Trends in Mathematics
Research Perspectives

Karl-Olof Lindahl, Torsten Lindström,
Luigi G. Rodino, Joachim Toft,
Patrik Wahlberg
Editors

Analysis, Probability, Applications, and Computation

Proceedings of the 11th ISAAC Congress, Växjö (Sweden) 2017

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Trends in Mathematics

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ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
Research Perspectives
ISBN 978-3-030-04458-9 ISBN 978-3-030-04459-6 (eBook)
https://doi.org/10.1007/978-3-030-04459-6

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The 11th International ISAAC Congress was held on 14–18 August 2017 at Linnæus University, Växjö, Sweden. The congress continued the successful series of biennial meetings previously held in the USA (1997), Japan (1999), Germany (2001), Canada (2003), Italy (2005), Turkey (2007), the UK (2009), Russia (2011), Poland (2013) and P. R. China (2015). The total number of participants of the congress was 304 coming from 43 countries, including the special guests and the organizing committee from Växjö. There were 12 plenary speakers. Totally, the congress had 17 sessions spanning 5 working days. One afternoon was assigned to excursions. The congress was sponsored by academic institutions, local municipality and the host university. One of the features of the congress was the invitation of applied subjects like electrical engineering and mathematics in biology. The 11th International ISAAC Congress was an important scientific event during which mathematicians from different parts of the world had an opportunity to present new results and ideas. It was also a great possibility for young mathematicians to contact experts in a variety of fields.

The atmosphere during the congress was warm and friendly. The social events included a banquet at Glass country (Glasriket) and excursions with a steam boat to the largest lake near Växjö.

It is a well-established tradition within the community to award one or several outstanding young researchers during the ISAAC Congress. The ISAAC award of the 11th International ISAAC Congress was presented to

Tuomas Hytönen (University of Helsinki, Finland)

for his strong contributions to harmonic analysis, geometric analysis, functional analysis and singular integral operators. Though he is a young scientist, he has already achieved several results of high quality, published in top journals. One of his major achievements is the proof of the A2 conjecture for Calderón–Zygmund operators, published in *Annals of Mathematics* 2012. He was also an invited speaker at ICM 2014.

At the ISAAC board meeting during the congress, several decisions of fundamental importance for the organization were taken:

1. Professor Michael Reissig of Freiberg University of Mining and Technology, Germany, was elected as the new ISAAC president; Professor Joachim Toft at Linnæus University, Sweden, as the vice-president; and Professor Irene Sabadini at Politecnico di Milano, Italy, as the new secretary and treasurer. Professor Reissig succeeded Professor Luigi Rodino of Turin University, Italy, who finished his 4-year service, and Professor Sabadini succeeded Professor Heinrich Begehr, Freie Universität Berlin, Germany. The board expressed their gratitude to Professors Heinrich Begehr and Luigi Rodino for all their efforts and contributions for ISAAC over the last years.
2. The decisions from the board meeting at the 10th ISAAC Congress in Macao, China, concerning organizations of accounts were renewed. At the same time, it was decided that a reregistration of ISAAC should be performed. (The decisions were implemented shortly after the congress.)
3. The decision from the board meeting at the 9th ISAAC Congress in Krakow, Poland, on modernizing the home page was renewed. (The decision was implemented shortly after the congress.)
4. The venue for the following 12th International ISAAC Congress in 2019 was decided to be at the University of Aveiro, Portugal.

The plenary lectures given at the congress appear not here but in the independent volume:

L. Rodino, J. Toft (Eds.) *Mathematical Analysis and Applications - Plenary Lectures, ISAAC 2017, Växjö, Sweden* Springer Proceedings in Mathematics & Statistics, Springer, to appear 2018 or 2019.

This volume contains the texts of a selection of talks delivered at the congress. As in the previous years, some of the sessions or interest groups decided to publish independently their own volumes of proceedings and are therefore excluded from the present collection. The work of the congress was spread over the following sessions:

- *Applications of dynamical systems theory in biology*, organized by Torsten Lindström, Amira Asta, Lucia Tamburino
- *Approximation theory and special functions*, organized by Oktay Duman, Esra Erkus-Duman
- *Complex analysis and convex optimization and their applications in wave physics*, organized by Sven Nordebo, Yevhen Ivanenko
- *Complex and functional analytic methods for differential equations*, organized by Heinrich Begehr, Okay Celebi, J.Y. Du
- *Complex-analytic and Wiener-Hopf methods in the applied sciences*, organized by Gennady Mishuris, Sergei Rogosin
- *Special interest group: IGCVPT, Complex variables and potential theory*, organized by Tahir Aliyev Azeroglu, Anatoly Golberg, Massimo Lanza de Cristoforis, Sergiy Plaksa
- *Fixed point theory and its applications*, organized by Erdal Karapinar

- *Special interest group: IGPDE, Harmonic analysis and partial differential equations*, organized by Michael Ruzhansky, Jens Wirth
- *Special interest group: IGPDE, Nonlinear PDE*, organized by Vladimir Georgiev, Tohru Ozawa
- *P-adic analysis*, organized by Alain Escassut, Andrei Khrennikov, Karl-Olof Lindahl
- *Special interest group: IGPDO, Pseudo-differential operators*, organized by Shahla Molahajloo, Patrik Wahlberg, M. W. Wong
- *Special interest group: IGCQA, Quaternionic and Clifford analysis*, organized by Swanhild Bernstein, Uwe Kähler, Irene Sabadini, Franciscus Sommen
- *Special interest group: IGPDE, Recent progress in evolution equations*, organized by Marcello D'Abbicco, Marcelo Rempel Ebert, Michael Reissig
- *Special interest group: IGGF, Generalized functions and applications*, organized by Michael Kunzinger, Michael Oberguggenberger, Stevan Pilipović
- *Theory and applications of boundary-domain integral and pseudodifferential operators*, organized by Sergey E. Mikhailov, David Natroshvili
- *Wavelet theory and its related topics*, organized by Keiko Fujita, Akira Morimoto
- *Contributed talks*, organized by Jonas Fransson, Joachim Toft

We thank the organizers of all the sessions of the congress for their work. They spent an enormous amount of time inviting participants, arranging their sessions, providing chairmen and creating a familiar and workshop-like atmosphere within their meetings. The session organizers were also responsible for collecting contributions to this proceedings volume and for the refereeing process of the papers. Finally, we would like to thank Dr. Elmira Nabizadeh for all her unselfish efforts on this volume as well as during the congress.

Växjö, Sweden — Karl-Olof Lindahl
Torsten Lindström
Joachim Toft
Patrik Wahlberg
Turin, Italy — Luigi G. Rodino
October 2018

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Part I
Applications of Dynamical Systems Theory in Biology

Session Organizers: Torsten Lindström, Amira Asta, and Lucia Tamburino

The session was primarily aimed for talks that are using dynamical systems theory in order to analyze various models that arise in biological applications. The models analyzed may be mechanistically formulated, fitted to data, deterministic, or stochastic. Various relations between such models that arise in different modeling approaches and under different simplifying assumptions can be analyzed. Possible biological applications can include ecology, epidemiology, pharmacokinetics, evolution, physiology, pattern formation, and resource distribution, but are not limited to these topics. A part of the session was reserved for stakeholders with rich contact networks outside academia.

Analysis of State-Control Optimality System for Invasive Species Management

Angela Martiradonna, Fasma Diele, and Carmela Marangi

Abstract Mathematical modeling and optimization provide decision-support tools of increasing popularity to the management of invasive species. In this chapter, we investigate problems formulated in terms of optimal control theory. A free terminal time optimal control problem is considered for minimizing the costs and the duration of an abatement program. Here, we introduce a discount term in the objective function that destroys the nonautonomous nature of the state–costate system. We show that the alternative state-control optimality system is autonomous and its analysis provides the complete qualitative description of the dynamics of the discounted optimal control problem. By using the expression of its invariant, we deduce several insights for detecting the optimal control solution for an invasive species obeying a logistic growth.

1 Introduction

At least 12 billion of euros per year are spent by the countries of the European Union for the management of invasive species [11]. This figure includes costs for key economic sectors, such as agriculture, fisheries, aquaculture, forestry, and health sectors as well as damages and management costs. Moreover, invasive species are commonly deemed as responsible of global biodiversity loss [15]. The human element affects invasive species in many different ways: by inadvertently introducing alien species in ecosystems or when they disrupt a territory with the result of a possible response growth in invasive species. Recently, some aspects of the problem have become controversial: both the identification of invasive species with no-native ones and the negative impact on the hosting habitat are currently debated in the scientific community [9]. There are cases of native species which become invasive due to environmental changes, either due to anthropic pressures or

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_1

to climate changes, and cases of alien species used to restore a somehow perturbed ecosystem equilibrium of the hosting environment [14], like feral cats, an alien species which has been introduced in Australia for keeping in control other invasive species (rodents) [12]. Despite the changing perspective of the role of invasive species, in protected areas with severe conservation issues, the total eradication is still an objective to be perceived. We are not entering here the debate on the soundness of the approach to invasive species from an ecological point of view, since it is out of the purpose of the present chapter. We will focus instead on the mathematical tools that may provide a solution to the management issue of the containment of those species which halter the ecosystem and the ecosystem services of a protected area because of their fast spread, regardless of their origin. Many studies [3, 6–8, 16] have indicated that, despite their high costs, intensive control strategies can be optimal since they are capable of minimizing the infestation area, halting future spread and associated damages.

In [13], an in-depth overview about the link between the optimal control theory and biology can be found. Therein, a discrete time model has been described as an example of optimal control approach to the management of invasive plant species. In this chapter, we follow the continuous dynamical approach contained in many papers by C. Baker and his coauthors on the specific topic of management of invasive species [1–5]. More specifically, we suppose that the dynamics of the invasive species population is described by the following ordinary differential equation:

$$\dot{u}(t) = r\,u(t)\left(1 - \frac{u(t)}{k}\right) - u\,(\mu\,E(t))^q\,, \tag{1}$$

where $u(t)$ represents the population density at time $t \in [0, T]$ and $\dot{u} = du/dt$. The term $r\,u(t)\left(1 - \frac{u(t)}{k}\right)$ describes the logistic population growth, with intrinsic growth rate $r > 0$ and carrying capacity $k > 0$. The effect of the control actions on the population is modeled by the term $-u\,(\mu\,E(t))^q$, where $E(t)$ is the control function, $\mu > 0$ is a scaling parameter, accounting for the control effectiveness, and $q \in \mathbf{Q} \cap \left[\frac{1}{2}, 1\right)$ is a diminishing return parameter in the set of rational numbers. Low values of the parameter q indicate control actions that are not cost effective at high intensity, since the related marginal returns decrease very quickly.

We require that the invasive population has to be reduced from the initial density value $u(0) = u_0$ to the threshold density $u(T) = u_T < u_0$ at exactly T units of time, where $T > 0$ represents the program length.

We assume that the allocation of resources for the abatement is evaluated by the objective functional:

$$\mathscr{J}(E, T) = \int_0^T e^{-\delta\,t}\,E(t)\,dt,$$

where $\delta \in (0, 1)$ represents the discount factor. We define the set of positive bounded Lebesgue integrable control functions as:

$$U = \left\{ E \in L^1(0, T) : 0 \leq E \leq b \right\},$$

with $b > 0$ a fixed constant. We also fix the constant $\bar{T}$ and seek for an optimal control pair $(E^*, T^*) \in U \times [0, \bar{T}]$ such that

$$\mathscr{J}(E^*, T^*) = \min_{E \in U,\, T \in [0, \bar{T}]} \mathscr{J}(E, T). \tag{2}$$

The invasive species dynamics governed by the model (1) and a nondiscounted version of the objective functional (2) have been introduced in [3], for the management of feral cats in Australia semiarid ecosystems. In [4], a theoretical analysis has been performed for the nondiscounted model in [3], by means of a dynamical system approach. Moreover, in that paper the authors provide the theoretical expression for the optimal control values as well as for the optimal abatement program length when the objective function does not depend explicitly on time. Here, we make a further step by introducing a discount factor for the abatement cost. This introduction has the effect of destroying the autonomous nature of the (Hamiltonian) optimality system describing first-order necessary equations. As a consequence, the tool of phase-plane analysis cannot be used to achieve theoretical results for the optimal solution. In this chapter, following the approach in [4], we deduce the alternative optimality system that describes the invasive density evolution in conjunction with the control effort. The state-control system results to be autonomous and can be analyzed by means of phase-space analysis. In so doing, we are able to provide some qualitative characterizations of the solution of the discounted optimal control problem (1)–(2).

The chapter is structured as follows: In Sect. 2, we apply the Pontryagin's Maximum Principle to set the necessary conditions for the optimal solution and we introduce the time as additional variable to build a new conserved quantity on the Hamiltonian of the original nonautonomous system. Then, in Sect. 3 we move from a state–costate representation to a state-control one and analyze the dynamics of resulting autonomous system. From the phase diagram generated with the parameters of the feral cats example and the properties of the invariant, we derive useful insights into the optimal solution in Sect. 4. Finally, in Sect. 5 we draw our conclusions.

2 Necessary Conditions for Optimality

To characterize the optimal solution, the following necessary conditions are standard results from Pontryagin's Maximum Principle as stated in [17].

Theorem 1 *Let* $(E^*,\ T^*) \in U \times [0, \bar{T}]$ *be a solution of the optimal control problem (2), with density* u^* *satisfying the state equation:*

$$\dot{u} = r\,u\,\left(1 - \frac{u}{k}\right) - u\,\mu^q\,E^{*q}, \tag{3}$$

with constraints:

$$u(0) = u_0, \quad u(T) = u_T. \tag{4}$$

Then, there exists a piecewise differentiable adjoint variable $\lambda(t)$ *such that*

$$H(t, u^*(t), E^*(t), \lambda(t)) \le H(t, u^*(t), E(t), \lambda(t))$$

for all the admissible controls E *at each time* t*, where the Hamiltonian* H *is*

$$H(t, u, E, \lambda) = e^{-\delta t}\,E\, + r\,\lambda\,u\,\left(1 - \frac{u}{k}\right) - \lambda\,u\,\mu^q\,E^q \tag{5}$$

and

$$\dot{\lambda} \,=\, -r\,\lambda\left(1 - \frac{2\,u^*}{k}\right) + \,\lambda\,\mu^q\,E^{*q}. \tag{6}$$

Furthermore,

$$H(T^*,\ u^*(T^*),\ E^*(T^*),\ \lambda(T^*)) \,=\, 0. \tag{7}$$

Let us check the concavity conditions of the Hamiltonian function at E^* to characterize controls that minimize the objective function [13]. Let $E^* \in U$ be a solution of the optimal control problem (2) with density u^* satisfying the state Eq. (3), and λ a piecewise differentiable function with $\lambda > 0$ for all t. Then,

$$\frac{\partial^2 H}{\partial E^2}\left(t, u^*(t), E^*(t), \lambda(t)\right) \,=\, (1-q)\,q\,\mu^q\,\lambda(t)\,u^*(t) \,>0.$$

Therefore, whenever the existence of the optimal solution is guaranteed, necessary conditions stated in Theorem 1 can be applied to solve the optimal control (2) subject to (3) with constraints (4). Let the triplet $(u(t), \lambda(t), E(t))$, with $E(t) > 0$, solve the equation:

$$\frac{\partial H}{\partial E}\,(t, u(t), E(t), \lambda(t)) \,=\, e^{-\delta t} \,-\, \lambda(t)\,u(t)\,\mu^q\,q\,E(t)^{q-1} = 0, \tag{8}$$

at $t \in [0, T]$. The solution is given by:

$$E(t) = \varphi(t, u(t), \lambda(t)) \; = (q\, \mu^q\, \lambda(t)\, u(t))^{\frac{1}{1-q}} \;\; e^{\frac{\delta t}{1-q}}.$$

We search for those $u(t)$ and $\lambda(t)$ which satisfy the following *state–costate problem*:

$$\dot{u} \; = r\, u \left(1 - \frac{u}{k}\right) - \; u\, \mu^q \; \left(E^*_{u,\lambda}\right)^q , \tag{9}$$

$$\dot{\lambda} \; = -\lambda\, r \left(1 - \frac{2\,u}{k}\right) + \lambda\, \mu^q \; \left(E^*_{u,\lambda}\right)^q \; , \tag{10}$$

for $0 \le t \le T$ and $u(0) = u_0$, $u(T) = u_T$, where $E^*_{u,\lambda}(t) = \min\{\varphi(t, u(t), \lambda(t)), b\}$. Notice that, since $\varphi(t, u(t), \lambda(t))$ depends explicitly on t, the previous system is not autonomous and a phase-space analysis cannot be performed.

The system (9)–(10) is a time-dependent Hamiltonian system which does not preserve the Hamiltonian function:

$$H(t, u, \varphi(t, u, \lambda), \lambda) = \frac{q-1}{q}\, e^{\frac{\delta q}{1-q}\, t} \; \left(q\, \mu^q\, \lambda\, u\right)^{\frac{1}{1-q}} + r\, \lambda\, u \left(1 - \frac{u}{k}\right). \tag{11}$$

In this case, a different conserved quantity $\widehat{H}$ can be identified, as in [10], by considering the time as an additional variable:

$$\widehat{H}(t, u, \lambda) = H(t, u, \varphi(t, u, \lambda), \lambda) + \delta \int_0^t e^{\frac{\delta q}{1-q}\, s} \; \left(q\, \mu^q\, \lambda(s)\, u(s)\right)^{\frac{1}{1-q}} \; ds. \tag{12}$$

3 Analysis of the State-Control Optimality System

Following the approach in [4], a system of differential equations given in terms of the population density $u(t)$ and the control $E(t)$ can be considered. From (8), it results that

$$\lambda(t)\, u(t) = \frac{1}{q\, \mu^q}\, E^{1-q}(t)\, e^{-\delta t}, \tag{13}$$

is well defined for all $E(t) \ge 0$. Moreover, by totally differentiating the condition (13) with respect to time, we get

$$q\,(1-q)\,\mu^q\,\lambda\, u\, E^{q-2}\, \dot{E} = -\,\delta\, e^{-\delta\, t} + \left(q\, \mu^q\, \lambda\, E^{q-1}\right) \dot{u} + q\, \mu^q\, u\, E^{q-1}\, \dot{\lambda}. \tag{14}$$

Then, accounting for Eqs. (3)–(6) and plugging (13) into (17) we obtain the following set of differential equations:

$$\dot{u} = r\,u\left(1-\frac{u}{k}\right) - u\,\mu^q\,E^{*q},$$

$$\dot{E} = \frac{r\,u + \delta k}{k\,(1-q)}\,E.$$

To localize the optimal solution, we observe that for all t such that $\frac{\partial H}{\partial E}\,(t, u(t), b, \lambda(t))$ is strictly negative, then $E^*(t) = b$. By using (13), let us evaluate

$$\frac{\partial H}{\partial E}\,(t, u(t), b, \lambda(t)) = e^{-\delta t}(1 - E^{1-q}(t)\; b^{q-1}). \tag{15}$$

It is easy to see that for all t such that $\frac{\partial H}{\partial E}\,(t, u(t), b, \lambda(t)) < 0$ it results $E(t) > b$. Hence, we will refer to the following system as to the *state-control optimality system*:

$$\dot{u} = r\,u\left(1-\frac{u}{k}\right) - u\,\mu^q\,\min(E^q,\; b^q), \tag{16}$$

$$\dot{E} = \frac{r\,u + \delta k}{k\,(1-q)}\,E. \tag{17}$$

Differently from the state–costate system, the state-control system is autonomous and can be analyzed by means of the phase-space analysis for $0 \le u \le k$ and $E \ge 0$. As the bound of the effort is here introduced only to guarantee the existence of an optimal solution, we limit our dynamical considerations to trajectories for which the constraint $E(t) < b$ is always verified. We hence focus on the following model:

$$\dot{u} = r\,u\left(1-\frac{u}{k}\right) - u\,\mu^q\,E^q, \tag{18}$$

$$\dot{E} = \frac{r\,u + \delta k}{k\,(1-q)}\,E. \tag{19}$$

We start the analysis of the state-control system (18)–(19) by observing that the first quadrant is an invariant set for the dynamics since system trajectories never cross the u and E axis. By looking at the zero-growth isoclines, we observe that $\dot{u} = 0$ on the axis $u = 0$ and along the curve:

$$E(u) = \frac{r^{1/q}}{\mu}\left(1-\frac{u}{k}\right)^{\frac{1}{q}}. \tag{20}$$

On the other hand, in the first quadrant, the equation $\dot{E} = 0$ defines the zero-growth isoclines $E = 0$. Consequently, the system admits as equilibria the points $P_k = (k, 0)$ and $P_0 = (0, 0)$. The Jacobian matrix of the state-control model (18)–(19):

$$J(u, E) = \begin{bmatrix} r\left(1 - \dfrac{2u}{k}\right) - \mu^q E^q & -u\,\mu^q\, q\, E^{q-1} \\ \\ \dfrac{r}{k(1-q)}E & \dfrac{r\,u + \delta k}{k(1-q)} \end{bmatrix}, \tag{21}$$

evaluated at P_0 has eigenvalues r and $\dfrac{\delta}{1-q}$ both positive, this implying that the origin is a repulsive node. The eigenvalues of the Jacobian matrix (21) evaluated at the equilibrium P_k are $-r < 0$ and $\dfrac{r + \delta}{1-q} > 0$, and hence P_k turns out to be a saddle. The loci $\dot{u} = 0$, defined by (20), intersect the E axis at the point $P_1 = \left(0, \dfrac{r^{1/q}}{\mu}\right)$ and the u axis at the saddle point $P_k = (k, 0)$. For value of u below the carrying capacity k, it results that isocline $E = E(u)$ defined in (20) lies in the first quadrant. Consequently, it partitions the $(u,\ E)$ positive plane into two regions, labeled I and II, lying below or above the curve, respectively. In Region I, the trajectories are featured by values of $\dot{u} > 0$, while in Region II it results $\dot{u} < 0$. Moreover, from (17), it can be checked that $\dot{E} > 0$ for positive values of E. As consequence in Region I, the solution trajectories are increasing in both u and E direction. After crossing the curve $\dot{u} = 0$, they enter into Region II, decrease in the u direction, and finally approach the increasing exponential dynamics on the E axis, described by $\dot{E} = \dfrac{\delta}{1-q} E$.

The above phase-plane analysis is a useful tool to provide some qualitative characterizations of the optimal solution, as we show in the following section. In Fig. 1, for illustrative purpose, we draw the phase diagram in correspondence to parameters related to the case study of feral cats analyzed in [3], i.e., $r = 0.55$, $k = 100$, $\mu = 2.21$, and $q = 0.64$; moreover, we set $\delta = 0.005$.

4 Optimal Paths for the State-Control Model

In order to detect the optimal solution, we notice that the function $\widehat{H}$, written in terms of the state-control variables:

$$\mathscr{H}(t, u, E) = \frac{e^{-\delta t}}{q}\left[(q-1)\,E + \frac{r}{\mu^q}\left(1 - \frac{u}{k}\right)E^{1-q}\right] + \delta \int_0^t e^{-\delta s}\, E(s)\, ds, \tag{22}$$

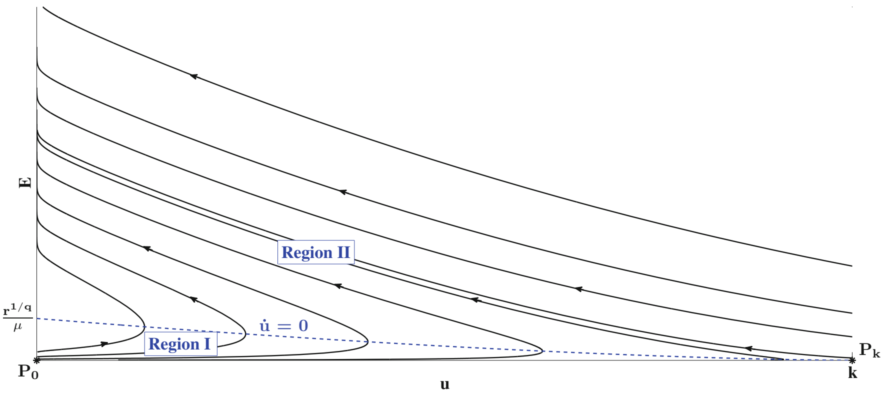

Fig. 1 System trajectories in the (u, E) plane for a variety of initial conditions in the interior of the first quadrant. Parameters: $r = 0.55$, $k = 100$, $\mu = 2.21$, $q = 0.64$, and $\delta = 0.005$

is an invariant for the state-control dynamics. The final optimal control value $E^*(T^*)$ is uniquely determined by the necessary condition (7):

$$\frac{e^{-\delta T^*}}{q} \left[(q-1)\, E^*(T^*) + \frac{r}{\mu^q} \left(1 - \frac{u_T}{k}\right) E^*(T^*)^{1-q} \right] = 0.$$

It follows that $E^*(T^*) = 0$ or

$$E^*(T^*) \;=\; \frac{1}{\mu} \left(\frac{r}{1-q}\right)^{\frac{1}{q}} \left(1 - \frac{u_T}{k}\right)^{\frac{1}{q}}. \tag{23}$$

As in [4], we call *curve of minimal effort* the curve in the plane $(u,\ E)$:

$$E(u) \;=\; \frac{1}{\mu} \left(\frac{r}{1-q}\right)^{\frac{1}{q}} \left(1 - \frac{u}{k}\right)^{\frac{1}{q}}. \tag{24}$$

With the above notations, Eq. (23) ensures that the final optimal control value $E^*(T^*)$ lies on the curve of minimal effort. Moreover, by imposing

$$\mathscr{H}(T^*, u_T, E^*(T^*)) \;=\; \mathscr{H}(0, u_0, E_0^*)$$

we have

$$\delta \int_0^{T^*} e^{-\delta s}\, E^*(s)\, ds \;=\; \frac{1}{q} \left[(q-1)\, E_0^* + \frac{r}{\mu^q} \left(1 - \frac{u_0}{k}\right) {E_0^*}^{1-q} \right] \tag{25}$$

From the above relation, we have several insights about the optimal solution. Firstly, observe that if $\delta = 0$, then E_0^* lies on the curve of minimal effort defined in (24).

For $\delta > 0$ (apart from the trivial case when $u_T = u_0$, corresponding to $T^* = 0$), the integral term is strictly positive, and hence E_0^* should lie below the curve of minimal effort (25).

The left-hand term represents the objective function to be minimized, up to the positive multiplicative constant $\delta > 0$. By dividing both terms in (25) by δ, we obtain an equivalent relation that can be written as follows:

$$\mathscr{J}(E^*, T^*) = \frac{1}{\delta\, q}\left[q\, E_0^* - E_0^{*\,1-q}\left(E_0^{*q} - \frac{r}{\mu^q}\left(1 - \frac{u_0}{k}\right)\right)\right]. \tag{26}$$

By using both (26) and the qualitative behavior of the dynamics resulting from the phase-space analysis in Sect. 3, we are able to characterize and localize the optimal control solution. To this aim, let us consider the values spanned by the following parametrization:

$$\mathscr{E}(\epsilon) = \frac{r^{1/q}}{\mu\,(1-\epsilon)^{1/q}}\left(1 - \frac{u_0}{k}\right)^{1/q} \qquad 0 \le \epsilon \le q. \tag{27}$$

Taking into account that the final optimal control value $E^*(T^*)$ must lie on the curve of minimal effort, we detect the optimal trajectories starting form $(u_0,\ E_0^*)$ with $E_0^* = \mathscr{E}(\epsilon)$, as follows:

- If $E_0^* = \mathscr{E}(q)$, then E_0^* lies on the curve of minimal effort (24). In this case, relation (25) is satisfied with $T^* = 0$ and the optimal control $E^* = E_0^*$ corresponds to the largest density target value $u^*(T^*) = u_0$. The objective function assumes the minimum value $\mathscr{J}_q = \mathscr{J}(\mathscr{E}(q), 0) = 0$, as can be deduced from (26);
- If $E_0^* = \mathscr{E}(0)$, then E_0^* lies on the isocline defined in (20). From (26), the objective value is $\mathscr{J}_0 = \dfrac{\mathscr{E}(0)}{\delta}$. The trajectory is optimal for the density target value $u^*(T^*) = u_T^{min}$, where u_T^{min} is the intersection between the trajectory starting from $(u_0, \mathscr{E}(0))$ and the curve of minimal effort (24);
- If $E_0^* = \mathscr{E}(\epsilon)$, with $0 < \epsilon < q$, then E_0 lies between the curve of minimal effort (24) and the isocline defined in (20). The trajectory is optimal for a target value u_T s.t. $u_T^{min} < u_T < u_0$ and the objective value is

$$\begin{aligned}
\mathscr{J}_\epsilon &= \frac{1}{\delta q}\left[(q-1)\,\mathscr{E}(\epsilon) + \frac{r}{\mu^q}\left(1 - \frac{u_0}{k}\right)\mathscr{E}(\epsilon)^{1-q}\right] \\
&= \frac{1}{\delta}\left[\frac{q-1}{q(1-\epsilon)^{1/q}}\,\mathscr{E}(0) + \frac{r}{\mu^q}\left(1 - \frac{u_0}{k}\right)\frac{\mathscr{E}(0)^{1-q}}{q(1-\epsilon)^{(1-q)/q}}\right] \\
&= \frac{\mathscr{E}(0)}{\delta}\left[\frac{q-1}{q(1-\epsilon)^{1/q}} + \frac{r}{\mu^q}\left(1 - \frac{u_0}{k}\right)\frac{\mathscr{E}(0)^{-q}}{q(1-\epsilon)^{(1-q)/q}}\right] \\
&= \frac{\mathscr{E}(0)}{\delta}\left[\frac{q-1}{q(1-\epsilon)^{1/q}} + \frac{1}{q(1-\epsilon)^{(1-q)/q}}\right] = \mathscr{J}_0\,\frac{(q-\epsilon)}{q(1-\epsilon)^{1/q}}.
\end{aligned}$$

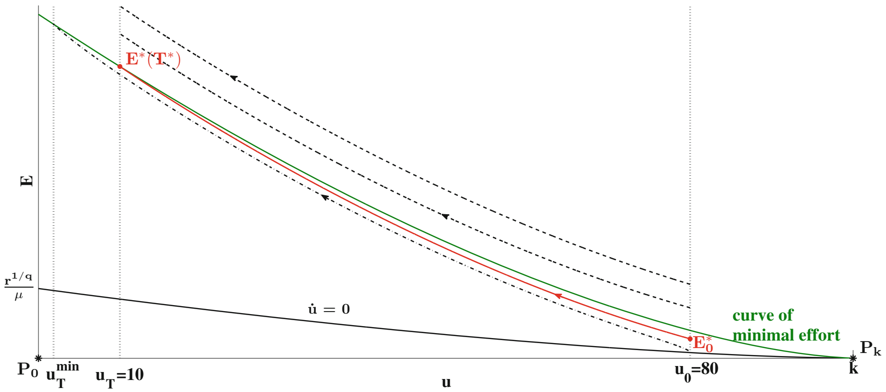

Fig. 2 System trajectories in the (u, E) plane starting from $u_0 = 80$ and reaching $u_T = 10$ (dotted black lines). Parameters: $r = 0.55$, $k = 100$, $\mu = 2.21$, $q = 0.64$, and $\delta = 0.005$. The optimal solution (red continuous line) is achieved in correspondence to $T^* \approx 3.972$, $E^*(T^*) \approx 0.7442$, and $E_0^* \approx 0.0495$. The dash-dotted black curve represents the optimal solution reaching the threshold value of the final density $u_T^{min} \approx 1.856$

Denoting with $s(\epsilon) = \dfrac{(q-\epsilon)}{q(1-\epsilon)^{1/q}}$, we have that $s(0) = 1$, $s(q) = 0$, and $\dot{s}(\epsilon) < 0$, hence we deduce that $\mathscr{J}_q < \mathscr{J}_\epsilon < \mathscr{J}_0$.

In conclusion, if $u_T^{min} \leq u_T \leq u_0$, the initial point of the optimal solution will belong to the interval $[\mathscr{E}(0), \mathscr{E}(q)]$. As a consequence, the optimal trajectory will lie between the trajectory starting from $(u_0, \mathscr{E}(0))$ and the curve of minimal effort. In Fig. 2, for illustrative purposes, the resulting optimal trajectory for $u_0 = 80$ and $u_T = 10$ is shown.

5 Conclusions

We considered an optimal control problem with free terminal time for the management of invasive species. With respect to recent literature, we made the model more realistic by introducing a discount term in the objective function. We showed that the alternative state-control optimality system, defined as in [4], is autonomous and can be analyzed with a dynamical system approach. We deduced the expression of its invariant that suggested several insights on the optimal solution. Further work will be devoted to theoretically establishing ranges of parameters that guarantee existence and uniqueness results for the optimal control solution.

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Part II
Approximation Theory and Special Functions: Fourth Series

Session Organizers: Oktay Duman and Esra Erkus-Duman

This session was the fourth edition of a series of mini-symposia which bring together researchers from all areas of Approximation Theory and Special Functions. The first one was organized within the international conference ICNAAM 2013 in Greece, the second one in MDS 2014 in Bulgaria, and the third one in ETAMM 2016 in France.

The highlighted topics (but not limited to) were: Classical approximation, Korovkin-type approximation, Statistical approximation, Interpolation, Fuzzy approximation, Summability, Timescales, Constructive approximation, Orthogonal polynomials, Generating functions, Matrix-valued polynomials, q-Analysis, Fractional analysis, General orthogonal systems, and Fourier analysis.

Extended Multivariable Hypergeometric Functions

Duriye Korkmaz-Duzgun and Esra Erkuş-Duman

Abstract In this chapter, we define an extension of multivariable hypergeometric functions. We obtain a generating function for these functions. Furthermore, we derive a family of multilinear and multilateral generating functions for these extended multivariable hypergeometric functions.

1 Introduction

Nowadays, there is a growing interest in extensions, including new extra parameter, of some special functions especially hypergeometric and multivariable hypergeometric functions [1, 11].

In this study, the extended beta, hypergeometric, Appell, Lauricella, Horn, and multivariable Horn functions, which are introduced below, have been used for defining a new extension of multivariable hypergeometric functions.

Definition 1 Let a function $\Theta(\{\kappa_l\}_{l\in\mathbb{N}_0}; z)$ be analytic within the disk $|z| < R$ $(0 < R < \infty)$ and let its Taylor–Maclaurin coefficients be explicitly denoted by the sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$. Suppose also that the function $\Theta(\{\kappa_l\}_{l\in\mathbb{N}_0}; z)$ can be continued analytically in the right half-plane $Re(z) > 0$ with the asymptotic property given as

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_2

follows [11]:

$$\Theta(\kappa_l; z) \equiv \Theta(\{\kappa_l\}_{l\in\mathbb{N}_0}; z)$$
$$= \begin{cases} \sum \kappa_l \frac{z^l}{l!} & (|z| < R;\ 0 < R < \infty;\ \kappa_0 = 1) \\ M_0 z^w \exp(z)\left[1 + O\left(\frac{1}{z}\right)\right] & (Re(z) \to \infty;\ M_0 > 0;\ w \in \mathbb{C}) \end{cases} \tag{1}$$

for some suitable constants M_0 and w depending essentially on the sequence $\{\kappa_l\}_{l\in\mathbb{N}_0}$.

By means of the function $\Theta(\{\kappa_l\}_{l\in\mathbb{N}_0}; z)$ defined by (1), Srivastava et al. defined the following extended beta function $B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\alpha,\beta)$ [11]:

$$B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\ \Theta\left(\kappa_l; -\frac{p}{t} - \frac{q}{(t-1)}\right)dt,$$
$$(\min\{Re(\alpha), Re(\beta)\} > 0, \quad \min\{Re(p), Re(q)\} \geq 0). \tag{2}$$

If we set $\kappa_l = \frac{(\rho)_l}{(\sigma)_l}$, $\rho = \sigma$, and $p = q = 0$ in (2), then (2) becomes the classical beta function [10].

By a similar idea, they had extended hypergeometric and confluent hypergeometric functions, respectively, as follows [11]:

$$F_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\alpha,\beta;\gamma;x) = \sum_{n=0}^{\infty}(\alpha)_n \frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)}\frac{x^n}{n!},$$
$$(|t| < 1, \quad Re(\gamma) > Re(\beta) > 0, \quad \min\{Re(p), Re(q)\} \geq 0) \tag{3}$$

and

$$\Phi_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta;\gamma;x) = \sum_{n=0}^{\infty}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta+n,\gamma-\beta)}{B(\beta,\gamma-\beta)}\frac{x^n}{n!},$$
$$(Re(\gamma) > Re(\beta) > 0, \quad \min\{Re(p), Re(q)\} \geq 0). \tag{4}$$

If we set $\kappa_l = \frac{(\rho)_l}{(\sigma)_l}$, $\rho = \sigma$, and $p = q = 0$ in (3), (4) then (3) and (4) become classical hypergeometric and confluent hypergeometric functions, respectively (see [10]).

On the other hand, they defined the following extended second kind Appell functions [11]:

$$F_2^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}(\alpha,\beta_1,\beta_2;\gamma_1,\gamma_2;x_1,x_2)$$

$$=\sum_{m,r=0}^{\infty}(\alpha)_{m+r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta_1+m,\gamma_1-\beta_1)}{B(\beta_1,\gamma_1-\beta_1)}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta_2+r,\gamma_2-\beta_2)}{B(\beta_2,\gamma_2-\beta_2)}\frac{x_1^m}{m!}\frac{x_2^n}{n!}, \tag{5}$$

$$(|x_1|+|x_2|<1,\quad \min\{Re(p),Re(q)\}\geq 0).$$

If we set $\kappa_l=\frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$,and $p=q=0$ in (5), then (5) becomes the second kind Appell functions [2].

After short period of time by using a similar method, Minjie defined the extended Lauricella functions [8]:

$$F_{A,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}^{(r)}(\alpha,\beta_1,\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;x_1,\ldots,x_r)$$

$$=\sum_{m_1,\ldots,m_r=0}^{\infty}(\alpha)_{m_1+\ldots+m_r}\prod_{j=1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{mr}}{m_r!}, \tag{6}$$

where $|x_1|+\ldots+|x_r|<1$ and $\min\{Re(p),Re(q)\}\geq 0$.

If we set $\kappa_l=\frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$,and $p=q=0$ in (6), then (6) becomes Lauricella functions [4].

Extended fourth kind Horn functions are defined by [6]:

$$H_4^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}(\alpha,\beta;\gamma_1,\gamma_2;x_1,x_2) \tag{7}$$

$$:=\sum_{m,r=0}^{\infty}\frac{(\alpha)_{2m+r}}{(\gamma_1)_m}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta+r,\gamma_2-\beta)}{B(\beta,\gamma_2-\beta)}\frac{x_1^m}{m!}\frac{x_2^r}{r!}.$$

where $2\sqrt{|x|}+|y|<1$ and $\min\{Re(p),Re(q)\}\geq 0$.

When $\kappa_l=\frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$, and $p=q=0$ in (7), then function (7) reduces to the fourth kind Horn functions [5].

The extension of multivariable fourth kind Horn functions was defined by [6]:

$$
{}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\alpha,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;x_1,\ldots,x_k,x_{k+1},\ldots,x_r\right)
$$

$$
:=\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(\alpha)_{2(m_1+\ldots+m_k)+m_{k+1}+\ldots+m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_k)_{m_k}}
$$

$$
\times\prod_{j=k+1}^{r}\frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!},
\tag{8}
$$

where $\left(2(\sqrt{|x_1|}+\ldots+\sqrt{|x_k|})+\ldots+|x_r|<1\right)$ and $\min\{Re(p),Re(q)\}\geq 0$.

When $\kappa_l=\frac{(\rho)_l}{(\sigma)_l}$, $\rho=\sigma$, and $p=q=0$ in (8), then (8) reduces to the multivariable Horn functions [4].

Finally, the multivariable hypergeometric functions are defined by [7]:

$$
{}^{(k)}E^{(r)}\left(\alpha,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;x_1,\ldots,x_r\right)
$$

$$
=\sum_{m_1,\ldots,m_r=0}^{\infty}(\alpha)_{\rho(m_1+\ldots+m_k)+m_{k+1}+\ldots+m_r}\frac{(\beta_{k+1})_{m_{k+1}}\cdots(\beta_r)_{m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_r)_{m_r}}\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_r^{m_r}}{m_r!},
\tag{9}
$$

$$
\rho\left(\sqrt[2]{|x_1|}+\ldots+\sqrt[2]{|x_k|}\right)+|x_{k+1}|+\ldots+|x_r|<1.
$$

The aim of this chapter is to define an extension of multivariable hypergeometric functions. We obtain a generating function for these functions. Then, we derive a family of multilinear and multilateral generating functions for these extended multivariable hypergeometric functions.

2 Generating Function

In this section, we define an extension of multivariable hypergeometric functions given in (9). Then, we obtain a generating function for these functions.

Definition 2 We define an extension of the multivariable hypergeometric functions by:

$$
{}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\alpha,\beta_{k+1},\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;x_1,\ldots,x_r\right)
\tag{10}
$$

$$
=\sum_{m_1,\ldots,m_r=0}^{\infty}\frac{(\alpha)_{\rho(m_1+\ldots+m_k)+m_{k+1}+\ldots+m_r}}{(\gamma_1)_{m_1}\ldots(\gamma_k)_{m_k}}
$$

$$\times \prod_{j=k+1}^{r} \frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta_j + m_j, \gamma_j - \beta_j)}{B(\beta_j, \gamma_j - \beta_j)} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!},$$

where $\rho\left(\sqrt{|x_1|} + \ldots + \sqrt{|x_k|}\right) + |x_{k+1}| + \ldots + |x_r| < 1,\ \min\{Re(p), Re(q)\} \geq 0$, and $\rho \geq 0$. If $\kappa_l = \frac{(\rho)_l}{(\sigma)_l}$, $\rho = \sigma$, and $p = q = 0$ in (10), then function (10) reduces to multivariable hypergeometric functions given by (9).

Theorem 1 *We have the following generating function for the extended multivariable hypergeometric functions defined by (10):*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \tfrac{(\lambda)_n}{n!}\ {}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}(\lambda+n, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r)\, t^n \\
&= (1-t)^{-\lambda} \\
&\quad \times {}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; \tfrac{x_1}{(1-t)^\rho}, \ldots, \tfrac{x_k}{(1-t)^\rho}, \tfrac{x_{k+1}}{(1-t)}, \ldots, \tfrac{x_r}{(1-t)}\right),
\end{aligned} \tag{11}$$

where $\lambda \in \mathbb{C}$ *and* $|t| < 1$.

Proof Let T denote the first member of assertion (11). From properties of Pochhammer symbol, we have

$$T = \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\lambda)_{\rho(m_1+\ldots+m_k)+\ldots+m_{k+1}+\ldots+m_r}}{(\gamma_1)_{m_1}\ldots(\gamma_k)_{m_k}} \prod_{j=k+1}^{r} \frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta_j + m_j, \gamma_j - \beta_j)}{B(\beta_j, \gamma_j - \beta_j)}$$

$$\times \frac{(x_1)^{m_1}}{m_1!} \cdots \frac{(x_r)^{m_r}}{m_r!} \sum_{n=0}^{\infty} \frac{(\lambda + \rho(m_1 + \ldots + m_k) + m_{k+1} + \ldots + m_r)_n\, t^n}{(n)!}$$

$$= (1-t)^{-\lambda} \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{(\lambda)_{\rho(m_1+\ldots+m_k)+\ldots+m_{k+1}+\ldots+m_r}}{(\gamma_1)_{m_1}\cdots(\gamma_k)_{m_k}} \prod_{j=k+1}^{r} \frac{B_{p,q}^{\left(\{K_l\}_{l\in\mathbb{N}_0}\right)}(\beta_j+m_j,\gamma_j-\beta_j)}{B(\beta_j,\gamma_j-\beta_j)}$$

$$\times \left(\tfrac{x_1}{(1-t)^\rho}\right)^{m_1} \cdots \left(\tfrac{x_k}{(1-t)^\rho}\right)^{m_k} \left(\tfrac{x_{k+1}}{(1-t)}\right)^{m_{k+1}} \cdots \left(\tfrac{x_r}{(1-t)}\right)^{m_r} \tfrac{1}{m_1!} \cdots \tfrac{1}{m_r!}$$

$$\begin{aligned}
&= (1-t)^{-\lambda} \\
&\quad \times {}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; \tfrac{x_1}{(1-t)^\rho}, \ldots, \tfrac{x_k}{(1-t)^\rho}, \tfrac{x_{k+1}}{(1-t)}, \ldots, \tfrac{x_r}{(1-t)}\right),
\end{aligned}$$

which completes the proof.

If we take $\rho = 2$ in Theorem 1, we have the following conclusion for the extended multivariable fourth kind Horn functions defined by (8):

Corollary 1 *We have the following relation for the extended multivariable fourth kind Horn functions:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda+n, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r\right) t^n$$

$$= (1-t)^{-\lambda} \times {}^{(k)}H^{(r)}_{4,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; \frac{x_1}{(1-t)^2}, \ldots, \frac{x_k}{(1-t)^2}, \frac{x_{k+1}}{(1-t)}, \ldots, \frac{x_r}{(1-t)}\right),$$

where $\lambda \in \mathbb{C}$ *and* $|t| < 1$.

If we choose $\rho = 2$, $k = 1$, and $r = 2$ in Theorem 1, we immediately have the following conclusion for the extended fourth kind Horn functions defined by (7):

Corollary 2 *We have the following generating function for the extended fourth kind Horn functions:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} H_4^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda+n, \beta; \gamma_1, \gamma_2; x_1, x_2\right) t^n$$

$$= (1-t)^{-\lambda}\, H_4^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda, \beta; \gamma_1, \gamma_2; \frac{x_1}{(1-t)^2}, \frac{x_2}{(1-t)}\right),$$

where $\lambda \in \mathbb{C}$ *and* $|t| < 1$.

If we set $\rho = 2$ and $k = 0$ in Theorem 1, we have the following conclusion for the extended Lauricella functions given by (6):

Corollary 3 *We have*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, F^{(r)}_{A,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda+n, \beta_1, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r\right) t^n$$

$$= (1-t)^{-\lambda}\, F^{(r)}_{A,\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}\left(\lambda, \beta_1, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; \frac{x_1}{(1-t)}, \ldots, \frac{x_r}{(1-t)}\right),$$

where $\lambda \in \mathbb{C}$ *and* $|t| < 1$.

If we choose $\rho = 2$, $k = 0$, and $r = 2$ in Theorem 1, we have the following conclusion for the extended second kind Appell functions given by (5):

Corollary 4 *We have the following generating function for the extended second kind Appell functions:*

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F_2^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)} (\lambda+n, \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2)\, t^n$$

$$= (1-t)^{-\lambda}\; F_2^{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)} \left(\lambda, \beta_1, \beta_2; \gamma_1, \gamma_2; \tfrac{x_1}{(1-t)}, \tfrac{x_2}{(1-t)}\right),$$

where $\lambda \in \mathbb{C}$ *and* $|t| < 1$.

3 Multilinear and Multilateral Generating Functions

In this section, we derive a family of multilinear and multilateral generating functions for the extended multivariable hypergeometric functions defined by (10), by using the similar method considered in [3, 9].

Theorem 2 *Corresponding to an identically nonvanishing function* $\Omega_\mu(y_1, \ldots, y_s)$ *of* s *complex variables* $y_1, \ldots, y_s$ $(s \in \mathbb{N})$ *and of complex order* μ*, let*

$$\Lambda_{\mu,\psi}(y_1, \ldots, y_s; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \ldots, y_s)\zeta^k,$$

where $a_k \neq 0$, $\mu, \psi \in \mathbb{C}$ *and*

$$\Theta_{n,b}^{\mu,\psi}(x_1, x_2; y_1, \ldots, y_s; \xi)$$

$$:= \sum_{k=0}^{[n/b]} a_k\, (\lambda)_{n-bk}\ \ {}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)}$$
$$\times (\lambda+n-bk, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r)$$

$$\times \Omega_{\mu+\psi k}(y_1, \ldots, y_s) \frac{\xi^k}{(n-bk)!}.$$

Then, for $b \in \mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty} \Theta_{n,b}^{\mu,\psi}\left(x_1, x_2; y_1, \ldots, y_s; \tfrac{\eta}{t^b}\right) t^n = \Lambda_{\mu,\psi}(y_1, \ldots, y_s; \eta)(1-t)^{-\lambda}$$

$$\times\, {}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)} \left(\lambda, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; \tfrac{x_1}{(1-t)^\rho}, \ldots \tfrac{x_k}{(1-t)^\rho}, \tfrac{x_{k+1}}{(1-t)} \ldots, \tfrac{x_r}{(1-t)}\right) \tag{12}$$

provided that each member of (12) exists.

Proof For convenience, let S denote the first member of the assertion (12). Then,

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/b]} a_k (\lambda)_{n-bk}$$

$$\times {}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)} (\lambda + n - bk, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r)$$

$$\times \Omega_{\mu+\psi k}(y_1, \ldots, y_s)\eta^k \frac{t^{n-bk}}{(n-bk)!}.$$

Replacing n by $n + bk$, we may write that

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k (\lambda)_n \; {}^{(k)}E^{(r)}_{\left(\{K_l\}_{l\in\mathbb{N}_0};p,q\right)} (\lambda + n, \beta_{k+1}, \ldots, \beta_r; \gamma_1, \ldots, \gamma_r; x_1, \ldots, x_r)$$

$$\times \Omega_{\mu+\psi k}(y_1, \ldots, y_s)\eta^k \frac{t^n}{n!}.$$

Using Theorem 1 on the last equality, one can get the desired result.

Furthermore, for every suitable choice of the coefficients a_k ($k \in \mathbb{N}_0$), if the multivariable functions $\Omega_{\mu+\psi k}(y_1, \ldots, y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 2 can be applied in order to derive various families of multilinear and multilateral generating functions for the extended multivariable hypergeometric functions defined by (10).

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Cubature of Multidimensional Schrödinger Potential Based on Approximate Approximations

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Abstract We report here on some recent results obtained in collaboration with Maz'ya and Schmidt (Appl Anal 98:408–429, 2019). We derive semi-analytic cubature formulas for the solution of the Cauchy problem for the Schrödinger equation which are fast and accurate also if the space dimension is greater than or equal to 3. We follow ideas of the method of approximate approximations, which provides high-order semi-analytic cubature formulas for many important integral operators of mathematical physics. The proposed method is very efficient in high dimensions if the data allow separated representations.

1 Introduction

In this chapter, we propose a numerical method, both fast and precise, for solving the following time-dependent multi-dimensional Schrödinger equation of free particles:

$$i\frac{\partial u}{\partial t} + \Delta_{\mathbf{x}} u = f(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}_+ \tag{1}$$

$$u(\mathbf{x}, 0) = g(\mathbf{x}), \qquad \mathbf{x} = (x_1, , \ldots, x_n) \in \mathbb{R}^n, \tag{2}$$

with $u = u(\mathbf{x}, t)$ the wave function depending on the spatial variables $\mathbf{x} \in \mathbb{R}^n$ and the time $t \in \mathbb{R}_+$, and $\Delta_{\mathbf{x}}$ the usual Laplacian with respect to the variables $\mathbf{x}$.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_3

We suppose that g and f are supported with respect to $\mathbf{x}$ in a hyper-rectangle $[\mathbf{P},\mathbf{Q}] = \{\mathbf{x} = (x_1,\dots,x_n) \in \mathbb{R}^n : P_j \le x_j \le Q_j,\ j = 1,\dots,n\}$, $\operatorname{supp} g \subseteq [\mathbf{P},\mathbf{Q}]$, $\operatorname{supp} f \subseteq [\mathbf{P},\mathbf{Q}] \times \mathbb{R}_+$. Under suitable integrability conditions on g and f, the solution of (1)–(2) is given by:

$$u(\mathbf{x},t) = \mathscr{S}g(\mathbf{x},t) + \Pi f(\mathbf{x},t)$$

$$\mathscr{S}g(\mathbf{x},t) = \frac{1}{(4\pi i t)^{n/2}} \int_{[\mathbf{P},\mathbf{Q}]} \mathrm{e}^{\,i\,|\mathbf{x}-\mathbf{y}|^2/(4t)}\, g(\mathbf{y})\, d\mathbf{y}\,, \tag{3}$$

$$\Pi f(\mathbf{x},t) = -i \int_0^t \frac{ds}{(4\pi i s)^{n/2}} \int_{[\mathbf{P},\mathbf{Q}]} \mathrm{e}^{i|\mathbf{x}-\mathbf{y}|^2/(4\,s)} f(\mathbf{y},t-s) d\mathbf{y}\,. \tag{4}$$

Our goal is to derive semi-analytic cubature formulas for $\mathscr{S}g$ and Πf of an arbitrary high-order which are fast and accurate also if the space dimension $n \ge 3$. We follow ideas of the method of approximate approximations, which provides high-order semi-analytic cubature formulas for many important integral operators of mathematical physics (cf.[15] and the reference therein). They are based on the use of approximating functions with the property that, on one hand, simple linear combinations provide high-order approximations up to a small negligible saturation error and, on the other hand, the action of the integral operators on these functions can be taken analytically. Examples of those cubature formulas for volume potentials over $\mathbb{R}^n$ and on bounded domains have been studied in [14] and [9]. We combine this approach with the strategy of tensor decomposition. The representation of data in tensor product form is a fundamental tool for managing the dimensionality issue [5], and methods based on tensor decompositions have become well established in a wide range of applications (cf. e.g. [3] and [2]). The aim is the separation of dimensions, which implies that expensive high-dimensional operations are decomposed into a series of one-dimensional operations. As a consequence, the complexity scales linearly in the dimension rather than exponentially.

In [7, 8], we applied this procedure for the cubature of high-dimensional Newton potential over the full space and over half-spaces. The new approach was extended to stationary advection–diffusion operators $-\Delta + 2\mathbf{b}\cdot\nabla + c$ with $\mathbf{b} \in \mathbb{C}^n$ and $c \in \mathbb{C}$ in [10] and [6], and to parabolic problems in [11]. For the Schrödinger equation, the situation is more difficult because the fundamental solution does not decay exponentially, and numerical approaches to solving (1)–(2) are very expensive due to the oscillatory solutions, especially in multi-dimensional case. The application of approximate approximations to this equation, combined with separated representations, reduces these problems and provides new very efficient semi-analytic cubature formulas [13]. In [12], we show that this approach is successful also for the diffraction potential.

2 Approximate Quasi-Interpolants

Our method uses quasi-interpolation formulas of the type:

$$(\mathcal{M}_{h\sqrt{\mathcal{D}}}\, g)(\mathbf{x}) = \mathcal{D}^{-n/2} \sum_{\mathbf{m}\in\mathbb{Z}^n} g(h\mathbf{m})\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\Big), \tag{5}$$

where η is a sufficiently smooth and rapidly decaying function, and h and $\mathcal{D}$ are positive fixed parameters. Under the assumption that

$$\int_{\mathbb{R}^n} \eta(\mathbf{x})\, \mathbf{x}^{\alpha}\, d\mathbf{x} = \delta_{0,\alpha}, \quad 0 \le |\alpha| < N \tag{6}$$

the following error estimate was proved.

Theorem 1 ([15, Theorem 2.28]) *Let $g \in W_p^N(\mathbb{R}^n)$ with $N > n/p$, $1 \le p \le \infty$. For any $\epsilon > 0$, there exists $\mathcal{D} > 0$ such that*

$$\|g - \mathcal{M}_{h\sqrt{\mathcal{D}}}\, g\|_{L^p} \le c(h\sqrt{\mathcal{D}})^N + \epsilon \sum_{k=0}^{N-1} \frac{(h\sqrt{\mathcal{D}})^k}{(2\pi)^k} \|\nabla_k g\|_{L^p},$$

where the constant c depends only on η and $\|\nabla_k g\|_{L^p} = \sum_{|\boldsymbol{\alpha}|=k} \frac{\|\partial^{\boldsymbol{\alpha}} g\|_{L^p}}{\boldsymbol{\alpha}!}$.

Here, we use the standard multi-index notations for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\ge 0}$, and we write $\|g\|_{L^p}$ the Lebesgue norm of a function g in $L^p = L^p(\mathbb{R}^n)$. Thus, the error consists of a term ensuring $\mathcal{O}(h^N)$ convergence and of the so-called *saturation error*, which in general does not converge to zero as h goes to 0 but it can be made arbitrarily small if $\mathcal{D}$ is sufficiently large.

For the approximation of $f \in L^p(\mathbb{R}^{n+1})$, we use the approximate quasi-interpolant:

$$\Big(\mathcal{N}_{h\sqrt{\mathcal{D}},\tau\sqrt{\mathcal{D}_0}}\Big) f(\mathbf{x},t) = \mathcal{D}_0^{-1/2}\mathcal{D}^{-n/2} \sum_{\substack{\ell\in\mathbb{Z}\\ \mathbf{m}\in\mathbb{Z}^n}} f(h\mathbf{m},\tau\ell)\, \psi\Big(\frac{t-\tau\ell}{\tau\sqrt{\mathcal{D}_0}}\Big)\eta\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}\Big) \tag{7}$$

where h, τ, $\mathcal{D}$, and $\mathcal{D}_0$ are positive fixed parameters, and ψ and η are sufficiently smooth and rapidly decaying functions. If also ψ fulfils the moment conditions (6), then $\mathcal{N}_{h\sqrt{\mathcal{D}},\tau\sqrt{\mathcal{D}_0}} f$ approximates f with order $\mathcal{O}((h\sqrt{\mathcal{D}} + \tau\sqrt{\mathcal{D}_0})^N)$ up to the saturation error in $L^p(\mathbb{R}^{n+1})$ if $N > (n+1)/p$.

In order to construct a cubature for $\mathcal{S}g$ and Πf, we approximate g and f such that $\mathcal{S}$ and $\mathcal{P}$ applied to it can be computed, analytically or at least efficiently. The error estimates for $\mathcal{M}_{h\sqrt{\mathcal{D}}} g$ and $\mathcal{N}_{h\sqrt{\mathcal{D}},\tau\sqrt{\mathcal{D}_0}} f$ are proved for functions on

$\mathbb{R}^n$ and $\mathbb{R}^{n+1}$, respectively, whereas g is given only on $[\mathbf{P}, \mathbf{Q}]$ and f on $[\mathbf{P}, \mathbf{Q}] \times \mathbb{R}_+$. If we extend g and f equal to zero, we don't obtain good approximations because $\mathscr{M}_{h\sqrt{\mathscr{D}}} g$ approximates g only in a subdomain of $[\mathbf{P}, \mathbf{Q}]$ and $\mathscr{N}_{h\sqrt{\mathscr{D}}, \tau\sqrt{\mathscr{D}_0}} f$ approximates f only in a subdomain of $[\mathbf{P}, \mathbf{Q}] \times \mathbb{R}_+$. To avoid this difficulty, we extend g and f with preserved smoothness, such that the extensions $\widetilde{g}$ and $\widetilde{f}$ satisfy

$$\|\widetilde{g}\|_{W^N_\infty(\mathbb{R}^n)} \leq C\|g\|_{W^N_\infty([\mathbf{P},\mathbf{Q}])}, \quad \|\widetilde{f}\|_{W^N_\infty(\mathbb{R}^n\times\mathbb{R})} \leq C\|f\|_{W^N_\infty([\mathbf{P},\mathbf{Q}]\times\mathbb{R}_+)}, \qquad C > 0 .$$

These extensions can be obtained, for example, by using Hestenes reflection principle (cf. [4] and [13]). Since η and ψ are smooth and of rapid decay, for any $\epsilon > 0$ one can fix $r > 0$, $r_0 > 0$ and positive parameters $\mathscr{D}$, $\mathscr{D}_0$ such that the quasi-interpolant:

$$\left(\mathscr{M}^{(r)}_{h\sqrt{\mathscr{D}}} g\right)(\mathbf{x}) = \mathscr{D}^{-n/2} \sum_{h\mathbf{m}\in\Omega_{rh}} \widetilde{g}(h\mathbf{m})\, \eta\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\right)$$

approximates g for all $\mathbf{x} \in [\mathbf{P}, \mathbf{Q}]$ with the same error estimate of (5), and

$$\left(\mathscr{N}^{(r,r_0)}_{h\sqrt{\mathscr{D}},\tau\sqrt{\mathscr{D}_0}} f\right)(\mathbf{x}, t) = \mathscr{D}_0^{-1/2}\mathscr{D}^{-n/2} \sum_{\substack{h\mathbf{m}\in\Omega_{rh} \\ \tau\ell\in\widetilde{\Omega}_{r_0\tau}}} \widetilde{f}(h\mathbf{m}, \tau\ell)\, \psi\left(\frac{t-\tau\ell}{\tau\sqrt{\mathscr{D}_0}}\right) \eta\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}\right)$$

approximates f for all $\mathbf{x} \in [\mathbf{P}, \mathbf{Q}]$ and for all $t \in [-T, T]$, $T > 0$ with the same error estimate of (7). Here $\widetilde{\Omega}_{r_0\tau} = (-r_0\tau\sqrt{\mathscr{D}_0} - T, T + r_0\tau\sqrt{\mathscr{D}_0})$ and $\Omega_{rh} = \prod_{j=1}^n I_j$ with $I_j = (P_j - rh\sqrt{\mathscr{D}}, Q_j + rh\sqrt{\mathscr{D}})$.

3 Approximation of $\mathscr{S} g$

Cubature formula for (3) is derived by replacing the density g with the quasi-interpolant $\mathscr{M}^{(r)}_{h\sqrt{\mathscr{D}}} g$ with appropriately chosen function η. For different basis functions η, the integrals $\mathscr{S}\eta$ allow analytic representations. Here, we assume that η is the product of univariate basis functions of the form Gaussians times special polynomials:

$$\eta(\mathbf{x}) = \prod_{j=1}^n \chi_{2M}(x_j), \;\; \chi_{2M}(x) = \frac{(-1)^{M-1}}{2^{2M-1}\sqrt{\pi}(M-1)!} \frac{H_{2M-1}(x)\mathrm{e}^{-x^2}}{x}, \;\; N = 2M, \tag{8}$$

where H_k are the Hermite polynomials $H_k(x) = (-1)^k \mathrm{e}^{x^2}\left(\frac{d}{dx}\right)^k \mathrm{e}^{-x^2}$. Then, the sum:

$$(\mathscr{S}_h g)(\mathbf{x}, t) := \mathscr{D}^{-n/2} \sum_{h\mathbf{m}\in\Omega_{rh}} \widetilde{g}(h\mathbf{m})\Phi^{[\mathbf{P_m},\mathbf{Q_m}]}_{2M}\left(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathscr{D}}}, \frac{4t}{h^2\mathscr{D}}\right), \quad t \neq 0$$

with $\mathbf{P_m} = (\mathbf{P} - h\mathbf{m})/(h\sqrt{\mathscr{D}})$, $\mathbf{Q_m} = (\mathbf{Q} - h\mathbf{m})/(h\sqrt{\mathscr{D}})$, and

$$\Phi_{2M}^{[\mathbf{P},\mathbf{Q}]}(\mathbf{x}, t) = \prod_{j=1}^{n} \frac{1}{(\pi\, i\, t)^{1/2}} \int_{P_j}^{Q_j} \mathrm{e}^{i\,(y_j - x_j)^2/t} \chi_{2M}(y_j) dy_j,$$

provides an approximation of $\mathscr{S}g$. The function $\Phi_{2M}^{[\mathbf{P},\mathbf{Q}]}(\mathbf{x}, 4\,t)$ gives the solution of the initial problem:

$$i\,\partial_t v + \Delta_{\mathbf{x}} v = 0, \qquad v(\mathbf{x}, 0) = \prod_{j=1}^{n} I_{(P_j, Q_j)}(x_j)\chi_{2M}(x_j), \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in \mathbb{R}. \tag{9}$$

Here, $I_{(P_j, Q_j)}$ is the characteristic function of the interval (P_j, Q_j).

Theorem 2 ([11, Theorem 3.1]) *The solution of the initial value problem* (9) *can be expressed by the tensor product:*

$$v(\mathbf{x}, t) = \prod_{j=1}^{n} \big(\Psi_M(x_j, 4\,t, P_j) - \Psi_M(x_j, 4\,t, Q_j)\big),$$

$$\Psi_M(x, t, y) = \frac{1}{2\sqrt{\pi}} \mathrm{e}^{-x^2/(1+it)} \left(\operatorname{erfc}(F(x, it, y))\mathscr{P}_M(x, it) - \frac{\mathrm{e}^{-F^2(x,it,y)}}{\sqrt{\pi}} \mathscr{Q}_M(x, it, y) \right)$$

with the complementary error function erfc*, the argument function:*

$$F(x, t, y) = \sqrt{\frac{t+1}{t}} \Big(y - \frac{x}{t+1} \Big), \tag{10}$$

and $\mathscr{P}_M$, $\mathscr{Q}_M$ are polynomials in x of degree $2M-2$ and $2M-3$, respectively:

$$\mathscr{P}_M(x, t) = \sum_{s=0}^{M-1} \frac{(-1)^s}{s!4^s} \frac{1}{(1+t)^{s+1/2}} H_{2s}\left(\frac{x}{\sqrt{1+t}}\right); \quad \mathscr{Q}_1(x, t, y) = 0,$$

$$\mathscr{Q}_M(x, t, y) = 2 \sum_{k=1}^{M-1} \frac{(-1)^k}{k!\,4^k} \sum_{\ell=1}^{2k} \frac{(-1)^\ell}{t^{\ell/2}} \bigg(H_{2k-\ell}(y) H_{\ell-1}\Big(\frac{y-x}{\sqrt{t}}\Big)$$
$$- \binom{2k}{\ell} H_{2k-\ell}\Big(\frac{x}{\sqrt{1+t}}\Big) \frac{H_{\ell-1}\big(F(t, x, y)\big)}{(1+t)^{k+1/2}} \bigg), \; M > 1.$$

From Theorem 2, we deduce the following semi-analytic cubature formula for $\mathscr{S}g$:

$$(\mathscr{S}_h g)(\mathbf{x},t) = \mathscr{D}^{-n/2} \sum_{h\mathbf{m}\in\Omega_{rh}} \widetilde{g}(h\mathbf{m}) \times$$

$$\prod_{j=1}^{n} \left(\Psi_M \left(\frac{x_j - hm_j}{h\sqrt{\mathscr{D}}}, \frac{4t}{h^2\mathscr{D}}, \frac{P_j - hm_j}{h\sqrt{\mathscr{D}}} \right) - \Psi_M \left(\frac{x_j - hm_j}{h\sqrt{\mathscr{D}}}, \frac{4t}{h^2\mathscr{D}}, \frac{Q_j - hm_j}{h\sqrt{\mathscr{D}}} \right) \right).$$

The cubature error follows immediately from the quasi-interpolation error due to the relation:

$$\mathscr{S}g(\cdot,t) - \mathscr{S}_h g(\cdot,t) = \mathscr{S}(I - \mathscr{M}^{(r)}_{h\sqrt{\mathscr{D}}})\, g(\cdot,t),$$

and Strichartz-type estimates valid for any Schrödinger-admissible pairs (q,r):

$$\|\mathscr{S}g\|_{L^{r,q}(\mathbb{R}_+)} \le C\|g\|_{L^2}$$

with constant C independent of $g \in L^2(\mathbb{R}^n)$. We recall that the exponent pair (q,r) is called Schrödinger admissible if $q, r \ge 2$, $(q,r) \ne (2,\infty)$ and $2/q + n/r = n/2$. By $L^{r,q}(I)$, we denote the Banach space of $L^r(\mathbb{R}^n)$-valued q-summable functions over an interval I with the norm:

$$\|u\|_{L^{r,q}(I)} = \left(\int_I \left(\int_{\mathbb{R}^n} |u(\mathbf{x},t)|^r d\mathbf{x} \right)^{q/r} dt \right)^{1/q}.$$

Theorem 3 ([13, Theorem 2.1]) *Let $g \in W_p^N(\mathbb{R}^n)$, $1 \le p \le 2$, $N > n/p$, be the initial value for the homogeneous Schrödinger equation. For any $\varepsilon > 0$, there exists $\mathscr{D} > 0$ such that for $t > 0$ the cubature formula $\mathscr{S}_h g$ approximates $\mathscr{S}g$ in $L^{p'}(\mathbb{R}^n)$, $p' = p/(p-1)$, with*

$$\|\mathscr{S}g(\cdot,t) - \mathscr{S}_h g(\cdot,t)\|_{L^{p'}} \le \frac{C}{t^{n(1/2-1/p')}} \Big((h\sqrt{\mathscr{D}})^N |g|_{W_p^N} + \varepsilon \sum_{k=0}^{N-1} \frac{(h\sqrt{\mathscr{D}})^k}{(2\pi)^k} \|\nabla_k g\|_{L^p} \Big).$$

Here, $|g|_{W_p^N} = \sum_{|\boldsymbol{\alpha}|=N} \|\partial^{\boldsymbol{\alpha}} g\|_{L_p}$.

If $g \in W_2^N(\mathbb{R}^n)$, then the approximation on $\mathbb{R}^n \times \mathbb{R}$ with $\mathscr{S}_h g$ can be estimated in the mixed Lebesgue spaces $L^{r,q}(\mathbb{R}_+)$ for any Schrödinger-admissible pairs (q,r) by:

$$\|\mathscr{S}g - \mathscr{S}_h g\|_{L^{r,q}(\mathbb{R}_+)} \le C\Big((h\sqrt{\mathscr{D}})^N |g|_{W_2^N} + \varepsilon \sum_{k=0}^{N-1} \frac{(h\sqrt{\mathscr{D}})^k}{(2\pi)^k} \|\nabla_k g\|_{L^2} \Big).$$

4 Approximation of Πf

We construct an approximation for Πf using the quasi-interpolant $\mathcal{N}^{(r,r_0)}_{h\sqrt{\mathcal{D}},\tau\sqrt{\mathcal{D}_0}} f$. We assume the function η in (8) and $\psi(t) = \chi_{2M}(t)$. Then, Πf is approximated by:

$$\Pi_{h,\tau} f(\mathbf{x}, t) := \left(\Pi \mathcal{N}^{(r,r_0)}_{h\sqrt{\mathcal{D}},\tau\sqrt{\mathcal{D}_0}} f\right)(\mathbf{x}, t) = \frac{-i}{\mathcal{D}_0^{1/2}\mathcal{D}^{n/2}} \sum_{\substack{h\mathbf{m}\in\Omega_{rh} \\ \tau\ell\in\widetilde{\Omega}_{r_0\tau}}} \widetilde{f}(h\mathbf{m}, \tau\ell) K_{2M}(\mathbf{x}, t, h\mathbf{m}, \tau\ell)$$

with

$$K_{2M}(\mathbf{x}, t, h\mathbf{m}, \tau\ell) = \int_0^t \chi_{2M}\Big(\frac{t-s-\tau\ell}{\tau\sqrt{\mathcal{D}_0}}\Big) \Phi_{2M}^{[\mathbf{P_m},\mathbf{Q_m}]}\Big(\frac{\mathbf{x}-h\mathbf{m}}{h\sqrt{\mathcal{D}}}, \frac{4s}{h^2\mathcal{D}}\Big) ds \qquad (11)$$

and, keeping in mind Theorem 2,

$$\Phi_{2M}^{[\mathbf{P},\mathbf{Q}]}(\mathbf{x}, t) = \prod_{j=1}^{n} \big(\Psi_M(x_j, t, P_j) - \Psi_M(x_j, t, Q_j)\big).$$

The sum $\Pi_{h,\tau} f$ provides an approximation for the solution of (1) with null initial data. The cubature error follows from the quasi-interpolation error:

$$\Pi f(\mathbf{x}, t) - \Pi_{h,\tau} f(\mathbf{x}, t) = \Pi(I - \mathcal{N}_{h\sqrt{\mathcal{D}},\tau\sqrt{\mathcal{D}_0}}) f(\mathbf{x}, t)$$

and Strichartz-type estimate:

$$\|\Pi f\|_{L^{r,q}(\mathbb{R}_+)} \le C\|f\|_{L^{r',q'}(\mathbb{R}_+)} \text{ for any Schrödinger-admissible pairs } (q, r)\,.$$

Theorem 4 ([13, Theorem 2.2]) *Let (q, r) be a Schrödinger-admissible pair and $N > \frac{(n+1)}{\min(q',r')}$, $q' = q/(q-1)$, $r' = r/(r-1)$. Suppose that the right-hand side f of the inhomogeneous Schrödinger equation satisfies $\partial_t^k \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} f \in L^{r',q'}(\mathbb{R}_+)$ for all $0 \le k + |\boldsymbol{\alpha}| \le N$. Then, there exist a constant C and for any $\varepsilon > 0$ parameters $\mathcal{D}_0, \mathcal{D} > 0$, not depending on f, such that the cubature formula $\Pi_{h,\tau} f$ provides the approximation estimate:*

$$\|\Pi f - \Pi_{h,\tau} f\|_{L^{r,q}(\mathbb{R}_+)} \le C \sum_{k=0}^{N} \sum_{|\boldsymbol{\alpha}|=N-k} (\tau\sqrt{\mathcal{D}_0})^k (h\sqrt{\mathcal{D}})^{N-k} \|\partial_t^k \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} f\|_{L^{r',q'}(\mathbb{R}_+)}$$
$$+ \varepsilon \sum_{k=0}^{N-1} \sum_{j=0}^{N-1-k} \frac{(\tau\sqrt{\mathcal{D}_0})^k (h\sqrt{\mathcal{D}})^j}{(2\pi)^{k+j}} \sum_{|\boldsymbol{\alpha}|=j} \|\partial_t^k \partial_{\mathbf{x}}^{\boldsymbol{\alpha}} f\|_{L^{r',q'}(\mathbb{R}_+)}\,.$$

We note that $K_{2M}(\mathbf{x}, t, h\mathbf{m}, \tau\ell)$ in (11) involves additionally an integration and it cannot be taken analytically. Therefore, we use an efficient quadrature based on the classical trapezoidal rule which is exponentially converging for rapidly decaying smooth functions on the real line. In particular, a "double exponential" rate of decay on $\mathbb{R}$ yields a more rapidly approximation of the trapezoidal rule. First, we make the substitution introduced in [16]:

$$s = t\varphi(\xi), \qquad \varphi(\xi) = \frac{1}{2}\Big(1 + \tanh\big(\frac{a\pi}{2}\sinh\xi\big)\Big) = \frac{1}{1 + \mathrm{e}^{-a\pi\sinh\xi}},$$

with certain positive constant a, which transforms to an integral over $\mathbb{R}$ with *doubly exponentially decaying integrand* ($|f(s)| \le C\exp(-a\mathrm{e}^{b|s|})$). Then, we apply the classical trapezoidal rule with step κ and sufficiently large $R \in \mathbb{N}$

$$K_{2M}(\mathbf{x}, t, h\mathbf{m}, \tau\ell) \approx \frac{\pi a t\kappa}{2}\sum_{r=-R}^{R}\chi_{2M}\Big(\frac{t(1-\varphi(\kappa r)) - \tau\ell}{\tau\sqrt{\mathscr{D}_0}}\Big)\Phi_{2M}^{[\mathbf{P_m},\mathbf{Q_m}]}\Big(\frac{\mathbf{x} - h\mathbf{m}}{h\sqrt{\mathscr{D}}}, \frac{4\,t\varphi(\kappa r)}{h^2\mathscr{D}}\Big)\omega(\kappa r),$$

where we denote $\omega(\xi) = \dfrac{\cosh\xi}{1 + \cosh(a\pi\sinh\xi)}$. Then, $\Pi_{h,\tau} f$ is explicitly computable.

5 Tensor Product Formulas

The computation of the approximate solution $\mathscr{S}_h g$ is very efficient if the function $g(\mathbf{x})$ allows a *separated representation*; that is, within a prescribed accuracy, it can be represented as sum of products of univariate functions:

$$g(\mathbf{x}) = \sum_{p=1}^{P}\alpha_p\prod_{j=1}^{n} g_j^{(p)}(x_j) + \mathscr{O}(\varepsilon)$$

with suitable functions $g_j^{(p)}$ chosen such that the separation rank P is small. The advantage of separated representation is that algebraic operations on g can be separated into one-dimensional operations. We derive that, at the points of the uniform grid $\{(h\mathbf{k}, \tau s)\}$, the n-dimensional integral (3) is approximated by the product of one-dimensional sums:

$$(\mathscr{S}g)(h\mathbf{k}, \tau s) \approx \mathscr{D}^{-n/2}\sum_{p=1}^{P}\alpha_p\prod_{j=1}^{n} S_{j,h}^{(p)}(k_j, \tau s),$$

where

$$S_{j,h}^{(p)}(k,t) = \sum_{hm \in I_j} g_j^{(p)}(hm)\Big(\Psi_M\left(\frac{k-m}{\sqrt{\mathscr{D}}}, \frac{4t}{h^2\mathscr{D}}, \frac{P_j - hm}{h\sqrt{\mathscr{D}}}\right) - \Psi_M\left(\frac{k-m}{\sqrt{\mathscr{D}}}, \frac{4t}{h^2\mathscr{D}}, \frac{Q_j - hm}{h\sqrt{\mathscr{D}}}\right)\Big).$$

Suppose that also $f(\mathbf{x}, t)$ allows a separated representation; that is, within a prescribed accuracy ε:

$$f(\mathbf{x}, t) = \sum_{p=1}^{P} \beta_p \prod_{j=1}^{n} f_j^{(p)}(x_j, t) + \mathscr{O}(\varepsilon).$$

Thus, we get, at the points of the uniform grid $\{(h\mathbf{k}, \tau s)\}$, the following efficiently computable high-order approximation formula for Πf:

$$\begin{aligned}\Pi_{h,\tau} f(h\mathbf{k}, \tau s) \;&\approx \frac{-i\,\pi a \tau s \kappa}{2\mathscr{D}_0^{1/2}\mathscr{D}^{n/2}} \sum_{r=-R}^{R} \omega(\kappa r) \\ &\times \sum_{\tau\ell \in \widetilde{\Omega}_{r_0\tau}} \chi_{2M}\Big(\frac{s(1-\varphi(\kappa r)) - \ell}{\sqrt{\mathscr{D}_0}}\Big) \sum_{p=1}^{P} \beta_p \prod_{j=1}^{n} T_j^{(p)}(k_j, \tau s, \tau\ell, \kappa r)\,,\end{aligned}$$

where

$$\begin{aligned}T_j^{(p)}(k, \tau s, \tau\ell, \kappa r) = \sum_{hm \in I_j} f_j^{(p)}(hm, \tau\ell) \;&\Big(\Psi_M\Big(\frac{k-m}{\sqrt{\mathscr{D}}}, \frac{4\tau s\varphi(\kappa r)}{h^2\mathscr{D}}, \frac{P_j - hm}{\sqrt{\mathscr{D}}}\Big) \\ &-\Psi_M\Big(\frac{k-m}{\sqrt{\mathscr{D}}}, \frac{4\,\tau s\varphi(\kappa r)}{h^2\mathscr{D}}, \frac{Q_j - hm}{\sqrt{\mathscr{D}}}\Big)\Big).\end{aligned}$$

In [13], we verified numerically the convergence order and the accuracy of the proposed method on different examples, up to approximation order 6 and space dimension 200. We remark that, for an efficient implementation of Ψ_M, we express erfc with the Faddeeva function $W(z) = \mathrm{e}^{-z^2}\,\mathrm{erfc}(-iz)$ (see [1, 7.1.3]) and write

$$\Psi_M(x,t,y) = \frac{\mathrm{e}^{-y^2 + i(y-x)^2/t}}{2\sqrt{\pi}} \left(W\big(iF(x,it,y)\big)\mathscr{P}_M(x,it) - \frac{\mathscr{Q}_M(x,it,y)}{\sqrt{\pi}}\right),$$

where $F(x, it, y)$ is defined by (10). Efficient implementations of double precision computations of $W(z)$ are available if $\operatorname{Im} z \geq 0$. For $\operatorname{Im} z < 0$, overflow problems can occur. To derive a stable formula also for $\operatorname{Im} z < 0$, we used the relation $W(z) = 2e^{-z^2} - W(-z)$ and we get the efficient formula:

$$\Psi_M(x,t,y) = -\frac{e^{-y^2+i(y-x)^2/t}}{2\sqrt{\pi}} \frac{\mathscr{Q}_M(x,it,y)}{\sqrt{\pi}} +$$

$$\begin{cases} e^{-y^2+i(y-x)^2/t}\, W\big(iF(x,it,y)\big) \dfrac{\mathscr{P}_M(x,it)}{2\sqrt{\pi}} & \operatorname{Re} F(x,it,y) \geq 0, \\ \Big(2\,e^{-x^2/(1+it)} - e^{-y^2+i(y-x)^2/t}\, W\big(-iF(x,it,y)\big)\Big) \dfrac{\mathscr{P}_M(x,it)}{2\sqrt{\pi}} & \operatorname{Re} F(x,it,y) < 0. \end{cases}$$

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Generalized Kantorovich Operators on Convex Compact Subsets and Their Application to Evolution Problems

Vita Leonessa

Abstract In this short survey paper, we review some of recent results contained in Altomare et al. (Banach J Math Anal 11:591–614, 2017; J Math Anal Appl 458:153–173, 2018) and concerning with the generalized Kantorovich operators C_n defined on convex compact subsets of $\mathbb{R}^d$ ($d \geq 1$). Such operators constitute a positive approximation process for continuous functions and, in some cases, for integrable functions. Moreover, an asymptotic formula for such approximating operators leads to a differential operator which pregenerates a Markov semigroup on $C(K)$ for which we obtain an approximation formula, in terms of suitable powers of C_n, useful to infer some preservation properties of it and, as a consequence, of solutions to evolution problems associated with the generators.

1 Introduction

In this short survey paper, we review some of recent results concerning with a new class of positive linear operators defined on a convex compact subset K of $\mathbb{R}^d$ ($d \geq 1$), and their useful connection with approximation problems, not only for functions defined on such sets, but also for solutions to some classes of differential problems.

These studies fall within the scope of a research project, developed in the last 20 years, which combines methods from Real Analysis, Operator Theory, and Approximation Theory, in order to study some degenerate second-order elliptic–parabolic differential problems furnishing constructive approximations of the relevant solutions in terms of iterates of positive linear operators. In this direction, the monograph [9] provides a rather complete overview on the main results regarding the Bernstein–Schnabl operators B_n generated by a Markov operator T on the space $C(K)$ of all continuous functions on K (i.e. a positive linear operator T on $C(K)$ such that $T(\mathbf{1}) = \mathbf{1}$, $\mathbf{1}$ being the constant function of value 1 on K). For fixed K and T, the

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_4

B_n's are defined by setting, for every $n \geq 1$, $x \in K$ and $f \in C(K)$,

$$B_n(f)(x) := \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n}{n}\right) \tilde{\mu}_x^T(x_1) \cdots \tilde{\mu}_x^T(x_n),$$

where $(\tilde{\mu}_x^T)_{x \in K}$ is the continuous selection of Borel probability measures on K associated with T via the Riesz representation theorem, that is:

$$\int_K f \, d\tilde{\mu}_x^T = T(f)(x) \quad (f \in C(K), x \in K).$$

For each $n \geq 1$, B_n are linear, positive, map $C(K)$ into $C(K)$ and $\|B_n\| = 1$. For all properties shared by the B_n operators, we refer the reader to [9] (see also [8]). Here, we limit us to recall that the class of differential operators to which they are connected has the following aspect:

$$W_T(u) = \frac{1}{2} \sum_{i,j=1}^{d} (T(pr_i pr_j) - pr_i pr_j) \frac{\partial^2 u}{\partial x_i \partial x_j} \quad (u \in C^2(K)), \tag{1}$$

where, for every $i = 1, \ldots, d$, pr_i denotes the i-th coordinate function on K, i.e. $pr_i(x) = x_i$.

Subsequently, in [10] we introduce a new sequence of positive linear operators acting on $C(K)$ and, in some particular cases, also in other larger function spaces, for instance L^p-spaces, $1 \leq p < +\infty$. Their construction depends not only on a Markov operator T but also on a real parameter $a \geq 0$ and a sequence $(\mu_n)_{n \geq 1}$ in $M_1^+(K)$ (i.e. in the set of all Borel probability measures on K). Namely, for every $n \geq 1$, the *generalized Kantorovich operator* C_n is the positive linear operator defined by setting, for every $f \in C(K)$ and $x = (x_1, \ldots, x_n) \in K$,

$$C_n(f)(x) = \int_K \int_K \cdots \int_K f\left(\frac{x_1 + \ldots + x_n + ax_{n+1}}{n+a}\right) \mathrm{d}\tilde{\mu}_x^T(x_1) \cdots \mathrm{d}\tilde{\mu}_x^T(x_n) \mathrm{d}\mu_n(x_{n+1}).$$

Note that, if $a = 0$, then $C_n = B_n$ $(n \geq 1)$. Therefore, $C_n(f) \in C(K)$ and the operator $C_n : C(K) \to C(K)$, being linear and positive, is continuous with norm equal to 1, because $C_n(\mathbf{1}) = \mathbf{1}$. In fact, a relation between C_n and B_n holds for every $a \geq 0$. Namely, for every $n \geq 1$ and $f \in C(K)$,

$$C_n(f) = B_n(I_n(f)) \quad \text{with} \quad I_n(f)(x) = \int_K f\left(\frac{nx + at}{n+a}\right) \mathrm{d}\mu_n(t). \tag{2}$$

The class of operators C_n contains several approximation processes, in both univariate and multivariate settings, present in the literature, as well as new ones. For examples, in the case of the unit interval or the multidimensional hypercube and simplex, the operators C_n turn into the classical Kantorovich operators defined on these settings (see the examples below), together with several other wide-ranging generalizations (see [3, 5, 14–17, 20, 22, 23]).

In what follows, we briefly present the approximation and preservation properties of operators C_n. Subsequently, we establish an asymptotic formula for them, which leads to a class of differential operators of the form:

$$V_T(u) = W_T(u) + \sum_{i=1}^{d} a(b_i - pr_i)\frac{\partial u}{\partial x_i} \quad (u \in C^2(K)), \tag{3}$$

where $b = (b_1, \ldots, b_d) \in K$ is the barycentre of the weak limit of $(\mu_n)_{n\geq 1}$, i.e. $b_i = \int_K pr_i \,\mathrm{d}\mu$ for every $i = 1, \ldots, d$, where $\mu \in M_1^+(K)$ verifies $\lim_{n\to\infty} \int_K f \,\mathrm{d}\mu_n = \int_K f \mathrm{d}\mu$ for every $f \in C(K)$. Specializing the convex compact set K and the other parameters, we obtain several classes of differential operators which are of current interest in the research area of evolution equations (see, e.g., examples in Sect. 2).

The couple $(V_T, C^2(K))$ is the pregenerator of a Markov semigroup on $C(K)$ which is approximated in terms of suitable iterates of the C_n's. Moreover, in the particular case of the hypercube $[0, 1]^d$, the above semigroup can be extended to a contraction semigroup on $L^p([0, 1]^d)$ which in turn is approximated by the powers of the natural extension of the operators C_n to $L^p([0, 1]^d)$. All results are taken from [10] and [11].

We want to point out that the present exposition is limited to the framework of $\mathbb{R}^d$ for the sake of simplicity, but the above questions have been investigated also in the infinite-dimensional setting (see [10, 12] for all the details).

2 The Operators C_n: Examples and Properties

We begin to showing some examples of the C_n's constructed by specifying all the parameters and using (2). Other examples may be found in [10].

Example 1 Assume $K = [0, 1]$ and consider the Markov operator T_1 defined, for every $f \in C([0, 1])$ and $0 \leq x \leq 1$, by $T_1(f)(x) := (1 - x)f(0) + xf(1)$. Then, for $n \geq 1$, $f \in C([0, 1])$ and $x \in [0, 1]$, the operators C_n are

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \int_0^1 f\left(\frac{k + as}{n + a}\right) d\mu_n(s). \tag{4}$$

In particular, assume that all the μ_n are equal to the Borel–Lebesgue measure λ_1 on $[0, 1]$. Then, if $a = 0$, we obtain the Bernstein operators, whereas for $a = 1$, formula (4) gives the classical Kantorovich operators [16, 17]. Thus, (4) represents a link between these fundamental sequences of approximating operators in terms of a continuous parameter $a \in [0, 1]$. Special cases of operators (4) have been also considered in [3] and [15].

Example 2 Let $Q_d := [0,1]^d$, $d \geq 1$, and consider the Markov operator S_d defined by setting, for every $f \in C(Q_d)$ and $x \in Q_d$,

$$S_d(f)(x) := \sum_{h_1,\dots,h_d=0}^{1} f(\delta_{h_1 1},\dots,\delta_{h_d 1}) x_1^{h_1}(1-x_1)^{1-h_1}\cdots x_d^{h_d}(1-x_d)^{1-h_d}$$

(δ_{ij} being the Kronecker symbol). Then, for any $n \geq 1$, $f \in C(Q_d)$, $x \in Q_d$,

$$\begin{aligned} C_n(f)(x) = &\sum_{h_1,\dots,h_d=0}^{n} \prod_{i=1}^{d} \binom{n}{h_i} x_i^{h_i}(1-x_i)^{n-h_i} \\ &\times \int_{Q_d} f\left(\frac{h_1+as_1}{n+a},\dots,\frac{h_d+as_d}{n+a}\right) d\mu_n(s_1,\dots,s_d). \end{aligned} \tag{5}$$

When all the μ_n coincide with the Borel–Lebesgue measure λ_d on Q_d and $a = 1$, the operators C_n turn into a generalization of Kantorovich operators introduced in [23]. Another special case of (5) has been studied in [5].

Example 3 Let K_d be the canonical simplex in $\mathbb{R}^d$, $d \geq 1$, and consider the canonical Markov operator T_d defined, for each $f \in C(K_d)$, $x \in K_d$, as:

$$T_d(f)(x) := \left(1 - \sum_{i=1}^{d} x_i\right) f(0) + \sum_{i=1}^{d} x_i f(e_i),$$

where, for every $i = 1,\dots,d$, $e_i := (\delta_{ij})_{1\leq j\leq d}$. In this case,

$$\begin{aligned} C_n(f)(x) = &\sum_{\substack{h_1,\dots,h_d=0,\dots,n \\ h_1+\dots+h_d\leq n}} \frac{n!\,x_1^{h_1}\cdots x_d^{h_d}}{h_1!\cdots h_d!(n-h_1-\dots-h_d)!}\left(1-\sum_{i=1}^{d} x_i\right)^{n-\sum_{i=1}^{d} h_i} \\ &\times \int_{K_d} f\left(\frac{h_1+as_1}{n+a},\frac{h_2+as_2}{n+a},\dots,\frac{h_d+as_d}{n+a}\right) d\mu_n(s_1,\dots,s_d) \end{aligned} \tag{6}$$

($n \geq 1$, $f \in C(K_d)$, $x \in K_d$). When all the μ_n are equal to the Borel–Lebesgue measure λ_d on K_d and $a = 1$, these operators are referred to as the Kantorovich operators on $C(K_d)$ and were introduced in [23]. Another particular case of (6) has been investigated in [5, Section 3].

From now on, assume that the Markov operator T satisfies the following condition:

$$T(h) = h \quad \text{for every } h \in \{pr_1,\dots,pr_d\}. \tag{Hp1}$$

For every $n \geq 1$ and $i = 1, \ldots, d$, we have

$$C_n(pr_i) = \frac{1}{n+a}\left\{a\left(\int_K pr_i\, d\mu_n\right)\mathbf{1} + npr_i\right\} \tag{7}$$

and

$$C_n(pr_i^2) = \frac{1}{(n+a)^2}\left\{a^2\left(\int_K pr_i^2\, d\mu_n\right)\mathbf{1} + 2na\left(\int_K pr_i\, d\mu_n\right)pr_i \right. \\ \left. +n(n-1)pr_i^2 + nT(pr_i^2)\right\}. \tag{8}$$

Therefore, applying a Korovkin-type result, formulas (7)–(8), and observing that the sequences $\left(\int_K pr_i\, d\mu_n\right)_{n\geq 1}$ and $\left(\int_K pr_i^2\, d\mu_n\right)_{n\geq 1}$ are bounded for every $i = 1, \ldots, d$, we have the following approximation property in $C(K)$ (see [10, Theorem 4.2]).

Theorem 1 *Assume that the Markov operator T satisfies condition* (Hp1)*. Then, for every $f \in C(K)$, $\lim_{n\to\infty} C_n(f) = f$ uniformly on K.*

From the previous theorem, and thanks to the fact that the sequence $(C_n)_{n\geq 1}$ is equibounded from $L^p(K)$ into $L^p(K)$ when $K = Q_d$ or $K = K_d$ (see Examples 2 and 3 above), we get the following approximation result in L^p-spaces, $1 \leq p < +\infty$ (see [10, Theorems 5.1 and 5.4]).

Theorem 2 *Assume that $a > 0$ and T satisfies* (Hp1)*. If $f \in L^p(Q_d)$ (resp. $f \in L^p(K_d)$), then* $\lim_{n\to\infty} C_n(f) = f$ *in $L^p(Q_d)$ (resp. in $L^p(K_d)$).*

Remark 1 Consider the sequence $(C_n)_{n\geq 1}$ of operators defined on $C([0, 1])$ by formula (4) with $a = 1$, and call K_n the classical Kantorovich operators on $[0, 1]$. Since $C_n(\mathbf{1}) = \mathbf{1}$, we know that, for every $n \geq 1$, $f \in C([0, 1])$, $x \in [0, 1]$ and $\delta > 0$, the following pointwise estimate holds (see [18]):

$$|C_n(f)(x) - f(x)| \leq (1 + \delta^{-2}C_n(\psi_x^2)(x))\omega(f, \delta),$$

where, for a fixed $0 \leq x \leq 1$, $\psi_x(t) = t - x$ for any $t \in [0, 1]$. Note that, for $f \in C([0, 1])$ fixed, if we consider $(\mu_n)_{n\geq 1}$ in $M_1^+[0, 1])$ and the points $x \in [0, 1]$ such that

$$C_n(\psi_x^2)(x) < K_n(\psi_x^2)(x), \tag{9}$$

then the order of approximation to $f(x)$ by means of operators C_n and K_n may be compared (see, e.g. [21]). It is easy to show that (9) is equivalent to

$$x(1-2b_n) < \frac{1}{3}-c_n \quad \text{with} \quad b_n := \int_0^1 e_1\, d\mu_n \quad \text{and} \quad c_n := \int_0^1 e_2\, d\mu_n\,. \tag{10}$$

For instance, (10) holds for all $x \in [0,1]$ in one of the following cases: $b_n = \frac{1}{2}$ and $\frac{1}{4} \le c_n < \frac{1}{3}$, or $\frac{1}{2} < b_n \le 1$ and $\frac{1}{4} < c_n \le \frac{1}{3}$, or $0 \le b_n < \frac{1}{2}$, $0 \le c_n < \frac{1}{3}$ and $3c_n - 6b_n + 2 \le 0$. In such cases, the order of approximation of $C_n(f)(x)$ is at least as good as that of $K_n(f)(x)$ in the whole [0, 1].

Among other things, in [10, Sec. 6] it is showed that the operators C_n preserve Lipschitz continuity and convexity. Given $M \ge 0$, consider the space of all Lipschitz continuous functions with Lipschitz constant M:

$$Lip(M,1) = \{f \in C(K) \; : \; |f(x) - f(y)| \le M\|x - y\| \text{ for every } x, y \in K\}$$

($\|\cdot\|$ denotes an arbitrary norm on $\mathbb{R}^d$). First, we have that

Proposition 1 *Suppose that there exists $c \ge 1$ such that $T(Lip(1,1)) \subset Lip(c,1)$. Then, for every $n \ge 1$ and $M > 0$, $C_n(Lip(M,1)) \subset Lip(cM,1)$.*

Note that the Markov operators T_1, S_d and T_d in Examples 1–3 satisfy Proposition 1 with $c = 1$, by considering on [0, 1] the usual metric, and on Q_d and K_d the l_1-metric (see [9, p. 124]).

As far as the convexity is concerned, for a given $f \in C(K)$, set

$$\Delta(f; x, y) := B_2(f)(x) + B_2(f)(y) - 2\iint_{K^2} f\left(\frac{s+t}{2}\right) d\tilde{\mu}_x^T(s) d\tilde{\mu}_y^T(t) \quad (x, y \in K).$$

Proposition 2 *Suppose that T satisfies the following hypotheses:*

1. *T maps continuous convex functions into (continuous) convex functions;*
2. *$\Delta(f; x, y) \ge 0$ for every convex function $f \in C(K)$ and $x, y \in K$.*

Then, each C_n maps continuous convex functions into (continuous) convex functions.

Also in this case, the operators of Examples 1–3 satisfy Proposition 2. Other type of convexity may be also investigated (see [10, Prop. 6.3]).

3 Evolution Problems Associated with Operators C_n

The asymptotic formula for operators C_n, which leads to the class of differential operators V_T given in (3), is contained in the next result (see [9]).

Theorem 3 *Under assumptions (Hp1), for every $u \in C^2(K)$, we have*

$$\lim_{n\to\infty} n(C_n(u) - u) = V_T(u) \quad \text{uniformly on } K.$$

The operator V_T associated with the particular parameters considered in Examples 1–3, are, resp.,

$$V_{T_1}(u)(x) = \frac{\alpha(x)}{2}u''(x) + a(b-x)u'(x) \quad \text{with } a \geq 0, b \in [0,1]\,,$$

$$V_{S_d}(u)(x) = \frac{1}{2}\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) + a\sum_{i=1}^{d}(b_i - x_i)\frac{\partial u}{\partial x_i}(x) \text{ with } a \geq 0, b \in Q_d\,,$$

$$V_{T_d}(u)(x) = \frac{1}{2}\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 u}{\partial x_i^2}(x) - \sum_{1\leq i<j\leq d} x_i x_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x)$$
$$+ a\sum_{i=1}^{d}(b_i - x_i)\frac{\partial u}{\partial x_i}(x) \quad \text{with } b \in K_d, a \geq 0\,.$$

Such differential operators are the Fleming–Viot-type operators on the unit interval and on the multidimensional hypercube and simplex (see, e.g. [1, 2, 4, 6, 7, 9, 13, 19]).

In order to get that the desired generation/approximation result for $(V_T, C^2(K))$, we need to require that T satisfies also the following condition:

$$T(P_m(K)) \subset P_m(K) \quad \text{for every } m \geq 1, \tag{Hp2}$$

$P_m(K)$ being the space of (restriction to K of all) polynomials of degree at most m.

Theorem 4 *Assume that T satisfies conditions* (Hp1) *and* (Hp2)*. Then, the operator $(V_T, C^2(K))$ is closable and its closure $(A_T, D(A_T))$ generates a Markov semigroup $(S(t))_{t\geq 0}$ on $C(K)$ such that if $t \geq 0$, $f \in C(K)$ and $(k_n)_{n\geq 1}$ is a sequence of positive integers satisfying $\lim_{n\to\infty} k_n/n = t$, then*

$$S(t)(f) = \lim_{n\to\infty} C_n^{k_n}(f) \quad \textit{uniformly on } K, \tag{11}$$

where each $C_n^{k_n}$ denotes the iterate of C_n of order k_n.

Moreover, $P_\infty(K) := \bigcup_{m\geq 1} P_m(K)$, and hence $C^2(K)$, is a core for $(A_T, D(A_T))$ and $S(t)(P_m(K)) \subset P_m(K)$ for every $t \geq 0$ and $m \geq 1$.

Theorem 4 allows us to represent the (unique) solution of the initial-boundary value evolution problem governed by such a semigroup:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = A_T(u(\cdot,t))(x) & x \in K, \quad t \geq 0, \\ u(x,0) = u_0(x) & u_0 \in D(A_T), \quad x \in K\,, \end{cases}$$

in terms of the C_n's, i.e. $u(x,t) = T(t)(u_0)(x) = \lim_{n\to\infty} C_n^{k_n}(u_0)(x)$ uniformly w.r.t. $x \in K$, and to deduce some spatial regularity properties of the relevant solutions by means of similar ones held for the C_n's (see [11, Section 3]).

Let us end this section by noting that, if $a = 1$ and $K = Q_d$, the semigroup $(S(t))_{t\geq 0}$ may be extended to a positive contraction semigroup on $L^p(Q_d)$, and the representation formula (11) extends to $L^p(Q_d)$ [11, Section 4].

Also, the differential operator V_{T_d} for the simplex K_d generates a contraction semigroup on $L^p(K_d)$ which extends the semigroup $(S(t))_{t\geq 0}$ on $C(K_d)$ given by Theorem 4 to $L^p(K_d)$. Unfortunately, C_n operators are not able to approximate such semigroup on $L^p(K_d)$, essentially because they are not contractive on $L^p(K_d)$. This important question is still open.

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On the Generalized Sylvester Polynomials

Nejla Özmen and Esra Erkuş-Duman

Abstract In this study, we give some new properties for the generalized Sylvester polynomials. The results obtained here include various families of multilinear and multilateral generating functions and miscellaneous properties. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Sylvester polynomials and the Lauricella functions. Finally, we get several results of this theorem.

1 Introduction

The generalized Sylvester polynomials $\phi_n(x;c)$ are defined by the generating relation [1]:

$$\sum_{n=0}^{\infty} \phi_n(x;c)t^n = (1-t)^{-x} e^{cxt}. \tag{1}$$

The following generating function also holds true for these polynomials [1]:

$$\sum_{n=0}^{\infty} \binom{n+m}{n} \phi_{n+m}(x;c)t^n = (1-t)^{-x-m} e^{cxt} \phi_m(x;c-ct). \tag{2}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_5

It follows from (1) that

$$\phi_n(x;c) = \frac{(cx)^n}{n!}\,{}_2F_0\left[-n, x; -; -(cx)^{-1}\right], \tag{3}$$

where ${}_2F_0$ denotes the Gauss hypergeometric series. It is noted that the special case $c = 1$ of (3) reduces immediately to the Sylvester polynomials [2].

The main object of this study is to derive several properties of the generalized Sylvester polynomials. Various families of multilinear and multilateral generating functions, integral representation and recurrence relations are obtained for these polynomials. In addition, we derive a theorem giving certain families of bilateral generating functions for the generalized Sylvester polynomials and the Lauricella functions.

2 Generating Functions and Miscellaneous Properties

In this section, we derive several families of bilinear and bilateral generating functions for the generalized Sylvester polynomials $\phi_n(x;c)$ given by (3).

Lemma 1 *The following addition formula holds for the generalized Sylvester polynomials:*

$$\phi_n(x_1 + x_2; c) = \sum_{m=0}^{n} \phi_{n-m}(x_1; c)\phi_m(x_2; c). \tag{4}$$

Proof Replacing x by $x_1 + x_2$ in (1), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_n(x_1 + x_2; c)t^n &= (1-t)^{-x_1-x_2} e^{(x_1+x_2)ct} \\ &= \sum_{n=0}^{\infty} \phi_n(x_1; c)t^n \sum_{m=0}^{\infty} \phi_m(x_2; c)t^m \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \phi_{n-m}(x_1; c)\phi_m(x_2; c)t^n.\end{aligned}$$

From the coefficients of t^n on the both sides of the last equality, one can get the desired result.

Theorem 1 *Corresponding to an identically non-vanishing function $\Omega_\mu(y_1, \ldots, y_r)$ of r complex variables $y_1, \ldots, y_r$ $(r \in \mathbb{N})$ and of complex order μ, ψ, let*

$$\Lambda_{\mu,\psi}(y_1, \ldots, y_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \ldots, y_r)\zeta^k$$

and

$$\Theta_{n,p}^{\mu,\psi}(x; c; y_1, \ldots, y_r; \xi) := \sum_{k=0}^{[n/p]} a_k \phi_{n-pk}(x; c)\Omega_{\mu+\psi k}(y_1, \ldots, y_r)\xi^k,$$

where $a_k \neq 0$, $n, p \in \mathbb{N}$ *and the notation* $[n/p]$ *means the greatest integer less than or equal* n/p.

Then, we have

$$\sum_{n=0}^{\infty} \Theta_{n,p}^{\mu,\psi}\left(x; c; y_1, \ldots, y_r; \frac{\eta}{t^p}\right) t^n = (1-t)^{-x} e^{cxt} \Lambda_{\mu,\psi}(y_1, \ldots, y_r; \eta) \tag{5}$$

provided that each member of (5) exists.

Proof For convenience, let S denote the first member of the assertion (5) of Theorem 1. Then,

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k \phi_{n-pk}(x; c)\Omega_{\mu+\psi k}(y_1, \ldots, y_r)\eta^k t^{n-pk}.$$

Replacing n by $n + pk$, we may write that

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \, \phi_n(x; c)\Omega_{\mu+\psi k}(y_1, \ldots, y_r)\eta^k t^n \\ &= \sum_{n=0}^{\infty} \phi_n(x; c)t^n \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(y_1, \ldots, y_r)\eta^k \\ &= (1-t)^{-x} e^{cxt} \Lambda_{\mu,\psi}(y_1, \ldots, y_r; \eta) \end{aligned}$$

which completes the proof.

By using a similar idea, we also get the next results immediately.

Theorem 2 *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1, \ldots, y_r)$ *of* r *complex variables* $y_1, \ldots, y_r$ $(r \in \mathbb{N})$ *and of complex order* μ, ψ, *let*

$$\Lambda_{\mu,\psi}^{n,p}(x_1 + x_2; c; y_1, \ldots, y_r; t) := \sum_{k=0}^{[n/p]} a_k \phi_{n-pk}(x_1 + x_2; c)\Omega_{\mu+\psi k}(y_1, \ldots, y_r)t^k.$$

Then, for $a_k \neq 0$, $n, p \in \mathbb{N}$, *we have*

$$\sum_{k=0}^{n} \sum_{l=0}^{[k/p]} a_l \phi_{n-k}(x_1, c)\phi_{k-pl}(x_2, c)\Omega_{\mu+\psi l}(y_1, \dots, y_r)t^l = \Lambda_{\mu,\psi}^{n,p}(x_1 + x_2; c; y_1, \dots, y_r; t) \tag{6}$$

provided that each member of (6) exists.

Theorem 3 *Corresponding to an identically non-vanishing function* $\Omega_\mu(y_1, \dots, y_r)$ *of* r *complex variables* $y_1, \dots, y_r$ $(r \in \mathbb{N})$ *and of complex order* μ, *let*

$$\Lambda_{\mu,p,q}(x; c; y_1, \dots, y_r; t) := \sum_{n=0}^{\infty} a_n \phi_{m+qn}(x; c)\Omega_{\mu+pn}(y_1, \dots, y_r)t^n$$

where $a_n \neq 0$ *and*

$$\theta_{n,p,q}(y_1, \dots, y_r; z) := \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k \Omega_{\mu+pk}(y_1, \dots, y_r)z^k.$$

Then, for $p, q \in \mathbb{N}$, *we have*

$$\sum_{n=0}^{\infty} \phi_{m+n}(x; c)\theta_{n,p,q}(y_1, \dots, y_r; z)t^n$$
$$= (1-t)^{-x-m} e^{cxt} \Lambda_{\mu,p,q}\left(x; c - ct; y_1, \dots, y_r; z(\frac{t}{1-t})^q\right) \tag{7}$$

provided that each member of (7) exists.

For every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(y_1, \dots, y_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorems 1–3 can be applied in order to derive various families of multilinear and multilateral generating functions for the generalized Sylvester polynomials.

Now, we give an integral representation and several recurrence relations for the generalized Sylvester polynomials.

Theorem 4 *The generalized Sylvester polynomials* $\phi_n(x; c)$ *have the following integral representation:*

$$\phi_n(x; c) = \frac{1}{n!\Gamma(x)} \int_0^{\infty} e^{-u} u^{x-1} (cx+u)^n du, \quad (Re(x) > 0).$$

Proof If we use the identity

$$a^{-v} = \frac{1}{\Gamma(v)} \int_0^\infty e^{-at} t^{v-1} dt, \quad (Re(v) > 0)$$

on the left-hand side of the generating function (1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_n(x;c) t^n &= \frac{1}{\Gamma(x)} \int_0^\infty e^{-(1-t)u} u^{x-1} e^{cxt} du \\ &= \frac{1}{\Gamma(x)} \int_0^\infty e^{-u} u^{x-1} e^{(cx+u)t} du \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n!\Gamma(x)} \int_0^\infty e^{-u} u^{x-1} (cx+u)^n du \right) t^n.\end{aligned}$$

From the coefficients of t^n on the both sides of the last equality, we get the desired result.

On the other hand, by differentiating each member of the generating function relation (1) with respect to x and using

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k),$$

we have the differential recurrence relation for the generalized Sylvester polynomials as follows:

$$\frac{\partial}{\partial x} \phi_n(x;c) = x\phi_n(x;c) + \sum_{m=0}^{n-1} \frac{1}{(m+1)} \phi_{n-m-1}(x;c).$$

Besides, by differentiating each member of the generating function relation (1) with respect to t, we have the following another recurrence relation for these polynomials:

$$(n+1)\phi_{n+1}(x;c) = x \left(c\phi_n(x;c) + \sum_{m=0}^{n} \phi_{n-m}(x;c) \right).$$

3 A Bilateral Generating Function Involving the Lauricella Functions

In the present section, we derive various families of bilateral generating functions for the generalized Sylvester polynomials and the Lauricella functions [7] by using the similar method in [3–5].

For a suitable bounded non-vanishing multiple sequence $\{\Omega(m_1, \ldots, m_s)\}_{m_1,\ldots,m_s \in \mathbb{N}_0}$ of real or complex parameters, let $\varphi_n(u_1; u_2, \ldots, u_s)$ of s (real or complex) variables $u_1, \ldots, u_s$ be defined by:

$$\varphi_n(u_1; u_2, \ldots, u_s) := \sum_{m_1=0}^{n} \sum_{m_2,\ldots,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \times \Omega\left(f(m_1, \ldots, m_s), m_2, \ldots, m_s\right) \frac{u_1^{m_1}}{m_1!} \cdots \frac{u_s^{m_s}}{m_s!}$$

where, for convenience,

$$((b))_{m_1\phi} = \prod_{j=1}^{B} (b_j)_{m_1\phi_j} \quad \text{and} \quad ((d))_{m_1\delta} = \prod_{j=1}^{D} (d_j)_{m_1\delta_j}.$$

Theorem 5 *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} \phi_n(x; c)\varphi_n(u_1; u_2, \ldots, u_s)t^n$$

$$= (1-t)^{-x} e^{cxt} \sum_{k,m_1,\ldots,m_s=0}^{\infty} \frac{((b))_{(m_1+k)\phi}(x)_k}{((d))_{(m_1+k)\delta}} \times \Omega\left(f\left(m_1+k\right), m_2, \ldots, m_s\right) \frac{(-u_1 cxt)^{m_1}}{m_1!} \frac{\left(\frac{u_1 t}{t-1}\right)^k}{k!} \frac{u_2^{m_2}}{m_2!} \cdots \frac{u_s^{m_s}}{m_s!}.$$

Proof By using the relation (2), it is easily observed that

$$\sum_{n=0}^{\infty} \phi_n(x; c)\varphi_n(u_1; u_2, \ldots, u_s)t^n$$

$$= \sum_{n=0}^{\infty} \phi_n(x; c) \sum_{m_1=0}^{n} \sum_{m_2,\ldots,m_s=0}^{\infty} \frac{(-n)_{m_1}((b))_{m_1\phi}}{((d))_{m_1\delta}} \times \Omega\left(f(m_1, \ldots, m_s), m_2, \ldots, m_s\right) \frac{u_1^{m_1}}{m_1!} \cdots \frac{u_s^{m_s}}{m_s!} t^n$$

$$= (1-t)^{-x} e^{cxt} \sum_{m_1,m_2,\ldots,m_s=0}^{\infty} \frac{((b))_{m_1\phi}}{((d))_{m_1\delta}} \Omega\left(f(m_1,\ldots,m_s), m_2,\ldots,m_s\right)$$

$$\times (-\frac{u_1 t}{1-t})^{m_1} \frac{u_2^{m_2}}{m_2!} \cdots \frac{u_s^{m_s}}{m_s!} \frac{(x(c-ct))^{m_1}}{m_1!} \sum_{k=0}^{m_1} (-m_1)_k (x)_k \frac{-(cx-cxt)^{-k}}{k!}$$

$$= (1-t)^{-x} e^{cxt}$$

$$\times \sum_{k,m_1,\ldots,m_s=0}^{\infty} \frac{((b))_{(m_1+k)\phi}}{((d))_{(m_1+k)\delta}} \Omega\left(f\left(m_1+k\right), m_2,\ldots,m_s\right)(x)_k$$

$$\times \frac{(-u_1 cxt)^{m_1}}{m_1!} \frac{(\frac{u_1 t}{t-1})^k}{k!} \frac{u_2^{m_2}}{m_2!} \cdots \frac{u_s^{m_s}}{m_s!}.$$

By appropriately choosing the multiple sequence $\Omega(m_1,\ldots,m_s)$ in Theorem 5, we obtain several interesting results as follows which give bilateral generating functions for the generalized Sylvester polynomials and the Lauricella functions.

By letting

$$\Omega\left(f(m_1,\ldots,m_s), m_2,\ldots,m_s\right) = \frac{(a)_{m_1+\ldots+m_s}(b_2)_{m_2}\ldots(b_s)_{m_s}}{(c_1)_{m_1}\ldots(c_s)_{m_s}}$$

and

$$\phi = \delta = 0 \quad (\text{that is, } \phi_1 = \ldots = \phi_B = \delta_1 = \ldots = \delta_D = 0)$$

in Theorem 5, we obtain the following result:

Corollary 1 *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} \phi_n(x;c) F_A^{(s)}\left[a,-n,b_2,\ldots,b_s;c_1,\ldots,c_s;u_1,u_2,\ldots,u_s\right] t^n$$

$$= (1-t)^{-x} e^{cxt}$$

$$\times F^{1:0;1;1;\ldots;1}_{1:0;0;1;\ldots;1}\left(\begin{array}{l} [(a):1,\ldots,1]: \quad -;\ [x:1];\ [b_2:1];\ \ldots;\ [b_s:1]; \\ \left[(c_1):\psi^{(1)},\ldots,\psi^{(s+1)}\right]: -; \quad -; \quad [c_2:1];\ \ldots;\ [c_s:1]; \end{array}\right.$$

$$\left. (-u_1 cxt), (\tfrac{u_1 t}{t-1}), u_2,\ldots,u_s \right),$$

where $F_A^{(s)}$ *is the Lauricella function,* $F_{1:0;0;1;\ldots;1}^{1:0;1;1;\ldots;1}$ *is the generalized Lauricella function [6, 7] and the coefficients* $\psi^{(\eta)}$ *are given by:*

$$\psi^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq s+1) \end{cases}.$$

If we put

$$\Omega\left(f(m_1,\ldots,m_s), m_2,\ldots,m_s\right) = \frac{(a_1^{(1)})_{m_2}\ldots(a_1^{(s-1)})_{m_s}(a_2^{(1)})_{m_2}\ldots(a_2^{(s-1)})_{m_s}}{(c)_{m_1+\ldots+m_s}}$$

and

$$B = 1,\ b_1 = b,\ \phi_1 = 1 \text{ and } \delta = 0$$

in Theorem 5, we obtain the next corollary.

Corollary 2 *The following bilateral generating function holds true:*

$$\begin{aligned} &\sum_{n=0}^{\infty} \phi_n(x;c) F_B^{(s)}\left[-n, a_1^{(1)},\ldots,a_1^{(s-1)}, b, a_2^{(1)},\ldots,a_2^{(s-1)}; c; u_1,u_2,\ldots,u_s\right] t^n \\ &= (1-t)^{-x} e^{cxt} \\ &\times F_{1:0;0;0;\ldots;0}^{1:0;1;2;\ldots;2}\left(\begin{array}{l} \left[(b):\theta^{(1)},\ldots,\theta^{(s+1)}\right]:-;\ [x:1];\ \left[a^{(1)}:1\right];\ \ldots;\ \left[a^{(s-1)}:1\right]; \\ [(c):1,\ldots,1]:\quad -;\quad -;\quad -;\quad \ldots;\quad -; \end{array}\right. \\ &\qquad \left. (-u_1cxt), (\tfrac{u_1t}{t-1}), u_2,\ldots,u_s \right), \end{aligned}$$

where $F_B^{(s)}$ *is the Lauricella function and the coefficients* $\theta^{(\eta)}$ *are given by:*

$$\theta^{(\eta)} = \begin{cases} 1, & (1 \leq \eta \leq 2) \\ 0, & (2 < \eta \leq s+1) \end{cases}.$$

By letting

$$\Omega\left(f(m_1,\ldots,m_s), m_2,\ldots,m_s\right) = \frac{(a)_{m_1+\ldots+m_s}(b_2)_{m_2}\ldots(b_s)_{m_s}}{(c)_{m_1+\ldots+m_s}}$$

and

$$\phi = \delta = 0,$$

in Theorem 5,we obtain the following result.

Corollary 3 *The following bilateral generating function holds true:*

$$\sum_{n=0}^{\infty} \phi_n(x;c) F_D^{(s)}\left[a, -n, b_2, \ldots, b_s; c; u_1, u_2, \ldots, u_s\right] t^n$$

$$= (1-t)^{-x} e^{cxt} F_D^{(s+1)}\left[a, 0, x, b_2, \ldots, b_s; c; (-u_1 cxt), (\frac{u_1 t}{t-1}), u_2, \ldots, u_s\right],$$

where $F_D^{(s)}$ is the Lauricella function.

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Durrmeyer-Type Bernstein Operators Based on (p, q)-Integers with Two Variables

Tuba Vedi-Dilek

Abstract The purpose of this study is to introduce the Durrmeyer-type Bernstein operators based on (p, q)-integers with two variables. Then, we compute the error of approximation by using modulus of continuity and the degree of approximation by means of Lipschitz class. Finally, we obtain the numerical results in detail.

1 Introduction

Recently, one of the most interesting areas of research in approximation theory is the application of (p, q)-calculus. Mursaleen et al. initiated the (p, q)-type generalisation of linear positive operators, describing the (p, q)-analogue of Bernstein operators [8] as:

$$B_{n,p,q}(f; x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} \left(p^s - q^s x\right) f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}}\right),$$

where $x \in [0, 1]$, $0 < q < p \leq 1$, the (p, q)-integers are given as [5]:

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}.$$

For each $k \in \mathbb{N}_0$, the (p, q)-factorial is represented by:

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q} \ldots [1]_{p,q}, & k = 1, 2, 3, \ldots, \\ 1, & k = 0 \end{cases}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_6

and (p,q)-binomial coefficients are defined as:

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}$$

where $n \geq k \geq 0$.

Then, Sidharth and Agrawal [12] introduced the (p,q)-analogue of Bernstein–Schurer operators as:

$$\bar{B}_{n,s}(f;p,q;x) = \sum_{k=0}^{n+s} p^{\frac{k(k-1)}{2} - \frac{(n+s)(n+s-1)}{2}} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n+s-k} f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}}\right)$$

where $x \in [0, 1+s]$, $s \in \mathbb{N}_0$, $0 < q \leq p < 1$ and $n \in \mathbb{N}$.

Then, Gemikonakli and Vedi-Dilek constructed the Chlodowsky variant of Bernstein–Schurer operators based on (p,q)-integers in [4] as:

$$\bar{C}_{n,s}(f;p,q;x) = \sum_{k=0}^{n+s} f\left(p^{n-k}\frac{[k]_{p,q}}{[n]_{p,q}} b_n\right) p^{\frac{k(k-1)}{2} - \frac{(n+s-1)(n+s)}{2}} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} \left(\frac{x}{b_n}\right)^k (1-x)_{p,q}^{n+s-k}$$

where $x \in [0, b_n]$, $n, s \in \mathbb{N}$, $0 < q < p \leq 1$ and (b_n) is the positive increasing sequence with $b_n \to \infty$ and $\frac{b_n}{n} \to 0$ as $n \to \infty$.

Over the past 2 years, there has been a considerable amount of research on the (p,q)-analogue of Bernstein operators (see [1–3, 5–7, 9–12, 14]).

In 2016, Durrmeyer-type generalisation of (p,q)-Bernstein operators was defined by Sharma in [13] as:

$$D_n^{(p,q)}(f;x) = [n+1]_{p,q}\, p^{-n^2} \sum_{k=0}^{n} b_{n,k}^{(p,q)}(x) \left(\frac{q}{p}\right)^{-k} \int_0^1 b_{n,k}^{(p,q)}(qt)\, f(t)\, d_{p,q}t, \quad x \in [0,1]$$

where $b_{n,k}^{(p,q)}(x) = p^{k(k-1)/2} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n-k}$ and $0 < q < p \leq 1$, $n \in \mathbb{N}$.

To giving the estimations, Sharma proved Lemmas 1 and 2, respectively:

Lemma 1 *For* $s = 0, 1, 2, 3, \ldots$, *we have*

$$\int_0^1 b_{n,k}^{(p,q)}(qt)\, t^s d_{p,q}t = \left(\frac{q}{p}\right)^k p^{-ks} p^{n(n+2s+1)/2} \frac{[n]_{p,q}!\,[k+s]_{p,q}!}{[k]_{p,q}!\,[n+s+1]_{p,q}!}.$$

Lemma 2 *Let $D_n^{(p,q)}(f; x)$ be given in Lemma 1 in [13]. Then,*

(i) $D_n^{(p,q)}(1; x) = 1$,
(ii) $D_n^{(p,q)}(t; x) = \frac{1}{[n+2]_{p,q}}\left(p^n + q\,[n]_{p,q}\,x\right)$,
(iii) $D_n^{(p,q)}\left(t^2; x\right) = \frac{(p+q)p^{2n}+(p+q)^2qp^{n-1}[n]_{p,q}x+q^4[n]_{p,q}[n-1]_{p,q}x^2}{[n+2]_{p,q}[n+3]_{p,q}}$.

Obtaining the proof of Lemmas 1 and 2, we need the following definition and corollary:

Definition 1 Let $s,t \in \mathbb{R}$ and for $0 < q < p \leq 1$, (p, q)-beta integral is defined by

$$\int_0^1 x^{t-1}\,(1-qx)_{p,q}^{s-1}\,d_{p,q}x.$$

Corollary 1 *For $0 \leq k \leq n$ and $0 < q < p \leq 1$, we have relation between (p, q)-integers and q-integers:*

$$[n]_{p,q} = p^{n-1}\,[n]_{q/p}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{k(n-k)}\begin{bmatrix} n \\ k \end{bmatrix}_{q/p}.$$

This chapter is structured in the following way:

Section 2 introduces the Durrmeyer-type Bernstein operators based on (p, q)-integers with two variables and investigates the moments of the operator. In Sect. 3, we obtain the order of convergence of the Durrmeyer-type Bernstein operators based on (p, q)-integers with two variables by means of Lipschitz class functions and the full and partial modulus of continuities. Finally, in Sect. 4, numerical results to illustrate the contribution of the Durrmeyer-type Bernstein operators based on (p, q)-integers with two variables are presented.

2 Construction of the Operators

In this section, we introduce Durrmeyer-type Bernstein operators based on (p, q)-integers with two variables. Henceforth, let $I = [0, 1]$ and for $I^2 = I \times I$, let $C\left(I^2\right)$ denote the space of all real-valued functions on I^2 endowed with the norm $\|f\|_I = \sup_{(x,y)\in I^2} |f(x, y)|$. Now, if $f \in C\left(I^2\right)$ and $0 < q_1, q_2 < p_1, p_2 \leq 1$, then we construct Durrmeyer-type Bernstein operators based on (p, q)-integers with two variables by:

$$D_{n,m}\left(f;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right) \tag{2.1}$$

$$:=[n+1]_{p_n,q_n}[m+1]_{p_m,q_m}p_1^{-n^2}p_2^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)$$

$$\times\mathscr{R}_{m,l}(p_2,q_2;y)\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)f(t,s)\,d_{p_1,q_1}t\,d_{p_2,q_2}s,$$

where $n,m\in\mathbb{N}$, $(x,y)\in I^2$, and $\mathscr{R}_{n,k}(p_1,q_1;x)=p_1^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p_1,q_1}x^k(1-x)_{p_1,q_1}^{n-k}$.

Now, for giving our estimations we need that the following equalities hold for Eq. (2.1):

Lemma 3 *Using the Lemma 2, directly, we have*

$$D_{n,m}\left(1;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)=1,$$

$$D_{n,m}\left(t;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)=\frac{1}{[n+2]_{p_n,q_n}}\left(p_n^n+q_n[n]_{p_1,q_1}x\right),$$

$$D_{n,m}\left(s;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)=\frac{1}{[m+2]_{p_2,q_2}}\left(p_2^m+q_2[m]_{p_2,q_2}y\right),$$

$$D_{n,m}\left(t^2;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$$

$$=\frac{(p_1+q_1)p_1^{2n}+(p_1+q_1)^2q_1p_1^{n-1}[n]_{p_1,q_1}x+q_1^4[n]_{p_1,q_1}[n-1]_{p_1,q_1}x^2}{[n+2]_{p_1,q_1}[n+3]_{p_1,q_1}},$$

$$D_{n,m}\left(s^2;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$$

$$=\frac{(p_2+q_2)p_2^{2m}+(p_2+q_2)^2q_2p_2^{m-1}[m]_{p_2,q_2}y+q_2^4[m]_{p,q}[m-1]_{p,q}y^2}{[m+2]_{p_2,q_2}[m+3]_{p_2,q_2}}$$

$$D_{n,m}\left(t^2+s^2;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$$

$$=\frac{(p_1+q_1)p_1^{2n}+(p_1+q_1)^2q_1p_1^{n-1}[n]_{p_1,q_1}x+q_1^4[n]_{p_1,q_1}[n-1]_{p_1,q_1}x^2}{[n+2]_{p_1,q_1}[n+3]_{p_1,q_1}}$$

$$+\frac{(p_2+q_2)p_2^{2m}+(p_2+q_2)^2q_2p_2^{m-1}[m]_{p_2,q_2}y+q_2^4[m]_{p,q}[m-1]_{p,q}y^2}{[m+2]_{p_2,q_2}[m+3]_{p_2,q_2}}.$$

Using the Korovkin's theorem, we can obtain the following theorem.

Theorem 1 *For all $f \in C\left(I^2\right)$ and assuming that $q_1 : (q_n)$, $q_2 : (q_m)$; $p_1 : (p_n)$, $p_2 : (p_m)$ with $0 < q_n, q_m < p_n, p_m \leq 1$ such that $q_n \to 1$ and $q_m \to 1$ as $n, m \to \infty$, we get*

$$\lim_{n,m\to\infty} \left\| D_{n,m}\left(f; \left(p_n, q_n\right), \left(p_m, q_m\right); \cdot, \cdot\right) - f\left(\cdot, \cdot\right)\right\|_I = 0.$$

3 Order of Convergence

In this section, we compute the rate of convergence of the operators in terms of the modulus of continuity and then the degree of approximation in terms of Lipschitz-type space.

Let $f \in C\left(I^2\right)$ and $x, y \in I$. Then, the definition of the modulus of continuity of f is given by:

$$\omega(f; \delta) = \max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \delta \\ x,y\in C(I)}} \left|f\left(x_1, y_1\right) - f\left(x_2, y_2\right)\right|. \tag{3.1}$$

It is known that for any $\delta > 0$

$$\left|f\left(x_1, y_1\right) - f\left(x_2, y_2\right)\right| \leq \omega(f; \delta)\left(\frac{\sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2}}{\delta} + 1\right).$$

Theorem 2 *For any $f \in C\left(I^2\right)$ and let the following inequalities*

$$\left|D_{n,m}\left(f; \left(p_1, q_1\right), \left(p_2, q_2\right); x, y\right) - f(x, y)\right| \leq 2\left[\omega^{(1)}\left(f; \delta_n\left(x\right)\right) + \omega^{(2)}\left(f; \delta_m\left(y\right)\right)\right] \tag{3.2}$$

$$\left|D_{n,m}\left(f; \left(p_1, q_1\right), \left(p_2, q_2\right); x, y\right) - f(x, y)\right| \leq 2\omega\left(f; \sqrt{\delta_n^2 + \delta_m^2}\right) \tag{3.3}$$

be satisfied where

$$\delta_n^2\left(x\right) := \frac{\left(p_1 + q_1\right) p_1^{2n} + \left(p_1 + q_1\right)^2 q_1 p_1^{n-1}\left[n\right]_{p_1,q_1} x + q_1^4\left[n\right]_{p_1,q_1}\left[n-1\right]_{p_1,q_1} x^2}{\left[n+2\right]_{p_1,q_1}\left[n+3\right]_{p_1,q_1}} \tag{3.4}$$

and

$$\delta_m^2\left(y\right) := \frac{\left(p_2 + q_2\right) p_2^{2m} + \left(p_2 + q_2\right)^2 q_2 p_2^{m-1}\left[m\right]_{p_2,q_2} y + q_2^4\left[m\right]_{p,q}\left[m-1\right]_{p,q} y^2}{\left[m+2\right]_{p_2,q_2}\left[m+3\right]_{p_2,q_2}}. \tag{3.5}$$

Proof We directly have

$$
\begin{aligned}
&D_{n,m}\left(f;\left(p_{1,}q_{1}\right),\left(p_{2},q_{2}\right);x,y\right)-f(x,y)\\
&=[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_1^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\left[f(t,s)-f(x,y)\right]d_{p_{1,}q_1}t\,d_{p_{2,}q_2}s\\
&=[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_1^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\left[f(t,s)-f(x,s)+f(x,s)-f(x,y)\right]d_{p_{1,}q_1}t\,d_{p_{2,}q_2}s
\end{aligned}
$$

By linearity and positivity of the operators, we get

$$
\begin{aligned}
&\left|D_{n,m}\left(f;\left(p_{1,}q_{1}\right),\left(p_{2},q_{2}\right);x,y\right)-f(x,y)\right|\\
&\le[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_2^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\left|f(t,s)-f(x,s)\right|d_{p_{1,}q_1}t\,d_{p_{2,}q_2}s\\
&+[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_2^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\left|f(t,s)-f(x,y)\right|d_{p_{1,}q_1}t\,d_{p_{2,}q_2}s\\
&\le[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_1^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\omega^{(1)}\left(f;|t-x|\right)d_{p_{1,}q_1}t\,d_{p_{2,}q_2}s\\
&+[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_1^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\omega^{(2)}\left(f;|s-y|\right)d_{p_{1,}q_1}t\,d_{p_{2,}q_2}s\\
&=\Omega_1(x,y)+\Omega_2(x,y).
\end{aligned}
$$

Using Lemma 1, we have

$$\begin{aligned}
&\Omega_1(x,y)\\
&=[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_1^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\omega^{(1)}(f;|t-x|)\,d_{p_1,q_1}t\,d_{p_2,q_2}s\\
&\leq\omega^{(1)}\left(f;\delta_n^2(x)\right)\\
&\times\left[1+\frac{[n+1]_{p_1,q_1}}{\delta_n^2(x)}p_1^{-n^2}\sum_{k=0}^{n}\left(\frac{q_1}{p_1}\right)^{k}\mathscr{R}_{n,k}(p_1,q_1;x)\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\,(f;|t-x|)\,d_{p_1,q_1}t\right].
\end{aligned}$$

Using the Cauchy–Schwarz inequality, we get

$$\Omega_1(x,y)\leq 2\omega^{(1)}\left(f;\delta_n^2(x)\right)\tag{3.6}$$

where we chose $\delta_n^2(x)$ as in Eq. (3.4).

In the same way, we obtain

$$\Omega_2(x,y)\leq 2\omega^{(2)}\left(f;\delta_m^2(y)\right)\tag{3.7}$$

where $\delta_m^2(y)$ is given in Eq. (3.5). Combining Eqs. (3.6) and (3.7), we get Eq. (3.2).

Now, by using linearity and the monotonicity of the operators, and taking into account Eq. (3.3), we have

$$\begin{aligned}
&\left|D_{n,m}\left(f;\left(p_1,q_1\right),(p_2,q_2);x,y\right)-f(x,y)\right|\\
&\leq[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_1^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\omega\left(f;\sqrt{(t-x)^2+(s-y)^2}\right)d_{p_1,q_1}t\,d_{p_2,q_2}s\\
&\leq[n+1]_{p_1,q_1}[m+1]_{p_2,q_2}\,p_1^{-n^2}p_1^{-m^2}\sum_{k=0}^{n}\sum_{l=0}^{m}\left(\frac{q_1}{p_1}\right)^{-k}\left(\frac{q_2}{p_2}\right)^{-l}\mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)\\
&\times\int_0^1\int_0^1\mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\omega\left(f;\sqrt{(t-x)^2+(s-y)^2}\right)d_{p_1,q_1}t\,d_{p_2,q_2}s
\end{aligned}$$

$$\leq [n+1]_{p_1,q_1} [m+1]_{p_2,q_2} p_1^{-n^2} p_1^{-m^2} \sum_{k=0}^{n}\sum_{l=0}^{m} \left(\frac{q_1}{p_1}\right)^{-k} \left(\frac{q_2}{p_2}\right)^{-l} \mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)$$

$$\times \int_0^1 \int_0^1 \mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\,|f(t,s)-f(x,y)|\, d_{p_1,q_1}t d_{p_2,q_2}s$$

$$\leq 1+\frac{1}{\delta_n(x)} [n+1]_{p_1,q_1} [m+1]_{p_2,q_2} p_1^{-n^2} p_1^{-m^2} \sum_{k=0}^{n}\sum_{l=0}^{m} \left(\frac{q_1}{p_1}\right)^{-k}$$

$$\times \left(\frac{q_2}{p_2}\right)^{-l} \mathscr{R}_{n,k}(p_1,q_1;x)\mathscr{R}_{m,l}(p_2,q_2;y)$$

$$\times \int_0^1 \int_0^1 \mathscr{R}_{n,k}(p_1,q_1;qt)\mathscr{R}_{m,l}(p_2,q_2;qs)\omega\left(f;\sqrt{(t-x)^2+(s-y)^2}\right) d_{p_1,q_1}t d_{p_2,q_2}s.$$

Using Lemma 1 and Cauchy–Schwartz inequality, we have Eq. (3.3). □

Now, for $0 < \mu_1 \leq 1$ and $0 < \mu_2 \leq 1$, we give the Lipschitz class $Lip_M(\mu_1,\mu_2)$ for the bivariate case as follows:

$$|f(t,s)-f(x,y)| \leq M\,|t-x|^{\mu_1}\,|s-y|^{\mu_2}$$

where $(t,s), (x,y) \in I^2$.

Theorem 3 *Let $f \in Lip_M(\mu_1,\mu_2)$ and $(q_n,q_m) \in (0,1)$ such that $\lim_{n\to\infty} q_n = 1$ and $\lim_{m\to\infty} q_m = 1$. Then for all $(x,y) \in I^2$, we get*

$$\left|D_{n,m}\left(f;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)-f(x,y)\right| \leq M\delta_n^{\mu_1/2}(x)\,\delta_m^{\mu_2/2}(y),$$

where $\delta_n(x)$ and $\delta_m(x)$ are defined as in Theorem 2.

Proof Because $f \in Lip_M(\mu_1,\mu_2)$, we can write

$$\begin{aligned}
&\left|D_{n,m}\left(f;\left(p_n,q_n\right),\left(p_m,q_m\right);x,y\right)-f(x,y)\right| \\
&\leq D_{n,m}\left(|f(t,s)-f(x,y)|;\left(p_n,q_n\right),\left(p_m,q_m\right);x,y\right) \\
&\leq MD_{n,m}\left(|t-x|^{\mu_1}|s-y|^{\mu_2};\left(p_n,q_n\right),\left(p_m,q_m\right);x,y\right) \\
&\leq MD_n\left(|t-x|^{\mu_1};\left(p_n,q_n\right);x\right)D_m\left(|s-y|^{\mu_2};\left(p_m,q_m\right);y\right).
\end{aligned}$$

Now, by the Hölder's inequality with $\bar{p} = \dfrac{2}{\mu_1}, \bar{q} = \frac{2}{2-\mu_1}$ and $\bar{p} = \frac{2}{\mu_2}, \bar{q} = \frac{2}{2-\mu_2}$, respectively, we have

$$\begin{aligned}
&\left|D_{n,m}\left(f; \left(p_{n,}q_n\right), (p_m, q_m); x, y\right) - f(x, y)\right| \\
&\leq M\left\{D_n\left((t-x)^2; \left(p_{n,}q_n\right); x\right)\right\}^{\mu_1}\left\{D_n\left(1; \left(p_{n,}q_n\right); x\right)\right\}^{\frac{2-\mu_1}{2}} \\
&\times\left\{D_m\left((s-y)^2; \left(p_{m,}q_m\right); y\right)\right\}^{\mu_2}\left\{D_m\left(1; \left(p_{m,}q_m\right); y\right)\right\}^{\frac{2-\mu_2}{2}}.
\end{aligned}$$

This completes the proof. □

4 Numerical Results

In order to show the effectiveness and accuracy of $D_{n,m}\left(f; \left(p_{1,}q_1\right), (p_2, q_2); x, y\right)$ to $f(x, y)$ with different values of parameters, numerical results are presented in this section. Sensitivity analysis is carried out to minimise the error of approximation of $D_{n,m}\left(f; \left(p_{1,}q_1\right), (p_2, q_2); x, y\right)$ to the function $f(x, y) = \cos\left(x^2 + y^2\right)$ for minimum n and m values by taking into account different q_1 and q_2 values.

In Fig. 1, $D_{n,m}\left(f; \left(p_{1,}q_1\right), (p_2, q_2); x, y\right)$ results are given as a function of $f(x, y) = \cos\left(x^2 + y^2\right)$ for different q_1 and q_2 values. Figure 2 demonstrates the convergence of $D_{n,m}\left(f; \left(p_{1,}q_1\right), (p_2, q_2); x, y\right)$ to $f(x, y)$ but this time considering different n and m values, when $q_1 = q_2 = 0.1$ and $p_1 = p_2 = 0.9$.

In Fig. 2a, b, as n and m values are increased, the error of the approximation of $D_{n,m}\left(f; \left(p_{1,}q_1\right), (p_2, q_2); x, y\right)$ to $f(x, y)$ is minimised given $q_1 = 0.2, q_2 = 0.5$ and $p_1 = 0.9, p_2 = 0.9$ values.

On the other hand, comparative results are given in Tables 1 and 2, for the errors of the approximation of $D_{n,m}\left(f; \left(p_{1,}q_1\right), (p_2, q_2); x, y\right)$, considering each for different n, m values. However, using $n = 20$, $m = 15$ for

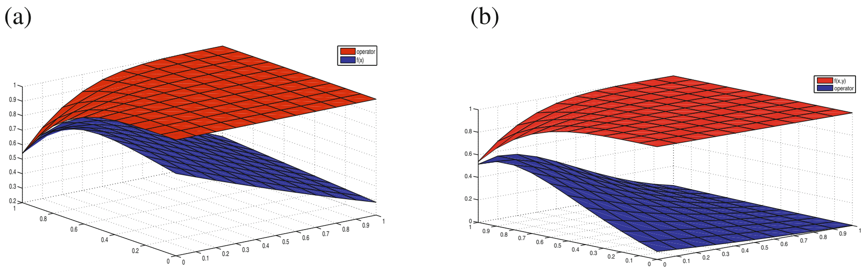

Fig. 1 Convergence of $D_{n,m}\left(f; \left(p_{1,}q_1\right), (p_2, q_2); x, y\right)$ for different q_1 and q_2 values. (**a**) $q_1 = 0.2, q_2 = 0.5, p_1 = 0.9, p_2 = 0.9$. (**b**) $q_1 = 0.8, q_2 = 0.8, p_1 = 0.9, p_2 = 0.9$

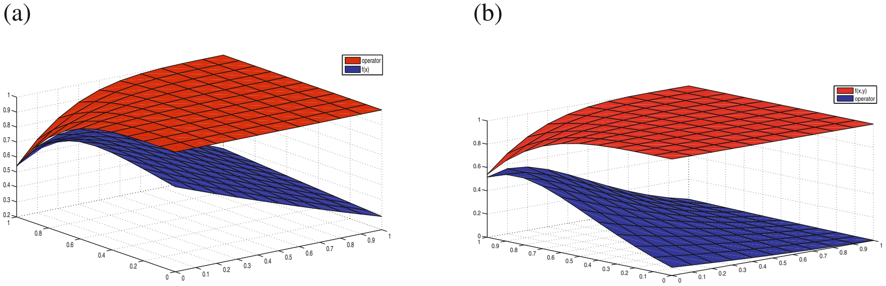

Fig. 2 Convergence of $D_{n,m}\left(f;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$ for different n and m values. (**a**) $n = 20$, $m = 15$. (**b**) $n = 1$, $m = 1$

Table 1 Errors of approximation $D_{n,m}\left(f;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$ to $f(x,y)$

x	y	$f(x,y)$	$\|f(x,y) - D_{20,15}(f;(p_1,q_1),(p_2,q_2);x,y)$	$\|f(x,y) - D_{1,1}(f;(p_1,q_1),(p_2,q_2);x,y)$
0.1	0.1	0.98346	0.0310	0.0150
0.2	0.2	0.98300	0.0432	0.0098
0.3	0.3	0.97653	0.0200	0.0049
0.4	0.4	0.87522	0.0411	0.0504
0.5	0.5	0.87100	0.1224	0.1052
0.6	0.6	0.67491	0.1998	0.0992
0.7	0.7	0.55902	0.2723	0.0632
0.8	0.8	0.28672	0.3627	0.0041

Table 2 Errors of approximation $D_{n,m}\left(f;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$ to $f(x,y)$

x	y	$f(x,y)$	$\|f(x,y) - D_{20,15}(f;(p_1,q_1),(p_2,q_2);x,y)$	$\|f(x,y) - D_{1,1}(f;(p_1,q_1),(p_2,q_2);x,y)$
0.1	0.1	0.99396	0.0190	0.0460
0.2	0.2	0.9968	0.0207	0.0720
0.3	0.3	0.98384	0.0049	0.0543
0.4	0.4	0.94924	0.0544	0.0180
0.5	0.5	0.87758	0.1050	0.1369
0.6	0.6	0.75181	0.1280	0.2913
0.7	0.7	0.55702	0.1210	0.4942
0.8	0.8	0.28672	0.1242	0.7627

$D_{n,m}\left(f;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$ rather than $n = 1$, $m = 1$ gives better approximation results. Therefore, the effect of increasing n and m values further than $n = 20$ and $m = 15$ is less evident for $x < 0.5$ and $y < 0.5$ for the convergence of $D_{n,m}\left(f;\left(p_1,q_1\right),\left(p_2,q_2\right);x,y\right)$ to the function $f(x,y)$.

On the other hand, it is required to increase the values of n and m further than $n = 20$ and $m = 15$ for $x > 0.5$ and $y > 0.5$ in order to have more accurate results.

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Part III
Complex Analysis and Convex Optimization and Their Applications in Wave Physics

Session Organizers: Sven Nordebo and Yevhen Ivanenko

The session focused primarily on theory of complex analysis and convex optimization, and their applications in wave physics. Complex analysis in one or several variables was considered as well as convex or nonconvex optimization. Typical research areas include various representation theorems and moment problems involving Herglotz–Nevanlinna and Stieltjes functions and their applications to linear, time translational invariant, causal, and passive systems to derive performance bounds in wave physics. Optimization can be used to analyze the realizability of performance bounds, as well as for automated design of acoustic or electromagnetic structures. Other research areas involving complex analysis and convex or nonconvex optimization were also highly appreciated in the session.

On the Passivity of the Delay-Rational Green's-Function-Based Model for Transmission Lines

Giulio Antonini, Maria De Lauretis, Jonas Ekman, and Elena Miroshnikova

Abstract In this paper, we study the delay-rational Green's-function-based (DeRaG) model for transmission lines. This model is described in terms of impedance representation and it contains a rational and a hyperbolic part. The crucial property of transmission lines models is to be passive. The passivity of the rational part has been studied by the authors in a previous work. Here, we extend the results to the rational part of the DeRaG model. Moreover, we prove the passivity of the hyperbolic part.

1 Introduction

Transmission lines (TLs) are of main interest in electrical engineering. They are mainly used for signal transmission, as interconnects in printed circuit boards (PCBs) and for energy distribution, as power transmission lines. Several port models exist for TLs in which only the terminal currents and voltage are related [1]. Among these models, the authors have recently proposed an improved version of the rational Green's-function-based model (RaG) presented in [2] called delay-RaG or "DeRaG" [5], where the line delay is explicitly included in the model. The DeRaG model is expressed by using the impedance representation Z. In fact, the Z representation allows an easy computation of the Green's function since a boundary value problem is solved by virtually enforcing the port currents at the ends of the line. This makes simple to compute the eigenfunctions and eigenvalues of the 1D propagation problem and the Green's function of the problem is identified as a series rational form. All the details can be found in [2]. Other well-known representations, such as the ABCD-matrix or cascade representation, can be obtained from the

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_7

impedance representation by virtue of transformation formulas [13]. TLs are passive circuit elements and their mathematical model should retain this essential property. However, it is not uncommon to find in literature TLs models that are not passive by construction and that, therefore, require a posteriori passivity enforcement. There is a vast literature devoted to the passivity of electrical models, and many articles have been written. The most interested reader can find more on [14–16], and the references therein. In this paper, we prove that the DeRaG model is passive by construction. The DeRaG model is described in terms of impedance representation and can be written as a finite sum of poles and residues, referred to as rational (delayless) part, and a hyperbolic matrix that accounts for the delay of the line. The rational delayless part contains asymptotic poles and residues that do not modify the positive-realness property of the impedance, as proved in [2]. In this paper, we check the passivity of the impedance matrix for single-conductor TLs, and the main result of the paper is the theorem about the positive realness of the hyperbolic matrix. The hyperbolic part consists of the asymptotic residue matrix, which is positive-definite, and the hyperbolic function matrix, which is proved to be passive.

2 Preliminaries

The notation used to describe the DeRaG model is summarized in Table 1. Matrices and vectors are written in bold. In the following, we review some basic definitions and properties of matrices that are important for our purposes; more can be found in the relevant literature (see, for example, [8]).

Table 1 Table of notation

ℓ	Length of conductors (m)
N	Number of conductors
x	Axis of the line in a rectangular coordinate system
p.u.l. or superscript $'$	Per-unit-length quantities
R', L', G', C'	p.u.l. resistance (Ω/m), inductance (H/m), conductance (S/m), and capacitance (F/m)
γ	Propagation constant
$\mathbf{Z}$	Impedance matrix (Ω)
$\mathbf{V}$	Voltage port vector (V)
$\mathbf{I}$	Current port vector (A)
$\mathscr{R}$	Residue matrix (Ω Hz)
$p = \alpha + i\beta$	Complex pole (Hz)
A_m	Positive coefficients equal to $\sqrt{1/\ell}$ for $m = 0$ and $\sqrt{2/\ell}$ for $m > 0$
$\mathbf{\Phi}$	Matrix of hyperbolic functions
T	Lossless propagation delay (s)
$\hat{}$	Asymptotic quantities

Let $s = x + \mathrm{i}y \in \mathbb{C}$ be any complex number, where $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are its real $\Re(s)$ and imaginary $\Im(s)$ parts correspondingly. By $\bar{s}$ we denote the conjugate number, i.e., $\bar{s} = x - \mathrm{i}y$. Given a complex matrix $\mathbf{M} = \{M_{ij}\}_{i,j=1}^{n} \in \mathbb{C}^{2n}$, the $\mathbf{M}^\star$ defines its conjugate transpose $\mathbf{M}^\star = \overline{\mathbf{M}^T}$ and $\mathbf{M}^{-1}$ its inverse—if exists. $\mathbf{M}^h = \frac{1}{2}(\mathbf{M} + \mathbf{M}^\star)$ denotes the Hermitian part of $\mathbf{M}$ ($\mathbf{M}^h = (\mathbf{M}^h)^\star$), and $|\mathbf{M}| = \det(\mathbf{M})$. $\|\mathbf{M}\| = \|\mathbf{M}\|_{\max}$ is the max-norm of $\mathbf{M}$, and $\mathbf{I}$ and $\mathbf{1}$ are the identity and all-ones $n \times n$ matrices.

Definition 1 A $n \times n$ complex matrix $\mathbf{M}$ is positive-(semi)definite if

$$\Re(\bar{\mathbf{s}}\mathbf{M}\mathbf{s}) > 0 \quad (\Re(\bar{\mathbf{s}}\mathbf{M}\mathbf{s}) \geq 0) \qquad \text{for any } \mathbf{s} \in \mathbb{C}^n \setminus \{\mathbf{0}\}\,.$$

Theorem 1 *The following statements are equivalent:*

1. $\mathbf{M}$ *is positive-definite;*
2. $\mathbf{M}^h$ *is positive-definite;*
3. *all leading principal minors of* $\mathbf{M}^h$ *are positive-definite (Silvester's criterion).*

In the present paper we deal with 2×2 matrices.

3 The DeRaG Model for Transmission Lines: Background Formulation

In the following, we review some basic properties of TLs and we refer the more interested reader to the relevant literature [11, 13]. In the general case, TLs consist of $N + 1$ conductors (N conductor plus the ground conductor) of length ℓ that can transmit electrical signals. In this paper, we focus on the case $N = 1$, normally referred to as *single-conductor* transmission lines. Figure 1 represents a single-conductor transmission line as a system with input and output ports. The x value refers to the position along the axis of the line in a rectangular coordinate system. We consider that the line is excited by the port currents $I(0, s)$ and $I(\ell, s)$. The excitation manifests as port voltages $V(0, s)$ and $V(\ell, s)$. The TL can be described

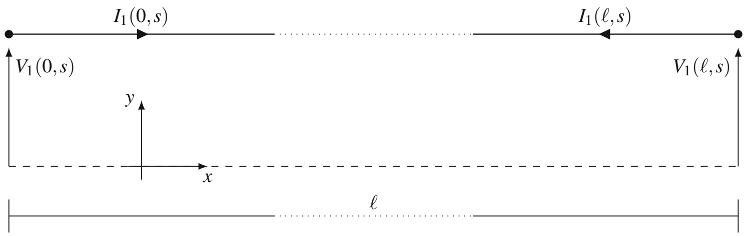

Fig. 1 Single-conductor transmission line. The axis of the line is the x axis of a rectangular coordinate system

in terms of the so-called per-unit-length (p.u.l.) parameters, namely, the resistance (R'), inductance (L'), capacitance (C'), and conductance (G'), which can be either dependent on or independent of the frequency. The superscript $'$ is used to be consistent with the notation found in literature for per-unit-length parameters. The p.u.l. parameters are positive for both homogeneous and inhomogeneous medium [13]. $R' = 0$ for perfect conductors, whereas $G' = 0$ if the surrounding medium is lossless. If the conductors and the medium are both lossless, then $R' = G' = 0$, which in the following is referred to as *lossless case*; otherwise, we talk about *lossy case*. The voltages and the currents along a transmission line can be uniquely defined only for a transverse electromagnetic (TEM) field structure, where the electric and magnetic fields are transverse to the line axis. However, non-ideal aspects of TLs can invalidate the TEM mode assumption, for example in case of inhomogeneous surrounding medium. The case of non-TEM modes that are negligible is referred to as the quasi-TEM mode assumption [13]. In [2], the telegraph equations, which describe the voltages and currents in a TL in terms of p.u.l. parameters, are formulated under the quasi-TEM mode assumption. Given the frequency-independent p.u.l. parameters, the impedance matrix representation in the Laplace domain reads as described in [2]

$$\begin{bmatrix} V(0,s) \\ V(\ell,s) \end{bmatrix} = \underbrace{\begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix}}_{\mathbf{Z}(s)} \begin{bmatrix} I(0,s) \\ I(\ell,s) \end{bmatrix}, \tag{1}$$

where s is the complex variable of the Laplace transform, and $\mathbf{Z}(s)$ is the 2×2 matrix-valued transfer function between the input (currents) and the output (voltages) and is symmetric. In [2], it was proven that the matrix $\mathbf{Z}(s)$ can be computed as

$$\mathbf{Z}(s) = \sum_{m=0}^{+\infty} \begin{bmatrix} Z_m & (-1)^m Z_m \\ (-1)^m Z_m & Z_m \end{bmatrix}, \quad Z_m = \left[\gamma^2 + \left(\frac{m\pi}{\ell}\right)^2\right]^{-1} A_m^2 (R' + sL')\,, \tag{2}$$

where $\gamma^2 = (R' + sL')(G' + sC')$ is the squared propagation constant γ, and A_m are positive coefficients defined as

$$A_m = \begin{cases} \sqrt{1/\ell}, & m = 0 \\ \sqrt{2/\ell}, & m > 0\,. \end{cases}$$

The impedance was proved to be passive by construction (no passivity enforcement) by resorting to the theorem for passive impedance matrices as given in Theorem 2, page 77. Equation (2) can be rewritten in a rational form as an infinite sum of the poles and residue matrices of $\mathbf{Z}(s)$, as explained in [2]. Specifically, each summation mode m generates a number of residues and poles equal to N for $m = 0$

and equal to $2N$ for $m > 0$ because poles and residues are in complex-conjugate pairs. In case of $N = 1$, we have

$$\mathbf{Z}(s) = \frac{\mathscr{R}_0}{s - p_0} + \sum_{m=1}^{\hat{m}-1} \left(\frac{\mathscr{R}_m}{s - p_m} + \frac{\overline{\mathscr{R}}_m}{s - \bar{p}_m} \right) . \tag{3}$$

p_m are the poles defined as

$$p_m = \begin{cases} \alpha_0, & m = 0 \\ \alpha_m \pm \mathrm{i}\beta_m, & m > 0 \end{cases} , \tag{4}$$

$\alpha_m \in (-\infty, 0]$, $\beta_m \in (0, +\infty)$, where the real parts are zero in the lossless case. The generic residue matrix corresponding to a mode m reads as

$$\mathscr{R}_m = \begin{bmatrix} R_{11,m} & R_{12,m} \\ R_{21,m} & R_{22,m} \end{bmatrix} = \begin{bmatrix} R_{11,m} & (-1)^m R_{11,m} \\ (-1)^m R_{11,m} & R_{11,m} \end{bmatrix} . \tag{5}$$

In the rest of the paper, we will avoid to use the subscript 11 for the elements of the residue matrix, i.e., $R_m = R_{11,m}$. The residues are real for $m = 0$ and complex for $m > 0$. Notice that the corresponding impedance elements $Z_{ij}(s)$ share the same poles. The sum in (3) is infinite and, in practice, a reasonable accuracy can be achieved only by using a large number of rational functions. Consequently, the method is computationally ineffective and the accuracy is limited because the *propagation delay*—the time that a signal spends in order to travel along the line, from the input to the output—is merely approximated. Additionally, if the delay is not properly considered but merely approximated, the causality condition is intrinsically violated [14]. Passive systems are causal systems [15]. Therefore, a model that intrinsically violates the causality condition cannot be passive. To overcome this limitation, in [5] the authors have proposed the DeRag model that reads as in Eq. (1) but it has an explicit delay extraction that modifies the rational sum in (3). The DeRaG model and its improved accuracy have been extensively studied in [3, 5, 6], and the reader is encouraged to read the aforementioned citations to gain a full understanding of the model. Here, we review the basic properties and expressions that allow to prove the passivity of the hyperbolic part. We proved that the sum can be truncated by introducing suitable tolerances to an optimal index $\hat{m}$, whereas the infinite behavior of (3) is retained by means of *hyperbolic functions* that account for the propagation delays and for the asymptotic values of residues and poles. In particular, for $m \to \infty$, the residues become asymptotically constant and real. At $\hat{m}$, we assume that the asymptotic behavior is fully established, and the asymptotic residue matrix, of dimension 2×2 (case $N = 1$), is real, $\hat{\mathscr{R}} = \hat{R}\mathbf{1}$, $\hat{R} > 0$. Notice that it is independent from the summation mode because it is computed for the optimal index $\hat{m}$. Similarly, for $m \to \infty$, the real part of the complex poles becomes constant, and the imaginary part shows a periodicity that allows us to extract the propagation delays, i.e., the following representation is valid:

$$\hat{p}_m = \hat{\alpha} + \mathrm{i}\left(m\hat{\beta}\right) \quad \text{for} \quad m \geq \hat{m} \ , \tag{6}$$

where $\hat{\beta} = \beta_{\hat{m}}/\hat{m}$ and $\alpha_m \underset{m\to\infty}{\longrightarrow} \hat{\alpha}$. The asymptotic real part $\hat{\alpha}$ of the poles can be either zero (lossless case) or negative (lossy case), e.g., $\hat{\alpha} \in (-\infty, 0]$. In [4] it is shown that the time delays can be computed as

$$T = \frac{2\pi}{\hat{\beta}}$$

and they correspond the well-known lossless delays. In [5], we proved that the asymptotic behavior of the model can be expressed with hyperbolic functions in the frequency domain and that the impedance in (3) can be expressed as a sum of a rational delayless part and a hyperbolic part as

$$\mathbf{Z}(s) = \frac{\mathscr{R}_0}{s - p_0} + \sum_{m=1}^{\hat{m}-1}\left(\frac{\mathscr{R}_m}{s - p_m} + \frac{\overline{\mathscr{R}}_m}{s - \bar{p}_m}\right) \tag{7}$$

$$- \frac{\hat{\mathscr{R}}}{s - \hat{\alpha}} - \sum_{m=1}^{\hat{m}-1}\left(\frac{\hat{\mathscr{R}}}{s - \hat{p}_m} + \frac{\hat{\mathscr{R}}}{s - \bar{\hat{p}}_m}\right) + \underbrace{\mathbf{H}(s)}_{\text{Hyperbolic part}} \ , \tag{8}$$

where the asymptotic poles and residues appear also for the modes $m < \hat{m}$ by means of mathematical manipulations necessary for the delay extraction. The asymptotic poles are properly adjusted such that they are periodic in the imaginary part starting from the mode $m = 1$. $\mathbf{H}(s)$ denotes the following matrix-valued function:

$$\mathbf{H}(s) = \frac{\hat{R}T}{2}\mathbf{\Phi}\left((s - \hat{\alpha})\frac{T}{2}\right) \ , \tag{9}$$

where $\hat{\alpha} < 0$ and

$$\mathbf{\Phi}(s) = \begin{bmatrix} \coth s & \operatorname{csch} s \\ \operatorname{csch} s & \coth s \end{bmatrix}, \qquad s \in \mathbb{C}, \tag{10}$$

accounts for the asymptotic delays. In [7, 10] it is proven that $\mathbf{H}$ represents a distortionless line. In particular, the term

$$\frac{\hat{R}T}{2} = \sqrt{\frac{L'}{C'}} \tag{11}$$

and it corresponds to the characteristic impedance matrix of the distortionless line associated to the original TL. Therefore, it is positive real, because the p.u.l. parameters are positive.

4 The Passivity Theorem

The passivity condition can be studied in both the time and frequency domain. In the Laplace domain, the following well-known theorem is valid for impedance representations [14]:

Theorem 2 *An impedance matrix* $\mathbf{Z} = \mathbf{Z}(s)$, $s \in \mathbb{C}$, *represents a passive linear system if and only if:*

1. *each element of* $\mathbf{Z}(s)$ *is defined and analytic in* $\Re(s) > 0$;
2. $\mathbf{Z}$ *is positive-semidefinite for* s, $\Re(s) > 0$;
3. $\mathbf{Z}(\bar{s}) = \bar{\mathbf{Z}}(s)$ *for any* $s \in \mathbb{C}$.

The first requirement means that there are no unstable poles in the system, and is related to both causality and stability. The second condition requires that the real part of the impedance must be positive, which is equivalent of saying that there are no negative resistors. The last one ensures that the associated impulse response is real. For systems with impedance matrix representation, the *passivity is equivalent to the positive-realness of the transfer function*, see [9] and the references therein.

5 Discussion on the Passivity of the DeRaG Model

5.1 Passivity of the Rational Part

We can start by considering the rational delayless part, which is the difference between the truncated sum of the original model and its asymptotic tail:

$$\mathbf{Z}_{dl}(s) = \frac{\mathscr{R}_0}{s - p_0} + \sum_{m=1}^{\hat{m}-1} \left(\frac{\mathscr{R}_m}{s - p_m} + \frac{\overline{\mathscr{R}_m}}{s - \bar{p}_m} \right) - \frac{\hat{\mathscr{R}}}{s - \hat{\alpha}} - \sum_{m=1}^{\hat{m}-1} \left(\frac{\hat{\mathscr{R}}}{s - \hat{p}_m} + \frac{\hat{\mathscr{R}}}{s - \hat{\bar{p}}_m} \right), \tag{12}$$

where the subscript *dl* is for "delayless." It satisfies the points 1 and 3 of the theorem as proved in [2]. Here, we show that it satisfies the second condition as well. The first (non-asymptotic) part is positive-semidefinite. In fact, it corresponds to (3) truncated to $\hat{m}-1$. As proved in [2], Eq. (3) is positive real and it retains this property also when truncated to a generic summation mode. The second (asymptotic) sum is equivalent to considering the $\mathbf{Z}$ matrix for $m \to +\infty$. As we have seen in Sect. 3, the asymptotic residues have imaginary part equal to zero. Additionally, the real part is the asymptotic one that converges to a value smaller than the initial one, such that

$$\|\hat{\mathscr{R}}\| < \|\mathscr{R}_m\|, \qquad \forall m < \hat{m} \ . \tag{13}$$

In respect to the poles, they have the same properties as the poles of the original line, but the real part is negative and fixed among the families, and their imaginary part is equally spaced. Therefore, the rational delayless part is still positive-semidefinite.

5.2 *Passivity of the Hyperbolic Part*

The following theorem holds:

Theorem 3 *The matrix* $\mathbf{H} = \mathbf{H}(s)$ *defined by* (9) *is a positive-real matrix.*

Proof

1. As it was mentioned before (see (11)), $\mathbf{H}(s)$ can be written as

$$\mathbf{H}(s) = \sqrt{\frac{L'}{C'}}\boldsymbol{\Phi}\left((s - \hat{\alpha})\frac{T}{2}\right),$$

where $\boldsymbol{\Phi}$ is as in (10). Since $L'/C' > 0$ and $\hat{\alpha} < 0$, it is sufficient to prove the theorem only for $\boldsymbol{\Phi}$.
2. By standard techniques of complex analysis one can show that

Lemma 1 *The complex-valued functions* coth *and* csch *allow the following* $\Re - \Im$ *representations:*

$$\coth(s) = \frac{\left(e^{2x} - e^{-2x}\right) - 4i \sin y \cos y}{\cos^2 y \left(e^x - e^{-x}\right)^2 + \sin^2 y \left(e^x + e^{-x}\right)^2} \tag{14}$$

and

$$\operatorname{csch}(s) = \frac{2\cos y \left(e^x - e^{-x}\right) - 2i \sin y \left(e^x + e^{-x}\right)}{\cos^2 y \left(e^x - e^{-x}\right)^2 + \sin^2 y \left(e^x + e^{-x}\right)^2}, \qquad s = x + iy \in \mathbb{C}\ , \tag{15}$$

correspondingly.

Corollary 1 *The matrix* $\boldsymbol{\Phi}$ (10) *satisfies* $\boldsymbol{\Phi}(\bar{s}) = \overline{\boldsymbol{\Phi}}(s) = \boldsymbol{\Phi}^{\star}(s)$ *for all* $s \in \mathbb{C}$.

3. The functions coth and csch satisfy the first condition of Theorem 2.
The third one follows directly from Corollary 1. In order to complete the proof of Theorem 3, we need to prove that $\boldsymbol{\Phi} = \boldsymbol{\Phi}(s)$ is positive-semidefinite for

$\Re(s) > 0$. We will show that it is positive-definite by using the Sylvester's criterion (see Theorem 1). Due to Corollary 1, the Hermitian part $\mathbf{\Phi}^h$ of the matrix $\mathbf{\Phi}$ has the form

$$\Phi_{ij}^h(s) = \frac{\Phi_{ij}(s) + \overline{\Phi_{ji}(s)}}{2} = \frac{\Phi_{ij}(s) + \Phi_{ij}(\bar{s})}{2} = \Re\left(\Phi_{ij}(s)\right), \qquad i, j = 1, 2,$$

where according to (14) and (15)

$$\Re(\Phi_{ij})(s) = \begin{cases} \frac{\left(e^{2x} - e^{-2x}\right)}{\cos^2 y\left(e^x - e^{-x}\right)^2 + \sin^2 y\left(e^x + e^{-x}\right)^2}, & i = j \\ \frac{2\cos y\left(e^x - e^{-x}\right)}{\cos^2 y\left(e^x - e^{-x}\right)^2 + \sin^2 y\left(e^x + e^{-x}\right)^2}, & i \neq j \end{cases}, \qquad s = x + \mathrm{i}y \in \mathbb{C}\,. \tag{16}$$

In order to use the Silvester's criterion (see Theorem 1), we check that

$$\Phi_{11}^h > 0 \qquad \text{and} \qquad |\mathbf{\Phi}^h| > 0, \quad \text{whenever} \quad x = \Re(s) > 0\,.$$

The first part $\Phi_{11}^h > 0$ follows directly from (16). The second one reads as

$$|\mathbf{\Phi}^h(s)| = \Re(\Phi_{11})^2 - \Re(\Phi_{12})^2 \tag{17}$$

$$= \left(\frac{e^{2x} - e^{-2x}}{\cos^2 y\left(e^x - e^{-x}\right)^2 + \sin^2 y\left(e^x + e^{-x}\right)^2}\right)^2 - \left(\frac{2\cos y\left(e^x - e^{-x}\right)}{\cos^2 y\left(e^x - e^{-x}\right)^2 + \sin^2 y\left(e^x + e^{-x}\right)^2}\right)^2. \tag{18}$$

Since $|\cos y| \leq 1$, it is sufficient to analyze the function

$$f(x) = (e^{2x} - e^{-2x})^2 - 2(e^x - e^{-x})^2.$$

The derivative reads as

$$f'(x) = \left((e^{2x} - e^{-2x})^2 - 4(e^x - e^{-x})^2\right)' = 4(e^{2x} - e^{-2x})\left((e^{2x} + e^{-2x}) - 1\right)$$

and has only one zero:

$$f'(0) = 0, \qquad f'(x) < 0,\ x < 0, \qquad f'(x) > 0,\ x > 0\,.$$

Since $f(0) = 0$, we can conclude that $|\mathbf{\Phi}^h| > 0$ for all s, $\Re(s) > 0$ and therefore $\mathbf{\Phi}$ is proved to be positive-definite for all $s \in \mathbb{C}$, $\Re(s) > 0$. It completes the proof of Theorem 3.

Remark 1 The term $\Phi_{11}(s) = \coth(s)$ can be as well analyzed by means of Herglotz–Nevanlinna functions, which are analytic functions $g(s)$, $s \in \mathbb{C}$, with the property $\Im(g(s)) \geq 0$ for $s \in \mathbb{C}$, $\Im(s) > 0$ (see [12]). The connection between the positive-real function f and the Herglotz–Nevanlinna function g is provided by the following formula:

$$f(s) = -\mathrm{i}g(\mathrm{i}s).$$

In [12], it was shown that $g(s) = \tan s$ is a Herglotz–Nevanlinna function. Using the formula above, we compute that $f(s) = \tanh s$ is a positive-real function. In fact,

$$-\mathrm{i}\tan(\mathrm{i}s) = -\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}(\mathrm{i}s)} - \mathrm{e}^{-\mathrm{i}(\mathrm{i}s)}}{\mathrm{i}(\mathrm{e}^{\mathrm{i}(\mathrm{i}s)} + \mathrm{e}^{-\mathrm{i}(\mathrm{i}s)})} = \frac{\mathrm{e}^{s} - \mathrm{e}^{-s}}{\mathrm{e}^{s} + \mathrm{e}^{-s}} = \tanh s.$$

Since $\coth s = \tanh^{-1} s$, $\Re(s) > 0$, it is also positive for $s \in \mathbb{C}$, $\Re(s) > 0$.

6 Conclusion and Future Work

In this paper, we have discussed the passivity properties of the DeRaG model for single-conductor transmission lines. In particular, the passivity of the hyperbolic part is analytically proved. However, the generalization to multiconductor case ($N > 1$) is not straightforward. In this case, the impedance matrix becomes a block matrix and therefore different techniques are required. It is the subject of interest for future works.

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Passive Approximation with High-Order B-Splines

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Abstract Convex optimization has emerged as a well-suited tool for passive approximation. Here, it is desired to approximate some pre-defined non-trivial system response over a given finite frequency band by using a passive system. This paper summarizes some explicit results concerning the Hilbert transform of general B-splines of arbitrary order and arbitrary partitions that can be useful with the convex optimization formulation. A numerical example in power engineering is included concerning the identification of some model parameters based on measurements on high-voltage insulation materials.

1 Introduction

Herglotz–Nevanlinna functions (also known as Nevanlinna, Herglotz, Pick, R-, and positive real (PR) functions) [1, 8, 17, 21, 27] can be used to represent admittance passive [4, 27] and scattering passive systems [4] with many applications in physics and engineering. In particular, moment relations for the generating measures (sum rules) can be used to derive physical bounds in electromagnetic applications such as with radar absorbers [22], high-impedance surfaces [12], passive metamaterials [11], scattering [4, 25], antennas [13], waveguides [26], etc., see also [4] for a general overview and an in-depth derivation of related sum rules. However, the application of sum rules is limited to some special cases where moment relations can be analyzed in detail and in some other applications the required sum rules may not even exist. This is typically the situation when it is desired to approximate some specific non-trivial system response over a given finite frequency band (such as with a general metamaterial, etc.) by using an admittance passive system. In this case, a numerical convex optimization approach provides an alternative tool which does not depend on the asymptotic expansion of the Herglotz–Nevanlinna functions.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_8

This approach has been used, e.g., with dispersion compensation for waveguides, passive metamaterials, passive radar absorbers in [19], and with optimal plasmonic resonances in [20]. A rigorous mathematical interpretation of the approach has been given in [16] showing that the convex cone of Herglotz–Nevanlinna functions generated by B-splines of any order and defined on uniform partitions is dense (in the sense of a weighted L^p-norm) in the larger cone generated by positive measures having Hölder continuous densities in a neighborhood of the approximation domain. In this paper, the convex optimization approach is complemented with some explicit results concerning the use of general B-splines of arbitrary order and arbitrary partitions. A numerical example in system identification is used to illustrate the theory.

2 Herglotz–Nevanlinna Functions

A Herglotz–Nevanlinna function $h(z)$ is a holomorphic function with the property $\Im\{h(z)\} \geq 0$ for $z \in \mathbb{C}^+ = \{z \in \mathbb{C} | \Im\{z\} > 0\}$, where $z = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$. It can be shown that the Herglotz–Nevanlinna functions have the following integral representation:

$$h(z) = b_1 z + c + \int_{-\infty}^{\infty} \left(\frac{1}{\xi - z} - \frac{\xi}{1 + \xi^2} \right) \mathrm{d}\beta(\xi), \tag{1}$$

for $z \in \mathbb{C}^+$ and where $b_1 \geq 0$, $c = \Re\{h(\mathrm{i})\}$, and β is the corresponding positive Borel measure with $\int_{\mathbb{R}} \mathrm{d}\beta(\xi)/(1 + \xi^2) < \infty$, see, e.g., [1, 3, 8, 17, 21]. When the measure β is absolutely continuous with density β', the differential can be expressed as $\mathrm{d}\beta(x) = \beta'(x)\mathrm{d}x = \frac{1}{\pi}\Im\{h(x + \mathrm{i}0)\}\mathrm{d}x$, where $\mathrm{d}x$ is the Lebesgue measure and the last expression indicates that the limit is taken from the upper half-plane, cf. [8, 17]. In the general case, the positive measure β is uniquely determined by the Herglotz function $h(z)$ from the Stieltjes inversion formula [17].

A symmetric Herglotz–Nevanlinna function satisfies the symmetry requirement $h(z) = -h(-z^*)^*$ for $z \in \mathbb{C}^+$ and where β is an even measure, i.e., $\mathrm{d}\beta(\xi) = \mathrm{d}\beta(-\xi)$. The integral representation (1) can then be simplified as

$$h(z) = b_1 z + \int_{-\infty}^{\infty} \frac{1}{\xi - z} \mathrm{d}\beta(\xi), \tag{2}$$

for $z \in \mathbb{C}^+$ and where the integral is taken as a symmetric limit at infinity.

Suppose that a symmetric Herglotz–Nevanlinna function has the following asymptotic expansions:

$$h(z) = \begin{cases} a_{-1}z^{-1} + a_1 z + \ldots + a_{2N_0-1}z^{2N_0-1} + o(z^{2N_0-1}) & z \hat{\to} 0, \\ b_1 z + b_{-1}z^{-1} + \ldots + b_{1-2N_\infty}z^{1-2N_\infty} + o(z^{1-2N_\infty}) & z \hat{\to} \infty, \end{cases} \tag{3}$$

where the expansion coefficients are real valued, $a_{-1} \leq 0$, $b_1 \geq 0$, and N_0 and N_∞ are non-negative integers where $1 - N_\infty \leq N_0$. Here, $z \hat{\to} 0$ and $z \hat{\to} \infty$ mean that $|z| \to 0$ and $|z| \to \infty$ in the Stolz cone $\phi \leq \arg z \leq \pi - \phi$, respectively, for any $\phi \in (0, \pi/2]$, see [4]. It is then possible to derive the following integral identities (sum rules) based on the representation (2):

$$\frac{2}{\pi}\int_{0+}^{\infty} \frac{\Im\{h(\xi)\}}{\xi^{2k}} \mathrm{d}\xi \overset{\text{def}}{=} \lim_{\varepsilon \to 0+} \lim_{y \to 0+} \frac{2}{\pi}\int_{\varepsilon}^{1/\varepsilon} \frac{\Im\{h(\xi + \mathrm{i}y)\}}{\xi^{2k}} \mathrm{d}\xi = a_{2k-1} - b_{2k-1}, \tag{4}$$

for $k = 1 - N_\infty, \ldots, N_0$, see, e.g., [4, 8].

3 Passive Approximation

Passive approximation based on the symmetric Herglotz–Nevanlinna function representation (2) can be used to approximate an arbitrary linear system function with a real-valued time convolution kernel, see [16, 19]. However, to formulate a well-defined convex optimization problem [5], it is necessary to first impose some a priori constraints on the target function as well as on the class of approximating Herglotz–Nevanlinna functions. In particular, we are interested here in Herglotz–Nevanlinna functions that are known to be locally Hölder continuous on some given intervals on the real line. Hence, a passive approximation problem is considered where the target function F is an arbitrary complex valued continuous function defined on an approximation domain $\Omega \subset \mathbb{R}$ consisting of a finite union of closed and bounded intervals of the real axis. The norms used, denoted by $\|\cdot\|_{\mathrm{L}^p(w,\Omega)}$, are weighted $\mathrm{L}^p(\Omega)$-norms [23] which are defined here by using a positive continuous weight function w on Ω, and where $1 \leq p \leq \infty$.

Let O denote an arbitrary neighborhood of the approximation domain Ω. Consider a Herglotz function h generated by a measure β which is absolutely continuous on $\overline{O}$ with Hölder continuous density β' on $\overline{O}$ (outside $\overline{O}$ the measure is arbitrary). In this case, the Herglotz–Nevanlinna function h can be Hölder continuously extended (with Hölder exponent $0 < \alpha < 1$) from $\mathbb{C}^+$ to $\mathbb{C}^+ \cup \Omega$ with boundary values

$$h(x) = b_1 x + \fint_{\mathbb{R}} \frac{1}{\xi - x} \mathrm{d}\beta(\xi) + \mathrm{i}\pi\beta'(x), \tag{5}$$

for $x \in \Omega$, and where the integral is taken as a Cauchy principal value, see [16, Theorem 2.2]. This result is readily obtained from the Sokhotski–Plemelj theorem [18, Theorem 7.6, p. 101] or the Plemelj–Privalov theorem [24, Theorem 5.7.21, p. 484] which is formulated for a bounded and simply connected domain in the complex plane.

The continuity of h on Ω implies that the norm $\|h\|_{L^p(w,\Omega)}$ is well-defined for $1 \leq p \leq \infty$. An approximation problem of interest can now be formulated in terms of the greatest lower bound on the approximation error, defined by

$$d := \inf_{h} \|h - F\|_{L^p(w,\Omega)}, \tag{6}$$

where the infimum is taken over all functions h with measures having Hölder continuous densities on $\overline{O}$ as represented in (5). In general, a best approximation achieving the bound d in (6) does not exist. In practice, however, the problem can be approached by using numerical algorithms such as CVX [10], solving finite-dimensional approximation problems where B-splines [6] are used to represent the generating measure β and where the number of basis functions N is fixed during the optimization, cf. [16, 19]. Moreover in [16, Theorems 3.2 and 3.3] it has been shown that the approximation error d defined in (6) can be achieved in the limit as $N \to \infty$ by using B-splines of arbitrary order (linear, quadratic, cubic, etc.) on uniform partitions. In the next sections are given some explicit results concerning the Hilbert transform of general B-splines of arbitrary order and arbitrary partitions which is useful in a formulation based on convex optimization.

4 B-Splines and Their Hilbert Transforms

The normalized B-spline $N_{0,m}(x)$ of order $m \geq 2$ consists of piecewise polynomial functions of order $m-1$, and is defined for $x_0 \leq x < x_m$, where $x_0 < x_1 < \cdots < x_m$ are called knots, or break-points. The B-spline $N_{0,m}(x)$ is uniquely defined by the formulas

$$N_{i,1}(x) = \begin{cases} 1 & x_i \leq x < x_{i+1}, \\ 0 & \text{otherwise}, \end{cases} \tag{7}$$

where $i = 0, 1, \ldots, m-1$ and

$$N_{i,k}(x) = \frac{x - x_i}{x_{i+k-1} - x_i} N_{i,k-1}(x) + \frac{x_{i+k} - x}{x_{i+k} - x_{i+1}} N_{i+1,k-1}(x), \tag{8}$$

where $k = 2, \ldots, m$ and $i = 0, \ldots, m-k$ and which is defined for $x_i \leq x < x_{i+k}$, cf. [6]. The B-spline $N_{0,m}(x)$ is a strictly positive function in the interval $x_0 < x < x_m$ and is defined to be zero elsewhere. The function $N_{0,m}(x)$ is $m-2$ times continuously differentiable and has piecewise constant derivatives of order $m-1$ with discontinuities at the knots [6]. An alternative way to define B-splines is by $m-1$ times repeated convolution of (7) with the normalized square pulse, see for details [16].

Let $p_n(x)$ denote a basis (B-spline) function $N_{0,m}(x)$ with distinct knots $\{x_i\}_{i=0}^m$ which depend on the finite index $n = 1, \ldots, N$. The (negative) Hilbert transform of the B-spline pulse function $p_n(x)$ is defined by

$$\hat{p}_n(x) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{1}{\xi - x} p_n(\xi) \mathrm{d}\xi, \tag{9}$$

for $x \in \mathbb{R}$ and where the integral is a Cauchy principal value. The result is most conveniently obtained by first calculating the corresponding Herglotz–Nevanlinna function $h_n(z)$ for $z \in \mathbb{C}^+$. A repeated integration by parts yields the following explicit formula for $m \geq 2$:

$$h_n(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\xi - z} p_n(\xi) \mathrm{d}\xi = (-1)^{m-1} \frac{1}{\pi} \sum_{i=0}^{m-1} p_n^{(m-1)}(x_i+)$$

$$\times \left[\frac{(x_{i+1} - z)^{m-1}(\ln(x_{i+1} - z) - C_m)}{(m-1)!} - \frac{(x_i - z)^{m-1}(\ln(x_i - z) - C_m)}{(m-1)!} \right], \tag{10}$$

where the constant C_m is given recursively by

$$C_m = C_{m-1} + \frac{1}{m-1}, \tag{11}$$

and where $C_1 = 0$. Hence, the sequence $\{C_2, C_3, C_4, \ldots\} = \{1, 3/2, 11/6, \ldots\}$ corresponds to linear, quadratic, and cubic B-splines, etc. It is observed that the result in (10) can be obtained by employing the following formula:

$$\frac{\mathrm{d}^m}{\mathrm{d}\zeta^m} \frac{\zeta^{m-1}}{(m-1)!} (\ln \zeta - C_m) = \frac{1}{\zeta}, \tag{12}$$

which can be proved by induction for $m \geq 2$ and where C_m is given by (11). In (10), the terms $p_n^{(m-1)}(x_i+)$ denote the right limits of the discontinuous $(m-1)$th derivative of $p_n(x)$ at the knots x_i. The discontinuity behavior of the linear, quadratic, and cubic B-splines is summarized in Appendix. Note that the polynomial zeros cancel the logarithmic singularities at the knots x_i in (10), where $i = 0, 1, \ldots, m$ and $m \geq 2$. In fact, since the B-spline functions are Hölder continuous on $\mathbb{R}$ (with exponent $0 < \alpha < 1$), it can be shown that the Herglotz–Nevanlinna function $h_n(z)$ in (10) can be Hölder continuously extended to $\mathbb{C}^+ \cup \mathbb{R}$, cf. [16, Theorem 2.2]. The (negative) Hilbert transform in (9) is finally obtained by taking the real part of (10) for $x \in \mathbb{R}$, i.e., by replacing $\ln(x_i - z)$ for $\ln |x_i - x|$.

In Fig. 1 are illustrated linear (l), quadratic (q) and cubic (c) B-splines over an interval $x \in [-0.5, 4]$, and their Hilbert transforms. Here, the dashed lines correspond to B-spline functions of order $m = 2, 3, 4$ with circles indicating their respective knots $x_\mathrm{l} \in \{0, 0.5, 1\}$, $x_\mathrm{q} \in \{0, 0.33, 0.67, 1\}$, and $x_\mathrm{c} \in \{0, 0.25, 0.5, 0.75, 1\}$, and the solid lines the corresponding Hilbert transforms.

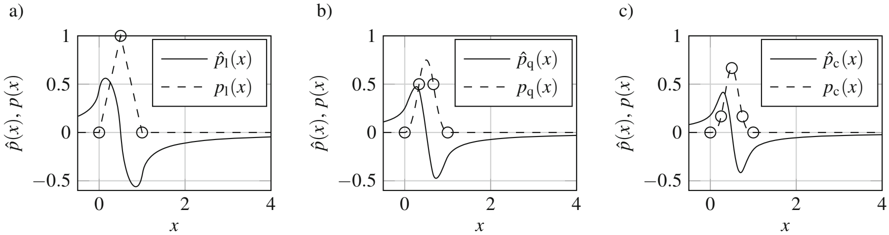

Fig. 1 Illustration of linear, quadratic, and cubic B-spline basis functions, $p_l(x)$, $p_q(x)$, and $p_c(x)$, their knots x_l, x_q, and x_c, and corresponding (negative) Hilbert transforms $\hat{p}_l(x)$, $\hat{p}_q(x)$, and $\hat{p}_c(x)$, respectively. (**a**) Linear B-spline. (**b**) Quadratic B-spline. (**c**) Cubic B-spline

5 Convex Optimization

Consider a discretization of the problem expressed in (6), which is based on an arbitrary, finite partition of the approximation domain Ω. Let

$$\Im\{h(x)\} = \sum_{n=1}^{N} c_n \left(p_n(x) + p_n(-x)\right), \tag{13}$$

for $x \in \mathbb{R}$ be a finite B-spline expansion of $\Im\{h(x)\} = \pi\beta'(x)$, where c_n are optimization variables for $n = 1, \ldots, N$, and $p_n(x)$ are the B-spline basis functions of fixed order m which have been defined on the given partition. The real part $\Re\{h(x)\}$ for $x \in \Omega$ is then given by (5), and can be expressed as

$$\Re\{h(x)\} = b_1 x + \sum_{n=1}^{N} c_n \left(\hat{p}_n(x) - \hat{p}_n(-x)\right), \tag{14}$$

for $x \in \Omega$ and where $\hat{p}_n(x)$ is the (negative) Hilbert transform of the B-spline function $p_n(x)$. Consider now the following convex optimization problem:

$$\begin{aligned} &\text{minimize} && \|h - F\|_{\mathrm{L}^p(w,\Omega)} \\ &\text{subject to} && c_n \geq 0, \quad \text{for } n = 1, \ldots N, \\ &&& b_1 \geq 0, \end{aligned} \tag{15}$$

where the optimization is over the variables $(c_1, \ldots, c_N, b_1)$. As mentioned above, the minimum approximation error obtained from (15) will approach the same minimum as defined in (6) in the limit as $N \to \infty$, cf. [16, Theorems 3.2 and 3.3].

The uniform continuity of all functions involved implies that the solution to (15) can be approximated within an arbitrary accuracy by discretizing the approximation domain Ω (and the computation of the norm) using only a finite number of sample points. The corresponding numerical problem (15) can now be solved efficiently by using the CVX Matlab software for disciplined convex programming [10].

When there is a priori information available regarding the asymptotic properties of the system to be approximated, the sum rules (4) can sometimes be used as convex constraints to supplement (15).

Another situation is when the large argument asymptotic coefficient b_{2k-1} is a priori known or estimated, and it is required to estimate the corresponding small argument asymptotic coefficient a_{2k-1} based on the solution to (15). In this case, the estimated coefficient a_{2k-1} can be calculated from

$$a_{2k-1} = b_{2k-1} + \frac{2}{\pi}\int_{0+}^{\infty} \frac{\Im\{h(\xi)\}}{\xi^{2k}} \mathrm{d}\xi = b_{2k-1} + \frac{2}{\pi}\sum_{n=1}^{N} c_n \int_{0+}^{\infty} \frac{p_n(\xi)}{\xi^{2k}} \mathrm{d}\xi, \tag{16}$$

where the last equality is due to the finite-dimensional approximation (13). Explicit formulas for the last integral can be readily obtained, similarly as in Sect. 4, and as exemplified in Sect. 6 below.

6 Numerical Example

As an engineering application example, we consider the problem to estimate the conductivity parameter of high-voltage insulation materials based on its dielectric response in the range of very low frequencies. To measure such responses, a dielectric spectroscopy measurement technique has been developed [7, 9] and the Havriliak–Negami (HN) model [2, 14] is commonly used to identify the conductivity parameter. However, the HN-models only constitute a certain subclass of dispersion models which have been chosen on empirical grounds. Moreover, to identify the parameters of the HN-model, one must in general solve a non-convex optimization problem which requires an exhaustive global search, and which becomes particularly cumbersome if several resonances are involved. In contrast, the modeling based on B-splines and their Hilbert transforms provides a general passive dispersion model, and a convex optimization problem that can be solved efficiently using, e.g., the CVX Matlab software for disciplined convex programming [10].

Here, the dielectric spectroscopy data is simulated by using the following Havriliak–Negami (HN) model:

$$\epsilon_{\mathrm{HN}}(x) = \epsilon_{\infty} + \frac{\Delta\epsilon}{(1+(-\mathrm{i}x\tau)^{\alpha})^{\beta}} + \mathrm{i}\frac{\sigma_{\mathrm{HN}}}{x\epsilon_0}, \tag{17}$$

where $\Delta\epsilon = \epsilon_s - \epsilon_{\infty} > 0$ and where ϵ_{s} and $\epsilon_{\infty} \geq 1$ are the static and the instantaneous dielectric responses, respectively, the parameters $\alpha, \beta \in (0, 1]$, $\tau > 0$ the relaxation coefficient, $\sigma_{\mathrm{HN}} > 0$ the static conductivity, and ϵ_0 the permittivity of free space. In the numerical example below, the HN-model has been generated with $\epsilon_\infty = 2.227$, $\Delta\epsilon = 0.0267$, $\alpha = 0.68$, $\beta = 0.5$, $\tau = 59.99$, and $\sigma_{\rm HN} = 8.9616 \cdot 10^{-2}$ fS/m based on a global optimization with respect to some given measurement data, see [15]. Here, the frequency f (in units Hz) is defined by $x = 2\pi f$ and the approximation domain is given by $\Omega = 2\pi[f_{\rm L}, f_{\rm U}]$, where $f_{\rm L}$ and $f_{\rm U}$ are the corresponding lower and upper frequency limits, respectively.

The convex optimization formulation for this problem is given by

$$\begin{array}{ll} \text{minimize} & \|h - h_{\rm HN}\|_{{\rm L}^2(w,\Omega)} \\ \text{subject to} & c_n \geq 0, \quad n = 1, \ldots, N, \\ & \epsilon_\infty \geq 1, \\ & \sigma \geq 0, \end{array} \tag{18}$$

where $w(x) = 1/x$ is the weight function, $h_{\rm HN}(x) = x\epsilon_{\rm HN}(x)$, and the approximating Herglotz function $h(x) = x\epsilon(x)$ is given by

$$h(x) = h_1(x) + {\rm i}\frac{\sigma}{\epsilon_0}, \tag{19}$$

for $x \in \Omega$ and where $h_1(x)$ is represented as in (13) and (14) with $b_1 = \epsilon_\infty$. The parameters to be estimated by using (18) are denoted $\hat{\sigma}$ and $\hat{\epsilon}_\infty$, etc.

The low-frequency behavior of the passive material is estimated using the expression (16) based on the sum rule (4) for $k = 1$

$$\hat{\epsilon}_{\rm s} = \hat{\epsilon}_\infty + \frac{2}{\pi}\int_{0+}^{\infty} \frac{\Im\{h(\xi)\}}{\xi^2}{\rm d}\xi = \hat{\epsilon}_\infty + \frac{2}{\pi}\sum_{n=1}^{N} c_n \sum_{i=n-m/2}^{n+m/2} p_n^{(m-1)}(x_i+) \\ \times\left[\frac{x_{i+1}^{m-2}(\ln x_{i+1} - C_{m-1})}{(m-2)!} - \frac{x_i^{m-2}(\ln x_i - C_{m-1})}{(m-2)!}\right], \tag{20}$$

where $a_1 = \hat{\epsilon}_{\rm s}$ and $b_1 = \hat{\epsilon}_\infty$ are the estimated static and instantaneous responses, respectively, m is even and corresponds to the order of the B-splines used, and C_m is defined in (11).

As an example, the objective here is to determine a sufficient frequency bandwidth based on the lower frequency limit $f_{\rm L}$ to reach some predetermined relative error on the estimation of conductivity σ. The upper limit $f_{\rm U} = 10$ Hz is fixed, and the approximation domain is non-uniformly sampled with a logarithmic step $\Delta \log f = 1/3$. After the optimization (18), the error norm $E = \|\epsilon(x) - \epsilon_{\rm HN}(x)\|_{{\rm L}^2(\Omega)}$ and the interpolated and extrapolated parameter $\epsilon(x) = h(x)/x$ are evaluated on a grid that is ten times denser than the one used for optimization ($\Delta \log f = 1/30$).

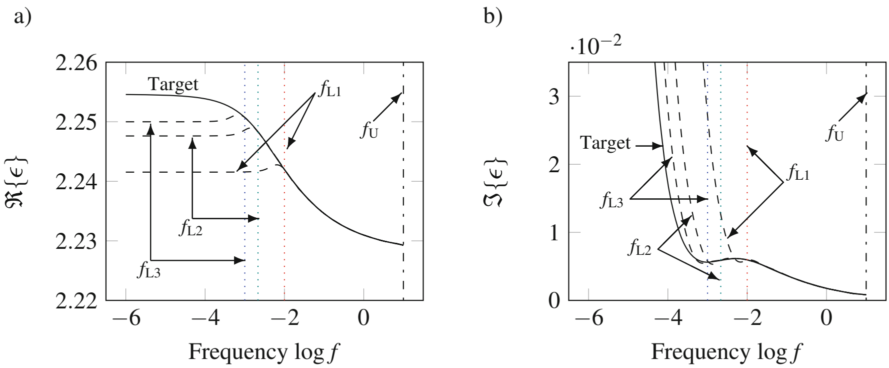

Fig. 2 Interpolation and extrapolation of the generated dielectric spectroscopy data. Dashed and dotted lines are plotted in the order of increasing the bandwidth parameter f_L, where the approximation domain is $\Omega = [f_L, 10]$ Hz. (**a**) Real part of $\epsilon(f)$. (**b**) Imaginary part of $\epsilon(f)$

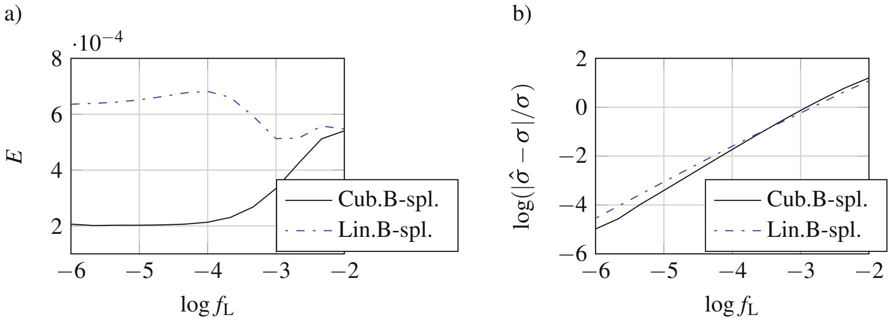

Fig. 3 (**a**) Approximation error E as a function of the lower frequency limit f_L. (**b**) Relative error on the estimation of conductivity $\hat{\sigma}$. The approximation domain is $\Omega = [f_L, 10]$ Hz

In Fig. 2a, b are illustrated interpolation and extrapolation of the generated dielectric spectroscopy data. Here, the solid lines correspond to the target data generated via the HN-model (17), and the dashed lines correspond to interpolation and extrapolation based on optimized high-order (cubic) B-spline approximation (18) on $\Omega = [f_L, \; f_U]$, where $f_L \in \{10, 2.2, 1\}$ mHz and $f_U = 10$ Hz. The lower and upper bounds of the approximation domain are illustrated via the vertical dotted and dash-dotted lines, respectively.

The investigation on the sufficient frequency bandwidth for an accurate estimation of conductivity σ is illustrated in Fig. 3a, b. Figure 3a shows the approximation error E and Fig. 3b the relative error of the conductivity estimate $\hat{\sigma}$, where the solid and the dash-dotted lines correspond to cubic and linear B-spline approximations, respectively. It can be concluded, e.g., that an accurate estimation of σ_{HN} within 1% error requires that $f_L < 0.03$ mHz. It is also seen that the approximation based on high-order B-splines can provide a more accurate solution in comparison to the linear B-splines in this example.

7 Summary

A convex optimization approach for passive approximation of electromagnetic systems based on Herglotz functions, general B-splines, and sum rules has been developed. A numerical example based on an approximation of the Havriliak–Negami dispersion model over a large frequency bandwidth has been studied. It is found that the data can be better represented using high-order B-splines in comparison to the linear B-splines. Moreover, the cubic B-splines provide a better estimator of the conductivity parameter which in general is very difficult to estimate based on finite bandwidth data.

Acknowledgements This work was supported by the Swedish Foundation for Strategic Research (SSF) under the program Applied Mathematics and the project Complex analysis and convex optimization for EM design.

Appendix

The discontinuity behavior of linear B-splines $N_{0,2}(x)$ with knot values $N_{0,2}(x_1)=1$ and $N_{0,2}(x_0) = N_{0,2}(x_2) = 0$ is given by

$$N_{0,2}^{(1)}(x_0+) = \frac{1}{x_1 - x_0}, \tag{21}$$

$$N_{0,2}^{(1)}(x_1+) = -\frac{1}{x_2 - x_1}. \tag{22}$$

The discontinuity behavior of quadratic B-splines $N_{0,3}(x)$ with knot values $N_{0,3}(x_1) = (x_1 - x_0)/(x_2 - x_0)$, $N_{0,3}(x_2) = (x_3 - x_2)/(x_3 - x_1)$, and $N_{0,3}(x_0) = N_{0,3}(x_3) = 0$ is given by

$$N_{0,3}^{(2)}(x_0+) = \frac{2}{(x_2 - x_0)(x_1 - x_0)}, \tag{23}$$

$$N_{0,3}^{(2)}(x_1+) = -\frac{2}{(x_2 - x_0)(x_2 - x_1)} - \frac{2}{(x_3 - x_1)(x_2 - x_1)}, \tag{24}$$

$$N_{0,3}^{(2)}(x_2+) = \frac{2}{(x_3 - x_1)(x_3 - x_2)}. \tag{25}$$

The discontinuity behavior of cubic B-splines $N_{0,4}(x)$ with knot values

$$N_{0,4}(x_1) = \frac{(x_1 - x_0)^2}{(x_3 - x_0)(x_2 - x_0)}, \tag{26}$$

$$N_{0,4}(x_2) = \frac{(x_2 - x_0)(x_3 - x_2)}{(x_3 - x_0)(x_3 - x_1)} + \frac{(x_4 - x_2)(x_2 - x_1)}{(x_4 - x_1)(x_3 - x_1)}, \tag{27}$$

$$N_{0,4}(x_3) = \frac{(x_4 - x_3)^2}{(x_4 - x_1)(x_4 - x_2)}, \tag{28}$$

and $N_{0,4}(x_0) = N_{0,4}(x_4) = 0$ is given by

$$N_{0,4}^{(3)}(x_0+) = \frac{6}{(x_3 - x_0)(x_2 - x_0)(x_1 - x_0)}, \tag{29}$$

$$N_{0,4}^{(3)}(x_1+) = -\frac{6}{(x_3 - x_0)(x_2 - x_0)(x_2 - x_1)} - \frac{6}{(x_3 - x_0)(x_3 - x_1)(x_2 - x_1)} - \frac{6}{(x_4 - x_1)(x_3 - x_1)(x_2 - x_1)}, \tag{30}$$

$$N_{0,4}^{(3)}(x_2+) = \frac{6}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} + \frac{6}{(x_4 - x_1)(x_3 - x_1)(x_3 - x_2)} + \frac{6}{(x_4 - x_1)(x_4 - x_2)(x_3 - x_2)}, \tag{31}$$

$$N_{0,4}^{(3)}(x_3+) = -\frac{6}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}. \tag{32}$$

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Part IV
Complex and Functional Analytic Methods for Differential Equations

Session Organizers: Heinrich Begehr, A. Okay Çelebi, and J.Y. Du

This session was organized at all the ISAAC congresses since the first one in 1997 at the University of Delaware. It was always well accepted and had many participants. However, it became obvious that the session was more frequented when the site of the congress was in the eastern hemisphere due to the fact that complex analysis is nowadays more studied and applied there. Also scientists from the east are less able to afford expensive travel and lodging costs while western scientists have easier access to financial support. ISAAC is therefore well advised to held its congresses consecutively in eastern and western counties.

After Växyö there was the chance to have the 2019 congress in Novosibirsk, even in conjunction with a celebration of the 130th anniversary of the famous mathematician S.L. Sobolev at this place. Unfortunately this chance was not chosen. As in Växyö also in Avairo our session will be small again. If not the colleagues from Kazakhstan would come only very few will be participating again. In 2017 there were only 5 colleagues from western countries among altogether 12 members, mainly from former SU. No one from China did make it. From the 2019 congress on H. Begehr will not co-organize the session further on.

From the contributions to the session just some were chosen for publication in this proceeding volume. Cialdea's study on conjugate differential forms is a rich and promising theory, where complex concepts are transferred to higher dimensions avoiding Clifford analysis. Differential and difference operators of second order with unbounded coefficients are investigated by Ospanov. Nino Manjavidze and collaborators are working on uniqueness results in complex analysis. Dirichlet boundary value problems in polydiscs are treated from Çelebi and Begehr's reports on a proper formulation of the Robin problem.

Some New Applications of the Theory of Conjugate Differential Forms

Alberto Cialdea

Abstract In this survey we describe two applications of the concept of conjugate differential forms. Namely, after describing the concept of conjugate and self-conjugate differential forms, we consider an extension of the Brothers Riesz theorem to higher real dimension and Riesz-type inequalities for differential forms.

1 Introduction

Many years ago, looking for a generalization of the Brothers Riesz theorem in higher real dimension, I was led to consider the concept of conjugate differential forms [5]. Such concept has been already used in the previous paper [4], in which I had to construct a reducing operator for a particular singular integral operator. This is why I began to study in detail conjugate differential forms and self-conjugate (non-homogeneous) differential forms [7].

Later on I have used such forms in several different problems. They concern, besides the extension of the Brothers Riesz theorem in higher real dimension, the concept of conjugate Laplace series in $\mathbb{R}^n$ [2, 3, 6, 9], potential theory with applications to several BVPs for different PDEs [1, 11, 13–20], and Riesz-type inequalities for differential forms [12].

In this brief survey I will just consider the Brothers Riesz theorem and Riesz-type inequalities. The first section is devoted to the concept of conjugate and self-conjugate differential forms.

For a survey on the applications in potential theory connected to BVPs I refer to [10].

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_9

2 Self-Conjugate Differential Forms

The idea of considering conjugate differential forms in order to extend the concept of conjugate harmonic functions dates back to Volterra [31]. Following this order of ideas, we say that a k-form u (i.e., a differential form of degree k) and a $(k+2)$-form v are *conjugate* in $\Omega \subset \mathbb{R}^n$ if

$$du = \delta v, \quad \delta u = 0, \quad dv = 0, \tag{1}$$

where d is the differential operator and δ is the co-differential (actually this concept is slightly different from the one given by Volterra: in fact u and v are conjugate in the sense of Volterra if $du = \delta v$, see [31], pp. 87–90). If $n = 2$, $f(z) = u(x, y) + iv(x, y)$ is a holomorphic function and we identify v with a 2-form, then $du = \delta v$ is just the Cauchy–Riemann equation, while $\delta u = 0$ and $dv = 0$ are automatically satisfied.

The system (1) includes several real generalizations of the Cauchy system.

For example, this concept of conjugate forms is more general than the concept of *harmonic vectors* considered by Stein and Weiss in the paper [28], i.e., of vectors $(w_1, \dots, w_n)$ satisfying the system

$$\sum_{i=1}^{n} \frac{\partial w_i}{\partial x_i} = 0, \qquad \frac{\partial w_i}{\partial x_j} = \frac{\partial w_j}{\partial x_i} \ (i \neq j)\ . \tag{2}$$

In fact, if we identify $(w_1, \dots, w_n)$ with the 1-form $u = w_h dx^h$, the system (2) is nothing but $du = 0$, $\delta u = 0$. In other words Stein and Weiss have considered only the forms which are of degree 1 and conjugate to $v = 0$.

More generally, the k-form

$$u_k = \frac{1}{k!}\, w_{s_1} \dots {}_{s_k}\, dx^{s_1} \dots dx^{s_k}$$

is conjugate to $u_{k+2} \equiv 0$ if, and only if, $du_k = 0$ and $\delta u_k = 0$. These are the so-called *harmonic forms*.

If we consider $n = 3$ and $u_0 \equiv u, u_2 = v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2$, we have that u_0 and u_2 are conjugate if, and only if $\operatorname{div}(v_1, v_2, v_3) = 0$, $\operatorname{grad} u = \operatorname{curl}(v_1, v_2, v_3)$, i.e., if, and only if, the vector (u, v_1, v_2, v_3) satisfies the *Moisil-Theodorescu system*.

The concept of conjugate differential forms can be further generalized. Let us consider a non-homogeneous differential form belonging to $C^1_0(\Omega) \oplus \dots \oplus C^1_n(\Omega)$

$$U = \sum_{k=0}^{n} u_k$$

where u_k is a differential form of degree k. We say that U is self-conjugate if $dU = \delta U$, i.e., if $\delta u_1 = 0$, $du_k = \delta u_{k+2}$ $(k = 0, \dots, n-2)$, *and* $du_{n-1} = 0$.

It is clear that if $U = u_k + u_{k+2}$, then U is *self-conjugate* if and only if u_k and u_{k+2} are conjugate in the sense of (1).

If $n = 4$ and $U = u_0 + u_2 + u_4$, where

$$u_0 = f_0, \quad u_4 = f_0 dx^0 dx^1 dx^2 dx^3$$

$$u_2 = f_1(dx^0 dx^1 - dx^2 dx^3) + f_2(dx^0 dx^2 - dx^3 dx^1) + f_3(dx^0 dx^3 - dx^1 dx^2),$$

the non-homogeneous form U is self-conjugate if, and only if,

$$\begin{cases} \frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0 \\ \frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} - \frac{\partial f_2}{\partial x_3} + \frac{\partial f_3}{\partial x_2} = 0 \\ \frac{\partial f_0}{\partial x_2} + \frac{\partial f_1}{\partial x_3} + \frac{\partial f_2}{\partial x_0} - \frac{\partial f_3}{\partial x_1} = 0 \\ \frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0} = 0 \, . \end{cases}$$

This shows that U is self-conjugate if, and only if, the vector (f_0, f_1, f_2, f_3) satisfies the *Fueter system*.

A similar computation shows that the form $U = u_0 + u_2 + u_4$, where

$$u_0 = f_0, \quad u_4 = -f_0 dx^0 dx^1 dx^2 dx^3$$

$$u_2 = f_1(dx^0 dx^1 + dx^2 dx^3) - f_2(dx^0 dx^2 + dx^3 dx^1) + f_3(dx^0 dx^3 + dx^1 dx^2),$$

is self-conjugate if and only if the vector (f_0, f_1, f_2, f_3) satisfies the *Cimmino system* (see [1]).

In what follows we shall use also the concept of *k-measure*, which was introduced by Fichera (see [22, 23]). Roughly speaking a k-measure is a differential form whose coefficients are measures and we refer to Fichera's papers for the precise definition and for several properties.

3 The Brothers Riesz Theorem

In their only joint paper [26] F. Riesz and M. Riesz proved this famous result:

Theorem 1 *If a trigonometric series and its conjugate series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\vartheta + b_k \sin k\vartheta) \, , \qquad \sum_{k=1}^{\infty} (a_k \sin k\vartheta - b_k \cos k\vartheta)$$

are both Fourier–Stieltjes series, then they are ordinary Fourier series.

In other words, if we have two real measures α, β defined on the Borel sets of $[0, 2\pi]$ such that

$$\int_0^{2\pi} \cos k\vartheta \, d\alpha = \int_0^{2\pi} \sin k\vartheta \, d\beta, \int_0^{2\pi} \sin k\vartheta \, d\alpha = -\int_0^{2\pi} \cos k\vartheta \, d\beta \qquad (k = 1, 2, \ldots), \tag{3}$$

then these measures have to be absolutely continuous, i.e., there exist two real valued L^1 functions f and g such that

$$\alpha(E) = \int_E f(\vartheta)\, d\vartheta, \qquad \beta(E) = \int_E g(\vartheta)\, d\vartheta$$

for any Borel set $E \subset [0, 2\pi]$. The interest of this result in the theory of Fourier series is evident. Theorem 1 can be easily rewritten in a "complex" form:

Theorem 2 *If μ is a complex measure defined on the Borel sets of the unit circle $C = \{z \in \mathbb{C} \mid |z| = 1\}$ such that*

$$\int_C e^{ik\vartheta} d\mu = 0 \qquad k = 1, 2, \ldots,$$

then μ is absolutely continuous, i.e., there exists a function $f \in L^1(C)$ such that

$$\mu(E) = \int_E f(\vartheta) d\vartheta$$

for any Borel set E of C.

This beautiful theorem gave rise to a long series of papers and "*in its direct applications as well as the generalizations it has inspired, this has proved to be one of the more important theorems of the century*" (R. B. Burckel, *Math. Rev.*, 96k:43009). For a survey of several results connected to the Brothers Riesz theorem, see [8] and the references therein.

The classical Brothers Riesz theorem can be stated also in the following way: if $u(x, y)$ and $v(x, y)$ are two conjugate real harmonic functions in a domain Ω and both of them have traces on $\partial\Omega$ in the sense of measures, then these measures have to be absolutely continuous.

Such a result was proved for conjugate differential forms and—more generally—for non-homogeneous self-conjugate differential forms in [5]. The result is the following. Here Ω is a bounded domain in $\mathbb{R}^n$ with a Lyapunov boundary and $M_k(\Sigma)$ denotes the space of k-measures defined on the Borel sets of Σ.

Theorem 3 *If* $U \in C^1_0(\Omega) \oplus \ldots \oplus C^1_n(\Omega)$ *is self-conjugate and* U *and* $*U$ *admit traces on* $\Sigma = \partial\Omega$ *in the sense of k-measures:*

$$\begin{cases} U\big|_{\Sigma} = \alpha \in M_0(\Sigma) \oplus \ldots \oplus M_{n-1}(\Sigma) \\ *U\big|_{\Sigma} = \widetilde{\alpha} \in M_0(\Sigma) \oplus \ldots \oplus M_{n-1}(\Sigma), \end{cases}$$

then the k-measures α *and* $\widetilde{\alpha}$ *have to be absolutely continuous.*

4 Conjugate Laplace Series

Given a trigonometric series, the conjugate trigonometric series can be considered as the "trace" of the harmonic function conjugate to the harmonic function whose trace is the given trigonometric series. Following this definition and hinging on the theory of conjugate differential forms, a new definition of conjugate Laplace series was given in [6].

Let us recall it. Consider a harmonic function u defined in the unit ball $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$, it is well known that it can be expanded by means of harmonic polynomials:

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}\left(\frac{x}{|x|}\right), \tag{4}$$

where $p_{nh} = (2h+n-2)\frac{(h+n-3)!}{(n-2)!h!}$ and $\{Y_{hk}\}$ is a complete system of ultraspherical harmonics. We suppose $\{Y_{hk}\}$ orthonormal, i.e.,

$$\int_{\Sigma} Y_{hk} Y_{rs}\, d\sigma \begin{cases} = 1 & \text{if } h = r \text{ and } k = s \\ = 0 & \text{otherwise.} \end{cases}$$

The "trace" of u on $\Sigma = \{x \in \mathbb{R}^n \mid |x| = 1\}$ is given by the expansion

$$\sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}(x) \qquad (|x| = 1). \tag{5}$$

Let us consider the 2-form

$$v = \sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)}\, dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2}) \tag{6}$$

and its adjoint

$$* v = \sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2}) \right) . \tag{7}$$

It is possible to show that $dv = 0$ *and* $\delta v = du$ in B, i.e., the non-homogeneous form $u + v$ is self-conjugate.

If $n = 2$ the series which is obtained by taking $|x| = 1$ in (7) is just the trigonometric series conjugate to (5). In general, for any n, we say that

$$\sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk}\left(\frac{x}{|x|}\right) \wedge d(|x|^{h+2}) \right) \Bigg|_{|x|=1} \tag{8}$$

is the series conjugate to (4); it represents the “restriction” of $*v$ on Σ, while the “restriction” of v, provided it does exist, is equal to 0, as it follows immediately from (6).

Several properties of the conjugate Laplace series (8) have been obtained (see [2, 3, 9]). They concern the Abel convergence, the pointwise convergence, and the convergence in L^p norm.

Here we mention a result (see [6]) which extends the original Brothers Riesz Theorem 1 to Laplace series and which is a consequence of Theorem 3:

Theorem 4 *Let* (5) *be a Laplace series of a measure* $\mu \in M(\Sigma)$*, i.e.,*

$$a_{hk} = \int_{\Sigma} Y_{hk}\, d\mu .$$

If there exists an $(n-2)$*-measure* $\beta \in M_{n-2}(\Sigma)$ *such that*

$$\int_{+\Sigma} Y_{hk}\, d\mu = \frac{1}{h} \int_{+\Sigma} \beta \wedge dY_{hk} \qquad (h = 1, 2, \ldots; k = 1, \ldots, p_{nh}) \tag{9}$$

and

$$\int_{+\Sigma} \beta \wedge \underset{\Sigma}{*} \gamma = 0 \tag{10}$$

for any $\gamma \in C^{\infty}_{n-2}(\mathbb{R}^n)$ *such that* $d\gamma = 0$ *on* Σ*, then* μ *and* β *are absolutely continuous.*

We remark that in the case $n = 2$, conditions (9) are nothing but (3), while (10) is not restrictive (the only closed 0-forms on the unit circle are the constants). However, if $n \geq 3$ condition (10) cannot be omitted.

5 Riesz-Type Inequalities

The classical Riesz inequality is well known:

$$\|g\|_{L^p(S)} \le C\|f\|_{L^p(S)}, \tag{11}$$

the function $f + ig$ being holomorphic in the unit disc D, continuous up to the boundary $S = \partial D$, and $g(0) = 0$ $(1 < p < \infty)$.

Another inequality, which—as we shall see—is related to (11), concerns normal derivative $\frac{\partial}{\partial \nu}$ and tangential gradient $\mathrm{grad}_{\partial\Omega}$ of a harmonic function defined on a sufficiently smooth bounded domain $\Omega \subset \mathbb{R}^n$. Namely, we have

$$\left\|\frac{\partial \omega}{\partial \nu}\right\|_{L^p(\partial\Omega)} \le C\|\mathrm{grad}_{\partial\Omega}\,\omega\|_{L^p(\partial\Omega)}, \tag{12}$$

for any harmonic function $\omega \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. Inequality (12) was proved by Vishik [30] for $p = 2$ when $\partial\Omega$ is a sphere, conjectured by Mikhlin in [24, p. 210] for $1 < p < \infty$, and established by De Vito [21] in the general case $1 < p < \infty$ when $\partial\Omega$ is the boundary of a $C^{2,\lambda}$-domain. Later Verchota [29] proved (12) on Lipschitz domains $(1 < p \le 2)$.

In [12] inequalities of this type have been obtained in the frame of conjugate differential forms. Namely, let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain and let u_k and v_{k+2} be two C^1 conjugate differential forms defined in Ω, continuous up to the boundary Σ. The following inequalities hold:

$$\inf_{\alpha\in\mathcal{N}_k^+}\|u_k+\alpha\|_{L^p_k(\Sigma)} \le C\Big\{\|*u_k\|_{L^p_{n-k}(\Sigma)} + \|*v_{k+2}\|_{L^p_{n-k-2}(\Sigma)}\Big\},$$

$$\inf_{\beta\in\mathcal{N}_{k+2}^+}\|v_{k+2}+\beta\|_{L^p_{k+2}(\Sigma)} \le C\Big\{\|u_k\|_{L^p_k(\Sigma)} + \|*v_{k+2}\|_{L^p_{n-k-2}(\Sigma)}\Big\},$$

$$\inf_{\alpha\in\mathcal{N}_{n-k}^+}\|*u_k+\alpha\|_{L^p_{n-k}(\Sigma)} \le C\Big\{\|u_k\|_{L^p_k(\Sigma)} + \|*v_{k+2}\|_{L^p_{n-k-2}(\Sigma)}\Big\},$$

$$\inf_{\beta\in\mathcal{N}_{n-k-2}^+}\|*v_{k+2}+\beta\|_{L^p_{n-k-2}(\Sigma)} \le C\Big\{\|u_k\|_{L^p_k(\Sigma)} + \|v_{k+2}\|_{L^p_{k+2}(\Sigma)}\Big\}.$$

Here $\mathcal{N}_k^+$ is the kernel of the singular integral equation:

$$-\frac{1}{2}\phi_k(x) + \int_\Sigma \phi_k(y) \wedge \underset{y}{*} d_y s_k(x,y) = 0, \quad \text{a.e. } x \in \Sigma,$$

where $s_k(x, y)$ is the Hodge double form

$$s_k(x, y) = \sum_{j_1<\ldots<j_k} s(x, y) dx_{j_1} \ldots dx_{j_k} dy_{j_1} \ldots dy_{j_k} .$$

As proved in [25, 27], the dimension of $\mathcal{N}_k^+$ is equal to b_k^-, the kth Betti number of Ω. It is clear that such inequalities generalize (11).

If ω is an harmonic k-form, we have that $\delta\omega$ and $-d\omega$ are conjugate. Therefore the inequalities we have obtained for conjugate differential forms lead to

$$\inf_{\alpha\in\mathcal{N}_{k-1}^+} \|\delta\omega_k + \alpha\|_{L^p_{k-1}(\Sigma)} \leq C\Big\{\|d * \omega_k\|_{L^p_{n-k+1}(\Sigma)} + \| * d\omega_k\|_{L^p_{n-k-1}(\Sigma)}\Big\},$$

$$\inf_{\beta\in\mathcal{N}_{k+1}^+} \|d\omega_k + \beta\|_{L^p_{k+1}(\Sigma)} \leq C\Big\{\|\delta\omega_k\|_{L^p_{k-1}(\Sigma)} + \| * d\omega_k\|_{L^p_{n-k-1}(\Sigma)}\Big\},$$

$$\inf_{\eta\in\mathcal{N}_{n-k+1}^+} \|d * \omega_k + \eta\|_{L^p_{n-k+1}(\Sigma)} \leq C\Big\{\| * d\omega_k\|_{L^p_{n-k-1}(\Sigma)} + \|\delta\omega_k\|_{L^p_{k-1}(\Sigma)}\Big\},$$

$$\inf_{\gamma\in\mathcal{N}_{n-k-1}^+} \| * d\omega_k + \gamma\|_{L^p_{n-k-1}(\Sigma)} \leq C\Big\{\|d\omega_k\|_{L^p_{k+1}(\Sigma)} + \|\delta\omega_k\|_{L^p_{k-1}(\Sigma)}\Big\}.$$

Suppose $b_{n-1}^- = 0$; the last inequality for $k = 0$ reads as follows:

$$\| * d\omega_0\|_{L^p_{n-1}(\Sigma)} \leq C\|d\omega_0\|_{L^p_1(\Sigma)} \tag{13}$$

for any scalar harmonic function ω_0. This is nothing but the Vishik–Mikhlin–De Vito formula (12).

We remark that, if $b_{n-1}^- \neq 0$, inequality (13) does not hold. Consider $\Omega = \{x \in \mathbb{R}^n : r < |x| < R\}$ and take

$$\omega_0(x) = \begin{cases} \log|x| & \text{if } n = 2 , \\ |x|^{2-n} & \text{if } n \geq 3 . \end{cases}$$

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On Maximal Regularity of Differential and Difference Operators

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Abstract In this paper we investigate a linear degenerate second-order difference operator and we find conditions that are sufficient for its bounded invertibility and separability in Hilbert space. We apply these results to prove the solvability of an infinite quasilinear difference system. We also give one result on the separability of its continuous analogue (a degenerate differential operator of second order) and show that the second-order discrete operator is separable under much weaker conditions.

1 Introduction

Let $h \in (0, h_0)$ (h_0 is a fixed positive number). We put $Z_h = \{x_j : x_j = jh, \quad \forall j \in Z\}$.

In what follows, instead of $a_{x_j} = a_{jh}$ we briefly write a_j. That is, for example, $\{a_j\}_{j=-\infty}^{+\infty} := \{a_{jh}\}_{j=-\infty}^{+\infty}$.

Let us consider the following operator:

$$l_0(h)y = -h^{-2}\Delta_h^{(2)}y + h^{-1}r\Delta_{h,-}y$$

acting on the space $l_2(h)$, where

$$l_2(h) = \left\{ y = \{y_j\}_{j=-\infty}^{+\infty} : \|y\|_{2,h} = \left(\sum_{j=-\infty}^{+\infty} |y_j|^2 h \right)^{\frac{1}{2}} < \infty \right\},$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_10

and $y = \{y_j\}_{j=-\infty}^{+\infty}$, $\Delta_{h,-}y = \{\Delta_{h,-}y_j\}_{j=-\infty}^{+\infty} = \{y_j - y_{j-1}\}_{j=-\infty}^{+\infty}$, $\Delta_h^{(2)}y = \left\{\Delta_h^{(2)}y_j\right\}_{j=-\infty}^{+\infty} = \{y_{j+h} - 2y_j + y_{j+h}\}_{j=-\infty}^{+\infty}$, and $r = diag\,\{r_j,\, j \in Z\}$ is a diagonal matrix.

l_0 is an unbounded operator. For l_0 we study such problems as the bounded invertibility of l_0, compactness and other spectral properties of l_0^{-1}, and the weighted estimates for the norms of elements of the domain $D(l_0)$.

The operator l_0 is the discrete analogue of the following differential operator:

$$\tilde{l} : L_2(R) \to L_2(R), \quad \tilde{l}y = -y'' + r(x)y', \quad R = (-\infty, +\infty).$$

So, $l_0 y = f$ and $\tilde{l}y = \tilde{f}$ are discrete and continuous representations, respectively, of one and the same mathematical model. The continuous model $\tilde{l}y = \tilde{f}$ and its three-member generalisation

$$Ay = -y'' + r(x)y' + q(x)y = F$$

have been mainly studied for a long time.

The operator A appears in the theory of stochastic processes and stochastic differential equations, in the study of the dynamics of a stratified compressible fluid and of the vibrational motion in media with resistance proportional to the velocity. The relevant information can be found in the works [1–7] and references therein.

Properties of the operator $Ay = -y'' + r(x)y' + q(x)y$ differ from the known ones of the Sturm–Liouville operator $-y'' + q(x)y$. For the operator A the growth of the function $|r|$ at infinity is important. If the growth of $|r|$ is weaker than the growth of some power of $|q|$, then the properties of A are the properties of the known cases of the Sturm–Liouville operator (see [8]).

Our purpose is to study another case: the case of A where $|r|$ is a fast increasing function. This case was studied relatively rarely.

The invertibility properties of the second-order degenerate differential operator A with unbounded intermediate coefficient r were studied in the papers [9] (the case $q = 0$) and [10] (the case of the space $L_1(R)$). The symmetric second- and higher-order differential operators without lower-order terms were investigated by A.G. Kostyuchenko, M.G. Gasymov, B.Ya. Skachek, M. Otelbaev, and O.D. Apyshev. The Sturm–Liouville difference operator was studied for several years (see, for example, [11, 12] and others).

The operator $l_0 y = -h^{-2}\Delta_h^{(2)}y + h^{-1}r\Delta_{h,-}y$ studied by us does not contain the free term. In addition, since the matrix r is not bounded, so the operator $h^{-1}r\Delta_{h,-}$ does not obey to $-h^{-2}\Delta_h^{(2)}$ in the operator sense. So, there are new difficulties in the study of l_0. One of them is that if the matrix r is bounded, then the domain $D(l_0)$ of l_0 may not belong to $l_2(h)$.

2 Main Results

We consider the operator $l_0 y = -h^{-2}\Delta_h^{(2)} y + h^{-1} r \Delta_{h,-} y$ defined in $D(l_0) = \Phi$, where $\Phi = \left\{ \{w_{jh}\}_{j=-\infty}^{+\infty} : \exists N,\ w_{jh} = 0,\ |j| \geq N \right\}$ is the set of the finite sequences.

Theorem 1 *Let $r_j = r_{jh}$ be the elements of the matrix r. Assume that $r_{jh} \geq 1$ $(j \in Z)$ and the following conditions hold:*

$$F^* := \sup_{n=0,1,2,\ldots} \left[n \sum_{j=n}^{+\infty} r_{jh}^{-1} \right] < \infty, \quad F^{**} := \sup_{k=-1,-2,\ldots} \left[(-k) \sum_{j=-\infty}^{k} r_{jh}^{-1} \right] < \infty,$$

then the operator l_0 is closable in $l_2(h)$.

We prove Theorem 1 by a standard method: for any sequence $\{v_n\}_{n=1}^{\infty} \subset \Phi$ such that $\|v_n\|_{2,h} \to 0$ and $\|l_0 v_n - w\|_{2,h} \to 0$ as $n \to +\infty$, we show that $w = 0$.

We denote by l the closure of l_0 in $l_2(h)$.

Theorem 2 *Let the matrix r satisfy the conditions of Theorem 1. Then the operator l is invertible, and its inverse l^{-1} is defined on all of $l_2(h)$. Furthermore, for any $y \in D(l)$ the following inequality holds:*

$$\left\| -\Delta_h^{(2)} y \right\|_{2,h} + \left\| r \Delta_{h,-} y \right\|_{2,h} + \|y\|_{2,h} \leq C(h) \|ly\|_{2,h}. \tag{1}$$

Sketch of Proof

(a) We transform the scalar product $\langle l_0 y, \Delta_{h,-} y \rangle$, where $y = \{y_j\}_{j=-\infty}^{+\infty} \in \Phi$, obtaining the following inequality:

$$\| -\Delta_h^{(2)} y\|_{2,h}^2 = \sum_{j=-\infty}^{+\infty} (-\Delta_h^{(2)} y_j)^2 h \leq C_1^2(h) \|l_0 y\|_{2,h}^2.$$

From this estimate by presentation of ly, we obtain that

$$\sum_{j=-\infty}^{+\infty} r_j^2 (\Delta_{h,-} y_j)^2 h \leq C_2^2(h) \|l_0 y\|_{2,h}^2.$$

And using conditions of Theorem 1, by some weighted Hardy type inequality, we receive the estimate $\|y\|_{2,h} \leq C_3(h) \|l_0 y\|_{2,h}$.

By the definition of l, from last inequalities it follows (1) for any $y \in D(l)$.

(b) From the estimate (1) follows that there exists the inverse l^{-1} of l.

(c) We prove the boundedness of l^{-1} using the definition of l and the general theory of linear operators. The theorem is proved.

The estimate (1) is exact. Indeed, if (1) holds, then $y \in D(l)$ is an element of the discrete Sobolev space $w^2_{2,h}(r)$ with the norm

$$\|y\|_w = \|\Delta_h^{(2)} y\|_{2,h} + \|r\Delta_{h,-} y\|_{2,h} + \|y\|_{2,h}.$$

Then $D(l) \subset w^2_{2,h}(r)$. On the other hand, it is clear that $w^2_{2,h}(r) \subset D(l)$. So, $D(l) = w^2_{2,h}(r)$.

The inequality (1) is called the coercive estimate, or the separability estimate for l.

Theorem 3 *Let r satisfy the conditions of Theorem 1 and the following conditions:*

$$\lim_{n\to+\infty}\left[n\sum_{j=n}^{+\infty} r_{jh}^{-2}\right] = 0, \quad \lim_{k\to-\infty}\left[(-k)\sum_{j=-\infty}^{k} r_{jh}^{-2}\right] = 0.$$

Then the resolvent l^{-1} is a completely continuous operator in $l_2(h)$.

Proof By Theorem 2, $D(l) = w^2_{2,h}(r)$. So, l^{-1} is a continuous operator from $l_{2,h}$ to $w^2_{2,h}(r)$. By the conditions of Theorem 3 and the results of the paper [13], we obtain that the space $w^2_{2,h}(r)$ is compactly embedded in $l_2(h)$. It means that l^{-1} is a compact operator. The theorem is proved.

Remark 1 The statements of Theorems 1–3 are fulfilled if $r_j = r_{jh} \le -1 \; \forall j \in Z$.

Example 1 Let $h = 1$. We consider the operator

$$\tilde{l}_0 y = -\Delta^{(2)} y + r\Delta_- y,$$

where $y = \{y_j\}_{j=-\infty}^{\infty} \in \Phi$, $r = diag\{r_j = \sqrt{1+j^2}, \quad j \in Z\}$, $\Delta^{(2)} = \Delta_1^{(2)}$, and $\Delta_- = \Delta_{1,-}$.

It is easy to see that the conditions of Theorem 1 are fulfilled. Then by Theorems 1 and 2, the operator $\tilde{l}_0$ is closable in the space l_2, and its closure, which we denote by $\tilde{l}$, is bounded invertible. Moreover, for any $y = \{y_j\}_{j=-\infty}^{\infty} \in D(\tilde{l})$ the following estimate holds:

$$\left\{\sum_{j=-\infty}^{+\infty} \left|\Delta^{(2)} y_j\right|^2\right\}^{1/2} + \left\{\sum_{j=-\infty}^{+\infty} \left|\sqrt{1+j^2}\Delta_- y_j\right|^2\right\}^{1/2} +$$

$$+\left\{\sum_{j=-\infty}^{+\infty} |y_j|^2\right\}^{1/2} \le C_4 \left\{\sum_{j=-\infty}^{+\infty} \left|(\tilde{l}y)_j\right|^2\right\}^{1/2}.$$

So, the operator $\tilde{l}$ is separable.

Example 2 Let us consider the operator $m_0 y = \{(m_0 y)_j\}_{j=-\infty}^{\infty}$, where $y = \{y_j\}_{j=-\infty}^{\infty} \in \Phi$ and $(m_0 y)_j = \Delta^{(2)} y_j + (1+j^2)\Delta_- y_j$, $j \in Z$.

We see that here $h = 1$, and r is the diagonal matrix with elements $r_j = 1 + j^2$. Then it is easy to show that the conditions of Theorem 1 hold. So, the operator m_0 is closable in l_2, and if we denote by m its closure, then m is bounded invertible. For $y = \{y_j\}_{j=-\infty}^{\infty} \in D(m)$ the following estimate holds:

$$\|\Delta^{(2)} y\|_2 + \|(1+j^2)\Delta_- y\|_2 + \|y\|_2 \le C_5 \|my\|_2.$$

Here $\|\cdot\|_2 = \|\cdot\|_{2,1}$, moreover

$$\lim_{s\to+\infty}\left[n\sum_{j=n}^{+\infty}\left(1+j^2\right)^{-2}\right] = 0, \quad \lim_{k\to-\infty}\left[(-k)\sum_{j=-\infty}^{k}\left(1+j^2\right)^{-2}\right] = 0.$$

So, by Theorem 3, the resolvent m^{-1} of m is a compact operator in l_2.

3 Some Applications

Now, we give some statements, which are proved using Theorems 1–3.

First we consider the following “three-member operator”:

$$L_0 y = -h^{-2}\Delta_h^{(2)} y + h^{-1} r \Delta_{h,-} y + qy, \quad y \in \Phi,$$

where $q = diag\left\{q_j : q_j = q_{jh},\ j \in Z\right\}$ is a diagonal matrix. We denote

$$A_{q,r}(s) = \left(\sum_{k=0}^{s} q_{kh}^2\right)^{\frac{1}{2}} \left(\sum_{k=s}^{+\infty} r_{kh}^{-2}\right)^{\frac{1}{2}}, \quad s = 0,\ 1,\ 2,\ \dots,$$

$$B_{q,r}(\tau) = \left(\sum_{j=\tau}^{-1} q_{jh}^2\right)^{\frac{1}{2}} \left(\sum_{j=-\infty}^{\tau} r_{jh}^{-2}\right)^{\frac{1}{2}}, \quad \tau = -1,\ -2,\ \dots.$$

Theorem 4 *Let r satisfy the conditions of Theorem 1 and the matrix q be such that*

$$\max\left[\sup_{s=0,\,1,\,2,\,\dots} A_{q,r}(s), \quad \sup_{\tau=-1,\,-2,\,\dots} B_{q,r}(\tau)\right] < \infty.$$

Then

(a) *L_0 is a closable operator in $l_2(h)$;*
(b) *the closure L of L_0 is bounded invertible (i.e. L is invertible and its inverse L^{-1} is defined on all of $l_2(h)$);*
(c) *for any $y \in D(L)$ the following estimate holds:*

$$\left\|-\Delta_h^{(2)} y\right\|_{2,h} + \left\|r\Delta_{h,-}y\right\|_{2,h} + \|qy\|_{2,h} \le C_6(h)\,\|Ly\|_{2,h}\,,$$

i.e. L is a separable operator.

To prove Theorem 4 we use Theorems 1 and 2, a weighted Hardy type inequality, and known results on the perturbations of linear operators (see, in addition, [14]).

Theorem 5 *Assume that r and q satisfy the conditions of Theorem 4. Let one of the following conditions (2) and (3) hold:*

$$\lim_{s\to+\infty} A_{q,r}(s) = 0, \qquad \lim_{\tau\to-\infty} B_{q,r}(\tau) = 0, \tag{2}$$

$$\lim_{|m|\to+\infty} q_m = +\infty. \tag{3}$$

Then the resolvent L^{-1} is a completely continuous operator in $l_2(h)$.

This theorem follows from Theorem 4 by results of [15].

Now we consider the following nonlinear system:

$$\tilde{L}_0 y = -\Delta_h^{(2)} y + r(y)\Delta_{h,-}y = f, \tag{4}$$

where $r(y) = diag\left\{r_{jh}(y),\, j \in Z\right\}$ is a diagonal matrix, which depends on the unknown element y, and $f = \{f_{jh}\}_{j=-\infty}^{+\infty} \in l_2(h)$.

Definition 1 $y \in l_2(h)$ is called a solution of equation (4), if there exists a sequence $\{w_n\}_{n=1}^{+\infty} \subset \Phi$ such that $\|\psi(w_n - y)\|_2 \to 0$, $\|\psi(\tilde{L}_0 w_n - f)\|_{2,h} \to 0$ $(n \to +\infty)$ for any $\psi = diag\{\psi_{jh},\ \ j \in Z\}$ $\ \ (\{\psi_{jh}\}_{j=-\infty}^{+\infty} \in \Phi)$.

Theorem 6 *Let $r_{jh}(v)$ $\ (j \in Z)$ be continuous with respect to v and satisfy the following inequality:*

$$r_{jh}(v) \ge 1 + j^2.$$

Then for any $f \in l_2(h)$ there exists a solution y of the nonlinear system (4) such that

$$\| - \Delta_h^{(2)} y\|_{2,h} + \|r(y)\Delta_{h,-}y\|_{2,h} + \|y\|_{2,h} < +\infty. \tag{5}$$

Proof We consider the following linearised system:

$$M_v y = -\Delta_h^{(2)} y + r(v)\Delta_{h,-}y = f, \quad f \in l_2(h),$$

where v belongs to some closed ball T in the space of bounded sequences with a specially selected radius. By Theorem 2 for any $f \in l_2(h)$ there exists a unique solution of this equation. We introduce the operator P putting $Pv = M_v^{-1} f$, where f is fixed. By Theorem 6, we obtain that P is a compact operator in the Banach space of bounded sequences and $P(T) \subset T$. Then by the known Schauder theorem there exists a fixed point u of P, $Pu = u$. We show that u is a solution of the system (4) and for u holds (5). The theorem is proved.

Example 3 Let $h = 1$. We consider the following nonlinear system:

$$-\Delta^{(2)} y_j + \left(3 + j^4 + \sum_{k=j-5}^{j+5} y_k^2\right) \Delta_- y_j = f_j \ \ (j \in Z).$$

It is easy to see that for this system the conditions of Theorem 6 are fulfilled. So, for any $f = \{f_j\}_{j=-\infty}^{+\infty} \in l_2$ there exists a solution $y = \{y_j\}_{j=-\infty}^{+\infty}$ of this system and

$$\sum_{j=-\infty}^{+\infty} \left(-\Delta^{(2)} y_j\right)^2 + \sum_{j=-\infty}^{+\infty} \left[\left(3 + j^4 + \sum_{k=j-5}^{j+5} y_k^2\right) \Delta_- y_j\right]^2 + \|y\|_2 < +\infty.$$

Now we give the separability condition for the differential operator $Ay = -y'' + r(x)y' + q(x)y$. We assume that $D(A)$ is the set $C_0^{(2)}(R)$ of twice continuously differentiable functions with compact support.

For given $g(x)$ and $h(x)$, we put

$$\alpha_{g,h}(t) := \|g\|_{L_2(0,\,t)} \left\|h^{-1}\right\|_{L_2(t,\,+\infty)} \quad (t > 0),$$

$$\beta_{g,h}(\tau) := \|g\|_{L_2(\tau,\,0)} \left\|h^{-1}\right\|_{L_2(-\infty,\,\tau)} \ (\tau < 0),$$

$$\gamma_{g,h} := \max\left(\sup_{t>0} \alpha_{g,h}(t),\ \sup_{\tau<0} \beta_{g,h}(\tau)\right).$$

The following theorem was proved in the work [16].

Theorem 7 *Let r be a continuously differentiable function, and q be a continuous function satisfying the following conditions:*

i) $|r| \geq 1,\ \gamma_{1+|q|,\sqrt{|r|}} < \infty$,
ii)

$$\sup_{x,z\in R:|x-z|\leq 1} \frac{r(x)}{r(z)} < +\infty. \tag{6}$$

Then

(a) *the operator A is closable in $L_2(R)$;*
(b) *the closure $\bar{A}$ of A is invertible and its inverse $\bar{A}^{-1}$ is defined on the whole space $L_2(R)$. Moreover, for any $y \in D(\bar{A})$ the following coercive estimate holds:*

$$\| - y''\|_2 + \|ry'\|_2 + \|qy\|_2 \le C_0\|\bar{A}y\|_2,$$

where $\| \cdot \|_2$ is the norm in $L_2(R)$.

We see that there is a difference between Theorems 7 and 4. The difference is that Theorem 7 has an additional condition (6) on r. The expression (6) is a condition on the oscillation of r. In addition, the condition (6) holds for the function $r_1(x) = (1 + x^2)^s \quad (s > 0)$. Note that the following strongly oscillating function $r_2(x) = (1 + x^2 sin^2 x)$ does not satisfy the condition (6).

The main result of this work is Theorem 4 on the operator $Ly = -\Delta_h^{(2)} y + r\Delta_{h,-}y + qy$. With respect to this theorem we add some comments.

1. Everitt, Giertz, and Waidmann in 1976 gave an example of a strongly oscillating function $q(x) \ge 1$, such that the Sturm–Liouville operator $Sy = -y'' + q(x)y$ is not separable, i.e. there is no C such that the estimate $\| - y''\|_2 + \|qy\|_2 \le C(\|Sy\|_2 + \|y\|_2)$ holds. Later a number of works were published, where different sufficient conditions are imposed on the oscillation of the function $q(x)$ such that S is a separable operator in $L_2(R)$.

   If $r(x)$ is a rapidly increasing function, then by Theorem 7, the three-member operator $Ay = -y'' + r(x)y' + q(x)y$ also is separable in $L_2(R)$, only if some conditions on the oscillation of $r(x)$ hold.
2. In 1984, Otelbaev and Grinshpun showed that, for the separability of the Sturm–Liouville operator $Sy = -y'' + q(x)y$ in the space $L_1(R)$ only the following natural condition $q(x) \ge 1$ is sufficient. In this case we say that S is an unconditionally separable operator. The discrete Sturm–Liouville operator $SDy = -\Delta_h^{(2)} y + ry$, where $r = diag\{r_j, j \in Z\}, \quad r_j \ge 1$, is a diagonal matrix and also is an unconditionally separable operator in the space l_1 [17].
3. At the same time, Theorem 4 shows that the difference operator $Ly = -\Delta_h^{(2)} y + r\Delta_{h,-}y + qy$ is a separable operator, already in the space $l_2(h)$, if the drift coefficient r is increasing and can strongly fluctuate, and q obey to them by the condition

$$\max \left[\sup_{s=0,\,1,\,2,\,\dots} A_{q,r}(s), \quad \sup_{\tau=-1,\,-2,\,\dots} B_{q,r}(\tau) \right] < \infty.$$

Acknowledgement This research was supported by the grant AP05131649 of the Ministry of Education and Science of the Republic of Kazakhstan.

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On the Generalized Liouville Theorem

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Abstract In this paper a generalization of the classical Liouville theorem for the solutions of special type elliptic systems and some nonclassical interpretations of this theorem are obtained.

1 Introduction

It is well known that the study of the real-world physical, technical, biological, and economical processes is actually reduced to the finding and investigation of the relations between given and desired functional values. These connections are often expressed by (ordinary as well as partial) differential equations and systems. The most important direction of investigation of the obtained differential equations is the construction of their solutions. It should be mentioned that the effective construction of the general solution in most cases is not possible and if possible, then the obtained results are less valuable in view of the investigation of the original problem. Therefore for the obtained equations the problem of the construction of the solutions, directly describing the studied processes, becomes actual. As it is well known in order to single out the concrete solution from the general class of solutions the so-called initial-boundary conditions are used, and if the mentioned additional conditions play the role of natural filters, then we get correctly posed boundary

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_11

problem. In purely theoretical (and also practical) view it is undoubtedly interesting to investigate the intrinsic qualitative and quantitative relation between the equations (systems of equations) and additional initial-boundary conditions, which constitute correctly posed boundary problems. It is evident that the mentioned intrinsic quantitative relation, in contrast to the qualitative, always exists. To illustrate this, note that in the classical Dirichlet problem there is a deep connection between the Laplace equation and the Dirichlet boundary condition, while the quantitative relation cannot be observed, because the Laplace equation has got no quantitative (numerical) parameters at all. Our work deals with the generalization of the classical Liouville theorem of complex analysis for a sufficiently wide class of elliptic systems. The most important result is the solution of the abovementioned problem. More precisely, the classical Liouville theorem can be interpreted in a somewhat specific way: all solutions of the classical Cauchy–Riemann system (1) satisfying the asymptotic condition (2) have the form (3) (see below). It is clear that there is no quantitative relation between the system (1) and the boundary condition (2), while the essential qualitative relation is evident (compare with the classical Dirichlet problem). One of the obtained results is that for a sufficiently wide generalization of the system (1) a generalization of the boundary condition (2) was found such that, on the one hand, we get a complete analogy of the classical Liouville theorem, and on the other, what is more important, the rigid quantitative connection between the generalized elliptic system and additional boundary condition is clearly visible. The following question is quite natural: why is the relation between the elliptic system (the Cauchy–Riemann system) and the corresponding boundary condition in case of the classical Liouville theorem not explicit? That is our point: that in this important theorem this relation is hidden. It appears in our generalization of the mentioned theorem. This work consists of two sections. In the first section a brief historical overview of the issue, some well-known facts, formulation of the problem, and some theorems, a priori related to Liouville type theorems are presented. In the second section the analogues of the Liouville type theorems and their proofs are studied.

2 Background

1^0 In 1847 Cauchy proved one of the fundamental regularities of complex analysis—every non-constant entire function is unbounded. This most important characteristic property of entire functions is referred in the literature as classical Liouville theorem (the only bounded entire functions are the constant functions). The most important applications of this theory (from purely theoretical as well as from practical point of view) in order to investigate actual problems of mathematical analysis are well known. From this theorem directly follows that an entire function $u + iv = f(z)$ of the complex variable $z = x + iy$, i.e., if the solution in classical sense of the first order differential system (Cauchy–Riemann system) on the whole complex plane C

$$\begin{aligned}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}&=0,\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}&=0\end{aligned}\tag{1}$$

satisfies the asymptotic condition

$$f(z)=O(z^N),\quad z\to\infty,\tag{2}$$

for some non-negative number N, then it is a polynomial of the variable z at most of order N

$$f(z)=a_0z^N+a_1z^{N-1}+\cdots+a_N,\tag{3}$$

where $a_0, a_1, \dots, a_N$ are complex constants. This result is sometimes referred to as Liouville theorem also. In other words the general solution of problem (1), (2) is given by formula (3). More significant is the fact that there is no connection between the parameter N, mentioned in the theorem and the system (1).

As we will see below, a somewhat unexpectedly important role plays the following trivial conclusion ensuing from Liouville theorem: an entire (analytic) bounded function of a complex variable is either nowhere equal to zero or identically equal to zero.

2⁰ The abovementioned theorems for analytical functions are enhanced and generalized by different authors in different directions. One of them is the most important direction of contemporary complex analysis, the theory of generalized analytic functions. In the classical monograph [1] of the Georgian founder of this theory, I. Vekua, are collected results of systematized authors, his disciples, and followers during many years of their investigations. One of them is the analogue of classical Liouville theorem for generalized analytic functions. Let some notations be introduced before formulating the corresponding result.

3⁰ Let us fix on the complex C plane a pair of regular coefficients (functions) A, B from the $L_{p,2}$, $p>2$, class (by definition the class $L_{p,2}$ consists of all functions U, defined on the entire plane, satisfying the conditions

$$\iint_G |U(\xi)|^p\,dG<\infty,\quad \iint_G \frac{1}{|z|^{2p}}\left|U\left(\frac{1}{z}\right)\right|^p dG<\infty,$$

where $G=\{|z|<1\}$ is a unit disc) and consider the Carleman–Vekua equation

$$\partial_{\overline{z}}\omega+A\omega+B\overline{\omega}=0,\tag{C–V}$$

where the operator $\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ is understood in generalized (Sobolev) sense. The generalized solutions of this equation are called generalized analytic functions [1].

The general representation of generalized analytic functions through analytic functions is the strongest tool in order to investigate the solutions of the equation (C–V). In particular, there exists the formula

$$\omega = f \exp\{T\}, \tag{4}$$

where f is an arbitrary entire function and the function $T = T(z)$ is evaluated by the formula

$$T(z) = -\frac{1}{\pi}\int\int_{C}\left(A(\xi) + B(\xi)\frac{\overline{\omega(\xi)}}{\omega(\xi)}\right)\frac{1}{\xi - z}\,dC_{\xi}.$$

The obtained representation of the solutions by means of entire functions is crucial for the solutions of (C–V) in order to get the analogue of the classical Liouville theorem. By the regularity of the coefficients of the equation, the factor $\exp\{T\}$ is continuous on the whole plane, never equals zero and $\exp\{T\} \to 1$ as $z \to \infty$.

Respectively, if the coefficients of equation (C–V) are regular, if the solution ω is bounded, and if it is zero at some point of the plane, then it is identically zero (in the abovementioned monograph [1] this statement is called as analogue of the classical Liouville theorem; besides, in the same monograph a very interesting geometric interpretation of this result is given). Thus, every bounded solution of the equation (C–V) with regular coefficients has exactly the same property as an entire function of a complex variable, in particular there is an alternative: a solution is not zero anywhere or identically equal to zero. At the same time one principal difference should be mentioned. A bounded entire function is constant, whereas a bounded solution of the (C–V) equation does not need to be constant. This makes it complicated to describe effectively the solutions of the equation (C–V) with $O(z^N)$ asymptotic at infinity, in particular to obtain a representation of type (3).

As was mentioned above for the solution of the equation (C–V) an obtained analogue of the classical Liouville theorem is essentially based on the principal limitation for the coefficients A and B of the equation (C–V)—they must be regular. Theoretically (and also very important for the analysis of applied problematics) the largest interest is attracted to find analogues of the classical Liouville theorem for the equation (C–V) when these coefficients are not regular. The most notable seems to be the simplest case of the coefficients

$$A = const \neq 0, \quad B = const \neq 0.$$

Exactly for these coefficients V. Vinogradov has obtained (probably for the first time in the literature in this direction) the most important results related with the Liouville type theorem (see [2]). For the equations (C–V) with non-regular coefficients the Liouville type theorems are obtained in the works of several authors, among them are the works [3, 4].

Note that for the Liouville theorem the fact that the parameter n is equal to 1 or is greater than 1, i.e., (4) for one complex equation or a system of complex equations is essential. In particular, K. Habetha (see [5]) has constructed a system for which the Liouville theorem is violated because it has a solution that is non-trivial, continuous on whole plane and is zero at infinity.

In the present paper (see Sect. 3 below) sufficiently wide classes of elliptic singular (with irregular-coefficients) systems (the number of equations and unknown functions is an arbitrary natural number n) are also studied and for them the generalizations of the classical Liouville type theorem are obtained.

3 Generalized Liouville Theorem

1^0 Consider the following system of first order differential equations of the complex variable $z = x + iy$ on C

$$\begin{aligned} \partial_{\overline{z}}\omega_1 &= \alpha_{11}\omega_1 + \alpha_{12}\omega_2 + \cdots + \alpha_{1n}\omega_n, \\ \partial_{\overline{z}}\omega_2 &= \alpha_{21}\omega_1 + \alpha_{22}\omega_2 + \cdots + \alpha_{2n}\omega_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \partial_{\overline{z}}\omega_n &= \alpha_{n1}\omega_1 + \alpha_{n2}\omega_2 + \cdots + \alpha_{nn}\omega_n, \end{aligned} \tag{5}$$

where $\partial_{\overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$, n is a natural number,

$$\alpha_{kp} = \rho(z)\alpha_{k,p}, \quad 1 \le k, p \le n,$$

where $\alpha_{k,p}$ are complex numbers; $\rho(z)$ is a given function

$$\rho(z) = |z|^{\nu} \exp\{im \arg z\}, \tag{6}$$

m is an integer and $\nu > 0$ is a real number; $\omega_k(z)$, $1 \le k \le n$, are unknown functions.

Obviously, the system of equations (5) is elliptic and under a solution we mean a system of functions $\omega_1, \omega_2, \ldots, \omega_n$ of the class $C^1(C)$ satisfying the equalities (5) for every finite point on the complex plane C.

Rewrite the system (5) in the following matrix form:

$$\partial_{\overline{z}} W = \rho(z) A W, \tag{7}$$

where

$$A = \left(\alpha_{i,k}\right)_{n\times n}, \quad W = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}. \tag{8}$$

Everywhere below we assume that

$$\det A \neq 0, \tag{9}$$

and the numerical parameters m and ν satisfy the condition

$$m \neq 1, \quad |m-1| \neq \nu + 1. \tag{10}$$

Denote by T a nondegenerate matrix, for which the matrix $B = T^{-1}AT$ has the following Jordan canonical form:

$$B = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots\cdots & 0 & 0 & 0 \\ b_1 & \lambda_2 & 0 & \cdots\cdots & 0 & 0 & 0 \\ 0 & b_2 & \lambda_3 & \cdots\cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \cdots\cdots & b_{n-2} & \lambda_{n-1} & 0 \\ 0 & \cdots & \cdots & \cdots\cdots & 0 & b_{n-1} & \lambda_n \end{pmatrix}. \tag{11}$$

The diagonal elements of the matrix B are the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_3$ of the matrix A (it is possible that some of them are equal). All the remained (non-diagonal) elements of the matrix B are equal to zero, perhaps except of those elements that are located left to the diagonal elements. Assume also that the following inequalities are fulfilled:

$$|\lambda_n| \leq |\lambda_{n-1}| \leq \cdots \leq |\lambda_1|. \tag{12}$$

From the conditions (10) it follows that $\nu - m + 2 \neq 0$, correspondingly by virtue of condition (9) and by means of the following formulas:

$$\tau = \frac{2|\lambda_n|}{|\nu - m + 2|}, \quad \delta_k = \frac{2\lambda_k}{\nu - m + 2}, \quad k = 1, 2, \ldots, n, \tag{13}$$

non-zero numbers are defined. Consider the system of functions

$$\omega_k = \delta_k |z|^{\nu+1} \exp\{i(m-1)\arg z\}, \quad k = 1, 2, \ldots, n. \tag{14}$$

Denote by $\Omega(N,\delta,\sigma)$ the set of all vector-functions $W(z)$ representing the solutions of the system (7) and satisfying the asymptotic conditions at infinity

$$\max_{1\le k\le n} |W_k(z)| = O\left(|z|^N \exp\left\{\delta|z|^\sigma\right\}\right), \quad z\to\infty. \tag{15}$$

Here N is a non-negative integer, δ and σ are non-negative real numbers. It is clear that for each fixed numbers N, δ, and σ, $\Omega(N,\delta,\sigma)$ represents a linear vector space over the complex numbers field. The dimension of this vector space is investigated below.

For arbitrary non-negative integer N the following theorems are valid:

Theorem 1 $\dim\Omega(N,\delta,\nu+1) = 0$ *if* $\delta < \tau$.

Theorem 2 $\dim\Omega(N,\delta,\sigma) = 0$ *if* $\sigma < \nu+1$, $\delta \ge 0$.

Theorem 3 $\dim\Omega(N,\delta,\nu+1) = \infty$ *if* $\sigma > \tau$.

Theorem 4 $\dim\Omega(N,\delta,\sigma) = \infty$ *if* $\sigma > \nu+1$, $\delta > 0$.

Proof Let $W(z)\in\Omega(N,\delta,\nu+1)$ for some number $\delta<\tau$ then the vector-function

$$Y \equiv \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix} = T^{-1}W \tag{16}$$

(T^{-1} is the matrix that transforms the matrix A in its canonical Jordan form) is the solution of the system

$$\begin{aligned} \partial_{\bar z}Y_1 &= \lambda_1\rho(z)Y_1, \\ \partial_{\bar z}Y_2 &= \lambda_2\rho(z)Y_2 + b_1\rho(z)Y_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \partial_{\bar z}Y_n &= \lambda_n\rho(z)Y_n + b_{n-1}\rho(z)Y_{n-1}, \end{aligned} \tag{17}$$

satisfying the asymptotic condition

$$\max_{1\le k\le n} |Y_k(z)| = O\left(|z|^N \exp\left\{\delta|z|^{\nu+1}\right\}\right), \quad z\to\infty. \tag{18}$$

Direct checking shows that the following equality takes place

$$\partial_{\bar z}\omega_1 = \lambda_1\rho(z),$$

by virtue of which the general solution of the first equation of the system (15) is given by the formula

$$Y_1(z) = \Phi_1 \exp\{\omega_1(z)\}, \tag{19}$$

where $\Phi_1(z)$ is an entire function.

Using (16) we get the following estimation for the entire function $\Phi_1(z)$:

$$\Phi_1(z) = O\Big(|z|^N \exp\big\{\delta|z|^{\nu+1} - \mathrm{Re}\,\omega_1(z)\big\}\Big), \quad z \to \infty. \tag{20}$$

From (18) it is clear that $\frac{\Phi_1(z)}{z^N}$ satisfies the conditions

$$\frac{\Phi_1(z)}{z^N} = O\Big(\exp\big\{2|\delta_1|\,|z|^{\nu+1}\big\}\Big), \quad z \to \infty, \tag{21}$$

and is bounded on every ray

$$\big\{z : \ \arg z = \psi, \ |z| \geq r_0\big\}, \tag{22}$$

where ψ is an arbitrary solution of the equation

$$\cos\big[(m-1)\psi + \arg\delta_1\big] - \frac{\delta}{|\delta_1|} = 0 \tag{23}$$

and r_0 is a sufficiently large number.

By virtue of the conditions $\delta < \tau \leq |\delta_1|$, $m \neq 1$, $|m-1| \neq \nu + 1$ and the abovementioned facts we obtain that $\Phi_1 \equiv 0$ and therefore $Y_1 \equiv 0$. Hence from the second equation of system (15) (similarly to the above considerations) we conclude that $\Phi_2 \equiv 0$, etc. So finally we obtain $Y_1 \equiv Y_2 \equiv \cdots \equiv Y_n \equiv 0$ and thus $W(z) \equiv 0$, i.e., Theorem 1 is proved.

Proof of Theorem 2 It follows directly from Theorem 1 by taking into account that under the conditions of Theorem 2 the following inclusion:

$$\Omega(N, \delta, \sigma) \subset \Omega(N, \delta, \nu + 1)$$

holds.

To *prove Theorem 3* it is sufficient to construct a linearly independent system of vector-functions of the space $\Omega(N, \delta, \nu + 1)$, $\delta > \tau$. For this purpose take the following system of vector-functions:

$$W^{(k)}(z) = T^{-1} Y^{(k)}(z), \quad k = 0, 1, 2, \ldots, \tag{24}$$

where

$$Y^{(k)}(z) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ z^k \exp\{\omega_n(z)\} \end{pmatrix}.$$

It is obvious that any arbitrary finite subsystem of the constructed system of functions is linearly independent and therefore the theorem is proved.

The *proof of Theorem 4* directly follows from Theorem 3 by taking into account that under the conditions of Theorem 3 the following inclusion:

$$\Omega(N, \delta, \nu + 1) \subset \Omega(N, \delta, \sigma)$$

holds.

From the above obtained results it follows that the system (5) has only the trivial solution satisfying

$$\max_{1 \le k \le n} |W_k(z)| = O(|z|^N), \quad z \to \infty, \tag{25}$$

for an arbitrary real number N.

2^0 In this subsection for one of the classes of systems of type (5) explicit values of parameters δ and σ, for which the solutions space $\Omega(N, \delta, \sigma)$ is non-trivial and finite-dimensional are obtained.

For this the inequality

$$|m - 1| > 2(\nu + 1), \tag{26}$$

instead of conditions (9) should be fulfilled and all eigenvalues of the matrix A should have one and the same modulus. In addition, we demand that the matrix A should have a so-called simple structure, i.e., A is similar to a diagonal matrix. Various criteria which guarantee simple structure for the matrix A are well known, in particular an n-dimensional square matrix is similar to a diagonal matrix if and only if it has n numbers of linearly independent eigenvectors. Besides, it is known that the matrix A has a simple structure if and only if the algebraic multiplicity of every of its eigenvalues (i.e., the multiplicity of the root of the characteristic equation $\det[A - \lambda E] = 0$, where E is the unit matrix of the corresponding dimension) coincides with the geometric multiplicity of the same eigenvalue (i.e., with the number of linearly independent eigenvectors corresponding to the eigenvalue of the matrix A).

The following theorem holds.

Theorem 5 *If the condition (24) is satisfied and the matrix A has a simple structure, then for any integer non-negative number N*

$$\dim \Omega(N, \tau, \nu + 1) = n(N + 1); \tag{27}$$

besides, the following system of vector-functions

$$W^{(k,i)}(z) = T^{-1} Y^{(k,i)}(z), \quad k = 0, 1, 2, \dots, N; \quad i = 1, 2, \dots, n, \tag{28}$$

is a basis for the space $\Omega(N, \delta, \nu + 1)$, *where* T^{-1} *is matrix that transforms the matrix A into a diagonal type matrix and*

$$Y^{(k,1)}(z) = \begin{pmatrix} z^k \exp\{\omega_n(z)\} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad Y^{(k,2)}(z) = \begin{pmatrix} 0 \\ z^k \exp\{\omega_n(z)\} \\ \vdots \\ 0 \end{pmatrix}, \dots,$$

$$Y^{(k,n)}(z) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ z^k \exp\{\omega_n(z)\} \end{pmatrix}, \tag{29}$$

where $k = 0, 1, 2, \dots, N$.

The proof is similar to the proof of Theorem 1 with the difference that according to the conditions of Theorem 5, the relation (18) does not provide the triviality of the entire functions $\Phi_1(z), \Phi_2(z), \dots, \Phi_n(z)$. They are polynomials of the complex variable z at most of order N.

Note that if the inequality (26) is not fulfilled, then in general the equality (25) is not valid, in particular if

$$|m - 1| < \nu + 1, \tag{30}$$

then

$$\dim \Omega(N, \tau, \nu + 1) = \infty. \tag{31}$$

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Neumann Problem in Polydomains

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Abstract In this presentation, we will discuss the Neumann problem for higher-order model equations in the unit polydisc of $\mathbb{C}^2$. We derive the integral representations of the functions defined in the unit polydisc of $\mathbb{C}^2$ which may particularly be suitable for Neumann problems.

1 Introduction

Boundary value problems in $\mathbb{C}^n$, $n \geq 1$, attracted many researchers in the last several decades. Riemann and Riemann–Hilbert problems and their particular cases known as Schwarz, Dirichlet, Neumann, and Robin problems have been investigated by many researchers in the case of $n = 1$ [1, 3–7, 9–11, 13, 14, 21, 22] and $n \geq 2$ [8, 16–20].

In this presentation we will concentrate on Neumann problems in $\mathbb{C}^2$ which can be easily generalized into $\mathbb{C}^n$, $n \geq 2$. In the case of $n = 1$, Neumann problem is discussed for the model equations, that is, for the equations having the leading parts with holomorphic and harmonic operators in several different types of domains.

In the next section of this paper, we summarize some relevant information for the problems in $\mathbb{C}$. The last section is reserved for integral representations of functions in $\mathbb{C}^2$.

2 Preliminaries

In this section we collect results obtained previously for the Neumann function in $\mathbb{C}$ [3, 12, 15, 22, 23].

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_12

The Neumann function for the unit disc $\mathbb{D}$ is given by

$$N_1(z,\zeta) = -\log|(\zeta - z)(1 - z\bar{\zeta})|^2 \tag{1}$$

for $z, \zeta \in \mathbb{D}$, $z \neq \zeta$, [12] which is slightly different from the one given previously [15, 22, 23]. (1) satisfies

$$\partial_{\nu_z} N_1(z,\zeta) = (z\partial_z + \bar{z}\partial_{\bar{z}})N_1(z,\zeta) = -2 \tag{2}$$

for $z \in \partial\mathbb{D}$, $\zeta \in \mathbb{D}$. A second order Neumann function [5, 7] for z and ζ in $\bar{\mathbb{D}}$ with $z \neq \zeta$ is given by

$$N_2(z,\zeta) = |\zeta - z|^2 \left[\log|(\zeta - z)(1 - z\bar{\zeta})|^2 - 4 + 4\sum_{k=2}^{\infty} \frac{1}{k^2}[(z\bar{\zeta})^k + (\bar{z}\zeta)^k]\right]$$

$$+2(z\bar{\zeta} + \bar{z}\zeta)\log(1 - z\bar{\zeta})^2 + (1 + |z|^2)(1 + |\zeta|^2)\left[\frac{\log(1 - z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1 - \bar{z}\zeta)}{\bar{z}\zeta}\right].$$

Nevertheless, the higher-order Neumann functions are not easy to derive in their explicit forms but they may be defined iteratively for $n \in \mathbb{N}$ where $n \geq 2$, as

$$N_n(z,\zeta) = -\frac{1}{\pi}\iint_{\mathbb{D}} N_1(z,\tilde{\zeta})N_{n-1}(\tilde{\zeta},\zeta)\,d\tilde{\xi}\,d\tilde{\eta}\,. \tag{3}$$

These functions satisfy

$$\partial_z\partial_{\bar{z}} N_n(z,\zeta) = N_{n-1}(z,\zeta) \tag{4}$$

in $\mathbb{D}$,

$$\partial_{\nu_z} N_n(z,\zeta) = \frac{2}{(n-1)!^2}(|\zeta|^2 - 1)^{n-1}$$

$$- \sum_{\mu=[\frac{n}{2}]}^{n-2} \frac{\mu!^2}{(n-1)!(n-1-\mu)!^2(2\mu - n + 1)!}\partial_{\nu_z} N_{\mu+1}(z,\zeta)$$

on $\partial\mathbb{D}$ and the normalization condition

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} N_n(z,\zeta)\frac{dz}{z} = 0\,,$$

see [12, Theorem 4.5]. Using the higher-order Neumann functions and higher-order Cauchy–Pompeiu representations, Neumann problems for Poisson and n-Poisson equations are solved uniquely under some normalization and solvability conditions [5, 7, 12].

It is known that [12]

$$w(z) = -\sum_{\mu=0}^{n-1} \left\{ c_\mu p_{\mu+1}(z) - \frac{1}{4\pi i} \int_{\partial\mathbb{D}} N_{\mu+1}(z,\zeta)\gamma_\mu(\zeta)\frac{d\zeta}{\zeta} \right\} \tag{5}$$

$$-\frac{1}{\pi} \iint_{\mathbb{D}} N_n(z,\zeta) g(\zeta) d\xi d\eta \tag{6}$$

is the unique solution of the Neumann-n problem

$$(\partial_z \partial_{\bar{z}})^n w = g \text{ in } \mathbb{D}\,,\ \ g \in L^p(\mathbb{D}) \text{ for } 1 < p < +\infty\,,$$

$$\partial_\nu (\partial_z \partial_{\bar{z}})^\sigma w = \gamma_\sigma \text{ on } \partial\mathbb{D}\,,\ \ \gamma_\sigma \in C(\partial\mathbb{D}) \text{ for } 0 \le \sigma \le n-1$$

satisfying

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} (\partial_\zeta \partial_{\bar{\zeta}})^\sigma w(\zeta) \frac{d\zeta}{\zeta} = c_\sigma\,,\ \ c_\sigma \in \mathbb{C} \text{ for } 0 \le \sigma \le n-1$$

iff for $|z| = 1$

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \gamma_\sigma(\zeta) \frac{d\zeta}{\zeta} = \sum_{\mu=\sigma+1}^{n-1} \alpha_{\mu-\sigma} c_\mu + \frac{1}{\pi} \iint_{\mathbb{D}} \partial_{\nu_z} N_{n-\sigma}(z,\zeta) g(\zeta) d\xi d\eta \tag{7}$$

for $\alpha_1 = 2$ and for $3 \le k$

$$\alpha_{k-1} = -\sum_{\mu=[\frac{k}{2}]}^{k-2} \frac{\mu!^2}{(k-1)!(k-1-\mu)!^2(2\mu-k+1)!} \alpha_\mu \tag{8}$$

where $\zeta \in \partial\mathbb{D}$

$$p_\mu(z) = \frac{1}{2} \partial_{\nu_\zeta} N_\mu(z,\zeta) \ \text{ for } \ 1 \le \mu \le n \ \text{ and } \ z \in \mathbb{D}.$$

In the article [3], a particular solution of the above problem and its derivatives are defined as an integral operator family:

Definition 1 For $n \in \mathbb{N}$, $k, l \in \mathbb{N}_0$ with $(k,l) \ne (n,n)$ and $k + l \le 2n$, we define

$$(S_{n,k,l}F)(z) := S_{n,k,l}F(z) = \frac{1}{\pi} \iint_{\mathbb{D}} \partial_z^k \partial_{\bar{z}}^l N_n(z,\zeta) F(\zeta) d\xi d\eta$$

for a suitable complex valued function F given in $\mathbb{D}$.

Let us note that the operators $S_{n,k,l}$ are weakly singular for $k+l<2n$ and strongly singular for $k+l=2n$. Particularly, $S_{1,0,0}$ and $S_{1,1,0}$ are modified forms of the operators $\hat{\Pi}_0$, $\hat{\Pi}_1$ and $S_{1,2,0}$ is the operator $\hat{\Pi}_2$ and all three operators are given by Vinogradov [23] and Vekua [22]. Besides, the properties of $S_{2,k,l}=P_{k,l}$ are investigated in [2, 3]. The boundedness, continuity of the operators $S_{n,k,l}$ in the case of $k+l<2n$, and L^p boundedness of them in the case of $k+l=2n$ are also proved in [3].

Using the integral representation formula related to the Neumann-n problem given by Begehr and Vanegas in [12], we can write any $w\in C^{2n}(\mathbb{D})$ as:

$$w(z)=P_{\mathbb{D},n}(w)+\partial\tilde{N}_{\mathbb{D},n}(w)+\tilde{N}_{\mathbb{D},n}(w)$$

where

$$P_{\mathbb{D},n}(w)=-\sum_{\mu=0}^{n-1}p_{\mu+1}(z)\left[\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(\partial_\zeta\partial_{\bar{\zeta}})^\mu w(\zeta)\frac{d\zeta}{\zeta}\right]$$

$$\partial\tilde{N}_{\mathbb{D},n}(w)=\frac{1}{4\pi i}\sum_{\mu=0}^{n-1}\int_{\partial\mathbb{D}}N_{\mu+1}(z,\zeta)\partial_{\nu_\zeta}(\partial_\zeta\partial_{\bar{\zeta}})^\mu w(\zeta)\frac{d\zeta}{\zeta}$$

$$S_{n,0,0}((\partial_\zeta\partial_{\bar{\zeta}})^n w)(z)=\tilde{N}_{\mathbb{D},n}(w)=-\frac{1}{\pi}\iint_{\mathbb{D}}N_n(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})^n w(\zeta)d\xi d\eta.$$

3 Integral Representations in $\mathbb{C}^2$

This section is devoted to the study of the integral representations of functions in $\mathbb{C}^2$ which is suitable to discuss the solutions of Neumann problems. We will employ the unit polydisc $\mathbb{D}^2$ of $\mathbb{C}^2$, which is the Cartesian product of two unit discs in $\mathbb{C}$:

$$\mathbb{D}^2:=\{z=(z_1,z_2):|z_k|<1,k=1,2\}=\mathbb{D}_1\times\mathbb{D}_2$$

with the corresponding distinguished boundary

$$\partial\mathbb{D}^2:=\{z=(z_1,z_2):|z_k|=1,k=1,2\}=\partial\mathbb{D}_1\times\partial\mathbb{D}_2$$

where $\mathbb{D}_k=\{z_k:|z_k|<1\}$ and $\partial\mathbb{D}_k=\{z_k:|z_k|=1\}$, $k=1,2$.

We will start with the integral representation defined above for the variable $\hat{z}_k$ in $\mathbb{C}$.

$$w(\hat{z}_k) = -\sum_{\mu=0}^{n-1} p_{\mu+1}(\hat{z}_k)\left[\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_k}(\partial_{\hat{\zeta}_k}\partial_{\bar{\hat{\zeta}}_k})^{\mu} w(\hat{\zeta}_k)\frac{d\hat{\zeta}_k}{\hat{\zeta}_k}\right]$$
$$-\frac{1}{4\pi i}\sum_{\mu=0}^{n-1}\int\limits_{\partial\mathbb{D}_k} N_{\mu+1}(z_k,\hat{\zeta}_k)\partial_{\nu_{\hat{\zeta}_k}}(\partial_{\hat{\zeta}_k}\partial_{\bar{\hat{\zeta}}_k})^{\mu} w(\hat{\zeta}_k)\frac{d\hat{\zeta}_k}{\hat{\zeta}_k}$$
$$-\frac{1}{\pi}\iint\limits_{\mathbb{D}_k} N_n(z_k,\hat{\zeta}_k)(\partial_{\hat{\zeta}_k}\partial_{\bar{\hat{\zeta}}_k})^{n} w(\hat{\zeta}_k)d\hat{\xi}_k d\hat{\eta}_k.$$

Using the above notations we have

$$w(\hat{z}_k) = P_{\mathbb{D},n}(w(\hat{z}_k)) + \partial\tilde{N}_{\mathbb{D},n}(w(\hat{z}_k)) + \tilde{N}_{\mathbb{D},n}((\partial_{\hat{z}_k}\partial_{\bar{\hat{z}}_k})^n w(\hat{z}_k)).$$

In the rest of the article we will concentrate on the problems in $\mathbb{C}^2$. Thus we get

$$\begin{aligned} w(\hat{z}_1, z_2) = {} & P_{\mathbb{D}_1,n}(w(\hat{z}_1, z_2)) + \partial\tilde{N}_{\mathbb{D}_1,n}(w(\hat{z}_1, z_2)) \\ & +\tilde{N}_{\mathbb{D}_1,n}((\partial_{\hat{z}_1}\partial_{\bar{\hat{z}}_1})^n w(\hat{z}_1, z_2)) \end{aligned} \tag{9}$$

$$\begin{aligned} w(z_1, \hat{z}_2) = {} & P_{\mathbb{D}_2,n}(w(z_1, \hat{z}_2)) + \partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1, \hat{z}_2)) \\ & +\tilde{N}_{\mathbb{D}_2,n}((\partial_{\hat{z}_2}\partial_{\bar{\hat{z}}_2})^n w(z_1, \hat{z}_2)). \end{aligned} \tag{10}$$

Substituting (10) in (9) gives

$$\begin{aligned} w(z_1, z_2) = {} & P_{\mathbb{D}_1,n}(P_{\mathbb{D}_2,n}(w(z_1, z_2)) + \partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1, z_2)) + \tilde{N}_{\mathbb{D}_2,n}((\partial_{z_1}\partial_{\bar{z}_1})^n w(z_1, z_2))) \\ & +\partial\tilde{N}_{\mathbb{D}_1,n}(P_{\mathbb{D}_2,n}(w(z_1, z_2)) + \partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1, z_2)) + \tilde{N}_{\mathbb{D}_2,n}((\partial_{z_1}\partial_{\bar{z}_1})^n w(z_1, z_2))) \\ & +\tilde{N}_{\mathbb{D}_1,n}(P_{\mathbb{D}_2,n}(w(z_1, z_2)) + \partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1, z_2)) + \tilde{N}_{\mathbb{D}_2,n}((\partial_{z_1}\partial_{\bar{z}_1})^n w(z_1, z_2))) \\ = {} & P_{\mathbb{D}_1,n}w(z_1, z_2) + P_{\mathbb{D}_2,n}w(z_1, z_2) - P_{\mathbb{D}_1,n}(P_{\mathbb{D}_2,n}w(z_1, z_2)) \\ & +\partial\tilde{N}_{\mathbb{D}_1,n}(w(z_1, z_2)) + \partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1, z_2)) - P_{\mathbb{D}_1,n}(\partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1, z_2))) \\ & -P_{\mathbb{D}_2,n}(\partial\tilde{N}_{\mathbb{D}_1,n}(w(z_1, z_2))) - \partial\tilde{N}_{\mathbb{D}_1,n}(\partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1, z_2))) \\ & +\tilde{N}_{\mathbb{D}_1,n}(\tilde{N}_{\mathbb{D}_2,n}(\partial_{z_1}\partial_{\bar{z}_1})^n(\partial_{z_2}\partial_{\bar{z}_2})^n w(z_1, z_2)). \end{aligned} \tag{11}$$

Now we may give the explicit forms of the abbreviations in (11). We already know the representation of the notations $P_{\mathbb{D},n}(w(\hat{z}_k))$, $\partial\tilde{N}_{\mathbb{D},n}(w(\hat{z}_k))$, and $\tilde{N}_{\mathbb{D},n}((\partial_{\hat{z}_k}\partial_{\bar{\hat{z}}_k})^n w(\hat{z}_k))$. Let us compute the others:

(i)

$$P_{\mathbb{D}_1,n}(P_{\mathbb{D}_2,n}w(z_1,z_2)) = \sum_{\mu_1=0}^{n-1} p_{\mu_1+1}(z_1)\left[\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}(\partial_{\zeta_1}\partial_{\bar{\zeta}_1})^{\mu_1}\right.$$

$$\left.\times\left\{\sum_{\mu_2=0}^{n-1} p_{\mu_2+1}(z_2)\left[\frac{1}{2\pi i}\int_{\partial\mathbb{D}_2}(\partial_{\zeta_2}\partial_{\bar{\zeta}_2})^{\mu_2}w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\right]\right\}\frac{d\zeta_1}{\zeta_1}\right]$$

$$= \left(\frac{1}{2\pi i}\right)^2 \sum_{\mu_1,\mu_2=0}^{n-1} p_{\mu_1+1}(z_1)p_{\mu_2+1}(z_2)$$

$$\times \int_{\partial\mathbb{D}_1\times\partial\mathbb{D}_2}(\partial_{\zeta_1}\partial_{\bar{\zeta}_1})^{\mu_1}(\partial_{\zeta_2}\partial_{\bar{\zeta}_2})^{\mu_2}w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\frac{d\zeta_1}{\zeta_1}.$$

Using the notation

$$c_{\mu_1\mu_2} = \left(\frac{1}{2\pi i}\right)^2 \int_{\partial\mathbb{D}_1\times\partial\mathbb{D}_2}(\partial_{\zeta_1}\partial_{\bar{\zeta}_1})^{\mu_1}(\partial_{\zeta_2}\partial_{\bar{\zeta}_2})^{\mu_2}w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\frac{d\zeta_1}{\zeta_1}$$

we get

$$P_{\mathbb{D}_1,n}(P_{\mathbb{D}_2,n}w(z_1,z_2)) = \sum_{\mu_1,\mu_2=0}^{n-1} c_{\mu_1\mu_2}p_{\mu_1+1}(z_1)p_{\mu_2+1}(z_2).$$

(ii)

$$P_{\mathbb{D}_j,n}(\partial\tilde{N}_{\mathbb{D}_k,n}(w(z_1,z_2))) = -\sum_{\mu_j=0}^{n-1} p_{\mu_j+1}(z_j)\left[\frac{1}{2\pi i}\int_{\partial\mathbb{D}_1}(\partial_{\zeta_j}\partial_{\bar{\zeta}_j})^{\mu_j}\right.$$

$$\left.\times\left\{\frac{1}{4\pi i}\sum_{\mu_k=0}^{n-1}\int_{\partial\mathbb{D}_k} N_{\mu_k+1}(z_k,\zeta_k)\partial_{\nu_{\zeta_k}}(\partial_{\zeta_k}\partial_{\bar{\zeta}_k})^{\mu_k}w(\zeta_1,\zeta_2)\frac{d\zeta_k}{\zeta_k}\right\}\right]\frac{d\zeta_j}{\zeta_j}$$

$$= \left(-\frac{1}{4\pi i}\right)\frac{1}{2\pi i}\sum_{\mu_j,\mu_k=0}^{n-1} p_{\mu_j+1}(z_j)$$

$$\times \int\limits_{\partial\mathbb{D}_1\times\partial\mathbb{D}_2} N_{\mu_k+1}(z_k,\zeta_k)\partial_{\nu_{\hat{\zeta}_k}}(\partial_{\zeta_k}\partial_{\bar{\zeta}_k})^{\mu_k}(\partial_{\zeta_j}\partial_{\bar{\zeta}_j})^{\mu_j}w(\zeta_1,\zeta_2)\frac{d\zeta_k}{\zeta_k}\frac{d\zeta_j}{\zeta_j}$$

for $j, k \in \{1, 2\}$ and $j \neq k$.

(iii)

$$\partial\tilde{N}_{\mathbb{D}_1,n}(\partial\tilde{N}_{\mathbb{D}_2,n}(w(z_1,z_2))) = \frac{1}{4\pi i}\sum_{\mu_1=0}^{n-1}\int\limits_{\partial\mathbb{D}_1} N_{\mu_1+1}(z_1,\zeta_1)\partial_{\nu_{\zeta_1}}(\partial_{\zeta_1}\partial_{\bar{\zeta}_1})^{\mu_1} \tag{12}$$

$$\times\left[\frac{1}{4\pi i}\sum_{\mu_2=0}^{n-1}\int\limits_{\partial\mathbb{D}_2} N_{\mu_2+1}(z_2,\zeta_2)\partial_{\nu_{\zeta_2}}(\partial_{\zeta_2}\partial_{\bar{\zeta}_2})^{\mu_2}w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\right]\frac{d\zeta_1}{\zeta_1} \tag{13}$$

$$= \left(\frac{1}{4\pi i}\right)^2\sum_{\mu_1,\mu_2=0}^{n-1}\int\limits_{\partial\mathbb{D}_1\times\partial\mathbb{D}_2} N_{\mu_1+1}(z_1,\zeta_1)N_{\mu_2+1}(z_2,\zeta_2) \tag{14}$$

$$\times\partial_{\nu_{\zeta_1}}(\partial_{\zeta_1}\partial_{\bar{\zeta}_1})^{\mu_1}\partial_{\nu_{\zeta_2}}(\partial_{\zeta_2}\partial_{\bar{\zeta}_2})^{\mu_2}w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\frac{d\zeta_1}{\zeta_1}. \tag{15}$$

(iv)

$$\tilde{N}_{\mathbb{D}_1,n}(\tilde{N}_{\mathbb{D}_2,n}(\partial_{z_1}\partial_{\bar{z}_1})^n(\partial_{z_2}\partial_{\bar{z}_2})^n w(z_1,\hat{z}_2)w(z_1,z_2)) = \frac{1}{\pi}\iint\limits_{\mathbb{D}_1} N_n(z_1,\zeta_1)(\partial_{\hat{\zeta}_1}\partial_{\bar{\hat{\zeta}}_1})^n$$

$$\times\left[\frac{1}{\pi}\iint\limits_{\mathbb{D}_2} N_n(z_2,\zeta_2)(\partial_{\hat{\zeta}_2}\partial_{\bar{\hat{\zeta}}_2})^n w(\zeta)d\xi_2 d\eta_2\right]d\xi_1 d\eta_1$$

$$= \left(\frac{1}{\pi}\right)^2\iint\limits_{\mathbb{D}^2} N_n(z_1,\zeta_1)N_n(z_2,\zeta_2)(\partial_{\hat{\zeta}_1}\partial_{\bar{\hat{\zeta}}_1})^n(\partial_{\hat{\zeta}_2}\partial_{\bar{\hat{\zeta}}_2})^n w(\zeta_1,\zeta_2)d\xi_2 d\eta_2 d\xi_1 d\eta_1.$$

Let us call

$$N_{\mathbb{D}^2,n}(z_1,z_2;\zeta_1,\zeta_2) = \prod_{j=1}^{2} N_n(z_j,\zeta_j)$$

as the Neumann function for the problem. Hence combining the above results we obtain the following formula:

$$w(z_1,z_2)=\sum_{j=1}^{2}\sum_{\mu_j=0}^{n-1}p_{\mu_j+1}(z_j)\left[\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_j}(\partial_{\zeta_j}\partial_{\bar\zeta_j})^{\mu_j}w(\zeta_1,\zeta_2)\frac{d\zeta_j}{\zeta_j}\right]$$

$$-\sum_{\mu_1,\mu_2=0}^{n-1}c_{\mu_1\mu_2}p_{\mu_1+1}(z_1)p_{\mu_2+1}(z_2)$$

$$-\frac{1}{4\pi i}\sum_{j=1}^{2}\sum_{\mu_j=0}^{n-1}\int\limits_{\partial\mathbb{D}_j}N_{\mu_j+1}(z_j,\zeta_j)\partial_{\nu_{\zeta_j}}(\partial_{\zeta_j}\partial_{\bar\zeta_j})^{\mu_j}w(\zeta_1,\zeta_2)\frac{d\zeta_j}{\zeta_j}$$

$$+\frac{1}{4\pi i}\frac{1}{2\pi i}\sum_{\mu_j,\mu_k=0}^{n-1}p_{\mu_j+1}(z_j)\int\limits_{\partial\mathbb{D}^2}N_{\mu_k+1}(z_k,\zeta_k)\partial_{\nu_{\zeta_k}}(\partial_{\zeta_k}\partial_{\bar\zeta_k})^{\mu_k}(\partial_{\zeta_j}\partial_{\bar\zeta_j})^{\mu_j}w(\zeta_1,\zeta_2)\frac{d\zeta_k}{\zeta_k}\frac{d\zeta_j}{\zeta_j}$$

$$+\left(\frac{1}{4\pi i}\right)^2\sum_{\mu_1,\mu_2=0}^{n-1}\int\limits_{\partial\mathbb{D}_1\times\partial\mathbb{D}_2}N_{\mu_1+1}(z_1,\zeta_1)N_{\mu_2+1}(z_2,\zeta_2)$$

$$\times\partial_{\nu_{\zeta_1}}(\partial_{\zeta_1}\partial_{\bar\zeta_1})^{\mu_1}\partial_{\nu_{\zeta_2}}(\partial_{\zeta_2}\partial_{\bar\zeta_2})^{\mu_2}w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\frac{d\zeta_1}{\zeta_1}$$

$$+\left(\frac{1}{\pi}\right)^2\iint\limits_{\mathbb{D}^2}N_{\mathbb{D}^2,n}(z_1,z_2;\zeta_1,\zeta_2)(\partial_{\hat\zeta_1}\partial_{\bar\zeta_1})^n(\partial_{\zeta_2}\partial_{\bar\zeta_2})^n w(\zeta_1,\zeta_2)d\xi_2d\eta_2d\xi_1d\eta_1. \quad (16)$$

4 Neumann-n Problem in $\mathbb{C}^2$

Firstly we state the Neumann-n problem for the model equation in $\mathbb{C}^2$.

Statement of the Problem
Find a function in the Sobolev space $W^{2n,p}(\overline{\mathbb{D}}^2)$ satisfying

$$(\partial_{z_1}\partial_{\bar z_1})^n(\partial_{z_2}\partial_{\bar z_2})^n w(z_1,z_2)=f(z_1,z_2)\ \text{ in }\ \mathbb{D}^2 \quad (17)$$

satisfying the boundary conditions

$$\partial_{\nu_{z_1}}(\partial_{z_1}\partial_{\bar z_1})^{\mu_1}w(z_1,z_2)=\gamma_{\mu_1}(z_1,z_2)\ \text{ on }\partial\mathbb{D}_1,0\le\mu_1\le n-1 \quad (18)$$

$$\partial_{\nu_{z_2}}(\partial_{z_2}\partial_{\bar z_2})^{\mu_2}w(z_1,z_2)=\gamma_{\mu_2}(z_1,z_2)\ \text{ on }\partial\mathbb{D}_2,0\le\mu_2\le n-1 \quad (19)$$

subject to the compatibility conditions

$$\partial_{\nu_{z_2}}(\partial_{z_2}\partial_{\bar z_2})^{\mu_2}\gamma_{\mu_1}(z_1,z_2)=\partial_{\nu_{z_1}}(\partial_{z_1}\partial_{\bar z_1})^{\mu_1}\gamma_{\mu_2}(z_1,z_2)=\gamma_{\mu_1\mu_2}(z_1,z_2) \quad (20)$$

on the distinguished boundary. The corresponding normalization conditions are

$$\left(\frac{1}{2\pi i}\right)^2 \int\limits_{\partial\mathbb{D}_1}\int\limits_{\partial\mathbb{D}_2} (\partial_{\zeta_1}\partial_{\bar{\zeta}_1})^{\mu_1}(\partial_{\zeta_2}\partial_{\bar{\zeta}_2})^{\mu_2} w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\frac{d\zeta_1}{\zeta_1}$$

$$= \frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_1} (\partial_{\zeta_1}\partial_{\bar{\zeta}_1})^{\mu_1}\left[\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}_2}((\partial_{\zeta_2}\partial_{\bar{\zeta}_2})^{\mu_2} w(\zeta_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\right]\frac{d\zeta_1}{\zeta_1} = c_{\mu_1\mu_2} \tag{21}$$

for $0 \le \mu_1, \mu_2 \le n-1$.

We employ the integral representation (16) given in the previous section to derive the solution of this problem. To simplify the notations we assume homogeneous normalization conditions.

Theorem 1 *The Neumann-n problem in* $\mathbb{C}^2$ *given by Eq. (17) with the Neumann conditions (18)–(20) subject to the homogeneous normalization conditions is uniquely solvable if and only if*

$$2\gamma_{\sigma_1\sigma_2}(z_1,z_2) - \frac{1}{\pi i}\int\limits_{\partial\mathbb{D}_1}\gamma_{\sigma_1\sigma_2}(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1} - \frac{1}{\pi i}\int\limits_{\partial\mathbb{D}_2}\gamma_{\sigma_1\sigma_2}(z_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}$$

$$+\frac{1}{(2\pi i)^2}\int\limits_{\partial\mathbb{D}^2}\gamma_{\sigma_1\sigma_2}(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}$$

$$+\frac{1}{\pi^2}\int_{\mathbb{D}^2}\partial_{\nu_{z_1}}N_{n-\sigma_1}(z_1,\zeta_1)\partial_{\nu_{z_2}}N_{n-\sigma_2}(z_2,\zeta_2)f(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2 = 0$$

on the distinguished boundary for every $0 \le \sigma_1 \le n-1$ *and* $0 \le \sigma_2 \le n-1$.

Proof Using Eq. (16) with homogeneous normalization conditions we easily get the following form of the solution:

$$w(z_1,z_2) = \frac{1}{4\pi i}\sum_{\mu_1=0}^{n-1}\int\limits_{\partial\mathbb{D}_1} N_{\mu_1+1}(z_1,\zeta_1)\gamma_{\mu_1}(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1}$$

$$+\frac{1}{4\pi i}\sum_{\mu_2=0}^{n-1}\int\limits_{\partial\mathbb{D}_2} N_{\mu_2+1}(z_2,\zeta_2)\gamma_{\mu_2}(z_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}$$

$$-\left(\frac{1}{4\pi i}\right)^2\sum_{\mu_1,\mu_2=0}^{n-1}\int\limits_{\partial\mathbb{D}^2} N_{\mu_1+1}(z_1,\zeta_1)N_{\mu_2+1}(z_2,\zeta_2)\gamma_{\mu_1\mu_2}(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}$$

$$+\frac{1}{\pi^2}\iint\limits_{\mathbb{D}^2} N_{\mathbb{D}^2,n}(z_1,z_2;\zeta_1,\zeta_2)f(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2. \tag{22}$$

Now we need to find the solvability condition in order to complete the proof of the theorem. Thus we want to see under which relations the function given by (22) satisfies the boundary conditions. We apply the differential operator

$$\partial_{\nu_{z_1}}(\partial_{z_1}\partial_{\bar{z}_1})^{\sigma_1}\partial_{\nu_{z_2}}(\partial_{z_2}\partial_{\bar{z}_2})^{\sigma_2} \;\; 0 \le \sigma_1 \le n-1, 0 \le \sigma_2 \le n-1$$

to (22), keeping in mind the results given by Begehr and Vanegas [12], afterwards restricting it to the distinguished boundary, that is substituting $|z_1| = 1$ and $|z_2| = 1$, we get

$$\begin{aligned}\partial_{\nu_{z_1}}(\partial_{z_1}\partial_{\bar{z}_1})^{\sigma_1}\partial_{\nu_{z_2}}(\partial_{z_2}\partial_{\bar{z}_2})^{\sigma_2}w(z_1,z_2) &= 3\gamma_{\sigma_1\sigma_2}(z_1,z_2) - \frac{1}{\pi i}\int\limits_{\partial\mathbb{D}_1}\gamma_{\sigma_1\sigma_2}(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1}\\ &-\frac{1}{\pi i}\int\limits_{\partial\mathbb{D}_2}\gamma_{\sigma_1\sigma_2}(z_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\\ &+\frac{1}{(2\pi i)^2}\int\limits_{\partial\mathbb{D}^2}\gamma_{\sigma_1\sigma_2}(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}\\ &+\frac{1}{\pi^2}\int_{\mathbb{D}^2}\partial_{\nu_{z_1}}N_{n-\sigma_1}(z_1,\zeta_1)\partial_{\nu_{z_2}}N_{n-\sigma_2}(z_2,\zeta_2)f(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2.\end{aligned}$$

The solvability condition is evaluated as

$$\begin{aligned}&2\gamma_{\sigma_1\sigma_2}(z_1,z_2) - \frac{1}{\pi i}\int\limits_{\partial\mathbb{D}_1}\gamma_{\sigma_1\sigma_2}(\zeta_1,z_2)\frac{d\zeta_1}{\zeta_1} - \frac{1}{\pi i}\int\limits_{\partial\mathbb{D}_2}\gamma_{\sigma_1\sigma_2}(z_1,\zeta_2)\frac{d\zeta_2}{\zeta_2}\\ &+\frac{1}{(2\pi i)^2}\int\limits_{\partial\mathbb{D}^2}\gamma_{\sigma_1\sigma_2}(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1}\frac{d\zeta_2}{\zeta_2}\\ &+\frac{1}{\pi^2}\int_{\mathbb{D}^2}\partial_{\nu_{z_1}}N_{n-\sigma_1}(z_1,\zeta_1)\partial_{\nu_{z_2}}N_{n-\sigma_2}(z_2,\zeta_2)f(\zeta_1,\zeta_2)d\xi_1 d\eta_1 d\xi_2 d\eta_2 = 0\end{aligned}$$

on the distinguished boundary for every $0 \le \sigma_1 \le n-1$ and $0 \le \sigma_2 \le n-1$.

Remarks

1. The results we have obtained may be extended to the Neumann problems for model equations in $\mathbb{C}^n$.
2. All the above discussions may be extended to linear elliptic differential equations in $\mathbb{C}^n$.

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Green and Neumann Functions for a Plane Degenerate Circular Domain

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Abstract Harmonic Green and Neumann functions are constructed using the parqueting-reflection principle for a simply connected domain in the complex plane with two touching circles as the boundary.

1 Introduction

The parqueting-reflection principle serves to determine explicit formulas for harmonic Green and Neumann functions for certain plane domains D the boundary ∂D of which are composed by segments of circles and lines such that the repeated reflections of the domain at these segments provide a parqueting of the entire complex plane $\mathbb{C}$. The principle is described in detail in [8, 11]. Examples for such domains are plane and disc sectors [3, 7, 11, 12, 15, 23], triangles [8, 9, 24], hexagons [18], some domains in hyperbolic geometry [2, 4, 6], lens and lune [1, 10, 11, 19], circular rings [17, 20–22], etc. The principle, however, does not work for any such domain [5, 13].

This work was completed with the support of our TEX-pert.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_13

It is used in [14] to construct in an explicit way the harmonic Green and Neumann functions for a certain degenerate ring domain

$$D_0 = D = \big\{\frac{1}{2} < \big|z - \frac{1}{2}\big|, |z| < 1\big\} = \big\{1 < |2z - 1|, |z| < 1\big\},$$

the boundary of which consists of two touching circles

$$\big|z - \frac{1}{2}\big| = \frac{1}{2} \quad and \quad |z| = 1.$$

As this domain is simply connected in principle the Green and Neumann functions can be found by their conformal invariance. But constructing the Riemann mapping function of D onto the unit disc or the upper half plane [16] seems more involved than just applying the parqueting-reflection principle.

2 Green Function for D

Reflecting the domain D at the inner boundary circle $\big|z - \frac{1}{2}\big| = \frac{1}{2}$ repeatedly produces a parqueting of the unit disc $\mathbb{D} = \{|z| < 1\}$ by the image domains

$$D_k = \big\{\frac{1}{k+2} < |z - \frac{k+1}{k+2}|, |z - \frac{k}{k+1}| < \frac{1}{k+1}\big\}, k \in \mathbb{N}_0.$$

The $D_k{}'s$ shrink to the point $z = 1$ and the parqueting for the unit disc is $\overline{\mathbb{D}} = \bigcup_{k\in\mathbb{N}_0} \overline{D_k}$. Reflecting D at the unit circle $|z| = 1$ produces

$$\widetilde{D}_0 = \big\{\frac{1}{2} < |\frac{1}{z_{re}} - \frac{1}{2}|, \frac{1}{|z_{re}|} < 1\big\} = \big\{1 < |z|, z + \overline{z} < 2\big\}$$

as the image. It is the complement of the closure of the unit disc with respect to the half plane $z + \overline{z} < 2$. Moreover, the half plane $\big\{z + \overline{z} \leq 2\big\} = \overline{\widetilde{D}_0} \bigcup \overline{\mathbb{D}}$. To complete the parqueting for the plane $\mathbb{C}$ reflection at the line $z + \overline{z} = 2$ is used mapping the domains D_k onto

$$\widehat{D}_k = \big\{\frac{1}{k+2} < |z - \frac{k+3}{k+2}|, |z - \frac{k+2}{k+1}| < \frac{1}{k+1}\big\}, k \in \mathbb{N}_0,$$

while $\widetilde{D}_0$ is reflected onto

$$\widehat{\widetilde{D}}_0 = \big\{1 < |z - 2|, 2 < z + \overline{z}\big\}.$$

Hence,

$$\mathbb{C} = \overline{\widetilde{D}}_0 \cup \overline{\widehat{\widetilde{D}}}_0 \cup_{k \in \mathbb{N}_0} \left(\overline{D}_k \cup \overline{\widehat{D}}_k\right)$$

is the parqueting of the complex plane.

Tracing now the orbit of a point $z \in D$ during this procedure leads to the sequence

$$z_{2k} = \frac{(k-1)z - k}{kz - (k+1)} \in D_{2k} = \Big\{\frac{1}{2k+2} < \Big|z - \frac{2k+1}{2k+2}\Big|, \Big|z - \frac{2k}{2k+1}\Big| < \frac{1}{2k+1}\Big\}$$

which reflected at $\left\{\left|z - \frac{2k+1}{2k+2}\right| = \frac{1}{2k+2}\right\}$ is mapped onto

$$z_{2k+1} = \frac{(k+1)\overline{z} - k}{(k+2)\overline{z} - (k+1)} \in D_{2k+1},$$

$$D_{2k+1} = \Big\{\frac{1}{2k+3} < \Big|z - \frac{2k+2}{2k+3}\Big|, \Big|z - \frac{2k+1}{2k+2}\Big| < \frac{1}{2k+2}\Big\}.$$

Also $z \in D$ reflected at $|z| = 1$ gives $\widetilde{z}_0 = \frac{1}{\overline{z}} \in \widetilde{D}_0$.

Finally reflection at the vertical line $z + \overline{z} = 2$ is required. The original point $z \in D_0$ is mapped onto $\widehat{z} = 2 - \overline{z}$ and $\widetilde{z} = \frac{1}{\overline{z}} \in \widetilde{D}_0$ onto $\widehat{\widetilde{z}} = \frac{2z-1}{z} \in \widehat{\widetilde{D}}_0$, similarly, $z_{2k} \in D_{2k}$ and $z_{2k+1} \in D_{2k+1}$, $k \in \mathbb{N}_0$, onto

$$\widehat{z}_{2k} = \frac{(k+1)\overline{z} - (k+2)}{k\overline{z} - (k+1)} = \frac{1}{\overline{z_{2k+2}}} \in \widehat{D}_{2k},$$

$$\widehat{z}_{2k+1} = \frac{(k+3)z - (k+2)}{(k+2)z - (k+1)} = \frac{1}{\overline{z_{2k+3}}} \in \widehat{D}_{2k+1}.$$

The parqueting-reflection principle requires to choose $z \in D$ as a simple pole of a meromorphic function $P_1(z, \cdot)$ to be constructed and to determine the direct reflection of a pole to become a simple zero and the direct reflection of a zero to become a simple pole of P_1. Hence,

$$P_1(z, \zeta) = \frac{z}{\overline{z}} \frac{1 - \overline{z}\zeta}{\zeta - z} \frac{\zeta + \overline{z} - 2}{z\zeta + 1 - 2z} \prod_{k=0}^{\infty} \frac{\zeta - z_{2k+1}}{\zeta - z_{2k+2}} \frac{\zeta - \widehat{z}_{2k+2}}{\zeta - \widehat{z}_{2k+1}}. \tag{2.1}$$

Lemma 1 *The infinite product P_1 from (2.1) converges for $z, \zeta \in D$, $\zeta \neq z$.*

For a proof see [14], where also the symmetry $|P_1(z, \zeta)| = |P_1(\zeta, z)|$ for $z, \zeta \in D$, $\zeta \neq z$, is shown.

Theorem 1 *The function* $G_1(z,\zeta) = \log|P_1(z,\zeta)|^2, z,\zeta \in D, \zeta \neq z$, *is the harmonic Green function for the domain* D.

Proof Because $P_1(z,\cdot)$ is meromorphic in D with a simple pole at the point z, the function $G_1(z,\cdot)$ is harmonic in $D \setminus \{z\}$. Moreover, $G_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in D. To check its boundary behavior one observes for $|z| = 1$

$$P_1(z,\zeta) = \frac{\overline{z}}{z}\frac{z-\zeta}{\overline{z}\zeta - 1}\frac{z\zeta + 1 - 2z}{2-\overline{z}-\zeta}\prod_{k=0}^{\infty}\frac{\zeta - z_{2k+2}}{\zeta - z_{2k+1}}\frac{\zeta - \widehat{z}_{2k+1}}{\zeta - \widehat{z}_{2k+2}} = \frac{1}{P_1(z,\zeta)},$$

i.e., $|P_1(z,\zeta)| = 1$, as for $|z| = 1$ the relations $z_{2k+2} = z_{2k+1}, \widehat{z}_{2k+1} = \widehat{z}_{2k+2}$ hold.

Inserting $|z - \frac{1}{2}| = \frac{1}{2}$, i.e., $2|z|^2 = z + \overline{z}$ or $z = z_1$ into the expression for $P_1(z,\zeta)$ and observing $z_{2k+1} = z_{2k}, \widehat{z}_{2k+1} = \widehat{z}_{2k}$ shows $P_1(z_1,\zeta) = \frac{1}{P_1(z,\zeta)}$, so that again $|P_1(z,\zeta)| = 1$.

3 Poisson Kernel for *D*

The Green function $G_1(z,\zeta)$ provides the Poisson kernel as

$$g_1(z,\zeta) = -\frac{1}{2}\partial_{\nu_\zeta}G_1(z,\zeta), z \in D, \zeta \in \partial D,$$

where ∂_{ν_ζ} is the outward normal derivative on ∂D. On $|\zeta| = 1$ it is given by $\partial_{\nu_\zeta} = \zeta\partial_\zeta + \overline{\zeta}\partial_{\overline{\zeta}}$, so that applied to real functions $\partial_{\nu_\zeta} = 2\mathrm{Re}\zeta\partial_\zeta$. On the part $|\zeta - \frac{1}{2}| = \frac{1}{2}$ similarly $\partial_{\nu_\zeta} = -(2\zeta - 1)\partial_\zeta - (2\overline{\zeta} - 1)\partial_{\overline{\zeta}}$, so that for real functions $\partial_{\nu_\zeta} = -2\mathrm{Re}(2\zeta - 1)\partial_\zeta$.

Besides the above representation the Green function can also be rewritten as

$$G_1(z,\zeta) = \log\left|\frac{z}{\overline{z}}\frac{1-\overline{z}\zeta}{\zeta - z}\frac{\zeta + \overline{z} - 2}{z\zeta + 1 - 2z}\frac{\zeta - z_1}{\zeta - \widehat{z}_1}\prod_{k=1}^{\infty}\frac{\zeta - z_{2k+1}}{\zeta - z_{2k}}\frac{\zeta - \widehat{z}_{2k}}{\zeta - \widehat{z}_{2k+1}}\right|^2. \tag{3.1}$$

Thus either

$$\partial_\zeta G_1(z,\zeta) = -\frac{1}{\zeta - z} - \frac{\overline{z}}{1 - \overline{z}\zeta} - \frac{z}{z\zeta + 1 - 2z} + \frac{1}{\zeta + \overline{z} - 2} + \sum_{k=0}^{\infty}\left[\frac{1}{\zeta - z_{2k+1}} - \frac{1}{\zeta - z_{2k+2}} + \frac{1}{\zeta - \widehat{z}_{2k+2}} - \frac{1}{\zeta - \widehat{z}_{2k+1}}\right] \tag{3.2}$$

or

$$\partial_\zeta G_1(z,\zeta) = -\frac{1}{\zeta - z} + \frac{1}{\zeta - z_1} - \frac{\overline{z}}{1-\overline{z}\zeta} - \frac{z}{z\zeta + 1 - 2z} + \frac{1}{\zeta + \overline{z} - 2} - \frac{1}{\zeta - \widehat{z}_1}$$
$$+ \sum_{k=1}^{\infty} \Big[\frac{1}{\zeta - z_{2k+1}} - \frac{1}{\zeta - z_{2k}} + \frac{1}{\zeta - \widehat{z}_{2k}} - \frac{1}{\zeta - \widehat{z}_{2k+1}}\Big]. \tag{3.3}$$

Theorem 2 *The Poisson kernel satisfies on* $|\zeta| = 1$

$$g_1(z,\zeta) = \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta - z}} - 1 + O(1-|z|^2) \quad for \quad |z| \to 1,$$

$$g_1(z,\zeta) = O(2|z|^2 - z - \overline{z}) \quad for \quad |z - \frac{1}{2}| \to \frac{1}{2}.$$

For $|\zeta - \frac{1}{2}| = \frac{1}{2}$

$$g_1(z,\zeta) = O(1-|z|^2) \quad for \quad |z| \to 1,$$

$$g_1(z,\zeta) = -\frac{2\zeta - 1}{\zeta - z} - \frac{2\overline{\zeta} - 1}{\overline{\zeta - z}} + 2 + O(2|z|^2 - z - \overline{z}) \quad for \quad |z - \frac{1}{2}| \to \frac{1}{2}.$$

As well for $|\zeta| = 1$ as for $|\zeta - \frac{1}{2}| = \frac{1}{2}$ one part of the proof is based on (3.2) and the relations

$$\frac{1}{\zeta + \overline{z} - 2} - \frac{z}{z\zeta + 1 - 2z} = \frac{1 - |z|^2}{(\zeta + \overline{z} - 2)(z\zeta + 1 - 2z)},$$
$$\frac{1}{\zeta + \overline{z} - 2} - \frac{z}{z\zeta + 1 - 2z} = 0 \quad for \quad z = 1, \tag{3.4}$$

$$z_{2k+1} - z_{2k+2} = \frac{|z|^2 - 1}{[(k+1)z - (k+2)][(k+2)\overline{z} - (k+1)]}, \tag{3.5}$$

$$\widehat{z}_{2k+2} - \widehat{z}_{2k+1} = \frac{4(|z|^2 - 1)}{[(k+2)z - (k+1)][(k+1)\overline{z} - (k+2)]}, \tag{3.6}$$

and when $|\zeta| \neq 1$ also

$$\frac{1}{\zeta - z} + \frac{\overline{z}}{1 - \overline{z}\zeta} = \frac{1 - |z|^2}{(\zeta - z)(1 - \overline{z}\zeta)}. \tag{3.7}$$

Similarly, for both boundary parts $|\zeta| = 1$ and $|\zeta - \frac{1}{2}| = \frac{1}{2}$ the remaining parts are deduced from (3.3) together with

$$\frac{1}{\zeta - z_1} - \frac{1}{\zeta - z} = \frac{z + \overline{z} - 2|z|^2}{(2\overline{z}\zeta - \zeta - \overline{z})(\zeta - z)}, \tag{3.8}$$

$$\frac{\overline{z}}{1 - \overline{z}\zeta} + \frac{z}{z\zeta + 1 - 2z} = \frac{z + \overline{z} - 2|z|^2}{(1 - \overline{z}\zeta)(2z\zeta + 1 - 2z)}, \tag{3.9}$$

$$\frac{1}{\zeta + \overline{z} - 2} - \frac{1}{\zeta - \widehat{z}_1} = \frac{z + \overline{z} - 2|z|^2}{(\zeta + \overline{z} - 2)(2z\zeta + 2 - 2z - \zeta)}, \tag{3.10}$$

$$z_{2k+1} - z_{2k} = \frac{2|z|^2 - z - \overline{z}}{[(k+2)\overline{z} - (k+1)][kz - (k+1)]}, \tag{3.11}$$

$$\widehat{z}_{2k} - \widehat{z}_{2k+1} = \frac{2|z|^2 - z - \overline{z}}{[k\overline{z} - (k+1)][(k+2)z - (k+1)]}, \tag{3.12}$$

where for $|\zeta - \frac{1}{2}| = \frac{1}{2}$ also

$$\frac{2\zeta - 1}{\zeta - z_1} = 2 + \frac{1}{2\overline{z}\zeta - \overline{z} - \zeta} = 2 - \overline{\frac{2\overline{\zeta} - 1}{\zeta - z}} \tag{3.13}$$

is used. Also all the denominators appearing are not vanishing at the boundary. In particular the following lemma is proved in [14].

Lemma 2

1. For $|\zeta| = 1, |z| = 1$ *but* $z \neq 1$ *then*

$$\zeta + \overline{z} - 2 \neq 0, z\zeta + 1 - 2z \neq 0.$$

2. For $|\zeta - \frac{1}{2}| = \frac{1}{2}, |z - \frac{1}{2}| = \frac{1}{2}$ *but* $z \neq 1$ *then*

$$\zeta + \overline{z} - 2 \neq 0, z\zeta + 1 - 2z \neq 0.$$

4 Neumann Function for *D*

The parqueting-reflection principle suggests to introduce the infinite product

$$Q_1(z,\zeta) = z\overline{z}\frac{\zeta - z}{\zeta - 1}\frac{1 - \overline{z}\zeta}{1 - \zeta}\frac{z\zeta + 1 - 2z}{\zeta - 1}\frac{\zeta + \overline{z} - 2}{\zeta - 1}$$
$$\times \prod_{k=0}^{\infty}\Big[\frac{\zeta - z_{2k+1}}{\zeta - 1}\frac{\zeta - z_{2k+2}}{\zeta - 1}\frac{\zeta - \widehat{z}_{2k+1}}{\zeta - 1}\frac{\zeta - \widehat{z}_{2k+2}}{\zeta - 1}\Big], \quad z, \zeta \in D\,. \tag{4.1}$$

Lemma 3 *The infinite product Q_1 from (4.1) converges for $z, \zeta \in D$.*

For details compare [14].

Theorem 3 *The function $N_1(z,\zeta) = -\log|Q_1(z,\zeta)|^2, z, \zeta \in D, \zeta \neq z$, is a harmonic Neumann function for D satisfying, in particular $\partial_{\nu_\zeta} N_1(z,\zeta) = 0$ for $\zeta \in \partial D \setminus \{1\}, z \in D$.*

Proof As $Q_1(z,\cdot)$ is analytic in D with a simple zero at the point z the function $N_1(z,\zeta)$ is harmonic in $D \setminus \{z\}$ and continuously differentiable on $\overline{D}$ up to the point $\zeta = 1$. Moreover, $N_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic in D.

$$\partial_\zeta N_1(z,\zeta) = -\frac{1}{\zeta - z} + \frac{\overline{z}}{1 - \overline{z}\zeta} - \frac{z}{z\zeta + 1 - 2z} - \frac{1}{\zeta + \overline{z} - 2} + \frac{4}{\zeta - 1}$$

$$-\sum_{k=0}^{\infty}\Big[\frac{1}{\zeta - z_{2k+1}} + \frac{1}{\zeta - z_{2k+2}} + \frac{1}{\zeta - \widehat{z}_{2k+1}} + \frac{1}{\zeta - \widehat{z}_{2k+2}} - \frac{4}{\zeta - 1}\Big]$$

$$= -\frac{1}{\zeta - z} + \frac{\overline{z}}{1 - \overline{z}\zeta} + \frac{2}{\zeta - 1} + \frac{1}{\zeta - 1}\Big[\frac{1 - z}{z\zeta + 1 - 2z} - \frac{1 - \overline{z}}{\zeta + \overline{z} - 2}\Big]$$

$$-\frac{1}{\zeta - 1}\sum_{k=0}^{\infty}\Big[\frac{1 - \overline{z}}{[(k+2)\overline{z} - (k+1)]\zeta - [(k+1)\overline{z} - k]}$$

$$+\frac{1 - z}{[(k+1)z - (k+2)]\zeta - [kz - (k+1)]} - \frac{1 - z}{[(k+2)z - (k+1)]\zeta - [(k+3)z - (k+2)]}$$

$$-\frac{1 - \overline{z}}{[(k+1)\overline{z} - (k+2)]\zeta - [(k+2)\overline{z} - (k+3)]}\Big],$$

so that for $|\zeta| = 1, \zeta \neq 1$

$$\text{Re}\zeta\,\partial_\zeta N_1(z,\zeta) = \text{Re}\frac{\zeta + 1}{\zeta - 1} = \frac{|\zeta|^2 - 1}{|\zeta - 1|^2}.$$

Inserting for $2|\zeta|^2 = \zeta + \overline{\zeta}$ the relations

$$|\zeta|^2 - \zeta = \overline{\zeta} - |\zeta|^2, \overline{\zeta}(2\zeta - 1) = \zeta, \frac{\zeta}{\zeta - 1} = \frac{\overline{\zeta}}{1 - \overline{\zeta}},$$

into

$$\partial_\zeta N_1(z,\zeta) = \frac{1 - z}{\zeta - 1}\frac{1}{\zeta - z} + \frac{1 - \overline{z}}{\zeta - 1}\frac{1}{1 - \overline{z}\zeta} + \frac{z - 1}{\zeta - 1}\frac{1}{2z - z\zeta - 1} + \frac{\overline{z} - 1}{\zeta - 1}\frac{1}{\zeta + \overline{z} - 2}$$

$$-\frac{1}{\zeta - 1}\sum_{k=0}^{\infty}\Big[\frac{z_{2k+1} - 1}{\zeta - z_{2k+1}} + \frac{z_{2k+2} - 1}{\zeta - z_{2k+2}} + \frac{\widehat{z}_{2k+1} - 1}{\zeta - \widehat{z}_{2k+1}} + \frac{\widehat{z}_{2k+2} - 1}{\zeta - \widehat{z}_{2k+2}}\Big]$$

shows for $2|\zeta|^2 = \zeta + \overline{\zeta}$

$$\mathrm{R}e(2\zeta - 1)\partial_\zeta N_1(z, \zeta) = 0$$

as long as $\zeta \neq 1$. See for details [14].

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Part V
Special Interest Group: IGCVPT Complex Variables and Potential Theory

Session Organizers: Tahir Aliyev Azeroglu, Anatoly Golberg, Massimo Lanza de Cristoforis, and Sergiy Plaksa

This session was devoted to the wide range of directions of complex analysis, potential theory, their applications, and related topics.

Biharmonic Monogenic Functions and Biharmonic Boundary Value Problems

Serhii V. Gryshchuk and Sergiy A. Plaksa

Abstract We consider a commutative algebra B over the field of complex numbers with a basis $\{e_1, e_2\}$ satisfying the conditions $(e_1^2 + e_2^2)^2 = 0$, $e_1^2 + e_2^2 \neq 0$. We consider a Schwarz-type boundary value problem for "analytic" B-valued functions in a simply connected domain. This problem is associated with BVPs for biharmonic functions. Using a hypercomplex analog of the Cauchy type integral, we reduce these BVPs to a system of integral equations on the real axes. We establish sufficient conditions under which this system has the Fredholm property.

1 Biharmonic Monogenic Functions

An associative commutative two-dimensional algebra B with the unit 1 over the field of complex numbers C is called *biharmonic* (see [8, 10]) if in B there exists a basis $\{e_1, e_2\}$ satisfying the conditions

$$(e_1^2 + e_2^2)^2 = 0, \qquad e_1^2 + e_2^2 \neq 0\,. \tag{1}$$

In the paper [10] Mel'nichenko proved that there exists the unique biharmonic algebra B. Note that the algebra B is isomorphic to four-dimensional over the field of real numbers R algebras considered by Douglis [1] and Sobrero [14].

In what follows, we consider a basis $\{e_1, e_2\}$ of the type (1) with the following multiplication table (see [8]):

$$e_1 = 1, \qquad e_2^2 = e_1 + 2i e_2\,,$$

where i is the imaginary complex unit.

Henceforth we assumed that $\zeta = x\, e_1 + y\, e_2$, $z = x + iy$, and x, y are real.

Consider the *biharmonic plane* $\mu := \{\zeta = x\, e_1 + y\, e_2 : (x, y) \in \mathbb{R}^2\}$.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_14

For a domain $D \subset \mathbb{R}^2$ consider the corresponding domains $D_\zeta := \{\zeta = xe_1 + ye_2 : (x, y) \in D\} \subset \mu$ and $D_z := \{z = x + iy : (x, y) \in D\} \subset \mathbb{C}$. We will assume by default that the domain D is bounded and simply connected.

We use the Euclidian norm $\|a\| := \sqrt{|z_1|^2 + |z_2|^2}$ in the algebra $\mathbb{B}$, where $a = z_1e_1 + z_2e_2$ and $z_1, z_2 \in \mathbb{C}$.

We say that a function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ and denote as $\Phi \in \mathscr{M}(D_\zeta)$, if the derivative $\Phi'(\zeta) \in \mathbb{B}$ exists at every point $\zeta \in D_\zeta$:

$$\Phi'(\zeta) := \lim_{h\to 0,\, h\in\mu} \big(\Phi(\zeta + h) - \Phi(\zeta)\big)\, h^{-1}.$$

It is proved in [8] that a function $\Phi : D_\zeta \longrightarrow \mathbb{B}$ is monogenic in D_ζ if and only if its each real-valued component-function $U_k : D \longrightarrow \mathbb{B}$, $k = \overline{1, 4}$, in the expansion

$$\Phi(\zeta) = U_1(x, y)\, e_1 + U_2(x, y)\, ie_1 + U_3(x, y)\, e_2 + U_4(x, y)\, ie_2 \tag{2}$$

is differentiable in the domain D and the following analogue of Cauchy–Riemann's conditions is fulfilled:

$$\frac{\partial\Phi(\zeta)}{\partial y}\, e_1 = \frac{\partial\Phi(\zeta)}{\partial x}\, e_2 \quad \forall\, \zeta \in D_\zeta\, .$$

Every $\Phi \in \mathscr{M}(D_\zeta)$ has the derivative of any order in D_ζ (cf., e.g., [2, 4]) and satisfies the equalities

$$\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}\right)\Phi(\zeta) = \Phi^{(4)}(\zeta)\,(e_1^2 + e_2^2)^2 = 0 \quad \forall\, \zeta \in D_\zeta\, .$$

due to the conditions (1). Therefore, we shall also term such a function Φ by *biharmonic monogenic* function in D_ζ.

Every biharmonic in D function $U(x, y)$ is the first component $U_1 \equiv U$ in the expression (2) of a certain function $\Phi \in \mathscr{M}(D_\zeta)$ and, moreover, all such functions Φ are found in an explicit form (cf., e.g., [2, 4]).

2 BVPs for Biharmonic Monogenic Functions and Biharmonic Functions

Consider the following boundary value problem: to find a function $\Phi \in \mathscr{M}(D_\zeta)$ which is continuously extended on the boundary ∂D_ζ of domain D_ζ when values of the first and the third component-functions in (2) are given on ∂D_ζ, i.e., the following boundary conditions are satisfied:

$$U_1(x, y) = u_1(\zeta)\,, \quad U_3(x, y) = u_3(\zeta) \qquad \forall\, \zeta \in \partial D_\zeta\,, \tag{3}$$

where $u_k\colon \partial D_\zeta \longrightarrow \mathbb{R}$, $k \in \{1, 3\}$ are given real-valued continuous functions. We call this problem as *(1-3)-problem* (cf., e.g., [5, 6]).

The *principal biharmonic problem* (cf., e.g., [15, p. 194] and [11, p. 13]) consists of finding a function $W\colon \overline{D} \longrightarrow \mathbb{R}$ which is continuous together with partial derivatives of the first order in the closure $\overline{D}$ of domain D and is biharmonic in D, when its values and values of its outward normal derivative are given on the boundary ∂D:

$$W(x_0, y_0) = \omega_0(s), \quad \frac{\partial W}{\partial \mathbf{n}}(x_0, y_0) = \omega_2(s) \qquad \forall\, (x_0, y_0) \in \partial D\,, \tag{4}$$

where s is an arc coordinate of the point $(x_0, y_0) \in \partial D$.

In the case where ω_1 is a continuously differentiable function, the principal biharmonic problem is equivalent to the following *biharmonic problem* (cf., e.g., [15, p. 194] and [11, p. 13]) on finding a biharmonic function $V\colon D \longrightarrow \mathbb{R}$ with the following boundary conditions:

$$\begin{aligned} \lim_{(x,y)\to(x_0,y_0),\,(x,y)\in D} \frac{\partial V(x,y)}{\partial x} &= \omega_1(s)\,, \\ \lim_{(x,y)\to(x_0,y_0),\,(x,y)\in D} \frac{\partial V(x,y)}{\partial y} &= \omega_3(s) \qquad \forall\,(x_0,y_0) \in \partial D\,. \end{aligned} \tag{5}$$

Let $\mathbf{s}$ and $\mathbf{n}$ denote unit vectors of the tangent and the outward normal to the boundary ∂D, respectively, and $\angle(\cdot,\cdot)$ denotes an angle between an appropriate vector ($\mathbf{s}$ or $\mathbf{n}$) and the positive direction of coordinate axis (x or y) indicated in the parenthesis.

A necessary condition for solving the biharmonic problem (5) is the following (cf., e.g., [15]):

$$\int_{\partial D} \Big(\omega_1(s)\cos\angle(\mathbf{s}, x) + \omega_3(s)\cos\angle(\mathbf{s}, y)\Big)\, ds = 0\,. \tag{6}$$

The (6) is rewriting in the form:

$$\int_{\partial D} v_1(x,y)\, dx + v_3(x,y)\, dy = 0, \tag{7}$$

where $v_k(x,y) := \omega_k(s)$, $x = x(s)$, $y = y(s)$, $k \in \{1, 3\}$.

Boundary functions ω_1, ω_3 have relations with given functions ω_0, ω_2 of the problem (4) (cf., e.g., [7, p. 554]), viz.,

$$\omega_1(s) = \omega_0'(s)\cos\angle(\mathbf{s}, x) + \omega_2(s)\cos\angle(\mathbf{n}, x)\,,$$

$$\omega_3(s) = \omega_0'(s)\cos\angle(\mathbf{s}, y) + \omega_2(s)\cos\angle(\mathbf{n}, y).$$

Furthermore, solutions of the problems (4) and (5) are related by the equality $V(x, y) = W(x, y) + c$, where $c \in \mathrm{R}$.

For finding a solution V of the biharmonic problem in D it is enough to know derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ in D, which we can find by solving (1-3)-problem with the boundary data: $u_k := \omega_k$, $k \in \{1, 3\}$, (cf., e.g., [6]). Therefore, the biharmonic problem is reduced to this (1-3)-problem and we focus our attention to the latter problem.

A technique of using analytic functions of the complex variable for solving the biharmonic problem is based on an expression of biharmonic functions by the Goursat formula. This expression allows to reduce the biharmonic problem to a certain boundary value problem for a pair of analytic functions. In the case where the boundary ∂D is a Lyapunov curve, the mentioned system can be reduced to a system of Fredholm equations. Such a scheme is developed (cf., e.g., [9, 12, 13]) for solving the main problems of the plane elasticity theory using a special biharmonic function which is called the Airy stress function.

Our alternative approach is based on expressions of monogenic functions via biharmonic Cauchy type integrals. Using their relations to the boundary value problems for biharmonic functions, we obtain a system of integral equations in the general case and establish the Fredholm property of this system in the case where the boundary of domain belongs to a class being wider than the class of Lyapunov curves.

3 Biharmonic Cauchy Type Integral

Let the boundary ∂D_ζ be a closed smooth Jordan curve, and

$$\varphi(\zeta) = \varphi_1(\zeta)\, e_1 + \varphi_3(\zeta)\, e_2 \quad \forall\, \zeta \in \partial D_\zeta, \tag{8}$$

where $\varphi_k \colon \partial D_\zeta \longrightarrow \mathrm{R}$, $k \in \{1, 3\}$ are continuous real-valued functions.

We use the modulus of continuity of the function φ:

$$\omega(\varphi, \varepsilon) := \sup_{\tau_1, \tau_2 \in \partial D_\zeta,\, \|\tau_1 - \tau_2\| \le \varepsilon} \|\varphi(\tau_1) - \varphi(\tau_2)\|\,.$$

We assume that $\omega(\varphi, \varepsilon)$ satisfies the Dini condition

$$\int_0^1 \frac{\omega(\varphi, \eta)}{\eta}\, d\eta < \infty\,. \tag{9}$$

Consider the *biharmonic* Cauchy type integral

$$\mathscr{B}[\varphi](\zeta) := \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta)^{-1}\, d\tau \quad \forall\, \zeta \in \mu \setminus \partial D_\zeta\,.$$

For every $\zeta_0 \in \partial D_\zeta$, a singular integral is understood in the sense of its Cauchy principal value:

$$\mathscr{B}_0\,[\varphi]\,(\zeta_0) \equiv \int_{\partial D_\zeta} \varphi(\tau)(\tau-\zeta_0)^{-1}d\tau := \lim_{\varepsilon\to 0} \int\limits_{\{\tau\in\partial D_\zeta:\|\tau-\zeta_0\|\geq\varepsilon\}} \varphi(\tau)(\tau-\zeta_0)^{-1}d\tau.$$

Let us compactify the biharmonic plane μ by means of addition of an infinite point ∞. Denote $D_\zeta^+ := D_\zeta$, $D_\zeta^- := \mu \setminus (D_\zeta \cup \partial D_\zeta)$.

Theorem 1 (cf., e.g., [3, 6]) *If a function φ satisfies the condition (9), then $\mathscr{B}\,[\varphi]$ is a monogenic function in D_ζ^+ and D_ζ^-, separately, and*

$$\mathscr{B}\,[\varphi]\,(\infty) = 0.$$

Moreover, the limiting values

$$\mathscr{B}^{\pm}\,[\varphi]\,(\zeta_0) := \lim_{\tau\to\zeta_0,\,\tau\in D_\zeta^{\pm}} \mathscr{B}\,[\varphi]\,(\tau)$$

exist at every point $\zeta_0 \in \partial D_\zeta$ and are represented by the Sokhotski–Plemelj formulas:

$$\Phi^+(\zeta_0) = \frac{1}{2}\,\varphi(\zeta_0) + \mathscr{B}_0\,[\varphi]\,(\zeta_0)\,,$$

$$\Phi^-(\zeta_0) = -\frac{1}{2}\,\varphi(\zeta_0) + \mathscr{B}_0\,[\varphi]\,(\zeta_0)\,.$$

If φ has the integrable contour derivative φ', then the following equality is fulfilled:

$$\frac{d}{d\zeta}\left(\mathscr{B}\,[\varphi]\,(\zeta)\right) = \mathscr{B}\left[\varphi'\right](\zeta) \quad \forall\,\zeta \in \mu \setminus \partial D_\zeta.$$

4 Reducing the (1-3)-Problem to a System of Fredholm Integral Equations

We assume that the boundary functions u_k, $k \in \{1, 3\}$, satisfy conditions of the type (9). We seek solutions of (1-3)-problem in a class of functions represented in the form $\Phi(\zeta) = \mathscr{B}\,[\varphi]\,(\zeta)$ with a function (8) satisfying the condition (9).

We use a conformal mapping $z = \tau(t)$ of the upper half-plane $\{t \in \mathrm{C} : \mathrm{Im}\, t > 0\}$ onto the domain D_z. Denote $\tau_1(t) := \mathrm{Re}\,\tau(t)$, $\tau_2(t) := \mathrm{Im}\,\tau(t)$.

Inasmuch as the mentioned conformal mapping is continued to a homeomorphism between the closures of corresponding domains, the function

$$\widetilde{\tau}(s) := \tau_1(s)e_1 + \tau_2(s)e_2 \qquad \forall\, s \in \overline{\mathrm{R}}$$

generates a homeomorphic mapping of the extended real axis $\overline{\mathrm{R}} := \mathrm{R} \cup \{\infty\}$ onto the curve ∂D_ζ.

Using Theorem 1, we rewrite the boundary data (3) in the form (cf. [6]) of the following system of integral equations:

$$\begin{aligned}
&\frac{1}{2}\, g_1(t) + \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} g_1(s)\Big(\mathrm{Im}\, k_1(t,s) + 2\mathrm{Re}\, k_2(t,s)\Big)\, ds - \\
&\qquad\qquad - \frac{1}{\pi} \int\limits_{-\infty}^{\infty} g_3(s)\mathrm{Im}\, k_2(t,s)\, ds = \widetilde{u}_1(t), \\
&\frac{1}{2}\, g_3(t) + \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} g_3(s)\Big(\mathrm{Im}\, k_1(t,s) - 2\mathrm{Re}\, k_2(t,s)\Big)\, ds - \\
&\qquad\qquad - \frac{1}{\pi} \int\limits_{-\infty}^{\infty} g_1(s)\mathrm{Im}\, k_2(t,s)\, ds = \widetilde{u}_3(t) \qquad \forall\, t \in \mathrm{R},
\end{aligned} \tag{10}$$

where $g_k(s) := \varphi_k\,(\widetilde{\tau}(s))$, $\widetilde{u}_k(t) := u_k\big(\widetilde{\tau}(t)\big)$, $k \in \{1, 3\}$, $k_1(t,s) := \frac{\tau'(s)}{\tau(s)-\tau(t)} - \frac{1+st}{(s-t)(s^2+1)}$, $k_2(t,s) := \frac{\tau'(s)\big(\tau_2(s)-\tau_2(t)\big)}{2\big(\tau(s)-\tau(t)\big)^2} - \frac{\tau_2'(s)}{2\big(\tau(s)-\tau(t)\big)}$.

Consider the conformal mapping $\sigma(T)$ of the unit disk $\Gamma := \{T \in \mathrm{C} : |T| < 1\}$ onto the domain D_z such that $\tau(t) = \sigma\left(\frac{t-i}{t+i}\right)$ for all $t \in \{t \in \mathrm{C} : \mathrm{Im}\, t > 0\}$.

For a function $g : \gamma \longrightarrow \mathrm{C}$ which is continuous on the curve $\gamma \subset \mathrm{C}$, a modulus of continuity is defined by the equality

$$\omega_\gamma(g, \varepsilon) := \sup_{t_1, t_2 \in \gamma,\; |t_1 - t_2| \le \varepsilon} |g(t_1) - g(t_2)|\,.$$

Let $C(\overline{\mathrm{R}})$ denote the Banach space of functions $g : \overline{\mathrm{R}} \longrightarrow \mathrm{C}$ that are continuous on the extended real axis $\overline{\mathrm{R}}$ with the norm $\|g\|_{C(\overline{\mathrm{R}})} := \sup_{t \in \mathrm{R}} |g(t)|$.

For any function $g \in C(\overline{\mathrm{R}})$ we use the local centered (with respect to the infinitely remote point) modulus of continuity

$$\omega_{\mathrm{R},\infty}(g, \varepsilon) := \sup_{t \in \mathrm{R}\,:\, |t| \ge 1/\varepsilon} |g(t) - g(\infty)|\,.$$

Let $\mathscr{D}(\overline{\mathrm{R}})$ denote the class of functions $g \in C(\overline{\mathrm{R}})$ whose moduli of continuity satisfy the Dini conditions

$$\int_0^1 \frac{\omega_{\mathrm{R}}(g,\eta)}{\eta}\, d\eta < \infty, \qquad \int_0^1 \frac{\omega_{\mathrm{R},\infty}(g,\eta)}{\eta}\, d\eta < \infty\,.$$

We use the notations $U_k\,[a] := a_k$, $k = \overline{1,4}$, where $a_k \in \mathrm{R}$ is the coefficient in the decomposition of element $a = a_1 e_1 + a_2 i e_1 + a_3 e_2 + a_4 i e_2 \in \mathrm{B}$ with respect to the basis $\{e_1, e_2\}$.

Let the next functions are well defined:

$$\Psi_+\,[\varphi]\,(\zeta) := \mathrm{U}_1\left[\mathscr{B}\left[\varphi'\right](\zeta)\right] - \mathrm{U}_4\left[\mathscr{B}\left[\varphi'\right](\zeta)\right]\ \forall \zeta \in D_\zeta^+, \tag{11}$$

$$\Psi_-\,[\varphi]\,(\zeta) := \mathrm{U}_2\left[\mathscr{B}\left[\varphi'\right](\zeta)\right] + \mathrm{U}_3\left[\mathscr{B}\left[\varphi'\right](\zeta)\right]\ \forall \zeta \in D_\zeta^-, \tag{12}$$

$$\varphi(\tau) := g_1(s)\, e_1 + g_3(s)\, e_2, \quad \tau = \widetilde{\tau}(s), \quad \forall\, s \in \mathrm{R}, \tag{13}$$

Theorem 2 *Let the functions* $u_1 \colon \partial D_\zeta \longrightarrow \mathrm{R}$, $u_3 \colon \partial D_\zeta \longrightarrow \mathrm{R}$ *satisfy conditions of the type* (9). *Let the conformal mapping* $\sigma(T)$ *have the nonvanishing continuous contour derivative* $\sigma'(T)$ *on the circle* Γ, *and its modulus of continuity* $\omega_\Gamma(\sigma', \varepsilon)$ *satisfies the condition*

$$\int_0^2 \frac{\omega_\Gamma(\sigma',\eta)}{\eta}\, \ln\frac{3}{\eta}\, d\eta < \infty.$$

Then all functions $g_1, g_3 \in C(\overline{\mathrm{R}})$ *satisfying the system of Fredholm integral equations* (10) *belong to the class* $\mathscr{D}(\overline{\mathrm{R}})$, *and the corresponding functions* φ *in* (13) *satisfy the condition* (9).

Let $\mathscr{B}\left[\varphi'\right]$ *exist in* D_ζ^+ *and* D_ζ^- *separately, the functions* (11), (12) *are bounded, and, in addition, all solutions* $(g_1, g_3) \in C(\overline{\mathrm{R}}) \times C(\overline{\mathrm{R}})$ *of the homogeneous system of equations* (10) (*with* $\widetilde{u}_k \equiv 0$ *for* $k \in \{1,3\}$) *are differentiable on* R. *Then the following assertions are true:*

(i) the number of linearly independent solutions of the homogeneous system of equations (10) *is equal to* 1;

(ii) the non-homogeneous system of equations (10) *is solvable if and only if the condition* (7) *with* $v_k := u_k$, $k \in \{1,3\}$, *is satisfied.*

Theorem 2 generalizes Theorem 6.13 [6]. It proves similar to the latter with the use of Theorem 1.

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Composition Operators of α-Bloch Spaces on Bounded Symmetric Domains in $\mathbb{C}^n$

Hidetaka Hamada and Gabriela Kohr

Abstract Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball $\mathbb{B}_X$ of a finite dimensional JB*-triple X. In this paper, we continue the work related to α-Bloch mappings on $\mathbb{B}_X$. We first show that α-Bloch spaces on $\mathbb{B}_X$ are complex Banach spaces. Next, we give sufficient conditions for the composition operator from the α-Bloch space into the β-Bloch space to be bounded or compact. In the case that the α-Bloch space is a Bloch space, then these conditions are also necessary. Particular cases of interest will also be mentioned.

1 Introduction

Let $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the unit disc in $\mathbb{C}$. A holomorphic function $f : \mathbb{U} \to \mathbb{C}$ is said to belong to the Bloch space $\mathscr{B}$ if $\sup_{\zeta\in\mathbb{U}}(1 - |\zeta|^2)|f'(\zeta)| < \infty$.

For Bloch functions $f \in \mathscr{B}$ with normalization $f'(0) = 1$, Ahlfors [1] , Bonk [4], Chen and Gauthier [5], Liu and Minda [18], Minda [24], and others studied the distortion theorem and the estimation for the Bloch constant. The distortion theorem and the estimation for the Bloch constant have been generalized to Bloch mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ by Liu [17] and on the unit polydisc in $\mathbb{C}^n$ by Wang and Liu [30]. Chen et al. [6] proved the distortion theorem and the estimation of the Bloch constant for α-Bloch mappings from $\mathbb{B}^n$ into $\mathbb{C}^n$. Hamada [11], Hamada and Kohr [13] generalized these results to Bloch mappings or α-Bloch mappings on bounded symmetric domains in $\mathbb{C}^n$.

From the point of view of the Riemann mapping theorem, a homogeneous unit ball of a complex Banach space is a natural generalization of the open unit disc. Every bounded symmetric domain in a complex Banach space is biholomorphically

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_15

equivalent to a homogeneous unit ball (see [15]). A complex Banach space X is a JB*-triple iff the open unit ball of X is homogeneous. These arguments give the motivation for studying the Bloch mappings on the unit balls of JB*-triples.

Another interesting study on the Bloch space or α-Bloch space is that on the composition operators between these spaces. In this paper we continue the work started in [13] related to α-Bloch mappings on the open unit ball $\mathbb{B}_X$ of a finite dimensional JB*-triple X. We first show that α-Bloch spaces on $\mathbb{B}_X$ are complex Banach spaces (Proposition 1). Next, we obtain a boundedness result (Theorem 1) and a compactness result (Theorem 2) for composition operators between α-Bloch type spaces on the unit balls $\mathbb{B}_X$ and $\mathbb{B}_Y$ of finite dimensional JB*-triples X and Y, respectively. On the unit disc in $\mathbb{C}$, Ohno et al. [25] studied this problem (cf. Madigan and Matheson [21]). On the Euclidean unit ball $\mathbb{B}^n$, this problem was studied by Zhang and Xu [31] (cf. Shi and Luo [28]). Note that their definition of α-Bloch functions on $\mathbb{B}^n$ is different from ours. This is one of the motivations for the study of α-Bloch mappings and compact composition operators on α-Bloch type spaces on the unit balls of finite dimensional JB*-triples. The main results of this paper are generalizations of these results to the unit balls $\mathbb{B}_X$ and $\mathbb{B}_Y$ of any finite dimensional JB*-triple X and Y, respectively.

Composition operators from H^∞ to Bloch spaces of infinite dimensional bounded symmetric domains have been studied in [10]. Other recent contributions related to composition operators between Bloch spaces on the Euclidean unit ball in $\mathbb{C}^n$ and in infinite dimensional spaces may be found in [3] and [9] (see also [23] and [32]).

2 Preliminaries

Let $\mathbb{B}_X$ be the unit ball of a complex Banach space X. Let Y be a complex Banach space. Let $H(\mathbb{B}_X, Y)$ denote the set of holomorphic mappings from $\mathbb{B}_X$ to Y. A holomorphic mapping $f \in H(\mathbb{B}_X, Y)$ is said to be biholomorphic if $f(\mathbb{B}_X)$ is a domain in Y, f^{-1} exists and is holomorphic on $f(\mathbb{B}_X)$. Let $L(X, Y)$ denote the set of continuous linear operators from X into Y. Let I_X be the identity in $L(X) = L(X, X)$. For a linear operator $A \in L(X, Y)$, let

$$\|A\|_{X,Y} = \sup\{\|Az\|_Y : \|z\|_X = 1\},$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on X and Y, respectively. In the case $Y = \mathbb{C}^n$ is the Euclidean space, we write $\|A\|_{X,e}$ for $A \in L(X, \mathbb{C}^n)$. For $x \in X \setminus \{0\}$, the set

$$T(x) = \{\ell_x \in X^* : \ \ell_x(x) = \|x\|_X, \ \|\ell_x\|_{X^*} = 1\}$$

of support functionals of x is nonempty by the Hahn–Banach theorem.

A complex Banach space X is called a JB*-triple (see, e.g., [7]), if it is a complex Banach space equipped with a continuous Jordan triple product

$$X \times X \times X \to X \qquad (x, y, z) \mapsto \{x, y, z\}$$

satisfying

(J$_1$) $\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,
(J$_2$) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
(J$_3$) $x\Box x \in L(X, X)$ is a Hermitian operator with spectrum $\geqq 0$,
(J$_4$) $\|\{x, x, x\}\| = \|x\|^3$

for $a, b, x, y, z \in X$, where the *box operator* $x\Box y : X \to X$ is defined by $x\Box y(\cdot) = \{x, y, \cdot\}$ and $\|\cdot\|$ is the norm on X.

Example 1 (see, e.g., [7])

(i) A complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ is a JB*-triple with the triple product $\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$.
(ii) The unit polydisc $\mathbb{U}^n$ is the unit ball of the JB*-triple $\mathbb{C}^n$ with the triple product $\{x, y, z\} = (x_i\overline{y_i}z_i)_{1\le i\le n}$, $x = (x_i)_{1\le i\le n}$, $y = (y_i)_{1\le i\le n}$, $z = (z_i)_{1\le i\le n} \in \mathbb{C}^n$.

We refer to [7, 19] and [27] for relevant details of JB*-triples and references. We recall some of them which will be needed later.

For every $x, y \in X$, let

$$B_X(x, y) = I_X - 2x\Box y + Q_x Q_y$$

be the Bergman operator $B_X(x, y) \in L(X)$, where $Q_a : X \to X$ is the conjugate linear operator defined by $Q_a(x) = \{a, x, a\}$. Throughout this section, we write $B(x, y)$ instead of $B_X(x, y)$ for simplicity. When $\|x\Box y\| < 1$, the fractional power $B(x, y)^r \in L(X)$ exists for every $r \in \mathbb{R}$, since the spectrum of $B(x, y)$ lies in $\{\zeta \in \mathbb{C} : |\zeta - 1| < 1\}$ (cf. [15, p.517]).

Let $\mathbb{B}_X$ be the unit ball of a JB*-triple X. For each $a \in \mathbb{B}_X$, let

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x\Box a)^{-1}x$$

be the Möbius transformation. Then g_a is a biholomorphic mapping of $\mathbb{B}_X$ onto itself with $g_a(0) = a$, $g_a(-a) = 0$ and $g_{-a} = g_a^{-1}$.

In the rest of this paper, we assume that $\dim X < \infty$. The authors [13, Lemma 2.2] obtained the following estimate for $\|B(a, a)^{-\alpha/2}\|$ on the unit ball of a finite dimensional JB*-triple (see [16, Corollary 3.6] and [14], in the case $\alpha = 1$).

Lemma 1 *Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X. Then, for any $\alpha > 0$, we have $\|B(a, a)^{-\alpha/2}\| = \frac{1}{(1-\|a\|^2)^\alpha}$, $a \in \mathbb{B}_X$.*

Definition 1 (see [13]) Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X and let $\alpha > 0$. A function $f \in H(\mathbb{B}_X, \mathbb{C})$ is called an α-Bloch function if

$$\|f\|_\alpha + |f(0)| < +\infty,$$

where $\|f\|_\alpha = \sup_{z\in\mathbb{B}_X} \|Df(z)B(z,z)^{\alpha/2}\|_{X,e}$ is the α-Bloch semi-norm of f.

The following lemma was proved in [13, Lemma 2.6]. This result is a generalization of [11, Lemma 2.8] to the case of α-Bloch functions. For bounded holomorphic functions on $\mathbb{B}_X$, see also [12, Theorem 4.6] and [9, Lemma 3.12].

Lemma 2 *Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X and let $\alpha > 0$. If $f \in H(\mathbb{B}_X, \mathbb{C})$ is an α-Bloch function, then $\|Df(z)\|_{X,e} \le \frac{\|f\|_\alpha}{(1-\|z\|^2)^\alpha}$, $z \in \mathbb{B}_X$.*

Remark 1 (see [13])

(i) Any α-Bloch function on $\mathbb{B}_X$ is also a β-Bloch function on $\mathbb{B}_X$ for $\alpha \le \beta$. Since 1-Bloch functions are equivalent to Bloch functions [11], cf. [29], it follows that any Bloch function is also an α-Bloch function, for $\alpha \ge 1$.
(ii) Taking into account the Cauchy integral formula for holomorphic functions, it is not difficult to deduce that the bounded functions in $H(\mathbb{B}_X, \mathbb{C})$ are Bloch functions, so they are also α-Bloch functions for $\alpha \ge 1$.
(iii) In view of Lemma 2, α-Bloch functions are bounded on $\mathbb{B}_X$ for $\alpha \in (0, 1)$.

Definition 2 Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X and let $\alpha > 0$. Let $f \in H(\mathbb{B}_X, \mathbb{C}^n)$. The mapping f is called an α-Bloch mapping in the sense of Chen et al. [6] if $\|f(0)\|_e + \sup_{z\in\mathbb{B}^n}(1-\|z\|^2)^\alpha \|Df(z)\|_{X,e} < \infty$.

Remark 2 Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X and let $\alpha > 0$. Let $f \in H(\mathbb{B}_X, \mathbb{C})$ be an α-Bloch function in the sense of Definition 1. Then, by Lemma 2, f is also an α-Bloch function in the sense of Definition 2 (see also [13]).

Let $\mathscr{B}_X^\alpha$ be the space of α-Bloch functions $f : \mathbb{B}_X \to \mathbb{C}$. We obtain the following result (see [2, 9] and [29] in the case $\alpha = 1$).

Proposition 1 *Let $\alpha > 0$ and $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple X. Then, $\mathscr{B}_X^\alpha$ is a complex Banach space with respect to the norm $\|\cdot\|_{\mathscr{B}_X^\alpha}$ given by*

$$\|f\|_{\mathscr{B}_X^\alpha} = |f(0)| + \|f\|_\alpha, \quad f \in \mathscr{B}_X^\alpha.$$

Proof Let (f_k) be a Cauchy sequence in $\mathscr{B}_X^\alpha$. By using Lemma 2, we deduce that

$$\begin{aligned}|f_k(z) - f_p(z)| &\le |f_k(0) - f_p(0)| + \int_0^1 \|Df_k(tz) - Df_p(tz)\|_{X,e}\|z\|dt \\ &\le |f_k(0) - f_p(0)| + \frac{\|f_k - f_p\|_\alpha}{(1-\|z\|^2)^\alpha},\end{aligned}$$

for all $z \in \mathbb{B}_X$ and $k, p \in \mathbb{N}$. Then, (f_k) is a Cauchy sequence in $H(\mathbb{B}_X, \mathbb{C})$. Hence there is a function $f \in H(\mathbb{B}_X, \mathbb{C})$ such that $f_k \to f$ locally uniformly on $\mathbb{B}_X$ as $k \to \infty$.

Next, we prove that $\|f_k - f\|_{\mathscr{B}_X^\alpha} \to 0$ as $k \to \infty$. To this end, fix $\varepsilon > 0$. Since (f_k) is a Cauchy sequence in $\mathscr{B}_X^\alpha$, there is $k_0 \in \mathbb{N}$ such that $|f_k(0) - f_p(0)| + \|f_k - f_p\|_\alpha < \varepsilon$, for $k, p \geq k_0$. Then

$$|f_k(0) - f_p(0)| + \|(Df_k(z) - Df_p(z))B(z,z)^{\alpha/2}\|_{X,e} < \varepsilon,\ z \in \mathbb{B}_X,\ k, p \geq k_0.$$

On the other hand, since $f_k \to f$ locally uniformly on $\mathbb{B}_X$ as $k \to \infty$, letting $k \to \infty$ in the above inequality, we have

$$|f(0) - f_p(0)| + \|(Df(z) - Df_p(z))B(z,z)^{\alpha/2}\|_{X,e} \leq \varepsilon, \quad p \geq k_0, \quad z \in \mathbb{B}_X.$$

Consequently, $\|f_p - f\|_{\mathscr{B}_X^\alpha} \leq \varepsilon$, $p \geq k_0$, and thus $\lim_{p\to\infty} \|f_p - f\|_{\mathscr{B}_X^\alpha} = 0$.

Finally, if $p \geq k_0$, then

$$\|f\|_{\mathscr{B}_X^\alpha} \leq \|f - f_p\|_{\mathscr{B}_X^\alpha} + \|f_p\|_{\mathscr{B}_X^\alpha} \leq \varepsilon + \|f_p\|_{\mathscr{B}_X^\alpha} < \infty.$$

Therefore, we have proved that $f \in \mathscr{B}_X^\alpha$ and $f_k \to f$ in the α-Bloch norm. Hence $\mathscr{B}_X^\alpha$ is a complex Banach space, as desired. This completes the proof. □

3 Composition Operators

In this section we are concerned with composition operators associated with α-Bloch functions on the unit ball $\mathbb{B}_X$ of a finite dimensional JB*-triple X. The problem related to the preservation of boundedness or compactness in the case of composition operators between Bloch type spaces on the unit disc in $\mathbb{C}$ has been intensively studied by many authors [21, 25] and [22]. On the unit ball in $\mathbb{C}^n$, this problem has been investigated in [28] and [31] (cf. [32]). We shall obtain sufficient conditions for a composition operator to be bounded or compact between Bloch type spaces on the unit balls $\mathbb{B}_X$ and $\mathbb{B}_Y$ of finite dimensional JB*-triples X and Y, respectively.

For a holomorphic mapping $\varphi : \mathbb{B}_X \to \mathbb{B}_Y$, we define the composition operator C_φ, induced by φ, by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(\mathbb{B}_Y, \mathbb{C}).$$

We first obtain the following result related to the boundedness of the operator C_φ. In the case $\mathbb{B}_X = \mathbb{B}_Y = \mathbb{U} \subset \mathbb{C}$, we have $B_{\mathbb{C}}(z,z) = (1 - \|z\|^2)^2$. Therefore, the following theorem reduces to the corresponding results in [20, 25] and [26].

Theorem 1 *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be the unit balls of finite dimensional JB*-triples X and Y, respectively. Let $\varphi : \mathbb{B}_X \to \mathbb{B}_Y$ be a holomorphic mapping and let $\alpha > 0$, $\beta > 0$.*

(*i*) *If*

$$\sup_{z\in\mathbb{B}_X} \left\| B_Y(\varphi(z),\varphi(z))^{-\alpha/2} D\varphi(z) B_X(z,z)^{\beta/2} \right\|_{X,Y} = M < +\infty, \quad (1)$$

then C_φ maps $\mathscr{B}_Y^\alpha$ boundedly into $\mathscr{B}_X^\beta$.

(*ii*) *If C_φ maps $\mathscr{B}_Y^1$ boundedly into $\mathscr{B}_X^\beta$, then*

$$\sup_{z\in\mathbb{B}_X} \left\| B_Y(\varphi(z),\varphi(z))^{-1/2} D\varphi(z) B_X(z,z)^{\beta/2} \right\|_{X,Y} = M < +\infty. \quad (2)$$

Proof First, assume that the relation (1) holds. Let $f \in \mathscr{B}_Y^\alpha$. By Lemma 2, we have

$$|f(\varphi(0))| \le |f(0)| + \int_0^1 |Df(t\varphi(0))\varphi(0)|dt \le |f(0)| + A_\varphi \|f\|_\alpha,$$

where $A_\varphi = \int_0^1 \frac{\|\varphi(0)\|_Y}{(1-t^2\|\varphi(0)\|_Y^2)^\alpha} dt$. Also, we have

$$\begin{aligned}\|f\circ\varphi\|_\beta &= \sup_{z\in\mathbb{B}_X} \|Df(\varphi(z))D\varphi(z)B_X(z,z)^{\beta/2}\|_{X,e} \\ &\le M \sup_{z\in\mathbb{B}_X} \|Df(\varphi(z))B_Y(\varphi(z),\varphi(z))^{\alpha/2}\|_{Y,e} \le M\|f\|_\alpha.\end{aligned}$$

Thus, $C_\varphi : \mathscr{B}_Y^\alpha \to \mathscr{B}_X^\beta$ is bounded.

Next, assume that $C_\varphi : \mathscr{B}_Y^1 \to \mathscr{B}_X^\beta$ is bounded. Let $z \in \mathbb{B}_X \setminus \{0\}$ be fixed and let $g_{-a} = g_a^{-1}$ be the Möbius transformation of $\mathbb{B}_Y$ such that $Dg_{-a}(a) = B_Y(a,a)^{-1/2}$, where $a = \varphi(z)$. Let $F = g_{-a} - g_{-a}(0)$. Since

$$\|DF(w)B_Y(w,w)^{1/2}\|_{Y,Y} = \|D(F\circ g_w)(0)\|_{Y,Y} \le 2, \quad w \in \mathbb{B}_Y,$$

by the Schwarz lemma, $\ell_b \circ F$ belongs to $\mathscr{B}_Y^1$ for all $\ell_b \in T(b)$ and all $b \in Y \setminus \{0\}$. Since $C_\varphi : \mathscr{B}_Y^1 \to \mathscr{B}_X^\beta$ is bounded, there exists a constant $c > 0$ such that $\|l_b \circ F \circ \varphi\|_\beta \le c\|l_b \circ F\|_1 \le 2c$ for all $\ell_b \in T(b)$ and all $b \in Y \setminus \{0\}$. Therefore, we have

$$\left\| l_b \circ B_Y(a,a)^{-1/2} D\varphi(z) B_X(z,z)^{\beta/2} \right\|_{X,e} = \left\| D(l_b \circ F \circ \varphi)(z) B_X(z,z)^{\beta/2} \right\|_{X,e} \le 2c.$$

Since $\ell_b \in T(b)$, $b \in Y \setminus \{0\}$ and $z \in \mathbb{B}_X$ are arbitrary and c is independent from $\ell_b \in T(b)$, $b \in Y \setminus \{0\}$ and $z \in \mathbb{B}_X$, we obtain (2), as desired. The proof is complete. □

Remark 3 Let $\varphi : \mathbb{B}_X \to \mathbb{B}_Y$ be a holomorphic mapping with $L = \sup_{z\in\mathbb{B}_X} \|\varphi(z)\|_Y < 1$. Also, let $\alpha > 0$ and $\beta \geq 1$. Then the condition (1) holds. Especially, if f is an α-Bloch function on $\mathbb{B}_Y$, then $C_\varphi(f)$ is also an α-Bloch function on $\mathbb{B}_X$ for $\alpha \geq 1$ by Theorem 1.

Proof Since $\beta \geq 1$, by Chu et al. [9, Proposition 3.14] and Remark 1(i), if follows that $l_b \circ \varphi$ is a β-Bloch function on $\mathbb{B}_X$ such that $\|l_b \circ \varphi\|_\beta \leq L$ for any $\ell_b \in T(b)$, $b \in Y \setminus \{0\}$. Since $\ell_b \in T(b)$ and $b \in Y \setminus \{0\}$ are arbitrary, we have $\|D\varphi(z)B_X(z,z)^{\beta/2}\|_{X,Y} \leq L$. Taking into account Lemma 1, we obtain that

$$\begin{aligned}&\|B_Y(\varphi(z),\varphi(z))^{-\alpha/2}D\varphi(z)B_X(z,z)^{\beta/2}\|_{X,Y}\\&\leq \|B_Y(\varphi(z),\varphi(z))^{-\alpha/2}\|\cdot\|D\varphi(z)B_X(z,z)^{\beta/2}\|_{X,Y} \leq \frac{L}{(1-L^2)^\alpha}.\end{aligned}$$

Hence, $\sup_{z\in\mathbb{B}_X} \|B_Y(\varphi(z),\varphi(z))^{-\alpha/2}D\varphi(z)B_X(z,z)^{\beta/2}\| \leq \frac{L}{(1-L^2)^\alpha}$, as desired. □

Next, we consider the compactness of the composition operator C_φ. In the case $\mathbb{B}_X = \mathbb{B}_Y = \mathbb{U} \subset \mathbb{C}$, the following theorem reduces to the corresponding results in [21] and [25].

Theorem 2 *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be the unit balls of finite dimensional JB*-triples X and Y, respectively. Let $\varphi : \mathbb{B}_X \to \mathbb{B}_Y$ be a holomorphic mapping and let $\beta > 0$.*

(*i*) *If $\alpha > 0$ and the condition (1) holds, and if*

$$\left\|B_Y(\varphi(z),\varphi(z))^{-\alpha/2}D\varphi(z)B_X(z,z)^{\beta/2}\right\|_{X,Y} \to 0 \text{ as } \varphi(z) \to \partial\mathbb{B}_Y, \tag{3}$$

then C_φ is a compact operator of $\mathscr{B}^\alpha_Y$ into $\mathscr{B}^\beta_X$.

(*ii*) *If $\mathbb{B}_Y = \mathbb{B}^n$ and C_φ is a compact operator of $\mathscr{B}^1 = \mathscr{B}^1_{\mathbb{C}^n}$ into $\mathscr{B}^\beta_X$, then*

$$\left\|B(\varphi(z),\varphi(z))^{-1/2}D\varphi(z)B_X(z,z)^{\beta/2}\right\|_{X,e} \to 0 \text{ as } \varphi(z) \to \partial\mathbb{B}^n. \tag{4}$$

Proof First, assume that the conditions (1) and (3) hold. Let $f_k \in \mathscr{B}^\alpha_Y$ be such that $\|f_k\|_{\mathscr{B}^\alpha_Y} = 1$ for $k = 1, 2, \ldots$. By Lemma 2 and Montel's theorem, by choosing a subsequence, we may assume that the sequence (f_k) converges to a holomorphic function f locally uniformly on $\mathbb{B}_Y$. Then $h_k = f_k - f \to 0$ locally uniformly on $\mathbb{B}_Y$ and

$$\|h_k\|_{\mathscr{B}^\alpha_Y} \leq 2 \text{ for } k = 1, 2, \ldots. \tag{5}$$

Let $\varepsilon > 0$ be given. By the assumption (3), there exists an $r_0 \in (0,1)$ such that

$$\left\|B_Y(\varphi(z),\varphi(z))^{-\alpha/2}D\varphi(z)B_X(z,z)^{\beta/2}\right\|_{X,Y} < \varepsilon \text{ if } \|\varphi(z)\|_Y > r_0. \tag{6}$$

Since $h_k(w) \to 0$ uniformly for $\|w\|_Y \le (r_0+1)/2$, there exists a K such that

$$\|Dh_k(w)B_Y(w,w)^{\alpha/2}\|_{Y,e} < \varepsilon \text{ for } k > K, \|w\|_Y \le r_0. \tag{7}$$

If $\|\varphi(z)\|_Y > r_0$, then we obtain from (5) and (6) that

$$\|Dh_k(\varphi(z))D\varphi(z)B_X(z,z)^{\beta/2}\|_{X,e} \le \varepsilon \|h_k\|_{\mathscr{B}_Y^\alpha} \le 2\varepsilon.$$

If $\|\varphi(z)\|_Y \le r_0$, then, from (1) and (7), for $k > K$, we have

$$\|Dh_k(\varphi(z))D\varphi(z)B_X(z,z)^{\beta/2}\|_{X,e} \le M\|Dh_k(\varphi(z))B_Y(\varphi(z),\varphi(z))^{\alpha/2}\|_{Y,e} \le M\varepsilon.$$

Therefore, $\|h_k \circ \varphi\|_{\mathscr{B}_X^\beta} = |h_k(\varphi(0))| + \|h_k \circ \varphi\|_\beta \to 0$ as $k \to \infty$. Thus C_φ is compact.

Next, assume that $C_\varphi : \mathscr{B}^1 \to \mathscr{B}_X^\beta$ is compact. Suppose the condition (4) does not hold. Then there exist $\delta > 0$ and $z_k \in \mathbb{B}_X$ such that $a_k = \varphi(z_k) \to \partial\mathbb{B}^n$ as $k \to \infty$ and

$$\left\| B(\varphi(z_k),\varphi(z_k))^{-1/2} D\varphi(z_k) B_X(z_k,z_k)^{\beta/2} \right\|_{X,e} > \delta \text{ for } k = 1, 2, \dots. \tag{8}$$

Let $F_k = g_{-a_k} - g_{-a_k}(0)$, where g_{-a_k} is the Möbius transformation of $\mathbb{B}^n$. Then, $F_k(z) = B(-a_k,-a_k)^{1/2}(I_n - z\square a_k)^{-1}z$. Since $\|B(-a_k,-a_k)^{1/2}\| \le \sqrt{1-\|a_k\|^2}$ by Chu et al. [8, Lemma 2.3], the sequence (F_k) converges to 0 locally uniformly on $\mathbb{B}^n$ and thus, $(F_k \circ \varphi)$ converges to 0 locally uniformly on $\mathbb{B}_X$. Let $F_k = (F_k^1, \dots, F_k^n)$. Since C_φ is compact and (F_k^j) is a bounded sequence in $\mathscr{B}^1$, by choosing a subsequence, there exists $f^j \in \mathscr{B}_X^\beta$ such that $(F_k^j \circ \varphi)$ converges to f^j in $\mathscr{B}_X^\beta$ for each $1 \le j \le n$. Then f^j must be identically equal to 0 because the sequence $(F_k \circ \varphi)$ converges to 0 locally uniformly on $\mathbb{B}_X$. Thus, $\|F_k^j \circ \varphi\|_{\mathscr{B}_X^\beta} \to 0$ as $k \to \infty$. Therefore, we have

$$\begin{aligned} &\left\| B(\varphi(z_k),\varphi(z_k))^{-1/2} D\varphi(z_k) B_X(z_k,z_k)^{\beta/2} \right\|_{X,e} \\ &= \left\| D(F_k \circ \varphi)(z_k) B_X(z_k,z_k)^{\beta/2} \right\|_{X,e} \to 0 \text{ as } k \to \infty. \end{aligned}$$

This is a contradiction. The proof is complete. □

Acknowledgements The authors would like to give thanks to the referee for useful suggestions which improved the paper. H. Hamada was partially supported by JSPS KAKENHI Grant Number JP16K05217. G. Kohr was partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0899.

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Monogenic Functions in Commutative Algebras

Vitalii Shpakivskyi

Abstract Let $\mathbf{A}_n$ be an arbitrary n-dimensional commutative associative algebra over the field of complex numbers. Let $e_1 = 1$, e_2, e_3 be elements of $\mathbf{A}_n$ which are linearly independent over the field of real numbers. We consider monogenic (i.e., continuous and differentiable in the sense of Gateaux) functions of the variable $xe_1 + ye_2 + ze_3$, where x, y, z are real, and obtain a constructive description of all mentioned monogenic functions by means of holomorphic functions of complex variables. It follows from this description that monogenic functions have Gateaux derivatives of all orders. The relations between monogenic functions and partial differential equations are investigated.

1 Introduction

Probably, Ketchum (see [1]) made the first attempt to use analytic functions in commutative algebras for constructing solutions of the three-dimensional Laplace equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0\,. \tag{1}$$

He showed that every analytic function $\Phi(\zeta)$ of the variable $\zeta = xe_1 + ye_2 + ze_3$ satisfies Eq. (1) in the case where the elements e_1, e_2, e_3 of the commutative algebra satisfy the condition

$$e_1^2 + e_2^2 + e_3^2 = 0\,, \tag{2}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_16

because

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}\equiv\Phi''(\zeta)\,(e_1^2+e_2^2+e_3^2)=0\,,\tag{3}$$

where $\Phi'':=(\Phi')'$ and $\Phi'(\zeta)$ is defined by the equality $d\Phi=\Phi'(\zeta)d\zeta$.

Generalized by Ketchum, Roşculeţ (see [2, 3]) used analytic functions in commutative algebras for investigating the equations of the form

$$\mathscr{L}_N U(x,y,z):=\sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\,\frac{\partial^N U}{\partial x^\alpha\,\partial y^\beta\,\partial z^\gamma}=0,\quad C_{\alpha,\beta,\gamma}\in\mathbf{R}.\tag{4}$$

For a mixed derivative the following equality is true:

$$\frac{\partial^{\alpha+\beta+\gamma}\Phi}{\partial x^\alpha\,\partial y^\beta\,\partial z^\gamma}=e_1^\alpha\,e_2^\beta\,e_3^\gamma\,\Phi^{(\alpha+\beta+\gamma)}(\zeta)=e_1^\alpha\,e_2^\beta\,e_3^\gamma\,\Phi^{(N)}(\zeta),\tag{5}$$

$$\zeta=xe_1+ye_2+ze_3\,.$$

Substituting (5) into Eq. (4), we have

$$\mathscr{L}_N\Phi(\zeta)=\Phi^{(N)}(\zeta)\sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\,e_1^\alpha\,e_2^\beta\,e_3^\gamma$$

and for satisfying the equality $\mathscr{L}_N\Phi(\zeta)=0$, we have the condition:

$$\sum_{\alpha+\beta+\gamma=N} C_{\alpha,\beta,\gamma}\,e_1^\alpha\,e_2^\beta\,e_3^\gamma=0\,.\tag{6}$$

Thus, under the condition (6) every analytic function Φ with values in an arbitrary commutative associative algebra satisfies Eq. (4), and, respectively, all real-valued components of the function Φ are solutions of Eq. (4).

Problem Which form has analytic functions in an arbitrary commutative algebra? How to describe it constructively?

What means a *constructive description of analytic functions*?

2 Examples

Example 1 Let $\mathbf{P}:=\{x+jy\,:\,j^2:=1, x,y\in\mathbf{R}\}$ be the algebra of double (or hyperbolic) numbers over the field $\mathbf{R}$. In the algebra $\mathbf{P}$ there exists a basis $\{I_1,I_2\}$ such that $I_1^2=I_1$, $I_2^2=I_2$, $I_1I_2=0$ and $I_1+I_1=1$. In this case, $z=x+jy=(x+y)I_1+(x-y)I_2$.

The following result is known (see [4–6]):

Every analytic function $\Phi : D \to \mathbf{P}$ *of the form* $\Phi(z) = u(x, y) + jv(x, y)$ *with analytic in* D *components* u, v *can be represented in the form*

$$\Phi(z) = F_1(x - y)I_2 + F_2(x + y)I_1, \tag{7}$$

where $F_1(x - y)$ *and* $F_2(x + y)$ *are certain real analytic functions on the intervals* $\Delta_1 := \{x - y \in \mathbf{R} : x + jy \in D\}$ *and* $\Delta_2 = \{x + y \in \mathbf{R} : x + jy \in D\}$*, respectively.*

The formula (7) is a constructive description of analytic function in the algebra $\mathbf{P}$.

Corollary 1 *If a domain* D *is convex in the direction of the straight lines* $y = x$*,* $y = -x$*, then every analytic function* $\Phi : D \to \mathbf{P}$ *can be continued to a function analytic in the domain*

$$\Pi := \{x + jy \in \mathbf{P} : x - y \in \Delta_1\} \cap \{x + jy \in \mathbf{P} : x + y \in \Delta_2\}.$$

Example 2 Let $\mathbf{D} := \{z := x + \delta y : \delta^2 := 0, x, y \in \mathbf{R}\}$ be the algebra of dual numbers over the field $\mathbf{R}$. The following result is known (see, e.g., [7]):

Every analytic function $\Phi : \Omega \to \mathbf{D}$ *of the form* $\Phi(z) = u(x, y) + \delta v(x, y)$ *with analytic in* Ω *components* u, v *can be represented in the form*

$$\Phi(z) = u(x) + \big(yu'(x) + k(x)\big)\delta\,, \tag{8}$$

where $u(x)$ *and* $k(x)$ *are certain real analytic functions on the interval* $\Delta := \{x \in \mathbf{R} : x + \delta y \in \Omega\}$*.*

The formula (8) is a constructive description of analytic function in the algebra $\mathbf{D}$.

Corollary 2 *If a domain* Ω *is convex in the direction of the axis* Oy*, then every analytic function* $\Phi : \Omega \to \mathbf{D}$ *can be continued to a function analytic in the domain* $\Pi := \{x + \delta y \in \mathbf{D} : x \in \Delta\}$*.*

Now, we consider the same algebras over the field $\mathbf{C}$.

Example 3 Let $\mathbf{BC} := \{\zeta := z_1 I_1 + z_2 I_2 : z_1, z_2 \in \mathbf{C}\}$ be the algebra of bicomplex numbers (or commutative Segre's quaternions):

$\cdot$	I_1	I_2
I_1	I_1	0
I_2	0	I_2

It is known (see, e.g., [8]) that every analytic function $\Phi : \Omega \to \mathbf{BC}$ can be represented in the form

$$\Phi(\zeta) = F_1(z_1)\, I_1 + F_2(z_2)\, I_2\,, \tag{9}$$

where F_k are holomorphic functions in certain domains $D_k \subset \mathbf{C}$.

The formula (9) is a constructive description of analytic function in the algebra **BC**.

This result is generalized in the paper [9] for the n-dimensional semi-simple algebra $\mathbf{A}_n$:

$\cdot$	I_1	I_2	$\dots$	I_n
I_1	I_1	0	$\dots$	0
I_2	0	I_2	$\dots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
I_n	0	0	$\dots$	I_n

In $\mathbf{A}_n$ every analytic function Φ is of the form

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + \cdots + F_n(\xi_n)I_n\,,$$

where F_k are holomorphic functions in certain domains $D_k \subset \mathbf{C}$, and $\zeta = \xi_1\,I_1 + \xi_2\,I_2 + \cdots \xi_n\,I_n, \quad \xi_k \in \mathbf{C},\ k = 1, 2, \dots, n.$

Example 4 Consider the algebra **B** over the field **C** with the following multiplication table:

$\cdot$	1	ρ
1	1	ρ
ρ	ρ	0

In this algebra every analytic function Φ is of the form (see [10]):

$$\Phi(\zeta) = F(\xi_1) + \Big[\xi_2 F'(\xi_1) + F_0(\xi_1)\Big]\rho,$$

where F, F_0 are holomorphic functions in a certain domain in **C**, and where $\zeta = \xi_1 + \xi_2\,\rho, \quad \xi_1, \xi_2 \in \mathbf{C}$. This result was established in relation with two-dimensional biharmonic equation

$$\frac{\partial^4 \Phi}{\partial x^4} + 2\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \Phi}{\partial y^4} = 0.$$

Example 5 Consider the algebra $\mathbf{A}_3$ over the field **C** with the following multiplication table:

$\cdot$	1	ρ_1	ρ_2
1	1	ρ_1	ρ_2
ρ_1	ρ_1	ρ_2	0
ρ_2	ρ_2	0	0

In this algebra every analytic function Φ is of the form (see [11]):

$$\Phi(\zeta) = F(\xi_1) + \Big[\xi_2 F'(\xi) + F_1(\xi_1)\Big]\rho_1 +$$

$$+\Big[\xi_3 F'(\xi_1) + \frac{\xi_1^2}{2} F''(\xi_1) + \xi_2 F_1'(\xi_1) + F_2(\xi_1)\Big]\rho_2,$$

where F, F_1, F_2 are holomorphic functions in a certain domain in $\mathbf{C}$, and $\zeta = \xi_1 + \xi_2\,\rho_1 + \xi_3\,\rho_2,\quad \xi_1, \xi_2, \xi_3 \in \mathbf{C}$.

Example 6 Consider the following algebra over the field $\mathbf{C}$:

$\cdot$	I_1	I_2	ρ
I_1	I_1	0	0
I_2	0	I_2	ρ
ρ	0	ρ	0

In this algebra every analytic function Φ is of the form (see [12]):

$$\Phi(\zeta) = F_1(\xi_1)I_1 + F_2(\xi_2)I_2 + \Big[\xi_3 F_2'(\xi_2) + F_0(\xi_2)\Big]\rho,$$

where F_0, F_1, F_2 are holomorphic functions in certain domains in $\mathbf{C}$, and $\zeta = \xi_1\,I_1 + \xi_2\,I_2 + \xi_3\,\rho,\quad \xi_1, \xi_2, \xi_3 \in \mathbf{C}$.

3 Main Result

Mel'nichenko [13] proposed for describing solutions of Eq. (4) to use hypercomplex functions differentiable in the sense of Gateaux. Such functions will be called *monogenic*. We obtain a constructive description of all monogenic functions in an arbitrary commutative associative algebra.

Now we will give an exact definition of *monogenic function*.

3.1 Monogenic Function

Let $\mathbf{A}$ be an arbitrary n-dimensional ($2 \le n < \infty$) commutative associative algebra with unit over the field of complex number $\mathbf{C}$. By Cartan's theorem [14] in $\mathbf{A}$ there exist the basis $\{I_k\}_{k=1}^n$ such that first m vectors $\{I_u\}_{u=1}^m$ are idempotents and form a semi-simple subalgebra of $\mathbf{A}$, and the vectors $\{I_r\}_{r=m+1}^n$ are nilpotents and form a nilpotent subalgebra of $\mathbf{A}$.

The algebra $\mathbf{A}$ contains m maximal ideals

$$\mathscr{I}_u := \left\{ \sum_{k=1,\, k\neq u}^{n} \lambda_k I_k : \lambda_k \in \mathbf{C} \right\}, \quad u = 1, 2, \ldots, m.$$

We define m linear functionals $f_u : \mathbf{A} \to \mathbf{C}$ by the equalities

$$f_u(I_u) = 1, \quad f_u(\omega) = 0 \quad \forall\, \omega \in \mathscr{I}_u\,, \quad u = 1, 2, \ldots, m.$$

Inasmuch as the kernel of functional f_u is the maximal ideal $\mathscr{I}_u$, this functional is also continuous and multiplicative.

Now we consider in the algebra $\mathbf{A}$ a triple of vectors $e_1 = 1, e_2, e_3$ which are linearly independent over the field of real numbers $\mathbf{R}$. Let

$$e_1 = 1, \quad e_2 = \sum_{r=1}^{n} a_r I_r\,, \quad e_3 = \sum_{r=1}^{n} b_r I_r$$

where $a_r\,, b_r \in \mathbf{C}$.

Consider a linear span $E_3 := \{\zeta = x + ye_2 + ze_3 : x, y, z \in \mathbf{R}\}$ over the field $\mathbf{R}$, generated by the vectors $1, e_2, e_3$.

Let Ω be a domain in $\mathbf{R}^3$. Associate with Ω the domain $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \Omega\}$ in E_3.

Definition 1 The continuous function $\Phi : \Omega_\zeta \to \mathbf{A}$ is *monogenic* in Ω_ζ if Φ is differentiable in the sense of Gateaux in every point of Ω_ζ, i. e. if for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbf{A}$ such that

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\Phi'(\zeta) \quad \forall\, h \in E_3\,. \tag{10}$$

$\Phi'(\zeta)$ is the *Gateaux derivative* of Φ at the point ζ.

3.2 Constructive Description of Monogenic Functions

Consider the decomposition of a function $\Phi : \Omega_\zeta \to \mathbf{A}$ with respect to the basis $\{I_k\}_{k=1}^n$:

$$\Phi(\zeta) = \sum_{k=1}^{n} U_k(x, y, z)\, I_k\,.$$

In the case where the components $U_k : \Omega \to \mathbf{C}$ are $\mathbf{R}$-differentiable in Ω, the function Φ is monogenic in the domain Ω_ζ if and only if the following Cauchy – Riemann conditions are satisfied:

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \Phi}{\partial x} e_2 , \qquad \frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial x} e_3 .$$

We note that the points $(x, y, z) \in \mathbf{R}^3$ corresponding to the noninvertible elements $\zeta = xe_1 + ye_2 + ze_3$ form the straight lines in $\mathbf{R}^3$:

$$L_u : \quad x + y \operatorname{Re} a_u + z \operatorname{Re} b_u = 0, \quad y \operatorname{Im} a_u + z \operatorname{Im} b_u = 0.$$

Let a domain $\Omega \subset \mathbf{R}^3$ be convex in the direction of the straight lines L_u, $u = 1, 2, \dots, m$. By D_u we denote that domain in $\mathbf{C}$ onto which the domain Ω_ζ is mapped by the functional f_u.

Theorem 1 (cf., e.g., [15, 16]) *Let a domain $\Omega \subset \mathbf{R}^3$ be convex in the direction of the straight lines L_u and $f_u(E_3) = \mathbf{C}$ for all $u = 1, 2, \dots, m$. Then every monogenic function $\Phi : \Omega_\zeta \to \mathbf{A}$ can be expressed in the form*

$$\Phi(\zeta) = \sum_{u=1}^{m} I_u \frac{1}{2\pi i} \int_{\Gamma_u} F_u(t)(te_1 - \zeta)^{-1} \, dt + \sum_{s=m+1}^{n} I_s \frac{1}{2\pi i} \int_{\Gamma_{u_s}} G_s(t)(te_1 - \zeta)^{-1} \, dt,$$

where F_u and G_s are certain holomorphic functions in the domains D_u and D_{u_s}, respectively, and Γ_q is a closed Jordan rectifiable curve in D_q which surrounds the point ξ_q and contains no points ξ_ℓ, $\ell, q = 1, 2, \dots, m$, $\ell \neq q$.

Thus, Theorem 1 specifies method to construct explicitly any monogenic function $\Phi : \Omega_\zeta \to \mathbf{A}$ using n corresponding holomorphic functions of complex variables.

Remark All constructive descriptions that were considered in *Examples* are partial cases of Theorem 1.

The following statement follows immediately from Theorem 1.

Corollary 3 *Let a domain Ω be convex in the directions of the straight lines L_u and $f_u(E_3) = \mathbf{C}$ for all $u = 1, 2, \dots, m$. Then every monogenic function $\Phi : \Omega_\zeta \to \mathbf{A}$ can be continued to a function monogenic in the domain*

$$\Pi_\zeta := \{\zeta \in E_3 : f_u(\zeta) = D_u \, , \; u = 1, 2, \dots, m\}.$$

The next statement is true for an arbitrary domain Ω_ζ.

Corollary 4 *Let $f_u(E_3) = \mathbf{C}$ for all $u = 1, 2, \dots, m$. Then for every monogenic function $\Phi : \Omega_\zeta \to \mathbf{A}$ in an arbitrary domain Ω_ζ, the Gateaux r-th derivatives $\Phi^{(r)}$ are monogenic functions in Ω_ζ for all r.*

In the case where a domain Ω is convex in the directions of the straight lines L_u, $u = 1, 2, \ldots, m$, we obtain the following expression for the Gateaux r-th derivative $\Phi^{(r)}$:

$$\Phi^{(r)}(\zeta) = \sum_{u=1}^{m} I_u \frac{r!}{2\pi i} \int_{\Gamma_u} F_u(t)\Big((te_1 - \zeta)^{-1}\Big)^{r+1} dt +$$

$$+ \sum_{s=m+1}^{n} I_s \frac{r!}{2\pi i} \int_{\Gamma_{u_s}} G_s(t)\Big((te_1 - \zeta)^{-1}\Big)^{r+1} dt \qquad \forall \zeta \in \Omega_\zeta .$$

Acknowledgement This research is partially supported by Grant of Ministry of Education and Science of Ukraine (Project No. 0116U001528).

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Part VI
Special Interest Group: IGPDE Harmonic Analysis and Partial Differential Equations

Session Organizers: Michael Ruzhansky and Jens Wirth

The session focused on the interplay between harmonic analysis, operator theory, and applications to partial differential equations.

On the Solvability of Tracking Problem with Nonlinearly Distributed Control for the Oscillation Process

Elmira Abdyldaeva

Abstract In the paper we investigate the unique solvability of the tracking problem with the distributed optimal control for the elastic oscillations described by Fredholm integro-differential equations. The sufficient conditions are found for existence of a unique solution to the boundary value problem, also the class of functions of external influence for which the optimization problem has a solution. The algorithm was developed for constructing the complete solution of the tracking problem of nonlinear optimization.

1 Introduction

There are many applied problems described by integro-differential equations [1–3]. The work of Egorov [4] made it possible to investigate the optimal control problems for the systems with distributed parameters described by integro-differential equations. Later A. Kerimbekov has developed the method of solving the nonlinear optimization problem [5]. The solvability of the tracking problem of nonlinear optimization is investigated with the distributed optimal control for the elastic oscillations described by Fredholm integro-differential equations in this paper.

2 Formulation of the Optimal Control Problem and Optimality Conditions

We consider the optimization problem in which it is required to minimize the integral functional

$$J[u(t,x)] = \int_0^T \int_Q [V(t,x) - \xi(t,x)]^2\, dx dt + \beta \int_0^T \int_Q M^2[t,x,u(t,x)] dx dt, \ \ \beta > 0, \tag{1}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_17

on the set of solutions to the boundary value problem

$$V_{tt} - AV = \lambda \int_0^T K(t,\tau)V(\tau,x)d\tau + f(t,x,u(t,x)),\ x \in Q \subset R^n,\ 0 < t \leq T, \tag{2}$$

$$V(0,x) = \psi_1(x),\ V_t(0,x) = \psi_2(x),\ x \in Q, \tag{3}$$

$$\Gamma V(t,x) = \sum_{i,j=1}^{n} a_{ij}(x)V_x(t,x)\cos(\delta,x_i) + a(x)V(t,x) = 0,\ x \in \gamma,\ 0 < t \leq T. \tag{4}$$

Here A is an elliptic operator defined by the formula

$$AV(t,x) = \sum_{i,j=1}^{n} (a_{ij}(x)V_{x_j}(t,x))_{x_i} - c(x)V(t,x), a_{ij}(x) = a_{ji}(x),$$

$$\sum_{i,j=1}^{n} a_{ij}(x)\alpha_i\alpha_j \geq c_0 \sum_{i=1}^{n} \alpha_i^2,\ c_0 > 0,$$

δ is the normal vector, outgoing from the point $x \in \gamma$; $K(t,\tau)$ is a given function with domain $D = \{0 \leq t \leq T, 0 \leq \tau \leq T\}$ and satisfying the condition

$$\int_0^T \int_0^T K^2(t,\tau)dtd\tau = K_0 < \infty,$$

i.e., $K(t,s)$ is element of Hilbert space $H(D)$;

$$\psi_1(x) \in H_1(Q),\ \psi_2(x) \in H(Q),\ f_u[t,x,u(t,x)] \neq 0, \forall t \in (0,T), x \in Q, \tag{5}$$

are given functions, $a(x) \geq 0, c(x) \geq 0$ are known measurable functions; $H(Q)$ is the Hilbert space of square integrable functions defined on the set Q; $H_1(Q)$ is the Sobolev space of first-order; $f[t,x,u(t,x)]$ is the boundary source function which varies nonlinearly depending on the control functions $u(t,x) \in H(Q_T)$ and is an element of $H(0,T)$; λ is a parameter, T is a fixed moment of time; and $\alpha > 0$ is a constant. $\xi(t,x) \in H(Q_T)$, $Q_T = Q \times (0,T)$; $M[t,x,u(t,x)] \in H(Q_T)$ is a given function satisfying the Lipschitz condition with respect to functional argument $u(t,x) \in H(Q_T)$ and $M_u[t,x,u(t,x)] \neq 0$.

In this problem we need to find a control $u^0(t,x) \in H(Q_T)$ for which the corresponding solution $V^0(t,x)$ to the boundary value problem (2)–(4) deviates little from the given trajectory $\xi(t,x) \in H(Q_T)$ during the entire control time $t \in [0,T]$. Here $u^0(t,x)$ is called optimal control, and $V^0(t,x)$ is an optimal process.

We will seek the solution to problem (2)–(4) in the form

$$V(t,x)=\sum_{n=1}^{\infty}V_n(t)z_n(x), \tag{6}$$

where $V_n(t)=\langle V(t,x),z_n(x)\rangle$ are Fourier coefficients system of eigenfunctions $z_n(x)$, where $z_n(x)$ are eigenfunctions of boundary value problem $Az_n(x)=-\lambda_n^2 z_n(x),\ x\in Q,\ \Gamma z_n(x)=0,\ x\in\gamma$. Following [6], they form a complete orthonormal system in the Hilbert space $H(Q)$, and the corresponding eigenvalues λ_n satisfy $\lambda_n\le\lambda_{n+1}, \forall n=1,2,3,\ldots$, and $\lim_{n\to\infty}\lambda_n=\infty$.

Definition 1 A generalized solution of problem (2)–(4) is a function $V(t,x)\in H(Q_T)$ that satisfies the initial condition in a weak sense i.e., for any function $\phi_0(x)\in H(Q), \phi_1(x)\in H(Q)$ we have the equalities

$$\lim_{t\to+0}\int_Q V(t,x)\phi_0(x)dx=\int_Q \psi_1(x)\phi_0(x)dx,$$
$$\lim_{t\to+0}\int_Q V_t(t,x)\phi_1(x)dx=\int_Q \psi_2(x)\phi_1(x)dx,$$

and the Fourier coefficients $V_n(t)$ satisfy the linear Fredholm integral equation of the second type

$$\begin{aligned}V_n(t)=\psi_{1n}\cos\lambda_n t+\frac{1}{\lambda_n}\psi_{2n}\sin\lambda_n t+\\ +\frac{1}{\lambda_n}\int_0^t\sin\lambda_n(t-\tau)\left(\lambda\int_0^T K(\tau,s)V_n(s)ds+f_n(\tau,u)\right)d\tau,\end{aligned} \tag{7}$$

where ψ_{1n}, ψ_{2n}, and $g_n(t)$ are the Fourier coefficients of the functions $\psi_1(x)$, $\psi_2(x)$, $f(t,x,u(t,x))$, respectively.

To determine the Fourier coefficients $V_n(t)$ Eq. (7) can be rewritten as

$$V_n(t)=\lambda\int_0^T K_n(t,s)V_n(s)ds+a_n(t), \tag{8}$$

where

$$K_n(t,s)=\frac{1}{\lambda_n}\int_0^t\sin\lambda_n(t-\tau)K(\tau,s)d\tau,\ \ K_n(0,s)=0, \tag{9}$$

$$a_n(t)=\psi_{1n}\cos\lambda_n t+\frac{1}{\lambda_n}\psi_{2n}\sin\lambda_n t+\frac{1}{\lambda_n}\int_0^t\sin\lambda_n(t-\tau)f_n[\tau,u]d\tau. \tag{10}$$

We will find the solution of the integral equation (8) by the formula [7, 8]:

$$V_n(t) = \lambda \int_0^T R_n(t, s, \lambda) a_n(s) ds + a_n(t), \tag{11}$$

where

$$R_n(t, s, \lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} K_{n,i}(t, s), \; n = 1, 2, 3, \ldots, \tag{12}$$

is the resolvent of the kernel $K_n(t, s) \equiv K_{n,1}(t, s)$, and the iterated kernels $K_{n,i}(t, s)$ are defined by the formula

$$K_{n,i+1}(t, s) = \int_0^T K_n(t, \eta) K_{n,i}(\eta, s) d\eta, \; i = 1, 2, 3, \ldots, \; K_{n,1}(t, s) = K_n(t, s), \tag{13}$$

for each $n = 1, 2, 3, \ldots$. We investigate the convergence of Neumann series (12). According to (9) and (13) by direct calculation the following estimates are established:

$$|K_{n,i}(t, s)|^2 \leq \frac{T^{2i-1}}{(\lambda_n^2)^i} K_0^{i-1} \int_0^T K^2(\tau, s) d\tau, \; i = 1, 2, 3, \ldots. \tag{14}$$

Convergence of the Neumann series (12) follows from the inequality

$$|R_n(t, s, \lambda)| \leq \sum_{i=1}^{\infty} |\lambda|^{i-1} |K_{n,i}(t, s)| \leq \sqrt{T} \left(\int_0^T K^2(y, s) ds \right)^{1/2} \frac{1}{\lambda_n - |\lambda| T \sqrt{K_0}},$$

which converges for the values of the parameter that satisfy $|\lambda| \frac{T}{\lambda_n} \sqrt{K_0} < 1$. By direct calculation we establish the following inequality:

$$\int_0^T |R_n(t, s, \lambda)|^2 ds \leq \int_0^T \int_0^T K^2(y, s) dy ds \frac{1}{(\lambda_n - |\lambda| T \sqrt{K_0})^2} = \frac{K_0 T}{(\lambda_n - |\lambda| T \sqrt{K_0})^2},$$

which is later used repeatedly. Note that the Neumann series for values of the parameter λ satisfying $|\lambda| < \frac{\sqrt{2}}{\sqrt{K_0 T}} \lambda_n \xrightarrow{n \to \infty} \infty$ converges absolutely for each $n = 1, 2, 3, \ldots$, i.e., the radius of convergence increases when n is growing. In this case, as the sum of an absolutely convergent series, the resolvent $R_n(t, s, \lambda)$ is a continuous function and satisfies the following estimates:

$$|R_n(t, s, \lambda)| \leq \frac{T K_0}{(\lambda_n - |\lambda| T \sqrt{K_0})^2}, \; n = 1, 2, 3, \ldots. \tag{15}$$

Thus, we find the solution to problem (2)–(4) by formula (6), where $V_n(t)$ is defined by (11) as the unique solution of integral equation (8). It is easy to verify that this solution satisfies initial condition (2) and it is an element of Hilbert space $H(Q_T)$.

According to the condition (5), each control $u(t)$ uniquely defines the controlled process $V(t,x)$. And so the control $u(t)+\Delta u(t)$ corresponds to the solution $V(t,x)+\Delta V(t,x)$ of the boundary value problem (2)–(4), where $\Delta V(t,x)$ is the increment corresponding to increment $\Delta u(t)$. By the maximum principle [9, 10] the increment of functional (1) can be written as

$$\begin{aligned}\Delta J &= J[u+\Delta u]-J[u]\\ &= -\int_0^T\int_Q \Delta\Pi[t,x,V(t,x),\omega(t,x),u(t,x)]dxdt+\int_0^T\int_Q \Delta V^2(t,x)dxdt,\end{aligned} \tag{16}$$

where

$$\Pi[t,x,V(t,x),\omega(t,x),u(t,x)] = f[t,x,u(t,x)]\omega(t,x)-\beta M^2[t,x,u(t,x)], \tag{17}$$

and the function $\omega(t,x)$ is the solution of the adjoint boundary value problem

$$\begin{aligned}&\omega_{tt}-A\omega=\lambda\int_0^T K(t,\tau)\omega(\tau,x)d\tau-2\left[V(t,x)-\xi(t,x)\right],\\ &\omega(T,x)=0, \omega_t(T,x)=0, \quad x\in Q,\\ &\Gamma\omega(t,x)=0, \quad x\in\gamma, 0<t<T.\end{aligned} \tag{18}$$

According to the maximum principle for systems with distributed parameters [9], the optimal control is determined by the relations

$$\omega(t,x)=\frac{2\beta M(t,x,u(t,x))M_u(t,x,u(t,x))}{f_u(t,x,u(t,x))}, \tag{19}$$

$$f_u(t,x,u(t,x))\left(\frac{M(t,x,u(t,x))M_u(t,x,u(t,x))}{f_u(t,x,u(t,x))}\right)_u>0, \tag{20}$$

which are called *optimality conditions*.

3 Solution of the Adjoint Boundary Value Problem

We are looking for a solution of the adjoint boundary value problem (18) in the form of the series

$$\omega(t,x)=\sum_{n=1}^{\infty}\omega_n(t)z_n(x). \tag{21}$$

The Fourier coefficients $\omega_n(t)$ for each fixed $n = 1, 2, 3, \ldots,$ satisfy the linear nonhomogeneous Fredholm integral equation of the second type

$$\omega_n(t)=\lambda\int_0^T B_n(s,t)\omega_n(s)ds-\frac{2}{\lambda_n}\int_t^T \sin\lambda_n(\tau-t)[V_n(\tau)-\xi_n(\tau)]d\tau, \tag{22}$$

where

$$B_n(s,t)=\frac{1}{\lambda_n}\int_t^T \sin\lambda_n(\tau-t)K(s,\tau)d\tau. \tag{23}$$

We find [3] the solution to Eq. (22) using the following formulas:

$$\omega_n(t)=\lambda\int_0^T P_n(s,t,\lambda)\left(-\frac{2}{\lambda_n}\int_s^T \sin\lambda_n(\tau-s)[V_n(\tau)-\xi_n(\tau)]d\tau\right)ds-$$
$$-\frac{2}{\lambda_n}\int_t^T \sin\lambda_n(\tau-t)[V_n(\tau)-\xi_n(\tau)]d\tau, \tag{24}$$

where the resolvent $P_n(s,t,\lambda)$ of the kernel $B_n(s,t)$ is given by

$$P_n(s,t,\lambda)=\sum_{i=1}^{\infty}\lambda^{i-1}B_{n,i}(s,t), \qquad B_{n,1}(s,t)=B_n(s,t),$$
$$B_{n,i+1}(s,t)=\int_0^t B_n(\eta,t)B_{n,i}(s,\eta)d\eta,\ i=1,2,3,\ldots, \tag{25}$$

and it is continuous function when $|\lambda| < \frac{\lambda_1}{T\sqrt{K_0}},\ n = 1, 2, 3, \ldots$, and satisfies

$$\int_0^T P_n^2(t,\tau,\lambda)d\tau \le \frac{TK_0}{(\lambda_n-|\lambda|T\sqrt{K_0})^2}. \tag{26}$$

Further, taking into account (18) and (24) the solution of the adjoint boundary value problem can be written as

$$\omega_n(t) = 2\int_0^T E_n(\eta, t, \lambda) l_n(\eta) d\eta - $$
$$- 2\int_0^T \left(\int_0^T E_n(\eta, t, \lambda)\varepsilon_n(\eta, \tau, \lambda) d\eta \right) f_n(\tau, u) d\tau, \quad (27)$$

where

$$E_n(\eta, t, \lambda) = \begin{cases} \lambda \int_0^\eta P_n(s, t, \lambda) \frac{1}{\lambda_n} \sin \lambda_n(\eta - s) ds, \ 0 \le \eta \le t, \\ \frac{1}{\lambda_n} \sin \lambda_n(\eta - t) + \lambda \int_0^\eta P_n(s, t, \lambda) \frac{1}{\lambda_n} \sin \lambda_n(\eta - s) ds, t \le \eta \le T, \end{cases} \quad (28)$$

$$l_n(t) = \xi_n(t) - \psi_{1n}[\cos \lambda_n t + \lambda \int_0^T R_n(t, s, \lambda) \cos \lambda_n s ds] - $$
$$- \frac{\psi_{2n}}{\lambda_n}[\sin \lambda_n t - \lambda \int_0^T R_n(t, s, \lambda) \sin \lambda_n s ds], \quad (29)$$

$$\varepsilon_n(t, \tau, \lambda) = \begin{cases} \frac{\sin \lambda_n(t-\tau)}{\lambda_n} + \lambda \int_\tau^T R_n(t, s\lambda) \frac{\sin \lambda_n(s-\tau)}{\lambda_n} ds, \ 0 \le \tau \le t, \\ \lambda \int_\tau^T R_n(t, s\lambda) \frac{\sin \lambda_n(s-\tau)}{\lambda_n} ds, \ t \le \tau \le T. \end{cases} \quad (30)$$

By means of direct calculations we have proved the following lemma.

Lemma 1 *The solution to adjoint boundary value problem* (18) *is an element of the space* $H(Q_T)$.

4 Nonlinear Integral Equation of Optimal Control

We find the optimal control according to optimality conditions (19) and (20). We substitute into (19) the solution to the adjoint boundary value problem (18) defined by (27)

$$\beta \frac{M(t, x, u(t, x)) M_u(t, x, u(t, x))}{f_u(t, x, u(t, x))} = \sum_{n=1}^{\infty} \left\{ \int_0^T E_n(\eta, t, \lambda) l_n(\eta) d\eta - \right.$$
$$\left. - \int_0^T \left(\int_0^T E_n(\eta, t, \lambda) \varepsilon(\eta, \tau, \lambda) d\eta \right) f_n(\tau, u) d\tau \right\} z_n(x), \quad (31)$$

where

$$f_n(\tau, u) = \int_Q f(t, x, u(t, x)) z_n(x) dx. \tag{32}$$

Thus, the optimal control is defined as the solution of a nonlinear integral equation (31). Condition (20) restricts the class of functions $f[t, x, u(t, x)]$ of external influences. Therefore, we assume that the function $f[t, x, u(t, x)]$ satisfies (20) for any control $u(t, x) \in H(Q_T)$, i.e., the optimization problem is considered in class $\{f(t, x, u(t, x))\}$ of functions satisfying (20).

Nonlinear integral equation (31) is solved according to [5]. We set

$$\beta \frac{M(t, x, u(t, x)) M_u(t, x, u(t, x))}{f_u(t, x, u(t, x))} = p(t, x). \tag{33}$$

According to condition (20) the control function $u(t, x)$ is uniquely determined from equality (31), i.e., there is a function φ such that

$$u(t, x) = \varphi(t, p(t, x), \beta). \tag{34}$$

By (33) and (34) we rewrite Eq. (32) in the operator form

$$p(t, x) + G[p(t, x), \lambda] = h(t, x, \lambda), \tag{35}$$

where

$$G[p(t, x), \lambda] = \sum_{n=1}^{\infty} \int_0^T \left\{ \int_0^T E_n(\eta, t, \lambda) \varepsilon_n(\eta, \tau, \lambda) d\eta \right\} \cdot \cdot \int_Q f[\tau, y, \varphi(\tau, y, p(\tau, y), \beta)] z_n(y) dy z_n(x), \tag{36}$$

$$h(t, x, \lambda) = \sum_{n=1}^{\infty} \int_0^T E_n(\eta, t, \lambda) l_n(\eta) d\eta z_n(x). \tag{37}$$

Now we investigate the questions of unique solvability of the operator equation (35).

By means of the direct calculations we have proved the following lemmas.

Lemma 2 *The operator $G[p(t, x), \lambda]$ defined by the formula (36) maps the space $H(Q_T)$ into itself, i.e., it is an element of the space $H(Q_T)$.*

Lemma 3 *Suppose that the conditions*

$$\|f[t, u(t, x)] - f[t, \bar{u}(t, x)]\|_{H(Q_T)} \leq f_0 \|u(t, x) - \bar{u}(t, x)\|_{H(Q_T)}, \; f_0 > 0 \tag{38}$$

$$\|\varphi[t, p(t, x), \beta] - \varphi[t, \bar{p}(t, x), \beta]\|_{H(Q_T)} \leq \varphi_0(\beta) \|p(t, x) - \bar{p}(t, x)\|_{H(Q_T)}, \; \varphi_0(\beta) > 0 \tag{39}$$

are satisfied. Then if the condition

$$\gamma = C_0 f_0 \varphi_0(\beta) < 1 \tag{40}$$

is met, the operator $G[p(t,x),\lambda]$ *is contracting.*

Here $C_0 = const > 0$, $f_0 > 0$, $\varphi_0(\beta) > 0$.

Proof The proof of this theorem follows from Lemma 2 by the following inequality, i.e., the following inequality is fulfilled:

$$\|G[p(t,x),\lambda]-G[\tilde{p}(t,x),\lambda]\|^2_{H(Q_T)} \le C_0^2\|f(t,x,u(t,x))-f(t,x,\tilde{u}(t,x))\|^2_{H(Q_T)} \le$$

$$\le C_0^2 f_0^2\|\varphi(t,x,p(t,x),\beta)-\varphi(t,x,\tilde{p}(t,x),\beta)\|^2_{H(Q_T)} \le C_0^2 f_0^2\|\bar{p}(t,x)-\tilde{p}(t,x)\|^2_{H(Q_T)}.$$

Theorem 1 *Suppose that conditions* (5)*,* (19)*,* (20)*, and equations* (38)–(40) *are satisfied. Then operator equation* (35) *has a unique solution in the space* $H(Q_T)$*.*

Proof According to Lemma 2, operator equation (35) can be considered in the space $H(Q_T)$. According to Lemma 3 operator $G(p(t,x),\lambda)$ is contracting. Since the Hilbert space $H(Q_T)$ is a complete metric space, by the theorem on principle of contracting mappings [11] the operator $G(p(t,x),\lambda)$ has a unique fixed point, i.e., operator equation (35) has unique solution.

The solution of operator equation (35) can be found by the method of successive approximations, i.e., nth approximation of the solution is found by the formula:

$$p_k(t,x) = h - G[p_{k-1}(t,x),\lambda], \ k = 1,2,3,\ldots,$$

where $p_0(t,x)$ is an arbitrary element of the space $H(Q_T)$. For the exact solution $\bar{p}(t,x) = \lim_{n\to\infty} p_n(t,x)$ we have the following estimate:

$$\|\bar{p}(t,x)-p_n(t,x)\|_{H(Q_T)} \le \frac{\gamma^n}{1-\gamma}\|h-G[p_0(t,x),\lambda]-p_0(t,x)\|_{H(Q_T)}, \tag{41}$$

or when $h = p_0(t,x)$

$$\|\bar{p}(t,x) - p_k(t,x)\|_{H(Q_T)} \le \frac{\gamma^k}{1-\gamma}\|G[p_0(t,x),\lambda]\|_{H(Q_T)},$$

where $0 < \gamma < 1$ is the contraction constant.

The exact solution $\bar{p}(t,x)$ can be found as the limit of the approximate solutions, i.e., substituting this solution into (36) we find the required optimal control

$$u^0(t,x) = \varphi[t,x,\bar{p}(t,x),\beta]. \tag{42}$$

We find the optimal process $V^0(t, x)$, i.e., the solution of boundary value problem (2)–(4), corresponding to the optimal control $u^0(t, x)$, according to (6) by the formula

$$V^0(t, x) = \sum_{n=1}^{\infty} \left(\lambda \int_0^T R_n(t, s, \lambda) a_n(s) ds + a_n(t) \right) z_n(x). \tag{43}$$

The minimum value of the functional (1) is calculated by the formula

$$J[u^0(t, x)] = \int_0^T \int_Q [V^0(T, x) - \xi(t, x)]^2 dx + \beta \int_0^T \int_Q M^2(t, x, u^0(t, x)) dx dt. \tag{44}$$

The found triple $(u^0(t, x), V^0(t, x), J[u^0(t, x)])$ is a complete solution of the nonlinear optimization problem.

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On a Class of Solutions of the Nonlinear Integral Fredholm Equation

Akylbek Kerimbekov

Abstract Sufficient conditions for the existence and uniqueness of a new class of solutions are found for the nonlinear integral Fredholm equation, and an algorithm for their construction has been developed.

1 The Method of Constructing the Solution of the Nonlinear Integral Fredholm Equation

The theory of nonlinear integral equations is much less developed than the theory of linear integral equations. There are only a small number of nonlinear integral equations that have been more or less fully investigated [1–4]. Among them, the most studied equation is Hammerstein equation

$$\varphi(t) = \int_a^b K(t,s)F(s,\varphi(s))ds,$$

which was investigated under the following assumptions:

1. For the linear integral equation with kernel $K(t,s)$, Fredholm theorems hold and the iterated kernel $K_2(t,s)$ is continuous function;
2. Kernel function $K(t,s)$ is symmetric, i.e., $K(t,s) = K(s,t)$;
3. The kernel $K(t,s)$ is positive definite kernel, i.e., all its characteristic numbers are positive;
4. $F(s,z)$ is a continuous function of arguments $a \leq s \leq b$, $|z| \leq M$, where M is a sufficiently large constant.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_18

Another example is the Urysohn equation

$$\varphi(t) = \int_a^b K(t, s, \varphi(s), \lambda) ds,$$

with one parameter λ. If we assume that this equation has a solution $\varphi_0(t)$ with parameter $\lambda = \lambda_0$ and one isn't a characteristic number of the kernel $K_\varphi(t, s, \varphi_0(s), \lambda_0)$, then the solution to Urysohn equation can be constructed by the method of the small parameter, representing it in the form of an expansion

$$\varphi(t, \lambda) = \varphi_0(t) + (\lambda - \lambda_0)\varphi_1(t) + .. + (\lambda - \lambda_0)^k \varphi_k(t) + o(|\lambda - \lambda_0|^k),$$

where the function $\varphi_k(t)$, for each $k = 1, 2, 3, \ldots,$, is defined as the solution to the linear Fredholm integral equation of the second kind with kernel $K_\varphi(t, s, \varphi_0(s), \lambda_0)$. In practice, this method leads to rather complex calculations.

Other nonlinear integral equations were studied by such mathematicians as R. Iglish, J. Leray, E. Holder, V. Nemytskii, E. Schmidt, A. Hammerstein, L. Lichtenstein, and the others until the mid-1930s.

However, up to the present time, constructive methods for solving nonlinear integral equations have not been sufficiently developed.

In Ref. [5], based on Lagrange finite-increments formula, a method is given that makes it possible to find solutions of nonlinear integral Fredholm equations in the form of a sum of two functions.

In this paper we consider the nonlinear integral Fredholm equation of the form

$$\varphi(x) = \lambda \int_a^b K(x, t, \varphi(t)) dt + f(x), \tag{1}$$

where *a and b* are given numbers, $\lambda \in (-\infty, +\infty)$ is a parameter, $f(x)$ is a given continuous function defined on the interval [a, b], $K(x, t, \varphi(t))$ is a given continuous function with respect to the set of arguments in the domain $Q = \{a \le x \le b; a \le t \le b; \varphi_1 \le \varphi \le \varphi_2\}$, and this function has the continuous derivative $K_\varphi(x, t, \varphi(t))$, where φ_1 *and* φ_2 are constants.

In the paper we suggest the method that makes it possible to find solutions to Eq. (1) in the form of the sum

$$\varphi(x) = \varphi_0(x) + \lambda u(x), \tag{2}$$

where $\varphi_0(x)$ *and* $u(x)$ are the functions to be determined.

To find necessary and sufficient conditions for which Eq. (1) has solutions of the form (2), we substitute function (2) in (1) and we obtain the identity

$$\varphi_0(x) + \lambda u(x) = \lambda \int_a^b K(x, t, \varphi_0(t) + \lambda u(t)) dt + f(x). \tag{3}$$

Suppose that function $K(x, t, \varphi(t))$ as the function of φ, $\varphi_1 \leq \varphi \leq \varphi_2$ satisfies the Lagrange finite-increments formula for each fixed (x, t) [6] and the following equality holds:

$$K[x, t, \varphi_0(t) + \lambda u(t)] = K[x, t, \varphi_0(t)] + K_\varphi[x, t, \overline{\varphi}(t)]\lambda u(t), \tag{4}$$

where $\overline{\varphi}(t)$ is an unknown function satisfying condition

$$\begin{gathered} \varphi_1 \leq \varphi_0(t) < \overline{\varphi}(t) < \varphi_0(t) + \lambda u(t) \leq \varphi_2, \\ \varphi_1 \geq \varphi_0(t) > \overline{\varphi}(t) > \varphi_0(t) + \lambda u(t) \geq \varphi_2, \forall t \in [a, b]. \end{gathered} \tag{5}$$

Taking into account (4), we rewrite identity (3) in the form

$$\varphi_0(x) + \lambda u(x) = \lambda \int_a^b K(x, t, \varphi_0(t))dt - \lambda^2 \int_a^b K_\varphi(x, t, \overline{\varphi}(t))u(t)dt + f(x), \tag{6}$$

from which we obtain the following identities:

$$\varphi_0(x) \equiv f(x), \tag{6_1}$$

$$u(x) \equiv \int_a^b K(x, t, \varphi_0(t))dt \equiv \int_a^b K(x, t, f(t))dt, \tag{6_2}$$

$$0 \equiv \int_a^b K_\varphi[x, t, \overline{\varphi}(t)]u(t)dt. \tag{6_3}$$

From the relations (6_1) and (6_2) we find the functions $\varphi_0(x)$ and $u(x)$. However, identity (6) is satisfied only when identity (6_3) is satisfied, and (6_3) contains an unknown function $\overline{\varphi}(t)$.

An unknown function $\overline{\varphi}(t)$ can be found from relation

$$K_\varphi[x, t, \overline{\varphi}(t)]\lambda u(t) = K[x, t, f(t) + \lambda u(t)] - K[x, t, f(t)], \tag{7}$$

which according to Lagrange's theorem has at least one solution, for each value of the parameter. The solution to Eq. (7) we denote by function $\psi(t, \lambda)$, i.e., $\overline{\varphi}(t) = \psi(t, \lambda)$. Then the identity (6_3) has the following form:

$$\int_a^b K_\varphi[x, t, \psi(t, \lambda)]u(t)dt \equiv 0, \tag{8}$$

where $\psi(t, \lambda)$ is known function. Identity (8) takes place when the following conditions are fulfilled:

1. $K_\varphi[x, t, \psi(t, \lambda)]$ and $u(t)$ are orthogonal functions for $\forall x$ and $\forall \lambda$;
2. λ is the root of equation

$$\int_a^b K_\varphi[x, t, \psi(t, \lambda)]u(t)dt = 0,$$

for any $x \in [a, b]$.

These conditions are sufficient for the existence of the solution to Eq. (1) in the form (2). Then we consider the condition for the uniqueness of this solution. Suppose that Eq. (1) is different from each other $\varphi_1(t)$ and $\varphi_2(t)$. Then, for their difference, we have a linear integral equation of a Fredholm type

$$\varphi_1(t) - \varphi_2(t) = \lambda \int_a^b K_\varphi(x, t, \overline{\psi}(t))(\varphi_1(t) - \varphi_2(t))dt,$$

which has a trivial solution for any $\varphi_1(t)$ and $\varphi_2(t)$ when the kernel $K_\varphi(x, t, \overline{\psi}(t))$ has no characteristic numbers. This condition ensures the uniqueness of the solution in the form of the sum (2).

2 Solution of the Nonlinear Integral Fredholm Equation with a Polynomial Nonlinearity

We consider the nonlinear integral Fredholm equation with a polynomial nonlinearity of the form

$$\varphi(x) = \lambda \int_a^b \sum_{i=0}^{n} \alpha_i(x) b_i(t) \varphi^{n-i}(t)dt + f(x), \tag{9}$$

and we seek its solution as the sum

$$\varphi(x) = \varphi_0(x) + \lambda u(x). \tag{10}$$

By substituting the function (10) into Eq. (9), we obtain the identity

$$\varphi_0(x) + \lambda u(x) = \lambda \int_a^b \Big(\alpha_0(x)b_0(t)\Big[\varphi_0(t) + \lambda u(t)\Big]^n + \ldots +$$
$$\alpha_{n-1}(x)b_{n-1}(t)\Big[\varphi_0(t) + \lambda u(t)\Big] + \alpha_n(x)b_n(t)\Big)dt + f(x)$$

$$= \lambda \int_a^b \Big(\Big\{\alpha_0(x)b_0(t)\Big[\varphi_0^n(t) + n\varphi_0^{n-1}\lambda u(t) + \ldots +$$

$$n\varphi_0(t)\big(\lambda u(t)\big)^{n-1} + \big(\lambda u(t)\big)^n\Big\} + \ldots$$

$$+\alpha_1(x)b_1(t)\Big[\varphi_0^{n-1}(t) + (n-1)\varphi_0^{n-2}\lambda u(t) + \ldots +$$

$$(n-1)\varphi_0(t)\big(\lambda u(t)\big)^{n-2} + \big(\lambda u(t)\big)^{n-1}\Big\} + \ldots +$$

$$+\alpha_k(x)b_k(t)\Big[\varphi_0^{n-k}(t) + (n-k)\varphi_0^{n-k-1}\lambda u(t) + \ldots +$$

$$(n-k)\varphi_0(t)\big(\lambda u(t)\big)^{n-k-1} + \big(\lambda u(t)\big)^{n-k}\Big\} + \ldots +$$

$$+\alpha_{n-2}(x)b_{n-2}(t)\Big[\varphi_0^2(t) + 2\varphi_0\lambda u(t) + \big(\lambda u(t)\big)^2\Big] +$$

$$+\alpha_{n-1}(x)b_{n-1}(t)\Big[\varphi_0(t) + 1\lambda u(t)\Big] + \alpha_n(x)b_n(t)\Big\}dt + f(x),$$

thence

$$\begin{aligned}
\varphi_0(x) &= f(x),\\
u(x) &= \int_a^b \Big\{\alpha_0(x)b_0(t)\varphi_0^n(t)\\
&\quad + \alpha_1(x)b_1(t)\varphi_0^{(n-1)}(t) + \ldots +\\
&\quad + \alpha_k(x)b_k(t)\varphi_0^{n-k}(t) + \ldots +\\
&\quad + \alpha_{n-2}(x)b_{n-2}(t)\varphi_0^2(t) +\\
&\quad + \alpha_{n-1}(x)b_{k-1}(t)\varphi_0(t) +\\
&\quad + \alpha_n(x)b_n(t)\Big\}dt =\\
&= P_{0,n}\alpha_0(x) + P_{1,n-1}\alpha_1(x) + \ldots +\\
&\quad + P_{k,n-k}\alpha_k(x) + \ldots +\\
&\quad + P_{n-1,1}\alpha_{n-1}(x) + P_{n,0}\alpha_n(x),
\end{aligned} \tag{11}$$

where

$$P_{i,n-i} = \int_a^b b_i(t)\varphi_0^{n-i}(t)dt = \int_a^b b_i(t)f^{n-i}(t)dt < \infty, i = 0, 1, \ldots, n,$$

and we have the identity

$$\alpha_0(x)P_{0,n-1}(\lambda)+\alpha_1(x)P_{1,n-2}(\lambda)+\ldots+\alpha_k(x)P_{k,n-(k+1)}(\lambda)+\ldots$$
$$+\alpha_{n-2}(x)P_{n-2,1}(\lambda)+\alpha_{n-1}(x)P_{n-1,0}(\lambda)=0, \qquad (12)$$

where

$$P_{k,n-(k+1)}(\lambda)=\int_a^b b_k(t)u(t)\Big[(n-k)f^{n-k-1}(t)+\ldots+$$
$$+(n-k)f(t)\lambda^{n-k-2}u^{n-k-2}(t)+\lambda^{n-k-1}u^{n-k-1}(t)\Big]dt.$$

Linear independence of the systems $\alpha_0(x),\ldots,\alpha_{n-1}(x)$ plays an important role in the solvability of Eq. (9).

Theorem 1 *Let* $\alpha_0(x),\alpha_1(x),\ldots,\alpha_{n-1}(x)$ *be linearly independent functions on the interval* $[a,b]$. *If the intersection of the sets of roots of equations*

$$p_{k,n-k-1}(\lambda)=0,\quad k=0,1,\ldots,n-1,$$

is not empty, then Eq. (9) has at least one solution in the form of sum (10).

Theorem 2 *Let* $\alpha_k(x),\alpha_{k+1}(x),\ldots,\alpha_{n-1}(x)$ *be linearly dependent functions on the interval* $[a,b]$. *Then Eq. (9) has as many solutions as sum (10), how many roots of the algebraic equation*

$$\beta_k p_{k,n-(k+1)}(\lambda)+\ldots+\beta_{n-1}p_{n-1,0}(\lambda)=0,\ \beta_k=const,$$

satisfy the system of equations

$$p_{j,n-(j+1)}(\lambda)=0,\quad j=0,1,\ldots,k-1.$$

The proofs of the theorems are verified by direct computations.

Corollary 1 *Let* $\alpha_0(x),\alpha_1(x),\ldots,\alpha_{n-1}(x)$ *be linearly dependent functions on the interval* $[a,b]$. *Then Eq. (9) has* $n-1$ *solutions in the form of sum (10), where* λ *are the roots of the algebraic equation degree* $n-1$ $p(\lambda)=0$.

Corollary 2 *Let* $\alpha_0(x),\alpha_1(x),\ldots,\alpha_{n-1}(x)$ *be linearly independent functions on the interval* $[a,b]$ *and*

$$p_{n-1,0}=\int_a^b b_{n-1}(t)u(t)dt=0.$$

Then Eq. (9) has a unique solution in the form of sum (10) when the root λ^* *of equation*

$$p_{n-2,1}(\lambda) = 0$$

is a root of equations

$$p_{j,n-(j+1)}(\lambda) = 0, \quad j = 0, 1, \ldots, n-3.$$

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On Conditional Stability of Inverse Scattering Problem on a Lasso-Shaped Graph

Kiyoshi Mochizuki and Igor Trooshin

Abstract We investigate the conditional stability of the inverse scattering problem on a lasso-shaped graph using the fundamental equation of inverse scattering theory.

1 Introduction

Time-dependent and stationary equations on graphs arise as simplified models in mathematics, physics, chemistry, and engineering (nanotechnology), when one considers the propagation of waves of different natures in thin, tube-like domains (for details see books of Pokornyi et al. [16], Berkolaiko and Kuchment [1], and the references therein).

Among several problems in this field, the scattering problems have been studied by many authors (e.g., Gerasimenko and Pavlov [6], Exner and Seba [3], Gerasimenko [5], Kostrykin and Schrader [8], Harmer [7], Kurasov and Stenberg [9], Boman and Kurasov [2], Pivovarchik [15], Latushkin and Pivovarchik [10], and Freiling and Ignatyev [4]) because of the general importance of their applications.

In this paper we investigate the case of a "lasso-shaped" graph, i.e., a graph Γ which consists of a half line $\gamma = \{x \mid 0 < x < \infty\}$ and a loop $\kappa = \{z \mid 0 < z < 2\pi\}$, joined at the point $\{x = 0\} = \{z = 0\} = \{z = 2\pi\}$. We consider on Γ

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_19

the following spectral problem describing one-dimensional scattering of a quantum particle:

$$-u'' + \left\{q(X) - \lambda^2\right\} u = 0, \quad X \in \Gamma, \tag{1}$$

$$u(x = 0) = u(z = 0) = u(z = 2\pi), \tag{2}$$

$$u'(x = 0 + 0) + u'(z = 0 + 0) - u'(z = 2\pi - 0) = 0. \tag{3}$$

Here differentiation with respect to the variable X is understood as differentiation with respect to x, when $X \in \gamma$, and as differentiation with respect to z, when $X \in \kappa$. Differentiation is not defined at the vertex. The potential $q(X)$ is real-valued, and is required to satisfy $q(x) \in L^1_{loc}(\Gamma)$, $(1+x)q(x) \in L^1(\gamma)$. Parameter λ is a complex number such that $\operatorname{Im} \lambda \geq 0$.

For any real $\lambda \neq 0$ there exists a solution $\Phi(X, \lambda)$ of problem (1)–(3) which is represented on γ uniquely as

$$\Phi(x, \lambda) = e(x, -\lambda) - S(\lambda)e(x, \lambda).$$

Here $e(x, \lambda)$ are so-called Jost functions, which behave on the closed upper half-plane of the spectral parameter λ as

$$e(x, \lambda) = e^{i\lambda x}\{1 + o(1)\}.$$

The function $S(\lambda)$ is the scattering function for the boundary value problem (1)–(3).

There are two types of eigenvalues of problem (1)–(3): "visible at infinity" if there exist corresponding eigenfunctions which are not identically vanishing on γ and "invisible at infinity" otherwise. Problem (1)–(3) possess at most a finite number of "visible at infinity" eigenvalues $-\lambda_j^2$, $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$, which are all negative and simple.

We call

$$m_j = \|E(\cdot, i\lambda_j)\|^{-1}_{L^2(\Gamma)}, \; j = 1, \ldots, n$$

the "weight numbers" of problem (1)–(3). Here $E(X, i\lambda_j)$ is an eigenfunction of operator L, corresponding to the eigenvalue "visible at infinity" $-\lambda_j^2$, which is normalized as $E(x, i\lambda_j) = e(x, i\lambda_j)$ for $x \in \gamma$.

The scattering data is then given by $\{S(\lambda),\; \lambda_j,\; m_j |\, \lambda \in \Re \; j = 1, \ldots, n\}$, where $-\lambda_j^2$, $j = 1, \ldots, n$, are the eigenvalues "visible at infinity," and m_j, $j = 1, \ldots, n$, are the corresponding "weight numbers."

Our inverse scattering problem is the following:

IScP: Given the scattering data $\{S(\lambda),\; \lambda_j,\; m_j |\, \lambda \in \Re \setminus \{0\},\; j = 1, \ldots, n\}$, recover the potential $q(X)$ on γ.

We have proved (see [12–14]) that scattering data defines $q(X)$ on γ uniquely and provided the reconstruction procedure. Now we investigate the stability of such an inverse scattering problem.

2 Conditional Stability

Let $\alpha(x) \in C([0,\infty)) \cap L^1([0,\infty))$ be a non-increasing function.

We denote by $V_0^1(\alpha)$ the set of potentials $q(X)$ such that $q(z) \equiv 0,\ z \in \kappa$, $q(x) \in C^1(\Gamma)$, and $(1+|\cdot|)q(\cdot)) \in L^1(\gamma)$, $\int_{t\in\gamma, t\geq x} q(t)dt \leq \alpha(x)$. Let us mention that for every potential $q(x)$ under consideration such function $\alpha(x)$ exists.

We introduce the following notations:

$$\alpha_1(x) = \int_x^\infty \alpha(t)dt, \quad \Delta(x) = \int_0^{x^{-1}} \alpha(t)dt, \quad x > 0,$$

$$\sigma = \min_{0\leq\lambda\leq 2} |2\sin\lambda\pi + i\cos\lambda\pi|,$$

$$\delta(\lambda) = \frac{1}{|\lambda|}\alpha(0)e^{\alpha_1(0)} + \frac{2\Delta(|\lambda|)}{1-\Delta(|\lambda|)} \text{ for } |\lambda| > 0,\ \Delta(|\lambda|) < 1. \tag{4}$$

Theorem 1 *Let $q^1(X),\ q^2(X) \in V_0^1(\alpha)$ and $\lambda_j^1 = \lambda_j^2,\ m_j^1 = m_j^2$, for $j = 1,\ldots,n$, and $S^1(\lambda) = S^2(\lambda), \lambda \in (-N,N)$, where $\left\{\lambda_j^i,\ m_j^i,\ S^i(\lambda)\right\}$ denotes the corresponding scattering data and N is a fixed number.*

For any N and $h > N^{-1}$ such that $\Delta(N) < 1/2$ and $\delta(N) < \sigma$ the following estimate holds:

$$|q^1(x) - q^2(x)| \leq 2\pi h D(x,h) + \frac{16(1+9h\alpha(x))}{3h^2\pi(\sigma-\delta(N))N}\left[\frac{\alpha(0)+\pi N\Delta(N))}{1-\Delta(N)} + \alpha(0)e^{\alpha_1(0)}\right], \tag{5}$$

where

$$D(x,h) = \max_{k=1,2} \sup_{x\leq y\leq x+\pi h}\left|\frac{d}{dy}q^k(y)\right|.$$

Proof It is well known (see [11]) that the solution $e(x,\lambda)$ of Eq. (1) on γ can be represented as

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \tag{6}$$

where the kernel $K(x,t)$ is differentiable with respect to both variables on $0 \le x \le t < \infty$ and satisfies the equation

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)dt, \quad x > 0. \tag{7}$$

$$|K_x(x,t) + \frac{1}{4}q(\frac{x+t}{2})| \le \frac{1}{2}\sigma(x)\sigma(\frac{x+t)}{2}) \exp\left\{\sigma_1(x) - \sigma_1(\frac{x+t)}{2})\right\}, \tag{8}$$

with

$$\sigma(x) = \int_x^\infty |q(t)|dt,\ \sigma_1(x) = \int_x^\infty \sigma(t)dt, \quad x > 0.$$

For every fixed $x \in \gamma$ the kernel $K(x,t)$ of transformation operator (6) satisfies the fundamental equation of inverse scattering theory (for its derivation in the case of the lasso-shaped graph see [12–14]):

$$F(x+t) + K(x,t) + \int_x^\infty K(x,y)F(t+y)dy = 0, \quad 0 < x < t < \infty,$$

where

$$F(x) = \sum_{k=1}^n m_k^2 e^{-\lambda_k x} + F_S(x),$$

$$F_S(x) = \frac{1}{2\pi}\int_{-\infty}^\infty (S_0(\lambda) - S(\lambda))e^{i\lambda x}d\lambda.$$

Function $S_0(\lambda)$ is the scattering function of (1)–(3) in the case $q(X) \equiv 0,\ X \in \Gamma$ and function $F_S(x)$ is understood as the Fourier transform of the function from $L^2(-\infty,\infty)$.

In the case $q(z) \equiv 0,\ z \in \kappa$ scattering functions $S(\lambda)$ and $S_0(\lambda)$ have representations (see [14])

$$S(\lambda) = \frac{\overline{n(\lambda)}}{n(\lambda)},\ n(\lambda) = 2\sin\lambda\pi e(0,\lambda) + \frac{1}{\lambda}\cos\lambda\pi e'(0,\lambda),$$

$$S_0(\lambda) = \frac{\overline{n_0(\lambda)}}{n_0(\lambda)},\ n_0(\lambda) = 2\sin\lambda\pi + i\cos\lambda\pi.$$

To estimate $|S(\lambda) - S_0(\lambda)|$ we remark that

$$n(\lambda) = n_0(\lambda) + \epsilon(\lambda),\ \ \epsilon(\lambda) = \epsilon_1(\lambda) + \epsilon_2(\lambda),$$

$$\epsilon_1(\lambda) = 2\sin\lambda\pi[e(0,\lambda) - 1],$$

$$\epsilon_2(\lambda) = \frac{1}{\lambda}\cos\lambda\pi[e'(0,\lambda) - i\lambda].$$

It is proved (see [11, pp. 374–375]) that

$$|e(0,\lambda) - 1| \leq \frac{\Delta(|\lambda|)}{1 - \Delta(|\lambda|)},$$

and consequently

$$|\epsilon_1(\lambda)| \leq \frac{2\Delta(|\lambda|)}{1 - \Delta(|\lambda|)}.$$

It follows from (6) and (7) that

$$\epsilon_2(\lambda) = \frac{1}{\lambda}\cos\lambda\pi[-\frac{1}{2}\int_0^\infty q(t)dt + \int_0^\infty K_x(0,t)e^{i\lambda t}dt].$$

So estimate (8) gives us

$$|\epsilon_2(\lambda)| \leq \frac{1}{|\lambda|}\sigma(0)e^{\sigma_1(0)} \leq \frac{1}{|\lambda|}\alpha(0)e^{\alpha_1(0)}. \tag{9}$$

As a result

$$|\epsilon(\lambda)| \leq \delta(\lambda)$$

with $\delta(\lambda)$ defined by formula (4).

The function $n_0(\lambda)$ is a 2−periodic continuous which does not vanish on the real axis function. So we have the following formula:

$$S(\lambda) - S_0(\lambda) = \frac{\overline{\epsilon(\lambda)}n_0(\lambda) - \epsilon(\lambda)\overline{n_0(\lambda)}}{(n_0(\lambda) + \epsilon(\lambda))n_0(\lambda)}$$

and the estimate

$$|S(\lambda) - S_0(\lambda)| \leq \frac{2}{\sigma - \delta(\lambda)}|\epsilon(\lambda)|.$$

Then we follow the arguments ([11, pp. 374–375]) to come to

$$\int_{|\lambda|\geq N}\left|\frac{\epsilon_1(\lambda)}{\lambda}\right|d\lambda \leq \frac{2}{1-\Delta(N)}\left[\frac{\alpha(0)}{N}+\pi\Delta(N)\right].$$

Estimate (9) allows us to obtain the following estimate:

$$\int_{|\lambda|\geq N}\left|\frac{\epsilon_2(\lambda)}{\lambda}\right|d\lambda \leq \frac{2}{N}\alpha(0)e^{\alpha_1(0)},$$

and consequently

$$\int_{|\lambda|\geq N}\left|\frac{S(\lambda)-S_0(\lambda)}{\lambda}\right|d\lambda \leq \frac{4}{(\sigma-\delta(N))N}\left[\frac{\alpha(0)+\pi N\Delta(N)}{1-\Delta(N)}+\alpha(0)e^{\alpha_1(0)}\right]. \tag{10}$$

We repeat the arguments ([11, pp. 364–380]) to prove

$$|q^1(x)-q^2(x)| \leq 2\pi h D(x,h) + \frac{2(1+9h\alpha(x))}{3h^2\pi}\times \int_{|\lambda|\geq N}\left(\left|\frac{S^1(\lambda)-S_0(\lambda)}{\lambda}\right|+\left|\frac{S^2(\lambda)-S_0(\lambda)}{\lambda}\right|\right)d\lambda. \tag{11}$$

Combining this estimate with (10) we obtain the desired estimate (5).

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On Solvability of Tracking Problem Under Nonlinear Boundary Control

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Abstract In the paper a nonlinear boundary optimal control problem is investigated for thermal process described by Volterra integro-differential equation. Sufficient conditions are established for unique solvability of a nonlinear optimization problem. An algorithm is developed for constructing a complete solution of the nonlinear optimization problem.

1 Introduction

Applied problems described by integro-differential equations are often use in practice [1–3]. Optimal control problems were widely investigated for processes described by integro-differential equations in partial derivatives of parabolic or hyperbolic types when control is linearly included in the equation [4–7].

Many applied problems are usually nonlinear. The unique solvability of the tracking problem for nonlinear boundary control of a thermal process described by the Volterra integro-differential equation is investigated when the control function nonlinearly enters into boundary condition. A quadratic functional is the optimality criterion.

In the research process:

- a weak generalized solution of the boundary-value problem was constructed for control process, in which Fourier coefficients are defined as the solution of a linear inhomogeneous Volterra integral equation;
- optimality conditions are found by the maximum principle for systems with distributed parameters [3] and they contained a weak generalized solution of the adjoint boundary-value problem;
- nonlinear integral equation of optimal control was obtained with the additional condition in the form of a differential inequality with respect to the functions of a boundary source, and unique solvability of this problem is studied;

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_20

- a sufficient condition was found for the unique solvability of the nonlinear optimization problem, and an algorithm for constructing its solution was developed in the form of a triple, consisting of an optimal control, an optimal process, and a minimum value of the functional.

Optimal control problem under consideration for Volterra operator possesses some specific properties which are not fulfilled when Volterra operator is replaced by Fredholm operator. For example,

- if solutions of both basic and adjoint boundary-value problems for the Volterra operator exist for any parameter λ, then solution of the boundary-value problems for Fredholm operator exists only for values of parameter λ varied only on a finite interval;
- boundary-value problems for the Volterra operator have the property of continuability with respect to the time variable t, whereas boundary-value problems for the Fredholm operator does not have this property, and it significantly affects the solvability of the optimal control problem.

Therefore, boundary optimal control problem under consideration for a thermal process is of theoretical and practical interest.

2 Formulation of the Optimal Control Problem: Optimality Conditions

Consider an optimization problem in which it is required to find the minimum value of the integral quadratic functional

$$J[u(t)] = \int_0^T \int_0^1 [V(t,x) - \xi(t,x)]^2 dx dt + \beta \int_0^T u^2(t) dt, \quad \beta > 0. \tag{1}$$

Here $V(t,x)$ characterizes the state of the controlled process, $\xi(t,x)$ describes the desired state of the controlled process for a given time, and $u(t)$ is the control function. Optimal control problem is in the definition of the control function $u^0(t)$, for which together with the corresponding to its solution $V^0(t,x)$ of following boundary-value problem:

$$\begin{aligned} V_t = V_{xx} + \lambda \int_0^t K(t,\tau) V(\tau,x) d\tau, \quad 0 < x < 1, \quad 0 < t \le T, \\ V(0,x) = \psi(x), \quad 0 < x < 1, \\ V_x(t,0) = 0, \quad V_x(t,1) + \alpha V(t,1) = p[t,u(t)], \quad 0 < t \le T, \end{aligned} \tag{2}$$

which it minimizes the functional (1). Here $K(t,\tau)$ is a given function defined in region $D = \{0 \le t \le T, \quad 0 \le \tau \le T\}$ and satisfies the condition

$$\sup_{(t,\tau)\in D} |K(t,\tau)| = K_0,$$

$$\xi(t,x) \in H(Q), \quad Q = \{0 < x < 1, \quad 0 < t \le T\}, \quad \psi(t,x) \in H(0,1), \quad p[t,u(t)] \in H(0,T) \tag{3}$$

are given functions and function $p[t,u(t)]$ is nonlinearly dependent on control function $u(t) \in H(0,T)$ and satisfies the condition, i.e.,

$$p_u[t,u(t)] \ne 0, \quad \forall t \in (0,T), \tag{4}$$

λ is a parameter, T is fixed time moment, $\alpha > 0$, and $H(X)$ is a Hilbert space of quadratically summable functions defined on the set X.

Control $u^0(t)$ is called *an optimal control*, and the corresponding to its solution $V^0(t,x)$ is *an optimal process*. Note condition (4) ensures a one-to-one correspondence between elements of space $\{u(t)\}$ of controls and space $\{V(t,x)\}$ of solutions of the boundary-value problem (2).

To determine the optimal control we calculate the increment of functional (1). Since to each control $u(t) \in H(0,T)$ corresponds uniquely a unique solution $V(t,x) \in H(Q)$ of the boundary-value problem, the control $u(t) + \Delta u(t) \in H(0,T)$ corresponds to the solution of the boundary-value problem (2) of form $V(t,x) + \Delta V(t,x) \in H(Q)$, where $\Delta u(t)$ is increment, i.e., the solution of the boundary-value problem has an increment $\Delta V(t,x)$ corresponding to the increment $\Delta u(t)$. Similar to [3], by means of the direct calculations the increment of the functional can be represented in the form

$$\Delta J[u(t)] = J[u(t) + \Delta u(t)] - J[u(t)] =$$
$$= -\int_0^T \Delta\Pi[t, V(t,x), \omega(t,x), u(t)]dt + \int_0^T \int_0^1 \Delta V^2(t,x)dxdt,$$

where

$$\Delta\Pi[t,\cdot,u(t)] = \Pi[t,\cdot,u(t) + \Delta u(t)] - \Pi[t,\cdot,u(t)],$$

$$\Pi[t, V(t,x), \omega(t,x), u(t)] = p[t,u(t)]\omega(t,1) - \beta u^2(t), \tag{5}$$

and $\omega(t, x) \in H(Q)$ is the unique weak generalized solution (the corresponding control $u(t) \in H(0, T)$) of boundary-value problem of the form

$$\begin{aligned} \omega_t + \omega_{xx} + \lambda \int_t^T K(\tau, t)\omega(\tau, x)d\tau = 2[V(t, x) - \xi(t, x)], \\ 0 < x < 1, \quad 0 \leq t < T, \\ \omega(T, x) = 0, \quad 0 < x < 1, \\ \omega_x(t, 0) = 0, \quad \omega_x(t, 1) + \alpha\omega(t, 1) = 0, \quad 0 \leq t < T. \end{aligned} \tag{6}$$

(6) is called adjoint boundary-value problem.

As

$$\int_0^T \int_0^1 \Delta V^2(t, x)dxdt$$

is nonnegative, the following relations hold:

1. If

$$\Delta J[u(t)] = J[u(t) + \Delta u(t)] - J[u(t)] \geq 0, \tag{7}$$

then function $\Pi[t, V(t, x), \omega(t, x), u(t)]$ must satisfy inequality

$$\Delta \Pi[t, \cdot, u(t)] = \Pi[t, \cdot, u(t) + \Delta u(t)] - \Pi[t, \cdot, u(t)] \leq 0. \tag{8}$$

2. If conditions (8) hold, then the functional satisfies condition (7).

On basis of these relations we obtain the maximum principle for function $\Pi[t, V(t, x), \omega(t, x), u^0(t)]$, the essence of which lies in the fact that for optimality of control it is necessary and sufficient that condition

$$\Pi[t, V(t, x), \omega(t, x), u^0(t)](=) \sup_{u \in Z} \Pi[t, V(t, x), \omega(t, x), u]$$

is satisfied almost everywhere on interval $[0, T]$, where Z is acceptable values set of function $u(t)$ for each fixed T.

Solving for function (5) the extremal problem for the unconstrained maximum we obtain necessary conditions in the form of the following relations:

$$\Pi_u[t, V(t, x), \omega(t, x), u(t)] = p_u[t, u(t)]\omega(t, 1) - 2\beta u(t) = 0, \tag{9}$$

$$\Pi_{uu}[t, V(t, x), \omega(t, x), u(t)] = p_{uu}[t, u(t)]\omega(t, 1) - 2\beta < 0, \tag{10}$$

which are called *optimality conditions*.

The optimality conditions contain solution $\omega(t, x)$ of the adjoint boundary-value problem, which make difficult the verification of condition (10). However, eliminating the function $\omega(t, x)$ from (10), we obtain the following optimality condition:

$$p_u[t, u(t)]\left(\frac{u}{p_u[t, u(t)]}\right)_u > 0. \tag{11}$$

Thus, we find optimal control according to conditions (9) and (11). We note that condition (11) restricts the class of given functions $\{p[t, u(t)]\}$, which essentially affects the solvability of the nonlinear optimization problem. Therefore, in the subsequent arguments it is assumed that condition (11) is satisfied for $\forall u(t) \in H(0, T)$. Then, to find the optimal control it suffices to consider only relation

$$\frac{2\beta u(t)}{p_u[t, u(t)]} = \omega(t, 1). \tag{12}$$

We note that the solution of the adjoint boundary-value problem $\omega(t, x)$ can be found only after determining the function $V(t, x)$ according to (6), i.e., solution of the basic boundary-value problem.

3 Solution of the Basic Boundary-Value Problem

We consider the boundary-value problem (2), where the function $p[t, u(t)]$ satisfies conditions (4) and (11) for any control $u(t) \in H(0, T)$.

We are looking for a solution of problem (2) in the form

$$V(t, x) = \sum_{n=1}^{\infty} V_n(t) z_n(x), \quad V_n(t) = \int_0^1 V(t, x), z_n(x) dx, \tag{13}$$

where

$$z_n(x) = \sqrt{\frac{2(\lambda_n^2 + \alpha^2)}{\lambda_n^2 + \alpha^2 + \alpha}} \cos \lambda_n x, \quad n = 1, 2, 3, \ldots,$$

are the eigenfunctions of the boundary-value problem

$$z''(x) + \lambda_0^2 z(x) = 0, \quad z'(0) = 0, \quad z'(1) + \alpha z(1) = 0,$$

and the corresponding eigenvalues λ_n are defined as the positive roots of the transcendental equation $\lambda \tan \lambda = \alpha$ and satisfy the conditions

$$(n-1)\pi < \lambda_n < \frac{\pi}{2}(2n-1), \quad \lambda_n < \lambda_{n+1}, \quad n = 1, 2, 3, \ldots, \quad \lim_{n\to\infty} \lambda_n = \infty.$$

Further, by means of the direct calculations we established that the Fourier coefficients $V_n(t)$ are defined as the solution of the linear inhomogeneous Volterra integral equation of the second type

$$V_n(t) = \int_0^t K_n(t,s)V_n(s)ds + a_n(t) \tag{14}$$

for each fixed $n = 1, 2, 3, \ldots$, where the function

$$K_n(t,s) = \int_s^t e^{-\lambda_n^2(t-\tau)}K(\tau,s)d\tau$$

is a kernel, and the function

$$a_n(t) = e^{-\lambda_n^2 t}\psi_n + \int_0^t e^{-\lambda_n^2(t-\tau)}z_n(1)p[\tau, u(\tau)]d\tau$$

is a free term of the integral equation.

Similar to [8], we find the solution of Eq. (14) by formula

$$V_n(t) = \lambda \int_0^t R_n(t,s,\lambda)a_n(s)ds + a_n(t), \tag{15}$$

where the resolvent $R_n(t,s,\lambda)$ of the kernel $K_n(t,s)$ is determined by Neumann series

$$R_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1}K_{n,i}(t,s), \;\; n = 1, 2, 3, \ldots. \tag{16}$$

The iterated kernels $K_{n,i}(t,s)$ are found by formulas

$$K_{n,i+1}(t,s) = \int_s^t K_n(t,\eta)K_{n,i}(\eta,s)d\eta, \;\;\; K_{n,1}(t,s) \equiv K_n(t,s), \;\;\; i = 1, 2, 3, \ldots,$$

and satisfy inequalities $\left|K_{n,i}(t,s)\right| \leq \left(\frac{K_0}{\lambda_n^2}\right)^i \frac{(t-s)^{i-1}}{(i-1)!}$. According to the estimate

$$\left|R_n(t,s,\lambda)\right| \leq \sum_{i=1}^{\infty} \left|\lambda\right|^{i-1}\left|K_{n,i}(t,s)\right| \leq \sum_{i=1}^{\infty} \left|\lambda\right|^{i-1}\left(\frac{K_0}{\lambda_n^2}\right)^i \frac{(t-s)^{i-1}}{(i-1)!} \leq$$

$$\leq \frac{K_0}{\lambda_n^2}\sum_{i=1}^{\infty}\frac{1}{(i-1)!}\left(\frac{\left|\lambda\right|K_0(t-s)}{\lambda_n^2}\right)^{i-1} = \frac{K_0}{\lambda_n^2}e^{\frac{|\lambda|K_0(t-s)}{\lambda_n^2}}, \;\; n = 1, 2, 3, \ldots, \;\;\; 0 \leq s \leq t \leq T,$$

$R_n(t,s,\lambda)$ is a continuous function for any value of the parameter λ for each $n = 1, 2, 3, \ldots$.

Taking into account (13) and (15), we find the solution of boundary-value problem (2) by formula

$$V(t,x)=\sum_{n=1}^{\infty}\Big\{\psi_n(t,\lambda)+\int_0^t S_n(t,\tau,\lambda)z_n(1)p[\tau,u(\tau)]d\tau\Big\}z_n(x), \tag{17}$$

where

$$\psi_n(t,\lambda)=\psi_n\Big[e^{-\lambda_n^2 t}+\lambda\int_0^t R_n(t,s,\lambda)e^{-\lambda_n^2 t}ds\Big],$$

$$S_n(t,\tau,\lambda)=e^{-\lambda_n^2(t-\tau)}+\lambda\int_\tau^t R_n(t,s,\lambda)e^{-\lambda_n^2(s-\tau)}ds.$$

Further, by means of the direct calculations we have proved that the function $V(t,x)$ is an element of the space $H(Q)$ and this function is called *a weak generalized solution of the boundary-value problem* (2).

4 Solution of the Adjoint Boundary-Value Problem

We are looking for a solution of the adjoint boundary-value problem (6) in the form

$$\omega(t,x)=\sum_{n=1}^{\infty}\omega_n(t)z_n(x),\quad \omega_n(t)=\int_0^1 \omega(t,x),z_n(x)dx. \tag{18}$$

Fourier coefficients $\omega_n(t)$ are defined as solution of linear inhomogeneous Volterra integral equation of the second type

$$\omega_n(t)=\lambda\int_t^T B_n(s,t)\omega_n(s)ds-2\int_t^T e^{-\lambda_n^2(\tau-t)}[V_n(\tau)-\xi_n(\tau)]d\tau, \tag{19}$$

$$n=1,2,3,\ldots,$$

where

$$B_n(s,t)=\int_t^s e^{-\lambda_n^2(\tau-t)}K(s,\tau)d\tau.$$

Here $V_n(t)$ *and* $\xi_n(t)$ are the Fourier coefficients of functions $V(t,x)$ and $\xi(t,x)$.

The solution of Eq. (19) is determined by the formula

$$\omega_n(t) = -2\lambda \int_t^T L_n(s,t,\lambda)\left(2\int_s^T e^{-\lambda_n^2(\tau-s)}[V_n(\tau)-\xi_n(\tau)]d\tau\right)ds - \\ -2\int_t^T e^{-\lambda_n^2(\tau-t)}[V_n(\tau)-\xi_n(\tau)]d\tau, \quad (20)$$

where resolvent $L_n(s,t,\lambda)$ is a continuous function, as the sum of an absolutely convergent Neumann series of the form

$$L_n(t,s,\lambda) = \sum_{i=1}^{\infty} \lambda^{i-1} B_{n,i}(s,t). \quad (21)$$

For iterated kernels $B_{n,i}(s,t)$

$$B_{n,i+1}(s,t) = \int_s^t B_n(s,\eta)B_{n,i}(\eta,t)d\eta, \quad B_n(s,t) \equiv B_{n,1}(s,t), \quad i = 1,2,3,\ldots,$$

estimates $\left|B_{n,i}(s,t)\right| \le \left(\frac{K_0}{\lambda_n^2}\right)^i \frac{(s-t)^{i-1}}{(i-1)!}$, $i = 1,2,3,\ldots$, are established. It ensures the convergence of the Neumann series (21) for any value of the parameter λ for each $n = 1,2,3,\ldots$.

Taking into account (15), (18), and (20) we find the solution of adjoint boundary-value problem by formula

$$\omega(t,x) = -2\sum_{n=1}^{\infty}\left(\int_0^T \int_t^T E_n(t,\tau,\lambda)S_n(\tau,y,\lambda)d\tau z_n(1)p[y,u(y)]dy - \right. \quad (22) \\ \left. -\int_t^T E_n(t,\tau,\lambda)b_n(\tau,\lambda)d\tau\right)z_n(x),$$

where

$$E_n(t,\tau,\lambda) = \lambda\int_t^\tau L_n(t,s,\lambda)e^{-\lambda_n^2(\tau-s)}ds + e^{-\lambda_n^2(\tau-t)},$$

$$b_n(\tau,\lambda) = \big(\xi_n(t) - \psi_n(\tau,\lambda)\big).$$

Taking into account the inequalities

$$|L_n(t,s,\lambda)| \le \frac{K_0}{\lambda_n^2} e^{\frac{|\lambda|K_0(s-t)}{\lambda_n^2}},$$

it is not difficult to prove that function $\omega(t, x)$ is an element of the space $H(Q)$ and this function is called *a weak generalized solution of adjoint boundary-value problem* (6).

5 Nonlinear Integral Equation of Optimal Control

In the optimality condition (12) substituting the function (22), we obtain the relation

$$\frac{\beta u(t)}{p_u[t, u(t)]} + \sum_{n=1}^{\infty} z_n(1) \int_0^T \left(\int_t^T E_n(t, \tau, \lambda) S_n(\tau, y, \lambda) d\tau \right) z_n(1) p[y, u(y)] dy = \\ = \sum_{n=1}^{\infty} z_n(1) \int_t^T E_n(t, \tau, \lambda) b_n(\tau, \lambda) d\tau, \quad (23)$$

where only control function $u(t)$ is unknown. This relation is called *the nonlinear integral equation of optimal control.*

Unique solvability of nonlinear integral equation (23) is investigated according to the procedure of work [9] tested in several studies of nonlinear optimal control problems [10–12]. Let's assume that

$$\frac{\beta u(t)}{p_u[t, u(t)]} = \nu(t). \quad (24)$$

We consider this equality as implicit function with respect to control function $u(t)$. Then, according to the optimality condition (11), Eq. (24) is uniquely resolved with respect to the function $u(t)$, i.e., there is such a function $\mu(\cdot)$ that

$$u(t) = \mu[t, \nu(t), \beta]. \quad (25)$$

According to (24)–(25), we reduce Eq. (23) to the following form:

$$\nu(t) + \sum_{n=1}^{\infty} z_n(1) \int_0^T \left(\int_t^T E_n(t, \tau, \lambda) S_n(\tau, y, \lambda) d\tau \right) z_n(1) p\big[y, \mu[y, \nu(y), \beta]\big] = \\ = \sum_{n=1}^{\infty} z_n(1) \int_t^T E_n(t, \tau, \lambda) b_n(\tau, \lambda) d\tau. \quad (26)$$

We introduce the notation

$$L[\nu(t)] = \sum_{n=1}^{\infty} z_n(1) \int_0^T \left(\int_t^T E_n(t,\tau,\lambda) S_n(\tau,y,\lambda) d\tau \right) z_n(1) p\big[y, \mu[y,\nu(y),\beta]\big] dy,$$

$$h = h(t,1) = \sum_{n=1}^{\infty} z_n(1) \int_t^T E_n(t,\tau,\lambda) b_n(\tau,\lambda) d\tau,$$

and we rewrite Eq. (26) in the operator form

$$\nu + L[\nu] = h. \tag{27}$$

Further, by means of the direct calculations we have proved the following lemmas:

Lemma 1 *The function* $\nu(t)$ *is an element of Hilbert space* $H(0,T)$.

Lemma 2 *The function* $h(t,1)$ *is an element of Hilbert space* $H(0,T)$.

Lemma 3 *The operator* $L[\nu]$ *maps spaces* $H(0,T)$ *into itself, i.e., it is an element of Hilbert space* $H(0,T)$ *for any* $\nu(t) \in H(0,T)$.

Lemma 4 *Suppose that for the function* $p[t,u(t)]$ *the Lipschitz condition is satisfied with respect to functional variable u, i.e.,*

$$\big|p[t,u(t)] - p[t,\overline{u}(t)]\big| \le p_0 \big|u(t) - \overline{u}(t)\big|, \quad p_0 > 0,$$

and for the function $\mu[t,\nu(t),\beta]$ *it is satisfied with respect to the functional variable* ν*, i.e.,*

$$\big|\mu[t,\nu(t),\beta] - \mu[t,\overline{\nu}(t),\beta]\big| \le \mu_0(\beta) \big|\nu(t) - \overline{\nu}(t)\big|, \quad \mu_0(\beta) > 0.$$

Then if the condition

$$\gamma = C_0 p_0 \mu_0(\beta) < 1$$

is met, operator $L[\nu]$ *is a contracting operator. Here constants* $C_0,\ p_0,\ \mu_0(\beta)$ *are positive numbers.*

Theorem 1 *Suppose that (4) and (11) and the conditions of Lemma 1–4 are satisfied. Then operator equation (27) has a unique solution in the Hilbert space* $H(0,T)$.

Proof Under the conditions of *Lemmas 1–4* the contracting mapping principle is valid, i.e., the operator $L[\nu]$ maps a complete metric space $H(0,T)$ into itself and it is the contracting operator. Therefore, by the theorem on the contracting mapping principle [13] there exists a unique fixed point for the operator $L[\nu]$, which is a solution of operator equation (27). □

Approximate solutions of operator equation (27) are constructed by the method of successive approximations

$$\nu_k = L[\nu_{k-1}] + h, \quad k = 1, 2, 3, \ldots .$$

The exact solution $\overline{\nu}(t)$ of the operator equation (27) is defined as the limit of the sequence $\{\overline{\nu}(t)\}$, i.e., $\overline{\nu}(t) = \lim_{k\to\infty} \nu_k(t)$ and it satisfies the estimate [13]

$$\|\overline{\nu}(t) - \nu_k(t)\|_{H(0,T)} \le \frac{\gamma^k}{1-\gamma}\|L[\nu_0(t)] + h(t,1) - \nu_0(t)\|_{H(0,T)},$$

where $\nu_0(t)$ is an arbitrary element of space $H(0, T)$.

Substituting the found solution $\overline{\nu}(t)$ into (27) we find the required optimal control

$$u^0(t) = \mu[t, \overline{\nu}(t), \beta], \tag{28}$$

which is a solution of the nonlinear integral equation (23).

6 Construction of the Complete Solution to the Nonlinear Optimization Problem

Substituting the optimal control (28) in (17) instead of the control $u(t)$ we obtain the optimal process

$$V^0(t,x) = \sum_{n=1}^{\infty}\left\{\psi_n(t,\lambda) + \int_0^T S_n(t,\tau,\lambda)z_n(1)p[\tau, u^0(\tau)]d\tau\right\}z_n(x),$$

i.e., the solution of the boundary-value problem (2) corresponding to the optimal control $u^0(t)$.

After determining the optimal control and the optimal process, we calculate the minimum value of the functional (1) by the formula

$$J[u^0(t)] = \int_0^T \int_0^1 [V^0(t,x) - \xi(t,x)]^2 dx dt + \beta\int_0^T [u^0(t)]^2 dt.$$

Thus, the found triple $\left(u^0(t),\ \ V^0(t,x),\ \ J[u^0(t)]\right)$ is called *a complete solution* to nonlinear optimization problem (1)–(4).

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Part VII
Nonlinear PDE

Session Organizers: Vladimir Georgiev and Tohru Ozawa

In this session, various nonlinear partial differential equations in mathematical physics were considered. Among possible arguments the following ones were discussed: existence and qualitative properties of the solutions, existence of wave operators and scattering for these problems, stability of solitary waves, and other special solutions.

Exponential Mixing and Ergodic Theorems for a Damped Nonlinear Wave Equation with Space-Time Localised Noise

Ridha Selmi and Rim Nasfi

Abstract This paper is devoted to study a damped nonlinear wave equation driven by a space-time localised noise, in a bounded domain with a smooth boundary. The equation is supplemented with the Dirichlet boundary conditions. It is assumed that the random perturbation is non-degenerate. We prove that the Markov process generated by the solution possesses a unique stationary distribution which is exponentially mixing. A strong law of large numbers and the central limit theorem are derived for this Markov process and used to estimate the corresponding rates of convergence.

1 Introduction

We consider the following damped nonlinear wave equation perturbed by a random force, in a bounded domain $D \subset \mathbb{R}^3$ with a smooth (e.g. class C^2) boundary ∂D:

$$\partial_t^2 v + \gamma \partial_t v - \Delta v + f(v) = h(x) + \eta(t,x), \quad (t,x) \in \mathbb{R}_+ \times D, \tag{1}$$

supplemented with the Dirichlet boundary condition

$$v \mid_{\partial D} = 0 \tag{2}$$

and the initial conditions

$$v(0,x) = v_0(x), \quad \partial_t v(0,x) = v_1(x), \quad x \in D. \tag{3}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_21

Here, t is the time variable, x is the spatial variable, $v = v(t, x)$ is the unknown real valued scalar function at the point (t, x), $\gamma > 0$ is the dissipation parameter and $f : \mathbb{R} \to \mathbb{R}$ is a nonlinear function having a quintic growth rate as v goes to $+\infty$ ($f(v) \sim v^5$) and satisfying conditions given in Sect. 3. The functions h and η are scalar real valued, respectively, deterministic and stochastic external forces. We shall assume that $h \in H_0^1(D)$ and η is sufficiently smooth and bounded, and its restriction to any cylinder of the form $[k-1, k] \times D$, $k \in \mathbb{N}^*$, is localised in both space and time. The initial data (v_0, v_1) belongs to the phase space $\mathcal{H} = H_0^1(D) \times L^2(D)$, endowed with the norm $\|u\|_{\mathcal{H}}^2 = \|\nabla u_1\|^2 + \|u_2\|^2 = \|u_1\|_1^2 + \|u_2\|^2$, where $u = (u_1, u_2) \in \mathcal{H}$. The Cauchy problem for (1)–(3) is well posed (see [6]), for any $u_0 = (v_0, v_1) \in \mathcal{H}$ there is a unique stochastic process $u = (v, \partial_t v)$ whose almost every trajectory coincides with u_0 for $t = 0$. The ergodicity of the stochastic nonlinear PDEs with a random external force was studied by many researchers (see [5] and [7]). The problem of ergodicity of a nonlinear wave equation driven by a white noise was studied by Barbu and Da Prato in [3] and by Martirosyan in [8].

In this work, we show that the coupling approach from [7] applies to the damped nonlinear wave equation (1) driven by the space-time localised noise, whenever the nonlinear function f satisfies the dissipative conditions (5) and (6), and the growth assumptions (4). The main result of this article is: when η is a non-degenerate random force with a space-time localised support and under suitable controllability properties of the nonlinear wave equation (see [1]), the discrete-time Markov process associated with our problem has a unique stationary measure μ in $\mathcal{P}(\mathcal{H})$, and the law of any solution converges exponentially fast to μ in the Kantorovich-Wasserstein (Lipschitz-dual) metric. An exact formulation of this result is given in Sect. 3. The proof of the existence of a stationary measure is related to the Bogolyubov–Krylov argument, which ensures the existence, under the condition that the process $u(t) = (v(t), \partial v(t))$ has a uniformly bounded moment in some $\mathcal{H}$-compact space. To achieve such a bound, we follow an argument of the theory of attractors (see [2]). The proof of the exponential mixing is based on a property of stabilisation to a non-stationary solution of problem (1–3) and a general criterion for mixing of Markov chains that provided a coupling approach relies on Theorem 3.1.7 in [7]. Furthermore, we show that the uniformly mixing Markov processes satisfy the strong law of large numbers (SLLN) and the central limit theorem (CLT) for a large class of Hölder continuous functionals with polynomial growth at infinity.

2 Notation

For an open set D of an Euclidean space, a closed interval $J \subset \mathbb{R}$ and Banach separable spaces X and Y, we introduce the following functional spaces:

- For $1 < p < \infty$, $L^p(D)$ is the Lebesgue space of measurable functions on D whose pth power is Lebesgue integrable. If $p = 2$, we denote the corresponding norm by $\|.\|$ and the corresponding scalar product by $(., .)$.

- $H^s(D)$ is the Sobolev space of order s with the usual norm $\|.\|_s$.
- $H_0^s(D)$ is the closure in H^s of infinitely smooth functions with compact support.
- $L^p(J, X)$ is the space of Borel-measurable functions $f : J \to X$, such that $\|f\|_{L^p(J,X)} = \left(\int_J \|f(t)\|_X^p dt\right)^{\frac{1}{p}} < \infty$. In the case $p = \infty$, this norm is $\|f\|_\infty = ess\ sup\ \|f(t)\|_X$.
- $C_b(X)$ denotes the space of continuous bounded functions $f : X \to \mathbb{R}$ endowed with the norm of uniform convergence: $\|f\|_\infty = \sup_{u \in X} |f(u)|$.
- $L_b(X)$ is the space of bounded Lipschitz functions $f : X \to \mathbb{R}$, i.e. of functions $f \in C_b(X)$, such that $\|f\|_L := \|f\|_\infty + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|_X} < \infty$.
- $\mathcal{C}^{0,\alpha}(X)$ is the space of locally Hölder continuous functions $f : X \to \mathbb{R}$, for $\alpha \in]0, 1]$ there is $C_\alpha \geq 0$, such that $|f(u) - f(v)| \leq C_\alpha \|u - v\|_X^\alpha, \quad u, v \in X$.
- $\mathcal{W}$ is the space of increasing continuous functions w such that $w(r) > 0, r \geq 0$. For the next definitions, we fix an arbitrary weight function $w \in \mathcal{W}$.
- $\mathcal{C}^{0,\alpha}(X, w)$ is the space of functions $f \in \mathcal{C}^{0,\alpha}(X)$, such that

$$\|f\|_w := \sup_{u \in X} \frac{|f(u)|}{w(\|u\|_X)} < \infty$$

$$\|f\|_{w,\alpha} := \|f\|_w + \sup_{u \neq v} \frac{|f(u) - f(v)|}{\|u - v\|_X^\alpha (w(\|u\|_X) + w(\|v\|_X))} < \infty.$$

- $\mathcal{B}(X)$ is the Borel σ-algebra in X and $\mathcal{P}(X)$ denotes the space of probability measures on $\mathcal{B}(X)$. If $f : X \to \mathbb{R}$ is a $\mathcal{B}(X)$-measurable function and $\mu \in \mathcal{P}(X)$, then $\langle f, \mu \rangle = \int_X f(x)\mu(dx)$.
- The space $\mathcal{P}(X)$ is endowed with the topology of weak convergence, which is generated by the Kantorovich-Wasserstein (dual Lipschitz) metric,

$$\|\mu_1 - \mu_2\|_L^* := \sup_{\|f\|_L \leq 1} |\langle f, \mu_1 \rangle - \langle f, \mu_2 \rangle|, \quad \mu_1, \mu_2 \in \mathcal{P}(X).$$

- For $T > 0$, we set $J_T = [0, T]$ and $D_T = J_T \times D$.

3 Main Results and Scheme of Their Proofs

In what follows, we suppose that the nonlinear function $f \in C^2(\mathbb{R})$, such that $f(0) = 0$, satisfies

$$|f'(v)| \leq C(1 + |v|^4), \text{ and } |f''(v)| \leq C(1 + |v|^3), \ v \in \mathbb{R}, \tag{4}$$

where C is a positive constant, and the dissipativity conditions

$$F(v) \geq -C + \kappa |v|^6, \quad v \in \mathbb{R}, \quad \kappa > 0, \tag{5}$$

$$f(v)v - 4F(v) \geq -C, \quad f(v)v \geq -C, \quad v \in \mathbb{R}, \tag{6}$$

where $F(w) := \int_0^w f(v)dv$. Also, we assume that η is a stochastic process of the form

$$\eta(t,x) = \sum_{k=1}^{\infty} I_k(t)\eta_k(t-k+1,x), \quad t \geq 0, \tag{7}$$

where I_k is the characteristic function of $[k-1,k]$, for $k \geq 1$, and $\{\eta_k\}$ is a sequence of independent and identically distributed (i.i.d) random variables in $L^2(D_1)$, defined by (9), with zero value if $t \notin J_1$. Let V_0 be the support of η and we shall always assume that there is $x_0 \in \mathbb{R}^3 \setminus \bar{D}$ and $\delta > 0$ such that $D_\delta(x_0) = \left\{x \in D \; and \text{ there is } x' \in \partial D(x_0) \text{ such that } |x - x'| < \delta\right\} \subset V_0$, where $\partial D(x_0) = \left\{x' \in \partial D, \langle x' - x_0, \mathbf{n}_{x'}\rangle > 0\right\}$ and $\mathbf{n}_{x'}$ is the outward unit normal to ∂D at the point x'. Namely, we assume that the above conditions hold. Thus, for any initial data $u_0 = (v_0, v_1) \in \mathcal{H}$, there exists a unique process v satisfying Eq. (1) and the initial conditions (3) such that $v \in C(\mathbb{R}_+, H_0^1(D)) \cap C^1(\mathbb{R}_+, L^2(D)) \cap L^4(\mathbb{R}_+, L^{12}(D))$, and

$$\begin{aligned} &\|v(t)\|_{L^4([0,1],L^{12}(D))} + \|v(t)\|_1 + \|\partial_t v(t)\| \\ &\qquad \leq \Theta((v_0,v_1))e^{-\alpha t} + \Theta(\|h + \eta(t)\|), \; t \in J_1, \end{aligned}$$

where the monotone function Θ and the positive constant α are independent of (v_0, v_1) and t. The well-posedness of the initial boundary value problem (1) and the regularity of solutions are proved in Section 3 of [6]. For the proof of the global solvability, we repeat exactly the arguments in [4] of the particular case $f(v) = v^5$ and $\gamma = h = \eta = 0$. Let us denote by $S : \mathcal{H} \times L^2(D_1) \to \mathcal{H}$ a continuous operator that takes $(u_0, h + \eta)$ to $u(1)$, where $u(t) = (v(t), \partial_t v(t))$ and v is the solution of (1)–(3). We denote $u(k)$ by u_k and we consider the discrete-time random dynamical system (RDS) in $\mathcal{H}$:

$$u_k = S(u_{k-1}, h + \eta_k), \quad k \geq 1. \tag{8}$$

Since η_k are i.i.d random variables in $L^2(D_1)$, Eq. (8) defines a homogeneous family of Markov chains in $\mathcal{H}$, denoted by $(u_k, \mathbb{P}_u)$.

Let Q be an open set of D_1 and $\{\varphi_j\} \subset H^1(Q)$ an orthonormal basis in $L^2(Q)$. Let $\chi \in C_0^\infty(D_1)$ such that $supp\chi \cap Q \subset V_0$ and $\chi(t,x) = 1$ if $(t,x) \in J_1 \times D_{\delta/2}(x_0)$ and we set $\psi_j = \chi\varphi_j$. We shall always assume that $\{\psi_j\}$ are linearly independent and the process η satisfies the following hypotheses:

- **(H1) Structure of the noise:** The i.i.d random variables η_k can be represented in the form

$$\eta_k(t,x) = \sum_{j=1}^{\infty} b_j \xi_{jk} \psi_j(t,x), \tag{9}$$

where ξ_{jk} are independent scalar random variables, such that $|\xi_{jk}| \leq 1$ with probability 1 and $\{b_j\} \subset \mathbb{R}$ is a non-negative sequence, such that $B := \sum_{j=1}^{\infty} b_j \|\psi_j\|_1 < \infty$. Let $\mathcal{J} \subset L^2(Q)$ be the support of the law of η_k.
- **(H2) Approximate controllability:** There is $\tilde{u} \in \mathcal{H}$, such that for any $R,\ \varepsilon \geq 0$, an integer $l \geq 1$ exists and verifies that for a given ρ_0 in $B_{\mathcal{H}}(R)$, there exists $\zeta_1, \ldots, \zeta_l \in \mathcal{J}$, such that $\|u_l - \tilde{u}\|_{\mathcal{H}} \leq \varepsilon$, where $u_l = S_l(\rho_0, \zeta_1, \ldots, \zeta_l)$ defined by (8), with $\eta_k = \zeta_k$ and $u_0 = \rho_0$.

Theorem 1 *Under conditions **(H1)** and **(H2)**, there is an integer $N \geq 1$, depending on $\|h\|_1$, B and ρ_0, such that if*

$$b_j \neq 0, \quad \textit{for } j = 1, \ldots..N, \tag{10}$$

then the following assertions hold:

- ***Existence and uniqueness:*** *The Markov family $(u_k, \mathbb{P}_u)$ generated by (8) has a unique stationary measure $\mu \in \mathcal{P}(\mathcal{H})$.*
- ***Exponential mixing:*** *There are positive constants C and ϱ, such that*

$$\|P_k(u,.) - \mu\|_L^* \leq C(1 + \|u\|_{\mathcal{H}})e^{-\varrho k}, \quad \textit{for } u \in \mathcal{H},\ k \geq 0. \tag{11}$$

Here, $P_k(u, \Gamma)$, $\Gamma \in \mathcal{B}(X)$ is the transition function associated with $(u_k, \mathbb{P}_u)$ and the constant C does not depend on u and k. We recall that μ is stationary measure for $(u_k, \mathbb{P}_u)$ if $\mu = \int_X P_k(u,.)\mu(du)$. Condition (10) implies the space-time non-degeneracy of the noise. For the condition (H2) to be checked, it is sufficient that $\mathcal{J}$ contains the zero element and problem (1) and (2) with $\eta = 0$ has a globally stable stationary solution.

In what follows, we outline the proof of Theorem 1. It will be based on two key ingredients: a coupling approach developed in [7] in the context of stochastic PDEs and a property of stabilisation to a non-stationary solution of a NLW equation in [1]. First, we recall an abstract result established in [7] (see Theorem 3.1.7). We assume here that X is a separable Banach space with a norm $\|.\|_X$ and let $(u_k, \mathbb{P}_u)$ be a family of Markov chains in X parameterised by the initial point $u \in X$. We denote by $P_k(u,.)$ its transition function. Let $(\mathbf{U}_k, \mathbb{P}_{\mathbf{U}})$ be another family of Markov family chains in the extended phase space $\mathbf{X} = X \times X$, such that

$$\begin{cases} \Pi_* \mathbf{P}_k(\mathbf{U},.) = P_k(u,.), & \text{for } \mathbf{U} = (u,u') \in \mathbf{X},\ k \geq 0, \\ \Pi'_* \mathbf{P}_k(\mathbf{U},.) = P_k(u',.), & \text{for } \mathbf{U} = (u,u') \in \mathbf{X},\ k \geq 0, \end{cases} \tag{12}$$

where $\Pi, \Pi^{'} : \mathbf{X} \to X$ stand for the natural projections to the components of $\mathbf{U} = (u, u')$ and $\mathbf{P}_k(\mathbf{U}, \mathbf{\Gamma})$ denotes the transition function for $(\mathbf{U}_k, \mathbb{P}_{\mathbf{U}})$.

In other words, relations (12) mean that, for any integer $k \geq 0$, the random variable $\mathbf{U}_k$ considered under the law $\mathbb{P}_{\mathbf{U}}$ is a coupling for the pair of measures $(P_k(u, .), P_k(u', .))$. We shall say that $(u_k, \mathbb{P}_u)$ satisfies the mixing hypothesis if there are a closed subset $G \subset X$ and $C, \varrho \geq 0$ such that it has an extension $(\mathbf{U}_k, \mathbb{P}_{\mathbf{U}})$ possessing the following properties:

- **Recurrence:** Let $\tau(\mathbf{G})$ be the first hitting time of the set $\mathbf{G}$, such that $\tau(\mathbf{G}) = \min\{k \geq 0 : \mathbf{U}_k \in \mathbf{G}\}$. Then, $\tau(\mathbf{G})$ is $\mathbb{P}_{\mathbf{U}}$ a.s. finite for any $\mathbf{U} \in \mathbf{X}$, and there are $C_1, \delta_1 \geq 0$, such that

$$\mathbb{E}_{\mathbf{U}} \exp(\delta_1 \tau(\mathbf{G})) \leq C_1, \quad \text{for } \mathbf{U} \in \mathbf{X}, \tag{13}$$

  where $\mathbb{E}_{\mathbf{U}}$ is the expectation with respect to $\mathbb{P}_{\mathbf{U}}$.
- **Exponential squeezing:** We set $\sigma = \min\{k \geq 0, \|u_k - u_k'\|_X > Ce^{-\varrho k}\}$. So, there are $C_2, \delta_2, \delta_3 \geq 0$, such that for any $\mathbf{U} \in \mathbf{G}$ we have

$$\mathbb{P}_{\mathbf{U}}\{\sigma = \infty\} \geq \delta_3, \tag{14}$$

$$\mathbb{E}_{\mathbf{U}}(I_{\{\sigma < \infty\}} \exp(\delta_2 \sigma)) \leq C_2. \tag{15}$$

The following proposition is a particular case of a more general result established in [7] (see Theorem 3.1.7).

Proposition 1 *Let $(u_k, \mathbb{P}_u)$ be a family of the Markov chains for which there exists another Markov family $(\boldsymbol{U}_k, \mathbb{P}_{\boldsymbol{U}})$ in extended space $\boldsymbol{X}$ that satisfies relation (12) and the mixing hypothesis. Then, $(u_k, \mathbb{P}_u)$ has a unique stationary distribution $\mu \in \mathcal{P}(X)$, and there are positive constants C and ϱ, such that*

$$\|P_k(u, .) - \mu\|_L^* \leq Ce^{-\varrho k}, \quad \textit{for all } u \in X, \; k \geq 0. \tag{16}$$

To prove Theorem 1, we first observe that the RDS defined by (8) possesses a compact absorbing invariant set $X \subset \mathcal{H}$. So, it suffices to prove the uniqueness of an invariant measure and the property of exponential mixing for the restriction of $(u_k, \mathbb{P}_u)$ to X, for which we maintain the same notation. We shall prove that $(u_k, \mathbb{P}_u)$ satisfies the hypotheses of Proposition 1. The recurrence property follows from the approximate controllability (hypothesis **(H2)**), while the exponential squeezing is established by the following result.

Proposition 2 *Under the hypothesis of Theorem 1, there exists $d > 0$, such that for any points $u, u' \in X$ satisfying the inequality $\|u - u'\|_{\mathcal{H}} \leq d$, the pair $(P_1(u, .), P_1(u', .))$ admits a coupling $(V(u, u'), V'(u, u'))$ verifying*

$$\mathbb{P}\left\{\|V(u, u') - V'(u, u')\|_{\mathcal{H}} > \frac{1}{2}\|u - u'\|_{\mathcal{H}}\right\} \leq C\|u - u'\|_{\mathcal{H}} \leq Cd, \tag{17}$$

where $C > 0$ is a constant independent of $u, u' \in X$.

For the proof of this proposition, we used controllability property for the NLW equation and an approach of an optimal coupling introduced in the Appendix of [10]. Now, we define a coupling operator $\boldsymbol{\mathcal{R}} = (\mathcal{R}, \mathcal{R}')$ by

$$\boldsymbol{\mathcal{R}}(u, u'; \omega) = \begin{cases} (V(u, u'), (V'(u, u')), & \text{for } \|u - u'\|_{\mathcal{H}} \le d, \\ (S(u, h + \zeta), S(u', h + \zeta')), & \text{for } \|u - u'\|_{\mathcal{H}} > d, \end{cases} \tag{18}$$

where ζ and ζ' are independent random variables defined on the same probability space as V and V'. The required Markov family $(\mathbf{U}_k, \mathbb{P}_{\mathbf{U}})$ is constructed by iterations of $\boldsymbol{\mathcal{R}}$. Let $(\Omega_k, \mathcal{F}_k, \mathbb{P}_k), k \ge 1$, be countably many copies of the probability space, where $\boldsymbol{\mathcal{R}}$ is defined and let $(\Omega, \mathcal{F}, \mathbb{P})$ be the direct product of these spaces. We set $\mathbf{U}_0 = (u, u')$, $\mathbf{U}_k = \boldsymbol{\mathcal{R}}(\mathbf{U}_{k-1}, \omega_k)$, $k \ge 1$. Thus, we defined a Markov chain $(\mathbf{U}_k, \mathbb{P}_{\mathbf{U}})$ in the extended phase space $\mathbf{X} = X \times X$. This Markov chain is an extension of $(u_k, \mathbb{P}_u)$ and possesses the recurrence and the exponential squeezing properties and therefore the hypotheses of Proposition 1 are satisfied. This will complete the proof of Theorem 1.

4 Ergodic Theorems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(u_k, \mathbb{P}_u)$ the Markov process associated with the RDS (8). We denote by $P_k(u, .)$ the corresponding transition function and by $\mathfrak{P}_k$ and $\mathfrak{P}_k^*$ the Markov semi-groups defined by the formulas

$$\mathfrak{P}_k : C_b(\mathcal{H}) \to C_b(\mathcal{H}), \qquad \mathfrak{P}_k \mathbf{f}(u) = \int_{\mathcal{H}} P_k(u, dv)\mathbf{f}(v),$$

$$\mathfrak{P}_k^* : \mathcal{P}(\mathcal{H}) \to \mathcal{P}(\mathcal{H}), \qquad \mathfrak{P}_k^* \mu(\Gamma) = \int_{\mathcal{H}} P_k(u, \Gamma)\mu(du).$$

Recall that a measure μ is stationary for the family $(u_k, \mathbb{P}_u)$ if $\mathfrak{P}_t^* \mu = \mu,\ t \le 0$.

Definition 1 We shall say that the family $(u_k, \mathbb{P}_u)$ is uniformly mixing if it has a unique stationary measure $\mu \in \mathcal{P}(\mathcal{H})$ and there exists a continuous function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ and a sequence $\{\gamma_k\}$ of positive numbers, such that $\gamma_k \to 0$ as $k \to \infty$, and for $\alpha \in (0, 1]$, $w \in \mathcal{W}$ and $\mathbf{f} \in C^{0,\alpha}(\mathcal{H}, w)$, we have

$$|\mathfrak{P}_k \mathbf{f}(u) - \langle \mathbf{f}, \mu \rangle| \le \gamma_k \rho(\|u\|_{\mathcal{H}}) \|\mathbf{f}\|_{w,\alpha}, \quad \text{for } k \ge 0, \quad u \in \mathcal{H}. \tag{19}$$

Let us fix an arbitrary constant $p > 0$ and set $w_p(r) = (1 + r)^p$, $r \ge 0$. For any $\mathbf{f} \in C^{0,\alpha}(\mathcal{H}, w_p)$ such that $\langle \mathbf{f}, \mu \rangle = 0$, we set a positive constant $\sigma_{\mathbf{f}}$, such that $\sigma_{\mathbf{f}}^2 = 2\langle \sum_{k=0}^{\infty} \mathfrak{P}_k \mathbf{f}(u), \mu \rangle$. The following theorem describes the ergodic theorems for the family $(u_k, \mathbb{P}_u)$ with an estimate of the rate of convergence.

Theorem 2 *Under the above hypotheses, we assume that* $(u_k, \mathbb{P}_u)$ *is uniformly mixing, and* $\bar{\gamma} := \sum_{k=0}^{\infty} \gamma_k < \infty$. *If there is a continuous function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, *such that* $\mathfrak{P}_k \rho(u) := \mathbb{E}_u \rho(\|u_k\|_{\mathcal{H}}) \le \psi(\|u\|_{\mathcal{H}})$, *for all* $k \ge 0$. *Then, the following statements hold*:

(i) ***Strong law of large numbers***. *There exists* $D > 0$, *such that for any* $\mathbf{f} \in C^{0,\alpha}(\mathcal{H}, w_0)$ *and* $\delta > 0$, *where* $w_0 \in \mathcal{W}$ *and* $\lim_{r\to+\infty} w_0(r)e^{\delta r^2} = 0$, *there is a* $\mathbb{P}_u$*-a.s. finite random integer* $K(\omega)$, *for any* $u \in \mathcal{H}$, *such that*

$$\left| \frac{1}{k}\sum_{l=0}^{k-1} \mathbf{f}(u_l) - \langle \mathbf{f}, \mu\rangle \right| \le D\|\mathbf{f}\|_w k^{-\frac{1}{3}+\delta}, \quad \text{for} \quad k \ge K(\omega).$$

(ii) ***Central limit theorem***. *For any* $\varepsilon \in (0, \frac{1}{4})$ *and* $\bar{\sigma} \ge 0$ *there exists a positive function* $\Upsilon_{\bar{\sigma},\varepsilon}$ *defined on* $\mathbb{R}_+ \times \mathbb{R}_+$ *and increasing in both arguments, such that for any function* $\mathbf{f} \in C^{0,\alpha}(\mathcal{H}, w_p)$ *satisfying the conditions* $\sigma_{\mathbf{f}} \ge \bar{\sigma}$ *and* $\langle \mathbf{f}, \mu\rangle = 0$, *we have*

$$\sup_{z\in\mathbb{R}} \left| \mathbb{P}_u\{k^{-\frac{1}{2}} \sum_{l=0}^{k-1} \mathbf{f}(u_l) \le z\} - \Phi_{\sigma_{\mathbf{f}}}(z) \right| \le \Upsilon_{\bar{\sigma},\varepsilon}(\|u\|_{\mathcal{H}}, \|\mathbf{f}\|_{w_p,\alpha}) k^{-\frac{1}{4}+\varepsilon},$$

where $k \ge 1$, $u \in \mathcal{H}$ *and* $\Phi_\sigma(r) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{r} exp(-\frac{s^2}{2\sigma^2}) ds$, $\sigma > 0$. *For* $\sigma = 0$, *we set* $\Phi_0 = \mathbb{1}_{[0,+\infty[}$.

The proof of this theorem repeats essentially the argument used in [9].

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Part VIII
P-adic Analysis

Session Organizers: Alain Escassut, Andrei Khrennikov, and Karl-Olof Lindahl

The session gathered works using ultrametric analysis: spaces of functions and operators, p-adic analytic functions, applications to Statistics and Physics, Levi-Civita fields, applications to Dynamical systems, Number Theory, and computers.

On the Injective Embedding of p-Adic Integers in the Cartesian Product of p Copies of Sets of 2-Adic Integers

Ekaterina Yurova Axelsson

Abstract We study an injective embedding of p-adic integers in the Cartesian product of p copies of sets of 2-adic integers. This embedding allows to explicitly specify any p-adic integer through p specially selected 2-adic numbers. This representation can be used in p-adic mathematical physics, for example, in justifying choice of the parameter p.

1 Introduction

In this paper we establish the possibility of injective embedding of p-adic integers (p is a prime and $p \geq 3$) in the Cartesian product of p copies of sets of the 2-adic numbers. Injective image of p-adic integers is a subset of the hyperplane from a Cartesian product of the p copies of sets of the 2-adic numbers. We show that it is possible to explicitly introduce arbitrary p-adic integer through suitable 2-adic numbers, using constructed injective embedding. The main results are presented in Proposition 1. We also present the numerical illustrative example.

The possibility of an injective embedding of p-adic integers into the Cartesian product p copies of sets of the 2-adic numbers has already been indirectly used in the description of ergodic p-adic functions, see, for example [8]. In addition, the representation of p-adic integers through 2-adic numbers can be used to study p-adic models in mathematical physics in order to support the choice of the number p for these models, see [1, 3–7, 11, 12].

We recall some definitions related to the p-adic numbers and introduce the necessary notations. For any prime number p the p-adic norm $|\cdot|_p$ is defined in the following way. For every nonzero integer n let $ord_p(n)$ be the highest power of p which divides n, i.e., $n \equiv 0 \pmod{p^{ord_p(n)}}$, $n \not\equiv 0 \pmod{p^{ord_p(n)+1}}$.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_22

Then we define $|n|_p = p^{-ord_p(n)}$, $|0|_p = 0$. For rationals $\frac{n}{m} \in \mathbb{Q}$ we set $|\frac{n}{m}|_p = p^{-ord_p(n)+ord_p(m)}$.

The completion of $\mathbb{Q}$ with respect to the p-adic metric $\rho_p(x, y) = |x - y|_p$ is called the field of p-adic numbers $\mathbb{Q}_p$. The metric ρ_p is the so-called strong triangle inequality $|x \pm y|_p \leq \max(|x|_p; |y|_p)$ where equality holds if $|x|_p \neq |y|_p$. The set $\{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is called the set of p-adic integers. It is denoted by $\mathbb{Z}_p$. In some sense, the set of p-adic integers is an analogue of the interval [0, 1] for real numbers. More detailed information about p-adic numbers can be found, for example, in the works [2, 9, 10].

Hereinafter, we will consider only the p-adic integers $\mathbb{Z}_p$.

Every $x \in \mathbb{Z}_p$ can be expanded in canonical form, namely in the form of a series which converges with respect to the p-adic norm:

$$x = x_0 + px_1 + \ldots + p^k x_k + \ldots, \qquad x_k \in \{0, 1, \ldots, p-1\}, k \geq 0.$$

We denote by $\delta_k(x)$ the value of p-ary digit from the canonical representation of the number $x \in \mathbb{Z}_p$ with the number k, i.e., $\delta_k(x) = x_k$, $k \geq 0$.

If necessary, we can identify every p-adic integer with a sequence of digits $(x_0, x_1, \ldots, x_k, \ldots)$.

Let $\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{p} = \mathbb{Z}_2^p$ be a Cartesian product of p copies of sets of the 2-adic numbers. In $\mathbb{Z}_p$ we allocate a hyperplane $S_2(p)$:

$$S_2(p) = \left\{ \left(A^{(0)}, A^{(1)}, \ldots, A^{(p-1)} \right) \in \mathbb{Z}_2^p \ : \ A^{(0)} + A^{(1)} + \ldots + A^{(p-1)} + 1 = 0 \right\} \tag{1}$$

The set $\mathbb{Z}_p$ is embedded in this hyperplane.

Let us define the following functions:

$$\Lambda : \mathbb{Z}_2 \to \mathbb{Z}_p, \quad \Lambda(x) = \Lambda(x_0 + 2x_1 + \ldots + 2^k x_k + \ldots) = \sum_{k=0}^{\infty} p^k x_k \quad x_k \in \{0, 1\}; \tag{2}$$

and $\psi_s : \mathbb{Z}_p \to \mathbb{Z}_2$, $s = 0, 1, 2, \ldots, p-1$

$$\psi_s(x) = \sum_{k=0}^{\infty} 2^k u_{s,k} \tag{3}$$

with $u_{s,k}(x) = 1$ whenever $\delta_k(x) = s$ and $u_{s,k}(x) = 0$ whenever $\delta_k(x) \neq s$.

2 An Injective Embedding of $\mathbb{Z}_p$ in $\mathbb{Z}_2^p$

In this section, we establish the fact of an injective embedding of $\mathbb{Z}_p$ in the hyperplane $S_2(p)$ of the Cartesian product of p copies of sets of the 2-adic numbers $\mathbb{Z}_2^p$. The injective image $Z(p)$ of $\mathbb{Z}_p$ coincides with some subset $S_2(p)$. Using the embedding $\mathbb{Z}_p$ in $\mathbb{Z}_2^p$, explicit expressions are obtained for the representation of p-adic numbers in terms of the sum of 2-adic numbers that are represented in the form (6). The values of the p-adic norm of the whole p-adic number in terms of the norms of 2-adic numbers in $Z(p)$ are also calculated explicitly, see second statement in Proposition 1. Proposition 1 also shows that the rationality property of a number from $\mathbb{Z}_p$ is preserved under its injective embedding in $\mathbb{Z}_2^p$, see third statement in Proposition 1. At the end of the section, numerical examples of the representation of p-adic numbers in terms of 2-adic numbers are given.

We define the set $Z(p) \subset \mathbb{Z}_2^p$ in the following way. Let

$$A^{(s)} = A_0^{(s)} + 2A_1^{(s)} + 2^2 A_2^{(s)} + \ldots + 2^k A_k^{(s)} + \ldots, \quad s = 0, 1, \ldots p-1$$

be a canonical representation of 2-adic numbers $A^{(0)}, A^{(1)}, \ldots, A^{(p-1)}$, $A_k^{(s)} \in \{0, 1\}$, $k \geq 0$. An element $\left(A^{(0)}, A^{(1)}, \ldots, A^{(p-1)}\right)$ from the Cartesian product $\mathbb{Z}_2^p$ belongs to $Z(p)$ if and only if

$$A_k^{(0)} + A_k^{(1)} + A_k^{(2)} + \ldots A_k^{(p-1)} = 1, \ k \geq 0. \tag{4}$$

In other words, $\left(A^{(0)}, A^{(1)}, \ldots, A^{(p-1)}\right) \in Z(p)$ is determined by the fact that for all $k \geq 0$ among the binary coordinates with the number k from the canonical representation of these 2-adic numbers there is exactly one unit.

Note that the set $Z(p)$ and the hyperplane $S_2(p) \subset \mathbb{Z}_2^p$ are given in a similar way, but here $Z(p) \subsetneqq S_2(p)$. Indeed, if

$$\Omega = \left(A^{(0)}, A^{(1)}, \ldots, A^{(p-1)}\right) \in Z(p),$$

then $A^{(0)} + A^{(1)} + \ldots + A^{(p-1)} = -1$ and $\Omega \in S_2(p)$. Let us consider the collection of 2-adic numbers $A^{(0)}, A^{(1)}, \ldots, A^{(p-1)}$ for some $r \geq 0$ such, that

$$A_k^{(0)} + A_k^{(1)} + A_k^{(2)} + \ldots + A_k^{(p-1)} = \begin{cases} 3, & \text{if } k = r; \\ 0, & \text{if } k = r+1; \\ 1, & \text{for other } k \geq 0. \end{cases}$$

It is clear that $\left(A^{(0)}, A^{(1)}, \dots, A^{(p-1)}\right) \notin Z(p)$, but

$$A^{(0)} + A^{(1)} + \ldots + A^{(p-1)} =$$

$$= \sum_{k=0}^{r-1} 2^k \sum_{s=0}^{p-1} A_k^{(s)} + 2^r \sum_{s=0}^{p-1} A_r^{(s)} + 2^{r+1} \sum_{s=0}^{p-1} A_{r+1}^{(s)} + \sum_{k=r+2}^{\infty} 2^k \sum_{s=0}^{p-1} A_k^{(s)} =$$

$$= \sum_{k=0}^{r-1} 2^k + 2^r \cdot 3 + \sum_{k=r+2}^{\infty} 2^k = -1,$$

that is, $\left(A^{(0)}, A^{(1)}, \dots, A^{(p-1)}\right) \in S_2(p)$, then $Z(p) \subsetneqq S_2(p)$.

Proposition 1 *Let $p \geq 3$ and $Z(p) \subset \mathbb{Z}_2^p$ be the subset of the Cartesian product of p copies of sets of the 2-adic numbers defined by the relations (4). Then*

1. *there exists an injective mapping $\Psi \colon \mathbb{Z}_p \to Z(p)$, that is, the set of p-adic integers $\mathbb{Z}_p$ is injective embedded in the Cartesian product $\mathbb{Z}_2^p$ of p-adic integers;*
2. *if $A \in \mathbb{Z}_p$ and $\Psi(A) = \left(A^{(0)}, A^{(1)}, \dots, A^{(p-1)}\right) \in Z(p)$, then*

$$ord_p A = \min_{s=1,2,\cdots,p-1} ord_2 A^{(s)} = ord_2(1 + A^{(0)}),$$

that is, $|A|_p = p^{-ord_2(1+A^{(0)})}$;

3. *$A \in \mathbb{Z}_p$ is a rational integer if and only if every 2-adic number*

$$A^{(0)}, A^{(1)}, \dots, A^{(p-1)}$$

is rational.

Proof Let us consider the map

$$\Psi \colon \mathbb{Z}_p \to Z(p), \quad \Psi(A) = \left(\psi_0(A), \psi_1(A), \cdots, \psi_{p-1}(A)\right),$$

where $\psi_s \colon \mathbb{Z}_p \to \mathbb{Z}_2$, $s = 0, 1, \dots, p-1$ are given by relations (3).

We show that the mapping Ψ is injective.

Let $\left(B^{(0)}, B^{(1)}, \dots, B^{(p-1)}\right) \in Z(p)$. We consider $B \in \mathbb{Z}_p$, such that

$$B = \Lambda(B^{(1)}) + 2\Lambda(B^{(2)}) + \ldots + (p-1)\Lambda(B^{(p-1)}), \tag{5}$$

where $\Lambda \colon \mathbb{Z}_2 \to \mathbb{Z}_p$ is defined in (2). Then

$$\psi_s(B) = \psi_s(\Lambda(B^{(1)}) + 2 \cdot \Lambda(B^{(2)}) + \ldots + s \cdot \Lambda(B^{(s)}) + \ldots +$$
$$+ (p-1) \cdot \Lambda(B^{(p-1)})) = B^{(s)}, \quad s = 1, 2, \dots, p-1.$$

By definition of the set $Z(p)$, we have $\delta_k(B^{(0)}) = 1$ if and only if $\delta_k(B^{(s)}) = 0$ for $s = 1, 2, \dots, p-1$ taking into account the canonical representation of the corresponding 2-adic numbers. Then from (5), it follows that $\delta_k(B^{(0)}) = 1$ as soon as $\delta_k(B) = 0$, i.e., $\psi_0(B) = B^{(0)}$ and $\Psi(B) = \left(B^{(0)}, B^{(1)}, \dots, B^{(p-1)}\right)$.

Suppose that there exist $A, B \in \mathbb{Z}_p$ such that $\Psi(A) = \Psi(B)$. Then $\psi_s(A) = \psi_s(B)$, $s = 0, 1, \dots, p-1$ and

$$A = \sum_{s=1}^{p-1} \psi_s(A) = \sum_{s=1}^{p-1} \psi_s(B) = B.$$

This contradiction shows that Ψ is injective.

The inverse mapping $\Psi^{(-1)}\colon \Psi(\mathbb{Z}_p) \to \mathbb{Z}_p$ is defined as follows:

$$\Psi^{(-1)}\left(A^{(0)}, A^{(2)}, \dots, A^{(p-1)}\right) = \\ = \Lambda(A^{(1)}) + 2\Lambda(A^{(2)}) + \dots + (p-1)\Lambda(A^{(p-1)}) \tag{6}$$

Let us prove the second statement. Let $A \in \mathbb{Z}_p$ and

$$A = \Lambda(A^{(1)}) + 2\Lambda(A^{(2)}) + \dots + (p-1)\Lambda(A^{(p-1)}),$$

where $\Psi(A) = \left(A^{(0)}, A^{(2)}, \dots, A^{(p-1)}\right)$. Then

$$|A|_p = \max\left(|\Lambda(A^{(1)})|_p, \dots, |\Lambda(A^{(p-1)})|_p\right).$$

From (2) it follows that $ord_p\left(\Lambda(A^{(s)})\right) = ord_2\left(A^{(s)}\right)$, $s = 1, 2, \dots, p-1$, i.e.,

$$ord_p A = \min_{s=1,2,\cdots,p-1} ord_2 A^{(s)}.$$

Since $\delta_k\left(A^{(0)}\right) = 1$, $k \geq 0$ as soon as $\delta_k\left(A^{(s)}\right) = 0$, $s = 1, 2, \dots, p-1$ by the method of construction of the set $Z(p)$. Therefore,

$$ord_2(1 + A^{(0)}) = \min_{s=1,2,\cdots,p-1} ord_2 A^{(s)} = ord_p A \ \text{ or } \ |A|_p = p^{-ord_2(1+A^{(0)})}.$$

Let us prove the third statement.

Let $A = A_0 + pA_1 + \dots + p^k A_k + \dots \in \mathbb{Z}_p$ be a rational number, i.e., the sequence $\{A_k\}_{k=0}^{\infty}$ has a certain period T (so $A_{k+T} = A_k$ for any $k > N$). Then, by the method of construction the mapping Ψ, the sequences $\{\delta_k(\psi_s(A))\}_{k=0}^{\infty}$, $s \in 0, 1, \dots, p-1$ are also periodic, i.e., 2-adic numbers $\psi_s(A)$ are rational.

Let us prove the assertion in the opposite direction. Let

$$\Psi(A) = \left(A^{(0)}, A^{(1)}, \ldots, A^{(p-1)}\right) \in Z(p)$$

and $A^{(s)}$, $s = 0, 1, \ldots, p-1$ be rational numbers given by the periodical sequences $\{\delta_k(A^{(s)})\}_{k=0}^{\infty}$ with the period length T_s, respectively. Since

$$A = \Lambda(A^{(1)}) + 2\Lambda(A^{(2)}) + \ldots + (p-1)\Lambda(A^{(p-1)}),$$

then the sequence $\{\delta_k(A)\}_{k=0}^{\infty}$ has a period $T = LCM(T_1, T_2, \ldots, T_{p-1})$, i.e., A is a rational number. Note that due to the method of setting the set $Z(p)$ in (4), T_0 divides T.

To illustrate the results of Proposition 1, we give a numerical example.

Example 1 Let $p = 5$. We define 5-adic number $A = -\frac{73250}{624}$ and we define by a periodic sequence of 5-ary coordinates with a period length of 4 (that is, we consider a rational 5-adic integer)

$$(000\ 1234\ 1234\ 1234\ \ldots).$$

Then

$$\psi_0(A) = 7; \quad \psi_1(A) = -\frac{8}{15}; \quad \psi_2(A) = -\frac{16}{15}; \quad \psi_3(A) = -\frac{32}{15}; \quad \psi_4(A) = -\frac{64}{15}$$

and 2-adic numbers $\psi_0(A)$, $\psi_1(A)$, $\psi_2(A)$, $\psi_3(A)$, $\psi_4(A)$ are given by the following periodic sequences with a length of period 4:

$$\psi_0(A) : (111\ 0000\ 0000\ 0000\ \ldots);$$
$$\psi_1(A) : (000\ 1000\ 1000\ 1000\ \ldots);$$
$$\psi_2(A) : (000\ 0100\ 0100\ 0100\ \ldots);$$
$$\psi_3(A) : (000\ 0010\ 0010\ 0010\ \ldots);$$
$$\psi_4(A) : (000\ 0001\ 0001\ 0001\ \ldots).$$

Note that

$$\Lambda(\psi_1(A)) = -\frac{5^3}{5^4 - 1} = -\frac{125}{624}; \qquad \Lambda(\psi_2(A)) = -\frac{5^4}{5^4 - 1} = -\frac{625}{624}$$
$$\Lambda(\psi_3(A)) = -\frac{5^5}{5^4 - 1} = -\frac{3125}{624}; \qquad \Lambda(\psi_4(A)) = -\frac{5^6}{5^4 - 1} = -\frac{15625}{624}.$$

Then

$$A = \Lambda(\psi_1(A)) + 2 \cdot \Lambda(\psi_2(A)) + 3 \cdot \Lambda(\psi_3(A)) + 4 \cdot \Lambda(\psi_4(A)) =$$
$$= -\left(\frac{125}{624} + 2 \cdot \frac{625}{624} + 3 \cdot \frac{3125}{624} + 4 \cdot \frac{15625}{624}\right) = -\frac{73250}{624}.$$

Since $ord_2(1 + \psi_0(A)) = 3$, then $|A|_5 = \left|-\frac{73250}{624}\right|_5 = 5^{-3}$.

Also note that $\Psi(A) = (\psi_0(A), \psi_1(A), \psi_2(A), \psi_3(A), \psi_4(A)) \in \mathbb{Z}_2^p$ is contained in the hyperplane $S_2(p)$, see (1). Indeed,

$$\psi_0(A) + \psi_1(A) + \psi_2(A) + \psi_3(A) + \psi_4(A) + 1 =$$
$$= 7 + \left(-\frac{8}{15}\right) + \left(-\frac{16}{15}\right) + \left(-\frac{32}{15}\right) + \left(-\frac{64}{15}\right) + 1 = 0.$$

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Description of (Fully) Homomorphic Cryptographic Primitives Within the p-Adic Model of Encryption

Ekaterina Yurova Axelsson and Andrei Khrennikov

Abstract In this paper we consider a description of homomorphic and fully homomorphic cryptographic primitives in the p-adic model. This model describes a wide class of ciphers (including substitution ciphers, substitution ciphers streaming, keystream ciphers in the alphabet of p elements), but certainly not all. Homomorphic and fully homomorphic ciphers are used to ensure the credibility of remote computing, including cloud technology. Within considered p-adic model we describe all homomorphic cryptographic primitives with respect to arithmetic and coordinate-wise logical operations in the ring of p-adic integers $\mathbb{Z}_p$. We show that there are no fully homomorphic cryptographic primitives for each pair of the considered set of arithmetic and coordinate-wise logical operations on $\mathbb{Z}_p$.

1 Introduction

Our novel approach to cloud computing is based on the use of p-adic numbers [12, 20] and it was motivated by our previous works on p-adic dynamical systems [6, 13, 14] especially connection of measure-preserving of dynamics with the possibility to use such dynamics as the basis for ciphers, see also for pioneer papers of Anashin [1, 2, 4] and monograph [3] (see also works [5, 9, 10, 15, 18] on general theory of p-adic dynamical system and more generally interrelation between number theory and dynamical systems).

Cloud computing and storage solutions provide users and enterprises with various capabilities to store and process their data in third-party data centers, see [11]. Homomorphic encryption is a form of encryption that allows computations to be carried out on ciphertext, thus generating an encrypted result which, when decrypted, matches the result of operations performed on the plaintext. A cryptosystem that supports arbitrary computation on ciphertexts is known as fully

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_23

homomorphic encryption (FHE). The existence of an efficient and fully homomorphic cryptosystem would have great practical implications in the outsourcing of private computations, for instance, in the context of cloud computing, see [16].

A brief review of the known homomorphic encryption algorithms is presented in [17]. Examples of fully homomorphic ciphers one can see in, for instance, [21].

The general idea of fully homomorphic encryption is as follows (see, for example, [19]). Suppose we have a set of data M. The operations $g_1 : M \times M \to M$, $g_2 : M \times M \to M$ are defined on the set M. It is necessary to find the value of an expression $W(d_1, \ldots, d_n)$, which is defined through the operations g_1 and g_2 on the data $d_1, \ldots, d_n \in M$.

We understand a cipher as a family of bijective transformations f_a of the set M, where each transformation is identified by a certain parameter a—the encryption key. Suppose that f_a is a homomorphism with respect to the operations g_1 and g_2. Then, $f_a(W(d_1, \ldots, d_n)) = W(f_a(d_1), \ldots, f_a(d_n))$. This means that the remote computations are performed on encrypted data $f_a(d_1), \ldots, f_a(d_n)$ and the result of calculations $W(d_1, \ldots, d_n)$ is obtained in encrypted form $f_a(W)$. That is, only the user has access to the data $d_1, \ldots, d_n$. In general, this approach provides complete trust in remote computing.

In this paper, we consider a p-adic cryptographic primitives which can be used for the construction of encryption functions for ciphers in the usual sense. P-adic cryptographic primitives consists of a family of p-adic functions that map a set of p-adic integers into itself. On the other hand, a p-adic cryptographic primitives is a "continuous" analogue of the family of ciphers $\mathcal{C}_p$. For such ciphers $\mathcal{C}_p$, the sets of plain and cipher texts are words of a finite length in the alphabet with p elements $\{0, 1, \ldots, p-1\}$. Note that this family of ciphers includes: substitution ciphers, substitution ciphers streaming, and keystream ciphers (in the alphabet with p elements).

Moreover, we present a description of all homomorphic and fully homomorphic cryptographic primitives with respect to a given set of operations, namely arithmetic ("+" and "·") and coordinate-wise logical ("XOR" and "AND"), defined on the set of p-adic integers $\mathbb{Z}_p$.

In Sect. 2, we describe a p-adic model of ciphers from the family $\mathcal{C}_p$ (p-adic cryptographic primitives correspond to ciphers from $\mathcal{C}_p$ within the framework of this model). Moreover, we show that the problem of description of homomorphic (fully homomorphic) cryptographic primitives is reduced to the description of the measure-preserving 1-Lipschitz functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$, which defines a homomorphism with respect to a given operation (relatively, to a given pair of operations) on $\mathbb{Z}_p$.

In Theorems 1 and 2, we describe all homomorphic cryptographic primitives with respect to arithmetic operations "+", "·" and coordinate-wise logical operations "XOR" and "AND" on $\mathbb{Z}_p$. Using these results, we show that the p-adic fully homomorphic cryptographic primitives with respect to any pair of the operations {"+", "·", "XOR", "AND"} do not exist (see Proposition 1). And, therefore, there are no fully homomorphic ciphers in the family of ciphers $\mathcal{C}_p$. Thus, using the apparatus of p-adic analysis, we were able to show the absence of fully homomorphic ciphers in a certain family of ciphers. The proofs of presented Theorems are given in the paper [8].

We start our paper by recalling definitions that are related to the p-adic analysis, as well as introducing the necessary notations.

1.1 P-adic Dynamical Systems

For any prime number p the p-adic norm $|\cdot|_p$ is defined on $\mathbb{Q}$ in the following way. For every nonzero integer n let $ord_p(n)$ be the highest power of p which divides n. Then we define $|n|_p = p^{-ord_p(n)}$, $|0|_p = 0$, and $|\frac{n}{m}|_p = p^{-ord_p(n)+ord_p(m)}$.

The completion of $\mathbb{Q}$ with respect to the p-adic metric $\rho_p(x, y) = |x - y|_p$ is called the field of p-adic numbers $\mathbb{Q}_p$. The metric ρ_p satisfies the so-called strong triangle inequality $|x \pm y|_p \leq \max(|x|_p; |y|_p)$. The set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is called the set of p-adic integers.

Hereinafter, we will consider only the p-adic integers. Every $x \in \mathbb{Z}_p$ can be expanded in canonical form, namely in the form of a series that converges for the p-adic norm: $x = x_0 + px_1 + \ldots + p^k x_k + \ldots, \quad x_k \in \{0, 1, \ldots, p-1\}, k \geq 0$.

In this paper, we consider functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$, which satisfy the Lipschitz condition with constant 1 (i.e., 1-Lipschitz functions). Recall that $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a 1-Lipschitz function if $|f(x) - f(y)|_p \leq |x - y|_p$, for all $x, y \in \mathbb{Z}_p$. This condition is equivalent to the following: $x \equiv y \pmod{p^k}$ follows $f(x) \equiv f(y) \pmod{p^k}$ for all $k \geq 1$.

Let functions $\delta_k(x), k = 0, 1, 2, \ldots$ be k-th digit in a p-base expansion of the number $x \in \mathbb{Z}_p$, i.e., $\delta_k : \mathbb{Z}_p \to \{0, 1, \ldots, p-1\}$, $\delta_k(x) = x_k$. Any map $f : \mathbb{Z}_p \to \mathbb{Z}_p$ can be represented in the form:

$$f(x) = \delta_0(f(x)) + p\delta_1(f(x)) + \ldots + p^k \delta_k(f(x)) + \ldots.$$

According to Proposition 3.33 in [3], f is a 1-Lipschitz function if and only if for every $k \geq 1$ the k-th coordinate function $\delta_k(f(x))$ does not depend on $\delta_{k+s}(x)$ for all $s \geq 1$, i.e., $\delta_k(f(x + p^{k+1}\mathbb{Z}_p)) = \delta_k(f(x))$ for all $x \in \{0, 1, \ldots, p^{k+1} - 1\}$. We consider the following functions of p-valued logic

$$\varphi_k : \underbrace{\{0, \ldots, p-1\} \times \ldots \times \{0, \ldots, p-1\}}_{k+1} \to \{0, \ldots, p-1\},$$

and $\varphi_k : (x_0, x_1, \ldots, x_k) \mapsto \delta_k(f(x))$. Then any 1-Lipschitz function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ can be represented as

$$f(x) = f(x_0 + \ldots + p^k x_k + \ldots) = \sum_{k=0}^{\infty} p^k \varphi_k(x_0, \ldots, x_k).$$

Dynamical system theory studies trajectories (orbits), i.e., sequences of iterations: $x_0,\ x_1 = f(x_0), \ldots, x_{i+1} = f(x_i) = f^{(i+1)}(x_0), \ldots,$

$$f^{(s)}(x) = \underbrace{f(f(\ldots f(x))\ldots)}_{s}.$$

We consider a p-adic autonomous dynamical system $\langle \mathbb{Z}_p, \mu_p, f \rangle$ (for more details see, for example, [1–7, 9, 10, 13, 14, 18]). The space $\mathbb{Z}_p$ is equipped with a natural probability measure, namely the Haar measure μ_p normalized so that $\mu_p(\mathbb{Z}_p) = 1$. Balls $B_{p^{-r}}(a) = \{x \in \mathbb{Z}_p : |x - a|_p \leq p^{-r}\} = a + p^r\mathbb{Z}_p$ of nonzero radii constitute the base of the corresponding σ-algebra of measurable subsets, $\mu_p(B_{p^{-r}}(a)) = p^{-r}$. The function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is continuous on $\mathbb{Z}_p$. A measurable mapping $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is called measure-preserving if

$$\mu_p(f^{-1}(U)) = \mu_p(U)$$

for each measurable subset $U \subset \mathbb{Z}_p$. In accordance with Corollary 3.4. from [7], a 1-Lipschitz function f is measure-preserving if and only if f is bijective on $\mathbb{Z}_p$.

2 Model

Let us remind that a cipher is a set $\langle X, R, Y, h_r, r \in R\rangle$, where X is a set of plain texts, Y is a set of ciphertexts, R is a set of keys, encryption functions h_r are defined by the parameter $r \in R$ and define an injective map $X \to Y$. Here we assume that all map h_r are surjective.

A family of ciphers $\mathcal{C}_p = \langle X, R, Y, h_r, r \in R\rangle$ we set in the following way:

1. $X = Y$ be a set of all words (as sequence of finite length) in the alphabet $\Omega = \{0, 1, \ldots, p-1\}$ for prime number p;
2. let $X^{(k)}$ be a set of all words of the length k in the alphabet Ω. Then $h_r : X^{(k)} \to X^{(k)}$ and h_r are bijective on $X^{(k)}$ for any $r \in R$, $k \geq 1$;
3. if $h_r(\{x_0, x_1, \ldots, x_s, x_{s+1}, \ldots, x_k\}) = \{y_0, y_1, \ldots, y_s, y_{s+1}, \ldots, y_k\}$, then $h_r(\{x_0, x_1, \ldots, x_s\}) = \{y_0, y_1, \ldots, y_s\}$ for any $1 \leq s \leq k,\ \ k \geq 1$ and $r \in R$.

Note that the family $\mathcal{C}_p$ contains substitution ciphers, substitution ciphers streaming, keystream ciphers (in the alphabet of p elements). On the other hand, there are no ciphers in $\mathcal{C}_p$ with different parameters of the sets of plaintext and ciphertext (for example, when the number of elements in the alphabet is a composite integer).

For ciphers from the family $\mathcal{C}_p$, we define operations on the set $X^{(\infty)}$. Let $x = \{x_0, x_1, \ldots, x_{k-1}\}$, $y = \{y_0, y_1, \ldots, y_{k-1}\} \in X^{(\infty)},\ \ k \geq 1$

$$\tau_k : X^{(k)} \to \{0, 1, \ldots, p^k - 1\},\ \ \tau_k(x) = x_0 + px_1 + \ldots + p^{k-1}x_{k-1}$$

The following operations are defined on the set $X^{(k)},\ k \geq 1$:

$$x + y = \tau_k^{-1}\left(\tau_k(x) + \tau_k(y) (\bmod p^k)\right);$$

$$x \cdot y = \tau_k^{-1}\left(\tau_k(x) \cdot \tau_k(y) (\bmod p^k)\right);$$

$$x \text{AND} y = \tau_k^{-1}\left(\tau_k(x) \text{AND} \tau_k(y) (\bmod p^k)\right);$$

$$x \text{XOR} y = \tau_k^{-1}\left(\tau_k(x) \text{XOR} \tau_k(y) (\bmod p^k)\right).$$

The set of such operations we denote as $\overline{O}_p$.

A family of ciphers $\mathcal{C}_p$ is embedded in the "continuous" p-adic model $\mathcal{M}_p = \langle \mathbb{Z}_p, f^r, r \in R \rangle$, where $f^r : \mathbb{Z}_p \to \mathbb{Z}_p$ is a 1-Lipschitz measure-preserving function. The functions f_r we will call cryptographic primitives for the family of ciphers $\mathcal{C}_p$. The choice of such a model is determined by the following circumstances:

1. The set $\cup_{k \geq 1} X^{(k)}$ we, naturally, associate with the projective limit of residue rings $\mathbb{Z}/p^k\mathbb{Z}$ with respect to the natural projections $\mathbb{Z}/p^{k+1}\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$. Since $\varprojlim \mathbb{Z}/p^k\mathbb{Z} = \mathbb{Z}_p$, then the set $\cup_{k \geq 1} X^{(k)}$ is associated with the ring of p-adic integers $\mathbb{Z}_p$ (in particular, $\mathbb{Z}_p$ corresponds to the sets of plain and encrypted text for $\mathcal{C}_p$).
2. A function h_r is modeled by a function $f^r : \mathbb{Z}_p \to \mathbb{Z}_p$ that is defined in the following way. Let $h_r(\{x_0, x_1, \ldots, x_k\}) = \{y_0, y_1, \ldots, y_k\}$, $k \geq 1$ and

$$f_k^{(r)}(x_0 + px_1 + \ldots + p^k x_{k-1}) \equiv y_0 + py_1 + \ldots + p^{k-1} y_{k-1} (\bmod p^k).$$

   The functions $f_k^{(r)} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Z}/p^k\mathbb{Z}$, $k \geq 1$ are bijective and, by Theorem 4.23 [3], define 1-Lipschitz measure-preserving function $f^r : \mathbb{Z}_p \to \mathbb{Z}_p$.
3. It is clear that operations from $\overline{O}_p$ can be extended by continuity on $\mathbb{Z}_p$, and these extensions correspond to the arithmetic and coordinate-wise logical operations on $\mathbb{Z}_p$ - $O_p = \{$"+," "·," "XOR," "AND"$\}$.

We say that some cipher from $\mathcal{C}_p$ be a homomorphic cipher with respect to the operation "g" on the sets of plain and encrypted text (for $\mathcal{C}_p$ these sets coincide), if $h_r(g(x, y)) = g(h_r(x), h_r(y))$ for any pair of plain text x, y and $r \in R$. If this property holds for the two operations, then such a cipher is called fully homomorphic cipher.

It is clear that for p-adic model of ciphers $\mathcal{M}_p$, a homomorphism condition for the ciphers from $\mathcal{C}_p$ corresponds to the fact that the functions $f_r, r \in R$ define homomorphisms but already on $\mathbb{Z}_p$ with respect to operations from the set O_p.

In this way, the problem of describing homomorphic (fully homomorphic) ciphers from $\mathcal{C}_p$ with respect to operations from $\overline{O}_p$ is reduced to the description of all functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$ such that f is a 1-Lipschitz function, f preserves the measure and f defines a homomorphism with respect operations from O_p.

3 Homomorphic Cryptographic Primitives

In this section we give a description of 1-Lipschitz functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$, which preserve the measure, and define the homomorphism relative to a binary operation on $\mathbb{Z}_p$ from the set O_p.

Theorem 1 (Arithmetic Operations) *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a 1-Lipschitz function. Then*

1. *f defines a homomorphism with respect to the operation "$+$" if and only if $f(x) = Ax$, $A \in \mathbb{Z}_p$. Such a function preserves the measure if and only if $A \not\equiv 0 \pmod p$;*
2. *f defines a homomorphism with respect to the operation "$\cdot$" if and only if*

$$f(x) = \begin{cases} p^k A^k \theta^s (1 + p\,t)^a, & \text{if } x = p^k \theta(1 + tp), \\ 0, & \text{if } x = 0, \end{cases}$$

where $k \geq 0$, $t, a, A \in \mathbb{Z}_p$, $s \in \{1, \ldots, p-1\}$ and $\theta \in \mathbb{Z}_p$, $\theta^{p-1} = 1$. Such a function preserves the measure if and only if $A \not\equiv 0 \pmod p$, $a \not\equiv 0 \pmod p$, $GCD\ (s, p-1) = 1$.

Remark 1 If in 2 we set $a = n$, $s = n$, $A = p^{n-1}$ for some $n \in \mathbb{N}$, then $f(x) = x^n$. That is, all such polynomials define a homomorphism with respect to multiplication on $\mathbb{Z}_p$. Functions of the form $f(x) = x^n$ for $n > 1$ do not preserve the measure.

Theorem 2 (Logical Operations) *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be a 1-Lipschitz function defined in the coordinate form, i.e.,*

$$f(x) = f(x_0 + \ldots + p^k x_k + \ldots) = \sum_{k=0}^{\infty} p^k \varphi_k(x_0, \ldots, x_k),$$

where $\varphi_k(x_0, \ldots, x_k)$ are p-valued logical functions. Then

1. *f defines a homomorphism with respect to the operation "XOR" if and only if*

$$\varphi_k(x_0, \ldots, x_k) = \alpha_0^{(k)} x_0 + \alpha_1^{(k)} x_1 + \ldots + \alpha_k^{(k)} x_k,$$

where $\alpha_i^{(k)} \in \{0, \ldots, p-1\}$, $0 \leq i \leq k$, $k \geq 0$. Such functions preserve the measure if and only if $\alpha_k^{(k)} \not\equiv 0 \pmod p$, $k \geq 0$;

2. *f defines a homomorphism with respect to the operation "AND" if and only if $\varphi_k(x_0, \ldots, x_k) = x_0^{s_0^{(k)}} \cdot x_1^{s_1^{(k)}} \cdots x_k^{s_k^{(k)}}$, where $s_i^{(k)} \in \{0, 1, \ldots, p-1\}$, $0 \leq i \leq k$, $k \geq 0$. Such functions preserve the measure if and only if $s_i^{(k)} = 0$ for $0 \leq i \leq k-1$ and GCD $(s_k^{(k)}, p-1) = 1$, $k \geq 0$.*

Now let us describe fully homomorphic cryptographic primitives with respect to each pair of operations from the set O_p.

Let $\mathcal{H}(*)$ be the set of all 1-Lipschitz functions, which define a homomorphism with respect to the operation "$*$" on $\mathbb{Z}_p$ and preserve the measure, "$*$"$\in O_p$.

The set of functions, consisting of identical function $f(x) = x$, is denoted by I.

Proposition 1 *The following relations hold:*

$$\begin{aligned}\mathcal{H}(+) \cap \mathcal{H}(\cdot) = \mathcal{H}(+) \cap \mathcal{H}(\text{XOR}) = \\ = \mathcal{H}(+) \cap \mathcal{H}(\text{AND}) = \mathcal{H}(\cdot) \cap \mathcal{H}(\text{XOR}) = \\ = \mathcal{H}(\cdot) \cap \mathcal{H}(\text{AND}) = \mathcal{H}(\text{XOR}) \cap \mathcal{H}(\text{AND}) = I.\end{aligned}$$

Proposition 1 shows that there are no non-trivial (different from identical) fully homomorphic cryptographic primitives for operations from O_p. It means that in the family of ciphers $\mathcal{C}_p$ there are no fully homomorphic with respect to the operations from $\overline{O}_p$.

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Spectrum of Ultrametric Banach Algebras of Strictly Differentiable Functions

Alain Escassut and Nicolas Maïnetti

Abstract Let $\mathbb{K}$ be an ultrametric complete field and let E be an open subset of $\mathbb{K}$. Let $\mathscr{D}(E)$ be the Banach $\mathbb{K}$-algebra of bounded strictly differentiable functions from E to $\mathbb{K}$. It is shown that each element of $\mathscr{D}(E)$ has a continuous derivative and that all functions that are bounded and analytic in all open disks of diameter r are strictly differentiable. Maximal ideals and continuous multiplicative semi-norms on $\mathscr{D}(E)$ are studied by recalling the relation of contiguity on ultrafilters. Every prime ideal of $\mathscr{D}(E)$ is included in a unique maximal ideal and every prime closed ideal of $\mathscr{D}(E)$ is a maximal ideal, hence every continuous multiplicative semi-norm on $\mathscr{D}(E)$ has a kernel that is a maximal ideal. Every maximal ideal of $\mathscr{D}(E)$ of finite codimension is of codimension 1. Every maximal ideal of $\mathscr{D}(E)$ is the kernel of a unique continuous multiplicative semi-norm and every continuous multiplicative semi-norm is the limit along an ultrafilter on E. The Shilov boundary of $\mathscr{D}(E)$ is equal to the whole set of continuous multiplicative semi-norms.

1 Introduction and Preliminaries

Let $\mathbb{K}$ be a field which is complete with respect to an ultrametric absolute value that will be denoted by $| \,.\, |$.

Consider a Banach $\mathbb{K}$-algebra T. Many studies were made on continuous multiplicative semi-norms on algebras of analytic functions, analytic elements, and their applications to holomorphic functional calculus [1, 3, 4]. Continuous multiplicative semi-norms of the Banach algebras of bounded continuous functions and those of bounded uniformly continuous functions were studied in [2, 7]. After these studies, it seems interesting to consider strictly differentiable functions.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_24

Definitions and Notations Throughout the paper, we denote by E an open subset of $\mathbb{K}$. Given $a \in \mathbb{K}$ and $r > 0$, we set $d(a, r) = \{x \in \mathbb{K} \mid |x - a| \leq r\}$, $d(a, r^-) = \{x \in \mathbb{K} \mid |x - a| < r\}$, $d_E(a, r) = \{x \in E \mid |x - a| \leq r\}$, and $d_E(a, r^-) = \{x \in E \mid |x - a| < r\}$.

We denote by δ the distance between two subsets of $\mathbb{K}$: given two subsets B_1, B_2 of $\mathbb{K}$, we set $\delta(B_1, B_2) = \inf\{|x - y| \mid x \in B_1,\ y \in B_2\}$. We denote by diam the diameter of a subset B of $\mathbb{K}$ and we set $\text{codiam}(B) = \delta(B, \mathbb{K} \setminus B)$. Similarly, given a subset B of E, we set $\text{codiam}_E(B) = \delta(B, E \setminus B)$. A subset B of E will be said to be *uniformly open in E* or *uniformly open subset of E* if $\text{codiam}_E(B) > 0$.

Given a bounded function f from E to $\mathbb{K}$, we put $\|f\|_0 = \sup_{x \in E} |f(x)|$.

Let $D = \{(x, x) \mid x \in E\}$ and let $\mathscr{D}(E)$ be the $\mathbb{K}$-vector space of bounded functions f from E to $\mathbb{K}$ such that the mapping ϕ defined in $(E \times E) \setminus D$ into $\mathbb{K}$ as $\phi(x, y) = \dfrac{f(x) - f(y)}{x - y}$ is bounded by a bound M_f and expands to a continuous function from $E \times E$ to $\mathbb{K}$. The functions $f \in \mathscr{D}(E)$ will be called *the strictly differentiable functions from E to* $\mathbb{K}$. Given $f \in \mathscr{D}(E)$, we put $\|f\|_1 = \sup_{(x,y) \in (E \times E) \setminus D} \phi(x, y)$ and we check that $\| \,.\, \|_1$ is another $\mathbb{K}$-vector space norm on $\mathscr{D}(E)$. Finally we put $\|f\| = \max(\|f\|_0, \|f\|_1)$.

Remark 1 Suppose $\mathbb{K}$ is algebraically closed and let $E = d(0, 1)$. For every $r \in [0, 1] \cap |\mathbb{K}|$, we denote by $\xi(r)$ an element b of E such that $|b|^2 = r$. The set $E \setminus \{0\}$ obviously admits a partition of the form $\{d(a_j, |a_j|^-)_{j \in I}\}$.

Now, let f be the function defined in E in the following way. Given $x \in d(a_j, |a_j|^-)$, we put $f(x) = \xi(|a_j|)$ and $f(0) = 0$. In this way, f is constant in each disk $d(a_j, |a_j|^-)$, $a \in E$ and therefore f has a derivative equal to 0 at each point $a \in E \setminus \{0\}$ but f has no derivative at 0 because

$$\lim_{x \to 0} \left| \frac{f(x)}{x} \right| = \lim_{x \to 0} \frac{\sqrt{|x|}}{|x|} = +\infty.$$

Now, let us take a decreasing sequence $(r_n)_{n \in \mathbb{N}}$ in $|\mathbb{K}|$, of limit 0 and for each $n \in \mathbb{N}$, let f_n be the function defined in E by $f_n(x) = f(x)$ for every $x \in E \setminus d(0, r_n)$ and $f_n(x) = \xi(a_j)$ for every $x \in d(0, r_n)$, with $a_j \in d(0, r_n)$. Thus, we can check that f_n has a derivative equal to 0 in all E. Therefore, the sequence $(f_n')_{n \in \mathbb{N}}$ trivially is uniformly convergent to the function that is identically zero in all E.

On the other hand, consider $\|f_n - f\|_0$. By construction, we check $|f(x)| \leq \sqrt{r_n}$ for every $x \in d(0, r_n)$ and hence $|f(x) - f_n(x)| \leq \sqrt{r_n}$ for every $x \in d(0, r_n)$. But since $f_n(x) = f(x)$ for every $x \in E \setminus d(0, r_n)$, we derive $\|f_n - f\|_0 \leq \sqrt{r_n}$. Consequently, the sequence $(f_n)_{n \in \mathbb{N}}$ is uniformly convergent in E to f. And f is not derivable at 0, although the sequence $(f_n')_{n \in \mathbb{N}}$ is also uniformly convergent in E.

Theorems 1 and 2 are designed to recall classical properties.

Theorem 1 *Every function $f \in \mathscr{D}(E)$ is uniformly continuous, derivable in E and f' is bounded and continuous in E. Moreover, if E is compact, then a function from E to $\mathbb{K}$ belongs to $\mathscr{D}(E)$ if and only if for every $a \in E$, $\dfrac{f(x)-f(y)}{x-y}$ has a limit when x and y tend to a separately while being distinct.*

Remark 2 A function f from E to $\mathbb{K}$ which is derivable with a continuous derivative is not automatically strictly differentiable. Indeed, suppose E has an infinite residue class field. We can find a sequence $(a_n)_{n\in\mathbb{N}}$ such that $|a_n| = |a_n - a_m| = 1$ for all $n \neq m$. Now take a sequence $(r_n)_{n\in\mathbb{N}}$ of $\mathbb{R}^+$ with $\lim\limits_{n\to+\infty} r_n = 0$. Then the set $E \setminus \bigcup\limits_{n=0}^{\infty} d(a_n, r_n^-)$ is open. Now, take $b_n \in d(a_n, r_n)$ such that $|a_n - b_n| = r_n$ for every $n \in \mathbb{N}$.

We can define a function f from E to $\mathbb{K}$ such that $f(x) = 0$ for all $x \in \bigcup_{n=0}^{\infty} d(a_n, r_n^-)$ and $f(x) = 1$ for every $x \in E \setminus \bigcup_{n=0}^{\infty} d(a_n, r_n^-)$. Of course, f is derivable and f' is continuous in E. However, $\left|\dfrac{f(a_n)-f(b_n)}{a_n-b_n}\right| = \dfrac{1}{r_n}$ and therefore $\left|\dfrac{f(x)-f(y)}{x-y}\right|$ is not bounded in E.

Theorem 2 *$\|\,.\,\|$ is a norm of $\mathbb{K}$-vector space on $\mathscr{D}(E)$ and $\mathscr{D}(E)$ is complete for that norm.*

Theorem 3 *$\mathscr{D}(E)$ is a $\mathbb{K}$-algebra and $\|\,.\,\|$ is a norm of $\mathbb{K}$-algebra.*

Corollary 3a *$\mathscr{D}(E)$ is a Banach $\mathbb{K}$-algebra.*

Theorem 4 *Let $r \in]0, s]$. Suppose E is uniformly open in $\mathbb{K}$ of codiameter s and let $r \in]0, s]$. Then $\mathscr{A}_b(E, r)$ provided with the norm $\|\,.\,\|_E$ is a Banach $\mathbb{K}$-algebra included in $\mathscr{D}(E)$.*

The role of ultrafilters here is essential as in a few previous works [6].

Notations and Definitions Let $\mathscr{F}$ be a filter on E. Given a function f from E to $\mathbb{K}$ admitting a limit along $\mathscr{F}$, we will denote by $\lim\limits_{\mathscr{F}} f(x)$ that limit.

Let $Ul(E)$ be the set of ultrafilters on E. Two filters $\mathscr{F}, \mathscr{G}$ on E will be said to be *contiguous* if for every $H \in \mathscr{F},\ L \in \mathscr{G}$, we have $\delta(H, L) = 0$. We shall denote by $(\mathscr{T})$ the relation defined on $Ul(E)$ as $\mathscr{U}(\mathscr{T})\mathscr{V}$ if $\mathscr{U}$ and $\mathscr{V}$ are contiguous.

An ultrafilter $\mathscr{U}$ on the set E is said to be *principal* if it converges to a point $a \in E$.

Remark 3 Let $\mathscr{U}, \mathscr{V}$ be contiguous ultrafilters on E and assume $\mathscr{U}$ is convergent. Then $\mathscr{V}$ is convergent and has the same limit as $\mathscr{U}$.

Theorem 5 *Let $X \subset E$ be uniformly open in E and let u be defined as $u(x) = 1$ for every $x \in X$ and $u(x) = 0$ for every $x \notin X$. Then u belongs to $\mathscr{D}(E)$.*

Notation Let $f_1, \ldots, f_q \in \mathcal{D}(E)$ and let $\epsilon > 0$. We set $W(f_1, \ldots, f_q, \epsilon) = \{x \in E \mid |f(x)| \leq \epsilon\}$.

Theorem 6 *Let $f_1, \ldots, f_q \in \mathcal{D}(E)$, let $\epsilon > 0$. Then if $W(f_1, \ldots, f_q, \epsilon)$ is not empty, it is uniformly open in E.*

Theorems 7 and 8 were proven in [6].

Theorem 7 *Let $\mathcal{U}$, $\mathcal{V}$ be two ultrafilters on E that are not contiguous. There exist uniformly open subsets $H \in \mathcal{U}$, $L \in \mathcal{V}$ of E and $f \in \mathcal{D}(E)$ such that $f(x) = 1$ for every $x \in H$, $f(x) = 0$ for every $x \in L$.*

Notation We denote by E' another open subset of $\mathbb{K}$. Let f be a mapping from E to E' and let $\mathcal{U}$ be an ultrafilter on E. We denote by $\overline{f}(\mathcal{U})$ the ultrafilter admitting the basis $f(\mathcal{U})$.

Notation Given a uniformly continuous mapping f from E to E' and a class of contiguity H on E, we will denote by $\overline{f}(H)$ the class of contiguity on E': $\{\overline{f}(\mathcal{U}) \mid \mathcal{U} \in H\}$.

Given a filter $\mathcal{F}$ on E, we will denote by $\mathcal{I}(\mathcal{F})$ the ideal of the $f \in \mathcal{D}(E)$ such that $\lim_{\mathcal{F}} f(x) = 0$. We will denote by $\mathcal{I}^*(\mathcal{F})$ the ideal of the $f \in \mathcal{D}(E)$ such that there exists a subset $L \in \mathcal{F}$ such that $f(x) = 0$ for every $x \in L$. Given $a \in E$ we will denote by $\mathcal{I}(a)$ the ideal of the $f \in \mathcal{D}(E)$ such that $f(a) = 0$.

We will denote by $Max(\mathcal{D}(E))$ the set of maximal ideals of $\mathcal{D}(E)$ and by $Max_E(\mathcal{D}(E))$ the set of maximal ideals of the form $\mathcal{I}(a)$, $a \in E$.

The proof of Theorem 8 is easy and is not specific to the algebra $\mathcal{D}(E)$ [6].

Theorem 8 *Given an ultrafilter $\mathcal{U}$ on E, $\mathcal{I}(\mathcal{U})$, $\mathcal{I}^*(\mathcal{U})$ are prime ideals of $\mathcal{D}(E)$.*

Notation We will denote by $|\cdot|_\infty$ the Archimedean absolute value of $\mathbb{R}$.

Theorem 9 *Let $\mathcal{U}$, $\mathcal{V}$ be two ultrafilters on E. Then $\mathcal{I}(\mathcal{U}) = \mathcal{I}(\mathcal{V})$ if and only if $\mathcal{U}$ and $\mathcal{V}$ are contiguous.*

Corollary 9a *Given an ultrafilter $\mathcal{U}$ on E, $\mathcal{I}(\mathcal{U})$ is a maximal ideal of $\mathcal{D}(E)$.*

Remark 4 As noticed in [6], relation $(\mathcal{T})$ is not transitive in the case of the set of all filters on E. However, given a topological space X satisfying the normality axiom, particularly, given a metric space X, then $(\mathcal{T})$ is transitive for ultrafilters and therefore is an equivalence relation on $Ul(X)$ [6].

Notation We will denote by $Y_{(\mathcal{T})}(E)$ the set of equivalence classes on $Ul(E)$ with respect to relation $(\mathcal{T})$. Given $H \in Y_{(\mathcal{T})}(E)$, we will denote by $\mathcal{I}(H)$ the ideal $\mathcal{I}(\mathcal{U})$, $\mathcal{U} \in H$.

Let $f \in \mathcal{D}(E)$ and let ϵ be > 0. We set $B(f, \epsilon) = \{x \in E \mid |f(x)| \leq \epsilon\}$.

Theorem 10 looks like certain Bezout–Corona statements [5, 6]. The proof is close to that given in [6] but here the functions in $\mathcal{D}(E)$ have more properties, allowing for a more specific proof.

Theorem 10 *Let* $f_1, \dots, f_q \in \mathscr{D}(E)$ *satisfying* $\inf_{x \in E}(\max_{1 \le j \le q} |f_j(x)|) > 0$. *Then there exist*

$$g_1, \dots, g_q \in \mathscr{D}(E) \text{ such that } \sum_{j=1}^{q} f_j(x) g_j(x) = 1 \text{ for all } x \in E.$$

Corollary 10a *Let* I *be an ideal of* $\mathscr{D}(E)$ *different from* $\mathscr{D}(E)$. *The family* $B(f, \epsilon)$, $f \in I$, $\epsilon > 0$, *generates a filter* $\mathscr{F}_I$ *on* E *such that* $I \subset \mathscr{I}(\mathscr{F}_I)$.

2 Main Results

Theorem 11 *Let* $\mathscr{M}$ *be a maximal ideal of* $\mathscr{D}(E)$. *There exists an ultrafilter* $\mathscr{U}$ *on* E *such that* $\mathscr{M} = \mathscr{I}(\mathscr{U})$.

Corollary 11a *For every maximal ideal* $\mathscr{M}$ *of* $\mathscr{D}(E)$ *there exists a unique* $\mathscr{H} \in Y_{(\mathscr{T})}(E)$ *such that* $\mathscr{M} = \mathscr{I}(\mathscr{U})$ *for every* $\mathscr{U} \in \mathscr{H}$.

Moreover, the mapping Ψ *that associates with each* $\mathscr{M} \in Max(\mathscr{D}(E))$ *the unique* $\mathscr{H} \in Y_{(\mathscr{T})}(E)$ *such that* $\mathscr{M} = \mathscr{I}(\mathscr{U})$ *for every* $\mathscr{U} \in \mathscr{H}$, *is a bijection from* $Max(\mathscr{D}(E))$ *onto* $Y_{(\mathscr{T})}(E)$.

Theorem 12 *Let* $\mathscr{U}$ *be an ultrafilter on* E *such that, for every* $f \in \mathscr{D}(E)$, $f(x)$ *converges on* $\mathscr{U}$ *in* $\mathbb{K}$. *Then* $\mathscr{I}(\mathscr{U})$ *is of codimension* 1.

Corollary 12a *Let* $\mathscr{U}$ *be a Cauchy ultrafilter on* E. *Then* $\mathscr{I}(\mathscr{U})$ *is of codimension* 1.

Corollary 12b *Let* $\mathbb{K}$ *be a locally compact field. Then every maximal ideal of* $\mathscr{D}(E)$ *is of codimension 1.*

We will now examine prime closed ideals of $\mathscr{D}(E)$.

Theorem 13 *Let* $\mathscr{U}$ *be an ultrafilter on* E *and let* $\mathscr{P}$ *be a prime ideal included in* $\mathscr{I}(\mathscr{U})$. *Let* $L \in \mathscr{U}$ *be uniformly open in* E *and let* $H = E \setminus L$. *Let* u *be the function defined on* E *by* $u(x) = 1$ *for every* $x \in H$, $u(x) = 0$ *for every* $x \in L$. *Then* u *belongs to* $\mathscr{P}$.

Corollary 13a *Let* $\mathscr{U}$ *be an ultrafilter on* E. *The ideal of the* $f \in \mathscr{D}(E)$ *such that there exists a uniformly open subset* $H \in \mathscr{U}$ *of* E *such that* $f(x) = 0$ *for every* $x \in H$ *is included in every prime ideal* $\mathscr{P} \subset \mathscr{I}(\mathscr{U})$.

Theorem 14 *The closure of a prime ideal of* $\mathscr{D}(E)$ *with respect to the norm* $\| \,.\, \|_0$ *is a maximal ideal.*

Corollary 14a *Let* $\mathscr{P}$ *be a prime ideal of* $\mathscr{D}(E)$. *There exists a unique maximal ideal* $\mathscr{M}$ *of* $\mathscr{D}(E)$ *containing* $\mathscr{P}$.

Corollary 14b *Every prime closed ideal of* $\mathscr{D}(E)$ *with respect to the norm* $\| \,.\, \|_0$ *is a maximal ideal.*

Let us recall the main definitions concerning multiplicative semi-norms [2, 6, 7].

Notation and Definition We denote by $Mult(\mathscr{D}(E), \|\ .\ \|)$ the set of multiplicative semi-norms of $\mathscr{D}(E)$ provided with the topology of pointwise convergence. Given $\phi \in Mult(\mathscr{D}(E), \|\ .\ \|)$, the set of the $f \in \mathscr{D}(E)$ such that $\phi(f) = 0$ is a closed prime ideal called *the kernel of* ϕ. It is denoted by $Ker(\phi)$.

We denote by $Mult_m(\mathscr{D}(E), \|\ .\ \|)$ the set of multiplicative semi-norms of $\mathscr{D}(E)$ whose kernel is a maximal ideal and by $Mult_1(\mathscr{D}(E), \|\ .\ \|)$ the set of multiplicative semi-norms of $\mathscr{D}(E)$ whose kernel is a maximal ideal of codimension 1.

Let $a \in E$. The mapping φ_a from $\mathscr{D}(E)$ to $\mathbb{R}$ defined by $\varphi_a(f) = |f(a)|$ belongs to $Mult(\mathscr{D}(E), \|\ .\ \|)$. Let $\mathscr{U}$ be an ultrafilter on E. By Urysohn's theorem, given $f \in \mathscr{D}(E)$, the mapping from E to $\mathbb{R}$ that sends x to $|f(x)|$ admits a limit along $\mathscr{U}$. We set $\varphi_{\mathscr{U}}(f) = \lim_{\mathscr{U}} |f(x)|$. Moreover, we denote by $Mult_E(\mathscr{D}(E), \|\ .\ \|)$ the set of multiplicative semi-norms of $\mathscr{D}(E)$ of the form $\varphi_a,\ a \in E$. Consequently, by definition, $Mult_E(\mathscr{D}(E), \|\ .\ \|)$ is a subset of $Mult_1(\mathscr{D}(E), \|\ .\ \|)$.

The following Theorems 15 and 16 are immediate and well known:

Theorem 15 *Let $a \in E$. Then $\mathscr{I}(a)$ is a maximal ideal of $\mathscr{D}(E)$ of codimension* 1 *and φ_a belongs to $Mult_1(\mathscr{D}(E), \|\ .\ \|)$.*

Corollary 15a *$Mult_E(\mathscr{D}(E), \|\ .\ \|)$ is included in $Mult_1(\mathscr{D}(E), \|\ .\ \|)$.*

Theorem 16 *Let $\mathscr{U}$ be an ultrafilter on E. Then $\varphi_{\mathscr{U}}$ belongs to the closure of $Mult_E(\mathscr{D}(E), \|\ .\ \|)$.*

Corollary 16a *$Mult(\mathscr{D}(E), \|\ .\ \|) = Mult_m(\mathscr{D}(E), \|\ .\ \|)$. Furthermore, if $\mathbb{K}$ is locally compact, then $Mult(\mathscr{D}(E), \|\ .\ \|) = Mult_1(\mathscr{D}(E), \|\ .\ \|)$.*

Remark 5 Suppose $\mathbb{K}$ is locally compact and E is a disk in an algebraically closed complete ultrametric field. There do exist ultrafilters on E that do not converge. Let $\mathscr{U}$ be such an ultrafilter. Then $\varphi_{\mathscr{U}}$ belongs to $Mult_1(\mathscr{D}(E), \|\ .\ \|)$ but does not belong to $Mult_E(\mathscr{D}(E), \|\ .\ \|)$.

Remark 6 In $\mathscr{H} \in Y_{(\mathscr{T})}(E)$ the various ultrafilters $\mathscr{U} \in \mathscr{H} \in Y_{(\mathscr{T})}(E)$ define various prime ideals of $\mathscr{D}(E)$ and it is not clear whether these ideals are minimal among the set of prime ideals of $\mathscr{D}(E)$.

Theorem 17 *The topology induced on E by the one of $Mult_E(\mathscr{D}(E), \|\ .\ \|)$ is equivalent to the metric topology induced on E by the field $\mathbb{K}$.*

Theorem 18 was proven in [6] for the algebra of bounded continuous functions. Here we adapt that proof to the algebra $\mathscr{D}(E)$.

Theorem 18 *Let $\mathscr{M}$ be a maximal ideal of $\mathscr{D}(E)$. Let T be the field $\dfrac{\mathscr{D}(E)}{\mathscr{M}}$ and let θ be the canonical surjection from $\mathscr{D}(E)$ onto T. Given any ultrafilter $\mathscr{U}$ such that $\mathscr{I}(\mathscr{U}) = \mathscr{M}$, the quotient norm $\|\ .\ \|'$ on T is defined by $\|\theta(f)\|' = \lim_{\mathscr{U}} |f(s)|$ and hence is multiplicative.*

Definition Recall that given a commutative Banach $\mathbb{K}$-algebra T with unity, every maximal ideal of T is the kernel of at least one continuous multiplicative semi-norm [2, 3]. The algebra T is said to be *multbijective* if every maximal ideal is the kernel of only one continuous multiplicative semi-norm. (There exist ultrametric Banach $\mathbb{K}$-algebras that are not multbijective [1, 2].)

Theorem 19 *$\mathscr{D}(E)$ is multbijective.*

By Theorem 19 and Corollary 14b, we can now state Corollary 19a:

Corollary 19a *The mapping that associates with each $\phi \in Mult(\mathscr{D}(E), \|\ .\ \|)$ its kernel $Ker(\phi)$ is a bijection from $Mult(\mathscr{D}(E), \|\ .\ \|)$ onto $Max(\mathscr{D}(E))$.*

By Theorem 11, Corollary 11a, and Theorem 19, we have Corollary 19b:

Corollary 19b *For every $\phi \in Mult(\mathscr{D}(E), \|\ .\ \|)$ there exists a unique $\mathscr{H} \in Y_{(\mathscr{T})}(E)$ such that $\phi(f) = \lim_{\mathscr{U}} |f(x)|$ for every $f \in \mathscr{D}(E)$, for every $\mathscr{U} \in \mathscr{H}$.*

Moreover, the mapping $\widetilde{\Psi}$ that associates with each $\phi \in Mult(\mathscr{D}(E), \|\ .\ \|)$ the unique $\mathscr{H} \in Y_{(\mathscr{T})}(E)$ such that $\phi(f) = \lim_{\mathscr{U}} |f(x)|$ for every $f \in \mathscr{D}(E)$, for every $\mathscr{U} \in \mathscr{H}$, is a bijection from $Mult(\mathscr{D}(E), \|\ .\ \|)$ onto $Y_{(\mathscr{T})}(E)$.

Now, by Theorems 15 and 16, we have Corollary 19c:

Corollary 19c *$Mult_E(\mathscr{D}(E), \|\ .\ \|)$ is dense in $Mult(\mathscr{D}(E), \|\ .\ \|)$.*

Theorem 20 *For every $\phi \in Mult(\mathscr{D}(E), \|\ .\ \|)$, ϕ satisfies $\phi(f) \le \|f\|_0$ for every $f \in \mathscr{D}(E)$.*

Notation On $\mathscr{D}(E)$ we denote by $\|\ .\ \|_{si}$ the semi-norm of $\mathscr{D}(E)$ defined as $\|f\|_{si} = \lim_{n\to+\infty} \sqrt[n]{\|f^n\|}$. Then we can state Theorem 21:

Theorem 21 *$\|f\|_{si} = \|f\|_0$ for every $f \in \mathscr{D}(E)$.*

Theorem 22 *Suppose $\mathbb{K}$ is algebraically closed. Let $\mathscr{U}$ be an ultrafilter on $\mathbb{K}$ and suppose there exists $P \in \mathbb{K}[x]$, $P \neq 0$ satisfying $\lim_{\mathscr{U}} P(x) = 0$. Then $\mathscr{U}$ is a principal ultrafilter.*

As a consequence, we have Theorem 23:

Theorem 23 *Suppose that $\mathbb{K}$ is algebraically closed and that E is a closed bounded subset of $\mathbb{K}$ with infinitely many points and let $\mathscr{M}$ be a maximal ideal of $\mathscr{D}(E)$ of the form $\mathscr{I}(\mathscr{U})$ where $\mathscr{U}$ is not principal. Then $\mathscr{M}$ is of infinite codimension.*

Theorem 24 *Let Ł be an algebraic extension of $\mathbb{K}$ of degree t of the form $\mathbb{K}[a]$ provided with the absolute value which expands that of $\mathbb{K}$. Let f be a strictly differentiable function from E to Ł. There exists $f_0, \dots, f_{t-1} \in \mathscr{D}(E)$ such that $f = \sum_{j=0}^{t-1} a^j f_j$.*

We can now prove Theorem 25 that will let us show Theorem 26:

Theorem 25 *Let Ł be a finite algebraic extension of* $\mathbb{K}$ *provided with the absolute value which expands that of* $\mathbb{K}$*. Suppose there exists a morphism of* $\mathbb{K}$*-algebra,* χ*, from* $\mathscr{D}(E)$ *onto Ł. Let* $\widehat{\mathscr{D}}(E)$ *be the Ł-algebra of strictly differentiable functions from* E *to Ł. Then* χ *has continuation to a morphism of Ł-algebra* $\widehat{\chi}$ *from* $\widehat{\mathscr{D}}(E)$ *to Ł.*

Theorem 26 *Every maximal ideal of finite codimension of* $\mathscr{D}(E)$ *is of codimension* 1.

By Theorem 26 and Corollary 12a, we can state this corollary:

Corollary 26a *Let* $\mathscr{U}$ *be an ultrafilter on* E*. The following* 3 *statements are equivalent:*

(i) $\mathscr{I}(\mathscr{U})$ *is of codimension* 1*,*
(ii) $\mathscr{I}(\mathscr{U})$ *is of finite codimension,*
(iii) for every $f \in \mathscr{D}(E)$*, the filter generated by* $f(\mathscr{U})$ *converges in* $\mathbb{K}$*.*

Theorem 27 *Suppose that* E *is separable and that* $\mathbb{K}$ *is not locally compact. Let* $\mathscr{U}$ *be a non-convergent ultrafilter on* E*. Then* $\mathscr{I}(\mathscr{U})$ *is of infinite codimension.*

Theorem 28 *The algebra* $\mathscr{D}(E)$ *admits maximal ideals of infinite codimension if and only if* $\mathbb{K}$ *is not locally compact.*

Given a norm of $\mathbb{K}$-algebra, we call *Shilov boundary of* T a closed subset S of $Mult(T, \|\,.\,\|)$ that is minimum with respect to inclusion, such that, for every $x \in T$, there exists $\phi \in S$ such that $\phi(x) = \|x\|_{si}$. By Escassut and Maïnetti [4] it is known that every normed $\mathbb{K}$-algebra admits a Shilov boundary.

Theorem 29 *The Shilov boundary* S *of* $\mathscr{D}(E)$ *is equal to* $Mult(\mathscr{D}(E), \|\,.\,\|)$*.*

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p-Adic Nevanlinna Theory

Alain Escassut and Ta Thi Hoai An

Abstract After recalling the classical p-adic Nevanlinna theory, we describe the same theory in the complement of an open disk and examine various immediate applications: uniqueness, Picard's values, branched values, small functions.

1 Meromorphic Functions

Let $\mathbb{K}$ be a complete ultrametric algebraically closed field of characteristic 0 whose ultrametric absolute value is denoted by $| \,.\, |$. The Nevanlinna theory was examined over $\mathbb{K}$ by Ha [9] and was finally constructed by Boutabaa [3]. Next, in [4] a similar theory was developed for unbounded meromorphic functions in an "open" disk of $\mathbb{K}$, taking into account Lazard's problem [6]. In [10], Hanyak and Kondratyuk constructed a Nevanlinna theory for meromorphic functions in a *punctured complex plane*, i.e., in the set $\mathbb{C} \setminus \{a_1, \ldots, a_m\}$, where we understand that the meromorphic functions can admit essential singularities at $a_1, \ldots, a_m$ [10].

Here we describe a Nevanlinna theory for meromorphic functions in the complement of an open disk [7] by using the Motzkin factorization [13]. Once the Nevanlinna theory is established for such functions, we can obtain applications.

Notations Given $r > 0$, $a \in \mathbb{K}$ we denote by $d(a, r)$ the disk $\{x \in \mathbb{K} \mid |x-a| \leq r\}$, by $d(a, r^-)$ the disk $\{x \in \mathbb{K} \mid |x - a| < r\}$, and by $C(a, r)$ the circle $\{x \in \mathbb{K} \mid |x - a| = r\}$. Given $r'' > r'$, we put $\Delta(0, r', r'') = d(0, r'') \setminus d(0, r'^-)$.

Henceforth, we fix $R > 0$. We denote by S the disk $d(0, R^-)$ and put $D = \mathbb{K} \setminus S$.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_25

Given a bounded function f in D, we put $\|f\| = \sup_D |f(x)|$. Given a subset E of $\mathbb{K}$ having infinitely many points, we denote by $R(E)$ the $\mathbb{K}$-algebra of rational functions $h \in \mathbb{K}(x)$ having no pole in E. We then denote by $H(E)$ the $\mathbb{K}$-vector space of analytic elements on E [12], i.e., the completion of $R(E)$ with respect to the topology of uniform convergence on E. We know that given a circle $C(a, r)$ and an element f of $H(C(a, r))$, i.e., a Laurent series $f(x) = \sum_{-\infty}^{+\infty} c_n (x-a)^n$ converging whenever $|x| = r$, then $|f(x)|$ is equal to $\sup_{n\in\mathbb{Z}} |c_n| r^n$ in all classes of the circle $C(a, r)$ except maybe in finitely many. When $a = 0$, we put $|f|(r) = \sup_{n\in\mathbb{Z}} |c_n| r^n$. Then $|f|(r)$ is a multiplicative norm on $H(C(0, r))$ (Chapters 13 and 19, Proposition 19.1 [6]).

We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$, by $\mathcal{A}(d(a, R^-))$ the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converging in all $d(a, R^-)$, and by $\mathcal{A}(D)$ the $\mathbb{K}$-algebra of Laurent series $\sum_{-\infty}^{\infty} c_n (x-a)^n$ converging in D. Similarly, we will denote by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e., the field of fractions of $\mathcal{A}(\mathbb{K})$, by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions in $d(a, R^-)$, i.e., the field of fractions of $\mathcal{A}(d(a, R^-))$, and by $\mathcal{M}(D)$ the field of meromorphic functions in D, i.e., the field of fractions of $\mathcal{A}(D)$.

Next, we will denote by $\mathcal{A}_b(d(a, R^-))$ the set of $f \in \mathcal{A}(d(a, R^-))$ that are bounded in $d(a, R^-)$ and we put $\mathcal{A}_u(d(a, R^-)) = \mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. We will denote by $\mathcal{A}^w(D)$ the set of $f \in \mathcal{A}(D)$ admitting finitely many zeros in D and we put $\mathcal{A}^*(D) = \mathcal{A}(D) \setminus \mathcal{A}^w(D)$ and similarly, we denote by $\mathcal{M}^w(D)$ the field of fraction of $\mathcal{A}^w(D)$ and we put $\mathcal{M}^*(D) = \mathcal{M}(D) \setminus \mathcal{M}^w(D)$. So, $\mathcal{M}^*(D)$ is the set of meromorphic functions in D having at least infinitely many zeros or infinitely many poles in D.

Let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(D)$). Given $r < R$ (resp. $r > R$), we know that $|f(x)|$ admits a limit denoted by $|f|(r)$ when $|x|$ tends to r while remaining different from r.

Let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(D)$) and let $\alpha \in d(a, R^-)$, (resp. $\alpha \in D$). If f admits a zero of order q at α, we set $\omega_\alpha(f) = q$ and if $f(\alpha) \neq 0$, we set $\omega_\alpha(f) = 0$.

Let $f = \dfrac{h}{l} \in \mathcal{M}(d(a, R^-))$, (resp. $f \in \mathcal{M}(D)$). For each $\alpha \in \mathbb{K}$ (resp. $\alpha \in d(a, R^-)$, resp. $\alpha \in D$) the number $\omega_\alpha(h) - \omega_\alpha(l)$ does not depend on the functions $h,\ l$ chosen to make $f = \dfrac{h}{l}$. Thus, we can generalize the notation by setting $\omega_\alpha(f) = \omega_\alpha(h) - \omega_\alpha(l)$.

If $\omega_\alpha(f)$ is an integer $q > 0$, α is called *a zero of f of order q*.
If $\omega_\alpha(f)$ is an integer $q < 0$, α is called *a pole of f of order -q*.
If $\omega_\alpha(f) \geq 0$, f will be said to be *holomorphic* at α.

Definitions Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(a, R^-))$, resp. $f \in \mathcal{M}(D)$) and let $b \in \mathbb{K}$. Then b will be said to be *an exceptional value for f* if $f - b$ has no zero in $\mathbb{K}$ (resp. in $d(a, R^-)$, resp. in D)). Moreover, if $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. if $f \in \mathcal{M}_u(d(a, R^-))$, resp. $f \in \mathcal{M}(D)$)), b will be said to be *a quasi-exceptional value for f* if $f - b$ has finitely many zeros in $\mathbb{K}$ (resp. in $d(a, R^-)$, resp in D).

Theorem 1.1 *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$, (resp. $f \in \mathcal{M}_u(d(a, R^-))$, resp. $f \in \mathcal{M}^*(D)$). Then f admits at most one quasi-exceptional value. Moreover, if f has finitely many poles in $\mathbb{K}$ (resp. in $d(a, R^-)$, resp. in D), then f has no quasi-exceptional value.*

Definitions and Notations Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$ have a pole α of order q and let $f(x) = \sum_{k=-q}^{-1} a_k(x - \alpha)^k + h(x)$ with $a_{-q} \neq 0$ and $h \in \mathcal{M}(\mathbb{K})$ (resp. $h \in \mathcal{M}(d(0, R^-))$ and h holomorphic at α. According to usual notations the coefficient a_{-1} is called *residue of f at α* and denoted by $\text{res}(f, \alpha)$.

Theorem 1.2 *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-)$, resp. $f \in \mathcal{M}(D)$). Then f admits primitives if and only if all residues of f are null.*

Conjecture Let $f \in \mathcal{M}(\mathbb{K})$. Then f' has no quasi-exceptional value.

Theorem 1.3 ([2]) *Let $f \in \mathcal{M}(\mathbb{K})$ and for each $r > 0$, let $\gamma(f, r)$ be the number of multiple poles of f in $d(0, r)$. If there exists $c > 0$ and $s \in \mathbb{N}$ such that $\gamma(r, f) \leq cr^s$ $\forall r > 1$, then f' admits no quasi-exceptional value.*

2 Nevanlinna Theory in the Classical p-Adic Context

The Nevanlinna theory was developed by Rolf Nevanlinna on complex functions in the 1920s. It consists of defining counting functions of zeros and poles of a meromorphic function f and giving an upper bound for multiple zeros and poles of various functions $f - b$, $b \in \mathbb{C}$.

A similar theory for functions in a p-adic field was constructed by Boutabaa [3].

Throughout the next paragraphs, we denote by I the interval $[t, +\infty[$, by J an interval of the form $[t, R[$ with $t > 0$, and by L the interval $[R, +\infty[$. We denote by f a function that belongs either to $\mathcal{M}(\mathbb{K})$ or to $\mathcal{M}(S)$.

Definitions We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way.

Let (a_n), $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n.

We set $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n(\log r - \log |a_n|)$ and so, $Z(r, f)$ is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity*.

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega_0}(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega_0}(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f without multiplicity:

$\overline{Z}(r, f) = \overline{\omega_0}(f)\log r + \sum_{n=1}^{\sigma(r)}(\log r - \log|a_n|)$ and so, $\overline{Z}(r, f)$ is called *the counting function of zeros of f in* $d(0, r)$ *ignoring multiplicity*.

In the same way, considering the finite sequence $(b_n),\ 1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n, we put

$N(r, f) = \max(-\omega_0(f), 0)\log r + \sum_{n=1}^{\tau(r)} t_n(\log r - \log|b_n|)$ and then $N(r, f)$ is called *the counting function of the poles of f, counting multiplicity*.

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\overline{\omega_0}}(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\overline{\omega_0}}(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$\overline{N}(r, f) = \overline{\overline{\omega_0}}(f)\log r + \sum_{n=1}^{\tau(r)}(\log r - \log|b_n|)$ and then $\overline{N}(r, f)$ is called *the counting function of the poles of f, ignoring multiplicity*.

Now, we can define the Nevanlinna function $T(r, f)$ in I or J as $T(r, f) = \max(Z(r, f), N(r, f))$ and the function $T(r, f)$ is called *characteristic function of f or Nevanlinna function of f*.

Finally, if Y is a subset of $\mathbb{K}$ we will denote by $Z^Y(r, f')$ the counting function of zeros of f', excluding those which are zeros of $f - a$ for any $a \in Y$.

Remark If we change the origin, the functions $Z,\ N,\ T$ are not changed, up to an additive constant.

Theorem 2.1 *Let* $f \in \mathcal{M}(\mathbb{K})$ *(resp.* $f \in \mathcal{M}(d(0, R^-))$*) have no zero and no pole at* 0. *Then*

$$\log(|f|(r)) = \log(|f(0)|) + Z(r, f) - N(r, f).$$

Theorem 2.2 (First Main Theorem) *Let* $f,\ g \in \mathcal{M}(\mathbb{K})$ *(resp.* $f,\ g \in \mathcal{M}(S)$*). Then*

$$Z(r, fg) \leq Z(r, f) + Z(r, g),\ N(r, fg) \leq N(r, f) + N(r, g),$$

$T(r, f + b) = T(r, f) + O(1)$, $T(r, fg) \leq T(r, f) + T(r, g)$, $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$, $T(r, cf) = T(r, f)\ \forall c \in \mathbb{K}^*$, $T(r, \frac{1}{f}) = T(r, f)$,

$$T(r, \frac{f}{g}) \leq T(r, f)) + T(r, g).$$

Let $P(X) \in \mathbb{K}[X]$. *Then* $T(r, P(f)) = \deg(P)T(r, f) + O(1)$ *and* $T(r, f'P(f)) \geq T(r, P(f))$.

Suppose now $f, g \in \mathscr{A}(\mathbb{K})$ *(resp.* $f,\ g \in \mathscr{A}(S)$*). Then* $Z(r, fg) = Z(r, f) + Z(r, g),\ T(r, f) = Z(r, f)\ T(r, fg) = T(r, f) + T(r, g) + O(1)$ *and* $T(r, f + g) \leq \max(T(r, f), T(r, g))$.

Moreover, if $\lim_{r\to+\infty}(T(r, f) - T(r, g)) = +\infty$ *then* $T(r, f+g) = T(r, f)$ *when* r *is big enough.*

Theorem 2.3 *Let* $f \in \mathscr{M}(\mathbb{K})$. *Then* f *belongs to* $\mathbb{K}(x)$ *if and only if* $T(r, f) = O(\log r)$.

Theorem 2.4 *Let* $f \in \mathscr{M}(S)$. *Then* f *belongs to* $\mathscr{M}_b(S)$ *if and only if* $T(r, f)$ *is bounded in* $[0, R[$.

Theorem 2.5 *Let* $f \in \mathscr{M}(\mathbb{K})$ *(resp.* $f \in \mathscr{M}(S)$*). Then for all* $k \in \mathbb{N}^*$, *we have* $N(r, f^{(k)}) = N(r, f) + k\overline{N}(r, f)$ *and* $Z(r, f^{(k)}) \leq Z(r, f) + k\overline{N}(r, f) + O(1)$.

Theorem 2.6 *Let* $f \in \mathscr{M}(\mathbb{K})$ *(resp.* $f \in \mathscr{M}(S)$*) and let* $a_1, \ldots, a_q \in \mathbb{K}$ *be distinct. Then*

$$(q-1)T(r, f) \leq \max_{1\leq k\leq q}\Big(\sum_{j=1, j\neq k}^{q} Z(r, f - a_j)\Big) + O(1).$$

Theorem 2.7 (Second Main Theorem) *Let* $\alpha_1, \ldots, \alpha_q \in \mathbb{K}$, *with* $q \geq 2$, *let* $Y = \{\alpha_1, \ldots, \alpha_q\}$ *and let* $f \in \mathscr{M}(\mathbb{K})$ *(resp.* $f \in \mathscr{M}_u(S)$*). Then*

$$(q-1)T(r, f) \leq \sum_{j=1}^{q} \overline{Z}(r, f - \alpha_j) + \overline{N}(r, f) - Z_0^Y(r, f') - \log r + O(1) \quad \forall r \in I$$

(resp. $\forall r \in J$*).*

Moreover, if $f \in \mathscr{A}(\mathbb{K})$ *(resp.* $f \in \mathscr{A}(S)$*), then*

$$(q-1)T(r, f) \leq \sum_{j=1}^{q} \overline{Z}(r, f - \alpha_j) - Z_0^S(r, f') + O(1) \quad \forall r \in I \text{ (resp. } \forall r \in J).$$

3 Nevanlinna Theory Out of a Hole

A Nevanlinna theory was made by Hanyak and Kondratyuk in 2007 for meromorphic functions in the complex plane except at finitely many points where they can have an essential singularity.

In this part, we will give some relations between the characteristic function and Motzkin factors.

Definition Let $f \in H(D)$ have no zero in D, $f(x) = \sum_{-\infty}^{q} a_n x^n$ with $|a_q|R^q > |a_n|R^n$ for all $n < q$ and $a_q = 1$. Then f will be called *a Motzkin factor associated with* S and the integer q will be called *the Motzkin index of* f and will be denoted by $m(f, S)$ (see [6, 8, 13]).

Theorem 3.1 *Let $f \in \mathcal{M}(D)$. We can write f in a unique way in the form $f^S f^0$ with $f^S \in H(D)$ a Motzkin factor associated with S and $f^0 \in \mathcal{M}(\mathbb{K})$, having no zero and no pole in S.*

Given $f \in \mathcal{M}(D)$, for $r > R$. If $\alpha_1, \ldots, \alpha_m$ are the distinct zeros of f in $\Delta(0, R, r)$, with respective multiplicity u_j, $1 \le j \le m$, then *the counting function of zeros* $Z_R(r, f)$ of f between R and r will be denoted by

$$Z_R(r, f) = \sum_{j=1}^{m} u_j(\log(r) - \log(|\alpha_j|)).$$

Similarly, if $\beta_1, \ldots, \beta_n$ are the distinct poles of f in $\Delta(0, R, r)$, with respective multiplicity v_j, $1 \le j \le m$, then *the counting function of poles* $N_R(r, f)$ of f between R and r is denoted by

$$N_R(r, f) = \sum_{j=1}^{n} v_j(\log(r) - \log(|\beta_j|)).$$

We put

$$T_R(r, f) = \max\big(Z_R(r, f), N_R(r, f)\big).$$

The counting function of zeros without counting multiplicity $\overline{Z}_R(r, f)$ is defined as:

$$\overline{Z}_R(r, f) = \sum_{j=1}^{m}(\log(r) - \log(|\alpha_j|)),$$

where $\alpha_1, \ldots, \alpha_m$ are the distinct zeros of f in $\Delta(0, R, r)$. Similarly, *the counting function of poles without counting multiplicity* $\overline{N}_R(r, f)$ is defined as:

$$\overline{N}_R(r, f) = \sum_{j=1}^{n}(\log(r) - \log(|\beta_j|)),$$

where $\beta_1, \ldots, \beta_n$ are the distinct poles of f in $\Delta(0, R, r)$.

Finally, putting $Y = \{a_1, \ldots, a_q\}$, we denote by $Z_R^Y(r, f')$ the counting function of zeros of f' on points x where $f(x) \notin Y$.

In the following, we denote by L the interval $[R, +\infty[$.

Theorem 3.2 *Let $f \in \mathscr{M}(D)$. Then, for all $r \in L$,*

$$\log(|f|(r)) - \log(|f|(R)) = Z_R(r, f) - N_R(r, f) + m(f, S)(\log r - \log R).$$

Theorem 3.3 *Let $f \in \mathscr{A}(D)$. Then, for $r \in L$,*

$$Z_R(r, f') \leq Z_R(r, f) - \log(r) + O(1).$$

Theorem 3.4 *Let $f \in \mathscr{M}(D)$. The three following statements are equivalent:*

(i) $\lim_{r \to +\infty} \dfrac{T_R(r, f)}{\log(r)} = +\infty$ *for $r \in L$,*
(ii) $\dfrac{T_R(r, f)}{\log(r)}$ *is unbounded,*
(iii) *f belongs to $\mathscr{M}^*(D)$.*

Properties in $\mathscr{M}(D)$ are similar to those on $\mathscr{M}(\mathbb{K})$ and we get again a second main theorem:

Theorem 3.5 (Second Main Theorem) *Let $f \in \mathscr{M}(D)$, let $\alpha_1, \ldots, \alpha_q \in \mathbb{K}$, with $q \geq 2$ and let $W = \{\alpha_1, \ldots, \alpha_q\}$. Then, for $r \in L$,*

$$(q-1)T_R(r, f) \leq \sum_{j=1}^{q} \overline{Z}_R(r, f - \alpha_j) + \overline{N}_R(r, f) - Z_R^W(r, f') + O(\log(r)).$$

4 Applications

Theorem 4.1 *Let $a_1, a_2 \in \mathbb{K}$ (with $a_1 \neq a_2$) and let $f, g \in \mathscr{A}^*(\mathbb{K})$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2$). Then $f = g$.*

Theorem 4.2 *Let $a_1, a_2, a_3 \in \mathbb{K}$ (with $a_i \neq a_j$ $\forall i \neq j$) and let $f, g \in \mathscr{A}_u(d(0, R^-))$ (resp. $f, g \in \mathscr{A}^*(D)$) satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3$). Then $f = g$.*

Theorem 4.3 *Let $a_1, a_2, a_3, a_4 \in \mathbb{K}$ (with $a_i \neq a_j$ $\forall i \neq j$) and let $f, g \in \mathscr{M}^*(\mathbb{K})$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3, 4$). Then $f = g$.*

Theorem 4.4 *Let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{K}$ (with $a_i \neq a_j$ $\forall i \neq j$) and let $f, g \in \mathscr{M}_u(d(0, R^-))$ (resp. $f, g \in \mathscr{M}^*(D)$) satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3, 4, 5$). Then $f = g$.*

Theorem 4.5 *Let Λ be a non-degenerate elliptic curve of equation*
$y^2 = (x - a_1)(x - a_2)(x - a_3)$.

There do not exist $g, f \in \mathscr{M}(\mathbb{K})$ such that $g(t) = y$, $f(t) = x$, $t \in \mathbb{K}$.

There do not exist $g,\ f \in \mathscr{A}_u(d(0, R^-))$ *such that* $g(t) = y,\ f(t) = x,\ t \in d(0, R^-)$.

There do not exist $g,\ f \in \mathscr{A}^*(D)$ *such that* $g(t) = y,\ f(t) = x,\ t \in D$.

Theorem 4.6 *Let* Λ *be a curve of equation* $y^q = P(x)$, $q \geq 2$, *with* $P \in \mathbb{K}[x]$ *admitting* n *distinct zeros of order* 1 *with* $n \geq 4$. *There do not exist* $g,\ f \in \mathscr{M}(\mathbb{K})$ *such that* $g(t) = y,\ f(t) = x,\ t \in \mathbb{K}$. *Moreover, if* $n \geq 5$, *there do not exist* $g,\ f \in \mathscr{M}_u(d(0, R^-))$ *(resp.* $g,\ f \in \mathscr{M}^*(D)$*) such that* $g(t) = y,\ f(t) = x,\ t \in d(0, R^-)$ *(resp.* $t \in D$*).*

Theorem 4.7 *Let* $f,\ g \in \mathscr{M}(K)$ *satisfy* $g^m + f^n = 1$, *with* $\min(m, n) \geq 2$ *and* $\max(m, n) \geq 3$. *Then* f *and* g *are constant.*

Theorem 4.8 *Let* $f,\ g \in \mathscr{M}(d(0, R^-))$ *(resp.* $f,\ g \in \mathscr{M}(D)$*) satisfy* $g^m + f^n = 1$, *with* $\min(m, n) \geq 3$ *and* $\max(m, n) \geq 4$). *Then* f *and* g *belong to* $\mathscr{M}_b(d(0, R^-))$ *(resp. to* $\mathscr{M}^w(D)$*). Moreover, if* $f,\ g \in \mathscr{A}(d(0, R^-))$ *(resp. if* $f,\ g \in \mathscr{A}(D)$*) satisfy* $g^m + f^n = 1$, *with* $\min(m, n) \geq 2$ *and* $(m, n) \neq (2, 2)$, *then* f *and* g *belong to* $\mathscr{A}_b(d(0, R^-)$, *(resp to* $\mathscr{A}^w(D)$*).*

Concerning the famous Hayman conjecture completely proven for complex meromorphic functions [1, 11], we have a similar conjecture on $\mathbb{K}$:

Theorem 4.9 ([8, 14]) *Let* $f \in \mathscr{M}^*(\mathbb{K})$, *(resp.* $f \in \mathscr{M}_u(S)$, *resp.* $f \in \mathscr{M}^*(D)$. *Then for every* $n \geq 3$, $f^n f'$ *takes every value* $b \in \mathbb{K}$ *infinitely many times. Moreover, if* $f \in \mathscr{M}^*(\mathbb{K})$, *then* $f^2 f'$ *takes every value* $b \in \mathbb{K}$ *infinitely many times.*

5 Small Functions

Definitions For each $f \in \mathscr{M}(\mathbb{K})$, (resp. $f \in \mathscr{M}(S)$, resp. $f \in \mathscr{M}(D)$), we will denote by $\mathscr{M}_f(\mathbb{K})$ (resp. $\mathscr{M}_f(S)$, $\mathscr{M}_f(D)$) the set of functions $h \in \mathscr{M}(\mathbb{K})$ (resp. $h \in \mathscr{M}(S)$, $h \in \mathscr{M}(D)$) such that $T_R(r, h) = o(T_R(r, f))$, $r \in I$, (resp. $r \in J$, resp. $r \in L$). Similarly, if $f \in \mathscr{A}(\mathbb{K})$ (resp. $f \in \mathscr{A}(S)$, resp. $f \in \mathscr{A}(D)$) we will denote by $\mathscr{A}_f(\mathbb{K})$ (resp. $\mathscr{A}_f(S)$, resp. $\mathscr{A}_f(D)$) the set $\mathscr{M}_f(\mathbb{K}) \cap \mathscr{A}(\mathbb{K})$ (resp. $\mathscr{M}_f(S) \cap \mathscr{A}(S)$, resp.$\mathscr{M}_f(D) \cap \mathscr{A}(D)$).

The elements of $\mathscr{M}_f(\mathbb{K})$ (resp. $\mathscr{M}_f(S)$, resp. $\mathscr{M}_f(D)$) are called *small meromorphic functions with respect to f, small functions* in brief. Similarly, if $f \in \mathscr{A}(\mathbb{K})$ (resp. $f \in \mathscr{A}(S)$, resp. $f \in \mathscr{A}(D)$), these functions are called *small analytic functions with respect to f, small functions in brief.*

A value $b \in \mathbb{K}$ will be called *a quasi-exceptional value for a function* $f \in \mathscr{M}(\mathbb{K})$ (resp. $f \in \mathscr{M}(S)$, resp. $f \in \mathscr{M}(D)$) if $f - b$ has finitely many zeros. In the same way, a small function w with respect to a function $f \in \mathscr{M}(\mathbb{K})$ (resp. $f \in \mathscr{M}(S)$, $f \in \mathscr{M}(D)$) will be called *a quasi-exceptional small function for f* if $f - w$ has finitely many zeros in D.

Theorem 5.1 *Let $f \in \mathscr{M}^*(\mathbb{K})$ (resp. $f \in \mathscr{M}_u(S)$, resp. $f \in \mathscr{M}^*(D)$). Then f admits at most one quasi-exceptional small function. Moreover, if f has finitely many poles, then f admits no quasi-exceptional small function.*

Corollary 5.1a *Let $f \in \mathscr{A}^*(\mathbb{K})$ (resp. $f \in \mathscr{A}_u(S)$, resp. $f \in \mathscr{A}^*(D)$). Then f has no quasi-exceptional small function.*

Theorem 5.2 *Let $f \in \mathscr{M}^*(\mathbb{K})$, (resp. $f \in \mathscr{M}_u(S)$, resp. $f \in \mathscr{M}^*(D)$ and let $w_1,\ w_2, w_3 \in \mathscr{M}_f(\mathbb{K})$ (resp. $\in \mathscr{M}_f(d(0, R^-))$, resp. $\in \mathscr{M}_f(D)$) be pairwise distinct. Then:*

$T(r, f) \le \sum_{j=1}^{3} \overline{Z}(r, f - w_j) + o(T(r, f))$

(resp. $T(r, f) \le \sum_{j=1}^{3} \overline{Z}(r, f - w_j) + o(T(r, f))$,

resp. $T_R(r, f) \le \sum_{j=1}^{3} \overline{Z}_R(r, f - w_j) + o(T_R(r, f))$.

Corollary 5.2a *Let $f \in \mathscr{M}^*(\mathbb{K})$ (resp. $f \in \mathscr{M}_u(S)$ resp. $f \in \mathscr{M}^*(D)$) and let $w_1,\ w_2 \in \mathscr{A}_f(\mathbb{K})$ (resp. $w_1,\ w_2 \in \mathscr{A}_f(S)$ resp. $w_1,\ w_2 \in \mathscr{A}_f(D)$) be distinct. Then*

$T(r, f) \le \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r, f) + o(T(r, f))$

(resp. $T(r, f) \le \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r, f) + o(T_R(r, f))$,

resp. $T_R(r, f) \le \overline{Z}_R(r, f - w_1) + \overline{Z}_R(r, f - w_2) + \overline{N}_R(r, f) + o(T_R(r, f))$).

Definitions Let $f \in \mathscr{M}^*(\mathbb{K})$ (resp. $f \in \mathscr{M}_u(d(0, R^-))$, resp. $f \in \mathscr{M}^*(D)$) and let $w \in \mathscr{M}_f(\mathbb{K})$ (resp. $w \in \mathscr{M}_f(d(0, R^-))$, resp. $w \in \mathscr{M}_f(D)$). Then w is called *a perfectly branched function with respect to f* if all zeros of $f - w$ are multiple except maybe finitely many. Particularly, the definition applies to constants [5].

Theorem 5.3 *Let $f \in \mathscr{M}^*(\mathbb{K})$ (resp. $f \in \mathscr{M}_u(d(0, R^-))$, resp. $f \in \mathscr{M}^*(D)$). Then f admits at most four perfectly branched values. Moreover, if f has finitely many poles and if $f \in \mathscr{M}^*(\mathbb{K})$ (resp. $f \in \mathscr{M}^*(D)$), then f admits at most one perfectly branched rational function.*

Theorem 5.4 *Let $f \in \mathscr{M}_u(d(0, R^-))$, having finitely many poles. Then f admits at most two perfectly branched rational functions.*

Corollary 5.3a *Let $f \in \mathscr{A}_u(d(0, R^-))$. Then f admits at most two perfectly branched rational functions.*

Acknowledgements The second named author is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2017.320 and Vietnam Institute Advanced Study in Mathematics.

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On an Operator Theory on a Banach Space of Countable Type over a Hahn Field

Khodr Shamseddine and Changying Ding

Abstract This paper is a survey of the results in Aguayo et al. (J Math Phys 54(2), 2013; Indag Math (N.S.) 26(1):191–205, 2015; p-Adic Num Ultrametr Anal Appl 9(2):122–137, 2017) but generalized to the case when the complex Levi-Civita field $\mathcal{C}$ is replaced by a Hahn field $\mathbb{K}((G))$ for particular choices of the field $\mathbb{K}$ and the abelian group G. In particular, we consider the Banach space of countable type c_0 consisting of all null sequences of $\mathbb{K}((G))$, equipped with the supremum norm $\|\cdot\|_\infty$ and we define a natural inner product on c_0 which induces the norm of c_0. Then we present characterizations of normal projections and of compact and self-adjoint operators on c_0. As a new result in this paper, we apply the Hahn–Banach theorem to show the existence of the dual operator of a given continuous linear operator on c_0 and to show that the dual operator and the adjoint operator coincide.

We present some B*-algebras of operators, including those mentioned above, then we define an inner product on such algebras which induces the usual norm of operators. Finally, we present a study of positive operators on c_0 and use that to introduce a partial order on the set of compact and self-adjoint operators on c_0.

1 Introduction

Throughout this paper, $\mathbb{K}((G))$ will denote the Hahn field defined as follows.

Definition 1 Let G be a subgroup of $(\mathbb{R}, +)$, let $\mathbb{F}$ be a formally real field, and let $\mathbb{K} = \mathbb{F}(i)$, where $i^2 = -1$. The Hahn field $\mathbb{K}((G))$ is

$$\mathbb{K}((G)) := \{z : G \to \mathbb{K} | \text{supp}(z) \text{ is well-ordered}\}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_26

with the addition and multiplication defined as follows: for every $z, w \in \mathbb{K}((G))$ and $x \in G$,

(1) $(z+w)(x) = z(x) + w(x)$, and
(2) $(zw)(x) = \sum_{a+b=x} z(a)w(b)$.

Given $z \in \mathbb{K}((G))$, we define

$$\lambda(z) = \begin{cases} \min\{\text{supp}(z)\} & \text{if } z \neq 0 \\ \infty & \text{if } z = 0. \end{cases}$$

Moreover, we can define an ultrametric absolute value on the field $\mathbb{K}((G))$, $|\cdot| : \mathbb{K}((G)) \to \mathbb{R}$ as follows:

$$|z| = \begin{cases} e^{-\lambda(z)} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

which makes $\mathbb{K}((G))$ into a spherically complete (and hence Cauchy complete) non-Archimedean valued field [8, A.9 pp. 288–292].

Note that every $z \in \mathbb{K}((G))$ can be written as $z = x + iy$, where $x, y \in \mathbb{F}((G))$, and consequently $|z| = \max\{|x|, |y|\}$.

The space $c_0 := \{z = \{z_n\}_{n=1}^{\infty} : z_n \in \mathbb{K}((G)), \lim_{n\to\infty} z_n = 0_{\mathbb{K}((G))}\}$ equipped with the natural norm $\|z\|_\infty := \sup_{n\in\mathbb{N}} |z_n|$ is a Banach space of countable type over $\mathbb{K}((G))$ (see page 28 in [7]).

2 Characterization of Compact and Self-adjoint Operators on c_0

In this section we generalize the results of [2] when the complex Levi-Civita field is replaced by a Hahn field.

2.1 An Inner Product on c_0

We consider the following form $\langle \cdot, \cdot \rangle : c_0 \times c_0 \to \mathbb{K}((G))$ given by

$$\langle z, w \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n},$$

where $\overline{w_n} = x_n - \mathrm{i}y_n$ is the complex conjugate of $w_n = x_n + \mathrm{i}y_n$. This form is well defined since $\lim_{n\to\infty} z_n\overline{w_n} = 0$ and satisfies Definition 2.4.1 for an inner product on page 38 in [7].

Let $\|z\| = \sqrt{|\langle z, z\rangle|}$ be the norm induced by the above inner product. Since 1, the multiplicative identity in $\mathbb{K}((G))$, has the property that $|1 + 1| = |2| = 1$, Theorem 2.4.2 in [7] entails that $\|\cdot\|$ is a non-Archimedean norm on c_0. It follows easily that if $\langle z, w\rangle = 0$ for all $z \in c_0$, then $w = 0$, which is referred to as the non-degeneracy condition.

The next theorem was proved in [6] and tells us when the non-Archimedean norm in a Banach space is induced by an inner product.

Theorem 1 *Let $(E, \|\cdot\|)$ be a $\mathbb{K}$-Banach space. Then if $\|E\| \subset |\mathbb{K}|^{1/2}$ and if every one-dimensional subspace of E admits a normal complement, then E has at least one inner product that induces the norm $\|\cdot\|$.*

The above conditions are satisfied for the $\mathbb{K}((G))$-Banach space $E = c_0$. In fact, for $z \in c_0$, $z \neq 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|z\|_\infty = \sup_{n\in\mathbb{N}} |z_n| = \max_{n\in\mathbb{N}} |z_n| = |z_{n_0}| \in |\mathbb{K}((G))|.$$

Note that for any $z \in \mathbb{K}((G))$, $|\mathbb{K}((G))| \subset |\mathbb{K}((G))|^{1/2}$ is guaranteed by the fact that $\lambda(z) \in G$ implies that $2\lambda(z) \in G$, and consequently $\|c_0\| \subset |\mathbb{K}((G))|^{1/2}$. The other condition is guaranteed by Lemma 2.3.19 on page 34 in [7].

Since $\mathbb{F}$ is formally real, the conclusion that $\|\cdot\|_\infty$ is induced by $\langle\cdot,\cdot\rangle$ could be obtained in a similar way as in [2] by using the following lemma, and we omit the proof here.

Lemma 1 *If $\{z_1, \ldots, z_n\} \subset \mathbb{K}((G))$, then* $|z_1\overline{z_1} + \ldots + z_n\overline{z_n}| = \max_{1\le j\le n} |z_j\overline{z_j}|$.

Definition 2 A subset D of c_0 such that for all $x, y \in D$, $x \neq y \Rightarrow \langle x, y\rangle = 0$, is called a normal family. A countable normal family $\{x_n : n \in \mathbb{N}\}$ of unit vectors is called an orthonormal sequence.

If $A \subset c_0$, then $[A]$ and $cl[A]$ will denote the linear span and the closed linear span of A, respectively. If M is a subspace of c_0, then M^p will denote the subspace of all $y \in c_0$ such that $\langle y, x\rangle = 0$, for all $x \in M$. Since the definition of the inner product given in [7], page 38, is implied by the definition of the inner product given here, the Gram–Schmidt procedure can be applied.

Theorem 2 *If $\{z_n\}$ is a sequence of linearly independent vectors in c_0, then there exists an orthonormal sequence $\{y_n\}$ such that $[\{z_1, \ldots, z_j\}] = [\{y_1, \ldots, y_j\}]$ for every $j \in \mathbb{N}$.*

Definition 3 A sequence $\{z_n\}_{n=1}^{\infty}$ of non-null vectors of c_0 is said to have the Riemann–Lebesgue property (RLP) if for all $z \in c_0$ $\lim_{n\to\infty}\langle z_n, z\rangle = 0$.

It follows immediately that any orthonormal basis of c_0 has this property. The following theorem has been proved in [6].

Theorem 3 *If $S \subset c_0$ is a finite orthonormal subset, say $\{z_1, \ldots, z_n\}$, or is an orthonormal sequence $\{z_n\}_{n=1}^{\infty}$ which satisfies the RLP, then S can be extended to an orthonormal basis for c_0: that is, there exists a countable orthonormal sequence $\{w_n\}_{n=1}^{\infty}$ (possibly finite) such that $S \bigcup \{w_n : n \in \mathbb{N}\}$ is an orthonormal basis for c_0.*

The next lemma, which could be obtained by using Lemma 1, shows that a normal sequence in c_0 is also an orthogonal sequence in the van Rooij's sense (see [9], page 57).

Lemma 2 *Let $\{x_n\}_{n=1}^{\infty}$ be a normal sequence in c_0. Then, for all $k \in \mathbb{N}$, we have that*

$$\left\| \sum_{j=1}^{k} \alpha_j x_{n_j} \right\|^2 = \max_{1 \le j \le k} \left\| \alpha_j x_{n_j} \right\|^2$$

Using the Gram–Schmidt process, we have the following theorem which was proved in [1].

Theorem 4 *Every closed subspace D of c_0 admits a countable orthonormal basis, that is, an orthonormal sequence $\{y_n\}$ such that $D = cl[\{y_n : n \in \mathbb{N}\}]$.*

If E and F are normed spaces over $\mathbb{K}$, then $\mathcal{L}(E, F)$ will be the normed space consisting of all continuous linear maps from E into F. $\mathcal{L}(E, \mathbb{K})$ will be denoted by E^* and $\mathcal{L}(E, E)$ will be denoted simply by $\mathcal{L}(E)$. For an operator $T \in \mathcal{L}(E, F)$, $\operatorname{Ker} T$ and $\operatorname{Im} T$ will denote the kernel and the image of T, respectively. It is well known that $c_0^* \cong \ell^{\infty}$, where c_0^* is the dual of c_0. Moreover, it was proved in [9] that if E and F are Banach spaces, then T is compact if and only if for each $\epsilon > 0$, there exists a continuous linear operator S with finite-dimensional $\operatorname{Im} S$ such that $\|T - S\| \le \epsilon$.

Definition 4 A functional $f \in c_0^*$ is called a Riesz functional if there exists $z \in c_0$ such that $f = \langle \cdot, z \rangle$. The space of all Riesz functionals of c_0^* will be denoted by $(c_0)_{RF}$, that is

$$(c_0)_{RF} = \{f \in c_0^* : f = \langle \cdot, z \rangle \text{ for some } z \in c_0\}$$

The following proposition is an immediate consequence of the definition of the inner product $\langle \cdot, \cdot \rangle$ and of the Riemann–Lebesgue property of $\{e_n\}_{n=1}^{\infty}$, where e_j is the element of c_0 whose jth entry is equal to 1 and with the rest of the entries all equal to zero.

Proposition 1 *Let $f \in c_0^*$. Then $f \in (c_0)_{RF}$ if and only if $\lim_{n \to \infty} f(e_n) = 0$. Moreover, in this case, $f = \langle \cdot, z \rangle$ where $z = \{\overline{f(e_n)}\}_{n=1}^{\infty}$.*

It is worth noting that for a given $f \in (c_0)_{RF}$, there is a corresponding element in c_0, say $\Phi(f) = \{\overline{f(e_n)}\}_{n=1}^{\infty} = z \in c_0$. Moreover, for any $x \neq 0$ in c_0, we have that

$$\frac{|f(x)|}{\|x\|} = \frac{|\langle x, z\rangle|}{\|x\|} \leq \frac{\|x\|\|z\|}{\|x\|} = \|z\| = \|\Phi(f)\|$$

from which we infer that

$$\|f\| \leq \|\Phi(f)\|.$$

On the other hand, we have that

$$\|\Phi(f)\| = \|z\| = \frac{|\langle z, z\rangle|}{\|z\|} = \frac{|f(z)|}{\|z\|} \leq \|f\|.$$

Thus, $\|\Phi(f)\| = \|f\|$ and hence Φ is a linear norm-preserving bijection.

2.2 *Normal Projections and Self-adjoint Operators*

Definition 5 An operator $P \in \mathcal{L}(c_0)$ is said to be a normal projection if $P^2 = P$ and $\langle z, w\rangle = 0$ for all $z \in \operatorname{Ker} P$ and all $w \in \operatorname{Im} P$.

Using the Cauchy–Schwarz inequality, we obtain that $\|Px\| \leq \|x\|$ for all $x \in c_0$, which implies that P is an orthoprojection in the van Rooij sense (see [9], page 63), while $\|Py\| = \|y\|$ for all $y \in \operatorname{Im} P$. It follows that $\|P\| = 1$.

Theorem 5 *Let P be a normal projection. If $\{z_n\}_{n=1}^{\infty}$ is an orthonormal basis of* Ker P*, then it has the Riemann–Lebesgue property.*

Proof It suffices to prove that $\lim_{n\to\infty} \langle z, z_n\rangle = 0$ for any $z \notin \operatorname{Ker} P$. Recall that P is a normal projection, which implies that $z - Pz \in \operatorname{Ker} P$ and $\langle Pz, z_n\rangle = 0$. It follows that

$$\langle z, z_n\rangle = \langle z - Pz, z_n\rangle + \langle Pz, z_n\rangle = \langle z - Pz, z_n\rangle \to 0 \text{ as } n \to \infty.$$

We can prove the following proposition using Corollary 8.2 in [6].

Proposition 2 *Let M be a closed subspace of* c_0*. If M has an orthonormal basis with the Riemann–Lebesgue property, then there exists a normal projection P such that $M =$* Ker P.

For $i, j \in \mathbb{N}$, we define $e_j^* \otimes e_i \in \mathcal{L}(c_0)$ by $e_j^* \otimes e_i(z) = \langle z, e_j \rangle e_i$, and it can be shown easily that $\|e_j^* \otimes e_i\| = 1$. Diarra in [5] proved the following lemma.

Lemma 3 *Suppose that* $\{\alpha_{ij}\}_{i,j\geq 1}$ *is a bounded sequence of elements in* $\mathbb{K}((G))$ *such that* $\lim_{i\to\infty} \alpha_{ij} = 0$*, for each* $j \in \mathbb{N}$*. Then the operator* $T : c_0 \to c_0$ *defined by* $T = \sum_{i,j\geq 1} \alpha_{ij} e_j^* \otimes e_i$ *is a continuous linear operator. Conversely, if* $T \in \mathcal{L}(c_0)$*, then* $T = \sum_{i,j\geq 1} \alpha_{ij} e_j^* \otimes e_i$ *for some bounded sequence* $\{\alpha_{ij}\}_{i,j\geq 1}$ *in* $\mathbb{K}((G))$ *such that* $\alpha_{ij} \to 0$ *as* $i \to \infty$*, for each* $j \in \mathbb{N}$*. Therefore, any* $T \in \mathcal{L}(c_0)$ *can be identified with the following matrix whose columns converge to 0:*

$$(T) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1j} & \dots \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2j} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ \alpha_{i1} & \alpha_{i2} & \dots & \alpha_{ij} & \dots \\ \downarrow & \downarrow & & \downarrow & \ddots \\ 0 & 0 & \dots & 0 & \dots \end{pmatrix}$$

Definition 6 A linear operator $T^\dagger : c_0 \to c_0$ is said to be an adjoint of a given operator $T \in \mathcal{L}(c_0)$ if $\langle Tx, y\rangle = \langle x, T^\dagger y\rangle$, for all $x, y \in c_0$. We also say T is self-adjoint if $T = T^\dagger$.

As in the classical case, the adjoint operator of a continuous linear operator, if it exists, is unique and continuous.

Proposition 3 *If a continuous linear operator* T *has an adjoint, then it is unique and continuous.*

Proof First, assume that $T^\dagger$ and $\tilde{T}^\dagger$ are adjoint operators of T. Then by the definition of an adjoint operator, we have that $\langle x, T^\dagger y - \tilde{T}^\dagger y\rangle = 0$ for all $x, y \in c_0$. The non-degeneracy condition of $\langle \cdot, \cdot \rangle$ then implies that $T^\dagger = \tilde{T}^\dagger$.

Now let $y \in c_0$, $y \neq 0$, be given. Then we have that

$$\|T^\dagger y\|^2 = |\langle T^\dagger y, T^\dagger y\rangle| = |\langle T(T^\dagger y), y\rangle| \leq \|T(T^\dagger y)\|\|y\| \leq \|T\|\|T^\dagger y\|\|y\|,$$

which implies that $\|T^\dagger y\| \leq \|T\|\|y\|$ for all $y \neq 0$ in c_0 and hence $\|T^\dagger\| \leq \|T\|$. Therefore, $T^\dagger \in \mathcal{L}(c_0)$.

We can check whether an operator has an adjoint or not by looking at its associated matrix.

Lemma 4 *Let $T \in \mathcal{L}(c_0)$ with associated matrix $(T) = (\alpha_{ij})$. Then T admits an adjoint operator $T^\dagger$ if and only if $\lim_{j\to\infty} \alpha_{ij} = 0$ for each $i \in \mathbb{N}$, i.e.,*

$$(T) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1j} & \to 0 \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2j} & \to 0 \\ \vdots & \vdots & \ddots & \vdots & \\ \alpha_{i1} & \alpha_{i2} & \dots & \alpha_{ij} & \to 0 \\ \downarrow & \downarrow & & \downarrow & \ddots \\ 0 & 0 & \dots & 0 & \dots \end{pmatrix}$$

However, unlike in the classical Hilbert space theory, not every continuous linear operator admits an adjoint, for example, see [2].

The next theorem shows the connection between self-adjoint operators and normal projections.

Theorem 6 *Let $P \in \mathcal{L}(c_0)$. If P is a normal projection, then P is self-adjoint. Conversely if P is self-adjoint and $P^2 = P$, then it is a normal projection.*

Proof First suppose P is a normal projection. Then for any $x, y \in c_0$, we have that $x - Px, y - Py \in \operatorname{Ker} P$. Consequently, $\langle x, Py\rangle = \langle Px, Py\rangle$ and $\langle y, Px\rangle = \langle Py, Px\rangle$. By conjugate symmetry, it follows that

$$\langle Px, y\rangle = \overline{\langle y, Px\rangle} = \overline{\langle Py, Px\rangle} = \langle Px, Py\rangle = \langle x, Py\rangle.$$

Therefore P is self-adjoint.

Now suppose that $P^2 = P$ and $\langle x, Py\rangle = \langle Px, y\rangle$ for any $x, y \in c_0$. Let $x \in \operatorname{Ker} P$ and $y \in \operatorname{Im} P$ and let $z \in c_0$ be such that $Pz = y$. Then $\langle x, y\rangle = \langle x, Pz\rangle = \langle Px, z\rangle = \langle 0, z\rangle = 0$, and hence P is a normal projection.

The next result provides a characterization for normal projections.

Theorem 7 *If $P \in \mathcal{L}(c_0)$ is a normal projection with $\operatorname{Im} P = cl[\{y_1, y_2, \dots\}]$, where $\{y_1, y_2, \dots\}$ is either an orthonormal finite subset of c_0 or an orthonormal sequence with the Riemann–Lebesgue property, then*

$$Px = \sum_{n=1}^{\infty} \langle x, y_n\rangle y_n / \langle y_n, y_n\rangle.$$

Proof let $x \in c_0$ be given. Then $P(x) = \sum_{n=1}^{\infty} \alpha_n(x) y_n$, where $\alpha_n \in c_0^*$ for each $n \in \mathbb{N}$. Now for this x, we have that $\langle x, y_j\rangle = \langle x, Py_j\rangle = \langle Px, y_j\rangle = \langle \sum_{n=1}^{\infty} \alpha_n(x) y_n, y_j\rangle = \alpha_j(x)\langle y_j, y_j\rangle$. Thus $\alpha_j(x) = \langle x, y_j\rangle / \langle y_j, y_j\rangle$.

Besides the connection between self-adjoint operators and normal projections that makes it possible to characterize all normal projections, for each $T \in \mathcal{L}(c_0)$ we can define T^*, the dual operator of T, which, as we will see below, coincides with $T^\dagger$, the adjoint operator of T.

The following theorem, proved in [7] page 171, guarantees the existence of dual operators of $\mathcal{L}(c_0)$. Note that the $\mathbb{K}((G))$-Banach space c_0 that we are considering in this paper meets all the conditions of the theorem.

Theorem 8 (Hahn–Banach Theorem) *Let E be a vector space over a spherically complete field $\mathbb{K}$, let p be a seminorm on E. Then for every subspace D of E and every $f_0 \in D^*$ with $|f_0| \leq p$ on D, there is an extension $f \in E^*$ of f_0 such that $|f| \leq p$ on E.*

We now discuss duality of linear operators. In particular, given a linear operator $T : c_0 \to c_0$, it induces a dual operator $T^* : c_0^* \to c_0^*$ that can be defined as follows.

Suppose $f_2 \in c_0^*$, then $f_1 = T^*(f_2) \in c_0^*$, is defined by $f_1(x_1) = f_2(T(x_1))$, whenever $x_1 \in c_0$. More succinctly

$$T^*(f_2)(x_1) = f_2(T(x_1)).$$

The following proposition is crucial to prove that the dual operator T^* has the same norm as T.

Proposition 4 *Let $x_0 \in c_0$ be given with $\|x_0\| = M \in \mathbb{R}$. Then there exists $f \in c_0^*$ such that $|f(x_0)| = M$ and $\|f\| = 1$.*

Proof First, note that if we write $x_0 = \{x_{0,n}\}_{n=1}^{\infty}$, then we have $\|x_0\| = M = \|x_0\|_\infty = |x_{0,N}|$ for some $N \in \mathbb{N}$.

Define f_0 on the one-dimensional subspace $D := \{\alpha x_0 : \alpha \in \mathbb{K}((G))\}$ by $f_0(\alpha x_0) = \alpha x_{0,N}$, for each $\alpha \in \mathbb{K}((G))$. Note that if we set $p(x) = \|x\|$ for every $x \in c_0$, the function p clearly satisfies the basic sub-linear property, that is, p is a seminorm. We also observe that

$$|f_0(\alpha x_0)| = |\alpha||x_{0,N}| = |\alpha|\|x_0\| = p(\alpha x_0),$$

and hence $|f_0(x)| \leq p(x)$ on the subspace D. By the Hahn–Banach theorem, we can extend f_0 to an f defined on c_0 with $|f(x)| \leq p(x) = \|x\|$, for all $x \in c_0$, and thus $\|f\| \leq 1$. The fact that $\|f\| \geq 1$ then follows from the defining property $f(x_0) = \|x_0\|$, thereby proving the proposition.

Theorem 9 *For a given $T \in \mathcal{L}(c_0)$, the dual operator T^* defined above is a bounded linear operator, and hence it is in $\mathcal{L}(c_0^*)$. Furthermore, we have $\|T^*\| = \|T\|$.*

Proof First, if $\|x_1\| \leq 1$, we have that

$$|T^*(f_2)(x_1)| = |f_1(x_1)| = |f_2(T(x_1))| \leq \|f_2\|\|T(x_1)\| \leq \|f_2\|\|T\|.$$

Thus taking the supremum over all $x_1 \in c_0$ with $\|x_1\| \leq 1$, we see that $\|T^*\| \leq \|T\|$.

To prove the other inequality, let $\epsilon > 0$ in $\mathbb{R}$ be given, then we can find $x_1 \in c_0$ with $\|x_1\| = 1$ and $\|T(x_1)\| \geq \|T\| - \epsilon$. Let $x_2 = T(x_1) \in c_0$. Then by Proposition 4 there is an f_2 in c_0^* so that $\|f_2\| = 1$ and $|f_2(x_2)| = \|x_2\| \geq \|T\| - \epsilon$. Thus by the definition of dual operator, we have that $|T^*(f_2)(x_1)| \geq \|T\| - \epsilon$; and since $\|x_1\| = 1$, we conclude $\|T^*(f_2)\| \geq \|T\| - \epsilon$. This shows that $\|T^*\| \geq \|T\| - \epsilon$. Since this holds for every $\epsilon > 0$, it follows that $\|T^*\| \geq \|T\|$.

The next commutative diagram, where $T^\dagger$ denotes the adjoint operator of T and Φ is given in Proposition 1, shows that the adjoint operator actually coincides with the dual operator.

$$\begin{array}{ccc} c_0 & \overset{T}{\underset{T^\dagger}{\longleftrightarrow}} & c_0 \\ \Phi \downarrow & & \downarrow \Phi \\ (c_0)_{RF} & \underset{T^*}{\longleftarrow} & (c_0)_{RF} \end{array}$$

To elaborate further on this, we observe that

$$T^*(\Phi(x_2))(x_1) = \Phi(x_2)(T(x_1)) = \langle T(x_1), x_2 \rangle = \langle x_1, T^\dagger(x_2) \rangle = \Phi(T^\dagger(x_2))(x_1),$$

for every $x_1, x_2 \in c_0$. Thus, $T^* \circ \Phi = \Phi \circ T^\dagger$.

2.3 *Characterization of Compact Operators*

Recall that a continuous linear operator T is compact if and only if T is the uniform limit of continuous linear operators of finite-dimensional range (see [9]). Unfortunately, as the argument in [2] shows, a normal projection P is not compact if $\dim(\operatorname{Im} P) = \infty$, and it follows that the convergence of $P(\cdot) = \sum_{n=1}^{\infty} \langle \cdot, y_n \rangle y_n / \langle y_n, y_n \rangle$ is only pointwise.

The following theorem provides a way to construct compact operators starting from an orthonormal sequence.

Theorem 10 *Let $\{y_n\}_{n=1}^{\infty}$ be an orthonormal sequence in c_0. Then for any $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in c_0$ such that $\alpha_n \in \mathbb{F}((G))$, the operator $T : c_0 \to c_0$ defined by*

$$T = \sum_{n=1}^{\infty} \alpha_n \langle \cdot, y_n \rangle y_n / \langle y_n, y_n \rangle = \sum_{n=1}^{\infty} \alpha_n P_n(\cdot)$$

is a compact and self-adjoint operator.

Proof It is clear that T is linear. Note that for $x \in c_0$, we have that

$$\left\| Tx - \sum_{j=1}^{n} \alpha_j P_j x \right\| = \left\| \sum_{j=n+1}^{\infty} \alpha_j \frac{\langle x, y_j \rangle}{\langle y_j, y_j \rangle} y_j \right\|$$
$$\leq \max_{j \geq n+1} |\alpha_j| \left| \frac{\langle x, y_j \rangle}{\langle y_j, y_j \rangle} \right| \|y_j\| \text{ (strong triangle inequality)}$$
$$\leq (\max_{j \geq n+1} |\alpha_j|) \|x\|,$$

and therefore

$$\|T - \sum_{i=1}^{n} \alpha_i P_i\| \leq \max_{j \geq n+1} |\alpha_j| \to 0 \text{ as } n \to \infty.$$

Since Im P_j is finite-dimensional for each $j \in \mathbb{N}$, it follows that T is compact. The self-adjoint condition follows easily from the fact that $\alpha_n \in \mathbb{F}((G))$ for each $n \in \mathbb{N}$.

The converse is also true, as stated in the next theorem whose proof is the same as that of the corresponding result (Theorem 10) in [2].

Theorem 11 *Let $T : c_0 \to c_0$ be a compact and self-adjoint linear operator. Then there exist an element $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in c_0$, where $\alpha_n \in \mathbb{F}((G))$ for each $n \in \mathbb{N}$, and an orthonormal sequence $\{y_n\}_{n=1}^{\infty}$ in c_0 such that*

$$T = \sum_{n=1}^{\infty} \alpha_n P_n,$$

where $P_n = \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$ is the normal projection defined by y_n.

The uniqueness of α is shown in the following proposition whose proof is the same as that of Proposition 6 in [2].

Proposition 5 *Let $T = \sum \alpha_n \frac{\langle \cdot, y_n \rangle}{\langle y_n, y_n \rangle} y_n$ be a compact and self-adjoint operator and let $\mu \neq 0$ in $\mathbb{K}((G))$ be an eigenvalue of T. Then $\mu = \alpha_n$ for some n.*

3 B^*-algebras of Operators on c_0

In this section we generalize the results of [3], replacing the complex Levi-Civita field by $\mathbb{K}((G))$. Recall that in Lemma 3 we show that each $T \in \mathcal{L}(c_0)$ has an associated matrix (α_{ij}) such that

(1) $\sup_{i,j \in \mathbb{N}} |\alpha_{ij}| < \infty$;
(2) $\lim_{i \to \infty} \alpha_{ij} = 0$ for each $j \in \mathbb{N}$;

(3) $T = \sum\limits_{i,j\in\mathbb{N}} \alpha_{ij} e_j^* \otimes e_i$;

(4) $\|T\| = \sup\limits_{i,j\in\mathbb{N}} |\alpha_{ij}| = \sup\limits_{n\in\mathbb{N}} \|Te_n\|$.

If we let

$$\mathcal{M}(c_0) = \{(\alpha_{ij}) : \sup_{i,j\in\mathbb{N}} |\alpha_{ij}| < \infty; \lim_{i\to\infty} \alpha_{ij} = 0 \text{ for each } j \in \mathbb{N}\}$$

equipped with the natural supremum norm, then $\mathcal{L}(c_0)$ and $\mathcal{M}(c_0)$ are isometrically isomorphic.

Lemma 5 *Let $T \in \mathcal{L}(c_0)$. Then for a given $y \in c_0$, the following conditions are equivalent:*

(1) There exists $y^\dagger \in c_0$ such that $\langle Tx, y\rangle = \langle x, y^\dagger\rangle$, for any $x \in c_0$;

(2) $\lim\limits_{n\to\infty} \langle Te_n, y\rangle = 0$.

Proof First, we assume that there exists $y^\dagger \in c_0$ such that $\langle Tx, y\rangle = \langle x, y^\dagger\rangle$, for any $x \in c_0$. Then $\langle Te_n, y\rangle = \langle e_n, y^\dagger\rangle \to 0$ as $n \to \infty$ since $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis.

Now assume $\lim\limits_{n\to\infty} \langle Te_n, y\rangle = 0$, then $\{\langle Te_n, y\rangle\}_{n=1}^\infty \in c_0$. Let $y^\dagger = \sum_{n=1}^\infty \overline{\langle Te_n, y\rangle} e_n$. Then a straightforward computation leads to $\langle x, y^\dagger\rangle = \langle Tx, y\rangle$ for any $x \in c_0$.

Let us denote by $\mathcal{A}_0(c_0)$, or simply $\mathcal{A}_0$, the set of all $T \in \mathcal{L}(c_0)$ such that $\lim\limits_{n\to\infty} \langle Te_n, y\rangle = 0$ for all $y \in c_0$. It is not hard to show, as was done in [3], that $\mathcal{A}_0$ is the set of all continuous linear operators which admit adjoints and consequently $\mathcal{A}_0$ contains normal projections. Moreover, if we define $T^\dagger : c_0 \to c_0$ by $T^\dagger(y) = y^\dagger$ for $T \in \mathcal{A}_0$, then $\langle Tx, y\rangle = \langle x, y^\dagger\rangle = \langle x, T^\dagger(y)\rangle$, for any $x, y \in c_0$, and hence $T^\dagger$ is exactly the adjoint operator we have already defined in Sect. 2.2. Since $(T^\dagger)^\dagger = T$, it follows that $T \in \mathcal{A}_0$ if and only if $T^\dagger \in \mathcal{A}_0$.

Let $\mathcal{A}_1 := \{T \in \mathcal{L}(c_0) : \lim\limits_{n\to\infty} Te_n = 0\}$. Then it follows that $\varnothing \neq \mathcal{A}_1 \subsetneq \mathcal{A}_0$ (see [3]).

Theorem 12 below gives a characterization of the elements of $\mathcal{A}_1$ and, for its proof, we need the following result given in [5].

Proposition 6 *A continuous linear operator $T = \sum\limits_{i,j\in\mathbb{N}} \alpha_{ij} e_j^* \otimes e_i$ is compact if and only if $\lim\limits_{i\to\infty} \sup\limits_{j\in\mathbb{N}} |\alpha_{ij}| = 0$.*

Theorem 12 *$T \in \mathcal{A}_1$ if and only if T is compact and $T \in \mathcal{A}_0$.*

Proof First, suppose $T \in \mathcal{A}_1$. For each $j \in \mathbb{N}$, we define $T_j : c_0 \to c_0$ by $T_j x = \sum\limits_{i=1}^{j} x_i Te_i$. Clearly T_j is a continuous linear operator and $\operatorname{Im} T_j$ is finite-dimensional, and hence it is compact. Moreover, it could be easily verified that $T_j \in \mathcal{A}_0$. Note that

$$\|Tx - T_j x\| = \left\| \sum_{i=j+1}^{\infty} x_i T e_i \right\| \leq \max_{i \geq j+1} \|x_i T e_i\| \leq \|x\| \max_{i \geq j+1} \|T e_i\|,$$

and $\lim_{i\to\infty} T e_i = 0$ by the definition of $\mathcal{A}_1$. It follows that T is compact.

Conversely, let $T = \sum_{i,j\in\mathbb{N}} \alpha_{ij} e_j^* \otimes e_i \in \mathcal{A}_0$ be a compact operator. Then $T^\dagger \in \mathcal{A}_0$. Since T is compact, we have that $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} |\alpha_{ij}| = 0$. Note that $\|T^\dagger e_i\| = \sup_{j\in\mathbb{N}} |\alpha_{ij}| \to 0$ as $i \to \infty$ and hence $T^\dagger \in \mathcal{A}_1$. Applying the first part of the proof, we conclude that $T^\dagger$ is also compact. Thus, $\lim_{i\to\infty} \sup_{j\in\mathbb{N}} |\beta_{ij}| = 0$, where $T^\dagger = \sum_{i,j\in\mathbb{N}} \beta_{ij} e_j^* \otimes e_i$, with $\beta_{ij} = \overline{\alpha_{ji}}$. Using the fact that

$$\|T e_i\| = \sup_{j\in\mathbb{N}} |\alpha_{ji}| = \sup_{j\in\mathbb{N}} |\beta_{ij}| \to 0 \text{ as } i \to \infty,$$

we conclude that $T \in \mathcal{A}_1$.

An immediate consequence of this theorem is that for $T \in \mathcal{A}_1$, the associated matrix (α_{ij}) satisfies $\lim_{(i,j)\in\mathbb{N}^2} \alpha_{ij} = 0$ and that every compact and self-adjoint operator is in $\mathcal{A}_1$. To be precise,

$$\mathcal{A}_2 := \{T \in \mathcal{A}_1 : T = T^\dagger\} \subsetneq \mathcal{A}_1.$$

Moreover, as was shown in [3], we can show that $\mathcal{A}_1$ is a closed subalgebra of $\mathcal{A}_0$ and $\mathcal{A}_2$ is a closed subset of $\mathcal{A}_1$.

3.1 Inner Product in $\mathcal{A}_1$

Since $Te_n, Se_n \to 0 \in c_0$ as $n \to \infty$ for $T, S \in \mathcal{A}_1$, the mapping

$$\langle \cdot, \cdot \rangle : \mathcal{A}_1 \times \mathcal{A}_1 \to \mathbb{K}((G))$$
$$(S, T) \mapsto \sum_{i=1}^{\infty} \langle Se_i, Te_i \rangle$$

is well defined, linear in the first variable and linear conjugate in the second variable, and $\langle S, T\rangle = \overline{\langle T, S\rangle}$ for all $S, T \in \mathcal{A}_1$.

It is clear that if $w \in c_0$ then $\langle w, w\rangle \in \mathbb{F}((G))$, and since $\mathbb{F}$ is formally real, we have that

$$|\langle w_1, w_1\rangle + \langle w_2, w_2\rangle + \ldots + \langle w_n, w_n\rangle| = \max_{1\le i\le n} |\langle w_i, w_i\rangle|.$$

We can then use the above equation to prove that $\langle\cdot,\cdot\rangle$ is an inner product in $\mathcal{A}_1$ according to Definition 2.4.1 in [7]. Moreover, $\sqrt{\langle S, S\rangle}$ is a norm on $\mathcal{A}_1$ and

$$|\langle S, T\rangle|^2 \le |\langle S, S\rangle||\langle T, T\rangle| \text{ for all } S, T \in \mathcal{A}_1.$$

The next proposition shows that the norm $\|\cdot\|$ on $\mathcal{A}_1$ is induced by the above inner product.

Proposition 7 *Let* $T \in \mathcal{A}_1$, $T \neq 0$. *Then* $|\langle T, T\rangle| = \|T\|^2$.

Proof Since $T \in \mathcal{A}_1$, there exists $N \in \mathbb{N}$ such that $\|Te_i\|^2 < \|T\|^2$ for $i \ge N$. It follows that

$$\begin{aligned}\left|\sum_{i=N}^{\infty}\langle Te_i, Te_i\rangle\right| &\le \max_{i\ge N} |\langle Te_i, Te_i\rangle| \\ &< \|T\|^2 = \max_{i\in\mathbb{N}} |\langle Te_i, Te_i\rangle| = \max_{i\le N-1} |\langle Te_i, Te_i\rangle| = \left|\sum_{i=1}^{N-1}\langle Te_i, Te_i\rangle\right|.\end{aligned}$$

Therefore,

$$\begin{aligned}|\langle T, T\rangle| &= \left|\sum_{i=1}^{\infty}\langle Te_i, Te_i\rangle\right| = \left|\sum_{i=1}^{N-1}\langle Te_i, Te_i\rangle + \sum_{i=N}^{\infty}\langle Te_i, Te_i\rangle\right| \\ &= \left|\sum_{i=1}^{N-1}\langle Te_i, Te_i\rangle\right| = \|T\|^2.\end{aligned}$$

The proof of the next theorem in [3] still works in the context of this paper, and we omit the proof here.

Theorem 13 c_0 *is isometrically isomorphic to a closed subspace of* $\mathcal{A}_1$. *Moreover, the restriction of the inner product on* $\mathcal{A}_1$ *to this closed subspace coincides with the inner product defined in* c_0.

Observe that if $T \in \mathcal{A}_1$, then $\langle\cdot, T\rangle : \mathcal{A}_1 \to \mathbb{K}((G))$, $S \mapsto \langle S, T\rangle$ is a continuous linear functional, i.e., $\langle\cdot, T\rangle \in \mathcal{A}_1^*$ and such functionals are called Riesz functionals. However, not all functionals in $\mathcal{A}_1^*$ are Riesz functionals. For example, see [3]. The next proposition proved in [3] gives a necessary and sufficient condition for Riesz functionals.

Proposition 8 *For $f \in \mathcal{A}_1^*$, f is a Riesz functional if and only if the double sequence $\{f(e_j^* \otimes e_i)\}_{(i,j)\in\mathbb{N}^2}$ is convergent to* 0.

Definition 7 Let M be a closed subspace of $\mathcal{A}_1$. We shall say that M admits a normal complement if $\mathcal{A}_1 = M \oplus M^p$, where $M^p = \{S \in \mathcal{A}_1 : \langle S, T\rangle = 0 \text{ for all } T \in M\}$.

We have the following result which has the same proof as that of the corresponding one in [3].

Theorem 14 *If M is an infinite-dimensional closed subspace of $\mathcal{A}_1$, then the following statements are equivalent:*

(1) M has a normal complement;
(2) M has an orthonormal basis with the Riemann–Lebesgue property;
(3) There exists a normal projection P such that Ker $P = M$.

Finally, we note that although $(\mathcal{A}_1, \langle\cdot,\cdot\rangle)$ is not orthomodular, there exist closed subspaces of $\mathcal{A}_1$ which admit normal complements, for example, see [3].

4 Positive Operators on c_0

So far we have only required $\mathbb{F}$ to be formally real. In order to work on positive operators on the Banach space c_0 over $\mathbb{K}((G))$, we will assume in this section that $\mathbb{F}$ is a totally ordered field.

For $x, y \in \mathbb{F}((G))$, we say $x \geq y$ if $x = y$ or $x \neq y$ and $(x-y)(\lambda(x-y)) > 0_{\mathbb{F}}$. Then $\mathbb{F}((G))$ with $\geq$ is a totally ordered field. It is clear that $\mathbb{F}((G))$ is a subfield of $\mathbb{K}((G))$.

Definition 8 For $T \in \mathcal{A}_1$, we say that T is positive and write $T \geq 0$ if $\langle Tx, x\rangle \in \mathbb{F}((G))$ and $\langle Tx, x\rangle \geq 0$ for all $x \in c_0$.

We have the following lemmas which have similar proofs to those of the corresponding ones in [4].

Lemma 6 *Let $T \in \mathcal{A}_1$ be positive. Then T is self-adjoint, that is, $T \in \mathcal{A}_2$. Moreover, all eigenvalues of T are in $\mathbb{F}((G))$ and non-negative.*

Lemma 7 *Let $S, T \geq 0$ in $\mathcal{A}_1$ and $\alpha \geq 0$ in $\mathbb{F}((G))$ be given. Then $\alpha S + T \geq 0$.*

Lemma 8 *For all $T \in \mathcal{A}_1$, both $T^\dagger T$ and $TT^\dagger$ are positive.*

In order to obtain a fundamental property of the inner product and be able to generalize the results on positive operators in [4], we now restrict our attention to the case that $\mathbb{F}$ is real closed and G is divisible so that $\mathbb{K}$ and $\mathbb{K}((G))$ are algebraically closed and thus the proofs could be carried over without any changes.

Proposition 9 *Let $T \in \mathcal{A}_1$ be positive. Then $|\langle Tx, y\rangle|^2 \leq |\langle Tx, x\rangle||\langle Ty, y\rangle|$ for all $x, y \in c_0$, where $|\cdot|$ denotes the ultrametric absolute value on $\mathbb{K}((G))$.*

Theorem 15 *For $T \in \mathcal{A}_1$, the following are equivalent:*

(1) $T \geq 0$;
(2) T is self-adjoint and all of its eigenvalues are in $\mathbb{F}((G))$ and non-negative;
(3) There exists $S \geq 0$ in $\mathcal{A}_1$ such that $T = S^2$;
(4) There exists $S \in \mathcal{A}_1$ such that $T = SS^\dagger$;
(5) There exists $M \in \mathcal{A}_1$ such that $T = M^\dagger M$.

Remark 1 Let T and S be as in Theorem 15: $T \geq 0$ and $S \geq 0$ in $\mathcal{A}_1$ such that $T = S^2$. Then S is unique. We say that S is the positive square root of T and write $S = \sqrt{T}$. Moreover, if $T = \sum_{n=1}^{\infty} \alpha_n P_n$, then $S = \sum_{n=1}^{\infty} \sqrt{\alpha_n} P_n$ where $\alpha_n \in \mathbb{F}((G))$ and $\alpha_n \geq 0$, for each $n \in \mathbb{N}$. It follows that $\|S\| = \|T\|^{1/2}$.

Following are results proved in [4] and we verified that the proofs are still valid in the context of this paper.

Proposition 10 *Let $S, T \in \mathcal{A}_1$ be positive. Then $ST \geq 0$ if and only if $ST = TS$.*

Proposition 11 *Let $T \in \mathcal{A}_2$ be given. Then there exist unique positive operators $A, B \in \mathcal{A}_2$ such that $T = A - B$ and $AB = BA = 0$. Moreover, we have that $\|T\| = \max\{\|A\|, \|B\|\}$.*

Proposition 12 *The set $\mathcal{P} = \{T \in \mathcal{A}_2 : T \geq 0\}$ is closed in $\mathcal{A}_2$.*

4.1 Partial Order on $\mathcal{A}_2$

In this section we introduce a relation on $\mathcal{A}_2$, which is a partial order, and some of its properties are presented.

Definition 9 For $S, T \in \mathcal{A}_2$, we say that $S \geq T$ if $S - T \geq 0$.

This defines a partial order on $\mathcal{A}_2$: the reflexivity, antisymmetry and transitivity of $\geq$ can be easily verified, and Example 3.3 in [4] shows that it is not a total order.

Proposition 13 *Let $S, T \in \mathcal{A}_2$ be given. Then $S \geq T$ if and only if $\langle Sx, x\rangle \geq \langle Tx, x\rangle$ for all $x \in c_0$.*

Proof First note that since $S, T, S - T$ are in $\mathcal{A}_2$ and hence self-adjoint, we have that $\langle Sx, x\rangle$, $\langle Tx, x\rangle$ and $\langle (S - T)x, x\rangle$ are all in $\mathbb{F}((G))$ for all $x \in c_0$. Therefore

$$S \geq T \Leftrightarrow S - T \geq 0 \Leftrightarrow \langle (S - T)x, x\rangle \geq 0 \Leftrightarrow \langle Sx, x\rangle \geq \langle Tx, x\rangle.$$

Moreover, we have the following result.

Proposition 14 *Let $S, T \in \mathcal{A}_2$ be such that $S \geq T \geq 0$. Then $\|S\| \geq \|T\|$.*

We finish this paper with the following result (for the proof see [4]) which gives equivalent conditions for two normal projections $P_1, P_2 \in \mathcal{A}_2$ to be related by the order relation defined above.

Theorem 16 *Let* $P_1, P_2 \in \mathcal{A}_2$ *be normal projections and let* $M_1 = \operatorname{Im} P_1$ *and* $M_2 = \operatorname{Im} P_2$. *Then the following are equivalent.*

(1) $P_2 \geq P_1$*;*
(2) $M_2 \supseteq M_1$*;*
(3) $P_2 P_1 = P_1$*;*
(4) $P_1 P_2 = P_1$.

Acknowledgements The second author "Changying Ding" was at the time of submission of this paper an undergraduate student supported by a MITACS Globalink summer research internship.

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Part IX
Special Interest Group: IGPDE Recent Progress in Evolution Equations

Session Organizers: Marcello D'Abbicco, Marcelo Rempel Ebert, and Michael Reissig

The main concern of the session was to explain the influence of low-regular coefficients, of different type of damping terms and power nonlinearities, on the qualitative properties of solutions of evolution equations.

Conditional Stability for Backward Parabolic Equations with Osgood Coefficients

Daniele Casagrande, Daniele Del Santo, and Martino Prizzi

Abstract The interest of the scientific community for the existence, uniqueness and stability of solutions to PDEs is testified by the numerous works available in the literature. In particular, in some recent publications on the subject (Del Santo et al. Nonlinear Anal 121:101–122, 2015; Del Santo and Prizzi, Math Ann 345:213–243, 2009) an inequality guaranteeing stability is shown to hold provided that the coefficients of the principal part of the differential operator are Log-Lipschitz continuous. Herein this result is improved along two directions. First, we describe how to construct an operator, whose coefficients in the principal part are not Log-Lipschitz continuous, for which the above-mentioned inequality does not hold. Second, we show that the stability of the solution is guaranteed, in a suitable functional space, if the coefficients of the principal part are Osgood-continuous.

1 Introduction

Backward parabolic equations are known to generate ill-posed (in the sense of Hadamard [6, 7]) Cauchy problems. Due to the smoothing effects of the parabolic operator, in fact, it is not possible, in general, to guarantee existence of the solution for initial data in any reasonable function space. In addition, even when solutions possibly exist, uniqueness does not hold without additional assumptions on the operator. Nevertheless, also for problems which are not well-posed the study of the conditional stability of the solution—the surrogate of the notion of "continuous dependence" when existence of a solution is not guaranteed—is interesting. Such kind of study can be performed by resorting to the notion of *well behaving* introduced by John [9]: a problem is *well-behaved* if "only a fixed percentage of the

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_27

significant digits need be lost in determining the solution from the data". In other words, a problem is well behaved if its solutions in a space $\mathscr{H}$ depend continuously on the data belonging to a space $\mathscr{K}$, provided they satisfy a prescribed bound in a space $\mathscr{H}'$ (possibly different from $\mathscr{H}$). In this paper we give a contribution to the study of the (*well*) behaviour of the Cauchy problem associated with a backward parabolic operator. In particular, we consider the operator $\mathscr{L}$ defined, on the strip $[0, T] \times \mathbb{R}^n$, by

$$\mathscr{L}u = \partial_t u + \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{i,j}(t,x)\partial_{x_j} u\right) + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j} u + c(t,x)u \,, \tag{1}$$

where all the coefficients are bounded. We suppose that $a_{i,j}(t,x) = a_{j,i}(t,x)$ for all $i, j = 1, \ldots, n$ and for all $(t,x) \in [0,T] \times \mathbb{R}^n$. We also suppose that $\mathscr{L}$ is backward parabolic, i.e. there exists $k_A \in]0,1[$ such that, for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$k_A|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(t,x)\xi_i\xi_j \le k_A^{-1}|\xi|^2 \,. \tag{2}$$

We show that if the coefficients of the principal part of $\mathscr{L}$ are at least Osgood regular, then there exists a function space in which the associated Cauchy problem

$$\begin{cases} \mathscr{L}u = f \,, & \text{in } (0,T) \times \mathbb{R}^n \,, \\ u|_{t=0} = u_0 \,, & \text{in } \mathbb{R}^n \,, \end{cases} \tag{3}$$

has a stability property.

To collocate the new result in the framework of the existing literature, we first recall the contents of some interesting publications on the subject which show that, as one could expect, the function space in which the stability property holds is related to the degree of regularity of the coefficients of $\mathscr{L}$. Weaker requirements on the regularity of the coefficients must be balanced, for the stability property to hold, by stronger a priori requirements on the regularity of the solution, hence stability holds in a smaller function space.

The overview on available works helps to lead the reader to the new result, claimed in the final part of the paper, concerning operators with Osgood-continuous coefficients. This kind of regularity is critical since it is the minimum required regularity that guarantees uniqueness of the solution and can therefore be considered as a sort of lower limit. The complete proof of the claim is rather cumbersome and is not reported here; instead, we provide a detailed discussion on the fact that, although the core reasoning is based on the theoretical scheme followed to achieve previous results [3], the modifications needed to obtain an analogous proof in the case of Osgood coefficients are by no means trivial.

2 Uniqueness and Non-uniqueness Results

We begin by recalling some results on the uniqueness and non-uniqueness of the solution of the problem (3). Consider the space

$$\mathcal{H}_0 = C([0,T], L^2) \cap C([0,T[, H^1) \cap C^1([0,T[, L^2)\,. \tag{4}$$

One of the first results concerning uniqueness is due to Lions and Malgrange [10]. They achieve a uniqueness result for an equation associated with a sesquilinear operator defined in a Hilbert space. With respect to the space (4), this result can be read as follows.

Theorem 21 *If the coefficients of the principal part of $\mathcal{L}$ are Lipschitz continuous with respect to t and x, if $u \in \mathcal{H}_0$ and if $u_0 = 0$, then $\mathcal{L}u = 0$ implies $u \equiv 0$.* □

The Lipschitz continuity of the coefficients is a crucial requirement for the claim, as shown some years later by Pliś [11] in the following theorem.

Theorem 22 *There exist u, b_1, b_2 and $c \in C^\infty(\mathbb{R}^3)$, bounded with bounded derivatives and periodic in the space variables and there exist $l : [0,T] \to \mathbb{R}$, Hölder-continuous of order δ for all $\delta < 1$ but not Lipschitz continuous, such that $1/2 \le l(t) \le 3/2$ and the support of the solution u of the Cauchy problem*

$$\begin{cases} \partial_t^2 u(t,x_1,x_2) + \partial_{x_1}^2 u(t,x_1,x_2) + l(t)\partial_{x_2}^2 u(t,x_1,x_2) + \\ \qquad + b_1(t,x_1,x_2)\partial_{x_1} u(t,x_1,x_2) + b_2(t,x_1,x_2)\partial_{x_2} u(t,x_1,x_2) + \\ \qquad\qquad + c(t,x_1,x_2)u(t,x_1,x_2) = 0 & \text{in } \mathbb{R}^3\,, \\ u(t,x_1,x_2)|_{t=0} = 0 & \text{in } \mathbb{R} \end{cases} \tag{5}$$

is the set $\mathbb{R} \times \mathbb{R} \times \{t \ge 0\}$. □

Note that the differential operator in (5) is elliptic. However, the same idea developed by Pliś to prove the claim can be exploited to obtain a counterexample for the backward parabolic operator

$$\mathcal{L}_P = \partial_t + \partial_{x_1}^2 + l(t)\partial_{x_2}^2 + b_1(t,x_1,x_2)\partial_{x_1} + b_2(t,x_1,x_2)\partial_{x_2} + c(t,x_1,x_2)\,.$$

Moreover, the result can be extended to the operator $\mathcal{L}$ in (1) by considering the problem solved by $u(t,x_1,x_2)e^{-x_1^2-x_2^2}$, thus obtaining the following theorem.

Theorem 23 *There exist coefficients $a_{i,j}$, depending only on t, which are Hölder continuous of every order but not Lipschitz continuous and there exist $u \in \mathcal{H}_0$ such that the solution of problem (3) with $u_0 = 0$ and $f = 0$ is not identically zero.* □

In view of the previous results, a question naturally arises: which is the *minimal* regularity with respect to t (between Lipschitz continuity and Hölder continuity) of the coefficients of the principal part of $\mathscr{L}$ guaranteeing uniqueness of the solution of (3)? To answer to this question, we recall the definition of *modulus of continuity* that can be exploited to measure the degree of regularity of a function.

Definition 24 A *modulus of continuity* is a function $\mu : [0,1] \to [0,1]$ which is continuous, increasing, concave and such that $\mu(0) = 0$. A function $f : \mathbb{R} \to \mathbb{R}$ has *regularity* μ if

$$\sup_{0<|t-s|<1} \frac{|f(t) - f(s)|}{\mu(|t-s|)} < +\infty\,.$$

The set of all functions having regularity μ is denoted by C^{μ}.

As particular cases, the Lipschitz continuity, the τ-Hölder continuity ($\tau \in]0,1[$) and the *logarithmic Lipschitz* (in short *Log-Lipschitz*) continuity are obtained for $\mu(s) = s$, $\mu(s) = s^{\tau}$ and $\mu(s) = s\log(1 + 1/s)$, respectively.

A further characterization of the modulus of continuity is the *Osgood condition* which is crucial in most of the results on uniqueness and continuity that are described in the rest of the article. A modulus of continuity μ satisfies the Osgood condition if

$$\int_0^1 \frac{1}{\mu(s)} ds = +\infty\,.$$

This characterization is used, for instance, in [2] to obtain the following result.

Theorem 25 *Le μ be a modulus of continuity that satisfies the Osgood condition. Let*

$$\mathscr{H}_1 \triangleq H^1([0,T], L^2(\mathbb{R}^n)) \cap L^2([0,T], H^2(\mathbb{R}^n)) \tag{6}$$

and let the coefficients $a_{i,j}$ in (1) be such that, for all $i, j = 1, \ldots, n$,

$$a_{i,j} \in C^{\mu}([0,T], \mathscr{C}_b(\mathbb{R}^n)) \cap \mathscr{C}([0,T], \mathscr{C}_b^2(\mathbb{R}^n))\,,$$

where $\mathscr{C}_b^2$ is the space of the bounded functions whose first and second derivatives are bounded. If $u \in \mathscr{H}_1$, if $\mathscr{L}u = 0$ on $[0,T] \times \mathbb{R}^n$ and if $u(0,x) = 0$ on $\mathbb{R}^n$, then $u \equiv 0$ on $[0,T] \times \mathbb{R}^n$.

More recently, by using Bony's para-multiplication, the result has been improved as far as the regularity with respect to x is concerned, i.e. replacing $\mathscr{C}^2$ regularity with Lipschitz regularity [4].

Note that the claim of Theorem 25 refers to the function space defined by (6), however, it is not difficult to extend it to the function space $\mathscr{H}_0$ defined by (4).

3 Conditional Stability Results

As mentioned in the introduction, for Cauchy problems related to the backward parabolic differential operators, which in general are not well posed, the notion of continuous dependence from initial data is replaced by the notion of (conditional) stability which is associated with the property of a problem to be well behaved, as defined by John [9]. The question about the conditional stability can be stated as follows. Suppose that two functions u and v, defined in $[0, T] \times \mathbb{R}^n$, are solutions of the same equation; suppose, in addition, that u and v satisfy a fixed bound in a space $\mathscr{K}$ and that $\|u(0, \cdot) - v(0, \cdot)\|_{\mathscr{H}}$ is small (less than some ϵ). Given these assumptions can we say something on the quantity $\sup_{t\in[0,T']} \|u(t, \cdot) - v(t, \cdot)\|_{\mathscr{K}}$ for some $T' < T$? Does it remain small as well (e.g. less than a value related to ϵ)? In this section we report some results that give an answer to the above questions.

3.1 Stability with Lipschitz-Continuous (with Respect to t) Coefficients

One of the first results on conditional stability has been proven by Hurd [8] in the same theoretical framework considered by Lions and Malgrange.

Theorem 31 *Suppose that the coefficients $a_{i,j}$ are Lipschitz continuous in t and in x. For every $T' \in]0, T[$ and for every $D > 0$ there exist $\rho > 0$, $\delta \in]0, 1[$ and $M > 0$ such that if $u \in \mathscr{H}_0$ is a solution of $\mathscr{L}u = 0$ on $[0, T]$ with $\|u(t, \cdot)\|_{L^2} < D$ on $[0, T]$ and $\|u(0, \cdot)\|_{L^2} < \rho$, then*

$$\sup_{t\in[0,T']} \|u(t, \cdot)\|_{L^2} \leq M\|u(0, \cdot)\|^{\delta}_{L^2} . \tag{7}$$

The constants ρ, δ and M depend only on T' and D, on the ellipticity constant of $\mathscr{L}$, on the L^∞ norms of the coefficients $a_{i,j}$, b_j, c and of their spatial derivatives, and on the Lipschitz constant of the coefficients $a_{i,j}$ with respect to time. □

The result expressed by Eq. (7) implies uniqueness of the solution to the Cauchy problem, so that a necessary condition to this kind of conditional stability is that the coefficients $a_{i,j}$ fulfil the Osgood condition with respect to time. Hence a natural question arises: is Osgood condition also a sufficient condition? Del Santo and Prizzi [3] have given a negative answer to this question. In particular, mimicking Pliś counterexample, they have shown that if the coefficients $a_{i,j}$ are not Lipschitz continuous but only Log-Lipschitz continuous then Hurd's result does not hold. Moreover, they have proven that if the coefficients are Log-Lipschitz continuous then a conditional stability property, although weaker than (7), does hold. More recently, the result has been further improved [5].

3.2 Stability with Log-Lipschitz-Continuous (with Respect to t) Coefficients

As mentioned above, Osgood condition is not a sufficient condition for conditional stability of the solution. The following paragraph specifies this claim.

3.2.1 Counterexample for the Lipschitz Continuity Case

The counterexample relies on the fact that it is possible [3] to construct

- a sequence $\{\mathscr{L}_k\}_{k\in\mathbb{N}}$ of backward uniformly parabolic operators with uniformly Log-Lipschitz-continuous coefficients (not depending on the space variables) in the principal part and space-periodic uniformly bounded smooth coefficients in the lower order terms,
- a sequence $\{u_k\}_{k\in\mathbb{N}}$ of space-periodic smooth uniformly bounded solutions of $\mathscr{L}_k u_k = 0$ on $[0, 1] \times \mathbb{R}^2$,
- a sequence $\{t_k\}_{k\in\mathbb{N}}$ of real numbers, with $t_k \to 0$ as $k \to \infty$,

such that

$$\lim_{k\to\infty} \|u_k(0, \cdot, \cdot)\|_{L^2([0,2\pi]\times[0,2\pi])} = 0$$

and

$$\lim_{k\to\infty} \frac{\|u_k(t_k, \cdot, \cdot)\|_{L^2([0,2\pi]\times[0,2\pi])}}{\|u_k(0, \cdot, \cdot)\|^{\delta}_{L^2([0,2\pi]\times[0,2\pi])}} = +\infty$$

for every $\delta > 0$. We remark that this situation is exactly what is needed to show that for backward operators with Log-Lipschitz continuous coefficient a result similar to Theorem 31 cannot hold.

3.2.2 Stability Result in the Log-Lipschitz Case

In the case of Log-Lipschitz coefficients a result weaker than (7) is valid.

Consider the equation $\mathscr{L}u = 0$ on $[0, T] \times \mathbb{R}^n$ and suppose that

1. for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and for all $i, j = 1, \dots, n$, $a_{i,j}(t, x) = a_{j,i}(t, x)$;
2. there exists $k > 0$ such that, for all $(t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$,

$$k|\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(t, x)\xi_i\xi_j \le k^{-1}|\xi|^2 ;$$

3. for all $i, j = 1, \dots, n$, $a_{i,j} \in \mathrm{LogLip}([0,T], L^\infty(\mathbb{R}^n)) \cap L^\infty([0,T], \mathscr{C}_b^2(\mathbb{R}^n))$, in particular

$$\sup_{x\in\mathbb{R}^n, 0<|\tau|<1} \frac{|a_{i,j}(t+\tau,x) - a_{i,j}(t,x)|}{|\tau|\left(\log\left(1+\frac{1}{|\tau|}\right)\right)} < +\infty\,;$$

4. for all $j = 1, \dots, n$, $b_j \in L^\infty([0,T], \mathscr{C}_b^2(\mathbb{R}^n))$;
5. $c \in L^\infty([0,T], \mathscr{C}_b^2(\mathbb{R}^n))$.

Theorem 32 ([3]) *Suppose that the above hypotheses 1–5 hold. For all $T' \in]0,T[$ and for all $D > 0$ there exist $\rho > 0$, $M > 0$, $N > 0$ and $0 < \beta < 1$ such that, if $u \in \mathscr{H}_0$ is a solution of $\mathscr{L}u = 0$ on $[0,T]$ with*

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^2} \le D$$

and $\|u(0,\cdot)\|_{L^2} \le \rho$, then

$$\sup_{t\in[0,T']} \|u(t,\cdot)\|_{L^2} \le M e^{-N|\log\|u(0,\cdot)\|_{L^2}|^\beta}\,, \tag{8}$$

where the constants ρ, β, M and N depend only on T', on D, on the ellipticity constant of $\mathscr{L}$, on the L^∞ norms of the coefficients $a_{i,j}$ and of their spatial first derivatives, and on the Log-Lipschitz constant of the coefficients $a_{i,j}$ with respect to time.

Using Bony's para-product the result can be extended to the case in which the coefficients are not necessarily $\mathscr{C}_b^2$-continuous with respect to x but only Lipschitz [5].

3.3 Stability with Osgood-Continuous (with Respect to Time) Coefficients

Let us finally come to the new result contained in this paper. As in the previous section we first present a counterexample to the stability condition (8) and then a new weaker stability result.

3.3.1 Counterexample for the Log-Lipschitz Case

Consider the modulus of continuity ω defined by

$$\omega(s) = s\log\left(1+\frac{1}{s}\right)\log\left(\log\left(1+\frac{1}{s}\right)\right)$$

and note that ω satisfies the Osgood condition but is not Log-Lipschitz continuous. Analogously to Sect. 3.2.1, it is possible to construct (see [1])

- a sequence $\{\mathscr{P}_k\}_{k\in\mathbb{N}}$ of backward uniformly parabolic operators with uniformly $\mathscr{C}^\omega$-continuous coefficients (not depending on the space variables) in the principal part and space-periodic uniformly bounded smooth coefficients in the lower order terms,
- a sequence $\{u_k\}_{k\in\mathbb{N}}$ of space-periodic smooth uniformly bounded solutions of $\mathscr{L}_k u_k = 0$ on $[0, 1] \times \mathbb{R}^2$,
- a sequence $\{t_k\}_{k\in\mathbb{N}}$ of real numbers, with $t_k \to 0$ as $k \to \infty$,

such that

$$\lim_{k\to\infty} \|u_k(0, \cdot, \cdot)\|_{L^2([0,2\pi]\times[0,2\pi])} = 0$$

but (8) does not hold for all k; more precisely

$$\lim_{k\to\infty} \frac{\|u_k(t_k, \cdot, \cdot)\|_{L^2([0,2\pi]\times[0,2\pi])}}{e^{-N|\log \|u_k(0,\cdot,\cdot)\|_{L^2([0,2\pi]\times[0,2\pi])}|^\delta}} = +\infty$$

for every $\delta > 0$.

3.3.2 Stability Result in the Osgood-Continuous Case

Let $\mathscr{L}$ be a backward parabolic operator whose coefficients depend only on t, i.e. let

$$\mathscr{L}u = \partial_t u + \sum_{i,j=1}^{n} a_{i,j}(t)\partial_{x_i}\partial_{x_j} u + \sum_{j=1}^{n} b_j(t)\partial_{x_j} u + c(t)u$$

on the strip $[0, T] \times \mathbb{R}^n$. Suppose that $a_{i,j}(t) = a_{j,i}(t)$ for all $i, j = 1, \ldots, n$ and for all $t \in [0, T]$. Let $a_{i,j}, b_j, c \in L^\infty([0, T])$, for all $i, j = 1, \ldots, n$. Let μ be a modulus of continuity satisfying the Osgood condition. Let $a_{i,j}$ $\mathscr{C}^\mu$-continuous, i.e.

$$\sup_{0<|\tau|<1} \frac{|a_{i,j}(t+\tau) - a_{i,j}(t)|}{\mu(|\tau|)} < +\infty\,.$$

Theorem 33 *For all $T' \in]0, T[$ and for all $D > 0$ there exist $\rho > 0$, and there exists an increasing continuous function $\Psi : [0, +\infty) \to [0, +\infty)$, with $\Psi(0) = 0$ such that, if $u \in \mathscr{H}_0$ is a solution of $\mathscr{L}u = 0$ on $[0, T]$ with $\|u(t, \cdot)\|_{L^2} \le D$ on $[0, T]$ and $\|u(0, \cdot)\|_{L^2} \le \rho$, then*

$$\sup_{t\in[0,T']} \|u(t, \cdot)\|_{L^2} \le \Psi(\|u(0, \cdot)\|_{L^2})\,. \tag{9}$$

The constant ρ and the function Ψ depend only on T', on D, on the ellipticity constant of $\mathscr{L}$, on the L^∞ norms of the coefficients $a_{i,j}$ and of their spatial first derivatives, and on the Osgood constant of the coefficients $a_{i,j}$. □

3.3.3 Comments on the Result and Its Proof

The complete proof of Theorem 33 is beyond the aims of this paper and is not reported here. However, to provide the reader with some insights about the proof, in the following we comment on the analogies and the differences between the new result and the previous ones [3, 5]. We begin by recalling that Theorem 32 is a consequence of the "energy" estimate (see Proposition 1 in [3])

$$\int_0^s e^{2\gamma t} e^{-2\beta\phi_\lambda((t+\tau)/\beta)} \|u(t,\cdot)\|^2_{H^{1-\alpha t}} dt \leq$$
$$\leq M\Big((s+\tau)e^{2\gamma s} e^{2\beta\phi_\lambda((s+\tau)/\beta)} \|u(s,\cdot)\|^2_{H^{1-\alpha s}} +$$
$$+ \tau\phi'_\lambda(\tau/\beta) e^{-2\beta\phi_\lambda(\tau/\beta)} \|u(0,\cdot)\|^2_{L^2}\Big), \qquad (10)$$

where ϕ_λ is the solution of the differential equation

$$y\phi''_\lambda(y) = -\lambda\phi'_\lambda(y)(1 + |\log\phi'(y)|). \qquad (11)$$

and the constants depend, in particular[1] on the Log-Lipschitz constant of the coefficients $a_{i,j}$ with respect to time. Now, the novelty of Theorem 33 is that the coefficients $a_{i,j}$ are supposed to be only Osgood-continuous, hence there is no Log-Lipschitz constant to be taken as a reference. On the other hand, the energy estimate will necessarily contain information on the modulus of continuity (which is assumed to verify the Osgood condition). Indeed, the energy estimate is

$$\frac{1}{4}\left(k_A|\xi|^2 + \gamma\right)\int_0^\sigma e^{(1-\alpha t)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{2\gamma t} e^{-2\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)} |\hat{u}(t,\xi)|^2 dt \leq$$
$$\leq \phi'_\lambda\left(\frac{\tau}{\beta}\right)\tau e^{|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)} |\hat{u}(0,\xi)|^2 +$$
$$+ (\sigma+\tau)(\gamma + k_A^{-1}|\xi|^2) e^{2\gamma\sigma} e^{-2\beta\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)} |\hat{u}(\sigma,\xi)|^2, \qquad (12)$$

[1] They also depend, as specified in the claim of the theorem, on T', on D, on the ellipticity constant of $\mathscr{L}$, on the L^∞ norms of the coefficients $a_{i,j}$ and of their spatial first derivatives. The parameter λ also depends on these quantities.

where, in particular, $\hat{u}$ denotes the Fourier transform of u with respect to x, ω is the modulus of continuity of the coefficients $a_{i,j}$, k_A is the ellipticity constant of the principal part of $\mathscr{L}$ and ϕ_λ is now the solution of the differential equation

$$y\phi_\lambda''(y) = -\lambda(\phi_\lambda'(y))^2\omega\left(\frac{k_A}{\phi_\lambda'(y)}\right), \tag{13}$$

where, again, the modulus of continuity appears. By comparing (10)–(11) with (12)–(13) one can see that Theorem 33 is not a trivial generalization of Theorem 32. In addition, (12) leads by integrating in ξ to the estimate

$$\sup_{z\in[0,\bar{\sigma}]} \|u(z,\cdot)\|^2_{H^1_{\frac{1}{2},\omega}} \leq Ce^{-\sigma\phi_\lambda'\left(\frac{\sigma+\tau}{\beta}\right)}\left[\phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|^2_{H^0_{1,\omega}} + \|u(\sigma,\cdot)\|_{H^1}\right], \tag{14}$$

where, in particular, the function spaces $H^1_{\frac{1}{2},\omega}$ and $H^0_{1,\omega}$ come into the scene. These spaces, defined by

$$\|u\|^2_{H^d_{a,\omega}} \triangleq \int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^d e^{a|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)}|\hat{u}(\xi)|^2 d\xi < +\infty,$$

are tailored on the modulus of continuity ω and, although comparable with Gevrey–Sobolev spaces, they do not coincide with any of them.

The final estimate (9), which is written with respect to the L^2 norm, can be obtained from (14) by exploiting the regularizing properties of the (forward) parabolic operator.

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Self-similar Asymptotic Profile for a Damped Evolution Equation

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Abstract In this note, we describe the self-similar asymptotic profile for evolution equations with strong, effective damping. We assume that initial data are in weighted Sobolev spaces, and the second data verifies suitable moment conditions. The asymptotic profile is obtained by means of the application of a differential operator given by a linear combination of Riesz potentials to the fundamental solution of a (polyharmonic) diffusive problem.

1 Introduction

In this note, we consider the Cauchy problem for the strongly damped polyharmonic evolution equation:

$$\begin{cases} u_{tt} + (-\Delta)^{m+1}u - \Delta u_t = 0, & t \geq 0, \ x \in \mathbb{R}^n, \\ (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

with $m > 1$, in space dimension $n \geq 1$, assuming $u_0 \in L^{1,1}$ and $u_1 \in L^{1,3}$, where $L^{1,\gamma}$ is the weighted space defined by:

$$L^{1,\gamma} \doteq L^1((1 + |x|)^\gamma \, dx).$$

We assume that u_1 verifies the moment conditions:

$$\int_{\mathbb{R}^n} u_1(x)\, dx = 0, \qquad \int_{\mathbb{R}^n} x_j u_1(x)\, dx = 0, \quad j = 1, \ldots, n. \tag{2}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_28

We denote by P_0 the moment of u_0 and by P_{jk} the second-order moments of u_1:

$$P_0 = \int_{\mathbb{R}^n} u_0(x)\,dx, \qquad P_{jk} = \int_{\mathbb{R}^n} x_j x_k u_1(x)\,dx, \quad j = 1, \dots, n. \tag{3}$$

We will show that the asymptotic profile of the solution to (1) is described by:

$$\varphi(t,x) = \Big(P_0 + \frac{1}{2}\sum_{j,k=1}^{n} P_{jk}\,R_j\,R_k\Big)\,G_m(t,x), \tag{4}$$

where G_m is the fundamental solution to the polyharmonic diffusive equation:

$$\begin{cases} v_t + (-\Delta)^m v = 0, & t \geq 0, x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{5}$$

and the operators R_j denote the j-th Riesz transform, that is:

$$R_j f = -i\mathfrak{F}^{-1}\big(|\xi|^{-1}\xi_j \hat{f}\big).$$

Our main result is the following.

Theorem 1 *Let $u_0 \in L^{1,1} \cap L^2$ and $u_1 \in L^{1,3} \cap L^2$. Assume that u_1 verifies* (2)*. Then, the solution to* (1) *satisfies*

$$\|u(t,\cdot) - \varphi(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n+2}{4m}}\big(\|u_0\|_{L^{1,1}\cap L^2} + \|u_1\|_{L^{1,3}\cap L^2}\big), \tag{6}$$

for any $t \geq 0$, where φ is as in (4)*, exception given for the case $n = 1$ and $m = 2$. In this latter case,* (6) *is replaced by:*

$$\|u(t,\cdot) - \varphi(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{1}{4}}\big(\|u_0\|_{L^{1,1}\cap L^2} + \|u_1\|_{L^{1,3}\cap L^2}\big). \tag{7}$$

By Theorem 1, it follows that $u(t,\cdot) \sim \varphi(t,\cdot)$ in L^2, provided that $\varphi(1,\cdot)$ is nontrivial, due to

$$\|\varphi(t,\cdot)\|_{L^2} = t^{-\frac{n}{4m}}\|\varphi(1,\cdot)\|_{L^1}.$$

Indeed, $\varphi(t,x) = t^{-\frac{n}{2m}}\varphi(1, t^{-\frac{1}{2m}}x)$. We also recall that the Riesz transform is a bounded operator in L^2.

In order to prove Theorem 1, we will first prove the diffusion phenomenon; that is, the asymptotic profile of the solution to (1) is described by the solution to (5), with initial data:

$$v_0 = u_0 + I_2 u_1, \tag{8}$$

where $I_2 = (-\Delta)^{-1}$ is the Riesz potential defined by:

$$I_2 f = \mathfrak{F}^{-1}\big(|\xi|^{-2}\,\hat{f}\big).$$

Theorem 2 *Let $u_0 \in L^1 \cap L^2$ and $u_1 \in L^{1,2} \cap L^2$. Assume (2). Then, the solution to (1) satisfies*

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n-4}{4m}-1}\big(\|u_0\|_{L^1\cap L^2} + \|u_1\|_{L^{1,2}\cap L^2}\big), \tag{9}$$

for $n \geq 4$, or

$$\|u(t,\cdot) - v(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n}{4}}\big(\|u_0\|_{L^1\cap L^2} + \|u_1\|_{L^{1,2}\cap L^2}\big), \tag{10}$$

in space dimension $n = 1, 2, 3$, for any $t \geq 0$, where v is the solution to (5) with initial data as in (8).

The diffusion phenomenon for evolution equations with structural damping, the asymptotic profile of the solution and the applications of the estimates to study nonlinear problems have been investigated in a recent series of paper [1–10, 13]. For corresponding results in the case of classical damping, we address the interested reader to [11, 12, 14, 15].

The novelty of this chapter is that assuming suitable moment conditions for initial data in weighted L^1 spaces, we describe the asymptotic profile of the solution by means of the application of a differential operator given by a linear combination of Riesz potentials to the fundamental solution of a (polyharmonic) diffusive problem.

2 Diffusion Phenomenon

We first present a useful lemma.

Lemma 1 *Let $f \in L^{1,\gamma}$, for some positive integer γ. Then,*

$$\Big|\hat{f}(\xi) - \sum_{|\alpha|\leq\gamma-1} (-i)^{|\alpha|}\,\frac{1}{\alpha!}\,\xi^\alpha\,P_\alpha(f)\Big| \leq C\,|\xi|^\gamma\,\|f\|_{L^{1,\gamma}},$$

where

$$P_\alpha = \int_{\mathbb{R}^n} x^\alpha f(x)\,dx.$$

Proof The proof of this lemma follows by the Taylor expansion:

$$\begin{aligned}\hat{f}(\xi) &= \hat{f}(0) + \sum_{|\alpha|\leq\gamma-1} \frac{1}{\alpha!}\xi^\alpha(\partial_\xi^\alpha \hat{f})(0) + \sum_{|\alpha|=\gamma} \frac{1}{\alpha!}\xi^\alpha(\partial_\xi^\alpha \hat{f})(\eta) \\ &= \sum_{|\alpha|\leq\gamma-1} (-i)^{|\alpha|}\frac{1}{\alpha!}\xi^\alpha P_\alpha(f) + \sum_{|\alpha|=\gamma} \frac{1}{\alpha!}\xi^\alpha(\partial_\xi^\alpha \hat{f})(\eta)\end{aligned}$$

for some η with $|\eta| \leq |\xi|$. This concludes the proof. □

We are now ready to prove Theorem 2.

Proof After applying the Fourier transform with respect to x to (1), we obtain

$$\begin{cases} u_{tt} + |\xi|^{2m+2}\hat{u} + |\xi|^2\hat{u}_t = 0, & t \geq 0, \\ (\hat{u}, \hat{u}_t)(0, x) = (\hat{u}_0, \hat{u}_1)(\xi). \end{cases} \tag{11}$$

For any $|\xi| < 4^{-2m+4}$, the solution to (11) is given by:

$$\hat{u}(t,\xi) = \hat{K}_0(t,\xi)\,\hat{u}_0 + \hat{K}_1(t,\xi)\hat{u}_1$$

where

$$\hat{K}_0 = \frac{\lambda_+ e^{\lambda_- t} - \lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-}, \qquad \hat{K}_1 = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \qquad \lambda_\pm = \frac{-|\xi|^2 \pm |\xi|^2\sqrt{1-4|\xi|^{2m-2}}}{2}.$$

In particular,

$$\begin{aligned}\lambda_- &= -|\xi|^2 + |\xi|^{2m} + |\xi|^{4m-2} + O(|\xi|^{6m-4}), &&\text{as } \xi \to 0, \\ \lambda_+ &= -|\xi|^{2m} - |\xi|^{4m-2} + O(|\xi|^{6m-4}), &&\text{as } \xi \to 0.\end{aligned}$$

Let $|\xi| < \delta$, for a sufficiently small $\delta > 0$. Then,

$$\begin{aligned}\Big|\frac{e^{\lambda_+ t}}{\lambda_+ - \lambda_-} - \frac{e^{-|\xi|^{2m}t}}{|\xi|^2}\Big| &\leq \Big|\frac{e^{\lambda_+ t} - e^{-|\xi|^{2m}t}}{\lambda_+ - \lambda_-}\Big| + \Big|e^{-|\xi|^{2m}t}\Big(\frac{1}{\lambda_+ - \lambda_-} - \frac{1}{|\xi|^2}\Big)\Big| \\ &\lesssim e^{-|\xi|^{2m}t}\Big|\frac{t(\lambda_+ - |\xi|^{2m})}{\lambda_+ - \lambda_-}\Big| + e^{-|\xi|^{2m}t}\Big|\frac{1}{\lambda_+ - \lambda_-} - \frac{1}{|\xi|^2}\Big| \\ &\lesssim e^{-|\xi|^{2m}t}\big(t|\xi|^{4m-4} + |\xi|^{2m-4}\big) \lesssim |\xi|^{2m-4}\,e^{-|\xi|^{2m}t}.\end{aligned}$$

On the other hand,

$$\left|\frac{e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \leq \left|\frac{e^{\lambda_- t} - e^{-t|\xi|^2}}{\lambda_+ - \lambda_-}\right| + \frac{e^{-t|\xi|^2}}{|\lambda_+ - \lambda_-|} \lesssim |\xi|^{-2} e^{-|\xi|^2 t}(t|\xi|^{2m} + 1) \lesssim |\xi|^{-2} e^{-|\xi|^2 t}.$$

Similarly, we obtain

$$\left|\frac{-\lambda_- e^{\lambda_+ t}}{\lambda_+ - \lambda_-} - e^{-|\xi|^{2m} t}\right| \lesssim |\xi|^{2m-2} e^{-|\xi|^{2m} t}, \qquad \left|\frac{\lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-}\right| \lesssim |\xi|^{2m-2} e^{-|\xi|^2 t}$$

By applying Lemma 1 to $f = u_1$ with $\gamma = 2$, we derive

$$|\hat{u}(t,\xi) - \hat{v}(t,\xi)| \lesssim \left(|\xi|^{2m-2} e^{-|\xi|^{2m} t} + e^{-|\xi|^2 t}\right)\left(\|u_0\|_{L^1} + \|u_1\|_{L^{1,2}}\right),$$

for any $|\xi| < \delta$, with a suitable small $\delta > 0$. By Plancherel's theorem, using the change of variable $\eta = t^{\frac{1}{2m}}\xi$, or $\eta = t^{\frac{1}{2}}\xi$, for $t \geq 1$, we obtain

$$\left(\int_{|\xi|<\delta} |\hat{u}(t,\xi) - \hat{v}(t,\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \lesssim \left(t^{-\frac{n-4}{4m}-1} + t^{-\frac{n}{4}}\right)\left(\|u_0\|_{L^1} + \|u_1\|_{L^{1,2}}\right).$$

On the other hand, for $t \in [0, 1]$, we trivially obtain

$$\left(\int_{|\xi|<\delta} |\hat{u}(t,\xi) - \hat{v}(t,\xi)|^2 \, d\xi\right)^{\frac{1}{2}} \lesssim \left(\|u_0\|_{L^1} + \|u_1\|_{L^{1,2}}\right).$$

For $|\xi| > \delta$, it is sufficient to employ

$$|\hat{u}(t,\xi)| + |\hat{v}(t,\xi)| \lesssim e^{-ct}\left(|\hat{u}_0(\xi)| + |\hat{u}_1(\xi)|\right)$$

for some $c = c(\delta) > 0$, together with Plancherel's theorem, to derive

$$\left(\int_{|\xi|>\delta} \left(|\hat{u}(t,\xi)| + |\hat{v}(t,\xi)|\right)^2 d\xi\right)^{\frac{1}{2}} \lesssim e^{-ct}\left(\|u_0\|_{L^2} + \|u_1\|_{L^2}\right).$$

This concludes the proof. □

3 Asymptotic Profile

In view of Theorem 2, the proof of Theorem 1 follows from

$$\|v(t,\cdot) - \varphi(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n+2}{4m}}\left(\|u_0\|_{L^{1,1}\cap L^2} + \|u_1\|_{L^{1,3}\cap L^2}\right). \tag{12}$$

Proof Let P_0, P_{jk} be as in (3). We notice that

$$\hat{\varphi}(t,\xi) = e^{-t|\xi|^{2m}}\Big(P_0 - \frac{1}{2|\xi|^2}\sum_{j,k}\xi_j\xi_k\,P_{jk}\Big).$$

By applying Lemma 1 with $f = u_0$ and $\gamma = 1$, and with $f = u_1$ and $\gamma = 3$, we derive

$$|\hat{v}(t,\xi) - \hat{\varphi}(t,\xi)| \lesssim |\xi|\,e^{-t|\xi|^{2m}}\big(\|u_0\|_{L^{1,1}} + \|u_1\|_{L^{1,3}}\big).$$

By Plancherel's theorem, using the change of variable $\eta = t^{\frac{1}{2m}}\xi$ for $t \geq 1$, we obtain

$$\Big(\int_{\mathbb{R}^n} |\hat{v}(t,\xi) - \hat{\varphi}(t,\xi)|^2\,d\xi\Big)^{\frac12} \lesssim t^{-\frac{n+2}{4m}}\big(\|u_0\|_{L^{1,1}\cap L^2} + \|u_1\|_{L^{1,3}\cap L^2}\big).$$

On the other hand, for $t \in [0,1]$, we estimate

$$\Big(\int_{|\xi|<1} |\hat{v}(t,\xi) - \hat{\varphi}(t,\xi)|^2\,d\xi\Big)^{\frac12} \lesssim \|u_0\|_{L^{1,1}} + \|u_1\|_{L^{1,3}},$$

and

$$\Big(\int_{|\xi|>1} \big(|\hat{v}(t,\xi)| + |\hat{\varphi}(t,\xi)|\big)^2\,d\xi\Big)^{\frac12} \lesssim \|u_0\|_{L^2} + \|u_1\|_{L^2}.$$

This concludes the proof. □

Acknowledgements The author has been supported by the INdAM—GNAMPA Project 2017 *Equazioni di tipo dispersivo e proprietà asintotiche.*

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On One Control Problem for Zakharov–Kuznetsov Equation

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Abstract An initial-boundary value problem for Zakharov–Kuznetsov equation $u_t + bu_x + u_{xxx} + u_{xyy} + uu_x = f_0(t)g(t, x, y)$ posed on a rectangle $(0, R) \times (0, L)$ for $t \in (0, T)$ under certain initial and boundary conditions is considered. Here, the function f_0 is unknown and is referred as a control, and the function g is given. The problem is to find a pair (f_0, u), satisfying the additional condition $\int_0^T u(t, x, y)\omega(x, y)\, dxdy = \varphi(t)$, where the functions ω, φ are given and u is the solution to the corresponding initial-boundary value problem. It is shown that under certain smallness assumptions on input data such a pair exists and is unique. For the corresponding linearized equation, a similar result is obtained without any smallness assumptions.

In the present chapter, we study control properties of an initial-boundary value problem for two-dimensional Zakharov–Kuznetsov equation with a special right side:

$$u_t + bu_x + u_{xxx} + u_{xyy} + uu_x = f_0(t)g(t, x, y), \tag{1}$$

$u = u(t, x, y)$, $b \in \mathbb{R}$, g is a given function, f_0 is an unknown control function, posed on a rectangle $\Omega = (0, R) \times (0, L)$ for $t \in (0, T)$ (R, L and T are arbitrary positive numbers) with an initial condition:

$$u(0, x, y) = u_0(x, y), \quad (x, y) \in \Omega, \tag{2}$$

boundary conditions for $(t, y) \in (0, T) \times (0, L)$:

$$u(t, 0, y) = u(t, R, y) = u_x(t, R, y) = 0 \tag{3}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_29

and boundary conditions for $(t, x) \in (0, T) \times (0, R)$ of one of the following four types:

$$\begin{aligned} &\text{whether} \quad && a)\ u(t,x,0) = u(t,x,L) = 0, \\ &\text{or} && b)\ u_y(t,x,0) = u_y(t,x,L) = 0, \\ &\text{or} && c)\ u(t,x,0) = u_y(t,x,L) = 0, \\ &\text{or} && d)\ u \text{ is an } L\text{-periodic function with respect to } y. \end{aligned} \tag{4}$$

We use the notation "problem (1)–(4)" for each of these four cases. The problem is to find a pair (f_0, u), satisfying the additional condition of integral overdetermination:

$$\int_\Omega u(t,x,y)\omega(x,y)\,dxdy = \varphi(t), \tag{5}$$

where the functions ω, φ are given and u is the solution to the corresponding problem (1)–(4). The main result of the chapter is that under certain assumptions (in particular, u_0 and φ are small) such a pair exists and is unique.

Zakharov–Kuznetsov equation is one of the variants of multi-dimensional generalizations of Korteweg–de Vries equation $u_t + bu_x + u_{xxx} + uu_x = f(t,x)$ and for the first time was derived in [1] for description of ion-acoustic waves in magnetized plasma. Results on global solubility and well-posedness of initial-boundary value problems, posed on a rectangle, for this equation were established in [2–4].

In [4], a result on boundary controllability for Zakharov–Kuznetsov equation, posed on a rectangle Ω, with the condition of final overdetermination $u(T,x,y) = u_T(x,y)$ similar to the one for Korteweg–de Vries equation from [5] was obtained. At the same time, it is shown in the book [6] that control problems with conditions of integral overdetermination, in particular of the type (5), are also very important and deserve investigation.

Let $L_p = L_p(\Omega)$. In accordance with boundary conditions (3) and (4) for $j = 1$ and $j = 2$, introduce special functional spaces $\widetilde{H}^j = \widetilde{H}^j(\Omega)$ as subspaces of $H^j(\Omega)$, consisting of functions ψ, such that $\psi|_{x=0} = \psi|_{x=R} = 0$ in all cases, $\psi|_{y=0} = \psi|_{y=L} = 0$ in the case a), $\psi_y|_{y=0} = \psi_y|_{y=L} = 0$ for $j = 2$ in the case b), $\psi|_{y=0} = 0$ and $\psi_y|_{y=L} = 0$ for $j = 2$ in the case c) and $\psi|_{y=0} = \psi|_{y=L}$ and $\psi_y|_{y=0} = \psi_y|_{y=L}$ for $j = 2$ in the case d).

Let $Q_T = (0,T) \times \Omega$. We consider weak solutions to problem (1)–(4) in the space:

$$X(Q_T) = C([0,T]; L_2) \cap L_2(0,T; \widetilde{H}^1)$$

(equipped with the natural norm) in the following sense.

Definition 1 Let $u_0 \in L_2$, $f_0 g \in L_1(0,T;L_2)$. A function $u \in X(Q_T)$ is called a weak solution to problem (1)–(4) if for any function $\phi \in L_2(0,T;\widetilde{H}^2)$, such that $\phi_t, \phi_{xxx}, \phi_{xyy} \in L_2(Q_T)$, $\phi\big|_{t=T} \equiv 0$, $\phi_x\big|_{x=0} \equiv 0$, the following equality holds:

$$\iiint_{Q_T} \Big[u(\phi_t + b\phi_x + \phi_{xxx} + \phi_{xyy}) + \frac{1}{2}u^2\phi_x + f_0 g\phi\Big]\,dxdydt + \iint_{\Omega} u_0\phi\big|_{t=0}\,dxdy = 0. \tag{6}$$

In the sequel, we need the following result from [4] on solubility of the initial-boundary value problem in Q_T with initial and boundary conditions (2)–(4) for a linear equation:

$$u_t + bu_x + u_{xxx} + u_{xyy} = f_1(t,x,y) + f_{2x}(t,x,y). \tag{7}$$

The notion of a weak solution here is similar to Definition 1.

Theorem 1 *Let $u_0 \in L_2$, $f_1 \in L_1(0,T;L_2)$, $f_2 \in L_2(Q_T)$. Then, there exists a unique solution to problem (7), (2)–(4) $u \in X(Q_T)$. Moreover, the operator $S : (u_0, f_1, f_2) \mapsto u$ is continuous in corresponding norms.*

Introduce the particular cases of the operator S, defined in the hypothesis of Theorem 1 and related to solutions of linear equation (7):

$$S_0 u_0 = S(u_0, 0, 0), \quad S_1 f_1 = S(0, f_1, 0), \quad S_2 f_2 = S(0, 0, f_2).$$

Remark 1 It is proved in [4] that the initial-boundary value problem for Zakharov–Kuznetsov equation itself

$$u_t + bu_x + u_{xxx} + u_{xyy} + uu_x = f(t,x,y) \tag{8}$$

with initial and boundary conditions (2)–(4) for $u_0 \in L_2$, $f \in L_1(0,T;L_2)$ is globally well-posed in the space $X(Q_T)$. Moreover, such a result is established there in the case of non-homogeneous boundary conditions (3). In this chapter, we use this result further only in the part concerning continuous dependence of solutions.

Further, we always assume that

$$\omega \in \widetilde{H}^2, \quad \omega_{xxx}, \omega_{xyy} \in L_2, \quad \omega_x\big|_{x=0} \equiv 0 \tag{9}$$

and

$$g \in C([0,T];L_2), \quad \Big|\iint_{\Omega} g(t,x,y)\omega(x,y)\,dxdy\Big| \ge g_0 = \text{const} > 0 \ \forall t \in [0,T]. \tag{10}$$

The main result of the chapter is the following theorem.

Theorem 2 *Let assumptions* (9), (10) *be satisfied,* $u_0 \in L_2$, $\varphi \in W_p^1(0,T)$ *for certain* $p \in [1, +\infty]$ *and*

$$\varphi(0) = \iint_\Omega u_0(x,y)\omega(x,y)\,dxdy. \tag{11}$$

Let

$$c_0 = \|u_0\|_{L_2} + \|\varphi'\|_{L_1(0,T)}. \tag{12}$$

Then, there exists $\delta > 0$*, such that if* $c_0 \le \delta$ *there exist a unique function* $f_0 \in L_p(0,T)$ *and the corresponding unique solution* $u \in X(Q_T)$ *to problem* (1)–(4)*, verifying* (5).

The proof of this result is presented further. First of all, we establish certain auxiliary linear results.

For $p \in [1, +\infty]$, let $\widetilde{W}_p^1(0,T) = \{\varphi \in W_p^1(0,T) : \varphi(0) = 0\}$ (obviously the equivalent norm in $\widetilde{W}_p^1(0,T)$ is $\|\varphi'\|_{L_p(0,T)}$). On the space of functions $u(t,x,y) \in L_1(\Omega)$ $\forall t \in [0,T]$, define the following linear operator Q:

$$(Qu)(t) = q(t) \equiv \iint_\Omega u(t,x,y)\omega(x,y)\,dxdy, \quad t \in [0,T].$$

Lemma 1 *Let the hypothesis of Theorem 1 be satisfied and, in addition,* $f_1 \in L_p(0,T;L_2)$*,* $f_2 \in L_p(0,T;L_1)$ *for certain* $p \in [1,+\infty]$*. Then for the function* $u = S(u_0, f_1, f_2)$ *and the function* ω*, satisfying* (9)*, the corresponding function* $q = Qu \in W_p^1(0,T)$ *and for a.e.* $t \in (0,T)$*:*

$$q'(t) = \iint_\Omega \big[u(b\omega_x + \omega_{xxx} + \omega_{xyy}) + f_1\omega - f_2\omega_x\big]\,dxdy. \tag{13}$$

Moreover,

$$\|q'\|_{L_p(0,T)} \le c(T)\big(\|u_0\|_{L_2} + \|f_1\|_{L_p(0,T;L_2)} + \|f_2\|_{L_2(Q_T)} + \|f_2\|_{L_p(0,T;L_1)}\big). \tag{14}$$

Proof For an arbitrary function $\nu \in C_0^\infty(0,T)$, let $\phi(t,x,y) \equiv \nu(t)\omega(x,y)$. This function verifies the hypothesis of Definition 1 and according to the corresponding analog of (6):

$$\int_0^T \nu'(t)q(t)\,dt = -\int_0^T \nu(t)r(t)\,dt$$

for $r(t)$ equal to the right side of (13). In particular, $q'(t) = r(t)$ (in the sense of distributions). Note that $\omega_x \in L_\infty$. Therefore,

$$|r(t)| \le c\big(\|u(t,\cdot,\cdot)\|_{L_2} + \|f_1(t,\cdot,\cdot)\|_{L_2} + \|f_2(t,\cdot,\cdot)\|_{L_1}\big).$$

Theorem 1 yields that

$$\|u\|_{C([0,T];L_2)} \le c\big(\|u_0\|_{L_2} + \|f_1\|_{L_1(0,T;L_2)} + \|f_2\|_{L_2(Q_T)}\big).$$

As a result, $r \in L_p(0,T)$ and can be estimated by the right side of (14). □

Lemma 2 *Let $u = S_1 f_1$ for $f_1 \in L_1(0,T;L_2)$, then for $t \in [0,T]$:*

$$\iint_\Omega u^2(t,x,y)\,dxdy \le 2\int_0^t \iint_\Omega f_1 u\,dxdyd\tau. \tag{15}$$

Proof For smooth solutions (see [4]), multiplying the corresponding Eq. (7) by $2u(t,x,y)$ and integrating yield:

$$\frac{d}{dt}\iint_\Omega u^2\,dxdy + \int_0^L u_x^2\big|_{x=0}\,dy = 2\iint_\Omega f_1 u\,dxdy,$$

whence (15) obviously follows. The general case is obtained via closure. □

The next assertion is crucial for our study.

Lemma 3 *Let assumptions (9), (10) be satisfied, $\varphi \in \widetilde W^1_p(0,T)$ for certain $p \in [1,+\infty]$. Then, there exists a unique function $f_0 = \Gamma\varphi$, such that the function $u = S_1(f_0 g)$ verifies (5). Moreover, the linear operator $\Gamma : \widetilde W^1_p(0,T) \to L_p(0,T)$ is bounded.*

Proof For any function f_0, defined on $(0,T)$, we set $Gf_0 \equiv f_0 g$. On the space $L_p(0,T)$, we define a linear operator $\Lambda = Q \circ S_1 \circ G$. Then according to Lemma 1 and continuity of the operator S_1 (in particular, as the operator from $L_1(0,T;L_2)$ into $C([0,T];L_2)$), the operator Λ maps $L_p(0,T)$ into $\widetilde W^1_p(0,T)$ and is bounded.

Note that an equality $\varphi = \Lambda f_0$ for $f_0 \in L_p(0,T)$ is obviously equal to the fact that the function f_0 is the desired control function. Let

$$g_1(t) \equiv \iint_\Omega g(t,x,y)\omega(x,y)\,dxdy, \quad |g_1(t)| \ge g_0. \tag{16}$$

Following the ideas from [6], introduce an operator $A : L_p(0,T) \to L_p(0,T)$ in the following way:

$$(Af_0)(t) \equiv \frac{\varphi'(t)}{g_1(t)} - \frac{1}{g_1(t)}\iint_\Omega u(t,x,y)(b\omega_x + \omega_{xxx} + \omega_{xyy})\,dxdy, \quad u = (S_1 \circ G)f_0.$$

It is easy to see that equality (13) ensures that $\varphi = \Lambda f_0$ iff $f_0 = A f_0$. Now, we show that the operator A is a contraction in $L_p(0,T)$, if we choose a special norm in this space.

Let $f_{01}, f_{02} \in L_p(0,T)$, $u_j = (S_1 \circ G) f_{0j}$, then according to (15) for $t \in [0,T]$:

$$\|u_1(t,\cdot,\cdot) - u_2(t,\cdot,\cdot)\|_{L_2} \le 2\|g\|_{C([0,T];L_2)} \|f_{01} - f_{02}\|_{L_1(0,t)}.$$

Let $\gamma > 0$, then for $p < +\infty$:

$$\begin{aligned} &\|e^{-\gamma t}(A f_{01} - A f_{02})\|_{L_p(0,T)} \\ &\le \frac{1}{g_0}\big(|b|\|\omega_x\|_{L_2} + \|\omega_{xxx}\|_{L_2} + \|\omega_{xyy}\|_{L_2}\big)\Big(\int_0^T e^{-p\gamma t}\|u_1 - u_2\|_{L_2}^p\,dt\Big)^{1/p} \\ &\le c\Big[\int_0^T e^{-p\gamma t}\int_0^t |f_{01}(\tau) - f_{02}(\tau)|^p\,d\tau dt\Big]^{1/p} \\ &\le \frac{c}{(p\gamma)^{1/p}}\|e^{-\gamma t}(f_{01} - f_{02})\|_{L_p(0,T)}, \end{aligned}$$

while for $p = +\infty$:

$$\begin{aligned} &\|e^{-\gamma t}(A f_{01} - A f_{02})\|_{L_\infty(0,T)} \\ &\le \frac{1}{g_0}\big(|b|\|\omega_x\|_{L_2} + \|\omega_{xxx}\|_{L_2} + \|\omega_{xyy}\|_{L_2}\big)\sup_{t\in[0,T]} e^{-\gamma t}\|u_1 - u_2\|_{L_2} \\ &\le c\sup_{t\in[0,T]} e^{-\gamma t}\|f_{01} - f_{02}\|_{L_1(0,t)} \le \frac{c}{\gamma}\|e^{-\gamma t}(f_{01} - f_{02})\|_{L_\infty(0,T)}, \end{aligned}$$

where the constants c are independent on γ. Thus, in both cases for sufficiently large γ the operator A is a contraction and, therefore, for any function $\varphi \in \widetilde{W}_p^1(0,T)$ there exists a unique function $f_0 \in L_p(0,T)$, verifying $f_0 = A f_0$, that is $\varphi = \Lambda f_0$. It means that the operator Λ is invertible; moreover according to Banach theorem, the inverse operator $\Gamma = \Lambda^{-1} : \widetilde{W}_p^1(0,T) \to L_p(0,T)$ is bounded. In particular,

$$\|\Gamma\varphi\|_{L_p(0,T)} \le c(T)\|\varphi'\|_{L_p(0,T)}. \tag{17}$$

□

Now, we can formulate a result on control properties in the linear case.

Theorem 3 *Let assumptions (9), (10) be satisfied,* $u_0 \in L_2$, $f_2 \in L_p(0,T;L_1) \cap L_2(Q_T)$, $\varphi \in W_p^1(0,T)$ *for certain* $p \in [1,+\infty]$ *and equality (11) holds true. Then, there exists a unique function* $f_0 \in L_p(0,T)$*, such that the function* $u = S(u_0, f_0 g, f_2)$ *verifies (5).*

Proof Let

$$\widetilde{\varphi} \equiv \varphi - Q(S_0 u_0 + S_2 f_2).$$

By virtue of Lemma 1 and equality (11), the function $\widetilde{\varphi} \in \widetilde{W}_p^1(0, T)$. Then, Lemma 3 provides that $f_0 = \Gamma\widetilde{\varphi}$ is the desired control. In particular, note that the function $u = S(u_0, f_0 g, f_2)$ can be expressed by the formula:

$$u = S_0 u_0 + (S_1 \circ G \circ \Gamma)(\varphi - Q(S_0 u_0 + S_2 f_2)) + S_2 f_2. \tag{18}$$

Uniqueness of the function f_0 also follows from Lemma 3. □

Now, we can pass to the proof of the main result.

Proof (Proof of Theorem 2) For an arbitrary function $v \in X(Q_T)$, let $f_2 \equiv -v^2/2$. It is obvious that $f_2 \in C([0, T], L_1)$. Next, the following interpolating inequality from [7]: for $\psi \in H^1$

$$\|\psi\|_{L_4} \le c\|\psi\|_{H^1}^{1/2}\|\psi\|_{L_2}^{1/2},$$

provides that

$$\|v^2\|_{L_2(Q_T)} \le c\|v\|_{C([0,T];L_2)}\|v\|_{L_2(0,T;H^1)} \le c\|v\|^2_{X(Q_T)}. \tag{19}$$

In particular, the hypothesis of Theorem 3 (for such function f_2) is satisfied.

On the space $X(Q_T)$, consider a map:

$$u = \Theta v \equiv S_0 u_0 + (S_1 \circ G \circ \Gamma)\big(\varphi - Q(S_0 u_0 - S_2(v^2/2))\big) - S_2(v^2/2).$$

Note that if $q(t) \equiv Q(S_0 u_0 - S_2(v^2/2))$, then according to Lemma 1 and (19):

$$\|q'\|_{L_\infty(0,T)} \le c(T)\big(\|u_0\|_{L_2} + \|v\|^2_{X(Q_T)}\big).$$

Moreover, $q(0) = \varphi(0)$. Therefore, the map Θ is well-defined and by virtue of Theorem 1 and Lemma 3:

$$\|\Theta v\|_{X(Q_T)} \le c(T)\big(c_0 + \|v\|^2_{X(Q_T)}\big),$$

$$\|\Theta v_1 - \Theta v_2\|_{X(Q_T)} \le c(T)\big(\|v_1\|_{X(Q_T)} + \|v_2\|_{X(Q_T)}\big)\|v_1 - v_2\|_{X(Q_T)}.$$

Choose

$$r = \frac{1}{4c(T)}, \quad \delta = \frac{1}{8c^2(T)},$$

then $c(T)\delta \le r/2$, $c(T)r \le 1/4$ and for $c_0 \le \delta$ the map Θ is a contraction on the ball $\overline{U}_r(0)$ in $X(Q_T)$. Theorem 3 provides that the unique fixed point $u = \Theta u \in X(Q_T)$ verifies (1)–(5) for $f_0 \equiv \Gamma\big(\varphi - Q(S_0 u_0 - S_2(u^2/2))\big) \in L_p(0,T)$.

In order to prove global uniqueness, we apply the following continuous dependence result from [4]: if $u, \widetilde{u} \in X(Q_T)$ are weak solutions to initial-boundary value problems for Eq. (8) with right sides $f, \widetilde{f} \in L_1(0,T;L_2)$ and the same initial-boundary conditions (2)–(4), then

$$\|u - \widetilde{u}\|_{X(Q_T)} \le c\|f - \widetilde{f}\|_{L_1(0,T;L_2)}, \tag{20}$$

where the constant c depends on T and the norms of $u, \widetilde{u}$ in $X(Q_T)$ and does not decrease with respect to them.

Let two pairs $(f_{01}, u_1), (f_{02}, u_2) \in L_1(0,T) \times X(Q_T)$ solve the same problem (1)–(5). Applying Lemma 1 for $u \equiv u_1 - u_2$, $f_1 \equiv (f_{01} - f_{02})g$, $f_2 \equiv -(u_1^2 - u_2^2)/2$, we derive from (13), that for $t \in [0,T]$:

$$\iint_\Omega \big[(u_1 - u_2)(b\omega_x + \omega_{xxx} + \omega_{xyy}) + (f_{01} - f_{02})g\omega + (u_1^2 - u_2^2)\omega_x/2\big]\,dxdy = 0.$$

With the use of (16) and (20), this equality yields that

$$\big(f_{01}(t) - f_{02}(t)\big)g_1(t) = -\iint_\Omega (u_1 - u_2)\big(b\omega_x + \omega_{xxx} + \omega_{xyy} + (u_1 + u_2)\omega_x/2\big)\,dxdy$$

and so,

$$\big|f_{01}(t) - f_{02}(t)\big| \le \frac{c}{g_0}\|u_1(t,\cdot,\cdot) - u_1(t,\cdot,\cdot)\|_{L_2} \le c_1\|f_{01} - f_{02}\|_{L_1(0,t)},$$

where the constant c_1 is uniform with respect to t. Therefore, for any $t_0 \in (0,T]$:

$$\|f_{01} - f_{02}\|_{L_1(0,t_0)} \le c_1 t_0 \|f_{01} - f_{02}\|_{L_1(0,t_0)}.$$

As a result, $f_{01}(t) = f_{02}(t)$ for $t \in [0, 1/(2c_1)]$ and, therefore, $u_1 = u_2$ in Q_{t_0}. Transferring the time origin to the point t_0, we obtain the same result for $t \in [t_0, 2t_0]$ and so on. □

Remark 2 In the case $p = +\infty$ in Theorems 2 and 3, the spaces $L_p(0,T;L_1)$, $W_1^p(0,T)$ and $L_p(0,T)$ can be simultaneously substituted by $C([0,T];L_1)$, $C^1[0,T]$ and $C[0,T]$.

Remark 3 The corresponding control results, similar to Theorems 2 and 3, can be obtained also under non-homogeneous boundary data (3).

Acknowledgements The publication was supported by the Ministry of Education and Science of the Russian Federation (Project 1.962.2017/PCh) and RFBR grants 17-01-00849, 17-51-52022 and 18-01-00590.

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The Self-interacting Scalar Field Propagating in FLRW Model of the Contracting Universe

Anahit Galstian and Karen Yagdjian

Abstract We present a condition on the self-interaction term that guaranties the existence of the global-in-time solution of the Cauchy problem for the semilinear Klein–Gordon equation in the FLRW model of the contracting universe. For the equation with the Higgs potential, we give an estimate for the lifespan of solution.

1 Introduction and Statement of Results

In the present chapter, we prove the global-in-time existence of the solutions of the Cauchy problem for the semilinear Klein–Gordon equation in the FLRW (Friedmann–Lemaître–Robertson–Walker) space–time of the contracting universe for the self-interacting scalar field.

The metric g in the FLRW space–time of the contracting universe in the Lamaître–Robertson coordinates (see, e.g., [10]) is defined as follows, $g_{00} = g^{00} = -1$, $g_{0j} = g^{0j} = 0$, $g_{ij}(x,t) = e^{-2t}\sigma_{ij}(x)$, $i, j = 1, 2, \dots, n$, where $\sum_{j=1}^{n} \sigma^{ij}(x)\sigma_{jk}(x) = \delta_{ik}$, and δ_{ij} is Kronecker's delta. The metric $\sigma^{ij}(x)$ describes the time slices. The covariant Klein–Gordon equation in that space–time in the coordinates is

$$\psi_{tt} - \frac{e^{2t}}{\sqrt{|\det\sigma(x)|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\sqrt{|\det\sigma(x)|}\sigma^{ij}(x)\frac{\partial}{\partial x^j}\psi\right) - n\psi_t + m^2\psi = F(\psi)\,. \tag{1.1}$$

It is obvious that the properties of this equation and of its solutions are not time invertible. In the present chapter, we are interested in the Cauchy problem, which, in fact, is not equivalent to the time backward problem for the equation with the

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_30

reflected time $t \to -t$:

$$\psi_{tt} - \frac{e^{-2t}}{\sqrt{|\det \sigma(x)|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{|\det \sigma(x)|} \sigma^{ij}(x) \frac{\partial}{\partial x^j} \psi \right) + n\psi_t + m^2 \psi = F(\psi) \,. \tag{1.2}$$

The last equation is the semilinear Klein–Gordon equation in the de Sitter space–time. Equation (1.2) is well investigated, and the conditions for the existence of small data global-in-time solutions for some important σ are discovered [1, 2, 5–8, 11, 18, 19].

Equation (1.1) is a special case of the equation:

$$\psi_{tt} - e^{2t} A(x, \partial_x)\psi - n\psi_t + m^2 \psi = F(\psi) \,, \tag{1.3}$$

where $A(x, \partial_x) = \sum_{|\alpha| \le 2} a_\alpha(x) \partial_x^\alpha$ is a second-order elliptic partial differential operator. We also assume that the mass m can be a complex number, $m^2 \in \mathbb{C}$.

In the present chapter, we study also the class of equations containing, in particular, the Higgs boson equation with the Higgs potential, that is the equation:

$$\psi_{tt} - e^{2t} A(x, D)\psi - n\psi_t = \mu^2 \psi - \lambda \psi^3, \tag{1.4}$$

with $\lambda > 0$ and $\mu > 0$, while $n = 3$.

To formulate the main theorem of this chapter, we need a characterization of the nonlinear term F. We want to stress here that the explicit form of F is not used merely. There are estimates of the form $\|F(\psi)\|_X < C\|\psi\|_{X'}^\alpha \|\psi\|_{X''}$, for some function spaces X, X', and X''. Furthermore, since we prove that for small data the solution is bounded in $L^{p'}$-norm, we are only concerned with the behavior of F at the origin. Let $B_p^{s,q}$ denote the Besov space.

Condition ($\mathscr{L}$) *The smooth in x function $F = F(x, \psi)$ is said to be Lipschitz continuous with exponent $\alpha \ge 0$ in the space $B_p^{s,q}$ if there is a constant $C \ge 0$ such that*

$$\|F(x, \psi_1(x)) - F(x, \psi_2(x))\|_{B_p^{s,q}} \le C\|\psi_1 - \psi_2\|_{B_{p'}^{s,q}} \left(\|\psi_1\|_{B_{p'}^{s,q}}^\alpha + \|\psi_2\|_{B_{p'}^{s,q}}^\alpha \right)$$

for all $\psi_1, \psi_2 \in B_{p'}^{s,q}$, *where* $1/p + 1/p' = 1$.

The polynomial in ψ functions $F(x, \psi) = \pm|\psi|^{\alpha+1}$ and $F(\psi) = \pm|\psi|^\alpha \psi$ are important examples of the Lipschitz continuous with exponent $\alpha > 0$ in the Lebesgue spaces $L^p(\mathbb{R}^n)$ and the Sobolev space $H_{(s)}(\mathbb{R}^n)$, $s > n/2$, functions.

Define also the metric space:

$$X(R, B_p^{s,q}, \gamma) := \left\{ \psi \in C([0,\infty); B_p^{s,q} \;\middle|\; \| \psi \|_X := \sup_{t\in[0,\infty)} e^{\gamma t} \| \psi(x,t) \|_{B_p^{s,q}} \le R \right\},$$

where $\gamma \in \mathbb{R}$, with the metric $d(\psi_1, \psi_2) := \sup_{t\in[0,\infty)} e^{\gamma t} \| \psi_1(x,t) - \psi_2(x,t) \|_{B_p^{s,q}}$.

We study the Cauchy problem through the integral equation. To determine that integral equation, we appeal to the operator:

$$G := \mathscr{K} \circ \mathscr{E}\mathscr{E}$$

($\mathscr{E}\mathscr{E}$ stands for the evolution equation) that is designed as follows. For the function $f(x,t)$, we define

$$v(x,t;b) := \mathscr{E}\mathscr{E}[f](x,t;b),$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem:

$$\partial_t^2 v - A(x,D)v = 0, \quad x \in \mathbb{R}^n, \quad t \ge 0, \tag{1.5}$$

$$v(x,0;b) = f(x,b), \quad v_t(x,0;b) = 0, \quad x \in \mathbb{R}^n, \tag{1.6}$$

while the integral transform $\mathscr{K}$ is introduced by:

$$\mathscr{K}[v](x,t) := 2e^{\frac{n}{2}t} \int_0^t db \int_0^{e^t - e^b} dr\, e^{-\frac{n}{2}b} v(x,r;b) E(r,t;0,b;M).$$

Here, the principal square root $M := (n^2/4 - m^2)^{\frac{1}{2}}$ is the main parameter that controls estimates and solvability of the integral equation. The last will be obtained by means of the integral transform. The kernel $E(r,t;0,b;M)$ was introduced in [16, 20] (see also (2.2)). Hence,

$$G[f](x,t) = 2e^{\frac{n}{2}t} \int_0^t db \int_0^{e^t - e^b} dr\, e^{-\frac{n}{2}b}\, \mathscr{E}\mathscr{E}[f](x,r;b) E(r,t;0,b;M).$$

Obviously, the Cauchy problem for Eq. (1.3) leads to the following integral equation:

$$\Phi(x,t) = \Phi_0(x,t) + G[e^{-\Gamma \cdot} F(\cdot, \Phi)](x,t) \tag{1.7}$$

where $\Gamma = 0$. Φ_0 is generated by the initial value problem (1.8), (1.9) with $F \equiv 0$.

We define the solution of the Cauchy problem through the last integral equation. For the real numbers γ and Γ, we define

$$I(t) := e^{t(\frac{n}{2}+\Re M+\gamma)} \int_0^t e^{-(\frac{n}{2}+\Re M+\gamma(\alpha+1)+\Gamma)b}\, db\,.$$

The main result of this chapter is the following theorem. The next theorem states the existence of global-in-time solution for small initial data in Sobolev spaces.

Theorem 1 *Assume that $A(x,\partial_x)$ is the Laplace operator on $\mathbb{R}^n$, and the nonlinear term $F(\Phi)$ is Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$, $s > n/2 \geq 1$, $F(x,0) \equiv 0$, and $\alpha > 0$.*
(GS) *Assume also that $\Re M > 0$ and that $M = \Re M$ if $\Re M = 1/2$, one of the following three conditions is fulfilled:*

$$(i)\ \ \frac{n}{2}+\Re M+\gamma(\alpha+1)+\Gamma > 0, \quad \frac{n}{2}+\max\{\frac{1}{2},\Re M\}+\gamma \leq 0,$$

$$(ii)\ \ \frac{n}{2}+\Re M+\gamma(\alpha+1)+\Gamma = 0, \quad \frac{n}{2}+\max\{\frac{1}{2},\Re M\}+\gamma < 0,$$

$$(iii)\ \ \frac{n}{2}+\Re M+\gamma(\alpha+1)+\Gamma < 0, \quad \frac{n}{2}+\max\{\frac{1}{2},\Re M\}+\gamma \leq 0, \quad \gamma\alpha+\Gamma \geq 0.$$

Then, there exists $\varepsilon_0 > 0$ such that, for every given functions $\varphi_0, \varphi_1 \in H_{(s)}(\mathbb{R}^n)$, satisfying estimate:

$$\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon, \qquad \varepsilon < \varepsilon_0\,,$$

there exists a solution $\Phi \in C([0,\infty); H_{(s)}(\mathbb{R}^n))$ of the Cauchy problem:

$$\Phi_{tt} - n\Phi_t - e^{2t}A(x,\partial_x)\Phi + m^2\Phi = e^{-\Gamma t}F(x,\Phi)\,, \tag{1.8}$$

$$\Phi(x,0) = \varphi_0(x)\,, \quad \Phi_t(x,0) = \varphi_1(x)\,. \tag{1.9}$$

The solution $\Phi(x,t)$ belongs to the space $X(2\varepsilon, H_{(s)}(\mathbb{R}^n),\gamma)$, that is:

$$\sup_{t\in[0,\infty)} e^{\gamma t}\|\Phi(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \leq 2\varepsilon\,.$$

(LS) *If $\Re M > 0$, $\frac{n}{2}+\max\{\frac{1}{2},\Re M\}+\gamma \leq 0$, and neither of the three conditions (i)–(iii) is fulfilled, then the lifespan T_{ls} of the solution can be estimated from below as follows:*

$$T_{ls} \geq \mathscr{I}\left(C_0(M,n,\alpha,\gamma,\Gamma)^{-1}\left(\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)}\right)^{-\alpha}\right).$$

with some constant $C_0(M,n,\alpha,\gamma,\Gamma)$ when $\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)}$ is sufficiently small. Here, $\mathscr{I}$ is the function inverse to $I = I(t)$.

The theorem covers the equations with $F(\Phi) = \pm|\Phi|^{\alpha+1}$ and $F(\Phi) = \pm|\Phi|^{\alpha}\Phi$. The last theorem implies also the existence of the energy class solution. The sharpness of the condition on α is an interesting open problem that will not be discussed here. In particular, the theorem states an estimate for the lifespan T_{ls} of the solution of the Higgs boson equation (1.4) in the contracting universe.

In order to prove theorem, we establish the estimates in the Besov spaces for the linear equation. For the equation without source term, these estimates for large time t imply the limitation for the rate of growth as follows:

$$\|\Phi(x,t)\|_{X'} \le \|\varphi_0\|_X e^{(\frac{n}{2}+a+\Re M)t} \begin{cases} 1 & \text{if} \quad \Re M > 1/2 \\ t^{\operatorname{sgn}|\Im M|} + e^{(\frac{1}{2}-\Re M)t} & \text{if} \quad \Re M \le 1/2 \end{cases}$$
$$+\|\varphi_1\|_X e^{(\frac{n}{2}+a+\Re M)t},$$

where if $X = B_p^{s,q}$, then $X' = B_{p'}^{s',q}$, $a := s - s' - 2n\,(1/p - 1/2)$, $1/p + 1/p' = 1$, while $X' = L^{p'}$ if $X = L^p$. In the case of Sobolev spaces, $X = X' = H_{(s)}(\mathbb{R}^n)$, $p = 2$.

The integral transform $\mathscr{K}$ allows us to avoid consideration in the phase space and to apply immediately the well-known decay estimates for the solution of the wave equation (operator $\mathscr{E}\mathscr{E}$) (see, e.g., [3]).

Ebert and do Nascimento [4] study the long-time behavior of the energy of solutions for a class of linear equations with time-dependent mass and speed of propagation. They introduce a classification of the potential term, which clarifies whether the solution behaves like the solution to the wave equation or Klein–Gordon equation. For the equation:

$$u_{tt} - e^{2t}\Delta u + m^2 u = |u|^p,$$

with $n \le 4$, $m > 0$, $2 \le p \le \frac{n}{[n-2]_+}$ they establish the existence of energy class solution for small data. Their proof is based on the splitting of the phase space into pseudo-differential and hyperbolic zones. That method of zones was invented for the hyperbolic operators with multiple characteristics (see [15]) and then modified and successfully used to study equations in the unbounded time domain (see [9, 12–14], and references therein).

In the next section, we will give outline of the proof of Theorem 1. The complete proof and sharpness of the obtained results will be published in the forthcoming paper.

2 Outline of the Proof of Theorem 1

The following partial Liouville transform (change of unknown function) $u = e^{-\frac{n}{2}t}\psi$, $\psi = e^{\frac{n}{2}t}u$, eliminates the term with time derivative of Eq. (1.8). We obtain

$$u_{tt} - e^{2t}A(x, \partial_x)u + \left(m^2 - \frac{n^2}{4}\right)u = e^{(-\frac{n}{2}-\Gamma)t}F(e^{\frac{n}{2}t}u)\,,$$

which can be written as follows:

$$u_{tt} - e^{2t}A(x, \partial_x)u - M^2u = e^{(-\frac{n}{2}-\Gamma)t}F(e^{\frac{n}{2}t}u)\,,$$

where $M = \left(n^2 - 4m^2\right)^{\frac{1}{2}}/2$. We consider the linear part of the equation:

$$u_{tt} - e^{2t}A(x, D)u - M^2u = -e^{-\frac{n}{2}t}V'(e^{\frac{n}{2}t}u), \tag{2.1}$$

with $M \in \mathbb{C}$. Equation (2.1) covers two important cases. The first one is the Higgs boson equation, which has $V'(\phi) = \lambda\phi^3$ and $M^2 = n^2/4 + \mu^2$ with $\lambda > 0$, $\mu > 0$, and $n = 3$. This includes also equation of *tachyonic scalar fields* living on the de Sitter universe. The second case is the case of the small physical mass (the light scalar field), that is $0 \le m \le \frac{n}{2}$. For the last case, $M = \sqrt{n^2 - 4m^2}/2$.

We introduce the kernel functions $E(x, t; x_0, t_0; M)$, $K_0(z, t; M)$, and $K_1(z, t; M)$ (see also [16, 20]). First, for $M \in \mathbb{C}$ we define the function:

$$\begin{aligned} E(x, t; x_0, t_0; M) = {} & 4^{-M}e^{-M(t_0+t)}\Big((e^t + e^{t_0})^2 - (x - x_0)^2\Big)^{-\frac{1}{2}+M} \\ & \times F\Big(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t - e^{t_0})^2 - (x - x_0)^2}{(e^t + e^{t_0})^2 - (x - x_0)^2}\Big). \end{aligned} \tag{2.2}$$

Next, we define also the kernels $K_0(z, t; M)$ and $K_1(z, t; M)$ by:

$$K_0(z, t; M) := -\left[\frac{\partial}{\partial b}E(z, t; 0, b; M)\right]_{b=0} \quad \text{and} \quad K_1(z, t; M) := E(z, t; 0, 0; M)\,.$$

The solution $u = u(x, t)$ to the Cauchy problem:

$$u_{tt} - e^{2t}A(x, D)u - M^2u = f, \qquad u(x, 0) = \varphi_0(x)\,, \quad u_t(x, 0) = \varphi_1(x)\,,$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, is given in [17] by the next expression:

$$u(x,t) = 2\int_0^t db \int_0^{e^t - e^b} dr\, v(x,r;b)E(r,t;0,b;M) \tag{2.3}$$
$$+e^{-\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) + 2\int_0^1 v_{\varphi_0}(x,\phi(t)s)K_0(\phi(t)s,t;M)\phi(t)\,ds$$
$$+2\int_0^1 v_{\varphi_1}(x,\phi(t)s)K_1(\phi(t)s,t;M)\phi(t)\,ds, \quad x \in \mathbb{R}^n,\ t > 0,$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem (1.5) and (1.6), while $\phi(t) := e^t - 1$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem for Eq. (1.5) with the first initial datum $\varphi(x)$, while the second datum is zero. Thus, for the solution Φ of the Cauchy problem:

$$\Phi_{tt} - n\Phi_t - e^{2t} \triangle \Phi + m^2\Phi = f, \quad \Phi(x,0) = \varphi_0(x), \quad \Phi_t(x,0) = \varphi_1(x),$$

due to the relation $u = e^{-\frac{n}{2}t}\Phi$, we obtain from (2.3)

$$\Phi(x,t) = 2e^{\frac{n}{2}t} \int_0^t db \int_0^{e^t-e^b} dr\, e^{-\frac{n}{2}b} v(x,r;b)E(r,t;0,b;M) + e^{\frac{n-1}{2}t} v_{\varphi_0}(x,\phi(t))$$
$$+ e^{\frac{n}{2}t} \int_0^1 v_{\varphi_0}(x,\phi(t)s)\big(2K_0(\phi(t)s,t;M) - nK_1(\phi(t)s,t;M)\big)\phi(t)\,ds$$
$$+ 2e^{\frac{n}{2}t} \int_0^1 v_{\varphi_1}(x,\phi(t)s)K_1(\phi(t)s,t;M)\phi(t)\,ds, \quad x \in \mathbb{R}^n,\ t > 0,$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem (1.5) and (1.6), while the function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem for Eq. (1.5) with the initial datum $\varphi(x)$, while the second datum is zero.

$B_p^{s,q} - B_{p'}^{s',q}$ Estimates for Equation Without Source Let $\varphi_j = \varphi(2^{-j}\xi)$, $j > 0$, and $\varphi_0 = 1 - \sum_{j=1}^\infty \varphi_j$, where $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\varphi \geq 0$ and $\operatorname{supp}\varphi \subseteq \{\xi \in \mathbb{R}^n\,;\ 1/2 < |\xi| < 2\}$, is that $\sum_{-\infty}^\infty \varphi(2^{-j}\xi) = 1,\ \xi \neq 0$. The norm $\|g\|_{B_p^{s,q}}$ of the Besov space $B_p^{s,q}$ is defined as follows
$\|v\|_{B_p^{s,q}} = \left(\sum_{j=0}^\infty \left(2^{js}\|\mathscr{F}^{-1}\left(\varphi_j\hat{v}\right)\|_p\right)^q\right)^{1/q}$, where $\hat{v}$ is the Fourier transform of v.

Theorem 2 *Assume that $A(x,\partial_x)$ is the Laplace operator on $\mathbb{R}^n$ and that $s, s' \geq 0$, $q \geq 1$, $1 \leq p \leq 2$, $1/p + 1/p' = 1$, and $\delta = 1/p - 1/2$, $(n+1)\delta \leq s - s'$, $-1 < s - s' - 2n\delta$. Denote $a := s - s' - 2n\delta$. The solution $\Phi = \Phi(x,t)$ of the*

Cauchy problem:

$$\Phi_{tt} - n\Phi_t - e^{2t}A(x,D)\Phi + m^2\Phi = 0\,, \quad \Phi(x,0) = \varphi_0(x)\,, \quad \Phi_t(x,0) = \varphi_1(x)\,, \tag{2.4}$$

with $\Re M > 0$ *satisfies the following estimate:*

$$\begin{aligned}\|\Phi(x,t)\|_{B^{s',q}_{p'}} \lesssim \|\varphi_0\|_{B^{s,q}_p} e^{\frac{n}{2}t}\Bigg(& e^{-\frac{1}{2}t}(e^t-1)^a + (e^t-1)^{a+1}\Bigg[e^{-\Re Mt}(e^t+1)^{2\Re M-1} \\ & + (e^t+1)^{\Re M-1}\begin{cases}1 & \text{if} \quad \Re M > 1/2 \\ t^{1-\operatorname{sgn}|\frac{1}{2}-\Re M|} + e^{(\frac{1}{2}-\Re M)t} & \text{if} \quad \Re M \le 1/2\end{cases}\Bigg]\Bigg) \\ & + \|\varphi_1\|_{B^{s,q}_p} e^{\frac{n}{2}t} e^{-\Re Mt}(e^t-1)^{a+1}(e^t+1)^{2\Re M-1}, \quad \textit{for all} \;\; t > 0.\end{aligned}$$

Corollary 1 *For large t, the solution $\Phi = \Phi(x,t)$ of the Cauchy problem (2.4) satisfies the following estimate:*

$$\begin{aligned}\|\Phi(x,t)\|_{B^{s',q}_{p'}} & \lesssim \|\varphi_0\|_{B^{s,q}_p} e^{(\frac{n}{2}+a+\Re M)t}\begin{cases}1 & \text{if} \quad \Re M > 1/2 \\ t^{1-\operatorname{sgn}|\frac{1}{2}-\Re M|} + e^{(\frac{1}{2}-\Re M)t} & \text{if} \quad \Re M \le 1/2\end{cases} \\ & \quad + \|\varphi_1\|_{B^{s,q}_p} e^{(\frac{n}{2}+a+\Re M)t}, \quad \textit{for all} \;\; t \in (1,\infty), \\ \|\Phi(x,t)\|_{B^{s',q}_{p'}} & \lesssim t^a\|\varphi_0\|_{B^{s,q}_p} + t^{a+1}\|\varphi_1\|_{B^{s,q}_p} \quad \textit{for all } t \in (0,1)\,.\end{aligned}$$

$B^{s,q}_p - B^{s',q}_{p'}$ Estimates for Equation with Source

Theorem 3 *Let $\Phi = \Phi(x,t)$ be a solution of the Cauchy problem:*

$$\Phi_{tt} + n\Phi_t - e^{2t}\bigtriangleup\Phi + m^2\Phi = f\,, \quad \Phi(x,0) = 0\,, \quad \Phi_t(x,0) = 0\,.$$

Then, solution $\Phi = \Phi(x,t)$ for $\Re M > 0$ satisfies the following estimate:

$$\begin{aligned}\|\Phi(x,t)\|_{B^{s',q}_{p'}} \le C_M e^{t\left(\frac{n}{2}+\Re M+s-s'-n\left(\frac{1}{p}-\frac{1}{p'}\right)\right)} & \\ \times \int_0^t e^{-(\frac{n}{2}+\Re M)b}\|f(x,b)\|_{B^{s,q}_p}\,db\,. &\end{aligned}$$

for all $t > 0$, provided that $s, s' \ge 0$, $q \ge 1$, $1 \le p \le 2$, $1/p + 1/p' = 1$, and $\delta = 1/p - 1/2$, $(n+1)\delta \le s - s'$, $-1 < s - s' - 2n\delta$.

In order to complete the proof of Theorem 1, we appeal to the integral equation (1.7). Using Theorems 2 and 3 and Banach fixed-point theorem, we prove the existence of a unique solution of the integral equation (1.7) and obtain an estimate of the lifespan. Assumptions (i)–(iii) imply that corresponding operator is a contraction, and thus Banach fixed-point theorem is applicable.

Acknowledgements K. Yagdjian was supported by the University of Texas Rio Grande Valley College of Sciences 2016–2017 Research Enhancement Seed Grant.

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On the Energy Estimate for Klein–Gordon-Type Equations with Time-Dependent Singular Mass

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Abstract We consider the energy estimate of the solution to the Cauchy problem of Klein–Gordon-type equation with time-dependent mass $M(t)$, in particular $M(t)$ has a singularity. The main purpose of this chapter is to give sufficient conditions to $M(t)$ for the energy to be asymptotically stable.

1 Introduction

The energy conservation is a typical property for the wave equation, and it is a natural question whether a similar kind of property holds or not for some perturbed equations. We consider the following backward Cauchy problem of the wave equation with a mass term; we shall call such an equation Klein–Gordon-type equation:

$$\begin{cases} \partial_t^2 u - \Delta u + M(t)u = 0, & (t,x) \in (0,T) \times \mathbb{R}^n, \\ u(T,x) = u_0(x), \quad \partial_t u(T,x) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where T is a positive constant, $\Delta = \sum_{j=1}^n \partial_{x_j}^2$, and $M(t)$ is a real-valued function defined on $(0,T]$.

For $\mu \in L^\infty((0,T)) \cap C^1((0,T])$, we define $V = V(t,\xi;\mu(t))$ by:

$$V(t,\xi;\mu(t)) := {}^t\left(i|\xi|\mu(t)\hat{u}(t,\xi), \partial_t(\mu(t)\hat{u}(t,\xi))\right), \tag{2}$$

where $\hat{f}(\xi)$ denotes the partial Fourier transform of $f(x)$ with respect to x. Let $E(t,\xi)$ be the operator providing

$$E(t,\xi)V(T,\xi;\mu(T)) = V(t,\xi;\mu(t)). \tag{3}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_31

If $M(t) \equiv 0$, then $E(t,\xi)$ is a unitary matrix on $\mathbb{C}^2$ with $\mu(t) \equiv 1$; hence, the energy conservation $\|V(T,\cdot;1)\|_{L^2} \equiv \|V(t,\cdot;1)\|_{L^2}$ is established by Parseval's theorem. If $M(t) \not\equiv 0$, then one cannot expect the unitarity of $E(t,\xi)$ in general, but some estimates which ensure an equivalence of $V(t,\xi;\mu(t))$ and $V(T,\xi;\mu(T))$, such as the generalized energy conservation (cf. [1, 5]), are possible to be established. In this chapter, we shall consider the following estimates:

$$0 < \inf_{\substack{Y\in\mathbb{C}^2\setminus\{0\} \\ (t,\xi)\in(0,T]\times\mathbb{R}^n}} \left\{ \frac{\|E(t,\xi)Y\|_{\mathbb{C}^2}}{\|Y\|_{\mathbb{C}^2}} \right\} \quad \text{and} \quad \sup_{\substack{Y\in\mathbb{C}^2\setminus\{0\} \\ (t,\xi)\in(0,T]\times\mathbb{R}^n}} \left\{ \frac{\|E(t,\xi)Y\|_{\mathbb{C}^2}}{\|Y\|_{\mathbb{C}^2}} \right\} < \infty \tag{4}$$

with a suitable choice of $\mu(t)$ satisfying $\mu(t) \simeq 1$. Here, $f \lesssim g$ with nonnegative functions f and g denotes that there exists a positive constant C such that $f \le Cg$, and $f \simeq g$ denotes that both estimates $f \lesssim g$ and $g \lesssim f$ hold.

One can prove the estimate (4) without difficulty if $M(t)$ satisfies

$$\int_t^T |M(s)|\,ds \in L^1((0,T)), \tag{5}$$

hence we focus in the case that (5) does not hold. In [3], the concrete model $M(t) = M_0 t^{-2\beta}$ with $\beta \ge 1$ and $M_0 \in \mathbb{R}$ is studied precisely by applying the estimates of special functions. Actually, the conclusions imply that one cannot expect (4) in general if (5) does not hold. However, (5) is not a necessary condition for (4); indeed, there exists $M(t)$ which does not satisfy (5) but (4) is valid. The aim of this chapter is to determine some sufficient conditions to $M(t)$ which provide (4) without assuming (5), especially focusing on the oscillating properties of $M(t)$.

2 Main Theorem

Let $M \in C((0,T])$ satisfy the following conditions for some positive constants α and β:

$$|M(t)| \lesssim t^{-2\beta}, \tag{6}$$

$$\left| \int_t^T M(s)\,ds \right| \lesssim t^{-\beta} \tag{7}$$

and

$$\left| \int_0^t \int_s^T M(\tau)\,d\tau\,ds \right| \lesssim t^{\alpha}. \tag{8}$$

Then, our main theorem is represented as follows:

Theorem 1 *If $M(t)$ satisfies (6), (7), and (8) for*

$$\beta < 1 + \frac{\alpha}{2}, \tag{9}$$

then there exists $\mu \in C^2((0,T])$ satisfying $\mu(t) \simeq 1$ such that (4) is established.

We observe from Theorem 1 the following:

(i) If $M(t) = M_0 t^{-2\beta}$, then (8) requires $\beta < 1$, hence (5) is valid.
(ii) Theorem 1 does not require the positivity of $M(t)$, hence the estimate:

$$\limsup_{t \to +0} \frac{\int_t^T M(s)\,ds}{\int_t^T |M(s)|\,ds} = 0$$

is possible if $M(t)$ is changing its sign infinitely many times as $t \to 0$.
(iii) The estimate (8) with $\alpha = -\beta + 1$ is trivial by (7), but $\alpha > 0$ requires $\beta < 1$; hence, (8) is a nontrivial condition.

There are many results which study the influence of the mass $M(t)$ to the stability of the solution on Klein–Gordon-type equations, but not many results are known for time-dependent and singular mass, in particular $M(t)$ is oscillating infinitely many times by changing its sign. However, Theorem 1 gives us a new point of view because of such a singular behavior of $M(t)$. The method on the analysis of our problem is similar to the problems for the stability of the energy as $t \to \infty$ with the mass $M \in C([T,\infty))$ oscillating infinity many times as $t \to \infty$. In [6], it is studied some sufficient conditions to the error term $\delta(t)$ that the asymptotic behaviors of the both energies with the masses $M(t) = M_0(1+t)^{-2}$ and $M(t) = M_0(1+t)^{-2} + \delta(t)$ with $M_0 > 0$ are the same. Briefly, this problem is corresponding to the conditions to $\tilde{\delta}(t)$ that the asymptotic behaviors of the both energies with the masses $M(t) = M_0 t^{-2}$ and $M(t) = M_0 t^{-2} + \tilde{\delta}(t)$ are the same as $t \to 0$. However, the proof of [6] requires the condition $\alpha \ge -\beta + 1$ instead of (8), which corresponds to the trivial assumption in (iii).

Example 1 Let p and q satisfy $2 < p < q + 1$. Then, the following $M(t)$ is applicable to Theorem 1:

$$M(t) = t^{-p} \sin t^{-q+1}. \tag{10}$$

Indeed, noting that $q > 1$ and $p \le 2q$ are valid, we can prove the following estimates:

$$\left| \int_0^t \int_s^T M(\tau)\,d\tau\,ds \right| \lesssim t^{2q-p}, \quad \left| \int_t^T M(s)\,ds \right| \lesssim t^{-(p-q)} \tag{11}$$

and $|M(t)| \lesssim t^{-p}$. Here, we observe that the singularity of $M(t)$ as $t \to 0$ can be higher as q becomes larger. Indeed, setting $\alpha = 2q - p(> 0)$ and $\beta = p/2(> 1)$, we have

$$p - q = \beta - \frac{2q - p}{2} < \beta \text{ and } \beta = 1 + \frac{\alpha}{2} - (q + 1 - p) < 1 + \frac{\alpha}{2},$$

thus (6), (7), (8), and (9) are valid. Here, we remark that the method of [6] requires the stronger restriction $2 < p < (q + 3)/2$ for (4).

3 Reduction to a Dissipative Wave Equation

Klein–Gordon-type equation with time-dependent mass can be reduced to the following dissipative wave equation with time-dependent dissipation:

$$\partial_t^2 w - \Delta w + 2b(t)\partial_t w = 0 \tag{12}$$

by the transformation:

$$w := \exp\left(\int_t^T b(s)\, ds\right) u, \tag{13}$$

where $b(t)$ is a solution of

$$b'(t) + b(t)^2 + M(t) = 0. \tag{14}$$

Moreover, the dissipative wave equation (12) can be reduced to the wave equation with time-dependent propagation speed:

$$\partial_t^2 y - a(t)^2 \Delta y = 0. \tag{15}$$

The conditions to the coefficients $a(t)$ and $b(t)$ for the stabilities of the energies have been studied well, for instance in [2, 5, 7]; hence, we may expect that our main theorem follows immediately from the previous results for (12) and (15). However, it is not really trivial what conditions to $M(t)$ provide the corresponding conditions to $b(t)$ which are required in the previous papers, because $b(t)$ is given as a solution to the nonlinear equation (14). In this chapter, we overcome this problem to introduce an explicit representation of the solution $b(t)$ of (14), which is introduced in [4, 6], with more precise estimates.

The following proposition is essential for the proof of our main theorem:

Proposition 1 *Let $M(t)$ satisfy the conditions of Theorem 1. Then, $\int_0^t b(s)\,ds \in C([0,T]) \cap C^2((0,T])$ and the following estimates are established:*

$$|b(t)| \lesssim t^{-\beta} \quad \text{and} \quad \left|b'(t)\right| \lesssim t^{-2\beta}. \tag{16}$$

Moreover, there exists $\lambda \in C^1([0,T])$ satisfying $\lambda(0) > 0$ and $\lambda'(t) \geq 0$ such that

$$b(t) = \frac{\lambda'(t)}{\lambda(t)} - \int_t^T M(s)\,ds. \tag{17}$$

Remark 1 For any $T_0 \in (0,T]$ and $t \in [T_0,T]$, the estimates of Proposition 1 are trivial. Therefore, we can suppose that T is small from now on without loss of generality.

Let $\eta = \eta(t)$ be the solution to the following equation:

$$\eta'' + M(t)\eta = 0 \tag{18}$$

on $(0,T]$ with the initial data $(\eta(T), \eta'(T)) = (1,0)$. Then, a solution of (14) is represented by:

$$b(t) = \frac{\eta'(t)}{\eta(t)}.$$

For $k = 1, 2, \ldots$, we define $Q_k(t)$ and $q_k(t)$ on $(0,T]$ by $Q_k(t) := -\int_t^T q_k(s)\,ds$ and

$$q_1(t) := M(t), \quad q_k(t) := \sum_{j=1}^{k-1} Q_j(t)Q_{k-j}(t) \quad (k \geq 2).$$

Then, we have the following lemmas:

Lemma 1 *$\eta(t)$ is represented by the following convergent series on $(0,T]$:*

$$\eta(t) = \exp\left(\sum_{k=1}^{\infty}\int_t^T Q_k(s)\,ds\right).$$

Proof The proof is straightforward. For the convergence of $\eta(t)$, refer to [6]. □

Lemma 2 *Let γ_k be the k-th Catalan number defined by $\gamma_k := (2k)!/(k!(k+1)!)$ for $k = 0, 1, \ldots$. For any $\delta > 0$, there exist positive constants $C_{1,2}$ and T such that*

$$|Q_2(t)| \leq C_{1,2}\, t^{\alpha-2\beta+1} \tag{19}$$

and

$$|Q_k(t)| \le \gamma_{k-1}\delta^{k-2}|Q_2(t)| \tag{20}$$

for any $0 < t \le T$ *and* $k \ge 2$.

Proof We note that the following inequalities are valid:

$$\alpha-\beta+1 > \alpha-2\beta+2 > 0 \ \text{ and } \ \alpha-2\beta+1 = -1+2\left(1+\frac{\alpha}{2}-\beta\right) < 0 \tag{21}$$

choosing β near to $1+\alpha/2$ without loss of generality. For a given $\delta > 0$, we shall show that (20) is valid by choosing T small enough. By (7) and (8), there exist positive constants C_1 and C_2 such that $|Q_1(t)| \le C_1 t^{-\beta}$ and $|\int_0^t Q_1(s)\,ds| \le C_2 t^{\alpha}$. Here, (20) is trivial for $k = 2$ and $-Q_2(t) = |Q_2(t)|$ is monotone decreasing since $q_2(t) = Q_1(t)^2$. By integration by parts, for $k \ge 1$ we have

$$\int_t^T Q_1(s)Q_k(s)\,ds = -\int_0^t Q_1(s)\,ds\, Q_k(t) - \int_t^T \left(\int_0^s Q_1(\tau)\,d\tau\right) q_k(s)\,ds.$$

Therefore, we have

$$\left|\int_t^T Q_1(s)Q_k(s)\,ds\right| \le C_2\left(t^{\alpha}|Q_k(t)| + \int_t^T s^{\alpha}|q_k(s)|\,ds\right). \tag{22}$$

If $k = 1$, then there exists a positive constant $C_{0,0}$ such that

$$\int_t^T Q_1(s)^2\,ds \le C_2\left(C_1 t^{\alpha-\beta} + C_{0,0}\int_t^T s^{\alpha-2\beta}\,ds\right) \le C_{1,2}t^{\alpha-2\beta+1},$$

where $C_{1,2} = C_2(C_1 + C_{0,0}/(-\alpha+2\beta-1))$. It follows that (19) holds. Here, we suppose that (20) is valid for $j = 2, \ldots, k$. Noting the formula $\sum_{j=0}^{k}\gamma_j\gamma_{k-j} = \gamma_{k+1}$, we have the following estimates by choosing T small such that $T^{\alpha-\beta+1} \le C_1\delta/C_{1,2}$:

$$\begin{aligned}
|q_{l+1}(t)| &= \left|\sum_{j=1}^{l} Q_j(t)Q_{l+1-j}(t)\right| \le 2|Q_1(t)Q_l(t)| + \sum_{j=2}^{l-1}|Q_j(t)Q_{l+1-j}(t)| \\
&\le 2C_1\gamma_0\gamma_{l-1}\delta^{l-2}t^{-\beta}|Q_2(t)| + \sum_{j=2}^{l-1}\gamma_{j-1}\gamma_{l-j}\delta^{l-3}Q_2(t)^2 \\
&\le \left(2C_1\gamma_0\gamma_{l-1} + C_{1,2}\left(\gamma_l - 2\gamma_0\gamma_{l-1}\right)\delta^{-1}t^{\alpha-\beta+1}\right)\delta^{l-2}\,t^{-\beta}|Q_2(t)| \\
&\le \left(2C_1\gamma_0\gamma_{l-1} + C_{1,2}\left(\gamma_l - 2\gamma_0\gamma_{l-1}\right)\delta^{-1}T^{\alpha-\beta+1}\right)\delta^{l-2}\,t^{-\beta}|Q_2(t)| \\
&\le C_1\gamma_l\delta^{l-2}\,t^{-\beta}|Q_2(t)|
\end{aligned}$$

for any $2 \le l \le k$. Therefore, by (22), noting the inequalities $C_1 C_2 \le C_{1,2}$ and $T^{\alpha} \le T^{\alpha-\beta+1} \le T^{\alpha-2\beta+2}$ since $T \le 1$, we have

$$\begin{aligned}
|Q_{k+1}(t)| \le& 2\left|\int_t^T Q_1(s)Q_k(s)\,ds\right| + \sum_{j=2}^{k-1}\left|\int_t^T Q_j(s)Q_{k+1-j}(s)\,ds\right| \\
\le& 2C_2\left(t^{\alpha}|Q_k(t)| + \int_t^T s^{\alpha}|q_k(s)|\,ds\right) + \delta^{k-3}\sum_{j=2}^{k-1}\gamma_{j-1}\gamma_{k-j}\int_t^T Q_2(s)^2\,ds \\
\le& 2C_2\gamma_{k-1}\delta^{k-2}t^{\alpha}|Q_2(t)| + 2C_1C_2\gamma_{k-1}\delta^{k-3}\int_t^T s^{\alpha-\beta}|Q_2(s)|\,ds \\
&+ C_{1,2}\,(\gamma_k - 2\gamma_0\gamma_{k-1})\,\delta^{k-3}|Q_2(t)|\int_t^T s^{\alpha-2\beta+1}\,ds \\
\le& 2C_2\gamma_0\gamma_{k-1}\delta^{k-2}t^{\alpha}|Q_2(t)| + \frac{2C_{1,2}}{\alpha-\beta+1}\gamma_0\gamma_{k-1}\delta^{k-3}T^{\alpha-\beta+1}|Q_2(t)| \\
&+ \frac{C_{1,2}}{\alpha-2\beta+2}\,(\gamma_k - 2\gamma_0\gamma_{k-1})\,\delta^{k-3}T^{\alpha-2\beta+2}|Q_2(t)| \\
\le& \left(\frac{2C_2\gamma_0\gamma_{k-1}}{\delta} + \frac{2C_{1,2}\gamma_0\gamma_{k-1}}{\delta^2(\alpha-\beta+1)} + \frac{C_{1,2}\,(\gamma_k - 2\gamma_0\gamma_{k-1})}{\delta^2(\alpha-2\beta+2)}\right)T^{\alpha-2\beta+2}\delta^{k-1}|Q_2(t)| \\
\le& \left(\frac{C_{1,2}}{C_1\delta} + \frac{C_{1,2}}{\delta^2(\alpha-2\beta+2)}\right)T^{\alpha-2\beta+2}\gamma_k\delta^{k-1}|Q_2(t)| \\
\le& \gamma_k\delta^{k-1}|Q_2(t)|
\end{aligned}$$

for any $0 < t \le T$ with

$$T \le \min\left\{\left(\frac{C_{1,2}}{C_1\delta} + \frac{C_{1,2}}{\delta^2(\alpha-2\beta+2)}\right)^{-\frac{1}{\alpha-2\beta+2}}, \left(\frac{C_1\delta}{C_{1,2}}\right)^{\frac{1}{\alpha-\beta+1}}\right\}.$$

Thus, the estimate (20) holds for any $k \ge 2$. □

Lemma 3 *For any* $0 < \varepsilon < 1$*, there exists a positive constant* T *such that the following estimate is established:*

$$\sum_{k=3}^{\infty}|Q_k(t)| \le \varepsilon|Q_2(t)|$$

for any $0 < t \le T$.

Proof We note that for $0 \le \delta < 1/4$, the following estimates are established:

$$\sum_{k=0}^{\infty}\gamma_k\delta^k = \frac{1-\sqrt{1-4\delta}}{2\delta} \le 1 + \delta + 4\delta^2.$$

By applying Lemma 2 with $\delta = \varepsilon/4$, there exists a positive constant T such that

$$\sum_{k=3}^{\infty} |Q_k(t)| \le \sum_{k=3}^{\infty} \gamma_{k-1}\delta^{k-2}|Q_2(t)| = \delta^{-1}\left(\sum_{k=0}^{\infty}\gamma_k\delta^k - \delta - 1\right)|Q_2(t)|$$
$$\le 4\delta|Q_2(t)| = \varepsilon|Q_2(t)|$$

for any $0 < t \le T$. □

Proof of Proposition 1 We note that $b(t) = -\sum_{k=1}^{\infty} Q_k(t)$. Then, $\int_0^t b(s)\,ds \in C^2((0,T])$ is trivial by (14). By (8), (19), (21), and Lemma 3, we have

$$\left|\int_0^t b(s)\,ds\right| = \left|\sum_{k=1}^{\infty}\int_0^t Q_k(s)\,ds\right| \le C_2 t^{\alpha} + C_{1,2}(1+\varepsilon)\int_0^t s^{\alpha-2\beta+1}\,ds$$
$$\le \left(C_2 + \frac{C_{1,2}(1+\varepsilon)}{\alpha-2\beta+2}\right)t^{\alpha-2\beta+2} \to 0 \quad (t \to 0),$$

hence $\int_0^t b(s)\,ds \in C([0,T])$. Moreover, by (6), (7), and (14) we have (16) for any $0 < t \le T$. We define $\lambda(t)$ by:

$$\lambda(t) := \exp\left(\int_0^T Q_1(s)\,ds + \sum_{j=2}^{\infty}\int_t^T Q_j(s)\,ds\right). \tag{23}$$

Then, we have (17), and thus $\lambda \in C^1([0,T])$ is valid. By (19), Lemmas 2, and 3 with $0 < \varepsilon < 1$, we have

$$\frac{\lambda'(t)}{\lambda(t)} = -\sum_{j=2}^{\infty} Q_j(t) = |Q_2(t)| - \sum_{j=3}^{\infty} Q_j(t) \ge (1-\varepsilon)|Q_2(t)| \ge 0,$$

it follows that $\lambda'(t) \ge 0$. □

4 Proof of Theorem 1

By partial Fourier transform with respect to x, (12) is reduced to the following equation:

$$\partial_t^2\hat{w} + |\xi|^2\hat{w} + 2b(t)\partial_t\hat{w} = 0. \tag{24}$$

For a large constant N, we define the hypersurface t_ξ in $[0, T] \times \mathbb{R}^n$ by:

$$t_\xi = \min \left\{ \left(N \langle \xi \rangle^{-1} \right)^{\frac{1}{\alpha+1}}, T \right\}, \quad \langle \xi \rangle = \sqrt{1 + |\xi|^2}.$$

Then, we separate the phase space $[0, T] \times \mathbb{R}^n_\xi$ by t_ξ into two zones Z_Ψ and Z_H:

$$Z_\Psi := \left\{ (t, \xi) \; ; \; 0 \le t \le t_\xi \right\} \text{ and } Z_H := \left\{ (t, \xi) \; ; \; t_\xi \le t \le T \right\}. \tag{25}$$

Let $E_0 = E_0(t, s, \xi)$ be the fundamental solution of

$$\partial_t E_0 = A_0(t, \xi) E_0, \quad E_0(s, s, \xi) = I, \quad A_0(t, \xi) = \begin{pmatrix} 0 & i|\xi| \\ i|\xi| & -2b(t) \end{pmatrix}.$$

Then, $E_0(t, s, \xi)$ provides

$$E_0(t, s, \xi) W(s, \xi) = W(t, \xi), \quad W(t, \xi) = \begin{pmatrix} i|\xi|\hat{w}(t, \xi) \\ \partial_t \hat{w}(t, \xi) \end{pmatrix} = V\left(t, \xi; \eta(t)^{-1} \right),$$

where V is defined by (2). Then, we have the following proposition:

Proposition 2 *The following estimates are established uniformly with respect to $Y \in \mathbb{C}^2$:*

$$\| E_0(t, s, \xi) Y \|_{\mathbb{C}^2} \simeq \| Y \|_{\mathbb{C}^2} \text{ for } 0 \le s \le t \le t_\xi \tag{26}$$

and

$$\| E_0(T, t, \xi) Y \|_{\mathbb{C}^2} \simeq \| Y \|_{\mathbb{C}^2} \text{ for } t_\xi \le t \le T. \tag{27}$$

If Proposition 2 is proved, then we immediately conclude Theorem 1 by setting

$$E(t, \xi) := \begin{cases} E_0(T, t_\xi, \xi) E_0(t_\xi, t, \xi) & \text{in } Z_\Psi, \\ E_0(T, t, \xi) & \text{in } Z_H. \end{cases}$$

If $b(t) = \lambda'(t)/\lambda(t)$, where $\lambda(t)$ given in Proposition 1, then (4) can be proved by using standard arguments for the proof of energy estimates with noneffective and monotone dissipation (cf. [7]). Thus, we introduce the following lemma without proof.

Lemma 4 *Let $\eta_1(t) = \exp(2 \int_t^T Q_1(s)\, ds)$ and $\mathscr{E}_0 = \mathscr{E}_0(t, s, \xi)$ be the fundamental solution of*

$$\partial_t \mathscr{E}_0 = \mathscr{A}_0(t, \xi) \mathscr{E}_0, \quad \mathscr{E}_0(s, s, \xi) = I, \quad \mathscr{A}_0(t, \xi) = \begin{pmatrix} 0 & i\eta_1(0)^{-1}|\xi| \\ i\eta_1(0)|\xi| & -\frac{2\lambda'(t)}{\lambda(t)} \end{pmatrix}.$$

Then, the following estimates are established uniformly with respect to $0 < s \le t \le T$, $\xi \in \mathbb{R}^n$, *and* $Y \in \mathbb{C}^2$*:*

$$\|\mathscr{E}_0(t,s,\xi)Y\|_{\mathbb{C}^2} \simeq \|Y\|_{\mathbb{C}^2} \simeq \|\mathscr{E}_0^{-1}(t,s,\xi)Y\|_{\mathbb{C}^2}. \tag{28}$$

Proof of Proposition 2 Let $0 \le s \le t \le t_\xi$ We define $\Lambda_0(t) = \mathrm{diag}\{1, \eta_1(t)^{-1}\}$ and $E_1 = E_1(t,s,\xi) = \mathscr{E}_0(t,s,\xi)^{-1}\Lambda_0(t)^{-1}E_0(t,s,\xi)$. By Proposition 1, E_1 is a solution of the following equation:

$$\partial_t E_1 = A_1(t,s,\xi)E_1,$$

$$A_1 = \mathscr{E}_0(t,s,\xi)^{-1}\begin{pmatrix} 0 & i\left(\eta_1(t)^{-1} - \eta_1(0)^{-1}\right)|\xi| \\ i\left(\eta_1(t) - \eta_1(0)\right)|\xi| & 0 \end{pmatrix}\mathscr{E}_0(t,s,\xi).$$

Noting

$$\left|\eta_1(t)^{\pm 1} - \eta_1(0)^{\pm 1}\right| = \eta_1(0)^{\pm 1}\left|\exp\left(\pm 2\int_0^t Q_1(s)\,ds\right) - 1\right| \simeq \left|\int_0^t Q_1(s)\,ds\right|,$$

there exists a positive constant C_0 such that

$$\begin{aligned}\partial_t\|E_1Y\|_{\mathbb{C}^2}^2 =& 2\Re\left(A_1E_1Y, E_1Y\right)_{\mathbb{C}^2} \le 2\|A_1E_1Y\|_{\mathbb{C}^2}\|E_1Y\|_{\mathbb{C}^2} \\ \le& C_0\langle\xi\rangle\left|\int_0^t Q_1(s)\,ds\right|\|E_1Y\|_{\mathbb{C}^2}^2\end{aligned}$$

for any $Y \in \mathbb{C}^2$. Therefore, by (8) and Gronwall's inequality, we have

$$\begin{aligned}\|E_1(t,s,\xi)Y\|_{\mathbb{C}^2}^2 \le& \exp\left(C_0\langle\xi\rangle\int_s^t\left|\int_0^\tau Q_1(\sigma)\,d\sigma\right|\,d\tau\right)\|E_1(s,s,\xi)Y\|_{\mathbb{C}^2}^2 \\ \le& \exp\left(C_0C_2\langle\xi\rangle\int_0^{t_\xi}\tau^\alpha\,d\tau\right)\|E_1(s,s,\xi)Y\|_{\mathbb{C}^2}^2 \\ =& \exp\left(\frac{C_0C_2}{\alpha+1}\langle\xi\rangle t_\xi^{\alpha+1}\right)\|E_1(s,s,\xi)Y\|_{\mathbb{C}^2}^2 \le e^{\frac{C_0C_2N}{\alpha+1}}\|E_1(s,s,\xi)Y\|_{\mathbb{C}^2}^2.\end{aligned}$$

Analogously, we have

$$\|E_1(t,s,\xi)Y\|_{\mathbb{C}^2}^2 \ge e^{-\frac{C_0C_2N}{\alpha+1}}\|E_1(s,s,\xi)Y\|_{\mathbb{C}^2}^2.$$

Consequently, by Lemma 4, and noting $E_1(s,s,\xi) = \Lambda_0^{-1}(s)E_0(s,s,\xi) = \Lambda_0^{-1}(s)$ and $\eta_1(t) \simeq 1$, we have

$$\|E_1(t,s,\xi)Y\|_{\mathbb{C}^2} \simeq \|Y\|_{\mathbb{C}^2}. \tag{29}$$

By using Lemma 4 again, we obtain

$$\|E_0(t,s,\xi)Y\|_{\mathbb{C}^2} \simeq \|\mathscr{E}_0(t,s,\xi)E_1(t,s,\xi)Y\|_{\mathbb{C}^2} \simeq \|E_1(t,s,\xi)Y\|_{\mathbb{C}^2} \simeq \|Y\|_{\mathbb{C}^2}.$$

Thus, the proof of (26) is concluded. The estimate (27) can be proved by applying standard technique of diagonalization (see [6, 7]), thus we omit the proof. □

Acknowledgements This work was supported by JSPS KAKENHI Grant Number 26400170.

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Nonlinear Evolution Equations and Their Application to Chemotaxis Models

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Abstract Recently, we have investigated the global existence in time and asymptotic profile of solutions of some nonlinear evolution equations with strong dissipation and proliferation arising in mathematical biology. In this chapter, we improve the asymptotic behaviour of the solution to a simpler equation so that its derivative with respect to t converges exponentially to a constant steady state. We apply our result to a chemotaxis model and show the global existence in time and such exponential convergence property of the solution.

1 Introduction

We begin with a chemotaxis model proposed by Kubo and Tello in [7]:

$$
\text{(CM)} \begin{cases} u_t - \Delta u = -\nabla \cdot (\chi u \nabla w) + \mu_1 u(1 - u - a_1 w), & (1.1) \\ w_t = \mu_2 w(1 - a_2 u - w) & \text{in } \Omega \times (0, T), \ (1.2) \\ (\partial_\nu u - \chi u \partial_\nu w)\,|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_1(x) > 0, \quad w(x, 0) = w_0(x) > 0 & \text{in } \Omega, \end{cases}
$$

where Ω is a bounded domain in $\mathbf{R}^n$ with a smooth boundary $\partial\Omega$, ν is a unit outer normal vector on $\partial\Omega$, $\chi > 0$, $\mu_i > 0$ and a_i, for $i = 1, 2$ are constants. In (CM), a competitive system with respect to u and w is considered so that it describes the behaviour of two biological species. They obtain the existence and the asymptotic behaviour of the solution for $|a_i| < 1$, $i = 1, 2$.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_32

In order to investigate (CM) in other cases, we consider the initial and Neumann-boundary value problem of nonlinear evolution equations with a logistic term:

$$\text{(NE)}\begin{cases} u_{tt} - D\triangle u_t + \nabla\cdot(\kappa u_t e^{-\epsilon u}\nabla u) - \mu(1-u_t)u_t = 0 & \text{in } \Omega\times(0,T), \quad (1.3)\\ \partial_\nu u|_{\partial\Omega} = 0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases}$$

where $D > 0$, $\mu \geq 0$, $\epsilon > 0$ and κ are constants. In fact, we show a simple example that (1.1)–(1.2), so-called parabolic-ODE system, are transformed into the same type of equation as (1.3) for $a_2 = 2$ and $\mu_1 = 0$. From (1.2), it follows that

$$w(x,t) = w(x,0)\exp(\mu_2\int_0^t (1-2u-w)ds).$$

Put $\int_0^t uds = \tilde{u} + t$ and $\Theta = \exp(-\mu_2(t + \tilde{u} + (\tilde{u} + \int_0^t wds)))$, then (1.1) is rewritten as:

$$\tilde{u}_{tt} - \Delta\tilde{u}_t - \nabla\cdot\left(\chi(1+\tilde{u}_t)\Theta\left(w(x,0)\mu_2(2\nabla\tilde{u} + \int_0^t \nabla wds) - \nabla w(x,0)\right)\right) = 0.$$

Since the main term of Θ and the above equation are $\exp(-\mu_2(t+\tilde{u}))$ and $\tilde{u}$, respectively, omitting $\tilde{u} + \int_0^t wds$, $\int_0^t \nabla wds$ and putting $w(x,0) = 1$ for our convenience, the reduced equation is the same as (1.3) for $\mu = 0$ replacing $\tilde{u} + t$ by u again. Without such restriction and omission, this process shall be discussed in detail in Sect. 3.

In the present chapter, we obtain the solution of (NE) with the exponential convergence property as $t \to \infty$. For this purpose, we seek the solution in the form:

$$u = t + U(x,t), \quad U(x,t) = \int_0^t e^{-\delta s}v_s(x,s)ds + v_0(x) + a, \tag{1.4}$$

where a and δ are positive parameters and $v_0(x)$ is specified later. Hence, (1.3) is written as:

$$U_{tt} - D\Delta U_t + \nabla\cdot(\kappa(1+U_t)e^{-\epsilon u}\nabla u) + \mu U_t(U_t+1) = 0.$$

Multiplying both sides of the above equation by $e^{\delta t}$, we have for $u_t = 1 + e^{-\delta t}v_t$

$$Q[v] := v_{tt} - \delta v_t - D\Delta v_t + \nabla\cdot(\kappa(e^{\delta t}+v_t)e^{-\epsilon u}\nabla u) + \mu v_t(1+e^{-\delta t}v_t) = 0. \tag{1.5}$$

Then, (NE) is reduced to the problem:

$$\text{(RP)}\begin{cases} Q[v]=0 & \text{in} \quad \Omega\times[0,T),\\ \partial_\nu v|_{\partial\Omega}=0 & \text{on } \partial\Omega\times[0,T),\\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x):=u_1(x)-1 & \text{in} \quad \Omega. \end{cases}$$

On the other hand, in [4, 9] we established global existence in time and asymptotic behaviour of solutions to the problem replaced by the following (1.6), including (1.3) in (NE), which is denoted by $(\widetilde{\text{NE}})$ in the below:

$$u_{tt}=D\triangle u_t+\nabla\cdot(\chi(u_t,e^{-u})e^{-u}\nabla u)+\mu(1-u_t)u_t, \tag{1.6}$$

where $\chi(\cdot,\cdot)$ satisfies for a constant $r>0$ and any integer $m\geq[n/2]+3$,

$$\chi(s_1,s_2)\in C^m(\mathbf{B}_{r+}),\quad (s_1,s_2)\in\mathbf{B}_{r+},$$

$\mathbf{B}_{r+}=\mathbf{B}_r\cap(\mathbf{R}\times\mathbf{R}_+)$ and $\mathbf{B}_r$ is a ball of radius r at 0 in $\mathbf{R}^2$.

The eigenvalues of $-\triangle$ with the homogeneous Neumann boundary condition on $\partial\Omega$ are denoted by $\{\lambda_i|i=0,1,2,\cdots\}$ with $0=\lambda_0<\lambda_1\leq\cdots\to+\infty$, and $\varphi_i=\varphi_i(x)$ indicates the L^2 normalized eigenfunction corresponding to λ_i. Then, we put for functions $h(x),k(x)\in H^l(\Omega)$ and non-negative integer l:

$$(h,k)(t)=\int_\Omega h(x,t)k(x,t)dx,\quad \|h\|_l^2(t)=\sum_{|\beta|\leq l}\|\partial_x^\beta h(\cdot,t)\|^2(t),$$

$$(h,k)_l=(h,k)+(\mathscr{D}^l h,\mathscr{D}^l k),\quad |h|_l^2=(h,h)_l,$$

where we write $\|h\|_0(t)$ by $\|h\|(t)$ for simplicity, β is a multi-index for $\beta=(\beta_1,\cdots,\beta_n)$, and $\mathscr{D}^l=\Delta^j$ $(l=2j)$, $\mathscr{D}^l=\nabla\cdot\Delta^j$ $(l=2j+1)$ for a non-negative integer j.

We set $W^l(\Omega)$ as a closure of $\{\varphi_1,\varphi_2,\cdots\varphi_n,\cdots\}$ in $H^l(\Omega)$. It holds that $\int_\Omega h(x)=0$ for $h(x)\in W^l(\Omega)$, which enables us to use the Poincarè inequality. The equivalence of norms $|\cdot|_l, ||\cdot||_l$ shall be used frequently.

In [4, 9], we obtain the result for $(\widetilde{\text{NE}})$ by following the same reduction process from (NE) to (RP) for $\delta=0$.

Theorem 1 ([4, 9]) *Assume* $(v_0(x),v_1(x))\in W^{m+1}(\Omega)\times W^m(\Omega)$ *for* $v_0(x)=u_0(x)-a, v_1(x)=u_1(x)-1$, *and that* $\|v_1\|_{m+1}^2$ *is sufficiently small,* a *and* r *are sufficiently large. Then, there is a solution* $u(x,t)=a+t+v(x,t)\in\bigcap_{i=0}^1 C^i([0,\infty);\,W^{m-i}(\Omega))$ *to* $(\widetilde{\text{NE}})$ *such that*

$$\lim_{t\to\infty}||v_t||_{m-1}=0,\quad \lim_{t\to\infty}||u_t(x,t)-1||_{m-1}=0. \tag{1.7}$$

Levine and Sleeman [10] obtained explicit solutions of the form: $u=\gamma t+v$ $(\gamma>0)$ of the same type of equation of (1.3) for $\epsilon=\mu=0$ and $n=1$, which is applied to a parabolic-ODE system arising from a mathematical model of biology. In this line, [5, 6, 8] showed the existence of solutions of a special case of ($\widetilde{\text{NE}}$) without the logistic term for $n \geq 1$, which arises from mathematical biology and biomedicine (see [1, 10–12]). In [3, 4, 9], we can get the solution in more general form $u(x,t) = a + bt + v(x,t)$ for any $b > 0$, which enables us to deal with (1.6) for $b = 1$. In case $\delta = 0$, u satisfying (1.4) coincides with the solution for $b = 1$ obtained in [3, 4, 9].

In Theorem 1, we obtain the asymptotic behaviour (1.7) of the solution to (NE). However, the decay rate of (1.7) is unknown. In this chapter, we show that for the solution u of (NE) u_t converges exponentially to the constant steady state. Also in [4, 9], Theorem 1 was applied to the mathematical model of tumour invasion by Chaplain and Lolas [2], and the full proof was given in [9]. In order to obtain the solution of their model, we consider the initial and boundary value problem that required zero-Neumann boundary condition instead of no-flux condition for the following equations:

$$\begin{cases} \partial_t n = d_n \partial_x^2 n - \gamma \partial_x (n \partial_x f) + \mu_1 n(1 - n - f), & (1.8) \\ \partial_t f = -\eta m f + \mu_2 f(1 - n - f), & (1.9) \\ \partial_t m = d_m \partial_x^2 m + \alpha n - \beta m \qquad \text{in} \quad \Omega \times (0, T), & (1.10) \end{cases}$$

where for $(x,t) \in \Omega \times (0,T)$ $n := n(x,t)$ is the density of tumour cells, $m := m(x,t)$ is the concentration of degradation enzymes and $f := f(x,t)$ is the density of the extracellular matrix and d_n, γ, μ_1, η, μ_2, d_m, α and β are positive constants.

One of the interesting problems in ecology and biology is the coexistence of species or the extinction. Kubo and Tello obtain the coexistence of species in (CM) for $|a_i| < 1, i = 1, 2$, so-called weak coupled conditions, under the additional conditions (see [7]). In the present chapter, we study other cases, i.e. $a_1 \in \mathbf{R}$, $a_2 > 1$ and $a_1 < 1, a_2 = 0$, and then the problem for $|a_1| \geq 1, a_2 \leq 1$ remains open except for the case of $a_1 \leq -1, a_2 = 0$. It is remarked that in case of $a_1 = 1, a_2 > 1$ the system of (1.1) and (1.2) is the same as that of (1.8) and (1.9) with $m(x,t) \equiv n(x,t)$ and $d_n = 1$.

In Sect. 2, based on Theorem 1 we obtain the existence theorem and exponential convergence property of the solution to (NE) by making use of the energy estimates of (RP). In Sect. 3, applying this result to (CM) we have our desired results for $a_1 \in \mathbf{R}, a_2 > 1$ and $a_1 < 1, a_2 = 0$.

Remark 1 We can apply Theorem 1 with $\mu = 0$ to mathematical models of tumour angiogenesis proposed by Anderson and Chaplain [1] and Othmer and Stevens [11] (see [3–6, 8, 9]), which are in the form of parabolic-ODE system.

2 Existence and Asymptotic Behaviour of Solutions

In this section, we derive the energy estimate of (RP) and show the existence and asymptotic profile of the solution to (RP) which leads us to the desired result for (NE) going back through the reduction process.

Assume that

$$h(x,t) \in \bigcap_{i=0}^{2} C^i([0,\infty); W^{m+1-i}(\Omega)), \|h_t\|_m \le r. \tag{2.1}$$

The following two estimates are obtained in [3, 4, 9]. For h satisfying (2.1) with $m \ge M \ge [n/2]+1$, and $\int_0^\infty \|h_t\|_M^2(s)ds < r_M$, then it holds that

$$||h||_M(t) \le C_0\sqrt{t} + C_0', \tag{2.2}$$

where C_0 depends on r_M and C_0' depends on $h(x,0)$. Also, it holds that for a constant $b > 0$ and $i = 1, 2, \cdots, n$:

$$||e^{-bt}h_{x_i}||^2(t) + \int_0^t e^{-2bs}||h_{x_i}||^2(s)ds \le C(\int_0^t e^{-2bs}||h_{x_i s}||^2(s)ds + ||h_{x_i}||^2(0)). \tag{2.3}$$

Lemma 1 *Assume that $u(x,t)$ is defined by (1.4) with $v(x,t)$ satisfying (2.1), then it holds that*

$$\|e^{-\epsilon u}\|_{L^\infty(\Omega)} \le C_a e^{-\epsilon' t},$$

where $C_a \to 0$ as $t \to \infty$.

Proof By (1.4), we get for v satisfying (2.1):

$$e^{-\epsilon u} = \exp(-\epsilon(a + t + v_0(x)))\exp(-\epsilon\int_0^t e^{-\delta s}v_s ds),$$

using the same way as derived (2.2) for $\int_0^t e^{-\delta s}v_s ds$ with application of Sobolev's inequality to v_s:

$$\le C_a e^{-\epsilon t + C_0\sqrt{t}} \le C_a e^{-\epsilon' t},$$

where $0 < \epsilon' < \epsilon$ and $C_a \to 0$ as $t \to \infty$, which ends the proof. □

Lemma 2 (Basic Estimate of (RP)) *We have a basic energy estimate of (RP) for $v(x,t)$ satisfying (2.1), sufficiently large a and $\epsilon > \delta$*

$$||v_t||^2(t) + \int_0^t ||\nabla v_s||^2 ds \le CE[v](0),$$

where $E[v](t) = ||v_t||^2(t) + ||\nabla v||^2(t)$.

Sketch of the Proof We consider $(Q[v], v_t)$

$$= (v_{tt} - \delta v_t - D\Delta v_t + \nabla \cdot \kappa(e^{\delta t} + v_t)e^{-\epsilon u}\nabla u, v_t) + \mu(v_t(1 + e^{-\delta t}v_t), v_t) = 0.$$

By the integration by parts, we have

$$\frac{1}{2}\partial_t \|v_t\|^2 + D\|\nabla v_t\|^2 + \mu(v_t(1+e^{-\delta t}v_t), v_t) = \delta\|v_t\|^2 + (\kappa(e^{\delta t} + v_t)e^{-\epsilon u}\nabla u, \nabla v_t).$$

If $\|v_t\|_{L^\infty} \ll 1$ holds, then integrating the both sides of the above quality over $(0, t)$ shows

$$\|v_t\|^2(t) + \int_0^t \|\nabla v_s\|^2(s)(s)ds + \int_0^t \|v_s\|^2(s)ds$$

$$\le C\|v_t\|^2(0) + Ce^{-a}\left(\int_0^t e^{2\delta s}e^{-2\epsilon u}\|\nabla u\|^2(s)ds + \int_0^t \|\nabla v_s\|^2(s)ds\right), \tag{2.4}$$

by means of Lemma 1 and (2.3) for the second term of (2.4) with $\epsilon' > \delta$

$$\le CE[v](0) + C_a \int_0^t \|\nabla v_s\|^2(s)ds. \tag{2.5}$$

Since the last term of the right-hand side of (2.5) is negligible for sufficiently large a, we obtain a basic energy estimate of (RP). □

Considering $\nabla^k v$, $k \le M$ for a positive integer $M \ge [n/2] + 1$, instead of v in the above procedure, in the same way as in Lemma 2, we obtain the higher-order estimate.

Lemma 3 (Higher-Order Estimate for (RP)) *For $v(x,t)$ satisfying (2.1), sufficiently large a and $\epsilon > \delta$, the following higher-order energy estimate (RP) holds for $E_k[v](t) = E[\nabla^k v]$:*

$$\sum_{j=0}^{M} ||\nabla^j v_t||^2(t) + \sum_{j=0}^{M+1} \int_0^t ||\nabla^j v_s||^2(s)ds \le CE_M[v](0). \tag{2.6}$$

Remark 2 Lemma 1 implies that for $\delta < \epsilon' < \epsilon$ the coefficient $e^{-\epsilon u}e^{\delta t}$ of the quadratic nonlinear term of (1.5) decays exponentially. Thus, Theorem 1 holds for the solution in the form of (1.4) to (NE) even though $\delta > 0$.

Now, we state the result of (RP) which gives our main result of (NE).

Theorem 2 *Assume that* $(v_0(x), v_1(x)) \in W^{m+1}(\Omega) \times W^m(\Omega)$ *for* $v_0(x) = u_0(x) - a$ *and* $v_1(x) = u_1(x) - 1$*, and that* $\|v_1\|_m^2$ *is sufficiently small and* a *is large enough. Then for* $\epsilon > \delta > 0$*, there is a solution* $v(x,t) \in \bigcap_{i=0}^{1} C^i([0,\infty); W^{m-i}(\Omega))$ *to* (RP) *such that* $u(x,t) = a + t + \int_0^t e^{-\delta s} v_s ds + v_0(x) \in C^1([0,\infty); W^{m-1}(\Omega))$ *is the solution to* (NE) *satisfying*

$$\lim_{t\to 0} ||v_t||_{m-1} = 0, ||u_t - 1||_{m-1} \le Ce^{-\delta t}. \tag{2.7}$$

Sketch of the Proof The proof is given by the same way as used in [4, 9]. We consider the following iteration scheme and derive the energy estimate of it by the use of Lemma 3:

$$\text{(i+1)} \begin{cases} Q_i[v_{i+1}] = \partial_t^2 v_{i+1} - \delta\partial_t v_{i+1} - D\partial_t \Delta v_{i+1} \\ +\nabla\cdot(\kappa e^{\delta t}e^{-\epsilon u_i}\nabla u_{i+1}) + \nabla\cdot(\kappa\partial_t v_{i+1}e^{-\epsilon u_i}\nabla u_i) + \mu\partial_t v_{i+1}(1 + e^{\delta t}\partial_t v_{i+1}) = 0, \\ \partial_\nu v_{i+1}|_{\partial\Omega} = 0, \\ v_{i+1}(x,0) = v_0(x), \ \partial_t v_{i+1}(x,0) = v_1(x), \end{cases}$$

where $v_i = \sum_{j=1}^{\infty} f_{ij}(t)\varphi_j(x)$, $v_0(x) = \sum_{j=1}^{\infty} h_j\varphi_j(x)$, $v_1(x) = \sum_{j=1}^{\infty} h'_j\varphi_j(x)$. The energy estimate (2.6) guarantees the uniform estimate of each (i+1) for $i = 1, 2, \cdots$. We determine $f_{ij}(t)$ by the solution of the following system of ordinary differential equation with initial data for $j = 1, 2, \cdots$:

$$\begin{cases} (Q_i[v_{i+1}], \varphi_j) = 0, \\ f_{i+1j}(0) = h_{i+1}, \ f_{i+1jt}(0) = h'_{i+1}. \end{cases}$$

Thanks to the energy estimates, we can obtain the global existence in time of the solution v_i of (i) satisfying (2.1) and justification of the limiting process of v_i as $i \to \infty$ that converges strongly to the desired solution v by the standard method (see [5]). Since we have the asymptotic behaviour of v, which is the same as (1.7) due to Remark 2, we arrive at (2.7). □

3 Application to a Chemotaxis Model

Now, we apply Theorem 2 to (CM). We deal with (CM), first in Sect. 3.1 for $a_1 \in \mathbf{R}$ and $a_2 > 1$, next in Sect. 3.2 for $a_1 < 1$ and $a_2 = 0$. Also in [7] $\chi > 0$ is required; however, in this chapter χ is allowed to be any constant. In order to seek the solution of (CM), we imposed zero-Neumann boundary condition on u and w instead of no-flux condition: $\partial_\nu u - \chi u \partial_\nu w = 0$ on the boundary.

3.1 The Case $-\infty < a_1 < \infty$, $a_2 > 1$

Put $a_2 = 1 + \varepsilon > 1$ for a positive constant $\epsilon > 0$, then by dividing the both sides of (1.2) by w and integrating it over $(0, t)$ we get

$$w(x,t) = w(x,0)e^{\mu_2 \int_0^t (1-(1+\varepsilon)u - w)ds}. \tag{3.1}$$

We consider here that u has a form of $u = 1 + U_t$ with

$$U(x,t) = a + v_0(x) + \int_0^t e^{-\delta s} v_s(x,s)ds,$$

then we have

$$w(x,t) = \tilde{w}_0(x)e^{-\mu_2 \epsilon t}\Theta, \qquad \Theta := e^{-\mu_2((1+\varepsilon)U + \int_0^t w ds)}, \tag{3.2}$$

where $\tilde{w}_0(x) \in W^{m+1}(\Omega)$ and

$$\tilde{w}_0(x) = w_0(x)e^{\mu_2(1+\epsilon)(a+v_0(x))}, \tag{3.3}$$

so that we find that (3.1) is reduced to

$$U_{tt} = \triangle U_t - \nabla \cdot (\chi(1+U_t)\nabla w) - \mu_1 U_t(1 + U_t + a_1 w) - \mu_1 a_1 w. \tag{3.4}$$

Multiplying the both sides of (3.4) by $e^{\delta t}$, we see that (3.4) is rewritten as:

$$Q_1[v] := v_{tt} - \delta v_t - \Delta v_t + \nabla \cdot (\chi e^{\delta t}(1+U_t)\nabla w) + \mu_1 v_t(1 + U_t + a_1 w) + \mu_1 a_1 w = 0. \tag{3.5}$$

Hence, (CM) is reduced to the problem considering $U(x,0) = v_0(x) + a$ and $U_t(x,0) = v_t(x,0)$:

$$(\mathrm{CM})_1 \begin{cases} Q_1[v] = 0 & \text{in } \Omega \times (0,T), \\ \partial_\nu u|_{\partial\Omega} = \partial_\nu w|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) := u_1(x) - 1 & \text{in } \Omega. \end{cases}$$

Since we can easily see by (3.2) that w decreases exponentially for w and v satisfying (2.1), taking δ so small that $\mu_2\epsilon > \delta$, Eq. (3.5) can be essentially regarded as the same type of equation as (1.5). Therefore, w appeared in (3.5) is harmless for derivation of our desired estimate of $(CM)_1$ by the same argument as obtained in (2.6). In fact, we can derive the following same type of estimate as (2.6):

$$\sum_{j=0}^{M} ||\nabla^j v_t||^2(t) + \sum_{j=0}^{M+1} \int_0^t ||\nabla^j v_s||^2(s)ds \leq C E_M[v](0) + C_a,$$

where $C_a \to 0 (a \to \infty)$. Applying the same argument as used for Theorem 2 to (CM) and considering an appropriate iteration scheme of (3.2) and (3.5), we can show that Theorem 2 holds for $(CM)_1$. For more details of this procedure, refer to section 4 in [9]. Hence, we find the solution of (CM) as follows, going back through the reduction process.

Theorem 3 *Assume that* $a_2 > 1$, $a_1 \in \mathbf{R}$, $v_0(x) \in W^{m+1}(\Omega)$, $v_1(x) \in W^m(\Omega)$ *and* $w_0(x)$ *satisfies (3.3) for* $m \geq [n/2] + 3$. *Moreover, assume that* $\|u_1 - 1\|_{m+1}$ *is sufficiently small and* a *is large enough. Then, for* $\epsilon > \delta > 0$ *there exists the solution* $(u(x,t)\ w(x,t))$ *of (CM) such that* $u(x,t), w(x,t)$ *belong to* $C^1([0,\infty); W^{m-1}(\Omega))$ *and*

$$\|u(x,t) - 1\|_{m-1} \leq Ce^{-\delta t}, \quad \|w(x,t)\|_{m-1} \leq Ce^{-\mu_2\epsilon t}.$$

Remark 3 For $a_2 = 1+\epsilon$, (1.2) is written as $w_t = -\mu_2\epsilon wu + \mu_2(1-u-w)$, which is the same as (1.9) for $\eta = \mu_2\epsilon$, $m(x,t) \equiv n(x,t)$ and $a_2 = 1$. Hence, in this case it implies that we can deal with (CM) in the same way as used for the mathematical model in [4, 9].

3.2 The Case $a_1 < 1$, $a_2 = 0$

We divide the both sides of (1.2) with $a_2 = 0$ by $(1 - w)$ and by the same way as obtained in (3.1), so that

$$w(x,t) = 1 + (w(x,0) - 1)e^{-\mu_2 \int_0^t w ds}.$$

We consider that

$$w(x,t) = 1 + \tilde{\tilde{w}}_0(x)e^{-\mu_2(\int_0^t w ds + a)} := 1 + f(w),$$

where $\tilde{\tilde{w}}_0(x) \in W^{m+1}(\Omega)$ and

$$\tilde{\tilde{w}}_0(x) = e^{\mu_2 a}(w_0(x) - 1). \tag{3.6}$$

Furthermore, we consider $u = b + U_t$ for $U = \int_0^t e^{-\delta s} v_s(x,s)ds + v_0(x)$ with $b > 0$, so as to find that (3.1) is reduced to the following for $b = 1 - a_1 > 0$.

$$U_{tt} = D\triangle U_t - \nabla\cdot(\chi(b+U_t)\nabla w) - \mu_1 U_t(b+U_t+a_1 f(w)) - \mu_1 a_1 b f(w). \tag{3.7}$$

Multiplying the both sides of (3.7) by $e^{\delta t}$ and taking account of $U_t = e^{-\delta t} v_t$, from (3.7) it follows that

$$\begin{aligned} Q_2[v] := v_{tt} - \delta v_t - D\triangle v_t + \chi\nabla\cdot(e^{\delta t}(b + e^{-\delta t} v_t)\nabla w) \\ +\mu_1 v_t(b + e^{-\delta} v_t + a_1 f(w)) + \mu_1 a_1 b f(w) = 0. \end{aligned} \tag{3.8}$$

The problem (CM) is reduced to the following:

$$(\mathrm{CM})_2 \begin{cases} Q_2[v] = 0 & \text{in } \Omega \times (0,T), \\ \partial_\nu u|_{\partial\Omega} = \partial_\nu w|_{\partial\Omega} = 0 & \text{on } \partial\Omega \times (0,T), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x) := u_1(x) - 1 & \text{in } \Omega. \end{cases}$$

We see that for w satisfying (2.1)

$$\exp(-\mu_2 \int_0^t w ds) = \exp(-\mu_2 \int_0^t (1 + f(w))ds) \le C_a \exp(-\mu_2 t), \tag{3.9}$$

which means that w and $f(w)$ converge exponentially. Hence, applying the same argument as used for Theorem 3 to $(\mathrm{CM})_2$, we can show that global existence in time of the solution to $(\mathrm{CM})_2$ such that U_t and w converge exponentially to 0 and 1, respectively. Going back through the reduction process, we obtain the following result for (CM).

Theorem 4 *Assume that $a_1 < 1, a_2 = 0$, $v_0(x) \in W^{m+1}(\Omega)$, $v_1(x) \in W^m(\Omega)$ and $w_0(x)$ satisfies (3.6) for $m \ge [n/2] + 3$. Moreover, assume that $\|u_1 - 1\|_m^2$ is sufficiently small and a is large enough. Then for a positive constant $b = 1 - a_1$ and sufficiently small $\delta > 0$, there is the solution $(u(x,t)\ w(x,t))$ of (CM) such that $u(x,t)$, $w(x,t)$ belong to $C^1([0,\infty); W^{m-1}(\Omega))$ and*

$$\|u(x,t) - b\|_{m-1} \le Ce^{-\delta t}, \quad \|w(x,t) - 1\|_{m-1} \le Ce^{-\mu_2 t}.$$

Remark 4 The full proofs of our results obtained in this chapter shall be published elsewhere later.

Acknowledgements We would like to express our sincere gratitude to Professor J.I. Tello for his inestimable suggestions and comments. Also, we are very grateful to thank Professor H. Takamura for his valuable advice. We would like to thank the referee for fruitful suggestions. This work was supported in part by the Grant-in-Aide for Scientific Research (C) 16540176, 19540200, 22540208, 25400148, and 16K05214 from Japan Society for the Promotion of Science.

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A Toy Model of 4D Semilinear Weakly Hyperbolic Wave Equations

Sandra Lucente and Emanuele Marrone

Abstract In this chapter, we prove the large data almost global existence of the 4-dimensional weakly hyperbolic equation:

$$u_{tt} - (t_0 - t)^2 \Delta u = -(t_0 - t)^4 |u| u \, .$$

1 Introduction

Let us consider the semilinear wave equation:

$$\partial_{tt}^2 u(t,x) - |t_0 - t|^{\lambda_1} \Delta u(t,x) = -|t_0 - t|^{\lambda_2} |u(t,x)|^{p-1} u(t,x) \tag{1}$$

where $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$, with $t_0 > 0$, $\lambda_1, \lambda_2 \geq 0$, and $p > 1$. According to the heuristic argument in [9], the critical exponent of this equation with large data is

$$p_c(\lambda_1, \lambda_2, n) = 1 + 4(\lambda_2 + 2)/(n(\lambda_1 + 2) - 4) \tag{2}$$

with $n = 4$, this means

$$p_c(\lambda_1, \lambda_2, 4) = 1 + (\lambda_2 + 2)/(\lambda_1 + 1). \tag{3}$$

In particular for $\lambda_1 = 0$, one has $p_c(0, \lambda_2, 4) = 3 + \lambda_2$. In turn for $\lambda_2 = 0$, this gives the classical exponent $p_c(0,0,4) = 3$. The case $\lambda_1 = \lambda_2 = 0$ has a long history: starting from 1961 with the case $p < p_c(0,0,3) = 5$ studied in [7], this result was extended in higher dimensions 20 years later in [1]. The main difference

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_33

between $n = 3$ and $n \geq 4$ is the use of the $L^p - L^q$ estimates. The critical case $p = p_c(0, 0, 3) = 5$ was completely solved in [6], while in higher dimensions in [12] by $L^p - L^q$ estimates. An exception is given by the result of Kim Lee in [8], where $n = 4$ is covered directly by energy method. Coming back to (1), we recall that the weakly hyperbolic semilinear results start from the paper [2] where Jorgens's theorem is extended. The critical three-dimensional case $p = p_c(\lambda_1, \lambda_2, 3)$ has been studied in [4] and [9] with a smallness assumption on the initial data. In [10], such smallness assumption is removed, and the radial case is considered. Concerning other dimensions, we can quote [3] for the case $n = 1, 2$, though in the 2D case the exponent does not reach the conjectured critical one. In [5], one can find some results in one dimension. No result is known in 4D weakly hyperbolic case. To the best of our knowledge, it depends on the fact that Strichartz estimates for Grushin-type operator like $\partial_{tt} - t^\ell \Delta$ are known only for $\ell \in \mathbb{N}$ and $t \to \infty$ (see[11] and [13]). In this chapter, we will consider a toy model 4D case in which we can avoid Strichartz estimates: we take integer exponents $p = 2$, $\lambda_1 = 2$, and $\lambda_2 = 4$.

Theorem 1 *Let $(u_0, u_1) \in C^4(\mathbb{R}^4) \times C^3(\mathbb{R}^4)$ with compact support. For any $T > 0$, there exists a unique solution $u \in C^2([0, T] \times \mathbb{R}^4)$ of the Cauchy problem:*

$$\begin{cases} \partial_{tt}^2 u - (t_0 - t)^2 \Delta u = -(t_0 - t)^4 |u| u\,, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^4\,, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^4\,, \\ \partial_t u(0, x) = u_1(x) & x \in \mathbb{R}^4\,. \end{cases} \tag{4}$$

In a forthcoming paper, we will consider (1) in subcritical 4D case (3) assuming $\lambda_1 - 1 < \lambda_2 \leq 2\lambda_1$. The critical case is more delicate.

2 Preliminary Results

For a direct proof of the *local existence and uniqueness* for $u_{tt} - a(t)\Delta u = f(t, x, u)$ with $a(t) \geq 0$ a continuous piecewise C^2 function with zeros of finite order, see [3]. This equation obeys to the *finite speed of propagation property* (see [3] again), in particular the solution is compactly supported in space variable.

Let $T > 0$. If $T \in [0, t_0[$, then the equation in (4) is strictly hyperbolic in $[0, T]$ and classical theory applies: we are in subcritical p-range and according to [1], there exists a unique solution $u \in C^2([0, T] \times \mathbb{R}^4)$. Suppose we can prolong this solution up to $T = t_0$, then the strictly hyperbolic argument leads to the solution in any subinterval $[t_0, T_1]$ with $T_1 > t_0$. In order to prove Theorem 1, it remains to consider the case $T = t_0$. In particular, for any $x \in \mathbb{R}^4$, it is enough to prove that

$$\lim_{t \to t_0^-} |u(t, x)| < +\infty\,. \tag{5}$$

Our first tool will be the *energy estimate*. Let

$$e(u)(t,x) = \frac{1}{2}|\partial_t u(t,x)|^2 + (t_0 - t)^2 \frac{|\nabla u(t,x)|^2}{2} + (t_0 - t)^4 \frac{|u(t,x)|^3}{3} \tag{6}$$

be the energy density of the solution of (4). Multiplying our equation by $\partial_t u$, we have

$$\partial_t e(u) - (t_0 - t)^2 \mathrm{div}(\partial_t u \nabla u) = -(t_0 - t)|\nabla u|^2 - \frac{4}{3}(t_0 - t)^3 |u|^3 \tag{7}$$

with negative right side. After integration by parts from (7), we deduce that the energy is pointwise controlled by the initial energy:

$$\int_{\mathbb{R}^4} e(u)(t,x)dx := E(u)(t) \le E_0 := E(u)(0)\,. \tag{8}$$

In particular, $u_t(t,\cdot) \in L^2(\mathbb{R}^4)$, and $u(t,\cdot) \in L^3(\mathbb{R}^4)$. Due to the finite propagation speed, having compact supported data, we can deduce that $u(t,\cdot) \in L^2(\mathbb{R}^4)$. Combining this information with the Sobolev embedding theorems, the aim (5) reduces to finding a continuous positive function $C(t)$ defined on $[0, t_0]$ such that

$$||u(t,\cdot)||_{\dot{H}^3_x(\mathbb{R}^4)} \le C(t). \tag{9}$$

Let us introduce the s-energy:

$$E_s(t) = \frac{1}{2}||\partial_t u(t,\cdot)||^2_{\dot{H}^s_x(\mathbb{R}^4)} + \frac{1}{2}(t_0 - t)^2 ||\nabla u(t,\cdot)||^2_{\dot{H}^s_x(\mathbb{R}^4)} + \frac{1}{2}||u(t,\cdot)||^2_{\dot{H}^s_x(\mathbb{R}^4)}.$$

Differentiating in time, commuting Δ with $|D|^s$, the formal operator calculus gives

$$E_s'(t) = -(t_0 - t)||\nabla u(t,\cdot)||^2_{\dot{H}^s_x(\mathbb{R}^4)} + \int_{\mathbb{R}^4} |D|^s u_t \left(|D|^s u - (t_0 - t)^4 |D|^s (|u|u)\right) dx$$

and finally,

$$E_s'(t) \le ||\partial_t u(t,\cdot)||_{\dot{H}^s_x(\mathbb{R}^4)} \left(||u(t,\cdot)||_{\dot{H}^s_x(\mathbb{R}^4)} + (t_0 - t)^4 |||u(t,\cdot)|u(t,\cdot)||_{\dot{H}^s_x(\mathbb{R}^4)}\right).$$

We know that $|||f|f||_{\dot{H}^3(\mathbb{R}^4)} \le C\ ||f||_{\dot{H}^4(\mathbb{R}^4)} ||f||_{L^\infty(\mathbb{R}^4)}$. If for all $t \in [0, t_0[$ and $x \in \mathbb{R}^4$, it holds

$$|u(t,x)| \lesssim (t_0 - t)^{-\beta} \quad \text{for } \beta < 5\,, \tag{10}$$

then we get $E_s'(t) \lesssim E_s(t)(1 + (t_0 - t)^4 (t_0 - t)^{-\beta})$. By the Gronwall's lemma, we conclude

$$||u(t,\cdot)||^2_{\dot{H}^s_x(\mathbb{R}^4)} \le E_s(t) \le E_s(0) e^t e^{-\frac{(t_0-t)^{5-\beta}}{5-\beta}} .$$

This relation implies (9) and hence (5). Our aim is now the pointwise estimate (10).

Next crucial instrument is the *Liouville transformation*. We associate to $a(t) = (t_0 - t)^2$ the function ϕ which satisfies

$$\begin{cases} \phi'(S) = a(\phi(S))^{-1/2} & S \in [0, T_0) \quad T_0 = \frac{t_0^2}{2}, \\ \phi(0) = 0, \end{cases} .$$

Hence,

$$\phi(S) = t_0 - (2(T_0 - S))^{1/2} \qquad \phi'(S) = (2(T_0 - S))^{-1/2} . \tag{11}$$

Following [9], we can check that if u solves (4) in $[0, t_0)$, then the function:

$$w(T, x) = a(\phi(T))^{1/4} u(\phi(T), x) = (2(T_0 - T))^{1/4} u(\phi(T), x), \tag{12}$$

defined in $[0, T_0)$, solves the equation:

$$(\partial_{TT} - \Delta) w(T, x) = \\ -\frac{3}{4}(2(T_0 - T))^{-2} w(T, x) - (2(T_0 - T))^{3/4} (|w|w)(T, x) . \tag{13}$$

Concerning the initial data, we have

$$w(0, x) = t_0^{\frac{1}{2}} u_0(x), \qquad \partial_t w(0, x) = -\frac{1}{2} t_0^{-3/2} u_0(x) + t_0^{-1/2} u_1(x). \tag{14}$$

Moreover, for any $(T, x) \in [0, T_0[\times \mathbb{R}^4$, starting from (10), our aim becomes

$$|w(T, x)| \lesssim (t_0 - \phi(T))^{-\beta + \frac{1}{2}} \simeq (T_0 - T)^{-\frac{2\beta - 1}{4}}, \text{ for } \beta < 5 . \tag{15}$$

2.1 Representation Formula

Fixed $\overline{z} = (\overline{T}, \overline{x}) \in [0, T_0[\times \mathbb{R}^4$, we look for a representation formula of the solution of (13)–(14). By using the notation of [8], for $T \le \bar{T}$ and $y = x - \overline{x}$, we put

$$[w] = [w](T, y) = w(T - |y|, y + \overline{x}) .$$

Since $\nabla[w] = [\nabla w] - [\partial_T w]\frac{y}{|y|}$ and $\partial_T[w] = [\partial_T w]$, we have

$$\nabla[\partial_T w] = [\nabla \partial_T w] - [\partial^2_{TT} w]\frac{y}{|y|}\,,$$

$$\Delta[w] = [\Delta w] - 2[\nabla\partial_T w]\cdot\frac{y}{|y|} - \frac{3}{|y|}[\partial_T w] + [\partial^2_{TT} w]\,.$$

then,

$$\nabla\cdot\left\{\frac{1}{|y|^2}[\nabla w] + \frac{y}{|y|^3}[\partial_T w] + 2\frac{y}{|y|^4}[w]\right\} = \frac{[\Delta w]}{|y|^2} - \frac{[\partial^2_{TT} w]}{|y|^2} - \frac{1}{|y|^3}[\partial_T w]\,.$$

Given $\varepsilon < \overline{T}$, we integrate this relation, with $T = \bar{T}$, on $D = \{\varepsilon \le |x - \overline{x}| \le \overline{T}\}$. By using (13), we find

$$LHS := \int_D \nabla\cdot\left\{\frac{1}{|y|^2}[\nabla w] + \frac{y}{|y|^3}[\partial_T w] + 2\frac{y}{|y|^4}[w]\right\}dy \tag{16}$$

$$= \int_D -\frac{1}{|y|^3}[\partial_T w] + \frac{1}{|y|^2}\frac{3}{4}[(2(T_0 - \bar{T}))^{-2} w] + \frac{1}{|y|^2}\left[\left(2(T_0 - \bar{T})\right)^{\frac{3}{4}}|w|w\right]dy\,.$$

By divergence theorem, we have

$$LHS = \int_{|y|=\overline{T}} \frac{1}{|y|^2}\left\{\frac{y}{|y|}\cdot\nabla w(0, y+\overline{x}) + \partial_T w(0, y+\overline{x}) + \frac{2}{|y|}w(0, y+\overline{x})\right\}d\sigma_y$$

$$-\int_{|y|=\epsilon} \frac{1}{|y|^2}\left\{\frac{y}{|y|}\cdot\nabla w(\overline{T}-\epsilon, y+\overline{x}) + \partial_T w(\overline{T}-\epsilon, y+\overline{x}) + \frac{2}{|y|}w(\overline{T}-\epsilon, y+\overline{x})\right\}d\sigma_y$$

$$= I(\bar{T},\bar{x}) + II(\bar{T},\bar{x})$$

Since $II \to -4\pi^2 w(\overline{T},\overline{x})$ for $\epsilon \to 0$, from (16), we get

$$w(\overline{T},\overline{x}) = \frac{1}{4\pi^2}I(\overline{T},\overline{x}) + w_L(\overline{T},\overline{x}) + w_M(\overline{T},\overline{x}) + w_N(\overline{T},\overline{x})\,. \tag{17}$$

Here,

$$I(\overline{T},\overline{x}) = t_0^{\frac{1}{2}}\int_{|y|=\overline{T}} \frac{y}{|y|^3}\cdot\nabla u_0(y+\overline{x}) + \left(\frac{t_0^{-2}}{2} + \frac{2}{|y|}\right)\frac{u_0(y+\overline{x})}{|y|^2} + t_0^{-1}u_1(y+\overline{x})d\sigma_y\,;$$

the linear part is given by:

$$w_L(\overline{T},\overline{x}) = \frac{1}{4\pi^2}\int_{|y|\le\overline{T}} \frac{1}{|y|^3}\partial_T w(\overline{T}-|y|, y+\overline{x})dy$$
$$= \frac{1}{4\pi^2\sqrt{2}}\int_0^{\overline{T}}\int_{|x-\overline{x}|=\overline{T}-R} (\overline{T}-R)^{-3}\partial R w(R,x)d\sigma_x dR\,;$$

the mass term with time-singular coefficient is

$$w_M(\overline{T},\overline{x}) = -\frac{3}{64\pi^2}\int_{|y|\le\overline{T}} \frac{1}{|y|^2}(T_0-\overline{T}+|y|)^{-2} w(\overline{T}-|y|, y+\overline{x})dy$$
$$= -\frac{3}{16\pi^2\sqrt{2}}\int_0^{\overline{T}}\int_{|x-\overline{x}|=\overline{T}-R} (T_0-R)^{-2}(\overline{T}-R)^{-2} w(R,x)d\sigma_x dR\,;$$

and the nonlinear part is

$$w_N(\overline{T},\overline{x}) = \frac{-1}{4\pi^2}\int_{|y|\le\overline{T}} \frac{1}{|y|^2}\left(2(T_0-\overline{T}+|y|)\right)^{\frac{3}{4}} |w(\overline{T}-|y|, y+\overline{x})| w(\overline{T}-|y|, y+\overline{x})dy$$
$$= \frac{-1}{4\pi^2\sqrt{2}}\int_0^{\overline{T}}\int_{|x-\overline{x}|=\overline{T}-R} (2(T_0-R))^{\frac{3}{4}}\,(\overline{T}-R)^{-2}|w(R,x)|w(R,x)d\sigma_x dR\,.$$

3 Proof of Theorem 1

Main idea is to use the following Euler integral equation (see [2]).

Lemma 1 *Let* $\gamma > 0$ *and* $\delta > 1$. *Considered the integral equation:*

$$y(t) = \gamma + \delta(\delta-1)\int_0^t (t-s)(r_0-s)^{-2}y(s)ds\,,$$

its solution satisfies $y(t) \le C(\gamma,\delta,r_0)(r_0-t)^{1-\delta}$.

In order to prove (15), we set

$$\mu(R) := \sup\{|w(S,x)| \mid (S,x)\in[0,R]\times\mathbb{R}^4\}\,.$$

and prove

$$\mu(\overline{T}) \le C_1(\overline{T}) + C_2\int_0^{\overline{T}} (\overline{T}-R)(T_0-R)^{-2}\mu(R)\,dR \qquad (18)$$

for a positive function $C_1(\overline{T})$ bounded for $\bar{T} \to T_0$, and suitable $C_2 > 0$. Once we establish this, we have

$$\mu(\overline{T}) \lesssim (T_0 - \overline{T})^{1-\delta} \quad \delta = (1 + \sqrt{1 + 4C_2})/2\,.$$

In order to gain (15), we take $(2\beta - 1)/4 = \delta - 1$ and check that $\beta < 5$; this means $\delta < 13/4$; that is,

$$C_2 < 117/16. \tag{19}$$

Let us estimate the single terms in (17).

Step 1: Estimate for Initial Data Term It is trivial to find a continuous function $C_I(\bar{T}) > 0$ on $[0, T_0]$ such that $\left|I(\overline{T}, \overline{x})/(4\pi^2)\right| \le C_I(\overline{T})$.

Step 2: The Estimate for the Mass Term Directly, we have

$$|w_M(\overline{T}, \overline{x})| \le \frac{3}{8\sqrt{2}} \int_0^{\overline{T}} (T_0 - R)^{-2}(\overline{T} - R)\mu(R)dR\,.$$

We will take $C_2(\bar{T}) \ge C_M = 3/(8\sqrt{2})$.

Step 3: Estimates for the Nonlinear Part Let $\overline{z} = (\bar{t}, \overline{x}) \in [0, t_0] \times \mathbb{R}^4$. Fixed $t_1, t_2, t \in [0, \bar{t}]$, we put

$$K_{t_2}^{t_1}(\overline{z}) = \left\{z = (t, x) \in [t_1, t_2] \times \mathbb{R}^4 \mid \; |x - \overline{x}| \le \phi^{-1}(\bar{t}) - \phi^{-1}(t)\right\}\,,$$

$$M_{t_2}^{t_1}(\overline{z}) = \left\{z = (t, x) \in [t_1, t_2] \times \mathbb{R}^4 \mid \; |x - \overline{x}| = \phi^{-1}(\bar{t}) - \phi^{-1}(t)\right\}\,,$$

$$D(t : \overline{z}) = \{x \in \mathbb{R}^4 | \; z = (t, x) \in K_0^{\bar{t}}(\overline{z})\}\,.$$

Let us denote the local energy:

$$E(u : D(t : \overline{z})) := \int_{D(t:\overline{z})} e(u)(t, x)dx$$

and the flux:

$$d_{\overline{z}}(u)(t, x) := \frac{1}{2}\left|\partial_t u(t, x) - (t_0 - t)\frac{x - \overline{x}}{|x - \overline{x}|} \cdot \nabla u(t, x)\right|^2 + (t_0 - t)^4 \frac{|u(t, x)|^3}{3}\,.$$

Lemma 2 *Fixed* $\overline{z} = (\overline{t}, \overline{x}) \in [0, t_0] \times \mathbb{R}^4$, *let* $u \in C^2(K(\overline{z}))$ *be the solution of* (4). *For any* $0 \le t_1 < t_2 < \overline{t} \le t_0$, *it holds*

$$E(u : D(t_1 : \overline{z})) = E(u : D(t_2 : \overline{z})) + \int_{M_{t_1}^{t_2}(\overline{z})} \frac{d_{\overline{z}}(u)}{\sqrt{1 + (t_0 - t)^{-2}}} d\sigma_x dR$$

$$+ \int_{K_{t_1}^{t_2}(\overline{z})} (t_0 - t)|\nabla u|^2 + 4(t_0 - t)^3 \frac{|u|^3}{3} dx dt \, . \tag{20}$$

We can prove this lemma following [9]. We deduce that $t \in [0, \overline{t}[\to E(u : D(t : \overline{z}))$ decreases. Recalling (8), we have $E(u : D(t : \overline{z})) \le E_0$ for any $t \in [0, \overline{t}]$. Moreover,

$$\int_{\phi^{-1}(\overline{t})}^{0} \int_{|x - \overline{x}| = \phi^{-1}(\overline{t}) - R} d_{\overline{z}}(u)(\phi(R), x) d\sigma_x dR \le E_0 \, . \tag{21}$$

In particular for $T_1, T_2, \overline{T} \in [0, T_0]$, with $t_1 = \phi(T_1)$ and $t_2 = \phi(T_2)$, it holds

$$\int_{T_1}^{T_2} \int_{|x - \overline{x}| = \overline{T} - R} (2(T_0 - R))^{\frac{5}{4}} |w(R, x)|^3 d\sigma_x dR$$

$$= \int_{\phi^{-1}(t_1)}^{\phi^{-1}(t_2)} \int_{|x - \overline{x}| = \phi^{-1}(\overline{t}) - R} (t_0 - \phi(R))^4 |u(\phi(R), x)|^3 d\sigma_x dR \le 3E_0 \, .$$

We are ready to estimate w_N with Hölder inequality:

$$|w_N(\overline{T}, \overline{x})| \le \frac{(2\pi^2)^{\frac{2}{3}}}{4\pi^2\sqrt{2}} \mu(\overline{T}) \left(\int_0^{\overline{T}} (\overline{T} - R)^{3-3} \, (2(T_0 - R))^{1/2} \, dR \right)^{2/3} \times$$

$$\times \left(\int_0^{\overline{T}} \int_{|x - \overline{x}| = \overline{T} - R} (2(T_0 - R))^{5/4} \, |w(R, x)|^3 d\sigma_x dR \right)^{1/3}$$

$$\le \frac{\mu(\overline{T})}{(2\pi^2)^{\frac{1}{3}} \sqrt{2}} \left(\frac{E_0}{3} \right)^{1/3} \left(T_0^{\frac{3}{2}} - (T_0 - \bar{T})^{\frac{3}{2}} \right)^{2/3} .$$

In the sequel, we will choose $0 < \varepsilon < 1$ and find $\bar{T}_\varepsilon > 0$ such that splitting the integral domain as $[0, \bar{T}] = [0, \bar{T}_\varepsilon] \cup [\bar{T}_\varepsilon, \bar{T}]$, it holds

$$|w_N(\overline{T}, \overline{x})| \le \varepsilon \mu(\overline{T}) + C_N(T_\varepsilon) \, .$$

Step 4: The Estimate for the Linear Term First, we change variable setting $x = \bar{x} + (\bar{T} - R)z$, and then we use

$$(\partial_R w)(R, \bar{x} + (\bar{T} - R)z) = \partial_R (w(R, \bar{x} + (\bar{T} - R)z) - z \cdot \nabla w(R, \bar{x} + (\bar{T} - R)z) \, .$$

We have

$$
\begin{aligned}
w_L(\overline{T},\overline{x}) &= \frac{1}{4\pi^2\sqrt{2}}\int_0^{\bar T}\int_{|z|=1}\frac{\partial_R w(R,\bar x+(\bar T-R)z)}{(\bar T-R)^3}(\bar T-R)^3\,d\sigma_z\,dR\\
&= \frac{1}{4\pi^2\sqrt{2}}\int_0^{\bar T}\int_{|z|=1}\partial_R(w(R,\bar x+(\bar T-R)z)\,d\,\sigma_z\,dR\\
&- \frac{1}{4\pi^2\sqrt{2}}\int_0^{\bar T}\int_{|z|=1} z\cdot\nabla w(R,\bar x+(\bar T-R)z)d\sigma_z\,dR \;:= I+II.
\end{aligned}
$$

Changing order of integration, we get

$$
I=\frac{1}{4\pi^2\sqrt{2}}\int_{|z|=1} w(\bar T,\bar x)d\sigma_z+\frac{1}{4\pi^2\sqrt{2}}\int_{|z|=1} w(0,\bar T z+\bar x)d\sigma_z\le\frac{\mu(\bar T)}{2\sqrt{2}}+\frac{\mu(0)}{2\sqrt{2}}\,.
$$

We rewrite $\nabla w=(\phi')^{-\frac{1}{2}}\nabla u$, hence we combine Hölder inequality with (12) and conservation of energy, and we can conclude that

$$
\begin{aligned}
|II| &\le \frac{1}{4\pi}\int_0^{\bar T} a(\phi(R))^{\frac{1}{4}}\int_{|z|=1} z\cdot\nabla u(\phi(R),\bar x+(\bar T-R)z)d\sigma_z dR\\
&\le \frac{1}{4\pi}\left(\int_0^{\bar T}\int_{|z|=1} a(\phi(R))^{-\frac{1}{2}}d\sigma_z dR\right)^{1/2}\times\\
&\times\left(\int_0^{\bar T}\int_{|z|=1} a(\phi(R))|\nabla u|^2(\phi(R),\bar x+(\bar T-R)z)d\sigma_z dR\right)^{1/2}\\
&\le \frac{1}{4\pi}E_0^{\frac{1}{2}}\left(2\pi^2\int_0^{\bar T}(2(\bar T-R)^{-\frac{1}{2}}dR\right)^{\frac{1}{2}}.
\end{aligned}
$$

The last integral converges for $\bar T\to T_0$, as a conclusion:

$$
|w_L(\bar T,\bar x)|\le\frac{\mu(\bar T)}{2\sqrt{2}}+C_L
$$

with C_L depending on the initial data.

Final Step: Proof of Theorem 1 Summarizing for any fixed $\bar T<T_0$ and $0<\varepsilon<1$ there exists $T_\varepsilon>0$ such that (18) is satisfied with

$$
C_1=(C_I(\bar T)+C_N(T_\varepsilon)+C_L(\bar T))/(1-1/2\sqrt{2}-\varepsilon)\,,\quad C_2=3/(8\sqrt{2}(1-(1/2\sqrt{2})-\varepsilon)).
$$

It is possible to choose $0 < \varepsilon < 1$ such that (19) is satisfied. This concludes the proof.

Acknowledgements The Authors thank the anonymous referee for the available remarks. The first Author is grateful to the organizers of Special Interest Group IGPDE in the 11th ISAAC Congress at Linneuniversitetet in Sweden.

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Gevrey Well-Posedness of the Generalized Goursat–Darboux Problem for a Linear PDE

Jorge Marques and Jaime Carvalho e Silva

Abstract We consider the generalized Goursat–Darboux problem for a third-order linear PDE with real coefficients. Our purpose is to find necessary conditions for the problem to be well-posed in the Gevrey classes Γ^s with $s > 1$. It is proved that there exists some critical index s_0 such that if the Goursat–Darboux problem is well-posed in Γ^s for $s > s_0$, then some conditions should be imposed on the coefficients of the derivatives with respect to one of the variables. In order to prove our results, we first construct an explicit solution of a family of problems with data depending on a parameter $\eta > 0$ and then we obtain an asymptotic representation of a solution as η tends to infinity.

1 Introduction

The simplest generalized Goursat–Darboux problem for a third-order linear PDE with real constant coefficients in the classes of Gevrey functions was studied in [8]. Given an open set $\Omega \subseteq \mathbf{R^{3+m}}$, neighborhood of origin, the problem is defined on Ω by:

$$\begin{cases} \partial_t \partial_x \partial_y u(t,x,y,z) = \sum\limits_{0\leq|j|\leq 3} A_j \partial_z^j u(t,x,y,z) \\ u(0,x,y,z) = f_1(x,y,z) \\ u(t,0,y,z) = f_2(t,y,z) \\ u(t,x,0,z) = f_3(t,x,z) \end{cases} \tag{1}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_34

where initial data satisfy necessary compatibility conditions:

$$\begin{cases} f_1(0,y,z) = f_2(0,y,z) \\ f_1(x,0,z) = f_3(0,x,z) \\ f_2(t,0,z) = f_3(t,0,z) \\ f_1(0,0,z) = f_2(0,0,z) = f_3(0,0,z)\,, \end{cases} \tag{2}$$

on three characteristic hyperplanes $\Sigma_i : t = 0$, $x = 0$ and $y = 0$. Let us begin by introducing the Gevrey classes [5] and the concept of the well-posed problem in the sense of Hadamard [6].

Definition 1 (Gevrey Classes) Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R^n}$. The Gevrey class of index s on Ω, $\Gamma^s(\Omega)$, is the space of the all functions $f \in C^\infty(\Omega)$ such that for every compact $K \subset \Omega$ there exist constants $C > 0$ and $L > 0$ satisfying

$$\sup_{x \in K} \mid \partial^\alpha f(x) \mid \leq C L^{|\alpha|} \alpha!^s \text{ , for all multi-index } \alpha. \tag{3}$$

We choose a topology for $\Gamma^s(\Omega)$ according to Rodino [11].

Definition 2 (Problem Well-Posed in the Gevrey Classes) Let $s > 1$ be a real number and Ω be an open subset of $\mathbf{R^n}$, neighborhood of origin. We say that the problem (1)–(2) is $\Gamma^s(\Omega)$ well-posed on Ω if there exists a neighborhood $\mathcal{U} \subset \Omega$ such that:

- For every data $f_i \in \Gamma^s(\Omega \cap \Sigma_i)$, $i = 1, 2, 3$, the problem (1)–(2) has a solution $u \in \Gamma^s(\mathcal{U})$ and it is unique;
- It depends continuously on the data. It means that for every compact $K \subset \Omega$ and every constant $L > 0$ there exist compacts K_i and constants $L_i > 0$, $i = 1, 2, 3$, and $C > 0$ such that

$$\|u\|^s_{L,K} \leq C \left(\| f_1 \|^s_{L_1,K_1} + \| f_2 \|^s_{L_2,K_2} + \| f_3 \|^s_{L_3,K_3} \right) . \tag{4}$$

We are now interested in the so-called case I [2], for a more general class of PDEs. Our goal is to find necessary conditions for the problem to be well-posed in the Gevrey classes. We will try to find some critical index s_0 such that if the generalized Goursat–Darboux problem is well-posed in Γ^s for $s > s_0$, then some conditions should be imposed on the coefficients of the derivatives with respect to one of the variables.

2 Formulation of the Generalized Goursat–Darboux Problem

For simplicity, we suppose $m = 1$ but the formulation and solvability of our problem can be generalized to $m > 1$. Let $\Omega \subseteq \mathbf{R^4}$ be an open set, neighborhood of origin and let

$$P_i(\partial_z) = D_{2,i}\partial_z^2 + D_{1,i}\partial_z \, , \; i = 1, 2, 3 \wedge Q(\partial_z) = E_3\partial_z^3 + E_2\partial_z^2 + E_1\partial_z + E_0 \tag{5}$$

be differential operators with real constant coefficients.

We consider the following generalized Goursat–Darboux problem on Ω:

$$\begin{cases} \partial_t\partial_x\partial_y u(t,x,y,z) = \big(P_1(\partial_z)\partial_t + P_2(\partial_z)\partial_x + P_3(\partial_z)\partial_y + Q(\partial_z)\big)\, u(t,x,y,z) \\ u(0,x,y,z) = f_1(x,y,z) \\ u(t,0,y,z) = f_2(t,y,z) \\ u(t,x,0,z) = f_3(t,x,z) \end{cases} \tag{6}$$

where the initial data satisfy the necessary compatibility conditions (2) on three characteristic hyperplanes $t = 0$, $x = 0$, and $y = 0$.

It was showed in [2] that if the problem (2)–(6) is locally C^∞ well-posed in the neighborhood of origin, then the coefficients of the derivatives with respect to z are zero. So, we expect stronger results in the Gevrey framework.

From now on, we suppose that the problem (6) is Γ^s well-posed on Ω. As we have done in [8], the problem (6) can be reduced to the Cauchy problem following ideas of Bronshtein [1]. By linearity, if $u(t,x,y,z)$ is a solution of the problem (6) on Ω, then

$$v(t,x,y,z) = u(t,x,y,z) + u(x,y,t,z) + u(y,t,x,z) \tag{7}$$

is a solution of the corresponding problem on $\Omega' \subset \Omega$

$$\begin{cases} \partial_t\partial_x\partial_y v(t,x,y,z) = \big(P(\partial_z)\partial_t + P(\partial_z)\partial_x + P(\partial_z)\partial_y + Q(\partial_z)\big)\, v(t,x,y,z) \\ v(0,x,y,z) = f_1(x,y,z) + f_3(x,y,z) + f_2(y,x,z) \\ v(t,0,y,z) = f_2(t,y,z) + f_1(y,t,z) + f_3(y,t,z) \\ v(t,x,0,z) = f_3(t,x,z) + f_2(x,t,z) + f_1(t,x,z)\,. \end{cases} \tag{8}$$

where

$$P(\partial_z) = \frac{1}{3}\left(P_1(\partial_z) + P_2(\partial_z) + P_3(\partial_z)\right)\,. \tag{9}$$

We then reduce the number of the independent variables by setting $t = x = y$. For every parameter $\eta > 0$, taking

$$v(0, x, y, z) = v(t, 0, y, z) = v(t, x, 0, z) = e^{i\eta z}$$

we are looking for a unique solution depending continuously on the data. If v_η is the solution of the problem on Ω'

$$\begin{cases} \partial_t \partial_x \partial_y v(t, x, y, z) = \big(P(\partial_z)\partial_t + P(\partial_z)\partial_x + P(\partial_z)\partial_y + Q(\partial_z)\big)\, v(t, x, y, z) \\ v(0, x, y, z) = v(t, 0, y, z) = v(t, x, 0, z) = e^{i\eta z} \end{cases} \tag{10}$$

then $w_\eta(r, z) = v_\eta(r, r, r, z)$ is the solution of the Cauchy problem on $\tilde{\Omega} \subseteq \mathbf{R^2}$

$$\begin{cases} \partial_r^3 w(r, z) = 27\,\big((D_2\partial_z^2 + D_1\partial_z)\partial_r + (E_3\partial_z^3 + E_2\partial_z^2 + E_1\partial_z + E_0)\big)\, w(r, z) \\ \quad w(0, z) = e^{i\eta z}\,, \end{cases} \tag{11}$$

where $D_j = \dfrac{1}{3}\left(D_{j,1} + D_{j,2} + D_{j,3}\right)$, $j = 1, 2$. We remark that there are two arbitrary data $\partial_r w(0, z)$ and $\partial_r^2 w(0, z)$.

3 Solving the Cauchy Problem

If the Cauchy problem (11) is well-posed in the Gevrey classes, then necessarily

$$E_3^2 - 4D_2^3 \geq 0 \tag{12}$$

by applying the Lax–Mizohata theorem [9].

We determine a unique solution of the problem (11) in the form $w_\eta(r, z) = m_\eta(r)e^{i\eta z}$, hence $m_\eta(r)$ is the solution of the initial value problem:

$$\begin{cases} m'''(r) = 27(-D_2\eta^2 + iD_1\eta)m'(r) + 27(-E_3 i\eta^3 - E_2\eta^2 + iE_1\eta + E_0)m(r) \\ m(0) = 1 \\ m'(0) = \alpha \\ m''(0) = \beta \end{cases} \tag{13}$$

where α and β are unknown. In order to solve the corresponding linear ODE, we use its characteristic equation:

$$\lambda^3 + p(\eta)\lambda + q(\eta) = 0\,, \tag{14}$$

$p(\eta) = -27(-D_2\eta^2 + iD_1\eta)$ and $q(\eta) = -27(-E_3 i\eta^3 - E_2\eta^2 + iE_1\eta + E_0)$. That equation has a solution ζ_η which is given by $\zeta_\eta = z_\eta + \omega_\eta$. To obtain ζ_η, we proceed in three steps (Vieta's method):

1. Find $A_\eta \neq 0$ such that $A_\eta^2 = \Delta_\eta = \left(\frac{q(\eta)}{2}\right)^2 + \left(\frac{p(\eta)}{3}\right)^3$;
2. Find a solution $z_\eta \neq 0$ of the equation $z^3 = -\frac{q(\eta)}{2} + A_\eta$ by de Moivre's formula;
3. Calculate $\omega_\eta = -\dfrac{p(\eta)}{3z_\eta}$.

The other two solutions are $\zeta_\eta = \gamma z_\eta + \overline{\gamma}\omega_\eta$ and $\zeta_\eta = \overline{\gamma} z_\eta + \gamma\omega_\eta$.

Lemma 1 *Let γ and $\overline{\gamma}$ be conjugate complex roots of unity. If $\zeta_\eta = z_\eta + \omega_\eta$, $\zeta_\eta \neq 0$, is a solution of (14), then the solution of the problem (13) is given by:*

$$m_\eta(r) = \tfrac{1}{3}(1 + a_\eta + b_\eta)e^{(z_\eta+\omega_\eta)r} + \tfrac{1}{3}\left(1 + \overline{\gamma}a_\eta + \gamma b_\eta\right)e^{(\gamma z_\eta + \overline{\gamma}\omega_\eta)r} + \\ + \tfrac{1}{3}\left(1 + \gamma a_\eta + \overline{\gamma} b_\eta\right)e^{(\overline{\gamma} z_\eta + \gamma\omega_\eta)r} \tag{15}$$

where

$$a_\eta = \frac{\alpha z_\eta^2 - (\beta + 2p(\eta)/3)\omega_\eta}{z_\eta^3 - \omega_\eta^3} \quad \wedge \quad b_\eta = \frac{-\alpha\omega_\eta^2 + (\beta + 2p(\eta)/3)z_\eta}{z_\eta^3 - \omega_\eta^3}. \tag{16}$$

If ζ_η is a real root of the (14), we simplify (15) by using the Euler's formula.

Corollary 1 (Characteristic Equation with One Real Root)
If $\zeta_\eta = z_\eta + \omega_\eta \in \mathbf{R} - \{0\}$ and $\kappa_\eta = z_\eta - \omega_\eta \in \mathbf{R} - \{0\}$, then

$$m_\eta(r) = \tfrac{1}{3}(1 - c_\eta)e^{\zeta_\eta r} + \tfrac{1}{3}\left(2 + c_\eta\right)\cos(\sqrt{3}\kappa_\eta r/2)e^{-\zeta_\eta r/2} + \\ + \tfrac{\sqrt{3}}{3}d_\eta \sin(\sqrt{3}\kappa_\eta r/2)e^{-\zeta_\eta r/2} \tag{17}$$

where

$$c_\eta = -\frac{\alpha\zeta_\eta + \beta + 2p(\eta)/3}{\zeta_\eta^2 + p(\eta)/3} = -a_\eta - b_\eta$$

and

$$d_\eta = \frac{-i[\alpha(\zeta_\eta^2 + \kappa_\eta^2)/2 - (\beta + 2p(\eta)/3)\zeta_\eta]}{(\zeta_\eta^2 + p(\eta)/3)\kappa_\eta} = -i(a_\eta - b_\eta).$$

If ζ_η is a pure imaginary root of the (14), (15) can be written in a simpler expression.

Corollary 2 (Characteristic Equation with a Pure Imaginary Root)
If $\zeta_\eta = z_\eta + \omega_\eta = -iY_\eta$ *and* $\kappa_\eta = z_\eta - \omega_\eta = -iX_\eta$ *with* $X_\eta \in \mathbf{R} - \{0\}$ *and* $Y_\eta \in \mathbf{R} - \{0\}$*, then*

$$m_\eta(r) = \frac{1}{3}\left[(2 + c_\eta)\cosh(\sqrt{3}X_\eta r/2) + \sqrt{3}d_\eta \sinh(\sqrt{3}X_\eta r/2)\right]e^{iY_\eta r/2} + \\ + \frac{1}{3}(1 - c_\eta)e^{-iY_\eta r} \tag{18}$$

where $c_\eta = -a_\eta - b_\eta$ *and* $d_\eta = -i(a_\eta - b_\eta)$.

4 Results

In the next asymptotic estimates, we use big O, little o, and $\sim$ symbols to compare the growth of functions [10].

Definition 3 Let f and g be complex functions of the real variable η, $\eta > 0$. As $\eta \to \infty$, we say that

(i) f and g are asymptotically equal, $f(\eta) \sim g(\eta)$, if $\lim_{\eta\to\infty} \frac{f(\eta)}{g(\eta)} = 1$;
(ii) f is of order not exceeding g, $f(\eta) = O(g(\eta))$, if there exists a constant k such that $|f(\eta)| \leq k|g(\eta)|$ for all $\eta > 0$;
(iii) f is of order less than g, $f(\eta) = o(g(\eta))$, if $\lim_{\eta\to\infty} \frac{f(\eta)}{g(\eta)} = 0$.

In previous works [2, 3, 7], an explicit solution of the generalized Goursat–Darboux problem involves a hypergeometric function of several variables. However, some difficulties for obtaining asymptotic representations for these kinds of functions were pointed out in the paper [4].

In our work, we have a linear combination of complex exponential functions as solution of the Cauchy problem. In the next propositions, we provide asymptotic representations, as η tends to infinity, for the absolute value of complex functions m_η on a compact, which depends on η.

Proposition 1 *If* $p(\eta) = 0$, $q(\eta) = -27E_1\eta i$, $E_1 \neq 0$, *and* $s > 3$, *then there exist a constant* $c > 0$ *and a compact* K_η*, neighborhood of origin, such that*

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|E_1|}\eta^{1/s}} \tag{19}$$

as η *tends to infinity.*

Proposition 2 *If $q(\eta) = O(\eta)$, $p(\eta) = -27D_1\eta i$, $D_1 \neq 0$, and $s > 2$, then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that*

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|D_1|}\eta^{1/s}} \tag{20}$$

as η tends to infinity.

Proposition 3 *If $p(\eta) = O(\eta)$, $q(\eta) = 27E_2\eta^2$, $E_2 \neq 0$, and $s > 3/2$, then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that*

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|E_2|}\eta^{1/s}} \tag{21}$$

as η tends to infinity.

Proposition 4 *Let $p(\eta) = 27D_2\eta^2 + O(\eta)$, $q(\eta) = 27E_3 i\eta^3 + O(\eta^2)$ such that $E_3^2 - 4D_2^3 > 0$.*

(i) If $D_2 \neq 0$ and $s > 1$, then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{|\sqrt[3]{\rho^2}-9D_2|\eta^{1/s}} \tag{22}$$

as η tends to infinity, where $\rho = \frac{27}{2}\left(\sqrt{E_3^2 - 4D_2^3} + E_3\right) \neq 0$;

(ii) If $D_2 = 0 \wedge E_3 \neq 0$ and $s > 1$, then there exist a constant $c > 0$ and a compact K_η, neighborhood of origin, such that

$$\sup_{r\in K_\eta} \mid m_\eta(r) \mid \sim ce^{\sqrt[3]{|E_3|}\eta^{1/s}} \tag{23}$$

as η tends to infinity.

In the proofs of the propositions, our approach is based on asymptotic analysis of the initial data in order to have only one exponential function as dominant term; that is, when one exponential function tends to infinity and the others tend to zero.

Finally, we present main results.

Theorem 1 *If the problem (2)–(6) is Γ^s well-posed on Ω, then:*

(i):

$$s > 1 \quad \Rightarrow \quad 27E_3^2 = 4(D_{2,1} + D_{2,2} + D_{2,3})^3\,; \tag{24}$$

(ii):

$$s > \frac{3}{2} \quad \Rightarrow \quad E_2 = 0\,; \tag{25}$$

(iii):

$$s > 2 \quad \Rightarrow \quad D_{1,1} + D_{1,2} + D_{1,3} = 0\,; \tag{26}$$

(iv):

$$s > 3 \quad \Rightarrow \quad E_1 = 0\,. \tag{27}$$

Proof We suppose that the problem (2)–(6) is Γ^s well-posed on Ω with $s > 1$. Then for every $\eta > 0$, the corresponding problem (10) has a unique solution v_η on Ω'.

On one hand, we determine *a priori* an estimate for the Gevrey norm of v_η, an upper bound, from the initial data, $\|e^{i\eta z}\|^s_{L,K}$, for every compact $K \subset \Omega \subseteq \mathbf{R}^{\mathbf{3+m}}$ and every constant $L > 0$. The partial derivatives of $e^{i\eta z}$ with respect to multi-index (l, k, j, α), such that $l \neq 0$ or $k \neq 0$ or $j \neq 0$, are zero. Otherwise, it is clear that

$$\partial_z^\alpha (e^{i\eta z}) = (i\eta)^{|\alpha|} e^{i\eta z}\,,$$

it follows that

$$\sup_{(t,x,y,z)\in K} \mid \partial^\alpha (e^{i\eta z}) \mid = \eta^{|\alpha|}\,.$$

Using $\mid \alpha \mid ! \leq m^{|\alpha|}\alpha!$ and $\mid \alpha \mid^{|\alpha|} \leq e^{|\alpha|} \mid \alpha \mid !$, we get

$$\|e^{i\eta z}\|^s_{L,K} \leq \sup_\alpha \left(\mid \alpha \mid^{-s|\alpha|} L^{-|\alpha|} (m^s e^s \eta)^{|\alpha|} \right)\,.$$

Since the supremum is equal to $e^{smL^{-1/s}\eta^{1/s}}$, there exist constants $c_1 = smL^{-1/s}$ and $C > 0$ such that

$$\|v_\eta\|^s_{L,K} \leq C\|e^{i\eta z}\|^s_{L,K} \leq Ce^{c_1\eta^{1/s}} \tag{28}$$

for every $\eta > 0$. It is a condition for stability of solution.

On the other hand, let's see that if we suppose $E_3^2 - 4D_2^3 > 0$ in (1), $E_2 \neq 0$ in (2), $D_1 \neq 0$ in (3), $E_1 \neq 0$ in (4), then we obtain a contradiction with (28). By using previous propositions, we construct an asymptotic representation of a solution as η tends to infinity. For every neighborhood of the origin $\mathcal{O}$, there exist a compact K_η,

$K_\eta \subset \mathcal{O}$, and constants $C > 0$ and $c_2 > 0$ such that

$$\sup_{r \in K_\eta} \mid v_\eta(r, r, r, z) \mid \sim Ce^{c_2 \eta^{1/s}} \tag{29}$$

Notice that $K_\eta \subset \mathcal{O}$ only if $s_0 = 1$ in (1), $s_0 = 3/2$ in (2), $s_0 = 2$ in (3), and $s_0 = 3$ in (4). We have

$$\sup_{r \in K_\eta} \mid m_\eta(r) \mid = \sup_{r \in K_\eta} \mid w_\eta(r, z) \mid = \sup_{r \in K_\eta} \mid v_\eta(r, r, r, z) \mid$$

and

$$\|v_\eta\|^s_{L,K_\eta} > \sup_{r \in K_\eta} \mid v_\eta(r, r, r, z) \mid ,$$

for all $L > 0$. We can choose L with $L > \left(\frac{sm}{c_2}\right)^s$ such that

$$\|v_\eta\|^s_{L,K_\eta} > Ce^{c_2 \eta^{1/s}} \tag{30}$$

as η tends to infinity. We conclude that (30) contradicts (28) because $c_2 > c_1$.

Acknowledgements This work was partially supported by the Centre for Mathematics of the University of Coimbra—UID/MAT/00324/2013, funded by the Portuguese Government through FCT/MEC, and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

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On the Regularity of the Semilinear Term on the Cauchy Problem for the Schrödinger Equation

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Abstract The regularity assumption on the semilinear term of the Schrödinger equation is considered in the class of Sobolev spaces. Based on the modification of the Strichartz estimate, the assumption is improved to the half value from below compared with the known result for the scaling critical semilinear term.

1 Introduction

In this chapter, we complement and give several comments on the talk by the author at the session of Special interest group: IGPDE Recent progress in evolution equations (igpde2), and Nonlinear PDE (nlpde) in the 11th ISAAC congress at Linnaeus University, Sweden.

Let us consider the Cauchy problem for semilinear Schrödinger equations:

$$\begin{cases} \partial_t u(t,x) + i\Delta u(t,x) = f(u)(t,x) \ \text{ for } \ (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,\cdot) = u_0(\cdot) \in H^s(\mathbb{R}^n), \end{cases} \tag{1}$$

where $n \geq 1$, $\Delta := \sum_{j=1}^n \partial^2/\partial x_j^2$ is the Laplacian, $f(u) := \lambda|u|^{p-1}u$ or $f(u) := \lambda|u|^p$ with $\lambda \in \mathbb{C}$ for example, $1 < p < \infty$, and u_0 is a given initial datum in the Sobolev space $H^s(\mathbb{R}^n)$ for $0 \leq s < \infty$. Cazenave and Weissler [3] proved the existence of time global solutions of (1) for small data under the conditions:

$$0 \leq s < \frac{n}{2}, \quad [s]+1 < p = p(s) := 1 + \frac{4}{n-2s}, \tag{2}$$

where $p(s)$ is the critical number for (1) by the scaling $u_R(t,x) = R^{2/(p-1)}u(R^2t, Rx)$ for any $R > 0$, and $[s]$ denotes the largest integer less than or equal to s. The

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_35

condition $[s]+1<p$ is the required regularity for $f(u)$, and it can be improved to $s<p$ by the method of Ginibre, Ozawa, and Velo in [8] (see [12]). The aim of this chapter is mainly to improve this condition to

$$\frac{s}{2} < p = p(s), \tag{3}$$

and to that end, we also aim to refine the modified Strichartz estimate by Pecher [14]. Here, the special case $s=2$ was proved in [3, Theorem 1.4]. However, the other case has been left open for long time.

To describe the corresponding results, we define $p_0(s)$ by:

$$p_0(s) := \begin{cases} 1 & \text{for } s \le 2, \\ s-1 & \text{for } 2 < s < 4, \\ s-2 & \text{for } 4 \le s. \end{cases}$$

And, we consider the problem (1) under the condition:

$$0 \le s < \frac{n}{2}, \quad p_0(s) < p < p(s). \tag{4}$$

The condition $p_0(s) < p$ for $s \le 2$ and $s \ge 4$ is natural since $1 < p$ and the s-derivative of u by the spatial variables requires the $(s-2)$-derivative of $f(u)$ by the first equation in (1). The existence of time local solutions of (1) under (4) has been shown by Tsutsumi [17] for $s=0$, Ginibre and Velo [4, Theorem 3.1] for $s=1$ (see also [5]), and Tsutsumi [16] for $s=2$ for $f(u)=\lambda|u|^{p-1}u$ with $i\lambda \in \mathbb{R}$ mainly by the use of the L^p-L^q estimate and the regularization technique. Kato [9, 10] used the Strichartz estimate and gave alternative proofs for the cases $s=0,1,2$ both for $f(u)=\lambda|u|^{p-1}u$ and $f(u)=\lambda|u|^p$ with $\lambda \in \mathbb{C}$. Pecher [14] used the fractional Besov space for the time variable and proved the result when s is a real number with (4) and $s>1$. He has also shown the existence of time global solutions when the initial data are sufficiently small. The condition $p_0(s)<p$ was improved to $s/2<p$ for $2<s<4$ in [18], which seems to be natural since $p_0(s)$ is discontinuous at $s=4$ and by the property of the Schrödinger equation (one time derivative corresponds to two spatial derivatives). However, the methods in [14] and [18] are not applicable to time global solutions for the critical case $p=p(s)$ by the technical conditions on the Strichartz estimates there. Especially, the interpolation argument to construct the Strichartz estimates prevents us from treating the critical point $p(s)$ in its application to (1). In this chapter, we improve the Strichartz estimates in [14] and [18] using the auxiliary space $\ell^\alpha L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ defined by (5) below, and we show the time global solutions for $p=p(s)$. We refer to [13] for the main theorem and its full proof in this chapter.

To state our theorem, we prepare several function spaces. Let $\{\varphi_j\}_{j=-\infty}^{\infty}$ be the Littlewood–Paley decomposition of the unity on $\mathbb{R}$. Namely, let φ be a function whose Fourier transform $\widehat{\varphi}$ is a nonnegative function which satisfies supp $\widehat{\varphi} \subset \{\tau \in$

$\mathbb{R};\ 1/2 \le |\tau| \le 2\}$ and $\sum_{j=-\infty}^{\infty} \widehat{\varphi}(\tau/2^j) = 1$ for $\tau \neq 0$. We define ψ and φ_j for $j \in \mathbb{N}$ by $\widehat{\varphi_j}(\cdot) = \widehat{\varphi}(\cdot/2^j)$, $\widehat{\psi} = 1 - \sum_{j\ge 1} \widehat{\varphi_j}$. We define $\chi_j := \sum_{k=j-1}^{j+1} \varphi_k$ for $j \ge 1$, $\chi_0 := \psi + \varphi_1$. We put $\psi(x) := \mathscr{F}_\xi^{-1}\widehat{\psi}(|\xi|)$ and $\varphi_j(x) := \mathscr{F}_\xi^{-1}\widehat{\varphi_j}(|\xi|)$ for $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$. For $s \in \mathbb{R}$ and $1 \le r, \alpha \le \infty$, the Besov space is defined by $B^s_{r,\alpha}(\mathbb{R}^n) := \{u \in \mathscr{S}'(\mathbb{R}^n);\ \|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)} < \infty\}$, where $\mathscr{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$:

$$\|u\|_{B^s_{r,\alpha}(\mathbb{R}^n)} := \|\psi *_x u\|_{L^r(\mathbb{R}^n)} + \begin{cases} \left\{ \sum_{j\ge 1} \left(2^{sj} \|\varphi_j *_x u\|_{L^r(\mathbb{R}^n)} \right)^\alpha \right\}^{1/\alpha} & \text{if } \alpha < \infty, \\ \sup_{j\ge 1} 2^{sj} \|\varphi_j \ *_x\ u\|_{L^r(\mathbb{R}^n)} & \text{if } \alpha = \infty, \end{cases}$$

where $*_x$ denotes the convolution in the variables in $\mathbb{R}^n$. We prepare the Besov space of vector-valued functions (see [1, 15]). For functions $u = u(t,x)$ and $v = v(t,x)$, we denote their convolutions in t and x variables by $u *_t v$ and $u *_x v$, respectively. For $1 \le q, \ell \le \infty$, and a Banach space V, we denote the Lebesgue space for functions on $\mathbb{R}$ to V by $L^q(\mathbb{R}, V)$ and the Lorentz space by $L^{q,\ell}(\mathbb{R}, V)$. We define the Sobolev space $W^{1,q}(\mathbb{R}, V) := \{u \in L^q(\mathbb{R}, V);\ \partial_t u \in L^q(\mathbb{R}, V)\}$ and the Besov space $B^s_{q,\alpha}(\mathbb{R}, V) := \left\{u \in \mathscr{S}'(\mathbb{R}, V);\ \|u\|_{B^s_{q,\alpha}(\mathbb{R},V)} < \infty\right\}$, where

$$\|u\|_{B^s_{q,\alpha}(\mathbb{R},V)} := \|\psi *_t u\|_{L^q(\mathbb{R},V)} + \left\{ \sum_{j\ge 1} \left(2^{sj} \|\varphi_j *_t u\|_{L^q(\mathbb{R},V)} \right)^\alpha \right\}^{1/\alpha}$$

if $\alpha < \infty$ with trivial modification if $\alpha = \infty$. We define the space $\ell^\alpha L^q(\mathbb{R}, L^r(\mathbb{R}^n)) := \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}, L^r(\mathbb{R}^n));\ \|u\|_{\ell^\alpha L^q(\mathbb{R},L^r(\mathbb{R}^n))} < \infty\right\}$, where

$$\|u\|_{\ell^\alpha L^q(\mathbb{R},L^r(\mathbb{R}^n))} := \|\psi *_x u\|_{L^q(\mathbb{R},L^r(\mathbb{R}^n))} + \left(\sum_{j\ge 1} \|\varphi_j *_x u\|^\alpha_{L^q(\mathbb{R},L^r(\mathbb{R}^n))} \right)^{1/\alpha} \tag{5}$$

if $\alpha < \infty$ with trivial modification if $\alpha = \infty$. We also define $\ell^\alpha L^{q,\ell}(\mathbb{R}, L^r(\mathbb{R}^n))$ similarly.

We show the global well-posedness of the problem (1) for small initial data under the condition (3). For any function f from $\mathbb{C}$ to $\mathbb{C}$, we denote the derivatives $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ by f', where $\bar{z}$ is the complex conjugate of z. For $1 < p < \infty$, we say that f satisfies $N(p)$ if $f \in C^1(\mathbb{C}, \mathbb{C})$ in the sense of the derivatives by z and $\bar{z}$, $f(0) = f'(0) = 0$, and

$$|f'(z_1) - f'(z_2)| \le \begin{cases} C \max_{w=z_1,z_2} |w|^{p-2}|z_1 - z_2| & \text{if } p \ge 2, \\ C|z_1 - z_2|^{p-1} & \text{if } 1 < p < 2 \end{cases} \tag{6}$$

for any $z_1, z_2 \in \mathbb{C}$. We note that $f(z) = \lambda |z|^{p-1} z$ and $f(z) = \lambda |z|^p$ with $\lambda \in \mathbb{C}$ satisfy $N(p)$ (see [6, Remark 2.3′]). For $s \geq 0$, an admissible pair (according to the definition given in Sect. 2) (q, r) and $1 \leq \alpha \leq \infty$, we define a function space $X^s_{q,r,\alpha}$ by:

$$X^s_{q,r,\alpha} := C(\mathbb{R}, H^s(\mathbb{R}^n)) \cap C^1(\mathbb{R}, H^{s-2}(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}, H^s(\mathbb{R}^n)) \cap W^{1,\infty}(\mathbb{R}, H^{s-2}(\mathbb{R}^n)) \\ \cap B^{s/2}_{q,2}(\mathbb{R}, L^r(\mathbb{R}^n)) \cap L^q(\mathbb{R}, B^s_{r,\alpha}(\mathbb{R}^n)) \cap W^{1,q}(\mathbb{R}, B^{s-2}_{r,\alpha}(\mathbb{R}^n))$$

with the metric in $L^\infty(\mathbb{R}, L^2(\mathbb{R}^n)) \cap L^q(\mathbb{R}, L^r(\mathbb{R}^n))$, where we remove $C^1(\mathbb{R}, H^{s-2}(\mathbb{R}^n))$, $W^{1,\infty}(\mathbb{R}, H^{s-2}(\mathbb{R}^n))$, and $W^{1,q}(\mathbb{R}, B^{s-2}_{r,\alpha}(\mathbb{R}^n))$ when $s < 2$.

We have the following result on the Cauchy problem (1).

Theorem 1 *Let $n \geq 8$. Let $1 < s < 4$ with $s \neq 2$, and let $s/2 < p = p(s) \leq s$. If $s \geq 3$ with $p < 2$, or equivalently if $3 \leq s < (n-4)/2$, we further assume either of the following:*

$$\begin{aligned} &\text{(i)} \quad n = 11, \\ &\text{(ii)} \quad n = 12 \ \text{ and } \ 7 - \sqrt{15} \leq s < 5 - \sqrt{3}. \end{aligned} \tag{7}$$

Let f satisfy $N(p)$. Then, there exists an admissible pair (q, r) such that if $u_0 \in H^s(\mathbb{R}^n)$ is sufficiently small, then the Cauchy problem (1) has a unique global solution u in $X^s_{q,r,\alpha}$, where we have put $\alpha := 2$ for $s < 3$, and $\alpha := q$ for $3 \leq s$. Moreover, the solutions depend on the initial data continuously, namely, the flow mapping $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R}^n)$ to $X^s_{q,r,\alpha}$.

Throughout the chapter, we denote by $A \lesssim B$ the inequality $A \leq CB$ for some constant $C > 0$ which is not essential in our argument. For any function $f = f(t)$ or $f = f(x)$, its Fourier transform is denoted by $\widehat{f}$. For any function $f = f(t, x)$, $\widehat{f}$ and $\widetilde{f}$ denote its Fourier transform by x and (t, x) variables, respectively. We abbreviate $L^r(\mathbb{R}^n)$ by L^r, $L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ by $L^q L^r$, and $\ell^\alpha L^q(\mathbb{R}, L^r(\mathbb{R}^n))$ by $\ell^\alpha L^q L^r$ as long as no fear of confusion. We use the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^n)$ and Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ (see [2]).

2 Outline of the Proof of Theorem 1

In this section, we give the outline of the proof of Theorem 1. We only consider the case $1 < s < 2$ since this case is simple compared with other cases and we are able to show the essential part in it.

Let us recall the Strichartz estimates. For $1 \leq r \leq \infty$, we put $\delta(r) := n(1/2 - 1/r)$. We say that the pair (q, r) is admissible if $2 \leq q, r \leq \infty$ and $2/q = \delta(r)$ with $(q, r, n) \neq (2, \infty, 2)$. For $1 \leq r \leq \infty$, r' denotes its conjugate number defined by $1/r + 1/r' = 1$. We use the following Strichartz estimates (see, e.g., [7, 11, 13, 14, 18]).

Lemma 1 *Let $s \in \mathbb{R}$, and let (q, r) and (γ, ρ) be admissible pairs. Then, the solution u of*

$$\begin{cases} \partial_t u + i\Delta u = f \;\; on \;\; \mathbb{R} \times \mathbb{R}^n, \\ u(0, \cdot) = u_0(\cdot) \end{cases}$$

satisfies

$$\|u\|_{L^\infty(\mathbb{R}, H^s(\mathbb{R}^n)) \cap L^q(\mathbb{R}, B^s_{r,2}(\mathbb{R}^n))} \leq C\|u_0\|_{H^s(\mathbb{R}^n)} + C\|f\|_{L^{\gamma'}(\mathbb{R}, B^s_{\rho',2}(\mathbb{R}^n))},$$

where the constant $C > 0$ is independent of u, f, and u_0. Moreover, $u \in C(\mathbb{R}, H^s(\mathbb{R}^n))$.

Lemma 2 *Let $s > 0$, and let (q, r) be an admissible pair with $2 < q < \infty$. Then, the solution u of the problem:*

$$\begin{cases} \partial_t u + i\Delta u = 0 \;\; on \;\; \mathbb{R} \times \mathbb{R}^n, \\ u(0, \cdot) = u_0(\cdot) \end{cases}$$

satisfies

$$\|u\|_{B^{s/2}_{q,2}(\mathbb{R}, L^r(\mathbb{R}^n))} \leq C\|u_0\|_{H^s(\mathbb{R}^n)},$$

where the constant $C > 0$ is independent of u and u_0.

Lemma 3 *Let $n \geq 1$, $0 < \theta < 1$, $2 \leq \alpha \leq \infty$. Let (q, r) and (γ, ρ) be admissible pairs. Assume $\rho < \infty$ when $\alpha < \infty$. Let $1 \leq \bar{q}, \bar{r} \leq \infty$ satisfy $2/\bar{q} - \delta(\bar{r}) = 2(1 - \theta)$. For any fixed function f, let us consider the problem:*

$$\begin{cases} \partial_t u + i\Delta u = f \;\; on \;\; \mathbb{R} \times \mathbb{R}^n, \\ u(0, \cdot) = 0. \end{cases} \tag{8}$$

Then, there exists a constant $C > 0$ which is independent of u and f such that

$$(1) \;\; \|u\|_{B^\theta_{q,\alpha}(\mathbb{R}, L^r(\mathbb{R}^n))} \leq C\|f\|_{B^\theta_{\gamma',\alpha}(\mathbb{R}, L^{\rho'}(\mathbb{R}^n))} + C\|f\|_{\ell^\alpha L^{\bar{q}}(\mathbb{R}, L^{\bar{r}}(\mathbb{R}^n))}. \tag{9}$$

Moreover, if $\max\{\alpha, \bar{q}\} \leq q$, then

$$(2) \;\; \|u\|_{L^q(\mathbb{R}, B^{2\theta}_{r,\alpha}(\mathbb{R}^n))} \leq C\|f\|_{B^\theta_{\gamma',\alpha}(\mathbb{R}, L^{\rho'}(\mathbb{R}^n))} + C\|f\|_{\ell^\alpha L^{\bar{q}}(\mathbb{R}, L^{\bar{r}}(\mathbb{R}^n))}. \tag{10}$$

We regard the solution of the Cauchy problem (1) as the fixed point of the integral equation given by:

$$u(t) = \Phi(u)(t) := U(t)u_0 + \int_0^t U(t - t')f(u(t'))dt'$$

for $t \in \mathbb{R}$, where $u(t) := u(t, \cdot)$ and $U(t) := \exp(-it\Delta)$. Let n, s, p satisfy the assumption in the theorem. For any given $2 \le \gamma \le \infty$, we define ρ, q, and r by:

$$q := p\gamma', \quad \frac{2}{\gamma} - \delta(\rho) = \frac{2}{q} - \delta(r) = 0. \tag{11}$$

We note that (γ, ρ) and (q, r) form admissible pairs if $2 \le q \le \infty$. We put

$$\frac{1}{m(r,s)} := \frac{1}{r} - \frac{s}{n} > 0, \tag{12}$$

where the last inequality holds since $1/m(r, s) = 2(p/(p-1) - 1/\gamma')/np > 0$ by the assumption $p = 1 + 4/(n - 2s)$. Moreover, $m(r, s)$ satisfies

$$\frac{1}{\rho'} = \frac{p-1}{m(r,s)} + \frac{1}{r}. \tag{13}$$

For any $2 \le \alpha \le \infty$, we put $X := X^s_{q,r,\alpha}$ and $X(R) := \{u \in X\,;\, u(0) = u_0,\ \|u\|_X \le R\}$ for $R > 0$. We show that Φ is a contraction mapping on $X(R)$ for some $R > 0$. We separate the proof of the theorem into three cases $1 < s < 2$, $2 < s < 3$, and $3 \le s < 4$. The case $1 < s < 2$ is the simplest case to apply Lemma 3 to the proof of Theorem 1.

Let $1 < s < 2$. We put $\theta := s/2$, $\gamma := 2(n+2)/n$, $\alpha := 2$. Then, $\rho = 2(n+2)/n$, $2 < q < \infty$, where $2 < q$ holds since it is rewritten as $n - 2s < 2(n+2)$ by $p = 1 + 4/(n-2s)$. So that, $2 < r < 2n/(n-2)$.

We have the estimate:

$$\|\Phi(u)\|_{L^\infty L^2 \cap L^q L^r} \lesssim \|u_0\|_{L^2} + \|u\|^{p-1}_{L^q B^s_{r,m(r,s)}} \|u\|_{L^q L^r},$$
$$d(\Phi(u), \Phi(v)) := \|\Phi(u) - \Phi(v)\|_{L^\infty L^2 \cap L^q L^r} \lesssim \max_{w=u,v} \|w\|^{p-1}_{L^q B^s_{r,m(r,s)}} d(u, v)$$

for any u and v. Indeed, by Lemma 1, we have

$$\|\Phi(u)\|_{L^\infty L^2 \cap L^q L^r} \lesssim \|u_0\|_{L^2} + \|f(u)\|_{L^{\gamma'} L^{\rho'}}.$$

By (13), we have

$$\|f(u)\|_{L^{\gamma'} L^{\rho'}} \lesssim \|u\|^{p-1}_{L^q L^{m(r,s)}} \|u\|_{L^q L^r} \lesssim \|u\|^{p-1}_{L^q B^s_{r,m(r,s)}} \|u\|_{L^q L^r} \tag{14}$$

by the Hölder inequality and the Sobolev embedding theorem. So that, we obtain the first inequality. We also obtain the second inequality similarly by:

$$\|\Phi(u) - \Phi(v)\|_{L^\infty L^2 \cap L^q L^r} \lesssim \|f(u) - f(v)\|_{L^{\gamma'} L^{\rho'}} \lesssim \max_{w=u,v} \|w\|^{p-1}_{L^q B^s_{r,m(r,s)}} \|u - v\|_{L^q L^r}.$$

We put $\bar{q} := \gamma'$. We define $\bar{r}$ by the equation $2/\bar{q} - \delta(\bar{r}) = 2(1-\theta)$. Since $\rho < \infty$, $1 < \bar{q} \le \alpha \le q$, and $1 < \bar{r} < \infty$, we use Lemmas 1–3 to have

$$\|\Phi(u)\|_X \lesssim \|u_0\|_{H^s} + \|f(u)\|_{B^{\theta}_{\gamma',\alpha}L^{\rho'}} + \|f(u)\|_{\ell^\alpha L^{\bar{q}} L^{\bar{r}}}. \tag{15}$$

We estimate the second and third terms in the right-hand side, respectively.

We have the estimate:

$$\|f(u)\|_{\ell^\alpha L^{\bar{q}} L^{\bar{r}}} \lesssim \|f(u)\|_{L^{\gamma'} B^0_{\bar{r},\alpha}} \lesssim \|u\|^p_{L^q B^s_{r,\alpha}}.$$

Indeed, we have $\|f(u)\|_{\ell^\alpha L^{\bar{q}} L^{\bar{r}}} \lesssim \|f(u)\|_{L^{\gamma'} B^0_{\bar{r},\alpha}}$ by $\bar{q} = \gamma' \le \alpha$. Let $\varepsilon > 0$ be a sufficiently small number. By the Sobolev embedding $B^{\varepsilon}_{m(\bar{r},-\varepsilon),\alpha} \hookrightarrow B^0_{\bar{r},\alpha}$, the nonlinear estimate in [8, Lemma 3.4] with the equation $1/m(\bar{r},-\varepsilon) = (p-1)/m(r,s) + 1/m(r,s-\varepsilon)$, and the embedding $B^s_{r,\alpha} \hookrightarrow L^{m(r,s)} \cap B^{\varepsilon}_{m(r,s-\varepsilon),\alpha}$ by $\alpha \le m(r,s)$, we have

$$\|f(u)\|_{B^0_{\bar{r},\alpha}} \lesssim \|f(u)\|_{B^{\varepsilon}_{m(\bar{r},-\varepsilon),\alpha}} \lesssim \|u\|^{p-1}_{L^{m(r,s)}} \|u\|_{B^{\varepsilon}_{m(r,s-\varepsilon),\alpha}} \lesssim \|u\|^p_{B^s_{r,\alpha}}.$$

Since $\bar{q} = \gamma' = q/p$, we obtain the required inequality.

We have the estimate:

$$\|f\|_{B^{\theta}_{\gamma',\alpha}L^{\rho'}} \lesssim \|u\|^{p-1}_{L^q B^s_{r,\alpha}} \|u\|_{B^{\theta}_{q,\alpha}L^r}.$$

Indeed, we use the equivalent norm: (see [15] and [18, (2.3)])

$$\|f(u)\|_{B^{\theta}_{\gamma',\alpha}L^{\rho'}} = \|f(u)\|_{L^{\gamma'}L^{\rho'}} + \left\{ \int_0^\infty \left(\tau^{-\theta} \|f(u(\cdot)) - f(u(\cdot+\tau))\|_{L^{\gamma'}L^{\rho'}} \right)^\alpha \frac{d\tau}{\tau} \right\}^{1/\alpha}.$$

The first term in the right-hand side is bounded by $\|u\|^{p-1}_{L^q B^s_{r,\alpha}} \|u\|_{L^q L^r}$ by (14). The second term is bounded by $\|u\|^{p-1}_{L^q L^{m(r,s)}} \|u\|_{B^{\theta}_{q,\alpha}L^r}$ by the inequality:

$$|f(u(\cdot)) - f(u(\cdot+\tau))| \lesssim (|u(\cdot)| + |u(\cdot+\tau)|)^{p-1} |u(\cdot) - u(\cdot+\tau)|.$$

So that, we obtain the required inequality by the embedding $B^s_{r,\alpha} \hookrightarrow L^{m(r,s)}$.

By the above estimates, we have obtained

$$\|\Phi(u)\|_X \le C\|u_0\|_{H^s} + C\|u\|^p_X \le C\|u_0\|_{H^s} + CR^p,$$

$$d(\Phi(u), \Phi(v)) \le C \max_{w=u,v} \|w\|^{p-1}_X d(u,v) \le CR^{p-1} d(u,v)$$

for any $u, v \in X(R)$ for some constant $C > 0$. Taking R such that $CR^{p-1} \leq 1/2$ and $R \geq 2C\|u_0\|_{H^s}$ for sufficiently small u_0, Φ becomes a contraction mapping on X_R.

The last part of the theorem, the continuous dependence of the solutions to the initial data, follows easily. Indeed, for any solutions u and $v \in X$ for initial data u_0 and $v_0 \in H^s(\mathbb{R}^n)$, respectively, we have

$$d(u, v) \lesssim \|u_0 - v_0\|_{L^2} + \|f(u) - f(v)\|_{L^{\gamma'}L^{\rho'}} \lesssim \|u_0 - v_0\|_{L^2} + \max_{w=u,v} \|w\|_X^{p-1} d(u, v),$$

where the first inequality follows from Lemma 1, and the second inequality follows similarly to the above argument for $d(\Phi(u), \Phi(v))$. So that, the flow mapping $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R}^n)$ to X.

Acknowledgements I am thankful to the organizers of Special interest group: IGPDE Recent progress in evolution equations (igpde2), and Nonlinear PDE (nlpde). I am also thankful to the anonymous referee for some comments to revise the paper.

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The Maximum Principle and Sign-Changing Solutions of the Klein–Gordon Equation with the Higgs Potential in the de Sitter Spacetime

Karen Yagdjian and Andras Balogh

Abstract In this chapter, we discuss the maximum principle for the linear equation and the sign-changing solutions of the semilinear equation with the Higgs potential. Numerical simulations indicate that the bubbles for the semilinear Klein–Gordon equation in the de Sitter spacetime are created and apparently exist for all times.

1 Introduction

The Klein–Gordon equation with the Higgs potential in the de Sitter spacetime is the equation:

$$\psi_{tt} - e^{-2t}\Delta\psi + n\psi_t = \mu^2\psi - \lambda\psi^3, \tag{1}$$

where Δ is the Laplace operator in $x \in \mathbf{R}^n$, $n = 3$, $t > 0$, $\lambda > 0$, and $\mu > 0$, while $\psi = \psi(x,t)$ is a real-valued function.

We focus on the zeros of the solutions to the linear and semilinear hyperbolic equations. One motivation for the study of the zeros of the solutions to the linear and semilinear hyperbolic equation comes from the cosmological contents and quantum field theory. It is of considerable interest for particle physics and inflationary cosmology to study the so-called bubbles [3, 8]. In [3], bubble is defined as a simply connected domain surrounded by a wall such that the field approaches one of the vacuums outside of a bubble. The creation and growth of bubbles is an interesting mathematical problem [3, Chap. 7], [8]. In this paper, for the continuous solution $\psi = \psi(x,t)$ to the Klein–Gordon equation, for every given positive time t we define a bubble as a maximal connected set of points $x \in \mathbf{R}^n$ at which solution changes its sign.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_36

2 The Maximum Principle

For hyperbolic equations with variable coefficients, the maximum principle is known only in 1-dimensional case [10] and for Euler–Poisson–Darboux equation [13]. We consider the linear part of the equation:

$$u_{tt} - e^{-2t} \triangle u - M^2 u = -e^{\frac{n}{2}t} V'(e^{-\frac{n}{2}t} u), \tag{2}$$

with $M \geq 0$ and the potential function $V = V(\psi)$. If we denote the non-covariant Klein–Gordon operator in the de Sitter spacetime $L_{KGdS} := \partial_t^2 - e^{-2t} \triangle - M^2$, then (2) can be written as follows: $L_{KGdS}[u] = -e^{\frac{n}{2}t} V'(e^{-\frac{n}{2}t} u)$. Equation (2) covers two important cases. The first one is the Higgs boson Eq. (1) that leads to (2) if one applies change of unknown function $\psi = e^{-\frac{n}{2}t} u$. Here, $V'(\psi) = \lambda \psi^3$ and $M^2 = n^2/4 + \mu^2$ with $\lambda > 0$ and $\mu > 0$, while $n = 3$. The second case is the case of the covariant Klein–Gordon equation $\psi_{tt} + n\psi_t - e^{-2t} \triangle \psi + m^2 \psi = -V'(\psi)$, with small physical mass, that is $0 \leq m \leq n/2$. For the last case, $M^2 = n^2/4 - m^2$.

It is known that the Klein–Gordon quantum fields whose squared physical masses are negative (imaginary mass) represent tachyons [2]. The Klein–Gordon quantum fields on the de Sitter manifold with imaginary mass present scalar tachyonic quantum fields. Epstein and Moschella [5] give an exhaustive study of scalar tachyonic quantum fields which are linear Klein–Gordon quantum fields on the de Sitter manifold whose masses take discrete values $m^2 = -k(k+n), k = 0, 1, 2, \ldots$.

The next theorem gives a certain kind of maximum principle for the non-covariant Klein–Gordon equation in the de Sitter spacetime. Define the "forward light cone" $D_+^{dS}(x_0, t_0)$ and the "backward light cone" $D_-^{dS}(x_0, t_0)$, in the de Sitter spacetime for the point $(x_0, t_0) \in \mathbf{R}^{n+1}$, as follows:

$$D_{\pm}^{dS}(x_0, t_0) := \left\{ (x, t) \in \mathbf{R}^{n+1} \,;\, |x - x_0| \leq \pm(e^{-t_0} - e^{-t}) \right\}.$$

For the domain $D_0 \subseteq \mathbf{R}^n$, define the dependence domain of D_0 as follows:

$$D^{dS}(D_0) := \bigcup_{x_0 \in \mathbf{R}^n,\ t_0 \in [0,\infty)} \left\{ D_-^{dS}(x_0, t_0) \,;\, D_-^{dS}(x_0, t_0) \bigcap \{t = 0\} \subset D_0 \right\}.$$

Theorem 1 *Assume that $M > 1$ and the function u satisfies*

$$u_{tt} - e^{-2t} \Delta u - M^2 u \leq 0\,, \quad \textit{for all} \quad t \leq T\,, \tag{3}$$

and $L_{KGdS}[u] \in C^2$ is a superharmonic function in x. Suppose that $u(x, 0)$ and $u_t(x, 0)$ are superharmonic nonpositive functions in $D_0 \subseteq \mathbf{R}^3$. Then,

$$u(x, t) \leq 0 \quad \textit{for} \quad t \in [\ln(M/(M-1)), T], \tag{4}$$

in the domain of dependence of D_0*. If* $u(x,0) \equiv 0$*, then the statement (4) holds also for all* $t \in [0,T]$ *and each* $M \geq 0$*.*

Proof We apply the integral transform and the kernel functions $E(x,t;x_0,t_0;M)$, $K_0(z,t;M)$, and $K_1(z,t;M)$ from [16, 17]. First, we introduce the function:

$$E(x,t;x_0,t_0;M) = 4^{-M} e^{M(t_0+t)} \Big((e^{-t}+e^{-t_0})^2 - (x-x_0)^2 \Big)^{-\frac{1}{2}+M} \times F\Big(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^{-t_0}-e^{-t})^2-(x-x_0)^2}{(e^{-t_0}+e^{-t})^2-(x-x_0)^2} \Big).$$

Here, $F\big(a,b;c;\zeta\big)$ is the hypergeometric function (see [1]). Next, we define the kernels:

$$K_0(z,t;M) := -\left[\frac{\partial}{\partial b} E(z,t;0,b;M) \right]_{b=0} \quad \text{and} \quad K_1(z,t;M) := E(z,t;0,0;M)\,.$$

These kernels have been introduced and used in [14, 19]. The positivity of the kernels E, K_0, and K_1 is proved in the next section. The solution $u = u(x,t)$ to the Cauchy problem:

$$u_{tt} - e^{-2t} \Delta u - M^2 u = f, \quad u(x,0)=0, \quad u_t(x,0)=0,$$

with $f \in C^\infty(\mathbf{R}^{n+1})$ and with vanishing initial data is given in [16] by:

$$u(x,t) = 2 \int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, v(x,r;b) E(r,t;0,b;M), \tag{5}$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation:

$$v_{tt} - \triangle v = 0\,, \quad v(x,0;b) = f(x,b)\,, \quad v_t(x,0) = 0\,. \tag{6}$$

If the superharmonic function f is also nonpositive, then due to Theorem 1 from [11] we conclude $v(x,r;b) \leq f(x,b) \leq 0$ in the domain of dependence of D_0. It follows

$$\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, v(x,r;b) E(r,t;0,b;M)$$

$$\leq \int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, f(x,b) E(r,t;0,b;M) \leq 0\,,$$

provided that $E(r, t; 0, b; M) \geq 0$. The solution $u = u(x, t)$ to the Cauchy problem:

$$u_{tt} - e^{-2t} \triangle u - M^2 u = 0\,, \quad u(x, 0) = u_0(x)\,, \quad u_t(x, 0) = u_1(x)\,, \tag{7}$$

with $u_0, u_1 \in C_0^\infty(\mathbf{R}^n)$, $n \geq 2$, can be represented (see [16]) as follows:

$$\begin{aligned} u(x, t) &= e^{\frac{t}{2}} v_{u_0}(x, \phi(t)) + 2 \int_0^1 v_{u_0}(x, \phi(t)s) K_0(\phi(t)s, t; M)\phi(t)\, ds \\ &\quad + 2 \int_0^1 v_{u_1}(x, \phi(t)s) K_1(\phi(t)s, t; M)\phi(t)\, ds, \quad x \in \mathbf{R}^n,\ t > 0\,, \end{aligned}$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C_0^\infty(\mathbf{R}^n)$ and for $x \in \mathbf{R}^n$, the function $v_\varphi(x, \phi(t)s)$ coincides with the value $v(x, \phi(t)s)$ of the solution $v(x, t)$ of the Cauchy problem:

$$v_{tt} - \triangle v = 0\,, \quad v(x, 0) = \varphi(x)\,, \quad v_t(x, 0) = 0\,.$$

For the function u_1, which is superharmonic, by using Theorem 1 from [11] we conclude $v_{u_1}(x, r) \leq u_1(x)$. It follows

$$\int_0^{\phi(t)} v_{u_1}(x, r) K_1(r, t; M)\, dr \leq u_1(x) \int_0^{\phi(t)} K_1(r, t; M)\, dr = \frac{1}{2M}(e^{Mt} - e^{-Mt}) u_1(x).$$

since $K_1(r, t; M) \geq 0$. In particular, if $u_1(x) \leq 0$, then $\int_0^{\phi(t)} v_{u_1}(x, r) K_1(r, t; M)\, dr \leq 0$.

Further, if $u_0 \in C^2$ is superharmonic, that is $\Delta u_0 \leq 0$, then, according to Theorem 1 from [11], $(\partial_t^2 - \Delta) v_{u_0} = 0$ implies $v_{u_0}(x, t) \leq u_0(x)$. Consequently,

$$\begin{aligned} & e^{\frac{t}{2}} v_{u_0}(x, \phi(t)) + 2 \int_0^{\phi(t)} v_{\varphi_0}(x, r) K_0(r, t; M)\, dr \\ & \leq u_0(x) \left[e^{\frac{t}{2}} + 2 \int_0^{\phi(t)} K_0(r, t; M)\, dr \right] = \frac{1}{2}(e^{Mt} + e^{-Mt}) u_0(x)\,. \end{aligned}$$

In particular, if $M > 1$, $K_0(r, t; M) \geq 0$, and $u_0(x) \leq 0$, then

$$e^{\frac{t}{2}} v_{u_0}(x, \phi(t)) + 2 \int_0^{\phi(t)} v_{u_0}(x, r) K_0(r, t; M)\, dr \leq 0$$

for all $t \in [\ln(M/(M - 1)), T]$. Theorem 1 is proved. □

3 The Positivity of the Kernel Functions E, K_0, and K_1

Proposition 1 *Assume that $M \geq 0$. Then,*

$$E(r,t;0,b;M) > 0, \quad \textit{for all} \quad 0 \leq b \leq t,\ r \leq e^{-b} - e^{-t}, \quad \textit{and for all } t \in [0,\infty),$$
$$K_1(r,t;M) > 0 \quad \textit{for all} \quad r \leq 1 - e^{-t} \quad \textit{and for all} \quad t \in [0,\infty).$$

If we assume $M > 1$, then

$$K_0(r,t;M) > 0 \quad \textit{for all } r \leq 1 - e^{-t} \quad \textit{and for all} \ \ t > \ln\left(M/(M-1)\right).$$

Proof Indeed, if $0 \leq b \leq t$ and $r \leq e^{-b} - e^{-t}$, then we have

$$E(r,t;0,b;M) = 4^{-M} e^{M(b+t)} \Big((e^{-t} + e^{-b})^2 - r^2 \Big)^{-\frac{1}{2}+M}$$
$$\times F\Big(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2}\Big).$$

For $M \geq 0$, the parameters $a = b = 1/2 - M$ and $c = 1$ of the hypergeometric function $F(a,b;c;z)$ satisfy the relation $a + b \leq c$. It remains to check the sign of the hypergeometric function $F(a,a;1;z)$ with parameter $a \leq 1/2$ and $z \in (0,1)$. If a is not a nonpositive integer, then the series:

$$F(a,a;1;x) = \sum_{n=0}^{\infty} \frac{[(a)_n]^2}{[n!]^2} x^n, \quad (a)_n := a(a+1)\cdots(a+n-1),$$

is a convergent series for all $x \in [0,1)$. If a is negative integer, $a = -k$, then $F(a,a;1;x)$ is a polynomial with positive coefficients. Since $K_1(z,t;M) := E(z,t;0,0;M)$, the first two statements of the proposition are proved.

In order to verify the last statement, it suffices to verify the inequality $K_0(r,t;M) > 0$, where $r \in (0,1)$. Denote $M = (2k+1)/2$. Then, $1/2 - M = -k < 0$ and due to the relation (20) from [1, Sect. 2.8] we can write

$$4^{k+1} e^{-\left(k+\frac{1}{2}\right)t} \left((1+e^{-t})^2 - r^2\right)^{-k+2} K_0\left(r,t;\frac{1}{2}+k\right) \tag{8}$$
$$= 8k^2 e^t \left(\left(r^2+1\right)e^{2t} - 1\right) F\left(1-k, 1-k; 2; \frac{(1-e^{-t})^2 - r^2}{(1+e^{-t})^2 - r^2}\right)$$
$$+ \left[(1+e^t)^2 - r^2\right] \left(e^{2t}\left(2k\left(r^2+1\right) + r^2 - 1\right) - 2k - 2e^t - 1\right)$$
$$\times F\left(-k, -k; 1; \frac{(1-e^{-t})^2 - r^2}{(1+e^{-t})^2 - r^2}\right).$$

Consider the right-hand side of the last expression. The functions $F(-k, -k; 1; z)$ and $F(1-k, 1-k; 2; z)$ are defined as follows:

$$F(-k, -k; 1; z) = \sum_{n=0}^{\infty} \frac{[(-k)(-k+1)\cdots(-k+n-1)]^2}{[n!]^2} z^n ,$$

$$F(1-k, 1-k; 2; z) = \sum_{n=0}^{\infty} \frac{[(1-k)(2-k)\cdots(n-k)]^2}{[n!]^2(n+1)} z^n .$$

Here, we have denoted

$$z := \frac{(1-e^{-t})^2 - r^2}{(1+e^{-t})^2 - r^2} \in [0,1] \quad \text{for all} \quad t \in [0,\infty), \quad r \in (0, 1-e^{-t}) .$$

Thus, $F(-k, -k; 1; z) \geq 1$ and $F(1-k, 1-k; 2; z) \geq 1$ for all $z \in [0, 1)$. Then,

$$8k^2 e^t \left(\left(r^2 + 1 \right) e^{2t} - 1 \right) > 1 \quad \text{for all } r \in [0,1] \quad \text{and} \quad t \geq M/(M-1) .$$

Next, we check the sign of the function:

$$\left[(1+e^t)^2 - r^2 \right] \left(e^{2t} \left(2k \left(r^2 + 1 \right) + r^2 - 1 \right) - 2k - 2e^t - 1 \right) .$$

Since $\left[(1+e^t)^2 - r^2 \right] \geq 3$, we consider the second factor only. We set $x := e^t > 1$ and $y := r^2 \in [0, 1]$, then we have the polynomial $P(x, y) = x^2 (2k(y+1) + y - 1) - 2k - 2x - 1$. It follows $\partial_y P(x, y) = x^2(2k+1) = 2x^2 M > 0$. On the other hand, if we set

$$P(x,0) = x^2 (2k-1) - 2k - 2x - 1 = 2[x^2(M-1) - x - M] > 0, \quad (9)$$

then for $M = 1$ the last inequality becomes false since $x > 0$. For $M > 1$, the inequality (9) holds whenever $x > M/(M-1)$. It follows

$$P(e^t, r^2) > const > 0 \quad \text{for all } r^2 \in [0,1] \quad \text{and for all } t \in (\ln(M/(M-1)), \infty) .$$

Since all terms of (8) are positive, the proposition is proved. □

Conjecture 1 Assume that $M \in [0, 1/2]$. Then, $K_0(r, t; M) \leq 0$ for all $r \leq 1 - e^{-t}$ and for all $t > 0$.

4 Sign-Changing Solutions and Evolution of Bubbles

We are interested in sign-changing solutions of the equation for the Higgs real-valued scalar field in the de Sitter spacetime:

$$\psi_{tt} + 3\psi_t - e^{-2t}\Delta\psi = \mu^2\psi - \lambda\psi^3 \,. \tag{10}$$

Unlike the equation in the Minkowski spacetime, that is, the equation $\psi_{tt} - \Delta\psi = \mu^2\psi - \lambda\psi^3$, Eq. (10) has no time-independent solution. A global in time solvability of the Cauchy problem for (10) is not known. An estimate for the lifespan is given by Theorem 0.1 in [18]. The local (in time) solutions exist for every smooth initial data. A C^2 solution of the Cauchy problem for Eq. (10) is unique and obeys the finite speed of propagation property (see [6]). In order to make our discussion more transparent, we appeal to the function $u = e^{\frac{3}{2}t}\psi$. For $u = u(x,t)$, Eq. (10) implies

$$u_{tt} - e^{-2t} \bigtriangleup u - M^2 u = -\lambda e^{-3t} u^3, \tag{11}$$

where $M = \sqrt{9+4\mu^2}/2 > 0$. Next, we use the fundamental solution of the corresponding linear operator in order to reduce the Cauchy problem for the semilinear equation to the integral equation and to define a weak solution. We denote by G the resolving operator of the problem:

$$u_{tt} - e^{-2t} \bigtriangleup u - M^2 u = f, \quad u(x,0) = 0, \quad \partial_t u(x,0) = 0 \,.$$

Thus, $u = G[f]$. The operator G is explicitly given in [19] for the case of a real mass. The analytic continuation with respect to the parameter M of this operator allows us to use G in the case of an imaginary mass. More precisely, for $M \geq 0$ we define the operator G acting on $f(x,t) \in C^\infty(\mathbf{R}^3 \times [0,\infty))$ by (5). Let $u_0 = u_0(x,t)$ be a solution of the Cauchy problem:

$$\partial_t^2 u_0 - e^{-2t} \bigtriangleup u_0 - M^2 u_0 = 0, \quad u_0(x,0) = \varphi_0(x), \quad \partial_t u_0(x,0) = \varphi_1(x) \,. \tag{12}$$

Then, any solution $u = u(x,t)$ of Eq. (11), which takes initial value $u(x,0) = \varphi_0(x)$, $\partial_t u(x,0) = \varphi_1(x)$, solves the integral equation:

$$u(x,t) = u_0(x,t) - G[\lambda e^{-3\cdot} u^3](x,t) \,. \tag{13}$$

Definition 1 If u_0 is a solution of the Cauchy problem (12), then the solution $u = u(x,t)$ of (13) is said to be *a weak solution* of the Cauchy problem for Eq. (11) with the initial conditions $u(x,0) = \varphi_0(x)$, $\partial_t u(x,0) = \varphi_1(x)$.

It is suggested in [15] to measure a variation of the sign of the function ψ by the deviation from the Hölder inequality $\left|\int_{\mathbf{R}^n} u(x)\,dx\right|^3 \le C_{supp\,u} \int_{\mathbf{R}^n} |u(x)|^3\,dx$ of the inequality between the integral $\int_{\mathbf{R}^n} u^3(x)\,dx \ne 0$ and the self-interaction functional: $\left|\int_{\mathbf{R}^n} u(x)\,dx\right|^3 \le \nu_u \left|\int_{\mathbf{R}^n} u^3(x)\,dx\right|$. The next definition is a particular case of Definition 1.2 in [15].

Definition 2 The real-valued function $\psi \in C([0,\infty); L^1(\mathbf{R}^3) \cap L^3(\mathbf{R}^3))$ is said to be asymptotically time-weighted L^3-nonpositive (nonnegative), if there exist a number $C_\psi > 0$ and a positive nondecreasing function $\nu_\psi \in C([0,\infty))$ such that with $\sigma = 1$ ($\sigma = -1$) one has

$$\left|\int_{\mathbf{R}^n} \psi(x,t)\,dx\right|^3 \le -\sigma C_\psi \nu_\psi(t) \int_{\mathbf{R}^n} \psi^3(x,t)\,dx \quad \text{for all sufficiently large} \quad t.$$

An application of Theorem 1.3 from [15] to the Higgs real-valued scalar field Eq. (10) with $\mu > 0$ results in the following statement (see also Corollary 1.4 [15]). Let $\psi = \psi(x,t) \in C([0,\infty); L^q(\mathbf{R}^3))$, $2 \le q < \infty$, be a global weak solution of Eq. (10). Assume also that the initial values of $\psi = \psi(x,t)$ satisfy

$$\sigma\left((\sqrt{9+4\mu^2}+3)\int_{\mathbf{R}^3} \psi(x,0)\,dx + 2\int_{\mathbf{R}^3} \partial_t\psi(x,0)\,dx\right) > 0 \tag{14}$$

with $\sigma = 1$ ($\sigma = -1$), while $\sigma \int_{\mathbf{R}^3} \psi^3(x,t)\,dx \le 0$ is fulfilled for all t outside of a sufficiently small neighborhood of zero. Then, the global solution $\psi = \psi(x,t)$ cannot be an asymptotically time-weighted L^3-nonpositive (L^3-nonnegative) solution with the weight $\nu_\psi(t) = e^{a_\psi t} t^{b_\psi}$, where $a_\psi < \sqrt{9+4\mu^2} - 3$, $b_\psi \in \mathbf{R}$. A solution $\psi = \psi(x,t) \in C^2(\mathbf{R}^3 \times [0,\infty))$ with compactly supported smooth initial data $\psi(x,0)$, $\psi_t(x,0)$, and with $\mu > 0$ has its support in some cylinder $B \times [0,\infty)$, and consequently, if it is sign preserving, it is also an asymptotically time-weighted L^3-nonpositive (L^3-nonnegative) solution with the weight $\nu_\psi(t) \equiv 1$. Hence, *the global solution with data satisfying (14) and* $\psi(x,0) \le 0$ *must take a positive value at some point and, consequently, must take the value zero inside of some section* $t = const > 0$. It gives rise to the formation of a bubble.

Since the issue of global (in time) solutions for Eq. (10) is not resolved, we present some simulation that shows evolution of the bubbles in time. Our numerical approach uses a fourth-order finite difference method in space [7] along with an explicit fourth-order Runge–Kutta method in time [4] for the discretization of the Higgs boson equation. The numerical code has been programmed using the Community Edition of PGI CUDA Fortran [9] on NVIDIA Tesla K40c GPU Accelerators. The grid size in space was $n \times n \times n = 501 \times 501 \times 501$, resulting in a uniform spatial grid spacing of $\delta x_1 = \delta x_2 = \delta x_3 = 2 \times 10^{-3}$. The time step $\delta t = 10^{-4}$ ensured that the Courant–Friedrichs–Lewy (CFL) condition [12] for stability, $|\psi| < \delta x/(\sqrt{3}\delta t) \approx 11.54$, was satisfied for all times. As first initial data

ψ_0 we choose the combination of two bell-shaped, infinitely smooth exponential functions $\psi_0(x) = B_1(x) + B_2(x)$ for all $x = (x_1, x_2, x_3) \in \Omega$, where

$$B_i(\mathbf{x}) = \begin{cases} \exp\left(\frac{1}{R_i^2} - \frac{1}{R_i^2 - |x - C_i|^2}\right) & if \quad |x - C_i| < R_i \\ 0 \quad if \quad |x - C_i| \geq R_i \end{cases}, \quad i = 1, 2,$$

with the center of the bell-shapes at $C_1 = (0.4, 0.4, 0.4)$, $C_2 = (0.6, 0.6, 0.6)$, and the radii of the bell-shapes $R_1 = R_2 = 0.2$. Note that the initial data is nonnegative with a compact support. The finite cone of influence enables us to use zero boundary conditions on the unit box $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ as computational domain, since the solution's domain of support stayed inside the unit box. As second initial data ψ_1 we choose a constant multiple of the first initial data $\psi_1(x) = -5\psi_0(x)$ for all $x \in \Omega$. The parameter values are $\lambda = \mu^2 = 0.1$. Figure 1 shows the formation and interactions of bubbles. Initially, there is no bubble present. At time $t = 0, 2$ two bubbles exist, and their size grows continuously in time. Around time $t = 0.69$ the two bubbles touch, and from that time on they are attached to each other. An additional bubble is formed inside of each of the new merged bubbles (part (c) of Fig. 1), and later these inner bubbles disappear. The growth of the larger outer bubble slows down exponentially and it does not seem to change its shape after time $t = 3$ (part (d) of Fig. 1).

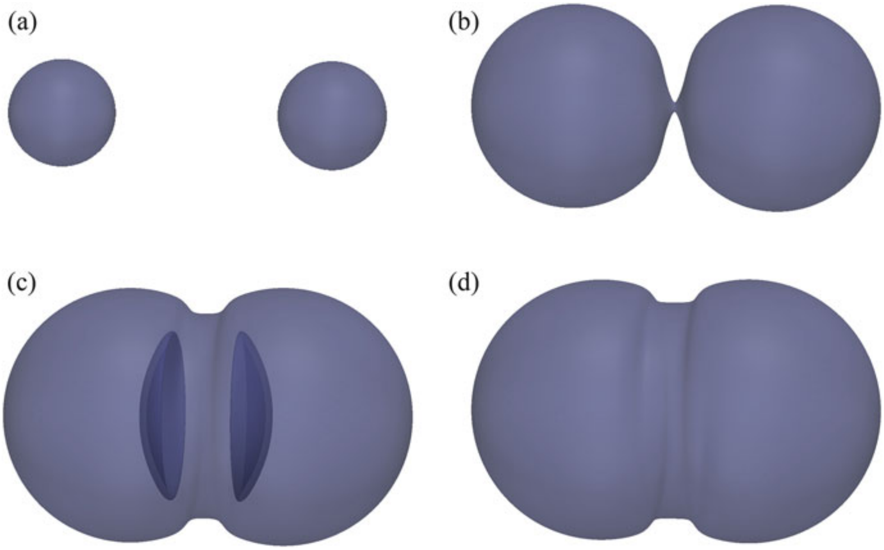

Fig. 1 Formation and interaction of two bubbles. (**a**) 3D bubbles at $t = 0.2$. (**b**) 3D bubbles at $t = 0.69$. (**c**) 3D bubbles at $t = 2$. (**d**) 3D bubbles at $t = 3$

Acknowledgements The authors acknowledge the Texas Advanced Computing Center (TACC) at the University of Texas at Austin for providing high-performance computing and visualization resources that have contributed to the research results reported within this paper. URL: http://www.tacc.utexas.edu. We also gratefully acknowledge the support of NVIDIA Corporation with the donation of the Tesla K40 GPU used for this research. K.Y. was supported by the University of Texas Rio Grande Valley College of Sciences 2016–17 Research Enhancement Seed Grant.

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A Remark on the Critical Exponent for the Semilinear Damped Wave Equation on the Half-Space

Yuta Wakasugi

Abstract In this short notice, we prove the non-existence of global solutions to the semilinear damped wave equation on the half-space, and we determine the critical exponent for any space dimension.

1 Introduction

Let $n \geq 1$ be an integer and let $\mathbb{R}^n_+$ be the n-dimensional half-space, namely,

$$\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,;\, x_n > 0\} \ (n \geq 2), \quad \mathbb{R}_+ = (0, \infty) \ (n = 1).$$

We consider the initial-boundary value problem for the semilinear damped wave equation on the half-space:

$$\begin{cases} u_{tt} - \Delta u + u_t = |u|^p & t > 0, x \in \mathbb{R}^n_+, \\ u(t,x) = 0, & t > 0, x \in \partial\mathbb{R}^n_+, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \mathbb{R}^n_+. \end{cases} \tag{1}$$

Here, u is a real-valued unknown function and u_0, u_1 are given initial data.

Our aim is to show the non-existence of global solutions and determine the critical exponent for any space dimension. Here, the *critical exponent* stands for the threshold of the exponent of the non-linearity for the global existence and the finite time blow-up of solution with small data.

For the semilinear heat equation $v_t - \Delta v = v^p$ on the whole space, Fujita [1] discovered that if $p > p_F(n) := 1 + 2/n$, then the unique global solution exists for every small positive initial data, while the local solution blows up in finite time for any positive data if $1 < p < p_F(n)$. Namely, the critical exponent of the semilinear heat equation on the whole space is given by $p_F(n)$, which is the so-called Fujita's

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_37

critical exponent. Later on, Hayakawa [3] and Kobayashi et al. [8] proved that the case $p = p_F(n)$ belongs to the blow-up region. Moreover, the initial-boundary value problem of the semilinear heat equation on the halved space $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n\,;\, x_{n-k+1} > 0, \dots, x_n > 0\}$ was studied in [9–12] and they determined the critical exponent as $p = p_F(n+k)$.

The critical exponent for the semilinear damped wave equation on the whole space was studied by many authors and it is determined as $p = p_F(n)$. We refer the reader to [14, 15] and the references therein.

Ikehata [5–7] studied the semilinear damped wave equation on the half-space (1) and proved that if $p_F(n+1) < p < \infty$ $(n = 1, 2)$, $p_F(n+1) < p \le \frac{n}{n-2}$ $(n \ge 3)$, $(u_0, u_1) \in H_0^1(\mathbb{R}^n_+) \times L^2(\mathbb{R}^n_+)$ have compact support in $\overline{\mathbb{R}^n_+}$ and $\|\nabla u_0\|_{L^2} + \|u_1\|_{L^2}$ is sufficiently small, then the problem (1) admits a unique global solution. When $n = 1$, Nishihara and Zhao [13] proved the blow-up of solutions when $1 < p \le p_F(2)$, namely, the critical exponent of (1) on the half-line is determined as $p = p_F(2)$. However, there is no blow-up result for (1) when $n \ge 2$. We also refer the reader to [4] for the asymptotic profile of global solutions for a critical absorbing-type non-linearity in one space dimension.

In this paper, we prove the non-existence of global classical solutions for (1) for all $n \ge 1$, and we determine the critical exponent of (1) as $p_F(n+1)$.

Theorem 1 *Let* $1 < p \le p_F(n+1) = 1 + \frac{2}{n+1}$. *We assume that the initial data satisfy* $x_n u_0, x_n u_1 \in L^1(\mathbb{R}^n_+)$ *and*

$$\int_{\mathbb{R}^n_+} x_n (u_0(x) + u_1(x))\, dx > 0 \tag{2}$$

(when $n = 1$, *we interpret* $x_n = x$*). Then, there is no global classical solution to* (1).

Our proof is based on the test function method by Zhang [15]. To apply it to the half-space, we employ the technique by Geng et al. [2]. Namely, we use the test function having the form $x_n \psi_R(t, x)$, where $\psi_R(t, x)$ is a test function supported on the rectangle $\{(t, x) \in [0, \infty) \times \mathbb{R}^n\,;\, t \le R^2, |x_j| \le R\ (j = 1, \dots, n)\}$.

2 Proof of Theorem 1

We suppose that the global classical solution u of the problem (1) exists and derive the contradiction. Let $\psi \in C_0^\infty([0, \infty) \times \mathbb{R}^n_+)$ be a test function. Using the integration by parts, we compute

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n_+} |u|^p \psi \, dxdt &= \int_0^\infty \int_{\mathbb{R}^n_+} (u_{tt} - \Delta u + u_t)\psi \, dxdt \\ &= \int_0^\infty \int_{\mathbb{R}^{n-1}} \partial_{x_n} u(t, x', 0)\psi(t, x', 0)\, dx'dt\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty \int_{\mathbb{R}^n_+} u(\psi_{tt} - \Delta\psi - \psi_t)\, dxdt \\
&- \int_{\mathbb{R}^n_+} ((u_0(x) + u_1(x))\psi(0,x) - u_0(x)\psi_t(0,x))\, dx,
\end{aligned}
\tag{3}
$$

where we used the notation $x' = (x_1, \ldots, x_{n-1})$. Now, we choose the test function ψ as follows. Let $\eta(t) \in C_0^\infty([0,\infty))$ be a non-increasing function satisfying

$$
\eta(t) = 1 \ (t \in [0, 1/2]), \quad \eta(t) = 0 \ (t \in [1, \infty)).
$$

We also define $\phi \in C_0^\infty(\mathbb{R}^n)$ by $\phi(x) := \eta(|x_1|)\eta(|x_2|)\cdots\eta(|x_n|)$. Let $R > 0$ be a parameter and let $\psi_R(t,x) := \phi(x/R)\eta(t/R^2)$. We denote the rectangle $D_R := \{x \in \mathbb{R}^n \,;\, |x_1| \le R, \ldots, |x_n| \le R\}$ and we put $D_R^+ = D_R \cap \mathbb{R}^n_+$. Then, it is obvious that $\operatorname{supp}(\partial_{x_j}\phi(\cdot/R)) \subset D_R \setminus D_{R/2}$. With the above notations, we choose our test function as $\psi(t,x) = x_n \psi_R(t,x)^l$ with sufficiently large integer l.

Let

$$
I_R := \int_0^\infty \int_{\mathbb{R}^n_+} |u|^p x_n \psi_R^l \, dxdt.
$$

We note that our choice of test function implies $x_n \psi_R(t,x)^l \Big|_{x_n=0} = 0$ and $x_n \partial_t(\psi_R(t,x)^l)\Big|_{t=0} = 0$. Moreover, by the assumption (2), we see that there exists $R_0 > 0$ such that

$$
\int_{\mathbb{R}^n_+} ((u_0(x) + u_1(x))x_n \psi_R(0,x)^l \, dx > 0
$$

holds for $R \ge R_0$. Therefore, we deduce from (3) that

$$
\begin{aligned}
I_R &\le \int_0^\infty \int_{\mathbb{R}^n_+} u(\partial_t^2(x_n\psi_R^l) - \Delta(x_n\psi_R^l) - \partial_t(x_n\psi_R^l))\, dxdt \\
&=: K_1 + K_2 + K_3
\end{aligned}
$$

for $R \ge R_0$. We estimate K_1, K_2 and K_3 individually. First, for K_1, we apply the Hölder inequality to obtain

$$
\begin{aligned}
K_1 &\le CR^{-4} \left(\int_{R^2/2}^{R^2} \int_{D_R^+} |u|^p x_n \psi_R^l \, dxdt \right)^{1/p} \left(\int_{R^2/2}^{R^2} \int_{D_R^+} x_n \, dxdt \right)^{1/p'} \\
&\le CR^{-4+(n+3)/p'} \hat{I}_R^{1/p},
\end{aligned}
$$

where p' stands for the Hölder conjugate of p and

$$\hat{I}_R := \int_{R^2/2}^{R^2} \int_{D_R^+} |u|^p x_n \psi_R^l \, dx dt.$$

Similarly, by using

$$\begin{aligned}
&\Delta(x_n \psi_R^l) \\
&= l R^{-2} x_n \left(\phi\left(\frac{x}{R}\right)^{l-1} (\Delta\phi)\left(\frac{x}{R}\right) + (l-1)\phi\left(\frac{x}{R}\right)^{l-2} \left|(\nabla\phi)\left(\frac{x}{R}\right)\right|^2 \right) \eta\left(\frac{t}{R^2}\right)^l \\
&\quad + 2l R^{-1} \phi\left(\frac{x}{R}\right)^{l-1} (\partial_{x_n}\phi)\left(\frac{x}{R}\right) \eta\left(\frac{t}{R^2}\right)^l,
\end{aligned}$$

we estimate K_2 as:

$$\begin{aligned}
K_2 &\le C R^{-2} \left(\int_0^{R^2} \int_{D_R^+ \setminus D_{R/2}^+} |u|^p x_n \psi_R^l \, dx dt \right)^{1/p} \left(\int_0^{R^2} \int_{D_R^+ \setminus D_{R/2}^+} x_n \, dx dt \right)^{1/p'} \\
&\quad + C R^{-1} \left(\int_0^{R^2} \int_{D_R^+ \setminus D_{R/2}^+} |u|^p x_n \psi_R^l \, dx dt \right)^{1/p} \\
&\quad \times \left(\int_0^{R^2} \int_{D_R^+ \cap \{x_n > R/2\}} x_n^{-p'/p} \, dx dt \right)^{1/p'} \\
&\le C R^{-2+(n+3)/p'} \tilde{I}_R^{1/p},
\end{aligned}$$

where

$$\tilde{I}_R = \int_0^{R^2} \int_{D_R^+ \setminus D_{R/2}^+} |u|^p x_n \psi_R^l \, dx dt$$

and we note that $(\partial_{x_n}\phi)(x/R) = 0$ on the set $\{x_n \le R/2\}$. The term K_3 is estimated in the same way as K_1 and we have

$$\begin{aligned}
K_3 &\le C R^{-2} \left(\int_{R^2/2}^{R^2} \int_{D_R^+} |u|^p x_n \psi_R^l \, dx dt \right)^{1/p} \left(\int_{R^2/2}^{R^2} \int_{D_R^+} x_n \, dx dt \right)^{1/p'} \\
&\le C R^{-2+(n+3)/p'} \hat{I}_R^{1/p}.
\end{aligned}$$

Combining the estimates above, we deduce

$$I_R \le C(R^{-4+(n+3)/p'}\hat{I}_R^{1/p} + R^{-2+(n+3)/p'}\tilde{I}_R^{1/p} + R^{-2+(n+3)/p'}\hat{I}_R^{1/p}). \tag{4}$$

In particular, using $\hat{I}_R \le I_R$ and $\tilde{I}_R \le I_R$, we have

$$I_R \le C(R^{-4+(n+3)/p'} + R^{-2+(n+3)/p'})I_R^{1/p}. \tag{5}$$

When $1 < p < p_F(n+1)$, letting $R \to \infty$, we see that $I_R \to 0$, which implies $u \equiv 0$. However, this contradicts $(u_0, u_1) \not\equiv 0$.

On the other hand, when $p = p_F(n+1)$, we have $-2 + (n+3)/p' = 0$ and hence, we see from (5) that $I_R \le C$ with a constant C independent of R. Thus, letting $R \to \infty$, we have $x_n|u|^p \in L^1([0,\infty) \times \mathbb{R}^n_+)$. Noting this and the integral region of $\hat{I}_R$ and $\tilde{I}_R$, we also deduce

$$\lim_{R\to\infty}(\hat{I}_R + \tilde{I}_R) = 0.$$

This and (4) imply

$$I_R \le C(\hat{I}_R^{1/p} + \tilde{I}_R^{1/p}) \to 0 \quad (R \to \infty),$$

and hence, $u \equiv 0$. This again contradicts $(u_0, u_1) \not\equiv 0$ and completes the proof.

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Part X
Special Interest Group: IGGF Special Session on Generalized Functions and Applications

Session Organizers: Michael Kunzinger, Michael Oberguggenberger, and Stevan Pilipović

The session was devoted to theory and application of generalized functions, which comprise, among others, distributions, ultradistributions, hyperfunctions, and algebras of generalized functions. Applications include, but are not restricted to, linear and nonlinear partial differential equations, asymptotic analysis, geometry, mathematical physics, stochastic processes, and harmonic analysis, both in theoretical and numerical aspects. The session was open to contributions on any aspect of generalized functions and their applications.

On Microlocal Regularity of Generalized Linear Partial Differential Operators

Chikh Bouzar and Tayeb Saidi

Abstract Given a set $\mathscr{R}$ of sequences of positive numbers, a subalgebra $\mathscr{G}^{\mathscr{R}}(\Omega)$ gives rise to local and microlocal $\mathscr{R}$-regularity of generalized functions, in this framework we study the microlocal regularity of solutions of linear partial differential equations with $\mathscr{R}$-regular generalized functions as coefficients.

1 Introduction

Let $P(x, D)$ be a classical linear partial differential operator with coefficients of $\mathscr{C}^{\infty}$ regularity, the first microlocal regularity result in the framework of distributions, due to L. Hörmander [4], is

$$WF(u) \subseteq WF(P(x, D)u) \cup Char(P), u \in \mathscr{D}'(\Omega), \tag{1}$$

where $WF(u)$ denotes the $\mathscr{C}^{\infty}$-wave front set of u.

When dealing with linear partial differential operators with discontinuous or even distributional coefficients, we need a larger context than the space of distributions. The Colombeau algebra of generalized functions $\mathscr{G}(\Omega)$ is a possible candidate for such a situation, as the space $\mathscr{D}'(\Omega)$ is linearly injected into $\mathscr{G}(\Omega)$ as a subspace and the space $\mathscr{C}^{\infty}(\Omega)$ is canonically embedded into $\mathscr{G}(\Omega)$ as a subalgebra, see for details [3, 8]. The notion of regularity in $\mathscr{G}(\Omega)$ is based on the subalgebra $\mathscr{G}^{\infty}(\Omega)$ which plays the same role as $\mathscr{C}^{\infty}(\Omega)$ in $\mathscr{D}'(\Omega)$, and it is the basis of the development of local and microlocal analysis within $\mathscr{G}(\Omega)$. However, the $\mathscr{G}^{\infty}$-regularity does not exhaust the regularity questions inherent to the algebra $\mathscr{G}(\Omega)$.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_38

Given a set $\mathscr{R}$ of sequences of positive numbers, a subalgebra $\mathscr{G}^{\mathscr{R}}(\Omega)$ of $\mathscr{G}(\Omega)$ gives rise to a new local and also a microlocal $\mathscr{R}$-regularity for generalized functions, see [1, 2, 9]. If we take $\mathscr{R}$ to be the set of all bounded sequences of positive numbers, we then obtain the $\mathscr{G}^{\infty}$-regularity associated with the subalgebra $\mathscr{G}^{\infty}(\Omega)$.

Linear partial differential equations with generalized coefficients from $\mathscr{G}^{\infty}(\Omega)$, see [7], are seen as equivalence classes of operators with regular coefficients that depend on a parameter ε satisfying a certain rule which is imposed by the studied problem. When this rule is defined by an asymptotic scale of type $\mathscr{R}$ we obtain a more general problem : an example of such a situation rising from applications to a concrete context is the subject of the paper [5].

This work aims to study the microlocal regularity of generalized solutions of linear partial differential equations with $\mathscr{R}$-regular generalized coefficients. In fact, we prove the following result:

$$WF_{\mathscr{R}}(u) \subseteq WF_{\mathscr{R}}(P(x,D)u) \cup \Sigma^{m'}_{\rho,\delta}(P), u \in \mathscr{G}(\Omega),$$

where $P(x,D)$ is a generalized linear partial differential operator with $\mathscr{R}$-regular functions as coefficients and $WF_{\mathscr{R}}(u)$ denotes the $\mathscr{R}$-wave front set of u. The set $\Sigma^{m'}_{\rho,\delta}(P) \subset \Omega \times (\mathbb{R}^n \setminus \{0\})$ characterizes the lack of $\mathscr{R}$-micro-hypoellipticity of the operator $P(x,D)$, it is seen as a generalization of its characteristic variety.

2 The $\mathscr{R}$-Regularity

Recall that $\mathbb{R}_+ :=]0,\infty[$, $\mathbb{Z}_+ := \{0,1,2,\dots\}$ and $\mathbb{R}_+^{\mathbb{Z}_+}$ is the set of all sequences of positive real numbers.

Let Ω be an open set of $\mathbb{R}^n$, $I :=]0,1]$ and define

$$\mathscr{M}(\Omega) := \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathscr{C}^{\infty}(\Omega)^I : \forall K \Subset \Omega, \forall \alpha \in \mathbb{Z}_+^n, \exists m \in \mathbb{R}, \\ \sup\limits_{x\in K} |\partial^\alpha u_\epsilon(x)| = O\left(\epsilon^{-m}\right), \epsilon \longrightarrow 0 \end{array} \right\},$$

and

$$\mathscr{N}(\Omega) := \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathscr{C}^{\infty}(\Omega)^I : \forall K \Subset \Omega, \forall \alpha \in \mathbb{Z}_+^n, \forall m \in \mathbb{Z}_+, \\ \sup\limits_{x\in K} |\partial^\alpha u_\epsilon(x)| = O\left(\epsilon^{m}\right), \epsilon \longrightarrow 0 \end{array} \right\},$$

then the Colombeau algebra of generalized functions $\mathscr{G}(\Omega)$ is by definition the quotient algebra

$$\mathscr{G}(\Omega) := \frac{\mathscr{M}(\Omega)}{\mathscr{N}(\Omega)}.$$

This algebra has been introduced in order to give a solution to the problem of multiplication of distributions.

The first regularity notion within $\mathcal{G}(\Omega)$ is based on the following subalgebra:

$$\mathcal{G}^{\infty}(\Omega) := \frac{\mathcal{M}^{\infty}(\Omega)}{\mathcal{N}(\Omega)},$$

where

$$\mathcal{M}^{\infty}(\Omega) = \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{C}^{\infty}(\Omega)^I : \forall K \Subset \Omega, \exists m \in \mathbb{R}, \forall \alpha \in \mathbb{Z}_+^n, \\ \sup\limits_{x \in K} |\partial^\alpha u_\epsilon(x)| = O\left(\epsilon^{-m}\right), \epsilon \longrightarrow 0 \end{array} \right\}.$$

The fundamental property of $\mathcal{G}^{\infty}(\Omega)$ is the following result:

$$\mathcal{D}'(\Omega) \cap \mathcal{G}^{\infty}(\Omega) = \mathcal{C}^{\infty}(\Omega),$$

which means that $\mathcal{G}^{\infty}(\Omega)$ plays in $\mathcal{G}(\Omega)$ the same role as $\mathcal{C}^{\infty}(\Omega)$ in $\mathcal{D}'(\Omega)$, see [3, 8].

Other subalgebras of $\mathcal{G}(\Omega)$ candidates for the regularity issue in $\mathcal{G}(\Omega)$ are introduced.

Definition 1 Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$, we define the algebra of $\mathcal{R}$-regular generalized functions on the open set Ω as the quotient algebra

$$\mathcal{G}^{\mathcal{R}}(\Omega) := \frac{\mathcal{M}^{\mathcal{R}}(\Omega)}{\mathcal{N}(\Omega)}, \tag{2}$$

where

$$\mathcal{M}^{\mathcal{R}}(\Omega) := \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{C}^{\infty}(\Omega)^I : \forall K \Subset \Omega, \exists N = (N_m)_{m \in \mathbb{Z}_+} \in \mathcal{R}, \\ \forall \alpha \in \mathbb{Z}_+^n, \sup\limits_{x \in K} |\partial^\alpha u_\epsilon(x)| = O\left(\epsilon^{-N_{|\alpha|}}\right), \epsilon \longrightarrow 0 \end{array} \right\}. \tag{3}$$

Remark 1 The subalgebra $\mathcal{G}^{\infty}$ corresponds to $\mathcal{G}^{\mathcal{B}}(\Omega)$, where $\mathcal{B}$ is the set of all bounded sequences of $\mathbb{R}_+^{\mathbb{Z}_+}$.

To be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$ is needed for obtaining the main properties of $\mathcal{G}^{\mathcal{R}}(\Omega)$.

Definition 2 A non-void subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{Z}_+}$ is called regular if it satisfies the following properties:

$\forall (N_m)_{m \in \mathbb{Z}_+} \in \mathcal{R}, \forall \left(k, k'\right) \in \mathbb{Z}_+^2, \exists \left(N'_m\right)_{m \in \mathbb{Z}_+} \in \mathcal{R}$ such that

$$N_{m+k} + k' \leq N'_m, \forall m \in \mathbb{Z}_+. \tag{R1}$$

$$\forall (N_m)_{m\in\mathbb{Z}_+}, \left(N'_m\right)_{m\in\mathbb{Z}_+} \in \mathscr{R}, \exists \left(N''_m\right)_{m\in\mathbb{Z}_+} \in \mathscr{R} \text{ such that}$$

$$\max\left(N_m, N'_m\right) \le N''_m, \forall m \in \mathbb{Z}_+. \tag{R2}$$

$$\forall (N_m)_{m\in\mathbb{Z}_+}, \left(N'_m\right)_{m\in\mathbb{Z}_+} \in \mathscr{R}, \exists \left(N''_m\right)_{m\in\mathbb{Z}_+} \in \mathscr{R} \text{ such that}$$

$$N_{l_1} + N'_{l_2} \le N''_{l_1+l_2}, \forall (l_1, l_2) \in \mathbb{Z}_+^2. \tag{R3}$$

The following results give some properties of $\mathscr{G}^{\mathscr{R}}(\Omega)$.

Proposition 1 *Let $\mathscr{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$, then*

(i) $\mathscr{M}^{\mathscr{R}}(\Omega)$ *is a subalgebra of* $\mathscr{M}(\Omega)$.
(ii) $\mathscr{N}(\Omega)$ *is an ideal of* $\mathscr{M}^{\mathscr{R}}(\Omega)$.
(iii) The algebra $\mathscr{G}^{\mathscr{R}}(\Omega)$ *is linearly injected into* $\mathscr{G}(\Omega)$.

Proof For the proof and more study see [1, 2].

The following examples of subsets of $\mathbb{R}_+^{\mathbb{Z}_+}$ appeared chronologically in different works on regularity questions of generalized functions, and they are the motivation of this work.

Example 1 The subalgebra $\mathscr{G}^{\infty}$, introduced by M. Oberguggenberger, corresponds to the set $\mathscr{B}$ of all bounded sequences of $\mathbb{R}_+^{\mathbb{Z}_+}$.

Example 2 The subalgebra $\mathscr{G}^{\mathscr{T}}$ of regular generalized functions with affine asymptotic scale, introduced by T. Tomikawa, corresponds to the set $\mathscr{T}$ defined by

$$\mathscr{T} := \left\{(N_m)_{m\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{Z}_+} : \exists a \ge 0, \exists b \in \mathbb{R}, \forall m \in \mathbb{Z}_+, N_m \le am + b\right\}.$$

Example 3 The asymptotic scale defining the Hölder–Zygmund regularity subalgebras $\mathscr{G}_*^s$ introduced by G. Hörmann.

Example 4 Let $a \ge 0$ and define the regular set $\mathscr{L}_a$ by

$$\mathscr{L}_a := \left\{(N_m)_{m\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{Z}_+} : \exists b \in \mathbb{R}, \forall m \in \mathbb{Z}_+, N_m \le am + b\right\},$$

then we obtain a scale of increasing subalgebras of regular generalized functions $\left(\mathscr{G}^{\mathscr{L}_a}\right)_{a\ge 0}$ such that

$$\mathscr{B} = \bigcap_{a\ge 0} \mathscr{L} \text{ and } \mathscr{T} = \bigcup_{a\ge 0} \mathscr{L}_a.$$

In [2], it is stated that the space of distributions $\mathscr{D}'(\Omega)$ is embedded into $\mathscr{G}^{\mathscr{L}_1}(\Omega)$.

For every regular subset $\mathcal{R}$, the functor $\Omega \longrightarrow \mathcal{G}^{\mathcal{R}}(\Omega)$ is a sheaf of differential subalgebras of $\mathcal{G}(\Omega)$, so the $\mathcal{R}$-singular support of $u \in \mathcal{G}(\Omega)$, denoted by $singsupp_{\mathcal{R}}(u)$, is well defined as the complementary set of the largest open $\Omega' \subset \Omega$ such that $u/\Omega' \in \mathcal{G}^{\mathcal{R}}(\Omega')$, i.e.,

$$singsupp_{\mathcal{R}}(u) = \Omega \setminus \cup \left\{ \Omega' \text{ open of } \Omega,\ u/\Omega' \in \mathcal{G}^{\mathcal{R}}\left(\Omega'\right) \right\}.$$

The microlocal aspect of the $\mathcal{R}$-regularity is studied in [1].

Definition 3 Let $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$, the $\mathcal{R}$-wave front set of $u \in \mathcal{G}(\Omega)$ is defined as follows : $(x_0, \xi_0) \notin WF_{\mathcal{R}}(u)$ if and only if there exist $\varphi \in \mathcal{D}(\Omega)$, $\varphi(x_0) \neq 0$, and a conic neighborhood Γ of ξ_0, such that

$$\exists N \in \mathcal{R}, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \Gamma} \left| \xi^\beta \widehat{\varphi u_\epsilon}(\xi) \right| = O\left(\epsilon^{-N_{|\beta|}}\right), \epsilon \longrightarrow 0. \tag{4}$$

The main properties of the $\mathcal{R}$-wave front set are summarized in the following Proposition.

Proposition 2 *Let $u \in \mathcal{G}(\Omega)$, $a \in \mathcal{G}^{\mathcal{R}}(\Omega)$, and $\alpha \in \mathbb{Z}^n$, then*

(i) *the projection of $WF_{\mathcal{R}}(u)$ on Ω is $singsupp_{\mathcal{R}}(u)$.*
(ii) $WF_{\mathcal{R}}(au) \subset WF_{\mathcal{R}}(u)$.
(iii) $WF_{\mathcal{R}}(D^\alpha u) \subset WF_{\mathcal{R}}(u)$.

Proof See [1].

3 The $\mathcal{R}$-Microlocal Regularity

We introduce the notion of $\mathcal{R}$-scale net which extends the well-known slow scale net studied in [6]. Recall the set of nets of numbers of moderate type

$$\mathcal{M}[\mathbb{K}] := \left\{ (z_\varepsilon)_\varepsilon \in \mathbb{K}^I : \exists m \in \mathbb{Z}_+, |z_\varepsilon| = O\left(\varepsilon^{-m}\right),\ \varepsilon \longrightarrow 0 \right\},$$

where $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$.

Definition 4 Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$, a net $(r_\varepsilon)_\varepsilon \in \mathcal{M}[\mathbb{K}]$ satisfying the following property:

$$\exists N \in \mathcal{R}, \exists \varepsilon_0 > 0, \forall l \in \mathbb{Z}_+, \exists C_l > 0, |r_\varepsilon|^l \leq C\varepsilon^{-N_l}, \forall \varepsilon \in]0, \varepsilon_0[$$

is called an $\mathcal{R}$-scale net.

A net $(r_\varepsilon)_\varepsilon \in \mathcal{M}[\mathbb{R}]$ is said an $\mathcal{R}$-positive scale net, if it is an $\mathcal{R}$-scale net satisfying $r_\varepsilon > 0, \forall \varepsilon \in]0, 1]$.

Example 5 The set of $\mathscr{B}$-scale nets is the set of slow scale nets introduced in [6].

Example 6 Let $\left(\frac{1}{\varepsilon}\right)_\varepsilon \in \mathscr{M}\left[\mathbb{R}\right]$, then $\left(\frac{1}{\varepsilon}\right)_\varepsilon$ is an $\mathscr{L}_1$-positive scale net which is not a slow scale net.

The proof of the next lemma is obtained directly from Definition 4.

Lemma 1 *Let $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ be $\mathscr{R}$-positive scale nets and $\lambda, a > 0$, then*

(i) $(r_\varepsilon + s_\varepsilon)_\varepsilon$ *is an $\mathscr{R}$-positive scale net.*
(ii) $(\lambda r_\varepsilon)_\varepsilon$ *is an $\mathscr{R}$-positive scale net.*
(iii) $\max\left\{(r_\varepsilon)_\varepsilon, (s_\varepsilon)_\varepsilon\right\}$ *is an $\mathscr{R}$-positive scale net.*
(iv) $(r_\varepsilon s_\varepsilon)_\varepsilon$ *is an $\mathscr{R}$-positive scale net.*
(v) $\left(r_\varepsilon^a\right)_\varepsilon$ *is an $\mathscr{R}$-positive scale net.*

A generalized linear differential operator of order m with $\mathscr{R}$-regular generalized coefficients denoted by

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u$$

is an equivalence class of a net of linear partial differential operators

$$P_\varepsilon(x, D) = \sum_{|\alpha| \le m} a_{\alpha,\varepsilon}(x)\, D^\alpha,$$

where $\left(a_{\alpha,\varepsilon}\right)_\varepsilon \in \mathscr{M}^{\mathscr{R}}(\Omega)$, see [7].

A consequence of Proposition 2 is that

$$WF_{\mathscr{R}}\left(P(x, D)\, u\right) \subset WF_{\mathscr{R}}(u),\, u \in \mathscr{G}(\Omega).$$

The reverse inclusion is an important issue of microlocal regularity problem within the algebra $\mathscr{G}(\Omega)$, it gives information about the localization of generalized $\mathscr{R}$-microlocal singularities of solutions of generalized linear partial differential equations with $\mathscr{R}$-regular generalized coefficients.

Definition 5 Let $\mathscr{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$ and $P(x, D)$ a generalized linear partial differential operator of order m with $\mathscr{R}$-regular generalized coefficients. Let $m' \in \mathbb{R}, 0 \le \delta < \rho \le 1$, and $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$, we define $(x_0, \xi_0) \notin \Sigma_{\rho,\delta}^{m',\mathscr{R}}(P)$, if there exists an open neighborhood U of x_0 in Ω and a conic neighborhood Γ of ξ_0 in $\mathbb{R}^n \setminus \{0\}$ such that for all $K \Subset U$, we have

(i) $\exists q > 0, \exists\, (r_\varepsilon)_\varepsilon$ an $\mathscr{R}$-positive scale net, $\exists \varepsilon_0 > 0$, such that

$$|P_\varepsilon(x, \xi)| \ge \varepsilon^q (1 + |\xi|)^{m'}, \tag{5}$$

$\forall (x, \xi) \in K \times \Gamma, |\xi| \ge r_\varepsilon, \forall \varepsilon \in]0, \varepsilon_0[\,.$

(ii) $\forall\alpha \in \mathbb{Z}_+^n$, $\exists \left(s_{\alpha,\varepsilon}\right)_\varepsilon$, $\exists \left(r_{\alpha,\varepsilon}\right)_\varepsilon$ two $\mathscr{R}$-positive scale nets, $\exists\varepsilon_\alpha > 0$, $\forall\beta \in \mathbb{Z}_+^n$,

$$\left|\partial_x^\alpha \partial_\xi^\beta P_\varepsilon (x, \xi)\right| \leq s_{\alpha,\varepsilon} \left|P_\varepsilon (x, \xi)\right| (1 + |\xi|)^{-\rho|\beta|+\delta|\alpha|}, \tag{6}$$

$\forall (x, \xi) \in K \times \Gamma$, $|\xi| \geq r_{\alpha,\varepsilon}$, $\forall\varepsilon \in]0, \varepsilon_\alpha[$.

The main result of this work is the following theorem.

Theorem 1 *Let $P(x, D)$ be a generalized linear partial differential operator with $\mathscr{R}$-regular generalized coefficients of order m, then*

$$WF_{\mathscr{R}}(u) \subseteq WF_{\mathscr{R}}(P(x, D)u) \cup \Sigma_{\rho,\delta}^{m',\mathscr{R}}(P), \; u \in \mathscr{G}(\Omega). \tag{7}$$

Proof The transpose ${}^tP(x, D)$ of the operator $P(x, D)$ is given by the representing net of linear partial differential operators with symbols

$${}^tP_\varepsilon (x, \xi) = \sum_{0\leq|\sigma|\leq m} \frac{(-1)^{|\sigma|}}{\sigma!} \partial_\xi^\sigma \partial_x^\sigma P_\varepsilon (x, -\xi).$$

We can get the following result:

$$(x_0, \xi_0) \in \Sigma_{\rho,\delta}^{m'}(P) \text{ if and only if } (x_0, -\xi_0) \in \Sigma_{\rho,\delta}^{m'}\left({}^tP\right).$$

Let $\Psi \in \mathscr{E}(\Omega)$, then it is not difficult, with the help of Leibniz formula, to obtain that

$${}^tP_\varepsilon (x, D) \left(e^{-ix\xi} \frac{\Psi}{{}^tP_\varepsilon (x, -\xi)}\right) = e^{-ix\xi} (\Psi - R_\varepsilon (\xi; x, D) \Psi), \tag{8}$$

where the net of linear partial differential operators $(R_\varepsilon (\xi; x, D))_\varepsilon$ is defined by

$$R_\varepsilon (\xi; x, D) = -\sum_{|\beta|\leq m} r_{\beta,\varepsilon} (x, \xi) D_x^\beta,$$

and

$$r_{\beta,\varepsilon} (x, \xi) = \sum_{0<|\gamma|+|\beta|\leq m} \frac{1}{\beta!\gamma!} \partial_\xi^{\beta+\gamma}\, {}^tP_\varepsilon (x, -\xi)\, D_x^\gamma \frac{1}{{}^tP_\varepsilon (x, -\xi)}.$$

So, we take $\varphi \in \mathscr{D}(\Omega)$, $L \in \mathbb{N}$, and define the net $\left(\psi_{\varepsilon,L}(x, \xi)\right)_\varepsilon$ in the following form:

$$\psi_{\varepsilon,L} (x, \xi) = \frac{v_{\varepsilon,L} (x, \xi)}{{}^tP_\varepsilon (x, -\xi)},$$

where

$$v_{\varepsilon,L}(x,\xi)=\sum_{k=0}^{L-1} R_\varepsilon(\xi;x,D)^k\,\varphi(x)\,.$$

Then from (8) we obtain

$$^{t}P_\varepsilon(x,D)\left(e^{-ix\xi}\psi_{\varepsilon,L}(x,\xi)\right)=e^{-ix\xi}\left(\varphi(x)-R_\varepsilon(\xi;x,D)^L\,\varphi(x)\right).$$

Consequently, we have

$$\widehat{\varphi u_\varepsilon}(\xi)=I_{\varepsilon,L}(\xi)+J_{\varepsilon,L}(\xi)\,, \tag{9}$$

where

$$I_{\varepsilon,L}(\xi)=\int u_\varepsilon(x)\,e^{-ix\xi}R_\varepsilon^L(\xi;x,D)\,\varphi(x)\,dx,$$

$$J_{\varepsilon,L}(\xi)=\int \psi_{\varepsilon,L}(x,\xi)\,e^{-ix\xi}P_\varepsilon(x,D)\,u_\varepsilon(x)\,dx.$$

What remains to end the proof of the theorem, due to (9), it is sufficient to show that $I_{\epsilon,L}$ and $J_{\epsilon,L}$ are $\mathscr{R}$-rapidly decreasing on a conic neighborhood of ξ_0 (in the sense of (4)) , and then we obtain

$$(x_0,\xi_0)\notin WF_{\mathscr{R}}(P(x,D)u)\cup\Sigma_{\rho,\delta}^{m',\mathscr{R}}(P)\Rightarrow(x_0,\xi_0)\notin WF_{\mathscr{R}}(u)\,.$$

This will be done by following carefully the steps of the proof of the main result of [7] in the spirit of the similar result of [4].

As a corollary, we obtain the following result.

Corollary 1 *If in the theorem $\mathscr{R}$ is the set of all bounded sequences of positive numbers $\mathscr{B}$, then we obtain the result of [7].*

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A Projective Description of Generalized Gelfand–Shilov Spaces of Roumieu Type

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Abstract We provide a projective description for a class of generalized Gelfand–Shilov spaces of Roumieu type. In particular, our results apply to the classical Gelfand–Shilov spaces and weighted L^∞-spaces of ultradifferentiable functions of Roumieu type.

1 Introduction

In general, there is no canonical way to find an explicit and useful system of seminorms describing a given inductive limit topology. However, in many concrete cases this is possible. For weighted (LB)-spaces of continuous and holomorphic functions, under quite general assumptions, the topology can be described in terms of weighted sup-seminorms. This problem of *projective description* goes back to the pioneer work of Bierstedt et al. [3] and plays an important role in Ehrenpreis' theory of analytically uniform spaces [2, 9]. On the other hand, an explicit system of seminorms describing the topology of the space of ultradifferentiable functions of Roumieu type was first found by Komatsu [12]. His proof was based on a structural theorem for the dual space and the same method was later employed by Pilipović [13] to obtain projective descriptions of Gelfand–Shilov spaces of Roumieu type. Such projective descriptions are indispensable for achieving topological tensor product representations of various important classes of vector-valued ultradifferentiable functions of Roumieu type [5, 12, 14].

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_39

The aim of this article is to provide a projective description of a general class of Gelfand–Shilov spaces of Roumieu type. More precisely, let $(M_p)_{p\in\mathbb{N}}$ be a sequence of positive real numbers and let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ be a pointwise decreasing sequence of positive continuous functions on $\mathbb{R}^d$. We study here the (LB)-space $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ consisting of all those $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha\in\mathbb{N}^d}\sup_{x\in\mathbb{R}^d}\frac{h^{|\alpha|}|\partial^\alpha\varphi(x)|v_n(x)}{M_{|\alpha|}} < \infty$$

for some $h > 0$ and $n \in \mathbb{N}$. Under rather general assumptions on M_p and $\mathscr{V}$, we shall give a projective description of the space $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ in terms of Komatsu's family $\mathfrak{R}$ [12] and the maximal Nachbin family associated with $\mathscr{V}$ [3]. We mention that we have already studied the problem in [4, Prop. 4.16], but we will present here a new approach. Our arguments are based on the mapping properties of the *short-time Fourier transform* on $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ and the projective description of weighted (LB)-spaces of continuous functions. We believe this is a transparent and flexible method. It has the advantage that one can work under very mild conditions on M_p and $\mathscr{V}$ and that it avoids duality theory; in fact, our result can be employed to more easily study dual spaces, e.g., one might easily deduce structural theorems from it without resorting to a rather complicated dual Mittag-Leffler argument.

Our general references are [3] for weighted inductive limits of spaces of continuous functions, [14] for Gelfand–Shilov spaces, and [10] for the short-time Fourier transform.

2 Weighted Inductive Limits of Spaces of Continuous Functions

In this section we recall a result of Bastin [1] concerning the projective description of weighted (LB)-spaces of continuous functions. This result will play a key role in the proof of our main theorem.

Let X be a completely regular Hausdorff space. Given a non-negative function v on X we write $Cv(X)$ for the seminormed space consisting of all $f \in C(X)$ such that $\|f\|_v := \sup_{x\in X}|f(x)|v(x) < \infty$. If v is positive, then $\|\cdot\|_v$ is actually a norm and if, in addition, $1/v$ is locally bounded, then $Cv(X)$ is complete and hence a Banach space. These requirements are fulfilled if v is positive and continuous. A (pointwise) decreasing sequence $\mathscr{V} := (v_n)_{n\in\mathbb{N}}$ of positive continuous functions on X is called a *decreasing weight system on X*. We define

$$\mathscr{V}C(X) := \varinjlim_{n\in\mathbb{N}} Cv_n(X),$$

a Hausdorff (LB)-space. The *maximal Nachbin family associated with $\mathscr{V}$*, denoted by $\overline{V} = \overline{V}(\mathscr{V})$, is given by the space of all non-negative upper semicontinuous functions v on X such that $\sup_{x\in X} v(x)/v_n(x) < \infty$ for all $n \in \mathbb{N}$. The *projective hull of* $\mathscr{V}C(X)$, denoted by $C\overline{V}(X)$, is defined as the space consisting of all $f \in C(X)$ such that $\|f\|_v < \infty$ for all $v \in \overline{V}$. The space $C\overline{V}(X)$ is endowed with the topology generated by the system of seminorms $\{\|\cdot\|_v : v \in \overline{V}\}$. An elementary argument by contradiction shows that $\mathscr{V}C(X)$ and $C\overline{V}(X)$ coincide algebraically and that these spaces have the same bounded sets (cf. [4, Lemma 4.11]). The problem of projective description in this context is to characterize the weight systems $\mathscr{V}$ for which the spaces $\mathscr{V}C(X)$ and $C\overline{V}(X)$ are equal as locally convex spaces. In this regard, there is the following result due to Bastin.

Theorem 1 ([1]) *Let* $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ *be a decreasing weight system on* X *satisfying condition* (V)*, i.e., for every sequence of positive numbers* $(\lambda_n)_{n\in\mathbb{N}}$ *there is* $v \in \overline{V}$ *such that for every* $n \in \mathbb{N}$ *there is* $N \in \mathbb{N}$ *such that* $\inf\{\lambda_1 v_1, \dots, \lambda_N v_N\} \leq \sup\{v_n/n, v\}$. *Then,* $\mathscr{V}C(X)$ *and* $C\overline{V}(X)$ *coincide topologically.*

Remark 1 Bastin also showed that if for every $v \in \overline{V}$ there is a positive continuous $\overline{v} \in \overline{V}$ such that $v \leq \overline{v}$, then condition (V) is also necessary for the topological identity $\mathscr{V}C(X) = C\overline{V}(X)$. We mention that if X is a discrete or a locally compact σ-compact Hausdorff space, then every decreasing weight system $\mathscr{V}$ on X satisfies the above condition [3, p. 112].

Remark 2 A decreasing weight system $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ is said to satisfy condition (S) if for every $n \in \mathbb{N}$ there is $m > n$ such that v_m/v_n vanishes at ∞. Every weight system satisfying (S) also satisfies (V), but the latter property also holds for constant weight systems (for which (S) obviously fails).

Let X and Y be completely regular Hausdorff spaces and let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ and $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ be decreasing weight systems on X and Y, respectively. We denote by $\mathscr{V} \otimes \mathscr{W} := (v_n \otimes w_n)_{n\in\mathbb{N}}$ the decreasing weight system on $X \times Y$ given by $v_n \otimes w_n(x, y) := v_n(x)w_n(y)$, $x \in X$, $y \in Y$.

Remark 3 Let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ and $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ be decreasing weight systems on X and Y, respectively. If both $\mathscr{V}$ and $\mathscr{W}$ satisfy (V), then also $\mathscr{V} \otimes \mathscr{W}$ satisfies (V).

Remark 4 Let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ and $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ be decreasing weight systems on X and Y, respectively. Then, for every $u \in \overline{V}(\mathscr{V} \otimes \mathscr{W})$ there are $v \in \overline{V}(\mathscr{V})$ and $w \in \overline{V}(\mathscr{W})$ such that $u \leq v \otimes w$.

3 Generalized Gelfand–Shilov Spaces of Roumieu Type

We now introduce the class of Gelfand–Shilov spaces of Roumieu type that we are interested in. They are defined via a weight sequence and a decreasing weight system $\mathscr{V}$ (on $\mathbb{R}^d$) and our aim is to give a projective description of these spaces in terms of Komatsu's family $\mathfrak{R}$ (defined below) and the maximal Nachbin family associated with $\mathscr{V}$.

Let $(M_p)_{p\in\mathbb{N}}$ be a *weight sequence*, that is, a positive sequence that satisfies $\lim_{p\to\infty} M_p/M_{p-1} = \infty$. For $h > 0$ and a non-negative function v on $\mathbb{R}^d$ we write $\mathscr{D}^{M_p,h}_{L^\infty_v}(\mathbb{R}^d)$ for the seminormed space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathscr{D}^{M_p,h}_{L^\infty_v}} := \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{h^{|\alpha|}|\partial^\alpha\varphi(x)|v(x)}{M_\alpha} < \infty,$$

where we write $M_\alpha = M_{|\alpha|}$, $\alpha \in \mathbb{N}^d$. Following Komatsu [12], we denote by $\mathfrak{R}$ the set of all positive increasing sequences $(r_j)_{j\in\mathbb{N}}$ tending to infinity. For $r_j \in \mathfrak{R}$ and a non-negative function v on $\mathbb{R}^d$ we write $\mathscr{D}^{M_p,r_j}_{L^\infty_v}(\mathbb{R}^d)$ for the seminormed space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathscr{D}^{M_p,r_j}_{L^\infty_v}} := \sup_{\alpha\in\mathbb{N}^d} \sup_{x\in\mathbb{R}^d} \frac{|\partial^\alpha\varphi(x)|v(x)}{M_\alpha \prod_{j=0}^{|\alpha|} r_j} < \infty.$$

Similarly as before, $\|\cdot\|_{\mathscr{D}^{M_p,h}_{L^\infty_v}}$ and $\|\cdot\|_{\mathscr{D}^{M_p,r_j}_{L^\infty_v}}$ are actually norms if v is positive, while $\mathscr{D}^{M_p,h}_{L^\infty_v}(\mathbb{R}^d)$ and $\mathscr{D}^{M_p,r_j}_{L^\infty_v}(\mathbb{R}^d)$ are complete and thus Banach spaces if additionally $1/v$ is locally bounded. Given a decreasing weight system $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$, we define, as in the introduction,

$$\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) := \varinjlim_{n\to\infty} \mathscr{D}^{M_p,1/n}_{L^\infty_{v_n}}(\mathbb{R}^d),$$

a Hausdorff (LB)-space. Moreover, we write $\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ for the space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathscr{D}^{M_p,r_j}_{L^\infty_v}} < \infty$ for all $r_j \in \mathfrak{R}$ and $v \in \overline{V}$, and endow it with the topology generated by the system of seminorms $\{\|\cdot\|_{\mathscr{D}^{M_p,r_j}_{L^\infty_v}} : r_j \in \mathfrak{R}, v \in \overline{V}\}$.

Lemma 1 *Let M_p be a weight sequence and let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing weight system. Then, $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ and $\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ coincide algebraically and the inclusion mapping $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) \to \widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ is continuous.*

Proof It is obvious that $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ is continuously included in $\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$. For the converse inclusion, we consider the decreasing weight system $\mathscr{W} = (w_n)_{n\in\mathbb{N}}$ on $\mathbb{N}^d$ (endowed with the discrete topology), given by $w_n(\alpha) := n^{-|\alpha|}$, $\alpha \in \mathbb{N}^d$. Now let $\varphi \in \widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ be arbitrary and define $f(x,\alpha) = \partial^\alpha\varphi(x)/M_\alpha$ for $x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d$. By Remark 4 and [12, Lemma 3.4(ii)] we have that $f \in C\overline{V}(\mathscr{V}\otimes\mathscr{W})(\mathbb{R}^d \times \mathbb{N}^d)$. Since $\mathscr{V}\otimes\mathscr{W}C(\mathbb{R}^d\times\mathbb{N}^d) = C\overline{V}(\mathscr{V}\otimes\mathscr{W})(\mathbb{R}^d\times\mathbb{N}^d)$ as sets (cf. Sect. 2), we obtain that $f \in \mathscr{V}\otimes\mathscr{W}C(\mathbb{R}^d\times\mathbb{N}^d)$, which precisely means that $\varphi \in \mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$. □

The rest of this article is devoted to showing that, under mild conditions on M_p and $\mathscr{V}$, the equality $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) = \widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ also holds topologically.

We will make use of the following two standard conditions for weight sequences: $(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \geq 1$, and $(M.2)'$ $M_{p+1} \leq C_0 H^p M_p$, $p \in \mathbb{N}$, for some $C_0, H \geq 1$. We also need the *associated function* of the sequence M_p in our considerations, which is given by

$$M(t) := \sup_{p \in \mathbb{N}} \log \frac{t^p M_0}{M_p}, \qquad t > 0,$$

and $M(0) := 0$. We define M on $\mathbb{R}^d$ as the radial function $M(x) = M(|x|)$, $x \in \mathbb{R}^d$. The assumption $(M.2)'$ implies that $M(H^k t) - M(t) \geq k \log(t/C_0)$, $t, k \geq 0$ [11, Prop. 3.4]. In particular, we have that $\mathrm{e}^{M(t)-M(H^{d+1}t)} \leq (2C_0)^{d+1}(1 + t^{d+1})^{-1}$, $t \geq 0$. Given $r_j \in \mathfrak{R}$, we denote by M_{r_j} the associated function of the weight sequence $M_p \prod_{j=0}^{p} r_j$.

As mentioned in the introduction, our arguments will rely on the mapping properties of the short-time Fourier transform, which we now introduce. The translation and modulation operators are denoted by $T_x f = f(\,\cdot\, - x)$ and $M_\xi f = \mathrm{e}^{2\pi \mathrm{i} \xi \cdot} f$, for $x, \xi \in \mathbb{R}^d$. The *short-time Fourier transform (STFT)* of a function $f \in L^2(\mathbb{R}^d)$ with respect to a window function $\psi \in L^2(\mathbb{R}^d)$ is defined as

$$V_\psi f(x, \xi) := (f, M_\xi T_x \psi)_{L^2} = \int_{\mathbb{R}^d} f(t)\overline{\psi(t - x)}\mathrm{e}^{-2\pi \mathrm{i} \xi t} \mathrm{d}t, \qquad (x, \xi) \in \mathbb{R}^{2d}.$$

We have that $\|V_\psi f\|_{L^2(\mathbb{R}^{2d})} = \|\psi\|_{L^2} \|f\|_{L^2}$. In particular, the mapping $V_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$ is continuous. The adjoint of V_ψ is given by the weak integral

$$V_\psi^* F = \int\int_{\mathbb{R}^{2d}} F(x, \xi) M_\xi T_x \psi \mathrm{d}x \mathrm{d}\xi, \qquad F \in L^2(\mathbb{R}^{2d}).$$

If $\psi \neq 0$ and $\gamma \in L^2(\mathbb{R}^d)$ is a synthesis window for ψ, that is, $(\gamma, \psi)_{L^2} \neq 0$, then

$$\frac{1}{(\gamma, \psi)_{L^2}} V_\gamma^* \circ V_\psi = \mathrm{id}_{L^2(\mathbb{R}^d)}\,. \tag{1}$$

We are interested in the STFT on the spaces $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ and $\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$. This requires to impose some further conditions on the weight system $\mathscr{V}$. Let A_p be a weight sequence with associated function A. A decreasing weight system $\mathscr{V} = (v_n)_{n \in \mathbb{N}}$ is said to be A_p*-admissible* if there is $\tau > 0$ such that for every $n \in \mathbb{N}$ there are $m \geq n$ and $C > 0$ such that $v_m(x + y) \leq C v_n(x)\mathrm{e}^{A(\tau y)}$, $x, y \in \mathbb{R}^d$. We start with two lemmas. As customary [14], given two weight sequences M_p and A_p, we denote by $\mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ the Gelfand–Shilov space of Beurling type. For the weight function $v = \mathrm{e}^{A(\tau\,\cdot)}$, $\tau > 0$, we use the alternative notation $\|\cdot\|_{\mathscr{S}^{M_p,h}_{A_p,\tau}} = \|\cdot\|_{\mathscr{D}^{M_p,h}_{L^\infty_v}}$

so that the Fréchet space structure of $\mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ is determined by the family of norms $\{\|\cdot\|_{\mathscr{S}^{M_p,h}_{A_p,\tau}} : h > 0, \tau > 0\}$.

Lemma 2 *Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$, let w and v be non-negative measurable functions on $\mathbb{R}^d$ such that*

$$v(x+y) \leq Cw(x)\mathrm{e}^{A(\tau y)}, \qquad x, y \in \mathbb{R}^d, \tag{2}$$

for some $C, \tau > 0$, and let $\psi \in \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$. Then, the mapping $V_\psi : \mathscr{D}^{M_p,h}_{L^\infty_w}(\mathbb{R}^d) \to Cv \otimes \mathrm{e}^{M(\pi h \cdot /\sqrt{d})}(\mathbb{R}^{2d}_{x,\xi})$ is well-defined and continuous.

Proof Let $\varphi \in \mathscr{D}^{M_p,h}_{L^\infty_w}(\mathbb{R}^d)$ be arbitrary. For all $\alpha \in \mathbb{N}^d$ and $(x,\xi) \in \mathbb{R}^{2d}$,

$$\begin{aligned}
&|\xi^\alpha V_\psi \varphi(x,\xi)|v(x) \\
&\leq C(2\pi)^{-|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^d} |\partial^\beta \varphi(t)|w(t)|\partial^{\alpha-\beta}\psi(t-x)|\mathrm{e}^{A(\tau(t-x))}\mathrm{d}t \\
&\leq C'\|\varphi\|_{\mathscr{D}^{M_p,h}_{L^\infty_w}} (\pi h)^{-|\alpha|} M_\alpha.
\end{aligned}$$

Hence

$$|V_\psi \varphi(x,\xi)|v(x) \leq C'\|\varphi\|_{\mathscr{D}^{M_p,h}_{L^\infty_w}} \inf_{p \in \mathbb{N}} \frac{M_p}{(\pi h|\xi|/\sqrt{d})^p} = C' M_0 \|\varphi\|_{\mathscr{D}^{M_p,h}_{L^\infty_w}} \mathrm{e}^{-M(\pi h \xi/\sqrt{d})}.$$

□

Lemma 3 *Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$, let w and v be non-negative measurable functions on $\mathbb{R}^d$ satisfying (2), and let $\psi \in \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$. Then, the mapping $V^*_\psi : Cw \otimes \mathrm{e}^{M(h\,\cdot)}(\mathbb{R}^{2d}_{x,\xi}) \to \mathscr{D}^{M_p,h/(4H^{d+1}\pi)}_{L^\infty_v}(\mathbb{R}^d)$ is well-defined and continuous.*

Proof Let $F \in Cw \otimes \mathrm{e}^{M(h\,\cdot)}(\mathbb{R}^{2d}_{x,\xi})$ be arbitrary and set $k = h/(2H^{d+1}\pi)$. For each $\alpha \in \mathbb{N}^d$ (we write $\|\cdot\|_{Cw\otimes \mathrm{e}^{M(h\,\cdot)}} = \|\cdot\|$)

$$\begin{aligned}
&\sup_{t \in \mathbb{R}^d} |\partial^\alpha V^*_\psi F(t)|v(t) \\
&\leq C \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \sup_{t \in \mathbb{R}^d} \int\int_{\mathbb{R}^{2d}} |F(x,\xi)|w(x)(2\pi|\xi|)^{|\beta|}|\partial^{\alpha-\beta}\psi(t-x)|\mathrm{e}^{A(\tau(t-x))}\mathrm{d}x\mathrm{d}\xi \\
&\leq CM_0^{-1}\|\psi\|_{\mathscr{S}^{M_p,k}_{A_p,H^{d+1}\tau}} \|F\| \frac{M_\alpha}{(k/2)^{|\alpha|}} \int\int_{\mathbb{R}^{2d}} \mathrm{e}^{M(2\pi k\xi)-M(h\xi)}\mathrm{e}^{A(\tau x)-A(H^{d+1}\tau x)}\mathrm{d}x\mathrm{d}\xi \\
&\leq C'\|F\| \frac{M_\alpha}{(h/(4H^{d+1}\pi))^{|\alpha|}}.
\end{aligned}$$

□

Lemmas 2 and 3 yield the following corollary.

Corollary 1 *Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$ and denote by X the Fréchet space consisting of all $F \in C(\mathbb{R}^{2d})$ such that*

$$\sup_{(x,\xi)\in\mathbb{R}^{2d}} |F(x,\xi)|\mathrm{e}^{A(nx)+M(n\xi)} < \infty$$

for all $n \in \mathbb{N}$. Let $\psi \in \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$. Then, the mappings $V_\psi : \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d) \to X$ and $V^*_\psi : X \to \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ are well-defined and continuous.

We are now able to establish the mapping properties of the STFT on $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$. Given a weight sequence M_p with associated function M, we define $\mathscr{V}_{\{M_p\}} := (\mathrm{e}^{M(\,\cdot\,/n)})_{n\in\mathbb{N}}$, a decreasing weight system on $\mathbb{R}^d$.

Proposition 1 *Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$, let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ be an A_p-admissible decreasing weight system, and let $\psi \in \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$. Then, the following mappings are continuous:*

$$V_\psi : \mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) \to \mathscr{V} \otimes \mathscr{V}_{\{M_p\}}C(\mathbb{R}^{2d}_{x,\xi})$$

and

$$V^*_\psi : \mathscr{V} \otimes \mathscr{V}_{\{M_p\}}C(\mathbb{R}^{2d}_{x,\xi}) \to \mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d).$$

Assume that $\mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d) \neq \{0\}$. If $\psi \neq 0$ and $\gamma \in \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ is a synthesis window for ψ, the following reconstruction formula holds

$$\frac{1}{(\gamma,\psi)_{L^2}} V^*_\gamma \circ V_\psi = \mathrm{id}_{\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)}\,. \tag{3}$$

Proof Since $\mathscr{V}$ is A_p-admissible, the continuity of V_ψ and V^*_ψ follows directly from Lemmas 2 and 3, respectively. We now show (3). Let $\varphi \in \mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ be arbitrary. As $V^*_\gamma(V_\psi\varphi)$ and φ are both $O(e^{A(\tau\cdot)})$-bounded continuous functions, it suffices to show that

$$\int_{\mathbb{R}^d} V^*_\gamma(V_\psi\varphi)(t)\chi(t)\mathrm{d}t = (\gamma,\psi)_{L^2}\int_{\mathbb{R}^d}\varphi(t)\chi(t)\mathrm{d}t$$

for all $\chi \in \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$. Formula (1) implies that

$$\begin{aligned}\int_{\mathbb{R}^d} V^*_\gamma(V_\psi\varphi)(t)\chi(t)\mathrm{d}t &= \int_{\mathbb{R}^d}\left(\int\int_{\mathbb{R}^{2d}} V_\psi\varphi(x,\xi)M_\xi T_x\gamma(t)\mathrm{d}x\mathrm{d}\xi\right)\chi(t)\mathrm{d}t \\ &= \int\int_{\mathbb{R}^{2d}}\left(\int_{\mathbb{R}^d}\varphi(t)M_{-\xi}T_x\overline{\psi}(t)\mathrm{d}t\right)V_{\overline{\gamma}}\chi(x,-\xi)\mathrm{d}x\mathrm{d}\xi \\ &= \int_{\mathbb{R}^d} V^*_{\overline{\psi}}(V_{\overline{\gamma}}\chi)(t)\varphi(t)\mathrm{d}t \\ &= (\gamma,\psi)_{L^2}\int_{\mathbb{R}^d}\varphi(t)\chi(t)\mathrm{d}t,\end{aligned}$$

where the switching of the integrals is permitted because of Corollary 1 and the first part of this proposition. □

In order to show the analogue of Proposition 1 for $\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$, we need the following technical lemma.

Lemma 4 *Let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ be an A_p-admissible decreasing weight system. For every $v \in \overline{V}$ there is $\overline{v} \in \overline{V}$ such that $v(x+y) \le \overline{v}(x)\mathrm{e}^{A(\tau y)}$, $x, y \in \mathbb{R}^d$.*

Proof Find a strictly increasing sequence of natural numbers $(n_j)_{j\in\mathbb{N}}$ such that $v_{n_{j+1}}(x+y) \le C_j v_{n_j}(x)\mathrm{e}^{A(\tau y)}$, $x, y \in \mathbb{R}^d$, for some $C_j > 0$. Pick $C'_j > 0$ such that $v \le C'_j v_{n_j}$ for all $j \in \mathbb{N}$. Set $\overline{v} = \inf_{j\in\mathbb{N}} C_j C'_{j+1} v_{n_j} \in \overline{V}$. We have that

$$v(x+y) \le \inf_{j\in\mathbb{N}} C'_{j+1}v_{n_{j+1}}(x+y) \le \mathrm{e}^{A(\tau y)}\inf_{j\in\mathbb{N}} C_j C'_{j+1}v_{n_j}(x) = \overline{v}(x)\mathrm{e}^{A(\tau y)}. \qquad \square$$

Proposition 2 *Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$, let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ be an A_p-admissible decreasing weight system, and let $\psi \in \mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d)$. Then, the following mappings are continuous:*

$$V_\psi : \widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) \to C\overline{V}(\mathscr{V}\otimes\mathscr{V}_{\{M_p\}})(\mathbb{R}^{2d}_{x,\xi})$$

and

$$V^*_\psi : C\overline{V}(\mathscr{V}\otimes\mathscr{V}_{\{M_p\}})(\mathbb{R}^{2d}_{x,\xi}) \to \widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d).$$

Proof Let $u \in \overline{V}(\mathscr{V}\otimes\mathscr{V}_{\{M_p\}})$ be arbitrary. By Remark 4 and [7, Lemma 4.5(i)] there is $v \in \overline{V}(\mathscr{V})$ and $r_j \in \mathfrak{R}$ such that $u \le v\otimes\mathrm{e}^{M_{r_j}}$. According to [15, Lemma 2.3] there is $r'_j \in \mathfrak{R}$ such that $r'_j \le r_j$ for j large enough and $r'_{j+1} \le 2^{j+1}r'_j$ for all $j \in \mathbb{N}$. Hence the sequence $M_p\prod_{j=0}^p r'_j$ satisfies $(M.2)'$. Next, by Lemma 4 there is $\overline{v} \in \overline{V}$ such that $v(x+y) \le \overline{v}(x)\mathrm{e}^{A(\tau y)}$ for all $x, y \in \mathbb{R}^d$. Therefore, Lemma 2 implies that the mapping $V_\psi : \mathscr{D}^{M_p,\pi r'_j/\sqrt{d}}_{L^\infty_{\overline{v}}}(\mathbb{R}^d) \to Cv \otimes \mathrm{e}^{M_{r'_j}}(\mathbb{R}^{2d})$ is well-defined and continuous. As the inclusion mapping $Cv \otimes \mathrm{e}^{M_{r'_j}}(\mathbb{R}^{2d}) \to Cu(\mathbb{R}^{2d})$ is continuous, we may conclude that V_ψ is continuous. Similarly, by using Lemma 3, one can show that V^*_ψ is continuous. □

We are ready to prove our main theorem.

Theorem 2 *Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$ such that $\mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d) \neq \{0\}$ and let $\mathscr{V} = (v_n)_{n\in\mathbb{N}}$ be an A_p-admissible decreasing weight system satisfying (V). Then, $\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ and $\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ coincide topologically.*

Proof By Lemma 1 it suffices to show that the inclusion mapping $\iota : \widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) \to \mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ is continuous. Since M_p satisfies $(M.2)'$, the decreasing weight system $\mathscr{V}_{\{M_p\}}$ satisfies (S) and thus condition (V) (see Remark 2). Hence Proposition 1 and Remark 3 imply that $\mathscr{V} \otimes \mathscr{V}_{\{M_p\}}C(\mathbb{R}^{2d}) = C\overline{V}(\mathscr{V} \otimes \mathscr{V}_{\{M_p\}})(\mathbb{R}^{2d})$ topologically. Choose $\psi, \gamma \in \mathscr{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$ such that $(\gamma, \psi)_{L^2} = 1$. By (3) the following diagram commutes:

$$
\begin{array}{ccc}
\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) & \xrightarrow{V_\psi} & C\overline{V}(\mathscr{V} \otimes \mathscr{V}_{\{M_p\}})(\mathbb{R}^{2d}) = \mathscr{V} \otimes \mathscr{V}_{\{M_p\}}C(\mathbb{R}^{2d}) \\
\iota \downarrow & \swarrow V^*_\gamma & \\
\mathscr{B}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) & &
\end{array}
$$

Propositions 1 and 2 imply that V_ψ and V^*_γ are continuous, whence ι is also continuous. □

Remark 5 Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$ such that $\mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d) \neq \{0\}$. By applying Theorem 2 to $\mathscr{V} = \mathscr{V}_{\{A_p\}}$ (and using [7, Lemma 4.5(i)]), we obtain the well-known projective description of the classical Gelfand–Shilov space $\mathscr{S}^{\{M_p\}}_{\{A_p\}}(\mathbb{R}^d)$ of Roumieu type [13, Lemma 4].

We end this article by stating an important particular case of Theorem 2. Given a positive function ω on $\mathbb{R}^d$ such that $1/\omega$ is locally bounded, we define

$$\mathscr{D}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d) := \varinjlim_{h\to 0^+} \mathscr{D}^{M_p,h}_{L^\infty_\omega}(\mathbb{R}^d),$$

a Hausdorff (LB)-space. Furthermore, we write $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d)$ for the space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_{\mathscr{D}^{M_p,r_j}_{L^\infty_\omega}} < \infty$ for all $r_j \in \mathfrak{R}$ and endow it with the topology generated by the system of seminorms $\{\|\cdot\|_{\mathscr{D}^{M_p,r_j}_{L^\infty_\omega}} : r_j \in \mathfrak{R}\}$.

Theorem 3 *Let M_p and A_p be weight sequences satisfying $(M.1)$ and $(M.2)'$ such that $\mathscr{S}^{(M_p)}_{(A_p)}(\mathbb{R}^d) \neq \{0\}$ and let ω be a positive measurable function on $\mathbb{R}^d$ such that $\omega(x+y) \leq C\omega(x)\mathrm{e}^{A(\tau y)}$, $x, y \in \mathbb{R}^d$, for some $C, \tau > 0$. Then, $\mathscr{D}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d)$ and $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d)$ coincide topologically.*

Proof We may assume without loss of generality that ω is continuous (for otherwise we may replace ω with the continuous weight $\tilde{\omega} = \omega * \varphi$, where $\varphi \in \mathscr{D}(\mathbb{R}^d)$ is non-negative and satisfies $\int_{\mathbb{R}^d} \varphi(t)\mathrm{d}t = 1$, since $\mathscr{D}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d) = \mathscr{D}^{\{M_p\}}_{L^\infty_{\tilde{\omega}}}(\mathbb{R}^d)$ and $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d) = \widetilde{\mathscr{D}}^{\{M_p\}}_{L^\infty_{\tilde{\omega}}}(\mathbb{R}^d)$ topologically). We set $\mathscr{V} = (\omega)_{n\in\mathbb{N}}$ and notice that $\mathscr{V}$ satisfies (V) (see Remark 2). Hence, by Theorem 2, we find that $\mathscr{D}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d) = \widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d)$ topologically. The result now follows from the fact that $\overline{V}(\mathscr{V}) = \{\lambda\omega : \lambda > 0\}$ and, thus, $\widetilde{\mathscr{B}}^{\{M_p\}}_{\mathscr{V}}(\mathbb{R}^d) = \widetilde{\mathscr{D}}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d)$ topologically. □

Remark 6 Theorem 3 was already shown in [8, Thm. 4.17] under much more restrictive conditions on M_p and A_p and with a more complicated proof.

In [6, Thm. 5.9] we have shown that, if M_p satisfies $(M.1)$ and $(M.2)$ (cf. [11]), the space $\mathscr{S}^{(M_p)}_{(p!)}(\mathbb{R}^d)$ is non-trivial if and only if $(\log p)^p \prec M_p$ (the latter means, as usual, that $M_p^{1/p}/\log p \to \infty$). Hence, we obtain the ensuing corollary.

Corollary 2 *Let M_p be a weight sequence satisfying $(M.1)$ and $(M.2)$ such that $(\log p)^p \prec M_p$ and let ω be a positive measurable function on $\mathbb{R}^d$ such that $\omega(x+y) \leq C\omega(x)\mathrm{e}^{\tau|y|}$, $x, y \in \mathbb{R}^d$, for some $C, \tau > 0$. Then, $\mathscr{D}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d)$ and $\widetilde{\mathscr{D}}^{\{M_p\}}_{L^\infty_\omega}(\mathbb{R}^d)$ coincide topologically.*

Acknowledgements A. Debrouwere acknowledges support by FWO-Vlaanderen, through the postdoctoral grant 12T0519N. The work of J. Vindas was supported by Ghent University, through the BOF-grants 01N01014 and 01J04017.

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Generalized Solutions and Distributional Shadows for Dirac Equations

Günther Hörmann and Christian Spreitzer

Abstract We discuss the application of recent results on generalized solutions to the Cauchy problem for hyperbolic systems to Dirac equations with external fields. In further analysis we focus on the question of existence of associated distributional limits and derive their explicit form in case of free Dirac fields with regularizations of initial values corresponding to point-like probability densities.

1 Introduction

The Dirac equation on Minkowski space M describes a relativistic spin-$\frac{1}{2}$ particle field $\psi\colon M \to \mathbb{C}^4$, involving the Dirac 4×4-matrices γ^α ($\alpha = 0, \ldots, 3$) as generators of (a representation of) the Clifford algebra. Two prominent examples of Dirac equations with external fields arise in the following models (cf. [12]):

1. If an electromagnetic potential one-form A (with components A_α) is given on M, then the action on a particle with charge e and mass m is described by

$$(\partial_t - ieA_0)\psi + \sum_{j=1}^{3} \gamma^0\gamma^j(\partial_j - ieA_j)\psi + im\gamma^0\psi = 0. \tag{1}$$

The distribution theoretic Cauchy problem with compactly supported smooth A and initial data given on arbitrary Cauchy surfaces is reviewed in detail in [2].

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_40

2. If F is a given electromagnetic field two-form (with components $F_{\alpha\beta}$), then the action on a neutral particle with magnetic moment μ and mass m is described by

$$\partial_t \psi + \sum_{j=1}^{3} \gamma^0 \gamma^j \partial_j \psi + \frac{\mu}{2} \sum_{\alpha,\beta=0}^{3} \gamma^0 \gamma^\alpha \gamma^\beta F_{\alpha\beta} \psi + i m \gamma^0 \psi = 0. \tag{2}$$

Both of the above cases will be seen to be combined in Sect. 2 below as special instances of a symmetric hyperbolic operator with generalized functions as coefficients. Moreover, this more general setting will enable us to approach the question of how to describe fields that are initially localized in a point-like sense. Recall that for a solution ψ of the Dirac equation, the quantity $|\psi(t,.)|^2$ is interpreted as a *spatial probability density at time t.* What could we then say about an initially prescribed probability density $|\psi(0,.)|^2$ modeling a δ-type concentration?

In terms of a representative $(\psi_\varepsilon)_{\varepsilon>0}$ of a generalized spinor field, this would mean that we solve the Dirac equation with an initial condition represented in the form $\psi_\varepsilon|_{t=0} = \varphi_\varepsilon$, where the regularization family $(\varphi_\varepsilon)_{\varepsilon>0}$ has the property that $|\varphi_\varepsilon|^2$ converges to δ in the sense of distributions as $\varepsilon \to 0$. In this sense, $(\varphi_\varepsilon)_{\varepsilon>0}$ is a regularization of $\sqrt{\delta}$. The corresponding Cauchy problem is well-posed in spaces of Colombeau generalized functions according to the results presented in Sect. 2. Moreover, in case of the free Dirac equation we will prove in Sect. 3 that the probability density associated with the generalized solution possesses a distributional limit and we will determine its explicit shape. Related questions for the Schrödinger equation have been discussed in [6, 7], where also some details on choices of regularizations of $\sqrt{\delta}$ can be found.

Why is it important to allow for a non-smooth potential A and a non-smooth electromagnetic field F? Recall that the Maxwell equations in terms of exterior differentials are $\mathrm{d}F = 0$, $*\mathrm{d}*F = J$, where J is the current one-form on M and $*$ denotes the Hodge-star operator according to the Lorentz metric g on M. For the electromagnetic field generated by a particle of charge e and with world line $s \mapsto c(s)$, we obtain the current J as the distributional one-form on M supported on c, acting on a compactly supported vector field v by

$$\langle J, v \rangle = e \int_{-\infty}^{\infty} g\big(\dot{c}(s), v(c(s))\big)\, \mathrm{d}s.$$

Remark 1

(i) The distributional nature of J (and hence F) is well-known to cause serious mathematical problems with self-interaction due to the non-smooth *Lorentz force* L generated in the form $\langle L, v \rangle = e \int_{-\infty}^{\infty} F_{(c(s))}\big(\dot{c}(s), v(c(s))\big)\, \mathrm{d}s$. Coupling with the Maxwell equations should include the *radiation-reaction* of the electron, but leads to systems of nonlinear differential equations with distributional data. A rigorous derivation of the Lorentz–Dirac equation in this context can be found in [4]. The generalized solutions to the coupled field equations and the equation of motion have been analyzed in [8, 9], where also the non-existence of distributional shadows has been shown. These difficulties do not disappear upon quantization, but some renormalization effects could be described via generalized numbers in [3, 5]. Discussing instead Dirac equations with external fields might thus be excused to some extent by the following folklore wisdom (e.g., from [13]): "A theory of particles in an 'external' field is a first step towards a description of a true interaction."

(ii) In terms of geometric structures, it is natural to consider the electromagnetic field as a curvature two-form on a space–time (M, g). The electromagnetic field F on M then stems from a connection on a principal $U(1)$-bundle P over M which is given in terms of a one-form ω on P. If M is contractible (e.g., Minkowski space), then $P = M \times U(1)$ is trivial and we have the simple description $\omega = \frac{\mathrm{d}z}{z} - iA$, where z denotes the coordinate on $\mathbb{C} \supset U(1)$ and A is a real one-form on M. If V is a complex vector space (with $U(1)$ action) and a field ψ is (locally) written as a map from M into V, then the covariant derivative in the direction of the tangent vector field h is $\mathrm{D}_h\psi = \mathrm{d}\psi(h) - iA(h)\psi$. We obtain the curvature two-form $R(h_1, h_2)(\psi) := \mathrm{D}_{h_1}(\mathrm{D}_{h_2}\psi) - \mathrm{D}_{h_2}(\mathrm{D}_{h_1}\psi) - \mathrm{D}_{[h_1,h_2]}\psi = -i\mathrm{d}A(h_1, h_2)\psi$, which means $F = iR = \mathrm{d}A$. All these geometric constructions can be carried out with non-smooth fields in the sense of generalized connections and curvature as developed in [11], where also an application to Dirac's theory of magnetic monopoles and basics of a Yang–Mills theory are given. A natural next step along these lines would be to discuss generalized spinor bundles and Dirac operators in this sense as well.

The contents of the remaining parts are as follows: Sect. 2 presents a brief review of the Cauchy problem and applications to Dirac equations with non-smooth external fields. Section 3 discusses distributional shadows of free Dirac fields with $\sqrt{\delta}$ as initial data in 1 + 1 dimensions and on Minkowski space.

2 The Cauchy Problem for the Dirac Equation with External Fields on Minkowski Space

All the results described in this section can be found in [12] or follow directly from statements proved there. Let $T > 0$ and $\Omega_T := [0, T] \times \mathbb{R}^3$. The two model Eqs. (1) and (2) may be combined in the form of the following first-order symmetric hyperbolic system for a Colombeau generalized spinor field $\psi \in \mathcal{G}_{L^2}(\Omega_T)^4$:

$$\partial_t \psi + \sum_{j=1}^{3} \gamma^0 \gamma^j \partial_j \psi + B\psi = 0, \quad \psi|_{t=0} = \varphi, \tag{3}$$

where B is a matrix with components $b_{kl} \in \mathcal{G}_{L^\infty}(\Omega_T)$, and initial value $\varphi \in \mathcal{G}_{L^2}(\mathbb{R}^3)^4$.

Theorem 1 *If $B + B^*$ is of $L^{1,\infty}$-log-type, that is, $\int_0^T \|(b_{kl,\varepsilon} + \overline{b_{lk,\varepsilon}})(t,.)\|_{L^\infty}\, dt = \mathcal{O}(\log(\frac{1}{\varepsilon}))$ as $\varepsilon \to 0$, then there exists a unique solution $\psi \in \mathcal{G}_{L^2}(\Omega_T)^4$ to (3).*

We point out that the theorem applies with B containing the field of a moving point particle, if logarithmic scaling is used in its regularization. We note in passing that there is also an intrinsic regularity property phrased in terms of $\mathcal{G}^\infty$, which requires uniform asymptotic bounds on all derivatives [10]: If $\varphi \in \mathcal{G}^\infty(\mathbb{R}^3)^4$ and B is of logarithmic slow scale in the $\mathcal{G}^\infty$-sense, then the solution satisfies $\psi \in \mathcal{G}^\infty(\Omega_T)^4$.

Regarding distributional aspects and compatibility with classical spaces, we have the following results. Recall that $u \in \mathcal{D}'$ is said to be a distributional shadow of $\psi \in \mathcal{G}$, in notation $\psi \approx u$, if $\psi_\varepsilon \to u$ ($\varepsilon \to 0$) holds in $\mathcal{D}'$ for any (hence every) representative.

Proposition 1

(i) If φ and B are smooth, then ψ equals the unique smooth solution.
(ii) If φ has components in H^s and B is smooth, then $\psi \approx v$, where v denotes the unique distributional solution.
(iii) If $\varphi \in H^1(\mathbb{R}^3)^4$ and B has components in $L^1([0,T], W^{1,\infty}(\mathbb{R}^3))$, then $\psi \approx u$ with $u \in C([0,T], H^1(\mathbb{R}^3))^4 \cap W^{1,1}(\Omega_T)^4$.
(iv) If $\varphi \in L^2(\mathbb{R}^3)^4$ and B has components in $L^1([0,T], H^2(\mathbb{R}^3))$, then $\psi \approx u$ with $u \in C([0,T], L^2(\mathbb{R}^3))^4$.

Unfortunately, none of these convergence results is applicable to the situation of a coefficient matrix B involving the field of a moving point particle or an initial value φ corresponding to a point-like concentrated field configuration.

3 Distributional Limits for Free Dirac Fields with $\sqrt{\delta}$ Initial Data

3.1 The Case of One Spatial Dimension

We consider the Cauchy problem for a free Dirac particle of mass m in 1+1 dimensions with generalized functions as initial data (and physical units with $\hbar = c = 1$)

$$i\partial_t\psi + i\sigma^1\partial_x\psi - m\,\sigma^3\psi = 0,$$
$$\psi|_{t=0} = \varphi,$$

where φ is represented by $(\varphi_\varepsilon)_{\varepsilon>0}$, $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We are interested in the distributional limit of $|\psi_\varepsilon|^2$, where $(\psi_\varepsilon)_{\varepsilon>0}$ represents the generalized solution, if $|\varphi_\varepsilon|^2$ converges to δ. For rapidly decaying initial data we may write

$$\psi_\varepsilon(t,x) = \int_{\mathbb{R}} \frac{e^{ikx}}{\sqrt{2\pi}} \Big(\langle u_{\text{pos}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle u_{\text{pos}}(k) e^{-it\lambda(k)} + \langle u_{\text{neg}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle u_{\text{neg}}(k) e^{it\lambda(k)} \Big) \mathrm{d}k,$$

where $\widehat{\varphi}$ denotes the Fourier transform of φ and $\lambda(k) := \sqrt{k^2+m^2}$. The functions

$$u_{\text{pos}}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1+\frac{m}{\lambda(k)}} \\ \operatorname{sgn}(k)\sqrt{1-\frac{m}{\lambda(k)}} \end{pmatrix}, \qquad u_{\text{neg}}(k) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\operatorname{sgn}(k)\sqrt{1-\frac{m}{\lambda(k)}} \\ \sqrt{1+\frac{m}{\lambda(k)}} \end{pmatrix}$$

are normalized eigenvectors of the matrix $-ik\sigma^1 + m\,\sigma^3$ with eigenvalues $\pm\lambda(k)$, corresponding to positive and negative energies. To model the initial wave-function in $\mathcal{G}_{L^2}(\mathbb{R})^2$, we pick $\rho_1, \rho_2 \in \mathcal{S}(\mathbb{R})$ such that $\|\rho_1\|^2_{L^2(\mathbb{R})} + \|\rho_2\|^2_{L^2(\mathbb{R})} = 1$ and set $\varphi(x) := \begin{pmatrix} \rho_1(x) \\ \rho_2(x) \end{pmatrix}$. The scaling $\varphi_\varepsilon(x) := \frac{1}{\sqrt{\varepsilon}}\varphi(\frac{x}{\varepsilon})$ yields $|\varphi_\varepsilon|^2 \to \delta$ as $\varepsilon \to 0$. Interpreting $\mu_t^\varepsilon(x) := |\psi_\varepsilon(t,x)|^2$ as a spatial probability density at time t, we consider its distributional action on a test function $h \in \mathcal{D}(\mathbb{R})$ and use Fubini's theorem and the fact that $\widehat{\varphi_\varepsilon}(k) = \sqrt{\varepsilon}\widehat{\varphi}(\varepsilon k)$ to obtain

$$\begin{aligned}
\langle \mu_t^\varepsilon, h\rangle = \frac{\varepsilon}{2\pi} \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} & e^{i(k'-k)x} h(x)\mathrm{d}x \\
&\Big(e^{-it(\lambda(k')-\lambda(k))} \overline{\langle u_{\text{pos}}(k)|\widehat{\varphi}(\varepsilon k)\rangle} \langle u_{\text{pos}}(k')|\widehat{\varphi}(\varepsilon k')\rangle \langle u_{\text{pos}}(k)|u_{\text{pos}}(k')\rangle \\
&+ 2\mathrm{Re}\Big(e^{it(\lambda(k')+\lambda(k))} \overline{\langle u_{\text{pos}}(k)|\widehat{\varphi}(\varepsilon k)\rangle} \langle u_{\text{neg}}(k')|\widehat{\varphi}(\varepsilon k')\rangle \langle u_{\text{pos}}(k)|u_{\text{neg}}(k')\rangle \Big) \\
&+ e^{it(\lambda(k')-\lambda(k))} \overline{\langle u_{\text{neg}}(k)|\widehat{\varphi}(\varepsilon k)\rangle} \langle u_{\text{neg}}(k')|\widehat{\varphi}(\varepsilon k')\rangle \langle u_{\text{neg}}(k)|u_{\text{neg}}(k')\rangle \Big) \mathrm{d}k\,\mathrm{d}k'.
\end{aligned}$$

By the change of variables $(k,k') \mapsto (\eta,\xi) := (\varepsilon k, k'-k)$ we may write this as

$$\begin{aligned}
\langle \mu_t^\varepsilon, h\rangle = \frac{\varepsilon}{\sqrt{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} & \mathcal{F}^{-1}(h)(\xi) \\
&\Big(e^{-it\Lambda_\varepsilon^-(\eta,\xi)} \overline{\langle u_{\text{pos}}(\tfrac{\eta}{\varepsilon})|\widehat{\varphi}(\eta)\rangle} \langle u_{\text{pos}}(\tfrac{\eta}{\varepsilon}+\xi)|\widehat{\varphi}(\eta+\varepsilon\xi)\rangle \langle u_{\text{pos}}(\tfrac{\eta}{\varepsilon})|u_{\text{pos}}(\tfrac{\eta}{\varepsilon}+\xi)\rangle \\
&+ 2\mathrm{Re}\Big(e^{it\Lambda_\varepsilon^+(\eta,\xi)} \overline{\langle u_{\text{pos}}(\tfrac{\eta}{\varepsilon})|\widehat{\varphi}(\eta)\rangle} \langle u_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)|\widehat{\varphi}(\eta+\varepsilon\xi)\rangle \langle u_{\text{pos}}(\tfrac{\eta}{\varepsilon})|u_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)\rangle \Big) \\
&+ e^{it\Lambda_\varepsilon^-(\eta,\xi)} \overline{\langle u_{\text{neg}}(\tfrac{\eta}{\varepsilon})|\widehat{\varphi}(\eta)\rangle} \langle u_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)|\widehat{\varphi}(\eta+\varepsilon\xi)\rangle \langle u_{\text{neg}}(\tfrac{\eta}{\varepsilon})|u_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)\rangle \Big) \mathrm{d}\eta\,\mathrm{d}\xi
\end{aligned}$$

where $\Lambda_\varepsilon^{\mp}(\eta,\xi) := \lambda(\frac{\eta}{\varepsilon}+\xi) \mp \lambda(\frac{\eta}{\varepsilon})$. The integrand is bounded uniformly in ε by the integrable function $g(\eta,\xi) := \|\widehat{\varphi}\|_{L^\infty(\mathbb{R})}|\widehat{\varphi}(\eta)||\mathcal{F}^{-1}(h)(\xi)|$ and thus the limit $\varepsilon \to 0$ commutes with integration by Lebesgue's dominated convergence. Writing $\lambda(k) = |k|(1+\frac{m^2}{k^2})^{1/2} = |k| + \mathcal{O}(|k|^{-1})$ as $|k| \to \infty$ and noting that for all $\eta \neq 0$,

$$\lim_{\varepsilon\to 0} |\tfrac{\eta}{\varepsilon}+\xi| - |\tfrac{\eta}{\varepsilon}| = \mathrm{sgn}(\eta)\xi,$$

it is easy to see that $\lim_{\varepsilon\to 0} e^{\mp it(\lambda(\frac{\eta}{\varepsilon}+\xi)-\lambda(\frac{\eta}{\varepsilon}))} = e^{\mp it\,\mathrm{sgn}(\eta)\xi}$. Moreover we have $u_{\mathrm{pos}}(\frac{\eta}{\varepsilon}) \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathrm{sgn}(\eta)\end{pmatrix}$ and $u_{\mathrm{neg}}(\frac{\eta}{\varepsilon}) \to \frac{1}{\sqrt{2}}\begin{pmatrix} -\mathrm{sgn}(\eta) \\ 1\end{pmatrix}$ as $\varepsilon \to 0$. Using these observations we can write the pointwise limit arranged in terms of $\widehat{\rho_1}$ and $\widehat{\rho_2}$, thereby obtaining

$$\lim_{\varepsilon\to 0}\langle \mu_t^\varepsilon, h\rangle = \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}^{-1}(h)(\xi)\Big(e^{-it\,\mathrm{sgn}(\eta)}|\widehat{\rho_1}(k) + \mathrm{sgn}(\eta)\widehat{\rho_2}(k)|^2 + e^{it\,\mathrm{sgn}(\eta)}|\widehat{\rho_1}(k) - \mathrm{sgn}(\eta)\widehat{\rho_2}(k)|^2\Big)\mathrm{d}\eta\,\mathrm{d}\xi,$$

which upon splitting the integral according to the sign of η and re-combining yields

$$\lim_{\varepsilon\to 0}\langle \mu_t^\varepsilon, h\rangle = \tfrac{1}{2}h(t)\|\rho_1+\rho_2\|^2_{L^2(\mathbb{R})} + \tfrac{1}{2}h(-t)\|\rho_1-\rho_2\|^2_{L^2(\mathbb{R})},$$

which by the normalization $\|\rho_1\|^2_{L^2} + \|\rho_2\|^2_{L^2} = 1$ implies the distributional limit

$$\lim_{\varepsilon\to 0}\mu_t^\varepsilon = \Big(\tfrac{1}{2} + \int_{\mathbb{R}} \mathrm{Re}(\overline{\rho}_1\rho_2)(k)\mathrm{d}k\Big)\delta_t + \Big(\tfrac{1}{2} - \int_{\mathbb{R}} \mathrm{Re}(\overline{\rho}_1\rho_2)(k)\mathrm{d}k\Big)\delta_{-t}.$$

Hence the distributional wave-function of the particle at time t is a convex combination of δ_t and δ_{-t}, where the coefficients are determined by $\mathrm{Re}\int(\overline{\rho_1}\rho_2)(k)\mathrm{d}k$. For example, if $\mathrm{Re}(\overline{\rho}_1\rho_2) = 0$, then $u_t^\varepsilon \to \frac{1}{2}(\delta_t + \delta_{-t})$ as $\varepsilon \to 0$. On the other hand, if $\rho_1 = \pm\rho_2$, then $u_t^\varepsilon \to \delta_{\pm t}$ as $\varepsilon \to 0$.

3.2 Minkowski Space

In physical units with $\hbar = c = 1$, we may write the Dirac equation in the "Hamiltonian operator form" (cf. [1, Chapter XIV])

$$i\partial_t\psi = -i\sum_{j=1}^{3}\alpha^j\partial_{x_j}\psi + m\beta\psi,$$

where $\psi \in C(\mathbb{R}, \mathscr{S}'(\mathbb{R}^3))^4$, and $\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$ and $\alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}$ $(j = 1, 2, 3)$ with I_2 denoting the (2×2)-identity matrix and with the Pauli matrices $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Writing $u = \exp(im\beta t)\psi$ we obtain $\partial_t u + A(\partial_x)u = 0$ with $A(\partial_x) := \sum_{j=1}^{3} \alpha^j \partial_{x_j}$. Given an initial condition $\psi\,|_{t=0} = \varphi \in \mathscr{S}'(\mathbb{R}^3)^4$ we obtain the unique solution in the form (cf. [1, Chapter XIV])

$$\exp(im\beta t)\psi = u(t) = E_0(t) * \varphi - E_1(t) * A(\partial_x)\varphi,$$

where the convolution is spatial componentwise and $E_0(t) := \mathcal{F}^{-1}(\cos(t|.|))$, $E_1(t) := \mathcal{F}^{-1}\left(\frac{\sin(t|.|)}{|.|}\right)$. If $\varphi(t) \in L^2(\mathbb{R}^3)^4$, then $\|\psi(t)\|_{L^2} = \|u(t)\|_{L^2} = \|\varphi\|_{L^2}$ for every t by unitarity of the time evolution in $L^2(\mathbb{R}^3)$ and also by unitarity of $\exp(im\beta t)$ on $\mathbb{C}^4$ which implies $|\psi(t,x)|^2 = |u(t,x)|^2$.

Let ψ_ε denote the unique solution with initial values $\varphi_\varepsilon(x) := \varepsilon^{-3/2}\varphi(x/\varepsilon)$, where $\varphi \in \mathscr{S}(\mathbb{R}^3)^4$ and $\|\varphi\|_{L^2} = 1$. We have $|\varphi_\varepsilon|^2 \to \delta$ in $\mathscr{S}'(\mathbb{R}^3)$ as $\varepsilon \to 0$.

Let $\mu_t^\varepsilon(x) := |\psi_\varepsilon(t,x)|^2$ denote the spatial probability density at time t corresponding to the solution ψ_ε. By unitarity of the time evolution, we have in terms of finite measures $\langle \mu_t^\varepsilon, 1\rangle = \|\psi_\varepsilon(t,.)\|_{L^2}^2 = \|\varphi_\varepsilon\|_{L^2}^2 = \|\varphi\|_{L^2}^2 = 1$, but in the sequel we are interested in the convergence properties of μ_t^ε in $\mathscr{S}'(\mathbb{R}^3)$ as $\varepsilon \to 0$.

Remark 2

(i) Note that with the initial value φ_ε specified above, a simple calculation yields $\varphi_\varepsilon \to 0$ in $\mathscr{S}'(\mathbb{R}^3)^4$ and therefore $\psi_\varepsilon(t,.) \to 0$ in $\mathscr{S}'(\mathbb{R}^3)^4$ by continuity of the solution map.

(ii) If instead we consider initial values $\varphi_\varepsilon(x) = \varepsilon^{-3}\varphi(x/\varepsilon)$ as regularizations of delta in the sense that $|\varphi_\varepsilon| \to \delta$ in $\mathscr{S}'$ as $\varepsilon \to 0$ (recall that $\varphi(x) \in \mathbb{C}^4$ and that $\|\varphi\|_{L^2} = 1$), e.g., with supp(φ) compact, and denote again by ψ_ε the corresponding unique solution, then $|\psi_\varepsilon(t,.)|^2$ diverges in $\mathscr{S}'(\mathbb{R}^3)$ for every t as $\varepsilon \to 0$: We have finite speed of propagation for the Dirac equation, hence $\mathrm{supp}(\psi_\varepsilon(t,.)) \subseteq \mathrm{supp}(\varphi) + B_t(0) =: K_t$ and picking $h \in \mathcal{D}(\mathbb{R}^3)$ such that $h \geq 0$ and $h = 1$ on K_t we deduce $\langle |\psi_\varepsilon(t,.)|^2, h\rangle \geq \|\psi_\varepsilon(t,.)\|_{L^2}^2 = \|\varphi_\varepsilon\|_{L^2}^2 = \frac{\|\varphi\|_{L^2}^2}{\varepsilon^3} \to \infty$ as $\varepsilon \to 0$.

Addressing now the question of convergence of μ_t^ε at fixed t as $\varepsilon \to 0$, we will represent the action of μ_t^ε as a distribution on a test function $h \in \mathcal{D}(\mathbb{R}^3)$ by employing the decomposition of the solution ψ_ε in terms of eigenfunctions corresponding to positive or negative energy, respectively, helicity as in [13, Chapter 1, Appendix 1.F]: Introducing $a_\pm(k) := \frac{1}{\sqrt{2}}\sqrt{1 \pm \frac{m}{\lambda(k)}}$, where $\lambda(k) = (k^2 + m^2)^{1/2}$, $h_+(k) := \frac{1}{\sqrt{2|k|(|k|-k_3)}}\begin{pmatrix} k_1 - ik_2 \\ |k| - k_3 \end{pmatrix}$, and $h_-(k) := \frac{1}{\sqrt{2|k|(|k|-k_3)}}\begin{pmatrix} k_3 - |k| \\ k_1 + ik_2 \end{pmatrix}$, we define

$$\omega_{\mathrm{pos},\pm}(k) := \begin{pmatrix} a_+(k)h_\pm(k) \\ \pm a_-(k)h_\pm(k) \end{pmatrix}, \qquad \omega_{\mathrm{neg},\pm}(k) := \begin{pmatrix} \mp a_-(k)h_\pm(k) \\ a_+(k)h_\pm(k) \end{pmatrix}.$$

Note that $h_\pm$ is homogeneous of degree 0. Moreover, $\lim_{|k|\to\infty} a_\pm(k) = 1/\sqrt{2}$ and thus $\lim_{\varepsilon\to 0}\omega_{\text{pos},\pm}(k/\varepsilon)$ and $\lim_{\varepsilon\to 0}\omega_{\text{neg},\pm}(k/\varepsilon)$ are homogeneous of degree 0 as well. The four complex-valued vectors $\omega_{\text{pos},\pm}(k)$, $\omega_{\text{neg},\pm}(k)$ form an orthonormal system with respect to the inner product $\langle\ |\ \rangle_{\mathbb{C}^4}$. For initial data $\varphi_\varepsilon \in L^2(\mathbb{R}^3)$, we may write

$$\begin{aligned}
&\psi_\varepsilon(t,x) = \\
&(2\pi)^{-3/2}\int_{\mathbb{R}^3}\Big(e^{i\langle k|x\rangle - i\lambda(k)t}\Big(\langle\omega^+_{\text{pos}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\omega^+_{\text{pos}}(k) + \langle\omega^-_{\text{pos}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\omega^-_{\text{pos}}(k)\Big) \\
&\qquad + e^{i\langle k|x\rangle + i\lambda(k)t}\Big(\langle\omega^+_{\text{neg}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\omega^+_{\text{neg}}(k) + \langle\omega^-_{\text{neg}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\omega^-_{\text{neg}}(k)\Big)\Big)\mathrm{d}k
\end{aligned}$$

and obtain

$$\begin{aligned}
&(2\pi)^3\langle\mu^\varepsilon_t, h\rangle = (2\pi)^3\langle|\psi_\varepsilon(t,\cdot)|^2, h\rangle = \int\limits_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\Big(e^{i\langle k-k'|x\rangle} \\
&\quad e^{it(\lambda(k)-\lambda(k'))}\big(\langle\omega^+_{\text{pos}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\overline{\langle\omega^+_{\text{pos}}(k')|\widehat{\varphi_\varepsilon}(k')\rangle}\langle\omega^+_{\text{pos}}(k')|\omega^+_{\text{pos}}(k)\rangle \\
&\quad + \langle\omega^-_{\text{pos}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\overline{\langle\omega^-_{\text{pos}}(k')|\widehat{\varphi_\varepsilon}(k')\rangle}\langle\omega^-_{\text{pos}}(k')|\omega^-_{\text{pos}}(k)\rangle\big) + \\
&\quad e^{-it(\lambda(k)-\lambda(k'))}\big(\langle\omega^+_{\text{neg}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\overline{\langle\omega^+_{\text{neg}}(k')|\widehat{\varphi_\varepsilon}(k')\rangle}\langle\omega^+_{\text{neg}}(k')|\omega^+_{\text{neg}}(k)\rangle \\
&\quad + \langle\omega^-_{\text{neg}}(k)|\widehat{\varphi_\varepsilon}(k)\rangle\overline{\langle\omega^-_{\text{neg}}(k')|\widehat{\varphi_\varepsilon}(k')\rangle}\langle\omega^-_{\text{neg}}(k')|\omega^-_{\text{neg}}(k)\rangle\big) + \mathcal{N}_\varepsilon(k,k')\Big)h(x)\mathrm{d}(k,k',x),
\end{aligned}$$

where the symbol $\mathcal{N}_\varepsilon(k,k')$ represents all terms involving "mixed" products such as $\langle\omega^+_{\text{pos}}(k')|\omega^+_{\text{neg}}(k)\rangle$ or $\langle\omega^+_{\text{pos}}(k')|\omega^-_{\text{pos}}(k)\rangle$. Using that $\widehat{\varphi_\varepsilon}(k) = \varepsilon^{3/2}\widehat{\varphi}(\varepsilon k)$ and changing variables according to $k \mapsto \frac{\eta}{\varepsilon}$ and $k' \mapsto \frac{\eta}{\varepsilon} + \xi$, we find

$$\begin{aligned}
&(2\pi)^3\lim_{\varepsilon\to 0}\langle\mu^\varepsilon_t, h\rangle = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3}\Big(\mathcal{N}_\varepsilon(\tfrac{\eta}{\varepsilon}, \tfrac{\eta}{\varepsilon}+\xi) + e^{-i\langle\xi|x\rangle - it\left(\lambda(\frac{\eta}{\varepsilon}+\xi)-\lambda(\frac{\eta}{\varepsilon})\right)} \\
&\quad\Big(\langle\omega^+_{\text{pos}}(\tfrac{\eta}{\varepsilon})|\widehat{\varphi}(\eta)\rangle\overline{\langle\omega^+_{\text{pos}}(\tfrac{\eta}{\varepsilon}+\xi)|\widehat{\varphi}(\eta+\varepsilon\xi)\rangle}\langle\omega^+_{\text{pos}}(\tfrac{\eta}{\varepsilon}+\xi)|\omega^+_{\text{pos}}(\tfrac{\eta}{\varepsilon})\rangle \\
&+ \langle\omega^-_{\text{pos}}(\tfrac{\eta}{\varepsilon})|\widehat{\varphi}(\eta)\rangle\overline{\langle\omega^-_{\text{pos}}(\tfrac{\eta}{\varepsilon}+\xi)|\widehat{\varphi}(\eta+\varepsilon\xi)\rangle}\langle\omega^-_{\text{pos}}(\tfrac{\eta}{\varepsilon}+\xi)|\omega^-_{\text{pos}}(\tfrac{\eta}{\varepsilon})\rangle\Big)\Big)h(x)\mathrm{d}(\eta,\xi,x)
\end{aligned}$$

$$+\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} e^{-i\langle\xi|x\rangle+it\left(\lambda(\frac{\eta}{\varepsilon}+\xi)-\lambda(\frac{\eta}{\varepsilon})\right)}$$

$$\Bigg(\langle\omega^{+}_{\text{neg}}(\tfrac{\eta}{\varepsilon})|\widehat{\varphi}(\eta)\rangle\overline{\langle\omega^{+}_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)|\widehat{\varphi}(\eta+\varepsilon\xi)\rangle}\langle\omega^{+}_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)|\omega^{+}_{\text{neg}}(\tfrac{\eta}{\varepsilon})\rangle$$

$$+\langle\omega^{-}_{\text{neg}}(\tfrac{\eta}{\varepsilon})|\widehat{\varphi}(\eta)\rangle\overline{\langle\omega^{-}_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)|\widehat{\varphi}(\eta+\varepsilon\xi)\rangle}\langle\omega^{-}_{\text{neg}}(\tfrac{\eta}{\varepsilon}+\xi)|\omega^{-}_{\text{neg}}(\tfrac{\eta}{\varepsilon})\rangle\Bigg)h(x)\mathrm{d}(\eta,\xi,x).$$

The integrand is uniformly dominated in L^1 and in the pointwise limit $\varepsilon\to 0$, the inner products of the eigenfunctions simplify by orthonormality and homogeneity (in particular, $\mathcal{N}_\varepsilon(\frac{\eta}{\varepsilon},\frac{\eta}{\varepsilon}+\xi)\to 0$), and the exponential terms converge thanks to

$$\lambda(\tfrac{\eta}{\varepsilon}+\xi)-\lambda(\tfrac{\eta}{\varepsilon}) = |\tfrac{\eta}{\varepsilon}+\xi|-|\tfrac{\eta}{\varepsilon}|+\mathcal{O}(\varepsilon) = \frac{|\frac{\eta}{\varepsilon}+\xi|^2-|\frac{\eta}{\varepsilon}|^2}{|\frac{\eta}{\varepsilon}+\xi|+|\frac{\eta}{\varepsilon}|}+\mathcal{O}(\varepsilon)$$

$$= \frac{2\langle\eta|\xi\rangle+\varepsilon|\xi|^2}{|\eta|+|\eta+\varepsilon\xi|}+\mathcal{O}(\varepsilon)\to\frac{2\langle\eta|\xi\rangle}{2|\eta|}=\frac{\langle\eta|\xi\rangle}{|\eta|},$$

which implies

$$\lim_{\varepsilon\to 0}\langle\mu_t^\varepsilon,h\rangle =$$

$$(2\pi)^{-3/2}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{-it\frac{\langle\eta|\xi\rangle}{|\eta|}}\Big(|\langle\omega^{+}_{\text{pos}}(\eta)|\widehat{\varphi}(\eta)\rangle|^2+|\langle\omega^{-}_{\text{pos}}(\eta)|\widehat{\varphi}(\eta)\rangle|^2\Big)\widehat{h}(\xi)\mathrm{d}\eta\mathrm{d}\xi$$

$$+(2\pi)^{-3/2}\int_{\mathbb{R}^3\times\mathbb{R}^3} e^{it\frac{\langle\eta|\xi\rangle}{|\eta|}}\Big(|\langle\omega^{+}_{\text{neg}}(\eta)|\widehat{\varphi}(\eta)\rangle|^2+|\langle\omega^{-}_{\text{neg}}(\eta)|\widehat{\varphi}(\eta)\rangle|^2\Big)\widehat{h}(\xi)\mathrm{d}\eta\mathrm{d}\xi$$

$$=\int_{\mathbb{R}^3}\Big(f_{\text{pos}}(\eta)h(-t\tfrac{\eta}{|\eta|})+f_{\text{neg}}(\eta)h(t\tfrac{\eta}{|\eta|})\Big)\mathrm{d}\eta=\int_{\mathbb{R}^3}\big(f_{\text{pos}}(-\eta)+f_{\text{neg}}(\eta)\big)h(t\tfrac{\eta}{|\eta|})\mathrm{d}\eta,$$

where $f_{\text{pos/neg}}(\eta) := |\langle\omega^{+}_{\text{pos/neg}}(\eta)|\widehat{\varphi}(\eta)\rangle|^2+|\langle\omega^{-}_{\text{pos/neg}}(\eta)|\widehat{\varphi}(\eta)\rangle|^2$. We finally obtain the limit as a distribution supported on the 2-sphere of radius t, namely,

$$\lim_{\varepsilon\to 0}\langle\mu_t^\varepsilon,h\rangle=\int_{\mathbb{S}^2} f(\theta)h(t\theta)\mathrm{d}\theta,\qquad \text{where } f(\theta):=\int_0^\infty r^2\big(f_{\text{pos}}(-r\theta)+f_{\text{neg}}(r\theta)\big)\mathrm{d}r.$$

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Modeling Abstract Stochastic Problems with White Noise Perturbations

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Abstract We consider construction of a vibrating string model under the influence of random impulses. The model is obtained in the form of the Cauchy problem for the second order difference equation with a Brownian sheet. Properties of random processes that reflect the stochastic influence are investigated. The connection of the Brownian sheet and its generalized derivative with Hilbert space valued Wiener processes is shown. As a result, the Cauchy problem for the stochastic equation with Ito integral with respect to a Wiener process is obtained.

1 Introduction

Study of various processes of the environment in the presence of incomplete information reduces to mathematical models in the form of stochastic problems. Among them an important place is occupied by models in the form of the abstract Cauchy problem with white noise $\mathscr{W}$:

$$X'(t) = AX(t)) + B\mathscr{W}(t),\ t \in [0, T], \qquad X(0) = \zeta, \tag{1}$$

where A is the generator of a semigroup in a Hilbert space H, operator B generally acts from H_1 to H, and ζ is an H-valued random value. For simplicity we consider $H_1 = H$. Due to irregular properties of $\mathscr{W}$ the problem (1) should be understood in the generalized sense. For the H-valued random process $X(t)$ the problem can be written in the integral form with infinite-dimensional Ito integral:

$$X(t) - \zeta = \int_0^t AX(s)ds + \int_0^t BdW(s),\ t \in [0, T] \tag{2}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_41

or in the form of differentials:

$$dX(t) = AX(t)dt + BdW(t),\ t \in [0, T], \quad X(0) = \zeta, \tag{3}$$

where $W = W(t)$, $t \geq 0$, is a Q-Wiener or cylindrical Wiener process. Being "primitives" of $\mathscr{W}$, Wiener processes are defined algorithmically (see, e.g., [5, 7]).

The challenge is to consider practical models that reduce to stochastic problems with Wiener processes. Presence of random processes can be considered as the result of superposition of different unpredictable changes connected with the environment and physical (economical, biological) nature of the model. The assumption of randomness of the processes is the way of avoiding unnecessary complication of the model that occurs when one tries to reject the interaction of different factors which are often hardly subject to formalization.

The aim of the paper is clarification of this question in order consistent with properties of the noise affecting the process. We start with a difference equation for the increment of the position of the string under the influence of random impulses on small segments Δx during a small period of time Δt. In Sect. 2 we show that a Brownian sheet naturally arises in modeling the random actuations. Crucial assumption here is independence between actuations at disjoint segments of the string and the time line. In Sect. 3 we show that the obstacle connected with non-differentiability of the Brownian sheet can be overcome with the help of the concept of cylindrical random variables defined in a generalized (weak) sense in a Hilbert space. Thus, we obtain a difference equation for the increments of the position of the string in the space of functions of x in $L^2[0; l]$ taken as H. In conclusion we show that the obtained second order stochastic equation for the string position under the influence of random impulses can be transformed to the form (3) with an operator-matrix A in $H \times H$.

2 Difference Equation with Brownian Sheet

We consider the problem of the position of a string under the influence of a stream of particles perpendicular to the string transmitting a random impulse of magnitude $\pm\gamma\sqrt{\Delta x}$ to segments of length Δx in time Δt with probability equal to $\lambda\Delta t$. Let $\mathtt{T}$ be the modulus of tension and $u(x, t)$ the displacement from the equilibrium position, then according to the influence of random impulses we have the following possible momentum changes :

$$\Delta M(x,t) = \Delta t(\mathtt{T}u_x(x+\Delta x,t) - \mathtt{T}u_x(x,t)) + \gamma\sqrt{\Delta x} \quad \text{with probability} \quad \lambda\Delta t,$$

$$\Delta M(x,t) = \Delta t(\mathtt{T}u_x(x+\Delta x,t) - \mathtt{T}u_x(x,t)) - \gamma\sqrt{\Delta x} \quad \text{with probability} \quad \lambda\Delta t,$$

$$\Delta M(x,t) = \Delta t(\mathtt{T}u_x(x+\Delta x,t) - \mathtt{T}u_x(x,t)) \qquad \text{with probability} \quad 1-2\lambda\Delta t.$$

According to the assumptions and classical laws of force balance on $[x, x+\Delta x]$ we can write the equalities

$$m(u_t(x, t+\Delta t) - u_t(x,t)) = \Delta M(x,t) = \Delta M_d(x,t) + \Delta M_{st}(x,t), \tag{4}$$

where $x \in [0, l]$, $t \in [0, T]$, and $m = \rho \Delta x$ is the mass of Δx. The deterministic part of the momentum changes is

$$\Delta M_d = \Delta t(\mathtt{T} u_x(x+\Delta x, t) - \mathtt{T} u_x(x,t))$$

and construction of the stochastic part ΔM_{st} is under consideration. Following the physical principles of the model, we suppose increments on disjoint rectangles $(t; t+\Delta t] \times (x; x+\Delta x]$ to be independent random values.

Consider a partition of $(x; x+\Delta x]$ into n equal parts. Following [1, 2], we suppose that the momentum changes ξ_{nk}, $1 \le k \le n$, obtained in the result of the partition are random values defined by the following series of distributions:

$$\xi_{nk} = \gamma\sqrt{\frac{\Delta x}{n}} \text{ with probability } \lambda \Delta t, \quad \xi_{nk} = -\gamma\sqrt{\frac{\Delta x}{n}} \text{ with probability } \lambda \Delta t,$$

and $\xi_{nk} = 0$ with probability $1 - 2\lambda\Delta t$. Then $\sum_{k=1}^{n} \xi_{nk}$ describes the total number of impulses arriving at $(x; x+\Delta x]$ during the time $(t; t+\Delta t]$. Supposing

$$\Delta M_{st} = \lim_{n\to\infty} \sum_{k=1}^{n} \xi_{nk},$$

we study properties of the stochastic part. First, we prove that the random variable ΔM_{st} has the normal law of distribution, second, that ΔM_{st}, up to a constant, can be described using a Brownian sheet. Finally, based on the representation of the Brownian sheet in the form of Fourier series in $H = L^2[0, l]$ we show that the random perturbations in this model can be described by means of a Wiener process. Due to this plan we begin with properties of ΔM_{st}.

Proposition 1 *The random value $\Delta M_{st}(t, x)$ has the normal law of distributions with parameters $\boldsymbol{E}[\Delta M_{st}(t,x)] = 0$ and $Var[\Delta M_{st}(t,x)] = 2\lambda\gamma^2\Delta t\Delta x$, that is,*

$$\Delta M_{st}(t,x) \sim \mathcal{N}(0; \sqrt{2\lambda\gamma^2 \Delta t \Delta x}). \tag{5}$$

Proof Consider the partition of $(x; x+\Delta x]$ into n equal parts and corresponding momentum changes $\{\xi_{nk}\}$. Due to their distributions and independence in each series we have

$$\mathbf{E}[\xi_{nk}] = 0, \quad Var[\xi_{nk}] = \frac{2}{n}\lambda\gamma^2\Delta t \Delta x, \qquad n \in \mathbb{N},\ 1 \le k \le n.$$

We verify that the introduced sequence $\{\xi_{nk}\}$ satisfies conditions of the CLT for series, formulated as follows [3].

Let $\{\xi_{nk},\ n \in \mathbb{N},\ 1 \le k \le n\}$ *be a sequence of random values independent in each series and* F_{nk} *be the distribution function of* ξ_{nk}. *If for any fixed* $\varepsilon > 0$ *and some* $\tau > 0$ *the following conditions are fulfilled*:

1. $\sum_{k=1}^{n} P(|\xi_{nk}| \ge \varepsilon) \to 0,\ n \to \infty,$
2. $\sum_{k=1}^{n} \int_{|y|<\tau} y\, dF_{nk}(y) \to a,\ n \to \infty,$
3. $\sum_{k=1}^{n} \left(\int_{|y|<\tau} y^2 dF_{nk}(y) - \left(\int_{|y|<\tau} y\, dF_{nk}(y) \right)^2 \right) \to b^2,\ n \to \infty,$

then $\sum_{k=1}^{n} \xi_{nk}$ *converges in distribution to* $\mathscr{N}(a; b)$ *as* $n \to \infty$.

The first condition is fulfilled since for any $\varepsilon > 0$ and $n \ge N(\varepsilon) = [\frac{\gamma^2 \Delta x}{\varepsilon^2}] + 1$, we have

$$P(|\xi_{nk}| \ge \varepsilon) \equiv 0,\ k = 1, 2, \ldots, n, \quad \Longrightarrow \quad \sum_{k=1}^{n} P(|\xi_{nk}| \ge \varepsilon) = 0.$$

Next, fix $\tau \ge \gamma \sqrt{\Delta x}$, then for any n

$$\sum_{k=1}^{n} \int_{|y|<\tau} y dF_{nk}(y) = \sum_{k=1}^{n} \left(\gamma \sqrt{\frac{\Delta x}{n}} \lambda \Delta t - \gamma \sqrt{\frac{\Delta x}{n}} \lambda \Delta t \right) = 0,$$

and

$$\sum_{k=1}^{n} \left(\int_{|y|<\tau} y^2 dF_{nk}(y) - \left(\int_{|y|<\tau} y dF_{nk}(y) \right)^2 \right) = \sum_{k=1}^{n} 2\gamma^2 \frac{\Delta x}{n} \lambda \Delta t = 2\gamma^2 \Delta x \lambda \Delta t.$$

Hence the second and third conditions hold as well. Thus, $\sum_{k=1}^{n} \xi_{nk}$ converges in distribution to the normal random value $\Delta M_{st}(t, x) \sim \mathscr{N}(0; \gamma \sqrt{2\lambda \Delta t \Delta x})$.

Now we show that the random value $\Delta M_{st}(t, x)$ coincides with increment of Brownian sheet within a constant. For this we use the following definition [3, 6].

A two-parameter random process $\mathbf{W} = \mathbf{W}(t, x),\ t, x \ge 0$, *is called a Brownian sheet if it satisfies the conditions:*

$(W1)$ $\mathbf{W}(0, x) = \mathbf{W}(t, 0) = 0$ a.e.;
$(W2)$ $\mathbf{W}$ *is a Gaussian process*;
$(W3)$ $\mathbf{E}[\mathbf{W}(t_1, x_1)\mathbf{W}(t_2, x_2)] = (t_1 \wedge t_2)(x_1 \wedge x_2),\ \mathbf{E}[\mathbf{W}(t, x)] = 0.$

There are different approaches to determining increments of the Brownian sheet. Taking into account the physical model of the process (namely, random variables $\Delta M_{st}(t, x)$ on disjoint rectangles $(t; t + \Delta t] \times (x; x + \Delta x]$ are considered independent) we use the definition of increments $\Delta \mathbf{W}(t, x)$ consistent with the independence property. Let the increments of the Brownian sheet on the rectangle

$(t; t + \Delta t] \times (x; x + \Delta x]$ be defined using the equality:

$$\Delta \mathbf{W}(t, x) := \mathbf{W}(t+\Delta t, x+\Delta x) - \mathbf{W}(t, x+\Delta x) - \mathbf{W}(t+\Delta t, x) + \mathbf{W}(t, x). \quad (6)$$

With this definition we show that $\Delta \mathbf{W}(t, x) \sim \mathcal{N}(0; \sqrt{\Delta t \Delta x})$. From properties of Gaussian processes and (W3) it follows that $\Delta \mathbf{W}$ is normally distributed and

$$Var[\Delta \mathbf{W}(t, x)] = \mathbf{E}[(\Delta \mathbf{W}(t, x))^2] - \mathbf{E}^2[\Delta \mathbf{W}(t, x)] = \mathbf{E}[(\Delta \mathbf{W}(t, x))^2]$$

$$= \mathbf{E}[(\mathbf{W}(t + \Delta t, x + \Delta x) - \mathbf{W}(t, x + \Delta x) - \mathbf{W}(t + \Delta t, x) + \mathbf{W}(t, x))^2] = \Delta t \Delta x.$$

In addition, we show that increments defined by (6) are independent on disjoint rectangles. To prove it, take two disjoint rectangles $(t_1; t_2] \times (x_1; x_2]$ and $(t_3; t_4] \times (x_3; x_4]$ and show that increments on these rectangles are uncorrelated:

$$\mathbf{E}[\Delta \mathbf{W}(t_1, x_1) \Delta \mathbf{W}(t_3, x_3)] = \mathbf{E}[(\mathbf{W}(t_2, x_2) - \mathbf{W}(t_2, x_1) - \mathbf{W}(t_1, x_2) + \mathbf{W}(t_1, x_1))$$

$$\times (\mathbf{W}(t_4, x_4) - \mathbf{W}(t_4, x_3) - \mathbf{W}(t_3, x_4) + \mathbf{W}(t_3, x_3))] = 0.$$

Hence, being uncorrelated normal random values $\Delta \mathbf{W}(t_1, x_1)$ and$\Delta \mathbf{W}(t_3, x_3)$ are independent.

Thus, the Brownian sheet is the process with independent increments $\Delta \mathbf{W}$ and $\Delta \mathbf{W}(t, x) \sim \mathcal{N}(0; \sqrt{\Delta t \Delta x})$. Comparing with (5) we obtain that $\Delta M_{st}(t, x)$ coincides with $\Delta \mathbf{W}(t, x)$ within the constant $\gamma \sqrt{2\lambda}$ and the equality (4) can be written as the difference equation with the Brownian sheet:

$$m(u_t(x, t + \Delta t) - u_t(x, t)) = \Delta t \mathrm{T}(u_x(x + \Delta x, t) - u_x(x, t)) + \gamma \sqrt{2\lambda}\, \Delta \mathbf{W}(t, x). \quad (7)$$

Now studying relations of the Brownian sheet with Wiener processes we transform (7) to a difference equation with a Wiener process.

3 Difference Equation with Cylindrical Wiener Process

Consider an H-valued Q-Wiener process W_Q. As is known, in a Hilbert space H

$$W_Q(t) = \sum_{n=0}^{\infty} \sqrt{\lambda_n} \beta_n(t) e_n, \quad t \geq 0, \quad (8)$$

where $\{e_n\}$ form an orthonormal basis in H, β_n are independent Brownian motions, and Q is a self-adjoint trace class operator in H: $Qe_n = \lambda_n e_n$, $\sum_{n=0}^{\infty} \lambda_n < \infty$ (see, e.g., [4, 8]).

In $H = L^2[0, l]$ any self-adjoint Hilbert–Schmidt operator, in particular a trace class operator, is an integral one:

$$Q\varphi(x) = \int_0^l g(x, y)\varphi(y)dy, \quad \varphi \in L^2[0, l].$$

We show that Q is the operator with special kernel $g(x_1, x_2) = x_1 \wedge x_2$. By the Hilbert–Schmidt theorem, we have

$$Q\varphi(x_1) = \sum_{n=0}^{\infty} \lambda_n(\varphi, e_n)e_n(x_1) = \sum_{n=0}^{\infty} \lambda_n e_n(x_1) \int_0^l \varphi(x_2)e_n(x_2)dx_2 =: \lambda_n \varphi_n e_n(x_1).$$

We formally change the order of summation and integration and consider the series obtained under the integral: $\sum_{n=0}^{\infty} \lambda_n e_n(x_1)e_n(x_2)$. Here the functions $e_n(x_1)e_n(x_2)$ form an orthonormal basis in $L^2(G)$, where $G = [0; l] \times [0; l]$. Since $\sum_{n=0}^{\infty} \lambda_n^2 < \infty$, there exists a symmetric function $g \in L^2(G)$ with the Fourier series

$$\sum_{n=0}^{\infty} \lambda_n e_n(x_1)e_n(x_2) = g(x_1, x_2).$$

Moreover, it is proved in [4] that g defines the spatial correlation of the process W_Q:

$$\mathbf{E}[W_Q(t, x_1)W_Q(t, x_2)] = tg(x_1, x_2), \qquad t \geq 0,\ x_1, x_2 \in [0; l].$$

In its turn spatial correlation of the process $\mathbf{W}$, due to the property (W3), is defined by the function:

$$\mathbf{E}[\mathbf{W}(t, x_1)\mathbf{W}(t, x_2)] = t(x_1 \wedge x_2), \qquad t \geq 0,\ x_1, x_2 \in [0; l].$$

Then, taking into account that a Gaussian process is uniquely defined by its correlation function and expectation, we conclude that Brownian sheet $\mathbf{W}$ coincides with $L^2[0; l]$-valued Q-Wiener process W_Q, where Q is the integral operator with kernel $g(x_1, x_2) = x_1 \wedge x_2$.

Next, eigenvectors and eigenfunctions in (8) for the integral operator Q with kernel $g(x_1, x_2) = x_1 \wedge x_2$ can be found as follows:

$$\lambda_n = \frac{4l^2}{\pi^2(2n+1)^2}, \quad e_n(x) = \sqrt{\frac{2}{l}} \sin \frac{\pi(2n+1)x}{2l}, \qquad x \in [0; l].$$

Hence in $L^2[0, l]$

$$\mathbf{W}(t, x) = W_Q(t, x) = \frac{2\sqrt{2l}}{\pi} \sum_{n=0}^{\infty} \frac{\beta_n(t)}{2n+1} \sin \frac{\pi(2n+1)x}{2l}, \quad t \geq 0. \tag{9}$$

In addition to (9) for $\mathbf{W}(t,x)$, we obtain the formal decomposition for the derivative

$$\frac{\partial \mathbf{W}(t,x)}{\partial x} = \sqrt{\frac{2}{l}} \sum_{n=0}^{\infty} \beta_n(t) \cos \frac{\pi(2n+1)x}{2l} =: W(t,x), \tag{10}$$

where functions $\widetilde{e}_n(x) := \sqrt{\frac{2}{l}} \cos \frac{\pi(2n+1)x}{2l}$ form an orthonormal basis in $L^2[0;l]$ and the series converges only weakly, that is, $W(t) = \sum_{n=0}^{\infty} \beta_n(t)\widetilde{e}_n$ weakly converges in $L^2[0;l]$. By definition of a cylindrical Wiener process, a weakly convergent series $\sum_{n=0}^{\infty} \beta_n(t)\widetilde{e}_n(\cdot)$ defines an $L^2[0;l]$-valued cylindrical Wiener process $W(t,\cdot)$.

Using the equality (6) for $\Delta\mathbf{W}$ and (10) for $\frac{\partial \mathbf{W}(t,x)}{\partial x}$ we obtain the relation between increments of the Brownian sheet and increments of the cylindrical Wiener process:

$$\Delta\mathbf{W}(t,x) = (W(t+\Delta t) - W(t))\,\Delta x + o(\Delta x), \quad t \geq 0,\ x \in [0;l].$$

Now using the obtained relations between the increments $\Delta\mathbf{W}(t,x)$ and $\Delta W(t,x)$ and supposing existence of corresponding derivatives we can write (7) as difference equations with Q-Wiener and cylindrical Wiener processes:

$$m(u_t(x,t+\Delta t)-u_t(x,t)) = \Delta t \mathrm{T}\,(u_x(x+\Delta x,t) - u_x(x,t)) + \gamma\sqrt{2\lambda}\,\Delta W_Q(t,x),$$

$$\rho\,(u_t(x,t+\Delta t) - u_t(x,t)) = \Delta t \mathrm{T} u_{xx}(x,t) + \gamma\sqrt{2\lambda}\,\Delta W(t,x). \tag{11}$$

4 Main Result

Summing increments in (11) and passing to limit as $\Delta t \to 0$ in $L^2(\Omega,\mathscr{F},P)$ weakly in $L^2[0,l]$ we arrive at the following result.

Theorem 1 *The stochastic Cauchy problem that describes string vibrations by taking into account the influence of a stream of particles perpendicular to the string transmitting random impulses of magnitude $\pm\gamma\sqrt{\Delta x}$ to segments of length Δx in time Δt with probability equal to $\lambda\Delta t$ is written as follows:*

$$\rho\,(u_t(t,x) - \zeta_2(x)) = \int_0^t \mathrm{T} u_{xx}(s,x)ds + \int_0^t \gamma\sqrt{2\lambda}\,dW(s),\ t \in [0;T], \quad u(0,x) = \zeta_1(x).$$

By replacing the variables $X(t) = (u(t), u_t(t))^{tr}$, the problem is reduced to the form (2) with $\zeta = (\zeta_1,\zeta_2)^{tr}$ and operator $A = \begin{pmatrix} 0 & I \\ a\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$ (generating an integrated semigroup in $L^2[0,l] \times L^2[0,l]$), with corresponding a and inhomogeneity term.

Acknowledgements This work was supported by the Russian Academic Excellence Project (Agreement No. 02.A03.21.0006 of August 27, 2013, between the RF Ministry of Education and Science and Ural Federal University).

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On Association in Colombeau Algebras Without Asymptotics

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Abstract A recent variant of Colombeau algebras does not employ asymptotic estimates for its definition. We discuss how the concept of association with distributions transfers to this setting and why it still needs to be based on asymptotics.

1 Introduction

Colombeau algebras of nonlinear generalized functions are given by factor spaces of suitable spaces of moderate and negligible functions (we refer to the standard literature [1, 3, 4, 8, 15, 16]). Usually, whether a function R in a certain basic space is moderate or negligible is decided by evaluating it on a suitable test object $(\vec{\varphi}_\varepsilon)_{\varepsilon\in(0,1]}$ (essentially an approximate identity) and examining the resulting asymptotic behavior as $\varepsilon \to 0$, see [6] for a unifying discussion. In special Colombeau algebras, moreover, evaluation on $\vec{\varphi}_\varepsilon$ is already built into the basic space in the sense that the embedded image of a distribution $u \in \mathscr{D}'$ is given by the net $\langle u, \vec{\varphi}_\varepsilon(x)\rangle$ of smooth functions in x. Hence, the basic space for the special algebra on an open subset $\Omega \subseteq \mathbb{R}^n$ is given by all nets $(u_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{(0,1]}$, and the growth or vanishing rate of derivatives of $u_\varepsilon(x)$ on compact sets determines whether this representative is moderate or negligible, respectively.

Two observations are in order: first, the properties of a given Colombeau algebra directly depend on the test objects $(\vec{\varphi}_\varepsilon)_\varepsilon$ used for its definition. Such properties are, for example, diffeomorphism invariance (cf. [9, 12]), the possibility to restrict to subspaces (cf. [15, Ch. III, §11, p. 100]) but also association properties, as we will see below. Second, so far the choice of test objects had to be made in advance because the basic definitions depend on it.

Recently, a novel construction of Colombeau algebras was given that is closer in spirit to the definition of distributions as those linear functionals on test functions

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_42

satisfying appropriate continuous seminorm estimates [14]. Because this change of perspective is similar to that encountered when switching from the sequential approach to distribution theory [11] to the classical approach based on the theory of locally convex spaces [18], these algebras were termed *Colombeau algebras without asymptotics*.

This formulation has several pleasing features. Most importantly, it decouples the definition of the Colombeau algebra itself from the choice of test objects. For this reason, the resulting space is close to being universal in the sense that one has canonical mappings into most of the classically used Colombeau algebras.

However, test objects necessarily reappear in the study of association. While there is no inherent notion of association anymore, we will see that the separation of association tests from the definition of Colombeau algebras makes them more flexible and to a certain degree even arbitrary.

After recalling the definition of a Colombeau algebra without asymptotics in Sect. 2 and previous notions of association in Sect. 3 we will discuss association in the new setting in Sect. 4.

2 The Colombeau Algebra

In short, the approach of [14] is based on two steps: first, one extends the domain of distributions from $\mathscr{D}(\Omega)$ to $C^\infty(\Omega, \mathscr{D}(\Omega))$ in a natural way, essentially by mapping $u \in \mathscr{D}'(\Omega)$ to $\mathrm{id}_{C^\infty(\Omega)} \otimes u$. One then still has seminorm estimates of the form $p(u(\vec{\varphi})) \le q(\vec{\varphi})$, where p and q are continuous seminorms of $C^\infty(\Omega)$ and $C^\infty(\Omega, \mathscr{D}(\Omega))$, respectively. This step amounts to representing distributions by their regularizations. In a second step one lets go of linearity and replaces the linear estimates in $q(\vec{\varphi})$ by polynomial ones. This allows in particular for products to be formed and gives the space of moderate functions. Similarly, a notion of negligible function is obtained by noticing that the prototypical negligible functions $\iota(f)\iota(g) - \iota(fg)$ or $\iota(f) - \sigma(f)$ with $f, g \in C^\infty(\Omega)$ and ι, σ the canonical embeddings have a uniformly continuous extension to $C^\infty(\Omega, \mathscr{E}'(\Omega))$ and take the value 0 if evaluated at the function $\vec{\delta} : x \mapsto \delta(. - x)$ [13, Lemma 3.1, p. 188].

In the following definition, the sheaves $C^\infty(-, \mathscr{D}(\Omega))$ and $C^\infty(-)$ are considered with values in the category of locally convex spaces with smooth mappings in the sense of convenient calculus [10] as morphisms. The *basic space* of nonlinear generalized functions on Ω then is the set of sheaf homomorphisms

$$\mathscr{B}(\Omega) := \mathrm{Hom}(C^\infty(-, \mathscr{D}(\Omega)), C^\infty(-)).$$

The embeddings $\iota : \mathscr{D}'(\Omega) \to \mathscr{B}(\Omega)$ and $\sigma : C^\infty(\Omega) \to \mathscr{B}(\Omega)$ are given by

$$(\iota u)(\vec{\varphi})(x) := \langle u, \vec{\varphi}(x)\rangle \qquad (u \in \mathscr{D}'(\Omega))$$

$$(\sigma f)(\vec{\varphi})(x) := f(x) \qquad (f \in C^\infty(\Omega))$$

for $\vec{\varphi} \in C^\infty(U, \mathscr{D}(\Omega))$ with $U \subseteq \Omega$ open and $x \in U$. For the present discussion we omit the discussion of derivatives [14, Def. 4].

The algebraic structure of moderate and negligible functions is based on the following semirings of polynomials with non-negative coefficients, $k \in \mathbb{N}_0$:

$$\mathscr{P}_k := \mathbb{R}^+[y_0, \ldots, y_k],$$
$$\mathscr{I}_k := \{\lambda \in \mathbb{R}^+[y_0, \ldots, y_k, z_0, \ldots, z_k] \mid \lambda(y_0, \ldots, y_k, 0, \ldots, 0) = 0\}.$$

For $K, L \subset\subset \Omega$, $m, l \in \mathbb{N}_0$ and $B \subseteq C^\infty(\Omega)$ bounded we set

$$\|f\|_{K,m} := \sup_{x\in K, |\alpha|\le m} |\partial^\alpha f(x)| \qquad (f \in C^\infty(\Omega)),$$
$$\|\vec{\varphi}\|_{K,m;L,l} := \sup_{\substack{x\in K,|\alpha|\le m\\ y\in L,|\beta|\le l}} |\partial_x^\alpha \partial_y^\beta \vec{\varphi}(x)(y)| \qquad (\vec{\varphi} \in C^\infty(\Omega, \mathscr{D}(\Omega))),$$
$$\|\vec{\varphi}\|_{K,m;B} := \sup_{\substack{x\in K,|\alpha|\le m\\ f\in B}} |\langle f(y), \partial_x^\alpha \vec{\varphi}(x)(y)\rangle| \qquad (\vec{\varphi} \in C^\infty(\Omega, \mathscr{E}'(\Omega))).$$

In the following definition, $\mathscr{U}_x$ denotes the filter base of open neighborhoods of x in Ω.

Definition 1 An element $R \in \mathscr{B}(\Omega)$ is called *moderate* if

$$(\forall x \in \Omega)\ (\exists U \in \mathscr{U}_x(\Omega))\ (\forall K, L \subset\subset U)\ (\forall m, k \in \mathbb{N}_0)$$
$$(\exists c, l \in \mathbb{N}_0)\ (\exists \lambda \in \mathscr{P}_k)\ (\forall \vec{\varphi}_0, \ldots, \vec{\varphi}_k \in C^\infty(U, \mathscr{D}_L(U))):$$
$$\|\mathrm{d}^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \ldots, \vec{\varphi}_k)\|_{K,m} \le \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \ldots, \|\vec{\varphi}_k\|_{K,c;L,l}).$$

The subset of all moderate elements of $\mathscr{B}(\Omega)$ is denoted by $\mathscr{M}(\Omega)$.

An element $R \in \mathscr{B}(\Omega)$ is called *negligible* if

$$(\forall x \in \Omega)\ (\exists U \in \mathscr{U}_x(\Omega))\ (\forall K, L \subset\subset U)\ (\forall m, k \in \mathbb{N}_0)\ (\exists c, l \in \mathbb{N}_0)$$
$$(\exists \lambda \in \mathscr{I}_k)\ (\exists B \subseteq C^\infty(\Omega) \text{ bounded})\ (\forall \vec{\varphi}_0, \ldots, \vec{\varphi}_k \in C^\infty(U, \mathscr{D}_L(U))):$$
$$\|\mathrm{d}^k R(\vec{\varphi}_0)(\vec{\varphi}_1, \ldots, \vec{\varphi}_k)\|_{K,m}$$
$$\le \lambda(\|\vec{\varphi}_0\|_{K,c;L,l}, \ldots, \|\vec{\varphi}_k\|_{K,c;L,l}, \|\vec{\varphi}_0 - \vec{\delta}\|_{K,c;B}, \|\vec{\varphi}_1\|_{K,c;B}, \ldots, \|\vec{\varphi}_k\|_{K,c;B}).$$

The subset of all negligible elements of $\mathscr{B}(\Omega)$ is denoted by $\mathscr{N}(\Omega)$.

We set $\mathscr{G}(\Omega) := \mathscr{M}(\Omega)/\mathscr{N}(\Omega)$.

To connect this definition to those of classical Colombeau algebras think of setting $k = 0$ and $\vec{\varphi}_0(x)(y) = \varepsilon^{-n}\varphi(\frac{y-x}{\varepsilon})$ for a mollifier $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Further details are given in [14].

3 Association in Previous Contexts

There are canonical mappings from $\mathscr{G}(\Omega)$ into the algebras $\mathscr{G}^s(\Omega)$ (the special algebra, cf. [8, Section 1.2]), $\mathscr{G}^e(\Omega)$ (the elementary full algebra, cf. [8, Section 1.4]), and $\mathscr{G}^d(\Omega)$ (the diffeomorphism invariant algebra, cf. [12, 13]). Because in each of these there is an intrinsic notion of association we first characterize when the image of an element of $\mathscr{G}(\Omega)$ is associated with a distribution there.

For the special algebra, let the canonical mapping $\Theta^s : \mathscr{G}(\Omega) \to \mathscr{G}^s(\Omega)$ be given by $(\Theta^s R)_\varepsilon(x) := R(\vec{\psi}_\varepsilon)(x)$, where the mollifier $\vec{\psi}_\varepsilon$ used for the embedding into $\mathscr{G}^s(\Omega)$ is as in [5], i.e., $\varphi_\varepsilon(x) = \chi(x|\ln\varepsilon|)\varepsilon^{-n}\rho(x/\varepsilon)$ with a cut-off function χ and a mollifier ρ and $\vec{\psi}_\varepsilon = \varphi_\varepsilon(x-y)$.

For the elementary full algebra, the canonical mapping $\Theta^e : \mathscr{G}(\Omega) \to \mathscr{G}^e(\Omega)$ is given by $(\Theta^e R)(\varphi, x) = R(\vec{\varphi})(x)$ for $\varphi \in \mathscr{D}(\Omega)$ and $x \in \Omega$ with $x + \operatorname{supp}\varphi \subseteq \Omega$, where $\vec{\varphi}$ is any element of $C^\infty(\Omega, \mathscr{D}(\Omega))$ such that $\vec{\varphi}(x)(y) = \varphi(y-x)$ for y in a neighborhood of x. Let $\mathscr{A}_q(\mathbb{R}^n)$ denote the space of test functions having integral one and vanishing moments of order up to q.

Finally, for the diffeomorphism invariant algebra we have a canonical mapping $\Theta^d : \mathscr{G}(\Omega) \to \mathscr{G}^d(\Omega)$ given by $\Theta^d(R)(\varphi)(x) := R(\vec{\varphi})(x)$ with $\vec{\varphi}(x') := \varphi$ for all $x' \in \Omega$ and $\varphi \in \mathscr{D}$. Moreover, $S(\Omega)$ denotes the space of test objects for $\mathscr{G}^d(\Omega)$ [13, p. 189].

From the respective definitions of association we immediately obtain:

Proposition 1 *Let $R \in \mathscr{G}(\Omega)$ and $u \in \mathscr{D}'(\Omega)$. Then*

$$\Theta^s(R) \approx u \iff \forall \psi \in \mathscr{D}(\Omega) : \lim_{\varepsilon\to 0} \langle R(\vec{\psi}_\varepsilon), \psi\rangle = \langle u, \psi\rangle,$$

$$\Theta^e(R) \approx u \iff \forall \psi \in \mathscr{D}(\Omega)\ \exists q > 0\ \forall \varphi \in \mathscr{A}_q(\mathbb{R}^n) : \lim_{\varepsilon\to 0} \langle R(S_\varepsilon\varphi, .), \psi\rangle = \langle u, \psi\rangle,$$

$$\Theta^d(R) \approx u \iff \forall \psi \in \mathscr{D}(\Omega)\ \forall (\vec{\varphi}_\varepsilon)_\varepsilon \in S(\Omega) : \lim_{\varepsilon\to 0} \langle R(\vec{\varphi}_\varepsilon), \psi\rangle = \langle u, \psi\rangle.$$

Here, $S_\varepsilon(\varphi)(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$. We list some possible generalizations occurring in the literature:

- Strong association requires convergence of order ε^β for some $\beta > 0$ uniformly for all ψ having support in a given compact set [16, Def. 1.38, p. 45].
- C^k-association requires convergence to take place in C^k [8, Def. 3.2.11, p. 287].
- s-association (for $s > 0$) requires convergence of order $o(\varepsilon^s)$ for all ψ, while for D-s-association one in addition takes ψ only from a test function space D [16, Def. 2.1, p. 92].
- Strong average association replaces the limit in strong association by an averaged limit [17].

Naturally, all of these can be formulated in the spirit of Proposition 1 as well.

4 Association in the Asymptotic-Free Algebra

A sensible notion of association of $R \in \mathscr{B}(\Omega)$ with $u \in \mathscr{D}'(\Omega)$, written $R \approx u$, requires at least the following properties:

$$\forall u \in \mathscr{D}'(\Omega) : \iota(u) \approx u, \tag{1}$$

$$\forall R \in \mathscr{N}(\Omega) : R \approx 0. \tag{2}$$

Condition (1), which ensures minimal compatibility with the distributional world, can be realized by calling R associated to u if $R(\vec{\varphi}) \to u$ in $\mathscr{D}'(\Omega)$ as $\vec{\varphi}$ converges to $\vec{\delta} : x \mapsto \delta_x$ (i.e., the identity in $\mathscr{L}(\mathscr{D}'(\Omega), \mathscr{D}'(\Omega))$ if we identify kernels and their operators here). Similarly, one can have stronger convergence (e.g., as in C^k-association) of $\iota(u)(\vec{\varphi})$ for $u \in \mathscr{H}$ if one supposes that $\vec{\varphi}$ converges to the identity on spaces of distributions $\mathscr{H} \subseteq \mathscr{D}'(\Omega)$. However, although convergence like $\vec{\varphi}_\varepsilon \to$ id in $\mathscr{L}(\mathscr{H}, \mathscr{H})$ gives compatibility with the linear theory, more structure on the test objects seems to be needed to incorporate certain nonlinear effects related to association. As an example, consider the property

$$x^k \delta^k \approx 0 \quad \text{in } \mathscr{G}^s(\mathbb{R}), \quad k \in \mathbb{N}, \tag{3}$$

which holds due to

$$\begin{aligned} \langle x^k \varphi_\varepsilon(x)^k, \psi(x) \rangle &= \int x^k \psi(x) \left(\chi(x|\ln \varepsilon|) \frac{1}{\varepsilon} \rho\left(\frac{x}{\varepsilon}\right)\right)^k \mathrm{d}x \\ &= \varepsilon \int z^k \psi(\varepsilon z) \chi(\varepsilon z |\ln \varepsilon|)^k \rho(z)\, \mathrm{d}z = O(\varepsilon). \end{aligned} \tag{4}$$

A similar calculation holds in the diffeomorphism invariant algebra $\mathscr{G}^d(\mathbb{R})$ of [7], where the convolution kernel $\vec{\psi}_\varepsilon(x)(y) := \varphi_\varepsilon(x - y)$ which is used for the embedding into $\mathscr{G}^s$ is replaced by $\vec{\varphi}_\varepsilon(x)(y) := \frac{1}{\varepsilon}\phi(\varepsilon, x)\left(\frac{y}{\varepsilon}\right)$ with $\phi \in C_b^\infty(I \times \mathbb{R}, \mathscr{A}_q(\mathbb{R}))$ (cf. [7, Definition 7.20]).

Relation (3) is crucially based on the fact that these kernels are obtained by scaling given test functions. Moreover, properties like (4) are essential in applications for calculating associated distributions, as is seen, for example, in [2], where it is shown that the curvature of a conical metric is proportional to the delta distribution at the apex of the cone.

Therefore, we are led to raise the following questions:

Q1. Which (nonlinear) association properties similar to (3) can one expect in general using the convolution kernels of $\mathscr{G}^s$ or $\mathscr{G}^d$?

Q2. Can the respective association tests be formulated in terms not involving asymptotics?

For condition (2), which is needed for association to be independent of representatives, we would like to use negligibility to show that $R(\vec{\varphi}) \to 0$ in $C^m(\Omega)$ ($m = 0$ is sufficient) and hence in $\mathscr{D}'(\Omega)$ if $\vec{\varphi} \to \vec{\delta}$ suitably. Negligibility as in Definition 1 implies that for given $K \subset\subset \Omega$, $m \in \mathbb{N}_0$, and $L \supset\supset K$ we have

$$\|R(\vec{\varphi}_\varepsilon)\|_{K,m} \le \lambda(\|\vec{\varphi}_\varepsilon\|_{K,c;L,l}, \|\vec{\varphi}_\varepsilon - \vec{\delta}\|_{K,c;B}) \tag{5}$$

whenever $\vec{\varphi}_\varepsilon(x) \in \mathscr{D}_L(\Omega)$ for $x \in K$. Suppose for simplicity that $\lambda(x, y) = Cx^a y^b$ for all $x, y \in \mathbb{R}_+$, some $C > 0$ and $a, b \in \mathbb{N}$. In general, (5) does not necessarily converge to zero even if $\vec{\varphi}_\varepsilon \to \vec{\delta}$; take, for example, in dimension $n = 1$ a model delta net $\vec{\varphi}_\varepsilon(x)(y) := \varepsilon^{-1}\varphi((y - x)/\varepsilon)$ where $\varphi \in \mathscr{D}(\Omega)$ has integral one and, say, vanishing moments of order up to q and nonvanishing $(q + 1)$th moment. Then

$$\|\vec{\varphi}_\varepsilon\|_{K,c;L,l} = O(\varepsilon^{-c-l-1})$$

for some constant c; however, taking $B = \{f\}$ with $f(x) := x^{q+1}$ we only obtain

$$\|\vec{\varphi}_\varepsilon - \vec{\delta}\|_{K,0;B} = \varepsilon^{q+1}|\int z^{q+1}\varphi(z)\,\mathrm{d}z|.$$

Hence, if $a(c+l+1) - b(q+1) > 0$ we cannot conclude that $R(\vec{\varphi}_\varepsilon) \to 0$. We can only do so if sufficiently many moments of φ vanish, or more generally, if $\vec{\varphi}_\varepsilon \to \delta$ fast enough. As above, we ask:

Q3. Can the conditions which ensure that (2) holds be formulated in terms not involving asymptotics?

5 Conclusion

We have seen that if one wants to formulate useful association tests in the asymptotic-free Colombeau algebra one still has to resort to the classically used association tests of full and special Colombeau algebras, which do employ asymptotics. On the one hand this is needed to prove that association does not depend on representatives; on the other hand, this helps in ensuring that association does not only give compatibility with the linear theory, but also is able to handle nonlinear effects as illustrated by the property $x^k\delta^k \approx 0$ ($k \in \mathbb{N}$), for example. It will be object of further research to investigate whether these association tests necessarily require a formulation in terms of convolution with scaled mollifiers, or whether a more general formulation is possible.

Acknowledgements This research was supported by grant P26859 of the Austrian Science Fund (FWF). The author thanks the anonymous referee for comments.

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Soliton Dynamics for the General Degasperis–Procesi Equation

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Abstract We consider the general Degasperis–Procesi model of shallow water outflows. This fife parametric family of conservation laws contains, in particular, KdV, Camassa–Holm, and Degasperis–Procesi equations. The main result consists of a criterion which guarantees the existence of a smooth soliton-type solution. We discuss also the scenario of soliton interaction for this model in the nonintegrable case.

1 Introduction

The general Degasperis–Procesi model [5] is the fife parametric family of conservation laws

$$\frac{\partial}{\partial t}\left\{u-\alpha^2\varepsilon^2\frac{\partial^2 u}{\partial x^2}\right\} \tag{1}$$

$$+\frac{\partial}{\partial x}\left\{c_0 u+c_1 u^2-c_2\varepsilon^2\Big(\frac{\partial u}{\partial x}\Big)^2+\varepsilon^2\big(\gamma-c_3 u\big)\frac{\partial^2 u}{\partial x^2}\right\}=0,\ \ x\in\mathbb{R}^1,\ \ t>0,$$

which describes, in particular, the dynamics of out-flows of shallow water. Here α, $c_0,\dots,c_3$, γ are real parameters and ε characterizes the dispersion.

It is known (see, e.g., [6]) that the family (1) contains only three special cases that satisfy the integrability condition. More in detail:

1. Obviously, if we set $\alpha=c_2=c_3=0$, then we obtain the KdV equation.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_43

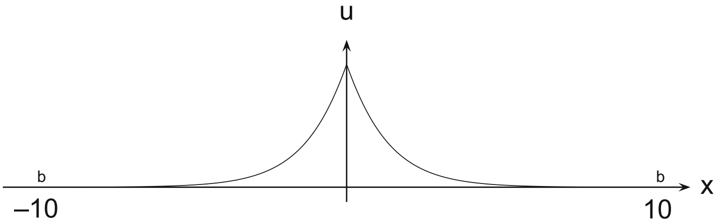

Fig. 1 Peacon solution of the Camassa–Holm equation for $c_0 = 0$

2. For $c_1 = 3c_3/(2\alpha^2)$, $c_2 = c_3/2$, $\gamma = 0$, and $v = c_3 u$ Eq. (1) becomes the Camassa–Holm equation

$$\frac{\partial}{\partial t}\left\{v - \alpha^2\varepsilon^2\frac{\partial^2 v}{\partial x^2}\right\} + \frac{\partial}{\partial x}\left\{c_0 v + \frac{3}{2\alpha^2}v^2 - \varepsilon^2\left(\frac{1}{2}\Big(\frac{\partial v}{\partial x}\Big)^2 + v\frac{\partial^2 v}{\partial x^2}\right)\right\} = 0. \tag{2}$$

3. For the special case $c_1 = 2c_3/\alpha^2$, $c_2 = c_3$, $c_0 = \gamma = 0$, and $v = c_3 u$, Eq. (1) is called the Degasperis–Procesi equation also,

$$\frac{\partial}{\partial t}\left\{v - \alpha^2\varepsilon^2\frac{\partial^2 v}{\partial x^2}\right\} + \frac{\partial}{\partial x}\left\{\frac{2}{\alpha^2}v^2 - \varepsilon^2\left(\Big(\frac{\partial v}{\partial x}\Big)^2 + v\frac{\partial^2 v}{\partial x^2}\right)\right\} = 0. \tag{3}$$

It is known that the KdV equation and the Camassa–Holm equation for $c_0 > 0$ admit smooth soliton solutions. Conversely, the Degasperis–Procesi equation and the Camassa–Holm equation for $c_0 = 0$ have continuous solitary wave solutions called "peacons," see Fig. 1. Moreover, solitary wave solutions of Eqs. (2) and (3) interact elastically, that is in the same manner as the KdV solitons (see, e.g., [6]).

However, the cases KdV, (2), and (3) exhaust that's all what is known about the family (1). So the first step of (1) investigation is the separation of the basic situations: smooth and non-smooth traveling wave solutions. We will do it in Sect. 2. The next question about the scenario of the solitary wave interaction we discuss in Sect. 3 for the case of solitons.

2 Solitary Wave Solution

Let us set the ansatz

$$u = A\omega\big(\beta(x - Vt)/\varepsilon\big), \tag{4}$$

where $\omega(\eta)$ is a smooth even function such that

$$\omega(0) = 1, \quad \omega(\eta) \to 0 \quad \text{as} \quad \eta \to \pm\infty, \tag{5}$$

$A > 0$, β, and V are free parameters. We assume that

$$\gamma \geq 0, \quad c_0 \geq 0, \quad \alpha > 0, \quad c_k > 0, \quad k = 1, 2, 3. \tag{6}$$

Substituting (4) into Eq. (1), integrating, and using (5), we obtain

$$\{1 - p\,\omega\} \frac{d^2\omega}{d\eta^2} = \frac{p}{c_4}\left(\frac{d\omega}{d\eta}\right)^2 + \frac{V - c_0}{\alpha^2 V + \gamma}\frac{\omega}{\beta^2} - \frac{c_1 p}{c_3 \beta^2}\omega^2 \tag{7}$$

with $\omega = \omega(\eta)$, $\eta = \beta(x - Vt)/\varepsilon$, $p = c_3 A/(\gamma + \alpha^2 V)$, $c_4 = c_3/c_2$. Next we set:

$$r = c_3/(c_2 + c_3), \quad q = c_3(V - c_0)/\big(c_1(\alpha^2 V + \gamma)\big), \tag{8}$$

and rescaling the function ω setting $W = p\omega$. Now, defining $\beta = \sqrt{c_1/c_3}$, assuming $V > c_0$, and using (8) we deduce that W satisfies the equation

$$(1 - W)\frac{d^2 W}{d\eta^2} = \frac{1 - r}{r}\left(\frac{dW}{d\eta}\right)^2 + qW - W^2. \tag{9}$$

The next step is the substitution

$$W(\eta) = 1 - g(\eta)^r, \tag{10}$$

which allows us to eliminate the first derivatives from the model equation (9). Taking into account the second condition in (5) and the property of being even, $g(-\eta) = g(\eta)$, we pass to the "boundary" problem

$$r\frac{d^2 g}{d\eta^2} = g - (2 - q)g^{1-r} + (1 - q)g^{1-2r}, \quad \eta \in (0, \infty), \tag{11}$$

$$g^r\big|_{\eta=0} = 1 - p, \quad g|_{\eta\to\infty} = 1. \tag{12}$$

Notice that the correctness of (12) implies the assumption

$$p < 1. \tag{13}$$

Now we integrate (11) and pass to the first order ODE

$$r(g')^2 = F(g), \quad \eta \in (0, \infty); \quad g|_{\eta=0} = g_*, \tag{14}$$

where prime denotes the derivative with respect to η and

$$F(g) = g^2 - 2\frac{2-q}{2-r}g^{2-r} + \frac{1-q}{1-r}g^{2-2r} - C, \tag{15}$$

$$C = r(r-q)/\{(1-r)(2-r)\}, \quad g_* = (1-p)^{1/r}. \tag{16}$$

Considering $\eta >> 1$ we write $g = 1 - w$ and obtain from (14)–(16)

$$(w')^2 = q\, w^2.$$

Thus $g \to 1 - \exp\{-\sqrt{q}\eta\}$ as $\eta \to \infty$. Therefore, the second condition in (12) is verified.

We now consider the even continuation $\widetilde{g}(\eta)$ of g for negative η. Obviously, $\widetilde{g} \in C^\infty(\mathbb{R})$ if and only if

$$g'|_{\eta=0} = 0. \tag{17}$$

Furthermore, since $F(1) = dF/dg|_{g=1} = 0$ and $d^2F/dg^2|_{g=1} > 0$, the equation

$$F(g) = 0 \tag{18}$$

has a solution $g_* \in (0, 1)$ if and only if $C > 0$. The last inequality is equivalent to the following assumption:

$$r > q. \tag{19}$$

On the other hand, the initial condition in (14) implies the relation

$$V = \alpha^{-2}\big(c_3 A/(1 - g_*^r) - \gamma\big). \tag{20}$$

This allows us to rewrite the coefficient q in (15) as a function of A and the parameters $\alpha, c_0, \ldots, c_3, \gamma$; therefore to find the solution g_* of Eq. (18) as a function of A and the parameters of the model (1). Representing (19) in the explicit form we obtain the conclusion

Theorem 1 *Under the assumptions (6), (13) we assume that*

$$c_3 - A^{-1}(\gamma + c_0\alpha^2)(1 - g_*^r) < c_1 r \alpha^2. \tag{21}$$

Then Eq. (1) has the classical soliton-type solution.

Example 1 For the Camassa–Holm equation (2) $r = 2/3$ and (18) is a cubic equation. Thus

$$g_* = \big(1 + c_3 A/(c_0\alpha^2)\big)^{-3/2} \quad \text{if} \quad c_0 > 0 \quad \text{and} \quad g_* = 0 \quad \text{if} \quad c_0 = 0. \tag{22}$$

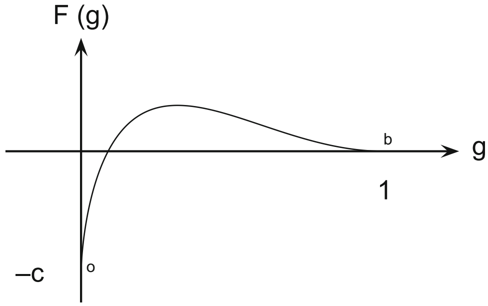

Fig. 2 Behavior of the function $F(g)$ for the Camassa–Holm equation with $c_0 = 1$

Respectively the condition (21) is satisfied for $c_0 > 0$ and it is broken for $c_0 = 0$. In the last case $\omega'|_{\eta=0} = -\sqrt{2(1-q)}/p \neq 0$, therefore $\omega(\eta)$ is a continuous function only. Figure 2 depicts the $F(g)$ graph in the case $c_0 = 1$, $A = 2$, $c_3 = 2$ and $\alpha = 1$. If $c_0 > 0$, then (20) and (22) imply the relation between the velocity and the amplitude:

$$V = c_0 + c_3 A/\alpha^2.$$

Example 2 For the Degasperis–Procesi equation (3) the condition (21) is violated and $\omega'|_{\eta=0} = -\sqrt{(1-q)}/p \neq 0$. Figure 1 demonstrates the graph of the peacon $\omega(\eta)$ for this equation.

Example 3 Now let $c_0 = \gamma = 0$ and $\alpha^2 c_1 > c_2 + c_3$. Then $q = c_3/\alpha^2 c_1 < r$ and g_* doesn't depend on V. Thus

$$V = c_3 A/\{(1 - {g_*}^r)\alpha^2\}. \tag{23}$$

Example 4 Let $c_3 = 4c_2$. Setting $z = g^r$, $r = 2/5$, we transform Eq. (18) to the form

$$F = (1-z)^2 f = 0, \quad f = z^3 + 2z^2 - \frac{1}{3}(1-5q)z - \frac{4}{5}(4-5q). \tag{24}$$

Solving the cubic equation $f = 0$ we find the root $z_* = z_*(V)$. This and (20) imply the equality

$$A = \mathfrak{A}(V), \quad \mathfrak{A} = c_3^{-1}(\gamma + \alpha^2 V)\big(1 - z_*(V)\big). \tag{25}$$

Simple calculations show that $d\mathfrak{A}/dV|_{q=0} > 0$. Thus, (25) allows us to define the velocity as a function of the amplitude at least for $V - c_0 << 1$.

Similar result can be obtained in the case $c_2 = 3c_3/2$.

3 Two-Soliton Asymptotic Solution

Nowadays, there is not any tool to construct neither an exact multi-soliton solution to (1) nor an asymptotics in the classical sense. So, we will treat ε as a small parameter and construct a weak asymptotic solution. The weak asymptotics method (see, e.g., [1–4, 7, 8] and references therein) takes into account the fact that soliton-type solutions which are smooth for $\varepsilon > 0$ become non-smooth in the limit as $\varepsilon \to 0$. Thus, it is possible to treat such solutions as a mapping $C^\infty(0, T; C^\infty(\mathbb{R}^1_x))$ for $\varepsilon = \text{const} > 0$ and only as $C(0, T; \mathscr{D}'(\mathbb{R}^1_x))$ uniformly in $\varepsilon \geq 0$. Accordingly, the remainder should be small in the weak sense. The main advantage of the method is such that we can ignore the real shape of the colliding waves but look for (and find) exceptionally their main characteristics. For Eq. (1) they are the amplitudes and trajectories of the waves.

Originally, such idea had been suggested by Danilov and Shelkovich for shock wave type solutions [3], and by Danilov and Omel'yanov for soliton-type solutions [2]. Later the method has been developed and adapted for many other problems (V. Danilov, G. Omel'yanov, V. Shelkovich, D. Mitrovic, M. Colombeau and others, see, e.g., [1–4, 7, 8] and references therein).

Notice finally that the treatment (see Omel'yanov [8]) of weak asymptotics as functions which satisfy some conservation or balance laws takes us back to the ancient Whitham's idea to construct one-phase asymptotic solution satisfying a Lagrangian. Now, for essentially nonintegrable equations and multi-soliton solutions, we use the appropriate number of the laws and satisfy them in the weak sense.

Let us apply these ideas for the problem of two-soliton interaction in the Degasperis–Procesi model (1). We set initial data

$$u|_{t=0} = \sum_{i=1}^{2} A_i \omega\big(\beta(x - x_i^0)/\varepsilon\big), \tag{26}$$

where $A_2 > A_1 > 0$, $x_1^0 > x_2^0$; $\beta = \sqrt{c_1/c_3}$, and V_i are defined in the same manner as in (20); and we assume that the trajectories $x = \varphi_{i0}(t) = V_i t + x_i^0$ have a joint point $x = x^*$ at a time instant $t = t^*$. To construct the weak asymptotic solution we start with the following definition of the smallness in the weak sense [2, 8]:

Definition 1 A function $v(t, x, \varepsilon)$ is said to be of the value $O_{\mathscr{D}'}(\varepsilon^\kappa)$ if the relation

$$\int_{-\infty}^{\infty} v(t, x, \varepsilon)\psi(x)dx = O(\varepsilon^\kappa)$$

holds uniformly in t for any test function $\psi \in \mathscr{D}(\mathbb{R}^1_x)$. The right-hand side here is a C^∞-function for $\varepsilon = \text{const} > 0$ and a piecewise continuous function uniformly in $\varepsilon \geq 0$.

Next we write two associated with (1) conservation and balance laws in the differential form:

$$\frac{\partial Q_j}{\partial t} + \frac{\partial P_j}{\partial x} + \varepsilon^{-1} K_j = O_{\mathscr{D}'}(\varepsilon^2), \quad j = 1, 2, \tag{27}$$

where

$$Q_1 = u, \quad P_1 = c_0 u + c_1 u^2 - (c_2 - c_3)(\varepsilon u_x)^2, \quad K_1 = 0, \tag{28}$$

$$Q_2 = u^2 + \alpha^2 (\varepsilon u_x)^2, \quad P_2 = \mathbb{P}_2 + 2\alpha^2 \varepsilon^2 u_x u_t, \quad K_2 = (2c_2 - c_3)(\varepsilon u_x)^3, \tag{29}$$

$$\mathbb{P}_2 = c_0 u^2 + \frac{4}{3} c_1 u^3 - \big(3\gamma + (2c_2 - 5c_3)u\big)(\varepsilon u_x)^2, \tag{30}$$

and subscripts denote partial derivatives.

Following [2, 8], we define two-soliton weak asymptotics:

Definition 2 A sequence $u(t, x, \varepsilon)$, belonging to $C^\infty(0, T; C^\infty(\mathbb{R}^1_x))$ for $\varepsilon =$ const > 0 and belonging to $C(0, T; \mathscr{D}'(\mathbb{R}^1_x))$ uniformly in $\varepsilon \geq 0$, is called a weak asymptotic mod $O_{\mathscr{D}'}(\varepsilon^2)$ solution of (1), (26) if the relations (27) hold uniformly in t with the accuracy $O_{\mathscr{D}'}(\varepsilon^2)$.

Next we present the ansatz as the sum of two distorted solitons, that is:

$$u = \sum_{i=1}^{2} G_i \omega\big(\beta(x - \varphi_i)/\varepsilon\big), \tag{31}$$

where

$$G_i = A_i + S_i(\tau), \; \varphi_i = \varphi_{i0}(t) + \varepsilon \varphi_{i1}(\tau), \; \tau = \beta_1\big(\varphi_{20}(t) - \varphi_{10}(t)\big)/\varepsilon, \tag{32}$$

$\varphi_{i0} = V_i t + x_{i0}$ describe the trajectories of the non-interacting waves (4) with the amplitudes A_i; and τ describes the distance between the non-interacting wave trajectories. Next we suppose that $S_i(\tau)$, $\varphi_{i1}(\tau)$ are smooth functions such that

$$S_i \to 0 \quad \text{as} \quad \tau \to \pm\infty, \tag{33}$$

$$\varphi_{i1} \to 0 \quad \text{as} \quad \tau \to -\infty, \quad \varphi_{i1} \to \varphi_{i1}^{\infty} = \text{const}_i \quad \text{as} \quad \tau \to +\infty. \tag{34}$$

To construct the asymptotics we should calculate the weak expansions of the terms from the left-hand sides of the relations (27). For any $\psi(x) \in \mathscr{D}(\mathbb{R}^1)$ we have

$$\int_{-\infty}^{\infty} u\psi(x)dx = \frac{\varepsilon}{\beta}\sum_{i=1}^{2} G_i \int_{-\infty}^{\infty} \omega(\eta)\psi\big(\varphi_i + \frac{\varepsilon}{\beta}\eta\big)d\eta = \frac{\varepsilon}{\beta}\sum_{i=1}^{2} G_i \int_{-\infty}^{\infty} \omega(\eta)\big\{\psi(\varphi_i)$$

$$+ \frac{\varepsilon}{\beta}\psi'(\varphi_i) + O(\varepsilon^2\eta^2)\big\}d\eta = \Big(a_1\frac{\varepsilon}{\beta}\sum_{i=1}^{2} G_i\delta(x-\varphi_i) + O_{\mathscr{D}'}(\varepsilon^3), \psi(x)\Big), \tag{35}$$

where $\delta(x)$ is the Dirac delta-function; here and in what follows, we use the notation

$$a_k \overset{\text{def}}{=} \int_{-\infty}^{\infty} \big(\omega(\eta)\big)^k d\eta, \quad k > 0, \qquad a_2' \overset{\text{def}}{=} \int_{-\infty}^{\infty} \big(\omega'(\eta)\big)^2 d\eta. \tag{36}$$

At the same time for any even $F(u, \varepsilon u_x) \in C^1$, $F\big(u(-x), \varepsilon u_x(-x)\big) = F\big(u(x), \varepsilon u_x(x)\big)$:

$$F(u, \varepsilon u_x) = \varepsilon \sum_{i=1}^{2} \frac{a_{F,i}}{\beta_i}\delta(x-\varphi_i) + \frac{\varepsilon}{\beta_2}\mathfrak{R}_F^{(0)}\delta(x-x^*)$$

$$- \frac{\varepsilon^2}{\beta_2}\big\{\chi_2\mathfrak{R}_F^{(0)} + {\beta_2}^{-1}\mathfrak{R}_F^{(1)}\big\}\delta'(x-x^*) + O_{\mathscr{D}'}(\varepsilon^3), \tag{37}$$

where $\chi_i = V_i\tau/\dot{\psi}_0 + \varphi_{i1}$, $\dot{\psi}_0 = \beta(V_2 - V_1)$,

$$\mathfrak{R}_F^{(n)} = \int_{-\infty}^{\infty} \eta^n \Big\{F\Big(\sum_{i=1}^{2} G_i\omega(\eta_{i2}), \beta\sum_{j=1}^{2} G_j\omega'(\eta_{j2})\Big)$$

$$- \sum_{i=1}^{2} F\big(A_i\omega(\eta_{i2}), \beta A_i\omega'(\eta_{i2})\big)\Big\}d\eta, \quad n = 1, 2,$$

$$a_{F,i} = \int_{-\infty}^{\infty} F\big(A_i\omega(\eta), \beta A_i\omega'(\eta)\big)d\eta, \quad \eta_{i2} = \eta - \delta_i^1\sigma, \quad \sigma = \beta(\varphi_1 - \varphi_2))/\varepsilon.$$

Substituting (35), (37) into (27) we obtain linear combinations of $\delta(x - x^*)$, $\varepsilon\delta'(x - \varphi_i)$, $i = 1, 2$, and $\varepsilon\delta'(x - x^*)$ (see also [2, 8]); therefore, we pass to the following system of equations:

$$\sum_{i=1}^{2} S_i = 0, \quad \dot{\psi}_0 \frac{d}{d\tau} \mathfrak{R}_{Q_2}^{(0)} + \mathfrak{R}_{K_2}^{(0)} = 0, \quad a_1 \dot{\psi}_0 \frac{d}{d\tau} \sum_{i=1}^{2} \Big\{ A_i \varphi_{i1} + \chi_i S_i \Big\} = f, \tag{38}$$

$$\dot{\psi}_0 \frac{d}{d\tau} \Big\{ \sum_{i=1}^{2} a_{Q_2,i} \varphi_{i1} + \chi_2 \mathfrak{R}_{Q_2}^{(0)} + \beta^{-1} \mathfrak{R}_{Q_2}^{(1)} \Big) = F, \tag{39}$$

where

$$f = \mathfrak{R}_{P_1}^{(0)}, \quad F = \mathfrak{R}_{P_2}^{(0)} - a_2' \mathfrak{L} - \chi_2 \mathfrak{R}_{K_2}^{(0)} - \beta^{-1} \mathfrak{R}_{K_2}^{(1)},$$

$$\mathfrak{L} = \dot{\psi}_0 \beta \sum_{i=1}^{2} \frac{d\varphi_{i1}}{d\tau} (G_i^2 - A_i^2) - \dot{\psi}_0 \left(G_1 \frac{dS_2}{d\tau} - G_2 \frac{dS_1}{d\tau} \right) \lambda_{(1,0)}$$

$$+ \beta G_1 G_2 (\dot{\varphi}_1 + \dot{\varphi}_1) \lambda_{(1,1)}, \quad \lambda_{(k,l)} = \frac{1}{a_2'} \int_{-\infty}^{\infty} \omega^{(k)}(\eta_{12}) \omega^{(l)}(\eta) d\eta,$$

Now we can formulate a formal result:

Theorem 2 *Let the assumptions (6), (13), (21) be satisfied. Presuppose also that Eqs. (38), (39) admit a solution with the properties (33), (34). Then the solitary wave collision in the problem (1), (26) preserves the elastic scenario with accuracy* $O_{\mathscr{D}'}(\varepsilon^2)$.

Acknowledgement The research was supported by SEP-CONACYT under grant 178690 (Mexico).

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Frame Expansions of Test Functions, Tempered Distributions, and Ultradistributions

Stevan Pilipović and Diana T. Stoeva

Abstract The paper is devoted to frame expansions in Fréchet spaces. First we review some results which concern series expansions in general Fréchet spaces via Fréchet and General Fréchet frames. Then we present some new results on series expansions of tempered distributions and ultradistributions, and the corresponding test functions, via localized frames and coefficients in appropriate sequence spaces.

1 Introduction

In this paper we present results devoted to frame expansions in Fréchet spaces. First we consider the general case, expansions via Fréchet frames and appropriate dual sequences in general, and then we aim at expansions of generalized functions via a proper class of frames. As the Hermite expansions are the basic ones for tempered distributions and ultradistributions, a suitable localization of a frame with respect to the Hermite basis enables us to extend the consideration of generalized functions using an appropriate class of frames instead of the Hermite basis.

Frames were introduced in Hilbert spaces [11]. They generalize the concept of an orthonormal basis allowing even redundancy, but still provide series expansions of all the elements of the space. Frames were extended to Banach spaces (atomic decompositions and Banach frames [15, 16, 21], p-frames [1], X_d-frames [8]) and furthermore to Fréchet spaces (pre-Fréchet, Fréchet, and General Fréchet frames, [26, 27, 29]). For other types of frame concepts in Banach spaces and Fréchet spaces, we refer to [7] and [5, 6], respectively.

While Hilbert frames always guarantee series expansions in Hilbert spaces, this is not always the case with Banach and Fréchet frames. In this paper we review

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_44

some results related to sufficient conditions for series expansions in general Fréchet spaces, as well as present new results devoted to series expansions in certain spaces of test functions and their duals via appropriate frames.

The paper is organized as follows. Section 2 contains the main definitions, notation, and basic needed facts. In Sect. 3 we review some results from [26, 27] which are related to series expansions in general Fréchet spaces via Fréchet and General Fréchet frames. Section 4 is devoted to new results which concern expansions in certain spaces of test functions, tempered distributions, and ultradistributions, via localized frames; the statements are given without proofs and the proofs are subject of a further extended paper [28].

2 Preliminaries

Throughout the paper, $(H, \langle\cdot,\cdot\rangle)$ denotes a separable Hilbert space. We consider countable sequences and for convenience of the writing we index them by the set $\mathbb{N}$. A sequence $(g_n)_{n=1}^\infty$ with elements from H is called: a *frame for* H if there exist positive constants A and B (called *frame bounds*) so that $A\|f\|^2 \leq \sum_{n=1}^\infty |\langle f, g_n\rangle|^2 \leq B\|f\|^2$ for every $f \in H$ [11]; a *Riesz basis for* H if its elements are the images of the elements of an orthonormal basis under a bounded bijective operator on H [4].

Let us recall some needed basic facts from frame theory (see, e.g., [10]). Let $G = (g_n)_{n=1}^\infty$ be a frame for H. Then there exists a frame $(f_n)_{n=1}^\infty$ for H so that $f = \sum_{n=1}^\infty \langle f, f_n\rangle g_n = \sum_{n=1}^\infty \langle f, g_n\rangle f_n$, $f \in H$. Such $(f_n)_{n=1}^\infty$ is called a *dual frame of* $(g_n)_{n=1}^\infty$. The *analysis operator* U_G, given by $U_G f = (\langle f, g_n\rangle)_{n=1}^\infty$, is bounded from H into ℓ^2, the *synthesis operator* T_G defined on finite sequences by $T_G(c) = \sum_n c_n e_n$ extends to a bounded operator from ℓ^2 into H, the *frame operator* $S_G = T_G U_G$ is a bounded bijection of H onto H, and the series in $S_G f = \sum_{n=1}^\infty \langle f, g_n\rangle g_n$ converges unconditionally. The sequence $(S_G^{-1} g_n)_{n=1}^\infty$ is a dual frame of $(g_n)_{n=1}^\infty$, called the *canonical dual of* $(g_n)_{n=1}^\infty$, and it will be denoted by $(\widetilde{g_n})_{n=1}^\infty$. When $(g_n)_{n=1}^\infty$ is a Riesz basis of H (and thus a frame for H), then $(\widetilde{g_n})_{n=1}^\infty$ is the only dual frame of $(g_n)_{n=1}^\infty$. Frames which are not Riesz bases have other dual frames in addition to the canonical dual one.

Recall also the localization-notions introduced in [22]. Given a Riesz basis $(g_n)_{n=1}^\infty$ for H, a frame $E = (e_n)_{n=1}^\infty$ for H is called: *polynomially localized with respect to* $(g_n)_{n=1}^\infty$ *with decay* $s > 0$ if there is a constant $C_s > 0$ so that $\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g_n}\rangle|\} \leq C_s(1 + |m-n|)^{-s}$, $m, n \in \mathbb{N}$; *exponentially localized with respect to* $(g_n)_{n=1}^\infty$ if for some $s > 0$ there is a constant $C_s > 0$ so that $\max\{|\langle e_m, g_n\rangle|, |\langle e_m, \widetilde{g_n}\rangle|\} \leq C_s \mathrm{e}^{-s|m-n|}$, $m, n \in \mathbb{N}$. Notice that in the literature there exist other ways to define localization of frames [2, 3, 17, 18], but for the purposes of the current paper it is relevant and enough to use the localization concepts according to [22].

Next, $(X, \|\cdot\|)$ denotes a Banach space and $(\Theta, |||\cdot|||)$ denotes a Banach sequence space. A Banach sequence space is a *BK-space* if the coordinate functionals are continuous. If the canonical vectors form a Schauder basis for Θ, then Θ is called a *CB-space* and it is clearly a BK-space. Given a BK-space Θ and a Riesz basis $G = (g_n)_{n=1}^{\infty}$ for H, one associates with Θ the following Banach space:

$$\mathfrak{H}_G^{\Theta} := \{f \in H \ : \ f = \sum_{n=1}^{\infty} c_n g_n \text{ with } (c_n)_{n=1}^{\infty} \in \Theta\} \text{ normed by } \|f\|_{\mathfrak{H}_G^{\Theta}} := |||(c_n)_{n=1}^{\infty}|||_{\Theta}.$$

Further, we consider Fréchet spaces which are projective limits of Banach spaces. Let $\{Y_k, |\cdot|_k\}_{k\in\mathbb{N}_0}$ be a sequence of separable Banach spaces such that

$$\{\mathbf{0}\} \neq \cap_{k\in\mathbb{N}_0} Y_k \subseteq \ldots \subseteq Y_2 \subseteq Y_1 \subseteq Y_0 \tag{1}$$

$$|\cdot|_0 \leq |\cdot|_1 \leq |\cdot|_2 \leq \ldots \tag{2}$$

$$Y_F := \cap_{k\in\mathbb{N}_0} Y_k \ \text{ is dense in } \ Y_k, \quad k \in \mathbb{N}_0. \tag{3}$$

Under the conditions (1)–(3), Y_F is a Fréchet space with the sequence of norms $|\cdot|_k$, $k \in \mathbb{N}_0$, and it is called the *projective limit* of Y_k, $k \in \mathbb{N}_0$. We will use such type of sequences in two cases:

1. $Y_k = X_k$ with norm $\|\cdot\|_k$, $k \in \mathbb{N}_0$;
2. $Y_k = \Theta_k$ with norm $|||\cdot|||_k$, $k \in \mathbb{N}_0$.

We use the term *operator* for a linear mapping. Given sequences of Banach spaces, $\{X_k\}_{k\in\mathbb{N}_0}$ and $\{\Theta_k\}_{k\in\mathbb{N}_0}$, which satisfy (1)–(3), an operator $G : \Theta_F \to X_F$ is called *F-bounded* if for every $k \in \mathbb{N}_0$, there exists a constant $C_k > 0$ such that $\|G(c_n)_{n=1}^{\infty}\|_k \leq C_k |||\{c_n\}_{n=1}^{\infty}|||_k$ for all $(c_n)_{n=1}^{\infty} \in \Theta_F$. Now we recall the definitions of Fréchet and General Fréchet frames.

Definition 2.1 ([27]) Let $\{X_k, \|\cdot\|_k\}_{k\in\mathbb{N}_0}$ be a sequence of Banach spaces satisfying (1)–(3) and let $\{\Theta_k, |||\cdot|||_k\}_{k\in\mathbb{N}_0}$ be a sequence of BK-spaces satisfying (1)–(3). A sequence $(g_n)_{n=1}^{\infty}$ of elements from X_F^* is called: a *General pre-Fréchet frame* (in short, *General pre-F-frame*) for X_F with respect to Θ_F if there exist sequences $\{\widetilde{s}_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{N}_0$, $\{s_k\}_{k\in\mathbb{N}_0} \subseteq \mathbb{N}_0$, which increase to ∞ with the property $s_k \leq \widetilde{s}_k$, $k \in \mathbb{N}_0$, and there exist constants $0 < A_k \leq B_k < \infty$, $k \in \mathbb{N}_0$, satisfying

$$(g_n(f))_{n=1}^{\infty} \in \Theta_F, \ \ f \in X_F, \tag{4}$$

$$A_k \|f\|_{s_k} \leq |||\{g_n(f)\}_{n=1}^{\infty}|||_k \leq B_k \|f\|_{\widetilde{s}_k}, \ \ f \in X_F, k \in \mathbb{N}_0; \tag{5}$$

a *General Fréchet frame* (in short, *General F-frame*) for X_F with respect to Θ_F if it is a General pre-F-frame for X_F with respect to Θ_F and there exists a continuous operator $V : \Theta_F \to X_F$ so that $V(g_n(f))_{n=1}^{\infty} = f$ for every $f \in X_F$.

Let $s_k = \widetilde{s}_k = k$, $k \in \mathbb{N}_0$. In this case the above definition of a General pre-F-frame reduces to the definition of a *pre-Fréchet frame* (in short *pre-F-frame*) [29], and if in addition the continuity of V is replaced by the stronger condition of F-boundedness of V, then one comes to the concept of a *Fréchet frame* (in short *F-frame*) [26]. In the particular case when $X_k = X$, and $\Theta_k = \Theta$, $k \in \mathbb{N}_0$: an F-frame (resp. pre-F-frame) for X_F with respect to Θ_F is actually a *Banach frame for X with respect to Θ* (resp. Θ-frame for X) as introduced in [21] (resp. [8]); when (4) and the upper inequality of (5) hold, one comes to the definition of a *Θ-Bessel sequence for X*.

When $(g_n)_{n=1}^{\infty}$ is a pre-F-frame for X_F with respect to Θ_F, then for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}_0$, g_n can be extended in a unique way to a continuous operator on X_k and this extension will be denoted by g_n^k.

Recall that a positive continuous function μ on $\mathbb{R}$ is called: a *k-moderate weight* if $k \geq 0$ and there exists a constant $C > 0$ so that $\mu(t+x) \leq C(1+|t|)^k \mu(x)$, $t, x \in \mathbb{R}$; a *sub-exponential* weight, if there exist constants $C > 0, \gamma > 0$ and $\beta \in (0,1)$ so that $\mu(t+x) \leq Ce^{\gamma |t|^{\beta}} \mu(x)$, $t, x \in \mathbb{R}$. Taking $\mu_k(x) = (1+|x|)^k$ (resp. $\beta \in (0,1)$ and $\mu_k(x) = e^{k|x|^{\beta}}$), $x \in \mathbb{R}$, $k \in \mathbb{N}_0$, the spaces $\Theta_k := \ell^2_{\mu_k}$, $k \in \mathbb{N}_0$, satisfy (1)–(3) and their projective limit $\cap_k \Theta_k$ is the so-called *space of rapidly decreasing sequences* $\mathbf{s}$ (resp. *space of sub-exponentially decreasing sequences* $\mathfrak{s}^{\beta}$). Further, we will use the following statement:

Lemma 2.2 *Let $G = (g_n)_{n=1}^{\infty}$ be a Riesz basis for H. For $k \in \mathbb{N}_0$, let μ_k be a k-moderate weight so that $1 = \mu_0(x) \leq \mu_1(x) \leq \mu_2(x) \leq \ldots$, for every $x \in \mathbb{R}$. Then $\{\Theta_k\}_{k\in\mathbb{N}_0} := \{\ell^2_{\mu_k}\}_{k\in\mathbb{N}_0}$ is a sequence of CB-spaces satisfying (1)–(3), the spaces $X_k := \mathfrak{H}_G^{\Theta_k}$, $k \in \mathbb{N}_0$, satisfy (1)–(3), and $g_n \in X_F$ for every $n \in \mathbb{N}$. The conclusions also hold if the assumptions "μ_k - k-moderate weight" are replaced by "μ_k - sub-exponential weight."*

Test function spaces and their duals under consideration in the paper are

$\mathscr{S}(\mathbb{R}) := \{f \in C^{\infty}(\mathbb{R}) : |f|_k := \sup_{x\in\mathbb{R}} \sup_{m\leq k} |f^{(m)}(x)|\,(1+|x|^2)^{k/2} < \infty,\ \forall k \in \mathbb{N}_0\}$, and its dual $\mathscr{S}'(\mathbb{R}) \subset C^{\infty}(\mathbb{R})$, the space of tempered distributions;

$\Sigma^{\alpha} := \{\phi \in C^{\infty}(\mathbb{R}) : |\phi|_{h,\alpha} := \sup_{n\in\mathbb{N}_0, x\in\mathbb{R}} \frac{h^n e^{m|x|^{1/\alpha}} |\phi^{(n)}(x)|}{n!^{\alpha}} < \infty,\ \forall h > 0\}$, and its dual $(\Sigma^{\alpha}(\mathbb{R}))'$, $\alpha > 1/2$, the space of Beurling tempered ultradistributions, cf. [9, 14, 19, 25].

In the sequel, $(h_n)_{n=1}^{\infty}$ is the Hermite orthonormal basis $(\mathbf{h}_n)_{n=0}^{\infty}$ of $L^2(\mathbb{R})$, re-indexed from 1 to ∞, i.e., $h_{n+1}(t) = \mathbf{h}_n(t) = (2^{(n)} n! \sqrt{\pi})^{-1/2} (-1)^n e^{t^2/2} \frac{d^n}{dt^n}(e^{-t^2})$, $n \in \mathbb{N}_0$. Recall that $h_n \in \mathscr{S}$ and $h_n \in \Sigma^{\alpha}$, $\alpha > 1/2$, $n \in \mathbb{N}$. Moreover, we know [30] the following:

- If $f \in \mathscr{S}$, then $(\langle f, h_n\rangle)_{n=1}^{\infty} \in \mathbf{s}$; conversely, if $(a_n)_{n=1}^{\infty} \in \mathbf{s}$, then $\sum_{n=1}^{\infty} a_n h_n$ converges in $\mathscr{S}$ to $f = \sum_{n=1}^{\infty} \langle f, h_n\rangle h_n$, $(\langle f, h_n\rangle)_{n=1}^{\infty} = (a_n)_{n=1}^{\infty}$.
- If $F \in \mathscr{S}'$, then $(b_n)_{n=1}^{\infty} := (F(h_n))_{n=1}^{\infty} \in \mathbf{s}'$ and $F(f) = \sum_{n=1}^{\infty} \langle f, h_n\rangle b_n$, $f \in \mathscr{S}$; conversely, if $(b_n)_{n=1}^{\infty} \in \mathbf{s}'$, then the mapping $F : f \to \sum_{n=1}^{\infty} \langle f, h_n\rangle b_n$ is well defined on $\mathscr{S}$, it determines F as an element of $\mathscr{S}'$ and $(F(h_n))_{n=1}^{\infty} = (b_n)_{n=1}^{\infty}$.

The above two statements also hold when $\mathscr{S}$, $\mathscr{S}'$, $\mathbf{s}$, and $\mathbf{s}'$ are replaced by Σ^α, $(\Sigma^\alpha)'$, $\mathfrak{s}^{1/(2\alpha)}$, and $(\mathfrak{s}^{1/(2\alpha)})'$ with $\alpha > 1/2$, respectively [9, 14, 19, 25]. Moreover, one can consider elliptic Shubin type polynomial operators, the corresponding orthonormal systems (given by eigenfunctions), and the corresponding eigenvalues in order to define Fréchet function spaces and corresponding sequence spaces (cf. [20, 31]), with the same aim as Hermite functions and the eigenvalues which correspond to the Harmonic oscillator.

3 Frame Expansions in Fréchet Spaces

In this section we recall some general statements about sufficient conditions for series expansions in Fréchet spaces via Fréchet and General Fréchet frames, and appropriate dual sequences. We start with the case of Fréchet frames.

Proposition 3.1 ([26]) *Let $(g_n)_{n=1}^\infty$ be an F-frame for X_F with respect to Θ_F. Then the following holds.*

(a) *For every $k \in \mathbb{N}_0$, the sequence $\{g_i^k\}_{i=1}^\infty$ is a Banach frame for X_s with respect to Θ_k.*
(b) *If Θ_k, $k \in \mathbb{N}_0$, are CB-spaces, then there exists $(f_n)_{n=1}^\infty \in (X_F)^{\mathbb{N}}$, which is Θ_k^*-Bessel sequence for X_k^* for every $k \in \mathbb{N}_0$ and such that*

$$f = \sum_{i=1}^\infty g_i(f) f_i, \ \ f \in X_F, \ (\text{in } X_F), \tag{6}$$

$$g = \sum_{i=1}^\infty g(f_i) g_i, \ \ g \in X_F^*, \ (\text{in } X_F^*), \tag{7}$$

$$f = \sum_{i=1}^\infty g_i^k(f) f_i, \ \ f \in X_k, \ k \in \mathbb{N}_0. \tag{8}$$

(c) *If Θ_k and Θ_k^*, $k \in \mathbb{N}_0$, are CB-spaces, then there exists $(f_n)_{n=1}^\infty \in (X_F)^{\mathbb{N}}$, which is a Θ_k^*-frame for X_k^* for every $k \in \mathbb{N}_0$ and such that (6)–(8) hold, and moreover,*

$$g = \sum_{i=1}^\infty g(f_i) g_i^k, \ \ g \in X_k^*, \ k \in \mathbb{N}_0. \tag{9}$$

(d) *If Θ_k, $k \in \mathbb{N}_0$, are reflexive CB-spaces, then there exists $(f_n)_{n=1}^\infty \in (X_F)^{\mathbb{N}}$, which is a Banach frame for X_k^* with respect to Θ_k^* for every $k \in \mathbb{N}_0$ such that (6)–(9) hold.*

As one can see in Proposition 3.1, the F-boundedness property of V leads to series expansions in all the spaces X_k, $k \in \mathbb{N}_0$. Now we continue with the case of General Fréchet frames and show that in this case the continuity property of V is enough to imply the existence of a subsequence $\{X_{\widetilde{w}_j}\}_{j=0}^{\infty}$ of the given sequence $\{X_{\widetilde{s}_k}\}_{k=0}^{\infty}$ according to Definition 2.1, so that one has series expansions in $X_{\widetilde{w}_j}$, $j \in \mathbb{N}_0$, with convergence in appropriate norms.

Theorem 3.2 ([27]) *Let $(g_n)_{n=1}^{\infty}$ be a General F-frame for X_F with respect to Θ_F and let Θ_k, $k \in \mathbb{N}_0$, be CB-spaces. Then there exist sequences $\{w_j\}_{j\in\mathbb{N}_0}$, $\{r_j\}_{j\in\mathbb{N}_0}$, and $\{\widetilde{w}_j\}_{j\in\mathbb{N}_0}$, which increase to ∞ and there exist constants $\widetilde{A}_j$, $\widetilde{B}_j$, $j \in \mathbb{N}_0$, such that for every $j \in \mathbb{N}_0$,*

$$\widetilde{A}_j \|f\|_{w_j} \le |||\{g_i(f)\}_{i=1}^{\infty}|||_{r_j} \le \widetilde{B}_j \|f\|_{\widetilde{w}_j}, \ \forall f \in X_F.$$

Moreover, there exists a sequence $(f_n)_{n=1}^{\infty} \in (X_F)^{\mathbb{N}}$ such that for every $j \in \mathbb{N}_0$, $(f_n)_{n=1}^{\infty}$ is a $\Theta_{r_j}^*$-Bessel sequence for $X_{w_j}^*$ and

$$f = \sum_{i=1}^{\infty} g_i^{\widetilde{w}_j}(f) f_i \text{ in } \|.\|_{w_j}\text{-norm}, \ f \in X_{\widetilde{w}_j}.$$

4 Expansions of Tempered Distributions and Ultradistributions by Localized Frames

In this section, we aim at expansions in a Fréchet space and its dual by localized frames and coefficients in a corresponding Fréchet sequence space. First we present a general result based on frames localized with respect to a Riesz basis and providing frame expansions in a corresponding Fréchet space, and then we apply it to obtain frame expansions in spaces of tempered distributions and ultradistributions, as well as in the corresponding test function spaces. To clarify some notation, when we take a frame element $e_n \in X_F (\subset H \subset X_F^*)$ and consider it as a functional in X_F^*, then we denote it by bold-style $\mathbf{e_n}$.

Let us start with a general theorem about expansions in Fréchet spaces via localized frames.

Theorem 4.1 *Let the assumptions and notation of Lemma 2.2 hold. Assume that $E = (e_n)_{n=1}^{\infty}$ is a sequence with elements from X_F which is a frame for H and polynomially localized with respect to G with decay s for every $s \in \mathbb{N}$ (resp. exponentially localized with respect to G). Then the following statements hold.*

(a) *$\widetilde{e_n} \in X_F$ for every $n \in \mathbb{N}$.*
(b) *The analysis operator U_E is F-bounded from X_F into Θ_F, the synthesis operator T_E is F-bounded from Θ_F into X_F, and the frame operator S_E is*

F-bounded and bijective from X_F onto X_F with unconditional convergence of the series in $S_E f = \sum_{n=1}^{\infty} \langle f, e_n \rangle e_n$.

(c) *For every $f \in X_F$,*

$$f = \sum_{n=1}^{\infty} \langle f, \widetilde{e_n} \rangle e_n = \sum_{n=1}^{\infty} \langle f, e_n \rangle \widetilde{e_n} \text{ (with convergence in } X_F\text{)}$$

with $(\langle f, \widetilde{e_n} \rangle)_{n=1}^{\infty} \in \Theta_F$ and $(\langle f, e_n \rangle)_{n=1}^{\infty} \in \Theta_F$.

(d) *If X_F and Θ_F have the following property with respect to $(g_n)_{n=1}^{\infty}$:*

$$\mathscr{P}_{(g_n)}\text{: For } f \in H \text{, one has } f \in X_F \text{ if and only if } (\langle f, g_n \rangle)_{n=1}^{\infty} \in \Theta_F.$$

then they also have the properties $\mathscr{P}_{(e_n)}$ and $\mathscr{P}_{(\widetilde{e_n})}$.

(e) *Both $(\mathbf{e_n})_{n=1}^{\infty}$ and $(\widetilde{\mathbf{e_n}})_{n=1}^{\infty}$ form Fréchet frames for X_F with respect to Θ_F.*

(f) *For every $g \in X_F^*$,*

$$g = \sum_{n=1}^{\infty} g(e_n)\, \widetilde{\mathbf{e_n}} = \sum_{n=1}^{\infty} g(\widetilde{e_n})\, \mathbf{e_n} \text{ (with convergence in } X_F^*\text{)} \tag{10}$$

with $(g(e_n))_{n=1}^{\infty} \in \Theta_F^*$ and $(g(\widetilde{e_n}))_{n=1}^{\infty} \in \Theta_F^*$.

(g) *If $(a_n)_{n=1}^{\infty} \in \Theta_F^*$, then $\sum_{n=1}^{\infty} a_n \mathbf{e}_n$ (resp. $\sum_{n=1}^{\infty} a_n \widetilde{\mathbf{e_n}}$) converges in X_F^*, i.e., the mapping $f \mapsto \sum_{n=1}^{\infty} \langle f, e_n \rangle a_n$ (resp. $f \mapsto \sum_{n=1}^{\infty} \langle f, \widetilde{e_n} \rangle a_n$) determines a continuous linear functional on X_F.*

Remark Note that in the setting of the above theorem, when G is an orthonormal basis of H or more generally, when G is a Riesz basis for H satisfying any of the following two conditions:

$(\mathscr{P}_1)$: $\forall s \in \mathbb{N}\ \exists C_s > 0\ :\ |\langle g_m, g_n \rangle| \le C_s (1 + |m - n|)^{-s},\ m, n \in \mathbb{N}$,
$(\mathscr{P}_2)$: $\exists s > 0\ \exists C_s > 0\ :\ |\langle g_m, g_n \rangle| \le C_s \mathrm{e}^{-s|m-n|},\ m, n \in \mathbb{N}$,

then the property $\mathscr{P}_{(g_n)}$ is satisfied.

Now we apply Theorem 4.1 to obtain series expansions in the spaces $\mathscr{S}$ and Σ^{α} (for $\alpha > 1/2$), and their duals, via frames localized by the Hermite orthonormal basis and coefficients in the corresponding sequence spaces. Furthermore, we extend the known characterizations of $\mathscr{S}$, Σ^{α}, $\alpha > 1/2$, and their dual spaces, based on the Hermite basis (see the end of Sect. 2), to characterizations based on a larger class of frame-functions.

Theorem 4.2 *Assume that $(e_n)_{n=1}^{\infty}$ is a sequence with elements from $\mathscr{S}(\mathbb{R})$ which is a frame for $L^2(\mathbb{R})$ and which is polynomially localized with respect to the Hermite basis $(h_n)_{n=1}^{\infty}$ with decay s for every $s \in \mathbb{N}$. Take $(g_n)_{n=1}^{\infty} := (h_n)_{n=1}^{\infty}$. Then $\mathscr{P}_{(g_n)}$ and the conclusions in Theorem 4.1 hold with X_F replaced by $\mathscr{S}$ and Θ_F replaced by* $\mathbf{s}$.

Theorem 4.3 *Let $\alpha > 1/2$. Assume that a sequence $(e_n)_{n=1}^{\infty}$ with elements from Σ^{α} is a frame for $L^2(\mathbb{R})$ which is exponentially localized with respect to the Hermite basis $(h_n)_{n=1}^{\infty}$. Take $(g_n)_{n=1}^{\infty} := (h_n)_{n=1}^{\infty}$. Then $\mathscr{P}_{(g_n)}$ and the conclusions in Theorem 4.1 hold with X_F replaced by Σ^{α} and Θ_F replaced by $\mathfrak{s}^{1/(2\alpha)}$.*

To illustrate Theorems 4.2 and 4.3, below we give an example based on appropriate linear combinations of Hermite functions. Further examples are to be given in an extended paper [28].

Example 4.4 Let $r \in \mathbb{N}$ and for $i = 1, 2, \dots, r$, take $\varepsilon_i \geq 0$ and a sequence $(a_n^i)_{n=1}^\infty$ of complex numbers satisfying $|a_n^i| \leq \varepsilon_i$ for $n \geq 2$, $\sum_{i=1}^r |a_1^i| \leq 1$, and $\sum_{i=1}^r \varepsilon_i < 1$. For $n \in \mathbb{N}$, consider $e_n := h_n + \sum_{i=1}^r a_n^i h_{n+i}$, which clearly belongs to $\mathscr{S}(\mathbb{R})$ and Σ^α, $\alpha > 1/2$. Then the sequence $(e_n)_{n=1}^\infty$ is a Riesz basis for $L^2(\mathbb{R})$ and it is s-localized with respect to the Hermite orthonormal basis $(h_n)_{n=1}^\infty$ for every $s > 0$, as well as exponentially localized with respect to $(h_n)_{n=1}^\infty$.

Remark 4.5 Having in mind the known expansions of tempered distributions $(\mathscr{S}(\mathbb{R}_+))'$ [12, 23] and Beurling ultradistributions $(G_\alpha^\alpha(\mathbb{R}_+))'$ [13, 24], and their test spaces, by the use of the Laguerre orthonormal basis l_n, $n \in \mathbb{N}$, and validity of the corresponding properties $\mathscr{P}_{(l_n)}$, we can transfer the above results to the mentioned classes of distributions and ultradistributions over $\mathbb{R}_+$.

Acknowledgements The authors acknowledge support from the Multilateral S&T Danube-Cooperation Project TIFMOFUS ("Time-Frequency Methods for Operators and Function Spaces"; MULT_DR 01/2017), the Austrian Science Fund (FWF) START-project FLAME ("Frames and Linear Operators for Acoustical Modeling and Parameter Estimation"; Y 551-N13), and the Project 174024 of the Serbian Ministry of Sciences. The second author is grateful for the hospitality of the University of Novi Sad, where most of the research on the presented topic was done. The authors express their gratitude to the referee for the valuable advices.

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Part XI
Theory and Applications of Boundary-Domain Integral and Pseudodifferential Operators

Session Organizers: Sergey E. Mikhailov and David Natroshvili

The goal of the session was to discuss recent progress in the theory of boundary-domain integral and pseudodifferential operators and their applications in Mathematical Physics, Solid and Fluid Mechanics, Wave Scattering Problems, Engineering Mathematics, etc.

Analysis of Boundary-Domain Integral Equations for Variable-Coefficient Mixed BVP in 2D

T. G. Ayele, T. T. Dufera, and S. E. Mikhailov

Abstract The direct segregated boundary-domain integral equations (BDIEs) for the mixed boundary value problem for a second order elliptic partial differential equation with variable coefficient in 2D is considered in this paper. An appropriate parametrix (Levi function) is used to reduce this BVP to the BDIEs. Although the theory of BDIEs in 3D is well developed, the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to insure the invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs is analysed.

1 Preliminaries

The direct segregated boundary-domain integral equations (BDIEs) for the mixed boundary value problem for a second order elliptic partial differential equation with variable coefficient in 2D is considered in this paper. An appropriate parametrix (Levi function) is used to reduce this BVP to the BDIEs. Although the theory of BDIEs in 3D is well developed, cf. [2, 3, 8, 9], the BDIEs in 2D need a special consideration due to their different equivalence properties. As a result, we need to set conditions on the domain or on the associated Sobolev spaces to insure the

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_45

invertibility of corresponding parametrix-based integral layer potentials and hence the unique solvability of BDIEs. The properties of corresponding potential operators are investigated. The equivalence of the original BVP and the obtained BDIEs is analysed.

Let Ω be a domain in $\mathbb{R}^2$ bounded by a smooth curve $\partial\Omega$, and let $n(x)$ be the exterior unit normal vector defined on $\partial\Omega$. The set of all infinitely differentiable functions on Ω with compact support is denoted by $\mathcal{D}(\Omega)$. The function space $\mathcal{D}'(\Omega)$ consists of all continuous linear functionals over $\mathcal{D}(\Omega)$. The space $H^s(\mathbb{R}^2)$, $s \in \mathbb{R}$, denotes the Bessel potential space, and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. We define $H^s(\Omega) = \{u \in \mathcal{D}'(\Omega) : u = U|_\Omega$ for some $U \in H^s(\mathbb{R}^2)\}$. The space $\widetilde{H}^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^2)$, and for $s \in (-\frac{1}{2}, \frac{1}{2})$, the space $H^s(\Omega)$ can be identified with $\widetilde{H}(\Omega)$, see, e.g., [7].

We shall consider the scalar elliptic differential equation

$$Au(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x) \qquad \text{in } \Omega,$$

where u is unknown function and f is a given function in Ω. We assume that

$$a \in C^\infty(\mathbb{R}^2), \quad 0 < a_{\min} \le a(x) \le a_{\max} < \infty, \quad \forall x \in \mathbb{R}^2. \tag{1}$$

For $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ if we put $h(x) = a(x)\frac{\partial u(x)}{\partial x_j}v(x)$ and apply the Gauss–Ostrogradsky theorem, we obtain the following *Green's first identity*:

$$\mathcal{E}(u, v) = -\int_\Omega (Au)(x)v(x)dx + \int_{\partial\Omega} T^{c+}u(x)\gamma^+ v(x)ds_x, \tag{2}$$

where $\mathcal{E}(u, v) := \sum_{i=1}^{2}\int_\Omega a(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_i}dx$ is the symmetric bilinear form, γ^+ is the trace operator and

$$T^{c+}u(x) := \sum_{i=1}^{2} n_i(x)\gamma^+\left[a(x)\frac{\partial}{\partial x_i}u(x)\right] \text{ for } x \in \partial\Omega, \tag{3}$$

is the *classical co-normal derivative*.

Remark 1.1 For $v \in \mathcal{D}(\Omega)$, $\gamma^+ v = 0$. If $u \in H^1(\Omega)$, then we can define Au as a distribution on Ω by $(Au, v) = -\mathcal{E}(u, v)$ for $v \in \mathcal{D}(\Omega)$.

The subspace $H^{1,0}(\Omega; A)$ is defined as in [5] (see also, [10])

$$H^{1,0}(\Omega; A) := \{g \in H^1(\Omega) : Ag \in L_2(\Omega)\},$$

with the norm $\|g\|^2_{H^{1,0}(\Omega;A)} := \|g\|^2_{H^1(\Omega)} + \|Ag\|^2_{L_2(\Omega)}$.

For $u \in H^1(\Omega)$ the classical co-normal derivative (3) is not well defined, but for $u \in H^{1,0}(\Omega; A)$, there exists the following continuous extension of this definition hinted by the first Green identity (2) (see, e.g., [5, 10] and the references therein).

Definition 1.2 For $u \in H^{1,0}(\Omega; A)$ the (canonical) co-normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ is defined in the following weak form:

$$\langle T^+u, w\rangle_{\partial\Omega} := \mathscr{E}(u, \gamma^+_{-1}w) + \int_\Omega (Au)\gamma^+_{-1}w\,dx \quad \text{for all } w \in H^{\frac{1}{2}}(\partial\Omega) \tag{4}$$

where $\gamma^+_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a continuous right inverse of the trace operator γ^+, which maps $H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, while $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denote the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, which extend the usual $L_2(\partial\Omega)$ inner product.

Remark 1.3 The first Green identity (2) also holds for $u \in H^{1,0}(\Omega; A)$ and $v \in H^1(\Omega)$ if we replace there T^{c+} by T^+, cf. [5, 10].

By interchanging the roles of u and v in the first Green identity and subtracting the result, we obtain *the second Green identity* for $u, v \in H^{1,0}(\Omega; A)$,

$$\int_\Omega (vAu - uAv)dx = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}. \tag{5}$$

Let $\partial\Omega = \overline{\partial\Omega_D} \cup \overline{\partial\Omega_N}$ where $\partial\Omega_D$ and $\partial\Omega_N$ are non-empty and non-intersecting parts of $\partial\Omega$. We shall derive and investigate BDIEs for the following mixed BVP: *Find a function $u \in H^1(\Omega)$ satisfying conditions*

$$Au = f \qquad \text{in } \Omega, \tag{6}$$

$$\gamma^+u = \varphi_0 \qquad \text{on} \quad \partial\Omega_D, \tag{7}$$

$$T^+u = \psi_0 \qquad \text{on} \quad \partial\Omega_N, \tag{8}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ and $f \in L_2(\Omega)$ are given functions. Equation (6) is understood in distributional sense as in Remark 1.1, Eq. (7) is understood in trace sense and Eq. (8) is understood in functional sense (4).

Theorem 1.4 *The homogeneous version of BVP* (6)–(8)*, i.e. with $f = 0$, $\varphi_0 = 0$, $\psi_0 = 0$ has only the trivial solution. Hence the nonhomogeneous problem* (6)–(8) *may possess at most one solution.*

Proof The proof follows from Green's formula (2) with $v = u$ as a solution of the homogeneous mixed BVP (cf. [2, Theorem 2.1]). □

2 Parametrix-Based Potential Operators

Definition 2.1 A function $P(x, y)$ is a parametrix (Levi function) for the operator A if

$$A_x P(x, y) = \delta(x - y) + R(x, y)$$

where δ is the Dirac-delta distribution, while $R(x, y)$ is a remainder possessing at most a weak singularity at $x = y$.

For 2D, the parametrix and hence the corresponding remainder can be chosen as in [8],

$$P(x, y) = \frac{\ln|x - y|}{2\pi a(y)}, \quad R(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \quad x, y \in \mathbb{R}^2.$$

Similar to [2, 8], we define the parametrix-based Newtonian and remainder potential operators as

$$\mathscr{P}g(y) := \int_{\Omega} P(x, y)g(x)dx, \quad \mathscr{R}g(y) := \int_{\Omega} R(x, y)g(x)dx. \tag{9}$$

The single and double layer potential operators corresponding to the parametrix $P(x, y)$, are defined for $y \notin \partial\Omega$ as

$$Vg(y) := -\int_{\partial\Omega} P(x, y)g(x)ds_x, \qquad Wg(y) := -\int_{\partial\Omega} T_x^+ P(x, y)g(x)ds_x, \tag{10}$$

where g is some scalar density function. The following boundary integral (pseudo-differential) operators are also defined for $y \in \partial\Omega$,

$$\mathscr{V}g(y) := -\int_{\partial\Omega} P(x, y)g(x)ds_x, \qquad \mathscr{W}g(y) := -\int_{\partial\Omega} T_x^+ P(x, y)g(x)ds_x, \tag{11}$$

$$\mathscr{W}'g(y) := -\int_{\partial\Omega} T_y^+ P(x, y)g(x)ds_x. \tag{12}$$

Let $V_\Delta, W_\Delta, \mathscr{V}_\Delta, \mathscr{W}_\Delta$ denote the potentials and the boundary operators corresponding to the Laplace operator $A = \Delta$. Then the relations similar to [1, Eq. (3.9)–(3.12)] hold (cf.[2] for the 3D case),

$$Vg = \frac{1}{a} V_\Delta g, \qquad Wg = \frac{1}{a} W_\Delta(ag) \tag{13}$$

$$\mathscr{V} g = \frac{1}{a} \mathscr{V}_\Delta g, \qquad \mathscr{W} g = \frac{1}{a} \mathscr{W}_\Delta(ag), \tag{14}$$

$$\mathscr{W}' g = \mathscr{W}'_\Delta g + \left[a \frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right] \mathscr{V}_\Delta g, \tag{15}$$

$$T^+ Wg = T^+_\Delta W_\Delta(ag) + \left[a \frac{\partial}{\partial n}\left(\frac{1}{a}\right)\right] W^+_\Delta(ag). \tag{16}$$

The mapping and jump properties of the operators (9)–(12) follow from relations (13)–(16) and are described in detail in [6, Theorems 1–3]. Particularly, we have the following jump relations.

Theorem 2.2 *Let $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$, $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$ and $y \in \partial\Omega$. Then*

$$\gamma^\pm V g_1(y) = \mathscr{V} g_1(y) \tag{17}$$

$$\gamma^\pm W g_2(y) = \mp\frac{1}{2}\, g_2(y) + \mathscr{W} g_2(y). \tag{18}$$

$$T^\pm V g_1(y) = \pm\frac{1}{2}\, g_1(y) + \mathscr{W}' g_1(y), \tag{19}$$

$$T^\pm W g_2(y) = \hat{\mathscr{L}}\, g_2(y) - \frac{\partial a}{\partial n}\left(\mp\frac{1}{2} I + \mathscr{W}\right) g_2(y), \tag{20}$$

where

$$\hat{\mathscr{L}} g_2 := T^+_\Delta W_\Delta(a g_2) = T^-_\Delta W_\Delta(a g_2) =: \hat{\mathscr{L}}_\Delta(a g_2) \quad \text{on} \quad \partial\Omega. \tag{21}$$

If $u \in H^{1,0}(\Omega; A)$, then substituting $v(x)$ by $P(x, y)$ in the second Green identity (5) for $\Omega \setminus B(y, \varepsilon)$, where $B(y, \varepsilon)$ is a disc of radius ε centred at y, and taking the limit $\varepsilon \to 0$, we arrive at the following parametrix-based third Green identity (cf. e.g. [2, 8, 11]:),

$$u + \mathscr{R}u - VT^+u + W\gamma^+u = \mathscr{P}Au \quad \text{in} \;\; \Omega. \tag{22}$$

Applying the *trace operator* to Eq. (22) and using the jump relations (17) and (18), we have

$$\frac{1}{2}\gamma^+u + \gamma^+\mathscr{R}u - \mathscr{V}T^+u + \mathscr{W}\gamma^+u = \gamma^+\mathscr{P}Au \quad \text{on}\; \partial\Omega. \tag{23}$$

Similarly, applying *co-normal derivative operator* to Eq. (22), and using the jump relation (19), we obtain

$$\frac{1}{2}T^+u + T^+\mathscr{R}u - \mathscr{W}'T^+u + T^+W\gamma^+u = T^+\mathscr{P}Au \quad \text{on } \partial\Omega. \tag{24}$$

For some functions f, Ψ and Φ let us consider a more general indirect integral relation associated with Eq. (22),

$$u + \mathscr{R}u - V\Psi + W\Phi = \mathscr{P}f \quad \text{in } \Omega. \tag{25}$$

Lemma 2.3 *Let $u \in H^1(\Omega)$, $f \in L_2(\Omega)$, $\Psi \in H^{-\frac{1}{2}}(\partial\Omega)$, $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ satisfy Eq. (25). Then u belongs to $H^{1,0}(\Omega; A)$ and is a solution of PDE (6), i.e. $Au = f$ in Ω, and $V(\Psi - T^+u)(y) - W(\Phi - \gamma^+u)(y) = 0, \quad y \in \Omega$.*

Proof The proof is similar to the one in 3D case in [2, Lemma 4.1]. □

For $s \in \mathbb{R}$ and $\Gamma_1 \subset \partial\Omega$, let us define the subspaces (cf. e.g. [12, p. 147])

$$H^s_{**}(\partial\Omega) := \{g \in H^s(\partial\Omega) : \langle g, 1\rangle_{\partial\Omega} = 0\}, \ \widetilde{H}^s_{**}(\Gamma_1) := \{g \in \widetilde{H}^s(\Gamma_1) : \langle g, 1\rangle_{\Gamma_1} = 0\}.$$

The following result is proved in [6, Theorem 4].

Theorem 2.4 *If $\psi \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$ satisfies $\mathscr{V}\psi = 0$ on $\partial\Omega$, then $\psi = 0$.*

Proof The theorem holds for the operator $\mathscr{V}_\Delta$ (see, e.g., [7, Corollary 8.11(ii)]), which due to (14) implies it for the operator $\mathscr{V}$ as well. □

The following theorem is proved in [7, Theorem 8.16].

Theorem 2.5

(i) The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, is $H^{-\frac{1}{2}}(\partial\Omega)$- elliptic, i.e. $\langle \mathscr{V}_\Delta\psi, \psi\rangle_{\partial\Omega} \geq c\|\psi\|_{H^{-\frac{1}{2}}(\partial\Omega)}$ for all $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, if and only if $Cap_{\partial\Omega} < 1$.

(ii) The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ has a bounded inverse if and only if $Cap_{\partial\Omega} \neq 1$.

The following result is proved in [6, Theorem 5].

Theorem 2.6 *Let $\Omega \subset \mathbb{R}^2$ have $\mathrm{diam}(\Omega) < 1$. Then the single layer potential $\mathscr{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

Proof Since $\mathrm{Cap}_{\partial\Omega} \leq \mathrm{diam}(\Omega)$, (see, [13, p. 553, properties 1 and 3]), then $\mathrm{diam}(\Omega) < 1$ implies $\mathrm{Cap}_{\partial\Omega} < 1$. The result follows from Theorem 2.5(ii) and the first relation in (14). □

Corollary 2.7 *Let Γ_1 be a non-empty part of the boundary curve $\partial\Omega$.*

(i) The operator

$$r_{\Gamma_1}\mathscr{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1) \tag{26}$$

is bounded and Fredholm of index zero.

(ii) If $\widetilde{\psi} \in \widetilde{H}_{**}^{-\frac{1}{2}}(\Gamma_1)$ satisfies $r_{\Gamma_1}\mathscr{V}\widetilde{\psi} = 0$ on Γ_1, then $\widetilde{\psi} = 0$.

Proof

(i) The operator $\mathscr{V} : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is bounded, which implies that operator (26) is bounded as well.

The operator $\mathscr{V}_\Delta : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ admits the decomposition $\mathscr{V}_\Delta = \mathscr{V}_0 + K$, where the operator $\mathscr{V}_0$ is positive and bounded below and K is a compact linear operator from $H^{-\frac{1}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ (cf. [7, Theorm 7.6], and [5, Theorem 2]). If $\widetilde{\psi} \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_1)$, then supp$\widetilde{\psi} \subset \overline{\Gamma}_1$ and

$$\langle r_{\Gamma_1}\mathscr{V}_0\widetilde{\psi}, \widetilde{\psi}\rangle_{\Gamma_1} = \langle \mathscr{V}_0\widetilde{\psi}, \widetilde{\psi}\rangle_{\partial\Omega} \geq c\|\widetilde{\psi}\|_{H^{-\frac{1}{2}}(\partial\Omega)} = c\|\widetilde{\psi}\|_{\widetilde{H}^{-\frac{1}{2}}(\Gamma_1)},$$

which means, the operator $r_{\Gamma_1}\mathscr{V}_0 : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is positive and bounded below. Also, the operator $r_{\Gamma_1}K : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is compact. Since $r_{\Gamma_1}\mathscr{V}_\Delta = r_{\Gamma_1}\mathscr{V}_0 + r_{\Gamma_1}K$, the operator $r_{\Gamma_1}\mathscr{V}\Delta : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is Fredholm of index zero (cf. [7, Theorem 2.33]). Since $\mathscr{V} = \frac{1}{a}\mathscr{V}_\Delta$ and the multiplication by $\frac{1}{a}$ is an isomorphism in $H^{\frac{1}{2}}(\Gamma_1)$ under condition (1), we obtain (cf. e.g. [7, Theorem 2.21]) that operator (26) is Fredholm of index zero as well.

To prove item (ii), suppose $\widetilde{\psi} \in \widetilde{H}_{**}^{-\frac{1}{2}}(\Gamma_1)$, i.e. $\langle\widetilde{\psi}, 1\rangle_{\Gamma_1} = \langle\widetilde{\psi}, 1\rangle_{\partial\Omega} = 0$, which implies $\widetilde{\psi} \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. For $\widetilde{\psi} \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$, we have $\langle\mathscr{V}_\Delta\widetilde{\psi}, \widetilde{\psi}\rangle_{\partial\Omega} \geq 0$, moreover, if $\langle\mathscr{V}_\Delta\widetilde{\psi}, \widetilde{\psi}\rangle_{\partial\Omega} = 0$, then $\widetilde{\psi} = 0$ on $\partial\Omega$ (cf. [7, Theorm 8.12]). Hence, if $r_{\Gamma_1}\mathscr{V}\widetilde{\psi} = 0$, then $r_{\Gamma_1}\mathscr{V}_\Delta\widetilde{\psi} = 0$ and $\langle\mathscr{V}_\Delta\widetilde{\psi}, \widetilde{\psi}\rangle_{\partial\Omega} = \langle r_{\Gamma_1}\mathscr{V}_\Delta\widetilde{\psi}, \widetilde{\psi}\rangle_{\Gamma_1} = 0$, which implies $\widetilde{\psi} = 0$. □

The following assertion can be proved similar to [7, Theorem 8.16].

Theorem 2.8 *Let Γ_1 be a non-empty part of the boundary curve $\partial\Omega$.*

(i) The operator $r_{\Gamma_1}\mathscr{V}_\Delta : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ is $\widetilde{H}^{-\frac{1}{2}}(\Gamma_1)$-elliptic if and only if $Cap_{\Gamma_1} < 1$.

(ii) The operators $r_{\Gamma_1}\mathscr{V}_\Delta : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ and $r_{\Gamma_1}\mathscr{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ are continuously invertible if and only if $Cap_{\Gamma_1} \neq 1$.

Corollary 2.9 *Let Γ_1 be a non-empty part of the boundary curve and* $\mathrm{diam}(\Gamma_1) < 1$. *Then the operator* $r_{\Gamma_1}\mathscr{V} : \widetilde{H}^{-\frac{1}{2}}(\Gamma_1) \to H^{\frac{1}{2}}(\Gamma_1)$ *has a bounded inverse.*

Proof Since $\mathrm{Cap}_{\Gamma_1} \leq \mathrm{diam}(\Gamma_1)$, (see, [13, p. 553, properties 1 and 3]), then $\mathrm{diam}(\Gamma_1) < 1$ implies $\mathrm{Cap}_{\Gamma_1} < 1$. The result follows from Theorem 2.8(ii). □

Theorem 2.10 *Let Γ_2 be a non-empty open part of the boundary curve $\partial\Omega$. The operator*

$$r_{\Gamma_2}\hat{\mathscr{L}}_\Delta := r_{\Gamma_2}T_\Delta^{\pm}W_\Delta : \widetilde{H}^{\frac{1}{2}}(\Gamma_2) \to H^{-\frac{1}{2}}(\Gamma_2) \tag{27}$$

is $\widetilde{H}^{\frac{1}{2}}(\Gamma_2)$-elliptic. Operator (27) and the operator

$$r_{\Gamma_2}\hat{\mathscr{L}} : \widetilde{H}^{\frac{1}{2}}(\Gamma_2) \to H^{-\frac{1}{2}}(\Gamma_2) \tag{28}$$

are continuously invertible.

Proof The ellipticity of operator (27) follows from inequality (6.39) in [12]. The continuity of this operator and the Lax–Milgram lemma then imply its invertibility. Together with relation (21) this implies the invertibility of operator (28).

The following result is proved in [6, Lemma 2].

Lemma 2.11

(i) Let either $\Psi^* \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\mathrm{diam}(\Omega) < 1$ *or $\Psi^* \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. If $V\Psi^*(y) = 0$ in Ω, then $\Psi^* = 0$ on $\partial\Omega$.*
(ii) Let $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$. If $W\Phi^*(y) = 0$ in Ω, then $\Phi^* = 0$ on $\partial\Omega$.

Lemma 2.12 *Let $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_1 and Γ_2 are non-empty non-intersecting parts of the boundary curve $\partial\Omega$. Let $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\Gamma_2)$ and either $\Psi^* \in \widetilde{H}_{**}^{-\frac{1}{2}}(\Gamma_1)$ or $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\Gamma_1)$ but* $\mathrm{diam}(\Gamma_1) < 1$. *If*

$$V\Psi^*(y) - W\Phi^*(y) = 0, \qquad y \in \Omega, \tag{29}$$

then $\Psi^* = 0$ and $\Phi^* = 0$.

Proof The proof follows from Theorems 2.8 (i) and 2.10 similar to [2, Lemma 4.2(iii)]. □

3 BDIEs for Mixed BVP

We shall use the following notations for product spaces:

$$\mathbb{X}^0 := H^{1,0}(\Omega; A) \times \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D) \times \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N),$$

$$\mathbb{Y}^{11,0} := H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega_D) \times H^{-\frac{1}{2}}(\partial\Omega_N),$$

$$\mathbb{Y}^{22,0} := H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega_D) \times H^{\frac{1}{2}}(\partial\Omega_N),$$

$$\mathbb{Y}^{12,0} := H^{1,0}(\Omega; A) \times H^{\frac{1}{2}}(\partial\Omega),$$

$$\mathbb{Y}^{21,0} := H^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega).$$

Let further in this section $u \in H^{1,0}(\Omega; A)$ be a solution of BVP (6)–(8) with $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ and $f \in L_2(\Omega)$.

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$ from $\partial\Omega_D$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$ from $\partial\Omega_N$ to $\partial\Omega$, respectively. Similar to [2], to reduce BVP (6)–(8) to one or another BDIE system, we shall use Eq. (22) in Ω, and restrictions of Eqs. (23) or (24) to appropriate parts of the boundary. We shall substitute f for Au, $\Phi_0 + \varphi$ for $\gamma^+ u$ and $\Psi_0 + \psi$ for $T^+ u$, where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ are considered as known, while ψ belongs to $\widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and φ to $\widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ due to the boundary conditions (7)–(8) and are to be found along with $u \in H^{1,0}(\Omega; A)$. This will lead us to four different *segregated* BDIE systems with respect to the unknown triplet $[u, \psi, \varphi]^\top =: \mathscr{U} \in \mathbb{X}^0 \subset \mathbb{X}$.

BDIE system (M11) is obtained from Eq. (22) in Ω, the restriction of Eq. (23) on $\partial\Omega_D$ and the restriction of Eq. (24) on $\partial\Omega_N$. Then we arrive at the following segregated system of BDIEs:

$$u + \mathscr{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega, \tag{30}$$

$$\gamma^+\mathscr{R}u - \mathscr{V}\psi + \mathscr{W}\varphi = \gamma^+ F_0 - \varphi_0 \quad \text{on} \quad \partial\Omega_D, \tag{31}$$

$$T^+\mathscr{R}u - \mathscr{W}'\psi + T^+W\varphi = T^+F_0 - \psi_0 \quad \text{on} \quad \partial\Omega_N, \tag{32}$$

where $F_0 := \mathscr{P}f + V\Psi_0 - W\Phi_0$.

System (30)–(32) can be written in the form $\mathscr{M}^{11}\mathscr{U} = \mathscr{F}^{11}$, where

$$\mathscr{M}^{11} := \begin{bmatrix} I + \mathscr{R} & -V & W \\ r_{\partial\Omega_D}\gamma^+\mathscr{R} & -r_{\partial\Omega_D}\mathscr{V} & r_{\partial\Omega_D}\mathscr{W} \\ r_{\partial\Omega_N}T^+\mathscr{R} & -r_{\partial\Omega_N}\mathscr{W}' & r_{\partial\Omega_N}T^+W \end{bmatrix}, \quad \mathscr{F}^{11} := \begin{bmatrix} F_0 \\ r_{\partial\Omega_D}\gamma^+F_0 - \varphi_0 \\ r_{\partial\Omega_N}T^+F_0 - \psi_0 \end{bmatrix}.$$

Due to the mapping properties of participating operators, $\mathscr{F}^{11} \in \mathbb{Y}^{11,0}$ and the operator $\mathscr{M}^{11} : \mathbb{X}^0 \to \mathbb{Y}^{11,0}$ is bounded.

Remark 3.1 $\mathscr{F}^{11} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Proof The proof follows in the similar way as in the corresponding proof in 3D case in [2, Remark 5.1]. □

BDIE system (M12), obtained using Eq. (22) in Ω and Eq. (23) on the whole boundary $\partial\Omega$, is:

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega, \tag{33}$$

$$\frac{1}{2}\varphi + \gamma^+\mathcal{R}u - \mathcal{V}\psi + \mathcal{W}\varphi = \gamma^+ F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega\,. \tag{34}$$

System (33)–(34) can be written in the form $\mathcal{M}^{12}\mathcal{U} = \mathcal{F}^{12}$, where

$$\mathcal{M}^{12} := \begin{bmatrix} I+\mathcal{R} & -V & W \\ \gamma^+\mathcal{R} & -\mathcal{V} & \frac{1}{2}I + \mathcal{W} \end{bmatrix}, \quad \mathcal{F}^{12} := \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \Phi_0 \end{bmatrix}.$$

Note that $\mathcal{F}^{12}$belongs to $\mathbb{Y}^{12,0}$ and due to the mapping properties of operators involved in $\mathcal{M}^{12}$, the operator $\mathcal{M}^{12} : \mathbb{X}^0 \to \mathbb{Y}^{12,0}$ is bounded.

Remark 3.2 Let $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ or $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ but $\operatorname{diam}(\Omega) < 1$. Then $\mathcal{F}^{12} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Proof Indeed, the latter equality evidently implies the former. Conversely, let $\mathcal{F}^{12} = (F_0, \gamma^+ F_0 - \Phi_0) = 0$. This implies $-V\Psi_0 + W\Phi_0 = \mathcal{P}f$ in Ω. Due to Lemma 2.3, $f = 0$ and $V\Psi_0 - W\Phi_0 = 0$ in Ω. The equality $\gamma^+ F_0 - \Phi_0 = 0$ implies $\Phi_0 = 0$ on $\partial\Omega$. Thus $V\Psi_0 = 0$, hence by Theorem 2.4 it follows $\Psi_0 = 0$. □

BDIE system (M21) is another system obtained using Eq. (22) in Ω and Eq. (24) on $\partial\Omega$, i.e.

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega, \tag{35}$$

$$\frac{1}{2}\psi + T^+\mathcal{R}u - \mathcal{W}'\psi + T^+W\varphi = T^+F_0 - \Psi_0 \quad \text{on} \quad \partial\Omega. \tag{36}$$

System (35)–(36) can be written in the form $\mathcal{M}^{21}\mathcal{U} = \mathcal{F}^{21}$, where

$$\mathcal{M}^{21} := \begin{bmatrix} I+\mathcal{R} & -V & W \\ T^+\mathcal{R} & \frac{1}{2}I - \mathcal{W}' & T^+W \end{bmatrix}, \quad \mathcal{F}^{21} := \begin{bmatrix} F_0 \\ T^+F_0 - \Psi_0 \end{bmatrix}.$$

Note that $\mathcal{F}^{21}$ belongs to $\mathbb{Y}^{21,0}$ and due to the mapping properties of operators involved in $\mathcal{M}^{21}$, the operator $\mathcal{M}^{21} : \mathbb{X}^0 \to \mathbb{Y}^{21,0}$ is bounded.

Remark 3.3 $\mathcal{F}^{21} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Proof The proof follows in the similar way as in Remark 3.2.

BDIE system (M22), a system of almost second kind (up to the spaces) obtained using Eq. (22) in Ω, the restriction of Eq. (24) to $\partial\Omega_D$ and the restriction of Eq. (23)

to $\partial\Omega_N$ is:

$$u + \mathscr{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega, \tag{37}$$

$$\frac{1}{2}\psi + T^+\mathscr{R}u - \mathscr{W}'\psi + T^+W\varphi = T_a^+F_0 - \Psi_0 \quad \text{on} \quad \partial\Omega_D, \tag{38}$$

$$\frac{1}{2}\varphi + \gamma^+\mathscr{R}_b u - \mathscr{V}_a\psi + \mathscr{W}_a\varphi = F_0^+ - \Phi_0 \quad \text{on} \quad \partial\Omega_N. \tag{39}$$

System (37)–(39) can be rewritten in the form $\mathscr{M}^{22}\mathscr{U} = \mathscr{F}^{22}$, where

$$\mathscr{M}^{22} := \begin{bmatrix} I + \mathscr{R} & -V & W \\ r_{\partial\Omega_D} T^+\mathscr{R} & r_{\partial\Omega_D}(\frac{1}{2}I - \mathscr{W}') & r_{\partial\Omega_D} T^+W \\ r_{\partial\Omega_N}\gamma^+\mathscr{R} & -r_{\partial\Omega_N}\mathscr{V} & r_{\partial\Omega_N}(\frac{1}{2}I + \mathscr{W}) \end{bmatrix}, \quad \mathscr{F}^{22} := \begin{bmatrix} F_0 \\ r_{\partial\Omega_D}\{T^+F_0 - \Psi_0\} \\ r_{\partial\Omega_N}\{\gamma^+F_0 - \Phi_0\} \end{bmatrix}.$$

Note that $\mathscr{F}^{22}$ belongs to $\mathbb{Y}^{22,0}$ and due to the mapping properties of operators involved in $\mathscr{M}^{22}$, the operator $\mathscr{M}^{22} : \mathbb{X}^0 \to \mathbb{Y}^{22,0}$ is bounded.

Remark 3.4 $\mathscr{F}^{22} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

Proof The proof follows in the similar way as in the corresponding proof in 3D case in [2, Remark 5.11]. □

4 Equivalence

In what follows, we shall prove the equivalence of the mixed BVP (6)–(8) to BDIE systems (M11), (M12), (M21) and (M22).

Theorem 4.1 *Let* $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega_D)$, $\psi_0 \in H^{-\frac{1}{2}}(\partial\Omega_N)$, $f \in L_2(\Omega)$ *and let* $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ *and* $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ *be some extensions of* φ_0 *and* ψ_0*, respectively.*

(i) If some $u \in H^{1,0}(\Omega; A)$ *solves the mixed BVP (6)–(8) in* Ω*, then the solution is unique and the triplet* $(u, \psi, \varphi)^T \in \mathbb{X}^0$*, where*

$$\psi = T^+u - \Psi_0, \quad \varphi = \gamma^+u - \Phi_0, \quad \text{on } \partial\Omega \tag{40}$$

solves the BDIE systems (M11), (M12), (M21) and (M22).

(ii) If $\text{diam}(\Omega) < 1$ *and a triplet* $(u, \psi, \varphi)^T \in \mathbb{X}^0$ *solves one of the BDIE systems (M11) or (M21) or (M12) or (M22), then this solution is unique and solves all the BDIE systems, while* u *solves BVP (6)–(8) and relations (40) hold.*

Proof

(i) Let $u \in H^{1,0}(\Omega; A)$ be a solution to BVP (6)–(8). Due to Theorem 1.4 it is unique. Set $\psi := T^+u - \Psi_0$ and $\varphi := \gamma^+u - \Phi_0$. Then $\psi \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$, $\varphi \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ and recalling how BDIE systems (M11), (M12), (M21) and (M22) were constructed, we obtain that the triplet $(u, \psi, \varphi)^T$ solves systems (M11), (M12), (M21) and (M22).

(ii) Let a triplet $(u, \psi, \varphi)^T \in \mathbb{X}^0$ solve BDIE system (M11) or (M12) or (M21) or (M22). The hypotheses of Lemma 2.3 are satisfied for the first equation in BDIE system, implying that u solves PDE (6) in Ω, while the following equation holds:

$$V\Psi^* - W\Phi^* = 0 \quad \text{in} \quad \Omega, \tag{41}$$

where $\Psi^* = \Psi_0 + \psi - T^+u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+u$.

Suppose first that the triplet $(u, \psi, \varphi)^T \in \mathbb{X}^0$ solves BDIE system (M11). Taking trace of Eq. (30) on $\partial\Omega_D$ using the jump relations (17)–(18), and subtracting Eq. (31) from it, we obtain

$$\gamma^+u = \varphi_0 \quad \text{on} \quad \partial\Omega_D, \tag{42}$$

i.e., u satisfies the Dirichlet condition (7). Taking the co-normal derivative of Eq. (30) on $\partial\Omega_N$, using the jump relations (19)–(20) and subtracting Eq. (32) from it, we obtain

$$T^+u = \psi_0 \quad \text{on} \quad \partial\Omega_N, \tag{43}$$

i.e., u satisfies the Neumann condition (8). Hence u solves the mixed BVP (6)–(8).

Taking into account $\varphi = 0$, $\Phi_0 = \varphi_0$ on $\partial\Omega_D$ and $\psi = 0$, $\Psi_0 = \psi_0$ on $\partial\Omega_N$, Eqs. (42) and (43) imply that the first equation in (40) is satisfied on $\partial\Omega_N$ and the second equation in (40) is satisfied on $\partial\Omega_D$. Thus we have $\Psi^* \in \widetilde{H}^{-\frac{1}{2}}(\partial\Omega_D)$ and $\Phi^* \in \widetilde{H}^{\frac{1}{2}}(\partial\Omega_N)$ in (41). Let $\Gamma_1 = \partial\Omega_D$, $\Gamma_2 = \partial\Omega_N$. Then $\text{diam}(\Gamma_1) \leq \text{diam}(\Omega) < 1$ and Lemma 2.12 implies $\Psi^* = \Phi^* = 0$, which completes the proof of conditions in (40). Uniqueness of the solution to BDIE systems (M11) follows from (40) along with Remark 3.1 and Theorem 1.4.

Finally, item (i) implies that triplet $(u, \psi, \varphi)^T \in \mathbb{X}^0$ solves also BDIE systems (M12), (M21) and (M22).

Similar arguments work if we suppose that instead of the BDIE systems (M11), the triplet $(u, \psi, \varphi)^T \in \mathbb{X}^0$ solves BDIE systems (M21) or (M12) or (M22).

□

5 Conclusion

In this paper, we considered the mixed BVP problem for variable-coefficient PDE in a two-dimensional bounded domain, where the right-hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial\Omega_D)$ and the Neumann data from the space $H^{-\frac{1}{2}}(\partial\Omega_N)$. The BVP was reduced to four systems of boundary-domain integral equations and their equivalence to the original BVP was shown. The invertibility of the associated operators in the corresponding Sobolev spaces can also be proved. In a similar way one can consider also the 2D versions of the BDIEs for mixed problem in exterior domains, united BDIEs as well as the localised BDIEs, which were analysed for 3D case in [2–4, 9].

Acknowledgements The work on this paper of the first author was supported by the Humboldt Foundation grant 3.4–ETH/1144171. He would like also to thank the Department of Mathematics at TU Kaiserslautern for hosting his research stay, the Department of Mathematics at Addis Ababa University for granting leave of absence and ISP (Sweden) for partial support. The work of the third author was supported by the grants EP/H020497/1 and EP/M013545/1, from the EPSRC, UK.

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Boundary-Domain Integral Equations for Variable Coefficient Dirichlet BVP in 2D Unbounded Domain

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Abstract In this paper, the Dirichlet boundary value problem for the second order stationary diffusion elliptic partial differential equation with variable coefficient is considered in unbounded (exterior) two-dimensional domain. Using an appropriate parametrix (Levi function), this problem is reduced to some direct segregated boundary-domain integral equations (BDIEs). We investigate the properties of corresponding parametrix-based integral volume and layer potentials in some weighted Sobolev spaces, as well as the unique solvability of BDIEs and their equivalence to the original BVP.

1 Basic Notations and Function Spaces

Let $\Omega = \Omega^+$ be an unbounded open domain in $\mathbb{R}^2$ such that the complement $\Omega^- := \mathbb{R}^2 \setminus \overline{\Omega}$ is bounded open domain. Let the boundary $\partial\Omega = \partial\Omega^-$ be closed and infinitely smooth curve. The space of infinitely differentiable functions having compact support in Ω is denoted by $\mathscr{D}(\Omega)$ and its dual space, the space of distributions, by $\mathscr{D}'(\Omega)$, while $\mathscr{D}(\bar{\Omega})$ is the set of restrictions on $\bar{\Omega}$ of functions from $\mathscr{D}(\mathbb{R}^2)$. The spaces $H^s(\Omega)$, $H^s(\partial\Omega)$ denote the Sobolev (Bessel potential) spaces.

We shall consider the following second order partial differential equation, with variable coefficient

$$Au(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(a(x)\frac{\partial u(x)}{\partial x_i}\right) = f(x) \quad x \in \Omega, \tag{1}$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_46

where u is unknown function; $f(x)$ and $a(x) > a_0 > 0$ are given functions in Ω.

We will further use the *weighted Sobolev spaces*. Let

$$\rho_2(x) := (1 + |x|^2)^{1/2} \ln(2 + |x|^2). \tag{2}$$

For any real β, we denote by $L_2(\rho_2^\beta; \Omega)$ the weighted Lebesgue space (see, e.g., [7]) consisting of all measurable functions $g(x)$ on Ω such that $g\rho_2^\beta \in L_2(\Omega)$, i.e.,

$$\|g\|^2_{L_2(\rho_2^\beta;\Omega)} = \int_\Omega \left| g(x)\rho_2^\beta(x) \right|^2 dx < \infty.$$

The space $L_2(\rho_2^\beta; \Omega)$, equipped with the norm $\|\cdot\|_{L_2(\rho_2^\beta;\Omega)}$ and appropriate inner product, is a Hilbert space.

The weighted Sobolev space $\mathscr{H}^1(\Omega)$ is defined by

$$\mathscr{H}^1(\Omega) := \left\{ g \in L_2(\rho_2^{-1}; \Omega) : \nabla g \in L_2(\Omega) \right\}, \tag{3}$$

and for its norm we have $\|g\|^2_{\mathscr{H}^1(\Omega)} := \|g\|^2_{L_2(\rho_2^{-1};\Omega)} + \|\nabla g\|^2_{L_2(\Omega)}$, while $|g|^2_{\mathscr{H}^1(\Omega)} := \sum_{i=1}^2 \int_\Omega |\frac{\partial g}{\partial x_i}|^2 dx = \|\nabla g\|^2_{L_2(\Omega)}$ is the square of the semi-norm. The space $\mathscr{D}(\mathbb{R}^2)$ is dense in $\mathscr{H}^1(\mathbb{R}^2)$, see, e.g., [1, Theorem 7.2]. This implies that the dual space of $\mathscr{H}^1(\mathbb{R}^2)$, denoted by $\mathscr{H}^{-1}(\mathbb{R}^2)$, is a space of distributions. Using the corresponding property for the space $H^1(\Omega)$, one can prove that $\mathscr{D}(\bar{\Omega})$ is dense in $\mathscr{H}^1(\Omega)$. The trace operator γ^+ on $\partial\Omega$ defined on functions from $\mathscr{H}^1(\Omega)$ satisfies the usual trace theorems. This allows to define in particular the subspace

$$\mathscr{H}_0^1(\Omega) = \left\{ g \in \mathscr{H}^1(\Omega) : \gamma^+ g = 0 \right\}.$$

It can be proved that $\mathscr{D}(\Omega)$ is dense in $\mathscr{H}_0^1(\Omega)$ and therefore its dual space is a space of distributions. Let us denote by $\widetilde{\mathscr{H}}^1(\Omega)$ a completion of $\mathscr{D}(\Omega)$ in $\mathscr{H}^1(\mathbb{R}^2)$, and $\widetilde{\mathscr{H}}^{-1}(\Omega) := [\mathscr{H}^1(\Omega)]'$, $\mathscr{H}^{-1}(\Omega) := [\widetilde{\mathscr{H}}^1(\Omega)]'$ are the corresponding dual spaces. The inclusion $L_2(\rho_2; \Omega) \subset \mathscr{H}^{-1}(\Omega)$ holds and a distribution f in the dual space $\widetilde{\mathscr{H}}^{-1}(\Omega)$ has the form $f = \sum_{i=1}^2 \frac{\partial g_i}{\partial x_i} + f_0$, where $g_i \in L_2(\mathbb{R}^2)$ and is zero outside Ω, $f_0 \in L_2(\rho_2; \Omega)$, cf., e.g., [12, Eq. (2.5.129)]. This implies that $\mathscr{D}(\Omega)$ is dense in $\widetilde{\mathscr{H}}^{-1}(\Omega)$ and $\mathscr{D}(\mathbb{R}^2)$ is dense in $\mathscr{H}^{-1}(\mathbb{R}^2)$.

From Definition (3) we obtain the following assertion.

Lemma 1 *The space $\mathscr{H}^1(\Omega)$ contains constant functions.*

Lemma 1 implies that the space of real constants, $\mathbb{R}$, is a closed subspace of $\mathscr{H}^1(\Omega)$. Thus we can define the quotient space $\mathscr{H}^1(\Omega)/\mathbb{R}$, which is a Banach space, and its norm is given by $\|u + \mathbb{R}\|_{\mathscr{H}^1(\Omega)/\mathbb{R}} = \inf_{c\in\mathbb{R}} \|u + c\|_{\mathscr{H}^1(\Omega)}$. The dual space $\left(\mathscr{H}^1(\Omega)/\mathbb{R}\right)'$ is identified with $\widetilde{\mathscr{H}}^{-1}(\Omega) \perp \mathbb{R}$, i.e., $\left(\mathscr{H}^1(\Omega)/\mathbb{R}\right)' = \widetilde{\mathscr{H}}^{-1}(\Omega) \perp$

$\mathbb{R}$ since they are isometrically isomorphic (see, e.g., [8, Lemma 2.12(ii)]). Similarly, $\big(\widetilde{\mathcal{H}}^1(\Omega)/\mathbb{R}\big)' = \mathcal{H}^{-1}(\Omega) \perp \mathbb{R}$.

The following Poincaré-type inequalities hold (cf. [2, Theorems 1.1 and 1.2]).

Theorem 1

(i) The semi-norm $|\cdot|_{\mathcal{H}^1(\Omega)}$ defined on $\mathcal{H}^1(\Omega)/\mathbb{R}$ is a norm equivalent to the quotient norm, i.e., there exist positive constants c_1, C_1 such that

$$c_1|v|_{\mathcal{H}^1(\Omega)} \le \|v\|_{\mathcal{H}^1(\Omega)/\mathbb{R}} \le C_1|v|_{\mathcal{H}^1(\Omega)}.$$

(ii) Moreover, the semi-norm $|\cdot|_{\mathcal{H}^1(\Omega)}$ is a norm on $\mathcal{H}^1_0(\Omega)$, equivalent to the norm $\|\cdot\|_{\mathcal{H}^1(\Omega)}$, i.e., there exist positive constants c_2, C_2 such that

$$c_2|v|_{\mathcal{H}^1(\Omega)} \le \|v\|_{\mathcal{H}^1_0(\Omega)} \le C_2|v|_{\mathcal{H}^1(\Omega)}.$$

For $u \in \mathcal{H}^1(\Omega)$ and the coefficient $a(x) \in L_\infty(\Omega)$, PDE (1) is well defined in the distributional sense as $\langle Au, v\rangle_\Omega := -\langle a\nabla u, \nabla v\rangle_\Omega = -\mathcal{E}(u, v)$, for any $v \in \mathcal{D}(\Omega)$, where $\mathcal{E}(u, v) := \int_\Omega E(u, v)(x)dx, \quad E(u, v)(x) := \nabla v(x) \cdot a(x)\nabla u(x)$. Unless stated otherwise we henceforth assume that there are some constants a_0, a_1 such that

$$a \in L_\infty(\mathbb{R}^2) \text{ and } 0 < a_0 < a(x) < a_1 < \infty \quad \text{for a.e } x \in \mathbb{R}^2. \tag{4}$$

To obtain boundary-domain integral equations, we will also always consider the coefficient a such that

$$a \in C^1(\mathbb{R}^2) \text{ and } \rho_2\nabla a \in L_\infty(\mathbb{R}^2). \tag{5}$$

If $u \in H^1(\Omega)$, then $u \in \mathcal{H}^1(\Omega)$, from the trace theorem it follows that, $\gamma^+u \in H^{\frac{1}{2}}(\partial\Omega)$, where $\gamma^+ = \gamma^+_{\partial\Omega}$ is the trace operator on $\partial\Omega$ from the exterior domain Ω^+.

For the operator A, similar to [4] for the three-dimensional case, we introduce the space, $\mathcal{H}^{1,0}(\Omega; A) := \{g \in \mathcal{H}^1(\Omega) : Ag \in L_2(\rho_2; \Omega)\}$, where the norm is given by its square, $\|g\|^2_{\mathcal{H}^{1,0}(\Omega;A)} := \|g\|^2_{\mathcal{H}^1(\Omega)} + \|Ag\|^2_{L_2(\rho_2;\Omega)}$. For $u \in \mathcal{H}^{1,0}(\Omega; A)$, as in the 3D case [4], we define the canonical co-normal derivative $T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ similar to, for example, in [5, Lemma 3.2] and [8, Lemma 4.3] as

$$\langle T^+u, \omega\rangle_{\partial\Omega} := \int_\Omega [(\gamma^+_{-1}\omega)Au + E(u, \gamma^+_{-1}\omega)]dx \qquad \forall\omega \in H^{\frac{1}{2}}(\partial\Omega), \tag{6}$$

where $\gamma^+_{-1} : H^{\frac{1}{2}}(\partial\Omega) \to \mathcal{H}^1(\Omega)$ is a bounded right inverse to the trace operator $\gamma^+ : \mathcal{H}^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$, and $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ which extends the usual $L_2(\partial\Omega)$ scalar product.

The operator $T^+ : \mathscr{H}^{1,0}(\Omega; A) \to H^{-\frac{1}{2}}(\partial\Omega)$ is continuous and gives the continuous extension to $\mathscr{H}^{1,0}(\Omega; A)$ of the classical co-normal derivative operator $a\frac{\partial}{\partial n}$, where $\frac{\partial}{\partial n} = \gamma^+\nabla \cdot n$ and $n = n^+$ is normal vector on $\partial\Omega$ *directed outward the exterior domain* Ω.

Similar to the proofs available in [5, Lemma 3.4] (see also [10] for the spaces $H^{s,t}(\Omega; A)$), one can prove that for $u \in \mathscr{H}^{1,0}(\Omega; A)$ the first Green identity

$$\langle T^+u, \gamma^+v\rangle_{\partial\Omega} = \int_\Omega [vAu + E(u, v)]dx \qquad \forall v \in \mathscr{H}^1(\Omega) \tag{7}$$

holds true. Then, for any functions $u, v \in \mathscr{H}^{1,0}(\Omega; A)$ we have the second Green identity,

$$\int_\Omega [vAu - uAv]dx = \langle T^+u, \gamma^+v\rangle_{\partial\Omega} - \langle T^+v, \gamma^+u\rangle_{\partial\Omega}. \tag{8}$$

Remark 1 If a satisfies condition (4) and the second condition in (5), then $\|ga\|_{\mathscr{H}^1(\Omega)} \le C_1\|g\|_{\mathscr{H}^1(\Omega)}$, $\|g\frac{1}{a}\|_{\mathscr{H}^1(\Omega)} \le C_2\|g\|_{\mathscr{H}^1(\Omega)}$, where the constant C_1 and C_2 are independent of $g \in \mathscr{H}^1(\Omega)$, this means, a and $1/a$ are multipliers in the space $\mathscr{H}^1(\Omega)$.

Let us introduce the following subspaces:

$$L_2(\rho_2; \Omega) \perp \mathbb{R} := \{f \in L_2(\rho_2; \Omega) : \langle f, 1\rangle_\Omega = 0\}$$

$$\mathscr{H}^{1,0\perp}(\Omega; A) := \{g \in \mathscr{H}^1(\Omega) : Ag \in L_2(\rho_2; \Omega) \perp \mathbb{R}\},$$

$$H_*^{-\frac{1}{2}}(\partial\Omega) := \left\{\psi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle\psi, 1\rangle_{\partial\Omega} = 0\right\}.$$

Employing the first Green identity (7) with $v = 1$, we arrive at the following assertion.

Lemma 2 *If* $u \in \mathscr{H}^{1,0\perp}(\Omega; A)$, *then* $T^+u \in H_*^{-\frac{1}{2}}(\partial\Omega)$.

2 Dirichlet BVP in Exterior Domain

Given $f \in L_2(\rho_2; \Omega)$ *and* $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, *find a function* $u \in \mathscr{H}^{1,0}(\Omega; A)$ *such that:*

$$Au = f \quad \text{in } \Omega, \tag{9}$$

$$\gamma^+u = \varphi_0 \quad \text{on } \partial\Omega. \tag{10}$$

Let us denote by $\mathscr{A}_D = \left[A, \gamma^+\right]^T : \mathscr{H}^{1,0}(\Omega; A) \to L_2(\rho_2; \Omega) \times H^{\frac{1}{2}}(\partial\Omega)$, the left-hand side operator, which is evidently continuous. Similar to the proof in [4] for the three-dimensional case, one can prove the following assertion in the 2D case.

Theorem 2 *Under conditions (4), the Dirichlet problem (9)–(10) is uniquely solvable and its solution can be written as* $u = \mathscr{A}_D^{-1}(f, \varphi_0)^T$, *where the operator* $\mathscr{A}_D^{-1} : L_2(\rho_2; \Omega) \times H^{\frac{1}{2}}(\partial\Omega) \to \mathscr{H}^{1,0}(\Omega)$ *is continuous.*

3 Parametrix-Based Potentials in Exterior Domain

A function $P(x, y)$ is a parametrix (Levi function) for the operator A if $A_x P(x, y) = \delta(x - y) + R(x, y)$, where δ is the Dirac-delta distribution, while $R(x, y)$ is a remainder possessing at most a weak (integrable) singularity at $x = y$. In particular, see, e.g., [9] the function

$$P(x, y) = \frac{\ln|x - y|}{2\pi a(y)}, \qquad x, y \in \mathbb{R}^2, \tag{11}$$

is a parametrix for the operator A and the corresponding remainder is given by

$$R(x, y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi a(y)|x - y|^2} \frac{\partial a(x)}{\partial x_i}, \qquad x, y \in \mathbb{R}^2. \tag{12}$$

Let $u \in \mathscr{D}(\bar{\Omega})$. For any fixed $y \in \Omega$, let $B_\varepsilon(y)$ be an open ball centered at y with a sufficiently small radius $\varepsilon > 0$, and let $B_r(0)$ be an open ball centered at the origin with a radius r large enough to contain $\partial\Omega$ and the support of u, put $\Omega_\epsilon := (\Omega \cap B_r(0)) \setminus B_\varepsilon(y)$, we have $R(\cdot, y) \in L_2(\rho_2; \Omega_\varepsilon)$ and $P(\cdot, y) \in \mathscr{H}^{1,0}(\Omega_\varepsilon)$. Applying the second Green identity (8) in Ω_ε with $v = P(y, \cdot)$ and taking usual limits as $\epsilon \to 0$, cf. [11], we get the *third Green identity* in $\Omega_r := \Omega \cap B_r(0)$,

$$u + \mathscr{R}u - V(T^+u) + W(\gamma^+u) = \mathscr{P}Au \tag{13}$$

for $u \in \mathscr{D}(\bar{\Omega})$. Here,

$$\mathscr{P}g(y) := \int_\Omega P(x, y)g(x)dx, \quad \mathscr{R}g(y) := \int_\Omega R(x, y)g(x)dx, \quad y \in \mathbb{R}^2, \tag{14}$$

are, respectively, the parametrix-based Newtonian and remainder potentials, while

$$Vg(y) := -\int_{\partial\Omega} P(x, y)g(x)dS_x, \quad Wg(y) := -\int_{\partial\Omega} [T_x P(x, y)]g(x)dS_x, \; x \in \mathbb{R}^2 \backslash \partial\Omega, \tag{15}$$

are the parametrix-based single layer and double layer potentials. Deducing (13) we took into account that $u \equiv 0$ in $\Omega \setminus B_r(0) \subset \Omega \setminus \operatorname{supp} u$. Since no term in (13) depends on r if r is sufficiently large, we obtain that (13) is valid in the whole domain Ω for any $u \in \mathscr{D}(\bar{\Omega})$.

From definitions (11)–(12) and (14)–(15) one can obtain representations of the parametrix-based potential operators in terms of their counterparts for $a = 1$ (i.e., associated with the Laplace operator Δ), cf. [3, 4],

$$\mathscr{P}g = \frac{1}{a}\mathscr{P}_\Delta g,\ \mathscr{R}g = -\frac{1}{a}\sum_{j=1}^{2}\partial_j[\mathscr{P}_\Delta(g\partial_j a)],\ Vg = \frac{1}{a}V_\Delta g,\ Wg = \frac{1}{a}W_\Delta(ag). \tag{16}$$

The Newtonian and the remainder potential operators given by (14) for $\Omega = \mathbb{R}^2$ will be denoted as $\mathbf{P}$ and $\mathbf{R}$, respectively, and the relations similar to (16) hold for them as well.

In addition to conditions (4) and (5) on the coefficient a, we will sometimes also need the condition

$$\rho_2^2 \Delta a \in L_\infty(\mathbb{R}^2). \tag{17}$$

Employing that the corresponding mapping properties hold true for the potentials associated with the Laplace operator Δ, cf., e.g., Section 8 in [13] and references therein, relations (16) lead to the following assertion.

Theorem 3 *The following operators are continuous under conditions (5).*

$$\boldsymbol{P} : \mathscr{H}^{-1}(\mathbb{R}^2) \perp \mathbb{R} \to \mathscr{H}^1(\mathbb{R}^2), \tag{18}$$

$$\mathscr{P} : \widetilde{\mathscr{H}}^{-1}(\Omega) \perp \mathbb{R} \to \mathscr{H}^1(\mathbb{R}^2), \tag{19}$$

$$\boldsymbol{R} : L_2(w; \mathbb{R}^2) \to \mathscr{H}^1(\mathbb{R}^2), \tag{20}$$

$$V : H_*^{-\frac{1}{2}}(\partial\Omega) \to \mathscr{H}^1(\Omega), \tag{21}$$

$$W : H^{\frac{1}{2}}(\partial\Omega) \to \mathscr{H}^1(\Omega), \tag{22}$$

while the following operators are continuous under conditions (5) and (17).

$$\mathscr{P} : L_2(\rho_2; \Omega) \perp \mathbb{R} \to \mathscr{H}^{1,0}(\mathbb{R}^2; A), \tag{23}$$

$$\mathscr{R} : \mathscr{H}^1(\Omega) \to \mathscr{H}^{1,0}(\Omega; A), \tag{24}$$

$$V : H_*^{-\frac{1}{2}}(\partial\Omega) \to \mathscr{H}^{1,0}(\Omega; A), \tag{25}$$

$$W : H^{\frac{1}{2}}(\partial\Omega) \to \mathscr{H}^{1,0}(\Omega; A). \tag{26}$$

Similar to [10, Theorem 3.12] one can prove that $\mathscr{D}(\bar{\Omega})$ is dense in $\mathscr{H}^{1,0}(\Omega; A)$ and in $\mathscr{H}^{1,0\perp}(\Omega; A)$. Then Theorem 3 and Lemma 2 imply the following assertion.

Corollary 1 *The third Green identity* (13) *holds true for any* $u \in \mathscr{H}^{1,0\perp}(\Omega; A)$.

The boundary integral (pseudo-differential) operators of the direct values and of the co-normal derivatives of the single and double layer potentials are defined by

$$\mathscr{V}g(y) := -\int_{\Gamma} P(x,y)g(x)ds_x, \quad \mathscr{W}g(y) := -\int_{\Gamma} T_x P(x,y)g(x)ds_x \qquad y \in \Gamma,$$

$$\mathscr{W}'g(y) := -\int_{\Gamma} T_y P(x,y)g(x)ds_x \quad \mathscr{L}^{\pm}g(y) := T_y^{\pm} W g(y) \qquad y \in \Gamma.$$

Applying the trace and co-normal derivative operators to the third Green identity (13), and using the jump relations for the potential operators we obtain for $u \in \mathscr{H}^{1,0\perp}(\Omega; A)$,

$$\frac{1}{2}\gamma^+ u + \gamma^+ \mathscr{R}u - \mathscr{V}T^+ u + \mathscr{W}\gamma^+ u = \gamma^+ \mathscr{P}Au \quad \text{on } \partial\Omega, \tag{27}$$

$$\frac{1}{2}T^+ u + T^+ \mathscr{R}u - \mathscr{W}'T^+ u + \mathscr{L}^+\gamma^+ u = T^+ \mathscr{P}Au \quad \text{on } \partial\Omega. \tag{28}$$

Conditions (5) are assumed to hold for (27) and conditions (5) and (17) for (28).

For some functions f, Ψ, and Φ let us consider a more general indirect integral relation associated with Eq. (13).

$$u + \mathscr{R}u - V\Psi + W\Phi = \mathscr{P}f \quad \text{in } \Omega. \tag{29}$$

Lemma 3 *Let* $u \in \mathscr{H}^{1,0\perp}(\Omega; A)$, $f \in L_2(\rho_2; \Omega) \perp \mathbb{R}$, $\Psi \in H_*^{-\frac{1}{2}}(\partial\Omega)$, *and* $\Phi \in H^{\frac{1}{2}}(\partial\Omega)$ *satisfy Eq. (29) and let conditions* (5), (17) *hold. Then, u is a solution of the equation*

$$Au = f \quad \textit{in } \Omega, \tag{30}$$

while

$$V(\Psi - T^+ u) - W(\Phi - \gamma^+ u) = 0, \quad \textit{in } \Omega. \tag{31}$$

Proof Since $u \in \mathscr{H}^{1,0\perp}(\Omega; A)$, we can write the third Green identity (13) for the function u. Then subtracting (29) from it, we obtain

$$-V\Psi^* + W\Phi^* = \mathscr{P}[Au - f] \quad \text{in } \Omega, \tag{32}$$

where $\Psi^* := T^+u - \Psi$ and $\Phi^* := \gamma^+u - \Phi$. Multiplying equality (32) by $a(y)$ we get

$$-V_\Delta \Psi^* + W_\Delta(a\Phi^*) = \mathscr{P}_\Delta[Au - f] \quad \text{in } \Omega.$$

Applying the Laplace operator Δ to the last equation and taking into consideration that both functions in the left-hand side are harmonic potentials, while the right-hand side function is the classical Newtonian potential, we arrive at Eq. (30). Substituting (30) back into (32) leads to (31). □

Lemma 4 *Let conditions (5) and (17) hold.*

(i) If $\Psi^* \in H_*^{-\frac{1}{2}}(\partial\Omega)$ and $V\Psi^* = 0$ in Ω, then $\Psi^* = 0$.
(ii) If $\Phi^* \in H^{\frac{1}{2}}(\partial\Omega)$ and $W\Phi^*(y) = 0$ in Ω, then $\Phi^*(x) = C/a(x)$, where C is a constant.

Proof The proof of item (i) coincides with the proof of its counterpart for interior domains in [6], while the proof of item (ii) is similar to the proof for the 3D case in [4, Lemma 4.2]. □

4 BDIEs for Exterior Dirichlet BVP

To reduce the variable coefficient Dirichlet BVP (9)–(10) to a *segregated* boundary-domain integral equation systems, let us denote the unknown co-normal derivative as $\psi := T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$ and further consider ψ as formally independent of u.

For a given function f in $L_2(\rho_2;\Omega) \perp \mathbb{R}$, assume that the function u satisfies the PDE $Au = f$ in Ω. Then by substituting the Dirichlet condition into the third Green identity (13) and either into its trace (27) or into its co-normal derivative (28) on $\partial\Omega$, we can reduce the BVP (9)–(10) to two different systems of boundary-domain integral equations for the unknown functions $u \in \mathscr{H}^1(\Omega; A)$ and $\psi := T^+u \in H^{-\frac{1}{2}}(\partial\Omega)$.

BDIE system (D1) obtained under conditions (5) from the third Green's identity (13) and its trace equation (27) is

$$u + \mathscr{R}u - V\psi = F_0 \quad \text{in } \Omega,$$
$$\gamma^+\mathscr{R}u - \mathscr{V}\psi = \gamma^+F_0 - \varphi_0 \quad \text{on } \partial\Omega,$$

where

$$F_0 := \mathscr{P}f - W\varphi_0 \quad \text{in } \Omega. \tag{33}$$

The system can be written in a matrix form as $\mathfrak{D}^1\mathscr{U} = \mathscr{F}^1$, where

$$\mathscr{U} := [u, \psi]^t \in \mathscr{H}^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega),$$

and

$$\mathfrak{D}^1 := \begin{bmatrix} I + \mathscr{R} & -V \\ \gamma^+\mathscr{R} & -\mathscr{V} \end{bmatrix}, \quad \mathscr{F}^1 = \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \varphi_0 \end{bmatrix}. \tag{34}$$

From the mapping properties of W and $\mathscr{P}$ in Theorem 3, we get the inclusion $F_0 \in \mathscr{H}^{1,0}(\Omega; A)$, and the trace theorem implies $\gamma^+ F_0 \in H^{\frac{1}{2}}(\partial\Omega)$. Therefore, $\mathscr{F}^1 \in \mathscr{H}^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$.

BDIE system (D2) obtained under conditions (5) and (17) from the third Green's identity (13) and its co-normal derivative equation (28) is

$$u + \mathscr{R}u - V\psi = F_0 \quad \text{in } \Omega,$$
$$\frac{1}{2}\psi + T^+\mathscr{R}u - \mathscr{W}'\psi = T^+F_0 \quad \text{on } \partial\Omega,$$

where F_0 is given by (33). In a matrix form it can be written as $\mathfrak{D}^2\mathscr{U} = \mathscr{F}^2$, where

$$\mathfrak{D}^2 = \begin{bmatrix} I + \mathscr{R} & -V \\ T^+\mathscr{R} & \frac{1}{2}I - \mathscr{W}' \end{bmatrix}, \qquad \mathscr{F}^2 = \begin{bmatrix} F_0 \\ T^+F_0 \end{bmatrix}.$$

Note that the operator $\mathfrak{D}^2 : \mathscr{H}^{1,0}(\Omega; A) \times H_*^{-\frac{1}{2}}(\partial\Omega) \to \mathscr{H}^{1,0}(\Omega; A) \times H^{-\frac{1}{2}}(\partial\Omega)$ is bounded.

5 Equivalence and Uniqueness Theorems

Theorem 4 *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in L_2(\rho_2; \Omega) \perp \mathbb{R}$, and conditions (5) and (17) hold.*

(i) If some $u \in \mathscr{H}^{1,0\perp}(\Omega; A)$ solves the BVP (9)–(10), then the pair (u, ψ), where

$$\psi = T^+u \in H_*^{-\frac{1}{2}}(\partial\Omega), \tag{35}$$

solves BDIE systems (D1) and (D2).

(ii) If a pair $(u, \psi) \in \mathscr{H}^{1,0\perp}(\Omega; A) \times H_*^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D1), then u solves BDIE system (D2) and BVP (9)–(10), this solution is unique, and ψ satisfies (35).

Proof

(i) Setting $\psi := T^+u$ and recalling how BDIE system (D1) and (D2) were constructed, we obtain that the couple (u, ψ) solves them.

(ii) Let now a pair $(u, \psi) \in \mathscr{H}^{1,0\perp}(\Omega; A) \times H_*^{-\frac{1}{2}}(\partial\Omega)$ solves system (D1). Due to the first equation in the BDIE systems, the hypotheses of Lemma 3 are satisfied implying that u solves PDE (9) in Ω and

$$V(\psi - T^+u) - W(\varphi_0 - \gamma^+u) = 0 \quad \text{in } \Omega. \tag{36}$$

Taking the trace of the first equation in (D1) and subtracting the second equation from it, we get $\gamma^+u = \varphi_0$ on $\partial\Omega$. Thus, the Dirichlet boundary condition is satisfied, and using this in (36), we obtain $V(\psi - T^+u) = 0$ in Ω. Lemma 4 (i) then implies $\psi = T^+u$.

The uniqueness of the BDIE system follows from the fact that the corresponding homogeneous BDIE systems can be associated with the homogeneous Dirichlet problem, which has only the trivial solution. Then the previous paragraph implies that the homogeneous BDIE system also has only the trivial solutions. □

Theorem 5 *Let $\varphi_0 \in H^{\frac{1}{2}}(\partial\Omega)$, $f \in L_2(\rho_2; \Omega) \perp \mathbb{R}$, and conditions (5) and (17) hold.*

(i) Homogeneous BDIE system (D2) admits only one linearly independent solution $(u^0, \psi^0) \in \mathscr{H}^{1,0\perp}(\Omega; A) \times H_*^{-\frac{1}{2}}(\partial\Omega)$, where u^0 is the solution of the Dirichlet BVP

$$Au^0 = 0 \quad \textit{in } \Omega, \tag{37}$$

$$\gamma^+u^0 = \frac{1}{a(x)} \quad \textit{on } \partial\Omega, \tag{38}$$

while

$$\psi^0 = T^+u^0 \quad \textit{on } \partial\Omega. \tag{39}$$

(ii) The non-homogeneous BDIE system (D2) is solvable, and any of its solutions $(u, \psi) \in \mathscr{H}^{1,0\perp}(\Omega; A) \times H_*^{-\frac{1}{2}}(\partial\Omega)$ can be represented as

$$u = \tilde{u} + Cu^0 \quad \textit{in } \Omega, \tag{40}$$

where $\tilde{u}$ solves BVP (9)–(10) and C is a constant, while

$$\psi = T^+\tilde{u} + C\psi^0 \quad \textit{on } \partial\Omega. \tag{41}$$

Proof Problem (37)–(38) is uniquely solvable in $\mathscr{H}^{1,0\perp}(\Omega; A)$ by Theorem 2. Consequently, the third Green identity (13) is applicable to u^0, leading to

$$u^0 + \mathscr{R}u^0 - V\psi^0 = 0 \quad \text{in } \Omega. \tag{42}$$

Taking the co-normal derivative of (42) and substituting (39) again, we arrive at

$$\frac{1}{2}\psi^0 + T^+\mathscr{R}u^0 - \mathscr{W}'\psi^0 = 0 \quad \text{on } \partial\Omega. \tag{43}$$

Equations (42) and (43) mean that the pair (u^0, ψ^0) solves the homogeneous BDIE system (D2).

To prove item (ii), we first remark that the solvability of non-homogeneous system (D2) follows from the solvability of the BVP (9)–(10) in $\mathscr{H}^{1,0\perp}(\Omega; A)$ and the deduction of system (D2).

Let now a pair $(u, \psi)^T \in \mathscr{H}^{1,0\perp}(\Omega) \times H_*^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE system (D2). Due to the first equation in the BDIE systems, Lemma 3 implies that u solves PDE (9) in Ω and relation (36) holds. Taking the co-normal derivative of the first equation in (D2) on $\partial\Omega$ and subtract it from the second equation in (D2), we obtain $\psi = T^+u$ on $\partial\Omega$. Then inserting this in (36) gives $W(\varphi_0 - \gamma^+u) = 0$, in Ω, and Lemma 4(ii) implies

$$\gamma^+u = \varphi_0 + C/a(x) \quad \text{on } \partial\Omega, \tag{44}$$

where C is a constant. Thus, u satisfies the Dirichlet condition (44) instead of (10). Introducing the notation $\tilde{u}$ by (40) in (44) and taking into account (37)–(38) prove the claim of item (ii). □

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A Boundary-Domain Integral Equation Method for an Elliptic Cauchy Problem with Variable Coefficients

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Abstract We consider an integral based method for numerically solving the Cauchy problem for second-order elliptic equations in divergence form with spacewise dependent coefficients. The solution is represented as a boundary-domain integral, with unknown densities to be identified. The given Cauchy data is matched to obtain a system of boundary-domain integral equations from which the densities can be constructed. For the numerical approximation, an efficient Nyström scheme in combination with Tikhonov regularization is presented for the boundary-domain integral equations, together with some numerical investigations.

1 Introduction

The Cauchy problem for elliptic equations is a classical example of an ill-posed inverse problem; there are numerous results and methods presented in the literature for this problem, and it is not possible to give an adequate overview of them in this work (some references are in the introduction in [5]).

In [6], a regularizing method based on a single-layer approach is described for the stable numerical solution to the Cauchy problem for the Laplace equation for two and three dimensional regions. That method builds on ideas given in [4, 11]. We continue the work of [6] by considering equations with spacewise dependent coefficients and using boundary-domain integrals. Boundary-domain formulations occur in the literature but mainly for direct problems, see, for example, [1, 2, 7, 8, 12, 13, 15, 16] and subsequent works by those authors.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_47

Let $D_0 \subset \mathbb{R}^2$ be a bounded simply connected domain with boundary curve $\Gamma_0 \in C^2$, and let $D_{-1} \subset D_0$ have a simple closed boundary curve $\Gamma_{-1} \in C^2$ lying wholly within D_0, and define the solution domain $D = D_0 \setminus \overline{D}_{-1}$. Let $u \in H^1(D)$ be a solution of the second-order elliptic equation

$$Lu(x) = \operatorname{div}(\sigma(x) \operatorname{grad} u(x)) = 0, \quad x \in D, \tag{1}$$

which satisfies the Dirichlet boundary condition

$$u = f_1 \text{ on } \Gamma_0 \tag{2}$$

and the Neumann boundary condition

$$\sigma \frac{\partial u}{\partial \nu} = f_2 \text{ on } \Gamma_0 \tag{3}$$

with given functions f_1 and f_2, $\sigma \in C^\infty(\bar{D})$, $\sigma > 0$, and ν the outward unit normal to the boundary.

The linear inverse problem we study is: find the function u satisfying (1)–(3), in particular, reconstruct, in a stable way, the Cauchy data u and $\frac{\partial u}{\partial \nu}$ on the inner boundary curve Γ_{-1}.

2 Reduction to a Boundary-Domain Integral Equation

The fundamental solution for Eq. (1) is in general not explicitly known unless σ is a constant, there is thus no feasible way to reduce (1)–(3) to a boundary integral equation. However, for Eq. (1) there is an alternative function called the parametrix.

Definition 1 A function $P(x, y)$, $x, y \in D$, is called a parametrix (or Levi function) of a differential operator L provided that

$$L_x P(x, y) = \delta(x - y) + R(x, y),$$

where δ is the Dirac function and the remainder function R has a weak singularity for $x = y$.

For the operator in (1), a Levi function (the Levi function is not unique [14]) can be given as

$$P(x, y) = \frac{\ln |x - y|}{2\pi\sigma(y)}, \quad x, y \in D, \quad x \neq y$$

with the remainder function being

$$R(x, y) = \frac{(x - y) \cdot \operatorname{grad} \sigma(x)}{2\pi \sigma(y)|x - y|^2}, \quad x, y \in D, \quad x \neq y.$$

The function R has clearly a weak singularity when $x = y$ as required in the above definition.

We use an indirect integral equation approach, based on the potential representation

$$\begin{aligned} u(x) = &\int_D \psi(y) P(x, y)\, dy + \int_{\Gamma_{-1}} \psi_{-1}(y) P(x, y)\, ds(y) \\ &+ \int_{\Gamma_0} \psi_0(y) P(x, y)\, ds(y), \quad x \in D \end{aligned} \tag{4}$$

with unknown densities $\psi \in C(D)$, $\psi_{-1} \in C(\Gamma_{-1})$, and $\psi_0 \in C(\Gamma_0)$.

From the representation (4) and the definition of a Levi function, (1)–(3) can be reduced to the following boundary-domain integral equations:

$$\begin{cases} \psi(x) + \displaystyle\int_D \psi(y) R(x, y)\, dy + \int_{\Gamma_{-1}} \psi_{-1}(y) R(x, y)\, ds(y) \\ + \displaystyle\int_{\Gamma_0} \psi_0(y) R(x, y)\, ds(y) = 0, \quad x \in D, \\ \displaystyle\int_D \psi(y) P(x, y)\, dy + \int_{\Gamma_{-1}} \psi_{-1}(y) P(x, y)\, ds(y) \\ + \displaystyle\int_{\Gamma_0} \psi_0(y) P(x, y)\, ds(y) = f_1(x), \quad x \in \Gamma_0, \\ -\dfrac{1}{2}\psi_0(x) + \displaystyle\int_D \psi(y)\sigma(x) \frac{\partial P(x, y)}{\partial \nu(x)}\, dy + \int_{\Gamma_{-1}} \psi_{-1}(y)\sigma(x) \frac{\partial P(x, y)}{\partial \nu(x)}\, ds(y) \\ + \displaystyle\int_{\Gamma_0} \psi_0(y)\sigma(x) \frac{\partial P(x, y)}{\partial \nu(x)}\, ds(y) = f_2(x), \quad x \in \Gamma_0. \end{cases} \tag{5}$$

Assume that the boundary curves Γ_0 and Γ_{-1} are homothetic with factor ξ_{-1} and have the parametric representations

$$\begin{aligned} \Gamma_0 &= \{x(t) = (x_1(t), x_2(t)) : t \in [0, 2\pi)\}, \\ \Gamma_{-1} &= \{x_{-1}(t) = (\xi_{-1} x_1(t), \xi_{-1} x_2(t)) : t \in [0, 2\pi)\}. \end{aligned} \tag{6}$$

Here, $\xi_{-1} \in (0, 1)$ is a fixed parameter. It is further assumed that D contains the origin.

We make the following change of variables in the double integrals in (5),

$$y_1 = p_1(\xi, \tau) = \xi x_1(\tau) \quad \text{and} \quad y_2 = p_2(\xi, \tau) = \xi x_2(\tau),$$

where $(\xi, \tau) \in \Pi = (\xi_{-1}, 1) \times [0, 2\pi)$ with Jacobian $J(\xi, \tau) = \xi(x_1(\tau)x_2'(\tau) - x_2(\tau)x_1'(\tau))$. We use the notation $p = (p_1, p_2)$ for the function mapping into Π. The system (5) can then be written in the parametrized form

$$\begin{cases} \varphi(\eta, t) + \dfrac{1}{2\pi}\displaystyle\int_\Pi \varphi(\xi, \tau)\widetilde{R}(\eta, t; \xi, \tau)\, d\tau d\xi + \dfrac{1}{2\pi}\int_0^{2\pi} \varphi_{-1}(\xi_{-1}, \tau)\widetilde{R}_{-1}(\eta, t; \xi_{-1}, \tau)\, d\tau \\ +\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \varphi_0(\tau)\widetilde{R}_0(\eta, t; \tau)\, d\tau = 0,\ (\eta, t) \in \Pi, \\ \dfrac{1}{2\pi}\displaystyle\int_\Pi \varphi(\xi, \tau)\check{P}(t; \xi, \tau)\, d\tau d\xi + \dfrac{1}{2\pi}\int_0^{2\pi} \varphi_{-1}(\xi_{-1}, \tau)\check{P}_{-1}(t; \xi_{-1}, \tau)\, d\tau \\ +\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \varphi_0(\tau)\check{P}_0(t; \tau)\, d\tau = \widetilde{f}_1(t),\ t \in [0, 2\pi), \\ -\dfrac{1}{2}\varphi_0(t) + \dfrac{1}{2\pi}\displaystyle\int_\Pi \varphi(\xi, \tau)\widehat{P}(t; \xi, \tau)\, d\tau d\xi + \dfrac{1}{2\pi}\int_0^{2\pi} \varphi_{-1}(\xi_{-1}, \tau)\widehat{P}_{-1}(t; \xi_{-1}, \tau)\, d\tau \\ +\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \varphi_0(\tau)\widehat{P}_0(t; \tau)\, d\tau = \widetilde{f}_2(t),\ t \in [0, 2\pi). \end{cases} \tag{7}$$

Here, we introduced the functions $\varphi(\eta, t) = \psi(p(\eta, t))$, $\varphi_{-1}(\xi_{-1}, t) = \psi_{-1}(p(\xi_{-1}, t))$, $\varphi_{-1}(t) = \psi_{-1}(x(t))$, $\varphi_0(t) = \psi_0(x(t))$, $\widetilde{f}_1(t) = f_1(x(t))$, $\widetilde{f}_2(t) = f_2(x(t))$, and the kernels

$$\widetilde{R}(\eta, t; \xi, \tau) = 2\pi R(p(\eta, t), p(\xi, \tau))J(\xi, \tau),$$

$$\widetilde{R}_{-1}(\eta, t; \xi_{-1}, \tau) = 2\pi R(p(\eta, t), \xi_{-1}x(\tau))\xi_{-1}|x'(\tau)|,$$

$$\widetilde{R}_0(\eta, t; \tau) = 2\pi R(p(\eta, t), x(\tau))|x'(\tau)|;$$

$$\check{P}(t; \xi, \tau) = 2\pi P(x(t), p(\xi, \tau))J(\xi, \tau),$$

$$\check{P}_{-1}(t; \xi_{-1}, \tau) = 2\pi P(x(t), \xi_{-1}x(\tau))\xi_{-1}|x'(\tau)|,$$

$$\check{P}_0(t; \tau) = 2\pi P(x(t), x(\tau))|x'(\tau)|;$$

$$\widehat{P}(t;\xi,\tau) = 2\pi\sigma(x(t))\frac{\partial P(x(t),\xi x(\tau))}{\partial\nu(x(t))}J(\xi,\tau),$$

$$\widehat{P}_{-1}(t;\xi_{-1},\tau) = 2\pi\sigma(x(t))\frac{\partial P(x(t),\xi_{-1}x(\tau))}{\partial\nu(x(t))}\xi_{-1}|x'(\tau)|,$$

$$\widehat{P}_0(t;\tau) = 2\pi\sigma(x(t))\frac{\partial P(x(t),x(\tau))}{\partial\nu(x(t))}|x'(\tau)|.$$

The kernels $\widetilde{R}$ and $\check{P}_0$ have various singularities, and they are handled using ideas from [3, 9, 10]. For example, the strong singularity in $\widetilde{R}(\eta,t;\eta,\tau)$ is split as $I_1(t) + I_2(t)$ using a vector function representation involving the normal and tangential vectors to the boundary, rendering I_2 as a Cauchy type integral dealt with using [10]. The logarithmic singularity in $\check{P}_0$ is split as in [9].

3 Numerical Solution of the Boundary-Domain Integrals

We use the following interpolation quadrature rules:

$$\frac{1}{2\pi}\int_{\Pi} g(\xi,\tau)\,d\tau d\xi \approx \frac{1}{2n}\sum_{k=1}^{N}\sum_{i=0}^{2n-1}\alpha_k g(\eta_k,t_i), \tag{8}$$

$$\frac{1}{2\pi}\int_{\Pi} g(\xi,\tau)\cot\frac{\tau-t}{2}\,d\tau d\xi \approx \sum_{k=1}^{N}\sum_{i=0}^{2n-1}\alpha_k T_i(t)g(\eta_k,t_i), \tag{9}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2n}\sum_{k=0}^{2n-1} f(t_k), \tag{10}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau \approx \sum_{k=0}^{2n-1} F_k(t)\,f(t_k), \tag{11}$$

with quadrature weights $\alpha_k \in \mathbb{R}$, quadrature points $\eta_k \in (0,1)$, $k = 1,\ldots,N$, and weight functions as in [9]. The midpoint quadrature has weights $\alpha_k = \frac{1-\xi_{-1}}{N}$, $k = 1,\ldots,N$, with quadrature points $\eta_k = 1 - \frac{1-\xi_{-1}}{2N}(2k-1)$.

Using (8)–(11) in (7) together with collocating the approximating equations at the quadrature points leads to the linear system

$$\begin{cases} \varphi_{mi} + \sum_{k=1}^{N}\sum_{j=0}^{2n-1} \alpha_k \varphi_{kj} \bar{R}(\eta_m, t_i; \eta_k, t_j) + \frac{1}{2n}\sum_{j=0}^{2n-1} \big[\varphi_{-1j} \widetilde{R}_{-1}(\eta_m, t_i; \xi_{-1}, t_j) \\ + \varphi_{0j} \widetilde{R}_0(\eta_m, t_i; t_j)\big] = 0, \\ \frac{1}{2n}\sum_{k=1}^{N}\sum_{j=0}^{2n-1} \alpha_k \varphi_{kj} \check{P}(t_i; \eta_k, t_j) + \frac{1}{2n}\sum_{j=0}^{2n-1} \varphi_{-1j} \check{P}_{-1}(t_i; \xi_{-1}, t_j) \\ + \sum_{j=0}^{2n-1} \varphi_{0j} \left[\check{P}_0^{(1)}(t_i, t_j) F_j(t_i) + \frac{1}{2n}\check{P}_0^{(2)}(t_i; t_j)\right] = \widetilde{f}_{1i}, \\ -\frac{1}{2}\varphi_{0i} + \frac{1}{2n}\sum_{k=1}^{N}\sum_{j=0}^{2n-1} \alpha_k \varphi_{kj} \widehat{P}(t_i; \eta_k, t_j) + \frac{1}{2n}\sum_{j=0}^{2n-1} \big[\varphi_{-1j} \widehat{P}_{-1}(t_i; \xi_{-1}, t_j) \\ + \varphi_{0j} \widehat{P}_0(t_i, t_j)\big] = \widetilde{f}_{2i}, \end{cases} \tag{12}$$

with

$$\bar{R}(\eta_m, t_i; \eta_k, t_j) = \begin{cases} \frac{1}{2n}\widetilde{R}(\eta_m, t_i; \eta_k, t_j) & \text{for} \quad m \neq k, \\ \frac{1}{2n}\widetilde{R}^{(1)}(\eta_m, t_i; \eta_k, t_j) & \\ +T_j(t_i)\widetilde{R}^{(2)}(\eta_m, t_i; \eta_k, t_j) & \text{for} \quad m = k, \end{cases}$$

and the right-hand side $\widetilde{f}_{1i} = \widetilde{f}_1(t_i)$ and $\widetilde{f}_{2i} = \widetilde{f}_2(t_i)$, where $\varphi_{mi} \approx \varphi(\eta_m, t_i)$, $\varphi_{0i} \approx \varphi_0(t_i)$ and $\varphi_{-1i} \approx \varphi_{-1}(\xi_{-1}, t_i)$, with $t_i = \frac{i\pi}{n}$ for $m = 1, \ldots, N$, and $i = 0, \ldots, 2n-1$, and $\tilde{R}^{(1)}$ and $\tilde{R}^{(2)}$ are smooth functions.

The matrix of the linear system (12) has a large condition number due to the ill-posedness of the Cauchy problem (1)–(3). Hence, to obtain a stable smooth solution, regularization is necessary. We employ Tikhonov regularization, rendering the solution x_α of the regularized normal equations

$$(A^T A + \alpha I)x_\alpha = A^T b, \tag{13}$$

where A^T is the transpose of A, b is the right-hand side of (12), and $\alpha > 0$ a regularization parameter.

An approximation of the solution to (1)–(3) and to the Cauchy data on Γ_{-1} is obtained by applying the above discretization to the boundary-domain representation (4) with coefficients from (12) calculated using (13).

4 Numerical Examples

The doubly connected solution domain D is bounded by the two boundary curves

$$\Gamma_0 = \{x(t) = (\cos(t), \sin(t)) : \ t \in [0, 2\pi)\},$$

$$\Gamma_{-1} = \{x_{-1}(t) = (0.5\cos(t), 0.5\sin(t)) : \ t \in [0, 2\pi)\},$$

where we have put $\xi_{-1} = 0.5$ (see (6)). The functions σ, f_1, and f_2 are given as

$$\sigma(x) = 4 - x_1^2 + x_2^2 \quad x \in D, \quad \text{and} \quad f_1 = x_1 x_2, \ f_2 = 2x_1x_2(4 - x_1^2 + x_2^2) \quad \text{on } \Gamma_0.$$

A straightforward calculation shows that the function $u_{ex} = x_1x_2$ is an exact solution to (1)–(3).

In Table 1 are the errors obtained for the Cauchy data on the inner boundary curve Γ_{-1} for the numerical solution of the Cauchy problem (1)–(3) calculated with the outlined boundary-domain approach for exact data generated from u_{ex}. In the table, N is the number of internal quadrature curves, and n is the number of quadrature points on each such curve. For graphical illustration, in Fig. 1 is the exact solution on Γ_{-1} and the numerical reconstructions with exact data, for function values and the normal derivative. Moreover, in Fig. 2 are the corresponding reconstructions with 3% noisy data.

The calculations and regularization are straightforward to use. It is anticipated that if the reader implements the outlined method for domains of the similar type, accuracy of the same order in the reconstruction will be obtained. The calculations only take a few seconds on an ordinary desktop computer. Thus, it has been demonstrated that boundary-domain integral equations can be applied for the stable numerical solution of the Cauchy problem for elliptic equations.

Table 1 The error in the case of exact data for the regularizing parameter $\alpha = 10e - 11$

N	n	$\|\tilde{u} - u_{ex}\|_{\infty,\Gamma_{-1}}$	$\|\frac{\partial \tilde{u}}{\partial \nu} - \frac{\partial u_{ex}}{\partial \nu}\|_{\infty,\Gamma_{-1}}$
3	32	5.278E-004	1.747E-002
5	64	1.348E-004	4.457E-003
7	128	6.617E-006	3.323E-004

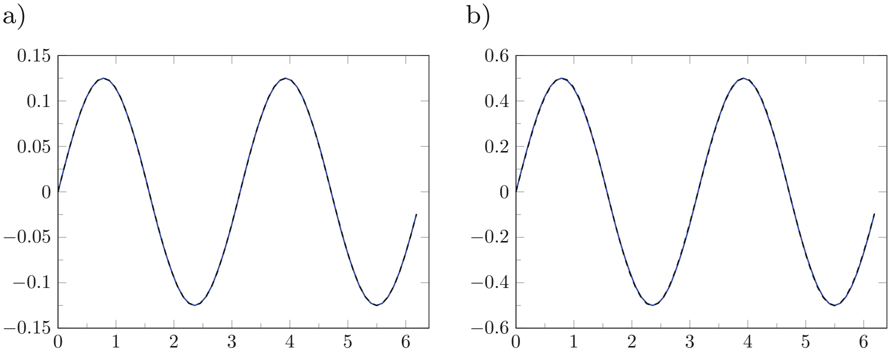

Fig. 1 The exact solution (*solid line*) and numerical reconstruction (*dashed line*) using the direct approach with Tikhonov regularization parameter $\alpha = 10e - 11$ $(N = 5,\ n = 64)$. (**a**) Boundary function on Γ_{-1}. (**b**) Normal derivative on Γ_{-1}

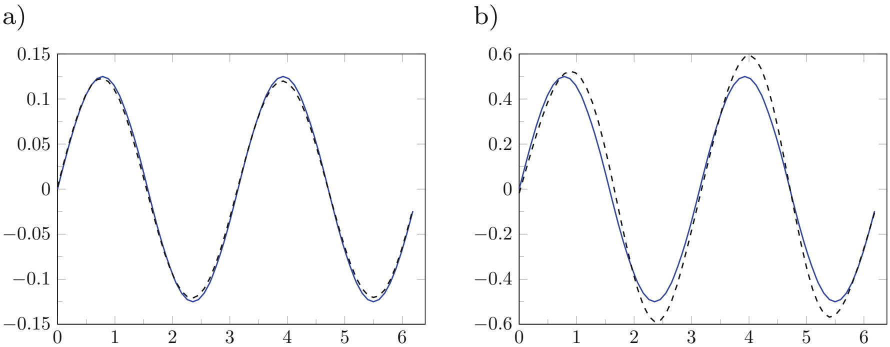

Fig. 2 The exact solution (*solid line*) and numerical reconstruction (*dashed line*) using the direct approach with Tikhonov regularization parameter $\alpha = 10e - 5$ in the case of 3% noisy data $(N = 7,\ n = 128)$. (**a**) Boundary function on Γ_{-1}. (**b**) Normal derivative on Γ_{-1}

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On Indirect Boundary Integral Equation Methods and Applications

Angelica Malaspina

Abstract In the classical indirect boundary integral equation method, the solution of the Dirichlet (Neumann) problem for Laplace equation is sought in the form of double (single) layer potential. The alternative method provides the choice of form for the solution of Dirichlet (Neumann) problem in terms of a single (double) layer potential. In this paper, we describe how to obtain these integral representations in multiply connected domains. An application of our results in the theory of conjugate differential forms is also given.

1 Introduction

A wide variety of BVPs (boundary value problems) in materials science, electrostatics and elasticity require the solution of Laplace equation $\Delta u = 0$ in multiply connected domains (see, e.g., [18] for a classic reference and [15] and the references therein for more recent works and applications). The boundary integral equation methods provide powerful and elegant tools for solving such problems, these procedures give the great opportunity to achieve the solution in closed form, which is very advantageous for numerical computations. As is well known, there are chiefly two procedures for the reduction to equivalent boundary integral equations: the direct approach (which is based on Green's representation formula for solutions of the BVPs) and the indirect approach (which rests on an appropriate layer ansatz).

In this note, we consider Laplace equation in a multiply connected domain Ω of $\mathbb{R}^n$ ($n \geq 2$) with either the Dirichlet boundary condition or the Neumann boundary condition on $\Sigma = \partial\Omega$. Under suitable assumptions, both BVPs have unique solutions (see, e.g., [14, 18]).

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_48

The classical indirect method for solving the Dirichlet problem by writing its solution as a double layer potential yields a second kind Fredholm integral equation. For bounded simply connected domains, the integral equation is uniquely solvable, while for multiply connected domains (see, e.g., [14, 18, 20]) it is necessary to add some further terms to double layer potential. In [5], we apply the alternative indirect method to solve the Dirichlet problem and we achieve a representation theorem for its solution only in terms of a single layer potential. The boundary integral equation method we employ has been introduced for the first time in [1] for simply connected domains; it strongly depends on the theory of reducible operators and the theory of differential forms. We remark that the use of differential forms leads to interesting results. In fact, we are able to study also the Neumann problem in a multiply connected domain (see [5, Sec. 5]). We recall that, in this case, when we seek the solution in terms of a double layer potential, by imposing the boundary condition we get a hypersingular integral. Differently from other methods (see, e.g., [17]), ours uses neither the theory of pseudodifferential operators nor the concept of hypersingular integrals. Moreover, the usage of the differential forms allows us to get necessary and sufficient conditions for the existence of a 2-form conjugate to a harmonic function in a multiply connected domain (see [5, Sec. 6]).

Subsequently, the study conducted for Laplace equation has laid the groundwork for other elliptic BVPs (in multiply connected domains see [4, 6] and in simply connected domains see [7, 8, 10, 11]).

2 Alternative Indirect Method to Dirichlet Problem

Throughout this paper Ω is a domain (open connected set) of $\mathbb{R}^n$ $(n \geq 2)$ bounded by several closed surfaces. In particular, Ω has the following form

$$\Omega = \Omega_0 \setminus \bigcup_{j=1}^{m} \overline{\Omega}_j \tag{1}$$

where $m \in \mathbb{N}$ and Ω_j $(j = 0, \dots, m)$ are $m + 1$ bounded domains of $\mathbb{R}^n$ such that $\overline{\Omega}_j \subset \Omega_0$ and $\overline{\Omega}_j \cap \overline{\Omega}_k = \emptyset$, $j, k = 1, \dots, m$, $j \neq k$. We suppose that its boundary $\Sigma = \bigcup_{j=0}^{m} \Sigma_j$ $(\Sigma_j = \partial\Omega_j \in C^{1,\lambda}$, $\lambda \in (0, 1])$ is a Lyapunov hypersurface. $\nu(x) = (\nu_1(x), \dots, \nu_n(x))$ denotes the outwards unit normal vector at the point $x = (x_1, \dots, x_n) \in \Sigma$.

The symbol $L^p_h(\Sigma)$ (for any non-negative integer h) denotes the vector space of all differential forms of degree h (briefly h-forms) defined on Σ such that their components are integrable functions belonging to $L^p(\Sigma)$ in a coordinate system of class C^1 (and consequently in every coordinate system of class C^1). We always consider p as a real number such that $p \in]1, +\infty[$.

A fundamental solution of the Laplace operator Δ is

$$s(x,y) = \begin{cases} \dfrac{1}{2\pi} \ln|x-y| & n=2, \\ \dfrac{1}{(2-n)c_n} |x-y|^{2-n} & n>2 \end{cases}$$

(c_n being the hypersurface measure of the unit sphere in $\mathbb{R}^n$).

2.1 Dirichlet Problem

Given $f \in W^{1,p}(\Sigma)$, we want to determine the solution of the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \Sigma, \end{cases} \tag{2}$$

in the form of a single layer potential

$$u(x) = \int_\Sigma \varphi(y)\, s(x,y)\, d\sigma_y, \quad x \in \Omega \tag{3}$$

with density $\varphi \in L^p(\Sigma)$.

We recall that for $n = 2$ there are some boundaries for which it is not possible to represent the solution of (2) in a simply connected domain $\Omega \subset \mathbb{R}^2$ by means of (3). In particular, it is not possible to represent the constant functions. In this case, we say that the boundary of such domain is exceptional. Also for multiply connected domains we have such particular cases and these occur if and only if $\Sigma_0 = \partial\Omega_0$ is exceptional (see [4, Theorem 3.1] for more details).

By imposing the boundary condition to (3), an integral equation of the first kind

$$\int_\Sigma \varphi(y) s(x,y)\, d\sigma_y = f(x), \quad x \in \Sigma \tag{4}$$

arises. Following [1], we take the differential d of both sides of Eq. (4) and the following singular integral equation

$$\int_\Sigma \varphi(y) d_x[s(x,y)] d\sigma_y = df(x), \qquad x \in \Sigma \tag{5}$$

comes out. Observe that in (5) the unknown is a function $\varphi \in L^p(\Sigma)$, while the data belongs to a different space: $df \in L^p_1(\Sigma)$. We show that Eq. (5) can be reduced to a Fredholm one. In fact, the left-hand side of (5) can be viewed as a linear and continuous operator $S : L^p(\Sigma) \to L^p_1(\Sigma)$. In [5, Lemma 4.1], we show that S can be reduced on the left by the following reducing operator $S' : L^p_1(\Sigma) \to L^p(\Sigma)$

$$S'\psi(x) = \underset{\Sigma}{*} \int_\Sigma \psi(y) \wedge d_x[s_{n-2}(x, y)], \qquad x \in \Sigma, \tag{6}$$

where $s_k(x, y) = \sum_{j_1<\ldots<j_k} s(x, y)dx^{j_1} \ldots dx^{j_k} dy^{j_1} \ldots dy^{j_k}$ is the double k-form introduced by Hodge, and the symbol $\underset{\Sigma}{*}$ has the following meaning: if w is an $(n-1)$-form on Σ and $w = w_0 d\sigma$, then $\underset{\Sigma}{*} w = w_0$. In fact we have

$$S'S\varphi = -\frac{1}{4}\varphi + K^2\varphi, \tag{7}$$

where $K\varphi(x) = \int_\Sigma \varphi(y)\frac{\partial}{\partial \nu_x} s(x, y)d\sigma_y$ is a compact operator from $L^p(\Sigma)$ into itself. We note that in the proof of (7), the following identity plays a key role (see [1, p. 186])[1]

$$\frac{\partial}{\partial \nu_x}\left(\int_\Sigma \frac{\partial}{\partial \nu_y} s(x, y)\psi(y)\, d\sigma_y\right) d\sigma_x = d_x \int_\Sigma d\psi(y) \wedge s_{n-2}(x, y), \quad \text{a.e. on } \Sigma \tag{8}$$

which holds for any $\psi \in W^{1,p}(\Sigma)$. (8) explains that the normal derivative of double layer potential is equal to the differential of single layer potential; it provides a generalization of formulas which usually can be found in the literature in a different form and only for dimension two and three (see, e.g., [16, (1.2.14), (1.2.15)]).

Since S is a reducible operator, its range is closed and we can apply the alternative theorem to Eq. (5). Hence, we deduce that (5) admits a solution if and only if the compatibility condition $\int_\Sigma \psi \wedge h = 0$ is satisfied for any solution $h \in L^q_{n-2}(\Sigma)$ $(q = p/(p-1))$ of the homogeneous adjoint equation: $S^*h(x) \equiv \int_\Sigma \psi(y) \wedge d_y[s(x, y)] = 0$, a.e. on Σ, $S^* : L^q_{n-2}(\Sigma) \to L^q(\Sigma)$ being the adjoint of S. One can prove that $S^*h = 0$ if and only if h is a weakly closed $(n-2)$-form (see [5, Theorem 4.2]). Consequently, the singular integral equation (5) is always solvable. We have thus the existence theorem.

Theorem 1 *For any $f \in W^{1,p}(\Sigma)$, there exists a solution of the following BVP*

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ dw = df & \text{on } \Sigma \end{cases}$$

[1]For a direct proof of (8), see [9, p. 70].

in the form of a single layer potential (3), where its density $\varphi \in L^p(\Sigma)$ *solves the singular integral equation* $S\varphi(x) \equiv \int_\Sigma \varphi(y) d_x[s(x,y)]\, d\sigma_y = df$.

In order to solve the Dirichlet problem (2) we also need the next result.

Lemma 1 ([5, Lemma 4.4]) *Given* $c_0, c_1, \ldots, c_m \in \mathbb{R}$, *there exists a solution of the following BVP*

$$\begin{cases} \Delta v = 0 & in\ \Omega, \\ v = c_h & on\ \Sigma_h,\ h = 0, \ldots, m, \end{cases}$$

which is given by

$$v(x) = \sum_{h=1}^{m} (c_h - c_0) \int_\Sigma \Psi_h(y) s(x,y)\, d\sigma_y + c_0, \qquad x \in \Omega, \tag{9}$$

where $\Psi_h \in L^p(\Sigma)$ *(h=1,...,m) are such that*

$$\frac{1}{2}\Psi_h(x) + \int_\Sigma \Psi_h(y) \frac{\partial}{\partial \nu_x} s(x,y)\, d\sigma_y = 0, \quad x \in \Sigma, \qquad h = 1, \ldots, m$$

and

$$\int_\Sigma \Psi_h(y) s(x,y)\, d\sigma_y = \delta_{hk}, \quad \forall\, x \in \overline{\Omega}_k,\ k = 1, \ldots, m.$$

We are now in a position to give our main claim.

Theorem 2 *Let* f *be a function belonging to* $W^{1,p}(\Sigma)$.

- *If* $n \geq 3$ *or* $n = 2$ *and* Σ_0 *is not exceptional, the Dirichlet problem* (2) *has a unique solution representable by means of a single layer potential* (3). *In particular, its density* φ *can be written as* $\psi = \varphi + \Psi$, *where* φ *solves the singular integral equation* (4) *and* Ψ *is the density of the single layer potential* (9) *which is constant on each connected component of* Σ.
- *If* $n = 2$ *and* Σ_0 *is exceptional, the solution of Dirichlet problem* (2) *can be represented as the sum of a single layer potential and a constant.*

Remark 1 We observe that the left reduction (7) is not an equivalent reduction.[2] However, as remarked in [3] in the case of a simply connected domain, we still have a kind of equivalence. Specifically, under the assumption that $N(S'S) = N(S)$, if β is such that the equation $S\alpha = \beta$ is solvable, then this equation is equivalent to $S'S\alpha = S'\beta$. Since $N(S'S) = N(S)$ and the equation $S\varphi = df$ admits always a

[2] A left reduction is said to be equivalent if $\mathrm{N}(S') = \{0\}$, where $\mathrm{N}(S')$ denotes the kernel of S' (see, e.g., [19, pp. 19–20]). This means that $S\alpha = \beta$ if, and only if, $S'S\alpha = S'\beta$.

solution, we deduce the equivalence between (5) and the Fredholm equation $S'S\varphi = S'(df)$.

3 Applications

The results of Sect. 1 can be applied to study the Neumann problem and they are also useful to get a property related to the theory of conjugate differential forms.

3.1 Neumann Problem

We consider the Neumann problem for Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = f & \text{on } \Sigma, \end{cases} \tag{10}$$

where f is an assigned function satisfying the following condition

$$\int_\Sigma f \, d\sigma = 0. \tag{11}$$

Theorem 2 allows us to represent the solution of (10) by means of a double layer potential with density ψ

$$u(x) = \int_\Sigma \psi(y) \frac{\partial}{\partial \nu_y} s(x, y) \, d\sigma_y, \quad x \in \Omega. \tag{12}$$

This holds true if and only if the datum f satisfies the following $m + 1$ conditions

$$\int_{\Sigma_j} f \, d\sigma = 0, \quad j = 0, 1, \ldots, m. \tag{13}$$

In fact, if u is a double layer potential with density $\psi \in W^{1,p}(\Sigma)$, in view of (8), the boundary condition $\dfrac{\partial u}{\partial \nu} = f$ turns into the following one

$$d_x \int_\Sigma d\psi(y) \wedge s_{n-2}(x, y) = f \, d\sigma$$

which can be rewritten as $S'(d\psi) = f$, S' being (6). Because of Theorem 2, any $\psi \in W^{1,p}(\Sigma)$ can be written as a single layer potential with density $\phi \in L^p(\Sigma)$ and since $d\psi = S\phi$, we infer $S'(d\psi) = S'S(\phi)$. Hence, by virtue of (7), we have

$$-\frac{1}{4}\phi + K^2\phi = f. \tag{14}$$

Therefore, there exists a solution of (10) if and only if (14) is solvable and this is possible if and only if f satisfies (13) (see [5, Theorem 5.2]). Then we get the next result.

Theorem 3 *Given $f \in L^p(\Sigma)$, the Neumann problem (10) admits solution (uniquely determined up to an additive constant) in terms of (12) if and only if f satisfies conditions* (13)*. In particular, the density $\psi \in W^{1,p}(\Sigma)$ of (12) is given by*

$$\psi(x) = \int_\Sigma \phi(y)s(x,y)\,d\sigma_y, \quad x \in \Sigma,$$

if $n \geq 3$ or $n = 2$ and Σ_0 is not exceptional and

$$\psi(x) = \int_\Sigma \phi(y)s(x,y)\,d\sigma_y + c, \quad x \in \Sigma,\ c \in \mathbb{R}$$

if $n = 2$ and Σ_0 is exceptional, $\phi \in L^p(\Sigma)$ being a solution of Eq. (14)*.*

Finally, if f satisfies only condition (11), we need to modify the representation of the solution.

Theorem 4 ([4, Theorem 5.4]) *Given $f \in L^p(\Sigma)$ satisfying* (11)*, the Neumann problem* (10) *admits a solution (uniquely determined up to an additive constant) representable as*

$$u(x) = \int_\Sigma \psi(y)\frac{\partial}{\partial\nu_y}s(x,y)\,d\sigma_y - \sum_{j=1}^m \frac{1}{|\Sigma_j|}\int_{\Sigma_j} f(t)\,d\sigma_t \int_{\Sigma_j} s(x,y)\,d\sigma_y,\ \psi \in W^{1,p}(\Sigma).$$

3.2 A Representation Theorem on Conjugate Differential Forms

It is well known that, if u is a harmonic function in a simply connected domain $\Omega \subset R^2$, there exists a conjugate harmonic function v (i.e., $u + iv$ is holomorphic in Ω). By means of differential forms it is possible to extend such a concept to real higher dimensions (see [2]). If u is a harmonic function, we say that the 2-form v is conjugate to u if

$$du = \delta v, \quad dv = 0, \tag{15}$$

where $\delta u = (-1)^{n+1} * d * u$ is co-differential of u and $*$ is the adjoint operator.[3]

We note that, in the case $n = 2$, $dv = 0$ is always satisfied, while $du = \delta v$ is nothing but the Cauchy-Riemann system. Then u and $v = v_0 dx^1 dx^2$ are conjugate differential forms if and only if $u + i v_0$ is holomorphic.

In the case $n = 3$, the 0-form u and the 2-form $v = v_1 dx^2 dx^3 + v_2 dx^3 dx^1 + v_3 dx^1 dx^2$ are conjugate if and only if the vector (u, v_1, v_2, v_3) is solution of the Moisil-Theodorescu system: $\nabla u = rot\,(v_1, v_2, v_3),\ div\,(v_1, v_2, v_3) = 0$.

The next result provides a representation theorem related to conjugate differential forms in a domain Ω of $\mathbb{R}^n$ bounded by several closed surfaces (see (1)).

Theorem 5 ([5, Theorem 6.1]) *Let u be a harmonic function of class $C^1(\overline{\Omega})$. There exists a 2-form v conjugate to u in Ω if and only if*

$$\int_{\Sigma_j} \frac{\partial u}{\partial \nu}\, d\sigma = 0, \quad j = 0, 1, \ldots, m. \tag{16}$$

Moreover, the 2-form v is given by

$$v(x) = - * \int_{\Sigma} \psi(y) \wedge d_y[s_{n-2}(x, y)] + \omega(x),$$

where $\psi \in W^{1,p}$ is the density of the double layer potential (12) representing u and ω is an arbitrary closed and co-closed 2-form (i.e., $d\omega = 0$ and $\delta\omega = 0$, respectively).

Proof If there exists a 2-form v which is conjugate to u, we can write $\dfrac{\partial u}{\partial \nu}\, d\sigma = -\delta * u = *du = *\delta v = d * v$. It follows that

$$\int_{\Sigma_j} \frac{\partial u}{\partial \nu}\, d\sigma = \int_{\Sigma_j} d * v = 0.$$

Thus the necessity of (16) is proved. Conversely, if (16) holds, Theorem 3 shows that u can be represented by a double layer potential (12) with density ψ. Set $v_0 = *w$; we have: $du = (-1)^{n-1} * dw = (-1)^{n-1} * d * v_0 = \delta v_0$. Moreover, since $d_y[s_{n-2}(x, y)] = \delta_x[s_{n-1}(x, y)]$ (see [12, p. 309]), we can write

$$v_0(x) = - * \delta \int_{\Sigma} \psi(y) \wedge s_{n-1}(x, y) = (-1)^n d * \int_{\Sigma} \psi(y) \wedge s_{n-1}(x, y)$$

[3] For definitions related to differential forms see, e.g., [13].

and also the second equation in (15) is satisfied by v_0. Finally, if $d\omega = 0$ and $\delta\omega = 0$, also $v_0 + \omega$ is conjugate to u. Conversely, if v_1 is conjugate to u, then $\omega = v_1 - v_0$ is closed and co-closed. □

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Part XII
Wavelet Theory and Its Related Topics

Session Organizers: Keiko Fujita and Akira Morimoto

The theory of the mathematics is important, but it is also important to apply it to real life. From this point of view, we organized the session "Wavelet theory and its related Topics" intended to discuss not only the pure mathematics, but also the applied mathematics related to the research in engineering, medicine, acoustics, and the other various fields. In the following, we will introduce the speakers (or the authors) in our session and their research briefly.

Unfortunately, Yoshihiro Aihara failed to attend, but he presents his research on the set of deficient divisors of an entire holomorphic curve as the paper "Holomorphic Curves and Linear Systems in Algebraic Manifolds." In his paper, he gives theorems on the structure of the set of deficient divisors of entire holomorphic curves and gives structure theorems for a family of linear systems of the set of deficient divisors.

Kensuke Fujinoki gave a talk on a two-dimensional directional lifting scheme on a frequency plane in the two-dimensional Euclidean space. It is useful for an efficient multidirectional wavelet expansion or transformation. He focuses on the lifting scheme as an elementary modification of a set of biorthogonal filters with biorthogonality. He proposes a method which is a straightforward extension of the original lifting scheme on the one-dimensional Euclidean spaces that offers a custom design of biorthogonal filters with directional properties.

Keiko Fujita gave a talk on Gabor wavelet transform of analytic functionals on the sphere in general dimension. To understand the Gabor wavelet transformation on the sphere, she treats Gabor wavelet transforms of the Delta function, the Gaussian function and a constant function.

Nobuko Ikawa gave a talk on a hearing test by using the complex continuous wavelet analysis. It is difficult to examine infants' hearing ability. Because the hearing test takes a long time and the infant must often be under anesthesia during that time. In order to solve such problems and to reduce inspection time, she proposes a new hearing test method.

Akira Morimoto gave a talk on an image separation problem in the case where the observed images are mixtures of rotated original images. He proposes two procedures to detect rotation angles of the original image and the rotated original images.

Holomorphic Curves and Linear Systems in Algebraic Manifolds

Yoshihiro Aihara

Abstract In this note we will give theorems on the set of deficient divisors of an entire holomorphic curve $f : \mathbf{C} \to M$, where M is a projective algebraic manifold. We first give an inequality of second main theorem type and a defect relation for f that improve the results in Aihara (Tohoku Math J 58:287–315, 2012). By making use of the defect relation, we give theorems on the structure of the set of deficient divisors of f. We also have structure theorems for a family of linear systems of the set of deficient divisors.

1 Introduction

Let M be a projective algebraic manifold and $L \to M$ an ample line bundle. We denote by $|L|$ the complete linear system of L and let $\Lambda \subseteq |L|$ be a linear system. In the previous paper [1], after the study of Nochka [3], we studied properties of the deficiencies of a holomorphic curve $f : \mathbf{C} \to M$ as functions on linear systems and gave the structure theorem for the set

$$\mathscr{D}_f = \{D \in \Lambda\,;\ \delta_f(D) > \delta_f(B_\Lambda)\}$$

of deficient divisors. For the definitions, see Sects. 2 and 3. In the proof of the structure theorem for $\mathscr{D}_f$, we used an inequality of the second main theorem type and a defect relation for f and Λ (Theorems 3.1 and 4.2 in [1]). In this note, we first give an improvement of the inequality of the second main theorem type and give a defect relation. We also give structure theorems for a family of linear systems of deficient divisors. Details will be published elsewhere.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_49

2 Preliminaries

We recall some known facts on Nevanlinna theory for holomorphic curves. For details, see [5] and [6]. Let z be the natural coordinate in $\mathbf{C}$ and set

$$\Delta(r) = \{z \in \mathbf{C};\ |z| < r\} \quad \text{and} \quad C(r) = \{z \in \mathbf{C};\ |z| = r\}.$$

For a (1,1)-current φ of order zero on $\mathbf{C}$, we set

$$N(r, \varphi) = \int_1^r \langle \varphi, \chi_{\Delta(t)} \rangle \frac{dt}{t},$$

where $\chi_{\Delta(r)}$ denotes the characteristic function of $\Delta(r)$. Let M be a compact complex manifold and let $L \to M$ be a line bundle over M. We denote by $\Gamma(M, L)$ the space of all holomorphic sections of $L \to M$ and by $|L| = \mathbf{P}(\Gamma(M, L))$ the complete linear system of L. Denote by $||\cdot||$ a Hermitian fiber metric in L and by ω its Chern form. Let $f : \mathbf{C} \to M$ be a holomorphic curve. We set

$$T_f(r, L) = N(r, f^*\omega)$$

and call it the characteristic function of f with respect to L. If

$$\liminf_{r\to+\infty} \frac{T_f(r, L)}{\log r} = +\infty,$$

then f is said to be *transcendental*. We define the order ρ_f of $f : \mathbf{C} \to M$ by

$$\rho_f = \limsup_{r\to+\infty} \frac{\log T_f(r, L)}{\log r}.$$

We notice that the definition of ρ_f is independent of a choice of positive line bundles $L \to M$. Let $D = (\sigma) \in |L|$ with $||\sigma|| \le 1$ on M. Assume that $f(\mathbf{C})$ is not contained in Supp D. We define the proximity function of D by

$$m_f(r, D) = \int_{C(r)} \log\left(\frac{1}{||\sigma(f(z))||}\right) \frac{d\theta}{2\pi}.$$

Then we have the following first main theorem for holomorphic curves.

Theorem 1 (First Main Theorem) *Let $L \to M$ be a line bundle over M and let $f : \mathbf{C} \to M$ be a non-constant holomorphic curve. Then*

$$T_f(r, L) = N(r, f^*D) + m_f(r, D) + O(1)$$

for $D \in |L|$ *with* $f(\mathbf{C}) \not\subseteq \operatorname{Supp} D$, *where* $O(1)$ *stands for a bounded term as* $r \to +\infty$.

Let f and D be as above. We define Nevanlinna's deficiency $\delta_f(D)$ by

$$\delta_f(D) = \liminf_{r \to +\infty} \frac{m_f(r, D)}{T_f(r, L)}.$$

It is clear that $0 \le \delta_f(D) \le 1$. Then we have a defect function δ_f defined on $|L|$. If $\delta_f(D) > 0$, then D is called *a deficient divisor in the sense of Nevanlinna.*

3 Value Distribution Theory for Coherent Ideal Sheaves

In this section we recall some basic facts in value distribution theory for coherent ideal sheaves and give Crofton type formula. For details, see [6, Chapter 2] and [7]. We use the same notation as in Sect. 2.

Let $f : \mathbf{C} \to M$ be a holomorphic curve and $\mathscr{I}$ a coherent ideal sheaf of the structure sheaf $\mathscr{O}_M$ of M. Let $\mathscr{U} = \{U_j\}$ be a finite open covering of M with a partition of unity $\{\eta_j\}$ subordinate to $\mathscr{U}$. We can assume that there exist finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathscr{I})$ such that every stalk $\mathscr{I}_p$ over $p \in U_j$ is generated by germs $(\sigma_{j1})_p, \ldots, (\sigma_{jl_j})_p$. Set

$$d_{\mathscr{I}}(p) = \left(\sum_j \eta_j(p) \sum_{k=1}^{l_j} \left| \sigma_{jk}(p) \right|^2 \right)^{1/2}.$$

We take a positive constant C such that $C d_{\mathscr{I}}(p) \le 1$ for all $p \in M$. Set

$$\phi_{\mathscr{I}}(p) = -\log C d_{\mathscr{I}}(p)$$

and call it the proximity potential for $\mathscr{I}$. It is easy to verify that $\phi_{\mathscr{I}}$ is well defined up to addition by a bounded continuous function on M. We now define the proximity function $m_f(r, \mathscr{I})$ of f for $\mathscr{I}$, or equivalently, for the complex analytic subspace (may be non-reduced)

$$Y = (\operatorname{Supp}(\mathscr{O}_M / \mathscr{I}),\ \mathscr{O}_M / \mathscr{I})$$

by

$$m_f(r, \mathscr{I}) = \int_{C(r)} \phi_{\mathscr{I}}(f(z)) \frac{d\theta}{2\pi},$$

provided that $f(\mathbf{C})$ is not contained in $\operatorname{Supp} Y$. For $z_0 \in f^{-1}(\operatorname{Supp} Y)$, we can choose an open neighborhood U of z_0 and a positive integer ν such that

$$f^*\mathscr{I} = ((z - z_0)^\nu) \quad \text{on } U.$$

Then we see

$$\log d_{\mathscr{I}}(f(z)) = \nu \log |z - z_0| + h_U(z) \quad \text{for } z \in U,$$

where h_U is a C^∞-function on U. Thus we have the counting function $N(r, f^*\mathscr{I})$ as in Sect. 2. Moreover, we set

$$\omega_{\mathscr{I},f} = -dd^c h_U \quad \text{on } U,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$. We obtain a well-defined smooth (1, 1)-form $\omega_{\mathscr{I},f}$ on $\mathbf{C}$. Define the characteristic function $T_f(r, \mathscr{I})$ of f for $\mathscr{I}$ by

$$T_f(r, \mathscr{I}) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} \omega_{\mathscr{I},f}.$$

We have the first main theorem in value distribution theory for coherent ideal sheaves due to Noguchi–Winkelmann–Yamanoi [7, Theorem 2.9]:

Theorem 2 (First Main Theorem) *Let $f : \mathbf{C} \to M$ and $\mathscr{I}$ be as above. Then*

$$T_f(r, \mathscr{I}) = N(r, f^*\mathscr{I}) + m_f(r, \mathscr{I}) + O(1).$$

When $\mathscr{I}$ defines an effective divisor D on M, it is easy to see that

$$T_f(r, \mathscr{I}) = T_f(r, \mathscr{O}_M(D)) + O(1) \quad \text{and} \quad m_f(r, \mathscr{I}) = m_f(r, D) + O(1).$$

Let $L \to M$ be an ample line bundle and $W \subseteq \Gamma(M, L)$ a subspace with $\dim W \geq 2$. Set $\Lambda = \mathbf{P}(W)$. The base locus $\operatorname{Bs} \Lambda$ of Λ is defined by

$$\operatorname{Bs} \Lambda = \bigcap_{D \in \Lambda} \operatorname{Supp} D.$$

We define a coherent ideal sheaf $\mathscr{I}_0$ in the following way. For each $p \in M$, the stalk $\mathscr{I}_{0,p}$ is generated by all germs $(\sigma)_p$ for $\sigma \in W$. Then $\mathscr{I}_0$ defines the base locus of Λ as a complex analytic subspace B_Λ, that is,

$$B_\Lambda = (\operatorname{Supp} (\mathscr{O}_M/\mathscr{I}_0), \mathscr{O}_M/\mathscr{I}_0).$$

Hence $\operatorname{Bs} \Lambda = \operatorname{Supp} (\mathscr{O}_M/\mathscr{I}_0)$.

We now give a Crofton type formula. Let $f : \mathbf{C} \to M$ be a non-constant holomorphic curve. If $f(\mathbf{C}) \not\subseteq \operatorname{Supp} D$ for all $D \in \Lambda$, then we say that f is *non-degenerate with respect to* Λ. Let μ be the invariant measure on $\mathbf{P}^l(\mathbf{C})$ normalized so that $\mu(\mathbf{P}^l(\mathbf{C})) = 1$. We have the following generalized Crofton's formula due to Kobayashi [6, Theorem 2.4.12].

Theorem 3 *Suppose that* $f : \mathbf{C} \to M$ *is non-degenerate with respect to* Λ *and* $f(\mathbf{C}) \not\subseteq \operatorname{Bs} \Lambda$. *Then*

$$\int_{D\in\Lambda} m_f(r, D) d\mu(D) = m_f(r, \mathscr{I}_0) + O(1)$$

and hence

$$T_f(r, L) = \int_{D\in\Lambda} N(r, f^*D) d\mu(D) + m_f(r, \mathscr{I}_0) + O(1).$$

We define the deficiency of B_Λ for f by

$$\delta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{m_f(r, \mathscr{I}_0)}{T_f(r, L)}.$$

Set

$$\mathscr{D}_f = \{D \in \Lambda;\ \delta_f(D) > \delta_f(B_\Lambda)\}.$$

We call $\mathscr{D}_f$ *the set of deficient divisors* in Λ. By making use of Theorem 2, we have the following proposition [1, Proposition 4.1].

Proposition 1 *The set* $\mathscr{D}_f$ *is a null set in the sense of the Lebesgue measure on* Λ. *In particular,*

$$\delta_f(D) = \delta_f(B_\Lambda)$$

for almost all $D \in \Lambda$.

This proposition plays an important role in the proof of theorems in Sect. 4.

4 Inequality of the Second Main Theorem Type

We will give an inequality of the second main theorem type for a holomorphic curve $f : \mathbf{C} \to M$ that improves Theorem 3.1 in [1]. For simplicity, we assume that f is of finite type. Let $W \subseteq \Gamma(M, L)$ be a linear subspace with $\dim W = l_0 + 1 \geq 2$ and set $\Lambda = \mathbf{P}(W)$. We call Λ a linear system included in $|L|$. Let $D_1, \ldots, D_q$ be divisors in Λ such that $D_j = (\sigma_j)$ for $\sigma_j \in W$. We first give a definition of *subgeneral position*. Set $Q = \{1, \dots, q\}$ and take a basis $\{\psi_0, \dots, \psi_{l_0}\}$ of W. We write

$$\sigma_j = \sum_{k=0}^{l_0} c_{jk}\psi_k \quad (c_{jk} \in \mathbf{C})$$

for each $j \in Q$. For a subset $R \subseteq Q$, we define a matrix A_R by $A_R = (c_{jk})_{j \in R, 0 \leq k \leq l_0}$.

Definition 1 Let $N \geq l_0$ and $q \geq N + 1$. We say that $D_1, \dots, D_q$ are in N-subgeneral position in Λ if

$$\operatorname{rank} A_R = l_0 + 1 \quad \text{for every subset } R \subseteq Q \text{ with } \sharp R = N + 1.$$

If they are in l_0-subgeneral position, we simply say that they are in general position.

Remark 1 The above definition is different from the usual one (cf. [6, p. 114]). In fact, the divisors $D_1, \dots, D_q$ are usually said to be in N-subgeneral position in Λ provided that

$$\bigcap_{j \in R} \operatorname{Supp} D_j = \emptyset \qquad \text{for every subset} \quad R \subseteq Q \quad \text{with} \quad \sharp R = N + 1.$$

However, the divisors $D_1, \dots, D_q$ may have a common point when they are in N-subgeneral position in the above sense.

Let $\Phi_\Lambda : M \to \mathbf{P}(W^*)$ be a natural meromorphic mapping, where W^* is the dual of W (cf. [5, p. 68]). Then we have the linearly non-degenerate holomorphic curve

$$F_\Lambda = \Phi_\Lambda \circ f : \mathbf{C} \to \mathbf{P}(W^*).$$

We let $W(F_\Lambda)$ denote the Wronskian of F_Λ.

Definition 2 If $\rho_f < +\infty$, then f is said to be of finite type.

By making use of the methods in [1] and [4], we have an inequality of the second main theorem type as follows.

Theorem 4 *Let $f : \mathbf{C} \to M$ be a transcendental holomorphic curve that is non-degenerate with respect to Λ. Let $D_1, \dots, D_q \in \Lambda$ be divisors in N-subgeneral position. Assume that f is of finite type. Then*

$$(q - 2N + l_0 - 1)\,(T_f(r, L) - m_f(r, \mathscr{I}_0)) \leq \sum_{j=1}^{q} N(r, f^* D_j) + E_f(r)$$

as $r \to +\infty$, *where*

$$E_f(r) = -(2N - n + 1)N(r,\ f^*\mathscr{I}_0) - \binom{N}{n} N(r,\ (W(F_\Lambda)_0) + o(T_f(r,\ L)).$$

We notice here that in the proof of the above theorem, we use an estimate for Nochka's weight improved by N. Toda (see [6, p. 118]). In order to get a defect relation from Theorem 4, we define a constant $\eta_f(B_\Lambda)$ by

$$\eta_f(B_\Lambda) = \liminf_{r\to+\infty} \frac{-(2N - n + 1)N(r,\ f^*\mathscr{I}_0) - (N/n)N(r,\ (W(F_\Lambda)_0)}{T_f(r,\ L)}.$$

It is clear that $\eta_f(B_\Lambda) \leq 0$. Now, by Theorem 4, we have a defect relation.

Theorem 5 *Let* Λ, f *and* $D_1, \ldots, D_q$ *be as in Theorem 4. Then*

$$\sum_{j=1}^{q} (\delta_f(D_j) - \delta_f(B_\Lambda)) \leq (1 - \delta_f(B_\Lambda))(2N - l_0 + 1) + \eta_f(B_\Lambda).$$

Remark 2 In the case where $\rho_f = +\infty$, by a suitable modification, we also have theorems similar to the above (cf. [1]).

5 Structure Theorems for the Set of Deficient Divisors

In this section we give theorems on the structure of the set of deficient divisors. Let $L \to M$ be an ample line bundle and $f : \mathbf{C} \to M$ a transcendental holomorphic curve of finite type. Let $\Lambda \subseteq |L|$ be a linear system. Let

$$\mathscr{D}_f = \{D \in \Lambda\,;\ \delta_f(D) > \delta_f(B_\Lambda)\}.$$

We summarize the basic facts on the set $\mathscr{D}_f$ (see [1, §5]).

Theorem 6 *The set* $\mathscr{D}_f$ *of deficient divisors is a union of at most countably many linear systems included in* Λ. *The set of values of deficiency of* f *is at most a countable subset* $\{e_i\}$ *of* $[0, 1]$. *For each* e_i, *there exist linear systems* $\Lambda_1(e_i), \ldots, \Lambda_s(e_i)$ *included in* Λ *such that* $e_i = \delta_f(B_{\Lambda_j(e_i)})$ *for* $j = 1, \ldots, s$.

By Theorem 6, there exist at most countably many linear systems $\{\Lambda_j\}$ in Λ such that $\mathscr{D}_f = \bigcup_j \Lambda_j$. Define $\mathscr{L}_f = \{\Lambda_j\} \cup \{\Lambda\}$. We call $\mathscr{L}_f$ *the fundamental family of linear systems for* f.

The set of all linear systems with dimension k included in Λ is parameterized by a Grassmann variety. For a transcendental holomorphic curve $f : \mathbf{C} \to M$, we will define a defect function $\delta_{\Lambda,f}$ on Grassmannians in the following way. Suppose that f is non-degenerate with respect to Λ. We fix a positive integer k with

$1 \le k \le \dim \Lambda$. We let $\mathbf{Gr}(k,\ \Lambda)$ denote the Grassmann variety of k-dimensional linear subvarieties of the projective space Λ. For a point $\mathbf{A} \in \mathbf{Gr}(k,\ \Lambda)$, we define

$$\delta_{\Lambda,f}(\mathbf{A}) = \inf\{\delta_f(D);\ D \in \mathbf{A}\}.$$

For a linear subsystem Π of Λ with $\dim \Pi > k$, we also define $\mathbf{Gr}(k,\ \Pi)$ that is the subvariety of $\mathbf{Gr}(k,\ \Lambda)$ (cf. [2, Lecture 6]). Set

$$\kappa_f(\mathbf{Gr}(k,\ \Pi)) = \inf\left\{\delta_{\Pi,f}(\mathbf{A});\ \mathbf{A} \in \mathbf{Gr}(k,\ \Pi)\right\}.$$

For a linear system Π_i in $\mathscr{L}_f$ with $\dim \Pi_i > k$, we get the Grassmannian $\mathbf{Gr}(k,\ \Pi_i)$ that is the subvariety of $\mathbf{Gr}(k,\ \Lambda)$. Let $\mathscr{G} = \{\mathbf{Gr}(k,\ \Pi_i)\}$ be the family of all such Grassmannians. Then we have the following structure theorem.

Theorem 7 *For a sufficiently small positive number* ϵ, *there exist finitely many subvarieties* $\mathbf{Gr}(k,\ \Pi_1), \ldots, \mathbf{Gr}(k,\ \Pi_t)$ *contained in* $\mathscr{G}$ *such that*

$$\{\mathbf{A} \in \mathbf{Gr}(k,\ \Lambda);\ \delta_{\Lambda,f}(\mathbf{A}) \ge \kappa_f(\mathbf{Gr}(k,\ \Lambda)) + \epsilon\} = \bigcup_{i=1}^{t} \mathbf{Gr}(k,\ \Pi_i).$$

In particular, the exceptional set

$$\{\mathbf{A} \in \mathbf{Gr}(k,\ \Lambda);\ \delta_{\Lambda,f}(\mathbf{A}) > \kappa_f(\mathbf{Gr}(k,\ \Lambda))\}$$

for $\mathbf{Gr}(k,\ \Lambda)$ *is the union of all Grassmannians in* $\mathscr{G}$.

For the function $\delta_{\Lambda,f} : \mathbf{Gr}(k,\ \Lambda) \to [0,\ 1]$, we have the following theorem.

Theorem 8 *The set of values of* $\delta_{\Lambda,f}$ *is an at most countable subset* $\{e_i\}$ *of* $[0,\ 1]$. *If* $\mathbf{Gr}(k,\ \Pi) \in \mathscr{G}$ *and if* $\mathbf{A} \in \mathbf{Gr}(k,\ \Pi)$ *is generic, then* $\kappa_f(\mathbf{Gr}(k,\ \Pi)) = \delta_f(\mathbf{A})$. *The set of non-generic points in* $\mathbf{Gr}(k,\ \Pi)$ *is contained in a union of at most countable Grassmannians in* $\mathscr{G}$. *In particular, the closure of the inverse image* $\delta^{-1}_{\Lambda,f}(e_i)$ *can be written*

$$\overline{\delta^{-1}_{\Lambda,f}(e_i)} = \bigcup_{j=1}^{t} \mathbf{Gr}(k,\ \Pi_{i_j})$$

for finitely many varieties $\mathbf{Gr}(k,\ \Pi_{i_1}), \ldots, \mathbf{Gr}(k,\ \Pi_{i_t})$ *in* $\mathscr{G}$.

Remark 3 Let $\Pi_1, \Pi_2 \in \mathscr{G}$. We notice that $\kappa_f(\mathbf{Gr}(k,\ \Pi_1)) < \kappa_f(\mathbf{Gr}(k,\ \Pi_2))$ if $\mathbf{Gr}(k,\ \Pi_2)$ is a proper subvariety of $\mathbf{Gr}(k,\ \Pi_1)$.

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Two-Dimensional Directional Lifting Schemes

Kensuke Fujinoki

Abstract We consider a two-dimensional directional lifting scheme on a frequency plane $\mathbf{R}^2$ that is useful for an efficient multidirectional wavelet expansion or transformation. In particular, we focus on the lifting scheme as an elementary modification of a set of biorthogonal filters that hold biorthogonality. The proposed method is a straightforward extension of the original lifting scheme on $\mathbf{R}$ that offers a custom design of biorthogonal filters with directional properties.

1 Introduction

With recent technological advances, we are managing large amounts of data, which requires efficient representations and processing. Applied harmonic analysis, whose purpose is to provide efficient representations of functions or data, has been developed not only for harmonic analysis but also for other scientific fields, such as electrical engineering and computer science, particularly signal processing.

A wavelet system is a typical example, and it has two main reasons for its success. The first is that it provides an optimally sparse approximation of a signal compared with traditional Fourier-based methods. The second is the existence of fast computation algorithms to precisely and efficiently manage digital data, thereby providing a range of applications in signal processing. For more details of wavelets and their applications, see [11, 12].

The appearance of the lifting scheme proposed by Sweldens [13, 14] is one of the milestones in the development of the theory of wavelet analysis. The lifting scheme has various aspects in both mathematics and signal processing; however, the main idea is to provide a custom design of biorthogonal wavelets as well as biorthogonal filters in any bounded domain (see [1, 3–5, 9, 10]).

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_50

In this paper, we focus on a custom filter design of the lifting scheme. In particular, we consider the lifting scheme on a frequency plane $\mathbf{R}^2$ to construct two-dimensional biorthogonal filters that have a freedom of directional properties.

2 Biorthogonal Filters

We first review one-dimensional biorthogonal filters. More detailed introductions are provided in [15, 16]. Following the signal processing literature, we refer to a sequence as a filter.

We consider two finite filters, a low-pass (LP) filter $\{h_n\}_{n\in\mathbf{Z}}$ and a high-pass (HP) filter $\{g_n\}_{n\in\mathbf{Z}}$, both of which have a finite number of nonzero coefficients and belong to the square summable sequence space $\ell^2(\mathbf{Z})$. For $\xi \in \mathbf{R}$, we define

$$h(\xi) = \sum_{n\in\mathbf{Z}} h_n\, e^{-i\xi n} \quad \text{and} \quad g(\xi) = \sum_{n\in\mathbf{Z}} g_n\, e^{-i\xi n},$$

which we call the Fourier transform of a sequence. Since $h(\xi)$ and $g(\xi)$ are periodic functions with period 2π, they belong to a space of square integrable functions $L^2(\mathbf{T})$, where $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Similarly, we define their dual functions $\tilde{h}(\xi),\ \tilde{g}(\xi) \in L^2(\mathbf{T})$ as

$$\tilde{h}(\xi) = \sum_{n\in\mathbf{Z}} \tilde{h}_n\, e^{-i\xi n} \quad \text{and} \quad \tilde{g}(\xi) = \sum_{n\in\mathbf{Z}} \tilde{g}_n\, e^{-i\xi n},$$

where $\{\tilde{h}_n\}_{n\in\mathbf{Z}}$ and $\{\tilde{g}_n\}_{n\in\mathbf{Z}}$ are the dual LP filter and the dual HP filter, respectively.

We assume that these functions are trigonometric polynomials with $h(0) = \tilde{h}(0) = 1$ and $g(\pi) = \tilde{g}(\pi) = 1$. We define a modulation matrix $M(\xi) \in (L^2(\mathbf{T}))^{2\times 2}$ and its dual modulation matrix $\widetilde{M}(\xi) \in (L^2(\mathbf{T}))^{2\times 2}$ as

$$M(\xi) := \begin{bmatrix} h(\xi) & h(\xi+\pi) \\ g(\xi) & g(\xi+\pi) \end{bmatrix} \quad \text{and} \quad \widetilde{M}(\xi) := \begin{bmatrix} \tilde{h}(\xi) & \tilde{h}(\xi+\pi) \\ \tilde{g}(\xi) & \tilde{g}(\xi+\pi) \end{bmatrix}.$$

The biorthogonality of the filters is then represented by the perfect reconstruction condition, which can be written as

$$\widetilde{M}(\xi)\, M(\xi)^* = I, \tag{1}$$

where M^* is the complex conjugate transpose of M and I is the 2×2 identity matrix.

Definition 1 (Perfect Reconstruction Filter) A set of filters $\{h, g, \tilde{h}, \tilde{g}\}$ is said to consist of perfect reconstruction filters if the perfect reconstruction condition (1) is satisfied.

Remark 1 Perfect reconstruction filters are also called biorthogonal filters because the condition (1) is also called the biorthogonality condition.

It is important to note that the condition (1) implies that

$$\tilde{h}(\xi)\,\overline{h(\xi)} + \tilde{g}(\xi)\,\overline{g(\xi)} = 1, \tag{2}$$

and

$$\tilde{h}(\xi)\,\overline{h(\xi+\pi)} + \tilde{g}(\xi)\,\overline{g(\xi+\pi)} = 0, \tag{3}$$

where $\overline{h(\xi)}$ denotes the complex conjugate of $h(\xi)$. Equation (2) is called the identity summation condition, which guarantees the conservation of a standard ℓ^2 norm; (3) is the alias cancellation condition. In general, biorthogonal wavelets are generated from biorthogonal filters that satisfy these conditions (see [2]).

3 Lifting Scheme

The lifting scheme corresponds to modifying biorthogonal filters without losing biorthogonality. Let $\ell(\xi)$ and $\tilde{\ell}(\xi)$ be trigonometric polynomials. We call them a lifting filter and dual lifting filter, respectively. We assume that a set $\{h, g, \tilde{h}, \tilde{g}\}$ is biorthogonal, and two HP filters $g(\xi)$ and $\tilde{g}(\xi)$ that satisfy (2) and (3) are defined by the standard choice:

$$g(\xi) = e^{-i\xi}\,\overline{\tilde{h}(\xi+\pi)} \quad \text{and} \quad \tilde{g}(\xi) = e^{-i\xi}\,\overline{h(\xi+\pi)}. \tag{4}$$

Then, we have the following:

Proposition 1 (Herley and Vetterli [17]) *Let $h(\xi)$ and $\tilde{h}(\xi)$ be finite biorthogonal LP filters in the sense of (2) and (4). A finite filter $\tilde{h}^{\ell}(\xi)$ is biorthogonal to $h(\xi)$ if and only if there exists a finite filter $\ell(\xi)$ such that*

$$\tilde{h}^{\ell}(\xi) = \tilde{h}(\xi) + e^{-i\xi}\,\overline{h(\xi+\pi)}\;\overline{\ell(2\xi)}.$$

The lifting scheme further extends Proposition 1. Together with Proposition 1 and the definition of HP filters in (4), we can generate a new primal HP filter $g^{\ell}(\xi)$ that is dual to $\tilde{g}(\xi)$. Similar results can be seen in $h(\xi)$ and $\tilde{g}(\xi)$. We summarize these observations as follows:

Definition 2 (Lifting Scheme [13]) The modifications of a set of biorthogonal filters $\{h, g, \tilde{h}, \tilde{g}\}$ defined by (5) and (6) are called lifting and dual lifting, respectively:

$$\tilde{h}^{\ell}(\xi) = \tilde{h}(\xi) + \tilde{g}(\xi)\,\overline{\ell(2\xi)}, \qquad g^{\ell}(\xi) = g(\xi) - h(\xi)\,\ell(2\xi), \tag{5}$$

$$h^{\tilde{\ell}}(\xi) = h(\xi) + g(\xi)\,\tilde{\ell}(2\xi), \qquad \tilde{g}^{\tilde{\ell}}(\xi) = \tilde{g}(\xi) - \tilde{h}(\xi)\,\overline{\tilde{\ell}(2\xi)}. \tag{6}$$

Remark 2 Lifting and dual lifting guarantee the biorthogonality of biorthogonal filters, which means that lifted filters $\{h, g^{\ell}, \tilde{h}^{\ell}, \tilde{g}\}$ and $\{h^{\tilde{\ell}}, g, \tilde{h}, \tilde{g}^{\tilde{\ell}}\}$ are both biorthogonal filters, and $\{h^{\tilde{\ell}}, g^{\ell}, \tilde{h}^{\ell}, \tilde{g}^{\tilde{\ell}}\}$ are also biorthogonal filters.

4 Two-Dimensional Directional Lifting

On the standard frequency plane $\mathbf{R}^2$, we consider a set of square lattice sites

$$\Gamma := \{\nu_i \mid i = 0, 1, 2, 3\},$$

where $\nu_0 = (0, 0)$, $\nu_1 = (0, \pi)$, $\nu_2 = (\pi, 0)$, $\nu_3 = (\pi, \pi)$. As in the one-dimensional case, we assume that $M(\xi)$ is a modulation matrix whose entries are biorthogonal filters with frequency shifts in Γ:

$$M(\xi) := \begin{bmatrix} h(\xi) & h(\xi+\nu_1) & h(\xi+\nu_2) & h(\xi+\nu_3) \\ g_1(\xi) & g_1(\xi+\nu_1) & g_1(\xi+\nu_2) & g_1(\xi+\nu_3) \\ g_2(\xi) & g_2(\xi+\nu_1) & g_2(\xi+\nu_2) & g_2(\xi+\nu_3) \\ g_3(\xi) & g_3(\xi+\nu_1) & g_3(\xi+\nu_2) & g_3(\xi+\nu_3) \end{bmatrix},$$

where $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ and $\nu \in \Gamma$. Note that we have three HP filters $g_k(\xi)$, $k = 1, 2, 3$. These filters are trigonometric polynomials defined as

$$h(\xi) = \sum_{n \in \mathbf{Z}^2} h_n\, e^{-i\xi\cdot n} \quad \text{and} \quad g_k(\xi) = \sum_{n \in \mathbf{Z}^2} g_{k,n}\, e^{-i\xi\cdot n},$$

which are normalized to $h(\nu_0) = \tilde{h}(\nu_0) = 1$ and $g_k(\nu_k) = \tilde{g}_k(\nu_k) = 1$ for $k = 1, 2, 3$. We assume that LP filters $h(\xi)$ and $\tilde{h}(\xi)$ are $\pi/2$-rotation invariant, whereas both HP filters $g_k(\xi)$ and $\tilde{g}_k(\xi)$, $k = 1, 2$ are defined as $\pm\pi/4$-rotations of the case of $k = 3$.

Similarly, we define a dual modulation matrix $\widetilde{M}(\xi)$ with dual filters $\tilde{h}(\xi)$ and $\tilde{g}_k(\xi)$, $k = 1, 2, 3$. Biorthogonality of the filters is now written as

$$\widetilde{M}(\xi)\, M(\xi)^* = I, \tag{7}$$

where I is the 4×4 identity matrix. In this setting, we have the following proposition, which generalizes (2) and (3).

Proposition 2 *A set of filters* $\{h, g_k, \tilde{h}, \tilde{g}_k\}_{k=1,2,3}$ *is a set of perfect reconstruction filters if and only if the following two conditions hold:*

$$\tilde{h}(\xi)\overline{h(\xi)} + \sum_{k=1}^{3} \tilde{g}_k(\xi)\overline{g_k(\xi)} = 1, \tag{8}$$

and

$$\tilde{h}(\xi)\overline{h(\xi+\nu)} + \sum_{k=1}^{3} \tilde{g}_k(\xi)\overline{g_k(\xi+\nu)} = 0, \qquad \nu \in \Gamma \setminus \{\nu_0\}. \tag{9}$$

Proposition 2 is a special case of Durand [6], and Yin and Daubechies [18], in which more general cases are found, that is, other types of frequency lattices. As in the one-dimensional case, (8) is the condition for ℓ^2 norm preservation; (9) is for alias cancellation.

From Propositions 1 and 2, we have the following results:

Theorem 1 *Let $k = 1, 2, 3$. Suppose that $\ell_k(\xi)$ and $\tilde{\ell}_k(\xi)$ are trigonometric polynomials, and $\{h, g_k, \tilde{h}, \tilde{g}_k\}$ is a set of finite biorthogonal filters that satisfies (8) and (9). Then, two new sets of finite biorthogonal filters $\{h, g_k^{\ell}, \tilde{h}^{\ell}, \tilde{g}_k\}$ and $\{h^{\tilde{\ell}}, g_k, \tilde{h}, \tilde{g}_k^{\tilde{\ell}}\}$ are defined as*

$$\tilde{h}^{\ell}(\xi) = \tilde{h}(\xi) + \sum_{k=1}^{3} \tilde{g}_k(\xi)\,\overline{\ell_k(2\xi)}, \qquad g_k^{\ell}(\xi) = g_k(\xi) - h(\xi)\,\ell_k(2\xi), \tag{10}$$

and

$$h^{\tilde{\ell}}(\xi) = h(\xi) + \sum_{k=1}^{3} g_k(\xi)\,\tilde{\ell}_k(2\xi), \qquad \tilde{g}_k^{\tilde{\ell}}(\xi) = \tilde{g}_k(\xi) - \tilde{h}(\xi)\,\overline{\tilde{\ell}_k(2\xi)}. \tag{11}$$

Proof Because of the two facts that the initial set of filters $\{h, g_k, \tilde{h}, \tilde{g}_k\}$ is biorthogonal, and both trigonometric polynomials $\ell_k(\xi)$ and $\tilde{\ell}_k(\xi)$ are periodic functions with period $(2\pi, 2\pi)$, we only need to show the biorthogonality in terms of $\ell_k(\xi)$ and $\tilde{\ell}_k(\xi)$.

We consider a matrix form of the lifting (10). We can rewrite it as

$$\begin{bmatrix} \tilde{h}^{\ell}(\xi) \\ \tilde{g}_1(\xi) \\ \tilde{g}_2(\xi) \\ \tilde{g}_3(\xi) \end{bmatrix} = \begin{bmatrix} 1 & \overline{\ell_1(2\xi)} & \overline{\ell_2(2\xi)} & \overline{\ell_3(2\xi)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{h}(\xi) \\ \tilde{g}_1(\xi) \\ \tilde{g}_2(\xi) \\ \tilde{g}_3(\xi) \end{bmatrix},$$

and

$$\begin{bmatrix} h(\xi) \\ g_1^{\ell}(\xi) \\ g_2^{\ell}(\xi) \\ g_3^{\ell}(\xi) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\ell_1(2\xi) & 1 & 0 & 0 \\ -\ell_2(2\xi) & 0 & 1 & 0 \\ -\ell_3(2\xi) & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h(\xi) \\ g_1(\xi) \\ g_2(\xi) \\ g_3(\xi) \end{bmatrix}.$$

From the perfect reconstruction condition (7), the biorthogonality follows from

$$\begin{bmatrix} 1 & \overline{\ell_1(2\xi)} & \overline{\ell_2(2\xi)} & \overline{\ell_3(2\xi)} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\ell_1(2\xi) & 1 & 0 & 0 \\ -\ell_2(2\xi) & 0 & 1 & 0 \\ -\ell_3(2\xi) & 0 & 0 & 1 \end{bmatrix}^* = I.$$

Similarly, in the case of the dual lifting (11), we show the biorthogonality using

$$\begin{bmatrix} h^{\tilde{\ell}}(\xi) \\ g_1(\xi) \\ g_2(\xi) \\ g_3(\xi) \end{bmatrix} = \begin{bmatrix} 1 & \tilde{\ell}_1(2\xi) & \tilde{\ell}_2(2\xi) & \tilde{\ell}_3(2\xi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h(\xi) \\ g_1(\xi) \\ g_2(\xi) \\ g_3(\xi) \end{bmatrix},$$

and

$$\begin{bmatrix} \tilde{h}(\xi) \\ \tilde{g}_1^{\tilde{\ell}}(\xi) \\ \tilde{g}_2^{\tilde{\ell}}(\xi) \\ \tilde{g}_3^{\tilde{\ell}}(\xi) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\overline{\tilde{\ell}_1(2\xi)} & 1 & 0 & 0 \\ -\overline{\tilde{\ell}_2(2\xi)} & 0 & 1 & 0 \\ -\overline{\tilde{\ell}_3(2\xi)} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{h}(\xi) \\ \tilde{g}_1(\xi) \\ \tilde{g}_2(\xi) \\ \tilde{g}_3(\xi) \end{bmatrix}.$$

□

Remark 3 As in the original lifting scheme on **R**, a set $\{h^{\tilde{\ell}}, g_k^{\ell}, \tilde{h}^{\ell}, \tilde{g}_k^{\tilde{\ell}}\}$ is also biorthogonal.

Remark 4 The three lifting filters $\ell_k(\xi)$ and their duals $\tilde{\ell}_k(\xi)$ can be freely chosen so that the resulting filters have specific directional properties. The biorthogonality is guaranteed regardless of how we choose the lifting filters.

By applying the inverse Fourier transformation to (10) and (11) in Theorem 1, we have convolution forms of the directional lifting scheme:

$$\tilde{h}_n^{\ell} = \tilde{h}_n + \sum_{k=1}^{3} \sum_{l \in \mathbf{Z}^2} \tilde{g}_{k,n-2l}\, \ell_{k,-l}, \qquad g_{k,n}^{\ell} = g_{k,n} - \sum_{l \in \mathbf{Z}^2} h_{n-2l}\, \ell_{k,l},$$

$$h_n^{\tilde{\ell}} = h_n + \sum_{k=1}^{3} \sum_{l \in \mathbf{Z}^2} g_{k,n-2l}\, \tilde{\ell}_{k,l}, \qquad \tilde{g}_{k,n}^{\tilde{\ell}} = \tilde{g}_{k,n} - \sum_{l \in \mathbf{Z}^2} \tilde{h}_{n-2l}\, \tilde{\ell}_{k,-l},$$

where $\{\ell_{k,l} \mid k = 1, 2, 3;\ l \in \mathbf{Z}^2\}$ and $\{\tilde{\ell}_{k,l} \mid k = 1, 2, 3;\ l \in \mathbf{Z}^2\}$ are finite lifting filters and dual lifting filters, respectively.

Remark 4 indicates an essential property of the two-dimensional directional lifting scheme. If an initial set of filters $\{h, g_k, \tilde{h}, \tilde{g}_k\}_{k=1,2,3}$ is biorthogonal, then lifting with $\ell_k(\xi)$ holds the biorthogonality so that we can always construct a new set of biorthogonal filters $\{h, g_k^{\ell}, \tilde{h}^{\ell}, \tilde{g}_k\}_{k=1,2,3}$. This is true for $\{h^{\tilde{\ell}}, g_k, \tilde{h}, \tilde{g}_k^{\tilde{\ell}}\}_{k=1,2,3}$ with dual lifting $\tilde{\ell}(\xi)$.

This means that we have remarkable freedom to design three biorthogonal HP filters $g_k^{\ell}(\xi)$ and $\tilde{g}_k^{\tilde{\ell}}(\xi)$ independently. For example, directional wavelet transformations for isotropic image representations introduced in [7, 8] use such a nature of the two-dimensional lifting scheme, where the HP filters are designed in an isotropic manner on an equilateral triangular lattice.

Acknowledgements The author is grateful to the reviewers for their valuable remarks and suggestions that helped to improve this paper. This work was partially supported by JSPS KAKENHI Grant Number 17K12716.

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Gabor Wavelet Transformation on the Sphere and Its Related Topic

Keiko Fujita

Abstract We studied the Gabor wavelet transform of analytic functionals on the sphere in general dimension. Then, we studied the Gabor wavelet transformation on the two-dimensional sphere and its inverse transformation. In this note, following our previous results, to understand the Gabor wavelet transformation on the sphere, we consider some Gabor wavelet transforms of analytic functionals and square integrable functions on the sphere.

1 Introduction

In [3], we studied the Gabor wavelet transform of analytic functional on the sphere in general dimension. Then to simplify the results in [3], we treated the Gabor wavelet transformation on the circle in [2]. And then, in [4], we constructed the inverse Gabor wavelet transformation concretely in the two-dimensional sphere by using our results in [1]. In this note, to see the Gabor wavelet transformation on the sphere, we consider some Gabor wavelet transforms of several simple functions on the sphere. Note that we call the Gabor wavelet transformation the Gabor transformation in [4].

1.1 Fourier Transformation on the Sphere

Let S_r^2 be the sphere with radius $r > 0$ in $\mathbf{R}^3$, that is,

$$S_r^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3; x_1^2 + x_2^2 + x_3^2 = r^2\}.$$

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*,
Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_51

For $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3) \in \mathbf{C}^3$, we set

$$z \cdot w = z_1 w_1 + z_2 w_2 + z_3 w_3, \quad z^2 = z \cdot z.$$

For an integrable function f on S_r^2, we define the Fourier transform of f by

$$(\mathscr{F} f)(\omega) = \int_{S_r^2} e^{-ix\cdot\omega} \overline{f(x)} d\Omega_x,$$

where $d\Omega$ is the normalized invariant measure on S_r^2. Note that $\mathrm{vol}(S_r^2) = 4\pi r^2$. For the square integrable functions f and g on S_r^2, we define a sesquilinear form $(f,\ g)_{S_r^2}$ by

$$(f,\ g)_{S_r^2} \equiv \int_{S_r^2} f(x)\overline{g(x)} d\Omega_x.$$

Then $(f,\ g)_{S_r^2}$ gives an inner product and we denote by $L^2(S_r^2)$ the space of square integrable functions on S_r^2 with the inner product $(f,\ g)_{S_r^2}$ and the norm $\|f\|_{S_r^2} = \sqrt{(f,\ f)_{S_r^2}}$.

1.2 Windowed Fourier Transformation on the Sphere

For $f \in L^2(S_r^2)$ and $\omega, \tau \in \mathbf{C}^3$, we define the windowed Fourier transformation $\mathscr{W}\mathscr{F}$ with the window function $\exp(-x^2/2)$ by

$$\begin{aligned}
\mathscr{W}\mathscr{F} \ :\ f \mapsto (\mathscr{W}\mathscr{F} f)(\tau, \omega) &= \int_{S_r^2} \exp(-ix \cdot \omega) \exp\left(-\frac{(x-\tau)^2}{2}\right) \overline{f(x)} d\Omega_x \\
&= e^{\frac{-r^2-\tau^2}{2}} \int_{S_r^2} \exp\left(-ix \cdot (\omega + i\tau)\right) \overline{f(x)} d\Omega_x \\
&= e^{\frac{-r^2-\tau^2}{2}} (\mathscr{F} f)(\omega + i\tau).
\end{aligned}$$

1.3 Gabor Wavelet Transformation on the Sphere

Let $\omega_0 \in \mathbf{R}^3$ be fixed. Put

$$G_{\omega_0}(x) = e^{-x^2/2} e^{-ix\cdot\omega_0}.$$

For $f \in L^2(S_r^2)$ and $a \in \mathbf{R}_+ = \{x : x > 0\}$, we define the Gabor wavelet transformation $\mathscr{G}_{\omega_0}$ by

$$\mathscr{G}_{\omega_0} : f \mapsto (\mathscr{G}_{\omega_0} f)(\tau, a) = \frac{1}{a} \int_{S_r^2} G_{\omega_0} \left(\frac{x-\tau}{a} \right) \overline{f(x)} d\Omega_x$$
$$= \frac{1}{a} \int_{S_r^2} \exp\left(-i \frac{x-\tau}{a} \cdot \omega_0 \right) \exp\left(-\frac{1}{2} \left(\frac{x-\tau}{a} \right)^2 \right) \overline{f(x)} d\Omega_x$$
$$= \frac{1}{a} e^{-\frac{r^2+\tau^2}{2a^2}} e^{i\tau \cdot \frac{\omega_0}{a}} \int_{S_r^2} \exp\left(-ix \cdot \frac{\omega_0}{a} \right) \exp\left(-\frac{x \cdot \tau}{a^2} \right) \overline{f(x)} d\Omega_x.$$

2 Gabor Wavelet Transform of Analytic Functional on the Sphere

Since $G_{\omega_o}(x)$ is an analytic function in $\mathbf{R}^3$, we can consider the Gabor wavelet transform of analytic functional. We treated the Gabor wavelet transform of analytic functional in [3] in general dimension. First we recall some notation.

We denote by $\mathscr{A}(S_r^2)$ the space of real analytic functions on S_r^2, and by $\mathscr{A}'(S_r^2)$ the space of analytic functionals (or hyperfunctions) on S_r^2. Let $\langle T , g \rangle$ be the canonical bilinear form of duality on $\mathscr{A}'(S_r^2) \times \mathscr{A}(S_r^2)$.

Since $\mathscr{A}(S^n) \subset L^2(S_r^2) \subset \mathscr{A}'(S_r^2)$, for $g \in L^2(S_r^2)$ we define $T_g \in \mathscr{A}'(S_r^2)$ by

$$\langle T_g , f \rangle = (f , g)_{S_r^2} = \int_{S_r^2} f(x) \overline{g(x)} d\Omega_x, \quad f \in \mathscr{A}(S_r^2).$$

Note the mapping $g \mapsto T_g$ is a continuous antilinear injection.

Let $T \in \mathscr{A}'(S_r^2)$. We will define the Fourier–Borel transform of T by

$$(\mathscr{F}T)(\omega) = \langle T_x, \exp(-ix \cdot \omega) \rangle, \quad \omega \in \mathbf{C}^3.$$

Note that we defined $(\mathscr{F}T)(\omega) = \langle T_x, \exp(x \cdot \omega) \rangle$ in [5].

For $\omega, \tau \in \mathbf{C}^3$, in [3], we defined a mapping $\mathscr{W}_G \mathscr{F}$ by

$$\mathscr{W}_G \mathscr{F} : T \mapsto (\mathscr{W}_G \mathscr{F} T)(\tau, \omega) = \left\langle T_x , \exp(-(x-\tau)^2/2) \exp(-ix \cdot \omega) \right\rangle,$$

and we call the mapping $T \mapsto (\mathscr{W}_G \mathscr{F} T)(\tau, \omega)$ the Gabor transformation. For $T \in \mathscr{A}'(S_r^2)$, we call the mapping

$$\mathscr{G}_{\omega_0} : T \mapsto (\mathscr{G}_{\omega_0} T)(\tau, a) = \frac{1}{a} \left\langle T_x , G_{\omega_0}\left(\frac{x-\tau}{a}\right) \right\rangle$$

the Gabor wavelet transformation. Since $T \in \mathscr{A}'(S_r^2)$,

$$(\mathscr{W}_G \mathscr{F} T)(\tau, \omega) = \exp((-r^2 - \tau^2)/2) \langle T_x \, , \, \exp(x \cdot (\tau - i\omega)) \rangle \ ,$$

$$(\mathscr{G}_{\omega_0} T)(\tau, a) = \frac{1}{a} \left\langle T_x \, , \, \exp(-i \frac{x - \tau}{a} \cdot \omega_0) \exp(-\frac{1}{2} (\frac{x - \tau}{a})^2) \right\rangle$$

$$= \frac{1}{a} e^{i \frac{\tau \cdot \omega_0}{a}} e^{-\frac{r^2 + \tau^2}{2a^2}} \left\langle T_x \, , \, \exp\left(-i \frac{x \cdot \omega_0}{a}\right) \exp\left(\frac{x \cdot \tau}{a^2}\right) \right\rangle .$$

The inverse mappings of $\mathscr{W}_G \mathscr{F}$ and $\mathscr{G}_{\omega_0}$ were given in [4].

3 Examples

3.1 Gabor Wavelet Transform of the Delta Function

Let $\delta(x)$ be the Delta function, that is, for $y \in S_r^2$ and $f \in \mathscr{A}(S_r^2)$

$$\langle \delta(x - y) \, , \, f(x) \rangle = f(y).$$

Therefore when $T_y(x) = \delta(x - y)$, we have

$$(\mathscr{W}_G \mathscr{F} T_y)(\tau, \omega) = \exp(-(y - \tau)^2/2) \exp(-iy \cdot \omega),$$

$$(\mathscr{G}_{\omega_0} T_y)(\tau, a) = \frac{1}{a} \exp(-i \frac{y - \tau}{a} \cdot \omega_0) \exp(-\frac{1}{2} (\frac{y - \tau}{a})^2)$$

$$= \frac{1}{a} e^{i \frac{(\tau - y) \cdot \omega_0}{a}} e^{-\frac{r^2 + \tau^2}{2a^2}} \exp\left(\frac{y \cdot \tau}{a^2}\right).$$

Thus

$$|(\mathscr{W}_G \mathscr{F} T_y)(\tau, \omega)| = \exp(-\frac{(y - \tau)^2}{2}), \quad |(\mathscr{G}_{\omega_0} T_y)(\tau, a)| = \frac{1}{a} \exp(-\frac{1}{2} (\frac{y - \tau}{a})^2).$$

When $\tau = y$, then $|(\mathscr{G}_{\omega_0} T_y)(\tau, a)| = 1/a$. For a fixed a, $|(\mathscr{G}_{\omega_0} T_y)(\tau, a)|$ is a Gaussian function.

Assume that $\tau \in S_r^2$. Let P be the point on S_r^2 represented by τ and let Q be the point on S_r^2 represented by y. We denote by $\overline{OX}$ the segment which is the straight line connecting the origin and the point X. And let θ be the angle between $\overline{OP}$ and

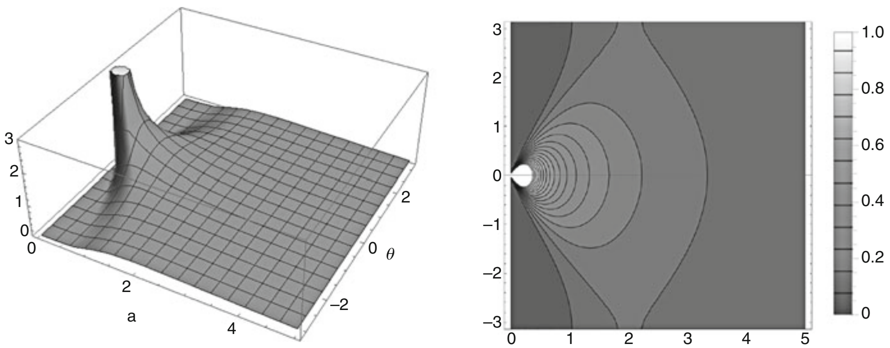


Fig. 1 $|(\mathscr{G}_{\omega_0}T_y)(\tau, a)| = \frac{1}{a}e^{-1/a^2}\exp(\cos\theta/a^2)$

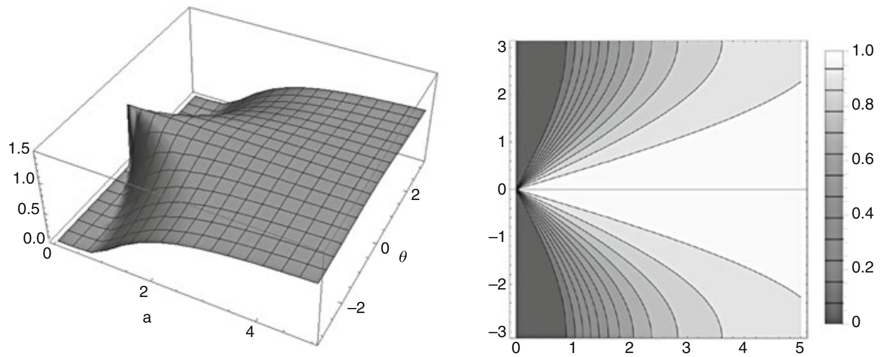


Fig. 2 $|G_{\omega_0}(\frac{x-\tau}{a})| = e^{-1/a^2}\exp(\cos\theta/a^2)$

$\overline{OQ}$. Then $(y-\tau)^2 = (2r\sin(\theta/2))^2$, and we have

$$|(\mathscr{W}_G \mathscr{F} T_y)(\tau, \omega)| = \exp(-2r^2\sin^2(\theta/2)),$$

$$|(\mathscr{G}_{\omega_0}T_y)(\tau, a)| = \frac{1}{a}\exp(-2r^2\sin^2(\theta/2)/a^2)$$

$$= \frac{1}{a}\exp(-r^2(1-\cos\theta)/a^2) = \frac{1}{a}e^{-r^2/a^2}\exp(r^2\cos\theta/a^2).$$

Therefore for $\tau, y \in S_1^2$, the graph of $|(\mathscr{G}_{\omega_0}T_y)(\tau, a)|$ is given by Fig. 1, and for $x = (0, 0, 1)$ and $\tau = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, $\varphi, \theta \in \mathbf{R}$, $|G_{\omega_0}(\frac{x-\tau}{a})|$ does not depend on ω_0 and on φ, and the graph of $|G_{\omega_0}(\frac{x-\tau}{a})|$ is given by Fig. 2.

More generally, let P be a point on $\mathbf{R}^3$ represented by $\tau = r'\tau'$, $\tau' \in S_1^2$, $r' > 0$ and Q_j be the point on S_r^2 represented by y_j, and let θ_j be the angle between the segments $\overline{OP}$ and $\overline{OQ_j}$. Then for $T = \sum_{j=1}^{m} c_j \delta(x - y_j)$,

$$|(\mathscr{G}_{\omega_0} T)(\tau, a)| \leq \frac{1}{a} e^{-\frac{r^2+(r')^2}{2a^2}} \sum_{j=1}^{m} c_j \exp\left(rr' \cos\theta_j / a^2\right).$$

3.2 Gabor Wavelet Transform of the Gaussian Function

Next we consider the Gabor wavelet transform of the Gaussian function. For $b > 0$, put $f_y(x) = \frac{1}{b}\exp(-(x-y)^2/(2b^2)), \quad y \in S_r^2$. For a sufficiently small $b > 0$ the Gaussian function looks like the Delta function. $(\mathscr{G}_{\omega_0} f_y)(\tau, a)$ is calculated as follows:

$$\begin{aligned}
(\mathscr{G}_{\omega_0} f_y)(\tau, a) &= \frac{1}{ab} \int_{S_r^2} G_{\omega_0}\left(\frac{x-\tau}{a}\right) e^{-(x-y)^2/2b^2} d\Omega_x \\
&= \frac{1}{ab} \int_{S_r^2} e^{-i(x-\tau)\cdot\frac{\omega_0}{a}} e^{-\frac{1}{2}\left(\frac{x-\tau}{a}\right)^2 - \frac{1}{2}\left(\frac{(x-y)}{b}\right)^2} d\Omega_x \\
&= \frac{1}{ab} e^{i\tau\cdot\frac{\omega_0}{a}} e^{-\frac{1}{2}\left(\frac{r^2+\tau^2}{a^2} + \frac{2r^2}{b^2}\right)} \int_{S_r^2} e^{-ix\cdot\frac{\omega_0}{a}} e^{\frac{x}{a^2}\cdot\left(\tau + \frac{a^2 y}{b^2}\right)} d\Omega_x. \qquad (1)
\end{aligned}$$

To consider the integral $\int_{S_r^2} \exp\left(-ix \cdot \frac{\omega_0}{a}\right) \exp\left(\frac{x}{a^2} \cdot \left(\tau + \frac{a^2 y}{b^2}\right)\right) d\Omega_x$, we recall some notation. Let $P_{k,2}(t)$ be the Legendre polynomial of degree k and of dimension 3:

$$P_{k,2}(t) = \left(\frac{-1}{2}\right)^2 \frac{1}{k!} \frac{d^k}{dt^k} (1-t^2)^k = \sum_{l=0}^{[k/2]} (-1)^l \frac{\Gamma(k-l+1/2)}{l!(k-2l)!\sqrt{\pi}} (2t)^{k-2l}.$$

We define the extended Legendre polynomial by

$$P_{k,2}(z, w) = (\sqrt{z^2})^k (\sqrt{w^2})^k P_{k,2}\left(\frac{z}{\sqrt{z^2}} \cdot \frac{w}{\sqrt{w^2}}\right), \quad z, w \in \mathbf{C}^3.$$

Let

$$J_\nu(t) = \left(\frac{t}{2}\right)^\nu \sum_{l=0}^{\infty} \frac{1}{l!\Gamma(\nu + l + 1)} \left(\frac{it}{2}\right)^{2l}, \quad \nu \neq -1, -2, \cdots,$$

be the Bessel function of order ν. We put

$$\tilde{J}_\nu(t) = \Gamma(\nu+1)\left(\frac{t}{2}\right)^{-\nu} J_\nu(t) = \sum_{l=0}^{\infty} \frac{\Gamma(\nu+1)}{l!\Gamma(\nu+l+1)}\left(\frac{it}{2}\right)^{2l}, \ \tilde{j}_k(t) = \tilde{J}_{k+1/2}(t). \tag{2}$$

When $\nu > 0$, $|\tilde{J}_\nu(t)| \le e^{|t|}$ for $t \in \mathbf{C}$ and $0 < \cos t < |\tilde{J}_\nu(t)|$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$. Further we know that $\lim_{\nu\to\infty} \tilde{J}_\nu(t) = 1$ for $t \in \mathbf{C}$ and $\nu > 0$. (See Lemma 5.13 in [5].)

Then by using the extended Legendre polynomials and the modified Bessel functions, the exponential function is represented as follows:

$$\exp(z \cdot w) = \sum_{k=0}^{\infty} \frac{\sqrt{\pi} N(k,2)}{2^{k+1}\Gamma(k+\frac{3}{2})} \tilde{j}_k(i\sqrt{z^2}\sqrt{w^2}) P_{k,2}(z,w),$$

where $N(k,2) = 2k+1$. Note that $\tilde{j}_k(-t) = \tilde{j}_k(t)$. For $\eta, \zeta \in \mathbf{C}^3$, we know

$$\begin{aligned}\int_{S_r^2} \exp(ix\cdot\eta)\exp(x\cdot\zeta)d\Omega_x &= \sum_{k=0}^{\infty} \frac{\pi N(k,2) r^{2k}}{2^{2k+2}\Gamma(k+\frac{3}{2})^2} \tilde{j}_k(r\sqrt{\eta^2})\tilde{j}_k(ir\sqrt{\zeta^2}) P_{k,2}(\eta,\zeta) \\ &= \tilde{j}_0\left(ir\sqrt{(\zeta+i\eta)^2}\right).\end{aligned}$$

For this calculation see [4] and [5], for example. By (2), we note that

$$\tilde{j}_0(t) = \frac{\sqrt{\pi}}{2} \sum_{l=0}^{\infty} \frac{1}{l!\Gamma(3/2+l)}\left(\frac{it}{2}\right)^{2l}.$$

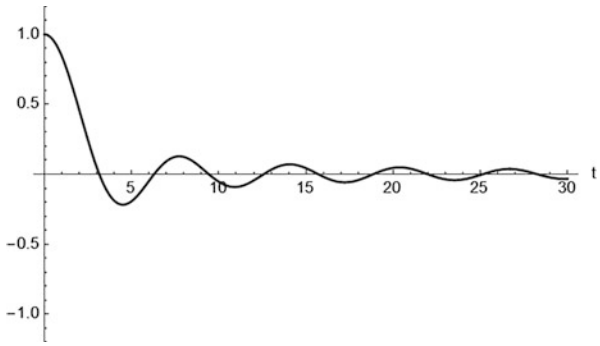
Therefore,

$$\begin{aligned}(\mathscr{G}_{\omega_0} f_y)(\tau,a) &= \frac{1}{ab}\int_{S_r^2} G_{\omega_0}\left(\frac{x-\tau}{a}\right) e^{-(x-y)^2/2b^2} d\Omega_x \\ &= \frac{1}{ab} e^{i\tau\cdot\frac{\omega_0}{a}} e^{-\frac{1}{2}\left(\frac{r^2+\tau^2}{a^2}+\frac{2r^2}{b^2}\right)} \tilde{j}_0\left(ir\sqrt{(\frac{\tau}{a^2}+\frac{y}{b^2}-i\frac{\omega_0}{a})^2}\right).\end{aligned}$$

By (1), we have

$$|(\mathscr{G}_{\omega_0} f_y)(\tau,a)| \le \frac{1}{ab} e^{-\frac{1}{2}\left(\frac{r^2+\tau^2}{a^2}+\frac{2r^2}{b^2}\right)} e^{\left(\frac{r\sqrt{\tau^2}}{a^2}+\frac{r^2}{b^2}\right)} = \frac{1}{ab}\exp\left(-\frac{(r-\sqrt{\tau^2})^2}{2a^2}\right).$$

For $t \in \mathbf{R}$, the graph of $\tilde{j}_0(t)$ is as follows:



3.3 Gabor Wavelet Transform of a Constant Function

When $f(x) = C \in L^2(S_r^2)$ is a constant function,

$$
\begin{aligned}
(\mathscr{G}_{\omega_0} f)(\tau, a) &= \frac{C}{a} \int_{S_r^2} G_{\omega_0}\left(\frac{x-\tau}{a}\right) d\Omega_x \\
&= \frac{C}{a} \int_{S_r^2} e^{-i(x-\tau)\cdot\frac{\omega_0}{a}} e^{-\frac{1}{2}\left(\frac{x-\tau}{a}\right)^2} d\Omega_x \\
&= \frac{C}{a} e^{i\tau\cdot\frac{\omega_0}{a}} e^{-\frac{1}{2}\left(\frac{r^2+\tau^2}{a^2}\right)} \int_{S_r^2} e^{-ix\cdot\frac{\omega_0}{a}} e^{x\cdot\frac{\tau}{a^2}} d\Omega_x. \qquad (3)
\end{aligned}
$$

Thus as we calculated in [4], we have

$$
(\mathscr{G}_{\omega_0} f)(\tau, a) = \frac{C}{a} e^{i\frac{\tau\cdot\omega_0}{a}} e^{-\frac{r^2+\tau^2}{2a^2}} \tilde{j}_0\left(r\frac{\sqrt{(a\omega_0 + i\tau)^2}}{a^2}\right).
$$

By (3), we have

$$
|\mathscr{G}_{\omega_0} f(\tau, a)| \leq \frac{C}{a} e^{-\frac{r^2+\tau^2}{2a^2}} e^{\frac{r\sqrt{\tau^2}}{a^2}} = \frac{C}{a} \exp\left(-\frac{(r-\sqrt{\tau^2})^2}{2a^2}\right).
$$

For $r, r' > 0$ and put $\tau_0 = r'\tau_0'$, $\tau_0' \in S_1^2$ and $x_0 = r\tau_0'$. Let f be a function on S_r^2 and let $(\mathscr{G}_{\omega_0} f)(\tau, a)$ be the Gabor wavelet transform of f. By examples as above, for a sufficiently small a, the graph of $|(\mathscr{G}_{\omega_0} f)(\tau, a)|$ near τ_0 tells us some behavior of f near x_0.

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Application of Complex Continuous Wavelet Analysis to Auditory Evoked Brain Responses

Nobuko Ikawa, Akira Morimoto, and Ryuichi Ashino

Abstract Most infants learn words by hearing. Hearing tests for infants are very important because infants who are hard of hearing often need special training to learn language skills. Unfortunately, hearing tests take 30 min, during which time the infant must often be under anesthesia. Therefore, reducing the test time would be beneficial. This paper proposes the use of a new hearing test based on the complex continuous wavelet analysis as a possible faster alternative.

1 Introduction

It is well known that Helen Adams Keller tried her utmost efforts to get language skills. For infants who are hard of hearing, the earlier the infants have hearing test, the more the language skills infants have. Nowadays, all the newborn infants have the automated ABR hearing test (see [6]). If an infant does not pass the automated ABR hearing test, he or she needs to have the ASSR hearing test further. These tests use EEG. Unfortunately, the automated ABR hearing test takes about 15 min, and the ASSR hearing test takes about 30 min.

All the newborn infants have the automated ABR hearing test during sleep. When an infant has the ASSR hearing test, the infant must often be under anesthesia. By this reason, reducing test times of the automated ABR hearing test and the ASSR hearing test is indispensable. We propose the use of a new hearing test based on the complex continuous wavelet analysis as a possible faster alternative.

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_52

Wavelet analysis is a new time-frequency analysis, more precisely, time-scale analysis (see [12]). Several applications of wavelet analysis in the neurosciences are given in [2]. The auditory nerve relays electrical nerve impulses from ears to the cerebrum. We record activities of the auditory neuron group at midbrain as electroencephalogram (EEG). This EEG is called an auditory evoked brain response (AEBR). AEBRs are used to assist human objective audiometry tests. For the tests, an auditory brainstem response (ABR) and an auditory steady-state response (ASSR) are commonly used.

Some earlier applications of wavelets to ABR were presented by Hanrahan [4, 5]. There are many papers to apply wavelet analysis to ABR and ASSR. Bradley and Wilson [1] applied wavelet analysis to auditory evoked potentials (AEPs). The proposed method by Bradley and Wilson uses the 2048 averaged ABR data.

In [7, 8, 10], we proposed a new method which uses only 10 averaged ABR data. Our proposed method reduces observation time. In this paper, we apply the one-dimensional complex continuous wavelet analysis (CCWA) to ABRs. Several experiments show that ABR consists of three groups in the time domain. By watching results of applying the CCWA to ASSRs, we propose a new averaging method based on the Galambos idea. Several experiments demonstrate that our proposed method is seven times faster than the conventional methods.

2 One-Dimensional Complex Continuous Wavelet Analysis

A mother wavelet is a function $\psi \in L^2(\mathbf{R})$ with zero average: $\int_{-\infty}^{+\infty} \psi(t)\,dt = 0$. Usually, it is normalized $\|\psi\| = 1$ and centers in the neighborhood of $t = 0$. For $a > 0$ and $b \in \mathbf{R}$, we define $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)$. Note that $\|\psi_{a,b}\| = 1$. The continuous wavelet transform of $x \in L^2(\mathbf{R})$ at (a, b) is defined by

$$W_\psi[x(t)](a,b) = C(a,b) = \langle x, \psi_{a,b} \rangle = \int_{-\infty}^{\infty} x(t)\psi_{a,b}^*(t)\,dt,$$

where ψ^* denotes the complex conjugate of ψ. For the mother wavelet, we choose the complex Morlet wavelet function $\psi(t)$ defined by $\psi(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} e^{i\omega_0 t}$, which is illustrated in Fig. 1. The real part $\Re\psi(t)$ of $\psi(t)$ is reflection symmetric at $t = 0$. The imaginary part $\Im\psi(t)$ of $\psi(t)$ is point symmetric at $(0, 0)$. Here, we put $\sigma = \frac{\sqrt{3}}{2}$, $\omega_0 = 2\pi$ and used the MATLAB(2016a) function:

```
[PSI,X] = cmorwavf(LB,UB,N,FB,FC)
```

for the complex Morlet wavelet with `LB=-5, UB=5, N=1000, FB=1.5, FC=1.`

In this paper, the wavelet analysis using the Morlet wavelet is called the one-dimensional complex continuous wavelet analysis (CCWA).

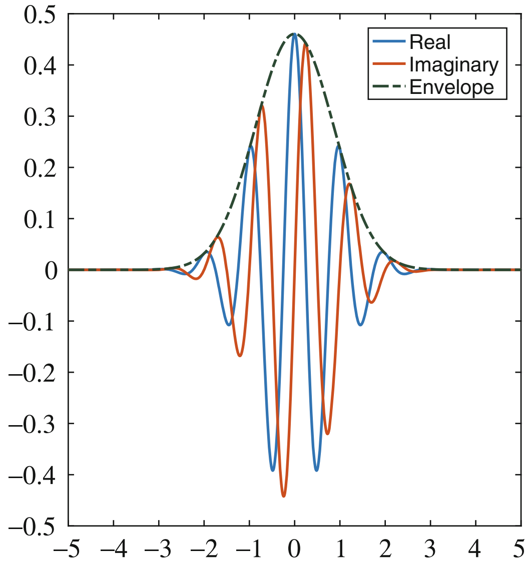

Fig. 1 The complex Morlet wavelet

3 Apply CCWA to ABR

The ABRs used in this paper were recorded in an acoustically quiet room. Subjects reclined in a comfortable chair or lay on a bed. An electrode was placed high on each subject's vertex. Two electrodes were placed on the earlobes of both ears. The ground electrode was placed on the forehead. The subjects were healthy 20-year-old male adults.

As input stimuli, acoustic stimuli composed of clicks with 70 dB nHL intensity, 0.1 ms duration, and 20 Hz frequency. Here, the decibel normal hearing level (dB nHL) values are referred to hearing thresholds of normal hearing subjects. We stored 512-points EEG data after the click stimulus. The 512-points data is called an epoch. The duration of one epoch is 10.24 ms, since the sampling rate is 50,000 Hz.

Definition 1 Denote by Epoch_k, the kth epoch. Define $\text{ABR}_N = \frac{1}{N}\sum_{k=1}^{N} \text{Epoch}_k$. We call ABR_N by N-average ABR.

In this paper, 2000-average ABR is simply called ABR. Figure 2 shows the results of applying the CCWA to two normal samples of ABRs. Each waveform (upper graph) shows the original waveform of ABR. Each intensity image (lower graph) shows the modulus of the continuous wavelet transform $C_{a,b} = C(a, b)$. Yellow indicates high modulus, and black indicates low modulus. The horizontal axis indicates time from 0 to 10 ms. These graphs of two examples show the typical characteristics of the peaks of the normal ABR waveforms. From the results of CCWA, we observed that ABR waveform has the three frequency bands. The first

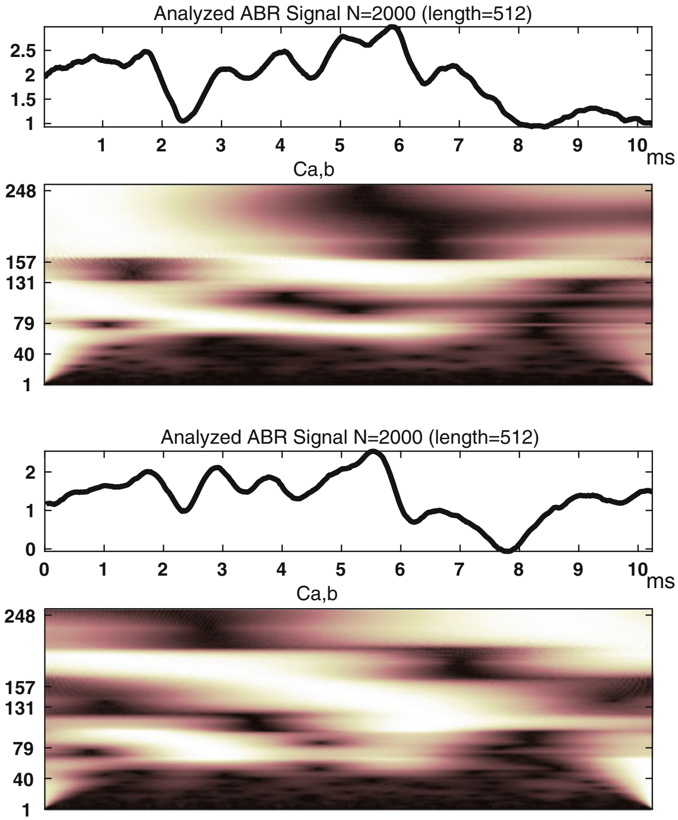

Fig. 2 Apply CCWA to ABRs. The upper ABR has the fourth peak at 5 ms. The lower ABR does not have this peak

group consists of the time interval from 0 to 3 ms. The second group consists of the time interval from 3 to 7 ms. The third group consists of the time interval from 7 to 10 ms. Furthermore, the second group has two yellow frequency bands. However, there is a difference between two modulus intensity images. The different frequency results around 5 ms suggest whether the fourth peak in ABR exists or not.

4 Apply CCWA to ASSR

For assessment of the clinical hearing level of infants, the MASTER (multiple auditory steady-state evoked response [11]) and Navigator PRO systems are useful. These systems use the 80-Hz ASSR. It takes 30 min to test the hearing level for five frequencies. Thirty minutes is quite a long time, especially for infants or very young children. Measuring the 80-Hz ASSR, we use a sedative, because subjects must be asleep.

On the other hand, the 40-Hz ASSR can be measured when subjects are awake. Therefore, a rapid objective audiometry test has been desired for the 40-Hz ASSR. The 40-Hz ASSR was recorded using our previous proposed hardware system shown in [9]. The input stimuli (sound conditions) were a sinusoidal amplitude-modulated (SAM) tone. We fixed a modulation frequency (MF) at 40 Hz. We selected a single carrier frequency (CF) of 1000 Hz. We started the sound intensity at 70 dB nHL. Then, we decreased the sound intensity to its threshold, in steps of 10 dB nHL. For normal hearing subjects, the 40-Hz ASSR is recorded as a waveform of the same frequency as the modulation frequency (40 Hz).

We recorded EEG for up to 30 s. The sampling frequency was 1024 Hz. We cut the digital data into epochs. One epoch consisted of 512-points data. The duration of one epoch was 500 ms. We needed to average at least 20 epochs with sound stimuli above 60 dB nHL. We needed to average at least 40 epochs with sound stimuli below 60 dB nHL. For 60 dB nHL, if we were unable to detect the 40-Hz ASSR by the average of 20 epochs, then we checked the average of 40 epochs. Since one epoch took 500 ms, we should measure EEG for 10 s or for 20 s.

In order to decrease time of measuring EEG for our original objective audiometry device, we design a new procedure of averaging waveforms of the 40-Hz ASSR. Our averaging method is based on the Galambos idea shown in Fig. 3. Since the sampling frequency is 1024 Hz, $1024/40 = 25.6 \approx 26$ points are shifted for one period of 40 Hz. For the sampling data $D = \{d[t] \mid t = 0, 1, 2, \ldots\}$ and $m \geq 1$, put

$$\mathbf{a}_k = (d[26(k-1)], d[26(k-1)+1], \ldots, d[26(k-1)+511]), \quad k = 1, \ldots, m.$$

Then, for $M \leq (m - 20)$, we define an averaged vector as $\mathbf{s}_M = \dfrac{1}{M} \sum_{k=21}^{M+20} \mathbf{a}_k$.

We have the following two observations. In the first step, we applied the CCWA to spontaneous EEG (non-evoked) waveforms. The CCWA results are shown in Fig. 4. The result of one epoch waveform $\mathbf{a}_1$ is shown in the left graph. The result of the averaged spontaneous EEG $\mathbf{S}_{40}$ is shown in the right graph. We can observe brain waves of periodic frequency, but we cannot observe 40 Hz waves. In Fig. 4, yellow indicates high modulus, and black indicates low modulus.

In the second step, we applied the CCWA to auditory evoked brain waveforms. The results are shown in Fig. 5. The CCWA results of one epoch waveform $\mathbf{a}_1$ with 70, 50, and 30 dB nHL are illustrated in the left side of Fig. 5. We cannot observe 40 Hz waves. The CCWA results of the averaged vectors are illustrated in the right side of Fig. 5. We can observe 40 Hz waves. We use $\mathbf{S}_{20}$ for 70 dB nHL and $\mathbf{S}_{40}$ for 50 or 30 dB nHL.

Several experiments demonstrate that we can detect the 40-Hz ASSR using $\mathbf{S}_{20}$ with sound stimuli above 60 dB nHL. Several experiments demonstrate that we can detect the 40-Hz ASSR using $\mathbf{S}_{40}$ below 60 dB nHL. We need to measure EEG

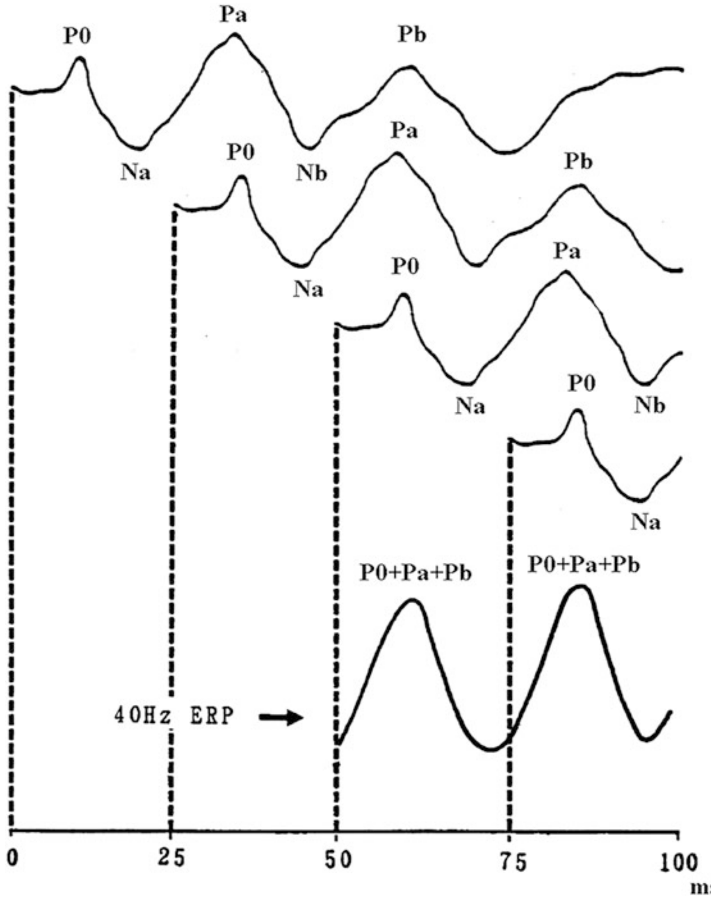

Fig. 3 Relationship of the 40-Hz event related potential (ERP) and middle latency response [3]

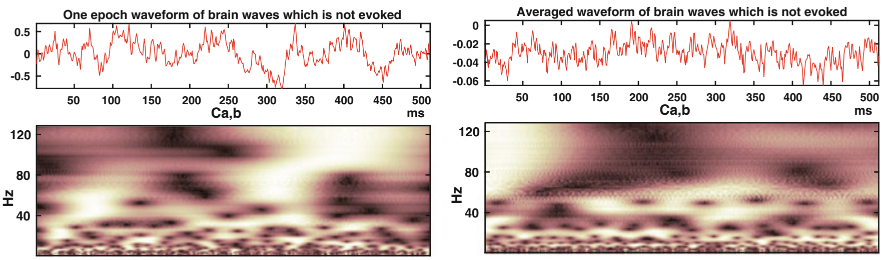

Fig. 4 Apply CCWA to non-evoked brain waveforms (spontaneous EEG). Left: one epoch $\mathbf{a}_1$. Right: the averaged vector $\mathbf{S}_{40}$

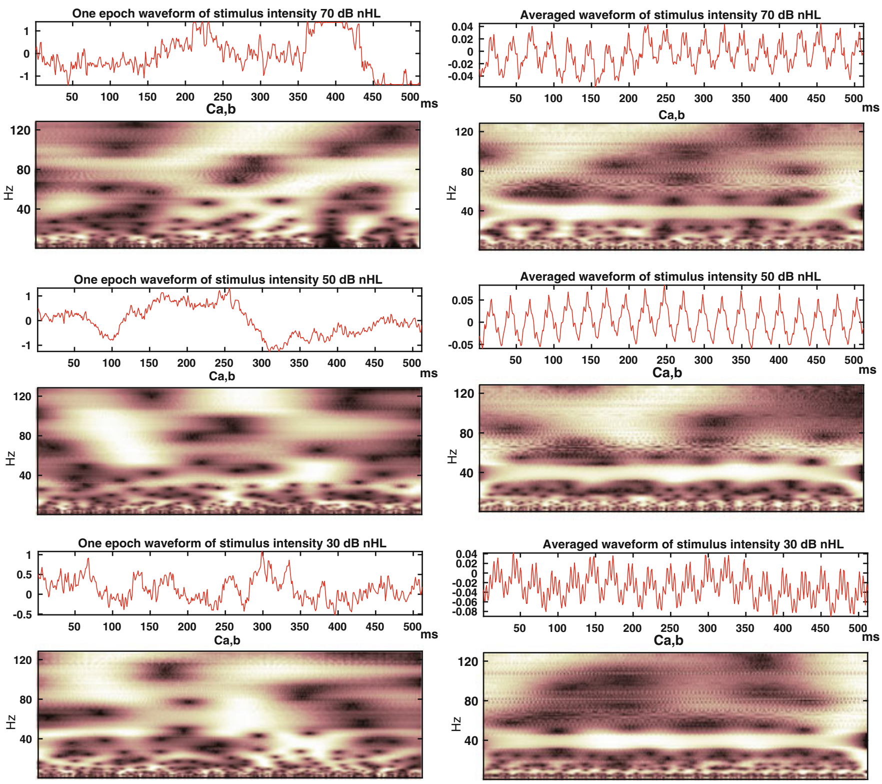

Fig. 5 Apply CCWA to 40-Hz ASSRs. Left: one epoch $\mathbf{a}_1$. Right: the averaged vector $\mathbf{S}_{20}$ for 70 dB nHL and $\mathbf{S}_{40}$ for 50 or 30 dB nHL

for 1.5 s or for 2 s. Then, our new averaging method is seven times faster than the conventional methods.

5 Conclusions

From the graphs of applying the CCWA to ABRs and ASSRs, we obtain the time-frequency characteristics of waveforms. The ABR consists of three groups in the time domain. Each group consists of multiple frequency bands.

In the case of 40-Hz ASSRs, for one epoch waveform, we have not observed the typical response around 40 Hz, when all subjects receive the sound stimuli. It means that the procedure of averaging epochs is essentially needed for the detection of the 40-Hz ASSR. We propose the averaging method based on the Galambos idea. We can detect the 40-Hz ASSR using $\mathbf{S}_{20}$ with sound stimuli above 60 dB nHL. We can detect the 40-Hz ASSR using $\mathbf{S}_{40}$ below 60 dB nHL. Our proposed method is seven times faster than the conventional methods.

The automated detection using the CCWA is a future work.

Acknowledgements We sincerely thank anonymous reviewers for their helpful comments. The investigation reported in this paper was supported by CFME of Chiba University. This work was partially supported by JSPS KAKENHI Grant Nos. JP17K05298 and JP17K05363.

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Detection of Rotation Angles on Image Separation Problem

Akira Morimoto, Ryuichi Ashino, and Takeshi Mandai

Abstract An image separation problem, where observed images are mixtures of rotated original images, is considered. Two procedures to detect rotation angles are proposed. Numerical experiments show the usefulness of proposed procedures.

1 Introduction

In 1953, Cherry studied the "cocktail party problem," that is, "how do we recognize what one person is saying when others are speaking at the same time?," in [1]. Cherry posed an open problem: "On what logical basis could one design a machine for carrying out such an operation?"

In 1991, Jutten et al. designed the machine in [2–4] and called their methods INdependent Components Analysis (INCA, now it is called ICA). In their situation, we observe several mixtures of original signals, then we separate the mixtures into original signals. This inverse problem is called a blind source separation (BSS). ICA is a powerful tool for solving BSS problems, for example, see [5]. There are methods which can solve BSS problems without using ICA, for example, see [6].

We have been interested in time-scale analysis using wavelet transforms, and we have been applying it to BSS problems. In [7–9], we used a pair of wavelet transforms to separate speech signals without time delay. In [10], we applied the analytic wavelet transform to solve speech separation problems with time delay. We considered image separation problems and proposed to use the continuous multi-wavelet transforms in [11, 12]. We proposed N-tree discrete wavelet transforms in [13]. We proposed a source reduction method by Gaussian elimination in [14]. We

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_53

treated image separation with translation in [15]. In this paper, we consider image separation problems where observed images are mixtures of rotated original images.

2 Image Mixing Model

Let us define the translation operator T_c and the rotation operator R_θ on $L^2(\mathbf{R}^2)$ by

$$T_c f(x) = f(x - c),$$
$$R_\theta f(x) = f(P(-\theta)x),$$

where $f \in L^2(\mathbf{R}^2)$, $c \in \mathbf{R}^2$, $\theta \in [0, 360)$ [degree], and $P(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a rotation matrix.

Let $s_n \in L^2(\mathbf{R}^2)$, $n = 1, \ldots, N$ be original images, and let $y_j \in L^2(\mathbf{R}^2)$, $j = 1, \ldots, J$ be observed images. Assume that the observed images are given by the mixing model:

$$y_j = \sum_{n=1}^{N} a_{j,n} T_{c_{j,n}} R_{\theta_{j,n}} s_n, \tag{1}$$

where N is the number of original images, $a_{j,n}$ are mixing coefficients, $c_{j,n}$ are translation parameters, and $\theta_{j,n}$ are rotation angles. A sample of observed images is illustrated in Fig. 1. From J observed images, we first estimate unknown parameters, N, $a_{j,n}$, $c_{j,n}$, and $\theta_{j,n}$ and second separate original images. We assume that the number J of observed images is larger than or equal to the number N of original images. Under this assumption, after estimating all parameters, we separate original images.

In [15], we estimated translation parameters when all $\theta_{j,n}$ vanish. In this paper, we estimate rotation angles when all $c_{j,n}$ vanish. We treat the following mixing model:

$$y_j = \sum_{n=1}^{N} a_{j,n} R_{\theta_{j,n}} s_n, \qquad j = 1, 2. \tag{2}$$

First, we estimate the number N of original images. Next, we estimate

$$(\theta_{1,n} - \theta_{2,n}) (\mathrm{mod}\ 360) \quad \text{[degree]}, \qquad n = 1, \ldots, N,$$

which are called relative rotation angles in this paper. In our numerical simulations, we set $N = 6$. The original images are illustrated in Fig. 2 left. Mixing coefficients are uniformly distributed random numbers in $[0.2, 0.8]$, and rotation angles are

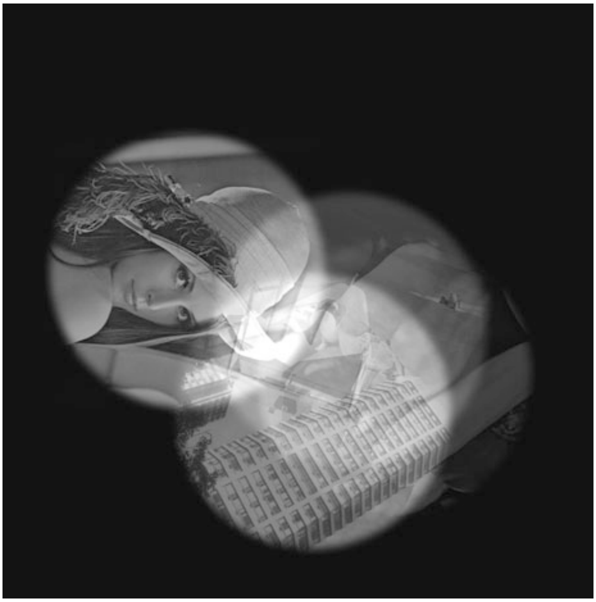


Fig. 1 A sample of observed images

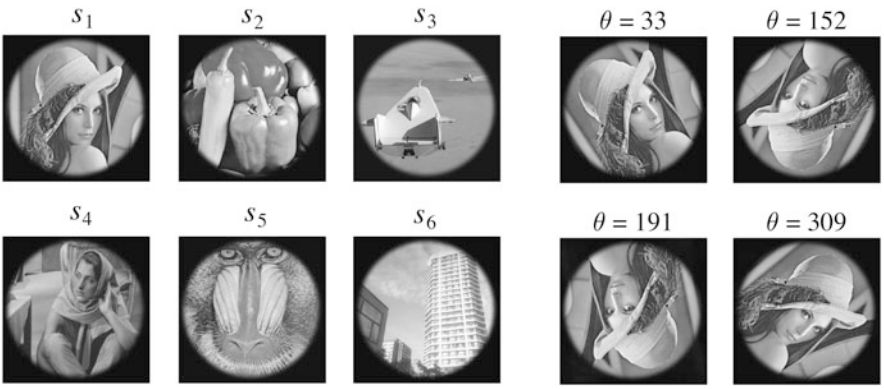


Fig. 2 Left: original images, s_n, $n = 1, \ldots, 6$. Right: rotated images of s_1 with rotation angles $\theta = 33$, 152, 191, and 309 [degree]

uniformly distributed random numbers between 0 and 360 [degree]. Let the mixing coefficients be

$$A = (a_{j,n}) = \begin{pmatrix} 0.7063 & 0.5064 & 0.7379 & 0.5986 & 0.2506 & 0.2107 \\ 0.4301 & 0.7653 & 0.4034 & 0.3448 & 0.2904 & 0.6540 \end{pmatrix}$$

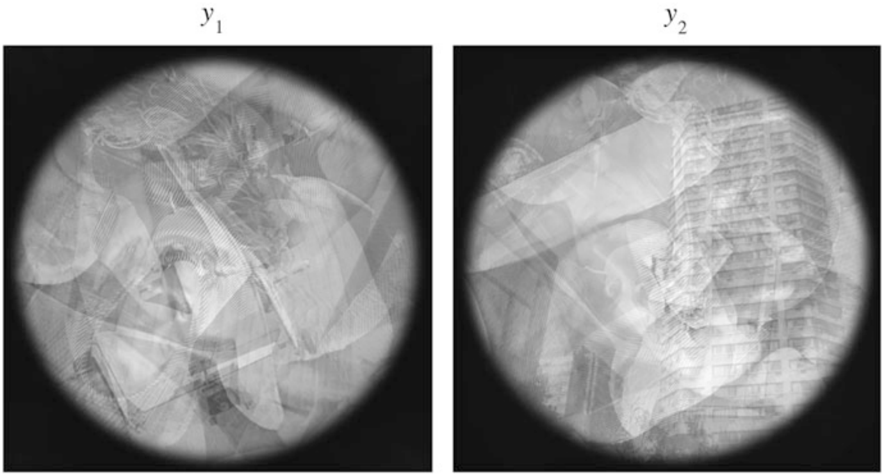


Fig. 3 A sample pair of observed images

and the rotation angles be

$$\Theta = (\theta_{j,n}) = \begin{pmatrix} 247.50 & 26.35 & 23.36 & 154.99 & 113.03 & 254.71 \\ 217.14 & 303.18 & 84.30 & 311.86 & 50.81 & 359.27 \end{pmatrix}.$$

Then we have observed images y_1 and y_2 illustrated in Fig. 3. Our purpose is to estimate the number N of original images and the relative rotation angles $(\theta_{1,n} - \theta_{2,n})(\text{mod } 360)$. In this simulation,

$$N = 6,$$
$$\left(\theta_{1,n} - \theta_{2,n}\right)_{n=1,\ldots,N} = \left(30.36\ 83.17\ 299.06\ 203.13\ 62.22\ 255.44\right). \quad (3)$$

3 Rotation Angle Detection

Let us introduce our procedure to estimate the number N of original images and the relative rotation angles from two observed images. In the following two procedures, $\tilde{N}$ denotes an estimate of N, and $\tilde{\theta}_n$, $n = 1, \ldots, \tilde{N}$ denote estimates of the relative rotation angles.

1. Apply the linear edge extraction filter to observed images.
2. Take an inner product between the edge image of the first observed image and the rotated edge image of the second observed image.
3. Plot the graph of the inner product versus rotation angle, see Fig. 6.
4. Let $\tilde{N}$ be the number of peaks in the graph.
5. Let $\tilde{\theta}_n$, $n = 1, \ldots, \tilde{N}$ be the rotation angles which attain peaks in the graph.

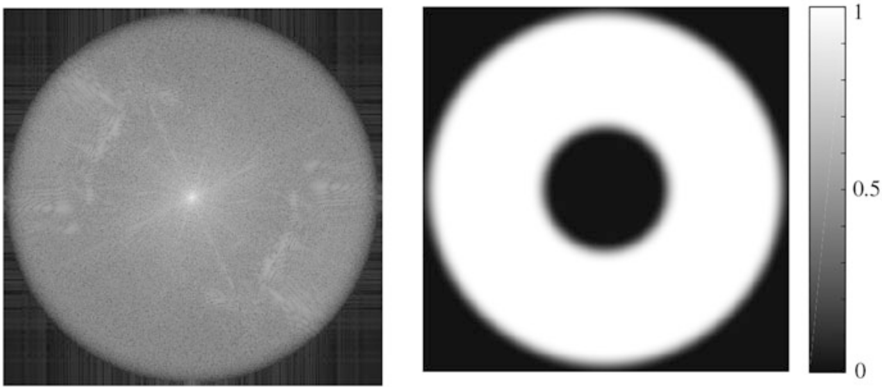


Fig. 4 Left: the Fourier image of y_1 (log scale). Right: the Fourier mask for edge extraction

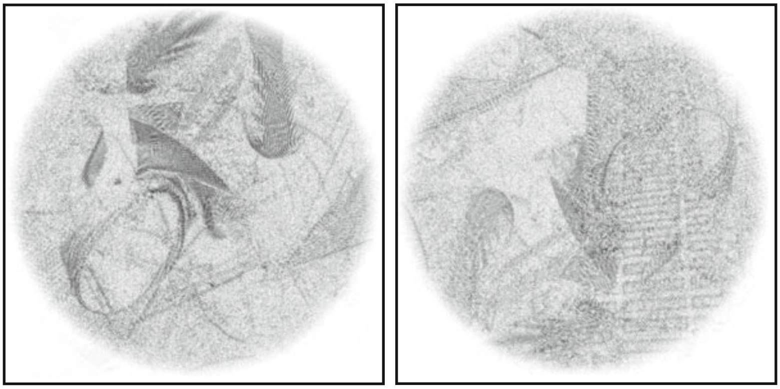


Fig. 5 Negatives of the absolute values of edge images. Left: y_1 and right: y_2

In the first step, take the Fourier transform of the observed image y_1 (see Fig. 4 left). Multiply the mask illustrated in Fig. 4 right, and apply the inverse Fourier transformation. Then, we have an edge image of y_1. Figure 5 shows negatives of the absolute values of edge images.

We illustrate the graph of the inner product versus rotation angle in Fig. 6. The number of peaks in the graph is six, which coincides with our estimation of the number N of original images. The angles which attain peaks in the graph indicate our estimations of the relative rotation angles.

We propose another procedure to estimate the number N of original images and the relative rotation angles in the Fourier space.

1. Apply the Fourier transformation to observed images.
2. Apply the mask illustrated in Fig. 4 right to the Fourier observed images.

Fig. 6 The graph of the inner product versus rotation angle. Crosses correspond to the relative rotation angles in (3)

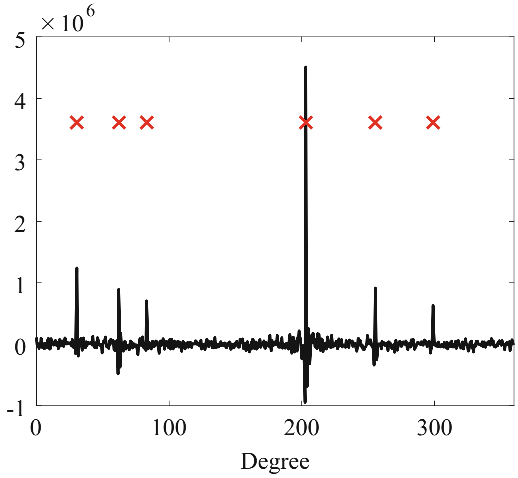
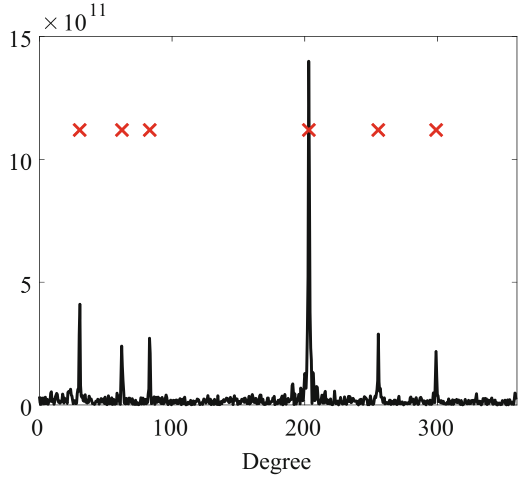


Fig. 7 The graph of the absolute value of the inner product versus rotation angle. The inner product is calculated in the Fourier space. Crosses correspond to the relative rotation angles in (3)



3. Take an inner product between the first masked Fourier image and the rotated second masked Fourier image.
4. Plot the graph of the absolute value of the above inner product versus rotation angle, see Fig. 7.
5. Let $\tilde{N}$ be the number of peaks in the graph.
6. Let $\tilde{\theta}_n, n = 1, \ldots, \tilde{N}$ be the rotation angles which attain peaks in the graph.

Figure 7 is the graph of the absolute value of the inner product versus rotation angle. We can estimate the number of original images and the relative rotation angles.

4 Conclusion and Feature Works

Under the condition described in Sect. 2, many numerical experiments suggest that our proposed procedures can estimate the number of original images and the relative rotation angles. We need a method to estimate ratios of mixing coefficients.

The computational costs of our proposed methods are rather high. To reduce the computational costs, we use the polar coordinate system of edge images illustrated in Fig. 8 and a circular convolution to make a similar graph of Fig. 6. This graph illustrated in Fig. 9 is noisier than Figs. 6 and 7. We cannot detect six peaks in Fig. 9. To treat the mixing model equation (1), more computational cost is needed. To reduce the computational cost, a modification of the polar coordinate system will be needed.

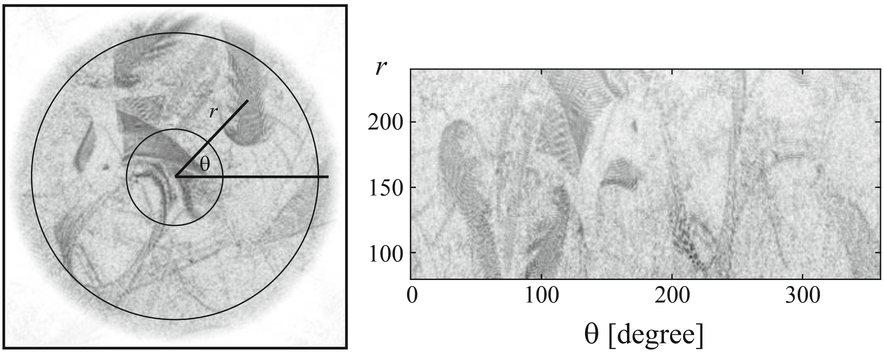


Fig. 8 Left: making the polar coordinate system. Right: the polar coordinate system of the first edge image

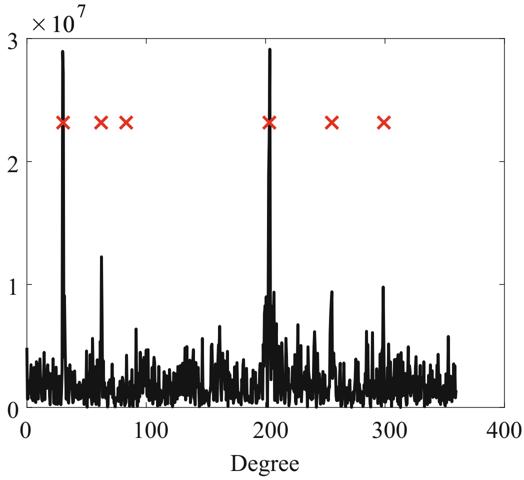


Fig. 9 The graph of the absolute value of the circular convolution versus rotation angle. Crosses correspond to the relative rotation angles in (3)

Acknowledgements We sincerely thank anonymous reviewers for their helpful comments. This work was partially supported by JSPS KAKENHI Grant Nos. JP16K05216, JP17K05298, andJP17K05363.

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Part XIII
Contributed Talks (Open Session)

Session Organizers: Jonas Fransson and Joachim Toft

This session is devoted for talks in analysis, its applications, and computations. The fields of the lectures were different. For example, some lectures dealt with questions on partial differential equations.

Uniform Boundary Stabilization of the Wave Equation with a Nonlinear Delay Term in the Boundary Conditions

Wassila Ghecham, Salah-Eddine Rebiai, and Fatima Zohra Sidiali

Abstract A wave equation in a bounded and smooth domain of $\mathbb{R}^n$ with a delay term in the nonlinear boundary feedback is considered. Under suitable assumptions, global existence and uniform decay rates for the solutions are established by adopting an approach due to Lasiecka and Tataru (Differ Integral Equ 6:507–533, 1993). The proof of existence of solutions relies on a construction of a suitable approximating problem for which the existence of solution will be established using nonlinear semigroup theory and then passage to the limit gives the existence of solutions to the original problem. The uniform decay rates for the solutions are obtained by proving certain integral inequalities for the energy function and by establishing a comparison theorem which relates the asymptotic behaviour of the energy and of the solutions to an appropriate dissipative ordinary differential equation.

1 Introduction

In [2], Datko et al. showed that an arbitrarily small time delay in the feedback may destabilize a wave system which is otherwise exponentially stable. Xu et al. [7] established sufficient conditions that guarantee the exponential stability of the one-dimensional wave equation with a delay term in the linear boundary feedback. Nicaise and Pignotti [6] extended this result to the multi-dimensional wave equation with a delay term in the linear boundary or internal feedback. In this paper, we study the problem of stability for the multi-dimensional wave equation with a delay term in the nonlinear boundary feedback.

Let Ω be an open bounded domain of $\mathbb{R}^n$ with smooth boundary Γ which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Let $\nu(.)$ denote the unit normal on Γ pointing towards the exterior of Ω. In Ω, we consider the

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K.-O. Lindahl et al. (eds.), *Analysis, Probability, Applications, and Computation*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-04459-6_54

wave equation with a nonlinear delay term in the boundary conditions

$$
\begin{aligned}
&u_{tt}(x,t) - \Delta u(x,t) = 0 \quad \text{in } \Omega \times (0,+\infty), \\
&u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) \text{ in } \Omega, \\
&u(x,t) = 0 \;\text{ on }\; \Gamma_1 \times (0,+\infty), \\
&\frac{\partial u}{\partial \nu}(x,t) = -\alpha_1 f(u_t(x,t)) - \alpha_2 g(u_t(x,t-\tau)) \;\text{ on } \Gamma_2 \times (0,+\infty), \\
&u_t(x,t-\tau) = f_0(x,t-\tau) \;\text{ on } \Gamma_2 \times (0,\tau),
\end{aligned} \tag{1}
$$

where $\frac{\partial u}{\partial \nu}$ is the normal derivative and $\tau > 0$ is the time delay. Moreover, α_1 and α_2 are positive constants, u_0, u_1, and f_0 are the initial data which belong to appropriate Hilbert spaces, and f and g are real-valued functions of class $C(\mathbb{R})$.

In absence of delay ($\alpha_2 = 0$), stability problems for (1) have been extensively treated in the literature (see [1, 3, 4, 8]), and the energy estimates obtained depend on the nonlinear function f.

The main purpose of this paper is to study the asymptotic behaviour of the solutions of (1) in the case where both α_1 and α_2 are different from zero. To this aim, we need to make the following assumptions:

(**H1**)

- (i) f is a continuous monotone increasing function on $\mathbb{R}$;
- (ii) $f(0) = 0$ and $sf(s) > 0$ for $s \neq 0$;
- (iii) $sf(s) \leq M_1 s^2$ for $|s| \geq 1$, for some $M_1 > 0$.

(**H2**)

- (i) g is an odd nondecreasing locally Lipschitz continuous function on $\mathbb{R}$;
- (ii) $g(0) = 0$ and $sg(s) > 0$ for $s \neq 0$;
- (iii) $sg(s) \leq M_2 s^2$ for $s \in \mathbb{R}$, for some $M_2 > 0$;
- (iv) $sg(s) \geq M_3 s^2$ for $|s| \geq 1$, for some $M_3 > 0$;
- (v) $a_1 sg(s) \leq G(s) \leq a_2 sf(s)$, where $G(s) = \int_0^s g(r)dr$, for some positive constants $a_1 and a_2$.

(**H3**) $\alpha_1 > \frac{a_2\alpha_2}{a_1}$.
(**H4**) There exists $x_0 \in \mathbb{R}^n$ such that with $m(x) = x - x_0$,

$$m(x).\nu(x) \leq 0 \quad \text{on } \Gamma_1.$$

Regarding the well-posedness of the solutions to system (1), we have the following result.

Theorem 1 *Assume* (**H1**) *and* (**H2**). *Then, for each* $(u_0, u_1, f_0) \in H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2, L^2(0, \tau))$, *system* (1) *has at least one solution*

$$u \in C_{loc}(0, +\infty; H^1_{\Gamma_1}(\Omega)) \cap C^1_{loc}(0, +\infty; L^2(\Omega)),$$

such that

$$u_t \in L^2_{loc}(0, +\infty; L^2(\Gamma_2)), \frac{\partial u}{\partial \nu} \in L^2_{loc}(0, +\infty; L^2(\Gamma_2)).$$

In order to state our stability result, we introduce as in [4] a real valued strictly increasing concave function $h(s)$ defined for $s \geq 0$ and satisfying

$$\begin{aligned} &h(0) = 0; \\ &h(sf(s)) \geq s^2 + f^2(s) \quad \text{for } |s| \leq N, \text{ for some } N > 0, \end{aligned}$$

and we define the following functions:

-
$$\widetilde{h}(s) = h(\frac{s}{mes\ \Sigma_2}), s \geq 0,$$

  where $\Sigma_2 = \Gamma_2 \times (0, T)$ and T is a given constant.
-
$$p(s) = (cI + \widetilde{h})^{-1} Ks,$$

  where c and K are positive constants. Then p is a positive, continuous, strictly increasing function with $p(0) = 0$.
-
$$q(s) = s - (I + p)^{-1}(s),\ s > 0,$$

  q is also a positive, continuous, strictly increasing function with $q(0) = 0$.

Let $E(t)$ be the energy function corresponding to the solution of (1) defined by

$$E(t) = \frac{1}{2} \int_\Omega \{|\nabla u(x, t)|^2 + |u_t(x, t)|^2\} dx + \frac{\xi}{2} \int_{\Gamma_2} \int_0^1 G(u_t(x, t - \rho\tau)) d\rho d\Gamma,$$

where the positive constant ξ is such that

(**H5**)

$$\frac{2\tau\alpha_2}{a_2}(1 - a_1) < \xi < \frac{2\tau}{a_2}(\alpha_1 - \alpha_2 a_2).$$

Theorem 2 *Assume hypotheses* $(H1)$–$(H5)$. *Let* (u, u_t) *be a solution of system* (1) *with the properties stated in Theorem 1. Then for some* $T_0 > 0$,

$$E(t) \leq S(\frac{t}{T_0} - 1)(E(0)) \qquad \textit{for } t > T_0,$$

where $S(t)$ *is the solution of the differential equation*

$$\frac{d}{dt}S(t) + q(S(t)) = 0, \quad S(0) = E(0).$$

If we additionally assume that the function $f(s)$ is of a polynomial growth at the origin, the following explicit decay rates are obtained.

Corollary 1 *Assume in addition to* $(H1)$–$(H5)$ *that there exists positive constants* b_1 *and* b_2 *such that*

$$f(s)s \leq b_1 s^2 \quad \textit{for all } s \in \mathbb{R},$$
$$f(s)s \geq b_2 |s|^{p+1} \quad \textit{for } |s| \leq 1, \textit{ for some } p \geq 1.$$

Then

$$E(t) \leq Me^{-\delta t} \quad \textit{if } p = 1,$$
$$E(t) \leq Mt^{\frac{2}{1-r}} \quad \textit{if } p > 1.$$

The rest of the paper is organized as follows: In Sect. 2, we sketch the proof of Theorem 1 and in Sect. 3, we outline the proof of Theorem 2.

2 Sketch of the Proof of Theorem 1

We first prove the theorem, by the arguments of nonlinear semigroup theory, for g Lipschitz continuous on $\mathbb{R}$ and f strongly monotone.

Then we consider the following approximation of problem (1) with l as the parameter of approximation:

$$\begin{aligned}
& u_{ltt}(x,t) - \Delta u_l(x,t) = 0 \quad \text{in } \Omega \times (0,+\infty), \\
& u_l(x,0) = u_0(x), \quad u_{lt}(x,0) = u_1(x) \text{ in } \Omega, \\
& u_l(x,t) = 0 \text{ on } \Gamma_1 \times (0,+\infty), \\
& \frac{\partial u_l}{\partial \nu}(x,t) = -\alpha_1 f_l(u_{lt}(x,t)) - \alpha_2 g_l(u_{lt}(x,t-\tau)) \quad \text{on } \Gamma_2 \times (0,+\infty), \\
& u_{lt}(x,t-\tau) = f_0(x,t-\tau) \quad \text{on } \Gamma_2 \times (0,\tau),
\end{aligned} \tag{2}$$

where

$$f_l(u_l(x,t)) = f(u_l(x,t)) + \frac{1}{l} u_{lt}(x,t)$$

and the functions g_l are defined by

$$g_l(s) = \begin{cases} g(s), & |s| \leq l \\ g(l), & s \geq l \\ g(-l), & s \leq -l. \end{cases}$$

Notice that for each value of the parameter l, the functions f_l are strongly monotone and the functions g_l are Lipschitz continuous on $\mathbb{R}$. Then there exists a solution (u_l, u_{lt}) of (2) such that

$$u_l \in C_{loc}(0, +\infty; H^1_{\Gamma_1}(\Omega)) \cap C^1_{loc}(0, +\infty; L^2(\Omega))$$

and

$$u_{lt} \in L^2_{loc}(0, +\infty; L^2(\Gamma_2)), \ \frac{\partial u_l}{\partial \nu} \in L^2_{loc}(0, +\infty; L^2(\Gamma_2)).$$

We prove that we can extract a subsequence from the above sequence of solutions u_l that has limit which is a solution of problem (1).

3 Sketch of the Proof of Theorem 2

By virtue of Lemma 2.2 in [4], it is enough to prove the theorem for smooth solutions

$$u \in C(0, T; H^2(\Omega)) \cap C^1(0, T; H^1_{\Gamma_1}(\Omega)).$$

We proceed in several steps.

Step 1
Differentiating $E(t)$ and applying Green's theorem, we conclude that the energy is decreasing and there exists a constant $C > 0$ such that

$$\frac{d}{dt} E(t) \leq -C \int_{\Gamma_2} \{u_t(x,t) f(u_t(x,t) + u_t(x,t-\tau) g(u_t(x,t-\tau)\} d\Gamma dt.$$

Here and throughout the rest of the paper C is a positive constant at different occurrences.

Step 2
Set

$$E(t) = \mathscr{E}(t) + E_d(t),$$

where

$$\mathscr{E}(t) = \frac{1}{2}\int_{\Omega}\{|\nabla u(x,t)|^2 + |u_t(x,t)|^2\}dx$$

and

$$E_d(t) = \frac{\xi}{2}\int_{\Gamma_2}\int_0^1 G(u_t(x,t-\rho\tau)d\rho d\Gamma.$$

From the mean value theorem and the monotonicity of g, we have

$$E_d(t) \le C\int_0^T\int_{\Gamma_2} u_t(x,t-\tau)g(u_t(x,t-\tau))d\Gamma dt.$$

Step 3
For $\mathscr{E}(t)$, we have the following estimate obtained by combining the multiplier techniques (multiplying both sides of the first equation in (1) by $m(x).\nabla u(x,t)$ and integrating over $\Omega \times (0,T)$) and the absorption of the tangential gradient by the normal derivative $\frac{\partial u}{\partial \nu}$ and u (Lemma 7.2 in [5]):

$$\int_0^T \mathscr{E}(t)dt \le C\{E(T) + \int_0^T\int_{\Gamma_2}\{u_t^2(x,t) + f^2(u_t(x,t)) + u_t(x,t-\tau)g(u_t(x,t-\tau))\}d\Gamma dt\} + C\,\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))},$$

where $0 < \epsilon < 1/2$ is small enough arbitrary but fixed.

Step 4
From Steps 2 and 3, we have

$$\int_0^T E(t)dt \le C\{E(T) + \int_0^T\int_{\Gamma_2}\{u_t^2(x,t) + f^2(u_t(x,t)) + u_t(x,t-\tau)g(u_t(x,t-\tau))\}d\Gamma dt\} + C\,\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))},$$

which in turn implies since $E(t)$ is nonincreasing

$$E(T) \leq C \int_0^T \int_{\Gamma_2} \{u_t^2(x,t) + f^2(u_t(x,t) + u_t(x,t-\tau)g(u_t(x,t-\tau))\} d\Gamma dt + C\,\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}.$$

Step 5
We drop the lower-order term on the right-hand side of the previous estimate by compactness/uniqueness argument to obtain

$$E(T) \leq C \int_0^T \int_{\Gamma_2} \{u_t^2(x,t) + f^2(u_t(x,t)) + u_t(x,t-\tau)g(u_t(x,t-\tau))\} d\Gamma dt.$$

Step 6
From the estimate established in Step 5 and the hypothesis $(H1)$, we get

$$p(E(T)) + E(T) \leq E(0).$$

Step 7
Now, from Step 6, we have for $m = 0, 1, 2, \ldots$

$$E(m(T+1)) + p(E(m(T+1)) \leq E(mT).$$

Set

$$s_m = E(mT), \quad s_0 = E(0).$$

It follows from Lemma 3.3 in [4] that

$$E(mT) \leq S(m), \quad m = 0, 1, 2, \ldots.$$

Let $t = mT + \tau$ and recall the evolution property, we obtain

$$E(t) \leq E(mT) \leq S(m) \leq S(\frac{t-\tau}{T}) \leq S(\frac{t}{T} - 1) \qquad \text{for } t > T.$$

This is the sought-after stability result.

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