# Some New Applications of the Theory of Conjugate Differential Forms

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**Abstract** In this survey we describe two applications of the concept of conjugate differential forms. Namely, after describing the concept of conjugate and self-conjugate differential forms, we consider an extension of the Brothers Riesz theorem to higher real dimension and Riesz-type inequalities for differential forms.

## 1 Introduction

Many years ago, looking for a generalization of the Brothers Riesz theorem in higher real dimension, I was led to consider the concept of conjugate differential forms [5]. Such concept has been already used in the previous paper [4], in which I had to construct a reducing operator for a particular singular integral operator. This is why I began to study in detail conjugate differential forms and self-conjugate (non-homogeneous) differential forms [7].

Later on I have used such forms in several different problems. They concern, besides the extension of the Brothers Riesz theorem in higher real dimension, the concept of conjugate Laplace series in $\mathbb{R}^n$ [2, 3, 6, 9], potential theory with applications to several BVPs for different PDEs [1, 11, 13–20], and Riesz-type inequalities for differential forms [12].

In this brief survey I will just consider the Brothers Riesz theorem and Riesz-type inequalities. The first section is devoted to the concept of conjugate and self-conjugate differential forms.

For a survey on the applications in potential theory connected to BVPs I refer to [10].

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## 2 Self-Conjugate Differential Forms

The idea of considering conjugate differential forms in order to extend the concept of conjugate harmonic functions dates back to Volterra [31]. Following this order of ideas, we say that a $k$-form $u$ (i.e., a differential form of degree $k$) and a $(k+2)$-form $v$ are *conjugate* in $\Omega \subset \mathbb{R}^n$ if

$$du = \delta v, \quad \delta u = 0, \quad dv = 0, \tag{1}$$

where $d$ is the differential operator and $\delta$ is the co-differential (actually this concept is slightly different from the one given by Volterra: in fact $u$ and $v$ are conjugate in the sense of Volterra if $du = \delta v$, see [31], pp. 87–90). If $n = 2$, $f(z) = u(x, y) + iv(x, y)$ is a holomorphic function and we identify $v$ with a 2-form, then $du = \delta v$ is just the Cauchy–Riemann equation, while $\delta u = 0$ and $dv = 0$ are automatically satisfied.

The system (1) includes several real generalizations of the Cauchy system.

For example, this concept of conjugate forms is more general than the concept of *harmonic vectors* considered by Stein and Weiss in the paper [28], i.e., of vectors $(w_1, \ldots, w_n)$ satisfying the system

$$\sum_{i=1}^{n} \frac{\partial w_i}{\partial x_i} = 0, \qquad \frac{\partial w_i}{\partial x_j} = \frac{\partial w_j}{\partial x_i} \; (i \neq j)\,. \tag{2}$$

In fact, if we identify $(w_1, \ldots, w_n)$ with the 1-form $u = w_h dx^h$, the system (2) is nothing but $du = 0$, $\delta u = 0$. In other words Stein and Weiss have considered only the forms which are of degree 1 and conjugate to $v = 0$.

More generally, the $k$-form

$$u_k = \frac{1}{k!}\, w_{s_1 \ldots s_k}\, dx^{s_1} \ldots dx^{s_k}$$

is conjugate to $u_{k+2} \equiv 0$ if, and only if, $du_k = 0$ and $\delta u_k = 0$. These are the so-called *harmonic forms*.

If we consider $n = 3$ and $u_0 \equiv u, u_2 = v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2$, we have that $u_0$ and $u_2$ are conjugate if, and only if $\operatorname{div}(v_1, v_2, v_3) = 0$, $\operatorname{grad} u = \operatorname{curl}(v_1, v_2, v_3)$, i.e., if, and only if, the vector $(u, v_1, v_2, v_3)$ satisfies the *Moisil-Theodorescu system*.

The concept of conjugate differential forms can be further generalized. Let us consider a non-homogeneous differential form belonging to $C^1_0(\Omega) \oplus \ldots \oplus C^1_n(\Omega)$

$$U = \sum_{k=0}^{n} u_k$$

where $u_k$ is a differential form of degree $k$. We say that $U$ is self-conjugate if $dU = \delta U$, i.e., if $\delta u_1 = 0$, $du_k = \delta u_{k+2}$ $(k = 0, \dots, n-2)$, *and* $du_{n-1} = 0$.

It is clear that if $U = u_k + u_{k+2}$, then $U$ is *self-conjugate* if and only if $u_k$ and $u_{k+2}$ are conjugate in the sense of (1).

If $n = 4$ and $U = u_0 + u_2 + u_4$, where

$$u_0 = f_0, \quad u_4 = f_0 dx^0 dx^1 dx^2 dx^3$$

$$u_2 = f_1(dx^0 dx^1 - dx^2 dx^3) + f_2(dx^0 dx^2 - dx^3 dx^1) + f_3(dx^0 dx^3 - dx^1 dx^2),$$

the non-homogeneous form $U$ is self-conjugate if, and only if,

$$\begin{cases} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0 \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0 \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} = 0 \\ \frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} = 0 \,. \end{cases}$$

This shows that $U$ is self-conjugate if, and only if, the vector $(f_0, f_1, f_2, f_3)$ satisfies the *Fueter system*.

A similar computation shows that the form $U = u_0 + u_2 + u_4$, where

$$u_0 = f_0, \quad u_4 = -f_0 dx^0 dx^1 dx^2 dx^3$$

$$u_2 = f_1(dx^0 dx^1 + dx^2 dx^3) - f_2(dx^0 dx^2 + dx^3 dx^1) + f_3(dx^0 dx^3 + dx^1 dx^2),$$

is self-conjugate if and only if the vector $(f_0, f_1, f_2, f_3)$ satisfies the *Cimmino system* (see [1]).

In what follows we shall use also the concept of *k-measure*, which was introduced by Fichera (see [22, 23]). Roughly speaking a $k$-measure is a differential form whose coefficients are measures and we refer to Fichera's papers for the precise definition and for several properties.

## 3 The Brothers Riesz Theorem

In their only joint paper [26] F. Riesz and M. Riesz proved this famous result:

**Theorem 1** *If a trigonometric series and its conjugate series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\vartheta + b_k \sin k\vartheta) \ , \qquad \sum_{k=1}^{\infty} (a_k \sin k\vartheta - b_k \cos k\vartheta)$$

*are both Fourier–Stieltjes series, then they are ordinary Fourier series.*

In other words, if we have two real measures $\alpha$, $\beta$ defined on the Borel sets of $[0, 2\pi]$ such that

$$\int_0^{2\pi} \cos k\vartheta \, d\alpha = \int_0^{2\pi} \sin k\vartheta \, d\beta, \int_0^{2\pi} \sin k\vartheta \, d\alpha = -\int_0^{2\pi} \cos k\vartheta \, d\beta \quad (k = 1, 2, \ldots), \tag{3}$$

then these measures have to be absolutely continuous, i.e., there exist two real valued $L^1$ functions $f$ and $g$ such that

$$\alpha(E) = \int_E f(\vartheta)\, d\vartheta, \qquad \beta(E) = \int_E g(\vartheta)\, d\vartheta$$

for any Borel set $E \subset [0, 2\pi]$. The interest of this result in the theory of Fourier series is evident. Theorem 1 can be easily rewritten in a "complex" form:

**Theorem 2** *If $\mu$ is a complex measure defined on the Borel sets of the unit circle $C = \{z \in \mathbb{C} \mid |z| = 1\}$ such that*

$$\int_C e^{ik\vartheta} d\mu = 0 \qquad k = 1, 2, \ldots,$$

*then $\mu$ is absolutely continuous, i.e., there exists a function $f \in L^1(C)$ such that*

$$\mu(E) = \int_E f(\vartheta) d\vartheta$$

*for any Borel set $E$ of $C$.*

This beautiful theorem gave rise to a long series of papers and "*in its direct applications as well as the generalizations it has inspired, this has proved to be one of the more important theorems of the century*" (R. B. Burckel, *Math. Rev.*, 96k:43009). For a survey of several results connected to the Brothers Riesz theorem, see [8] and the references therein.

The classical Brothers Riesz theorem can be stated also in the following way: if $u(x, y)$ and $v(x, y)$ are two conjugate real harmonic functions in a domain $\Omega$ and both of them have traces on $\partial\Omega$ in the sense of measures, then these measures have to be absolutely continuous.

Such a result was proved for conjugate differential forms and—more generally—for non-homogeneous self-conjugate differential forms in [5]. The result is the following. Here $\Omega$ is a bounded domain in $\mathbb{R}^n$ with a Lyapunov boundary and $M_k(\Sigma)$ denotes the space of $k$-measures defined on the Borel sets of $\Sigma$.

**Theorem 3** *If* $U \in C_0^1(\Omega) \oplus \ldots \oplus C_n^1(\Omega)$ *is self-conjugate and* $U$ *and* $*U$ *admit traces on* $\Sigma = \partial\Omega$ *in the sense of k-measures:*

$$\begin{cases} U\big|_{\Sigma} = \alpha \in M_0(\Sigma) \oplus \ldots \oplus M_{n-1}(\Sigma) \\ *U\big|_{\Sigma} = \widetilde{\alpha} \in M_0(\Sigma) \oplus \ldots \oplus M_{n-1}(\Sigma), \end{cases}$$

*then the k-measures* $\alpha$ *and* $\widetilde{\alpha}$ *have to be absolutely continuous.*

## 4 Conjugate Laplace Series

Given a trigonometric series, the conjugate trigonometric series can be considered as the "trace" of the harmonic function conjugate to the harmonic function whose trace is the given trigonometric series. Following this definition and hinging on the theory of conjugate differential forms, a new definition of conjugate Laplace series was given in [6].

Let us recall it. Consider a harmonic function $u$ defined in the unit ball $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$, it is well known that it can be expanded by means of harmonic polynomials:

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}\left(\frac{x}{|x|}\right), \tag{4}$$

where $p_{nh} = (2h+n-2)\frac{(h+n-3)!}{(n-2)!h!}$ and $\{Y_{hk}\}$ is a complete system of ultraspherical harmonics. We suppose $\{Y_{hk}\}$ orthonormal, i.e.,

$$\int_{\Sigma} Y_{hk} Y_{rs}\, d\sigma \begin{cases} = 1 & \text{if } h = r \text{ and } k = s \\ = 0 & \text{otherwise.} \end{cases}$$

The "trace" of $u$ on $\Sigma = \{x \in \mathbb{R}^n \mid |x| = 1\}$ is given by the expansion

$$\sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}(x) \qquad (|x| = 1). \tag{5}$$

Let us consider the 2-form

$$v = \sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)}\, dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2}) \tag{6}$$

and its adjoint

$$* v = \sum_{h=0}^{\infty}\sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2})\right). \tag{7}$$

It is possible to show that $dv = 0$ *and* $\delta v = du$ in $B$, i.e., the non-homogeneous form $u + v$ is self-conjugate.

If $n = 2$ the series which is obtained by taking $|x| = 1$ in (7) is just the trigonometric series conjugate to (5). In general, for any $n$, we say that

$$\sum_{h=0}^{\infty}\sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2})\right)\Bigg|_{|x|=1} \tag{8}$$

is the series conjugate to (4); it represents the "restriction" of $*v$ on $\Sigma$, while the "restriction" of $v$, provided it does exist, is equal to 0, as it follows immediately from (6).

Several properties of the conjugate Laplace series (8) have been obtained (see [2, 3, 9]). They concern the Abel convergence, the pointwise convergence, and the convergence in $L^p$ norm.

Here we mention a result (see [6]) which extends the original Brothers Riesz Theorem 1 to Laplace series and which is a consequence of Theorem 3:

**Theorem 4** *Let* (5) *be a Laplace series of a measure* $\mu \in M(\Sigma)$*, i.e.,*

$$a_{hk} = \int_{\Sigma} Y_{hk}\, d\mu\,.$$

*If there exists an* $(n-2)$*-measure* $\beta \in M_{n-2}(\Sigma)$ *such that*

$$\int_{+\Sigma} Y_{hk}\, d\mu = \frac{1}{h}\int_{+\Sigma} \beta \wedge dY_{hk} \qquad (h = 1, 2, \ldots; k = 1, \ldots, p_{nh}) \tag{9}$$

*and*

$$\int_{+\Sigma} \beta \wedge \underset{\Sigma}{*}\gamma = 0 \tag{10}$$

*for any* $\gamma \in C^{\infty}_{n-2}(\mathbb{R}^n)$ *such that* $d\gamma = 0$ *on* $\Sigma$*, then* $\mu$ *and* $\beta$ *are absolutely continuous.*

We remark that in the case $n = 2$, conditions (9) are nothing but (3), while (10) is not restrictive (the only closed 0-forms on the unit circle are the constants). However, if $n \geq 3$ condition (10) cannot be omitted.

## 5 Riesz-Type Inequalities

The classical Riesz inequality is well known:

$$\|g\|_{L^p(S)} \leq C\|f\|_{L^p(S)}, \tag{11}$$

the function $f + ig$ being holomorphic in the unit disc $D$, continuous up to the boundary $S = \partial D$, and $g(0) = 0$ $(1 < p < \infty)$.

Another inequality, which—as we shall see—is related to (11), concerns normal derivative $\frac{\partial}{\partial \nu}$ and tangential gradient $\mathrm{grad}_{\partial\Omega}$ of a harmonic function defined on a sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^n$. Namely, we have

$$\left\| \frac{\partial \omega}{\partial \nu} \right\|_{L^p(\partial\Omega)} \leq C\|\mathrm{grad}_{\partial\Omega}\, \omega\|_{L^p(\partial\Omega)}, \tag{12}$$

for any harmonic function $\omega \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Inequality (12) was proved by Vishik [30] for $p = 2$ when $\partial\Omega$ is a sphere, conjectured by Mikhlin in [24, p. 210] for $1 < p < \infty$, and established by De Vito [21] in the general case $1 < p < \infty$ when $\partial\Omega$ is the boundary of a $C^{2,\lambda}$-domain. Later Verchota [29] proved (12) on Lipschitz domains $(1 < p \leq 2)$.

In [12] inequalities of this type have been obtained in the frame of conjugate differential forms. Namely, let $\Omega \subset \mathbb{R}^n$ be a bounded $C^1$ domain and let $u_k$ and $v_{k+2}$ be two $C^1$ conjugate differential forms defined in $\Omega$, continuous up to the boundary $\Sigma$. The following inequalities hold:

$$\inf_{\alpha \in \mathscr{N}_k^+} \|u_k + \alpha\|_{L^p_k(\Sigma)} \leq C\Big\{\| * u_k\|_{L^p_{n-k}(\Sigma)} + \| * v_{k+2}\|_{L^p_{n-k-2}(\Sigma)}\Big\},$$

$$\inf_{\beta \in \mathscr{N}_{k+2}^+} \|v_{k+2} + \beta\|_{L^p_{k+2}(\Sigma)} \leq C\Big\{\|u_k\|_{L^p_k(\Sigma)} + \| * v_{k+2}\|_{L^p_{n-k-2}(\Sigma)}\Big\},$$

$$\inf_{\alpha \in \mathscr{N}_{n-k}^+} \| * u_k + \alpha\|_{L^p_{n-k}(\Sigma)} \leq C\Big\{\|u_k\|_{L^p_k(\Sigma)} + \| * v_{k+2}\|_{L^p_{n-k-2}(\Sigma)}\Big\},$$

$$\inf_{\beta \in \mathscr{N}_{n-k-2}^+} \| * v_{k+2} + \beta\|_{L^p_{n-k-2}(\Sigma)} \leq C\Big\{\|u_k\|_{L^p_k(\Sigma)} + \|v_{k+2}\|_{L^p_{k+2}(\Sigma)}\Big\}.$$

Here $\mathscr{N}_k^+$ is the kernel of the singular integral equation:

$$-\frac{1}{2}\phi_k(x) + \int_\Sigma \phi_k(y) \wedge \underset{y}{*} d_y s_k(x, y) = 0, \quad \text{a.e. } x \in \Sigma\,,$$

where $s_k(x, y)$ is the Hodge double form

$$s_k(x, y) = \sum_{j_1<\ldots<j_k} s(x, y) dx_{j_1} \ldots dx_{j_k} dy_{j_1} \ldots dy_{j_k} .$$

As proved in [25, 27], the dimension of $\mathcal{N}_k^+$ is equal to $b_k^-$, the $k$th Betti number of $\Omega$. It is clear that such inequalities generalize (11).

If $\omega$ is an harmonic $k$-form, we have that $\delta\omega$ and $-d\omega$ are conjugate. Therefore the inequalities we have obtained for conjugate differential forms lead to

$$\inf_{\alpha\in\mathcal{N}_{k-1}^+} \|\delta\omega_k + \alpha\|_{L^p_{k-1}(\Sigma)} \le C\Big\{\|d * \omega_k\|_{L^p_{n-k+1}(\Sigma)} + \| * d\omega_k\|_{L^p_{n-k-1}(\Sigma)}\Big\},$$

$$\inf_{\beta\in\mathcal{N}_{k+1}^+} \|d\omega_k + \beta\|_{L^p_{k+1}(\Sigma)} \le C\Big\{\|\delta\omega_k\|_{L^p_{k-1}(\Sigma)} + \| * d\omega_k\|_{L^p_{n-k-1}(\Sigma)}\Big\},$$

$$\inf_{\eta\in\mathcal{N}_{n-k+1}^+} \|d * \omega_k + \eta\|_{L^p_{n-k+1}(\Sigma)} \le C\Big\{\| * d\omega_k\|_{L^p_{n-k-1}(\Sigma)} + \|\delta\omega_k\|_{L^p_{k-1}(\Sigma)}\Big\},$$

$$\inf_{\gamma\in\mathcal{N}_{n-k-1}^+} \| * d\omega_k + \gamma\|_{L^p_{n-k-1}(\Sigma)} \le C\Big\{\|d\omega_k\|_{L^p_{k+1}(\Sigma)} + \|\delta\omega_k\|_{L^p_{k-1}(\Sigma)}\Big\}.$$

Suppose $b_{n-1}^- = 0$; the last inequality for $k = 0$ reads as follows:

$$\| * d\omega_0\|_{L^p_{n-1}(\Sigma)} \le C\|d\omega_0\|_{L^p_1(\Sigma)} \tag{13}$$

for any scalar harmonic function $\omega_0$. This is nothing but the Vishik–Mikhlin–De Vito formula (12).

We remark that, if $b_{n-1}^- \neq 0$, inequality (13) does not hold. Consider $\Omega = \{x \in \mathbb{R}^n : r < |x| < R\}$ and take

$$\omega_0(x) = \begin{cases} \log|x| & \text{if } n = 2\,, \\ |x|^{2-n} & \text{if } n \ge 3\,. \end{cases}$$

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