Bismut's Way of the Malliavin Calculus for Non-Markovian Semi-groups: An Introduction

Rémi Léandre

Abstract We give a review of our recent works related to the Malliavin calculus of Bismut type for non-Markovian generators. Part IV is new and relates the Malliavin calculus and the general theory of elliptic pseudo-differential operators.

Keywords Malliavin calculus · Large deviations estimates · Higher-order parabolic equation · Pseudo-differential operators

Mathematics Subject Classification (2000) Primary 60H07; Secondary 35K05, 35K25

1 Introduction

Let M be a compact Riemannian manifold endowed with its natural Riemannian measure dx (x is the generic element of M). In local coordinates, we can think at the linear space $\mathbb{R}^d$ endowed with the metric $g_{i,j}(x)dx^i \otimes dx^j$ where x are the local coordinates and $x \to (g_{.,.}(x))$ is a smooth function from $\mathbb{R}^d$ into the space of symmetric strictly positive matrix. The Riemannian measure associated is

$$dx = det(g_{.,.})^{-1/2} dx^1 .. \otimes dx^d \tag{1.1}$$

We consider a **linear** symmetric positive operator densely defined on $L^2(dx)$ acting on a space which separates the point on M. This means if f and g belong to this space,

$$\int_M g(x)Lh(x)ds = \int_M h(x)Lg(x)dx \tag{1.2}$$

R. Léandre (✉)
Laboratoire de Mathématiques, Université de Bourgogne-Franche-Comté, Besançon, France
e-mail: remi.leandre@univ-fcomte.fr

S. Molahajloo, M. W. Wong (eds.), *Analysis of Pseudo-Differential Operators*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-05168-6_7

$$\int_M h(x)Lh(x)dx \geq 0 \tag{1.3}$$

It has by abstract theory a self-adjoint extension on $L^2(dx)$, which generates a contraction semi-group P_t on $L^2(dx)$ which solves the heat equation for $t > 0$

$$\frac{\partial}{\partial t}P_t h = -LP_t h \tag{1.4}$$

with initial condition

$$P_0 h = h \tag{1.5}$$

It is a natural question to know if there is a heat kernel:

$$P_t h(x) = \int_M p_t(x, y)h(y)dy \tag{1.6}$$

There are several ways to solve this problem:

- The microlocal analysis [12, 18, 19], which uses as basic tool the Fourier transform and some regularity on the coefficients of L. In the case of a partial differential operator on $\mathbb{R}^d$, this means that $L = \sum a_{(\alpha)}(x)\frac{\partial^{(\alpha)}}{\partial x^{(\alpha)}}$ where (α) is a multiindex and $x \to a_{(\alpha)}(x)$ is smooth.
- The harmonic analysis, which uses as basic tools functional inequalities and does not need any regularity on the coefficients of L [3, 13, 51].
- The Malliavin calculus [20, 44, 49], which works for Markov semi-groups: $P_t f \geq 0$ if $f \geq 0$. The Malliavin calculus requires moreover that the semi-group is represented by a stochastic differential equation.

More precisely, the Malliavin calculus needs a probabilistic representation of the semi-group P_t by using the theory of stochastic differential equations where a flat Brownian motion or a Poisson process plays a fundamental role.

Let us recall the main idea of the Malliavin calculus in the case of the flat Brownian motion. Let us consider the Hilbert space $\mathbb{H}$ of finite energy maps starting from 0 from $[0, 1]$ into $\mathbb{R}^m$ $t \to r_t = (r_t^i)$ endowed with the Hilbert norm

$$\|r\|^2 = \sum_{i=1}^{m} \int_0^1 |d/dtr_t^i|^2 dt \tag{1.7}$$

We consider the formal Gaussian measure on $\mathbb{H}$ (written in the heuristic way of Feynman path integral)

$$d\mu(r) = 1/Z \exp| - \|r\|^2/2]dD(r) \tag{1.8}$$

where $dD(r)$ is the formal Lebesgue measure on $\mathbb{H}$. Haar measure satisfying all the axioms of measure theory on a group exists if and only if the group is locally compact. (We refer to [2] and [30] to define Haar measure in infinite dimension in a generalized way). This explains that we need to construct this measure on a bigger space, the space of continuous function $C([0, 1], \mathbb{R}^m)t \to B_t$ issued from 0 from [0, 1] into $\mathbb{R}^m$. There are a lot of Gaussian measures on $C([0, 1], \mathbb{R}^m)$ [48] but the law of the Brownian motion is related to the heat equation on $\mathbb{R}^m$

$$\frac{\partial}{\partial t} P_t f(x) = 1/2 \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} P_t f(x) \tag{1.9}$$

We have, namely,

$$P_t h(x) = E[h(B_t + x)] \tag{1.10}$$

if f is a bounded continuous function on $\mathbb{R}^m$. In such a case we have a semi-group operating on continuous function on $\mathbb{R}^m$.

We consider m smooth vector fields on $\mathbb{R}^d$ with bounded derivatives at each order. Vector fields here are considered as first order partial differential operators. We consider the operator

$$L = 1/2 \sum_{i=1}^{m} X_i^2 \tag{1.11}$$

We introduce the Stratonovich differential equation [20, 49] starting from x (vector fields here are considered as vectors which depend smoothly on x).:

$$dx_t(x) = \sum_{i=1}^{m} X_i(x_t(x)) dB_t^i \tag{1.12}$$

This is (and not the Itô equation) the correct equation associated to

$$dx_t(r)(x) = \sum_{i=1}^{m} X_i(x_t(h)(x)) dr_t^i \tag{1.13}$$

for $r \in \mathbb{H}$ endowed with the formal Gaussian measure $d\mu(r)$.

By Itô Calculus [20, 49], we can show that the semi-group P_t generated by $L = 1/2 \sum_{i=1}^{m} X_i^2$ is related to the diffusion $x_t(x)$ by the formula

$$P_t(h)(x) = E[h(x_t(x))] \tag{1.14}$$

if h is a continuous function on $\mathbb{R}^d$ (in such a case, the semi-group acts on continuous bounded functions on $\mathbb{R}^d$).

Malliavin idea is the following [44]: he differentiates in a generalized sense the Itô map $B_. \to x_t(x)$. If this Itô map is a submersion in a generalized sense (the inverse of the Malliavin matrix belongs to all the L^p), the law of $x_t(x)$ has a smooth density and therefore the semi-group has a heat kernel. Malliavin for that uses a heavy apparatus of differential operations on the Wiener space. Let us recall that there are several pioneering works of the Malliavin calculus [1, 6, 16] motivated by mathematical physics, but only Malliavin calculus is adapted to the study of stochastic differential equations and fits very well to the study of all measures of stochastic analysis.

Bismut [7] don't use this heavy apparatus of differential operations on the Wiener space, by using a suitable Girsanov transformation and a system of convenient stochastic differential equations in cascade associated to the original stochastic differential equation. This allows Bismut's way to get in a simpler way the Malliavin integration by parts for diffusions: if (α) is a multiindex, if $t > 0$,

$$E[h^{(\alpha)} x_t(x))] = E[h(x_t(x)) Q_t^{(\alpha)}] \tag{1.15}$$

where $Q_t^{(\alpha)}$ is a polynomial in the extra components of the system of stochastic differential equations in cascade and in the inverse of the Malliavin matrix.

The fact that only stochastic differential equations in cascade (therefore a system of semi-groups in cascade) appear in Bismut's approach of the Malliavin calculus allows us to interpret Bismut's way of the Malliavin calculus in the theory of semi-group by expulsating the probabilistic language in [31]. We refer to [32, 33] for reviews with some applications.

Léandre [31] uses an elementary integration by parts, which has to be optimized. The main remark is that we can adapt this elementary integration by parts for non-Markovian semi-groups. It is possible to adapt Bismut's way of the Malliavin calculus for non-Markovian semi-groups.

It is divided into two steps:

- An algebra on the semi-group. Only existences on the semi-group are required.
- Estimates on the enlarged semi-group, which are necessary because polynomial function appears in the Malliavin integration by parts which are not bounded, but are performed in the non-Markovian case by the Davies gauge transform (in the Markovian case, they were done by an adaptation in semi-group on the classic Burkholder-Davies-Gundy inequalities of stochastic analysis).

Moreover, Bismut in his seminal work [9] has done an intrinsic integration by part formula for the Brownian motion on a manifold, which overcame the problem that in the standard Malliavin calculus there are a lot of stochastic differential equations which represent the **same** semi-group. In Part IV we perform an intrinsic Malliavin calculus associated to a wide class of pseudo-differential elliptic operator, by performing a variation of the original pseudo-differential operator by a fractional power of it **intrinsically** associated to the original operator. We exhibit the relation between the Malliavin calculus of Bismut type and the general theory of elliptic pseudo-differential operators.

Bismut in his seminal work [9] pointed out the relation between the Malliavin calculus and the large deviation theory for the study of short time asymptotics of the heat-kernel associated to diffusion semi-groups. We refer to the reviews [26, 29, 53], the book [5], and the seminal work [47] for probabilistic methods in short time asymptotics of semi-groups.

Let us recall quickly the main goal of large deviation theory, here of Wentzel-Freidlin type [4, 52] and [54]. We introduce a small parameter and consider the stochastic differential equation with a small parameter starting from x:

$$dx_t^\epsilon(x) = \epsilon \sum_{i=1}^{m} X_i(x_t^\epsilon)(x) dB_t^i \tag{1.16}$$

Wentzel-Freidlin theory allows to get estimates of the type, when $\epsilon \to 0$

$$\lim 2\epsilon^2 Log[P[x_.^\epsilon(x) \in 0] = - \inf_{x_.(h)(x)\in O} \|r\|^2 \tag{1.17}$$

if O is an open subset of $C([0, 1], \mathbb{R}^d)$ equipped with the uniform norm. We don't give details of the lot of technicalities in this estimate.

It is possible to adapt [35, 37–40] Wentzel-Freidlin estimates to the case of non-Markovian semi-groups with the normalization of W.K.B. analysis of Maslov school [45] (see [17, 27] for seminal works on W.K.B. analysis). The main remark is that we can get only upper-bounds, because the semi-group does not preserve the positivity in this case. The second remark is that these estimates are valid only for the semi-group, because in this case path space functional integrals are not defined (see [36] for a review and the work [11, 25, 46]). The normalizations are standard in semi-classical analysis but the type of estimates is different. They work for the heat equation and not for the Schrodinger equation.

This allows to fulfill in this non-Markovian context the beautiful request of Bismut's book [5] and to do the marriage between the Malliavin calculus and Wentzel-Freidlin estimates. The main difference is that we have to consider the absolute value of the heat-kernel because in such a case the semi-group does not preserve the positivity such that we get only upper-bound in the studied Varadhan type estimates (Wentzel-Freidlin estimates are still valid for the heat-kernel).

This work is a review paper of several of our works. The main novelty is part IV, which is new.

2 The Case of a Formal Stochastic Differential Equation

Let us consider an elliptic differential operator of order l on a compact manifold M of dimension d. If we perturb it by a strictly lower order operator L_p, it results by the theory of pseudo-differential operator (which is given by the role of the principal symbol of an elliptic operator) that the qualitative behavior (hypoellipticity..) is the same than the qualitative behavior of $L + L_p$. See [12, 18, 19] for various textbooks in analysis about this problem.

Recently, we have introduced an elliptic operator of order $2k$ $L_0 = \sum f_i^{2k}$ where f_i is an orthonormal basis of the Lie algebra of a compact Lie group G of dimension m with generic element g. f_i are considered as right invariant vector fields. We have established the Malliavin calculus of Bismut type for L_0. We consider a polynomial Q of degree strictly smaller than $2k$ in the vector fields f_i with constant components. We consider the total operator

$$L = L_0 + Q \tag{2.1}$$

The goal of this part by using a small interpretation of [41] and [42] is to adapt in this present situation the strategy of [41] for diffusions. (Léandre [41, 42] used the machinery of the Malliavin calculus [7] translated in semi-group theory for diffusions in [31].) Malliavin matrix plays here a fundamental role in the optimization of the integration by parts in order to arrive to full Malliavin integration by parts. All formulas are **formally the same** if we add or do not add the perturbation of the main operator.

We consider the elliptic operator on $G \times \mathbb{R}$

$$Q + \sum_i f_i^{2k} + \sum r_{i,t} f_i \frac{\partial}{\partial u} + \frac{\partial^{2k}}{\partial u^{2k}} = \tilde{L}_t^r \tag{2.2}$$

It generates by elliptic theory a semi-group on $C_b(G \times \mathbb{R})$, the space of bounded continuous function on $G \times \mathbb{R}$ endowed with the uniform norm.

Theorem 2.1 (Elementary Integration by Parts Formula) *We have if h is smooth with compact support*

$$\int_0^t P_{t-s} \sum h_{s,i} f_i P_s[h] ds = \tilde{P}_t^h[uh](.,0) \tag{2.3}$$

Proof It is the same proof than the proof of Theorem 3 of [42]. □

Let $V = G \times M_d$. M_d is the space of symmetric matrices on $LieG$. $(x, v) \in V$. v is called the Malliavin matrix. We consider

$$\hat{X}_0 = (0, \sum < g^{-1} f_i, . >^2) \tag{2.4}$$

We consider the Malliavin generator (we skip the problems of signs)

$$\hat{L} = \sum f_i^{2k} - \hat{X}_0 \tag{2.5}$$

Theorem 2.2 *$\hat{L}$ spans a semi-group. $\hat{P}_t$ called the Malliavin semi-group on $C_b(M)$.*

Proof It is the same proof of theorem 4 of [42] since Q is a polynomial with constant components in the f_i and L generates a $C_b(G)$ semi-group. The proof leads to some difficulties because the Malliavin operator is not the perturbation of an elliptic operator and uses the Volterra expansion. □

The Malliavin semi-group will allow us to get suitable integration by parts formulas 2. We have the main theorem of this paper:

Theorem 2.3 (Malliavin) *If the Malliavin condition holds*

$$|\hat{P}_t][v^{-p}](g,0) < \infty \tag{2.6}$$

for all integer positive integer p, P_t has a heat-kernel.

Proof It is the same proof as in the beginning of the proof of theorem 6 of [42]. Under Malliavin assumption, we can optimize the elementary integration by part of Theorem 2, in order to get, according to the framework of the Malliavin calculus, the inequality for any smooth function h on G

$$|P_t[< dh, f_i >]| \leq C\|f\|_\infty \tag{2.7}$$

□

Remark Let us explain quickly the philosophy of this theorem, when there is no perturbation term. We consider a set of path in $\mathbb{R}^m$ denoted r_t^i which represent the semi-group associated to $\sum_i \frac{\partial^{2k}}{\partial u_i^{2k}}$. We don't enter into the problem of signs. We consider the formal stochastic differential equation

$$dx_t(r)(e) = \sum_i f_i dr_t^i \tag{2.8}$$

issued from e. Formally, this represents the semi-group P_t without the perturbation term

$$P_t[h](e) = \text{“}E\text{”}[f(x_t(e)] \tag{2.9}$$

Malliavin assumption expresses in some sense that the "Itô" map $r_. \to x_t(e)$ is a submersion.

By this inequality, we deduce according to the framework of the Malliavin calculus that

$$P_t[h](e) = \int_G h(g)p_t(e,g)dg \tag{2.10}$$

for a nonstrictly positive heat-kernel p_t (dg) denotes the normalized Haar measure on G), if the Malliavin assumption is satisfied.

Theorem 2.4 *Under the previous elliptic assumptions,*

$$|\hat{P}_t|[|v^{-p}|]](g_0, 0) < \infty \tag{2.11}$$

if $t > 0$

Proof It is the same proof than the proof of theorem 8 of [42]. It is based upon the initial strategy to invert the Malliavin matrix in stochastic analysis by slicing the time interval in small time intervals. Only the main part of the generator plays the main role in this strategy because we are in an elliptic case. □

We can iterate the integration by parts formulas, by introducing a system of semi-groups in cascade. We deduce the theorem:

Theorem 2.5 *If $t > 0$, the semi-group P_t has a smooth heat kernel*

$$P_t([h](g) = \int_G p_t(g, g')h(g')dg' \tag{2.12}$$

We remark that the heat kernel can change of sign. This theorem is classical in analysis [51] but it enters in our general strategy to implement stochastic tools in the general theory of linear semi-groups.

In order to simplify the computation, we have used the symmetry of the group. In the next part, we will use fully the symmetry of the group to simplify the computations.

3 The Full Use of the Symmetry of the Group

Let us recall what is a pseudo-differential operator on $\mathbb{R}^d$ [12, 17, 18]. Let be a smooth function $a(x, \xi)$ from $\mathbb{R}^d \times \mathbb{R}^d$ with values in $\mathbb{C}$. We suppose that

$$\sup_{x \in \mathbb{R}^d} |D_x^r D_\xi^l a(x, \xi)| \leq C|\xi|^{m-l} + C \tag{3.1}$$

We suppose that

$$\inf_{x \in \mathbb{R}^d} |a(x, \xi)| \geq C|\xi|^{m'} \tag{3.2}$$

for $|\xi| > C$ for a suitable $m' > 0$. Let $\hat{h}$ be the Fourier transform of the continuous function h. We consider the operator L defines on smooth function h by :

$$\hat{L}h(x) = \int_{\mathbb{R}^d} a(x, \xi)\hat{h}(\xi)d\xi \tag{3.3}$$

L is said to be a pesudodifferential operator elliptic of order larger than m' with symbol a. This property is invariant if we do a diffeomorphism on $\mathbb{R}^d$ with bounded derivatives at each order. This remark allows to define by using charts a pseudo-differential operator elliptic of order larger than m' on a compact manifold M.

Let f^i be a basis of T_eG. We can consider rightinvariant vector fields. This means that if we consider the action $R_{g_0}\, h \to (g \to h(gg_0))$ on smooth function h on G, we have

$$R_{g_0}(f^i h) = f^i(R_{g_0}h). \tag{3.4}$$

We consider a rightinvariant elliptic pseudo-differential positive operator L of order larger than $2k$ on G . It generates by elliptic theory a semi-group P_t on $L^2(dg)$ and even on $C_b(G)$ the space of continuous functions on G endowed with the uniform norm.

Theorem 3.1 *If $t > 0$,*

$$P_t h(g_0) = \int_G p_t(g_0, g)h(g)dg \tag{3.5}$$

where $g \to p_t(g_0, g)$ is smooth if h is continuous.

This theorem is classical in analysis , but it enters in our general program to implement stochastic analysis tool in the theory of non-Markovian semi-group. See the review [36] for that. See [41, 42] for another presentation where the Malliavin matrix plays a key role. Here we don't use the Malliavin matrix. See [43] for the case of rightinvariant differential operators. The proof is divided into two steps.

3.1 Algebraic Scheme of the Proof: Malliavin Integration by Parts

We consider the family of operators on $C^\infty(G \times \mathbb{R}^n)$:

$$\tilde{L}_t^n = L + \sum_{i=1}^n f^{j_i} \frac{\partial}{\partial u_i} \alpha_t^i + \sum_{i=1}^n \frac{\partial^{2k}}{\partial u_i^{2k}} \tag{3.6}$$

α_t^i are smooth function from $\mathbb{R}^+$ into $\mathbb{R}$. By elliptic theory, $\tilde{L}_t^n$ generates a semi-group $\tilde{P}_t^n$ on $C_b(G \times \mathbb{R}^n)$. This semi-group is time inhomogeneous.

$$\tilde{P}_t^{n+1}[h(g)h^n(u)v](.,.,0) = \int_0^t \tilde{P}_{t,s}^n[f^{j+1}\alpha_s^{n+1}\tilde{P}_s^n[h(g)h^n(u)](.,.) \tag{3.7}$$

Moreover

$$\tilde{P}_t^{n+1}[uh(.)h^n(.)](.,.,u_{n+1}) = \tilde{P}_t^{n+1}[uh(.)h^n(.)](.,.,0) + \tilde{P}_t^n[h(.)h^n(.)](.,.)u_{n+1} \tag{3.8}$$

h is a function of g, h^n a function of $u_1, \ldots, u_n$. This comes from the fact that $\frac{\partial}{\partial u_{n+1}}$ commute with the considered operator.

Therefore the two sides of (3.8) satisfy the same parabolic equation with second member. We deduce that

$$\tilde{P}_t^{n+1}[u_{n+1}\prod_{j=1}^n u_j h(.)](.,.,0) = \int_0^t ds \tilde{P}_{t,s}^n[f^{j_{n+1}}\alpha_s^{n+1}\tilde{P}_s^n[h\prod_{j=1}^n u_j]](.,.) \tag{3.9}$$

This is an integration by parts formula. We would like to present this formula in a more appropriate way for our object.

We consider the operator

$$\overline{L}^n = L + \sum_{j=1}^n \frac{\partial^{2k}}{\partial u_j^{2k}} \tag{3.10}$$

It generates a semi-group $\overline{P}_t^n$. In the sequel we will skip the problem of sign coming if k is even or not.

We introduce a suitable generator

$$\tilde{R}_t^{n+1} = \overline{L}^n + F_s \tag{3.11}$$

by taking care of the relation $[f^i, f^j] = \sum_k \lambda_k^{i,j} f^k$. It is an operator of the type studied. It generates therefore a time inhomogeneous semi-group $\tilde{Q}_t^n$. Therefore the integration by parts formula (3.9) can be written in a more suitable way

$$\tilde{P}_t^{n+1}[u_{n+1}\prod_{j=1}^n u_j h(.)](.,.,0) = \int_0^t \alpha_s^{n+1} ds \tilde{P}_t^n[f^{j_{n+1}}h\prod_{i=1}^n u_i](.,.)+ \int_0^t \alpha_s^{n+1} ds \tilde{P}_{t,s}^n \tilde{Q}_s^n[h\prod_{i=1}^n u_i](.,.) \tag{3.12}$$

We do the following recursion hypothesis on l:

Hypothesis (l) There exists a positive real r_l such that if (α) is a multiindex of length smaller than l

$$|\tilde{P}_t^n[f^{(\alpha)}h\prod_{i=n}^n u_i](g, v_.)| \le Ct^{-r_l}\|h\|_\infty(1 + \prod_{i=n}^n |v_i|) \tag{3.13}$$

where $\|.\|_\infty$ is the uniform norm of h.

It is true for $l = 1$ by (3.9) and the estimates which follow.

If it is true for l, it is still true for $l + 1$, by using (3.12) for $f^{(\alpha)}h$ and taking $\alpha_s^{n+1} = s^{r_l}$.

By choosing suitable α_t^j, we have according to the framework of the Malliavin calculus for any multiindex (α)

$$|P_t[f^{(\alpha)}h](g_0)| \leq C_{(\alpha)}\|h\|_\infty \tag{3.14}$$

in order to conclude.

3.2 Estimates: The Davies Gauge Transform

We do as in [43] (26). The problem is that in $\tilde{P}_t^n[h\prod_{j=1}^n u_j](.,.)$ the test function u_j are not bounded and that $\tilde{P}_t^n$ acts only on $C_b(G \times \mathbb{R}^n)$. We do as in [3] the Davies gauge transform $\prod_I^n g(u_i)$ where

$$g(u) = (|u|) \tag{3.15}$$

if u is big and g is smooth strictly positive.

This gauge transform acts on the original operator by the simple formula $(\prod_{i=1}^n g(u_i))^{-1}\tilde{L}_1^n((\prod_{i=1}^n g(u_i).)$. On the semi-group it acts as

$$(\prod_{i=1}^n g(.))^{-1}\tilde{P}_t^n[(\prod_{i=1}^n g(u_i)h(.)h^n(.)](.,.) \tag{3.16}$$

But

$$(g(u_i))^{-1}\frac{\partial}{\partial u_i}(g(u_i).) = \frac{\partial}{\partial u_i} + C(u_i) \tag{3.17}$$

where the potential $C(u_i)$ is smooth with bounded derivatives at each order. Therefore the transformed semi-group acts on $C_b(G \times R^n)$.

Remark We can consider a particular case [43] Let G be a compact connected Lie group, with generic element g endowed with its bi-invariant Riemannian structure and with its normalized Haar measure dg. e is the unit element of G.

Let f^i be a basis of T_eG. We can consider rightinvariant vector fields. This means that if we consider the action $R_{g_0}\ h \to (g \to h(gg_0))$ on smooth function h on G, we have

$$R_{g_0}(f^i h) = f^i(R_{g_0}h). \tag{3.18}$$

Let be $\xi^{(\alpha)} = \xi^{\alpha_1} \dots . \xi^{\alpha_{|\alpha|}}$ and let be $f^{(\alpha)} = f^{\alpha_1} .. f^{\alpha_{|\alpha|}}$. (α) is a multi-index of length $|\alpha|$.

We consider a matrix $a_{\alpha,\beta}$ for multiindices of length k, which is supposedly symmetric strictly positive.

We consider the operator

$$L = \sum_{(\alpha),(\beta)} f^{(\alpha)} a_{(\alpha),(\beta)} f^{(\beta)} \tag{3.19}$$

According to [51], $(-1)^k L$ is a positive symmetric densely elliptic defined operator on $L^2(G)$, which generates by elliptic theory a semi-group acting on $C_b(G)$, the space of continuous function on G. In such a case, we have a heat-kernel associated to the semi-group (See [43]). The case of a rightinvariant differential operator has exactly the same proof than the case of theorem 6, where the details will be presented elsewhere. See [14] for the general case.

4 The Case of an Intrinsic Variation

Let L be a strictly positive self-adjoint operator on a compact manifold M. We suppose that L is a pseudo-differential elliptic operator of order $l \geq 2k$ for an integer $k \geq 1$. It generates a contraction semi-group on $L^2(M)$ and by ellipticity a semi-group on $C_b(M)$. See [8] and [23, 24] in the Markovian case.

Theorem 4.1 *There is a heat-kernel $p_t(x, y)$ associated to P_t. If $t > 0$*

$$P_t(h)(x) = \int_M p_t(x, y)h(y)dy \tag{4.1}$$

where $y \to p_t(x, y)$ is smooth.

The proof is divided into two steps:

4.1 Algebraic Scheme of the Proof: Malliavin Integration by Parts

Let α belong to $]0, 1[$. The fractional power [50] L^α is still a strictly positive pseudo-differential operator elliptic of order αl, which commutes with L. We skip up later the problem if k is even or not. We consider the operator on $C^\infty(M \times \mathbb{R}^n)$

$$\tilde{L}_s^n = L + s^r L^\alpha \frac{\partial}{\partial u_n} + \sum_{i=1}^n \frac{\partial^{2k}}{\partial u_i^{2k}} \tag{4.2}$$

It is an elliptic operator of order $2k$ on $M \times \mathbb{R}^n$. The main part

$$\overline{L}^n = L + \sum_{i=1}^{n} \frac{\partial^{2k}}{\partial u_i^{2k}} \tag{4.3}$$

is positive and is essentially self-adjoint. Therefore the main part generates a semi-group on $C_b(M \times \mathbb{R}^n)$. This remains true for $\tilde{L}^n$ because $\tilde{L}^n$ is a perturbation of $\overline{L}^n$ by a strictly lower operator. We call this semi-group $\tilde{P}_t^n$.

The main remark is that L^α commutes with $\tilde{L}^n$ such that

$$L^\alpha \tilde{P}_t^n = \tilde{P}_t^n L^\alpha \tag{4.4}$$

According to the beginning of the previous part, we get the elementary integration by part

$$\tilde{P}_t^{n+1}[h \prod_{i=1}^{n} u_i u](x, v_i, 0) = \int_0^t P_{t-s}^n [s^r L^\alpha \tilde{P}_s^n [h \prod_{i=1}^{n} u_i]](x, v_i) =$$
$$\tilde{P}_t^n [L^\alpha h \prod_{i=1}^{n} u_i](x, u_i) \int_0^t s^r ds \tag{4.5}$$

Suppose by induction on l that

$$|\tilde{P}_t^n [(L^\alpha)^l h \prod_{i=1}^{n} u_i](x, v_i)| \leq C t^{-r(l)} \|h\|_\infty (1 + \prod_{i=1}^{n} |v_i|) \tag{4.6}$$

By applying the elementary integration by parts (4.5) to $(L^\alpha)^l)f$, and choosing $r = r(l)$, we deduce our result. Therefore we have the inequality

$$|P_t[(L^\alpha)^l h](x)| \leq C t^{-r(l)} \|h\|_\infty \tag{4.7}$$

The result follows from the fact that L^α is an elliptic operator.

4.2 Estimates: The Davies Gauge Transform

We do as in [43] (26). The problem is that in $\tilde{P}_t^n [h \prod_{j=1}^n u_j](.,.)$ the test function u_j are not bounded and that $\tilde{P}_t^n$ acts only on $C_b(G \times \mathbb{R}^n)$. We do as in [35] the Davies gauge transform $\prod_I^n g(u_i)$ where

$$g(u) = (|u|) \tag{4.8}$$

if u is big and g is smooth strictly positive.

This gauge transform acts on the original operator by the simple formula $(\prod_{i=1}^n g(u_i))^{-1}\tilde{L}_1^n((\prod_{i=1}^n g(u_i).)$. On the semi-group it acts as

$$(\prod_{i=1}^n g(.))^{-1}\tilde{P}_t^n[(\prod_{i=1}^n g(u_i)h(.)h^n(.)](.,.) \tag{4.9}$$

But

$$(g(u_i))^{-1}\frac{\partial}{\partial u_i}(g(u_i).) = \frac{\partial}{\partial u_i} + C(u_i) \tag{4.10}$$

where the potential $C(u_i)$ is smooth with bounded derivatives at each order. Therefore the transformed semi-group acts on $C_b(G \times R^n)$. It remains to choose

$$h^n(u_.) = \prod_{j=1}^n \frac{u_j}{g(u_j)} \tag{4.11}$$

in order to conclude. We deduce the bound:

$$|\tilde{P}_t^n|[h\prod_{j=1}^n |u_j|](.; v_.) \leq C(\|h\|_\infty(1 + \prod_{i=n}^n |v_i|) \tag{4.12}$$

where $|\tilde{P}_t^n|$ is the absolute value of the semi-group $\tilde{P}_t^n$.

Remark We could show that $(x, y) \to p_t(x, y)$ is smooth if $t > 0$ by the same argument.

Remark We can replace the hypothesis L strictly positive by the hypothesis L positive by replacing L^α by $(L + CI_d)^\alpha$ where $C > 0$.

5 Wentzel-Freidlin Estimates for the Semi-Group Only

We consider a differential operator of order $2k$ on the compact manifold M which is supposedly elliptic of order $2k$ and strictly positive. We suppose we can write it as

$$L = \sum_{j=0}^{2k}\sum_{i=0}^{r(j)}(X_{i,j})^j \tag{5.1}$$

where $X_{i,j}$ are smooth vector fields on M. The ellipticity assumption states that

$$\sum_{i=0}^{r(2k)} < X_{i,2k}, \xi >^{2k} = H(x, \xi) \geq C|\xi|^{2k} \tag{5.2}$$

To the Hamiltonian H, we introduce the Lagrangian

$$L(x, p) = \sup_{\xi}(< p, \xi > -H(x, \xi)) \tag{5.3}$$

We get the estimate

$$-C + C|p|^{\frac{2k}{2k-1}} \leq L(x, p) \leq C + |p|^{\frac{2k}{2k-1}} \tag{5.4}$$

for some strictly positive constants C.

If ϕ is a continuous piecewise differentiable path on M, we put:

$$S(\phi) = \int_0^1 L(\phi(t), d/dt\phi(t))dt \tag{5.5}$$

and we put

$$l(x, y) = \inf_{\phi(0)=x,\phi(1)=y} S(\phi) \tag{5.6}$$

By Ascoli theorem, $(x, y) \to l(x, y)$ is a continuous function on $M \times M$.

Theorem 5.1 (Wentzel-Freidlin) *If O is an open ball of M, we have when $t \to 0$*

$$\overline{\lim} t^{\frac{1}{2k-1}} \log |P_t|(1_O](x) \leq -\inf_{y\in O} l(x, y) \tag{5.7}$$

Proof We put $\epsilon = t^{\frac{1}{2k-1}}$. According to the normalization of Maslov school [37], we consider the semi-group P_s^ϵ associated to $L_\epsilon = \epsilon^{2k-1}L$. Moreover

$$P_t = P_1^t \tag{5.8}$$

where P_s^t is associated to tL ([10]). The result will arise if we show when $\epsilon \to 0$

$$\overline{\lim} \epsilon \log |P_1^\epsilon|(1_O](x) \leq -\inf_{y\in O} l(x, y) \tag{5.9}$$

The main ingredient is: □

Lemma 5.2 *For all $\delta > 0$, all C, there exists s_δ such that if $s < s_\delta$*

$$|P_s^\epsilon|[1_{B(x,\delta)^c}](x) \leq \exp[-C/\epsilon] \tag{5.10}$$

where $B(x, \delta)$ is the ball of radius δ and center x.

Proof We imbed M in a linear space. We consider the semi-group

$$Q_s^\epsilon(h)(x) = \exp[- < x, \xi > /\epsilon] P_s^\epsilon[\exp[< x', \xi > /\epsilon] h(x')](x) \tag{5.11}$$

Its generator is

$$\overline{L}_\epsilon + H(x, \xi)/\epsilon \tag{5.12}$$

$$\overline{L}_\epsilon = L_\epsilon + R_\epsilon \tag{5.13}$$

In the perturbation term R_ϵ, there are only differential operators of order l, $l \in]0, 2k[$. When a differential operator of degree l appears, there is a power of at least $l - 1$ of ϵ which appears and a power of ξ at most $2k$ which appears.

Let us consider in a small neighborhood of x the diffeomorphism

$$\Psi_\epsilon : y \to x + \frac{y - x}{\epsilon^{\frac{2k-1}{2k}}} \tag{5.14}$$

Outside a big neighborhood of x, Ψ_ϵ is the identity.

We consider the measure μ_ϵ

$$f \to P_1^\epsilon[F(\Psi_\epsilon(x))](x) \tag{5.15}$$

Under the transformation Ψ_ϵ, the vector fields $\epsilon^{\frac{2k-1}{2k}} X_{i,j}$ are transformed in the vector field $X_{i,j}(x + \epsilon^{\frac{2k-1}{2k}}(y - x))$. Therefore we can apply the machinery of the previous part in order to show that the measure μ_ϵ has a bounded density $q_\epsilon(x, .)$ when $\epsilon \to 0$.

Let R be a differential operator of order l. We have

$$\int_M g(x) R P_1^\epsilon[h](x) dx = \int_{M \times M} g(x) h(y) R_x p_1^\epsilon(x, y) dx dy \tag{5.16}$$

By symmetry

$$p_1^\epsilon(x, y) = p_1^\epsilon(y, x) \tag{5.17}$$

Then

$$\int_M g(x) R P_1^\epsilon[h](x) dx = \int_M h(y) P_1^\epsilon[Rg](y) dy \tag{5.18}$$

By the previous remark

$$|P_1^\epsilon[Rh](y)| \leq \frac{C}{\epsilon^{l\frac{2k-1}{2k}}} \|h\|_\infty \tag{5.19}$$

Therefore

$$|\int_M g(x) R P_1^\epsilon[h](x)dx| \leq \frac{C}{\epsilon^{l\frac{2k-1}{2k}}}\|g\|_\infty \|h\|_\infty \tag{5.20}$$

We deduce that

$$|RP_1^\epsilon[h](x)| \leq \frac{C}{\epsilon^{l\frac{2k-1}{2k}}}\|h\|_\infty \tag{5.21}$$

We deduce a bound of $R_\epsilon P_s^\epsilon$

$$|R_\epsilon P_s^\epsilon h(x)| \leq \frac{|\xi|^{2k-1}}{s^{\frac{l}{2k}}}\epsilon^{-1+1/k}\|h\|_\infty \tag{5.22}$$

We apply Volterra expansion to Q_s^ϵ. We get

$$|Q_s^\epsilon h| \leq |P_s^\epsilon h| + \sum_{i=1}^{\infty} |\int_{\Delta_l(s)} I_{s_1,..,s_l} ds_1 \ldots ds_l| \tag{5.23}$$

where $\Delta_l(s)$ is the simplex $0 < s_1 < .. < s_l < s$ and

$$I_{s_1,..,s_l} = P_{s_1}^\epsilon(R_\epsilon + H/\epsilon)\ldots P_{s_l - s_{l-1}}^\epsilon(R_\epsilon + H/\epsilon)P_{s-s_{l-1}}^\epsilon h \tag{5.24}$$

We deduce a bound of $|\int_{\Delta_l(s)} I_{s_1,..,s_l} ds_1 \ldots ds_l|$ by

$$\frac{|\xi|^{2lk}}{\epsilon^l}\int_{\Delta_l(s)}\prod (s_{i+1} - s_i)^{-\frac{2k-1}{2k}} ds_1..ds_l = \frac{|\xi|^{2lk}}{\epsilon^l} I_l(s) \tag{5.25}$$

We suppose by induction that

$$I_l(s) = \alpha_l s^{l(1+\beta_k)} \tag{5.26}$$

where $\beta_k \in]-1, 0[$. It is still true by the recursion formula

$$I_{l+1}(s) = \int_0^s I_l(u)(s-u)^{-\frac{2k-1}{2k}} du \tag{5.27}$$

We deduce the bound

$$\alpha_l \leq \frac{C^l}{l!} \tag{5.28}$$

Therefore

$$|Q_s^\epsilon h(x)| \leq \exp[Cs|\xi|^{2k}/\epsilon]\|h\|_\infty \tag{5.29}$$

It remains to remark that we have the bound

$$|P_s^{\xi}|[1_{B(x,\delta)^c}](x) \leq \exp[-\frac{C\delta|\xi|}{\epsilon} + Cs|\xi|^{2k}/\epsilon] \tag{5.30}$$

and to extremize in $|\xi|$ to conclude. □

End of the Proof of Theorem 5.1 We operate as in Freidlin-Wentzel book [54] and as in [35, 38] and [39]. We slice the time interval [0, , 1] in a finite number of time intervals $[s_i, s_{i+1}]$ where we can apply the previous lemma. We deduce a positive measure on the set of polygonal paths, where we can repeat exactly the considerations of [35].◇

Remark This estimate is a semi-classical estimate with different type of estimates of W.K.B. estimates a la Maslov and with a different method. We consider in W.K.B. estimate a symbol of an operator $a(x, \xi)$ and we consider the generator L_ϵ associated with the normalized symbol (a la Maslov) $1/\epsilon a(x, \epsilon\xi)$. Let us suppose that L_ϵ generates a semi-group P_t^ϵ. The object of WKB method is to get **precise** estimates of the semi-group P_1^ϵ when $\epsilon \to 0$. For that people look at a formal asymptotic expansion (we omit to write the initial conditions) of P_1^ϵ of the type

$$\epsilon^{-r}\exp[-l(y)/\epsilon]\sum \epsilon^i C_i(y) \tag{5.31}$$

The function l satisfy a highly non-linear equation (the Hamilton-Jacobi-Belman equation) and $c_i(y)$ satisfy formally a system of linear partial differential equation in cascade. The cost function in theorem $l(x, y)$ is the solution of the highly non-linear Hamilton-Jacobi-Belman equation, which is difficult to solve. We don't have precise asymptotics, we are interested by logarithmic estimates which are totally different with a method totally different. On the other hand, generally semi-classical asymptotics considers the case of the Schrodinger equation.

On $\mathbb{R}^d$ we can speak without any difficulty of the symbol of an operator. Poisson processes, Lévy processes, and jump processes are more or less generated by pseudo-differential operators whose generator satisfy the maximum principle (See [10, 13, 21, 22, 24, 28]). We will present pseudo-differential operators with a type of compensation of stochastic analysis which do not satisfy the maximum principle. The end of this part is extracted from [35] and [40]. Let us consider the generator on $C_\infty(\mathbb{R}^d)$

$$Lf(x) = (-)^{l+1}\int_{\mathbb{R}^d}(f(x+y) - f(x)) - \sum_{i=1}^{2l} < y^{\otimes i}, h^{\otimes i}(x) >)\frac{h(x,y)}{|y|^{2l+d+\alpha}}dy \tag{5.32}$$

$\alpha \in]-1, 0[$ $h(x, y) = 0$ if $|y| > C$ and $h \geq 0$. The measure $\frac{h(x,y)}{|y|^{2l+d+\alpha}}dy$ is called the Lévy measure.

Theorem 5.2 *If $h(x,0) = 1$, L is an elliptic pseudo-differential generator.*

Definition 5.3 If $h(x, y) = h(y)$, we will say that L is a generalized Lévy generator.

Theorem 5.4 *Suppose that L is of Lévy type and that $h(y) = h(-y)$. L is positive symmetric, and therefore admits by ellipticity a self-adjoint extension on $L^2(\mathbb{R}^d)$, which generates a contraction semi-group on $L^2(\mathbb{R}^d)$ which is still a semi-group on $C_b(\mathbb{R}^d)$.*

Remark The symbol $a(x, \xi)$ of the generator is given by

$$(-)^{l+1}\int_{\mathbb{R}^d}(\exp[\sqrt{-1} < y, \xi > - \sum_{i=1}^{2l}(\frac{(\sqrt{-1} < \xi, y >^i)}{i!}\frac{h(x,y)}{|y|^{2l+d+\alpha}}dy \tag{5.33}$$

The Hamiltonian associated is the symbol in real phase. Let us consider a generator of Lévy type of the previous theorem: it is

$$(-)^{l+1}\int_{\mathbb{R}^d}(\exp[< y, \xi > - \sum_{i=1}^{l}(\frac{< \xi, y >^{2i}}{2i!})\frac{h(x,y)}{|y|^{2l+d+\alpha}}dy \tag{5.34}$$

The Hamiltonian is smooth, convex, equal to 1 in 0. Associated to it, we consider the Lagrangian:

$$L(p) = \sup_{\xi}(< \xi, p > -H(\xi)) \tag{5.35}$$

If $t \to \phi_t$ is a piecewise differentiable continuous curve in $\mathbb{R}^d$, we consider its action $\int_0^1 dt L(\phi_t, d/dt\phi_t) = S(\phi_.)$ We introduce the control function

$$l(x, y) = \inf_{\phi_0 = x; \phi_1 = y} S(\phi) \tag{5.36}$$

Let us recall that $(x, y) \to l(x, y)$ is positive finite continuous.

We consider the generator associated to $1/\epsilon a(\epsilon\xi)$. This corresponds in the classical case of jump process where the compensation is only of one term to the case of a jump process with more and more jumps which are more and more small [54]. We consider the generator L^ϵ associated to $1/\epsilon a(\epsilon\xi)$. It generates a semi-group P_t^ϵ. We get:

Theorem 5.5 *[Wentzel-Freidlin [35, 40]] When $\epsilon \to 0$, we get if O is an open ball of $\mathbb{R}^d$ if $l + 1$ is even:*

$$\overline{\lim}\epsilon \log |P_1^\epsilon|[1_O](x) \le - \inf_{y \in O} l(x, y) \tag{5.37}$$

Remark For this type of operator, Wentzel-Freidlin estimates are not related to short time asymptotics.

6 Application: Some Varadhan Estimates

This part follows closely [43]. Only the mechanism of the integration by part is different from [39]. For large deviation estimates with respect to W.K.B normalization in the manner of Maslov [45] for non-Markovian operators, we refer to [38] for instance.

Let us consider the Hamiltonian function from $T^*(G)$ into $\mathbb{R}^+$

$$H(g,\xi)= \sum_{|\alpha|=k,|\beta|=k} < f^{(\alpha)1},\xi > \cdots < f^{(\alpha)_k},\xi > a_{(\alpha),(\beta)} < f^{(\beta)1},\xi > \cdots < f^{(\beta)_k},\xi > \tag{6.1}$$

$H(g,p)$ is positive convex in p. According to the theory of large deviation, we consider the associated Lagrangian

$$L(g,\xi) = \sup_p(<\xi, p> - H(g,\xi)) \tag{6.2}$$

If $t \to \phi_t$ is a curve in the group, we consider its action $\int_0^1 dt L(\phi_t, d/dt\phi_t) = S(\phi_.)$ We introduce the control function

$$l(g_0, g_1) = \inf_{\phi_0=g_0;\phi_1=g_1} S(\phi) \tag{6.3}$$

Let us recall that $(g_0, g_1) \to l(g_0, g_1)$ is positive finite continuous.

We have shown in the previous part that if we consider a small parameter ϵ and if we consider the generator $\epsilon^{2k-1}L$ and the semi-group P_t^ϵ associated and if g_0 and g_1 are not closed , we get for any small ball centered in g_1 uniformly:

$$\overline{Lim}_{\epsilon\to 0}\epsilon Log|P_1^\epsilon|[1_O](g_0) \leq - \inf_{g_1\in O} l(g_0, g_1) \tag{6.4}$$

where$|P_1^\epsilon|$ is the absolute value of the semi-group (See [38]). See for that the previous part.

But $P_t = P_1^t$ where P_s^t is the semi-group associated to tL (See [15]). We put $\epsilon = t^{1/2k-1}$ such that

$$\overline{Lim}_{t\to 0} t^{1/2k-1} Log|P_t|[1_O](g_0) \leq - \inf_{g_1\in O} l(g_0, g_1) \tag{6.5}$$

We consider a smooth positive function χ equal to 0 outside O and equal to 1 on a small open ball centered in g_1 smaller than 1.

We would like to apply the mechanism of Malliavin integration by parts to the measure

$$h \to P_t[h\chi](g_0) \tag{6.6}$$

such that

$$|P_t[\chi f^{(\alpha)}h](g_0)| \leq Ct^{(-r_{(\alpha)})} \exp[\frac{-l(g_0, g_1) + \delta}{t^{1/2k-1}}]\|h\|_\infty \tag{6.7}$$

for a small δ. Since (6.7) is true, we have:

Theorem 6.1 *When $t \to 0$*

$$\overline{Lim}_{t\to 0} t^{1/2k-1} Log|p_t(g_0, g_1)| \leq -l(g_0, g_1) \tag{6.8}$$

References

1. S. Albeverio, R. Hoegh-Krohn, Dirichlet forms and diffusion processes on rigged Hilbert spaces. Z.W. **40**, 1–57 (1977)
2. A. Asada, Regularized calculus: an application of zeta regularization to infinite dimensional geometry. Int. J. Geom. Methods Mod. Phys. **1**, 107–157 (2004)
3. P. Auscher, P. Tchamitchian, Square root problem for divergence operators and related topics. Asterisque **249** (1998) pp. 1–172
4. R. Azencott, Grandes déviations et applications, in *Ecole de probabilités de Saint-Flour*. Lecture Notes in Mathematics, vol. 778 (Springer, Berlin, 1978), pp. 2–176
5. F. Baudoin, *An Introduction to the Geometry of Stochastic Flows* (Imperial College Press, London, 2000)
6. Y. Berezanskii, The self-adjointes of elliptic operators with infinite number of variables. Ukr. Math. J. **27**, 729–742 (1975)
7. J.M. Bismut, Martingales, the Malliavin Calculus and hypoellipticity under general Hoermander's condition. Z.W. **56**, 469–505 (1981)
8. J.M. Bismut, Calcul des variations stochastiques et processus de sauts. Z. Wahr. Verw. Gebiete **63**, 147–235 (1983)
9. J.M. Bismut, *Large Deviations and the Malliavin Calculus* (Birkhauser, Boston, 1984)
10. N. Bouleau, L. Denis, *Dirichlet Forms Methods for Poisson Measures and Lévy Processes with Emphasis on the Creation-Annihilation Techniques* (Springer, London, 2015)
11. K. Burdzy, Some path properties of iterated Brownian motion, in *Seminar on Stochastic Processes 1992*. Progress in Probability, vol. 33 (Birkhauser, Boston, 1993), pp. 67–87
12. J. Chazarain, A. Piriou, *Introduction a la théorie des équations au dérivées partielles linéaires* (Gauthier-Villars, Paris, 1981)
13. E.B. Davies, *Heat Kernels and Spectral Theory* (Cambridge University Press, Cambridge, 1989)
14. E.B. Davies, Uniformly elliptic operators with measurable coefficients. J. Funct. Anal. **132**, 141–169 (1995)
15. P. Greiner, An asymptotic expansion for the heat equation. Arch. Ration. Mech. Anal. **41**, 163–218 (1971)
16. T. Hida, *Analysis of Brownian Functionals*. Carleton Mathematical Lecture Notes, vol. 13 (Carleton University Press, Ottawa, 1975)
17. L. Hörmander, Fourier integral operators. I. Acta. Math. **127**, 78–183 (1971)
18. L. Hörmander, *The Analysis of Linear Partial Operators III* (Springer, Berlin, 1984)
19. L. Hörmander, *The Analysis of Linear Partial Operators IV* (Springer, Berlin, 1984)
20. N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes* (North-Holland, Amsterdam, 1989)
21. Y. Ishikawa, *Stochastic Calculus of Variations for Jump Processes* (de Gruyter, Berlin, 2012)

22. N. Jacob, *Pseudo-Differential Operators and Markov Processes. I. Fourier Analysis and Semigroups* (Imperial College Press, London, 2001)
23. N. Jacob, *Pseudo-Differential Operators and Markov Processes. II. Generators and Their Potential Theory* (Imperial College Press, London, 2003)
24. N. Jacob, *Pseudo-Differential Operators and Markov Processes. III. Markov Processes and Their Applications* (Imperial College Press, London, 2005)
25. N. Kumano-Go, Phase space Feynman path integrals of higher order parabolic type with general functional as integrand. Bull. Sci. Math. **139**, 495–537 (2015)
26. S. Kusuoka, More recent theory of Malliavin Calculus. Sugaku Expositions **5**, 155–173 (1992)
27. P. Lax, Asymptotic solutions of oscillatory initial value problems. Duke Math. J. **25**, 627–646 (1957)
28. R. Léandre, Extension du théoreme de Hoemander a divers processus de sauts, PHD Thesis, Université de Besançon, 1984
29. R. Léandre, Applications quantitatives et qualitatives du Calcul de Malliavin, in *Séminaire Franco-Japonais*. Lectures Notes in Maths, vol. 1322 (Springer, Berlin, 1988), pp. 109–134. English translation: *Geometry of random motion*. Contemporary Maths, vol. 73. (AMS, 1988), pp. 173–196
30. R. Léandre, Path integrals in noncommutative geometry. *Encyclopedia of Mathematical Physics* (Elsevier, Amsterdam, 2006), pp. 8–12
31. R. Léandre, Malliavin Calculus of Bismut type without probability. Festchrift in honour of K. Sinha. Proc. Indian Acad. Sci. Math. Sci. **11**, 507–518 (2006)
32. R. Léandre, Applications of the Malliavin Calculus without probability. WSEAS Trans. Math. **5**, 1205–1206 (2006)
33. R. Léandre, Malliavin Calculus of Bismut type in semi-group theory. Far East J. Math. Sci. **30**, 1–26 (2008)
34. R. Léandre, Wentzel-Freidlin estimates for an operator of order four, in *2014 International Conference on Computational Science and Computational Intelligence* (IEEE Computer Society, Piscataway, 2014), pp. 360–364. (IEEE X-plore)
35. R. Léandre, Large deviation estimates for a non-Markovian generator of big order, in *4th International Conference on Mathematical Models in Physical Sciences*. Journal of Physics: Conference Series, **633** (2015) paper 012085
36. R. Léandre, Stochastic analysis for a non-Markovian generator: an introduction. Russ. J. Math. Phys. **22**, 39–52 (2015)
37. R. Léandre, *The Itô-Stratonovitch for an operator of order four*. *Festchrift in honour of A. Mukharjea, Ph. Feinsilver and S. Mohammed*. Contemporary Mathematics, vol. 668 (American Mathematical Society, Providence, 2016), pp. 165–169
38. R. Léandre, Large deviation estimates for an operator of order four with a potential, in *XII International Work Differential Geometry and Application*. Revue Roumaine de Math. Pures et Appliquées, **61** (2016), pp. 233–240
39. R. Léandre, Varadhan estimates for an operator of order four on a Lie group, in *Control, Decision and Information Technologies*. IEEE-Xplore.IEEE. (2016)
40. R. Léandre, A class of non-Markovian generator of Lévy type, in *Pseudo-Differential Operators: Groups, Geometry and Application* (Birkhauser, Basel, 2017), pp. 149–159
41. R. Léandre, Perturbation of the Malliavin Calculus of Bismut type of large order, in *Physical and Mathematical, Aspects of Symmetries* (Springer, Berlin, 2017), pp. 221–225
42. R. Léandre, Malliavin Calculus of Bismut type for an operator of order four on a Lie group. J. Pseudo-Differ. Oper. Appl. **9** (2018), pp. 419–430
43. R. Léandre, Bismut's way of the Malliavin Calculus of large order generators on a Lie group, in *6th International Eurasian Conference on Mathematical Sciences and Applications*. A.I.P. Proceedings, vol. 1926 (American Institute of Physics, Melville, 2018) 020026
44. P. Malliavin, Stochastic calculus of variations and hypoelliptic operators, in *Proceedings of the Symposium on Stochastic Differential Equations* (Kinokuyina, 1978), pp. 195–263
45. V.P. Maslov, M.V. Fedoriuk, *Semiclassical Approximation in Quantum Mechanics* (Reidel, Dordrecht, 1981)

46. S. Mazzucchi, Infinite dimensional oscillatory integrals with polynomial phase and application to higher-order heat equation. Potential Anal. **49** (2018), pp. 209–233
47. S. Molchanov, Diffusion processes and Riemannian geometry. Russ. Math. Surv. **30** 1–63 (1975)
48. J. Neveu, Processus aléatoires Gaussiens, in *Séminaire Maths Supérieures*, vol. 34 (The University Press of Montreal, Montreal, 1968)
49. D. Nualart, *The Malliavin Calculus and Related Topics* (Springer, Berlin, 1995)
50. R.T. Seeley, Complex powers of an elliptic operator, in *Singular Integrals*. Proceedings of Symposia in Pure Mathematics (American Mathematical Society, Providence, 1970), pp. 288–307
51. A. Ter Elst, D.W. Robinson, Subcoercive and subelliptic operators on Lie groups: variable coefficients. Publ. Res. Inst. Math. Sci. **29**, 745–801 (1993)
52. S.R.S. Varadhan, *Large Deviations and Applications* (Society for Industrial and Applied Mathematics, Philadelphia, 1984)
53. S. Watanabe, Stochastic analysis and its applications. Sugaku Expositions **5**, 51–69 (1992)
54. A.D. Wentzel, M.J. Freidlin, *Random Perturbations of Dynamical Systems* (Springer, Berlin, 1984)