Tree t-Spanners of a Graph: Minimizing Maximum Distances Efficiently

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Abstract. A tree t-spanner of a graph G is a spanning subtree T in which the distance between any two adjacent vertices of G is at most t. The smallest t for which G has a tree t-spanner is the tree stretch index. The problem of determining the tree stretch index has been studied by: establishing lower and upper bounds, based, for instance, on the girth value and on the minimum diameter spanning tree problem, respectively; and presenting some classes for which t is a tight value. Moreover, in 1995, the computational complexities of determining whether $t = 2$ or $t \geq 4$ were settled to be polynomially time solvable and NP-complete, respectively, while deciding if $t = 3$ still remains an open problem.

With respect to the computational complexity aspect of this problem, we present an inconsistence on the sufficient condition of tree 2-spanner admissible graphs. Moreover, while dealing with operations in graphs, we provide optimum tree t-spanners for 2 cycle-power graphs and for prism graphs, which are obtained from 2 cycle-power graphs after removing a perfect matching. Specifically, the stretch indexes for both classes are far from their girth's natural lower bounds, and surprisingly, the parameter does not change after such a matching removal. We also present efficient strategies to obtain optimum tree t-spanners considering threshold graphs, split graphs, and generalized octahedral graphs. With this last result in addition to vertices addition operations and the tree decomposition of a cograph, we are able to present the stretch index for cographs.

Keywords: Tree t-spanner · Stretch index · Lower bounds
Generalized octahedral graph · Cycle-power graph · Prism graph
Threshold graph · Split graph · Cograph

1 Introduction

The problems of obtaining subgraphs with special restrictions have been considered in several papers, with many motivations and applications in different fields,

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as message routing, computational geometry, and phylogenetic analysis [1–3]. In addition to the inherent difficulty of these problems, another challenge arises when we look for a spanning tree with constraints on the vertices' distances.

A *tree t-spanner* of a graph G is defined as a spanning subtree T of G in which the distance between every pair of vertices is at most t times their distance in G or, equivalently, as the subtree T in which the distance between two adjacent vertices of G is at most t (*cf.* [4]). If a graph has a tree t-spanner, then it is called a *tree t-spanner admissible graph*. The parameter t of a tree t-spanner is called the *tree stretch factor*, and the smallest t for which a graph G is tree t-spanner admissible is called the *tree stretch index* of G, denoted by $\sigma_T(G)$.

Note that the problem of determining the *tree stretch index* of G, called the *minimum stretch spanning tree problem* (MSST), is one of the interesting min-max problems, which are studied not only in graphs, but in several other combinatorial problems, in such a way that bounds, algorithms and computational complexity studies are widely developed [5,6].

An intriguing aspect comes when we want to determine if a graph is tree 3-spanner admissible. In terms of the computational complexity, this task is still the greatest breakthrough we aim to solve, since deciding if $\sigma_T(G) \geq 4$ is NP-complete, whereas for $\sigma_T(G) = 2$ it is polynomially time solvable [4]. There are also some classes for which this problem was settled to be NP-complete, as planar and chordal graphs [7,8], or classes for which the stretch index was proved to be bounded by specific values, as split and cographs (*cf.* [9]). Hence, it is also a great challenge to determine the stretch index even restricted to graph classes. Still in the computational complexity approach, in this work, we can observe that Cai and Corneil's characterization for tree 2-spanner admissible graphs [4], which deals with triconnected components of a graph, is not consistent with the usual definition of k-connected graphs, considering, for instance, complete graphs. In this sense, we present infinite families of split graphs that do not admit tree 2-spanners, but satisfy their sufficient condition, considering either, the convention for K_n graphs connectivity (see [10,11]) or that the connectivity concept does not apply to complete graphs (see [12]).

Studying bounds is an ordinary kind of approach for MSST. A natural lower bound arises when we consider the girth $g(G)$ of a graph G, i.e. the length of its minimum cycle. We have that, if G is a tree t-spanner admissible, then $t \geq g(G)-1$. Regarding this bound, it is possible to observe some optimum tree t-spanners for some families or classes, for instance complete graphs, cycle graphs, wheel graphs, or complete k-partite graphs, for $k \geq 2$. However, establishing lower bounds is challenging in general, and so it remains when we deal with the MSST problem restricted to graph classes, since the results on it often present tree t-admissible graphs (*cf.* [4]). Another kind of approach considers variant problems, for instance the *minimum diameter spanning tree*. In this problem, the solution tree minimizes the maximum distances between all pairs of vertices, which is polynomially time solvable, and the solution parameter is an upper bound for the MSST problem [13].

We focus on obtaining the stretch index for some graph classes and, although there are already known upper bounds for some of them, in this work we present minimum $t = \sigma_T(G)$ values considering these classes. We also present the stretch index for 2 cycle-power graphs, which is far from the girth's natural lower bound. Furthermore, we are also interested in the stretch index after vertices/edges operations, particularly for generalized octahedral graphs (complete graphs after removing a perfect matching), generalized octahedral graphs after non-universal vertices additions, and for prism graphs (2 cycle-power graphs after removing a perfect matching). Surprisingly, in this last case, the matching removal does not modify the stretch index of 2 cycle-power graphs.

This paper is organized as follows: In Sect. 2, we present basic definitions, considerations about Cai and Corneil's characterization for tree 2-spanner admissible graphs, and previous results. In Sect. 3, we present optimum tree t-spanner for some graph classes, such as 2 cycle-power graphs, prism graphs, generalized octahedral graphs, threshold graphs and their minimal superclasses, split graphs and cographs; In Sect. 4, we present final remarks by considering further investigation on other classes and their properties.

2 Preliminaries

Given a graph $G = (V, E)$, $d_G(x, y)$ denotes the distance between x and y in G and $d_G(v)$, the degree of v in G. We say that a *non-edge* of a spanning tree T is an edge of $G \setminus T$. A p-path is a path of length p.

A *tree t-spanner* of a graph G is a spanning subtree T of G in which the distance between every pair of vertices is at most t times their distance in G. Cai and Corneil proved that this problem is equivalent to the one that considers only adjacent vertices of G [4]. Moreover, they showed what follows.

Theorem 1. *A spanning tree T is a* tree t-spanner *of G if and only if for every edge $xy \in E(G) \setminus E(T)$ we have $d_T(x, y) \leq t$.*

The *minimum stretch spanning tree* of G (MSST) is an optimization problem of finding a tree t-spanner of G with minimum t. In this case, we say that $\sigma_T(G) = t$, and $\sigma_T(G)$ is called the *stretch index* of G. Upper bounds for $\sigma_T(G)$ can be obtained considering, for instance, the minimum diameter spanning tree, whose smallest parameter is $D_T(G)$, and some other problems [4,14]. In opposite, a natural lower bound can be obtained accordingly to the girth of G, i.e., the length of its minimum induced cycle. Therefore, Theorem 2 states the range of the stretch index of a given graph G.

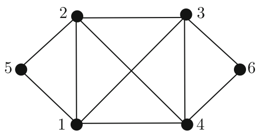
Theorem 2 [4,13]**.** *Given $g(G)$ the girth of G, we have that $g(G) - 1 \leq \sigma_T(G) \leq D_T(G)$.*

Consider, for instance, a tree $(n-1)$-spanner of the cycle graph C_n, i.e. a path P_n, and a tree 2-spanner of the complete graph K_n, i.e. a star S_{n-1}. Both spanning trees are optimum, and their associated stretch factors are tight with respect to Theorem 2.

On Cai and Corneil Tree 2-Spanner Characterization. Cai and Corneil [4] proposed a characterization to decide if $\sigma_T(G) = 2$, formulated as follows: a nonseparable graph G has a 2-spanner if and only if G contains a spanning tree T such that for each triconnected component H of G, $T \cap H$ is a spanning star of H.

Indeed, the statement above gives a necessary condition for a graph having a 2-spanner. However, we show in Fig. 1 a nonseparable graph G and a spanning tree T of G such that the intersection of T with the unique triconnected component of G ($H = K_4$) is a spanning star of H, but there is no tree 2-spanner for the split graph in Fig. 1, as a consequence of Proposition 3. Observe that, since the connectivity of a complete graph with n vertices is $n - 1$ [10,11], a K_4 is triconnected and, once this is the unique triconnected component of G, $H = K_4$. Thus, in order that G is tree 2-spanner admissible it must exist a spanning tree T of G such that $T \cap H$ is a star. Figure 1(b) exhibits such a tree. This example can be generalized, for instance, to a graph obtained from a K_{2k} adding k vertices adjacent to two vertices of K_{2k} with no common adjacent vertex. k-sun (see [15]) are also an example of split graphs that satisfy the sufficient condition mentioned above, and thus would be tree 2-spanner admissible graphs, but, accordingly to Proposition 3, they do not admit a tree 2-spanner.

Thus, the Cai and Corneil's sufficient condition for tree 2-spanner admissible graphs is not consistent with the usual definition of the connectivity for complete graphs. Even if we consider that the connectivity concept does not apply for such graphs, the condition does not hold in these families of examples.

(a) 

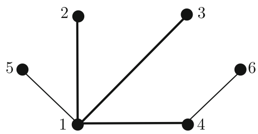
(b) 

Fig. 1. (a) A split graph G with only one triconnected component $H = K_4$. (b) A spanning tree T of G such that $T \cap H$ is a spanning star of H, but G is not a tree 2-spanner admissible graph (see Proposition 3, since there is no vertex of the set $\{1, 2, 3, 4\}$ adjacent to both vertices 5 and 6, then the stretch index of G is equal to 3).

3 Stretch Index for Graph Classes

Next we consider some related graph classes, for which we are able to obtain optimum tree t-spanners even when, in some cases, the lower bound of Theorem 2 is far from the stretch index we obtain. Moreover, seeing whether and how vertices/edges operations affect the stretch index is another goal of this section.

3.1 Cycle-Power Graphs

Any graph is a tree $(n-1)$-spanner admissible, but, in general, such a bound is far from the stretch index. However, for cycle graphs, $n-1$ is a tight value, since it reaches the lower bound of Theorem 2. Next, we present classes with extremal bounds, in such a way that $\sigma_T(G)$ is large and far from the lower bound given by Theorem 2.

A *cycle-power* graph [16], C_n^k, is obtained from a C_n by adding edges between two vertices with distance at most k in C_n. We call *external edges* the edges of the *external cycle* C_n, and *internal edges* the added edges. Since $g(C_n^k) = 3$, then $\sigma_T(C_n^k) \geq 2$. We restrict ourselves to $k = 2$ and show an optimum tree $\lfloor \frac{n}{2} \rfloor$-spanner.

Given a graph $G = C_n^2$, we define an *ℓ-come-go* path with respect to a pair of vertices u_i, u_{i+j}, for $j \in \{1, 2\}$, by a path of length ℓ from u_i to u_{i+j}, for $\ell \in \{2, \cdots, n-1\}$, following one of two directions, either: clockwise/counterclockwise direction; or counterclockwise/clockwise direction. When we are not interested in the length, we suppress such a value and refer an *ℓ-come-go* path by a *come-go* path.

A *spot edge* of a come-go path is an external edge that either: changes the way of the path, i.e. from clockwise to counterclockwise or from counterclockwise to clockwise; or immediately precedes an internal edge that changes the way of the path. Figure 2 illustrates the two 7-come-go paths with respect to u_1 and u_2.

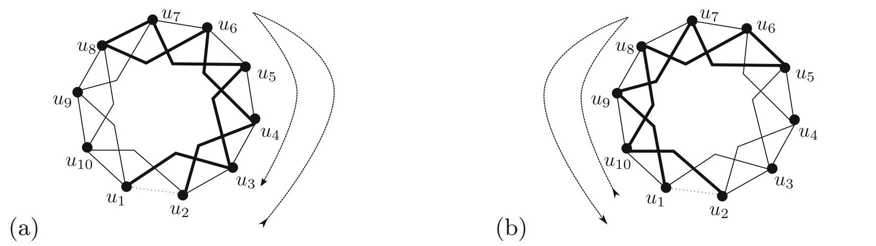


Fig. 2. Bold edges belong to 7-come-go paths with respect to u_1 and u_2, such that: (a) path using the counterclockwise/clockwise direction, where u_7u_8 is the spot edge; (b) path using the clockwise/counterclockwise direction, where u_5u_6 is the spot edge.

Lemma 1. *Given a graph $G = C_n^2$ and a come-go path P with respect to u_iu_{i+j}, for $j \in \{1, 2\}$, then P contains a unique spot edge.*

Proof. Once P is a come-go path, P must contain a spot edge. Suppose there are more than one of such edges. Following the path P from u_i to u_{i+j}, consider u_fu_{f+1} as the first reachable spot edge of P. After reaching the last spot edge of P, the unique way to achieve u_{i+j} is by crossing again u_f or u_{f+1}, once it is not possible to bypass two consecutive vertices of the external cycle. Since u_f and u_{f+1} already belong to P, then there is a cycle in the come-go path P. $\square$

A path between u_i, u_{i+j}, for $j \in \{1, 2\}$, of length greater than 2 which is not a *come-go* is called a *turn around* path, which is depicted in Fig. 3(a). If $j = 1$, then the length of a turn around path is at least $\lfloor \frac{n}{2} \rfloor$. If $j = 2$, then the length is

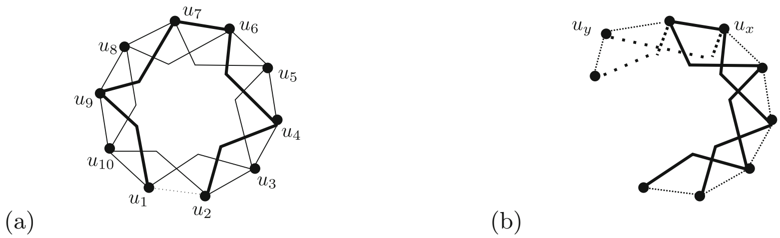


Fig. 3. (a) Bold edges belong to a turn around path with respect to u_1 and u_2. (b) An example of vertices u_x and u_y.

at least $\lfloor \frac{n}{2} \rfloor$ for n odd, and at least $\frac{n}{2} - 1$ for n even. Note that, when $j = 2$, we have between u_i and u_{i+j} either a turn around path or the 2-path $u_i u_{i+1} u_{i+2}$.

For any non-edge of a spanning tree T of a graph G, there is a path which is either: a come-go path, or a turn around path, or the 2-path $u_i u_{i+1} u_{i+2}$, for the non-edge $u_i u_{i+2}$.

Proposition 1. *Given an ℓ-come-go path with respect to u_i and u_{i+j}, for $j \in \{1, 2\}$, if $j = 1$, then there is a unique external edge, otherwise there are exactly two external edge.*

Proof. Considering $j = 1$, since the spot edge is external, let us suppose that there is at least one more external edge $u_f u_{f+1}$, for $i+1 < f < \ell$ in an ℓ-come-go path P. In this case, following the path from u_i to u_{i+1}, at least one of u_f and u_{f+1} will be reached, and after crossing the spot edge, it is necessary to reach u_f or u_{f+1} again, which implies that P is not a path. Similarly, considering $j = 2$, the unique external edges are $u_i u_{i+1}$ and the spot edge. □

Lemma 2. *Given a graph $G = C_n^2$ and an ℓ-come-go path P, with respect to $u_i u_{i+j}$, for $j \in \{1, 2\}$, then P is the unique ℓ-come-go path with respect to $u_i u_{i+j}$ following the same direction of P.*

Proof. Suppose there are at least two ℓ-come-go paths P_1 and P_2 following, w.l.g., the counterclockwise/clockwise direction with respect to $u_i u_{i+j}$, for $j \in \{1, 2\}$. In this case, there is a non-edge in P_1 which is external edge of P_2, and then it is a spot edge of P_2, by Proposition 1. Hence, the length of P_2 is distinct of ℓ. □

Lemma 3. *For any spanning tree T of $G = C_n^2$, there is at least a non-edge $u_i u_{i+j}$, for $j \in \{1, 2\}$, such that the unique path between u_i and u_{i+j} in T is a turn around path.*

Proof. Suppose the path between any pair of vertices of a non-edge of T is not a turn around path. Hence, if the non-edge is external, then the path is a come-go path. If the non-edge is internal, then the path between them is either a 2-path, or it is a come-go path.

Since T must contain an external non-edge, let $u_i u_{i+1}$ be such an external non-edge of T and thus, by hypothesis, there is a come-go path P_1 between u_i

and u_{i+1}, in which $u_f\, u_{f+1}$ is the spot edge. The paths between all pairs of vertices in P_1 consisting of non-edges in T are induced paths of P_1, hence, let us analyze the neighbors of u_f and u_{f+1} outside P_1.

If there is a vertex of G outside of P_1, at least one of the vertices u_f and u_{f+1} has a neighbor outside P_1 consisting of a non-edge in T, because if there were all edges from u_f and u_{f+1} to their neighbors outside P_1, it would have in T the cycle $u_f\, u_{f+1}\, u_y, u_f$, for $u_y \in \{N(u_f)\backslash P_1,\ N(u_{f+1})\backslash P_1\}$. Let $u_x u_y$ be a non-edge of T, for $x \in \{f, f+1\}$, Fig. 3(b).

If $u_x u_y$ is an internal non-edge, then we have two options of a path between u_x and u_y in T:

i. by the 2-path $u_x u_{x+1} u_y$. In this case, go to a non-edge where one of the vertices belongs to the 2-path. This non-edge belongs to: a 2-path, and in this case, we go to a non-edge and the analysis continues; a come-go path with respect to its extremities, and in this case, as it was done with P_1, go to its spot edge and continue the analysis; or $u_{x+1} u_y$ is the spot edge of a come-go path, P_2, following the opposite direction of P_1, with respect to vertices that do not belong to P_1, nor to the 2-path, either. Hence, go to the extremity vertices of P_2 and analyze a non-edge whose vertices are an extremity vertex of P_2 and a vertex that does not belong to P_2 nor to P_1;
ii. by a come-go path with respect to $u_x u_y$, which we call P_3. Note that P_3 must have the same direction of P_1, otherwise, we would visit vertices already in P_1, implying in a cycle. Hence, go to the spot edge of P_3 and consider it similarly as done considering P_1.

If $u_x u_y$ is an external non-edge, then u_x and u_y must be connected by come-go path in T. In this case, proceed as in the previous case ii.

Note that the procedures considered in i and ii. must be finished when we reach either the vertex u_{i-1} (whenever P_1 follows anticlockwise/clockwise direction), or the vertex u_{i+j+1}, for $j \in \{1, 2\}$ (whenever P_1 follows clockwise/anticlockwise direction). Let u_w be the last visited vertex, which is neighbor of u_i or u_{i+j}. In T, we have three possible paths between u_w and u_i or u_{i+j}: there is an edge; there is a come and go path; there is a 2-path. For any of such cases, we have created a cycle, because, by P_1, there is a path between u_i and u_{i+j}, which does not include u_w. Therefore, there is path, distinct of P_1, starting from either u_i or u_{i+j} passing through u_w. Thus, there is a turn around path between u_i and u_{i+j}. □

Lemma 4. *For any cycle-power graph C_n^2, $\sigma_T(C_n^2) \geq \lfloor \frac{n}{2} \rfloor$.*

Proof. Since there is at least a turn around path in any spanning tree T of $G = C_n^2$ (Lemma 3), and if n is odd, then there is a non-edge in T whose corresponding vertices' distance is at least $\lfloor \frac{n}{2} \rfloor$. Therefore, $\sigma_T(C_n^2) \geq \lfloor \frac{n}{2} \rfloor$, for n odd.

Since when n is even, a turn around path has length at least: $\frac{n}{2} - 1$, for an internal non-edge $u_i\ u_{i+2}$; or $\frac{n}{2}$, for an external non-edge. Hence, it remains to analyze the former case. Note that G contains two disjoint internal cycles I_1 and

I_2, each one of length $\frac{n}{2}$. Consider that u_i and u_{i+2} belong to I_1 and the distance between them in T is $\frac{n}{2} - 1$. Since the unique turn around path of length $\frac{n}{2} - 1$ between u_i and u_{i+2} in G includes each edge of the cycle I_1, all internal edges of I_1 must belong to T, except $u_i\ u_{i+2}$. On the other hand, at least one of $u_i\ u_{i+1}$ and $u_{i+1}\ u_{i+2}$ must be non-edge of T. Otherwise, if both edges belong to T, then there would be the 2-path $u_i\, u_{i+1}\, u_{i+2}$ in T, contradicting the assumption of the path between $u_i\, u_{i+2}$ is turn around.

1. If $u_i\ u_{i+1}$ is non-edge of T and $u_{i+1}\ u_{i+2}$ is edge of T (which is similar to the case of $u_i\, u_{i+1}$ being edge of T and $u_{i+1}\ u_{i+2}$ non-edge of T), then the path between u_i and u_{i+1} has length at least $\frac{n}{2}$ considering the path between u_i and u_{i+2}, and the edge $u_{i+2}\ u_{i+1}$. Otherwise, if there is a distinct path P between u_i and u_{i+1}, we would have another path between u_i and u_{i+2}, say $P \cup \{u_{i+1}u_{i+2}\}$.
2. If $u_i\ u_{i+1}$ and $u_{i+1}\ u_{i+2}$ are both non-edges of T, then at least one of the edges $u_{i-1}\ u_{i+1}$ and $u_{i+1}\ u_{i+3}$ must belong to T, otherwise u_{i+1} would be isolated of T. Hence, we analyze the two cases:
   - $u_{i-1}\ u_{i+1}$ is an edge of T and $u_{i+1}\ u_{i+3}$ is a non-edge of T. In this case, note that the path between u_{i+1} and u_{i+2} must be a turn around, because $u_{i+1}\ u_{i+2}$ and $u_{i+1}\ u_{i+3}$ are non-edges of T, Fig. 4(a). Since $u_{i+1}\ u_{i+2}$ is an external non-edge of T, then, the distance between u_{i+1} and u_{i+2} is at least $\frac{n}{2}$.
   - $u_{i-1}\ u_{i+1}$ and $u_{i+1}\ u_{i+3}$ are edges of T. Let us consider the distance between u_{i+1} and u_{i+2} in T. If it is given by a turn around path, then its length is at least $\frac{n}{2}$. Otherwise, it is an come-go path P^1. If P^1 follows the clockwise/anticlockwise direction, then the edge $u_i\ u_{i+2}$ must exist in T, but it contradicts the hypothesis. Hence, P^1 follows the anticlockwise/clockwise direction. Similarly, the path between u_i and u_{i+1} is a turn around path, implying that the distance between u_i and u_{i+1} is at least $\frac{n}{2}$, or it is a come-go path P^2 following the clockwise/anticlockwise direction. In this case, we have that:
     - if there is any path in T between the spot edges of the two come-go paths without passing through u_{i+1}, then we have created a cycle, since u_{i+1} belongs to the two come-go paths just settled;
     - Suppose there is no path in T between the spot edges of the two come-go paths without passing through u_{i+1}, and let P be the path composed by external edges in G that links the P^1 spot edge to the P^2 spot edge following the anticlockwise direction, Fig. 4(b). Note that P has at least one edge, because, otherwise, T would have a cycle. Clearly, there is an edge in P which is a non-edge in T. Thus, there is a path of length at least $\frac{n}{2}$.

Hence, we have that there is a pair of neighbors in G whose distance is at least $\lfloor \frac{n}{2} \rfloor$ in T. □

Lemma 5. *For any cycle-power graph C_n^2, $\sigma_T(C_n^2) \leq \lfloor \frac{n}{2} \rfloor$.*

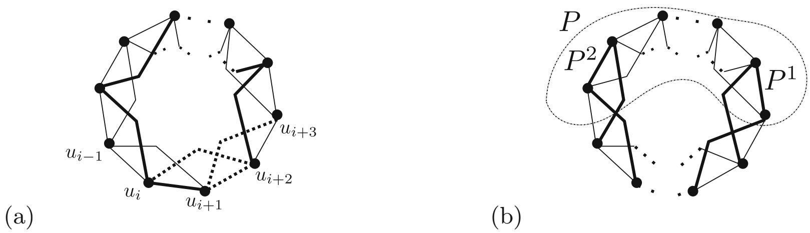


Fig. 4. (a) Turn around path between u_{i+1}, u_{i+2} in T. Note that $u_i u_{i+2}$, $u_{i+1}u_{i+2}$ and $u_{i+1}u_{i+3}$ are non-edges of T. (b) Bold edges compose two come-go paths P^1 and P^2, where the bold external edges are their spot edges. The path P is inside the dotted diagram.

Proof. We obtain a spanning tree T of C_n^2 with vertex set $\{u_1,\ u_2,\ \ldots, u_n\}$ as follows: add to T the vertex u_1 and its neighbors $u_2,\ u_3,\ u_n$ and u_{n-1}. Now, follow the direction in which the next vertex is u_2, set $i = 3$, and: (i) take the vertex u_i; (ii) Add to T the vertices adjacent to u_i which are not in T yet, following the same direction as established initially, i.e., $u_4,\ u_5$ in the first step. Increment $i+1$ and return to step (i) until reaching the last vertices not in T yet. It is easy to see that, between two adjacent vertices of C_n^2, the distance between them in T is either $1,\ 2,\ 3$ or $\frac{n}{2}$. Hence, $\sigma_T(C_n^2) \leq \lfloor \frac{n}{2} \rfloor$. □

Figure 5 depicts a tree $\lfloor \frac{n}{2} \rfloor$-spanner for C_{10}^2.

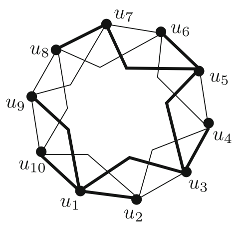


Fig. 5. Bold edges form the tree $\lfloor \frac{n}{2} \rfloor$-spanner T for C_{10}^2. There are: three turn around paths in T, with respect to the internal non-edges u_7u_9 and u_8u_{10}, and the external non-edge u_8u_9; a 2-path between the internal non-edge u_2u_{10}; 3-come-go paths with respect to the internal non-edges u_2u_4, u_4u_6, u_6u_8 and u_8u_{10}; and 2-come-go paths with respect to the external non-edges u_2u_3, u_4u_5, u_6u_7 and u_9u_{10}.

Theorem 3 follows from Lemmas 4 and 5.

Theorem 3. *For any cycle-power C_n^2 with $n > 5$, $\sigma_T(C_n^2) = \lfloor \frac{n}{2} \rfloor$.*

3.2 Stretch Index After Edges Removal

For several graph classes, we are able to determine the stretch index. But obtaining the stretch index after we consider operations on the vertex/edge sets regarding those classes is a challenge. In this section, we are particularly interested on a perfect matching removal considering 2 cycle-power and complete graphs, which are prism and generalized octahedral graphs, respectively. With this last result, in Sect. 3.3 we obtain the stretch index for cographs.

Removing a Perfect Matching of Cycle-Power Graphs. Considering 2 cycle-power graphs of even order after removing a perfect matching M with respect to external edges, one can note that a $C_{2p}^2 \backslash M$ is the *prism graph* with bases C_p. Lemma 6 presents a lower bound which is far from its girth's lower bound. Moreover, in Lemma 7 we prove that the stretch index is not affected by a perfect matching removal, differently from what happens with the complete graph and the octahedral graph, as proved in Theorem 5.

As in 2 cycle-power graphs, in prism graphs, we also have come-go and turn around paths.

Lemma 6. *Given $G = C_{2p}^2 \backslash M$, a cycle-power graph C_{2p}^2 after removing a perfect matching M with respect to the external edges, we have that $\sigma_T(G) \geq \frac{n}{2}$.*

Proof. Considering any tree t-spanner of G, we analyze two cases: all external edges belong to T; and there is at least an external non-edge in T.

1. All external edges belong to T: In this case, between two consecutive external edges $u_i\ u_{i+1}$ and $u_{i+2}\ u_{i+3}$ it is not possible to exist both internal edges $u_i\ u_{i+2}$ and $u_{i+1}\ u_{i+3}$ in T, otherwise the C_4, $u_i,\ u_{i+1},\ u_{i+3},\ u_{i+2},\ u_i$, would belong to T. Next, we analyze two possible subcases: $u_i\ u_{i+2}$ and $u_{i+1}\ u_{i+3}$ are both non-edges of T; only one of such edges belongs to T.
   1.1. $u_i\ u_{i+2}$ and $u_{i+1}\ u_{i+3}$ are both non-edges of T: Considering the edge $u_{i+4}\ u_{i+5}$, it must exist in T either $u_{i+2}\ u_{i+4}$, or $u_{i+3}\ u_{i+5}$, because otherwise, the edge $u_{i+2}\ u_{i+3}$ would be isolated in T.
   Consider, w.l.g., $u_{i+2}\ u_{i+4}$ is in T (and so, $u_{i+3}\ u_{i+5}$ is a non-edge). Although a turn around for an internal non-edge is at least $\frac{n}{2} - 1$, the distance between u_{i+1} and u_{i+3} is at least $\frac{n}{2} + 1$ by a turn around path with respect such an internal non-edge. Note that the length of the turn around path between u_{i+1} and u_{i+3} is equal to $\frac{n}{2} - 1$ only when all edges of $I_1 \backslash \{u_{i+1}u_{i+3}\}$ are in T, where I_1 is the internal cycle of G that contains vertices u_{i+1}, u_{i+3} and u_{i+5}. However, $u_{i+3}\ u_{i+5}$ is a non-edge in T, which implies in a exchange of $u_{i+3}\ u_{i+5}$ by the path $u_{i+5}, u_{i+4}, u_{i+2}, u_{i+3}$ in T. Hence, the length of the turn around path is at least $\frac{n}{2} + 1$ after the edges' exchange.
   1.2. Suppose, w.l.g., $u_i\ u_{i+2}$ is an edge of T and $u_{i+1}\ u_{i+3}$ is a non-edge of T. Now, we prove that in T it must exist a pair of non-edges $u_j\ u_{j+2}$ and $u_{j+1}\ u_{j+3}$ for some j, similarly to Case 1.1. Assume that one of such edges is in T and belongs to the internal cycle I_2 of G. Hence, we create a path starting by the edge $u_i\ u_{i+2}$, and after that we choose one of the two ways of reaching the external edge $u_{i+4}\ u_{i+5}$, by $u_{i+2}\ u_{i+4}$ or $u_{i+3}\ u_{i+5}$. If $u_{i+2}\ u_{i+4}$ is an edge of T, then we are making a path through I_2, otherwise, the path is $u_i,\ u_{i+2},\ u_{i+3},\ u_{i+5}$. Therefore, it is

always possible to reach two consecutive external edges by using I_1 or I_2 edges. So, if there is not a pair of non-edges similar to Case 1.1, we can continue this path through external edges of I_1 and I_2, creating then a cycle. Once it is necessary to have non-edges of Case 1.1, then we have the existence of vertices with distance at least $\frac{n}{2}+1$ in T.

2. There is at least an external non-edge in T: Suppose $u_i\ u_{i+1}$ is a non-edge of T. If the path between u_i and u_{i+1} in T is a turn around, then its length is at least $\frac{n}{2}+1$, because u_i belongs to I_2 and u_{i+1} belongs to I_1. Hence, assume that the path in T between u_i and u_{i+1} is a come-go.
   Note that we have at least one non-edge of I_1 and of I_2 in T, and similarly to Lemma 3 and Case 1.2 above, there is a turn around path between the corresponding vertices of an internal non-edge of I_1 and I_2.
   So, each turn around has length at least $\frac{n}{2}-1$, and such a path with respect to an internal non-edge in T is unique in G. Moreover, in T, a turn around with respect to a non-edge of I_2 (or I_1) has length $\frac{n}{2}-1$ or any greater value with the same parity, because the path between such vertices does not go through all edges of the corresponding internal cycle, and then we must move to the other internal cycle and return, increasing the path in at least two edges.
   Hence, in order to keep in T the distances equal to $\frac{n}{2}-1$ between the vertices of non-edges e^2 of I_2 and e^1 of I_1, all other edges of both internal cycles of G must belong to T. Let P^2 be the path $I_2 \backslash e^2$ and P^1 be the path $I_1 \backslash e^1$.
   Now, P^1 must be linked to P^2. The unique way to do that is by using only one external edge, otherwise, there would be a cycle in T by at least two ways to go through P^2 to P^1, each one using a distinct external edge. Therefore, in T, there is only one external edge of G.
   Since there is a come-go path between $u_i,\ u_{i+1}$, as well between all other $\frac{n}{2}-2$ external non-edges, all come-go paths between corresponding vertices of external non-edges must be composed by the same spot edge, say, the external edge we have used to link P^1 and P^2.
   Furthermore, the unique way to exist only come-go paths between corresponding vertices of external non-edges in T is by considering $u_{k-1}u_{k+1}$ and $u_{k-2}u_k$ internal non-edges of T. Otherwise, if the non-edges of T were u_ju_{j+2} and $u_{j+2s+1}\ u_{j+2s+3}$, there would be a turn around path with respect to the external edges $u_j\ u_{j+1}$ and $u_{j+2s+1}\ u_{j+2s+2}$.
   In this way, we have that the distances between the vertices of the non-edge u_{k-1} and u_k, and between the vertices of the non-edge u_{k+1} and u_{k+2} are $\frac{n}{2}-x$ and $\frac{n}{2}+x$, respectively, according to the place we have chosen the spot edge. Therefore, when $x=0$, we have that $\sigma_T(G) \geq \frac{n}{2}$. □

Accordingly to the arguments of Lemma 6, we are able to build a tree $\frac{n}{2}$-spanner as follows.

Lemma 7. *Given $G = C^2_{2p} \backslash M$, a cycle-power graph C^2_{2p} after removing a perfect matching with respect to the external edges M, we have that $\sigma_T(G) \leq \frac{n}{2}$.*

Proof. Consider I_1 and I_2 the internal cycles of G, in such a way that $I_1 = u_1, u_3, u_5, \ldots, u_{n-1}, u_1$, $I_2 = u_2, u_4, u_6, \ldots, u_n, u_2$ and $M = \{\{u_2u_3\}, \{u_4u_5\}, \{u_6u_7\}, \ldots, \{u_nu_1\}\}$. We create the spanning tree T by the edge set $\{\{u_3u_5\}, \{u_5u_7\}, \ldots, \{u_{n-3}u_{n-1}\} \cup \{u_2u_4\}, \{u_4u_6\}, \ldots, \{u_{n-2}u_n\} \cup \{u_{\frac{n}{2}}u_{\frac{n}{2}+1}\}\}$. Note that the unique external edge of G in T is $\{u_{\frac{n}{2}}u_{\frac{n}{2}+1}\}$. The paths between the external non-edges of T have length at most $\frac{n}{2}$, which is equal to this value for the non-edges $u_1\ u_2$ and $u_{n-1}\ u_n$. Furthermore, there are only two internal non-edges in T, which are u_nu_2 and $u_{n-1}u_1$, with distances equal to $\frac{n}{2} - 1$, because all other edges of I_2 and I_2 belong to T. □

Theorem 4 follows from Lemmas 6 and 7.

Theorem 4. *Given $G = C_{2p}^2 \backslash M$, a cycle-power graph C_{2p}^2 after removing a perfect matching with respect to the external edges M, we have that $\sigma_T(G) = \frac{n}{2}$.*

Generalized Octahedral Graphs. Generalized octahedral graphs figure in several well studied problems [17] because of their regularity and symmetry. A generalized octahedral graph, or simply *octahedral graph* O_k, is the $(2k-2)$-regular graph, which is exactly a complete graph K_{2k} after removing a perfect matching. This class sounds interesting in here when we deal with cographs in Sect. 3.3, even considering O_k after vertices addition, in Lemma 10.

Theorem 5. *Given an octahedral graph O_k, then $\sigma_T(O_k) = 3$, for $k > 2$.*

Proof. Consider the vertex set $\{u_1, v_1, \ldots, u_k, v_k\}$ in such a way that u_i and v_i are not neighbors, but they are adjacent to all other vertices of O_k. A tree T can be built by first considering two stars, with centers in u_1 and v_2, such that u_1 is adjacent to all u_i's and v_2 is adjacent to all v_i's. Now, we add to T the edge u_1v_2. The distances in T of two vertices of u_i's or of v_i's are equal to 2, and from distinct side are equal to 3, hence $\sigma_T(O_k) \leq 3$. In order to prove that $\sigma_T(O_k) = 3$, suppose we have an optimum tree spanner T for O_k that can be partitioned into two rooted trees, T_1 and T_2, each one with more than two vertices, such that at least one of them is not a star. Suppose, w.l.g., that T_1 is not a star. Let $l \in T_1$ and c be two vertices of T such that $lc \notin E(O_k)$. If $c \in T_1$, then l and c are adjacent to each vertex of T_2. Since T_1 is linked to T_2, there is an edge with one extreme in T_1 and another in T_2. If l is such an extreme, then $d_T(c, v) \geq 3, \forall v \in T_2$. Otherwise, there is a vertex $v \in T_2$ such that $d_T(l, v) \geq 3$. □

3.3 Threshold Graphs and Their Superclasses

Next, we establish the stretch index for three classes whose graphs are tree 3-spanner admissible (*cf.* [4]).

Threshold Graphs. Threshold graphs [18] can be defined as the intersection of two very well studied classes: split graphs and cographs. Thus, threshold graphs are $\{2K_2,\ P_4,\ C_4\}$-free graphs. Moreover, G is a *threshold graph* if G can constructed from the empty graph by repeatedly adding either an isolated vertex or a universal vertex.

Since to obtain spanning trees we only consider connected graphs, the last vertex of a threshold graph construction must be universal. Hence, a tree can be built as a star whose center is such a universal vertex. Thus we can state the following proposition.

Proposition 2. *If G is a threshold graph, then $\sigma_T(G) =\ 2$.*

Split Graphs. As just mentioned, split graphs are a superclass of threshold graphs. Formally, a graph $G = (X, Y)$ is a *split graph*, also called a $(1, 1)$-graph, if and only if it can be partitioned into a clique X and a stable set Y. In terms of forbidden subgraphs, they are $\{2K_2,\ C_4, C_5\}$-free graphs.

Lemma 8. *If G is a split graph, then $\sigma_T(G) \leq 3$.*

Proof. We obtain a spanning tree T for a split graph $G = (X, Y)$ as follows. Set any vertex x in X to be the center of a star which includes each other vertex of X. Next, for each vertex $y \in Y$, choose an edge incident to y, arbitrarily, and make y a pendant in T. It remains to show that the distance between two adjacent vertices v, w in G is at most 3 in T. (i) $v, w \in X$: since we have a star in T with respect to X, then $d(v, w) = 2$. (ii) $v \in X$, $w \in Y$: the worst case occurs when $d_G(w) \geq 2$ and v is a leaf of the star in T. In this case, $d(v, w) = 3$ by the path $vxx'w$, where $x'w$ belongs to T. □

Now, we characterize split graphs whose stretch indexes are 2 or 3.

Proposition 3. *Let $G = (X, Y)$ be a split graph which is not a tree. Thus, $\sigma_T(G)=2$ iff either: (i) $d_G(y) = 1, \forall\ y \in Y$, or (ii) $\exists\ x \in \bigcap_{y \in Y} N_G(y),\ x\ \in\ X$ such that $d_G(y) \geq 2$.*

Proof. If G satisfies (i) or (ii), then G contains a tree 2-spanner which can be constructed following Lemma 8, and, particularly in case (ii), consider any vertex x satisfying conditions required in (ii) to be center of the star. Conversely, by contradiction, since $\sigma_T(G) = 2$, for each pair of vertices in X there is in T either an edge or a P_3 centered in a vertex v of G. If $v \in X$, then the minimum stretch spanning subtree with respect to X is a star. Otherwise, $v \in Y$ and each vertex of the clique would be a leaf of the star centered in v. Once there are two vertices in Y with degree at least 2 without an adjacent vertex in common, in the first case, for any center of the star we have chosen regarding the clique's vertices, there is a vertex of the stable set such that all its neighbors are leaves of the star, which implies $\sigma_T(G) \geq 3$. In the second case, $\sigma_T(G) \geq 3$ anyway, because, by hypothesis, there exist at least two more vertices in Y with degree at least 2, and they will be adjacent only to the leaves of the star centered in v. □

Figure 1 exhibits a split graph G with $\sigma_T(G) = 3$. Another example of split graphs that have stretch index equal to 3 are the k-sun. Such graphs do not satisfy conditions of Proposition 3 either.

Cograph. A *cograph* is a P_4-free graph. A trivial graph is a cograph, and any other can be obtained by disjoint union or join operations of cographs. We can represent the union and join operations of a cograph by a tree decomposition, called *cotree* [19].

Theorem 6. *If G is a cograph, then $\sigma_T(G) \leq 3$.*

Proof. Since G must be connected, its cotree root's label is 1, implying that any vertex of G represented as a leaf node of a root's subtree is adjacent to all vertices of the other root's subtrees. We build a spanning tree T of G as follows. Let f be a leaf node of the leftmost root's subtree, F_1. Since f is adjacent to all vertices of the other root's subtrees, set T as a star with center f and make f adjacent to each vertex of all root's subtrees on F_1's right. Let lf be an edge of the star just obtained. Once the vertex l in G is adjacent to all vertices of F_1, hence we add to T each edge corresponding to a neighbor of l in F_1, except to the edge lf. Therefore, $\sigma_T(G) \leq 3$. □

Lemma 9. *Given a graph G, let k be the number of its cotree root's subtrees. If G does not contain a universal vertex, then G contains an octahedral O_k as an induced subgraph.*

Proof. Since G does not contain a universal vertex, then each root's son of its cotree has label 0. Hence, in each subtree there are at least two leaves corresponding to non-adjacent vertices in G, but these two vertices are adjacent to all vertices of the other cotree root's subtrees. So, the union of each two non-adjacent vertices per subtree induces an O_k in G. □

If a cograph G contains a universal vertex and there exist k' subtrees of the root with more than one leaf each, then there is an octahedral $O_{k'}$ as an induced subgraph of G. Moreover, if there were a universal vertex u with respect to $O_{k'}$, then $\sigma_T(O_{k'} \cup \{u\}) = 2$. However, such a vertex does not exist in a cograph without a universal vertex, because, in this case, all root's subtrees have label 0, and considering two $O_{k'}$ non-adjacent vertices, it does not exist a vertex of a same subtree adjacent to both vertices, otherwise their lowest common ancestor would be 1.

Lemma 10. *Let H be a cograph obtained from O_k by non-isolated vertices addition. If there is not a universal vertex in H with respect to O_k, then $\sigma_T(H) = 3$.*

Proof. Since O_k is an induced subgraph of H, by construction H is a triconnected component. If $\sigma_T(H) = 2$, then the tree 2-spanner of H would be a star. However, it is not possible since H does not have a universal vertex. □

Since a cograph without universal vertex does not contain a universal vertex with respect to some octahedral, then we have that, for cographs, containing a universal vertex is also a necessary condition so that $\sigma_T(G) = 2$.

Theorem 7. *Let G be a cograph. $\sigma_T(G) = 2$ iff G has a universal vertex.*

Proof. If G contains a universal vertex, then $\sigma_T(G) = 2$. Let us prove the converse by contrapositive. If there is no universal vertex in G, then by Lemma 9 we have that G contains an octahedral O_k as induced subgraph, and by Lemma 10 we have that the unique case for decreasing σ_T from 3 to 2 is when there is a universal vertex with respect to O_k, but in a cograph with no universal vertex, there is no universal vertex with respect to an O_k. □

4 Concluding Remarks and Further Work

In this work, we present an inconsistence on a well known sufficient condition for tree 2-spanner admissible graphs. Moreover, we establish optimum tree t-spanners for some graph classes by considering their characteristics, decompositions and by vertex/edges operations. Following the strategies proposed in this work, we intend to obtain optimum tree t-spanners for generalized split graphs, say (k, ℓ)-graphs, and also for graphs obtained by vertex/edges operations.

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