_{k\ell}, 1) \in \mathbb{B}\}$ of the Blascke group, a function $\varphi_{00} \in A^2_\alpha$, and to generate an adapted version of the multiresolution in the weighted Bergman space A^2_α using the images of this single function $\{U^\alpha_{a_{k\ell}}\varphi_{00}\}$ trough the representation?

9.2 New Results

9.2.1 Special Discrete Subsets in $\mathbb{B}$ and Their Sampling Property

In order to answer the formulated question first we construct a sampling set in the weighted Bergman space $A^2_\alpha(\mathbb{D})$, which is a discrete subset of the Blaschke group.

Let us consider the following one parameter subgroups of the Blaschke group:

$$\mathbb{B}_1 := \{(r, 1) : r \in (-1, 1)\}, \quad \mathbb{B}_2 := \{(0, \varepsilon) : \varepsilon \in \mathbb{T}\}. \tag{9.2.1}$$

These subgroups generate $\mathbb{B}$, i. e.

$$\mathbf{a} = (0, \varepsilon_2) \circ (0, \varepsilon_1) \circ (r, 1) \circ (0, \overline{\varepsilon}_1) \quad (\mathbf{a} = (r\varepsilon_1, \varepsilon_2) \in \mathbb{B},\ r \in [0, 1), \varepsilon_1, \varepsilon_2 \in \mathbb{T}). \tag{9.2.2}$$

$\mathbb{B}_1$ is the analog of the group of dilation, $\mathbb{B}_2$ is the analog of the group of translation (see [33]).

The group operation $(r, 1) = (r_1, 1) \circ (r_2, 1)$ in $\mathbb{B}_1$ can be expressed using the tangent hyperbolic and its inverse (ath) in the following way

$$r = \frac{r_1 + r_2}{1 + r_1 r_2} = \mathrm{th}(\mathrm{ath}\, r_1 + \mathrm{ath}\, r_2) \quad (r_1, r_2 \in (-1, 1)). \tag{9.2.3}$$

Let denote $r = \mathrm{th}\,\alpha$, $r_i = \mathrm{th}\,\alpha_i$, $i = 1, 2$. Then by

$$(r_1, 1) \circ (r_2, 1) = (\mathrm{th}\,\alpha_1, 1) \circ (\mathrm{th}\,\alpha_2, 1) = (\mathrm{th}\,(\alpha_1 + \alpha_2), 1),$$

it follows that $(\mathbb{B}_1, \circ)$ is isomorphic to $(\mathbb{R}, +)$. It is known that $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{R}, +)$, then $\overline{\mathbb{B}_1} = \{(\mathrm{th}\,k, 1),\ k \in \mathbb{Z}\}$ is an one parameter subgroup of $(\mathbb{B}_1, \circ)$ (see [34]).

Let $a > 1$, denote by

$$\mathbb{B}_3 = \left\{ (r_k, 1) : \ r_k = \frac{a^k - a^{-k}}{a^k + a^{-k}}, \ k \in \mathbb{Z} \right\}. \tag{9.2.4}$$

It can be proved that $(\mathbb{B}_3, \circ)$ is another subgroup of $(\mathbb{B}, \circ)$, and we have the following composition rule: $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$. The hyperbolic distance of the points $r_k, \ r_n$ has the following property:

$$\rho(r_k, r_n) := \frac{|r_k - r_n|}{|1 - r_k \overline{r_n}|} = \left| \frac{\frac{a^k - a^{-k}}{a^k + a^{-k}} - \frac{a^n - a^{-n}}{a^n + a^{-n}}}{1 - \frac{a^k - a^{-k}}{a^k + a^{-k}} \frac{a^n - a^{-n}}{a^n + a^{-n}}} \right| = |r_{k-n}|. \tag{9.2.5}$$

This property implies that the sequence $(r_k, k \in \mathbb{N})$ forms an equidistant division of the interval $[0, 1)$ in the pseudohyperbolic metric.

Let $N(a, k), k \geq 1, N(a, 0) := 1$, be an increasing sequence of natural numbers. Let us consider the following set of points $z_{00} := 0$,

$$\mathscr{A} = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N}}, \quad \ell = 0, 1, \ldots, N(a, k) - 1, \quad k = 0, 1, 2, \ldots\}. \tag{9.2.6}$$

For a fixed $k \in \mathbb{N}$, let the level k be the following set of uniformly distributed points on the circle with radius r_k

$$\mathscr{A}_k = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N(a,k)}}, \ \ell \in \{0, 1, \ldots, N(a, k) - 1\}\}. \tag{9.2.7}$$

The points of $\mathscr{A}$ determine a similar, decomposition to the Whitney cube decomposition of the unit disc (see for ex. [30] p.80).

The question is how to choose a and $N = N(a, k)$ such that $\mathscr{A}$ to be a sampling set in the weighted Bergman space $A^p_\alpha(\mathbb{D})$. For the unweighted case, for $\alpha = 0$, this question was studied by Pap in [28], where it was proved that for a convenient choice of a and $N(a, k)$

1. $\mathscr{A}$ is uniformly discrete,
2. $\mathscr{A}$ is an ε-net set for some $0 < \varepsilon < 1$.

In this paper, we will extend these results and we will show that there exist a and $N(a, k)$ such that $\mathscr{A}$ will be sampling sequence for weighted Bergman spaces A^2_α.

Theorem 9.2.1 *Let $a > 1$, $(N(a, k), k \geq 1)$ a sequence of increasing natural numbers, and consider the set of points $\mathscr{A}$ defined by (9.2.6). Suppose that $N(a, k)a^{-2k} = b$, for $k \geq 1$, and $0 < b < \infty$. Let us denote by $K := 1 + \frac{(a-a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2$. If*

$$\sqrt{1 - 1/K} < \frac{1}{1 + \sqrt{\frac{2(\alpha+1)}{p}}},$$

then $\mathscr{A}$ is a sampling set for A^p_α.

Proof In [28] it was proved that, if there exists $b = \lim_{k\to\infty} N(a,k)a^{-2k}$, and if $(N(a,k)a^{-2k}, k \geq 1)$ is increasing sequence and b is finite, then $\mathscr{A}$ is uniformly discrete and the separation constant satisfies

$$\delta \geq \min\left\{r_1, \frac{1}{\sqrt{1+b^2}}\right\}.$$

In [28] it was also proved, that if $(N(a,k)a^{-2k}, k \geq 1)$ is decreasing and $0 < b < \infty$, then the set A is ε_0-net, where $\varepsilon_0 = \sqrt{1-1/K}$, with $K := 1 + \frac{(a-a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2$.

Indeed for given $z = re^{i\theta} \in \mathbb{D}$ we take k and $j \in \{0, 1, \dots N(a,k)-1\}$ such that $r_k < r \leq r_{k+1}$, $\theta \in \left[\frac{2\pi j}{N(a,k)}, \frac{2\pi(j+1)}{N(a,k)}\right)$, $\theta_{kj} = \frac{2\pi j}{N(a,k)}$, then

$$\frac{1}{1-\rho^2(z,z_{kj})} = \frac{(1-rr_k)^2 + 4rr_k\sin^2\frac{\theta-\theta_{kj}}{2}}{(1-r^2)(1-r_k^2)} = 1 + \frac{(r-r_k)^2 + 4rr_k\sin^2\frac{\theta-\theta_{kj}}{2}}{(1-r^2)(1-r_k^2)} \leq$$

$$1 + \frac{(r-r_k)^2 + 4rr_k\frac{\pi^2}{N^2(a,k)}}{(1-r^2)(1-r_k^2)} = 1 + \frac{(a-a^{-1})^2}{4} + \frac{(a^{2k+2}-a^{-2k-2})(a^{2k}-a^{-2k})}{4}\frac{\pi^2}{N^2(a,k)}.$$

If $(N(a,k)a^{-2k}, k \geq 1)$ is decreasing and $b = \lim_{k\to\infty} N(a,k)a^{-2k} \in (0,\infty)$, then the last term in the previous inequality is upper bounded by

$$K := 1 + \frac{(a-a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2.$$

Then for $\varepsilon_0 = \sqrt{1-1/K}$, we have $\rho(z, z_{kj}) < \varepsilon_0$.

If $N(a,k)a^{-2k} = b$, for $k \geq 1$, and $0 < b < \infty$, then $\mathscr{A}$ is in the same time uniformly discrete and ε_0-net. In [8] it is shown that if $\mathscr{A}$ is ε_0-net, then the lower density of the set satisfies

$$D^-(\mathscr{A}) \geq \frac{(1-\varepsilon_0)^2}{2\varepsilon_0^2}.$$

If $\mathscr{A}$ is separated (is a uniformly discrete) and $D^-(\mathscr{A}) > (\alpha+1)/p$ then is a sampling set for A^p_α (see Theorem 5.23 of [16]).

Using this results we get that if

$$\varepsilon_0 = \sqrt{1-1/K} < \frac{1}{1+\sqrt{\frac{2(\alpha+1)}{p}}},$$

then

$$D^-(\mathscr{A}) \geq \frac{(1-\varepsilon_0)^2}{2\varepsilon_0^2} > (\alpha+1)/p,$$

which implies that $\mathscr{A}$ is a sampling set for A^p_α.

Remark

1. As it was showed in [28], for $\alpha = 0$, from this theorem we obtain that if $\mathscr{A}$ is a sampling set for the Bergman space A^p, then
$$(a - a^{-1})^2 < 2p,$$
therefore a must be in the interval $(1, \frac{\sqrt{2p}+\sqrt{2p+4}}{2})$. Then we can always choose $N = N(a, k)$ big enough, such that the the sampling condition to be satisfied.
2. From the point of view of computations and to have on every circle the less possible points, for $p = 2, \alpha = 0$ a convenient choice is $a = 2$, and $N(2, k) = 2^{2k+\beta}$ for $k \geq 1$ with β a fixed integer. Then $b = 2^\beta$, and the smallest value for β for which the sampling condition is satisfied is $\beta = 3$, then on the k-th circle we will have $N_1(2, k) = 2^{2k+3}$ equidistant points corresponding to the roots of order 2^{2k+3} of the unity. For $a = \sqrt{2}$ for sampling we need $N_1(\sqrt{2}, k) = 2^{k+2}$ points.
3. For $p = 2, \alpha > -1$ in order to have $\mathscr{A}$ a sampling set for A^2_α we have to choose a and on the level k the number of the points $N(a, k)$ such that for $N(a, k)a^{-2k} = b$ to have the following inequality
$$\frac{(a - a^{-1})^2}{4} + \frac{a^2}{4b^2}\pi^2 < \frac{1}{\sqrt{\alpha + 1}}.$$
From now on, we will concentrate on this case and using this special sampling set we will construct multiresoluion analysis in the A^2_α.

9.2.2 Multiresolution Analysis in the Weighted Bergman Space

Using the subgroup $\mathbb{B}_3$ of the Blaschke group, a discrete subgroup of $\mathbb{B}_2$ and the representation U^α_a we give a similar construction of the affine wavelet multiresolution in the weighted Bergman space. To show the analogy with the affine wavelet multiresolution, we first represent the levels V_n by non-orthogonal frames, and then we construct an orthonormal bases in V_n. We give also an orthogonal basis in W_n which is orthogonal to V_n. We will show that the analog of the Malmquist–Takenaka systems for weighted Bergman space, will span the resolution spaces and the density property will be fulfilled, i.e., $\overline{\bigcup_{k=1}^{\infty} V_k} = A^2_\alpha$ in norm. Similar multiresolution results, based on another discrete subset of the Blaschke group, were obtained by the author in [26] for the Hardy space of the unit disc, for upper half plane by Feichtinger, Pap in [11], and in the unweighted case, in Bergman space, by Pap in [28].

We show that the projection $P_n f$ on the n-th resolution level is an interpolation operator in the unit disc until the n-th level, which converges in A^2_α norm to f.

Let us consider $a > 1$, denote by $r_k = \frac{a^k - a^{-k}}{a^k + a^{-k}}$, $k \in \mathbb{N}$, and the concentric circles with radius r_k. On the circle with radius r_k let us consider $N_k = N(a,k)$ equidistantly situated points $z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N(a,k)}}$, such that $b = N(a,k)a^{-2k}$ satisfies

$$0 < b < \infty, \quad (a - a^{-1})^2 + \pi^2 \frac{a^2}{b^2} < 4\frac{1}{\sqrt{\alpha+1}}.$$

If these conditions are satisfied then, due to Theorem 2.1, $\mathscr{A}$ given by (9.2.6) is a sampling set for A_α^2. This implies that the set of normalized and localized weighted Bergman kernels in these points

$$\left\{ \varphi_{k\ell}(z) = \frac{(1 - r_k^2)^{\frac{\alpha+2}{2}}}{(1 - \overline{z_{k\ell}}z)^{2+\alpha}},\ \varphi_{00} = 1,\ k = 0, 1, \ldots,\ \ell = 0, 1, \ldots N(a,k) - 1 \right\}$$

will constitute a frame system for A_α^2. These system can be derived from a single function using the representation and a discrete subset of the Blaschke group

$$\varphi_{k\ell}(z) = (U^\alpha_{(z_{k\ell},1)^{-1}}\varphi_{00})(z).$$

Due to this observation, we can consider them as an analog of affine wavelet frames, and we call them hyperbolic wavelet frames.

From the frame theory (see for example in [14]), follows that every function f from A_α^2 can be represented

$$f(z) = \sum_{(k,\ell)} c_{k\ell}\varphi_{kl}(z)$$

for some $\{c_{k\ell}\} \in \ell^2$, with the series converging in A_α^2 norm. The determination of the coefficients it is related to the construction of the inverse frame operator (see [14]), which is not an easy task in general. This is the reason why we try to construct other approximation process for $f \in A_\alpha^2$ and to give an exactly defined algorithmic scheme for the determination of the coefficients.

Let us consider the function $\varphi_{00} = 1$ and let define $V_0 := \{c\varphi_{00},\ c \in \mathbb{C}\}$. Let us consider the non-orthogonal hyperbolic wavelets at the first level

$$\varphi_{1\ell}(z) = (U^\alpha_{(z_{1\ell},1)^{-1}}\varphi_{00})(z) = \frac{(1 - r_1^2)^{\frac{\alpha+2}{2}}}{(1 - \overline{z_{1\ell}}z)^{2+\alpha}}, \quad \ell = 0, 1, \ldots, N(a,1) - 1. \tag{9.2.8}$$

They can be obtained from φ_{10} using the analog of translation operator which in the unit disc is a multiplication by a unimodular complex number and from φ_{00} using first the representation operator $U_{(r_1,1)^{-1}}$ followed by the translation operator:

$$\varphi_{1\ell}(z) = \varphi_{10}(ze^{-\frac{2\pi i\ell}{N(a,1)}}) = (U^\alpha_{(r_1,1)^{-1}}\varphi_{00})(ze^{-\frac{2\pi i\ell}{N(a,1)}}). \tag{9.2.9}$$

Let us define the first resolution level as follows

$$V_1 := \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{1} \sum_{\ell=0}^{N(a,k)-1} c_{k\ell}\varphi_{k\ell},\ c_{k\ell} \in \mathbb{C} \right\}. \tag{9.2.10}$$

Let us consider the non-orthogonal wavelets on the n-th level

$$\varphi_{n\ell}(z) = (U^{\alpha}_{(z_{n\ell},1)^{-1}}\varphi_{00})(z) = \frac{(1-r_n^2)^{\frac{\alpha+2}{2}}}{(1-\overline{z_{n\ell}}z)^{\alpha+2}}, \quad \ell = 0, 1, \dots, N(a,n)-1, \tag{9.2.11}$$

which can be obtained from φ_{n0} using the translation operator, and from φ_{00} using the representation $U^{\alpha}_{((r_{n-1},1)\circ(r_1,1))^{-1}}$, and the translations

$$\varphi_{n,\ell}(z) = (U^{\alpha}_{((r_{n-1},1)\circ(r_1,1))^{-1}}\varphi_{00})(ze^{-i\frac{2\pi\ell}{N(a,n)}}). \tag{9.2.12}$$

Let us define the n-th resolution level by

$$V_n := \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(a,k)-1} c_{k\ell}\varphi_{k\ell},\ c_{k\ell} \in \mathbb{C} \right\}. \tag{9.2.13}$$

The closed subset V_n is spanned by

$$\{\varphi_{k\ell},\ \ell = 0, 1, \dots, N(a,k)-1,\ k = 0, \dots, n\}. \tag{9.2.14}$$

Continuing this procedure, we obtain a sequence of closed, nested subspaces of A^2_α for $z \in \mathbb{D}$

$$V_0 \subset V_1 \subset V_2 \subset \dots V_n \subset \dots A^2_\alpha. \tag{9.2.15}$$

Due to Theorem 9.2.1 the normalized kernels

$$\{\varphi_{kl}(z),\ k = 0, 1, \dots,\ \ell = 0, 1, \dots N(a,k)-1\}$$

form a frame system for A^2_α. This implies, that this is a complete and a closed set in norm, consequently the density property it is satisfied, i.e.,

$$\overline{\bigcup_{n\in\mathbb{N}} V_n} = A^2_\alpha. \tag{9.2.16}$$

From now on for simplicity, we consider $a = 2$ and $N(2,k)$ such that $b = N(2,k)2^{-2k}$ satisfies the following conditions

$$0 < b < \infty, \quad (2-2^{-1})^2 + \pi^2 \frac{2^2}{b^2} < 4\frac{1}{\sqrt{\alpha+1}}.$$

For $\alpha = 0$ a good choice is $N(2,k) = 2^{2k+3}$. In general on the circle k-th we will have $N(2,k) = 2^{2k}b$ points.

We show that, if a function $f \in V_n$, then $U^{\alpha}_{(r_1,1)^{-1}} f \in V_{n+1}$. This is the analog of the dilation. For this it is sufficient to show that, for $k = 0, 1, .., n,\ \ \ell = 0, 1, \ldots, 2^{2k}b - 1$, we have

$$U^{\alpha}_{(r_1,1)^{-1}}(\varphi_{k\ell})(z) = U^{\alpha}_{(r_1,1)^{-1}}[(U^{\alpha}_{(r_k,1)^{-1}}\varphi_{00})](ze^{-i\frac{2\pi\ell}{2^{2k}b}}) =$$

$$= [(U^{\alpha}_{(r_{k+1},1)^{-1}}\varphi_{00})](ze^{-i\frac{2\pi 4\ell}{2^{2(k+1)}b}}) = \varphi_{k+1\ell'} \in V_{n+1}, \tag{9.2.17}$$

for $\ell' = 4\ell \in \{0, 1, \ldots, 2^{2(k+1)}b - 1\}$.

Summarizing our construction we have constructed a sequence of subspaces $(V_j,\ \ j \in \mathbb{N})$ of A^2_α with following properties:

1. (nested) $V_j \subset V_{j+1} \subset A^2_\alpha$,
2. (density) $\overline{\cup V_j} = A^2_\alpha$
3. (analog of dilatation) $U^{\alpha}_{(r_1,1)^{-1}}(V_j) \subset V_{j+1}$
4. (basis) There exist $\{\varphi_{k\ell},\ k = 0, 1, .., n,\ \ \ell = 0, 1, \ldots, 2^{2k}b - 1\}$ (orthonormal or frame) bases in V_j.

This is the adapted definition of the multiresolution analysis in the weighted Bergman spaces. These four properties are required for $(V_j,\ \ j \in \mathbb{N})$ to form a hyperbolic wavelet multiresolution analysis (MRA) in the weighted Bergman spaces.

Because $\mathscr{A}$ is a sampling set, it follows that is a set of uniqueness for A^2_α. This means, that every function $f \in A^2_\alpha$ is uniquely determined by the values $\{f(z_{k\ell}),\ z_{k\ell} \in \mathscr{A}\}$. In [36] Zhu described in general, how can be recaptured a function from a Hilbert space, when the values of the function on a set of uniqueness are known, and developed in details this process in the Hardy space. At the beginning, we will follow the steps of the recapturation process, and we will combine this with the multiresolution analysis. The elements of the set $\mathscr{A}$ are different numbers, this implies that the localized weighted Bergman kernels

$$\left\{\frac{1}{(1-\overline{z_{k\ell}}z)^{2+\alpha}},\ \ell = 0, 1, \ldots, N(2,k) - 1,\ k = 0, 1, \ldots, n.\right\} \tag{9.2.18}$$

are linearly independent, and constitute a non-orthogonal basis in V_n.

Using Gram–Schmidt orthogonalization process they can be orthogonalized. Denote by $\psi_{k\ell}$ the resulting functions. They can be viewed as the analog of the Malmquist–Takenaka system in the Hardy space. This functions can be obtained using the following two methods. The first arises from the orthogonalization procedure. To describe this, let reindex the points of the set $\mathscr{A}$ as follows, $a_1 = z_{00}$, $a_2 = z_{10}$, $a_3 = z_{11}, \ldots, a_{N(2,1)+1} = z_{1N(2,1)-1}, \ldots, a_m = z_{k\ell} \ldots, k = 0, 1, \ldots, \ell = 0, 1,$

$\dots, N(2,k)-1$, and denote by $K(z, z_{k\ell}) = \frac{1}{(1-\overline{z_{k\ell}}z)^{2+\alpha}} := K(z, a_m)$. The resulted orthonormal system is

$$\phi_{00}(z) = \phi(a_1, z) = \frac{K(z, a_1)}{\|K(., a_1)\|},\ \ \phi_{k\ell}(z) = \phi(a_1, a_2, \dots, a_m, z) =$$

$$K(z, a_m) - \sum_{i=1}^{m-1} \phi(a_1, a_2, \dots, a_i, z) \frac{\langle K(., a_m), \phi(a_1, a_2, \dots, a_i, .)\rangle}{\|\phi(a_1, a_2, \dots, a_i, .)\|^2}.$$

Thus the normalized functions

$$\left\{\psi_{k\ell}(z) = \frac{\phi_{k\ell}(z)}{\|\phi_{k\ell}\|},\ k = 1, 2, \dots, \ell = 0, 1, \dots N(2,k) - 1\right\}$$

became an orthonormal system. Applying similar construction in Hardy space, with the Cauchy kernel as reproducing kernel, the result of the orthogonalization process can be written in closed form using the Blaschke products, and in this way we get the Malmquist–Takenaka system. Unfortunately in our situation the result of the ortogonalization can be not written in closed form, and the properties of the system cannot be seen from the previous construction.

Another approach for the construction is given by Zhu in [36]. He proved that, the result of the Gram–Schmidt process is connected to some reproducing kernels, and the contractive zero divisors. Let denote $A_m = \{a_1, a_2, \dots a_m\}$ a set of distinct points in the unit disc. Let H_{A_m} the subspace of A_α^2 consisted of all functions in A_α^2 which vanish on A_m. H_{A_m} is a closed subspace of A_α^2 and denote by K_{A_m} the reproducing kernel of H_{A_m}. These reproducing kernels satisfies the following recursion formula

$$K_{A_{m+1}}(z, w) = K_{A_m}(z, w) - \frac{K_{A_m}(z, a_{m+1}) K_{A_m}(a_{m+1}, w)}{K_{A_m}(a_{m+1}, a_{m+1})}, m \geq 0, \quad (9.2.19)$$

$$K_{A_0} := K(z, w) = \frac{1}{(1-\overline{w}z)^{2+\alpha}}.$$

The result of the Gram-Schmidt process can be expressed as

$$\frac{K(z, a_1)}{\sqrt{K(a_1, a_1)}},\ \frac{K_{A_1}(z, a_2)}{\sqrt{K_{A_1}(a_2, a_2)}},\ \dots \frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}}, \dots.$$

Then

$$\psi_{k\ell}(z) = \frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}}, \quad (9.2.20)$$

and is the solution of the following problem

$$\sup\{Ref(a_m) : f \in H_{A_{m-1}}, \|f\| \leq 1\}.$$

This extremal functions in the context of the Bergman spaces have been studied by Hedenmalm [15]. The main result in [15] is that the function

$$\frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}}$$

is a contractive divisor on the Bergman space, vanishes on A_{m-1}, and if $\mathscr{A}$ is not a zero set for A^2, as is in our case, the functions converge to 0 as $m \to \infty$. In Hardy space, the partial products of a Blaschke product corresponding to a nonzero set own all these nice properties.

From the Gram–Schmidt orthogonalization process it follows,that

$$V_n = span\{\psi_{k\ell},\ \ell = 0, 1, \ldots, N(2, k) - 1,\ k = 0, \ldots, n\}. \tag{9.2.21}$$

The wavelet space W_n is the orthogonal complement of V_n in V_{n+1}. We will prove that

$$W_n = span\{\psi_{n+1\ell}, \quad \ell = 0, 1, \ldots, N(2, n+1) - 1\}. \tag{9.2.22}$$

If $f \in V_n$, one has $f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} c_{k\ell}\varphi_{k\ell} \subset A^2_\alpha$, then using (1.4) we obtain that

$$\langle \psi_{n+1j}, f \rangle = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} c_{k\ell} \langle \psi_{n+1j}, \varphi_{k\ell} \rangle =$$

$$\sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} c_{k\ell}(1 - r_k^2)^{\frac{\alpha+2}{2}} \psi_{n+1\ell}(z_{k\ell}) = 0,\ \ j = 0, 1, \ldots N(2, n+1) - 1.$$

We have proved that for $f \in V_n$

$$\langle f, \psi_{n+1j} \rangle = 0, \tag{9.2.23}$$

which is equivalent with

$$\psi_{n+1j} \perp V_n, \quad (j = 0, 1, \ldots, N(2, n+1) - 1). \tag{9.2.24}$$

From

$$V_{n+1} = V_n \bigoplus span\{\varphi_{n+1,j},\ \ j = 0, 1, \ldots, N(2, n+1) - 1\} \tag{9.2.25}$$

it follows that, W_n is an $N(2, n+1)$ dimensional space and

$$W_n = span\{\psi_{n+1\ell}, \quad \ell = 0, 1, \ldots, N(2, n+1) - 1\}. \tag{9.2.26}$$

9.2.3 The Projection Operator Corresponding to the n-th Resolution Level

Let us consider the orthogonal projection operator of an arbitrary function $f \in A^2_\alpha$ on the subspace V_n given by

$$P_n f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,n)-1} \langle f, \psi_{k\ell} \rangle \psi_{k\ell}(z). \tag{9.2.27}$$

This operator is called the projection of f at nth scale or resolution level.

Theorem 9.2.2 *For $f \in A^2_\alpha$ the projection operator $P_n f$ is an interpolation operator in the points*

$$z_{k\ell} = r_k e^{i \frac{2\pi\ell}{N(2,k)}}, \ (\ell = 0, \ldots, N(2,k) - 1, \quad k = 0, \ldots, n),$$

is norm convergent in A^2_α to f i.e.,

$$\|f - P_n f\| \to 0, \quad n \to \infty,$$

uniformly convergent inside the unit disc on every compact subset.

Proof Let us consider $N = 1 + N(2, 1) + \cdots + N(2, n)$, and the corresponding kernel function of the projection operator

$$\mathbf{K}_N(z, \xi) = \sum_{k=0}^{n} \sum_{\ell=0}^{N(2,k)-1} \overline{\psi_{k\ell}(\xi)} \psi_{k\ell}(z) = \tag{9.2.28}$$

$$\sum_{m=1}^{N} \frac{K_{A_{m-1}}(z, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}} \overline{\left(\frac{K_{A_{m-1}}(\xi, a_m)}{\sqrt{K_{A_{m-1}}(a_m, a_m)}} \right)} = \sum_{m=1}^{N} \frac{K_{A_{m-1}}(z, a_m) K_{A_{m-1}}(a_m, \xi)}{K_{A_{m-1}}(a_m, a_m)}.$$

From the recursion relation (9.2.19) it follows that

$$\mathbf{K}_N(z, \xi) = \sum_{m=1}^{N} (K_{A_{m-1}}(z, \xi) - K_{A_m}(z, \xi)) = K(z, \xi) - K_{A_N}(z, \xi) \tag{9.2.29}$$

From this relation, it follows that the values of the kernel function in the points $z_{k\ell},\ (\ell = 0, \dots, N(2,k)-1,\quad k = 0, \dots, n)$ are equal to

$$K(z_{kl}, \xi) = \frac{1}{(1 - z_{k\ell}\overline{\xi})^{2+\alpha}}. \tag{9.2.30}$$

Using again formula (1.4) we have

$$P_n f(z_{k\ell}) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w} z_{k\ell})^{2+\alpha}} dA_\alpha(w) = f(z_{k\ell}) \tag{9.2.31}$$

for $(\ell = 0, \dots, N(2,k)-1,\quad k = 0, \dots, n)$. We obtain that $P_n f$ is interpolation operator for every $f \in A^2_\alpha$ on the set $\cup_{k=0}^n \mathscr{A}_k$.

Because of 2.16 and 2.21 $\{\psi_{k\ell},\ k = 0, \dots, \infty,\ \ell = 0, 1, \dots, N(2,k)-1\}$ is a closed set in the Hilbert space A^2_α, we have that that $\|f - P_n f\| \to 0$ as $n \to \infty$. Since convergence in A^2_α norm implies uniform convergence on every compact subset inside the unit disc, we conclude that $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc. From Theorem 5.3.1 of [30], there exists a unique $\hat{f}_n \in V_n$ with minimal norm such that

$$\hat{f}_n(z_{k\ell}) = f(z_{k\ell}),\quad (\ell = 0, \dots, N(2,k)-1,\quad k = 0, \dots, n), \tag{9.2.32}$$

$\hat{f}_n$ is uniquely determined by the interpolation conditions and is equal to the orthogonal projection of f on V_n, thus $\hat{f}_n(z) = P_n f(z)$.

9.2.4 Reconstruction Algorithm

In what follows, we propose a computational scheme for the best approximant in the wavelet base $\{\psi_{k\ell},\ \ell = 0, 1, \dots, N(2,k)-1,\ k = 0, \dots, n\}$.

The projection of $f \in A^2_\alpha$ onto V_{n+1} can be written in the following way:

$$P_{n+1} f = P_n f + Q_n f, \tag{9.2.33}$$

where

$$Q_n f(z) := \sum_{\ell=0}^{N(2,n+1)-1} \langle f, \psi_{n+1\ell} \rangle \psi_{n+1\ell}(z). \tag{9.2.34}$$

This operator has the following properties

$$Q_n f(z_{k\ell}) = 0,\quad k = 1, \dots, n,\ \ell = 0, 1, \dots, N(2,k)-1. \tag{9.2.35}$$

Consequently, P_n contains information on low resolution, i.e., until the level $\mathscr{A}_n$, and Q_n is the high resolution part. After n steps

$$P_{n+1}f = P_1 f + \sum_{k=1}^{n} Q_n f. \tag{9.2.36}$$

Thus

$$V_{n+1} = V_0 \bigoplus W_0 \bigoplus W_1 \bigoplus \ldots \bigoplus W_n. \tag{9.2.37}$$

The set of coefficients of the best approximant $P_n f$

$$\{b_{k\ell} = \langle f, \psi_{k\ell}\rangle,\ \ \ell = 0.1, \ldots, N(2,k) - 1\ \ k = 0, 1, \ldots, n\} \tag{9.2.38}$$

is the (discrete) hyperbolic wavelet transform of the function $f \in A^2_\alpha$. Thus, it is important to have an efficient algorithm for the computation of the coefficients.

The coefficients of the projection operator $P_n f$ can be computed, if we know the values of the functions on $\bigcup_{k=0}^{n} \mathscr{A}_k$. For this reason, we express first the function $\psi_{k\ell}$ using the bases $\{\varphi_{k'\ell'}\ \ell' = 0, 1, \ldots, N(2,k') - 1,\ \ k' = 0, \ldots, k\}$, i.e. we write the partial fraction decomposition of $\psi_{k\ell}$:

$$\psi_{k\ell}(\xi) = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{N(2,k')-1} c_{k'\ell'} \frac{1}{(1 - \overline{z_{k'\ell'}}\xi)^{2+\alpha}} + \sum_{j=0}^{\ell} c_{kj} \frac{1}{(1 - \overline{z_{kj}}\xi)^{2+\alpha}}. \tag{9.2.39}$$

Using the orthogonality of the functions $\{\psi_{k'\ell'}\ \ell' = 0, 1, \ldots, N(2,k') - 1,\ \ k' = 0, \ldots, k\}$ and the formula (1.4) we obtain that

$$\delta_{kn}\delta_{\ell m} = \langle \psi_{nm}, \psi_{k\ell}\rangle = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{N(2,k')-1} \overline{c_{k'\ell'}}\psi_{n,m}(z_{k'\ell'}) + \sum_{j=0}^{\ell} \overline{c_{kj}}\psi_{nm}(z_{kj}), \tag{9.2.40}$$

$$(m = 0, 1, \ldots N(2,n) - 1,\ \ n = 0, \ldots, k).$$

If we order these equalities so that we write first the relations (9.2.40) for $n = k$ and $m = \ell, \ell - 1, \ldots, 0$ respectively, then for $n = k - 1$ and $m = N(2, k-1) - 1,\ N(2, k-1) - 2, \ldots, 0$, etc., this is equivalent to

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} = \begin{pmatrix} \psi_{k\ell}(z_{k\ell}) & 0 & 0 & \ldots & 0 \\ \psi_{k\ell-1}(z_{k\ell}) & \psi_{k\ell-1}(z_{k\ell-1}) & 0 & \ldots & 0 \\ \psi_{k\ell-2}(z_{k\ell}) & \psi_{k\ell-2}(z_{k\ell-1} & 0 & \ldots & 0 \\ \vdots & & & \vdots & \\ \psi_{00}(z_{k\ell}) & \psi_{00}(z_{k\ell-1}) & \psi_{00}(z_{k\ell-2}) & \ldots & \psi_{00}(z_{00}) \end{pmatrix} \begin{pmatrix} \overline{c_{k\ell}} \\ \overline{c_{k\ell-1}} \\ \overline{c_{k\ell-2}} \\ \vdots \\ \overline{c_{00}} \end{pmatrix}. \tag{9.2.41}$$

Because on the main diagonal the elements of the matrix are different from zero, this system has a unique solution $(\overline{c_{k\ell}}, \overline{c_{k\ell-1}}, \overline{c_{k\ell-2}}, \dots, \overline{c_{00}})^T$. If we determine this vector, then we can compute the exact value of $\langle f, \psi_{k\ell}\rangle$ knowing the values of f on the set $\bigcup_{k=0}^{n} \mathscr{A}_k$.

Indeed, using again the partial fraction decomposition of $\psi_{k\ell}$ and the reconstruction formula formula we get that

$$\langle f, \psi_{k\ell}\rangle = \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{N(2,k')-1} \overline{c_{k'\ell'}} \left\langle f(\xi), \frac{1}{(1-\overline{z_{k'\ell'}}\xi)^{2+\alpha}} \right\rangle + \sum_{j=0}^{\ell} \overline{c_{kj}} \left\langle f(\xi), \frac{1}{(1-\overline{z_{kj}}\xi)^{2+\alpha}} \right\rangle =$$

$$= \sum_{k'=0}^{k-1} \sum_{\ell'=0}^{N(2,k')-1} \overline{c_{k'\ell'}} f(z_{k'\ell'}) + \sum_{j=0}^{\ell} \overline{c_{kj}} f(z_{kj}). \tag{9.2.42}$$

9.2.5 Conclusion

We introduced a new sampling set for A_α^p which is connected to the Blaschke group operation. We have generated a multiresolution in A_α^2 and we have constructed a rational orthogonal wavelet system which generates the levels of the multiresolution. Compared with the classical affine multiresolution, according to the obtained results, we can conclude the following advantages of the constructed hyperbolic multiresolution in A_α^p:

1. The levels of the multiresolution are finite dimensional, which makes easier to find a basis on every level, but in the same time the density condition remains valid.
2. We can compute the wavelet coefficients exactly measuring the values of the function f at the points of the set $\mathscr{A} = \bigcup_{k=0}^{\infty} \mathscr{A}_k \subset \mathbb{D}$. We can write exactly the projection operator $(P_n f,\ n \in \mathbb{N})$ on the n-th resolution level which is convergent in A_α^2 norm to f, and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc.
3. In same time: Note that $P_n f(z)$ is also the best approximant interpolation operator on the set the $\bigcup_{k=0}^{n} \mathscr{A}_k$ inside the unit circle for the analytic continuation of f.

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Chapter 10
Infinite Order Pseudo-Differential Operators

Stevan Pilipović and Bojan Prangoski

Abstract We present a class of global pseudo-differential operators of infinite order which are intrinsically related to the spaces of tempered ultradistributions as well as the symbolic calculus these operators enjoy. We also give the notion of hypoellipticity in this setting and consider the complex powers of non-negative hypoelliptic pseudo-differential operators. Finally, we give the construction of the heat parametrix and present an application of these results to semigroups whose infinitesimal generators are square roots of non-negative operators having hypoelliptic Weyl symbols.

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10.1 Introduction

Our aim in this expository article is to present a class of global symbols of infinite order, the corresponding pseudo-differential calculus as well as to give some applications. The motivation for our study comes from the classical theory of pseudo-differential operators of Shubin type as well as from mathematical physics where operators of infinite order appear; cf. [25, 30, 42].

The spaces of symbols and corresponding pseudo-differential operators involved in this approach were introduced by Prangoski in his thesis (see [37] for the symbolic calculus) and then extensively studied in several articles by himself and his coauthors. Similar symbol classes were considered by Cappiello [3, 4] for studying SG-hyperbolic problems of finite order. The definition of the symbols classes in our

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setting is linked to two Gevrey-type weight sequences A_p and M_p, $p \in \mathbb{N}$. The first one controls the smoothness, while the second one controls the growth at infinity of the symbols. These symbol classes are denoted by $\Gamma_{A_p,\rho}^{(M_p),\infty}$ and $\Gamma_{A_p,\rho}^{\{M_p\},\infty}$. The first one gives rise to operators acting continuously on Gelfand–Shilov-type spaces of Beurling class (i.e. of (M_p)-class) and the second one on Gelfand–Shilov-type spaces of Roumieu type (of $\{M_p\}$-class); we will employ $\Gamma_{A_p,\rho}^{*,\infty}$ as a common notation for both cases. Since the symbols are allowed to grow sub-exponentially (i.e. ultra-polynomially), the corresponding ΨDOs are of infinite order and the theory goes beyond the classical Weyl–Hörmander calculus.

We outline the content and the organisation of the article. Section 10.2 gives some basic background material about Gelfand–Shilov-type spaces of ultradifferentiable functions and ultradistributions $\mathscr{S}^*(\mathbb{R}^d)$ and $\mathscr{S}'^*(\mathbb{R}^d)$. We refer to [13, 18–20, 31–35, 38, 40, 43] although we know that this list does not contain a great number of papers devoted to studying these spaces of ultradistributions. In Sect. 10.3.1 and 10.3.2, we follow [35, 36] and collect and explain some useful properties of the symbol classes $\Gamma_{A_p,\rho}^{*,\infty}$ and the corresponding global pseudo-differential operators of infinite order $\mathrm{Op}_\tau(a)$ (the τ-quantisation of the symbol a) acting on $\mathscr{S}^*(\mathbb{R}^d)$ and $\mathscr{S}'^*(\mathbb{R}^d)$. When $\tau = 1/2$, that is, the Weyl quantisation, denoted as $a^w = \mathrm{Op}_{1/2}(a)$, the symbolic calculus is realised through the ring structure of the spaces of asymptotic expansions $FS_{A_p,\rho}^{*,\infty}$ where the product (i.e. the #-product) corresponds to the composition of operators. In our approach, we mainly follow the expositions of the global pseudo-differential operator calculus given in [32] and [41]: for the pseudo-differential calculus, we refer to [21–23] and especially, the four volumes of excellent and the most popular books of Hörmander [24].

Section 10.4 is devoted to hypoelliptic operators of infinite order, a subclass of $\Gamma_{A_p,\rho}^{*,\infty}$. Hypoellipticity in the Gevrey classes has been studied by several authors, see [21, 31, 41, 44] and the references therein. The functional setting in these papers allows the consideration of a class of general symbols $a(x,\xi)$ admitting exponential growth at infinity with respect to the covariable ξ. This was first noted in [44] and generalised in [15, 16] with applications to hyperbolic equations in Gevrey classes. The results of this section are published in [11, 12]. In Sect. 10.4.1, we consider a linear pseudo-differential equation of the form $a(x,D)u = v \in \mathscr{S}^*(\mathbb{R}^d)$ while in Sect. 10.4.2, we consider $a(x,D)u = f + F[u]$, where $a(x,\xi)$ is a symbol from our class of hypoelliptic symbols with $a(x,D)$ being its left quantisation (i.e. $a(x,D) = \mathrm{Op}_0(a)$), and F is a certain power series of an unknown ultradistribution u. Although the classical approach was followed, in our analysis we have had to use much more sophisticated methods in the construction of a parametrix and, in the semilinear case, the commutator technique already used in [1, 5–10] required more involved analysis as the operators in our setting are of infinite order and the nonlinear part is allowed to have sub-exponential growths. We also give several interesting instances of infinite order hypoelliptic operators (as well as several instances of $F[u]$ for the semilinear equation) where these results are applicable.

The last two sections are devoted to the complex powers of infinite order hypoelliptic operators and their applications. In Sect. 10.5.1, we recall some results on the realisations in $L^2(\mathbb{R}^d)$ of infinite order hypoelliptic operators as well as the semi-boundedness of the Weyl quantisation of positive hypoelliptic infinite order symbols (the proof is based on our results on the Anti-Wick quantisation [35]). These results, together with the additional assumption that a tends to infinity at infinity, implies that the spectrum of the closure $\overline{A}$ of the unbounded densely define operator $A = a^w_{|\mathscr{S}^*(\mathbb{R}^d)}$ on $L^2(\mathbb{R}^d)$ is given by a sequence of isolated eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$, each one with finite multiplicity. Section 10.5.3 contains the main theorem concerning the complex powers of the closure $\overline{A}$ in $L^2(\mathbb{R}^d)$ of $A = a^w_{|\mathscr{S}^*(\mathbb{R}^d)}$, where $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is a hypoelliptic symbol. Under some technical conditions on a, the theorem states that $\overline{A}^z$, where $\operatorname{Re} z > 0$, is given by a ΨDO modulo an ultra-smoothing operator (i.e. an operator that maps $\mathscr{S}'^*(\mathbb{R}^d)$ into $\mathscr{S}^*(\mathbb{R}^d)$ continuously; later on also called $*$-regularising). It furthermore gives estimates on the symbol of this ΨDO in terms of the original symbol a. The proof of this theorem can be found in [36]. The last section is devoted to the application of the complex powers to semigroups whose infinitesimal generators are of the form $-\overline{A}^{1/2}$ where $\overline{A}$ is non-negative, and it is the closure of $A = a^w_{|\mathscr{S}^*(\mathbb{R}^d)}$ with $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ being hypoelliptic. Under some assumptions on the symbol, the main result of this part states that such semigroup is comprised of a smooth family of ΨDOs modulo a smooth family of ultra-smoothing operators. Besides the theory on complex powers, the other key ingredient for this part is the construction of the heat parametrix. As this is of independent interest, we devote a separate subsection to it (Sect. 10.6.1). All of the results of this section are published in [36].

10.2 Preliminaries

In this section, we collect some basic background material on the Gelfand–Shilov-type spaces of ultradifferentiable functions and ultradistributions: $\mathscr{S}^*(\mathbb{R}^d)$ and $\mathscr{S}'^*(\mathbb{R}^d)$.

We start with a sequence of positive real numbers M_p, $p \in \mathbb{N}$, satisfying some of the conditions $(M.1)$, $(M.2)$, $(M.3)$, $(M.3)'$ and $(M.4)$ (cf. [27]); we always assume that such sequences satisfy $M_0 = M_1 = 1$. Recall,

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}, \quad p \in \mathbb{Z}_+$;

$(M.2)$ $M_p \leq c_0 H^p \min_{0\leq q\leq p}\{M_{p-q}M_q\}$, $p, q \in \mathbb{N}$, for some c_0, $H \geq 1$;

$(M.3)$ $\sum_{p=q+1}^{\infty} M_{p-1}/M_p \leq c_0 q M_q/M_{q+1}$, $q \in \mathbb{Z}_+$;

$(M.4)$ $\left(M_p/p!\right)^2 \leq M_{p-1}/(p-1)! \cdot M_{p+1}/(p+1)!$, for all $p \in \mathbb{Z}_+$.

In some assertions in the sequel, we could replace $(M.3)$ by the weaker assumption:

$(M.3)'$ $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$ (cf. [27]).

We observe moreover that $(M.4)$ implies $(M.1)$ and, by [33, Prop. 1.1], the condition $(M.3)$ on (M_p) implies $(M.4)$ for an equivalent sequence to M_p.

Let M_p and $\tilde{M}_p$, $p \in \mathbb{N}$, be two sequences of positive numbers. Then $M_p \subset \tilde{M}_p$ means that there are $C, L > 0$ such that $M_p \leq CL^p\tilde{M}_p$, $\forall p \in \mathbb{N}$, and $M_p \prec \tilde{M}_p$ means that this inequality holds for each $L > 0$ and a corresponding $C = C_L >$

0. Obviously, without losing generality, we can assume that the constant H from $(M.2)$ is the same for the sequences M_p and $\tilde{M}_p$. For a multi-index $\alpha \in \mathbb{N}^d$, $D^\alpha = D_1^{\alpha_1}...D_d^{\alpha_d}$, where $D_j^{\alpha_j} = i^{-\alpha_j}\partial^{\alpha_j}/\partial x_j{}^{\alpha_j}$; moreover, M_α denotes $M_{|\alpha|}$, $|\alpha| = \alpha_1 + ... + \alpha_d$. We use the notation ([27, Sect. 3]), $m_p = M_p/M_{p-1}$, $p \in \mathbb{Z}_+$, and if M_p satisfies $(M.1)$ and $m_p \to \infty$ (the latter always holds when M_p satisfies $(M.3)'$), then its associated function is defined by $M(\rho) = \sup_{p\in\mathbb{N}} \ln_+ \rho^p/M_p$, $\rho > 0$. It is a non-negative, continuous, monotonically increasing function, vanishes for sufficiently small $\rho > 0$ and increases more rapidly than $\ln \rho^n$ as $\rho \to \infty$, for any $n \in \mathbb{N}$. When $M_p = p!^s$, with $s > 0$, we have $c_1\rho^{1/s} \le M(\rho) \le c_2\rho^{1/s}$, for some $c_1, c_2 > 0$ and large ρ.

Let K be a regular compact subset of $\mathbb{R}^d$ (i.e. $\overline{\operatorname{int} K} = K$). Then, for $h > 0$,

$$\mathscr{E}^{\{M_p\},h}(K) = \{\varphi \in \mathscr{C}^\infty(K) |\ \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in K} |D^\alpha\varphi(x)|/(h^\alpha M_\alpha) < \infty\},$$

where $\mathscr{C}^\infty(K)$ stands for the space of all smooth functions on int K whose all partial derivatives extend to continuous functions on K. Then $\mathscr{E}^{\{M_p\},h}(K)$ is a Banach space (from now on abbreviated to (B)-space) and $\mathscr{D}_K^{\{M_p\},h}$ denotes its subspace of all smooth functions supported by K. We define as locally convex spaces (abbreviated to l.c.s.) $\mathscr{E}^{(M_p)}(\mathbb{R}^d)$, $\mathscr{E}^{\{M_p\}}(\mathbb{R}^d)$, $\mathscr{D}^{(M_p)}(\mathbb{R}^d)$, $\mathscr{D}^{\{M_p\}}(\mathbb{R}^d)$ and their strong duals, the corresponding spaces of ultradistributions of Beurling and Roumieu type; we refer to [27–29] for their properties.

We denote by $\mathfrak{R}$ the set of all positive sequences which monotonically increase to infinity. One can define a partial order on $\mathfrak{R}$ by $(r_p) \le (k_p)$ if $r_p \le k_p$, $\forall p \in \mathbb{Z}_+$, and with it $(\mathfrak{R}, \le)$ becomes a directed set.

Let $(r_p) \in \mathfrak{R}$, and $N_p = M_p \prod_{j=1}^p r_j$, $p \in \mathbb{N}$.[1] This sequence satisfies $(M.1)$ and $(M.3)'$ when M_p does so and its associated function is denoted by $N_{r_p}(\rho)$: $N_{r_p}(\rho) = \sup_{p\in\mathbb{N}} \ln_+ \rho^p/(M_p \prod_{j=1}^p r_j)$, $\rho > 0$. Note that for $(r_p) \in \mathfrak{R}$ and $k > 0$ there is $\rho_0 > 0$ such that $N_{r_p}(\rho) \le M(k\rho)$, for $\rho > \rho_0$.

A measurable function f on $\mathbb{R}^d$ is said to have ultrapolynomial growth of class (M_p) (resp. of class $\{M_p\}$) if $\|e^{-M(h|\cdot|)} f\|_{L^\infty(\mathbb{R}^d)} < \infty$ for some $h > 0$ (resp. for every $h > 0$). We have the following equivalent description of continuous functions of ultrapolynomial growth of class $\{M_p\}$. Let $B \subseteq \mathscr{C}(\mathbb{R}^d)$. Then (cf. [36, Lemma 2.1])): For every $h > 0$, there exists $C > 0$ such that $|f(x)| \le Ce^{M(h|x|)}$, for all $x \in \mathbb{R}^d$, $f \in B$, if and only if there exist $(r_p) \in \mathfrak{R}$ and $C > 0$ such that $|f(x)| \le Ce^{N_{r_p}(|x|)}$, for all $x \in \mathbb{R}^d$, $f \in B$.

An entire function $P(z) = \sum_{\alpha\in\mathbb{N}^d} c_\alpha z^\alpha$, $z \in \mathbb{C}^d$, is an ultrapolynomial of class (M_p) (resp. of class $\{M_p\}$), whenever the coefficients c_α satisfy the estimate $|c_\alpha| \le CL^{|\alpha|}/M_\alpha$, $\alpha \in \mathbb{N}^d$, for some $L > 0$ and $C > 0$ (resp. for every $L > 0$ and some $C = C(L) > 0$). The corresponding operator $P(D) = \sum_\alpha c_\alpha D^\alpha$ is an ultradifferential operator of class (M_p) (resp. of class $\{M_p\}$) and, when M_p satisfies $(M.2)$, it acts

[1] Here, and throughout the rest of the article, we use the principle of vacuous (empty) product; thus, $N_0 = \prod_{j=1}^0 r_j = 1$.

continuously on $\mathscr{E}^{(M_p)}(\mathbb{R}^d)$ and $\mathscr{D}^{(M_p)}(\mathbb{R}^d)$ (resp. on $\mathscr{E}^{\{M_p\}}(\mathbb{R}^d)$ and $\mathscr{D}^{\{M_p\}}(\mathbb{R}^d)$) and the corresponding ultradistribution spaces.

Let M_p satisfy $(M.1)$, $(M.3)'$ and $m > 0$. Then $\mathscr{S}_\infty^{M_p,m}(\mathbb{R}^d)$ denotes the (B)-space of all $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d)$ for which the norm $\sup_{\alpha\in\mathbb{N}^d} m^{|\alpha|}\|e^{M(m|\cdot|)}D^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)}/M_\alpha$ is finite. The spaces of sub-exponentially decreasing ultradifferentiable functions of Beurling and Roumieu type are defined as

$$\mathscr{S}^{(M_p)}(\mathbb{R}^d) = \varprojlim_{m\to\infty} \mathscr{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right) \quad \text{and} \quad \mathscr{S}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim_{m\to 0} \mathscr{S}_\infty^{M_p,m}\left(\mathbb{R}^d\right),$$

respectively. Their strong duals $\mathscr{S}'^{(M_p)}(\mathbb{R}^d)$ and $\mathscr{S}'^{\{M_p\}}(\mathbb{R}^d)$ are the spaces of tempered ultradistributions of Beurling and Roumieu type, respectively. When $M_p = p!^s$, $s > 1$, $\mathscr{S}^{\{M_p\}}(\mathbb{R}^d)$ is just the Gelfand–Shilov space $\mathscr{S}_s^s(\mathbb{R}^d)$ [32]. If M_p satisfies $(M.2)$, then the ultradifferential operators of class $*$ act continuously on $\mathscr{S}^*(\mathbb{R}^d)$ and $\mathscr{S}'^*(\mathbb{R}^d)$; these spaces are nuclear, and the Fourier transform is a topological isomorphism on them. We refer to [13, 34] for the topological properties of $\mathscr{S}^*(\mathbb{R}^d)$ and $\mathscr{S}'^*(\mathbb{R}^d)$. Here we recall that, when M_p satisfies $(M.2)$, the space $\mathscr{S}^{\{M_p\}}(\mathbb{R}^d)$ is topologically isomorphic to

$$\varprojlim_{(r_p)\in\mathfrak{R}} \mathscr{S}_\infty^{M_p,(r_p)}(\mathbb{R}^d),$$

where the projective limit is taken with respect to the partial order on $\mathfrak{R}$ defined above and $\mathscr{S}_\infty^{M_p,(r_p)}(\mathbb{R}^d)$ is the (B)-space of all $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d)$ for which the norm $\sup_{\alpha\in\mathbb{N}^d} \|e^{N_{r_p}(|\cdot|)}D^\alpha\varphi\|_{L^\infty(\mathbb{R}^d)}/(M_\alpha \prod_{j=1}^{|\alpha|} r_j)$ is finite.

We end this section with a few notations from functions analysis. When X and Y are two locally convex spaces, we denote by $\mathscr{L}(X, Y)$ the space of all continuous linear operators from X to Y; if $X = Y$ we will often abbreviate notations and simply write $\mathscr{L}(X)$ instead of $\mathscr{L}(X, X)$. We write $\mathscr{L}_b(X, Y)$ for the space $\mathscr{L}(X, Y)$ equipped with the topology of bounded convergence; similarly, $\mathscr{L}_p(X, Y)$ and $\mathscr{L}_\sigma(X, Y)$ stand for the space $\mathscr{L}(X, Y)$ equipped with the topology of precompact and simple convergence respectively. If $a, b \in \mathbb{R}$ and $0 \le k \le \infty$, $\mathscr{C}^k((a, b); X)$ stands for the space of all k-times continuously differentiable functions on (a, b) with values in E while $\mathscr{C}^k((a, b]; E)$ for the space of those on $(a, b]$ where the derivatives at b are to be understood as left derivatives. We use analogous notations when we consider functions over $[a, b)$ or $[a, b]$.

10.3 Global ΨDOs of Infinite Order

We collect in this section the basic properties of the classes of infinite order pseudo-differential operators that we shall consider in the article. In Sects. 10.3.1 and 10.3.2, we collect some useful facts about their symbolic calculus and the sharp product; we excerpt only the most important facts and refer to [11, 37] and [36, Sects. 2 and 3] for a complete account.

10.3.1 Spaces of Symbols. Symbolic Calculus

In the sequel, A_p and M_p will denote two weight sequences of positive numbers for which we assume:

M_p satisfies $(M.1)$, $(M.2)$ and $(M.3)$, and

A_p satisfies $(M.1)$, $(M.2)$, $(M.3)'$ and $(M.4)$.

We assume that $A_p \subset M_p$. Let $\rho_0 = \inf\{\rho \in \mathbb{R}_+ |\, A_p \subset M_p^{\rho}\}$; clearly $0 < \rho_0 \leq 1$. In the sequel, ρ will be a fixed number satisfying $\rho_0 \leq \rho \leq 1$, if the infimum can be reached, or, otherwise $\rho_0 < \rho \leq 1$.

Let $h, m > 0$. The basic ingredient is the (B)-space $\Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m)$ consisting of all $a \in \mathscr{C}^{\infty}(\mathbb{R}^{2d})$ for which the norm

$$\sup_{\alpha,\beta\in\mathbb{N}^d}\ \sup_{(x,\xi)\in\mathbb{R}^{2d}} \frac{\left|D^{\alpha}_{\xi} D^{\beta}_{x} a(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha|+|\beta|} A_{\alpha} A_{\beta}}.$$

is finite (see [37]) and then we define as l.c.s.

$$\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m) = \varprojlim_{h\to 0} \Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m); \quad \Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d})\ \varinjlim_{m\to\infty} \Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m);$$

$$\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h) = \varprojlim_{m\to 0} \Gamma^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h, m); \quad \Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}) = \varinjlim_{h\to\infty} \Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h).$$

The spaces $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m)$ and $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h)$ are (F)-spaces while $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ are barrelled and bornological.

Let $\tau \in \mathbb{R}$ and $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$. Then the τ-quantisation of a is the operator $\operatorname{Op}_{\tau}(a)$, continuous on $\mathscr{S}^*(\mathbb{R}^d)$ given by the iterated integral (first Fourier, then inverse Fourier transform):

$$\left(\operatorname{Op}_{\tau}(a)u\right)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy d\xi.$$

Furthermore, $\operatorname{Op}_{\tau}(a)$ naturally extends to a continuous operator on $\mathscr{S}'^*(\mathbb{R}^d)$. More precisely, we have the following result.

Proposition 10.1 ([36, Proposition 3.1]) *For each* $\tau \in \mathbb{R}$, *the bilinear mapping* $(a, \varphi) \mapsto \operatorname{Op}_{\tau}(a)\varphi$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \times \mathscr{S}^*(\mathbb{R}^d) \to \mathscr{S}^*(\mathbb{R}^d)$, *is hypocontinuous and it extends to the hypocontinuous bilinear mapping* $(a, T) \mapsto \operatorname{Op}_{\tau}(a)T$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \times$

$\mathscr{S}'^*(\mathbb{R}^d) \to \mathscr{S}'^*(\mathbb{R}^d)$. *The mappings* $a \mapsto \operatorname{Op}_\tau(a)$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \to \mathscr{L}_b(\mathscr{S}^*(\mathbb{R}^d))$, $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \to \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d))$ *are continuous.*

Since we frequently use the Weyl quantisation (when $\tau = 1/2$), we use a^w as a shorthand of $\operatorname{Op}_{1/2}(a)$. As standard, we denote by $a(x, D)$ the left quantisation (when $\tau = 0$) of a.

Next we define the spaces of asymptotic expansions corresponding to the symbol classes $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$. Let $t \geq 0$, $B \geq 0$ and $h, m > 0$. We put

$$Q_t = \left\{(x,\xi) \in \mathbb{R}^{2d} | \, \langle x\rangle < t, \langle \xi \rangle < t\right\} \text{ and } Q^c_t = \mathbb{R}^{2d} \backslash Q_t.$$

If $0 \leq t \leq 1$, then $Q_t = \emptyset$ and $Q^c_t = \mathbb{R}^{2d}$. Then [37], $FS^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, h, m)$ is the vector space of formal sums $\sum_{j=0}^{\infty} a_j(x,\xi)$ where $a_j \in \mathscr{C}^\infty(\operatorname{int} Q^c_{Bm_j})$ and $D^\alpha_\xi D^\beta_x a_j(x,\xi)$ can be extended to a continuous function on $Q^c_{Bm_j}$ for all $\alpha, \beta \in \mathbb{N}^d$ and

$$\sup_{j\in\mathbb{N}} \sup_{\alpha,\beta} \sup_{(x,\xi)\in Q^c_{Bm_j}} \frac{\left|D^\alpha_\xi D^\beta_x a_j(x,\xi)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2j\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha|+|\beta|+2j} A_\alpha A_\beta A_j A_j} < \infty.$$

Here, and throughout the rest of the article, we use the convention $m_0 = 0$, and thus $Q^c_{Bm_0} = \mathbb{R}^{2d}$. With this norm, $FS^{M_p,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}; B, h, m\right)$ becomes a (B)-space. As l.c.s., we define [37]

$$\begin{aligned}
FS^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, m) &= \varprojlim_{h\to 0} FS^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, h, m),\\
FS^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B) &= \varinjlim_{m\to \infty} FS^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, m),\\
FS^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, h) &= \varprojlim_{m\to 0} FS^{M_p,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, h, m),\\
FS^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B) &= \varinjlim_{h\to \infty} FS^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, h).
\end{aligned}$$

One easily verifies that $FS^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, m)$ and $FS^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, h)$ are (F)-spaces, and $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ is barrelled and bornological. Furthermore, the inclusion mapping $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ defined by

$$a \mapsto \sum_j a_j, \text{ where } a_0 = a, a_j = 0,\, j \geq 1,$$

is continuous. We call this inclusion the canonical one. For $B_1 \leq B_2$, the mapping $\sum_j p_j \mapsto \sum_j p_{j|_{Q^c_{B_2 m_j}}}$, $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B_1) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B_2)$, called canonical, is continuous.

Let $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) = \varinjlim_{B\to\infty} FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$, where the inductive limit is taken in an algebraic sense and the linking mappings are the canonical ones described above.

If $\sum_j a_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ and $n \in \mathbb{N}$, then $(\sum_j a_j)_n$ denotes the function $a_n \in \mathscr{C}^\infty(Q^c_{Bm_n})$, while $(\sum_j a_j)_{<n}$ denotes the function $\sum_{j=0}^{n-1} a_j \in \mathscr{C}^\infty(Q^c_{Bm_{n-1}})$. Furthermore, **1** denotes the element $\sum_j a_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ given by $a_0(x,\xi) = 1$ and $a_j(x,\xi) = 0$, $j \in \mathbb{Z}_+$.
Recall ([37, Definition 3]) that sums, $\sum_{j\in\mathbb{N}} a_j$, $\sum_{j\in\mathbb{N}} b_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, are said to be equivalent, in notation $\sum_{j\in\mathbb{N}} a_j \sim \sum_{j\in\mathbb{N}} b_j$, if there exist $m > 0$ and $B > 0$ (resp. there exist $h > 0$ and $B > 0$), such that for every $h > 0$ (resp. for every $m > 0$),

$$\sup_{n\in\mathbb{Z}_+}\sup_{\alpha,\beta}\sup_{(x,\xi)\in Q^c_{Bm_n}} \frac{\left|D^\alpha_\xi D^\beta_x \sum_{j<n}\left(a_j(x,\xi) - b_j(x,\xi)\right)\right| \langle (x,\xi)\rangle^{\rho|\alpha|+\rho|\beta|+2n\rho}}{h^{|\alpha|+|\beta|+2n} A_\alpha A_\beta A_n A_n e^{M(m|\xi|)} e^{M(m|x|)}} < \infty.$$

Let Λ be an index set and $\{f_\lambda \,|\, \lambda \in \Lambda\}$ be a set of positive continuous functions on $\mathbb{R}^{2d}$ each with ultrapolynomial growth of class $*$. Then a set $U^{(\Lambda)} = \left\{\sum_j a^{(\lambda)}_j \,\middle|\, \lambda \in \Lambda\right\} \subseteq FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B')$ is subordinated to $\{f_\lambda \,|\, \lambda \in \Lambda\}$ in $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, $U^{(\Lambda)} \precsim \{f_\lambda \,|\, \lambda \in \Lambda\}$, if the following estimate holds: there exists $B \geq B'$ such that for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\sup_{\lambda\in\Lambda}\sup_{j\in\mathbb{N}}\sup_{\alpha\in\mathbb{N}^{2d}}\sup_{w\in Q^c_{Bm_j}} \frac{\left|D^\alpha_w a^{(\lambda)}_j(w)\right| \langle w\rangle^{\rho(|\alpha|+2j)}}{h^{|\alpha|+2j} A_{|\alpha|+2j} f_\lambda(w)} \leq C.$$

(In the sequel, $w = (x,\xi) \in \mathbb{R}^{2d}$.) When $f_\lambda = f$, $\forall\lambda \in \Lambda$, we write $U \precsim f$, and then say that U is subordinated to f. Let $U \subseteq FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B_1)$ such that $U \precsim f$. Then, there exists $B \geq B_1$ such that the image of U under the canonical mapping $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B_1) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ is a bounded subset of $FS^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, m)$ for some $m > 0$ (resp. a bounded subset of $FS^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B, h)$ for some $h > 0$).

Given $U \subseteq FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B_1)$ with $U \precsim f$, we say that a bounded set V in $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m)$ for some $m > 0$ (resp. in $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h)$ for some $h > 0$) is subordinated to U under f, in notations $V \precsim_f U$, if there exists a surjective mapping $\Sigma : U \to V$ such that the following estimate holds: there exists $B \geq B_1$ such that for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that for all $\sum_j a_j \in U$ and the corresponding $\Sigma(\sum_j a_j) = a \in V$,

$$\sup_{n\in\mathbb{Z}_+}\sup_{\alpha\in\mathbb{N}^{2d}}\sup_{w\in Q^c_{Bm_n}} \frac{\left|D^\alpha_w\left(a(w) - \sum_{j<n} a_j(w)\right)\right| \langle w\rangle^{\rho(|\alpha|+2n)}}{h^{|\alpha|+2n} A_{|\alpha|+2n} f(w)} \leq C.$$

Let $V \precsim_f U$ and $\tilde{V}$ be the image of V under the canonical inclusion $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};0)$, $a \mapsto a + \sum_{j\in\mathbb{Z}_+} 0$. Then the above estimate for $n = 1$ together with the boundedness of V in $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d};m)$ for some $m > 0$ (resp. in $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d};h)$ for some $h > 0$) and the continuity and positivity of f, imply that $\tilde{V} \precsim f$ in $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};0)$. In such a case, we write $V \precsim f$. This estimate also implies $\Sigma(\sum_j a_j) \sim \sum_j a_j$.

If one starts with a formal sum $\sum_j a_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B)$, there always exists a symbol which is equivalent to $\sum_j a_j$. In fact, the construction of such symbols can be made uniform in the following sense: given $U \subseteq FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B)$ such that $U \precsim f$ there always exists $V \precsim_f U$. To see this, we proceed as follows. Let $\psi \in \mathscr{D}^{(A_p)}(\mathbb{R}^d)$ in the (M_p) case and $\psi \in \mathscr{D}^{\{A_p\}}(\mathbb{R}^d)$ in the $\{M_p\}$ case, respectively, be such that $0 \le \psi \le 1$, $\psi(\xi) = 1$ when $\langle\xi\rangle \le 2$ and $\psi(\xi) = 0$ when $\langle\xi\rangle \ge 3$. Set $\chi(x,\xi) = \psi(x)\psi(\xi)$, $\chi_{n,R}(w) = \chi(w/(Rm_n))$ for $n \in \mathbb{Z}_+$ and $R > 0$ and put $\chi_{0,R}(w) = 0$. For each $\sum_j a_j \in U$, let $R(\sum_j a_j)(w) = \sum_{j=0}^{\infty}(1-\chi_{j,R}(w))a_j(w)$. If $R > B$, this is a well defined smooth function on $\mathbb{R}^{2d}$, since the series is locally finite. We have the following result.

Proposition 10.2 ([36, Proposition 3.3]) *Let $U = \left\{\sum_j a_j^{(\lambda)} \,\middle|\, \lambda \in \Lambda\right\}$ be a subset of $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B')$ that is subordinated to $\{f_\lambda \,|\, \lambda\in\Lambda\}$ in $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$. There exists $R_0 > B'$ such that for each $R \ge R_0$, $U_R = \left\{R(\sum_j a_j^{(\lambda)}) \,\middle|\, \lambda\in\Lambda\right\} \subseteq \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ and the following estimate holds: there exists $B = B(R) \ge B'$ such that for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that*

$$\sup_{\lambda\in\Lambda}\sup_{n\in\mathbb{Z}_+}\sup_{\alpha\in\mathbb{N}^{2d}}\sup_{w\in Q^c_{Bm_n}} \frac{\left|D^\alpha_w\left(R(\sum_j a_j^{(\lambda)})(w) - \sum_{j<n} a_j^{(\lambda)}(w)\right)\right| \langle w\rangle^{\rho(|\alpha|+2n)}}{h^{|\alpha|+2n}A_{|\alpha|+2n}f_\lambda(w)} \le C.$$

If in addition $f_\lambda = f$, $\forall\lambda\in\Lambda$, then U_R is bounded in $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d};m)$ for some $m > 0$ (resp. bounded in $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d};h)$ for some $h > 0$) and hence $U_R \precsim_f U$.

We say that this U_R is canonically obtained from U by $\{\chi_{n,R}\}_{n\in\mathbb{N}}$; in this case, the mapping $\Sigma : U \to U_R$ is nothing else but $\sum_j a_j \mapsto R(\sum_j a_j)$.

If two symbols are equivalent, then the operators they define only differ by an operator in $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d))$ (see [37, Theorem 3]). In fact, if we start with a set of symbols which is bounded in $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d};\tilde{m})$ (resp. in $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d};\tilde{h})$) and they are all "uniformly equivalent" to 0 then they are in fact an equicontinuous subset of $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d))$. The following result makes this precise.

Proposition 10.3 ([36, Proposition 3.4]) *Let V be a bounded subset of $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d};\tilde{m})$ for some $\tilde{m} > 0$ (resp. of $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d};\tilde{h})$ for some $\tilde{h} > 0$). Assume that there exist $B, m > 0$ such that for every $h > 0$, there exists $C > 0$ (resp. there exist $B, h > 0$ such that for every $m > 0$ there exists $C > 0$) such that*

$$\sup_{a\in V}\sup_{n\in\mathbb{Z}_+}\sup_{\alpha\in\mathbb{N}^{2d}}\sup_{w\in Q^c_{Bm_n}}\frac{\left|D^\alpha_w a(w)\right|\langle w\rangle^{\rho(|\alpha|+2n)}}{h^{|\alpha|+2n}A_{|\alpha|+2n}e^{M(m|w|)}}\leq C.$$

Then, $\{\mathrm{Op}_\tau(a)|\, a\in U\}$ *is an equicontinuous subset of* $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d),\mathscr{S}^*(\mathbb{R}^d))$ *for each* $\tau\in\mathbb{R}$*.*

In particular, the proposition states that it does not matter how we produce a symbol out of $\sum_j a_j$ as long as it is equivalent to it since the difference between any two of them will always be an element of $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d),\mathscr{S}^*(\mathbb{R}^d))$. To simplify notation, we will often call the operators in $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d),\mathscr{S}^*(\mathbb{R}^d))$ $*$-regularising.

The next result states that one can always change the quantisation modulo a $*$-regularising operator; it also gives the asymptotic expansion of the new symbol.

Proposition 10.4 ([36, Proposition 3.5]) *Let* $U_1\subseteq FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B)$ *be such that* $U_1\precsim f$*, for some continuous positive function* f *with ultrapolynomial growth of class* $*$ *and let* $\tau,\tau_1\in\mathbb{R}$*. For each* $\sum_j a_j\in U_1$ *and* $j\in\mathbb{N}$*, define*

$$p_{j,a}(x,\xi)=\sum_{k+|\beta|=j}\frac{(\tau_1-\tau)^{|\beta|}}{\beta!}\partial^\beta_\xi D^\beta_x a_k(x,\xi),\ \ (x,\xi)\in Q^c_{Bm_j}.$$

Then, $U=\left\{\sum_j p_{j,a}\middle|\,\sum_j a_j\in U_1\right\}$ *is a subset of* $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B)$ *and* $U\precsim f$*. There exists* $R>0$*, which can be chosen arbitrarily large, such that*

$$\{\mathrm{Op}_{\tau_1}(R(\sum_j a_j))-\mathrm{Op}_\tau(R(\sum_j p_{j,a}))|\sum_j a_j\in U_1\}$$

is an equicontinuous subset of $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d),\mathscr{S}^*(\mathbb{R}^d))$*. Moreover,* $\{R(\sum_j a_j)|\,\sum_j a_j\in U_1\}\precsim_f U_1$ *and* $\{R(\sum_j p_{j,a})|\,\sum_j a_j\in U_1\}\precsim_f U$*.*

10.3.2 Weyl Quantisation. The Ring Structure of $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B)$

As in the distributional setting, in the infinite order case the composition of two pseudo-differential operators is again a pseudo-differential operator with symbol in the same class, but there is always an additional $*$-regularising operator; see [37, Theorem 7] (in the quoted result this is proved for the 0-quantisation, but because of Proposition 10.4 it holds for any quantisation). We recall in this subsection results from [36] about the Weyl quantisation of symbols (when $\tau=1/2$) and the composition of two such operators. The first line of discourse is to define the #-product on the spaces of asymptotic expansions: it is the operation that corresponds to the composition of operators on the symbolic level; that is, if a^w and b^w are two Weyl quantisations than their #-product gives the asymptotic expansion of the composition (as stated before, a^w stands for the Weyl quantisation $\mathrm{Op}_{1/2}(a)$).

For $\sum_j a_j, \sum_j b_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$, we define their sharp product, denoted as $\sum_j a_j \# \sum_j b_j$, via the formal series $\sum_j c_j = \sum_j a_j \# \sum_j b_j$ where

$$c_j(x,\xi) = \sum_{s+k+l=j} \sum_{|\alpha+\beta|=l} \frac{(-1)^{|\beta|}}{\alpha!\beta!2^l} \partial^\alpha_\xi D^\beta_x a_s(x,\xi) \partial^\beta_\xi D^\alpha_x b_k(x,\xi), \ \ (x,\xi) \in Q^c_{Bm_j};$$

a straightforward computation verifies that $\sum_j c_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$. If $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, then $a\# \sum_j b_j$ will denote the # product of the image of a under the canonical inclusion $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ and $\sum_j b_j$. The same convention applies if $b \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ or if both $a, b \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$.

Note, if $\sum_j a_j, \sum_j b_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ and $\sum_j c_j = \sum_j a_j \# \sum_j b_j$, then $\sum_j \overline{c_j} = \sum_j \overline{b_j} \# \sum_j \overline{a_j}$. In particular, if a_j and b_j are real valued for all $j \in \mathbb{N}$ and $\sum_j a_j \# \sum_j b_j = \sum_j b_j \# \sum_j a_j$, then c_j are real valued for all $j \in \mathbb{N}$.

As we mentioned before, there is always an additional $*$-regularising operator that appears when one composes two pseudo-differential operators. The next result proves that one can always control the resulting set of $*$-regularising operators when one composes the Weyl quantisations of two bounded sets of symbols; in fact, this set of $*$-regularising operators is always equicontinuous in $\mathcal{L}(\mathcal{S}'^*(\mathbb{R}^d), \mathcal{S}^*(\mathbb{R}^d))$.

Theorem 10.1 ([36, Theorem 4.2, Corollary 4.3]) *Let* $U_1, U_2 \subseteq FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ *be such that* $U_1 \precsim f_1$ *and* $U_2 \precsim f_2$ *in* $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ *for some continuous positive functions* f_1 *and* f_2 *with ultrapolynomial growth of class* $*$.
(*a*) *The following statements hold true.*

(i) $U_1 \# U_2 \precsim f_1 f_2$ *in* $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$.
(ii) Let $V_k \precsim_{f_k} U_k$, *with* $\Sigma_k : U_k \to V_k$ *the surjective mapping,* $k = 1, 2$. *There exists* $R > 0$, *which can be chosen arbitrarily large, such that*

$$\left\{ \mathrm{Op}_{1/2}\left(\Sigma_1(\sum_j a_j)\right) \mathrm{Op}_{1/2}\left(\Sigma_2(\sum_j b_j)\right) - \mathrm{Op}_{1/2}\left(R(\sum_j a_j \# \sum_j b_j)\right) \Big| \sum_j a_j \in U_1, \sum_j b_j \in U_2 \right\}$$

is an equicontinuous subset of $\mathcal{L}(\mathcal{S}'^*(\mathbb{R}^d), \mathcal{S}^*(\mathbb{R}^d))$ *and*

$$\left\{ R(\sum_j a_j \# \sum_j b_j) \Big| \sum_j a_j \in U_1, \sum_j b_j \in U_2 \right\} \precsim_{f_1 f_2} U_1 \# U_2. \quad (10.3.1)$$

(b) For $\sum_j a_j \in U_1$ *and* $\sum_j b_j \in U_2$*, denote* $\sum_j c_{j,a,b} = \sum_j a_j \# \sum_j b_j \in U_1 \# U_2$*. Then, there exists* $R > 0$*, which can be chosen arbitrarily large, such that*

$$\left\{ a^w b^w - c^w \,\middle|\, a = R(\sum_j a_j),\ b = R(\sum_j b_j),\ c = R(\sum_j c_{j,a,b}) \right\}$$

is an equicontinuous subset of $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d))$ *and* (10.3.1) *holds.*

Part (b) is applicable when U_1 and U_2 are bounded subsets of $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m)$ for some $m > 0$ (resp. of $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h)$ for some $h > 0$). In this case, part $b)$ reads: there exists $R > 0$, which can be chosen arbitrary large, such that $\{a^w b^w - \mathrm{Op}_{1/2}(R(a\#b)) | \, a \in U_1,\ b \in U_2\}$ is equicontinuous $*$-regularising set and $\{R(a\#b)| \, a \in U_1,\ b \in U_2\}$ is bounded in $\Gamma^{(M_p),\infty}_{A_p,\rho}(\mathbb{R}^{2d}; m)$ for some $m > 0$ (resp. of $\Gamma^{\{M_p\},\infty}_{A_p,\rho}(\mathbb{R}^{2d}; h)$ for some $h > 0$ (this is the fact we hinted before the statement of the theorem).

Now we have the main assertion of this section.

Theorem 10.2 ([36, Proposition 4.5]) *For each* $B \geq 0$*,* $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ *is a ring with the pointwise addition and multiplication given by #. The #-identity is given by* **1***. Moreover, the multiplication* $\# : FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B) \times FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; B)$ *is hypocontinuous.*

10.4 Hypoelliptic Operators of Infinite Order

In this section, we will consider hypoelliptic pseudo-differential operators of infinite order and then the corresponding linear and semilinear pseudo-differential equations. We will present our results implying in both cases the Gevrey hypoellipticity of the solutions. This will be done in Sects. 10.4.1 and 10.4.2 after the introduction of the hypoelliptic class of symbols within $\Gamma^{*,\infty}_{A_p,\rho}$.

Definition 10.1 ([11, Definition 1.1]) Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$. We say that a is $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic (or hypoelliptic) if

(i) there exists $B > 0$ such that there are $c, m > 0$ (resp. for every $m > 0$, there is $c > 0$) such that

$$|a(x,\xi)| \geq c e^{-M(m|x|)-M(m|\xi|)}, \quad (x,\xi) \in Q^c_B, \tag{10.4.1}$$

(ii) there exists $B > 0$ such that for every $h > 0$, there is $C > 0$ (resp. there are $h, C > 0$) such that

$$\left| D^\alpha_\xi D^\beta_x a(x,\xi) \right| \leq C \frac{h^{|\alpha|+|\beta|} |a(x,\xi)| A_\alpha A_\beta}{\langle (x,\xi) \rangle^{\rho(|\alpha|+|\beta|)}},\ \alpha, \beta \in \mathbb{N}^d,\ (x,\xi) \in Q^c_B. \tag{10.4.2}$$

The importance of $\Gamma^{*,\infty}_{A_p,\rho}$-hypoellipticity lies in the possibility to construct parametrices.

Proposition 10.5 ([36, Proposition 5.2]) *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be hypoelliptic. Define $q_0(w) = a(w)^{-1}$ on Q^c_B and inductively, for $j \in \mathbb{Z}_+$,*

$$q_j(x,\xi) = -q_0(x,\xi)\sum_{s=1}^{j}\sum_{|\alpha+\beta|=s}\frac{(-1)^{|\beta|}}{\alpha!\beta!2^s}\partial^\alpha_\xi D^\beta_x q_{j-s}(x,\xi)\partial^\beta_\xi D^\alpha_x a(x,\xi),\ \ (x,\xi)\in Q^c_B.$$

Then, for every $h > 0$ there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left|D^\alpha_w q_j(w)\right| \le C\frac{h^{|\alpha|+2j}A_{|\alpha|+2j}}{|a(w)|\langle w\rangle^{\rho(|\alpha|+2j)}},\ \ w\in Q^c_B,\ \alpha\in\mathbb{N}^{2d},\ j\in\mathbb{N}. \tag{10.4.3}$$

If $B \le 1$, then $(\sum_j q_j)\#a = \mathbf{1}$ in $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};0)$. If $B > 1$, one can extend q_0 to an element of $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ modifying it on $Q_{B'}\backslash Q_B$, for $B' > B$. In this case, $\sum_j q_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B')$, $((\sum_j q_j)\#a)_k = 0$ on $Q^c_{B'}$, $\forall k\in\mathbb{Z}_+$, and $((\sum_j q_j)\#a)_0 - 1 = q_0 a - 1$ belongs to $\mathscr{D}^{(A_p)}(\mathbb{R}^{2d})$ (resp. $\mathscr{D}^{\{A_p\}}(\mathbb{R}^{2d})$).

In particular, for $q \sim \sum_j q_j$ there exists $*$-regularising operator T such that $q^w a^w = \mathrm{Id} + T$.

A similar proposition holds for $\tilde{q} \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ which satisfies $a^w\tilde{q}^w = \mathrm{Id} + \tilde{T}$ with $\tilde{T} \in \mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d))$. Thus, we have that we can use the left parametrix q^w as a right one as well, i.e. there exists $T_1, T_2 \in \mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d))$ such that $q^w a^w = \mathrm{Id} + T_1$ and $a^w q^w = \mathrm{Id} + T_2$.

For hypoelliptic $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, we can construct a parametrix q out of $\sum_j q_j \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};B')$ in a specific way. Namely, applying part b) of Theorem 10.1 to $(\sum_j q_j)\#a$ together with (10.4.3) and Proposition 10.2, we conclude the existence of $R > 0$ and a $*$-regularising operator T such that $q^w a^w = \mathrm{Id} + T$, where $q = R(\sum_j q_j) \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ satisfies the following conditions: there exist $B'' \ge B'$ and $c'', C'' > 0$ such that

$$c''/|a(w)| \le |q(w)| \le C''/|a(w)|,\ \ \forall w\in Q^c_{B''}, \tag{10.4.4}$$

and for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left|D^\alpha_w q(w)\right| \le Ch^{|\alpha|}A_\alpha|a(w)|^{-1}\langle w\rangle^{-\rho|\alpha|},\ \ w\in Q^c_{B''},\ \alpha\in\mathbb{N}^{2d},\ j\in\mathbb{N}. \tag{10.4.5}$$

In particular, q is hypoelliptic. This estimate enables us to prove the next compactness result.

Assume that a is hypoelliptic and $|a(w)| \to \infty$ as $|w| \to \infty$ and let q be the parametrix for a constructed above. Take $\psi \in \mathscr{D}^{(A_p)}(\mathbb{R}^{2d})$ (resp. $\psi \in \mathscr{D}^{\{A_p\}}(\mathbb{R}^{2d})$) such that $0 \le \psi \le 1$, $\psi = 1$ on a compact neighbourhood of $Q_{B''}$ and $\psi = 0$ on the complement of a slightly larger neighbourhood. Then, for each $n \in \mathbb{Z}_+$, the function $b_n(w) = q(w)\psi(w/n)$ is in $\mathscr{D}^{(A_p)}(\mathbb{R}^{2d})$ (resp. in $\mathscr{D}^{\{A_p\}}(\mathbb{R}^{2d})$), and hence, b_n^w is $*$-regularising for each $n \in \mathbb{Z}_+$. Since $|a(w)| \to \infty$ as $|w| \to \infty$ and (10.4.5) holds, it follows that $b_n \to q$ in $\Gamma_\rho^0(\mathbb{R}^{2d})$. Thus, $b_n^w \to q^w$ in $\mathscr{L}_b(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d))$ (see [32, Theorem 1.7.14, p. 58]). As b_n^w, $n \in \mathbb{Z}_+$, are compact operators on $L^2(\mathbb{R}^d)$, the same holds for q^w.

If a symbol is hypoelliptic, then any $\tau-$quantisation will have a parametrix. This can be easily done by applying Proposition 10.4 since after the change of quantisation the new symbol remains hypoelliptic; we briefly outline the procedure. Start with hypoelliptic $b \in \Gamma_{A_p,\rho}^{*,\infty}(\mathbb{R}^{2d})$ and $\tau \in \mathbb{R}$. Proposition 10.4 gives the existence of a hypoelliptic $a \in \Gamma_{A_p,\rho}^{*,\infty}(\mathbb{R}^{2d})$ such that $c_1|b(w)| \leq |a(w)| \leq c_2|b(w)|$, $\forall w \in Q_{B_1}^c$, for some $c_1, c_2, B_1 > 0$, and $\operatorname{Op}_\tau(b) - a^w$ is $*$-regularising. We can find a hypoelliptic $\tilde{q} \in \Gamma_{A_p,\rho}^{*,\infty}(\mathbb{R}^{2d})$ such that

$$c_1'/|b(w)| \leq |\tilde{q}(w)| \leq c_2'/|b(w)|,\ \forall w \in Q_{B_1'}^c,$$

for some $c_1', c_2', B_1' > 0$, and $\operatorname{Op}_\tau(\tilde{q})\operatorname{Op}_\tau(b) - \operatorname{Id}$ is $*$-regularising. We infer

$$\operatorname{Op}_\tau(b)\operatorname{Op}_\tau(\tilde{q}) - \operatorname{Id} \in \mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)).$$

If $\tilde{q}_1 \in \Gamma_{A_p,\rho}^{*,\infty}(\mathbb{R}^{2d})$ is any other left τ-parametrix of b, i.e. $\operatorname{Op}_\tau(\tilde{q}_1)\operatorname{Op}_\tau(b) - \operatorname{Id} \in \mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d))$, then $\operatorname{Op}_\tau(\tilde{q}_1) - \operatorname{Op}_\tau(\tilde{q})$ is $*$-regularising, which, in turn, yields that we can use $\operatorname{Op}_\tau(\tilde{q}_1)$ as a right parametrix as well.

10.4.1 Hypoellipticity of a Linear Pseudo-Differential Equation

In [44], the hypoellipticity has been obtained by means of the construction of a parametrix. The results of [44] have been extended by Fernández et al. [17] to the space of ultradistributions of Beurling type and by the first author to the global frame of the Gelfand–Shilov spaces of type $\mathscr{S}$, see [2–4], allowing exponential growth for the symbols also with respect to the variables x and ξ.

It is then natural to study the same problem for pseudo-differential operators acting on tempered ultradistributions. The main result of this subsection is the following one from [11] on the global regularity of hypoelliptic operators; of course, this is an easy consequence of the existence of parametrices of such operators (given in the previous subsection).

Theorem 10.3 ([11, Theorem 1.2]) *Let $a \in \Gamma_{A_p,\rho}^{*,\infty}(\mathbb{R}^{2d})$ be $\Gamma_{A_p,\rho}^{*,\infty}$-hypoelliptic and let $v \in \mathscr{S}^*(\mathbb{R}^d)$. Then every solution $u \in \mathscr{S}^{*\prime}(\mathbb{R}^d)$ to the equation $a(x, D)u = v$ belongs to $\mathscr{S}^*(\mathbb{R}^d)$.*

Because of the comments we gave at the end of the last subsection, the result holds true for any τ-quantisation of the hypoelliptic symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$.

Example 10.1 We will give a couple of interesting instance of $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic symbols; because of the above, the corresponding operators have parametrices and Theorem 10.3 is valid for them.

(a) The symbols of the form $\langle(x,\xi)\rangle^k$, $k \in \mathbb{R}$, are $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic symbols [11].

(b) Let a be a positive elliptic Shubin symbol of order $m \geq 1$ (elliptic in the sense of the Shubin class, i.e. $c_1\langle w\rangle^m \leq a(w) \leq c_2\langle w\rangle^m$, $w \in \mathbb{R}^{2d}$) which additionally satisfies the following estimate: for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$|D^\alpha a(w)| \leq Ch^{|\alpha|} A_\alpha a(w)\langle w\rangle^{-\rho_1|\alpha|}, \text{ for all } w \in \mathbb{R}^{2d},\ \alpha \in \mathbb{N}^{2d}, \quad (10.4.6)$$

for some $\rho_1 > \rho$. Then, by taking $s > 1$ large enough such that $\rho_1 - \frac{1}{s} \geq \rho$ and the function $e^{\langle w\rangle^{1/s}}$ to be of ultrapolynomial growth of class $*$, the symbol $e^{\pm a(w)^{1/(sm)}}$ becomes $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic (see [36, Remark 7.6]; see also [11]). For example, if one takes $a(w) = \langle w\rangle$, then $e^{\pm\langle w\rangle^{1/s}}$ is $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic for an appropriate choice of s; notice that $\langle w\rangle$ satisfies (10.4.6) with $\rho_1 = 1$.

(c) Let $1 < v < l$ and fix $0 < \rho < 1$ such that $v \leq l\rho$. Take $s > l$ such that $1 - \frac{1}{s} \geq \rho$ and consider the function

$$a(w) = 1 + \sum_{n=1}^{\infty} \frac{h^n \langle w\rangle^n}{n!^s}, \quad w \in \mathbb{R}^{2d},$$

for some $h > 0$. Then a is a $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic symbol in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, when $M_p = p!^l$ and $A_p = p!^v$ (see [12, Sect. 3]).

10.4.2 *Hypoellipticity of a Semilinear Pseudo-Differential Equation*

We consider a class of semilinear equations and present a result on regularity in the spaces of tempered ultradistributions of Beurling and Roumieu type cf. [18, 34]. We present results of [12] related to the equation

$$Au = f + F[u]. \quad (10.4.7)$$

Here $A = a(x, D)$, f is a given test function in our setting and $F[u]$ is a nonlinear term given by a suitable infinite series of powers of u. For the semilinear equations of type (10.4.7), we adopted in [12] a more complicated calculus used for the already known commutator method for such nonlinearity.

The operator A is hypoelliptic, and it is of truly infinite order; i.e. its symbol is bounded from below by a function with a sub-exponential growth. The nonlinear term is of the form

$$F[u] = \sum_{|\beta|=2}^{\infty} p_\beta u^{|\beta|}, \tag{10.4.8}$$

where p_β are smooth functions which are allowed to have sub-exponential growth. The function u in the nonlinear term under consideration additionally satisfies that $u \in H^s(\mathbb{R}^d)$, $s > d/2$. Thus, $F[u]$ is also of sub-exponential growth. Roughly speaking, the main result states that all the solutions of (10.4.7) that are known to be in $H^s(\mathbb{R}^d)$ are in fact highly regular, i.e. they are sub-exponentially decaying and ultradifferentiable. This shows an intrinsic connection of the ΨDO calculus of [37] with the spaces of ultradistributions.

Before we give the result, we state precisely the conditions on A and $F[u]$.

Let A_p, M_p and ρ be as in Sect. 10.3.1. Let $\tilde{M}_p$ be another sequence that satisfies $(M.1)$, $(M.2)$, $(M.3)'$ and $(M.4)$ and so that $M_p \subset \tilde{M}_p$. Moreover, for $(k_p) \in \mathfrak{R}$, denote $\tilde{N}_p = \tilde{M}_p \prod_{j=1}^{p} k_j$.

A symbol $a \in \Gamma^{*,\infty}_{A_p,\rho}\left(\mathbb{R}^{2d}\right)$ is said to be $(\tilde{M}_p)$-hypoelliptic, (resp. $\{\tilde{M}_p\}$-hypoelliptic) if

(i) there exist $m, B, c > 0$ (resp. there exist $(k_p) \in \mathfrak{R}$ such that $\prod_{j=1}^{p} k_j$, $p \in \mathbb{N}$, satisfies $(M.2)$ and $B, c > 0$) such that

$$|a(x,\xi)| \geq c e^{\tilde{M}(m|\xi|)} e^{\tilde{M}(m|x|)} (\text{ resp. } |a(x,\xi)| \geq c e^{\tilde{N}_{k_p}(|\xi|)} e^{\tilde{N}_{k_p}(|x|)}), \quad (x,\xi) \in Q_B^c;$$

(ii) for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left|D_\xi^\alpha D_x^\beta a(x,\xi)\right| \leq C \frac{h^{|\alpha|+|\beta|} A_{\alpha+\beta} |a(x,\xi)|}{\langle (x,\xi)\rangle^{\rho(|\alpha|+|\beta|)}}, \quad (x,\xi) \in Q_B^c.$$

Notice that $(\tilde{M}_p)$-hypoelliptic (resp. $\{\tilde{M}_p\}$-hypoelliptic) symbol is just $\Gamma^{*,\infty}_{A_p,\rho}$-hypoelliptic with sub-exponential lower bound $e^{\tilde{M}(m|\xi|)} e^{\tilde{M}(m|x|)}$ (resp. $e^{\tilde{N}_{k_p}(|\xi|)} e^{\tilde{N}_{k_p}(|x|)}$).

Next, we introduce the class of nonlinear terms involved in the Eq. (10.4.7).

For $\beta \in \mathbb{N}^d$, let $p_\beta(x)$ be smooth functions on $\mathbb{R}^d$ such that for every $h > 0$, there exists $C > 0$ such that

$$\left|D_x^\alpha p_\beta(x)\right| \leq C \frac{h^{|\alpha|+|\beta|} A_\alpha e^{\tilde{M}(h|x|)}}{\tilde{M}_\alpha} \text{ for all } \alpha, \beta \in \mathbb{N}^d, \tag{10.4.9}$$

respectively

$$\left|D_x^\alpha p_\beta(x)\right| \le C\frac{h^{|\alpha|+|\beta|}A_\alpha e^{\tilde{N}_{k_p}(h|x|)}}{\tilde{M}_\alpha \prod_{j=1}^{|\alpha|} k_j} \text{ for all } \alpha,\beta\in\mathbb{N}^d. \tag{10.4.10}$$

For such a family of functions $p_\beta(x)$ and $u \in H^s(\mathbb{R}^d)$, $s > d/2$, we can consider the function $F[u]$ given by (10.4.8). The condition $s > d/2$ implies that $F[u]$ is well defined and continuous on $\mathbb{R}^d$ and $\left\|F[u]e^{-\tilde{M}(h|\cdot|)}\right\|_{L^\infty(\mathbb{R}^d)} < \infty$ (resp. $\left\|F[u]e^{-\tilde{N}_{k_p}(h|\cdot|)}\right\|_{L^\infty(\mathbb{R}^d)} < \infty$) for some h. This implies that $F[u]\in\mathscr{S}'^*(\mathbb{R}^d)$.

Our main result in [12] is the following one.

Theorem 10.4 ([12, Theorem 1.2]) *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be $(\tilde{M}_p)$-hypoelliptic (resp. $\{\tilde{M}_p\}$-hypoelliptic) and let $f \in \mathscr{S}^*(\mathbb{R}^d)$. Let $u \in H^s(\mathbb{R}^d)$, $s > d/2$, be a solution of the equation (10.4.7) with $F[u]$ defined by (10.4.9) and (10.4.8) (resp. (10.4.10) and (10.4.8)). Then the following properties hold:*

(i) For every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that $|u(x)| \le Ce^{-M(h|x|)}$. Moreover, $u \in \mathscr{C}^\infty(\mathbb{R}^d)$ with the following estimate on its derivatives: there exists $\tilde{h} > 0$ such that

$$\sup_\alpha \frac{\tilde{h}^{|\alpha|}\|D^\alpha u\|_{L^\infty}}{\tilde{M}_\alpha} < \infty, \left(\textit{resp. } \sup_\alpha \frac{\tilde{h}^{|\alpha|}\|D^\alpha u\|_{L^\infty}}{\tilde{M}_\alpha \prod_{j=1}^{|\alpha|} k_j} < \infty \right).$$

(ii) Furthermore, if $F[u]$ is a finite sum, then $u \in \mathscr{S}^*(\mathbb{R}^d)$.

One interesting instance where Theorem 10.4 is applicable is when one takes $A = a(x, D)$, where a is the symbol given in Example 10.1 (c); here $M_p = p!^l$, $A_p = p!^v$. Then one can prove that a is $(\tilde{M}_p)$-hypoelliptic with $\tilde{M}_p = p!^{l+l'}$ (resp. $\{\tilde{M}_p\}$-hypoelliptic with $\tilde{M}_p = p!^{l+l'/2}$) for some $l' > 0$ (see [12, Sect. 3]). An interesting non-trivial example of the nonlinear term $F[u]$ related to (the growth of) this operator can be given by taking $p_\beta(x) = \sum_{\alpha\in\mathbb{N}^d} c_{\alpha,\beta}x^{\alpha+\beta}$, $\beta\in\mathbb{N}^d$, where the coefficients $c_{\alpha,\beta}$ satisfy the following estimate: for every $h > 0$, there exists $C > 0$ such that $|c_{\alpha,\beta}| \le Ch^{|\alpha|+|\beta|}/\tilde{M}_{\alpha+\beta}$ (resp. $|c_{\alpha,\beta}| \le Ch^{|\alpha|+|\beta|}/(\tilde{M}_{\alpha+\beta}\prod_{j=1}^{|\alpha|+|\beta|} k_j)$, with $k_p = p^{l'/2}$); see [12, Sect. 3]. Another interesting instance of the nonlinear term is given by

$$F[u](x) = P(x)\sin(u(x)) = P(x)u(x) + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}P(x)}{(2n-1)!}\cdot (u(x))^{2n-1},$$

where P is smooth and satisfies the following condition: for every $h > 0$ there exists $C > 0$ such that $|D^\alpha P(x)| \le Ch^{|\alpha|}A_\alpha e^{(h|x|)^{1/(l+l')}}/\alpha!^{l+l'}$ (resp. $|D^\alpha P(x)| \le Ch^{|\alpha|}A_\alpha e^{(h|x|)^{1/(l+l'+\varepsilon)}}/\alpha!^{l+l'}$, with arbitrary but fixed $\varepsilon > 0$). Notice that the term $P(x)u(x)$ can be absorbed in Au, i.e. one can instead consider the operator A_1 with symbol $a(x,\xi) - P(x)$ which is again $(\tilde{M}_p)$-hypoelliptic (resp. $\{M_p\}$-hypoelliptic). Similarly, one can consider nonlinear terms of the form

$$P(x)(\cos(u(x)) - 1) \text{ or } P(x)(e^{(u(x))^k} - 1),\ k \in \mathbb{Z}_+,\ \text{fixed},$$

with $P(x)$ smooth with appropriate growth conditions on the derivatives and at infinity.

10.5 Complex Powers of Hypoelliptic Operators

10.5.1 Some Results from the Theory of Operators

The main goal of this subsection is to give several important facts about the $L^2(\mathbb{R}^d)$ realisations and the spectrum of pseudo-differential operators with symbols in our class. All the results, we mention here can be found in [36, 39].

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ and denote by A the unbounded operator on $L^2(\mathbb{R}^d)$ with domain $D(A) = \mathscr{S}^*(\mathbb{R}^d)$, defined by $A\varphi = a^w\varphi$, $\varphi \in \mathscr{S}^*(\mathbb{R}^d)$. Then A is closable since the restriction of a^w to $\{f \in L^2(\mathbb{R}^d)|\, a^w f \in L^2(\mathbb{R}^d)\}$ defines a closed extension; as customary, we call this closed extension the maximal realisation of A. The minimal realisation of A is by definition the closure of A, denoted by $\overline{A}$. The next result states that they are the same in the case of hypoelliptic a.

Proposition 10.6 ([36, Proposition 5.4]) *Let a be hypoelliptic and A be the corresponding unbounded operator on $L^2(\mathbb{R}^d)$ defined above. Then the minimal realisation $\overline{A}$ coincides with the maximal realisation. Moreover, $\overline{A}$ coincides with the restriction of a^w on the domain of $\overline{A}$.*

This is a known result in the case of finite order ΨDOs; the proposition claims that it remains true in our setting as well. To better appreciate the result, the reader should keep in mind that the notion of hypoellipticity in our setting allows the symbol to decay sub-exponentially at infinity. Having this result, one can deduce the following consequence.

Corollary 10.1 ([39, Proposition 4.4]) *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be a hypoelliptic real-valued symbol. Then its minimal (i.e. maximal) realisation $\overline{A}$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$.*

The last result we give is concerning the spectrum of ΨDOs with positive hypoelliptic symbols. However, before we can say anything meaningful about the spectrum of such operators, we need a result on their semi-boundedness. This is a well-known fact for finite order operators. But in the case of our class of pseudo-differential operators of infinite order one cannot use the classical Weyl–Hörmander calculus as the operators go beyond this calculus. The proof is given in [39] and is not a simple transfer from the well-known finite order case. Here we just quote the proposition and refer to [39] for its proof.

Proposition 10.7 ([39, Proposition 4.5]) *Let $b \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be positive hypoelliptic symbol. Then, there exists $C > 0$ such that $(b^w\varphi, \varphi) \geq -C\|\varphi\|^2_{L^2(\mathbb{R}^d)}$, $\forall \varphi \in \mathscr{S}^*(\mathbb{R}^d)$.*

This result on semi-boundedness allows us to prove the following fact on the spectrum of infinite order ΨDOs much like for the classical Shubin hypoelliptic operators (see, for example, the proof of [32, Theorem 4.2.9, p. 163]).

Proposition 10.8 ([39, Proposition 4.6]) *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be a hypoelliptic real-valued symbol such that $|a(w)| \to \infty$ as $|w| \to \infty$ and let A be the unbounded operator on $L^2(\mathbb{R}^d)$ defined by a^w. Then the closure $\overline{A}$ of A is a self-adjoint operator having spectrum given by a sequence of real eigenvalues either diverging to $+\infty$ or to $-\infty$ according to the sign of a at infinity. The eigenvalues have finite multiplicities and the eigenfunctions belong to $\mathscr{S}^*(\mathbb{R}^d)$. Moreover, $L^2(\mathbb{R}^d)$ has an orthonormal basis consisting of eigenfunctions of $\overline{A}$.*

10.5.2 Known Results from the Abstract Theory of Non-negative Operators and their Complex Powers

In this subsection, we recall several facts from the abstract theory of non-negative operators on (B)-spaces. In view of our goal to present in the next subsection results concerning the complex powers of infinite order ΨDOs, most of the facts we recall here are about the complex powers of non-negative densely defined operators on (B)-spaces. All results given here are borrowed from [14, 26].

Following Komatsu [26], given a (B)-space X, a closed operator $A : D(A) \subseteq X \to X$ is said to be non-negative if $(-\infty, 0)$ is contained in the resolvent set of A and

$$\sup_{\lambda\in\mathbb{R}_+} \lambda\|(A + \lambda\mathrm{Id})^{-1}\|_{\mathscr{L}_b(X,X)} < \infty.$$

In this case, for $z \in \mathbb{C}_+ = \{\zeta \in \mathbb{C} | \operatorname{Re}\zeta > 0\}$ and $v \in D(A^{[\operatorname{Re} z]+1})$, the function $\lambda \mapsto \lambda^{z-1}\left(A(A + \lambda\mathrm{Id})^{-1}\right)^k v$, $\mathbb{R}_+ \to X$, is Bochner integrable for all integers $k > \operatorname{Re} z$ and by defining

$$I^z_{A,k}v = \gamma_k(z)\int_0^\infty \lambda^{z-1}\left(A(A + \lambda\mathrm{Id})^{-1}\right)^k v d\lambda, \ \ v \in D(A^{[\operatorname{Re} z]+1}), \ \ k > \operatorname{Re} z,$$

where $\gamma_k(z) = \Gamma(k)/(\Gamma(z)\Gamma(k - z))$, we have that $I^z_{A,k+1}v = I^z_{A,k}v$, for all integers $k > \operatorname{Re} z$ (see [14, Proposition 3.1.3, p. 59]). The operator

$$J^z_A : D(J^z_A) = D(A^{[\operatorname{Re} z]+1}) \subseteq X \to X, \ \ J^z_A v = I^z_{A,k}v, \ \text{ for any } k > \operatorname{Re} z,$$

is closable (cf. [14, Theorem 3.1.8, p. 64]). Balakrishnan defines the power of A with exponent z as the operator $\overline{J^z_A}$. If in addition A is densely defined, then $A^{z+\zeta} = A^z A^\zeta$,

$\forall z, \zeta \in \mathbb{C}_+$ (in particular, $A^k = \underbrace{A \dots A}_{k}$) and $\sigma(A^z) = \{\zeta^z | \zeta \in \sigma(A)\}$; where ζ^z is defined by the principal branch of logarithm and we put $0^z = 0$ (cf. [14, Corollary 5.1.12, p. 110] and [14, Theorem 5.3.1, p. 116]).

10.5.3 Complex Powers of Infinite Order Hypoelliptic ΨDOs

Given a hypoelliptic $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, we know that the unbounded densely defined operator

$$A : \mathscr{S}^*(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\ \ A\varphi = a^w\varphi,$$

is closable and its closure coincides with its maximal realisation (cf. Sect. 10.5.1). The main goal of this part is to give sufficient conditions on a and $\overline{A}$ which will ensure that the complex power $\overline{A}^z$, $z \in \mathbb{C}_+$, as defined in Sect. 10.5.2 is in fact given by a ΨDO modulo a $*$-regularising operator and to find its symbol; incidentally, we also give precise estimates on this symbol involving the original symbol a. Before we give the main result, we precisely state the assumption we impose on a.

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be hypoelliptic, where the hypoellipticity conditions (10.4.1) and (10.4.2) hold for some $\tilde{B} > 0$. We impose the following conditions on a:

(I) $\operatorname{Re} a(w) \geq -\tilde{B}|\operatorname{Im} a(w)|$ for $w \in Q^c_{\tilde{B}}$;
(II) the densely defined operator $A : \mathscr{S}^*(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $A = a^w|_{\mathscr{S}^*(\mathbb{R}^d)}$, is such that $\overline{A}$ is non-negative.

Let $\tilde{\chi} \in \mathscr{D}^{(A_p)}(\mathbb{R}^{2d})$ in the (M_p) case and $\tilde{\chi} \in \mathscr{D}^{\{A_p\}}(\mathbb{R}^{2d})$ in the $\{M_p\}$ case respectively, be such that $\tilde{\chi} \geq 0$ and $\tilde{\chi}(w) > \max\{0, -\operatorname{Re} a(w)\}$ when $w \in \overline{Q_{\tilde{B}}}$. Denote $a_0 = a + \tilde{\chi}$. Possibly for a larger $\tilde{B}$, we infer the following estimate

$$\operatorname{Re} a_0(w) > -\tilde{B}|\operatorname{Im} a_0(w)|,\ \ \forall w \in \mathbb{R}^{2d}, \tag{10.5.1}$$

and consequently,

$$|a_0(w)| \leq \sqrt{1+\tilde{B}^2}|a_0(w)+\lambda|,\ \ \lambda \leq \sqrt{1+\tilde{B}^2}|a_0(w)+\lambda|, \tag{10.5.2}$$

for all $w \in \mathbb{R}^{2d}$ and $\lambda \geq 0$. As a consequence of (10.5.1), a_0 never vanishes and for any $z \in \mathbb{C}_+$, the function $w \mapsto (a_0(w))^z$, $\mathbb{R}^{2d} \to \mathbb{C}$, is a well-defined $\mathscr{C}^\infty$ function (in $(a_0(w))^z$ we use the principal branch of the logarithm). The inequalities (10.5.2) also give the existence of $c > 0$ such that

$$|a_0(w)| + \lambda \leq c|a_0(w)+\lambda|,\ \ w \in \mathbb{R}^{2d},\ \lambda \geq 0. \tag{10.5.3}$$

Thus, employing the identity

$$\int_0^\infty \frac{\lambda^{z-1}\zeta^k}{(\zeta+\lambda)^k}d\lambda = \frac{\zeta^z}{\gamma_k(z)}, \tag{10.5.4}$$

which is valid for $z \in \mathbb{C}_+$, $k \in \mathbb{N}$ with $k > \operatorname{Re} z$ and $\zeta \in \mathbb{C}\backslash\{0\}$ with $|\arg\zeta| < \pi$, we deduce that

$$\left|(a_0(w))^z\right| \le c^k|\gamma_k(z)| \int_0^\infty \frac{\lambda^{\operatorname{Re} z-1}|a_0(w)|^k}{(|a_0(w)|+\lambda)^k}d\lambda = \frac{c^k|\gamma_k(z)||a_0(w)|^{\operatorname{Re} z}}{\gamma_k(\operatorname{Re} z)},$$

for all $w \in \mathbb{R}^{2d}$, $z \in \mathbb{C}_+$, $k > \operatorname{Re} z$, $k \in \mathbb{Z}_+$. Thus, $e^{-\operatorname{Im} z \arg(a_0(w))} \le c^k|\gamma_k(z)|/\gamma_k(\operatorname{Re} z)$. Hence,

$$|a_0(w)|^{\operatorname{Re} z} = \left|(a_0(w))^z\right| e^{\operatorname{Im} z \arg(a_0(w))} \le c^k \left|(a_0(w))^z\right| |\gamma_k(\bar{z})|/\gamma_k(\operatorname{Re} z), \tag{10.5.5}$$

for all $w \in \mathbb{R}^{2d}$, $z \in \mathbb{C}_+$, $k > \operatorname{Re} z$, $k \in \mathbb{Z}_+$.

The main result is the following theorem.

Theorem 10.5 ([36, Theorem 6.1]) *Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be a hypoelliptic symbol that satisfies (I) and (II) and let a_0 and A be defined as above. Then, for every $z \in \mathbb{C}_+$ there exists $a_0^{\#z} \in FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};0)$ such that the following conditions hold.*

(i) $a_0^{\#z}\#a_0^{\#\zeta} = a_0^{\#(z+\zeta)} = a_0^{\#\zeta}\#a_0^{\#z}$, $\forall z,\zeta \in \mathbb{C}_+$.
(ii) *When $z = k \in \mathbb{Z}_+$, $a_0^{\#z}$ is just $\underbrace{a_0\#\ldots\#a_0}_{k} = a_0^{\#k}$. In particular, for $z=1$, $a_0^{\#z}$ is just a_0.*
(iii) *The mapping $z \mapsto a_0^{\#z}$, $\mathbb{C}_+ \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d};0)$ is continuous.*
(iv) $(a_0^{\#z})_0(w) = (a_0(w))^z$, $w \in \mathbb{R}^{2d}$, $z \in \mathbb{C}_+$.

For each fixed vertical strip $\mathbb{C}_{+,t} = \{\zeta \in \mathbb{C}_+ | \operatorname{Re}\zeta \le t\}$, $t > 0$, and $k = [t]+1 \in \mathbb{Z}_+$, the following estimate holds: for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left|D_w^\alpha (a_0^{\#z})_j(w)\right| \le C\frac{h^{|\alpha|+2j}A_{|\alpha|+2j}|\gamma_k(z)||a_0(w)|^{\operatorname{Re} z}}{\gamma_k(\operatorname{Re} z)\langle w\rangle^{\rho(|\alpha|+2j)}}, \tag{10.5.6}$$

for all $w \in \mathbb{R}^{2d}$, $\alpha \in \mathbb{N}^{2d}$, $j \in \mathbb{N}$, $z \in \mathbb{C}_{+,t}$. Furthermore, there exists $R_t > 0$ such that $a^{\underset{\sim}{z}} := R_t(a_0^{\#z})$, $z \in \mathbb{C}_{+,t}$, are hypoelliptic symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ and the following conditions hold:

(v) *There exists $B_t > 0$ such that for every $h > 0$ (resp. for some $h > 0$)*

$$\sup_{\substack{N\in\mathbb{Z}_+\\ z\in\mathbb{C}_{+,t}}}\sup_{\alpha\in\mathbb{N}^{2d}}\sup_{w\in Q^c_{B_t m_N}}\frac{\left|D^\alpha_w \overset{z}{\hat{a}}(w)-D^\alpha_w\sum_{j<N}(a_0^{\#z})_j(w)\right|\langle w\rangle^{\rho(|\alpha|+2N)}\gamma_k(\mathrm{Re}\, z)}{h^{|\alpha|+2N}A_{|\alpha|+2N}|a_0(w)|^{\mathrm{Re}\, z}|\gamma_k(z)|}<\infty,$$

$$\sup_{z\in\mathbb{C}_{+,t}}\sup_{\alpha\in\mathbb{N}^{2d}}\sup_{w\in\mathbb{R}^{2d}}\frac{\left|D^\alpha \overset{z}{\hat{a}}(w)\right|\langle w\rangle^{\rho|\alpha|}\gamma_k(\mathrm{Re}\, z)}{h^{|\alpha|}A_\alpha|a_0(w)|^{\mathrm{Re}\, z}|\gamma_k(z)|}<\infty.$$

(vi) $D(\overline{A}^z)=\left\{v\in L^2(\mathbb{R}^d)|\,(\overset{z}{\hat{a}})^w v\in L^2(\mathbb{R}^d)\right\}$ *and there exist* $*$*-regularising operators* $\overset{z}{\hat{S}}$, $z\in\mathbb{C}_{+,t}$, *such that* $\{\gamma_k(\mathrm{Re}\, z)(\gamma_k(z))^{-1}\overset{z}{\hat{S}}\,|\,z\in\mathbb{C}_{+,t}\}$ *is an equicontinuous subset of* $\mathscr{L}(\mathscr{S}'^*(\mathbb{R}^d),\mathscr{S}^*(\mathbb{R}^d))$ *and*

$$\overline{A}^z=\overline{(\overset{z}{\hat{a}})^w|_{\mathscr{S}^*(\mathbb{R}^d)}}+\overset{z}{\hat{S}}.$$

Moreover, for each $v\in L^2(\mathbb{R}^d)$, $z\mapsto \overset{z}{\hat{S}}v$, $\mathrm{int}\,\mathbb{C}_{+,t}\to L^2(\mathbb{R}^d)$, *is analytic and for each* $v\in D(A^k)$, $z\mapsto(\overset{z}{\hat{a}})^w v$, $\mathrm{int}\,\mathbb{C}_{+,t}\to L^2(\mathbb{R}^d)$, *is analytic.*

Remark 10.1 Notice that the second estimate in (v) together with (10.5.5) implies that for every $h>0$ (resp. for some $h>0$)

$$\sup_{z\in\mathbb{C}_{+,t}}\sup_{\alpha\in\mathbb{N}^{2d}}\sup_{w\in\mathbb{R}^{2d}}\frac{\left|D^\alpha \overset{z}{\hat{a}}(w)\right|\langle w\rangle^{\rho|\alpha|}(\gamma_k(\mathrm{Re}\, z))^2}{h^{|\alpha|}A_\alpha|(a_0(w))^z||\gamma_k(z)|^2}<\infty. \tag{10.5.7}$$

10.6 Semigroups Generated by Square Roots of Non-negative Infinite Order Operators

If A is a non-negative operator with a dense domain in $L^2(\mathbb{R}^d)$, then it is known that $-A^{1/2}$ is the infinitesimal generator of an analytic semigroup (see Sect. 10.6.2 below). The goal of this section is to apply Theorem 10.5 to prove that if a A is the $L^2(\mathbb{R}^d)$-closure of the Weyl quantisation of an appropriate hypoelliptic symbol in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, then all the operators in the semigroup are pseudo-differential with symbols in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ modulo $*$-regularising operators. The heat parametrix is the key ingredient in the proof.

10.6.1 The Heat Parametrix

We assume that b is a hypoelliptic symbol in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ for which the condition (10.4.2) holds on the whole $\mathbb{R}^{2d}$. Furthermore, we assume that there exists $c>0$ such that

$$\operatorname{Re} b(w) > c|\operatorname{Im} b(w)|, \ \forall w \in \mathbb{R}^{2d}; \tag{10.6.1}$$

hence $\operatorname{Re} b(w) > 0$, $\forall w \in \mathbb{R}^{2d}$. This implies that (10.4.1) also holds on the whole $\mathbb{R}^{2d}$. The goal is to find a smooth family of symbols $t \to \mathbf{u}(t)$, $[0, \infty) \to \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$, which solve

$$\begin{cases} (\partial_t + b^w)(\mathbf{u}(t))^w = \mathbf{K}(t), \ t \in [0, \infty), \\ (\mathbf{u}(0))^w = \mathrm{Id}, \end{cases} \tag{10.6.2}$$

for some $\mathbf{K} \in \mathscr{C}^\infty([0, \infty); \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)))$. For this purpose, we consider the system

$$\begin{cases} \partial_t u_j + \sum_{k+l=j} \sum_{|\mu+\nu|=l} \frac{(-1)^{|\nu|}}{\mu!\nu!2^l} \partial^\mu_\xi D^\nu_x b \cdot \partial^\nu_\xi D^\mu_x u_k = 0, \quad j \in \mathbb{N}, \\ u_0(0, x, \xi) = 1, \\ u_j(0, x, \xi) = 0, \ \ j \in \mathbb{Z}_+. \end{cases} \tag{10.6.3}$$

There exist $u_j \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R}^{2d})$, $j \in \mathbb{N}$, which solve (10.6.3); clearly, $u_0(t, x, \xi) = e^{-tb(x,\xi)}$. The important fact about u_j, $j \in \mathbb{N}$, is that they satisfy uniform estimates on the derivatives and growths which will make $\sum_j u_j(t, \cdot)$ a well-defined smooth function on $[0, \infty)$ with values in $FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; 0)$. In fact, one can prove the following estimate (see [36, Lemma 7.8]): for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$|D^n_t D^\alpha_w u_j(t, w)| \le Cn! h^{|\alpha|+2j} A_{|\alpha|+2j} \, (\operatorname{Re} b(w))^n \, \langle w\rangle^{-\rho(|\alpha|+2j)} e^{-\frac{t}{4}\operatorname{Re} b(w)},$$

for all $\alpha \in \mathbb{N}^{2d}$, $n \in \mathbb{N}$, $(t, w) \in [0, \infty) \times \mathbb{R}^{2d}$. Employing Taylor formula, this implies that for every $h > 0$, there exists $C > 0$ (there exist $h, C > 0$) such that

$$\begin{aligned} &|\partial^n_t D^\alpha_w u_j(t, w) - \partial^n_t D^\alpha_w u_j(t_0, w)| \\ &\qquad \le C|t - t_0|(n + 1)! h^{|\alpha|+2j} A_{|\alpha|+2j} \, (\operatorname{Re} b(w))^{n+1} \, \langle w\rangle^{-\rho(|\alpha|+2j)}, \end{aligned} \tag{10.6.4}$$

for all $\alpha \in \mathbb{N}^{2d}$, $n, j \in \mathbb{N}$, $w \in \mathbb{R}^{2d}$, $t, t_0 \in [0, \infty)$, which gives the continuity of the mapping $t \mapsto \sum_j \partial^n_t u_j(t, \cdot)$, $[0, \infty) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; 0)$, for each $n \in \mathbb{N}$. Similarly, by expanding $\partial^n_t D^\alpha_w u_j(t, w)$ at t_0 up to order 1, we can conclude that the above mappings are differentiable and one can deduce the following result.

Proposition 10.9 ([36, Lemmas 7.9 and 7.10]) *The following statements hold true.*
 (*a*) *The mapping*

$$t \mapsto \sum_j u_j(t, \cdot), \ \ [0, \infty) \to FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; 0),$$

is in $\mathscr{C}^\infty([0, \infty); FS^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}; 0))$ *and* $\partial^n_t(\sum_j u_j(t, \cdot)) = \sum_j \partial^n_t u_j(t, \cdot)$, $n \in \mathbb{N}$.

(*b*) *There exists* $R > 1$ *such that the* $\mathscr{C}^\infty$*-function*

$$u(t,w) = \sum_{n=0}^{\infty}(1-\chi_{n,R}(w))u_n(t,w) = R(\sum_j u_j)(t,w)$$

satisfies the following condition: for every $h > 0$*, there exists* $C > 0$ *(resp. there exist* $h, C > 0$*) such that*

$$|D_t^n D_w^\alpha u(t,w)| \leq Cn!h^{|\alpha|}A_\alpha\,(\operatorname{Re} b(w))^n\,\langle w\rangle^{-\rho|\alpha|}e^{-\frac{t}{4}\operatorname{Re} b(w)}, \tag{10.6.5}$$

for all $\alpha \in \mathbb{N}^{2d}$*,* $n \in \mathbb{N}$*,* $(t,w) \in [0,\infty) \times \mathbb{R}^{2d}$ *and*

$$\sup_{N\in\mathbb{Z}_+}\sup_{\substack{\alpha\in\mathbb{N}^{2d}\\ n\in\mathbb{N}}}\sup_{\substack{w\in Q^c_{3Rm_N}\\ t\in[0,\infty)}} \frac{\left|D_t^n D_w^\alpha\left(u(t,w)-\sum_{j<N}u_j(t,w)\right)\right|\langle w\rangle^{\rho(|\alpha|+2N)}}{n!h^{|\alpha|+2N}A_{|\alpha|+2N}\,(\operatorname{Re} b(w))^n\,e^{-\frac{t}{4}\operatorname{Re} b(w)}} \leq C.$$

We claim the mapping $\mathbf{u} : t \mapsto u(t,\cdot)$ solves (10.6.2). In fact, we have:

Theorem 10.6 ([36, Theorem 7.11]) *The* $\mathscr{C}^\infty$ *function* $u(t,w)$ *constructed in Proposition 10.9 is such that the mapping* $\mathbf{u} : t \mapsto u(t,\cdot)$*,* $[0,\infty) \to \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$*, belongs to* $\mathscr{C}^\infty([0,\infty); \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}))$*. The mapping* $t \mapsto (\mathbf{u}(t))^w$ *is in both*

$$\mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d))) \text{ and } \mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}'^*(\mathbb{R}^d))).$$

Moreover, $(\mathbf{u}(t))^w$ *satisfy*

$$\begin{cases} (\partial_t + b^w)(\mathbf{u}(t))^w = \mathbf{K}(t),\ t \in [0,\infty),\\ (\mathbf{u}(0))^w = \mathrm{Id}, \end{cases} \tag{10.6.6}$$

where $\mathbf{K} \in \mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)))$*.*

For each $t \geq 0$*,* $(\mathbf{u}(t))^w \in \mathscr{L}(L^2(\mathbb{R}^d))$ *and there exists* $C > 0$ *such that*

$$\|(\mathbf{u}(t))^w\|_{\mathscr{L}_b(L^2(\mathbb{R}^d))} \leq C, \quad \text{for all } t \geq 0.$$

The mapping $t \mapsto (\mathbf{u}(t))^w$*,* $(0,\infty) \to \mathscr{L}_b(L^2(\mathbb{R}^d))$*, is continuous and*

$$(\mathbf{u}(t))^w \to (\mathbf{u}(0))^w = \mathrm{Id}, \quad \text{as } t \to 0^+, \text{ in } \mathscr{L}_p(L^2(\mathbb{R}^d)).$$

Furthermore, for each $n \in \mathbb{Z}_+$*,* $(\partial_t^n \mathbf{u}(t))^w \in \mathscr{L}(L^2(\mathbb{R}^d))$*, for all* $t > 0$*. The mapping* $t \mapsto (\mathbf{u}(t))^w$*,* $(0,\infty) \to \mathscr{L}_b(L^2(\mathbb{R}^d))$*, is smooth and* $\partial_t^n(\mathbf{u}(t))^w = (\partial_t^n \mathbf{u}(t))^w$*.*

10.6.2 Semigroup Generated by the Square Root of a Non-negative Hypoelliptic Operator

Now we have all the ingredients we need to briefly outline the proof of the result, we announced at the beginning of Sect. 10.6.

Let $a \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be a hypoelliptic symbol which satisfies the assumptions in Theorem 10.5 and let $a_0 \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ be the symbol defined in Theorem 10.5. By this theorem, with $z = 1/2$, we concluded the existence of a hypoelliptic symbol $a^{\frown 1/2} \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ and a $*$-regularising operator S_1 such that

$$\overline{A}^{1/2} = \overline{\left(a^{\frown 1/2}\right)^w|_{\mathscr{S}^*(\mathbb{R}^d)}} + S_1 \ \text{ with } \ D(\overline{A}^{1/2}) = \left\{v \in L^2(\mathbb{R}^d) \middle| \left(a^{\frown 1/2}\right)^w v \in L^2(\mathbb{R}^d)\right\}$$

and the estimates in part (v) of Theorem 10.5 hold true with $k = 1$. By (v) for $N = 1$ and $\alpha = 0$, one obtains $\operatorname{Re} a^{\frown 1/2}(w) > c'\left|\operatorname{Im} a^{\frown 1/2}(w)\right|$, $\forall w \in Q^c_B$, for some $B, c' > 0$ (cf. (10.5.1)). Clearly, we can assume that $a^{\frown 1/2}$ satisfies (10.4.1) and (10.4.2) for this B. Take $\tilde{\tilde{\chi}} \in \mathscr{D}^{(A_p)}(\mathbb{R}^{2d})$ (resp. $\tilde{\tilde{\chi}} \in \mathscr{D}^{\{A_p\}}(\mathbb{R}^{2d})$) such that $0 \leq \tilde{\tilde{\chi}} \leq 1$, $\tilde{\tilde{\chi}} = 1$ on a small neighbourhood of Q_B and $\tilde{\tilde{\chi}} = 0$ on the complement of a slightly larger neighbourhood and define $b = \tilde{\tilde{\chi}} + (1 - \tilde{\tilde{\chi}})a^{\frown 1/2}$. Then b is a hypoelliptic symbol in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ that satisfies (10.4.1), (10.4.2) and (10.6.1) on $\mathbb{R}^{2d}$. Furthermore, for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that

$$\left|D^\alpha_w b(w)\right| \leq C h^{|\alpha|} A_\alpha |a_0(w)|^{1/2} \langle w\rangle^{-\rho|\alpha|}, \ \ w \in \mathbb{R}^{2d}, \ \alpha \in \mathbb{N}^{2d}. \tag{10.6.7}$$

Moreover, $\overline{A}^{1/2} = \overline{b^w|_{\mathscr{S}^*(\mathbb{R}^d)}} + S$ with S a $*$-regularising operator and

$$D(\overline{A}^{1/2}) = \left\{v \in L^2(\mathbb{R}^d) \middle| b^w v \in L^2(\mathbb{R}^d)\right\}.$$

Now, we apply Theorem 10.6 to b to obtain (10.6.6). Since $t \mapsto S(\mathbf{u}(t))^w$ belongs to $\mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)))$ ($\mathbf{u} \in \mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}'^*(\mathbb{R}^d)))$) we have

$$\begin{cases} (\partial_t + b^w + S)(\mathbf{u}(t))^w = \tilde{\mathbf{K}}(t), \ t \in [0,\infty), \\ (\mathbf{u}(0))^w = \mathrm{Id}, \end{cases} \tag{10.6.8}$$

for some $\tilde{\mathbf{K}} \in \mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)))$.

On the other hand, since $\overline{A}$ is non-negative and densely defined, [14, Theorem 5.5.2, p. 131] (cf. [14, Theorem 5.4.1, p. 123; Theorem A.7.6, p. 329]) implies that $-\overline{A}^{1/2}$ is the infinitesimal generator of an analytic semigroup $T(t)$ of amplitude less than $\pi/2$ and

$$T(t) = \frac{2}{\pi} \lim_{s\to\infty} \int_0^s \lambda \sin(t\lambda)(\overline{A} + \lambda^2 \mathrm{Id})^{-1} d\lambda, \ \ t > 0, \tag{10.6.9}$$

where the limit exists in $\mathscr{L}_b(L^2(\mathbb{R}^d))$. In fact, one can prove that

$$T(t) \in \mathscr{L}(\mathscr{S}^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)), \ \text{for all } t \geq 0,$$

and $t \mapsto T(t)$ belongs to $\mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)))$ (see [36, Lemma 7.14]); furthermore, $T(t)$ and $(\mathbf{u}(t))^w$ are the same modulo a smooth family of $*$-regularising operators. More precisely, we have the following result.

Theorem 10.7 ([36, Theorem 7.12]) *Let a be a hypoelliptic symbol in $\Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ that satisfies the assumptions of Theorem 10.5 and let $T(t)$, $t \geq 0$, be the analytic semigroup generated by $-\overline{A}^{1/2}$. There exists $u \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R}^{2d})$ such that the mapping $t \mapsto \mathbf{u}(t) = u(t,\cdot)$ belongs to $\mathscr{C}^\infty([0,\infty); \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d}))$ and $T(t) = (\mathbf{u}(t))^w + \mathbf{Q}(t)$, where the mapping $t \mapsto \mathbf{Q}(t)$ belongs to $\mathscr{C}^\infty([0,\infty); \mathscr{L}_b(\mathscr{S}'^*(\mathbb{R}^d), \mathscr{S}^*(\mathbb{R}^d)))$. Moreover, the function u satisfies the following estimate: there exists $0 < c_1 < 1$ such that for every $h > 0$, there exists $C > 0$ (resp. there exist $h, C > 0$) such that*

$$|D_t^n D_w^\alpha u(t,w)| \leq C n! h^{|\alpha|} A_\alpha |a_0(w)|^{n/2} \langle w\rangle^{-\rho|\alpha|} e^{-c_1 t |a_0(w)|^{1/2}}, \tag{10.6.10}$$

for all $\alpha \in \mathbb{N}^{2d}$, $n \in \mathbb{N}$, $(t,w) \in [0,\infty) \times \mathbb{R}^{2d}$, where $a_0 \in \Gamma^{*,\infty}_{A_p,\rho}(\mathbb{R}^{2d})$ is the symbol defined in the statement of Theorem 10.5.

Furthermore, $(\mathbf{u}(0))^w = \mathrm{Id}$, $(\mathbf{u}(t))^w \in \mathscr{L}(L^2(\mathbb{R}^d))$ for every $t \geq 0$, and there exists $C > 0$ such that $\|(\mathbf{u}(t))^w\|_{\mathscr{L}_b(L^2(\mathbb{R}^d))} \leq C$, for all $t \geq 0$. The mapping $t \mapsto (\mathbf{u}(t))^w$, $\mathbb{R}_+ \to \mathscr{L}_b(L^2(\mathbb{R}^d))$, is continuous and $(\mathbf{u}(t))^w \to \mathrm{Id}$, as $t \to 0^+$, in $\mathscr{L}_p(L^2(\mathbb{R}^d))$.

For each $n \in \mathbb{Z}_+$, $(\partial_t^n \mathbf{u}(t))^w \in \mathscr{L}(L^2(\mathbb{R}^d))$, $\forall t > 0$, and the mapping $t \mapsto (\mathbf{u}(t))^w$, belongs to $\mathscr{C}^\infty(\mathbb{R}_+; \mathscr{L}_b(L^2(\mathbb{R}^d)))$, with $\partial_t^n (\mathbf{u}(t))^w = (\partial_t^n \mathbf{u}(t))^w$, $n \in \mathbb{Z}_+$.

Note that with our semigroup related to A the unique bounded solution of $u_{tt} - Au = 0$, $u(0) \in L^2(\mathbb{R}^d)$, can be given as an action of pseudo-differential operators plus a smooth family of $*$-regularising operators on $u(0)$ (cf. [14, Theorem 6.3.2, p. 165]).

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Chapter 11
New Progress on Weighted Trudinger–Moser and Gagliardo–Nirenberg, and Critical Hardy Inequalities on Stratified Groups

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Abstract In this paper, we present a summary of our recent research on local and global weighted (singular) Trudinger–Moser inequalities with remainder terms, critical Hardy-type and weighted Gagliardo–Nirenberg inequalities on general stratified groups. These include the cases of $\mathbb{R}^n$ and Heisenberg groups. Moreover, the described critical Hardy-type inequalities give the critical case of the Hardy-type inequalities from [4].

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11.1 Introduction

In this short survey, we give a summary of our recent research on local and global weighted (singular) Trudinger–Moser inequalities with remainder terms, critical Hardy-type and weighted Gagliardo–Nirenberg inequalities on general stratified groups. The full proofs of the obtained results will appear elsewhere.

The classical Trudinger–Moser inequality takes the form

$$\sup_{f\in L_1^n(\Omega),\|\nabla f\|_{L^n(\Omega)}\leq 1}\int_{\Omega}\exp(\alpha_n|f(x)|^{\frac{n}{n-1}})dx<\infty, \tag{11.1.1}$$

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where ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$ and $\alpha_n = n\omega_{n-1}^{1/(n-1)}$. Here Ω is a bounded smooth domain in $\mathbb{R}^n$, and $L_1^n(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|f\|_{L^n(\Omega)} + \|\nabla f\|_{L^n(\Omega)}$, where ∇ is the usual gradient in $\mathbb{R}^n$. The Trudinger–Moser inequality (11.1.1) was obtained independently by Pohožaev [23], Yudovič [33], and Trudinger [30]. Then, the optimal constant α_n was found by Moser [18]. The inequality (11.1.1) was obtained in the higher-order Sobolev spaces by Adams [1].

Adimurthi and Sandeep [2] introduced the following weighted (singular) Trudinger–Moser inequality

$$\sup_{f\in L_1^n(\Omega), \|\nabla f\|_{L^n(\Omega)}\le 1} \frac{1}{|\Omega|}\int_\Omega \exp(\alpha|f(x)|^{\frac{n}{n-1}})\frac{dx}{|x|^\beta} \begin{cases} <\infty, & \alpha \le \frac{n-\beta}{n}\alpha_n; \\ =\infty, & \text{otherwise}, \end{cases} \tag{11.1.2}$$

where $\beta \in [0, n)$. In [8, 9], the authors proved that the supremum here is attained for $0 < \alpha \le 2\pi(2-\beta)$ and for any bounded domain $\Omega \subset \mathbb{R}^2$.

Since we are interested, in particular, in the Trudinger–Moser inequality on Lie groups, let us recall some results in this direction. In the setting of stratified groups, (11.1.1) was proven by Saloff-Coste in [29] with horizontal gradient instead of the Euclidean gradient in (11.1.1). Then, the sharp exponent α_Q for the Heisenberg groups was obtained in [5] and for the general stratified groups in [3]. However, when Ω has infinite volume, the above inequalities are meaningless.

Most of the existing proofs of the Trudinger–Moser inequalities on unbounded domains of the Euclidean space or of the Heisenberg group are based on rearrangement arguments, which are not readily available on the general stratified Lie groups. Therefore, we are interested in obtaining such inequalities on stratified groups. However, Yang [32] showed the following Trudinger–Moser inequalities on the entire Heisenberg group without using a rearrangement argument, namely by gluing local estimates with the help of cutoff functions. Let us recall this result: Let $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$. Let $\tau \in \mathbb{R}^+$, $Q' = Q/(Q-1)$ and ω_{2n-1} is the surface area of the unit sphere in $\mathbb{R}^{2n}$. Let $\alpha_Q = Q\sigma_Q^{1/(Q-1)}$ with $\sigma_Q = \Gamma(1/2)\Gamma(n+1/2)\omega_{2n-1}/n!$. Then for any $\beta \in [0, Q)$ and $\alpha \in (0, \alpha_Q(1-\beta/Q))$, we have

$$\sup_{\|f\|_{1,\tau}\le 1} \int_{\mathbb{H}^n} \frac{1}{(\rho(\xi))^\beta}\left(\exp(\alpha|f(\xi)|^{Q'}) - \sum_{k=0}^{Q-2}\frac{\alpha^k|f(\xi)|^{kQ'}}{k!}\right)d\xi < \infty, \tag{11.1.3}$$

where

$$\|f\|_{1,\tau} = \left(\int_{\mathbb{H}^n}(|\nabla_{\mathbb{H}^n} f(\xi)|^Q + \tau|f(\xi)|^Q)d\xi\right)^{1/Q},$$

and $\rho(\xi) = (|z|^4 + t^2)^{1/4}$ with $z = (x, y) \in \mathbb{R}^{2n}$ and $\xi = (z, t) \in \mathbb{H}^n$. Moreover, it was shown there that in the case $\alpha > \alpha_Q(1-\beta/Q)$ the integral in (11.1.3) is still finite for any $f \in L_1^Q(\mathbb{H}^n)$, but the supremum is infinite.

We also refer to [6, 7, 14–17] for the Trudinger–Moser inequalities on Heisenberg groups.

However, to obtain the weighted Trudinger–Moser inequality on fractional order Sobolev space even on Heisenberg groups, the simplest nontrivial stratified groups, is a much more delicate matter.

In this paper, we show the Trudinger–Moser inequality on fractional order Sobolev space on stratified groups using the strategy developed in [13, 22].

We are also interested in obtaining critical cases of the Hardy-type inequality, which have been obtained on stratified groups in [4]: Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q, and let $\mathscr{L}$ be a sub-Laplacian on $\mathbb{G}$ (see Sect. 11.2 for more details).

Hardy-type inequality [4, Theorem A]. Let $1 < p < \infty$ and $T_\gamma f = |\cdot|^{-\gamma}(-\mathscr{L})^{-\gamma/2} f$ with $0 < \gamma < Q/p$, where $|\cdot|$ is a homogeneous norm on $\mathbb{G}$. Then the operator T_γ extends uniquely to a bounded operator on $L^p(\mathbb{G})$, and we have

$$\|T_\gamma\|_{L^p(\mathbb{G})\to L^p(\mathbb{G})} \le 1 + C\gamma + O(\gamma^2) \tag{11.1.4}$$

for a particular choice of a homogeneous norm $|\cdot|_0$. We refer to [4] for the history of (11.1.4).

In this paper when $\gamma = Q/p$, we describe different types of the critical Hardy inequalities, which can be thought as a critical case of (11.1.4).

For the convenience of the reader let us now briefly recapture the main results of this short survey. Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q and let $\mathscr{L}$ be a sub-Laplacian on $\mathbb{G}$. Let $|\cdot|$ be homogeneous norm on $\mathbb{G}$. Let $B(x_0, r)$ is the quasi-ball of radius r in $\mathbb{G}$ centered at the origin x_0. Let $L^p_{Q/p}(B(x_0, r))$ with $1 < p < \infty$ be the completion of $C_0^\infty(B(x_0, r))$ with respect to the norm

$$\|f\|_{L^p_{Q/p}(B(x_0,r))} = \left(\int_{B(x_0,r)} (|(-\mathscr{L})^{\frac{Q}{2p}} f(x)|^p + |f(x)|^p)dx\right)^{1/p}. \tag{11.1.5}$$

Then, we have

- **(Local weighted Trudinger–Moser inequalities I)**. Let $1 < p < \infty$ and $\beta \in [0, Q)$. Let $r > 0$ be given, and let x_0 be any point of $\mathbb{G}$. Then there exist positive constants $C_1 = C_1(p, Q, \alpha, \beta, r)$ and $C_2 = C_2(p, Q, \beta)$, so that we have

$$\int_{B(x_0,r)} \frac{1}{|\cdot|^\beta}\left(\exp(\alpha|f(x)|^{p'}) - \sum_{0\le k<p-1,\ k\in\mathbb{N}} \frac{\alpha^k|f(x)|^{kp'}}{k!}\right)dx$$
$$\le C_1\|f\|^p_{L^p_{Q/p}(B(x_0,r))}, \tag{11.1.6}$$

for any $\alpha \in [0, C_2)$ and for all $f \in L^p_{Q/p}(B(x_0, r))$ satisfying $\|f\|_{L^p_{Q/p}(B(x_0,r))} \le 1$, where $1/p + 1/p' = 1$, and the space $L^p_{Q/p}(B(x_0, r))$ is defined in (11.1.5).

- **(Local weighted Trudinger–Moser inequalities II)**. Let $\beta \in [0, Q)$ and let α_Q be as in Theorem 11.3.8. Let $r > 0$ be given and let x_0 be any point of $\mathbb{G}$. Then there exists a positive constant $C = C(Q, r, \beta)$ such that

$$\int_{B(x_0,r)} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{Q'}) - \sum_{k=0}^{Q-2} \frac{\alpha^k |f(x)|^{kQ'}}{k!} \right) dx \leq C \|\nabla_H f\|^{Q}_{L^Q(B(x_0,r))} \quad (11.1.7)$$

holds for all $f \in L_1^Q(B(x_0,r))$ satisfying $\|\nabla f\|_{L^Q(B(x_0,r))} \leq 1$ and any $\alpha \in [0, \alpha_Q (1 - \beta/Q)]$, where $Q' = Q/(Q-1)$, and the space $L_1^Q(B(x_0,r))$ is defined in (11.1.5).

- **(Weighted Trudinger–Moser inequalities with remainder terms I)**. Let $1 < p < \infty$ and $Q/(Q-\beta) < \mu < \infty$ with $\beta \in [0, Q)$. Then, there exist positive constants $C_2 = C_2(p, Q, \beta, \mu)$ and $C_3 = C_3(p, Q, \alpha, \beta, \mu)$ such that

$$\int_{\mathbb{G}} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \leq k < p-1,\ k \in \mathbb{N}} \frac{\alpha^k |f(x)|^{kp'}}{k!} \right) dx$$

$$\leq C_3 (\|f\|^{p}_{L^p(\mathbb{G})} + \|f\|^{p/\mu}_{L^p(\mathbb{G})}) \quad (11.1.8)$$

holds for all $\alpha \in (0, C_2)$, and for all functions $f \in L^p_{Q/p}(\mathbb{G})$ satisfying $\|(-\mathscr{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbb{G})} \leq 1$, where $1/p + 1/p' = 1$.

- **(Weighted Trudinger–Moser inequalities with remainder terms II)**. Let α_Q be as in Theorem 11.3.8. Then we have

$$\sup_{\|f\|_{L_1^Q(\mathbb{G})} \leq 1} \int_{\mathbb{G}} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{Q'}) - \sum_{k=0}^{Q-2} \frac{\alpha^k |f(x)|^{kQ'}}{k!} \right) dx < \infty \quad (11.1.9)$$

for any $\beta \in [0, Q)$ and $\alpha \in (0, \alpha_Q(1 - \beta/Q))$, where $Q' = Q/(Q-1)$. When $\alpha > \alpha_Q(1 - \beta/Q)$, the integral in (11.1.9) is still finite for any $f \in L_1^Q(\mathbb{G})$, but the supremum is infinite.

- **(Critical Hardy-type inequalities I)**. Let $1 < p < \infty$ and $\beta \in [0, Q)$. Let $r > 0$ be given, and let x_0 be any point of $\mathbb{G}$. Then for any $p \leq q < \infty$, there exists a positive constant $C = C(p, Q, \beta, r, q)$ such that

$$\left\| \frac{f}{|\cdot|^{\frac{\beta}{q}}} \right\|_{L^q(B(x_0,r))} \leq C (\|f\|_{L^p(B(x_0,r))} + \|(-\mathscr{L})^{\frac{Q}{2p}} f\|_{L^p(B(x_0,r))}) \quad (11.1.10)$$

holds for all $f \in L^p_{Q/p}(B(x_0,r))$, where the space $L^p_{Q/p}(B(x_0,r))$ is defined in (11.1.5).

- **(Critical Hardy-type inequalities II)**. Let $\beta \in [0, Q)$. Let $r > 0$ be given, and let x_0 be any point of $\mathbb{G}$. Then for any $Q \le q < \infty$, there exists a positive constant $C = C(Q, \beta, r, q)$ such that

$$\left\| \frac{f}{|\cdot|^{\frac{\beta}{q}}} \right\|_{L^q(B(x_0,r))} \le C \|\nabla_H f\|_{L^Q(B(x_0,r))} \tag{11.1.11}$$

holds for all $f \in L_1^Q(B(x_0, r))$, where the space $L_1^Q(B(x_0, r))$ is defined in (11.1.5).

- **(Critical Hardy-type inequalities III)** Let $1 < p_1 < p_3 < \infty$ and $p_1 < p_2 < (p_3 - 1)p_1^{'}$. Then, there exists a positive constant $C = C(p_1, p_2, p_3, Q)$ such that

$$\left\| \frac{f}{\left(\log\left(e + \frac{1}{|\cdot|}\right)\right)^{\frac{p_3}{p_2}} |\cdot|^{\frac{Q}{p_2}}} \right\|_{L^{p_2}(\mathbb{G})} \le C(\|f\|_{L^{p_1}(\mathbb{G})} + \|(-\mathscr{L})^{\frac{Q}{2p_1}} f\|_{L^{p_1}(\mathbb{G})}) \tag{11.1.12}$$

holds for all $f \in L^{p_1}_{Q/p_1}(\mathbb{G})$, where $1/p_1 + 1/p_1^{'} = 1$.

- **(Weighted Gagliardo–Nirenberg inequalities)**. Let $1 < p < \infty$ and $\beta \in [0, Q)$ with $Q/(Q - \beta) < \mu < \infty$. Then for any $p \le q < \infty$, there exists a positive constant $C = C(p, Q, \beta, \mu, q)$ such that

$$\left\| \frac{f}{|\cdot|^{\frac{\beta}{q}}} \right\|_{L^q(\mathbb{G})} \le C(\|(-\mathscr{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbb{G})}^{1-p/q} \|f\|_{L^p(\mathbb{G})}^{p/q} + \|(-\mathscr{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbb{G})}^{1-p/(q\mu)} \|f\|_{L^p(\mathbb{G})}^{p/(q\mu)}) \tag{11.1.13}$$

holds for all $f \in L^p_{Q/p}(\mathbb{G})$.

We note that the inequalities (11.1.8) and (11.1.9) are generalized versions of the inequality (11.1.3). Moreover, (11.1.10), (11.1.11), and (11.1.12) give the critical cases of the Hardy-type inequalities (11.1.4). Similarly to (11.1.8), (11.1.12) and (11.1.13) were investigated in the Euclidean setting in [20]. For the so-called Caffarelli–Kohn–Nirenberg-type inequalities, similar to (11.1.13), we refer to [24] on stratified groups, to [25, 26] on general homogeneous groups, to [31] on Lie groups of polynomial growth and to [19, 28] on Riemannian manifolds as well as references therein.

The remaining part of this note is organized as follows. In Sect. 11.2, we briefly recall the necessary concepts of stratified Lie groups and fix the notation. The local and global weighted Trudinger–Moser inequalities with remainder terms on stratified groups are described in more detail in Sect. 11.3. Finally, in Sects. 11.4 and 11.5, we discuss the critical Hardy-type and weighted Gagliardo–Nirenberg inequalities on stratified groups.

11.2 Preliminaries

In this section, we very briefly recall the necessary notation concerning the setting of stratified groups.

We say that $\mathbb{G} = (\mathbb{R}^n, \circ)$ is a stratified group (or a homogeneous Carnot group) if it satisfies the following two conditions:

- Let $N_i \in \mathbb{N}$ for $i = 1, \dots, r$ with $N = N_1$. Let $x' \equiv x^{(1)} \in \mathbb{R}^N$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \dots, r$. Then we have the decomposition $\mathbb{R}^n = \mathbb{R}^N \times \dots \times \mathbb{R}^{N_r}$, and for every $\lambda > 0$ the dilation $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\delta_\lambda(x) = \delta_\lambda(x', x^{(2)}, \dots, x^{(r)}) := (\lambda x', \lambda^2 x^{(2)}, \dots, \lambda^r x^{(r)})$$

  is an automorphism of the group $\mathbb{G}$.
- Let N be as in above. Let $X_1, \dots, X_N$ be the left invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \dots, N$. Then we have

$$\operatorname{rank}(\operatorname{Lie}\{X_1, \dots, X_N\}) = n,$$

  for every $x \in \mathbb{R}^n$, that is, the iterated commutators of $X_1, \dots, X_N$ span the Lie algebra of the group $\mathbb{G}$.

Such groups have been thoroughly investigated by Folland [10]. We also refer to [12] (see also [11]) for more detailed discussions from the point of view of more general stratified Lie groups.

Recall that the homogeneous dimension of $\mathbb{G}$ is defined by

$$Q = \sum_{k=1}^{r} kN_k, \quad N_1 = N.$$

Let us also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g., [11, Proposition 1.6.6]). The (canonical) sub-Laplacian and horizontal gradient on the stratified group $\mathbb{G}$ are defined by

$$\mathscr{L} = \sum_{k=1}^{N} X_k^2$$

and

$$\nabla_H := (X_1, \dots, X_N),$$

respectively.

11.3 Weighted Trudinger–Moser Inequalities with Remainder Terms

In this section, we discuss local and global weighted Trudinger–Moser inequalities on stratified group $\mathbb{G}$.

First, let us recall the following unweighted Trudinger–Moser and Gagliardo–Nirenberg inequalities on graded groups, which include the cases of $\mathbb{R}^n$, Heisenberg, and general stratified groups:

Theorem 11.3.1 ([27, Theorem 3.3])] *Let $\mathbb{G}$ be a graded Lie group of homogeneous dimension Q, and let $\mathscr{R}$ be a positive Rockland operator of homogeneous degree ν. Then, there exists a positive constant $\widetilde{C_1} = \widetilde{C_1}(p, Q)$ such that*

$$\|f\|_{L^q(\mathbb{G})} \leq \widetilde{C_1} q^{1-1/p} \|\mathscr{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})}^{1-p/q} \|f\|_{L^p(\mathbb{G})}^{p/q}, \quad 1 < p < \infty, \tag{11.3.1}$$

holds for any q with $p \leq q < \infty$ and all functions $f \in L^p_{Q/p}(\mathbb{G})$.

Theorem 11.3.2 ([27, Theorem 3.5]) *Let $\mathbb{G}$ be a graded group of homogeneous dimension Q, and let $\mathscr{R}$ be a positive Rockland operator of homogeneous degree ν. Then, there exist positive constants α and $\widetilde{C_2}$ such that*

$$\int_{\mathbb{G}} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \leq k < p-1,\ k \in \mathbb{N}} \frac{1}{k!} (\alpha |f(x)|^{p'})^k \right) dx \leq \widetilde{C_2} \|f\|_{L^p(\mathbb{G})}^p, \quad 1 < p < \infty, \tag{11.3.2}$$

holds for any function $f \in L^p_{Q/p}(\mathbb{G})$ satisfying $\|\mathscr{R}^{\frac{Q}{\nu p}} f\|_{L^p(\mathbb{G})} \leq 1$, where $1/p + 1/p' = 1$.

Remark 11.3.3 ([27, Remark 3.6]) We note that the constant $\widetilde{C_2}$ in (11.3.2) can be expressed in terms of the constant $\widetilde{C_1} = \widetilde{C_1}(p, Q)$ from (11.3.1) as follows

$$\widetilde{C_2} = \widetilde{C_2}(\alpha) = \sum_{k \geq p-1,\ k \in \mathbb{N}} \frac{k^k}{k!} (p' \widetilde{C_1}^{p'} \alpha)^k.$$

Then, it is easy to see that the inequality (11.3.2) is valid for all $\alpha \in (0, (ep' \widetilde{C_1}^{p'})^{-1})$ and $\widetilde{C_2}(\alpha)$.

We also recall the following result:

Theorem 11.3.4 ([3, Theorem 2.6]) *Let $\mathbb{G}$ be a stratified group with homogeneous dimension Q, and let u_Q be a singular solution for the subelliptic Q-Laplacian with pole at $0 \in \mathbb{G}$. Then, there exists a positive constant a_Q such that the function*

$$N(x) = \exp(-a_Q u_Q(x)) \tag{11.3.3}$$

is a homogeneous norm on $\mathbb{G}$.

From Theorems 11.3.1 and 11.3.2, one can obtain the following corollary:

Corollary 11.3.5 *Let $\mathbb{G}$ be a stratified group of homogeneous dimension Q. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. Then, there exist some positive constants α and $\widetilde{C_3} = \widetilde{C_3}(p, Q, \alpha)$ such that*

$$\int_{\Omega} \exp(\alpha |f(x)|^{p'}) dx \leq \widetilde{C_3} \tag{11.3.4}$$

holds for any bounded smooth domain $\Omega \subset \mathbb{G}$, and for all functions $f \in L^p_{Q/p}(\Omega)$ satisfying $\|f\|_{L^p_{Q/p}(\Omega)} \leq 1$.

Remark 11.3.6 Remark 11.3.3 shows that the inequality (11.3.4) is valid for all $\alpha \in [0, (ep' \widetilde{C_1}^{p'})^{-1})$, where $\widetilde{C_1}$ is defined in (11.3.1). We also note that the smallest constant $\widetilde{C_1}$ (and hence also $\widetilde{C_2}$) can be expressed in the variational form as well as in terms of the ground state solutions of the nonlinear Schrödinger-type equations (see [27, Sect. 5]).

Using Corollary 11.3.5, we can obtain the following Trudinger–Moser inequalities on fractional order Sobolev space.

Theorem 11.3.7 *Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q, and let $|\cdot|$ be a homogeneous norm on $\mathbb{G}$. Let $\beta \in [0, Q)$ and $1 < p < \infty$. Let $r > 0$ be given, and let x_0 be any point of $\mathbb{G}$. Then there exist some positive constants $C_1 = C_1(p, Q, \alpha, \beta, r)$ and $C_2 = C_2(p, Q, \beta)$ such that*

$$\int_{B(x_0,r)} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \leq k < p-1,\, k \in \mathbb{N}} \frac{\alpha^k |f(x)|^{kp'}}{k!} \right) dx \leq C_1 \|f\|^p_{L^p_{Q/p}(B(x_0,r))} \tag{11.3.5}$$

holds for any $\alpha \in [0, C_2)$ and all $f \in L^p_{Q/p}(B(x_0, r))$ satisfying $\|f\|_{L^p_{Q/p}(B(x_0,r))} \leq 1$, where the space $L^p_{Q/p}(B(x_0, r))$ is defined in (11.1.5) and $1/p + 1/p' = 1$.

In the case $p = Q$, applying [3, Theorem 4.1], assuming $\|\nabla_H f\|_{L^Q(B(x_0,r))} \leq 1$ and using the strategy developed in [32], we can also obtain the following Theorems 11.3.8 and 11.3.9:

Theorem 11.3.8 *Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q, and let $|\cdot|$ be a homogeneous norm on $\mathbb{G}$. Let $\beta \in [0, Q)$, and let $r > 0$ be given, and let x_0 be any point of $\mathbb{G}$. Then, there exists a positive constant $C = C(Q, r, \beta)$ such that*

$$\int_{B(x_0,r)} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{Q'}) - \sum_{k=0}^{Q-2} \frac{\alpha^k |f(x)|^{kQ'}}{k!} \right) dx \leq C \|\nabla_H f\|^Q_{L^Q(B(x_0,r))} \tag{11.3.6}$$

holds for all $f \in L_1^Q(B(x_0,r))$ *satisfying* $\|\nabla f\|_{L^Q(B(x_0,r))} \le 1$ *and any* $\alpha \in [0, \alpha_Q(1-\beta/Q)]$ *with* $\alpha_Q = Q c_Q^{Q'-1}$, *where* $Q' = Q/(Q-1)$, *and the space* $L_1^Q(B(x_0,r))$ *is defined in* (11.1.5). *Here* $c_Q = \int_{\wp} |\nabla_H N(y)|^Q d\sigma(y)$, $N(x)$ *is a homogeneous norm on* $\mathbb{G}$ *(see* (11.3.3)*), and* $\wp := \{x \in \mathbb{G} : |x| = 1\}$ *is the unit sphere with respect to the homogeneous norm* N *from Theorem 11.3.4.*

Theorem 11.3.9 *Let* $\mathbb{G}$ *be a stratified Lie group of homogeneous dimension* Q, *and let* $|\cdot|$ *be a homogeneous norm on* $\mathbb{G}$. *Let* α_Q *be as in Theorem 11.3.8. Then, we have*

$$\sup_{\|f\|_{L_1^Q(\mathbb{G})} \le 1} \int_{\mathbb{G}} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{Q'}) - \sum_{k=0}^{Q-2} \frac{\alpha^k |f(x)|^{kQ'}}{k!} \right) dx < \infty \tag{11.3.7}$$

for any $\beta \in [0, Q)$ *and* $\alpha \in (0, \alpha_Q(1-\beta/Q))$, *where* $Q' = Q/(Q-1)$. *When* $\alpha > \alpha_Q(1-\beta/Q)$, *the integral in* (11.3.7) *is still finite for any* $f \in L_1^Q(\mathbb{G})$, *but the supremum is infinite.*

Remark 11.3.10 In the case when $\mathbb{G}$ is the Heisenberg group, and $|\cdot|$ is the Kaplan distance, the obtained Theorems 11.3.8 and 11.3.9 were established in [32].

Now we introduce the weighted Trudinger–Moser inequality with remainder terms on the entire stratified group.

Theorem 11.3.11 *Let* $\mathbb{G}$ *be a stratified Lie group of homogeneous dimension* Q, *and let* $|\cdot|$ *be a homogeneous norm on* $\mathbb{G}$. *Let* $1 < p < \infty$ *and* $Q/(Q-\beta) < \mu < \infty$ *with* $\beta \in [0, Q)$. *Then, there exist positive constants* $C_2 = C_2(p, Q, \beta, \mu)$ *and* $C_3 = C_3(p, Q, \alpha, \beta, \mu)$ *such that*

$$\int_{\mathbb{G}} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \le k < p-1,\, k \in \mathbb{N}} \frac{\alpha^k |f(x)|^{kp'}}{k!} \right) dx \le C_3 (\|f\|^p_{L^p(\mathbb{G})} + \|f\|^{p/\mu}_{L^p(\mathbb{G})}) \tag{11.3.8}$$

holds for all $\alpha \in (0, C_2)$ *and all functions* $f \in L^p_{Q/p}(\mathbb{G})$ *satisfying* $\|(-\mathscr{L})^{\frac{Q}{2p}} f\|_{L^p(\mathbb{G})} \le 1$, *where* $1/p + 1/p' = 1$.

If we take supremum over $\|f\|_{L^p_{Q/p}(\mathbb{G})} \le 1$ in (11.3.8), then we obtain the following weighted Trudinger–Moser inequality:

Corollary 11.3.12 *Let* $\mathbb{G}$ *be a stratified Lie group of homogeneous dimension* Q, *and let* $|\cdot|$ *be a homogeneous norm on* $\mathbb{G}$. *Let* $1 < p < \infty$ *and* $\beta \in [0, Q)$. *Then, there exist positive constants* $C_2 = C_2(p, Q, \beta)$ *and* $C_3 = C_3(p, Q, \alpha, \beta)$ *such that*

$$\sup_{\|f\|_{L^p_{Q/p}(\mathbb{G})} \le 1} \int_{\mathbb{G}} \frac{1}{|\cdot|^{\beta}} \left(\exp(\alpha |f(x)|^{p'}) - \sum_{0 \le k < p-1,\, k \in \mathbb{N}} \frac{\alpha^k |f(x)|^{kp'}}{k!} \right) dx \le C_3 \tag{11.3.9}$$

holds for all $\alpha \in (0, C_2)$, *where* $1/p + 1/p' = 1$.

Remark 11.3.13 In the special case, when $\mathbb{G}$ is the Heisenberg group and $|\cdot|$ is the Kaplan distance with $p = Q$, Corollary 11.3.12 was obtained in [32, Theorem 1.1]. We note that when $\beta = 0$, the unweighted Trudinger–Moser inequality similar to (11.3.8) was investigated in [21] on $\mathbb{G} = (\mathbb{R}^n, +)$ with $\mathscr{L} = \triangle$ the Laplacian, and in [27, Theorem 3.5] on the graded groups $\mathbb{G}$ with the Rockland operator.

11.4 Critical Hardy-Type Inequalities

In this section, we show critical Hardy-type inequalities, which are critical case of (11.1.4) when $\gamma = Q/p$.

Theorem 11.4.1 *Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q, and let $|\cdot|$ be a homogeneous norm on $\mathbb{G}$. Let $1 < p < \infty$ and $\beta \in [0, Q)$. Let $r > 0$ be given, and let x_0 be any point of $\mathbb{G}$. Then for any $p \le q < \infty$, there exists a positive constant $C = C(p, Q, \beta, r, q)$ such that*

$$\left\| \frac{f}{|\cdot|^{\frac{\beta}{q}}} \right\|_{L^q(B(x_0,r))} \le C \|f\|_{L^p_{Q/p}(B(x_0,r))} \tag{11.4.1}$$

holds for all $f \in L^p_{Q/p}(B(x_0, r))$, where the space $L^p_{Q/p}(B(x_0, r))$ is defined in (11.1.5).

In the case $p = Q$, we have the following improved version of Theorem 11.4.1.

Theorem 11.4.2 *Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q, and let $|\cdot|$ be a homogeneous norm on $\mathbb{G}$. Let $\beta \in [0, Q)$. Let $r > 0$ be given, and let x_0 be any point of $\mathbb{G}$. Then for any $Q \le q < \infty$, there exists a positive constant $C = C(Q, \beta, r, q)$ such that*

$$\left\| \frac{f}{|\cdot|^{\frac{\beta}{q}}} \right\|_{L^q(B(x_0,r))} \le C \|\nabla_H f\|_{L^Q(B(x_0,r))} \tag{11.4.2}$$

holds for all $f \in L^Q_1(B(x_0, r))$, where the space $L^Q_1(B(x_0, r))$ is defined in (11.1.5).

Remark 11.4.3 In the special case, the inequalities (11.4.1) with $q = p$ and (11.4.2) with $q = Q$ imply the critical case of (11.1.4) for all $f \in L^p_{Q/p}(B(x_0, r))$ and $f \in L^Q_1(B(x_0, r))$, respectively.

Now we introduce the critical case $\beta = Q$ of Theorem 11.4.1.

Theorem 11.4.4 *Let $\mathbb{G}$ be a stratified Lie group of homogeneous dimension Q, and let $|\cdot|$ be a homogeneous norm on $\mathbb{G}$. Let $1 < p_1 < p_3 < \infty$ and $p_1 < p_2 < (p_3 - 1)p_1'$. Then, there exists a positive constant $C = C(p_1, p_2, p_3, Q)$ such that*

$$\left\| \frac{f}{\left(\log\left(e+\frac{1}{|\cdot|}\right)\right)^{\frac{p_3}{p_2}} |\cdot|^{\frac{Q}{p_2}}} \right\|_{L^{p_2}(\mathbb{G})} \leq C\|f\|_{L^{p_1}_{Q/p_1}(\mathbb{G})} \tag{11.4.3}$$

holds for all $f \in L^{p_1}_{Q/p_1}(\mathbb{G})$*, where* $1/p_1 + 1/p_1' = 1$.

11.5 Weighted Gagliardo–Nirenberg Inequalities

In this section, we establish weighted Gagliardo–Nirenberg inequalities.

Theorem 11.5.1 *Let* $\mathbb{G}$ *be a stratified Lie group of homogeneous dimension* Q*, and let* $|\cdot|$ *be a homogeneous norm on* $\mathbb{G}$*. Let* $1 < p < \infty$ *and* $Q/(Q-\beta) < \mu < \infty$ *with* $\beta \in [0, Q)$*. Then for any* $p \leq q < \infty$*, there exists a positive constant* $C = C(p, Q, \beta, \mu, q)$ *such that*

$$\left\| \frac{f}{|\cdot|^{\frac{\beta}{q}}} \right\|_{L^q(\mathbb{G})} \leq C(\|(-\mathscr{L})^{\frac{Q}{2p}} f\|^{1-p/q}_{L^p(\mathbb{G})} \|f\|^{p/q}_{L^p(\mathbb{G})} + \|(-\mathscr{L})^{\frac{Q}{2p}} f\|^{1-p/(q\mu)}_{L^p(\mathbb{G})} \|f\|^{p/(q\mu)}_{L^p(\mathbb{G})}) \tag{11.5.1}$$

holds for any function $f \in L^p_{Q/p}(\mathbb{G})$.

Theorem 11.5.1 gives the following corollary when $\beta = 0$:

Corollary 11.5.2 *Let* $\mathbb{G}$ *be a stratified Lie group of homogeneous dimension* Q*, and let* $1 < p < \infty$*. Then,* $L^p_{Q/p}(\mathbb{G})$ *is continuously embedded in* $L^q(\mathbb{G})$ *for any* $p \leq q < \infty$.

Remark 11.5.3 We note that Corollary 11.5.2 was obtained in [21] on $\mathbb{G} = (\mathbb{R}^n, +)$ and in [32, Lemma 4.1] on the Heisenberg group for $p = Q$.

Remark 11.5.4 We also note that the Corollary 11.5.2 shows the critical case $a = Q/p$ of the continuous embedding $L^p_a(\mathbb{G}) \hookrightarrow L^q(\mathbb{G})$ [11, Proposition 4.4.13] on stratified groups, where $1/q = 1/p - a/Q$ and $0 < a < Q/p$.

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Chapter 12
Continuity Properties of Multilinear Localization Operators on Modulation Spaces

Nenad Teofanov

Abstract We introduce multilinear localization operators in terms of the short-time Fourier transform and multilinear Weyl pseudodifferential operators. We prove that such localization operators are in fact Weyl pseudodifferential operators whose symbols are given by the convolution between the symbol of the localization operator and the multilinear Wigner transform. To obtain such interpretation, we use the kernel theorem for the Gelfand–Shilov space $\mathscr{S}^{(1)}(\mathbb{R}^d)$ and its dual space of tempered ultra-distributions $\mathscr{S}^{(1)'}(\mathbb{R}^{2d})$. Furthermore, we study the continuity properties of the multilinear localization operators on modulation spaces. Our results extend some known results when restricted to the linear case.

12.1 Introduction

Multilinear localization operators were first introduced in [8], and their continuity properties are formulated in terms of modulation spaces. The key point is the interpretation of these operators as multilinear Kohn–Nirenberg pseudodifferential operators. The multilinear pseudodifferential operators were already studied in the context of modulation spaces in [1]; see also a more recent contribution [24] where such approach is strengthened and applied to the bilinear and trilinear Hilbert transforms.

Our approach is related to Weyl pseudodifferential operators instead, with another (Weyl) correspondence between the operator and its symbol. Both correspondences are particular cases of the so-called $\tau-$pseudodifferential operators, $\tau \in [0, 1]$. For $\tau = 1/2$ we obtain Weyl operators, while for $\tau = 0$ we recapture Kohn–Nirenberg operators. We refer to [7, 10] for the recent contribution in that context (see also the references given there).

The Weyl correspondence provides an elegant interpretation of localization operators as Weyl pseudodifferential operators. This is given by the formula that contains

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the Wigner transform which is, together with the short-time Fourier transform, the main tool in our investigations. We refer to [17, 41] for more details on the Wigner transform.

In signal analysis, different localization techniques are used to describe signals which are as concentrated as possible in general regions of the phase space. This motivated I. Daubechies to address these questions by introducing certain localization operators in the pioneering contribution [14]. Afterward, Cordero and Grochenig made an essential contribution in the context of time–frequency analysis [6]. Among other things, their results emphasized the role played by modulation spaces in the study of localization operators.

In this paper, we first recall the basic facts on modulation spaces in Sect. 12.2. Then, in Sect. 12.3, following the definition of bilinear localization operators given in [33] we introduce multilinear localization operators, Definition 12.2. Then we define the multilinear Weyl pseudodifferential operators and give their weak formulation in terms of the multilinear Wigner transform (Lemma 12.2). By using the kernel theorem for Gelfand–Shilov spaces, Theorem 12.1, we prove that the multilinear localization operators can be interpreted as multilinear Weyl pseudodifferential operators in the same way as in the linear case, Theorem 12.5.

In Sect. 12.4 we first recall two results from [9]: (multilinear version of) sharp integral bounds for the Wigner transform, Theorem 12.6, and continuity properties of pseudodifferential operators on modulation spaces, Theorem 12.8. These results, in combination with the convolution estimates for modulation spaces from [38], Theorem 12.3, are then used to prove the main result of the continuity properties of multilinear localization operators on modulation spaces, Theorem 12.9.

Notation. The Schwartz space of rapidly decreasing smooth functions is denoted by $\mathscr{S}(\mathbb{R}^d)$, and its dual space of tempered distributions is denoted by $\mathscr{S}'(\mathbb{R}^d)$. We use the brackets $\langle f, g\rangle$ to denote the extension of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$ to any pair of dual spaces. The Fourier transform is normalized to be

$$\hat{f}(\omega) = \mathscr{F} f(\omega) = \int f(t)e^{-2\pi i t\omega}dt.$$

The involution f^* is $f^*(\cdot) = \overline{f(-\cdot)}$, and the convolution of f and g is given by $f * g(x) = \int f(x-y)g(y)dy$, when the integral exists.

We denote by $\langle\cdot\rangle^s$ the polynomial weights

$$\langle(x,\omega)\rangle^s = (1+|x|^2+|\omega|^2)^{s/2}, \quad (x,\omega)\in\mathbb{R}^{2d}, \quad s\in\mathbb{R},$$

and $\langle x\rangle = \langle 1+|x|^2\rangle^{1/2}$, when $x\in\mathbb{R}^d$.

We use the notation $A \lesssim B$ to indicate that $A \le cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means that $c^{-1}A \le B \le cA$ for some $c \ge 1$.

The Gelfand–Shilov space and Weyl pseudodifferential operators. The Gelfand–Shilov-type space of analytic functions $\mathscr{S}^{(1)}(\mathbb{R}^d)$ is given by

$$f \in \mathscr{S}^{(1)}(\mathbb{R}^d) \Longleftrightarrow \sup_{x \in \mathbb{R}^d} |f(x)e^{h\cdot|x|}| < \infty \text{ and } \sup_{\omega \in \mathbb{R}^d} |\hat{f}(\omega)e^{h\cdot|\omega|}| < \infty, \quad \forall h > 0.$$

Any $f \in \mathscr{S}^{(1)}(\mathbb{R}^d)$ can be extended to a holomorphic function $f(x+iy)$ in the strip $\{x+iy \in \mathbb{C}^d \ : \ |y| < T\}$ some $T > 0$, [18, 25]. The dual space of $\mathscr{S}^{(1)}(\mathbb{R}^d)$ will be denoted by $\mathscr{S}^{(1)'}(\mathbb{R}^d)$.

The space $\mathscr{S}^{(1)}(\mathbb{R}^d)$ is nuclear, and we will use the following kernel theorem in the context of $\mathscr{S}^{(1)}(\mathbb{R}^d)$.

Theorem 12.1 *Let $\mathscr{L}_b(\mathscr{A}, \mathscr{B})$ denote the space of continuous linear mappings between the spaces $\mathscr{A}$ and $\mathscr{B}$ (equipped with the topology of bounded convergence). Then the following isomorphisms hold:*

1. $\mathscr{S}^{(1)}(\mathbb{R}^{d_1})\hat{\otimes}\mathscr{S}^{(1)}(\mathbb{R}^{d_2}) \cong \mathscr{S}^{(1)}(\mathbb{R}^{d_1+d_2}) \cong \mathscr{L}_b(\mathscr{S}^{(1)'}(\mathbb{R}^{d_1}), \mathscr{S}^{(1)}(\mathbb{R}^{d_2}))$,
2. $\mathscr{S}^{(1)'}(\mathbb{R}^{d_1})\hat{\otimes}\mathscr{S}^{(1)'}(\mathbb{R}^{d_2}) \cong \mathscr{S}^{(1)'}(\mathbb{R}^{d_1+d_2}) \cong \mathscr{L}_b(\mathscr{S}^{(1)}(\mathbb{R}^{d_1}), \mathscr{S}^{(1)'}(\mathbb{R}^{d_2}))$.

Theorem 12.1 is a special case of [31, Theorem 2.5], see also [27], so we omit the proof. We refer to the classical reference [40] for kernel theorems and nuclear spaces, and in particular to Theorem 51.6 and its corollary related to $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$, which will be used later on.

By the isomorphisms in Theorem 12.1 2. it follows that for a given kernel distribution $k(x, y)$ on $\mathbb{R}^{d_1+d_2}$ we may associate a continuous linear mapping k of $\mathscr{S}^{(1)}(\mathbb{R}^{d_2})$ into $\mathscr{S}^{(1)'}(\mathbb{R}^{d_1})$ as follows:

$$\langle k_\varphi, \phi\rangle = \langle k(x, y), \phi(x)\varphi(y)\rangle, \quad \phi \in \mathscr{S}^{(1)}(\mathbb{R}^{d_1}),$$

which is commonly written as $k_\varphi(\cdot) = \int k(\cdot, y)\varphi(y)dy$. The correspondence between $k(x, y)$ and k is an isomorphism and this fact will be used in the proof of Theorem 12.5.

Let $\sigma \in \mathscr{S}^{(1)}(\mathbb{R}^{2d})$. Then the Weyl pseudodifferential operator L_σ with the Weyl symbol σ can be defined as the oscillatory integral:

$$L_\sigma f(x) = \iint \sigma(\frac{x+y}{2}, \omega) f(y) e^{2\pi i(x-y)\cdot\omega} dy d\omega, \quad f \in \mathscr{S}^{(1)}(\mathbb{R}^d).$$

This definition extends to each $\sigma \in \mathscr{S}^{(1)'}(\mathbb{R}^{2d})$, so that L_σ is a continuous mapping from $\mathscr{S}^{(1)}(\mathbb{R}^d)$ to $\mathscr{S}^{(1)'}(\mathbb{R}^d)$, cf. [19, Lemma 14.3.1] If

$$W(f, g)(x, \omega) = \int f(x + \frac{t}{2})\overline{g(x - \frac{t}{2})} e^{-2\pi i \omega t}\, dt, \qquad f, g \in \mathscr{S}^{(1)}(\mathbb{R}^d), \tag{12.1.1}$$

denotes the Wigner transform, also known as the cross-Wigner distribution, then the following formula holds:

$$\langle L_\sigma f, g\rangle = \langle \sigma, W(g, f)\rangle, \qquad f, g \in \mathscr{S}^{(1)}(\mathbb{R}^d),$$

for each $\sigma \in \mathscr{S}^{(1)'}(\mathbb{R}^{2d})$; see e.g., [16, 19, 41].

12.2 Modulation Spaces

In this section, we collect some facts on modulation spaces which will be used in Sect. 12.4. First, we introduce the short-time Fourier transform in terms of duality between the Gelfand–Shilov space $\mathscr{S}^{(1)}(\mathbb{R}^d)$ and its dual space of tempered ultra-distributions $\mathscr{S}^{(1)'}(\mathbb{R}^{2d})$ as follows.

The short-time Fourier transform (STFT in the sequel) of $f \in \mathscr{S}^{(1)}(\mathbb{R}^d)$ with respect to the window $g \in \mathscr{S}^{(1)}(\mathbb{R}^d) \setminus 0$ is defined by

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle = \int_{\mathbb{R}^d} f(t)\,\overline{g(t-x)}\,e^{-2\pi i \omega t}\,dt, \tag{12.2.1}$$

where the translation operator T_x and the modulation operator M_ω are given by

$$T_x f(\cdot) = f(\cdot - x) \quad \text{and} \quad M_\omega f(\cdot) = e^{2\pi i \omega \cdot} f(\cdot) \quad x, \omega \in \mathbb{R}^d. \tag{12.2.2}$$

The map $(f, g) \mapsto V_g f$ from $\mathscr{S}^{(1)}(\mathbb{R}^d) \otimes \mathscr{S}^{(1)}(\mathbb{R}^d)$ to $\mathscr{S}^{(1)}(\mathbb{R}^{2d})$ extends uniquely to a continuous operator from $\mathscr{S}^{(1)'}(\mathbb{R}^d) \otimes \mathscr{S}^{(1)'}(\mathbb{R}^d)$ to $\mathscr{S}^{(1)'}(\mathbb{R}^{2d})$ by duality, cf. [12, Theorem 4.1], [37, proposition 1.8].

Moreover, for a fixed $g \in \mathscr{S}^{(1)}(\mathbb{R}^d) \setminus 0$ the following characterization holds:

$$f \in \mathscr{S}^{(1)}(\mathbb{R}^d) \quad \Longleftrightarrow \quad V_g f \in \mathscr{S}^{(1)}(\mathbb{R}^{2d}).$$

We recall the notation from [33] related to the bilinear case. For given $\varphi_1, \varphi_2, f_1, f_2 \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, we put

$$\begin{aligned} V_{\varphi_1 \otimes \varphi_2}(f_1 \otimes f_2)(x,\omega) &= \int_{\mathbb{R}^{2d}} f_1(t_1) f_2(t_2) \overline{M_{\omega_1} T_{x_1}\varphi_1(t_1) M_{\omega_2} T_{x_2}\varphi_2(t_2)} dt_1 dt_2 \\ &= \int_{\mathbb{R}^{2d}} (f_1 \otimes f_2)(t) \overline{(M_{\omega_1} T_{x_1}\varphi_1 \otimes M_{\omega_2} T_{x_2}\varphi_2)(t)} dt, \end{aligned} \tag{12.2.3}$$

where $x = (x_1, x_2)$, $\omega = (\omega_1, \omega_2)$, $t = (t_1, t_2)$, $x_1, x_2, \omega_1, \omega_2, t_1, t_2 \in \mathbb{R}^d$.

To give an interpretation of multilinear operators in the weak sense we note that, if $\mathbf{f} = (f_1, f_2, \dots, f_n)$ and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_n)$, $f_j, \varphi_j \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $j = 1, 2, \dots, n$, then (12.2.3) becomes

$$V_{\boldsymbol{\varphi}}\mathbf{f}(x,\omega) = \int_{\mathbb{R}^{nd}} \mathbf{f}(t) \prod_{j=1}^{n} \overline{M_{\omega_j} T_{x_j}\varphi_j(t_j)} dt, \tag{12.2.4}$$

see also (12.3.1) for the notation.

We refer to [23, 30–32, 37] for more details on STFT in other spaces of Gelfand–Shilov type. Since we restrict ourselves to weighted modulation spaces with polynomial weights in this paper, we proceed by using the duality between $\mathscr{S}$ and $\mathscr{S}'$ instead of the more general duality between $\mathscr{S}^{(1)}$ and $\mathscr{S}^{(1)'}$. Related results in the framework of subexponential and superexponential weights can be found in, e.g., [11, 12, 31, 37], and we leave the study of multilinear localization operators in that case for a separate contribution.

Modulation spaces [15, 19] are defined through decay and integrability conditions on STFT, which makes them suitable for time–frequency analysis, and for the study of localization operators in particular. They are defined in terms of weighted mixed-norm Lebesgue spaces.

In general, a weight $w(\cdot)$ on $\mathbb{R}^d$ is a nonnegative and continuous function. The weighted Lebesgue space $L^p_w(\mathbb{R}^d)$, $p \in [1,\infty]$, is the Banach space with the norm

$$\|f\|_{L^p_w} = \|fw\|_{L^p} = \left(\int |f(x)|^p w(x)^p dx\right)^{1/p},$$

and with the usual modification when $p = \infty$. When $w(x) = \langle x\rangle^t$, $t \in \mathbb{R}$, we use the notation $L^p_t(\mathbb{R}^d)$ instead.

Similarly, the weighted mixed-norm Lebesgue space $L^{p,q}_w(\mathbb{R}^{2d})$, $p, q \in [1,\infty]$, consists of (Lebesgue) measurable functions on $\mathbb{R}^{2d}$ such that

$$\|F\|_{L^{p,q}_w} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |F(x,\omega)|^p w(x,\omega)^p dx\right)^{q/p} d\omega\right)^{1/q} < \infty.$$

where $w(x,\omega)$ is a weight on $\mathbb{R}^{2d}$.

In particular, when $w(x,\omega) = \langle x\rangle^t\langle\omega\rangle^s$, $s, t \in \mathbb{R}$, we use the notation $L^{p,q}_w(\mathbb{R}^{2d}) = L^{p,q}_{s,t}(\mathbb{R}^{2d})$.

Now, modulation space $M^{p,q}_{s,t}(\mathbb{R}^d)$ consists of distributions whose STFT belongs to $L^{p,q}_{s,t}(\mathbb{R}^{2d})$:

Definition 12.1 Let $\phi \in \mathscr{S}(\mathbb{R}^d)\setminus 0$, $s, t \in \mathbb{R}$, and $p, q \in [1,\infty]$. The *modulation space* $M^{p,q}_{s,t}(\mathbb{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_{s,t}} \equiv \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_\phi f(x,\omega)\langle x\rangle^t\langle\omega\rangle^s|^p\, dx\right)^{q/p} d\omega\right)^{1/q} < \infty$$

(with obvious interpretation of the integrals when $p = \infty$ or $q = \infty$).

In special cases, we use the usual abbreviations: $M^{p,p}_{0,0} = M^p$, $M^{p,p}_{t,t} = M^p_t$, etc.

For the consistency, and according to (12.2.4), we denote by $\mathscr{M}^{p,q}_{s,t}(\mathbb{R}^{nd})$ the set of $\mathbf{f} = (f_1, f_2, \ldots, f_n)$, $f_j \in \mathscr{S}'(\mathbb{R}^d)$, $j = 1, 2, \ldots, n$, such that

$$\|\mathbf{f}\|_{\mathscr{M}^{p,q}_{s,t}} \equiv \left(\int_{\mathbb{R}^{nd}}\left(\int_{\mathbb{R}^{nd}} |V_{\boldsymbol{\varphi}}\mathbf{f}(x,\omega)\langle x\rangle^t\langle\omega\rangle^s|^p\, dx\right)^{q/p} d\omega\right)^{1/q} < \infty, \tag{12.2.5}$$

where $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$, $\varphi_j \in \mathscr{S}(\mathbb{R}^d)\setminus 0$, $j = 1, 2, \ldots, n$, is a given n-tuple of window functions.

The kernel theorem for $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ (see [40]) implies that there is an isomorphism between $\mathscr{M}^{p,q}_{s,t}(\mathbb{R}^{nd})$ and $M^{p,q}_{s,t}(\mathbb{R}^{nd})$ (which commutes with the operators from (12.2.2)). This allows us to identify $\mathbf{f} \in \mathscr{M}^{p,q}_{s,t}(\mathbb{R}^{nd})$ with (its isomorphic image) $F \in M^{p,q}_{s,t}(\mathbb{R}^{nd})$ (and vice versa). We will use this identification whenever convenient and without further explanation.

Remark 12.1 The original definition of modulation spaces given in [15] deals with more general *submultiplicative* weights. We restrict ourselves to the weights of the form $w(x, \omega) = \langle x\rangle^t \langle\omega\rangle^s$, $s, t \in \mathbb{R}$, since the convolution and multiplication estimates which will be used later on are formulated in terms of weighted spaces with such polynomial weights. As already mentioned, weights of exponential type growth are used in the study of Gelfand–Shilov spaces and their duals in cf. [11, 23, 30, 37]. We refer to [20] for a survey on the most important types of weights commonly used in time–frequency analysis.

The following theorem lists some basic properties of modulation spaces. We refer to [15, 19] for the proof.

Theorem 12.2 *Let $p, q, p_j, q_j \in [1, \infty]$ and $s, t, s_j, t_j \in \mathbb{R}$, $j = 1, 2$. Then:*

1. *$M^{p,q}_{s,t}(\mathbb{R}^d)$ are Banach spaces, independent of the choice of $\phi \in \mathscr{S}(\mathbb{R}^d) \setminus 0$;*
2. *if $p_1 \le p_2$, $q_1 \le q_2$, $s_2 \le s_1$ and $t_2 \le t_1$, then*

$$\mathscr{S}(\mathbb{R}^d) \subseteq M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \subseteq M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d) \subseteq \mathscr{S}'(\mathbb{R}^d);$$

3. $\cap_{s,t} M^{p,q}_{s,t}(\mathbb{R}^d) = \mathscr{S}(\mathbb{R}^d), \quad \cup_{s,t} M^{p,q}_{s,t}(\mathbb{R}^d) = \mathscr{S}'(\mathbb{R}^d);$
4. *For $p, q \in [1, \infty)$, the dual of $M^{p,q}_{s,t}(\mathbb{R}^d)$ is $M^{p',q'}_{-s,-t}(\mathbb{R}^d)$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.*

Modulation spaces include the following well-known function spaces:

1. $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and $M^2_{t,0}(\mathbb{R}^d) = L^2_t(\mathbb{R}^d)$;
2. The Feichtinger algebra: $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$;
3. Sobolev spaces: $M^2_{0,s}(\mathbb{R}^d) = H^2_s(\mathbb{R}^d) = \{f \mid \hat{f}(\omega)\langle\omega\rangle^s \in L^2(\mathbb{R}^d)\}$;
4. Shubin spaces: $M^2_s(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H^2_s(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$, cf. [28].

To deal with duality when $pq = \infty$ we observe that, by a slight modification of [1, Lemma 2.2] the following is true.

Lemma 12.1 *Let $L^0(\mathbb{R}^{2nd})$ denote the space of bounded, measurable functions on $\mathbb{R}^{2nd}$ which vanish at infinity and put*

$$\begin{aligned}
\mathscr{M}^{0,q}(\mathbb{R}^{nd}) &= \{\mathbf{f} \in \mathscr{M}^{\infty,q}(\mathbb{R}^{nd}) \mid V_{\boldsymbol{\varphi}}\mathbf{f} \in L^0(\mathbb{R}^{2nd})\}, \quad 1 \le q < \infty,\\
\mathscr{M}^{p,0}(\mathbb{R}^{nd}) &= \{\mathbf{f} \in \mathscr{M}^{p,\infty}(\mathbb{R}^{nd}) \mid V_{\boldsymbol{\varphi}}\mathbf{f} \in L^0(\mathbb{R}^{2nd})\}, \quad 1 \le p < \infty,\\
\mathscr{M}^{0,0}(\mathbb{R}^{nd}) &= \{\mathbf{f} \in \mathscr{M}^{\infty,\infty}(\mathbb{R}^{nd}) \mid V_{\boldsymbol{\varphi}}\mathbf{f} \in L^0(\mathbb{R}^{2nd})\},
\end{aligned}$$

equipped with the norms of $\mathscr{M}^{\infty,q}$, $\mathscr{M}^{p,\infty}$ and $\mathscr{M}^{\infty,\infty}$ respectively. Then,

1. *$\mathscr{M}^{0,q}$ is $\mathscr{M}^{\infty,q}$−closure of $\mathscr{S}$ in $\mathscr{M}^{\infty,q}$, hence is a closed subspace of $\mathscr{M}^{\infty,q}$. Likewise for $\mathscr{M}^{p,0}$ and $\mathscr{M}^{0,0}$.*
2. *The following duality results hold for $1 \le p, q < \infty$: $(\mathscr{M}^{0,q})' = \mathscr{M}^{1,q'}$, $(\mathscr{M}^{p,0})' = \mathscr{M}^{p',1}$, and $(\mathscr{M}^{0,0})' = \mathscr{M}^{1,1}$.*

From now on, we will use these duality relations in the cases $p = \infty$ and/or $q = \infty$ without further explanations.

For the results on multiplication and convolution in modulation spaces and in weighted Lebesgue spaces, we first introduce the *Young functional*:

$$\mathsf{R}(\mathrm{p}) = \mathsf{R}(p_0, p_1, p_2) \equiv 2 - \frac{1}{p_0} - \frac{1}{p_1} - \frac{1}{p_2}, \qquad \mathrm{p} = (p_0, p_1, p_2) \in [1, \infty]^3. \tag{12.2.6}$$

When $\mathsf{R}(\mathrm{p}) = 0$, the Young inequality for convolution reads as

$$\|f_1 * f_2\|_{L^{p_0'}} \le \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}}, \quad f_j \in L^{p_j}(\mathbb{R}^d), \quad j = 1, 2.$$

The following theorem is an extension of the Young inequality to the case of weighted Lebesgue spaces and modulation spaces when $0 \le \mathsf{R}(\mathrm{p}) \le 1/2$.

Theorem 12.3 *Let $s_j, t_j \in \mathbb{R}$, $p_j, q_j \in [1, \infty]$, $j = 0, 1, 2$. Assume that $0 \le \mathsf{R}(\mathrm{p}) \le 1/2$, $\mathsf{R}(\mathrm{q}) \le 1$,*

$$0 \le t_j + t_k, \qquad j, k = 0, 1, 2, \quad j \ne k, \tag{12.2.7}$$

$$0 \le t_0 + t_1 + t_2 - d \cdot \mathsf{R}(\mathrm{p}), \ \textit{and} \tag{12.2.8}$$

$$0 \le s_0 + s_1 + s_2, \tag{12.2.9}$$

with strict inequality in (12.2.8) when $\mathsf{R}(\mathrm{p}) > 0$ and $t_j = d \cdot \mathsf{R}(\mathrm{p})$ for some $j = 0, 1, 2$.

Then $(f_1, f_2) \mapsto f_1 * f_2$ on $C_0^\infty(\mathbb{R}^d)$ extends uniquely to a continuous map from

1. *$L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$ to $L^{p_0'}_{-t_0}(\mathbb{R}^d)$;*
2. *$M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \times M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d)$ to $M^{p_0',q_0'}_{-s_0,-t_0}(\mathbb{R}^d)$.*

For the proof, we refer to [38]. It is based on the detailed study of an auxiliary three-linear map over carefully chosen regions in $\mathbb{R}^d$ (see Sects. 3.1 and 3.2 in [38]). This result extends multiplication and convolution properties obtained in [26]. Moreover, the sufficient conditions from Theorem 12.3 are also necessary in the following sense.

Theorem 12.4 *Let $p_j, q_j \in [1, \infty]$ and $s_j, t_j \in \mathbb{R}$, $j = 0, 1, 2$. Assume that at least one of the following statements hold true:*

1. *The map $(f_1, f_2) \mapsto f_1 * f_2$ on $C_0^\infty(\mathbb{R}^d)$ is continuously extendable to a map from $L^{p_1}_{t_1}(\mathbb{R}^d) \times L^{p_2}_{t_2}(\mathbb{R}^d)$ to $L^{p_0'}_{-t_0}(\mathbb{R}^d)$;*

2. *The map* $(f_1, f_2) \mapsto f_1 * f_2$ *on* $C_0^\infty(\mathbb{R}^d)$ *is continuously extendable to a map from* $M^{p_1,q_1}_{s_1,t_1}(\mathbb{R}^d) \times M^{p_2,q_2}_{s_2,t_2}(\mathbb{R}^d)$ *to* $M^{p_0',q_0'}_{-s_0,-t_0}(\mathbb{R}^d)$*;*

Then (12.2.7) *and* (12.2.8) *hold true.*

12.3 Multilinear Localization Operators

In this section, we introduce multilinear localization operators in Definition 12.2 and show that they can be interpreted as particular Weyl pseudodifferential operators, Theorem 12.5. We also introduce multilinear Weyl pseudodifferential operators and prove their connection to the multilinear Wigner transform in Lemma 12.2. This is done in the context of the duality between $\mathscr{S}^{(1)}(\mathbb{R}^d)$ and $\mathscr{S}^{(1)'}(\mathbb{R}^d)$ and carried out verbatim to the duality between $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$ in the next section.

The localization operator $A_a^{\varphi_1,\varphi_2}$ with the symbol $a \in L^2(\mathbb{R}^{2d})$ and with windows $\varphi_1, \varphi_2 \in L^2(\mathbb{R}^d)$ can be defined in terms of the short-time Fourier transform (12.2.1) as follows:

$$A_a^{\varphi_1,\varphi_2} f(t) = \int_{\mathbb{R}^{2d}} a(x,\omega) V_{\varphi_1} f(x,\omega) M_\omega T_x \varphi_2(t)\, dx d\omega, \quad f \in L^2(\mathbb{R}^d).$$

To define multilinear localization operators, we slightly abuse the notation (as it is done in, e.g., [24]) so that $\mathbf{f}$ will denote both the vector $\mathbf{f} = (f_1, f_2, \dots, f_n)$ and the tensor product $\mathbf{f} = f_1 \otimes f_2 \otimes \cdots \otimes f_n$. This will not cause confusion, since the meaning of $\mathbf{f}$ will be clear from the context.

For example, if $t = (t_1, t_2, \dots, t_n)$, and $F_j = F_j(t_j)$, $t_j \in \mathbb{R}^d$, $j = 1, 2, \dots, n$, then

$$\prod_{j=1}^n F_j(t_j) = F_1(t_1) \cdot F_2(t_2) \cdot \dots \cdot F_n(t_n) = F_1(t_1) \otimes F_2(t_2) \otimes \cdots \otimes F_n(t_n) = \mathbf{F}(t). \tag{12.3.1}$$

Definition 12.2 Let $f_j \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $j = 1, 2, \dots, n$, and $\mathbf{f} = (f_1, f_2, \dots, f_n)$. The *multilinear localization operator* $A_a^{\varphi,\phi}$ with *symbol* $a \in \mathscr{S}^{(1)'}(\mathbb{R}^{2nd})$ and *windows*

$$\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \text{ and } \phi = (\phi_1, \phi_2, \dots, \phi_n), \quad \varphi_j, \phi_j \in \mathscr{S}^{(1)}(\mathbb{R}^d), \quad j = 1, 2, \dots, n,$$

is given by

$$A_a^{\varphi,\phi} \mathbf{f}(t) = \int_{\mathbb{R}^{2nd}} a(x,\omega) \prod_{j=1}^n \left(V_{\varphi_j} f_j(x_j, \omega_j) M_{\omega_j} T_{x_j} \phi_j(t_j) \right) dx d\omega, \tag{12.3.2}$$

where $x_j, \omega_j, t_j \in \mathbb{R}^d$, $j = 1, 2, \dots, n$, and $x = (x_1, x_2, \dots, x_n)$, $\omega = (\omega_1, \omega_2 \dots, \omega_n)$, $t = (t_1, t_2 \dots, t_n)$.

Remark 12.2 When $n = 2$ in Definition 12.2 we obtain the bilinear localization operators studied in [33]. (There is a typo in [33, Definition 1]; the integration in (9) should be taken over $\mathbb{R}^{4d}$.)

Let $\mathscr{R}$ denote the trace mapping that assigns to each function F defined on $\mathbb{R}^{nd}$ a function defined on $\mathbb{R}^d$ by the formula

$$\mathscr{R} : F \mapsto F\,\Big|_{t_1=t_2=\cdots=t_n}, \quad t_j \in \mathbb{R}^d,\ j = 1, 2, \ldots, n.$$

Then $\mathscr{R} A_a^{\varphi,\phi}$ is the multilinear operator given in [8, Definition 2.2].

By (12.2.4) it follows that the weak definition of (12.3.2) is given by

$$\langle A_a^{\varphi,\phi}\mathbf{f}, \mathbf{g}\rangle = \langle a V_{\boldsymbol{\varphi}}\mathbf{f}, V_{\boldsymbol{\phi}}\mathbf{g}\rangle = \langle a, \overline{V_{\boldsymbol{\varphi}}\mathbf{f}}\, V_{\boldsymbol{\phi}}\mathbf{g}\rangle, \tag{12.3.3}$$

and $f_j, g_j, \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $j = 1, 2, \ldots, n$. The brackets can be interpreted as duality between a suitable pair of dual spaces. Thus, $A_a^{\varphi,\phi}$ is well-defined continuous operator from $\mathscr{S}^{(1)}(\mathbb{R}^{nd})$ to $(\mathscr{S}^{(1)})'(\mathbb{R}^{nd})$.

Next, we introduce a class of multilinear Weyl pseudodifferential operators (ΨDO for short) and use the Wigner transform to prove appropriate interpretation of multilinear localization operators as multilinear Weyl pseudodifferential operators, Theorem 12.5.

Recall that in [8], multilinear localization operators are introduced in connection to Kohn–Nirenberg ΨDOs instead.

By analogy with the bilinear Weyl pseudodifferential operators given in [33], we define the multilinear Weyl pseudodifferential operator as follows:

$$L_\sigma(\mathbf{f})(x) = \int_{\mathbb{R}^{2nd}} \sigma\Big(\frac{x+y}{2}, \omega\Big)\mathbf{f}(y) e^{2\pi i \mathscr{I}(x-y)\cdot\omega} dy d\omega, \quad x \in \mathbb{R}^{nd}, \tag{12.3.4}$$

where $\sigma \in \mathscr{S}^{(1)'}(\mathbb{R}^{2nd})$, $\mathbf{f}(y) = \prod_{j=1}^n f_j(y_j)$, $f_j \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $j = 1, 2, \ldots, n$. Here $\mathscr{I}$ denotes the identity matrix in nd, that is, $\mathscr{I}(x-y)\cdot\omega = \sum_{j=1}^{n}(x_j - y_j)\omega_j$.)

Similarly, the bilinear Wigner transform from [33] extends to

$$W(\mathbf{f}, \mathbf{g})(x, \omega) = \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} \left(f_j(x_j + \frac{t_j}{2})\overline{g_j(x_j - \frac{t_j}{2})} \right) e^{-2\pi i \mathscr{I}\omega t}\, dt, \tag{12.3.5}$$

where $f_j, g_j \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $x_j, \omega_j, t_j \in \mathbb{R}^d$, $j = 1, 2, \ldots, n$, and $x = (x_1, x_2, \ldots, x_n)$, $\omega = (\omega_1, \omega_2 \ldots, \omega_n)$, $t = (t_1, t_2 \ldots, t_n)$.

It is easy to see that $W(\mathbf{f}, \mathbf{g}) \in \mathscr{S}^{(1)}(\mathbb{R}^{2nd})$, when $\mathbf{f}, \mathbf{g} \in \mathscr{S}^{(1)}(\mathbb{R}^{nd})$.

Lemma 12.2 *Let $\sigma \in \mathscr{S}^{(1)}(\mathbb{R}^{2nd})$ and $f_j, g_j \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $j = 1, 2, \ldots, n$. Then L_σ given by (12.3.4) extends to a continuous map from $\mathscr{S}^{(1)}(\mathbb{R}^{nd})$ to $(\mathscr{S}^{(1)})'(\mathbb{R}^{nd})$.*

$$\langle L_\sigma \mathbf{f}, \mathbf{g}\rangle = \langle \sigma, W(\mathbf{g}, \mathbf{f})\rangle.$$

Proof The proof follows by the straightforward calculation:

$$\begin{aligned}
\langle \sigma, W(\mathbf{g}, \mathbf{f})\rangle &= \int_{\mathbb{R}^{2nd}} \sigma(x,\omega) W(\mathbf{f}, \mathbf{g})(x,\omega) dx d\omega \\
&= \int_{\mathbb{R}^{3nd}} \sigma(x,\omega) \prod_{j=1}^{n} \left(f_j(x_j + \frac{t_j}{2}) \overline{g_j(x_j - \frac{t_j}{2})} \right) e^{-2\pi i \mathscr{I} \omega t} dt dx d\omega \\
&= \int_{\mathbb{R}^{6d}} \sigma(\frac{u+v}{2}, \omega) \prod_{j=1}^{n} \left(f_j(v_j) \overline{g_j(u_j)} \right) e^{-2\pi i \mathscr{I} (u-v)\omega} du dv d\omega \\
&= \langle \sigma(\frac{u+v}{2}, \omega) \mathbf{f}(v) e^{2\pi i \mathscr{I} (u-v)\omega}, \mathbf{g}(u)\rangle = \langle L_\sigma \mathbf{f}, \mathbf{g}\rangle,
\end{aligned}$$

where we used $W(\mathbf{g}, \mathbf{f}) = \overline{W(\mathbf{f}, \mathbf{g})}$ and the change of variables $u = x + \frac{t}{2}$, $v = x - \frac{t}{2}$. This extends to each $\sigma \in \mathscr{S}^{(1)'}(\mathbb{R}^{2nd})$, since $W(\mathbf{f}, \mathbf{g}) \in \mathscr{S}^{(1)}(\mathbb{R}^{2nd})$ when $f_j, g_j \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $j = 1, 2, \dots, n$. □

The so-called Weyl connection between the set of linear localization operators and Weyl ΨDOs is well known; we refer to, e.g., [4, 16, 32]. The corresponding Weyl connection in bilinear case is established in [33, Theorem 4]. The proof is quite technical and based on the kernel theorem for Gelfand–Shilov spaces (see, e.g., [27, 31, 39]) and direct calculations. Since the proof of the following Theorem 12.5 is its straightforward extension, here we only sketch the main ideas. The conclusion of Theorem 12.5 is that any multilinear localization operator can be viewed as a particular multilinear Weyl ΨDOs, as expected.

Theorem 12.5 *Let there be given $a \in \mathscr{S}^{(1)'}(\mathbb{R}^{2d})$ and let $\phi = (\phi_1, \phi_2, \dots, \phi_n)$, $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$, $\varphi_j, \phi_j \in \mathscr{S}^{(1)}(\mathbb{R}^d)$, $j = 1, 2, \dots, n$. Then the localization operator $A_a^{\varphi,\phi}$ is the Weyl pseudodifferential operator with the Weyl symbol*

$$\sigma = a * W(\boldsymbol{\phi}, \boldsymbol{\varphi}) = a * (\prod_{j=1}^{n} W(\phi_j, \varphi_j)).$$

Therefore, if $\mathbf{f} = (f_1, f_2, \dots, f_n)$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$, $f_j, g_j, \in \mathscr{S}^{(1)'}(\mathbb{R}^d)$, $j = 1, 2, \dots, n$, then

$$\langle A_a^{\varphi,\phi} \mathbf{f}, \mathbf{g}\rangle = \langle L_{a * W(\boldsymbol{\phi}, \boldsymbol{\varphi})} \mathbf{f}, \mathbf{g}\rangle.$$

Proof The formal expressions given below are justified due to the absolute convergence of the involved integrals and the standard interpretation of oscillatory integrals in distributional setting. We refer to [33, Sect. 5] for this and for detailed calculations.

The calculations from the proof of [33, Theorem 4] yield the following kernel representation of (12.3.3):

$$\langle A_a^{\varphi,\phi} \mathbf{f}, \mathbf{g}\rangle = \langle k, \prod_{j=1}^{n} \overline{f_j} \otimes \prod_{j=1}^{n} g_j \rangle,$$

where the kernel $k = k(t, s)$ is given by

$$k(t,s) = \int_{\mathbb{R}^{2nd}} a(x,\omega) \prod_{j=1}^{n} \overline{M_{\omega_j} T_{x_j} \varphi_j(t)} \cdot \prod_{j=1}^{n} M_{\omega_j} T_{x_j} \phi_j(s) dx d\omega, \qquad (12.3.6)$$

$t = (t_1, t_2, \ldots, t_n)$, $s = (s_1, s_2, \ldots, s_n)$, $t_j, s_j \in \mathbb{R}^d$, $j = 1, 2, \ldots, n$.

To calculate the convolution $a * (\prod_{j=1}^n W(\phi_j, \varphi_j)) = a * W(\boldsymbol{\phi}, \boldsymbol{\varphi})$, we use $W(g, f) = \overline{W(f, g)}$, the commutation relation $T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x$, and the covariance property of the Wigner transform:

$$W(T_{x_j} M_{\omega_j} \phi_j, T_{x_j} M_{\omega_j} \varphi_j)(p_j, q_j) = W(\phi_j, \varphi_j)(p_j - x_j, q_j - \omega_j), \quad j = 1, 2, \ldots, n.$$

Let $p = (p_1, p_2, \ldots, p_n)$, $q = (q_1, q_2 \ldots, q_n)$, $p_j, q_j \in \mathbb{R}^d$, $j = 1, 2, \ldots, n$. Then,

$$a * W(\boldsymbol{\phi}, \boldsymbol{\varphi})(p,q) = \int_{\mathbb{R}^{2nd}} a(x,\omega) \times \left(\int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} M_{\omega_j} T_{x_j} \phi_j (p_j + \frac{t_j}{2}) \cdot \prod_{j=1}^{n} \overline{M_{\omega_j} T_{x_j} \varphi_j} (p_j - \frac{t_j}{2}) e^{-2\pi i q\cdot t} dt \right) dx d\omega, \qquad (12.3.7)$$

where $q \cdot t$ denotes the scalar product of $q, t \in \mathbb{R}^d$, cf. [33, Sect. 5].

Therefore,

$$\langle L_{a*W(\boldsymbol{\phi},\boldsymbol{\varphi})} \mathbf{f}, \mathbf{g} \rangle = \langle a * \prod_{j=1}^{n} W(\phi_j, \varphi_j), W(\mathbf{g}, \mathbf{f}) \rangle = \int_{\mathbb{R}^{2nd}} a(x,\omega) \times \int_{\mathbb{R}^{nd}} \Big(\int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} M_{\omega_j} T_{x_j} \phi_j (p_j + \frac{t_j}{2}) \cdot \prod_{j=1}^{n} \overline{M_{\omega_j} T_{x_j} \varphi_j} (p_j - \frac{t_j}{2}) \times \prod_{j=1}^{n} f_j (p_j - \frac{t_j}{2}) \cdot \prod_{j=1}^{n} \overline{g_j} (p_j + \frac{t_j}{2}) dt \Big) dp dx d\omega,$$

Finally, after performing the change of variables we obtain

$$\langle L_{a*W(\boldsymbol{\phi},\boldsymbol{\varphi})} \mathbf{f}, \mathbf{g} \rangle = \langle k, \prod_{j=1}^{n} \overline{f_j} \otimes \prod_{j=1}^{n} g_j \rangle,$$

where the kernel k is given by (12.3.6). The theorem now follows from the uniqueness of the kernel representation, Theorem 12.1. □

12.4 Continuity Properties of Localization Operators

We first recall the sharp estimates of the modulation space norm for the cross-Wigner distribution given in [9]. There it is shown that the sufficient conditions for the continuity of the cross-Wigner distribution on modulation spaces are also necessary (in the unweighted case). Related results can be found elsewhere, e.g., in [32, 34, 35]. In many situations, such results overlap. For example, Proposition 10 in [33] coincides with certain sufficient conditions from [9, Theorem 1.1] when restricted to $\mathsf{R}(\mathrm{p}) = 0$, $t_0 = -t_1$, and $t_2 = |t_0|$.

Theorem 12.6 *Let there be given $s \in \mathbb{R}$ and $p_i, q_i, p, q \in [1, \infty]$, such that*

$$p \le p_i, q_i \le q, \quad i = 1, 2 \tag{12.4.1}$$

and

$$\min\left\{\frac{1}{p_1} + \frac{1}{p_2}, \frac{1}{q_1} + \frac{1}{q_2}\right\} \ge \frac{1}{p} + \frac{1}{q}. \tag{12.4.2}$$

If $f, g \in \mathscr{S}(\mathbb{R}^d)$, then the map $(f, g) \mapsto W(f, g)$ where W is the cross-Wigner distribution given by (12.1.1) extends to sesquilinear continuous map from $M^{p_1,q_1}_{|s|}(\mathbb{R}^d) \times M^{p_2,q_2}_s(\mathbb{R}^d)$ to $M^{p,q}_{s,0}(\mathbb{R}^{2d})$ and

$$\|W(f, g)\|_{M^{p,q}_{s,0}} \lesssim \|f\|_{M^{p_1,q_1}_{|s|}} \|g\|_{M^{p_2,q_2}_s}. \tag{12.4.3}$$

Viceversa, if there exists a constant $C > 0$ such that

$$\|W(f, g)\|_{M^{p,q}} \lesssim \|f\|_{M^{p_1,q_1}} \|g\|_{M^{p_2,q_2}}.$$

then (12.4.1) and (12.4.2) must hold.

Proof We omit the proof which is given in [9, Sect. 3] and recall here only the main formulas which highlight its most important parts.

The first formula is the well-known relation between the Wigner transform and the STFT (see [19, Lemma 4.3.1]):

$$W(f, g)(x, \omega) = 2^d e^{4\pi i x\cdot\omega} V_{\overline{g^*}} f(2x, 2\omega), \quad f, g \in \mathscr{S}(\mathbb{R}^d).$$

To estimate the modulation space norm of $W(f, g)(x, \omega)$, we fix $\psi_1, \psi_2 \in \mathscr{S}(\mathbb{R}^d) \setminus 0$ and use the fact that modulation spaces are independent on the choice of the window function from $\mathscr{S}(\mathbb{R}^{2d}) \setminus 0$, Theorem 12.2 1. By choosing the window to be $W(\psi_1, \psi_2)$, after some calculations we obtain:

$$(V_{W(\psi_1,\psi_2)} W(g, f))(z, \zeta) = e^{-2\pi i z_2\zeta_2} \overline{V_{\psi_1} f}(z_1 + \frac{\zeta_2}{2}, z_2 - \frac{\zeta_1}{2}) V_{\psi_2} g(z_1 - \frac{\zeta_2}{2}, z_2 + \frac{\zeta_1}{2}),$$

cf. the proof of [19, Lemma 14.5.1 (b)]. Consequently (cf. [9, Sect. 3]),

$$\|W(g,f)\|_{M^{p,q}_{s,0}} \asymp \left(\int_{\mathbb{R}^{2d}} (|V_{\psi_1} f|^p * |V_{\psi_2} g^*|^p)^{q/p}(\zeta_2, -\zeta_1) \langle(\zeta_2, -\zeta_1)\rangle^{sq} d\zeta \right)^{1/q}$$
$$= \| |V_{\psi_1} f|^p * |V_{\psi_2} g^*|^p \|_{L^{q/p}_{ps,0}}.$$

Then one proceeds with a careful case study to obtain (12.4.3) when (12.4.1) and (12.4.2) hold true. We refer to [9] for details. □

From the inspection of the proof of Theorem 12.6 given in [9, Sect. 3], the definition of $W(\mathbf{f}, \mathbf{g})$ given by (12.3.5), and the use of the kernel theorem, we conclude the following.

Corollary 12.1 *Let the assumptions of Theorem 12.6 hold. If* $\mathbf{f} = (f_1, f_2, \dots, f_n)$, $\mathbf{g} = (g_1, g_2, \dots, g_n)$ *and* $f_j, g_j \in \mathscr{S}(\mathbb{R}^d)$, $j = 1, 2, \dots, n$, *then the map* $(\mathbf{f}, \mathbf{g}) \mapsto W(\mathbf{f}, \mathbf{g})$, *where W is the cross-Wigner distribution given by* (12.3.5) *extends to a continuous map from* $\mathscr{M}^{p_1,q_1}_{|s|}(\mathbb{R}^d) \times \mathscr{M}^{p_2,q_2}_{s}(\mathbb{R}^d)$ *to* $\mathscr{M}^{p,q}_{s,0}(\mathbb{R}^{2d})$, *where the modulation spaces are given by* (12.2.5).

Next, we give an extension of [19, Theorem 14.5.2] and [33, Theorem 14] to the multilinear Weyl ΨDOs. Recall, if $\sigma \in M^{\infty,1}(\mathbb{R}^{2d})$ is the Weyl symbol of L_σ, then [19, Theorem 14.5.2] says that L_σ is bounded on $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$. This result has a long history starting from the Calderon–Vaillancourt theorem on boundedness of the pseudodifferential operators with smooth and bounded symbols on $L^2(\mathbb{R}^d)$, [5]. It is generalized by Sjöstrand in [29] where $M^{\infty,1}$ is used as appropriate symbol class. Sjöstrand's results were thereafter extended in [19, 21, 22, 34–36]. Moreover, we refer to [1–3] for the multilinear Kohn–Nirenberg ΨDOs and the recent contribution [10] related to $\tau{-}\Psi$DOs (these include both Kohn–Nirenberg (when $\tau = 0$) and Weyl operators (when $\tau = 1/2$)).

The following fact related to symbols $\sigma \in M^{\infty,1}(\mathbb{R}^{2nd})$ is a straightforward extension of [33, Theorem 14].

Theorem 12.7 *Let* $\sigma \in M^{\infty,1}(\mathbb{R}^{2nd})$ *and let* L_σ *be given by* (12.3.4). *The operator* L_σ *is bounded from* $\mathscr{M}^{p,q}(\mathbb{R}^{nd})$ *to* $\mathscr{M}^{p,q}(\mathbb{R}^{nd})$, $1 \le p, q \le \infty$, *with a uniform estimate* $\|L_\sigma\|_{op} \le \|\sigma\|_{M^{\infty,1}}$ *for the operator norm.*

On the other hand, Theorem 12.7 is a special case of [9, Theorem 5.1.] if L_σ is a linear operator. Here, we give the multilinear version of [9, Theorem 5.1.].

Theorem 12.8 *Let there be given* $s \ge 0$ *and* $p_i, q_i, r_i, p, q \in [1, \infty]$, *such that*

$$q \le \min\{p_1', q_1', p_2, q_2\} \tag{12.4.4}$$

and

$$\min\left\{ \frac{1}{p_1} + \frac{1}{p_2'}, \frac{1}{q_1} + \frac{1}{q_2'} \right\} \ge \frac{1}{p'} + \frac{1}{q'}. \tag{12.4.5}$$

Then the operator L_σ *given by* (12.3.4) *with symbol* $\sigma \in M^{p,q}_{s,0}(\mathbb{R}^{2nd})$*, from* $\mathscr{S}(\mathbb{R}^{nd})$ *to* $\mathscr{S}'(\mathbb{R}^{nd})$*, extends uniquely to a bounded operator from* $\mathscr{M}^{p_1,q_1}_{s,0}(\mathbb{R}^{nd})$ *to* $\mathscr{M}^{p_2,q_2}_{s,0}(\mathbb{R}^{nd})$*, with the estimate*

$$\|L_\sigma \mathbf{f}\|_{\mathscr{M}^{p_2,q_2}_{s,0}} \lesssim \|\sigma\|_{M^{p,q}_{s,0}} \|\mathbf{f}\|_{\mathscr{M}^{p_1,q_1}_{s,0}}. \tag{12.4.6}$$

In particular, when $\sigma \in M^{\infty,1}(\mathbb{R}^{2nd})$ *we have* $\|L_\sigma\|_{op} \le \|\sigma\|_{M^{\infty,1}}$ *for the operator norm.*

Vice versa, if (12.4.6) *holds for* $s = 0$*, and for every* $\mathbf{f} \in \mathscr{S}(\mathbb{R}^{nd})$*,* $\sigma \in \mathscr{S}'(\mathbb{R}^{2nd})$*, then* (12.4.1) *and* (12.4.2) *must be satisfied.*

Proof The proof is a straightforward extension of the proof of [9, Theorem 5.1.], and we give it here for the sake of completeness.

When $\mathbf{f} \in \mathscr{M}^{p_1,q_1}_{s,0}(\mathbb{R}^{nd})$ and $\mathbf{g} \in \mathscr{M}^{p_2',q_2'}_{s,0}(\mathbb{R}^{nd})$, their Wigner transform $W(\mathbf{g},\mathbf{f}) = \overline{W(\mathbf{f},\mathbf{g})}$ belongs to $M^{p',q'}_{-s,0}$ since the conditions (12.4.1) and (12.4.2) of Theorem 12.6 are transferred to (12.4.4) and (12.4.5), respectively.

Now, Lemma 12.2 and the duality of modulation spaces give

$$\begin{aligned} |\langle L_\sigma \mathbf{f}, \mathbf{g}\rangle| = |\langle \sigma, W(\mathbf{g},\mathbf{f})\rangle| &\le \|\sigma\|_{M^{p,q}_{s,0}} \|W(\mathbf{f},\mathbf{g})\|_{M^{p',q'}_{-s,0}} \\ &\le C\|\mathbf{f}\|_{\mathscr{M}^{p_1,q_1}_{s,0}} \|\mathbf{g}\|_{\mathscr{M}^{p_2',q_2'}_{s,0}}, \end{aligned}$$

for some constant $C > 0$ (and we used the fact that modulation spaces are closed under the complex conjugation).

We refer to [13, Theorem 1.1.] for the necessity of conditions (12.4.4) and (12.4.5) (in linear case). □

Next, we combine different results established so far to obtain an extension of [33, Theorem 15]. More precisely, we use the relation between the Weyl pseudodifferential operators and the localization operators (Lemma 12.5), the convolution estimates for modulation spaces (Theorem 12.3), and boundedness of pseudodifferential operators (Theorem 12.8) to obtain continuity results for $A_a^{\varphi,\phi}$ for different choices of windows and symbols.

Theorem 12.9 *Let there be given* $s \ge 0$ *and* $p_i, q_i, p, q \in [1,\infty]$, $i = 0,1,2$ *such that* (12.4.4) *and* (12.4.5) *hold. Moreover, let* $q_0 \le q$*, and*

$$p_0 \ge p \quad \text{if} \quad p \ge 2, \quad \text{and} \quad \frac{2p}{2-p} \ge p_0 \ge p \quad \text{if} \quad 2 > p \ge 1. \tag{12.4.7}$$

If $\boldsymbol{\varphi} \in \mathscr{M}^{r_1}_{2s,0}(\mathbb{R}^{nd})$*,* $\boldsymbol{\phi} \in \mathscr{M}^{r_2}_{2s,0}(\mathbb{R}^{nd})$*, where* $\frac{1}{r_1} + \frac{1}{r_2} \ge 1$*, and* $a \in M^{p_0,q_0}_{s_0,t_0}(\mathbb{R}^{2nd})$ *with* $s_0 \ge -s$*, and* $t_0 \ge d\left(\frac{1}{p} - \frac{1}{p_0}\right)$ *with the strict inequality when* $p_0 = p$*, then* $A_a^{\varphi,\phi}$ *is continuous from* $\mathscr{M}^{p_1,q_1}_{s,0}(\mathbb{R}^{nd})$ *to* $\mathscr{M}^{p_2,q_2}_{s,0}(\mathbb{R}^{nd})$ *with*

$$\|A_a^{\varphi,\phi}\|_{op} \lesssim \|a\|_{M^{p_0,q_0}_{s_0,t_0}} \|\boldsymbol{\varphi}\|_{\mathscr{M}^{r_1}_{2s,0}} \|\boldsymbol{\phi}\|_{\mathscr{M}^{r_2}_{2s,0}}.$$

Proof We first estimate $W(\boldsymbol{\phi}, \boldsymbol{\varphi})$. If $\boldsymbol{\varphi} \in \mathscr{M}^{r_1}_{2s,0}(\mathbb{R}^{nd})$, $\boldsymbol{\phi} \in \mathscr{M}^{r_2}_{2s,0}(\mathbb{R}^{nd})$, with $\frac{1}{r_1} + \frac{1}{r_2} \geq 1$, then Corollary 12.1 implies that

$$W(\boldsymbol{\phi}, \boldsymbol{\varphi}) \in \mathscr{M}^{1,\infty}_{2s,0}(\mathbb{R}^{2nd}).$$

Now, we use the result of Theorem 12.3 2. The Young functional (12.2.6) becomes $\mathsf{R}(\mathrm{p}) = \mathsf{R}(p', p_0, 1)$, and the condition $\mathsf{R}(\mathrm{p}) \in [0, 1/2]$ is equivalent to (12.4.7), while $\mathsf{R}(\mathrm{q}) = \mathsf{R}(q', q_0, \infty) \leq 1$ is equivalent to $q_0 \leq q$. Furthermore, (12.2.9) transfers to $s_0 \geq -s$, while (12.2.7) and (12.2.8) are equivalent to $t_0 \geq d\left(\dfrac{1}{p} - \dfrac{1}{p_0}\right)$ with the strict inequality when $p_0 = p$. Therefore, the conditions of by Theorem 12.3 2 are fulfilled, and we obtain

$$a * W(\boldsymbol{\phi}, \boldsymbol{\varphi}) \in M^{p_0,q_0}_{s_0,t_0}(\mathbb{R}^{2nd}) * \mathscr{M}^{1,\infty}_{2s,0}(\mathbb{R}^{2nd}) \subset M^{p,q}_{s,0}(\mathbb{R}^{2nd}).$$

Finally, by Theorem 12.7 with $\sigma = a * W(\boldsymbol{\phi}, \boldsymbol{\varphi})$, it follows that

$$\|A_a^{\boldsymbol{\varphi},\boldsymbol{\phi}}\|_{op} = \|L_\sigma\|_{op} \leq \|\sigma\|_{M^{p,q}_{s,0}} \leq \|a\|_{M^{p_0,q_0}_{s_0,t_0}} \|\boldsymbol{\varphi}\|_{\mathscr{M}^{r_1}_{2s,0}} \|\boldsymbol{\phi}\|_{\mathscr{M}^{r_2}_{2s,0}},$$

and the Theorem is proved. □

In particular, we recover (the linear case treated in) [9, Theorem 5.2] when $r_1 = r_2 = r$, $t_0 = 0$, $s_0 = -s$, $p_0 = p$ (i.e., $\mathsf{R}(p', p_0, 1) = 0$), and $q_0 = q$ (i.e., $\mathsf{R}(q', q_0, \infty) = 1$). Therefore, by [9, Remark 5.3], we obtain an extension of [6, Theorem 3.2] and [35, Theorem 4.11] for this particular choice of weights.

Note that conditions $\mathsf{R}(p', p_0, 1) \in (0, 1/2]$ which extend the possible choices of the Lebesgue parameters beyond the usual Young condition $\mathsf{R}(p', p_0, 1) = 0$ must be compensated by an additional condition to the weights, expressed by $t_0 \geq d\left(\dfrac{1}{p} - \dfrac{1}{p_0}\right)$.

Another result concerning the boundedness of (bilinear) localization operators on unweighted modulation spaces is given by [33, Theorem 15]. There we used different type of estimates, leading to the result which partially overlaps with Theorem 12.9. For example, both results give the same continuity property when the symbol a belongs to $a \in M^{\infty,1}(\mathbb{R}^{2nd})$.

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Chapter 13
Semi-continuous Convolution Estimates on Weakly Periodic Lebesgue Spaces

Joachim Toft

Abstract We deduce mixed quasi-norm estimates of Lebesgue types on semi-continuous convolutions between sequences and functions which may be periodic or possess a weaker form of periodicity in certain directions. In these directions, the Lebesgue quasi-norms are applied on the period instead of the whole axes.

13.1 Introduction

Continuous, discrete and semi-continuous convolutions appear naturally when searching for estimates between short-time Fourier transforms with different window functions. By straightforward application of Fourier's inversion formula, the short-time Fourier transform $V_\phi f$ of the function or (ultra-)distribution f with window function ϕ is linked to $V_{\phi_0} f$ by

$$|V_\phi f| \lesssim |V_\phi \phi_0| * |V_{\phi_0} f| \tag{13.1.1}$$

(cf. e. g. [12, Chap. 11]). Here $*$ denotes the usual (continuous) convolution and it is assumed that the window functions ϕ and ϕ_0 are fixed and belongs to suitable classes (see [12, 14] and Sect. 13.2 for notations).

Modulation spaces appear by imposing norm or quasi-norm estimates on the short-time Fourier transforms of (ultra-)distributions in Fourier-invariant spaces. In most situations, these (quasi-)norms are mixed norms of (weighted) Lebesgue type, given in Definition 13.2.1. More precisely, let $\mathscr{B}$ be a mixed quasi-Banach space of Lebesgue type with functions defined on the phase space, and let ω be a moderate weight. Then the modulation space $M(\omega, \mathscr{B})$ consists of all ultra-distributions f such that

$$\|f\|_{M(\omega,\mathscr{B})} \equiv \|V_\phi f \cdot \omega\|_{\mathscr{B}} \tag{13.1.2}$$

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is finite.

If $\mathscr{B}$ is a Banach space of mixed Lebesgue type, then the inequality (13.1.1) can be used to deduce:

(1) that $M(\omega, \mathscr{B})$ is invariant of the choice of window function ϕ in (13.1.2), and that different ϕ give rise to equivalent norms.
(2) that $M(\omega, \mathscr{B})$ increases with the Lebesgue exponents.
(3) that $M(\omega, \mathscr{B})$ is complete.

Essential parts of these basic properties for modulation spaces were established in the pioneering paper [7], but some tracks go back to [5, 6]. The theory has thereafter been developed in different ways, see e.g. [8–10, 12].

A more complicated situation appear when some of the Lebesgue exponents for $\mathscr{B}$ above are strictly smaller than one, since $\mathscr{B}$ is then merely a quasi-Banach space, but not a Banach space, because only a weaker form of the triangle inequality holds true. In such situations, $\mathscr{B}$ even fails to be a local convex topological vector space, and the analysis based on (13.1.1) to reach (1)–(3) in their full strength above seems not work. (Some partial properties can be achieved if for example it is required that the Fourier transform of ϕ and ϕ_0 should be compactly supported, see e.g [18]).

In [11], the more discrete approach is used to handle this situation, where a Gabor expansion of ϕ with ϕ_0 as Gabor window leads to that $|V_\phi f|$ can be estimated by

$$|V_\phi f| \lesssim a *_{[E]} |V_{\phi_0} f|, \tag{13.1.3}$$

for some nonnegative sequence a with enough rapid decay toward zero at infinity. Here $*_{[E]}$ denotes the semi-continuous convolution

$$a *_{[E]} F \equiv \sum_{j \in \Lambda_E} F(\cdot - j) a(j)$$

with respect to the basis E, between functions F and sequences a, and Λ_E is the lattice spanned by E. It follows that $*_{[E]}$ is similar to discrete convolutions.

For the discrete convolution $*$ both the classical Young's inequality

$$\|a * b\|_{\ell_0^p} \le \|a\|_{\ell^{p_1}} \|b\|_{\ell^{p_2}}, \quad \frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p_0}, \quad p_j \in [1, \infty], \tag{13.1.4}$$

as well as

$$\|a * b\|_{\ell^p} \le \|a\|_{\ell^p} \|b\|_{\ell^r}, \qquad r \le \min(1, p), \quad p, r \in (0, \infty], \tag{13.1.5}$$

hold true, and it is proved in [11] and extended in [15] that similar facts hold true for semi-continuous convolutions. In the end, the following restatement of [15, Proposition 2.1] is deduced. The result also extends [11, Lemma 2.6].

Theorem 13.1.1 *Let E be an ordered basis of $\mathbf{R}^d$, $\omega, v \in \mathscr{P}_E(\mathbf{R}^d)$ be such that ω is v-moderate, and let $p, r \in (0, \infty]^d$ be such that*

$$r_k \le \min_{m\le k}(1, p_m).$$

Then the map $(a, f) \mapsto a *_{[E]} f$ *from* $\ell_0(\Lambda_E) \times \Sigma_1(\mathbf{R}^d)$ *to* $L^p_{E,(\omega)}(\mathbf{R}^d)$ *extends uniquely to a linear and continuous map from* $\ell^r_{E,(v)}(\Lambda_E) \times L^p_{E,(\omega)}(\mathbf{R}^d)$ *to* $L^p_{E,(\omega)}(\mathbf{R}^d)$*, and*

$$\|a *_{[E]} f\|_{L^p_{E,(\omega)}} \lesssim \|a\|_{\ell^r_{E,(v)}} \|f\|_{L^p_{E,(\omega)}}, \qquad a \in \ell^r_{E,(v)}(\Lambda_E),\ f \in L^p_{E,(\omega)}(\mathbf{R}^d). \tag{13.1.6}$$

In [11], (13.1.3) in combination with [11, Lemma 2.6] is used to show that (1)–(3) still hold when $\mathscr{B} = L^{p,q}$ and ω is a moderate weight of polynomial type. In [15], (13.1.3) in combination with Theorem 13.1.1 are used to show (1)–(3) for an even broader class of mixed Lebesgue spaces $\mathscr{B}$ and weight functions ω.

The aim of the paper is to extend Theorem 13.1.1, so that f in some directions (variables) is allowed to be periodic, or a weaker form of periodicity, called *echo-periodic functions* (cf. Theorem 13.3.1 in Sect. 13.3). Such functions appear for example when applying the short-time Fourier transform on periodic or quasi-periodic functions. In fact, if f is E-periodic, then $x \mapsto |V_\phi f(x,\xi)|$ is E-periodic for every ξ. A function or distribution $F(x,\xi)$ is called quasi-periodic of order $\rho > 0$, if

$$\begin{aligned} F(x+\rho k, \xi) &= e^{2\pi i\rho\langle k,\xi\rangle} F(x,\xi), && k \in \mathbf{Z}^d,\\ F(x, \xi + \kappa/\rho) &= F(x,\xi), && \kappa \in \mathbf{Z}^d. \end{aligned}$$

and by straightforward computations it follows that

$$\begin{aligned} |(V_\Phi F)(x+\rho k, \xi, \eta, y)| &= |(V_\Phi F)(x, \xi, \eta, y - 2\pi k)|, && k \in \mathbf{Z}^d,\\ |(V_\Phi F)(x, \xi + \kappa/\rho, \eta, y)| &= |(V_\Phi F)(x, \xi, \eta, y)|, && \kappa \in \mathbf{Z}^d, \end{aligned} \tag{13.1.7}$$

for such F. Hence, the notion on echo-periodic functions comprises periodicity, quasi-periodicity and the weaker form of periodicity that $V_\Phi F$ possess in (13.1.7).

It is expected that the achieved extensions will be useful when performing local investigations of short-time Fourier transforms of periodic and quasi-periodic functions, e.g. in [16].

Finally, we remark that in the Banach space case, i.e., when $r_k \ge 1$ for every k, then Theorems 13.1.1, and 13.3.1 follow from [9].

13.2 Preliminaries

In this section, we recall some basic facts and introduce some notations. In the first part, we recall the notion of weight functions. Thereafter we discuss mixed quasi-norm spaces of Lebesgue types and modulation spaces. Finally, we consider periodic functions and distributions and introduce the notion of echo-periodic functions, which is a weaker form of periodicity which at the same time also include the notion of quasi-periodicity. The results in the section can be found in e.g. [5–13, 15].

13.2.1 Weight Functions

A *weight* on $\mathbf{R}^d$ is a positive function $\omega \in L^\infty_{loc}(\mathbf{R}^d)$ such that $1/\omega \in L^\infty_{loc}(\mathbf{R}^d)$. A usual condition on ω is that it should be *moderate*, or v-*moderate* for some positive function $v \in L^\infty_{loc}(\mathbf{R}^d)$. This means that

$$\omega(x+y) \lesssim \omega(x)v(y), \qquad x, y \in \mathbf{R}^d. \tag{13.2.1}$$

Here $f(\theta) \lesssim g(\theta)$ means that $f(\theta) \le cg(\theta)$ for some constant $c > 0$ which is independent of θ in the domain of f and g. We note that (13.2.1) implies that ω fulfills the estimates

$$v(-x)^{-1} \lesssim \omega(x) \lesssim v(x), \quad x \in \mathbf{R}^d. \tag{13.2.2}$$

We let $\mathscr{P}_E(\mathbf{R}^d)$ be the set of all moderate weights on $\mathbf{R}^d$.

It can be proved that if $\omega \in \mathscr{P}_E(\mathbf{R}^d)$, then ω is v-moderate for some $v(x) = e^{r|x|}$, provided the positive constant r is large enough (cf. [13]). In particular, (13.2.2) shows that for any $\omega \in \mathscr{P}_E(\mathbf{R}^d)$, there is a constant $r > 0$ such that

$$e^{-r|x|} \lesssim \omega(x) \lesssim e^{r|x|}, \quad x \in \mathbf{R}^d$$

(cf. [12]).

We say that v is *submultiplicative* if v is even and (13.2.1) holds with $\omega = v$. In the sequel, v and v_j for $j \ge 0$, always stand for submultiplicative weights if nothing else is stated.

13.2.2 Spaces of Mixed Quasi-Norm Spaces of Lebesgue Types

For the (ordered) basis $E = \{e_1, \dots, e_d\}$ of $\mathbf{R}^d$, the corresponding lattice Λ_E is given by

$$\Lambda_E = \{\, n_1e_1 + \cdots + n_de_d\, ;\ (n_1, \dots, n_d) \in \mathbf{Z}^d\, \},$$

and mixed (quasi-)normed space of Lebesgue types with respect to E is given in the following definition.

Definition 13.2.1 Let $E = \{e_1, \dots, e_d\}$ be an ordered basis of $\mathbf{R}^d$, $\kappa(E)$ be the parallelepiped spanned by E, $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ $p = (p_1, \dots, p_d) \in (0, \infty]^d$ and $r = \min(1, p)$. If $f \in L^r_{loc}(\mathbf{R}^d)$, then

$$\|f\|_{L^p_{E,(\omega)}} \equiv \|g_{d-1}\|_{L^{p_d}(\mathbf{R})}$$

where $g_k(z_k)$, $z_k \in \mathbf{R}^{d-k}$, $k = 0, \dots, d-1$, are inductively defined as

$$g_0(x_1, \dots, x_d) \equiv |f(x_1 e_1 + \dots + x_d e_d)\omega(x_1 e_1 + \dots + x_d e_d)|,$$

and

$$g_k(z_k) \equiv \|g_{k-1}(\,\cdot\,, z_k)\|_{L^{p_k}(\mathbf{R})}, \quad k = 1, \dots, d-1.$$

1. If $\Omega \subseteq \mathbf{R}^d$ is measurable, then $L^p_{E,(\omega)}(\Omega)$ consists of all $f \in L^r_{loc}(\Omega)$ with finite quasi-norm

$$\|f\|_{L^p_{E,(\omega)}(\Omega)} \equiv \|f_\Omega\|_{L^p_{E,(\omega)}(\mathbf{R}^d)}, \qquad f_\Omega(x) \equiv \begin{cases} f(x), & \text{when } x \in \Omega \\ 0, & \text{when } x \notin \Omega. \end{cases}$$

   The space $L^p_{E,(\omega)}(\Omega)$ is called *E-split Lebesgue space (with respect to ω, p, Ω and E)*, and p is called the Lebesgue exponents of $L^p_{E,(\omega)}(\Omega)$;
2. If $\Lambda \subseteq \mathbf{R}^d$ is a lattice such that $\Lambda_E \subseteq \Lambda$, then the quasi-Banach space $\ell^p_{E,(\omega)}(\Lambda)$ consists of all $a \in \ell'_0(\Lambda)$ such that

$$\|a\|_{\ell^p_{E,(\omega)}(\Lambda)} \equiv \left\| \sum_{j \in \Lambda} a(j)\chi_{j+\kappa(E)} \right\|_{L^p_{E,(\omega)}(\mathbf{R}^d)}$$

   is finite. The space $\ell^p_{E,(\omega)} \equiv \ell^p_{E,(\omega)}(\Lambda_E)$ is called the *discrete version of the space* $L^p_{E,(\omega)}(\mathbf{R}^d)$ (cf. [9]).

Evidently, $L^p_{E,(\omega)}(\Omega)$ and $\ell^p_{E,(\omega)}(\Lambda)$ in Definition 13.2.1 are quasi-Banach spaces of order $\min(p, 1)$. We set

$$L^p_E = L^p_{E,(\omega)} \quad \text{and} \quad \ell^p_E = \ell^p_{E,(\omega)}$$

when $\omega = 1$, and if $p = (p_0, \dots, p_0)$ for some $p_0 \in (0, \infty]$, then

$$L^p_{E,(\omega)} = L^{p_0}_{(\omega)}, \quad L^p_E = L^{p_0}, \quad \ell^p_{E,(\omega)} = \ell^{p_0}_{(\omega)} \quad \text{and} \quad \ell^p_E = \ell^{p_0}$$

with equivalent quasi-norms.

13.2.3 Echo-Periodic Functions

We recall that if $E = \{e_1, \dots, e_d\}$ is an ordered basis of $\mathbf{R}^d$, then the function or distribution f on $\mathbf{R}^d$ is called E-periodic, if $f(\,\cdot\, + v) = f$ for every $v \in E$. More generally, if $E_0 \subseteq E$, then f above is called E_0-periodic, if $f(\,\cdot\, + v) = f$ for every $v \in E_0$. We shall consider functions that possess conditions resembling periodic ones, which appear when dealing with, e.g., quasi-periodic functions and their short-time Fourier transforms.

Definition 13.2.2 Let $E = \{e_1, \dots, e_d\}$ be an ordered basis of $\mathbf{R}^d$, $E_0 \subseteq E$ and let f be a (complex-valued) function on $\mathbf{R}^d$. For every $k \in \{1, \dots, d\}$, let M_k be the set of all $l \in \{1, \dots, k\}$ such that $e_l \in E \setminus E_0$. Then f is called an *echo-periodic function with respect to* E_0, if for every $e_k \in E_0$, there is a vector

$$v_k = \sum_{l \in M_k} v_{k,l} e_l$$

such that

$$|f(\,\cdot\, + e_k)| = |f(\,\cdot\, + v_k)|. \tag{13.2.3}$$

Example 13.1 Evidently, any periodic function or quasi-periodic function is echo-periodic. The notion of echo-periodic functions is also related to the notion on almost periodic functions (cf. [2, 3, 17]). We also notice that if f is a periodic function or distribution on $\mathbf{R}^d$, then $x \mapsto |V_\phi f(x, \xi)|$ is also period, giving that $x \mapsto V_\phi f(x, \xi)$ is echo-periodic, but in general not periodic, for every fixed $\xi \in \mathbf{R}^d$.

A more sophisticated example on echo-periodic functions concern the short-time Fourier transform on quasi-periodic functions. In fact, in [16] it is observed that (13.1.7) hold true, when F is quasi-periodic of order ρ. If $e_1, \dots, e_{4d}$ is the standard basis of $\mathbf{R}^{4d}$,

$$E = \left\{ \frac{e_{d+1}}{\rho}, \dots, \frac{e_{2d}}{\rho}, e_{3d+1}, \dots, e_{4d}, \rho e_1, \dots, \rho e_d, e_{2d+1}, \dots, e_{3d} \right\}$$

and

$$E_0 = \left\{ \frac{e_{d+1}}{\rho}, \dots, \frac{e_{2d}}{\rho}, \rho e_1, \dots, \rho e_d \right\},$$

then it follows that $V_\Phi F$ is echo-periodic with respect to E_0.

We notice that in (13.1.7), relations of the form (13.2.3) appear.

Remark 13.2.3 Let E, E_0 and M_k be the same as in Definition 13.2.2, and let f be a (complex-valued) function on $\mathbf{R}^d$ such that (13.2.3) holds true. Also let

$$J_k = \begin{cases} \mathbf{R}, & k \in M_d, \\ [0,1], & k \notin M_d, \end{cases}$$
$$I_k = \{\, x e_k \,;\, x \in J_k \,\}, \qquad k \in \{1, \dots, d\}$$

and

$$\begin{aligned} I &= \{\, x_1 e_1 + \cdots x_d e_d \,;\, x_k \in J_k,\ k = 1, \dots, d \,\} \\ &\simeq I_1 \times \cdots \times I_d. \end{aligned}$$

Then evidently, $|f(\,\cdot\, + n e_k)| = |f(\,\cdot\, + n v_k)|$ for every integer n. Hence, if f is measurable and echo-periodic with respect to E_0, and $p \in (0, \infty]^d$, then it follows by straightforward computations that

$$\| f(\,\cdot\, + n e_k) \|_{L^p_E(I)} = \| f \|_{L^p_E(I)}$$

for every integer n and $e_k \in E_0$.

Definition 13.2.4 Let E, E_0 and $I \subseteq \mathbf{R}^d$ be the same as in Remark 13.2.3, $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ and let $p \in (0, \infty]^d$. Then $L^{p,E_0}_{E,(\omega)}(\mathbf{R}^d)$ denotes the set of all complex-valued measurable echo-periodic functions f with respect to E_0 such that

$$\| f \|_{L^{p,E_0}_{E,(\omega)}} \equiv \| f \|_{L^p_{E,(\omega)}(I)}$$

is finite.

In the next section, we shall deduce weighted $L^p_E(I)$ estimates of the *semi-discrete convolution*

$$(a *_{[E]} f)(x) \sum_{j \in \Lambda_E} a(j) f(x - j), \tag{13.2.4}$$

of the measurable function f on $\mathbf{R}^d$ and $a \in \ell_0(\Lambda_E)$, with respect to the ordered basis E.

13.3 Weighted Lebesgue Estimates on Semi-discrete Convolutions

In this section, we extend Theorem 13.1.1 from the introduction such that $L^p_E(I)$-estimates of echo-periodic functions are included.

Let E, E_0, M_k, I and J_k, $k = 1, \dots, d$, be the same as in Remark 13.2.3. In what follows we let $\Sigma_1^{E_0}(\mathbf{R}^d)$ be the set of all E_0-periodic $f \in C^\infty(\mathbf{R}^d)$ such that if

$$g(x_1, \dots, x_d) \equiv f(x_1 e_1 + \cdots + x_d e_d),$$

then

$$\sup_{\alpha,\beta\in\mathbf{N}^d} \frac{\|x^\alpha D^\beta g\|_{L^\infty(I)}}{h^{|\alpha+\beta|}\alpha!\beta!}$$

is finite for every $h > 0$. By the assumptions and basic properties due to [4] it follows that $\Sigma_1^{E_0}(\mathbf{R}^d) \subseteq L^{p,E_0}_{E,(\omega)}(\mathbf{R}^d)$ for every choice of $\omega \in \mathscr{P}_E(\mathbf{R}^d)$ and $p \in (0,\infty]^d$ such that

$$\omega(x) = \omega(x_0) \qquad \text{when} \qquad x = \sum_{k=1}^d x_k e_k, \quad x_0 = \sum_{k\in M_d} x_k e_k. \tag{13.3.1}$$

Our extension of Theorem 13.1.1 to include echo-periodic functions is the following, which is also our main result.

Theorem 13.3.1 *Let E be an ordered basis of $\mathbf{R}^d$, $E_0 \subseteq E$, $\omega, v \in \mathscr{P}_E(\mathbf{R}^d)$ be such that ω is v-moderate and satisfy (13.3.1), and let $p, r \in (0,\infty]^d$ be such that*

$$r_k \le \min_{m\le k}(1, p_m).$$

Also let $I \subseteq \mathbf{R}^d$ be as in Remark 13.2.3. Then the map $(a, f) \mapsto a *_{[E]} f$ from $\ell_0(\Lambda_E) \times \Sigma_1^{E_0}(\mathbf{R}^d)$ to $L^p_{E,(\omega)}(I)$ extends uniquely to a linear and continuous map from $\ell^r_{E,(v)}(\Lambda_E) \times L^{p,E_0}_{E,(\omega)}(\mathbf{R}^d)$ to $L^p_{E,(\omega)}(I)$, and

$$\|a *_{[E]} f\|_{L^p_{E,(\omega)}(I)} \lesssim \|a\|_{\ell^r_{E,(v)}(\Lambda_E)} \|f\|_{L^p_{E,(\omega)}(I)}, \qquad a \in \ell^r_{E,(v)}(\Lambda_E),\ f \in L^{p,E_0}_{E,(\omega)}(\mathbf{R}^d) \tag{13.3.2}$$

For the proof we recall that

$$\left(\sum_{j\in I} |b(j)|\right)^r \le \sum_{j\in I} |b(j)|^r \qquad 0 < r \le 1, \tag{13.3.3}$$

for any sequence b and countable set I (cf. [1]).

Proof Since $\ell^r_{E,(v)}$ increases with r, we may assume that r_k is equal to the smallest number of $1, p_1, \dots, p_k$. By letting

$$f_\omega(x_1,\dots,x_d) = |f(x_1e_1 + \cdots + x_de_d)\omega(x_1e_1 + \cdots + x_de_d)|,$$
$$a_v(l_1,\dots,l_d) = |a(l_1e_1 + \cdots + l_de_d)v(l_1e_1 + \cdots + l_de_d)|$$

and using the inequality

$$|a *_{[E]} f \cdot \omega| \lesssim a_v *_{[E]} f_\omega,$$

(with pointwise estimate in the inequality) we reduce ourselves to the case when E is the standard basis, $\omega = v = 1$ and $f, a \ge 0$. This implies that we may identify I_k in Remark 13.2.3 with J_k for every k.

Let

$$z_k = (x_{k+1}, \dots, x_d) \in \mathbf{R}^{d-k}, \qquad m_k = (l_{k+1}, \dots, l_d) \in \mathbf{Z}^{d-k}$$

for $k = 0, \dots, d-1$.

Then $z_{k-1} = (x_k, z_k)$ and $m_{k-1} = (l_k, m_k)$. It follows that $x_k \in I_k$ when applying the mixed quasi-norms of Lebesgue types, and that

$$\begin{aligned} 0 &\le (a *_{[E]} f)(x_1, \dots, x_d) \\ &\le \sum_{m_0 \in \mathbf{Z}^d} f(x_1 - \varphi_1(m_0), \dots, z_d - \varphi_d(m_{d-1}))a(m_0), \end{aligned} \tag{13.3.4}$$

for some linear functions φ_k from $\mathbf{R}^{d+1-k}$ to $\mathbf{R}$, which satisfy

$$\varphi_k(z_{k-1}) = \begin{cases} x_k + \psi_k(z_k), & J_k = \mathbf{R}, \\ 0, & J_k = [0,1], \end{cases} \tag{13.3.5}$$

for some linear forms ψ_k on $\mathbf{R}^{d-k}$, $k = 1, \dots, d$.

Let

$$f_0 = f, \qquad a_0 = a, \qquad g_0 = a *_{[E]} f.$$

and define inductively

$$f_k(z_k) = \| f_{k-1}(\,\cdot\,, z_k)\|_{L^{p_k}(J_k)}, \quad a_k(m_k) = \|a_{k-1}(\,\cdot\,, m_k)\|_{\ell^{r_k}(\mathbf{Z})},$$

and

$$g_k(z_k) = \|g_{k-1}(\,\cdot\,, z_k)\|_{L^{p_k}(J_k)}, \qquad k = 1, \dots d.$$

Also let

$$\boldsymbol{\varphi}_k(z_k) = (\varphi_{k+1}(z_k), \dots, \varphi_d(z_{d-1})), \quad k = 0, \dots, d-1.$$

Then (13.3.4) is the same as

$$0 \le (a *_{[E]} f)(x_1, \dots, x_d) \le \sum_{m_0 \in \mathbf{Z}^d} f(z_0 - \boldsymbol{\varphi}_0(m_0))a(m_0), \tag{13.3.6}$$

We claim

$$g_k(z_k) \lesssim \left(\sum_{m_k} f_k(x_{k+1} - \varphi_{k+1}(m_k), \dots, x_d - \varphi_d(m_{d-1}))^{r_k} a_k(m_k)^{r_k} \right)^{\frac{1}{r_k}},$$

which in view of the links between (13.3.4) and (13.3.6) is the same as

$$g_k(z_k) \lesssim \left(\sum_{m_k} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^{r_k} a_k(m_k)^{r_k} \right)^{\frac{1}{r_k}} \tag{13.3.7}$$

when $k = 0, \dots, d$. Here we set $r_0 = 1$, and interprete f_d, a_d, g_d and the right-hand side of (13.3.7) as $\|f\|_{L^p_E(I)}$, $\|a\|_{\ell^r_E(\mathbf{Z}^d)}$, $\|g_0\|_{L^p_E(I)}$ and $\|f\|_{L^p_E(I)}\|a\|_{\ell^r_E(\mathbf{Z}^d)}$, respectively. The result then follows by letting $k = d$ in (13.3.7).

We shall prove (13.3.7) by induction. The result is evidently true when $k = 0$. Suppose it is true for $k-1$, $k \in \{1, \dots, d-1\}$. We shall consider the cases when $p_k \ge r_{k-1}$ or $p_k \le r_{k-1}$, and $J_k = \mathbf{R}$ or $J_k = [0, 1]$ separately and for conveniency we set $q = r_{k-1}$ and $f_{k-1} = h$.

First assume that $p_k \ge r_{k-1}$. Then $r_k = r_{k-1}$. Also suppose $J_k = \mathbf{R}$. Then it follows from the induction hypothesis that

$$\begin{aligned} &g_k(z_k) \\ &\lesssim \left(\int_{-\infty}^{\infty} \left(\sum h(x_k - \varphi_k(m_{k-1}), z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{p_k}{q}} dx_k \right)^{\frac{1}{p_k}}, \end{aligned}$$

where the sum is taken over all $(l_k, m_k) \in \mathbf{Z} \times \mathbf{Z}^{d-k}$. By Minkowski's inequality we get

$$\begin{aligned} &g_k(z_k) \\ &\lesssim \left(\sum \left(\int_{-\infty}^{\infty} h(x_k - \varphi_k(m_{k-1}), z_k - \boldsymbol{\varphi}_k(m_k))^{p_k} \, dx_k \right)^{\frac{q}{p_k}} a_{k-1}(l_k, m_k)^q \right)^{\frac{1}{q}} \\ &= \left(\sum \left(\int_{-\infty}^{\infty} h(x_k, z_k - \boldsymbol{\varphi}_k(m_k))^{p_k} \, dx_k \right)^{\frac{q}{p_k}} a_{k-1}(l_k, m_k)^q \right)^{\frac{1}{q}} \\ &= \left(\sum f_k(z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{1}{q}} \\ &= \left(\sum_{m_k \in \mathbf{Z}^{d-k}} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^q \left(\sum_{l_k \in \mathbf{Z}} a_{k-1}(l_k, m_k)^q \right) \right)^{\frac{1}{q}} \\ &= \left(\sum_{m_k \in \mathbf{Z}^{d-k}} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^{r_k} a_k(m_k)^{r_k} \right)^{\frac{1}{r_k}}, \end{aligned} \tag{13.3.8}$$

and (13.3.7) follows in the case $p_k \ge r_{k-1}$ and $J_k = \mathbf{R}$ by combining these estimates.

Next we consider the case when $p_k \ge r_{k-1}$ and $J_k = [0, 1]$. Then $\varphi_k(m_{k-1}) = 0$, and by the induction hypothesis and Minkowski's inequality we get

$$
\begin{aligned}
g_k(z_k) &\lesssim \left(\int_0^1 \left(\sum_{m_{k-1}} h(x_k, z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{p_k}{q}} dx_k \right)^{\frac{1}{p_k}} \\
&\le \left(\sum_{m_{k-1}} \left(\int_0^1 h(x_k, z_k - \boldsymbol{\varphi}_k(m_k))^{p_k} \, dx_k \right)^{\frac{q}{p_k}} a_{k-1}(l_k, m_k)^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{m_{k-1}} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{1}{q}} \\
&= \left(\sum_{m_k \in \mathbf{Z}^{d-k}} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^{r_k} a_k(m_k)^{r_k} \right)^{\frac{1}{r_k}}, \qquad (13.3.9)
\end{aligned}
$$

and (13.3.7) follows in the case $p_k \ge r_{k-1}$ and $J_k = [0, 1]$ as well.

Next assume that $p_k \le r_{k-1}$ and $J_k = \mathbf{R}$. Then

$$
p_k / r_{k-1} = p_k / q \le 1 \quad \text{and} \quad r_k = p_k,
$$

and (13.3.3) gives

$$
\begin{aligned}
&g_k(z_k) \\
&\lesssim \left(\int_{-\infty}^{\infty} \left(\sum_{m_{k-1}} h(x_k - \varphi_k(m_{k-1}), z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{p_k}{q}} dx_k \right)^{\frac{1}{p_k}} \\
&\lesssim \left(\int_{-\infty}^{\infty} \sum_{m_{k-1}} \left(h(x_k - \varphi_k(m_{k-1}), z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{p_k}{q}} dx_k \right)^{\frac{1}{p_k}} \\
&= \left(\sum_{m_{k-1}} \left(\int_{-\infty}^{\infty} h(x_k - \varphi_k(m_{k-1}), z_k - \boldsymbol{\varphi}_k(m_k))^{p_k} \, dx_k \right) a_{k-1}(l_k, m_k)^{p_k} \right)^{\frac{1}{p_k}} \\
&= \left(\sum_{m_{k-1}} \left(\int_{-\infty}^{\infty} h(x_k, z_k - \boldsymbol{\varphi}_k(m_k))^{p_k} \, dx_k \right) a_{k-1}(l_k, m_k)^{p_k} \right)^{\frac{1}{p_k}} \\
&= \left(\sum_{m_k} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^{p_k} \left(\sum_{l_k} a_{k-1}(l_k, m_k)^{p_k} \right) \right)^{\frac{1}{p_k}} \\
&= \left(\sum_{m_k} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^{r_k} a_k(m_k)^{r_k} \right)^{\frac{1}{r_k}}, \qquad (13.3.10)
\end{aligned}
$$

and (13.3.7) follows in this case as well.

It remain to consider the case $p_k \le r_{k-1}$ and $J_k = [0, 1]$. Then $\varphi_k(m_{k-1}) = 0$, and by similar arguments as above we get

$$
\begin{aligned}
& g_k(z_k) \\
& \lesssim \left(\int_0^1 \left(\sum_{m_{k-1}} h(x_k, z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{p_k}{p}} dx_k \right)^{\frac{1}{p_k}} \\
& \lesssim \left(\int_0^1 \sum_{m_{k-1}} \left(h(x_k, z_k - \boldsymbol{\varphi}_k(m_k))^q a_{k-1}(l_k, m_k)^q \right)^{\frac{p_k}{q}} dx_k \right)^{\frac{1}{p_k}} \\
& = \left(\sum_{m_{k-1}} \left(\int_0^1 h(x_k, z_k - \boldsymbol{\varphi}_k(m_k))^{p_k} dx_k \right) a_{k-1}(l_k, m_k)^{p_k} \right)^{\frac{1}{p_k}} \\
& = \left(\sum_{m_k} f_k(z_k - \boldsymbol{\varphi}_k(m_k))^{r_k} a_k(m_k)^{r_k} \right)^{\frac{1}{r_k}}, \qquad (13.3.11)
\end{aligned}
$$

and (13.3.7) and thereby the result follow.

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Chapter 14
Almost Diagonalization of Pseudodifferential Operators

S. Ivan Trapasso

Abstract In this review we focus on the almost diagonalization of pseudodifferential operators and highlight the advantages that time-frequency techniques provide here. In particular, we retrace the steps of an insightful paper by Gröchenig, who succeeded in characterizing a class of symbols previously investigated by Sjöstrand by noticing that Gabor frames almost diagonalize the corresponding Weyl operators. This approach also allows to give new and more natural proofs of related results such as boundedness of operators or algebra and Wiener properties of the symbol class. Then, we discuss some recent developments on the theme, namely an extension of these results to a more general family of pseudodifferential operators and similar outcomes for a symbol class closely related to Sjöstrand's one.

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14.1 Introduction

The wide range of problems that one can tackle by means of time-frequency analysis bears witness to the relevance of this quite modern discipline stemmed from both pure and applied issues in harmonic analysis. There is no way to provide here a comprehensive bibliography on the theme, which would encompass studies in quantum mechanics and partial differential equations. We confine ourselves to list some references to be used as points of departure for a walk through the topic: see [1, 4, 6, 11, 25, 28]. Besides the countless achievements as tool for other fields, Gabor analysis is a fascinating subject in itself and it may happen to shed new light on established facts in an effort to investigate the subtle problems underlying its foundation. We report here the case of Gröchenig's work [17]: the author retrieved and extended well-known outcomes obtained by Sjöstrand within the realm of "hard" analysis—

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cf. [26, 27], and this was achieved using techniques from phase space analysis. We will give a detailed account in the subsequent sections, but let us briefly introduce here the main characters of this story.

The (cross-)Wigner distribution is a quadratic time-frequency representation of signals f, g in suitable function spaces (for instance $f, g \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz class) defined as

$$W(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i y\omega} f\left(x+\frac{y}{2}\right)\overline{g\left(x-\frac{y}{2}\right)}\,\mathrm{d}y. \tag{14.1.1}$$

It is possible to associate a pseudodifferential operator to this representation, namely the so-called Weyl transform—it is a quite popular quantization rule in Physics community. Given a tempered distribution $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$ as *symbol* (also *observable*, in physics vocabulary), the corresponding Weyl transform maps $\mathscr{S}(\mathbb{R}^d)$ into $\mathscr{S}'(\mathbb{R}^d)$ and can be defined via duality by

$$\langle \mathrm{Op_W}(\sigma) f, g\rangle = \langle \sigma, W(g,f)\rangle, \quad f,g \in \mathscr{S}(\mathbb{R}^d). \tag{14.1.2}$$

The Weyl transform has been thoroughly studied in [15, 32] among others. In his aforementioned works, Sjöstrand proved that Weyl operators with symbols of special type satisfy a number of interesting properties concerning their boundedness and algebraic structure as a set. In terms that will be specified later in Sect. 14.3, we can state that the set of such operators is a spectral invariant *-subalgebra of $\mathscr{B}(L^2(\mathbb{R}^d))$, the ($C*$-)algebra of bounded operators on $L^2(\mathbb{R}^d)$.

To be precise, given a Schwartz function $g \in \mathscr{S}(\mathbb{R}^{2d}) \setminus \{0\}$, we provisionally define the *Sjöstrand's class* as the space of tempered distributions $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$ such that

$$\int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |\langle \sigma, \pi(z,\zeta) g\rangle| \mathrm{d}\zeta < \infty.$$

As a rule of thumb, notice that a symbol in $M^{\infty,1}(\mathbb{R}^{2d})$ locally (i.e., for fixed $z \in \mathbb{R}^{2d}$) coincides with the Fourier transform of a L^1 function. Furthermore, it can be proved that this somewhat exotic symbol class contains classical Hörmander's symbols of type $S^0_{0,0}$, together with non-smooth ones.

The crucial remark here is that Sjöstrand's class actually coincides with a function space of a particular type, namely the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$. In more general terms, modulation spaces are Banach spaces defined by means of estimates on time-frequency concentration and decay of its elements—see Sect. 14.2 for the details. They were introduced by Feichtinger in the '80s, although the original approach was quite different from the simplified one adopted hereinafter (cf. the pioneering papers [7, 8]), and soon established themselves as the optimal environment for time-frequency analysis. Nevertheless, they also provide a fruitful context to set problems in harmonic analysis and PDEs—see for instance [12, 14, 31].

Gröchenig deeply exploited this connection with time-frequency analysis by proving that Sjöstrand's results extend to more general modulation spaces and, more importantly, he was able to completely characterize symbols in these classes by means of a property satisfied by the corresponding Weyl operators, namely approximate diagonalization. This is a classical problem in pure and applied harmonic analysis—a short list of references is [2, 21, 23, 24]. We will thoroughly examine Gröchenig's results in Sect. 14.3. Here, we limit ourselves to heuristically argue that the choice of a certain type of symbols assures that the corresponding Weyl operators preserve the time-frequency localization, since their "kernel" with respect to continuous or discrete time-frequency shifts satisfies a convenient decay condition.

In the subsequent Sect. 14.4, we report some results on almost diagonalization obtained by the author in a recent joint work with Elena Cordero and Fabio Nicola—see [5]. Mimicking the scheme which leads to define the Weyl transform, in [1] the authors consider a one-parameter family of time-frequency representations (τ-*Wigner distributions*) and also define the corresponding pseudodifferential operators Op_τ via duality. Precisely, for $\tau \in [0, 1]$, the (cross-)τ-Wigner distribution is given by

$$W_\tau(f, g)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i y \zeta} f(x + \tau y)\overline{g(x - (1 - \tau)y)}\, \mathrm{d}y, \quad f, g \in \mathscr{S}(\mathbb{R}^d), \tag{14.1.3}$$

whereas the corresponding τ-pseudodifferential operator is defined by

$$\langle \mathrm{Op}_\tau(a) f, g\rangle = \langle a, W_\tau(g, f)\rangle, \quad f, g \in \mathscr{S}(\mathbb{R}^d). \tag{14.1.4}$$

For $\tau = 1/2$, we recapture the Weyl transform and the usual Wigner distribution, while the cases $\tau = 0, 1$, respectively cover the classical theory of Kohn-Nirenberg and anti-Kohn-Nirenberg operators—whose corresponding distributions are also known as Rihaczek and conjugate-Rihaczek distributions, respectively.

Our contribution aims at enlarging the area of application of Gröchenig's result along two directions. First, one finds that symbols in the Sjöstrand's class are in fact characterized by almost diagonalization of the corresponding τ-pseudodifferential operators for any $\tau \in [0, 1]$. While this is not surprising for reasons that will be discussed later, it seems worthy of interest to get a similar result for symbols belonging to a function space closely related to $M^{\infty,1}$, namely the Wiener amalgam space $W(\mathscr{F}L^\infty, L^1)$. The connection between these spaces is established by Fourier transform: in fact, the latter exactly contains the Fourier transforms of symbols in the Sjöstrand's class. It is important to remark that even if the spirit of the result is the same, numerous differences occur and we try to clarify the intuition behind this situation in Sect. 14.4.

To conclude, we take advantage of this characterization in regards to boundedness results. We were able to study the boundedness of τ-pseudodifferential operator covering several possible choices among modulation and Wiener amalgam space for symbol classes and spaces on which they act. We mention that in a number of these outcomes, we have benefited from a strong linkage with the theory of Fourier integral operators. Besides, the latter condition also made possible to establish (or disprove) the algebraic properties considered by Sjöstrand for special classes of τ-pseudodifferential operators.

14.2 Preliminaries

Notation. We write $t^2 = t \cdot t$, for $t \in \mathbb{R}^d$, and $xy = x \cdot y$ is the scalar product on $\mathbb{R}^d$. The Schwartz class is denoted by $\mathscr{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathscr{S}'(\mathbb{R}^d)$. The brackets $\langle f, g\rangle$ denote both the duality pairing between $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d)$ and the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}\mathrm{d}t$ on $L^2(\mathbb{R}^d)$. In particular, we assume it to be conjugate-linear in the second argument. The symbol $\lesssim$ means that the underlying inequality holds up to a positive constant factor $C > 0$ on the RHS:

$$f \lesssim g \quad \Rightarrow \quad \exists C > 0 \, : \, f \le Cg.$$

The Fourier transform of a function f on $\mathbb{R}^d$ is normalized as

$$\mathscr{F} f(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\xi} f(x)\, \mathrm{d}x.$$

Given $x, \omega \in \mathbb{R}^d$, the modulation M_ω and translation T_x operators act on a function f (on $\mathbb{R}^d$) as

$$M_\omega f(t) = e^{2\pi i t\omega} f(t), \qquad T_x f(t) = f(t-x).$$

We write a point in phase space as $z = (x, \omega) \in \mathbb{R}^{2d}$, and the corresponding phase-space shift acting on a function or distribution as

$$\pi(z) f(t) = e^{2\pi i \omega t} f(t-x), \quad t \in \mathbb{R}^d. \tag{14.2.1}$$

Denote by J, the canonical symplectic matrix in $\mathbb{R}^{2d}$:

$$J = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} \end{pmatrix} \in \mathrm{Sp}(2d, \mathbb{R}),$$

where the symplectic group $\mathrm{Sp}(2d, \mathbb{R})$ is defined by

$$\mathrm{Sp}(2d, \mathbb{R}) = \left\{ M \in \mathrm{GL}(2d, \mathbb{R}) : \; M^\top J M = J \right\}.$$

Observe that, for $z = (z_1, z_2) \in \mathbb{R}^{2d}$, we have $Jz = J(z_1, z_2) = (z_2, -z_1)$, $J^{-1}z = J^{-1}(z_1, z_2) = (-z_2, z_1) = -Jz$, and $J^2 = -I_{2d\times 2d}$.

Short-time Fourier transform. Let $f \in \mathscr{S}'(\mathbb{R}^d)$ and $g \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$. The short-time Fourier transform (STFT) of f with window function g is defined as

$$V_g f(x,\omega) = \langle f, \pi(x,\omega)g\rangle = \mathscr{F}(f T_x g)(\omega) = \int_{\mathbb{R}^d} f(y)\, \overline{g(y-x)}\, e^{-2\pi i y\omega}\, dy. \tag{14.2.2}$$

We remark that the last expression has to be intended in formal sense, but it truly represents the integral corresponding to the inner product $\langle f, \pi(x,\omega)g\rangle$ whenever $f, g \in L^2(\mathbb{R}^d)$.

Recall the fundamental property of time-frequency analysis:

$$V_g f\,(x,\omega) = e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}\,(J(x,\omega)). \tag{14.2.3}$$

Gabor frames. Let $\Lambda = A\mathbb{Z}^{2d}$, with $A \in \mathrm{GL}(2d,\mathbb{R})$, be a lattice in the time-frequency plane. The set of time-frequency shifts $\mathscr{G}(\varphi,\Lambda) = \{\pi(\lambda)\varphi : \lambda \in \Lambda\}$ for a non-zero $\varphi \in L^2(\mathbb{R}^d)$ (the so-called window function) is called Gabor system. A Gabor system $\mathscr{G}(\varphi,\Lambda)$ is said to be a Gabor frame if the lattice is such thick that the energy content of a signal as sampled on the lattice by means of STFT is comparable with its total energy, that is, there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \le \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)\varphi\rangle|^2 \le B\|f\|_2^2, \qquad \forall f \in L^2(\mathbb{R}^d). \tag{14.2.4}$$

14.2.1 Function Spaces

Weight functions. Let us call *admissible weight function* any non-negative continuous function v on $\mathbb{R}^{2d}$ such that:

1. $v(0) = 1$ and v is even in each coordinate:

$$v(\pm z_1, \ldots, \pm z_{2d}) = v(z_1, \ldots, z_{2d}).$$

2. v is submultiplicative, that is,

$$v(w+z) \le v(w)\, v(z) \qquad \forall w, z \in \mathbb{R}^{2d}.$$

3. v satisfies the Gelfand–Raikov–Shilov (GRS) condition:

$$\lim_{n\to\infty} v(nz)^{\frac{1}{n}} = 1 \qquad \forall z \in \mathbb{R}^{2d}. \tag{14.2.5}$$

Examples of admissible weights are given by $v(z) = e^{a|z|^b}(1+|z|)^s \log^r(e+|z|)$, with real parameters $a, r, s \ge 0$ and $0 \le b < 1$. Functions of polynomial growth such as

$$v_s(z) = \langle z\rangle^s = \left(1+|z|^2\right)^{\frac{s}{2}}, \qquad z \in \mathbb{R}^{2d},\ s \ge 0 \tag{14.2.6}$$

are admissible weights too. From now on, v will denote an admissible weight function unless otherwise specified. We remark that the GRS condition is exactly the technical tool required to forbid an exponential growth of the weight in some direction. For further discussion on this feature, see [18].

Given a submultiplicative weight v, a positive function m on $\mathbb{R}^{2d}$ is called *v-moderate weight* if there exists a constant $C \geq 0$ such that

$$m(z_1 + z_2) \leq Cv(z_1)m(z_2) \ , \qquad z_1, z_2 \in \mathbb{R}^{2d}.$$

The set of all v-moderate weights will be denoted by $\mathscr{M}_v(\mathbb{R}^{2d})$.

In order to remain in the framework of tempered distributions, in what follows we shall always assume that weight functions m on $\mathbb{R}^d$ under our consideration satisfy the following condition:

$$m(z) \geq 1, \quad \forall z \in \mathbb{R}^d \quad \text{or} \quad m(z) \gtrsim \langle z \rangle^{-N}, \tag{14.2.7}$$

for some $N \in \mathbb{N}$. The same holds with suitable modifications for weights on $\mathbb{R}^{2d}$.

Modulation spaces. Given a non-zero window $g \in \mathscr{S}(\mathbb{R}^d)$, a v-moderate weight function m on $\mathbb{R}^{2d}$ satisfying (14.2.7), and $1 \leq p, q \leq \infty$, the *modulation space* $M^{p,q}_m(\mathbb{R}^d)$ consists of all tempered distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ such that $V_g f \in L^{p,q}_m(\mathbb{R}^{2d})$ (weighted mixed-norm space). The norm on $M^{p,q}_m$ is

$$\|f\|_{M^{p,q}_m} = \|V_g f\|_{L^{p,q}_m} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, \mathrm{d}x \right)^{q/p} \mathrm{d}\omega \right)^{1/q} ,$$

with suitable modifications if $p = \infty$ or $q = \infty$. If $p = q$, we write M^p_m instead of $M^{p,p}_m$, and if $m(z) \equiv 1$ on $\mathbb{R}^{2d}$, then we write $M^{p,q}$ and M^p for $M^{p,q}_m$ and $M^{p,p}_m$.

It can be proved (see [15]) that $M^{p,q}_m(\mathbb{R}^d)$ is a Banach space whose definition is independent of the choice of the window g—meaning that different windows provide equivalent norms on $M^{p,q}_m$. The window class can be extended to M^1_v, cf. [15, Thm. 11.3.7]. Hence, given any $g \in M^1_v(\mathbb{R}^d)$ and $f \in M^{p,q}_m$, we have

$$\|f\|_{M^{p,q}_m} \asymp \|V_g f\|_{L^{p,q}_m}. \tag{14.2.8}$$

We recall the inversion formula for the STFT (see [15, Proposition 11.3.2]). If $g \in M^1_v(\mathbb{R}^d) \setminus \{0\}$, $f \in M^{p,q}_m(\mathbb{R}^d)$, with m satisfying (14.2.7), then

$$f = \frac{1}{\|g\|_2^2} \int_{\mathbb{R}^{2d}} V_g f(z) \pi(z) g \, \mathrm{d}z \, , \tag{14.2.9}$$

and the equality holds in $M^{p,q}_m(\mathbb{R}^d)$.

The adjoint operator of V_g, defined by

$$V_g^* F(t) = \int_{\mathbb{R}^{2d}} F(z) \pi(z) g \mathrm{d}z \, ,$$

maps the Banach space $L^{p,q}_m(\mathbb{R}^{2d})$ into $M^{p,q}_m(\mathbb{R}^d)$. In particular, if $F = V_g f$ the inversion formula (14.2.9) reads

$$\mathrm{Id}_{M^{p,q}_m} = \frac{1}{\|g\|_2^2} V_g^* V_g. \tag{14.2.10}$$

Wiener Amalgam Spaces. Fix $g \in \mathscr{S}(\mathbb{R}^d) \setminus \{0\}$ and consider *even* weight functions u, w on $\mathbb{R}^d$ satisfying (14.2.7). The Wiener amalgam space $W(\mathscr{F}L^p_u, L^q_w)(\mathbb{R}^d)$ is the space of distributions $f \in \mathscr{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{W(\mathscr{F}L^p_u, L^q_w)(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p u^p(\omega)\, \mathrm{d}\omega \right)^{q/p} w^q(x) \mathrm{d}x \right)^{1/q} < \infty$$

with obvious modifications for $p = \infty$ or $q = \infty$.

It shall be underlined that this kind of spaces is a (very special, indeed) subclass of the more general family $W(B, C)$, where B and C are suitable functional spaces. Furthermore, the definition provided here sweeps under the carpet many non-trivial aspects of their structure. A thorough discussion of the subject is not necessary for our purposes and would lead us too far, thus we prefer to address the interested reader to the literature, cf. for instance [9, 15, 20].

Notice that the fundamental identity of time-frequency analysis (14.2.3) yields $|V_g f(x,\omega)| = |V_{\hat{g}} \hat{f}(\omega, -x)| = |\mathscr{F}(\hat{f}\, T_\omega \overline{\hat{g}})(-x)|$ and (since $u(x) = u(-x)$)

$$\|f\|_{M^{p,q}_{u\otimes w}} = \left(\int_{\mathbb{R}^d} \|\hat{f}\, T_\omega \overline{\hat{g}}\|^q_{\mathscr{F}L^p_u} w^q(\omega)\, \mathrm{d}\omega \right)^{1/q} = \|\hat{f}\|_{W(\mathscr{F}L^p_u, L^q_w)}.$$

Hence, the special Wiener amalgam spaces under our consideration are simply the image under Fourier transform of modulation spaces with weights of tensor product type, namely $m(x,\omega) = u \otimes w(x,\omega) = u(x)w(\omega)$:

$$\mathscr{F}(M^{p,q}_{u\otimes w}) = W(\mathscr{F}L^p_u, L^q_w). \tag{14.2.11}$$

This should not at all come as a surprise. In fact, it is exactly how modulation spaces have been originally designed by Feichtinger, i.e., as special Wiener amalgams on the Fourier transform side. We recommend to look at [10] for an intriguing conceptual and historical account on the whole matter.

14.2.2 τ-Pseudodifferential Operators

Let us introduce the τ-pseudodifferential operators as it is customary in time-frequency analysis, i.e., by means of superposition of time-frequency shifts:

$$\mathrm{Op}_{\tau}(\sigma) f(x)=\int_{\mathbb{R}^{2d}} \hat{\sigma}(\omega, u) e^{-2\pi i(1-\tau)\omega u}\left(T_{-u} M_{\omega} f\right)(x) \mathrm{d}u \mathrm{d}\omega, \qquad x \in \mathbb{R}^{d}, \tag{14.2.12}$$

for any $\tau \in [0,1]$. The symbol σ and the function f belong to suitable function spaces, to be determined in order for the previous expression to make sense. As an example, minor modifications to [15, Lem. 14.3.1] give that $\mathrm{Op}_{\tau}(\sigma)$ maps $\mathscr{S}\left(\mathbb{R}^{d}\right)$ to $\mathscr{S}'\left(\mathbb{R}^{2d}\right)$ whenever $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$.

Assuming that (14.2.12) is a well-defined absolutely convergent integral (for instance, it is enough to assume $\hat{\sigma} \in L^{1}\left(\mathbb{R}^{2d}\right)$), easy computations lead to the usual integral form of τ-pseudodifferential operators, namely

$$\mathrm{Op}_{\tau}(\sigma) f(x)=\int_{\mathbb{R}^{2d}} e^{2\pi i(x-y)\omega} \sigma((1-\tau) x+\tau y, \omega) f(y) \mathrm{d}y \mathrm{d}\omega.$$

We finally aim to represent $\mathrm{Op}_{\tau}(\sigma)$ as an integral operator of the form

$$\mathrm{Op}_{\tau}(\sigma) f(x)=\int_{\mathbb{R}^{2d}} k(x, y) f(y) \mathrm{d}y.$$

Let us introduce the operator $\mathfrak{T}_{\tau}$ acting on functions on $\mathbb{R}^{2d}$ as

$$\mathfrak{T}_{\tau} F(x, y)=F(x+\tau y, x-(1-\tau) y), \qquad \mathfrak{T}_{\tau}^{-1} F(x, y)=F((1-\tau) x+\tau y, x-y),$$

and denote by $\mathscr{F}_{i}$, $i=1,2$, the partial Fourier transform with respect to the i-th $d-$dimensional variable (it is then clear that $\mathscr{F}=\mathscr{F}_{1} \mathscr{F}_{2}$).

Since the operators $\mathfrak{T}_{\tau}$ and $\mathscr{F}_{i}$ are continuous bijections on $\mathscr{S}\left(\mathbb{R}^{2d}\right)$, the kernel k is well-defined (as a tempered distribution) also for symbols in $\mathscr{S}'\left(\mathbb{R}^{2d}\right)$ and we finally recover the representation by duality given in the Introduction according to [1].

Proposition 14.1 *For any symbol* $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and any real* $\tau \in [0,1]$*, the map* $\mathrm{Op}_{\tau}(\sigma): \mathscr{S}\left(\mathbb{R}^{d}\right) \rightarrow \mathscr{S}\left(\mathbb{R}^{d}\right)$ *is defined as integral operator with distributional kernel*

$$k=\mathfrak{T}_{\tau}^{-1} \mathscr{F}_{2}^{-1} \sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right),$$

meaning that, for any $f, g \in \mathscr{S}\left(\mathbb{R}^{d}\right)$*,*

$$\left\langle\mathrm{Op}_{\tau}(\sigma) f, g\right\rangle=\left\langle k, g \otimes \overline{f}\right\rangle.$$

In particular, since the representation

$$W_{\tau}(f, g)(x, \omega)=\mathscr{F}_{2} \mathfrak{T}_{\tau}(f \otimes \overline{g})(x, \omega)$$

holds for $f, g \in \mathscr{S}\left(\mathbb{R}^{d}\right)$*, we have*

$$\left\langle\mathrm{Op}_{\tau}(\sigma) f, g\right\rangle=\left\langle\sigma, W_{\tau}(g, f)\right\rangle.$$

As a consequence of the celebrated Schwartz's kernel theorem (see for instance [15, Theorem 14.3.4]), we are able to relate the representations for τ-pseudodifferential operators given insofar.

Theorem 14.1 *Let $T : \mathscr{S}\left(\mathbb{R}^d\right) \to \mathscr{S}'\left(\mathbb{R}^d\right)$ be a continuous linear operator. There exist tempered distributions $k, \sigma, F \in \mathscr{S}'\left(\mathbb{R}^d\right)$ and $\tau \in [0,1]$ such that T admits the following representations:*

(i) as an integral operator: $\langle Tf, g\rangle = \left\langle k, g \otimes \overline{f}\right\rangle$ for any $f, g \in \mathscr{S}\left(\mathbb{R}^d\right)$;
(ii) as a τ-pseudodifferential operator $T = \mathrm{Op}_\tau(\sigma)$ with symbol σ;
(iii) as a superposition (in a weak sense) of time-frequency shifts :

$$T = \int_{\mathbb{R}^{2d}} F(x,\omega)\, e^{2(1-\tau)\pi i x\omega} T_x M_\omega \mathrm{d}x\mathrm{d}\omega.$$

The relations among k, σ and F are the following:

$$\sigma = \mathscr{F}_2 \mathfrak{T}_\tau k, \qquad F = \mathscr{I}_2 \hat{\sigma},$$

where $\mathscr{I}_2$ denotes the reflection in the second d-dimensional variable (i.e., $\mathscr{I}_2 G(x,\omega) = G(x,-\omega)$, $(x,\omega) \in \mathbb{R}^{2d}$).

To conclude this anthology, since the algebraic properties of pseudodifferential operators' families will be considered, recall that the composition of Weyl transforms provides a bilinear form on symbols, the so-called *twisted product*:

$$\mathrm{Op_W}(\sigma) \circ \mathrm{Op_W}(\rho) = \mathrm{Op_W}(\sigma \sharp \rho).$$

Although explicit formulas for the twisted product of symbols can be derived (cf. [32]), we will not need them hereafter. Anyway, this is a fundamental notion in order to establish an algebra structure on symbol spaces: it is quite natural to ask if the composition of operators with symbols in the same class reveals to be an operator of the same type for some symbol in the same class. Also recall that taking the adjoint of a Weyl operator provides an involution on the level of symbols, since $(\mathrm{Op_W}(\sigma))^* = \mathrm{Op_W}(\overline{\sigma})$.

14.3 Time-Frequency Analysis of the Sjöstrand's Class

The study of pseudodifferential operators has a wide and long tradition in the field of mathematical analysis, starting from the monumental work of Hörmander. It has to be noticed that the classical symbol classes considered in these investigations are usually defined by means of differentiability conditions. In the spirit of time-frequency analysis, we hereby employ modulation and Wiener amalgam spaces as reservoirs of symbols for pseudodifferential operator and hence the short-time Fourier transform to shape the desired properties.

Recall that the Sjöstrand's class is the modulation space $M^{\infty,1}(\mathbb{R}^{2d})$ consisting of distributions $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$ such that

$$\int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |\langle \sigma, \pi(z,\zeta)g\rangle| \mathrm{d}\zeta < \infty.$$

The control on symbols can be improved by weighting the condition on their short-time Fourier transform, i.e., the modulation space norm. In the following, we will employ weight functions of type $1 \otimes v$, where v is an admissible weight on $\mathbb{R}^{2d}$, according to the properties assumed in the Preliminaries. Weighted Sjöstrand's classes of this type are thus defined as

$$M^{\infty,1}_{1\otimes v}\left(\mathbb{R}^{2d}\right) = \left\{ \sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right) : \int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |V_g\sigma(z,\zeta)| v(\zeta) \, \mathrm{d}\zeta < \infty \right\}.$$

A function space closely related to the previous one is the Wiener amalgam space $W(\mathscr{F}L^\infty, L^1_v)(\mathbb{R}^{2d})$. As discussed in the previous section, we have indeed $W(\mathscr{F}L^\infty, L^1_v)(\mathbb{R}^{2d}) = \mathscr{F}M^{\infty,1}_{1\otimes v}(\mathbb{R}^{2d})$. Heuristically, a symbol in $W(\mathscr{F}L^\infty, L^1)(\mathbb{R}^{2d})$ locally coincides with the Fourier transform of a $L^\infty(\mathbb{R}^{2d})$ signal and exhibits global decay of L^1 type. For instance, the δ distribution (in $\mathscr{S}'(\mathbb{R}^{2d})$) belongs to $W(\mathscr{F}L^\infty, L^1)(\mathbb{R}^{2d})$.

Although Sjöstrand's definition of the eponym symbol class was quite different from the one given here in terms of modulation spaces, in his works [26, 27] he proved three fundamental results on Weyl operators with symbols in $M^{\infty,1}$.

Theorem 14.2

(i) *(**Boundedness**) If $\sigma \in M^{\infty,1}\left(\mathbb{R}^{2d}\right)$, then* $\mathrm{Op_W}(\sigma)$ *is a bounded operator on* $L^2(\mathbb{R}^d)$.
(ii) *(**Algebra property**) If $\sigma_1, \sigma_2 \in M^{\infty,1}\left(\mathbb{R}^{2d}\right)$ and* $\mathrm{Op_W}(\rho) = \mathrm{Op_W}(\sigma_1)\mathrm{Op_W}(\sigma_2)$, *then $\rho = \sigma_1 \sharp \sigma_2 \in M^{\infty,1}\left(\mathbb{R}^{2d}\right)$.*
(iii) *(**Wiener property**) If $\sigma \in M^{\infty,1}\left(\mathbb{R}^{2d}\right)$ and* $\mathrm{Op_W}(\sigma)$ *is invertible on* $L^2(\mathbb{R}^d)$, *then* $\left[\mathrm{Op_W}(\sigma)\right]^{-1} = \mathrm{Op_W}(\rho)$ *for some $\rho \in M^{\infty,1}\left(\mathbb{R}^{2d}\right)$.*

For sake of conciseness, we can resume the preceding outcomes by saying that the family of Weyl operators with symbols in Sjöstrand's class (denoted by $\mathrm{Op_W}(M^{\infty,1})$) is an inverse-closed Banach *-subalgebra of $\mathscr{B}(L^2(\mathbb{R}^d))$.

Both these results and their original proofs might appear fairly technical at first glance. Nonetheless, they unravel a deep and fascinating analogy between Weyl operators with symbols in the Sjöstrand's class and Fourier series with ℓ^1 coefficients. Similarities of this kind come under the multifaceted problem of spectral invariance, a topic thoroughly explored by Gröchenig in his insightful lecture [18].

In view of the structure of τ-pseudodifferential operators as superposition of time-frequency shifts (cf. Eq. 14.2.12), it can be fruitful to study how operators interact with time-frequency shifts. A measure of this interplay is given by the entries of the infinite matrix which we are going to refer to as *channel matrix*, according to traditional nomenclature in applied contexts like data transmission. First, fix a non-zero window $\varphi \in M_v^1(\mathbb{R}^d)\left(\mathbb{R}^d\right)$ and a lattice $\Lambda = A\mathbb{Z}^{2d} \subseteq \mathbb{R}^{2d}$, where $A \in \mathrm{GL}(2d, \mathbb{R})$, such that $\mathscr{G}(\varphi, \Lambda)$ is a Gabor frame for $L^2\left(\mathbb{R}^d\right)$. Therefore, the entries of the channel matrix are given by

$$\langle \mathrm{Op_W}(\sigma)\pi(z)\varphi, \pi(w)\varphi\rangle, \qquad z, w \in \mathbb{R}^{2d},$$

or

$$M(\sigma)_{\lambda,\mu} := \langle \mathrm{Op_W}(\sigma)\pi(\lambda)\varphi, \pi(\mu)\varphi\rangle, \qquad \lambda, \mu \in \Lambda,$$

if we restrict to the lattice Λ. In this context, we could say that $\mathrm{Op_W}$ is almost diagonalized by the Gabor frame $\mathscr{G}(\varphi, \Lambda)$ if its channel matrix exhibits a suitable off-diagonal decay. The key result proved by Gröchenig in [17] is a characterization of this type: a symbol belongs to the (weighted) Sjöstrand's class if and only if time-frequency shifts are almost eigenvectors of the corresponding Weyl operator. More precisely, the claim is the following.

Theorem 14.3 *Let v be an admissible weight and fix a non-zero window $\varphi \in M_v^1(\mathbb{R}^d)\left(\mathbb{R}^d\right)$ such that $\mathscr{G}(\varphi, \Lambda)$ is a Gabor frame for $L^2\left(\mathbb{R}^d\right)$. The following properties are equivalent:*

(i) $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}\left(\mathbb{R}^{2d}\right)$.
(ii) $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and there exists a function* $H \in L_v^1\left(\mathbb{R}^{2d}\right)$ *such that*

$$|\langle \mathrm{Op_W}(\sigma)\,\pi(z)\,\varphi, \pi(w)\,\varphi\rangle| \le H(w - z), \qquad \forall w, z \in \mathbb{R}^{2d}.$$

(iii) $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and there exists a sequence* $h \in \ell_v^1(\Lambda)$ *such that*

$$|\langle \mathrm{Op_W}(\sigma)\,\pi(\mu)\,\varphi, \pi(\lambda)\,\varphi\rangle| \le h(\lambda - \mu), \qquad \forall \lambda, \mu \in \Lambda.$$

This characterization is very strong: in particular, by applying Schwartz's kernel theorem, we also have:

Corollary 14.1 *Under the hypotheses of the previous Theorem, assume that* $T : \mathscr{S}\left(\mathbb{R}^d\right) \to \mathscr{S}'\left(\mathbb{R}^d\right)$ *is continuous and satisfies one of the following conditions:*

(i) $|\langle T\pi(z)\,\varphi, \pi(w)\,\varphi\rangle| \le H(w - z), \quad \forall w, z \in \mathbb{R}^{2d}$ *for some* $H \in L^1$.
(ii) $|\langle T\pi(\mu)\,\varphi, \pi(\lambda)\,\varphi\rangle| \le h(\lambda - \mu), \quad \forall \lambda, \mu \in \Lambda$ *for some* $h \in \ell^1$.

Therefore, $T = \mathrm{Op_W}(\sigma)$ *for some symbol* $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}\left(\mathbb{R}^{2d}\right)$.

The proof of the main result heavily relies on a simple but crucial interplay between the entries of the channel matrix of $\mathrm{Op_W}$ and the short-time Fourier transform of the symbol, which will be discussed in complete generality in the subsequent section. We mention that at this point, Gröchenig establishes a strong link with matrix algebra, hence heading towards a more conceptual discussion of the almost diagonalization property. In particular, it is easy to prove that $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}$ if and only if its channel matrix $M(\sigma)$ belongs to the class $\mathscr{C}_v(\Lambda)$ of matrices $A = (a_{\lambda,\mu})_{\lambda,\mu\in\Lambda}$ such that there exists a sequence $h \in \ell^1_v$ which almost diagonalizes its entries, i.e.,

$$\|a_{\lambda,\mu}\| \leq h(\lambda - \mu), \qquad \lambda, \mu \in \Lambda.$$

It can be proved that $\mathscr{C}_v(\Lambda)$ is indeed a Banach *-algebra, and this insight allows a natural extension if one considers other matrix algebras and investigates the relation between symbols and the membership of their Gabor matrices in a matrix algebra. For further investigations in more general contexts, see for instance [19].

Thanks to this fresh new formulation, the proofs of Sjöstrand's results provided by Gröchenig are to certain extent more natural. Furthermore, they extend the previous ones since weighted spaces are considered. We summarize the main outcomes in the following claims.

Theorem 14.4 (Boundedness) *If* $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}$, *then* $\mathrm{Op_W}(\sigma)$ *is bounded on* $M^{p,q}_m$ *for any* $1 \leq p, q \leq \infty$ *and any* $m \in \mathscr{M}_v$. *In particular, if* $\sigma \in M^{\infty,1}$, $\mathrm{Op_W}(\sigma)$ *is bounded on* $L^2(\mathbb{R}^d)$ *and*

- *if* $1 \leq p \leq 2$, $\mathrm{Op_W}(\sigma)$ *maps* L^p *into* $M^{p,p'}$;
- *if* $2 \leq p \leq \infty$, $\mathrm{Op_W}(\sigma)$ *maps* L^p *into* M^p.

Theorem 14.5 (Algebra property) *If* v *is a submultiplicative on* $\mathbb{R}^{2d}$, *then* $M^{\infty,1}_v$ *is a Banach* $*$*-algebra with respect to the twisted product* $\sharp$ *and the involution* $\sigma \mapsto \overline{\sigma}$.

Theorem 14.6 (Wiener property) *Assume that* v *is a submultiplicative weight on* $\mathbb{R}^{2d}$. $\mathrm{Op_W}\left(M^{\infty,1}_v\right)$ *is inverse-closed in* $\mathscr{B}(L^2(\mathbb{R}^d))$ *(i.e., if* $\sigma \in \left(M^{\infty,1}_v\right)$ *and* $\mathrm{Op_W}(\sigma)$ *is invertible on* L^2, *then* $[\mathrm{Op_W}(\sigma)]^{-1} = \mathrm{Op_W}(\rho)$ *for some* $\rho \in \left(M^{\infty,1}_v\right)$*) if and only if* v *satisfies the GRS condition* (14.2.5).

Corollary 14.2 (Spectral invariance on modulation spaces) *Assume that* v *is an admissible weight,* $\sigma \in \left(M^{\infty,1}_v\right)$ *and* $\mathrm{Op_W}(\sigma)$ *is invertible on* L^2. *Then,* $\mathrm{Op_W}(\sigma)$ *is simultaneously invertible on every modulation space* $M^{p,q}_m(\mathbb{R}^d)$, *for any* $1 \leq p, q \leq \infty$ *and* $m \in \mathscr{M}_v$.

Remark 14.1 The intuition behind the last result is that the spectrum of an operator with suitably likable properties does not truly depend on the space on which it acts. In order to establish a link with Beals' theorem on spectral invariance in the context of classical pseudodifferential operators, notice that Hörmander's class

$$S^0_{0,0}(\mathbb{R}^{2d}) = \{\sigma \in C^\infty(\mathbb{R}^{2d}) : \partial^\alpha \sigma \in L^\infty(\mathbb{R}^{2d}) \forall \alpha \in \mathbb{N}^{2d}_0\}$$

can be recasted as intersection of Sjöstrand's classes with polynomial weights (cf. [19]), namely

$$S^0_{0,0}(\mathbb{R}^{2d}) = \bigcap_{s\geq 0} M^{\infty,1}_{v_s}(\mathbb{R}^{2d}).$$

The Wiener property of these spaces leads to the conclusion that $\mathrm{Op_W}\left(S^0_{0,0}\right)$ is inverse-closed in $\mathscr{B}(L^2)$ too.

14.4 Almost Diagonalization of τ-Pseudodifferential Operators

In a recent joint work of the author with E. Cordero and F. Nicola, an attempt has been made to follow the path outlined by Gröchenig. The two directions investigated are

1. the extension of the almost-diagonalization theorem to more general operators;
2. the search of an almost-diagonalization-like characterization of other symbol classes.

For what concerns the first point, τ-pseudodifferential operators were investigated instead of those of Weyl type. We already discussed in the Introduction how this general class of operators extends in a natural way the previous one, which can be recovered as the case $\tau = 1/2$. We were able to obtain an identical result with an identical proof—apart from the substantial modifications in the preliminary lemmas—see [5] for the details.

Theorem 14.7 *Let v be an admissible weight on $\mathbb{R}^{2d}$. Consider $\varphi \in M^1_v\left(\mathbb{R}^d\right)\setminus\{0\}$ and a lattice $\Lambda \subseteq \mathbb{R}^{2d}$ such that $\mathscr{G}(\varphi, \Lambda)$ is a Gabor frame for $L^2\left(\mathbb{R}^d\right)$. For any $\tau \in [0, 1]$, the following properties are equivalent:*

(i) $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}\left(\mathbb{R}^{2d}\right)$.

(ii) $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and there exists a function* $H_\tau \in L^1_v\left(\mathbb{R}^{2d}\right)$ *such that*

$$\left|\left\langle \mathrm{Op}_\tau(\sigma)\,\pi(z)\,\varphi, \pi(w)\,\varphi\right\rangle\right| \leq H_\tau(w-z) \qquad \forall w, z \in \mathbb{R}^{2d}.$$

(iii) $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and there exists a sequence* $h_\tau \in \ell^1_v(\Lambda)$ *such that*

$$\left|\left\langle \mathrm{Op}_\tau(\sigma)\,\pi(\mu)\,\varphi, \pi(\lambda)\,\varphi\right\rangle\right| \leq h_\tau(\lambda-\mu) \qquad \forall \lambda, \mu \in \Lambda.$$

This result is not surprising for at least two reasons. Looking at the mapping relating the symbols of different τ-quantizations, namely (see for instance [22, 29])

$$\mathrm{Op}_{\tau_1}(a_1) = \mathrm{Op}_{\tau_2}(a_2) \;\Leftrightarrow\; \widehat{a_2}(\xi_1,\xi_2) = e^{-2\pi i(\tau_2-\tau_1)\xi_1\xi_2}\widehat{a_1}(\xi_1,\xi_2),$$

we see that the map that relates a Weyl symbol to its τ-counterpart is bounded in the Sjöstrand's class. At a more fundamental level, it is instructive to give a look at the crucial ingredient of the proof, which is the relation between the channel matrix of the τ-pseudodifferential operator and the short-time Fourier transform of the symbol.

Proposition 14.2 *Fix a non-zero window $\varphi \in \mathscr{S}(\mathbb{R}^d)$ and set $\Phi_\tau = W_\tau(\varphi, \varphi)$ for $\tau \in [0, 1]$. Then, for $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$,*

$$\left|\left\langle \mathrm{Op}_\tau(\sigma)\,\pi(z)\,\varphi, \pi(w)\,\varphi\right\rangle\right| = \left|V_{\Phi_\tau}\sigma\left(\mathscr{T}_\tau(z, w), J(w - z)\right)\right| = \left|V_{\Phi_\tau}\sigma(x, y)\right| \tag{14.4.1}$$

and

$$\left|V_{\Phi_\tau}\sigma(x, y)\right| = \left|\left\langle \mathrm{Op}_\tau(\sigma)\,\pi(z(x, y))\,\varphi, \pi(w(x, y))\,\varphi\right\rangle\right|, \tag{14.4.2}$$

for all $w, z, x, y \in \mathbb{R}^{2d}$, where $\mathscr{T}_\tau$ is defined as

$$\mathscr{T}_\tau(z, w) = \begin{pmatrix} (1-\tau)\,z_1 + \tau w_1 \\ \tau z_2 + (1-\tau)\,w_2 \end{pmatrix} \quad z = (z_1, z_2),\ w = (w_1, w_2) \in \mathbb{R}^{2d}. \tag{14.4.3}$$

and

$$z(x, y) = \begin{pmatrix} x_1 + (1-\tau)\,y_2 \\ x_2 - \tau y_1 \end{pmatrix}, \qquad w(x, y) = \begin{pmatrix} x_1 - \tau y_2 \\ x_2 + (1-\tau)\,y_1 \end{pmatrix}. \tag{14.4.4}$$

The main remark here is that the controlling function $H_\tau \in L^1_v(\mathbb{R}^d)$ in the almost diagonalization theorem can be chosen as the so-called *grand symbol* associated to $\sigma \in M^{\infty,1}_{v\circ J^{-1}}$ (according to [16]): for the general τ-case, we have

$$H_\tau(v) = \sup_{u\in\mathbb{R}^{2d}} \left|V_{\Phi_\tau}\sigma(u, Jv)\right|.$$

The choice of the grand symbol is quite natural if one looks at the modulation norm in the Sjöstrand's class. However, it is clear that the dependence from τ is completely confined to the window function Φ_τ and does not affect the variable $v \in \mathbb{R}^{2d}$, which corresponds to the frequency variable for the short-time Fourier transform of the symbol. The proof of the general case can thus proceed exactly as the one for Weyl case. We remark that also Corollary 14.1 generalizes in the obvious way.

It is reasonable at this stage to ask what happens if a slight modification of the grand symbol is taken into account, that is, what happens if we look at the time dependence of $V_{\Phi_\tau}\sigma$? This is equivalent to wonder if similar arguments extend in some fashion to Fourier transform of symbols in the Sjöstrand's class, namely symbols in a suitably weighted version of Wiener amalgam space $W\left(\mathscr{F}L^\infty, L^1\right) = \mathscr{F}M^{\infty,1}$—hereinafter referred to $\mathscr{F}$-Sjöstrand's class. The main outcome we got is the following.

Theorem 14.8 *Let v be an admissible weight function on $\mathbb{R}^{2d}$. Consider $\varphi \in M^1_v\left(\mathbb{R}^d\right) \setminus \{0\}$. For any $\tau \in (0, 1)$, the following properties are equivalent:*

(i) $\sigma \in W\left(\mathscr{F}L^\infty, L^1_{v\circ\mathscr{B}_\tau}\right)\left(\mathbb{R}^{2d}\right)$.

(ii) $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and there exists a function* $H_\tau \in L^1_v\left(\mathbb{R}^{2d}\right)$ *such that*

$$\left|\left\langle \mathrm{Op}_\tau(\sigma)\,\pi(z)\,\varphi, \pi(w)\,\varphi\right\rangle\right| \leq H_\tau\,(w - \mathscr{U}_\tau z) \qquad \forall w, z \in \mathbb{R}^{2d}, \tag{14.4.5}$$

where the matrices $\mathscr{B}_\tau$ *and* $\mathscr{U}_\tau$ *are defined as*

$$\mathscr{B}_\tau = \begin{pmatrix} \frac{1}{1-\tau} I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & \frac{1}{\tau} I_{d\times d} \end{pmatrix}, \qquad \mathscr{U}_\tau = -\begin{pmatrix} \frac{\tau}{1-\tau} I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & \frac{1-\tau}{\tau} I_{d\times d} \end{pmatrix} \in \mathrm{Sp}\,(2d, \mathbb{R})\,. \tag{14.4.6}$$

If $\tau \in [0, 1]$*, the estimate in* (14.4.5) *weakens as follows:*

(ii') $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and there exists a function* $H_\tau \in L^1_v\left(\mathbb{R}^{2d}\right)$ *such that*

$$\left|\left\langle \mathrm{Op}_\tau(\sigma)\,\pi(z)\,\varphi, \pi(w)\,\varphi\right\rangle\right| \leq H_\tau\,(\mathscr{T}_\tau(w, z)) \qquad \forall w, z \in \mathbb{R}^{2d}. \tag{14.4.7}$$

A number of differences arise with respect to its counterpart for Sjöstrand's symbols. First, the almost diagonalization of the (continuous) channel matrix is lost, but this is still a well-organized matrix: in the favourable case $\tau = (0, 1)$, (14.4.5) can be interpreted as a measure of the concentration of the time-frequency representation of $\mathrm{Op}_\tau(\sigma)$ along the graph of the map $\mathscr{U}_\tau$. If we include the endpoints, the estimate loses this meaning too.

Furthermore, notice that the discrete characterization via Gabor frames is lost, the main obstruction being the following: for a given lattice Λ, the inclusion $\mathscr{U}_\tau \Lambda \subseteq \Lambda$ holds if and only if $\tau = 1/2$, i.e., $\mathscr{U}_\tau = \mathscr{U}_{1/2} = -I_{2d\times 2d}$. In this particular framework, the matrix $\mathscr{B}_{1/2}$ then becomes $\mathscr{B}_{1/2} = 2I_{2d\times 2d}$ and the symmetry of Weyl operators is rewarded by an additional characterization:

(iii') $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ and there exists a sequence $h \in \ell^1_v(\Lambda)$ such that

$$|\langle \mathrm{Op_W}(\sigma)\,\pi(\mu)\,\varphi, \pi(\lambda)\,\varphi\rangle| \leq h\,(\lambda + \mu) \qquad \forall \lambda, \mu \in \Lambda.$$

14.5 Consequences of Almost Diagonalization

14.5.1 Boundedness

We are now able to study the boundedness of τ-pseudodifferential operators covering several possible choices for symbol classes and spaces on which they act. If one considers the action of τ-pseudodifferential operators on modulation spaces, a Sjöstrand-type result for symbols in the Sjöstrand's class can be inferred by means of the same arguments applied in the Weyl case.

Theorem 14.9 *Consider* $m \in \mathscr{M}_v\left(\mathbb{R}^{2d}\right)$ *satisfying* (14.2.7)*. For any* $\tau \in [0, 1]$ *and* $\sigma \in M^{\infty,1}_{1\otimes v\circ J^{-1}}$ *the operator* $\mathrm{Op}_\tau(\sigma)$ *is bounded on* $M^{p,q}_m(\mathbb{R}^d)$*, and there exists a* *constant* $C_\tau > 0$ *such that*

$$\|\mathrm{Op}_\tau(\sigma)\|_{M^{p,q}_m} \le C_\tau \|\sigma\|_{M^{\infty,1}_{1\otimes v\circ J^{-1}}}. \tag{14.5.1}$$

In order to address the problem of boundedness of τ-pseudodifferential operators on modulation spaces with symbols in $\mathscr{F}$-Sjöstrand's class, a different strategy is needed. Following [3], the idea is to recast $\mathrm{Op}_\tau(\sigma)$ as the transformation (via the short-time Fourier transform and its adjoint) of an integral operator with the channel matrix as distributional kernel. Therefore, the almost diagonalization property allows to obtain the desired estimates and claim the following result.

Theorem 14.10 *Fix* $m \in \mathscr{M}_v$ *satisfying* (14.2.7). *For* $\tau \in (0,1)$ *consider a symbol* $\sigma \in W(\mathscr{F}L^\infty, L^1_{v\circ\mathscr{B}_\tau})\left(\mathbb{R}^{2d}\right)$, *with the matrix* $\mathscr{B}_\tau$ *defined in* (14.4.6). *Then the operator* $\mathrm{Op}_\tau(\sigma)$ *is bounded from* $M^{p,q}_m\left(\mathbb{R}^d\right)$ *to* $M^{p,q}_{m\circ\mathscr{U}^{-1}_{1-\tau}}\left(\mathbb{R}^d\right)$, $1 \le p, q \le \infty$.

We now turn to consider the boundedness of τ-pseudodifferential operators on Wiener amalgam spaces. Looking for a big picture and given that modulation and Wiener amalgam spaces are intertwined by the Fourier transform, it is natural to wonder if continuity properties of an operator acting on modulation spaces may still hold true when it acts on the corresponding amalgam spaces. In the case of τ-pseudodifferential operators, the answer is yes but heavily relies on the particular way Fourier transform and τ-pseudodifferential operators commute. This phenomenon is a special case of the symplectic covariance property of Shubin calculus, which we briefly recall—see [13] for a comprehensive discussion on the issue.

Lemma 14.1 *For any* $\sigma \in \mathscr{S}'\left(\mathbb{R}^{2d}\right)$ *and* $\tau \in [0,1]$,

$$\mathscr{F}\mathrm{Op}_\tau(\sigma)\mathscr{F}^{-1} = \mathrm{Op}_{1-\tau}\left(\sigma\circ J^{-1}\right).$$

This property, along with other preliminary results, allows to quickly prove the desired claims for symbols in both Sjöstrand's class and the corresponding amalgam space.

Theorem 14.11 *Consider* $m = m_1 \otimes m_2 \in \mathscr{M}_v\left(\mathbb{R}^{2d}\right)$ *satisfying* (14.2.7). *For any* $\tau \in [0,1]$ *and* $\sigma \in M^{\infty,1}_{1\otimes v}\left(\mathbb{R}^{2d}\right)$, *the operator* $\mathrm{Op}_\tau(\sigma)$ *is bounded on* $W\left(\mathscr{F}L^p_{m_1}, L^q_{m_2}\right)\left(\mathbb{R}^d\right)$ *with*

$$\|\mathrm{Op}_\tau(\sigma)\|_{W(\mathscr{F}L^p_{m_1}, L^q_{m_2})} \le C_\tau \|\sigma\|_{M^{\infty,1}_{1\otimes v}},$$

for a suitable $C_\tau > 0$.

Theorem 14.12 *Consider* $m = m_1 \otimes m_2 \in \mathscr{M}_v\left(\mathbb{R}^{2d}\right)$ *satisfying* (14.2.7). *For any* $\tau \in (0,1)$ *and* $\sigma \in W\left(\mathscr{F}L^\infty, L^1_{v\circ\mathscr{B}_\tau\circ J^{-1}}\right)\left(\mathbb{R}^{2d}\right)$, *the operator* $\mathrm{Op}_\tau\sigma$ *is bounded from* $W\left(\mathscr{F}L^p_{m_1}, L^q_{m_2}\right)\left(\mathbb{R}^d\right)$ *to* $W\left(\mathscr{F}L^p_{m_1\circ(\mathscr{U}^{-1}_{1-\tau})_1}, L^q_{m_2\circ(\mathscr{U}^{-1}_{1-\tau})_2}\right)\left(\mathbb{R}^d\right)$, $1 \le p, q \le \infty$, *where*

$$\left(\mathscr{U}_{1-\tau}^{-1}\right)_1(x) = -\frac{\tau}{1-\tau}x, \qquad \left(\mathscr{U}_{1-\tau}^{-1}\right)_2(x) = -\frac{1-\tau}{\tau}x, \qquad x \in \mathbb{R}^d.$$

We finally remark that even if the results with symbols in $\mathscr{F}$-Sjöstrand's class do not hold for the endpoint cases $\tau = 0$ and $\tau = 1$, it is still possible to use the weak characterization (14.4.7) to construct ad hoc examples of bounded operators.

Proposition 14.3 *Assume* $\sigma \in W(\mathscr{F}L^\infty, L^1)(\mathbb{R}^{2d})$.

1. *The Kohn-Nirenberg operator* $\mathrm{Op}_{\mathrm{KN}}(\sigma)$ $(\tau = 0)$ *is bounded on* $M^{1,\infty}(\mathbb{R}^d)$.
2. *The anti-Kohn-Nirenberg* $\mathrm{Op}_1(\sigma)$ $(\tau = 1)$ *is bounded on* $W(\mathscr{F}L^1, L^\infty)(\mathbb{R}^d)$.

14.5.2 Algebra and Wiener Properties

To conclude, we give a brief summary on the extension of the other properties studied by Sjöstrand, namely algebra and Wiener property, to τ-pseudodifferential operators. Wiener algebras of pseudodifferential operators have been already investigated by Cordero, Gröchenig, Nicola and Rodino in several occasions, see for instance [2, 3]. Let us recall the definition and the relevant properties of generalized metaplectic operators, introduced by the aforementioned authors.

Definition 14.1 Given $\mathscr{A} \in \mathrm{Sp}\,(2d, \mathbb{R})$, $g \in \mathscr{S}(\mathbb{R}^d)$, and $s \geq 0$, a linear operator $T : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$ belongs to the class $FIO(\mathscr{A}, v_s)$ of generalized metaplectic operators if

$$\exists H \in L^1_{v_s}(\mathbb{R}^{2d}) \text{ such that } |\langle T\pi(z)g, \pi(w)g\rangle| \leq H(w - \mathscr{A}z), \qquad \forall w, z \in \mathbb{R}^{2d}.$$

Theorem 14.13 *Fix* $\mathscr{A}_i \in \mathrm{Sp}\,(2d, \mathbb{R})$, $s_i \geq 0$, $m_i \in \mathscr{M}_{v_{s_i}}$, *and* $T_i \in FIO\left(\mathscr{A}_i, v_{s_i}\right)$, $i = 0, 1, 2$.

1. T_0 *is bounded from* $M^p_{m_0}\left(\mathbb{R}^d\right)$ *to* $M^p_{m_0 \circ \mathscr{A}_i^{-1}}\left(\mathbb{R}^d\right)$ *for any* $1 \leq p \leq \infty$.
2. $T_1 T_2 \in FIO\,(\mathscr{A}_1\mathscr{A}_2, v_s)$, *where* $s = \min\{s_1, s_2\}$.
3. *If* T_0 *is invertible in* $L^2\left(\mathbb{R}^d\right)$, *then* $T_0^{-1} \in FIO\left(\mathscr{A}_0^{-1}, v_{s_0}\right)$.

In short, the class

$$FIO(\mathrm{Sp}(2d, \mathbb{R}), v_s) = \bigcup_{\mathscr{A} \in \mathrm{Sp}(2d, \mathbb{R})} FIO(\mathscr{A}, v_s)$$

is a Wiener subalgebra of $\mathscr{B}(L^2(\mathbb{R}^d))$. In view of the defining property of operators in $FIO(\mathscr{A}, v_s)$, we immediately recognize that for any $\tau \in (0, 1)$, if $\sigma \in W(\mathscr{F}L^\infty, L^1_{v_s})$ then $\mathrm{Op}_\tau(\sigma) \in FIO(\mathscr{U}_\tau, v_s)$. Therefore, if we limit to consider admissible weights of polynomial type v_s on $\mathbb{R}^d$, $s \geq 0$, we are able to establish a fruitful connection and to derive a number of properties without any effort. For instance, we have another boundedness result.

Corollary 14.3 *If* $\sigma \in W(\mathscr{F}L^\infty, L^1_{v_s})(\mathbb{R}^{2d})$*,* $s \geq 0$*, then the operator* $\mathrm{Op}_\tau(\sigma)$ *is bounded on every modulation space* $M^p_{v_s}(\mathbb{R}^d)$*, for* $1 \leq p \leq \infty$ *and* $\tau \in (0,1)$.

For what concerns the algebra property, we in fact have a no-go result. By inspecting the composition properties of matrices $\mathscr{U}_\tau$, we notice that there is no $\tau \in (0,1)$ such that $\mathscr{U}_{\tau_1}\mathscr{U}_{\tau_2} = \mathscr{U}_\tau$. This implies that there is no τ-quantization rule such that composition of τ-operators with symbols in $W\left(\mathscr{F}L^\infty, L^1_{v_s}\right)$ has symbol in the same class. We can only state weaker algebraic results, such as the following property of "symmetry" with respect to the Weyl quantization.

Theorem 14.14 *For any* $a, b \in W\left(\mathscr{F}L^\infty, L^1_{v_s}\right)(\mathbb{R}^{2d})$ *and* $\tau \in (0,1)$*, there exists a symbol* $c \in M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$ *such that*

$$\mathrm{Op}_\tau(a)\,\mathrm{Op}_{1-\tau}(b) = \mathrm{Op}_{1/2}(c).$$

Also notice that, given $a \in W\left(\mathscr{F}L^\infty, L^1_{v_s}\right)$, $b \in M^{\infty,1}_{1\otimes v_s}$ and $\tau, \tau_0 \in (0,1)$, we have

$$\mathrm{Op}_{\tau_0}(b)\,\mathrm{Op}_\tau(a) = \mathrm{Op}_\tau(c_1), \qquad \mathrm{Op}_\tau(a)\,\mathrm{Op}_{\tau_0}(b) = \mathrm{Op}_\tau(c_2),$$

for some $c_1, c_2 \in W\left(\mathscr{F}L^\infty, L^1_{v_s}\right)$. This means that, for fixed quantization rules τ, τ_0, the amalgam space $W\left(\mathscr{F}L^\infty, L^1_{v_s}\right)(\mathbb{R}^{2d})$ is a bimodule over the algebra $M^{\infty,1}_{1\otimes v_s}(\mathbb{R}^{2d})$ under the laws

$$M^{\infty,1}_{1\otimes v_s} \times W\left(\mathscr{F}L^\infty, L^1_{v_s}\right) \to W\left(\mathscr{F}L^\infty, L^1_{v_s}\right) \;:\; (b,a) \mapsto c_1,$$

$$W\left(\mathscr{F}L^\infty, L^1_{v_s}\right) \times M^{\infty,1}_{1\otimes v_s} \to W\left(\mathscr{F}L^\infty, L^1_{v_s}\right) \;:\; (a,b) \mapsto c_2,$$

with c_1 and c_2 as before.

Finally, after noticing that $\mathscr{U}_\tau^{-1} = \mathscr{U}_{1-\tau}$ for any $\tau \in (0,1)$, a Wiener-like property comes at the price of passing to the complementary τ-quantization when inverting Op_τ.

Theorem 14.15 *For any* $\tau \in (0,1)$ *and* $a \in W\left(\mathscr{F}L^\infty, L^1_{v_s}\right)(\mathbb{R}^{2d})$ *such that* $\mathrm{Op}_\tau(a)$ *is invertible on* $L^2\left(\mathbb{R}^d\right)$*, we have*

$$\mathrm{Op}_\tau(a)^{-1} = \mathrm{Op}_{1-\tau}(b)$$

for some $b \in W\left(\mathscr{F}L^\infty, L^1_{v_s}\right)(\mathbb{R}^{2d})$.

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