Holomorphic Curves and Linear Systems in Algebraic Manifolds

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Abstract In this note we will give theorems on the set of deficient divisors of an entire holomorphic curve $f : \mathbf{C} \to M$, where M is a projective algebraic manifold. We first give an inequality of second main theorem type and a defect relation for f that improve the results in Aihara (Tohoku Math J 58:287–315, 2012). By making use of the defect relation, we give theorems on the structure of the set of deficient divisors of f. We also have structure theorems for a family of linear systems of the set of deficient divisors.

1 Introduction

Let M be a projective algebraic manifold and $L \to M$ an ample line bundle. We denote by $|L|$ the complete linear system of L and let $\Lambda \subseteq |L|$ be a linear system. In the previous paper [1], after the study of Nochka [3], we studied properties of the deficiencies of a holomorphic curve $f : \mathbf{C} \to M$ as functions on linear systems and gave the structure theorem for the set

$$\mathscr{D}_f = \{D \in \Lambda\,;\ \delta_f(D) > \delta_f(B_\Lambda)\}$$

of deficient divisors. For the definitions, see Sects. 2 and 3. In the proof of the structure theorem for $\mathscr{D}_f$, we used an inequality of the second main theorem type and a defect relation for f and Λ (Theorems 3.1 and 4.2 in [1]). In this note, we first give an improvement of the inequality of the second main theorem type and give a defect relation. We also give structure theorems for a family of linear systems of deficient divisors. Details will be published elsewhere.

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2 Preliminaries

We recall some known facts on Nevanlinna theory for holomorphic curves. For details, see [5] and [6]. Let z be the natural coordinate in $\mathbf{C}$ and set

$$\Delta(r) = \{z \in \mathbf{C};\ |z| < r\} \quad \text{and} \quad C(r) = \{z \in \mathbf{C};\ |z| = r\}.$$

For a (1,1)-current φ of order zero on $\mathbf{C}$, we set

$$N(r,\ \varphi) = \int_1^r \langle \varphi,\ \chi_{\Delta(t)} \rangle \frac{dt}{t},$$

where $\chi_{\Delta(r)}$ denotes the characteristic function of $\Delta(r)$. Let M be a compact complex manifold and let $L \to M$ be a line bundle over M. We denote by $\Gamma(M,\ L)$ the space of all holomorphic sections of $L \to M$ and by $|L| = \mathbf{P}(\Gamma(M,\ L))$ the complete linear system of L. Denote by $||\cdot||$ a Hermitian fiber metric in L and by ω its Chern form. Let $f : \mathbf{C} \to M$ be a holomorphic curve. We set

$$T_f(r,\ L) = N(r,\ f^*\omega)$$

and call it the characteristic function of f with respect to L. If

$$\liminf_{r \to +\infty} \frac{T_f(r,\ L)}{\log r} = +\infty,$$

then f is said to be *transcendental*. We define the order ρ_f of $f : \mathbf{C} \to M$ by

$$\rho_f = \limsup_{r \to +\infty} \frac{\log T_f(r,\ L)}{\log r}.$$

We notice that the definition of ρ_f is independent of a choice of positive line bundles $L \to M$. Let $D = (\sigma) \in |L|$ with $||\sigma|| \leq 1$ on M. Assume that $f(\mathbf{C})$ is not contained in Supp D. We define the proximity function of D by

$$m_f(r,\ D) = \int_{C(r)} \log\left(\frac{1}{||\sigma(f(z))||}\right) \frac{d\theta}{2\pi}.$$

Then we have the following first main theorem for holomorphic curves.

Theorem 1 (First Main Theorem) *Let $L \to M$ be a line bundle over M and let $f : \mathbf{C} \to M$ be a non-constant holomorphic curve. Then*

$$T_f(r,\ L) = N(r,\ f^*D) + m_f(r,\ D) + O(1)$$

for $D \in |L|$ *with* $f(\mathbf{C}) \not\subseteq \mathrm{Supp}\, D$*, where* $O(1)$ *stands for a bounded term as* $r \to +\infty$.

Let f and D be as above. We define Nevanlinna's deficiency $\delta_f(D)$ by

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r,\ D)}{T_f(r,\ L)}.$$

It is clear that $0 \leq \delta_f(D) \leq 1$. Then we have a defect function δ_f defined on $|L|$. If $\delta_f(D) > 0$, then D is called *a deficient divisor in the sense of Nevanlinna.*

3 Value Distribution Theory for Coherent Ideal Sheaves

In this section we recall some basic facts in value distribution theory for coherent ideal sheaves and give Crofton type formula. For details, see [6, Chapter 2] and [7]. We use the same notation as in Sect. 2.

Let $f : \mathbf{C} \to M$ be a holomorphic curve and $\mathscr{I}$ a coherent ideal sheaf of the structure sheaf $\mathscr{O}_M$ of M. Let $\mathscr{U} = \{U_j\}$ be a finite open covering of M with a partition of unity $\{\eta_j\}$ subordinate to $\mathscr{U}$. We can assume that there exist finitely many sections $\sigma_{jk} \in \Gamma(U_j,\ \mathscr{I})$ such that every stalk $\mathscr{I}_p$ over $p \in U_j$ is generated by germs $(\sigma_{j1})_p, \ldots, (\sigma_{jl_j})_p$. Set

$$d_{\mathscr{I}}(p) = \left(\sum_j \eta_j(p) \sum_{k=1}^{l_j} \left|\sigma_{jk}(p)\right|^2 \right)^{1/2}.$$

We take a positive constant C such that $Cd_{\mathscr{I}}(p) \leq 1$ for all $p \in M$. Set

$$\phi_{\mathscr{I}}(p) = -\log C d_{\mathscr{I}}(p)$$

and call it the proximity potential for $\mathscr{I}$. It is easy to verify that $\phi_{\mathscr{I}}$ is well defined up to addition by a bounded continuous function on M. We now define the proximity function $m_f(r,\ \mathscr{I})$ of f for $\mathscr{I}$, or equivalently, for the complex analytic subspace (may be non-reduced)

$$Y = (\mathrm{Supp}\,(\mathscr{O}_M/\mathscr{I}),\ \mathscr{O}_M/\mathscr{I})$$

by

$$m_f(r,\ \mathscr{I}) = \int_{C(r)} \phi_{\mathscr{I}}(f(z)) \frac{d\theta}{2\pi},$$

provided that $f(\mathbf{C})$ is not contained in $\operatorname{Supp} Y$. For $z_0 \in f^{-1}(\operatorname{Supp} Y)$, we can choose an open neighborhood U of z_0 and a positive integer ν such that

$$f^*\mathscr{I} = ((z - z_0)^\nu) \quad \text{on } U.$$

Then we see

$$\log d_{\mathscr{I}}(f(z)) = \nu \log |z - z_0| + h_U(z) \quad \text{for } z \in U,$$

where h_U is a C^∞-function on U. Thus we have the counting function $N(r, f^*\mathscr{I})$ as in Sect. 2. Moreover, we set

$$\omega_{\mathscr{I},f} = -dd^c h_U \quad \text{on } U,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar\partial - \partial)$. We obtain a well-defined smooth (1, 1)-form $\omega_{\mathscr{I},f}$ on $\mathbf{C}$. Define the characteristic function $T_f(r, \mathscr{I})$ of f for $\mathscr{I}$ by

$$T_f(r, \mathscr{I}) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} \omega_{\mathscr{I},f}.$$

We have the first main theorem in value distribution theory for coherent ideal sheaves due to Noguchi–Winkelmann–Yamanoi [7, Theorem 2.9]:

Theorem 2 (First Main Theorem) *Let $f : \mathbf{C} \to M$ and $\mathscr{I}$ be as above. Then*

$$T_f(r, \mathscr{I}) = N(r, f^*\mathscr{I}) + m_f(r, \mathscr{I}) + O(1).$$

When $\mathscr{I}$ defines an effective divisor D on M, it is easy to see that

$$T_f(r, \mathscr{I}) = T_f(r, \mathscr{O}_M(D)) + O(1) \quad \text{and} \quad m_f(r, \mathscr{I}) = m_f(r, D) + O(1).$$

Let $L \to M$ be an ample line bundle and $W \subseteq \Gamma(M, L)$ a subspace with $\dim W \geq 2$. Set $\Lambda = \mathbf{P}(W)$. The base locus $\operatorname{Bs} \Lambda$ of Λ is defined by

$$\operatorname{Bs} \Lambda = \bigcap_{D \in \Lambda} \operatorname{Supp} D.$$

We define a coherent ideal sheaf $\mathscr{I}_0$ in the following way. For each $p \in M$, the stalk $\mathscr{I}_{0,p}$ is generated by all germs $(\sigma)_p$ for $\sigma \in W$. Then $\mathscr{I}_0$ defines the base locus of Λ as a complex analytic subspace B_Λ, that is,

$$B_\Lambda = (\operatorname{Supp}(\mathscr{O}_M/\mathscr{I}_0), \mathscr{O}_M/\mathscr{I}_0).$$

Hence $\operatorname{Bs} \Lambda = \operatorname{Supp}(\mathscr{O}_M/\mathscr{I}_0)$.

We now give a Crofton type formula. Let $f : \mathbf{C} \to M$ be a non-constant holomorphic curve. If $f(\mathbf{C}) \not\subseteq \mathrm{Supp}\, D$ for all $D \in \Lambda$, then we say that f is *non-degenerate with respect to* Λ. Let μ be the invariant measure on $\mathbf{P}^l(\mathbf{C})$ normalized so that $\mu(\mathbf{P}^l(\mathbf{C})) = 1$. We have the following generalized Crofton's formula due to Kobayashi [6, Theorem 2.4.12].

Theorem 3 *Suppose that* $f : \mathbf{C} \to M$ *is non-degenerate with respect to* Λ *and* $f(\mathbf{C}) \not\subseteq \mathrm{Bs}\, \Lambda$. *Then*

$$\int_{D\in\Lambda} m_f(r,\ D) d\mu(D) = m_f(r,\ \mathscr{I}_0) + O(1)$$

and hence

$$T_f(r,\ L) = \int_{D\in\Lambda} N(r,\ f^*D) d\mu(D) + m_f(r,\ \mathscr{I}_0) + O(1).$$

We define the deficiency of B_Λ for f by

$$\delta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{m_f(r,\ \mathscr{I}_0)}{T_f(r,\ L)}.$$

Set

$$\mathscr{D}_f = \{D \in \Lambda;\ \delta_f(D) > \delta_f(B_\Lambda)\}.$$

We call $\mathscr{D}_f$ *the set of deficient divisors* in Λ. By making use of Theorem 2, we have the following proposition [1, Proposition 4.1].

Proposition 1 *The set* $\mathscr{D}_f$ *is a null set in the sense of the Lebesgue measure on* Λ. *In particular,*

$$\delta_f(D) = \delta_f(B_\Lambda)$$

for almost all $D \in \Lambda$.

This proposition plays an important role in the proof of theorems in Sect. 4.

4 Inequality of the Second Main Theorem Type

We will give an inequality of the second main theorem type for a holomorphic curve $f : \mathbf{C} \to M$ that improves Theorem 3.1 in [1]. For simplicity, we assume that f is of finite type. Let $W \subseteq \Gamma(M,\ L)$ be a linear subspace with $\dim W = l_0 + 1 \geq 2$ and set $\Lambda = \mathbf{P}(W)$. We call Λ a linear system included in $|L|$. Let $D_1, \ldots, D_q$ be divisors in Λ such that $D_j = (\sigma_j)$ for $\sigma_j \in W$. We first give a definition of *subgeneral position*. Set $Q = \{1, \ldots, q\}$ and take a basis $\{\psi_0, \ldots, \psi_{l_0}\}$ of W. We write

$$\sigma_j = \sum_{k=0}^{l_0} c_{jk}\psi_k \quad (c_{jk} \in \mathbf{C})$$

for each $j \in Q$. For a subset $R \subseteq Q$, we define a matrix A_R by $A_R = (c_{jk})_{j\in R, 0\le k\le l_0}$.

Definition 1 Let $N \geq l_0$ and $q \geq N+1$. We say that $D_1, \ldots, D_q$ are in N-subgeneral position in Λ if

$$\operatorname{rank} A_R = l_0 + 1 \quad \text{for every subset } R \subseteq Q \text{ with } \sharp R = N+1.$$

If they are in l_0-subgeneral position, we simply say that they are in general position.

Remark 1 The above definition is different from the usual one (cf. [6, p. 114]). In fact, the divisors $D_1, \ldots, D_q$ are usually said to be in N-subgeneral position in Λ provided that

$$\bigcap_{j\in R} \operatorname{Supp} D_j = \emptyset \qquad \text{for every subset} \quad R \subseteq Q \quad \text{with} \quad \sharp R = N+1.$$

However, the divisors $D_1, \ldots, D_q$ may have a common point when they are in N-subgeneral position in the above sense.

Let $\Phi_\Lambda : M \to \mathbf{P}(W^*)$ be a natural meromorphic mapping, where W^* is the dual of W (cf. [5, p. 68]). Then we have the linearly non-degenerate holomorphic curve

$$F_\Lambda = \Phi_\Lambda \circ f : \mathbf{C} \to \mathbf{P}(W^*).$$

We let $W(F_\Lambda)$ denote the Wronskian of F_Λ.

Definition 2 If $\rho_f < +\infty$, then f is said to be of finite type.

By making use of the methods in [1] and [4], we have an inequality of the second main theorem type as follows.

Theorem 4 *Let $f : \mathbf{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ. Let $D_1, \ldots, D_q \in \Lambda$ be divisors in N-subgeneral position. Assume that f is of finite type. Then*

$$(q - 2N + l_0 - 1)\,(T_f(r, L) - m_f(r, \mathscr{I}_0)) \leq \sum_{j=1}^{q} N(r, f^*D_j) + E_f(r)$$

as $r \to +\infty$, *where*

$$E_f(r) = -(2N - n + 1)N(r,\ f^*\mathscr{I}_0) - \binom{N}{n} N(r,\ (W(F_\Lambda)_0) + o(T_f(r,\ L)).$$

We notice here that in the proof of the above theorem, we use an estimate for Nochka's weight improved by N. Toda (see [6, p. 118]). In order to get a defect relation from Theorem 4, we define a constant $\eta_f(B_\Lambda)$ by

$$\eta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{-(2N - n + 1)N(r,\ f^*\mathscr{I}_0) - (N/n)N(r,\ (W(F_\Lambda)_0)}{T_f(r,\ L)}.$$

It is clear that $\eta_f(B_\Lambda) \leq 0$. Now, by Theorem 4, we have a defect relation.

Theorem 5 *Let* Λ, f *and* $D_1, \ldots, D_q$ *be as in Theorem 4. Then*

$$\sum_{j=1}^{q} (\delta_f(D_j) - \delta_f(B_\Lambda)) \leq (1 - \delta_f(B_\Lambda))(2N - l_0 + 1) + \eta_f(B_\Lambda).$$

Remark 2 In the case where $\rho_f = +\infty$, by a suitable modification, we also have theorems similar to the above (cf. [1]).

5 Structure Theorems for the Set of Deficient Divisors

In this section we give theorems on the structure of the set of deficient divisors. Let $L \to M$ be an ample line bundle and $f : \mathbf{C} \to M$ a transcendental holomorphic curve of finite type. Let $\Lambda \subseteq |L|$ be a linear system. Let

$$\mathscr{D}_f = \{D \in \Lambda\,;\ \delta_f(D) > \delta_f(B_\Lambda)\}.$$

We summarize the basic facts on the set $\mathscr{D}_f$ (see [1, §5]).

Theorem 6 *The set* $\mathscr{D}_f$ *of deficient divisors is a union of at most countably many linear systems included in* Λ. *The set of values of deficiency of* f *is at most a countable subset* $\{e_i\}$ *of* $[0, 1]$. *For each* e_i, *there exist linear systems* $\Lambda_1(e_i), \ldots, \Lambda_s(e_i)$ *included in* Λ *such that* $e_i = \delta_f(B_{\Lambda_j(e_i)})$ *for* $j = 1, \ldots, s$.

By Theorem 6, there exist at most countably many linear systems $\{\Lambda_j\}$ in Λ such that $\mathscr{D}_f = \bigcup_j \Lambda_j$. Define $\mathscr{L}_f = \{\Lambda_j\} \cup \{\Lambda\}$. We call $\mathscr{L}_f$ *the fundamental family of linear systems for* f.

The set of all linear systems with dimension k included in Λ is parameterized by a Grassmann variety. For a transcendental holomorphic curve $f : \mathbf{C} \to M$, we will define a defect function $\delta_{\Lambda,f}$ on Grassmannians in the following way. Suppose that f is non-degenerate with respect to Λ. We fix a positive integer k with

$1 \le k \le \dim \Lambda$. We let $\mathbf{Gr}(k, \Lambda)$ denote the Grassmann variety of k-dimensional linear subvarieties of the projective space Λ. For a point $\mathbf{A} \in \mathbf{Gr}(k, \Lambda)$, we define

$$\delta_{\Lambda, f}(\mathbf{A}) = \inf\{\delta_f(D);\ D \in \mathbf{A}\}.$$

For a linear subsystem Π of Λ with $\dim \Pi > k$, we also define $\mathbf{Gr}(k, \Pi)$ that is the subvariety of $\mathbf{Gr}(k, \Lambda)$ (cf. [2, Lecture 6]). Set

$$\kappa_f(\mathbf{Gr}(k, \Pi)) = \inf\left\{\delta_{\Pi, f}(\mathbf{A});\ \mathbf{A} \in \mathbf{Gr}(k, \Pi)\right\}.$$

For a linear system Π_i in $\mathscr{L}_f$ with $\dim \Pi_i > k$, we get the Grassmannian $\mathbf{Gr}(k, \Pi_i)$ that is the subvariety of $\mathbf{Gr}(k, \Lambda)$. Let $\mathscr{G} = \{\mathbf{Gr}(k, \Pi_i)\}$ be the family of all such Grassmannians. Then we have the following structure theorem.

Theorem 7 *For a sufficiently small positive number ϵ, there exist finitely many subvarieties $\mathbf{Gr}(k, \Pi_1), \ldots, \mathbf{Gr}(k, \Pi_t)$ contained in $\mathscr{G}$ such that*

$$\{\mathbf{A} \in \mathbf{Gr}(k, \Lambda);\ \delta_{\Lambda, f}(\mathbf{A}) \ge \kappa_f(\mathbf{Gr}(k, \Lambda)) + \epsilon\} = \bigcup_{i=1}^{t} \mathbf{Gr}(k, \Pi_i).$$

In particular, the exceptional set

$$\{\mathbf{A} \in \mathbf{Gr}(k, \Lambda);\ \delta_{\Lambda, f}(\mathbf{A}) > \kappa_f(\mathbf{Gr}(k, \Lambda))\}$$

for $\mathbf{Gr}(k, \Lambda)$ is the union of all Grassmannians in $\mathscr{G}$.

For the function $\delta_{\Lambda, f} : \mathbf{Gr}(k, \Lambda) \to [0, 1]$, we have the following theorem.

Theorem 8 *The set of values of $\delta_{\Lambda, f}$ is an at most countable subset $\{e_i\}$ of* $[0, 1]$. *If $\mathbf{Gr}(k, \Pi) \in \mathscr{G}$ and if $\mathbf{A} \in \mathbf{Gr}(k, \Pi)$ is generic, then $\kappa_f(\mathbf{Gr}(k, \Pi)) = \delta_f(\mathbf{A})$. The set of non-generic points in $\mathbf{Gr}(k, \Pi)$ is contained in a union of at most countable Grassmannians in $\mathscr{G}$. In particular, the closure of the inverse image $\delta^{-1}_{\Lambda, f}(e_i)$ can be written*

$$\overline{\delta^{-1}_{\Lambda, f}(e_i)} = \bigcup_{j=1}^{t} \mathbf{Gr}(k, \Pi_{i_j})$$

for finitely many varieties $\mathbf{Gr}(k, \Pi_{i_1}), \ldots, \mathbf{Gr}(k, \Pi_{i_t})$ in $\mathscr{G}$.

Remark 3 Let $\Pi_1, \Pi_2 \in \mathscr{G}$. We notice that $\kappa_f(\mathbf{Gr}(k, \Pi_1)) < \kappa_f(\mathbf{Gr}(k, \Pi_2))$ if $\mathbf{Gr}(k, \Pi_2)$ is a proper subvariety of $\mathbf{Gr}(k, \Pi_1)$.

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