Stabilization of Deterministic Control Systems Under Random Sampling: Overview and Recent Developments

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Abstract This chapter addresses the problem of stabilizing continuous-time deterministic control systems via a sample-and-hold scheme under random sampling. The sampling process is assumed to be a Poisson counter, and the open-loop system is assumed to be stabilizable in an appropriate sense. Starting from as early as mid-1950s, when this problem was studied in the Ph.D. dissertation of R.E. Kalman, we provide a historical account of several works that have been published thereafter on this topic. In contrast to the approaches adopted in these works, we use the framework of piecewise deterministic Markov processes to model the closed-loop system, and carry out the stability analysis by computing the extended generator. We demonstrate that for any continuous-time robust feedback stabilizing control law employed in the sample-and-hold scheme, the closed-loop system is asymptotically stable for all large enough intensities of the Poisson process. In the linear case, for increasingly large values of the mean sampling rate, the decay rate of the sampled process increases monotonically and converges to the decay rate of the unsampled system in the limit. In the second part of this article, we fix the sampling rate and address the question of whether there exists a feedback gain which asymptotically stabilizes the system in mean square under the sample-and-hold scheme. For the scalar linear case, the answer is in the affirmative and a constructive formula is provided here. For systems with dimension greater than 1 we provide an answer for a restricted class of linear systems, and we leave the solution corresponding to the general case as an open problem.

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1 Introduction

This chapter addresses the problem of stabilization of sampled-data control systems under random sampling. Let $(\tau_n)_{n\in\mathbb{N}}$ denote a monotonically increasing sequence in $[0, +\infty[$ with $\tau_0 := 0$. Consider a nonlinear control system

$$\dot{x}(t) = f\big(x(t), u(t)\big), \qquad x(0) \text{ given}, \quad t \geqslant 0, \tag{1}$$

where $f : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}^d$ is a continuously differentiable map, and the *control process* $t \mapsto u(t)$ is constant on each $[\tau_n, \tau_{n+1}[$ for each n. The corresponding solution $\big(x(t)\big)_{t\geqslant 0}$ of (1) is referred to as the *state process*. We shall comment on the precise properties of the solutions of (1) momentarily. Control systems where the control process gets updated at the discrete time instants $(\tau_n)_{n\in\mathbb{N}}$ are referred to as *sampled-data control systems* [2, 10, 24], and typically arise when implementing controllers using a computer [8, 18], or in the context of networked control systems [20, 33, 47].

Since any admissible control process $t \mapsto u(t)$ defined above can be written as

$$u(t) = \sum_{k=0}^{+\infty} u(\tau_k) \cdot 1_{[\tau_k, \tau_{k+1}[}(t) \quad \text{for } t \geqslant 0, \tag{2}$$

it is clear that the two key ingredients of sampled-data control systems are the sampling times $(\tau_k)_{k\in\mathbb{N}}$ and the control values $\big(u(\tau_k)\big)_{k\in\mathbb{N}}$. Different classes of these two ingredients are possible: the former may be periodic [35, 36, 46], state-dependent [7, 22, 38, 42] or random [23, 24]; and the latter may be a random sequence generated by a randomized Markovian policy as defined in [1] or just a feedback from the states at the sampling instants [20, 23], etc. One of the fundamental problems of interest is to provide a description of these two components (often in the form of an algorithm) that results in stability of the closed-loop system. Different approaches have been developed for the necessary analysis depending on how the sampling instants $(\tau_n)_{n\in\mathbb{N}}$ are chosen: see [2] for an overview of classical tools in linear systems with periodic sampling, the papers [30, 36, 37] provide tools specifically suited for nonlinear systems, and the approaches used for optimizing certain performance criterion can be found in [9, 10]. In this article, we are interested in the situation where the sampling times are generated *randomly*. Formally, we define N_t to be the number of sampling instants before (and including) time t as

$$N_t := \sup \big\{ n \in \mathbb{N} \big| \tau_n \leq t \big\} \quad \text{for } t \geqslant 0, \tag{3}$$

and stipulate that the *sampling process* $(N_t)_{t\geqslant 0}$ is a continuous-time stochastic process satisfying the basic requirement

$$\tau_{N_t} \xrightarrow[t\uparrow+\infty]{} +\infty \text{ almost surely.} \tag{4}$$

It is assumed that there is an underlying probability triplet $(\Omega, \mathcal{F}, \mathsf{P})$, sufficiently rich, that provides the substrate for these processes (i.e., each random variable considered here is defined on $(\Omega, \mathcal{F}, \mathsf{P})$), and in the sequel we shall denote the mathematical expectation with respect to the probability measure P by $\mathsf{E}[\cdot]$.

Due to our assumptions on the random sequence $(\tau_k)_{k\in\mathbb{N}}$ and the right-hand side f of (1), it follows that, P-almost everywhere on the sample space Ω, Carathéodory solutions of (1) exist for a sufficiently small interval of time containing $t = 0$. In addition, we *assume* that solutions of (1) exist for all times. Typically, the sampling process $(N_t)_{t\geqslant 0}$ is constructed by means of a renewal process [4, 20]: independent and identically distributed positive random variables $(S_n)_{n\in\mathbb{N}^*}$ are defined on $(\Omega, \mathcal{F}, \mathsf{P})$,[1] with the probability distribution function of S_1 being $\mathsf{F}_{\text{hld}}(t) := \mathsf{P}(S_1 \le t)$ for $t \geqslant 0$, and the sequence $(\tau_n)_{n\in\mathbb{N}}$ is defined according to $\tau_0 := 0$ and $\tau_k := \sum_{\ell=1}^{k} S_\ell$ for $k \in \mathbb{N}^*$. The random variable S_n is the nth *holding time*.

Typical control problems in this setting consist of the design of controllers (feedbacks) for stabilization [23, 49], optimal control [3, 10], state estimation[2] [32, 41], etc., and we will study the problem of stabilization in this article. A mapping $t \mapsto x(t)$ that satisfies (1) in the preceding setting is, naturally, a stochastic process, and consequently, a library of different notions of stochastic stability are available to us [25, 26]. We will restrict our attention mostly to the particularly important property of stability in the mean and mean-square—especially well-studied in the context of linear models [11, 28]—in the sequel.

Finally, we note a connection with the work of Roberto Tempo, to whom this article is dedicated, and his coworkers on randomized algorithms in control theory [43]. That work asks the question of how many random samples *in space* are needed to obtain a sufficient guarantee that a property of interest holds over the whole space, whereas here we are asking how frequently we should sample randomly *in time* so that the feedback is still stabilizing.

2 Connections with Piecewise Deterministic Markov Processes

This section serves the purpose of demonstrating that sampled-data control systems under random sampling can be readily recast as piecewise deterministic Markov processes (PDMPs); consequently, typical control problems can be immediately addressed under this rather general and well-established umbrella framework [13, 14].

To start our discussion, we recall that the sequence of holding times $(S_n)_{n\in\mathbb{N}^*}$ is, typically, independent and identically distributed. The assumption of S_1 being

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[1] For us $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

[2] In contrast to the continuous-time systems given in (1), the references indicated here in the context of state estimation problems deal with discrete times linear systems, and the arrival of observations is modeled as a random process.

an exponential random variable with a given positive intensity λ is fairly common, and the resulting sampling process $\big(N_t\big)_{t\geqslant 0}$ is, consequently, a Poisson process with intensity λ. Recall [39, Theorem 2.3.2] that the Poisson process of intensity $\lambda > 0$ is a continuous-time random process $\big(N_t\big)_{t\geqslant 0}$ taking values in $\mathbb{N}^*$, with $N_0 = 0$, for every $n \in \mathbb{N}^*$ and $0 =: t_0 < t_1 < \cdots < t_n < +\infty$ the increments $\{N_{t_k} - N_{t_{k-1}}\}_{k=1}^{n}$ are independent, and $N_{t_k} - N_{t_{k-1}}$ is distributed as a Poisson-$\lambda(t_k - t_{k-1})$ random variable for each k. The Poisson process is among the most well-studied processes, and standard results (see, e.g., [39, §2.3]) show that it is memoryless and Markovian. Nevertheless, the resulting state process $\big(x(t)\big)_{t\geqslant 0}$ obtained as a solution of (1) under Poisson sampling is *not* controlled Markovian in general. Recall that an $\mathbb{R}^{\nu}$-valued random process $\big(\widetilde{x}(t)\big)_{t\geqslant 0}$ controlled by an $\mathbb{R}^m$-valued random process $\big(\widetilde{u}(t)\big)_{t\geqslant 0}$ is *controlled Markov* [19, §III.6] if for every $t, h > 0$ and every Borel set $\mathcal{S} \subset \mathbb{R}^{\nu}$ we have

$$\mathsf{P}\left(\widetilde{x}(t+h) \in \mathcal{S} \middle| \widetilde{x}(s), \widetilde{u}(s) \text{ for } s \in [0,t]\right) = \mathsf{P}\left(\widetilde{x}(t+h) \in \mathcal{S} \middle| \widetilde{x}(t), \widetilde{u}(t)\right).$$

Indeed, suppose that we intend to employ feedback at sampling instants so that $u(t) = u(\tau_{N_t}) = \kappa\big(x(\tau_{N_t})\big)$ for some measurable map κ, fix $t, t' > 0$, and suppose that the history $\big\{\big(x(s), u(s)\big) \big| s \in [0,t]\big\}$ up to time t is available to us. Of course, any finite k samples may have occured during $[t, t+t']$. If $k = 0$, then $x(\tau_{N_t})$ is not needed to find the conditional distribution of $x(t+t')$ given $\big\{\big(x(s), u(s)\big) \big| s \in [0,t]\big\}$. If $k = 1$, then the conditional distribution of $x(t+t')$ depends on the value of $x(\tau_{N_t})$: since $\tau_{N_t+1} \in]t, t+t']$, the control action at τ_{N_t+1} depends on $x(\tau_{N_t+1})$, and influences $x(t+t')$. A similar reasoning holds for all $k \geqslant 2$.

The controlled Markovian property is extremely desirable in practice, and to arrive at a controlled Markov process in the context of (6), we proceed to adjoin an additional random vector by enlarging the state space. Corresponding to the state process $\big(x(t)\big)_{t\geqslant 0}$ that solves (1), we define the continuous-time *last-sample process* $\big(x(\tau_{N_t})\big)_{t\geqslant 0}$; at each time t, $x(\tau_{N_t})$ is the value of the vector of states at the last sampling time immediately preceding t. In other words, $\mathbb{R}^d$-valued process $\big(x(\tau_{N_t})\big)_{t\geqslant 0}$ attains the value of the states at each sampling instant and stays constant over the corresponding holding time. It turns out to be convenient to introduce the continuous-time *error* process $\big(e(t)\big)_{t\geqslant 0}$ defined by

$$e(t) := x(t) - x(\tau_{N_t}) \quad \text{for } t \geqslant 0. \tag{5}$$

With the joint stochastic process $\big(x(t), e(t)\big)_{t\geqslant 0}$ taking values in $\mathbb{R}^d \times \mathbb{R}^d$, we write the system of interest as a stochastic process described by the ordinary differential equation

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} f\big(x(t), u(t)\big) \\ f\big(x(t), u(t)\big) \end{pmatrix} \quad \text{for almost all } t \geqslant 0, \tag{6a}$$

and at each sampling time τ_{N_t} the process $\big(x(t), e(t)\big)_{t\geqslant 0}$ is reset according to

$$\begin{pmatrix} x(\tau_{N_t}) \\ e(\tau_{N_t}) \end{pmatrix} = \begin{pmatrix} x(\tau_{N_t}^-) \\ 0 \end{pmatrix} \quad \text{with the convention that } x(\tau_0^-) = x_0. \tag{6b}$$

It is readily observed that the joint process $\big(x(t), e(t)\big)_{t\geqslant 0}$ is controlled Markovian. We sometimes abbreviate the right-hand side of (6a) by

$$\mathbb{R}^d \times \mathbb{R}^m \ni (x, u) \mapsto F(x, u) := \begin{pmatrix} f(x, u) \\ f(x, u) \end{pmatrix} \in \mathbb{R}^d \times \mathbb{R}^d.$$

We shall be concerned exclusively with *feedback* controls in this article. In other words, we stipulate that there exists some measurable map

$$\mathbb{R}^d \times \mathbb{R}^d \ni (x, e) \mapsto \kappa(x, e) \in \mathbb{R}^m$$

such that our control process becomes, in the notation of (2),

$$u(t) = \sum_{k=0}^{+\infty} \kappa\big(x(\tau_k), e(\tau_k)\big) 1_{[\tau_k, \tau_{k+1}[}(t) \quad \text{for } t \geqslant 0.$$

In other words, with κ substituted into (6a), our *closed-loop* system becomes

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} f\big(x(t), \kappa\big(x(\tau_{N_t}), e(\tau_{N_t})\big)\big) \\ f\big(x(t), \kappa\big(x(\tau_{N_t}), e(\tau_{N_t})\big)\big) \end{pmatrix} \quad \text{for almost all } t \geqslant 0, \tag{7}$$

while the reset map (6b) stays intact.

With the class of admissible feedback control processes as described above, the description (6b)–(7) provides the basic ingredients to transit to the framework of PDMPs. Indeed, we see readily that the standard conditions for a PDMP [14, (24.8), p. 62] hold for the joint process $\big(x(t), e(t)\big)_{t\geqslant 0}$ described by (6b)–(7) with

- the vector field $\mathfrak{X}$ in [14, §24] being the map $(x, e) \mapsto F\big(x, \kappa(x, e)\big)$,
- the jump rate λ in [14, §24] being a nonnegative measurable function such that $\mathsf{F}_{\mathrm{hld}}(t) = \exp\big(\int_0^t \lambda(s)\, \mathrm{d}s\big)$, which can be readily derived for particular cases of probability distribution functions $\mathsf{F}_{\mathrm{hld}}$ as in [14, p. 37], and
- the stochastic kernel Q for the reset map in [14, §24, p. 58] is the Dirac measure $Q\big(B; (x, e)\big) := \delta_{\{(x,0)\}}(B) = 1_B(x, 0)$ for every Borel subset $B \subset \mathbb{R}^d \times \mathbb{R}^d$ in the context of (6b)–(7).

In this chapter, we will work exclusively under the assumption that the controller has access to perfect state measurements at sampling times. While, in general, it is of interest to consider feedbacks which depend on the measurement error at sampling times $e(\tau_{N_t})$, we can drop the dependence of feedback κ on $e(\tau_{N_t})$ in the case of perfect measurements since $e(\tau_{N_t}) = 0$, for each $t \in [0, +\infty[$, in such cases. In the sequel, we shall employ the feedback exclusively as a function of $x(\tau_{N_t})$, which is described in (5) by the difference between $x(t)$ and $e(t)$, i.e, we shall employ some measurable map $\kappa' : \mathbb{R}^d \to \mathbb{R}^m$ such that $\kappa(x, e) = \kappa'(x - e)$ for all $(x, e) \in \mathbb{R}^d \times \mathbb{R}^d$; we shall abuse notation and continue to use the symbol κ for κ' since there is no risk of confusion.

Remark 2.1 As a consequence of the preceding discussion, we observe that the techniques in [14, Chapters 4, 5] (including several results on stability and optimal control) carry over at once to the setting of sampled-data control systems under random sampling as special cases. In particular, the so-called *extended generator* of the PDMP (6b)–(7) is a particularly useful device for the purposes of analyzing stability and optimality, and we shall look at it in greater detail below in the context of stability.

The *extended generator* of the joint process $\big(x(t), e(t)\big)_{t\geqslant 0}$ is the linear operator $\psi \mapsto \mathcal{L}\psi$ defined by

$$\begin{aligned}\mathbb{R}^d \times \mathbb{R}^d \ni (y, z) \mapsto \mathcal{L}\psi(y, z) &:= \\ \lim_{h\downarrow 0} \frac{1}{h}\Big(\mathsf{E}\big[\psi\big(x(t+h), e(t+h)\big)\big|x(t) = y, e(t) = z\big] &- \psi(y, z)\Big) \in \mathbb{R} \qquad (8)\end{aligned}$$

for all maps $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that the limit is defined everywhere. It is possible to directly write down the extended generator of $\big(x(t), e(t)\big)_{t\geqslant 0}$ from [14, (26.15), p. 70]. We provide the following Proposition catering to the most standard special case of sampling process being Poisson; a direct proof of Proposition 2.2 is included in Appendix A for completeness.

Proposition 2.2 *If the sampling process* $\big(N_t\big)_{t\geqslant 0}$ *is Poisson with intensity* $\lambda > 0$*, then the joint process* $\big(x(t), e(t)\big)_{t\geqslant 0}$ *described above is Markovian. Moreover, for any function* $\mathbb{R}^d \times \mathbb{R}^d \ni (y, z) \mapsto \psi(y, z) \in [0, +\infty[$ *with at most polynomial growth as* $\|(y, z)\| \to +\infty$*, we have*

$$\mathcal{L}\psi(y, z) = \big\langle \nabla_y \psi(y, z) + \nabla_z \psi(y, z), f\big(y, \kappa(y - z)\big)\big\rangle + \lambda\big(\psi(y, 0) - \psi(y, z)\big). \qquad (9)$$

We submit that this extended generator serves as an important tool in most control-theoretic problems associated with this class of randomly sampled-data systems. In particular, (9) provides the following *Dynkin's formula*

$$\mathsf{E}\left[\psi(x(t), e(t))\right] = \mathsf{E}\left[\psi(x(0), e(0))\right] + \mathsf{E}\left[\int_0^t \mathcal{L}\psi(x(s), e(s))\,\mathrm{d}s\right], \qquad (10)$$

which allows us to establish connections with definitive results on stability.

In the sequel, while we provide an account of stability results obtained by different means in prior works, the focus is on using the extended generator to obtain conditions under which the sampled-data systems are asymptotically stable.

3 Lower Bounds on the Sampling Rate

We employ the tools from the previous section to study the following qualitative property of the closed-loop system (7)–(6b). The closed-loop system (7)–(6b) is *globally exponentially stable in the second moment* [25, Chapter 1, p. 23] if there exist two constants $C, \mu > 0$ such that

$$\text{for every } x(0) \in \mathbb{R}^d \text{and } t \geqslant 0, \quad \mathsf{E}\left[\|x(t)\|^2 \,\middle|\, x(0)\right] \leq C\,\|x(0)\|^2\,\mathrm{e}^{-\mu t}.$$

This particular property of stochastic stability is standard, and says that, on an average, the square norm of the system states converges exponentially fast to 0 uniformly from every initial condition.

As a first step in obtaining conditions which guarantee this property, we specify the class of feedback controls in (7). The natural candidates for feedback controls, for which we solve the sampled-data problem, are the ones which asymptotically stabilize the system when the measurements of the state are available in continuous time (without sampling), and possess some robustness properties with respect to errors in the measurement of state. To attribute these properties to the feedback law $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ appearing in (7), it is assumed that there is a function $U : \mathbb{R}^d \to [0, +\infty[$ such that

(**L1**) there exist $\underline{\alpha}, \overline{\alpha} > 0$ satisfying

$$\underline{\alpha}|x|^2 \leq U(x) \leq \overline{\alpha}|x|^2 \quad \text{for all } x \in \mathbb{R}^d;$$

(**L2**) there exist $\alpha, \gamma > 0$ which satisfy

$$\langle \nabla U(x), f(x, \kappa(x-e)) \rangle \leq -\alpha\, U(x) + \gamma U(e) \quad \text{for all } (x, e) \in \mathbb{R}^d \times \mathbb{R}^d;$$

(**L3**) there exist $\chi_x > 0$, $\chi_e \in \mathbb{R}$ satisfying

$$\langle \nabla U(e), f(x, \kappa(x-e)) \rangle \leq \chi_x\, U(x) + \chi_e U(e) \quad \text{for all } (x, e) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Restricting our attention to such a class of controllers, we are interested in addressing the following problem:

Problem 1 Consider the system (7)–(6b) with $\left(N_t\right)_{t\geqslant 0}$ in (3) a Poisson process of intensity λ. If the feedback law $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ is such that (L1)–(L3) hold for some function $U : \mathbb{R}^d \to [0, +\infty[$, does there exist $\lambda > 0$ such that the closed-loop system (7)–(6b) is globally exponentially stable in the second moment?

It is noted that, between two consecutive updates in the controller value, the process $\left(x, e\right)$ follows the differential equation

$$\dot{x} = f(x, \kappa(x - e)) \tag{11a}$$
$$\dot{e} = f(x, \kappa(x - e)). \tag{11b}$$

Assumptions (L1)–(L2) basically characterize the existence of a feedback controller which renders the system (11a) input-to-state stable (ISS) with respect to measurement errors e. Assumption (L3) is introduced to bound the growth of the error e which satisfies (11b). The notion of ISS, pioneered in [40], has been instrumental in the synthesis of control laws for nonlinear systems under actuation and measurement errors. While the general formulation of ISS property would involve nonlinear gains, here we choose to work with linear gains to simplify the presentation. Sampled-data problems in the deterministic setting, where the objective is to find upper bounds on the sampling period that guarantee asymptotic stability, employing feedback controllers with aforementioned robustness properties, have been studied in [36]. In fact, such tools have also been useful in a more general framework where errors in measurements may result from sources other than sampling (see, e.g., [30]). For our purposes, the existence of such robust static controllers allows us to compute a lower bound on the mean sampling rate that solves Problem 1.

Proposition 3.1 *Assume that there exist $\kappa : \mathbb{R}^d \to \mathbb{R}^m$ and $U : \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ such that (L1), (L2), and (L3) hold. If the sampling process $\left(N_t\right)_{t\geqslant 0}$ is Poisson with intensity $\lambda > 0$, then for each $\lambda > 0$ and $\delta \in [0, 1[$ satisfying*

$$\lambda > \chi_e + \frac{\gamma\chi_x}{\delta\alpha} \tag{12}$$

the system (7)–(6b) is exponentially stable in the second moment.

Proof Let us define the function $V : \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty[$

$$V(x, e) = U(x) + \beta U(e),$$

where $\beta > 0$ is to be specified momentarily. From Proposition 2.2 it follows that

$$\begin{aligned}\mathcal{L}V(x, e) &= \langle \nabla U(x) + \beta \nabla U(e), f(x, \kappa(x - e))\rangle - \lambda\beta U(e)\\ &\leq -\alpha U(x) + \gamma U(e) + \beta\chi_x U(x) + \beta\chi_e U(e) - \lambda\beta U(e).\end{aligned}$$

Pick $\delta \in [0, 1[$ and select $\beta = \delta\alpha\chi_x^{-1}$. Then for any $\lambda > 0$ satisfying (12), there exists $0 < \varepsilon < 1$ such that

$$\lambda\beta > \chi_e\beta + \gamma + \varepsilon\alpha(1 - \delta)\beta$$

so that

$$\mathcal{L}V(x, e) \leq -\varepsilon\alpha(1 - \delta)(U(x) + \beta U(e)) = -\varepsilon\alpha(1 - \delta)V(x, e).$$

Exponential stability in the second moment of the process $\left(x(t), e(t)\right)_{t\geqslant 0}$ now follows from Dynkin's formula (10). □

The main point of Proposition 3.1 is to show that, for controllers with certain robustness properties, the sampled-data system with random sampling is exponentially stable with large enough sampling rate, and this is done by using the extended generator for the controlled Markovian process $\big(x(t), e(t)\big)_{t\geqslant 0}$. This result can be generalized in several ways. Instead of requiring quadratic bounds on the function U in (L1), if for some $\underline{\alpha} > 0$, $p \geqslant 1$, $U(x)$ is lower (respectively, upper) bounded by $\underline{\alpha}|x|^p$ (resp. $\overline{\alpha}|x|^p$) for each $x \in \mathbb{R}^n$, then exponential stability in pth mean can be established. Other than the Poisson process, it is also possible to consider a different random process to determine the sampling instants. This of course changes the formula for the extended generator. Another level of generalization arises from introducing a diffusion term in the system dynamics (1), which would require us to work with a weaker notion of a solution, and consequently, the assumptions on function U need to be strengthened to be able to compute the extended generator. Stability analysis using extended generator for impulsive renewal systems with diffusion term in the differential equation has been carried out in [23].

So far, we have adopted a general approach to address the control of sampled-data nonlinear systems. Most of the results in the literature on stabilization with random sampling have been presented in the context of linear systems, and with the exception of [23], extended generators have not appeared elsewhere. We now focus our attention on linear systems: An overview of different approaches is presented and our eventual goal is to establish equivalence between some of these approaches and the extended generator approach for the case of Poisson sampling. In the process, we establish what may be regarded as a converse Lyapunov theorem for (6b)–(7) when the underlying renewal process $\big(N_t\big)_{t\geqslant 0}$ is Poisson with fixed intensity $\lambda > 0$.

4 Randomly Sampled Linear Systems: A Random Walk Down the History Lane

4.1 System Description

In the remainder of this chapter, we will restrict our attention to randomly sampled-data control of linear systems described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \text{ given}, \quad t \geqslant 0, \tag{13}$$

with the input u given by

$$u(t) = Kx(\tau_{N_t}) \quad \text{for all } t \geqslant 0,$$

where the pair (A, B) is assumed to be stabilizable, and the feedback gain K is assumed to be fixed a priori. With $\big(N_t\big)_{t\geqslant 0}$ the sampling process for the above control system, the resulting stochastic system for the joint process $\big(x(t), e(t)\big)_{t \geqslant 0}$ is described by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} A + BK & -BK \\ A + BK & -BK \end{pmatrix} \begin{pmatrix} x(t) \\ e(t) \end{pmatrix} \quad \text{for almost all } t \geqslant 0, \tag{14a}$$

and the reset equation at the sampling times is

$$\begin{pmatrix} x(\tau_{N_t}) \\ e(\tau_{N_t}) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(\tau_{N_t}^-) \\ e(\tau_{N_t}^-) \end{pmatrix}. \tag{14b}$$

For this class of systems, lower bounds on the sampling rates required for stability can be computed more explicitly. Also, this case has been studied in the literature over several epochs, and we provide an overview of the approaches that have been used for analyzing the stability of randomly sampled linear systems. To simplify notation, let us abbreviate the system in (14) as

$$\begin{aligned} \dot{\overline{x}} &= F\overline{x} \\ \overline{x}(\tau_{N_t}) &= G\,\overline{x}(\tau_{N_t}^-) \end{aligned} \tag{15}$$

where $\overline{x} := (x^\top, e^\top)^\top \in \mathbb{R}^{\overline{n}}$ and $\overline{n} = 2d$.

4.2 Early Efforts

It may appear surprising that the investigations into control of linear sampled-data control systems under random sampling started as early as the late 1950s. Indeed, Rudolf Kalman in his Ph.D. dissertation [24] studied sample-and-hold schemes for linear time-invariant control systems under random sampling. In particular, he studied several stochastic stability notions for both linear scalar systems and systems of higher dimensions: the definitions of *stability almost surely*, *stability in the mean*, *stability in mean-square*, and *stability in the mean sampling period* appear in his thesis. It is interesting to note that the key steps in his work were to first understand the asymptotic behaviour of the process $\big(x(\tau_k)\big)_{k \in \mathbb{N}}$ as $k \to +\infty$, and thereafter to derive certain inferences about the continuous-time process $\big(x(t)\big)_{t \geqslant 0}$. Only asymptotically stable system matrices A were considered by Kalman; this peculiar assumption was perhaps a natural consequence of his proof technique. The operator-theoretic approach à la extended generators pioneered by Dynkin [16, 17] was relatively less known at the time of Kalman's graduation.

About a decade later, Oskar Leneman at MIT published a sequence of short articles on control of linear time-invariant sample-and-hold systems under random sampling. Chief among this sequence is [29], where Leneman claimed that certain calculations in [24] did not quite lead to correct results. He focussed attention on scalar problems in [29], and derived his results following the same route as that of

Kalman: first getting estimates of the behavior of the sampled process $\big(x(\tau_k)\big)_{k\in\mathbb{N}}$, followed by inferring stability of the underlying continuous-time process $\big(x(t)\big)_{t\geqslant 0}$ via lengthy calculations involving some integral transform calculus. Once again, only scalar asymptotically stable systems were considered. Related problems of stability of linear control systems under random sample-and-hold schemes were almost concurrently investigated by Harold Kushner and his collaborators [27], and their techniques were also similar to those in [29]. To the best of our knowledge, it seems that this early period focussed attention only on open-loop asymptotically stable systems; even neutrally stable linear systems were perhaps considered too difficult to handle via these techniques. An admittedly speculative reason for this may have been that even for Poisson sampling, it was not clear how to deal with remarkably long holding times (that appear with probability 1) during which the process may deviate very far away from a given compact set since the right-hand side of the x-subsystem of (14) is *affine* in x during the holding times.

To anyone attempting to follow the footprints of Leneman, it is not difficult to appreciate the tediousness of the calculations involved in transitioning from estimates of the behavior of the sampled process. (In fact, [29] skips quite a few details and provides the readers with just the key steps of his proofs.) The first part of deriving estimates for the sampled process $\big(x(\tau_k)\big)_{k\in\mathbb{N}}$ is relatively simple:

Lemma 4.1 *Let $-\infty < t' < t'' < +\infty$. If $A \in \mathbb{R}^{d\times d}$, then*

$$\int_{t'}^{t''} \mathrm{e}^{tA}\,\mathrm{d}t = \frac{\big(\mathrm{e}^{t''A} - \mathrm{e}^{t'A}\big)}{A},$$

where the object on the right-hand side is defined by

$$\frac{\big(\mathrm{e}^{t''A} - \mathrm{e}^{t'A}\big)}{A} := \sum_{k=1}^{+\infty} \frac{(t'')^k - (t')^k}{k!} A^{k-1}.$$

Proof On the one hand, if $A \in \mathbb{R}^{d\times d}$ is non-singular, then (see also [6, p. 47])

$$\begin{aligned}\int_{t'}^{t''} \mathrm{e}^{tA}\,\mathrm{d}t &= \int_{t'}^{t''} \sum_{k=0}^{+\infty} \frac{A^k}{k!} t^k\,\mathrm{d}t = \sum_{k=0}^{+\infty} \frac{A^k}{(k+1)!}\big((t'')^{k+1} - (t')^{k+1}\big)\\ &= A^{-1}(\mathrm{e}^{t''A} - \mathrm{e}^{t'A})\\ &= \sum_{k=1}^{+\infty} \frac{(t'')^k - (t')^k}{k!} A^{k-1} = \frac{\big(\mathrm{e}^{t''A} - \mathrm{e}^{t'A}\big)}{A},\end{aligned}$$

where we have carried out the interchange of the summation and the integral under the shadow of Tonelli's theorem [15, Theorem 4.4.5]. In particular, we observe that the map

$$\mathbb{R}^{d\times d} \ni A \mapsto \frac{\big(\mathrm{e}^{t''A} - \mathrm{e}^{t'A}\big)}{A} \in \mathbb{R}^{d\times d}$$

is continuous. On the other hand, if $A \in \mathbb{R}^{d\times d}$ is singular, we pick a sequence of matrices $\left(A_n\right)_{n\in\mathbb{N}^*}$ with $A_n := A + \varepsilon_n I$ and $\varepsilon_n \downarrow 0$, such that each A_n is nonsingular. (For instance, we employ a similarity transformation to obtain the upper-triangular complex-Jordan form J of A; the eigenvalues of A are on the diagonal of J and since A is singular, there is at least one 0 on the diagonal of J; we pick the sequence $\varepsilon_n \downarrow 0$ such that $J + \varepsilon_n I$ is nonsingular for each n — this is possible since the spectrum of A is a finite set.) Since $A_n \xrightarrow[n\to+\infty]{} A$, we apply the assertion to the nonsingular matrix A_n instead of A, and the general formula follows at once from continuity. □

To simplify some calculations below, we assume that $A \in \mathbb{R}^{d\times d}$ is non-singular. Starting from (13) with a given initial condition $x(0)$, and

$$u(t) = Kx(\tau_i) \quad \text{whenever } t \in [\tau_i, \tau_{i+1}[, \quad i \in \mathbb{N}, \tag{16}$$

we arrive at

$$x(t) = \Big(\mathrm{e}^{(t-\tau_i)A} + \mathrm{e}^{tA}A^{-1}\big(\mathrm{e}^{-\tau_i A} - \mathrm{e}^{-tA}\big)BK\Big)x(\tau_i) \quad \text{for } t \in [\tau_i, \tau_{i+1}[, \tag{17}$$

or equivalently,

$$x(t) = \Big(\mathrm{e}^{(t-\tau_i)A}\big(I + A^{-1}BK\big) - A^{-1}BK\Big)x(\tau_i) \quad \text{for all } t \in [\tau_i, \tau_{i+1}[. \tag{18}$$

By continuity of solutions,

$$x(\tau_{i+1}) = \Big(\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big(I + A^{-1}BK\big) - A^{-1}BK\Big)x(\tau_i),$$

which is a recursive formula for the states at consecutive sampling instants. Multiplying out, for any $N \in \mathbb{N}^*$,

$$x(\tau_N) = \prod_{i=0}^{N-1}\Big(\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big(I + A^{-1}BK\big) - A^{-1}BK\Big)x(0), \tag{19}$$

where we remember that the product is directed.

In the scalar case ($d = 1$), by independence of the holding times,

$$\begin{aligned}\mathsf{E}\left[x(\tau_N)\middle|x(0)\right] &= \mathsf{E}\left[\prod_{i=0}^{N-1}\Big(\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big(1 + A^{-1}BK\big) - A^{-1}BK\Big)x(0)\middle|x(0)\right]\\ &= \prod_{i=0}^{N-1}\mathsf{E}\Big[\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big(1 + A^{-1}BK\big) - A^{-1}BK\Big]x(0)\end{aligned}$$

$$= \prod_{i=0}^{N-1} \Big(\mathsf{E}\big[\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big]\big(1 + A^{-1}BK\big) - A^{-1}BK \Big) x(0).$$

The quantity $\mathsf{E}[\mathrm{e}^{(\tau_{i+1}-\tau_i)A}]$ is simply the moment generating function $\mathcal{M}_S$ (if it exists) of $(\tau_{i+1} - \tau_i)$ evaluated at $A \in \mathbb{R}$, denoted hereafter by $\mathcal{M}_S(A)$.[3] Therefore,

$$\mathsf{E}\left[x(\tau_N)\big|x(0)\right] = \prod_{i=0}^{N-1} \Big(\mathcal{M}_S(A)\big(1 + A^{-1}BK\big) - A^{-1}BK \Big) x(0).$$

For convergence of the product on the right-hand side to 0 as $N \to +\infty$, it is necessary and sufficient that

$$|\mathcal{M}_S(A)(A + BK) - BK| < |A|, \tag{20}$$

from which we can immediately arrive at the range of permissible K's. The question of designing stabilizing feedback gains K is addressed in detail in Sect. 6; Merely assuming that $A + BK = A(1 + A^{-1}BK)$ is Hurwitz stable may not be enough!

Remark 4.2 ($A + BK$ *Hurwitz is necessary for the scalar case*) In the scalar case and an unstable open-loop system (that is, $A > 0$), if we select the feedback gain K such that $A + BK > 0$, then the condition (20) will not be satisfied. Indeed, $\mathcal{M}_S(A) > 1$ for every $A > 0$ whenever the former exists.

The multidimensional case is similar to the scalar one: by independence of the holding times,

$$\begin{aligned} \mathsf{E}\left[x(\tau_N)\big|x(0)\right] &= \mathsf{E}\left[\prod_{i=0}^{N-1} \Big(\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big(I + A^{-1}BK\big) - A^{-1}BK \Big) x(0) \big| x(0) \right] \\ &= \prod_{i=0}^{N-1} \mathsf{E}\Big[\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big(I + A^{-1}BK\big) - A^{-1}BK \Big] x(0) \\ &= \prod_{i=0}^{N-1} \Big(\mathsf{E}\big[\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big]\big(I + A^{-1}BK\big) - A^{-1}BK \Big) x(0). \end{aligned} \tag{21}$$

The matrix $\mathsf{E}\big[\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big]$ is well defined whenever $\mathcal{M}_S(\|A\|) = \mathsf{E}\big[\mathrm{e}^{(\tau_{i+1}-\tau_i)\|A\|}\big]$ exists; this follows from a standard application of the dominated convergence theorem [15, Theorem 4.3.5]. Now, the necessary and sufficient condition for convergence of the product on the right-hand side to 0 as $N \to +\infty$ is that

[3] Recall that the moment generating function $\mathcal{M}_X$, if it exists, of a random variable X is the function $\mathbb{R} \ni \xi \mapsto \mathcal{M}_X(\xi) := \mathsf{E}[\mathrm{e}^{\xi X}] \in \mathbb{R}$. The moment generating function may only be defined on a subset of $\mathbb{R}$, of course.

$$A^{-1}\Big(\mathsf{E}\big[\mathrm{e}^{(\tau_{i+1}-\tau_i)A}\big](A+BK)-BK\Big) \quad \text{is Schur stable.} \tag{22}$$

It is evident that straightforward calculations are enough to arrive at necessary and sufficient conditions for stability in the mean of the sampled process $\big(x(\tau_k)\big)_{k\in\mathbb{N}}$. A similar calculation can be carried out for $\big(\,\|x(\tau_k)\|\,\big)_{k\in\mathbb{N}}$ to arrive at convergence in mean-square of the process $\big(\,\|x(\tau_k)\|\,\big)_{k\in\mathbb{N}}$.

However, the preceding calculations do not shed much light on the inter-sample behavior of $\big(x(t)\big)_{t\geqslant 0}$. The transition from stability of the sampled process to that of $\big(x(t)\big)_{t\geqslant 0}$ is a nontrivial matter. A tiny calculation in this direction is to check whether the process $\big(x(\tau_{N_t})\big)_{t\geqslant 0}$ is stable, and to this end, our assumption (4) provides the necessary support, and one concludes that $\mathsf{E}\left[x(\tau_{N_t})\middle|x(0)\right] \xrightarrow[t\to+\infty]{} 0$. The next natural step is to compute $\mathsf{E}\left[x(t)\middle|x(0)\right]$ for a given time t, and finally to take the limit (if it exists), as $t\to+\infty$. However, at this stage matters start to become rather tedious and complicated. Indeed, if we proceed as Leneman does in [29], for the quadratic function $\mathbb{R}^d \ni x \mapsto \varphi(x) := \frac{1}{2}\langle x, Qx\rangle \in [0,+\infty[$ where $Q\in\mathbb{R}^{d\times d}$ is some symmetric and positive-definite matrix,

$$\begin{aligned}\mathsf{E}\left[\varphi\big(x(t)\big)\middle|x(0)\right] &= \mathsf{E}\left[\varphi\big(x(t)\big)\sum_{k=0}^{+\infty}1_{[\tau_k,\tau_{k+1}[}(t)\middle|x(0)\right]\\ &= \sum_{k=0}^{+\infty}\mathsf{E}\left[\varphi\big(x(t)\big)1_{[\tau_k,\tau_{k+1}[}(t)\middle|x(0)\right]\end{aligned}$$

where the second equality follows by the monotone convergence theorem. Since $1_{[\tau_k,\tau_{k+1}[}(t)=1$ if and only if $N_t=k$ and 0 otherwise, each summand on the right-hand side can be manipulated as

$$\mathsf{E}\left[\varphi\big(x(t)\big)1_{[\tau_k,\tau_{k+1}[}(t)\middle|x(0)\right] = \mathsf{P}\left(N_t=k\middle|x(0)\right)\,\mathsf{E}\left[\varphi\big(x(t)\big)\middle|x(0),\,N_t=k\right].$$

If the sampling process $\big(N_t\big)_{t\geqslant 0}$ is Poisson with intensity λ, we have the expression $\mathsf{P}\left(N_t=k\middle|x(0)\right)=\mathrm{e}^{-\lambda t}\frac{(\lambda t)^k}{k!}$ since the sampling process is independent of the state process, but for more general sampling (renewal) processes, such expressions are difficult to arrive at. Even if $\big(N_t\big)_{t\geqslant 0}$ is Poisson-λ, it is still not simple to compute the second term $\mathsf{E}\left[\varphi\big(x(t)\big)\middle|x(0),\,N_t=k\right]$. Indeed, one would naturally proceed, for the specific case of φ defined above, by employing (19) and then (17) and separating out terms consisting of terms involving $x(\tau_k)$ and $(t-\tau_k)$. The (quadratic) terms consisting only of $x(\tau_k)$ can be dealt with as discussed above, and those containing $(t-\tau_k)$ would need the probability distribution of $(t-\tau_k)$. By all indications, Leneman's calculations (which are not explicitly provided in [29]) completed the preceding steps for the case of $d=1$ and asymptotically stable A. It should be evident that for sampling processes more general than Poisson, this route quickly becomes intractable.

4.3 New Generation, Same Problem

Skipping a few decades, we arrive at [31] which presents stability conditions for several sampling routines, one of which is random sampling. Instead of computing $\mathsf{E}\left[\varphi\big(x(t)\big)\middle|x(0)\right]$ exactly, the authors of [31] obtain an upper bound and provide conditions which make this upper bound converge to zero asymptotically. However, the conditions given in their main result on random sampling [31, Theorem 5] are seen to hold only for open-loop stable systems. To see this, consider the scalar system

$$\dot{x} = ax + u$$

and by choosing $u = \kappa x(\tau_{N_t})$, we consider the system $\dot{x} = F\dot{x}$, where

$$F := \begin{pmatrix} a+\kappa & -\kappa \\ a+\kappa & -\kappa \end{pmatrix}.$$

Employing the transformation $T = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ and defining $\overline{x} = T\overline{z}$, we see that

$$\mathrm{e}^{Ft} = \begin{pmatrix} (1-\kappa/a)\mathrm{e}^{at} + \kappa/a & 0 \\ -(1-\kappa/a)\mathrm{e}^{at} - \kappa/a & 0 \end{pmatrix}.$$

Let

$$\overline{M} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathrm{e}^{Ft} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ((1-\kappa/a)\mathrm{e}^{at} + \kappa/a) & 0 \\ 0 & 0 \end{pmatrix}.$$

According to [31, Theorem 5], the sufficient condition for asymptotic stability in the second moment is

$$\left\| \mathsf{E}\left[\overline{M}^{\top}\overline{M}\right] \right\| < 1.$$

However, for Poisson sampling with intensity λ, it is seen that

$$\begin{aligned} \mathsf{E}\left[((1-\kappa/a)\mathrm{e}^{at} + \kappa/a)^2\right] &= \lambda \int_0^{+\infty} ((1-\kappa/a)\mathrm{e}^{at} + \kappa/a)^2 \mathrm{e}^{-\lambda t}\,\mathrm{d}t \\ &= (1-\kappa/a)^2 \frac{\lambda}{\lambda - 2a} + \frac{\kappa^2}{a^2} + 2(\kappa/a)(1-\kappa/a)\frac{\lambda}{\lambda - a}. \end{aligned}$$

Note that the term on right-hand side is greater than 1 for each $\lambda > 2a$.[4] In fact, it is a decreasing function of λ, and

$$\lim_{\lambda \to +\infty} \mathsf{E}\left[((1-\kappa/a)\mathrm{e}^{at} + \kappa/a)^2\right] = 1.$$

[4]The necessity of the condition $\lambda > 2a$ for scalar linear systems with Poisson sampling is discussed in Sect. 6.2.

This shows that, even in such simple cases, we do not get $\left\| \mathsf{E}\left[\overline{M}^\top \overline{M}\right]\right\| < 1$ for arbitrarily large values of λ. This demonstrates the conservatism in the sufficient condition proposed in [31, Theorem 5], and hence it can be presumed that the problem of computing $\mathsf{E}\left[\varphi\big(x(t)\big)\middle|x(0)\right]$ did not get a positive response until the first decade of this century. One positive response to this question has been provided in [4], which we treat in greater detail in the next section. The authors of [4] provide necessary and sufficient conditions for mean-square stability of linear systems under random sampling for a rather general class of random processes. We examine closely and comment on their main result in Sect. 5.1. The techniques involved in [4] are quite different from the ones that are mainstream.

Before moving on, we mention a couple of additional references dealing with random sampling. The article [3] deals with control under random sampling: An optimal control problem with a quadratic instantaneous cost for linear controlled diffusions was studied in this particular work, but under the assumption that there are only finitely many sampling instants. The authors of the recent article [49] also limited their scope to a Lyapunov stable matrix A.

The preceding efforts involve hands-on calculations that are specific to linear system models and/or specific (and simple) sampling processes, with the exception of [4]. The connection between PDMPs and sampled-data control under random sampling discussed in Remark 2.1 immediately opens up the possibility of employing generator-based ideas in this context; our agenda for the next section will focus on this connection closely. In particular, we shall demonstrate in Sect. 5.2 that the main results of [4] can also be derived by employing the extended generator (8).

5 Equivalence of Different Stability Conditions for Linear PDMPs

Turning our attention to (15), and looking at this joint system with state $\overline{x} = (x^\top, e^\top)^\top$, it is possible to find necessary and sufficient conditions for asymptotic stability in second moment by computing $\mathsf{E}\left[\psi\big(\overline{x}(t)\big)\middle|\overline{x}(0)\right]$ for system (15), with ψ quadratic in $\overline{x}$. This is done in an explicit manner in [4], where the authors use the recursive Volterra integral equation to compute $\mathsf{E}\left[\|\overline{x}(t)\|^2 \middle|\overline{x}(0)\right]$. Another tool for analyzing the stability in second moment for system (15) was already revealed in Sect. 3 in the form of extended generator. After providing a quick overview of how $\mathsf{E}\left[\|\overline{x}(t)\|^2 \middle|\overline{x}(0)\right]$ is computed, we show the equivalence between the two approaches, which essentially establishes a converse Lyapunov theorem for (15) with Poisson renewal process.

5.1 Volterra Integral Approach

To analyze stability in second moment for system (15), it is observed that we can write [4, Proposition 6]

$$\mathsf{E}\left[\overline{x}^\top(t)\overline{Q}\overline{x}(t)\right] = \overline{x}_0^\top W(t)\overline{x}_0 \tag{23}$$

where the matrix-valued function $W : [0, +\infty[\to \mathbb{R}^{\overline{n}\times\overline{n}}$ satisfies the Volterra integral equation

$$W(t) = \mathcal{K}(W)(t) + H(t), \tag{24}$$

with $H(t) = \mathrm{e}^{F^\top t}\overline{Q}\mathrm{e}^{Ft}\mathrm{e}^{-\lambda t}$ for some positive-definite and symmetric $\overline{Q} \in \mathbb{R}^{\overline{n}\times\overline{n}}$, and $\lambda > 0$ being the intensity of the Poisson sampling process $\left(N_t\right)_{t\geqslant 0}$ so that the jump times τ_{N_t} in (14) have the property that $\left(\tau_{N_t} - \tau_{N_t-1}\right) \sim \mathrm{Exp}(\lambda)$. In (24), the operator $\mathcal{K} : \mathcal{C}^1(\mathbb{R}_{\geqslant 0}; \mathbb{R}^{\overline{n}\times\overline{n}}) \to \mathcal{C}^1(\mathbb{R}_{\geqslant 0}; \mathbb{R}^{\overline{n}\times\overline{n}})$ is given by

$$\mathcal{K}(W)(t) := \lambda \int_0^t \mathrm{e}^{F^\top s} G^\top W(t-s) G \mathrm{e}^{Fs}\mathrm{e}^{-\lambda s}\, \mathrm{d}s. \tag{25}$$

Due to (23), stability of (14) can be formulated in terms of the asymptotic properties of the matrix-valued function $W(t)$. In [4, Theorem 3], depending upon the stability notion under consideration, several conditions are provided which are equivalent to convergence of W in appropriate norms. For example, conditions for stochastic stability are equivalent to absolute convergence of $\int_0^{+\infty} W(s)\, ds$, and the conditions given for mean-square stability are equivalent to $W(t) \to 0$.

5.2 Connections Between the Extended Generator and Volterra Integral Techniques

In Sect. 3, we used the extended generator to obtain sufficient conditions for stability of nonlinear PDMPs. In case of linear systems (14), the same approach can be adopted while restricting attention to quadratic test functions. Since we now have a characterization of stability in terms of the function W given in (24), it is natural to ask whether we can establish necessary conditions for stability in second moment using the extended generator. To show that these approaches are equivalent for linear dynamics (15) and Poisson renewal processes, we have the following result.

Theorem 5.1 *Consider system (14) with $(N_t)_{t\geqslant 0}$ a Poisson process of intensity $\lambda > 0$. The following statements are equivalent:*

(S1) *System (14) is exponentially stable in second moment.*

(S2) *There exists a symmetric positive-definite matrix* $\overline{Q} \in \mathbb{R}^{\overline{n} \times \overline{n}}$ *such that the matrix-valued function* W *satisfying* (24)*, with* $H(t) = \mathrm{e}^{F^\top t}\overline{Q}\mathrm{e}^{Ft}\mathrm{e}^{-\lambda t}$*, converges to zero exponentially as* $t \to +\infty$*.*

(S3) *There exists a symmetric, positive- definite matrix* $\overline{P} \in \mathbb{R}^{\overline{n} \times \overline{n}}$ *such that*

$$F^\top \overline{P} + \overline{P}\,F + \lambda(G^\top \overline{P}\,G - \overline{P}) < 0. \tag{26}$$

If we let $\psi(\overline{x}) := \overline{x}^\top \overline{P}\overline{x}$, then using the expression for $\mathcal{L}\psi(x, e)$ in (9), the inequality (26) is equivalently written as $\mathcal{L}\psi(x, e) < 0$, for each $(x, e) \in \mathbb{R}^{d \times d}$. A condition similar to (26) has also appeared in [5, Theorem 7]. Note that the result of Theorem 5.1 is of independent interest as it proves a converse Lyapunov theorem for a class of linear PDMPs which are exponentially stable in second moment. Establishing converse Lyapunov theorems for stochastic hybrid systems, in general, was identified as an open problem in [44, Section 8.4, Open Problem 4], and Theorem 5.1 provides a result in this direction for a particular class of stochastic hybrid systems. The nontrivial aspect of the proof of Theorem 5.1 relies on constructing $\overline{P}$ using the expression for W in (24).

Proof The equivalence between (S1) and (S2) follows directly from (23), where the latter is derived in [4, Proposition 6]. In the sequel, we prove the equivalence between (S2) and (S3), and for our purposes it is useful to recall that, using the properties of Volterra integral equation, W can be explicitly described by the expression

$$W(t) := \sum_{j=1}^{+\infty} \mathcal{K}^j(H)(t) + H(t). \tag{27}$$

Now, let us assume that (S3) holds, and from there we show that there is a matrix $\overline{Q}$ such that W satisfying (24), with $H(t) = \mathrm{e}^{F^\top t}\overline{Q}\mathrm{e}^{Ft}\mathrm{e}^{-\lambda t}$, converges to zero as t goes to infinity. Let $\overline{P}$ be the symmetric, positive-definite matrix satisfying (26), so there exists $\alpha > 0$ such that

$$F^\top \overline{P} + \overline{P}F + \lambda(G^\top \overline{P}G - \overline{P}) + \alpha\overline{P} < 0.$$

Take $\overline{Q} = \overline{P}$. Multiplying the last inequality by $\mathrm{e}^{F^\top t}$ from left, e^{Ft} from right, and the scalar $\mathrm{e}^{-\lambda t}$, we get

$$F^\top H + HF + \lambda(J - H) < -\alpha H \tag{28}$$

where we recall that $H(t) = \mathrm{e}^{F^\top t}\overline{Q}\mathrm{e}^{Ft}\mathrm{e}^{-\lambda t}$, and

$$J(t) := \mathrm{e}^{F^\top t} G^\top \overline{Q} G \mathrm{e}^{Ft}\mathrm{e}^{-\lambda t}.$$

With this choice of $\overline{Q}$ and H, let W be the function obtained from solving (27). To see that W converges to zero exponentially, we need the following lemma:

Lemma 5.2 *For the continuously differentiable matrix-valued function W given in (27), it holds that*

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \sum_{j=1}^{+\infty}\mathcal{K}^j(F^\top H + HF - \lambda H)(t) + \sum_{j=0}^{+\infty}\lambda\mathcal{K}^j(J)(t) \\ + F^\top H(t) + H(t)F - \lambda H(t). \qquad (29)$$

The proof of this lemma is given in Appendix B. Combined with (28), and using the expression for W in (27), this lemma immediately yields

$$\dot{W}(t) \le -\alpha W(t)$$

from which the exponential convergence of W follows.

Next, we show that (S2) implies the existence of matrix $\overline{P}$ such that (S3) holds. For this implication to hold, the important relation that we need to develop is

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = F^\top W(t) + W(t)F - \lambda W(t) + \lambda G^\top W(t)G, \quad t \geqslant 0. \qquad (30)$$

Indeed, if (30) holds, then by letting,

$$\overline{P} := \lim_{t\to+\infty}\int_0^t W(s)\,\mathrm{d}s,$$

it is seen that

$$\begin{aligned} F^\top\overline{P} + \overline{P}F + \lambda(G^\top\overline{P}G - \overline{P}) &= \lim_{t\to+\infty}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}W(s)\,\mathrm{d}s, \\ &= \lim_{t\to+\infty} W(t) - W(0) \\ &= -\overline{Q} \end{aligned}$$

where we used the fact that $\lim_{t\to+\infty} W(t) = 0$ because of (S2). The limit in the definition of the matrix $\overline{P}$ is well-defined because W converges to zero exponentially. The matrix $\overline{P}$ is also seen to be symmetric and positive definite. To show this, we first observe from (27) that, for each $s \geqslant 0$, $W(s)$ is symmetric and $W(s) \geqslant H(s)$. Suppose, ad absurdum, that $\overline{P}$ is not positive definite; then, there exists $\overline{x} \in \mathbb{R}^{\overline{n}}$, $\overline{x} \neq 0$, such that

$$\begin{aligned} 0 = \overline{x}^\top\overline{P}\overline{x} &= \lim_{t\to+\infty}\int_0^t \overline{x}^\top W(s)\overline{x}\,\mathrm{d}s \\ &\geqslant \lim_{t\to+\infty}\int_0^t \overline{x}^\top H(s)\overline{x}\,\mathrm{d}s = \lim_{t\to+\infty}\int_0^t \overline{x}^\top \mathrm{e}^{sF^\top}\overline{Q}\mathrm{e}^{sF}\mathrm{e}^{-\lambda s}\overline{x}\,\mathrm{d}s. \end{aligned}$$

Since $\overline{Q}$ is positive definite, the last inequality suggests that $\mathrm{e}^{sF}\overline{x} = 0$ for every $s \geqslant 0$, and hence $\overline{x} = 0$; a contradiction.

So, the focus in the remainder of the proof is on proving (30). We already have an expression for $\frac{\mathrm{d}}{\mathrm{d}t}W$ in Lemma 5.2. To simplify the terms on the right-hand side of (29), we introduce the following lemma:

Lemma 5.3 *For each $j \geqslant 1$, we have*

$$\mathcal{K}^j(F^\top H + HF - \lambda H) + \lambda\mathcal{K}^{j-1}(J)(t) = \lambda G^\top \mathcal{K}^{j-1}(H)(t)G + F^\top \mathcal{K}^j(H)(t) + \mathcal{K}^j(H)(t)F - \lambda\mathcal{K}^j(H)(t). \tag{31}$$

Again, the proof of this lemma is provided in Appendix B. Combining the statements of Lemmas 5.2 and 5.3, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \sum_{j=1}^{+\infty} \lambda G^\top \mathcal{K}^{j-1}(H)(t)G + F^\top \mathcal{K}^j(H)(t) + \mathcal{K}^j(H)(t)F - \lambda\mathcal{K}^j(H)(t) + F^\top H(t) + H(t)F - \lambda H(t). \tag{32}$$

On the other hand, it follows from the expression for W in (27) that

$$F^\top W(t) + W(t)F - \lambda W(t) = \sum_{j=1}^{+\infty} F^\top \mathcal{K}^j(H)(t) + \mathcal{K}^j(H)(t)F - \lambda\mathcal{K}^j(H)(t) + F^\top H(t) + H(t)F - \lambda H(t). \tag{33}$$

Substituting (33) in (32), and using the notation $\mathcal{K}^0$ to denote the identity operator, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = F^\top W(t) + W(t)F - \lambda W(t) + \lambda G^\top \left(\sum_{j=1}^{+\infty} \mathcal{K}^{j-1}(H)(t)\right) G.$$

The desired Eq. (30) now follows by recalling the definition of W from (27). □

5.3 *Exponential Stability Under Random Sampling*

Now that we have established the necessary and sufficient conditions for stability of the randomly sampled-data system (14) in Theorem 5.1, we can obtain refined estimates on the mean sampling rate λ for stability in second moment to solve Problem 1. We will only work out the estimates that can be obtained from the statement

(S3). A direct way to obtain a lower bound on the mean sampling rate is by solving the inequality (26) in λ and $\overline{P}$, for a given $K \in \mathbb{R}^{m\times d}$. But, since (26) is a bilinear matrix inequality, and hence nonconvex, it is difficult to obtain analytical bounds on λ for feasibility. To overcome this issue, we choose to work with a block diagonal $\overline{P}$ and proceed with computing the lower bounds on λ analytically with such $\overline{P}$. We fix K to be any matrix which makes $A + BK$ Hurwitz, and with this assumption, we show that by choosing λ large enough as a function of the matrices A, B, K, the resulting system is asymptotically stable in second moment.

Theorem 5.4 *Consider the system (14), with $(N_t)_{t\geqslant 0}$ a Poisson process of intensity λ. Assume that there exist $\alpha > 0$, a matrix $K \in \mathbb{R}^{d\times m}$ and a symmetric positive-definite matrix $P \in \mathbb{R}^{d\times d}$ satisfying*

$$(A + BK)^\top P + P(A + BK) \leq -\alpha P. \tag{34}$$

For $\mathbb{R}^d \ni y \mapsto V(y) := \langle y, Py\rangle$, there exist constants C_0, C_1, such that

$$\begin{aligned}&\text{for every } \rho \in \,]0, \alpha[, \text{ for every } \lambda > \rho + C_0 + \frac{C_1}{(\alpha - \rho)},\\ &\text{for every } x(0) \in \mathbb{R}^d, \text{ and for every } t \geqslant 0\end{aligned} \tag{35}$$

we have

$$\mathsf{E}\left[V(x(t))\middle|x(0)\right] \leq V(x(0)) \exp(-\rho t). \tag{36}$$

In particular, for all $\lambda > 0$ sufficiently large, the closed-loop system (14) is globally exponentially stable in the second moment.

Remark 5.5 It is seen from the statement of the theorem that, even if we choose the decay rate ρ to be close to α, it is possible to achieve it by choosing the sampling rate λ to be sufficiently large. In other words, with faster sampling rates, we approach the performance of the continuous-time system.

Remark 5.6 In the proof of Theorem 5.4, we compute the constants C_0 and C_1 in (35) as functions of the matrices A, B, K and P satisfying (34). By letting $\widetilde{Y} := P^{1/2}BKP^{-1/2}$, and $\widetilde{A} := P^{1/2}AP^{-1/2}$, it turns out that we can choose

$$C_0 := \sigma_{\max}\big(-\widetilde{Y} - \widetilde{Y}^\top\big) \quad \text{and} \tag{37a}$$

$$C_1 := \sigma_{\max}\Big(\big(\widetilde{Y}^\top - \widetilde{Y} - \widetilde{A}\big)\big(\widetilde{Y} - \widetilde{Y}^\top - \widetilde{A}^\top\big)\Big), \tag{37b}$$

where, for a given matrix M, $\sigma_{\max}(M)$ denotes the maximum eigenvalue of a matrix M. In fact, it is possible to show that the claim of Theorem 5.4 holds whenever

$$\lambda - \rho > \sigma_{\max}\Big(\frac{1}{\alpha - \rho}\big(\widetilde{Y}^\top - \widetilde{Y} - \widetilde{A}\big)\big(\widetilde{Y} - \widetilde{Y}^\top - \widetilde{A}^\top\big) - \widetilde{Y} - \widetilde{Y}^\top\Big).$$

Corollary 5.7 *Let $K = -R^{-1}B^\top P$, where R and P are symmetric positive-definite matrices which satisfy, for some $\alpha > 0$, the relation*

$$\left(A + \frac{\alpha}{2}I\right)^\top P + P\left(A + \frac{\alpha}{2}I\right) - 2PBR^{-1}B^\top P \leq 0. \tag{38}$$

For each $\rho \in]0, \alpha[$, if λ satisfies (35) with

$$C_0 := 2\sigma_{\max}\big(P^{1/2}BR^{-1}B^\top P^{1/2}\big) \quad \textit{and}$$
$$C_1 := \sigma_{\max}\big(P^{1/2}AP^{-1}A^\top P^{1/2}\big),$$

then (36) holds.

The bounds in Corollary 5.7 are obtained by observing that the choice of $K = -R^{-1}B^\top P$ leads to $\widetilde{Y} = \widetilde{Y}^\top$, which simplifies the expression for C_0 and C_1 to some extent.

Proof of Theorem 5.4 We choose a quadratic function $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_{\geqslant 0}$ of the form

$$(x, e) \mapsto \psi(x, e) := \langle x, P_x x\rangle + \langle e, P_e e\rangle\,, \tag{39}$$

where P_x and P_e are symmetric positive-definite matrices. Using (9) from Proposition 2.2, we obtain

$$\begin{aligned}
\mathcal{L}\psi(x, e) &= \big\langle (P_x + P_x^\top)x + (P_e + P_e^\top)e, (A + BK)x - BKe\big\rangle \\
&\quad + \lambda\left(\langle x, P_x x\rangle - \langle e, P_e e\rangle - \langle x, P_x x\rangle\right) \\
&= \big\langle (P_e + P_e^\top)e, (A + BK)x - BKe\big\rangle \\
&\qquad + \big\langle (P_x + P_x^\top)x, (A + BK)x - BKe\big\rangle - \lambda\langle e, P_e e\rangle \\
&= -\lambda\langle e, P_e e\rangle + \big\langle x, P_x(A + BK)x + (A + BK)^\top P_x x\big\rangle \\
&\quad - \big\langle e, (P_e BK + K^\top B^\top P_e)e\big\rangle + 2\langle e, P_e(A + BK)x\rangle \\
&\quad - 2\langle x, P_x BKe\rangle\,.
\end{aligned}$$

Letting $P_x = P_e = P$ and $A_K := A + BK$, we get

$$\begin{aligned}
&\mathcal{L}\psi(x, e) \\
&= -\left\langle \begin{pmatrix} x \\ e \end{pmatrix}, \begin{pmatrix} -PA_K - A_K^\top P & PBK - A_K^\top P \\ -PA_K + K^\top B^\top P & \lambda P + PBK + K^\top B^\top P \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} \right\rangle \\
&\leq -\left\langle \begin{pmatrix} x \\ e \end{pmatrix}, \underbrace{\begin{pmatrix} \alpha P & PBK - A_K^\top P \\ -PA_K + K^\top B^\top P & \lambda P + PBK + K^\top B^\top P \end{pmatrix}}_{=:M(\lambda)} \begin{pmatrix} x \\ e \end{pmatrix} \right\rangle.
\end{aligned}$$

We next analyze the matrix $M(\lambda)$ and show that for λ large enough, $M(\lambda)$ is positive definite and see how the minimum eigenvalue of $M(\lambda)$ varies with λ. We first write

$M(\lambda)$ as

$$M(\lambda) := M_0 + M_1(\lambda)$$

where for a fixed $\rho \in]0, \alpha[$,

$$M_0 := \begin{pmatrix} \rho P & 0 \\ 0 & \rho P \end{pmatrix} \tag{40}$$

and

$$M_1(\lambda) := \begin{pmatrix} (\alpha-\rho)P & PBK - A_K^\top P \\ -PA_K + K^\top B^\top P & (\lambda-\rho)P + PBK + K^\top B^\top P \end{pmatrix}.$$

Using Schur complements [48, §7.4] and introducing the notation $Y := PBK$ it is seen that

$$\begin{aligned} & M_1(\lambda) \geqslant 0 \\ \Leftrightarrow\ & (\lambda-\rho)P + Y + Y^\top \geqslant \frac{\big(Y^\top - Y - PA\big)P^{-1}\big(Y - Y^\top - A^\top P\big)}{\alpha-\rho}. \end{aligned}$$

Let $P^{1/2}$ denote the positive square root of P. Also, let $\widetilde{Y} := P^{1/2}BKP^{-1/2}$, and $\widetilde{A} := P^{1/2}AP^{-1/2}$. Then, conjugation by $P^{-1/2}$ yields

$$\begin{aligned} & M_1(\lambda) \geqslant 0 \\ \Leftrightarrow\ & (\lambda-\rho)I + \widetilde{Y} + \widetilde{Y}^\top \geqslant \frac{1}{\alpha-\rho}\big(\widetilde{Y}^\top - \widetilde{Y} - \widetilde{A}\big)\big(\widetilde{Y} - \widetilde{Y}^\top - \widetilde{A}^\top\big). \end{aligned}$$

Using Weyl's inequality [21, Theorem 4.3.1], we obtain

$$\begin{aligned} & \sigma_{\max}\left(\frac{1}{\alpha-\rho}\big(\widetilde{Y}^\top - \widetilde{Y} - \widetilde{A}\big)\big(\widetilde{Y} - \widetilde{Y}^\top - \widetilde{A}^\top\big) - (\widetilde{Y} + \widetilde{Y}^\top)\right) \\ & \leq \sigma_{\max}\left(\frac{1}{\alpha-\rho}\big(\widetilde{Y}^\top - \widetilde{Y} - \widetilde{A}\big)\big(\widetilde{Y} - \widetilde{Y}^\top - \widetilde{A}^\top\big)\right) + \sigma_{\max}(-\widetilde{Y} - \widetilde{Y}^\top) \\ & = \frac{1}{\alpha-\rho}\sigma_{\max}\left(\big(\widetilde{Y}^\top - \widetilde{Y} - \widetilde{A}\big)\big(\widetilde{Y} - \widetilde{Y}^\top - \widetilde{A}^\top\big)\right) + \sigma_{\max}(-\widetilde{Y} - \widetilde{Y}^\top) \\ & =: \frac{1}{\alpha-\rho}C_1 + C_0 \end{aligned}$$

where we introduced the constants C_0, C_1 given in (37). It is now observed that $M_1 \geqslant 0$ for each $\lambda > \rho + C_0 + C_1/(\alpha-\rho)$, and hence

$$\mathcal{L}\psi(x,e) \leq -\left\langle \begin{pmatrix} x \\ e \end{pmatrix}, M_0 \begin{pmatrix} x \\ e \end{pmatrix} \right\rangle = -\rho\,\psi(x,e).$$

The assertion of Theorem 5.4 follows. □

It must be noted that the condition (35) is only sufficient for stability in second moment because in the notation of (S3) of Theorem 5.1, the proof was worked out by choosing $\overline{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$. This choice indeed makes our estimates of λ conservative. In the next section, we study stability of closed-loop systems for smaller values of λ by addressing the converse question of designing static feedbacks for linear systems.

6 Converse Question and Feedback Design

In contrast to finding lower bounds on the sampling rate for a given feedback law in previous sections, we are now interested in designing the feedback laws for a fixed sampling rate. The problem of interest is thus formalized as follows:

Problem 2 Consider the system (14), with $\left(N_t\right)_{t \geqslant 0}$ a Poisson process of intensity λ. If $\lambda > 0$ is given, does there exist a matrix $K \in \mathbb{R}^{m \times d}$ such that (14) is globally exponentially stable in second moment?

Preparatory to addressing this problem, we first observe that the search space for the feedback gain K is constrained by the sampling rate even in the setting of deterministic sampling—see Sect. 6.1 for the relevant discussion. Moreover, in the setting of Poisson sampling, there is a lower bound on the sampling rate that must be satisfied for the expectation to be well defined; see Sect. 6.2 for the corresponding details. These two observations are then employed to provide a partial answer to Problem 2.

6.1 Using the Scalar Deterministic Case as a Guideline

Before addressing this question with random sampling, let us have a quick look at the deterministic sampling case and observe how one would choose a feedback gain in that case. Consider the scalar system

$$\dot{x}(t) = ax(t) + u(t), \quad t \geqslant 0,$$

with a given $a > 0$. Our objective is to asymptotically stabilize this system at the origin, and the state measurements are available only periodically at $(\tau_i)_{i \in \mathbb{N}^*} \subset [0, +\infty[$, where $\tau_{i+1} - \tau_i = T$ for some fixed $T > 0$; in other words, $\tau_n = nT$. We aim to design a controller $u(t) = \kappa x(\tau_{N_t})$, with an appropriately chosen κ depending on the sampling period T. Elementary calculations yield

$$x(\tau_{i+1}) = \exp(aT)x(\tau_i) + \int_{\tau_i}^{\tau_{i+1}} \exp(a(\tau_{i+1} - s))\kappa x(\tau_i)\,\mathrm{d}s$$
$$= \Big(\exp(aT) + \frac{\kappa}{a}\big(\exp(aT) - 1\big)\Big)x(\tau_i),$$

and for a fixed sampling period $T > 0$, the closed-loop system is asymptotically stable if and only if the sequence $(x(\tau_n))_{n\in\mathbb{N}^*}$ converges to 0. The latter holds if and only if

$$\left|\exp(aT) + \frac{\kappa}{a}\big(\exp(aT) - 1\big)\right| < 1,$$

or equivalently, if and only if

$$-a\Big(\frac{\exp(aT) + 1}{\exp(aT) - 1}\Big) < \kappa < -a.$$

We observe two key facts:

- The inequality $\kappa < -a$ is necessary for the stability of the continuous-time system. The other inequality gives a lower bound on the value of κ, and shows that for a fixed sampling rate, one can *not* choose $|\kappa|$ to be very large.
- On the one hand, as T goes to zero (the case of fast sampling), this lower bound goes to $-\infty$. On the other hand, as T grows large (the case of slow sampling), this lower bound approaches $-a$ from below, and the admissible set of the stabilizing gain κ becomes smaller.

In dimensions larger than 1, the problem of selecting a suitable control gain K becomes more delicate, as we shall momentarily see.

6.2 Necessary Lower Bounds for the Sampling Rate

We turn our attention back to the system

$$\dot{x}(t) = Ax(t) + BKx(\tau_{N_t}), \qquad x(0) \text{ given}, \quad t \geqslant 0, \tag{41}$$

where we recall that $\big(N_t\big)_{t\geqslant 0}$ defined in (3) is a Poisson process of intensity λ which determines the sampling times. We assume for the sake of simplicity that A is in its complex-Jordan normal form and that it is non-singular. It can be easily verified that, for each sample path, and $i \in \mathbb{N}^*$, we have

$$x(\tau_{i+1}) = A^{-1}\Big(\mathrm{e}^{A(\tau_{i+1}-\tau_i)}\big(A + BK\big) - BK\Big)x(\tau_i). \tag{42}$$

If the linear system (41) is exponentially stable in the second moment, then the discrete-time system (42) must also be exponentially stable in the second moment,[5] and therefore, there exist [34, Theorem 9.4.2] a symmetric positive definite matrix $P_{\rm d} \in \mathbb{R}^{d\times d}$ and $\gamma \in [0, 1[$ such that for each $i \in \mathbb{N}^*$,

$$\mathsf{E}\left[\langle x(\tau_{i+1}), P_{\rm d} x(\tau_{i+1})\rangle \,\middle|\, x(\tau_i)\right] \leq \gamma \mathsf{E}\left[\langle x(\tau_i), P_{\rm d} x(\tau_i)\rangle \,\middle|\, x(\tau_i)\right].$$

With $A_K := (A + BK)$ and $\tilde{P}_{\rm d} := A^{-1} P_{\rm d} A^{-1}$, time-invariance of the data leads to

$$A_K^\top \mathsf{E}\left[\mathrm{e}^{SA^\top} \tilde{P}_{\rm d} \mathrm{e}^{SA}\right] A_K - A_K^\top \mathsf{E}\left[\mathrm{e}^{SA^\top}\right] \tilde{P}_{\rm d} BK \\ - (BK)^\top \tilde{P}_{\rm d} \mathsf{E}\left[\mathrm{e}^{SA}\right] A_K + (BK)^\top \tilde{P}_{\rm d} BK \leq \gamma A \tilde{P}_{\rm d} A,$$

where S is an exponential random variable with parameter λ. The matrix on the left-hand side is well defined if and only if $\mathsf{E}\left[\mathrm{e}^{SA^\top} \tilde{P}_{\rm d} \mathrm{e}^{SA}\right]$ and $\mathsf{E}\left[\mathrm{e}^{SA}\right]$ are well-defined.

The (j, k)th entry of the matrix $\mathsf{E}\left[\mathrm{e}^{SA^\top} \tilde{P}_{\rm d} \mathrm{e}^{SA}\right]$ is

$$\mathsf{E}\left[\sum_{\ell=1}^{d} \sum_{m=1}^{d} (\mathrm{e}^{SA^\top})_{j\ell} \big(\tilde{P}_{\rm d}\big)_{\ell m} (\mathrm{e}^{SA})_{mk}\right].$$

Since e^{SA} is in the block-diagonal form with the eigenvalues of A on the diagonal, this expectation is of the form $\mathsf{E}\left[p_{jk}(S)\mathrm{e}^{S(\sigma_j+\sigma_k)}\right]$ for $1 \leq j, k \leq d$, where σ_j, σ_k are the jth and kth diagonal entries (eigenvalues) of A, and $p_{jk}(\cdot)$ is a polynomial of degree at most $2d$. This expectation is finite only if $\lambda > \Re\sigma_j + \Re\sigma_k$, and therefore, $\mathsf{E}\left[\mathrm{e}^{SA^\top} \tilde{P}_{\rm d} \mathrm{e}^{SA}\right]$ is well-defined whenever $\lambda > 2\max\{\Re\sigma_j(A) \mid j = 1, \ldots, d\}$. Similarly, $\mathsf{E}\left[\mathrm{e}^{SA}\right]$ is well-defined only for $\lambda > \max\{\Re\sigma_j(A) \mid j = 1, \ldots, d\}$.

We conclude from this discussion that

$$\lambda > 2\max\{\Re\sigma_j(A) \mid j = 1, \ldots, d\}$$

is a necessary condition for asymptotic stability in the second moment of the sampled process $\big(x(\tau_n)\big)_{n\in\mathbb{N}^*}$, and seek to resolve the following conjecture:

Conjecture 6.1 *Consider the system (41), where $\big(N_t\big)_{t\geqslant 0}$ is a Poisson process of given intensity $\lambda > 0$. For each $\lambda > 2\max\{\Re\sigma_j(A) \mid j = 1, \ldots, d\}$, there exists a feedback matrix $K \in \mathbb{R}^{m\times d}$ such that (41) is globally asymptotically stable in the second moment.*

[5] The definition of exponential stability in the second moment for the discrete-time case is analogous to the continuous-time version that we have quoted above.

6.3 The Scalar Case with Poisson Sampling

We proceed to verify that the Conjecture 6.1 holds in the scalar case.

Proposition 6.2 *Conjecture 6.1 holds when the system dimension* $d = 1$.

Proof Without loss of generality, we look at the scalar plant

$$\dot{x}(t) = ax(t) + u(t)$$

with $a > 0$ and are interested in choosing the scalar feedback gain κ such that $u(t) = \kappa x(\tau_{N_t})$, $t \geqslant 0$, results in mean-squared asymptotic stability. Recalling that $e(t) = x(t) - x(\tau_{N_t})$ for $t \geqslant 0$, we pick

$$\psi(x, e) := px^2 + e^2$$

for some $p > 0$ to be specified later. Using (9), we get

$$\mathcal{L}\psi(x, e) = -\left\langle \begin{pmatrix} x \\ e \end{pmatrix}, \underbrace{\begin{pmatrix} -2(a+\kappa)p & p\kappa - (a+\kappa) \\ p\kappa - (a+\kappa) & \lambda + 2\kappa \end{pmatrix}}_{=:M} \begin{pmatrix} x \\ e \end{pmatrix} \right\rangle.$$

If we show that there exist $p > 0$ and $\kappa < 0$ such that M is positive definite, our proof will be complete. Toward this end, we first look at the determinant of M:

$$\begin{aligned} \det(M) &= -2p(a+\kappa)(\lambda+2\kappa) - (a+\kappa)^2 - p^2\kappa^2 + 2p\kappa(a+\kappa) \\ &= -(p+1)^2(a+\kappa)^2 + 2p(a+\kappa)(ap+a-\lambda) - a^2p^2. \end{aligned}$$

Defining $\theta := -(a+\kappa)$, we observe that $\det(M) > 0$ if and only if

$$(p+1)^2\theta^2 - 2p\theta(ap+a-\lambda) + a^2p^2 < 0.$$

The left-hand side of the inequality is a convex function of θ, and it attains its global minimum at

$$\theta^* = \frac{p(\lambda - a(1+p))}{(p+1)^2}.$$

It is then readily verified that the value of $\det(M)$ with $\theta = \theta^*$ is

$$\det(M_{\theta=\theta^*}) = \frac{p^2(\lambda - a(p+1))^2}{(p+1)^2} - a^2p^2,$$

so that $\det(M_{\theta=\theta^*}) > 0$ whenever

$$0 < p < \frac{\delta}{2a}, \quad \text{where } \delta := \lambda - 2a. \tag{43}$$

Fixing $\theta = \theta^*$ and letting p satisfy (43), we next look at the trace of M:

$$\begin{aligned}\text{trace}(M_{\theta=\theta^*}) &= \lambda - 2a + 2\theta^*(p-1) \\ &= \delta + 2p\frac{\lambda - a(p+1)}{(p+1)^2}(p-1).\end{aligned}$$

Since $\text{trace}(M_{\theta=\theta^*})$ is a continuous function of p and $\text{trace}(M) = \delta > 0$ when $p = 0$, it follows that for $p > 0$ sufficiently small, it is possible to make both $\text{trace}(M)$ and $\det(M)$ strictly positive. The resulting feedback law is

$$\kappa = -a - \frac{p(2a - \delta - a(1+p))}{(p+1)^2},$$

with $p > 0$ chosen such that $\text{trace}(M_{\theta=\theta^*}) > 0$. The proof is complete. □

Remark 6.3 In the proof of Proposition 6.2 we selected the function ψ from (39) with $P_x = p$ and $P_e = 1$. An interesting observation is that if we select $P_x = P_e$ (as we did in the proof of Theorem 5.4), and λ is fixed, it is not possible to choose a feedback gain K such that $\mathcal{L}\psi(x, e) < 0$. To see this, we observe again in the scalar case that by letting $p_x = p_e = p$,

$$\mathcal{L}\psi(x, e) = -\left\langle \begin{pmatrix} x \\ e \end{pmatrix}, \begin{pmatrix} -2(a+\kappa)p & -ap \\ -ap & (\lambda + 2\kappa)p \end{pmatrix} \begin{pmatrix} x \\ e \end{pmatrix} \right\rangle.$$

We can choose $\kappa < -a$ so that both the diagonal terms of the matrix become negative, and by looking at the determinant of the matrix, it is seen that $\mathcal{L}\psi(x, e) < 0$, if and only if,

$$2\theta(\lambda - 2a - 2\theta) > a^2,$$

where $\theta = -(a + \kappa) > 0$. For a given value of a, one can find $\lambda > 2a$, such that the foregoing inequality is infeasible, regardless of the values of θ, or κ.

6.4 The Multidimensional Case

We employ the guidelines from the previous subsections to address Conjecture 6.1 for systems with dimension greater than 1. As already mentioned, our results here are not quite complete, and we require an additional assumption on the class of linear control systems:

Assumption 1 The matrix pair (A, B) is such that, there exist positive-definite matrices R and P, which solve the algebraic Riccati equation

$$A^\top P + PA - 2PBR^{-1}B^\top P = -\alpha P, \tag{44}$$

and $(A - BR^{-1}B^{\top}P)$ is Hurwitz. Moreover, the matrix P has the property that for some $C > 0$ and $p > \frac{2}{3}$,

$$\lim_{\alpha \downarrow 0} \frac{\sigma_{\max}(P)}{\alpha^p} \leq C. \tag{45}$$

Assumption 1 requires that $\sigma_{\max}(P) = O(\alpha^p)$ when $\alpha \downarrow 0$. There exist linear systems that satisfy this Assumption; indeed, consider A and B given by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{46}$$

and choose $R = 2\,I$, with I denoting the identity matrix (of appropriate dimension). Then (44) admits a unique solution P, with $(A - BR^{-1}B^{\top}P)$ Hurwitz, and the (i, j)th entry of P has the form

$$[P]_{ij} = \frac{p_{ij}(\alpha)}{1 + \alpha^4}$$

where p_{ij} are functions satisfying $\lim_{\alpha \downarrow 0} \frac{p_{ij}(\alpha)}{\alpha} = 0$ when $(i, j) \neq (3, 3)$, and for $(i, j) = (3, 3)$ we have $\lim_{\alpha \downarrow 0} \frac{p_{ij}(\alpha)}{\alpha} = 3$. A crisp characterization of the class of systems that satisfy Assumption 1 is under investigation.

Remark 6.4 System (46) is a particular example of null-controllable systems where the eigenvalues of A are on the imaginary axis. In general, we do not expect Assumption 1 to hold for systems with eigenvalues of A in open right-half complex plane. This can be seen for the scalar systems $\dot{x} = ax + u$, for which the solution of (44) with $R = 1$ is $p = 2a + \alpha$, and clearly (45) holds only with $a = 0$ for $0 < p \leq 1$.

The following Theorem provides a recipe for designing feedback controllers under Assumption 1.

Theorem 6.5 *Consider the system* (41) *where* $\left(N_t\right)_{t \geqslant 0}$ *is a Poisson process of given intensity* $\lambda > 0$*, and suppose that Assumption 1 holds. Then there exists* $\alpha > 0$ *(sufficiently small) such that the feedback gain*

$$K = -R^{-1}B^{\top}P \quad \textit{with } P \textit{ solving} (44)$$

renders the system (41) *globally asymptotically stable in the second moment.*

Proof of Theorem 6.5 For $\alpha > 0$ we let P denote the solution of (44), and choose

$$\psi(x, e) := \eta_e \langle e, Pe \rangle + \eta_x \langle x, Px \rangle \quad \text{for } (x, e) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where the positive scalars η_e, η_x will be specified later. The expression in (9) with the above choice of ψ yields

$$\mathcal{L}\psi(e,x) = -\left\langle \begin{pmatrix} x \\ e \end{pmatrix}, M(\alpha,\lambda) \begin{pmatrix} x \\ e \end{pmatrix} \right\rangle,$$

where

$$M(\alpha,\lambda) := \begin{pmatrix} -\eta_x(A_K^\top P + PA_K) & \eta_x PBK - \eta_e A_K^\top P \\ \eta_x K^\top B^\top P - \eta_e PA_K & \eta_e(\lambda P + PBK + K^\top B^\top P) \end{pmatrix},$$

in which $A_K = A + BK$, and dependence on α is through the matrix P. It follows that M is positive definite if M_0 and M_1 are positive definite, where

$$M_0 := \begin{pmatrix} \frac{\alpha}{2}\eta_x P & -\eta_e A_K^\top P \\ -\eta_e PA_K & \frac{\lambda}{2}\eta_e P \end{pmatrix}$$
$$M_1 := \begin{pmatrix} \frac{\alpha}{2}\eta_x P & \eta_x PBK \\ \eta_x K^\top B^\top P & \frac{\lambda}{2}\eta_e P + \eta_e PBK + \eta_e K^\top B^\top P \end{pmatrix}.$$

We first treat M_0. Using Schur complements [48, §7.4] followed by conjugation with $\eta_e^{-1/2} P^{-1/2}$, we get

$$M_0 > 0 \Leftrightarrow \frac{\lambda}{2} I > 2\frac{\eta_e}{\eta_x \alpha} P^{1/2}(A+BK)P^{-1}(A+BK)^\top P^{1/2}. \tag{47}$$

In view of Assumption 1, for a $p > \frac{2}{3}$ satisfying $\sigma_{\max}(P) = O(\alpha^p)$, we pick $\varepsilon > 0$ such that $0 < \varepsilon < p - \frac{2}{3}$, and select $\eta_e, \eta_x > 0$ such that

$$\frac{\eta_e}{\eta_x} = O(\alpha^{1+\varepsilon}). \tag{48}$$

By letting $\alpha \downarrow 0$, we see that $\sigma_{\max}(P^{1/2}) = O(\alpha^{p/2})$, which also yields that $P^{1/2}(A + BK)P^{-1}(A+BK)^\top P^{1/2} = O(1)$. Thus, the term on the right-hand side of the inequality (47) is bounded by $O(\alpha^\varepsilon)$. This shows that for α sufficiently small, $M_0 > 0$.

We next analyze M_1. Substituting $K = -R^{-1}B^\top P$ into M_1, using Schur complements [48, §7.4], and conjugating by $\eta_e^{-1/2} P^{-1/2}$, we get

$$\begin{aligned} &M_1 > 0 \quad \Leftrightarrow \\ &\frac{\lambda}{2} I > 2\alpha P^{1/2} BR^{-1} B^\top P^{1/2} + 2\frac{\eta_x}{\eta_e \alpha} P^{1/2} BR^{-1} B^\top P^2 BR^{-1} B^\top P^{1/2}. \end{aligned} \tag{49}$$

Letting $\alpha \downarrow 0$, in view of Assumption 1 we have $\sigma_{\max}(P) = O(\alpha^p)$. The first term on the right-hand side is $O(\alpha^{p+1})$. For our choice of η_e and η_x in (48), we get

$$\frac{\eta_x}{\eta_e \alpha} = O(\alpha^{-2-\varepsilon}). \tag{50}$$

This way, the second term on the right-hand side of the inequality (49) is $O(\alpha^{3p-2-\varepsilon})$, which under the assumption $p > \frac{2}{3} + \varepsilon$, converges to zero as $\alpha \downarrow 0$. We conclude that M_1, and hence $M = M_0 + M_1$, are positive definite for sufficiently small $\alpha > 0$.

7 Conclusions

This chapter provided an overview on the problem of stabilization of deterministic control systems under random sampling. Although the problem was first introduced almost 60 years ago, the earlier efforts did not create many inroads. The use of modern tools from the literature on stochastic systems has indeed brought a constructive solution to this problem. In particular, this chapter provided the solution to this problem using the extended generator and Volterra integral techniques, and also developed connections between these two approaches. One particular question that needs further investigation is the design of feedback laws for fixed sampling rates. In this direction, Conjecture 6.1 is shown to hold for scalar systems and to some extent for multidimensional systems under a strong assumption. Investigating design techniques for constructing feedback gains in linear case for given sampling rates is indeed relevant for several applications.

As it is naturally the case, the problem has been studied with more depth in the case of linear systems which lead to Theorem 5.1 and quantitative estimates in Theorem 5.4. Extending such results for the case when the sampling process in not necessarily Poisson, but governed by some other distribution needs to be investigated. In general, one can also apply the extended generator approach to the case where transition rates are state dependent and locally bounded [20], but the stability conditions need to be worked out more explicitly for such cases. Another set of problems that emerges from these results is to develop their analogue counterparts for nonlinear systems. It is not immediately clear how the Volterra integral technique used in Theorem 5.1 could be generalized in nonlinear setting. Hence, it needs to be seen whether a converse Lyapunov theorem can be proven for nonlinear PDMPs. Also, at this moment, Theorem 5.4 shows that faster sampling in the limit leads to the same convergence rate as one obtains for the unsampled system. To extend this line of thought, we are currently looking into whether for randomly sampled processes, the expected value of the random variable at each time converges to the value of the function obtained as a solution to the unsampled process, as the mean sampling rate grows.

While this chapter addressed the problem of stabilization with random sampling using static time-invariant state feedback controllers, one can also explore the possibility of considering dynamic controllers with output feedback. Going beyond the realm of conventional dynamic controllers, more recently in [45], the authors work with discontinuous, or hybrid controllers, and consider the effect of random perturbations in communication of discrete and continuous state to the controller. Addressing similar questions, as the ones confronted in this chapter, for a more general class of controllers is likely to bring significant contributions to the currently active field of stochastic hybrid systems [12, 20, 44].

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Appendix

A: Proof of Proposition 2.2

Proof The fact that $\big(x(t), e(t)\big)_{t\geqslant 0}$ is Markovian follows from the observation that the future of $x(t)$ depends on $x(\tau_{N_t})$ and, therefore, equivalently on $e(t)$.

Let $\mathbb{R}^d \times \mathbb{R}^d \ni (y, z) \mapsto \psi(y, z) \in \mathbb{R}$ denote a function with at most polynomial growth as $\|(y, z)\| \to +\infty$. Since the system under consideration is well-posed, we have, for $h > 0$ small,

$$\begin{aligned}
&\mathsf{E}\left[\psi\big(x(t+h), e(t+h)\big)\middle|x(t) = y, e(t) = z\right] \\
&= \mathsf{E}\Big[\psi\big(x(t+h), e(t+h)\big)\big(1_{\{N_{t+h}=N_t\}} + 1_{\{N_{t+h}=1+N_t\}} \\
&\quad + 1_{\{N_{t+h}-N_t\geqslant 2\}}\big)\,\Big|\, x(t), e(t)\Big].
\end{aligned}\tag{51}$$

We now compute the conditional probability distribution of $\big(x(t+h), e(t+h)\big)$ for small $h > 0$ given $\big(x(t), e(t)\big)$. Since the sampling process is independent of the joint process $\big(x(\tau_{N_t}), x(t)\big)_{t\geqslant 0}$, by definition of the sampling (Poisson) process we have, for $h \downarrow 0$,

$$\begin{cases}
\mathsf{P}\left(N_{t+h} - N_t = 0\middle|N_t, e(t), x(t)\right) = 1 - \lambda h + o(h),\\
\mathsf{P}\left(N_{t+h} - N_t = 1\middle|N_t, e(t), x(t)\right) = \lambda h + o(h),\\
\mathsf{P}\left(N_{t+h} - N_t \geqslant 2\middle|N_t, e(t), x(t)\right) = o(h).
\end{cases}$$

Using these expressions we develop (51) further for $h \downarrow 0$ as

$$\begin{aligned}
&\mathsf{E}\left[\psi\big(x(t+h), e(t+h)\big)\middle|x(t) = y, e(t) = z\right] \\
&= \mathsf{E}\Big[\psi\big(x(t+h), e(t+h)\big)\big(1_{\{N_{t+h}=N_t\}} + 1_{\{N_{t+h}=1+N_t\}}\big)\,\Big|\, x(t), e(t)\Big] + o(h) \\
&= \mathsf{E}\left[\psi\big(x(t+h), e(t+h)\big)\middle|x(t), e(t), N_{t+h} = N_t\right]\cdot\big(1 - \lambda h + o(h)\big) \\
&\quad + \mathsf{E}\left[\psi\big(x(t+h), e(t+h)\big)\middle|x(t), e(t), N_{t+h} = 1 + N_t\right]\big(\lambda h\big) + o(h).
\end{aligned}\tag{52}$$

The two significant terms on the right-hand side of (52) are now computed separately. For the event $N_{t+h} = N_t$, given $x(t) = y$, $e(t) = z$, we have for $h \downarrow 0$,

$$\begin{aligned}
\psi\big(x(t+h), e(t+h)\big) &= \psi\big(y, z\big) + h\left\langle \nabla_y\psi\big(y, z\big), f\Big(y, \kappa\big(x(\tau_{N_t})\big)\Big)\right\rangle \\
&\quad + h\left\langle \nabla_z\psi\big(y, z\big), f\Big(y, \kappa\big(x(\tau_{N_t})\big)\Big)\right\rangle + o(h),
\end{aligned}$$

leading to the first term on the right-hand side of (52) having the estimate

$$\begin{aligned}
&\mathsf{E}\left[\psi\big(x(t+h), e(t+h)\big)\big|N_{t+h} = N_t, x(t) = y, e(t) = z\right] \cdot \big(1 - \lambda h + o(h)\big)\\
&= \psi\big(y, z\big) + h\left\langle\nabla_y\psi\big(y, z\big) + \nabla_z\psi\big(y, z\big), f\left(y, \kappa\big(x(\tau_{N_t})\big)\right)\right\rangle\\
&\quad - (\lambda h)\psi\big(y, z\big) + o(h) \quad \text{for } h \downarrow 0.
\end{aligned}$$

Concerning the second term on the right-hand side of (52), we observe that conditional on $N_{t+h} = 1 + N_t$, the probability distribution of $\tau_{N_{t+h}}$ is [39, Theorem 2.3.7] uniform over $[t, t+h[$ by definition of the sampling (Poisson) process, i.e.,

$$\mathsf{P}\left(\tau_{N_{t+h}} \in [s, s+s'[\big|N_{t+h} = 1 + N_t\right) = \frac{1}{h}s' \quad \text{for } [s, s+s'[\subset [t, t+h[.$$

Since the sampling process is independent of the state process, the preceding conditional probability is equal to

$$\mathsf{P}\left(\tau_{N_{t+h}} \in [s, s+s'[\big|N_{t+h} = 1 + N_t, x(t) = y, e(t) = z\right).$$

We define $\theta \in [0, 1[$ such that $\tau_{N_{t+h}} = t + \theta h, x(t) = y, e(t) = z$; then θ is uniformly distributed on $[0, 1[$ given $N_{t+h} = 1 + N_t$. We also have, conditioned on the same event,

$$e(\tau_{N_{t+h}}) = e(t + \theta h) = 0,$$

and

$$x(\tau_{N_{t+h}}) = x(t + \theta h) = x(t) + \theta h f\left(x(t), \kappa\big(x(\tau_{N_t})\big)\right) + o(h).$$

The above expressions then lead to, conditioned on the event $N_{t+h} = 1 + N_t, x(t) = y, e(t) = z$ and for $h \downarrow 0$,

$$\begin{aligned}
x(t+h) &= x(t+\theta h) + (1-\theta)hf\big(x(t+\theta h), \kappa(x(t+\theta h))\big) + o(h)\\
&= x(t) + \theta h f\left(x(t), \kappa\big(x(\tau_{N_t})\big)\right) + (1-\theta)hf\big(x(t+\theta h), \kappa(x(t+\theta h))\big) + o(h)\\
&= x(t) + \theta h f\left(x(t), \kappa\big(x(\tau_{N_t})\big)\right) + (1-\theta)hf\big(x(t), \kappa(x(t))\big) + o(h).
\end{aligned}$$

Similarly, it can be verified directly from the differential equation governing e that conditioned on the same event,

$$e(t+h) = (1-\theta)hf(x(t), \kappa(x(t))) + o(h) \quad \text{for } h0.$$

Therefore, for $h \downarrow 0$,

$$\begin{aligned}
&\mathsf{E}\left[\psi\big(x(t+h), e(t+h)\big)\big|x(t) = y, e(t) = z, N_{t+h} = 1 + N_t\right] \cdot (\lambda h)\\
&= \int_0^1 \psi\Big(y + \theta h f\left(x(t), \kappa\big(x(\tau_{N_t})\big)\right) + (1-\theta)hf\big(x(t), \kappa(x(t))\big) + o(h),\\
&\qquad (1-\theta)hf\big(x(t), \kappa(x(t))\big) + o(h)\Big)\, \mathrm{d}\theta \cdot (\lambda h)
\end{aligned}$$

$$= \int_0^1 \Big(\psi\big(y,0\big) + h \left\langle \nabla_y \psi\big(y,0\big), \theta h f\big(x(t), \kappa\big(x(\tau_{N_t})\big)\big)\right\rangle + (1-\theta) h f\big(x(t), k(x(t),0)\big\rangle$$

$$+ h \left\langle \nabla_z \psi\big(y,0\big), (1-\theta) h f\big(x(t), \kappa(x(t))\big)\right\rangle + o(h) \Big) \,\mathrm{d}\theta \cdot (\lambda h)$$

$$= \Big(\psi\big(y,0\big) + O(h)\Big) \cdot (\lambda h)$$

$$= (\lambda h)\psi\big(y,0\big) + o(h).$$

Putting everything together, we arrive at

$$\begin{aligned} &\mathsf{E}\psi\big(x(t+h), e(t+h)\big)\big| x(t) = y, e(t) = z \\ &= \psi(y,z) + h\Big(\big\langle \nabla_y \psi(y,z) + \nabla_z \psi(y,z),\, f\big(y, \kappa\big(y-z\big)\big)\big\rangle\Big) \\ &\quad - (\lambda h)\big(\psi(y,z) - \psi(y,0)\big) + o(h). \end{aligned}$$

Substituting these expressions in (8), we see that for each $(y, z) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\begin{aligned} \mathcal{L}\psi(y,z) = \big\langle \nabla_y \psi(y,z) + \nabla_z \psi(y,z),\, f\big(y, \kappa\big(y-z\big)\big)\big\rangle \\ - \lambda\big(\psi(y,z) - \psi(y,0)\big), \end{aligned}$$

as asserted. □

B: Proofs of Lemmas 5.2 and 5.3

Proof of Lemma 5.2 The desired expression for $\frac{\mathrm{d}}{\mathrm{d}t} W(t)$ is obtained by differentiating

$$W(t) = \sum_{j=0}^{+\infty} \mathcal{K}^j(H)(t)$$

where we recall that $\mathcal{K}$ is given in (25) and $\mathcal{K}^0$ is the identity operator. To do so, we basically compute $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}^j(H)(t)$ for each $j \geqslant 0$. Since $\mathcal{K}^0(H)(t) = H(t)$, we first observe that

$$\frac{\mathrm{d}}{\mathrm{d}t} H(t) = F^\top H(t) + H(t) F - \lambda H(t).$$

Similarly, we compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(H)(t) &= \lambda \mathrm{e}^{tF^\top} G^\top (H)(0) G \mathrm{e}^{tF} \mathrm{e}^{-\lambda t} \\ &\quad + \lambda \int_0^t \mathrm{e}^{sF^\top} G^\top \left(\frac{\mathrm{d}}{\mathrm{d}t} H(t-s) \right) G \mathrm{e}^{sF} \mathrm{e}^{-\lambda s} \,\mathrm{d}s \\ &= \lambda J(t) + \mathcal{K}(F^\top H + HF - \lambda H)(t). \end{aligned}$$

Next, to compute $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}^j(H)(t)$, for $j \geqslant 2$, we use the induction principle. Let us assume that, for some $j \geqslant 2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}^{j-1}(H)(t) = \lambda\mathcal{K}^{j-2}(J)(t) + \mathcal{K}^{j-1}(F^\top H + HF - \lambda H)(t).$$

It then follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}^j(H)(t) &= \lambda \mathrm{e}^{tF^\top} G^\top \mathcal{K}^{j-1}(H)(0) G \mathrm{e}^{tF}\mathrm{e}^{-\lambda t} \\ &\quad + \lambda \int_0^t \mathrm{e}^{sF^\top} G^\top \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}^{j-1}(H)(t-s) G \mathrm{e}^{sF}\mathrm{e}^{-\lambda s}\,\mathrm{d}s \\ &= \lambda\mathcal{K}^{j-1}(J)(t) + \mathcal{K}^j(F^\top H + HF - \lambda H)(t).\end{aligned}$$

Using this last expression and recalling the definition of W from (27), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) = \sum_{j=1}^{+\infty}\mathcal{K}^j(F^\top H + HF + \lambda(J-H))(t) + (F^\top H + HF + \lambda(J-H))(t),$$

which is the desired statement. □

Proof of Lemma 5.3 We first verify the desired expression (31) for $j=1$. It is seen that

$$\begin{aligned}\lambda J(t) - \lambda G^\top H(t) G &= \lambda \mathrm{e}^{F^\top t} G^\top Q G \mathrm{e}^{Ft}\mathrm{e}^{-\lambda t} - \lambda G^\top H(t) G \\ &= \lambda \int_0^t \frac{\partial}{\partial s}\left(\mathrm{e}^{F^\top s} G^\top H(t-s) G \mathrm{e}^{Fs}\mathrm{e}^{-\lambda s}\right)\mathrm{d}s \\ &= F^\top \mathcal{K}(H)(t) + \mathcal{K}(H)(t)F - \lambda\mathcal{K}(H)(t) \\ &\quad + \lambda \int_0^t \left(\mathrm{e}^{F^\top s} G^\top \frac{\partial}{\partial s} H(t-s) G \mathrm{e}^{Fs}\mathrm{e}^{-\lambda s}\right)\mathrm{d}s \\ &= F^\top \mathcal{K}(H)(t) + \mathcal{K}(H)(t)F - \lambda\mathcal{K}(H)(t) \\ &\quad - \mathcal{K}(F^\top H + HF - \lambda H)(t),\end{aligned}$$

and hence (31) holds for $j=1$.

Proceeding by induction, we assume that for some $j \geqslant 1$

$$\begin{aligned}F^\top \mathcal{K}^j(H)(t) + \mathcal{K}^j(H)(t)F - \lambda\mathcal{K}^j(H)(t) &= -\lambda G^\top \mathcal{K}^{j-1}(H)(t) G \\ &\quad + \mathcal{K}^j(F^\top H + HF - \lambda H) + \lambda\mathcal{K}^{j-1}(J)(t).\end{aligned} \tag{53}$$

We then observe that

$$-\lambda G^\top \mathcal{K}^j(H(t)) G = \lambda \int_0^t \frac{\partial}{\partial s_j}\left(\mathrm{e}^{F^\top s_j} G^\top \mathcal{K}^j(H)(t-s_j) G \mathrm{e}^{Fs_j}\mathrm{e}^{-\lambda s_j}\right)\mathrm{d}s_j \tag{54}$$

because $\mathcal{K}^j(H)(0) = 0$ for each $j \geqslant 1$. To compute the expression in the integrand on the right-hand side, we observe that

$$\frac{\partial}{\partial s_j}\mathcal{K}^j(H)(t-s_j) = -\lambda\mathcal{K}^{j-1}(J)(t-s_j) - \mathcal{K}^j(F^\top H + HF - \lambda H)(t-s_j),$$

which results in

$$\begin{aligned}
&\frac{\partial}{\partial s_j}\big(\mathrm{e}^{F^\top s_j}G^\top\mathcal{K}^j(H)(t-s_j)G\mathrm{e}^{Fs_j}\mathrm{e}^{-\lambda s_j}\big)\mathrm{d}s_j \\
&\quad = -\lambda\mathrm{e}^{F^\top s_j}G^\top\mathcal{K}^{j-1}(J)(t-s)G\mathrm{e}^{Fs_j}\mathrm{e}^{-\lambda s_j} \\
&\qquad - \mathrm{e}^{F^\top s_j}G^\top\mathcal{K}^j(F^\top H + HF - \lambda H)(t-s_j)G\mathrm{e}^{Fs_j}\mathrm{e}^{-\lambda s_j} \\
&\qquad + F^\top(\mathrm{e}^{F^\top s_j}G^\top\mathcal{K}^j(H)(t-s_j)G\mathrm{e}^{Fs_j}\mathrm{e}^{-\lambda s_j}) \\
&\qquad + (\mathrm{e}^{F^\top s_j}G^\top\mathcal{K}^j(H)(t-s_j)G\mathrm{e}^{Fs_j}\mathrm{e}^{-\lambda s_j})F \\
&\qquad - \lambda(\mathrm{e}^{F^\top s_j}G^\top\mathcal{K}^j(H)(t-s_j)G\mathrm{e}^{Fs_j}\mathrm{e}^{-\lambda s_j}).
\end{aligned}$$

Substituting this last equality in (53), we get

$$\begin{aligned}
\lambda\mathcal{K}^j(J(t)) - \lambda G^\top\mathcal{K}^j(H(t))G = F^\top\mathcal{K}^{j+1}(H)(t) + \mathcal{K}^{j+1}(H)(t)F - \lambda\mathcal{K}^{j+1}(H)(t) \\
- \mathcal{K}^{j+1}(F^\top H + HF - \lambda H)(t),
\end{aligned}$$

and the assertion follows. □

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