Chapter 5
What Is Learned About the Roles of Mathematics in Physics While Learning Physics Concepts? A Mathematics Sensitive Look at Physics Teaching and Learning

Olaf Krey

5.1 Introduction

Physics is part of the human attempt to find the laws that govern nature. Doing physics implies a lot of activities, e.g.:

- Observing and experimenting
- Describing systems (identifying relevant entities and the relations between them)
- Establishing cause and effect relationships
- Defining observables and taking measurements
- Analysing relationships between observables in a lab
- Quantifying and mathematizing these relationships (thereby expressing them in a more precise way)
- Analysing the (mathematical) expressions and producing testable statements about the real or at least lab world or even predicting "new" entities in the real world
- Trying to systematize a body of knowledge, presented mathematically, etc.

This list is not exhaustive, but one would easily agree that all the items on this list are part of the "physics game". There are very different conceptions about the epistemology of physics (e.g. Losee 2001; Wenning 2009). Depending on a specific author's background, the field of physics, the physicist or the historical time under consideration, there are, of course, different emphases, but all of these approaches consider mathematics and experiments as core elements of today's physics endeavour.

O. Krey (✉)
Universität Augsburg, Augsburg, Germany
e-mail: olaf.krey@physik.uni-augsburg.de

In science education the role of experiment in physics has been considered extensively (e.g. Hegarty-Hazel 1990; Leach and Paulson 1999; Lunetta et al. 2007), and aspects from the philosophy and history of physics concerning the use and value of experiments have found its way into the mainstream discussions of the science education community (e.g. the explorative and affirmative style of experimentation (Steinle 1998)). Unfortunately, this is not the case for the role of mathematics, which has simply not been a popular field of research until recently. The interplay between experiments and mathematization in physics and the implications for teaching and learning physics – despite a few exceptions (see contribution by Mäntyla and Proonen) – are even less investigated.

For decades the nature of science has been a highly valued field of science education research and is today considered to be an important aspect of scientific literacy. However, in general the discussion culminates in lists of items that summarize important characteristics learners should know about the nature of science (Lederman et al. 2002; McComas et al. 1998; Osborne et al. 2003; Rutherford and Ahlgren 1990). None of them includes the role of mathematics in (any acceptable) depth (cf. Krey 2012). Accordingly, well-developed physics units that help teaching about the role of mathematics are rare to find, although there are thoughtfully developed teaching units offering great potential for teaching physics content and aspects of the nature of physics at the same time (e.g. the HIPST case studies (Höttecke 2012)).

This is in line with many students' (and teachers') views of using mathematics in physics as a rather artificial, distant, algorithmic activity not leaving much space for creativity, controversial discourse or individual taste. It is not unlikely that the way we use mathematics in our physics lessons contributes strongly to the naïve realistic view of physical theories as simplified descriptions of reality that is well documented for students and teachers in the literature (e.g. Abd-El-Khalick 2012). From this perspective, the lack of research about the role of mathematics in learning and teaching physics may have covered an influential factor for an adequate understanding of the nature of physics. Over the years, research in the field of teaching and learning about the nature of science has validated the assumption that aspects of the nature of science had to be addressed *explicitly* and *reflectively* in order to be learnt efficiently (e.g. Abd-El-Khalick and Lederman 2000). It can be assumed that the same holds to be true for teaching and learning about the role of mathematics in physics.

The twofold aim of this paper is (a) to help clarify the roles of mathematics in (learning) physics, not by means of an abstract philosophical analysis but by two case studies about (i) uniform motion and (ii) image formation by thin lenses and (b) to exemplify how intentionally or unintentionally teaching about the roles of mathematics in physics occurs in the average physics classroom. I have chosen the two cases to illustrate an inductive and a deductive way of teaching. By doing so, the author wishes to draw attention to the fact that learning about the roles of mathematics in physics occurs (usually implicitly) at most of our physics lessons

and most likely shapes our students' views on the nature of (learning) physics. To make this point, the case studies are used to illustrate certain characteristics of the use of mathematics in physics. Although they have been chosen to cover equations from different epistemological categories, the case studies are not everything that could or should be said about the use of mathematics in physics. It is not the aim of this chapter to come up with another list of possible characteristics of the use of mathematics in physics. Therefore, I will not even make an attempt to generate a complete list of those features. My modest aim is to illustrate a few of the messages (potentially and/or implicitly) inherent in average physics lessons. I thereby hope to help develop sensitivity towards potential learning opportunities for our students. Before considering the two cases, a more elaborated framework allowing to frame the chosen cases and to guide their analysis is necessary.

5.1.1 Thinking About the Roles of Mathematics in (Learning) Physics: A Basic Framework

As shown on the left side of Fig. 5.1, the author of this paper makes use of a more or less consensual distinction between reality (R) and mathematical theory (MT) (as suggested by, e.g. Ludwig 1985). This distinction is made for analytical purposes, and it can be discussed whether the separation of the two is adequate and empirically valid. Accepting this distinction allows to define a physical theory (PT) as a mapping ($\sim$) between reality and mathematical theory: $PT = R(\sim)MT$. (For details, see Ludwig 1985, 1990.) Obviously, mathematics can stand for itself, as an artificial creation by the human mind without a direct relation to the real world (pure mathematics). The world can also be considered independently of any mathematical theory, and humans can engage with this real world in many ways. Without going into further details, it becomes obvious that from this point of view, a large part of physics can be considered as the venture of building bridges between mathematics (MT) and the real word (R). This broad perspective allows to recognize otherwise isolated findings as aspects of a greater common – the attempt to build bridges between R and MT.

Ludwig's simple approach ($PT = R(\sim)MT$) serves well as an integrating perspective for many more elaborated descriptions of the roles of mathematics in physics. One of the limitations of this approach has been pointed out by philosophers (Cartwright 1983) and physicists, e.g. Einstein, who states: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality" (Einstein 2002). That's why on the right side of Fig. 5.1, a pre-mathematical model (M) is included. Elements of this simplified (e.g. by selection, isolation, idealization) model are the entities to which mathematical structures are mapped. Refining Ludwig's approach I therefore suggest to consider two mappings ($-$) and ($*$), one between the real world and the
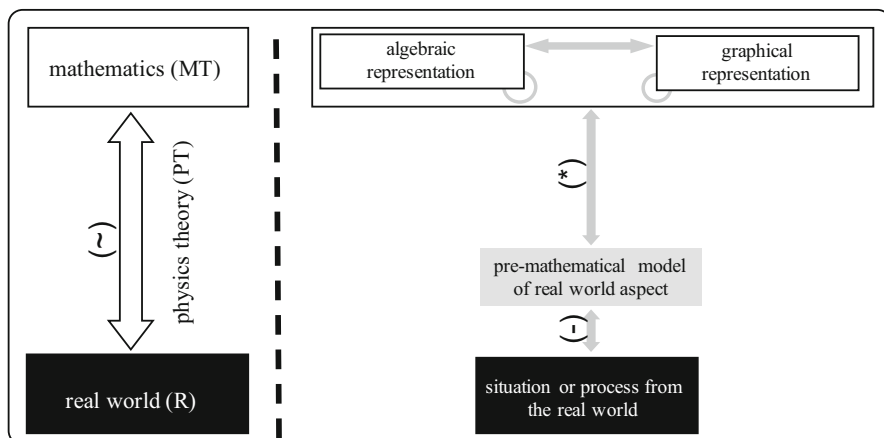
Fig. 5.1 Physics as building and walking on bridges between mathematics and (models of) the real world by the use of algebraic and graphical mathematical representation

pre-mathematical model $(-)$ and a second one between the pre-mathematical model and mathematics $(*)$. In total we have $PT = R(-)M(*)MT = R(\sim)MT$.[1] From the world of mathematics, the (arguably) two most important forms of representation (algebraic and graphical) have been selected to exemplify translations between them (straight arrows) as well as constructive and interpretive activities (straight and circular arrows) that go along with their usage.

This model is not meant to describe physics as a human activity as a whole. (I honestly doubt this is possible.) Its purpose here is to provide a framework for this chapter. Another weakness of this model is its lack to specifically address further important aspects, e.g. concerning the role of experiments or more precisely the great epistemological gap between experimental data and their algebraic representation (Falk 1990). This is something to keep in mind.

As mentioned before, this primary framework is a rather broad one, allowing to frame otherwise quite different attempts to describe how mappings between mathematics and the real word are established in science education. The mathematics-physics interplay can be described in many helpful ways, focusing on different aspects. (For a reconstruction of interplay patterns based on middle school physics classroom evidence, see Lehavi et al. 2017; for an overview on modelling concepts, see Pospiech in this book; for a related framework assuming a "physical-mathematical model" to be essential for an in-depth analysis of using mathematics in learning physics, see Uhden et al. 2012.)

---

[1]Ludwig is aware of this complexity and his approach is much more detailed than can be explicated here. (For details see Ludwig 1985, 1990.)

The way bridges between mathematics and the real world are built and used in physics instruction has been identified as a matter of concern quite early. Wagenschein likens the learners' empty recall of physics equations to "paper flowers" (Wagenschein 1995); others articulate their impression of physics being destroyed/decomposed/ripped by calculation ("zerrechnete Physik", Dittmann et al. 1989). More recently a distinction between a *technical* dimension and a *structural* dimension of mathematics in physics (Karam 2014) roughly corresponding with *low-level* and more *elaborated* mathematical activities on the student side (Krey 2012) has been suggested. The technical dimension refers to an instrumental role of mathematics, to mathematics as a calculation tool which implies a procedural focus on practices such as to blindly use equations for solving quantitative problems, to focus on algorithmic manipulation, etc. The structural dimension in contrary refers to an organizational role of mathematics, to mathematics as a reasoning instrument which suggests the focus on physical interpretation of mathematical representations and the derivation of equations from physical principles, etc. (cf. Karam 2014; Krey 2012).

A very simple and widely spread distinction (with far-reaching consequences) can be made between an inductive and a deductive approach to teaching physics content. An *inductive* approach starts from observations (real world), while a *deductive* approach starts from already established (e.g. algebraically represented) laws and principles. Although these are general teaching patterns not necessarily referring to a quantitative relationship to be taught, for this article the implications of their application are of interest as far as these patterns are used to frame ways to the laws of physics, usually represented mathematically (e.g. as an equation).

Working out more elaborated patterns of the mathematics-physics interplay in physics education is necessary to better understand how a misrepresentation of the roles of mathematics in physics can be avoided. On the one hand, this knowledge is likely to be more useful for (future) physics teachers than for actual physics learners, and at the same time, in-depth philosophical considerations about the role of mathematics in physics tend to become difficult and challenging, perhaps too difficult for the average high school learner. On the other hand, it is rather simple to define certain features that can also serve as fields of reflection for physics learners (Krey 2012). Here are a few questions that more or less directly lead to these features and provide first opportunities for learners to learn to appreciate the use of mathematics in physics: How do mathematical representations influence *cognitive load*? How does mathematics contribute to the *exactness* of physics? How can mathematics influence information exchange (*communication*) in physics? How is the *certainty* and *objectivity/intersubjectivity* of physics knowledge influenced? At what *cost* do we apply mathematics in physics? It has been shown that students have an inappropriate understanding of all the above questions (Krey 2012).

Karam and Krey (2015) suggest that physics equations can be categorized according to their (subjective) epistemological status. This leads to four categories:

principles (e.g. $\sum \vec{p} = const.$), definitions (e.g. $\vec{p} = m\vec{v}$), empirical regularities (e.g. Balmer's formula) and derivations (e.g. $a = v^2/r$). It is reasonable to assume that teaching and learning (about) physics equations implicitly or explicitly always go hand in hand with an underlying message about the roles of mathematics in physics. However, the message can be very different depending on what category a certain equation belongs to, e.g. whether it was introduced via an inductive or deductive way.

Since it must be assumed that most of the knowledge about the nature of science is taught and learned implicitly, the message sent by an inductive approach needs to be considered more carefully, and perhaps some explicit measures have to be taken to avoid a misrepresentation (see case study 1 on uniform motion). This becomes even more clear by (a) recognizing the fact that only a few equations can be obtained by a deductive approach (derivation) in secondary school physics and (b) knowing about a certain tendency by teachers to choose an inductive approach over a deductive one, even if a deductive one is easily accessible.[2] When physics equations are involved in the teaching of physics, deductions are often used in simple ways, e.g. to predict the final velocity of a falling object based on the equation for free fall. In contrast, the equation for free fall (or accelerated motion in general) itself is obtained by an inductive approach. From this, the introduced equation likely appears as the summarized and generalized result of a longer learning-teaching sequence, the only important "fact" of the lesson.

Although there is no empirical evidence (since a study focusing on the actual use of mathematics in the secondary classroom has not been conducted yet), it can be considered as common ground that the above-mentioned technical dimension of using mathematics in physics is by far over-represented. Calculation in the sense of given-sought problems calling for a plug and chug strategy and the corresponding low-level activities are assumed to be a widespread practice. For example, more recently learners' physics problem-solving has been observed, and their activities could often be described in a calculation frame (Bing and Redish 2009) or as following a recursive plug and chug strategy (Tuminaro and Redish 2007). Furthermore, analysing pupils' beliefs about the role of mathematics in physics supports the assumption of having experienced a lot of low-level activities in school (Krey 2012). Although this problem is known for decades, it has not been addressed efficiently so far. However, a few attempts to develop future teachers' awareness about the problem as well as their equation-related pedagogical content knowledge (Shulman 1986) have been suggested, implemented and evaluated (e.g. Karam and Krey 2015).

"We do students a disservice by treating conceptual understanding as separate from the use of mathematical notations" (Sherin 2001). This chapter is written in

---

[2]See Redfors et al. in this book for examples of deductive elements in physics textbooks that are not used by the teachers. Although there are many examples where equations could be deduced from theoretical or mathematical considerations in high school (Snell's law, centripetal acceleration, etc.), it is rather hard to find many examples for equations that could be dealt with in middle school (e.g. total resistance in series and parallel circuit).

the spirit of this quotation. The modest objective for this article is not to offer a solution for the mentioned problems, but rather provide a rather small contribution to a desired future solution, a small step on a long way to go. The author suggests that teaching physics, whether we are aware of it or not, at the same time means (at least implicitly) teaching about the role of mathematics in physics. In what follows, two cases covering content of most, if not all, secondary physics curricula are considered: uniform motion and image formation by thin (converging) lenses. These cases are used to identify a few of the messages usually sent implicitly to our students and to point out aspects of the mathematics-physics interplay that could be discussed explicitly in the physics classroom in some cases, but at least inform the teacher about a few implications and potentials of her or his everyday physics teaching. For this purpose, an inductive approach of introducing an equation (uniform motion) is complemented by a derivation of the thin lens equation (deductive approach). The author's hope is that the roles of mathematics in learning and teaching physics become clearer and a step towards a better balance between the technical and the structural role of mathematics can be envisioned. The way we use mathematics may play a decisive role in how our students perceive the nature of physics, and this is why I argue that any view about the nature of physics that does not consider the roles of mathematics is incomplete and missing one of the essential aspects.

By going through the two cases, typical, not necessarily observed, teaching scenarios are described from a meta-perspective. After considering a certain aspect of the teaching and learning about uniform motion or image formation, a potentially important or interesting insight about the roles of mathematics in physics is highlighted in a shaded box. The collection of these possible (and sometimes obvious) insights relates to the above-mentioned fields of reflection; by the help of which, one may find a way to help students learn about the roles of mathematics in physics explicitly and reflectively, as suggested for other aspects of the nature of science.

5.2   Case Study 1: Uniform Motion

Usually during their first year of science or physics in secondary school, pupils deal with motions. They learn to distinguish between straight line and curvilinear motion (rotary, circular, oscillatory motion) as well as between uniform and accelerated motion. The most simple (straight line uniform) motion is then analysed in more detail. While semi-quantitative approaches are possible and a helpful intermediate step, sooner or later, pupils start to measure distances and times and display and evaluate a functional dependency (see Box 5.1).

In Box 5.1 the widely used inductive approach to teaching physics at school (see Fig. 5.2) is exemplified. This inductive way of teaching usually starts with a real-world problem, phenomenon, a specific situation or process that is potentially

Fig. 5.2 Inductive way of
teaching physics
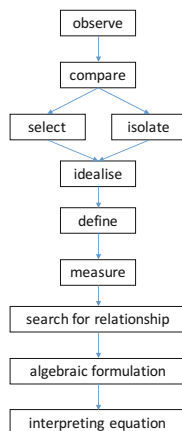
interesting to be investigated. The process under consideration (e.g. the motion of an
air bubble) is observed in more detail; relevant aspects are identified by comparison
and isolated. Usually an observable is identified or defined to allow measurement.
In measurements numerical data are produced (quantification), which (often but not
necessarily via a graphical representation) lead to an algebraic statement, usually
an equation (mathematization). The specific situation is encapsulated in a diagram
or equation. However, the equation stands for much more than just this specific
situation from which it was obtained, e.g. represents all uniform motions (with
velocity v). In total, a generalization is made and a mathematical statement usually
is the end result.[3] Thereby, the gap between the real world and the world of
mathematics has been bridged (cf., Fig. 5.1).

> Mathematisation comes with generalisation and downgrades specific situa-
> tions to examples of a more general insight.[4]

---

[3]Further examples of equations that are likely to be introduced inductively include $x(t) = vt$
(uniform motion), $x(t) = \frac{1}{2}at^2$ (accelerated linear motion), $R = \frac{U}{I} = const.$ (Ohm's law),
$R = \varrho \frac{l}{A}$ (Pouillet's law), $Q = mc\Delta T$ (heat equation), etc.

[4]Depending on your perspective, you may wish to emphasize the epistemological upgrade in terms
of generality, width of applicability, elegance and simplicity that comes with the mathematical
formulation of physical knowledge about the world. However, focusing on the concrete problem
(e.g. the motion of an air bubble) from which this exploration started, this specific problem is
downplayed and loses its relative importance as it becomes an example of a class of processes with
many other processes in this class that are basically the same in that they are just one of many
examples for a uniform motion. While none of these perspectives is more correct than the other, I
doubt that many young learners can truly appreciate the (surely valid) upgrade at a first encounter.

Box 5.1: Uniform Motion

Task Using the given glass tube filled with coloured water, investigate the motion of the air bubble in the tube, when one end is lifted a bit.

Expectations and Objectives Write down what kind of motion you expect. Give reasons!

_____

_____

Materials/Equipment glass tube filled with coloured water, marker pen, stop watch, ruler

Procedure Lift one end of the tube a little, so that the air bubble takes at least 30s from one end to another. Mark the position of the air bubble every 5s (marker pen). Take measures of the total distances travelled (total displacement) after 5s, 10s, 15s, ...
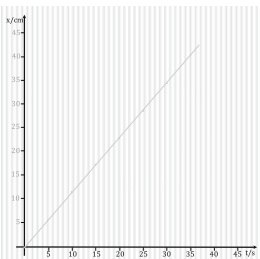
Observations and Measurements

t in s	0	5	10	15	20	25	30	35	40
x in cm	0	5,4	11,2	16,9	22,7	28,6	34,3	40,2	46,1

observations: _____

Data Analysis and Conclusion

1. Plot the measured distances over time.

2. Summarize your findings mathematically.

$$x \sim t$$

For every time t and the related distance x we have

$$v = \frac{x}{t} = constant.$$

This can be described as a uniform motion, since the air bubble travels approximately the same distance every 5s. The velocity therefore is constant.

To experts this feature of generality that comes with equations is what makes them "icons of knowledge" (Bais 2005), important nodes in an expert's widely branching web of knowledge. For learners, however, equations can (and often do) become isolated facts that are considered to be the only really important message, which leads to an "obsession" with formulas (Schecker 1985). For example, $v = \frac{x}{t}$ (v – velocity, x – displacement, t – time) is what is usually remembered, after the teaching/learning path described above. This can already be seen as a seed for the common practice of the subsequent instrumental use of equations described in the literature as calculation framing (Bing and Redish 2009) and recursive plug and chug (Tuminaro and Redish 2007), which are part of the technical dimension (Karam 2014) of the use of mathematics in physics, characterized by low-level activities (Krey 2012).

The above-illustrated inductive approach certainly is a valid one that also has its parallels in the history of physics as well as today's research. However, in school physics this approach is probably over-represented. Reasons (among others) can be seen in the assumed learners' lack of mathematical prerequisites that would allow

for alternatives (See Karam, Uhden, & Höttecke in this volume to clarify the validity of this assumption.) or the relatively easy planning by the teacher for given-sought scheme problems.

By teaching along this pattern, an implicit message about the role of mathematics in physics, and therefore the nature of physics, is delivered. This message may include the false (or at least one-sided) impressions that, e.g.:

- Experiments are the beginning of any knowledge generation process in physics.
- The main function of mathematics is to summarize experimental results and allow for calculations.
- Quantification and mathematization are more or less the same.
- Mathematics describes reality, using artificial signs and symbols.
- Making use of mathematics means to make statements in a largely arbitrary language to describe insights more easily to be expressed in our natural language.

That all the above-mentioned statements are problematic is well documented in the literature (Cartwright 1983; Koponen and Mäntylä 2006; Krey 2012).

However, there are valid lessons to be learned here as well. As mentioned before, in the example above, it is the formula $v = \frac{x}{t}$ (v – velocity, x – displacement, t – time) that is usually remembered by pupils. Quite often the equation stands for both the definition of velocity and the characterization of a uniform motion. From a teacher's perspective, this can be an intermediate step, but not a satisfying end result. The equation above is the definition of an average velocity for the time interval $[0; t]$, and for learners it is an additional step to understand that $v = \frac{x}{t} = const.$ is what makes this equation an appropriate description for a uniform motion. Even more precise would be a quantification like "for all t". This implies that the initially learned concept of velocity over the years becomes more refined, and formerly ignored or unseen details become accentuated and perhaps more meaningful. The language of mathematics in which statements about velocity are made allows (and forces) to express these refinements carefully. The distinction between momentary (limit concept) and average velocity (ratio concept) shows how this differentiation process occurs conceptually and mathematically.

> Mathematization requires and induces conceptual precision.

In the above example as well as in many other cases collecting data in table, representing them graphically and expressing regularities algebraically (and perhaps even retranslating into common language) are a quite common strategy. It develops pupils' abilities to deal with different types of representations. Learners can experience (often implicitly) that different representations have certain advantages and disadvantages. For example, a certain kind of relationship between observables may be recognized easier in a graph than in a table; an equation allows to chunk different dependencies between observables. The lesson to be learned:

> Different mathematical representations (of the same content) allow access to information in different ways and therefore make one aspect easier accessible than others.

To embrace this message fully, an understanding of (mathematical) representations, their different strength and weaknesses and their coordination in case multiple representations are used is required (Ainsworth 2006). However, at least in German physics lessons and especially in lab work activities, the three-step approach table-graph-equation to collect, visualize and summarize measurement readings often becomes a ritual instead of a rational choice. (See Geyer and Kuske-Janßen in this volume for a more detailed description of translation activities.) This again gives very good reason to students to think about the use of mathematics as something static, ritualized and algorithmic rather than a creative, question-driven and goal-oriented process.

The example of the introduction of the concept of constant velocity and uniform motion hopefully made visible a few of the usually missed learning opportunities about the role of mathematics in physics as a vital part of the nature of physics. Three non-trivial insights were emphasized. If made explicit to learners, those insights might be able to initiate thought processes that perhaps over several years in school may help to develop a more appropriate view on the role(s) of mathematics in physics.

In what follows, another basic example that is part of every secondary physics curriculum is considered. This example helps to illustrate other aspects of an attempt to establish a connection between the real world and the world of mathematics. As it is one of the rare cases in secondary school physics making use of a deductive approach, the example is meant to complement the view of mathematics as a mere language used to express experimental results more precisely and more efficiently as suggested implicitly by the inductive approach. For this purpose we consider a second field of physics, namely, ray optics and more specifically the magnification equation and the thin lens equation.

5.3 Case Study 2: Image Formation by Thin (Converging) Lenses

There are two equations that usually are presented to the students when teaching about ray optics and image formation by thin lenses. The magnification equation $\frac{h_i}{h_o} = \frac{d_i}{d_o}$ and the thin lens equation $\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$ are usually considered for convex lenses (h – height, d – distance to lens, o – object, i – image, f – focal length). Although they are closely related to one another, for many students they are two isolated facts that have to be learned. In the case of the lens equation, an immediate

interpretation is difficult, as the form of the equation is pretty unfamiliar to a learner of school physics. This may explain part of the difficulties to understand the full meaning of these equations and the relation between them.

5.3.1 Towards the Magnification Equation

A teaching sequence could start with an explorative experimental setting from which students learn that for a given object distance (d_o), a sharp image formation on the screen is only possible for a certain image distance (d_i). They may also come up with the hypothesis that the size of a sharp image (image height, h_i) is somehow related to the object distance (d_o). A more explanative experimental approach may follow in which the relation between object distance (d_o), object height (h_o), image distance (d_i) and image height (h_i) is investigated. Sooner or later pupils usually can be convinced that $\frac{h_i}{h_o} = \frac{d_i}{d_o}$ is a valid statement. So far everything that has been said about the role of mathematics in the case of an inductive approach before can be said here as well. A pattern is found within the measurements and students express that pattern more precisely by means of the magnification equation. However, something is different here already. The equation is valid only for a specific constellation, the case of a sharp image formation. That is why we usually write it as a proportion, knowing that the kind of continuity that can be found in the equation describing a uniform motion ($x = vt$) is not a part of the package so to speak. At a first glance however, it would be totally acceptable to write $h_i = \frac{h_o}{d_o} d_i$, and wishful thinking may suggest that for a given object height and object distance, any image height may be realized by adjusting the image distance (screen position). From a purely mathematical point of view, it is no problem to consider a function $h_i(d_i)$, and there is no reason why one would assume that the domain of this function would not be R^+, the positive real numbers. And given the formal analogy between the two equations $x(t) = v \cdot t$ and $h_i(d_i) = \left(\frac{h_o}{d_o} \right) \cdot d_i$, it is (from a mathematics point of view) surprising that the application for describing natural phenomena requires different constraints to more or less the same mathematical concept. For the physicist this may not be of much surprise, but for pupils this seems worth to be made explicit, as it points to an important feature of doing physics and the interplay between mathematics and physics.

> The often seemingly perfect mapping between real world and mathematics is not a guaranteed given and it is by far not self-evident.

In the example discussed, the mathematical structure is more comprehensive than the real-world situation does require. However, at least there is a mathematical structure that does fit our needs.

5.3.2   Towards the Thin Lens Equation

The second equation in the field of ray optics and image formation by thin lenses (lens equation) is usually motivated by the question about how image distance and object distance relate to one another for a given lens. A possible activity for students is presented in Box 5.2.

Box 5.2: Thin Lens Equation – Part 1

The photo shows a candle and a sharp inverted image of the candle appearing at a screen behind the converging lens.

Investigate for this lens (and two other converging lenses) how the image distance depends on the object distance.
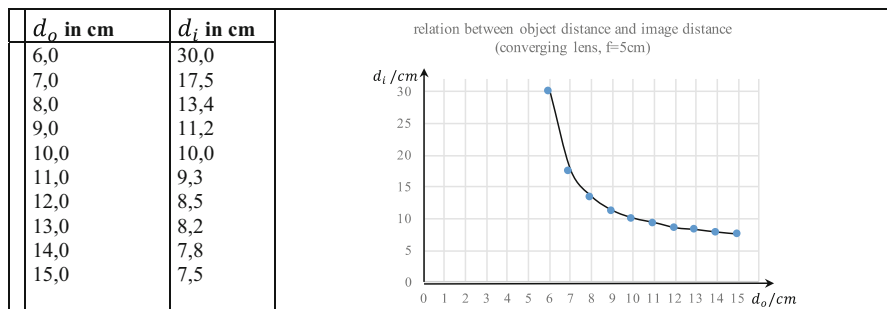
Generalize the relationship you discovered in terms of a functional dependency.

By doing a few explorative experimental moves, which is an important part of finding a problem solution, pupils usually find a few regularities. For example, they will find that by placing the candle too close to the lens ($d_o \leq f$), a sharp image formation cannot be observed on the screen. They may remember from constructing ray diagrams that a virtual image is formed. For candle positions outside the focal length ($d_o > f$), one finds that when moving the candle (object) closer to the lens, the image location will move away from the lens. So both the location of the object and the image move in the same direction.

For further investigation there are at least two different strategies available – an inductive experimental strategy and a deductive mathematical strategy. From my experience most pupils and (also students at university) will follow the experimental approach, which is not much of a surprise, since as mentioned before, this is what happens in physics lessons quite often, when a physical law is introduced. (One could argue that it is simply more fun to do an experiment. Well, I think this could be true, but what we consider to be fun is learned, at least to a large extent.) They vary the object distance, perhaps in 1 cm steps, and make measurements of the related image distance. For a converging lens ($f = 5cm$), this may lead to the table of measurements and their graphical representation presented in Box 5.3.

This way, no doubt, a regularity is found, and by plotting the graphs for different lenses, one may obtain a more general statement about the relationship. A few students may perhaps be convinced quickly that this is a linear relation by only considering object distances not too close to the focal length, which is not surprising, since most relations we consider in school physics are linear ones.

Box 5.3: Measurement Readings and Plot Showing the Relation Between d_o and d_i

d_o **in cm**	d_i **in cm**	
6,0	30,0	
7,0	17,5	
8,0	13,4	
9,0	11,2	
10,0	10,0	
11,0	9,3	
12,0	8,5	
13,0	8,2	
14,0	7,8	
15,0	7,5	

relation between object distance and image distance
(converging lens, f=5cm)

Those who make it to a more appropriate graph would be happy that they have found and documented a relation in an empirical way. How one could express the regularity found in an algebraic representation is a question that also bears an opportunity to learn *about* physics. Physicists strive for understanding – often by investigating specific phenomena in an experimental setting – and describe relations in mathematical terms, but in the end they want to have general laws from which they can form explanations and predictions. As Feynman puts it:

First we have an observation, then we have numbers that we measure, then we have a law which summarizes all the numbers. But the real *glory* of science is that *we can find a way of thinking* such that the law is *evident*. (Feynman et al. 1964; 26-3)

This is a rather deep reason for why a structural role of mathematics is essential for physicists, implying that mathematical representations help to make logical deductions which provide "a way of thinking such that the law is evident" and therefore are essential tools for the analysis of natural phenomena. This rather philosophical approach can be reduced to a more pragmatic argument that perhaps is more suitable to convince learners of the helpfulness of a deductive mathematical strategy, at least in the context of our example, as new measurements have to be taken for every lens and for every new object distance that has not been investigated before. This also points to an interesting feature of the physics enterprise. Apparently, on one hand asking "*Why*?", the search for reason and understanding is what keeps physicists going. On the other hand, however, their knowledge is a huge collection of *how* observables are related. And of course, this web of knowledge is structured, and there are statements of different epistemological and hierarchical status that allow to form explanations and predictions (cf. the different epistemo-

logical categories for physics equations described in the framework section), but whether this is the same as an answer to the question "Why?" can be argued.

Box 5.4 shows the main steps in deriving the lens equation from prior knowledge including mainly (a) the magnification equation,[5] (b) the ability to read and mentally manipulate ray diagrams, and (c) the fact that light travels in a straight line. The – in this case geometrical – representation of the situation allows an analysis and extended mental manipulation. Without visualizing the situation in a geometrically "correct" way, it is much more difficult to identify meaningful relations between the relevant observables.

> Mathematical representations (e.g. geometrical diagrams, tables, graphs, equations, symbols) materialise abstract ideas (entities and the relations between them). These ideas can be manipulated (literally) (Fischer 2006) and therefore serve as external cognitive tools.

As mentioned before, the derivation of the thin lens equation is one of the rather rare opportunities to demonstrate a deductive way to generate knowledge in the secondary physics classroom. Based on already established knowledge, a geometrical representation is generated, analysed and combined with already established pieces of knowledge represented algebraically (magnification equation). Through purely mathematical reasoning, a "new" equation is produced and interpreted. Perhaps predictions based on the derived equation are made and tested by experiment (Fig. 5.3).

**Fig. 5.3** Deductive way of teaching physics

---
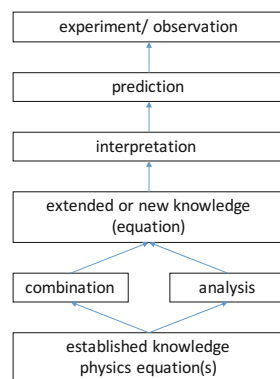
[5]Of course, the magnification equation can also be derived by using the intercept theorem, which shows the deep structural equivalence between geometry and light propagation based on the ray model of light, which allows the mapping between relevant aspects of the real world and geometric entities.

Box 5.4: Thin Lens Equation

Ray diagram

The necessary translation from the real world situation to this diagram is not an easy task. Physics knowledge and mathematical representational ability have to be coordinated in order to come up with a helpful representation.
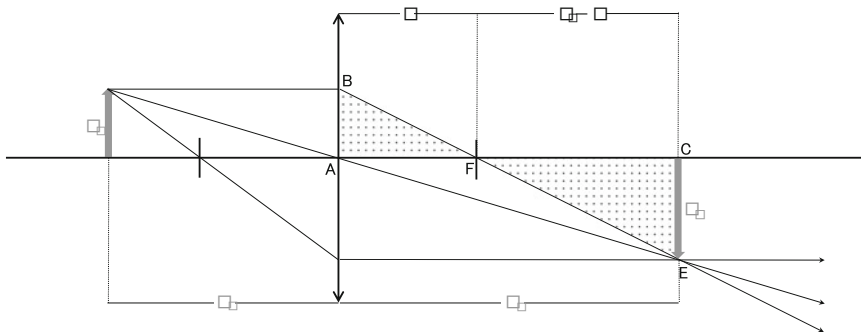
An analysis by geometrical means

1. Since the measure of each interior angle in ΔABF is equal to the measure of the corresponding angle in ΔFCE, the two triangles are similar.

2. The intercept theorem applies, since $\acute{A}B$ und $\acute{C}E$ are parallels. We can therefore establish a relation between $f, d_i - f, h_o, h_i$.

Finding an algebraic form of representation and combine it with prior knowledge

The intercept theorem gives $\frac{h_i}{h_o} = \frac{d_i - f}{f}$. With the magnification equation $\frac{h_i}{h_o} = \frac{d_i}{d_o}$ we get

$$\frac{d_i}{d_o} = \frac{d_i - f}{f} \therefore d_i f = d_o(d_i - f) \therefore d_i f - d_o d_i = -d_o f \therefore d_i = \frac{d_o f}{d_o - f}.$$

The result of the suggested derivation is $d_i = \frac{d_o f}{d_o - f}$. Most textbooks, however, will present $\frac{1}{f} = \frac{1}{d_o} + \frac{1}{d_i}$ as the lens equation. While the first is a direct answer to the problem/question that motivated the investigation, the latter seems to satisfy a purely aesthetic concern. It's a matter of taste to prefer a certain kind of simplicity here – and that actually is a bit of a surprise. Aesthetics and taste sometimes function as guiding principles or at least are influential to the physicist's work. This indeed is astonishing to most learners but also clearly indicates that physics is a human enterprise. The more so, as this aesthetic concern is not related to experimental equipment or phenomena that can be beautiful as well but to mathematical representations used in physics.

> Aesthetical aspects are influential to the physicist's work.

Of course, going through this derivation can be a challenge for pupils – for some of them, it involves barriers too high to overcome by themselves. For most students the connection between the symbols and their meaning can get lost, and they blindly follow the manipulation of mathematical symbols. In a way that is not much of a surprise, since this is another great feature of using mathematics in physics. The signs themselves are meaningless; they only carry the meaning that has been explicitly ascribed to them. To be meaningful these signs need to be embedded in a natural language, and they and the results achieved by their manipulation cannot be more precise than the precision with which one can say what these signs actually mean. By the use of the mathematical mechanism, one only has to be diligent in executing the manipulations to keep the same level of precision without actually thinking about it. Interpreting the manipulated equation, however, does require deeper thought (cf. von Weizsäcker 2004, p. 90). It is this conservation of the real-world relations within the mathematical calculus that allows the tool-like use of mathematics in physics. Whether this can be understood at all or whether one simply has to get used to it is not my concern here. However, I argue that to know about the roles and functions that mathematics plays in physics gives a more complete understanding of the nature of physics.

> The mathematical calculus that comes with a physics theory allows to find logical implications without content-related deeper thought (reduction of cognitive load). The derived statements are as precise as the definitions of all observables and operations allows them to be.

For the learner, lost in translation, the interpretation of the derived equation is of particular importance. Sometimes, and especially for younger pupils, it might be necessary or extremely helpful to demonstrate and check the validity of the new (intermediate) equation by prediction and experimental test.

In school we usually end here and one can wonder if all this is worth it. However, as far as I can see, by going through the whole process and pointing out the more or less obvious features of the use of mathematics in physics, by relating the world of mathematics to the experimental world and by bringing together experiences in manipulating the experimental setting and mathematical analysis, we create a learning opportunity worthwhile. This may help to make the thin lens equation not just another "paper flower" (Wagenschein 1995, S. 288) but the meaningful peak of a challenging learning path.

5.3.3   The Thin Lens Equation: Beyond the Secondary School Physics Curriculum

What follows goes beyond the average physics school curriculum, but illustrates an-other important feature of the use of mathematics in physics. This feature becomes visible by considering the implications of an already constructed mathematical representation, which sometimes helps to suggest new insights about the real world.

Above we only considered the case of $f < d_o$ which lead to a first quadrant branch of a hyperbola. By connecting the plotted measurement readings to a continuous graph, the idealization of "continuity" was already applied (and the usefulness of irrational values for the object and image distances was assumed, although those measurement readings obviously do not occur). But otherwise, the branch of the hyperbola conserves the observations made: $d_o \longrightarrow f \implies d_i \longrightarrow \infty$, $d_o = 2f \implies d_i = 2f$, $d_o \longrightarrow \infty \implies d_i \longrightarrow f$. Exploring the derived thin lens equation $d_i = \frac{d_o f}{d_o - f}$ from a mathematical point of view reveals that there must be a second branch of the hyperbola for $f > d_o$. A first part of this branch ($f > d_o > 0$) turns out to describe the formation of virtual images, for which is $d_i < 0$ by convention. So the full consideration of the mathematical solution suggests the existence of a real world entity – virtual images in this case. And also the second part of this hyperbola branch ($d_o < 0$, dotted line in Fig. 5.4) turns out to have a meaningful interpretation for the image formation for virtual objects.
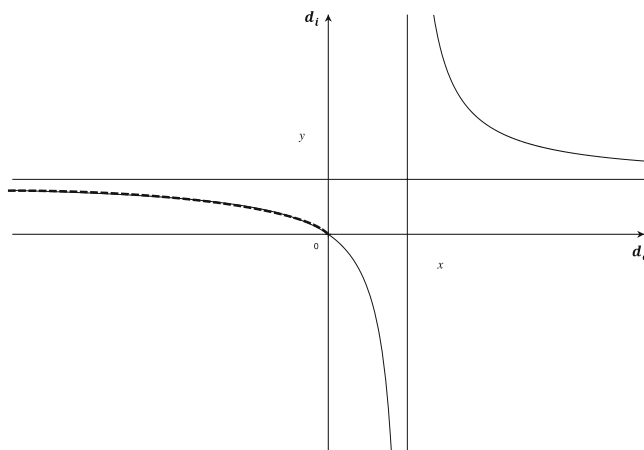
Fig. 5.4 Different parts of the hyperbola describing image formation for real objects and real images (first quadrant), but also for real objects and virtual images (fourth quadrant) and virtual objects (second quadrant)

> Mathematical descriptions can suggest the existence of real world entities or relations.

I am not suggesting that this has to become a necessary curricular content in high school physics or that this would correspond to a historical development; it does not. However, I consider this case to be a helpful example to illustrate an important feature of the mathematics physics interplay. While there are other correspondences that may help to point towards that feature (e.g. Snell's law of refraction and the critical angle), it is hard to find a historically valid and authentic example (such as the prediction of the positron) that can be discussed easily in a high school physics classroom.

5.4 Summary

Within an analytic framework based on Ludwig's conception of a physical theory that allows to view physics as an attempt to build bridges between the real world and the world of mathematics, two cases have been looked at in more detail, the uniform motion and the image formation by thin lenses. This helped to see more clearly which problems and potentials building and using these bridges may provide. The inductive way of introducing a physics equation and (even more so) the less usual deductive way (see Redfors et al. in this volume) are full of more or less implicit messages sent to our students (un)intentionally by the teacher.[6] The explicit consideration or even discussion of these messages bears great potential for learning about the role of mathematics in physics and therefore about the nature of science. It may be considered wishful thinking by some and a plausible hypothesis by others that using mathematics in physics is not necessarily a barrier for learning or limited to routine activities used in given-sought problems but actually a great cognitive tool that allows for analysis and understanding of physics. Perhaps we could even assume that the way we use mathematics in the physics classroom could have a direct influence on the content learning and conceptual understanding of physics. Physics is a difficult subject, and the use of mathematics does not make it easier, but

---

[6]It should be clarified that by no means the author intends to argue for a more deductive and against inductive teaching sequences. Both of them have their advantages and disadvantages. They are adequate to situations, outcomes and learners or they are not. For example, an arranged deductive way of introducing a law of physics (even the lens equation the way it was presented) can lead to lots of low-level activities on the learners' side and simply illustrate bad teaching. Furthermore, both approaches can help teach valuable lessons about the nature of science in general as well as the role of mathematics in (learning) physics more specifically. What the author wishes to say though, is that both approaches to teaching differ in the learning opportunities they can potentially provide with regards to the mathematics-physics-interplay.

it might well be possible that by learning about the nature of physics and about the role mathematics plays within this human endeavour, physics is perceived as more meaningful. If this article helped to generate or elaborate an idea of how the general roles of mathematics in physics can be found or identified in an everyday physics classroom – perhaps sometimes by verbalizing the obvious, the author would be delighted.

References

Abd-El-Khalick, F. (2012). Nature of science in science education: Toward a coherent framework for synergistic research and development. In B. J. Fraser, K. G. Tobin, & C. J. McRobbie (Eds.), *Second international handbook of science education* (pp. 1041–1060). Dordrecht/Heidelberg/London/New York: Springer.

Abd-El-Khalick, F., & Lederman, N. G. (2000). Improving science teachers' conceptions of nature of science: A critical review of the literature. *International Journal of Science Education, 22*(7), 665–701.

Ainsworth, S. (2006). DeFT: A conceptual framework for considering learning with multiple representations. *Learning and Instruction, 16*(3), 183–198.

Bais, S. (2005). *The equations. Icons of knowledge*. Cambridge, MA: Cambridge University Press.

Bing, T. J., & Redish, E. F. (2009). Analyzing problem solving using math in physics: Epistemological framing via warrants. *Physical Review Special Topics – Physics Education Research, 5*, 1–15.

Cartwright, N. (1983). *How the Laws of physics lie. English*. Oxford: Oxford University Press.

Dittmann, H., Näpfel, H., & Schneider, W. B. (1989). Die zerrechnete Physik. In W. B. Schneider (Ed.), *Wege in der Physikdidaktik. Band 1. Sammlung aktueller Beiträge aus der physikdidaktischen Forschung* (Vol. 1, pp. 41–46). Erlangen: Palm & Enke.

Einstein, A. (2002). Geometrie und Erfahrung. Zweite Fassung des Festvortrages gehalten an der Preussischen Akademie der Wissenschaften zu Berlin am 27. Januar 1921. In M. Janssen, R. Schulmann, J. Illy, C. Lehner, & D. K. Buchwald (Eds.), *The collected papers of Albert Einstein. Volume 7. The Berlin years: Writings, 1918–1921* (Vol. 7, pp. 382–388). Princeton: Princeton University Press (Reprint).

Falk, G. (1990). *Physik: Zahl und Realität. Die begrifflichen und mathematischen Grundlagen einer universellen quantitativen Naturbeschreibung*. Basel: Birkhäuser.

Feynman, R., Leighton, R., & Sands, M. (1964). *The Feynman lectures on physics, volume I. mainly mechanics, radiation, and heat*. Reading: Addison-Wesley. Retrieved from http://www.feynmanlectures.caltech.edu/.

Fischer, R. (2006). *Materialisierung und Organisation*. München/Wien: Profil Verlag.

Hegarty-Hazel, E. (Ed.). (1990). *The student laboratory and the science curriculum*. London/New York: Routledge.

Höttecke, D. (2012). HIPST—History and philosophy in science teaching: A European project. *Science & Education, 21*(9), 1229–1232.

Karam, R. (2014). Framing the structural role of mathematics in physics lectures: A case study on electromagnetism. *Physical Review Special Topics – Physics Education Research, 10*(1), 10119-1–10119-23.

Karam, R., & Krey, O. (2015). Quod erat demonstrandum: Understanding and explaining equations in physics teacher education. *Science & Education, 24*(5), 661–698.

Koponen, I. T., & Mäntylä, T. (2006). Generative role of experiments in physics and in teaching physics: A suggestion for epistemological reconstruction. *Science and Education, 15*(1), 31–54.

Krey, O. (2012). *Zur Rolle der Mathematik in der Physik. Wissenschaftstheoretische Aspekte und Vorstellungen Physiklernender*. Berlin: Logos Verlag.

Leach, J., & Paulson, A. (Eds.). (1999). *Practical work in science education. Recent research studies*. Roskilde: Roskilde University Press.

Lederman, N. G., Abd-El-Khalick, F., Bell, R. L., & Schwartz, R. S. (2002). Views of nature of science questionnaire: Toward valid and meaningful assessment of learners' conceptions of nature of science. *Journal of Research in Science Teaching, 39*(6), 497–521.

Lehavi, Y., Bagno, E., Eylon, B.-S., Mualem, R., Pospiech, G., Böhm, U., Krey, O., & Karam, R. (2017). Classroom evidence of teachers' PCK of the interplay of physics and mathematics. In *Key competences in physics teaching and learning* (pp. 95–104). Cham: Springer. https://doi.org/10.1007/978-3-319-44887-9.

Losee, J. (2001). *A historical introduction to the philosophy of science* (4th ed.). Oxford: Oxford University Press.

Ludwig, G. (1985). In Z. Raum (Ed.), *Einführung in die Grundlagen der Theoretischen Physik* (Vol. 1). Braunschweig/Wiesbaden: Vieweg.

Ludwig, G. (1990). *Die Grundstrukturen einer physikalischen Theorie*. Berlin: Springer.

Lunetta, V. N., Hofstein, A., & Clough, M. P. (2007). Learning and teaching in the school science laboratory: An analysis of research, theory, and practice. In S. K. Abell & N. G. Lederman (Eds.), *Handbook of research on science education* (pp. 393–441). New York/London: Routledge.

McComas, W. F., Clough, M. P., & Almazroa, H. (1998). The role and character of the nature of science in science education. In W. F. McComas (Ed.), *The nature of science in science education. Rationales and strategies* (pp. 3–40). Dordrecht: Kluwer Academic Publishers.

Osborne, J., Collins, S., Ratcliffe, M., Millar, R., & Duschl, R. (2003). What "ideas-about-science" should be taught in school science? A Delphi study of the expert community. *Journal of Research in Science Teaching, 40*(7), 692–720.

Rutherford, F. J., & Ahlgren, A. (1990). Science for all Americans. Retrieved from http://www.project2061.org/publications/sfaa/online/sfaatoc.htm

Schecker, H. (1985). *Das Schülervorverständnis zur Mechanik. Eine Untersuchung in der Sekundarstufe II unter Einbeziehung historischer und wissenschaftshistorischer Aspekte*. Bremen: Universität Bremen.

Sherin, B. L. (2001). How students understand physics equations. *Cognition and Instruction, 19*(4), 479–541.

Shulman, L. S. (1986). Those who understand. Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Steinle, F. (1998). Exploratives vs. theoriebestimmtes Experimentieren: Ampères frühe Arbeiten zum Elektromagnetismus. In M. Heidelberger & F. Steinle (Eds.), *Experimental Essays – Versuche zum Experiment* (pp. 272–297). Basden-Baden: Nomos Verlag.

Tuminaro, J., & Redish, E. F. (2007). Elements of a cognitive model of physics problem solving: Epistemic games. *Physical Review Special Topics – Physics Education Research, 3*, 1–22.

Uhden, O., Karam, R., Pietrocola, M., & Pospiech, G. (2012). Modelling mathematical reasoning in physics education. *Science & Education, 21*, 485–506. https://doi.org/10.1007/s11191-011-9396-6.

von Weizsäcker, C. F. (2004). *Der begriffliche Aufbau der theoretischen Physik*. Stuttgart/Leipzig: S. Hirzel.

Wagenschein, M. (1995). *Die pädagogische Dimension der Physik*. Aachen: Hahner Verlagsgesellschaft.

Wenning, C. J. (2009). Scientific epistemology: How scientists know what they know. *Journal of Physics Teacher Education Online, 5*(2), 3–15.