



Upward Planar Morphs

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Abstract. We prove that, given two topologically-equivalent upward planar straight-line drawings of an n -vertex directed graph G , there always exists a morph between them such that all the intermediate drawings of the morph are upward planar and straight-line. Such a morph consists of $O(1)$ morphing steps if G is a reduced planar st -graph, $O(n)$ morphing steps if G is a planar st -graph, $O(n)$ morphing steps if G is a reduced upward planar graph, and $O(n^2)$ morphing steps if G is a general upward planar graph. Further, we show that $\Omega(n)$ morphing steps might be necessary for an upward planar morph between two topologically-equivalent upward planar straight-line drawings of an n -vertex path.

1 Introduction

One of the definitions of the word *morph* that can be found in English dictionaries is “to gradually change into a different image”. The Graph Drawing community defines the morph of graph drawings similarly. Namely, given two drawings Γ_0 and Γ_1 of a graph G , a *morph* between Γ_0 and Γ_1 is a continuously changing family of drawings of G indexed by time $t \in [0, 1]$, such that the drawing at time $t = 0$ is Γ_0 and the drawing at time $t = 1$ is Γ_1 . Further, the way the Graph Drawing community adopted the word *morph* is consistent with its Ancient Greek root $\mu\omega\rho\phi\acute{\eta}$, which means “shape” in a broad sense. Namely, if both Γ_0 and Γ_1 have a certain geometric property, it is desirable that all the drawings of the morph also have the same property. In particular, we talk about a *planar*, a *straight-line*, an *orthogonal*, or a *convex morph* if all the intermediate drawings of the morph are *planar* (edges do not cross), *straight-line* (edges are straight-line segments), *orthogonal* (edges are polygonal lines composed of horizontal and vertical segments), or *convex* (the drawings are planar and straight-line, and the faces are delimited by convex polygons), respectively.

The state of the art on planar morphs covers more than 100 years, starting from the 1914/1917 works of Tietze [25] and Smith [23]. The seminal papers of Cairns [13] and Thomassen [24] proved the existence of a planar straight-line morph between any two topologically-equivalent planar straight-line drawings of a graph. In the last 10 years, the attention of the research community focused

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on algorithms for constructing planar morphs with few *morphing steps* (see, e.g., [1–7, 11, 12, 21, 26]). Each morphing step, sometimes simply called *step*, is a *linear* morph, in which the vertices move along straight-line (possibly distinct) trajectories at uniform speed. A *unidirectional* morph is a linear morph in which the vertex trajectories are all parallel. It is known [2, 4] that a planar straight-line morph with a linear number of unidirectional morphing steps exists between any two topologically-equivalent planar straight-line drawings of the same graph, and that this bound is the best possible.

Upward planarity is usually regarded as the natural extension of planarity to directed graphs; see, e.g., [9, 10, 15, 16, 18]. A drawing of a directed graph is *upward planar* if it is planar and the edges are represented by curves monotonically increasing in the vertical direction. Despite the importance of upward planarity, up to now, no algorithm has been devised to morph upward planar drawings of directed graphs. This paper deals with the following question: Given two topologically-equivalent upward planar drawings Γ_0 and Γ_1 of an upward planar directed graph G , does an *upward planar straight-line morph* between Γ_0 and Γ_1 always exist? In this paper we give a positive answer to this question.

Problems related to upward planar graphs are usually more difficult than the corresponding problems for undirected graphs. For example, planarity can be tested in linear time [20] while testing upward planarity is NP-complete [18]; all planar graphs admit planar straight-line grid drawings with polynomial area [22] while there are upward planar graphs that require exponential area in any upward planar straight-line grid drawing [17]. Quite surprisingly, we show that, from the morphing point of view, the difference between planarity and upward planarity is less sharp; indeed, in some cases, upward planar straight-line drawings can be morphed even more efficiently than planar straight-line drawings.

More in detail, our results are as follows. Let Γ_0 and Γ_1 be topologically-equivalent upward planar drawings of an n -vertex upward plane graph G . We show algorithms to construct upward planar straight-line morphs between Γ_0 and Γ_1 with the following number of unidirectional morphing steps:

- i. $O(1)$ steps if G is a reduced plane *st*-graph (see Sect. 4);
- ii. $O(n)$ steps if G is a plane *st*-graph (see Sect. 4);
- iii. $O(n)$ steps if G is a reduced upward plane graph (see Sect. 5);
- iv. $O(n \cdot f(n))$ steps if G is a general upward plane graph, assuming that an $O(f(n))$ -step algorithm exists to construct an upward planar morph between any two upward planar drawings of any n -vertex plane *st*-graph (see Sect. 5). This, together with Result ii., yields an $O(n^2)$ -step upward planar morph for general upward plane graphs.

Further, we show (Sect. 3) that there exist two topologically-equivalent upward planar drawings of an n -vertex upward plane path such that any upward planar morph between them consists of $\Omega(n)$ morphing steps.

In order to prove Result i. we devise a technique that allows us to construct a morph in which each morphing step modifies either only the x -coordinates or only the y -coordinates of the vertices. Result ii. builds on the techniques in

[2] and leverages on the arrangement of low-degree vertices in upward planar drawings in order to morph maximal plane st -graphs. We then exploit such morphs for general plane st -graphs. In order to prove Results **iii.** and **iv.** we use an inductive technique for reducing the geometric differences between Γ_0 and Γ_1 .

Because of space limitations, some proofs are omitted or sketched. They can be found in the full version of the paper.

2 Preliminaries

We assume familiarity with graph drawing [15] and related concepts.

In this paper we only consider straight-line drawings. Thus, where it leads to no confusion, we will omit the term “straight-line”. Let Γ be a drawing of a graph G and let H be a subgraph of G . We denote by $\Gamma[H]$ the restriction of Γ to the vertices and edges of H . Two planar drawings of a connected graph are *topologically equivalent* if they have the same circular order of the edges around each vertex and the same cycle bounding the outer face. A *planar embedding* is an equivalence class of planar drawings. A *plane graph* is a planar graph equipped with a planar embedding. In a planar straight-line drawing an internal face (the outer face) is *strictly convex* if its angles are all smaller (greater) than π . A planar straight-line drawing is *strictly convex* if each face is strictly convex.

A *y-assignment* $y_G : V(G) \rightarrow \mathbb{R}$ is an assignment of reals to the vertices of a graph G . A drawing Γ of G *satisfies* y_G if the y -coordinate in Γ of each vertex $v \in V(G)$ is $y_G(v)$. An *x-assignment* x_G for G is defined analogously.

In a directed graph G we denote by uv an edge directed from a vertex u to a vertex v ; then v is a *successor* of u , and u is a *predecessor* of v . A *directed path* consists of the edges $u_i u_{i+1}$, for $i = 1, \dots, n - 1$. The *underlying graph* of G is the undirected graph obtained from G by omitting the directions from its edges. A *transitive edge* in a directed graph G is an edge uv such that G contains a directed path from u to v different from the edge uv . A *reduced graph* is a directed graph that does not contain any transitive edges.

A drawing of a directed graph is *upward planar* if it is planar and each edge uv is drawn as a curve monotonically increasing in the y -direction from u to v . A directed graph is *upward planar* if it admits an upward planar drawing. Consider an upward planar drawing Γ of an upward planar graph G . Let u, v , and w be three vertices consecutive and in this clockwise order along the boundary of a face f of G . We denote by $\angle(u, v, w)$ the angle formed by the (undirected) edges (u, v) and (v, w) in the interior of f . Also, we say that v is a *sink-switch* (*source-switch*) of f if uv and wv (vu and vw) are edges of G . Furthermore, we say that v is a *switch* of G if it is either a sink-switch or a source-switch of some face of Γ . Two switches u and v of a face f are *clockwise* (*counter-clockwise*) *consecutive* if traversing f clockwise (counter-clockwise) no switch is encountered in between u and v . The drawing Γ determines a *large-angle assignment*, that is, a labeling, for each face f and each three clockwise consecutive switches u, v , and w for f of the corresponding angle $\angle(u, v, w)$ as **large**, if it is larger than π in Γ , or **small**, it is smaller than π in Γ [9].

Two upward planar drawings of an upward planar graph G have the same *upward planar embedding* if they have the same planar embedding and the same large-angle assignment. We denote by $\ell(G)$ the number of switches labeled **large** in G . A combinatorial characterization of upward planar embeddings in terms of large angles is given in [9]. An *upward plane graph* is an upward planar graph equipped with an upward planar embedding.

Let Γ_0 and Γ_1 be upward planar drawings of an upward plane graph G . An *upward planar morph* is a continuous transformation from Γ_0 to Γ_1 indexed by time $t \in [0, 1]$ in which the drawing at each time $t \in [0, 1]$ is upward planar.

A *plane st-graph* is an upward plane graph with a single source s and a single sink t , and with an upward planar embedding in which s and t are incident to the outer face. A plane *st-graph* always admits an upward planar straight-line drawing [16]. A cycle in an upward plane graph is an *st-cycle* if it consists of two directed paths. A face f of an upward plane graph is an *st-face* if it is delimited by an *st-cycle*; the directed paths delimiting an *st-face* f are called *left* and *right boundary*, where the edge of the left boundary incident to the source-switch s_f of f immediately precedes the edge of the right boundary incident to s_f in the clockwise order of the edges incident to s_f . The following is well-known.

Lemma 1. *An upward plane graph is a plane st-graph iff all its faces are st-faces.*

An internal vertex v of a maximal plane *st-graph* G is *simple* if the neighbors of v induce a cycle in the underlying graph of G .

Lemma 2 (Alamdari et al. [2]). *Any maximal plane st-graph contains a simple vertex of degree at most 5.*

3 Slow Morphs and Fast Morphs

We start this section by proving the following lower bound.

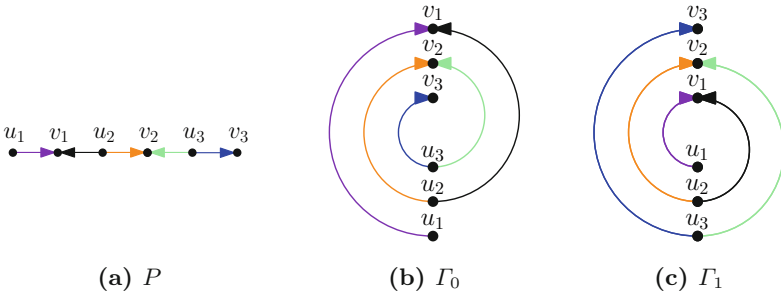


Fig. 1. Illustration for Theorem 1. (a) P ; (b) Γ_0 ; and (c) Γ_1 . For the sake of readability Γ_0 and Γ_1 have curved edges. However, the x -coordinates of the vertices can be slightly perturbed in order to make Γ_0 and Γ_1 straight-line.

Theorem 1. *There are two upward planar drawings of an n -vertex upward plane path such that any upward planar morph between them consists of $\Omega(n)$ steps.*

Proof sketch. Assume, for the sake of simplicity, that n is even, and let $n = 2k$. Consider the n -vertex upward plane path P defined as follows (refer to Fig. 1a). The path P contains vertices u_i and v_i , for $i = 1, \dots, k$, and directed edges $u_i v_i$, for $i = 1, \dots, k$, and $u_{i+1} v_i$, for $i = 1, \dots, k - 1$. We fix the upward planar embedding of P as in Fig. 1b and c.

Let Γ_0 and Γ_1 be two upward planar straight-line drawings of P in which the bottom-to-top order of the vertices is $u_1, \dots, u_k, v_k, \dots, v_1$ (see Fig. 1b) and $u_k, \dots, u_1, v_1, \dots, v_k$ (see Fig. 1c), respectively. Let $\langle \Gamma_0 = \Lambda_1, \Lambda_2, \dots, \Lambda_{h+1} = \Gamma_1 \rangle$ be any upward planar morph from Γ_0 to Γ_1 that consists of h morphing steps. We have the following.

Claim 1.1. *For each $j = 1, \dots, h + 1$, the vertices $u_j, u_{j+1}, \dots, u_{k-1}, u_k$ appear in this bottom-to-top order in Λ_j .*

By Claim 1.1 and since u_k, u_{k-1} appear in this bottom-to-top order in $\Gamma_1 = \Lambda_{h+1}$, we have that $h + 1 > k - 1$, hence $h \in \Omega(n)$. \square

We now establish a tool that will allow us to design efficient algorithms for morphing upward planar drawings. Consider two planar straight-line drawings Γ' and Γ'' of a plane graph G with the same y -assignment. Since the drawings are straight-line and have the same y -assignment, a horizontal line ℓ intersects a vertex or an edge of G in Γ' if and only if it intersects the same vertex or edge in Γ'' . We say that Γ' and Γ'' are *left-to-right equivalent* if, for any horizontal line ℓ , for any vertex or edge α of G , and for any vertex or edge β of G such that ℓ intersects both α and β (in Γ' and in Γ''), we have that the intersection of α with ℓ is to the left of the intersection of β with ℓ in Γ' if and only if the intersection of α with ℓ is to the left of the intersection of β with ℓ in Γ'' . The definition of *bottom-to-top equivalent* drawings is analogous. We have the following.

Lemma 3. *Any two upward planar drawings Γ' and Γ'' of a plane st -graph G with the same y -assignment are left-to-right equivalent.*

Proof. Since G is a plane st -graph, the drawings Γ' and Γ'' have the same faces. By Lemma 1 such faces are st -faces. Also, every horizontal line ℓ crosses an st -face f at most twice, and the left-to-right order of these crossings along ℓ is the same in Γ' and Γ'' because the left and right boundaries of f are the same in Γ' and Γ'' . The statement follows. \square

Lemma 4 is due to [2]. We extend it in Lemma 5.

Lemma 4 ([2], Corollary 7.2). *Consider a unidirectional morph acting on points p, q , and r . If p is on one side of the oriented line through \overline{qr} at the beginning and at the end of the morph, then p is on the same side of the oriented line through \overline{qr} throughout the morph.*

Lemma 5. *Let Γ' and Γ'' be two left-to-right or bottom-to-top equivalent planar drawings of a plane graph. Then the linear morph \mathcal{M} from Γ' to Γ'' is unidirectional and planar.*

Proof. Since Γ' and Γ'' have the same y -assignment (x -assignment), given that they are left-to-right (bottom-to-top) equivalent, it follows that all the vertices move along horizontal (vertical) trajectories. Thus, \mathcal{M} is unidirectional. Also, since Γ' and Γ'' are left-to-right (bottom-to-top) equivalent, each horizontal (vertical) line crosses the same sequence of vertices and edges in both Γ' and Γ'' . Thus, by Lemma 4, \mathcal{M} is planar. \square

Lemma 5 allows us to devise a simple morphing technique between any two upward planar drawings Γ_0 and Γ_1 of the same upward plane graph G , when a pair of upward planar drawings of G with special properties can be computed. We say that the pair (Γ_0, Γ_1) is an *hvh-pair* if there exist upward planar drawings Γ'_0 and Γ'_1 of G such that: (i) Γ_0 and Γ'_0 are left-to-right equivalent, (ii) Γ'_0 and Γ'_1 are bottom-to-top equivalent, and (iii) Γ'_1 and Γ_1 are left-to-right equivalent. Our morphing tool is expressed by the following lemma.

Lemma 6 (Fast morph). *Let (Γ_0, Γ_1) be an hvh-pair of upward planar drawings of an upward plane graph G . There is a 3-step upward planar morph from Γ_0 to Γ_1 .*

Proof sketch. We define the morph \mathcal{M} as $\langle \Gamma_0, \Gamma'_0, \Gamma'_1, \Gamma_1 \rangle$. The drawings Γ'_0 and Γ'_1 exist by hypothesis. Lemma 5 guarantees that \mathcal{M} is unidirectional and planar. We use Lemma 4 to prove that \mathcal{M} is upward. \square

The next lemma will allow us to restrict our attention to biconnected graphs.

Lemma 7. *Let Γ_0 and Γ_1 be two upward planar drawings of an n -vertex upward plane graph G whose underlying graph is connected. There exist upward planar drawings Γ'_0 and Γ'_1 of an $O(n)$ -vertex upward plane graph G' that is a supergraph of G , whose underlying graph is biconnected, and such that $\Gamma'_0[G] = \Gamma_0$ and $\Gamma'_1[G] = \Gamma_1$. Further, if G is reduced or an st -graph, then so is G' .*

Proof sketch. We iteratively apply the following procedure. Consider a cutvertex v of G and two edges that belong to distinct blocks of G and that are consecutive in the circular order of the edges incident to v . Let u and w be the end-vertices of such edges different from v . We add to G a vertex v' and two edges connecting v' with u and w ; these edges are oriented as the ones connecting v with u and w , respectively. By placing v' and its incident edges inside the face of G incident to v , u , and w , we obtain an upward plane supergraph of G with one block less than G . Upward planar drawings of this graph extending Γ_0 and Γ_1 can be easily obtained. The repetition of this procedure proves the lemma. \square

4 Plane st -Graphs

In this section, we show algorithms for constructing upward planar morphs between upward planar drawings of plane st -graphs.

4.1 Reduced Plane *st*-Graphs

We first consider plane *st*-graphs without transitive edges. We have the following.

Lemma 8. *Any two upward planar drawings Γ_0 and Γ_1 of a reduced plane *st*-graph G form an hvh-pair.*

Proof sketch. By Lemma 7 we can assume that G is biconnected. We construct two upward planar drawings Γ'_0 and Γ'_1 that, together with Γ_0 and Γ_1 , satisfy Conditions (i)–(iii) of the definition of hvh-pair. We construct Γ'_0 and Γ'_1 as follows. First, we draw the left boundary of the outer face of G so that each vertex has the same y -coordinate in Γ'_i as in Γ_i , for $i = 0, 1$. In both Γ'_0 and Γ'_1 the x -coordinates of all the vertices of this path are 0. Then, we add to Γ'_0 and Γ'_1 the right boundaries of the *st*-faces of G one by one, following a topological sorting of the oriented dual graph of G . In Γ'_0 (in Γ'_1) we assign to the internal vertices of each right boundary the same y -coordinates they have in Γ_0 (Γ_1); since G is reduced, the set of these vertices is non-empty. All the internal vertices of each right boundary get the same x -coordinate, which is used in both Γ'_0 and Γ'_1 ; this x -coordinate is sufficiently large so that no crossing is introduced. \square

Combining Lemma 6 with Lemma 8 we obtain the following result.

Theorem 2. *Let Γ_0 and Γ_1 be any two upward planar drawings of a reduced plane *st*-graph. There is a 3-step upward planar morph from Γ_0 to Γ_1 .*

4.2 General Plane *st*-Graphs

We now turn our attention to general plane *st*-graphs. We restate here, in terms of plane *st*-graphs, a result by Hong and Nagamochi [19] that was originally formulated in terms of hierarchical plane (undirected) graphs.

Theorem 3 ([19], **Theorem 8**). *Consider an internally 3-connected plane *st*-graph G and let y_G be a y -assignment of the vertices of G such that each vertex v is assigned a value $y_G(v)$ that is greater than those assigned to its predecessors. There exists a strictly-convex upward planar drawing of G satisfying y_G .*

We use Theorem 3 to prove the following theorem, which allows us to restrict our attention to maximal plane *st*-graphs.

Theorem 4. *Let Γ_0 and Γ_1 be two upward planar drawings of an n -vertex plane *st*-graph G . Suppose that an algorithm \mathcal{A} exists that constructs an $f(r)$ -step upward planar morph between any two upward planar drawings of an r -vertex maximal plane *st*-graph. Then there exists an $O(f(n))$ -step upward planar morph from Γ_0 to Γ_1 .*

Proof sketch. By Lemma 7 we can assume that G is biconnected. We augment G to a maximal plane *st*-graph G^* by inserting a vertex v_f into each face f of G and by inserting a directed edge from the source-switch s_f of f to v_f

and directed edges from v_f to every other vertex incident to f . We define a y -assignment $y_{G^*}^0$ for G^* by setting $y_{G^*}^0(v) = y_G^0(v)$ for each vertex $v \in V(G)$ and by setting, for each vertex $v_f \in V(G^*) \setminus V(G)$, a value for $y_{G^*}^0(v_f)$ that is larger than $y_{G^*}^0(s_f)$ and smaller than $y_{G^*}^0(v)$, for every other vertex v incident to f . We similarly define a y -assignment $y_{G^*}^1$ using the y -coordinates of Γ_1 . We use Theorem 3 to construct upward planar drawings Γ_0^* and Γ_1^* of G^* satisfying $y_{G^*}^0$ and $y_{G^*}^1$, respectively. By Lemma 3 we have that $\Gamma_0^*[G]$ and $\Gamma_1^*[G]$ are left-to-right equivalent. Therefore, by Lemma 5, the linear morph \mathcal{M}'_0 from Γ_0^* to $\Gamma_1^*[G]$ (\mathcal{M}'_1 from $\Gamma_1^*[G]$ to Γ_1^*) is unidirectional and planar. Such a morph is also upward since both Γ_0^* and $\Gamma_1^*[G]$ (Γ_1^* and $\Gamma_1^*[G]$) are upward planar and left-to-right equivalent. Then, we apply algorithm \mathcal{A} to construct a morph from Γ_0^* to Γ_1^* and restrict such a morph to a morph \mathcal{M}'' from $\Gamma_0^*[G]$ to $\Gamma_1^*[G]$. The morph from Γ_0 to Γ_1 is the concatenation of \mathcal{M}'_0 , \mathcal{M}'' , and \mathcal{M}'_1 . \square

The *kernel* of a polygon P is the set of points p inside or on P such that, for any point q on P , the open segment \overline{pq} lies inside P .

Lemma 9 (Convexify). *Let Γ be an upward planar drawing of an internally 3-connected plane st -graph G , let f be an st -face of G , and let P be the polygon representing f in Γ . There exists an upward planar drawing Γ' of G such that the polygon representing the boundary of f is strictly-convex and $\mathcal{M} = \langle \Gamma, \Gamma' \rangle$ is a unidirectional upward planar morph. Further, if v is a vertex incident to f that is in the kernel of P in Γ , then v is in the kernel of the polygon representing the boundary of f throughout \mathcal{M} .*

Proof. Denote by y_G the y -assignment for the vertices of G induced by Γ . By Theorem 3, there exists a strictly-convex upward planar drawing Γ' of G satisfying y_G . Thus, by Lemma 3 and since G is a plane st -graph, Γ and Γ' are left-to-right-equivalent drawings. By Lemma 5, the linear morph \mathcal{M} from Γ to Γ' is unidirectional and planar. Since Γ and Γ' are upward, \mathcal{M} is upward as well.

Consider now a vertex v incident to f that is in the kernel of P in Γ . Since the polygon representing the boundary of f in Γ' is strictly-convex, v is also in the kernel of such a polygon. Augment G to a graph G_* by introducing (suitably oriented) edges connecting v to the vertices incident to f that are not already adjacent to v . Since v is in the kernel of the polygon representing the boundary of f both in Γ and in Γ' , this results in two left-to-right equivalent upward planar drawings Γ_* and Γ'_* of G_* . By the same arguments used for \mathcal{M} , we have that the linear morph $\mathcal{M}_* = \langle \Gamma_*, \Gamma'_* \rangle$ is planar. Hence, v is in the kernel of the polygon representing the boundary of f throughout \mathcal{M} . \square

Given two upward planar straight-line drawings Γ_0 and Γ_1 of a maximal plane st -graph G , our strategy for constructing an upward planar morph from Γ_0 to Γ_1 is as follows: (1) we find a simple vertex v of G of degree at most 5; (2) we remove v and its incident edges from G , Γ_0 , and Γ_1 , obtaining upward planar drawings Γ'_0 and Γ'_1 of an upward plane graph G' ; (3) we triangulate G' , Γ'_0 , and Γ'_1 by inserting edges incident to a former neighbor u of v , obtaining upward planar

drawings Γ_0'' and Γ_1'' of a maximal plane st -graph G'' ; (4) we apply induction in order to construct an upward planar morph \mathcal{M}'' from Γ_0'' to Γ_1'' ; and (5) we remove the edges incident to u that are not in G and insert v and its incident edges in \mathcal{M}'' , thus obtaining an upward planar morph \mathcal{M} from Γ_0 to Γ_1 . In order for this strategy to work, we need u to satisfy certain properties, which are expressed in the upcoming definition of *distinguished neighbor*; further, we need to perform one initial (and one final) unidirectional upward planar morph so to convexify the polygon representing what will be called a *characteristic cycle*.

Let v be a simple vertex with degree at most 5 in a maximal plane st -graph G . Let $G(v)$ be the subgraph of G induced by v and its neighbors. A predecessor u of v in G is a *distinguished predecessor* if, for each predecessor w of v , there is a directed path in $G(v)$ from w to v through u . A successor u of v in G is a *distinguished successor* if, for each successor w of v , there is a directed path in $G(v)$ from v to w through u . A neighbor of v is a *distinguished neighbor* if it is a distinguished predecessor or successor of v . Examples of distinguished neighbors are in Fig. 2.

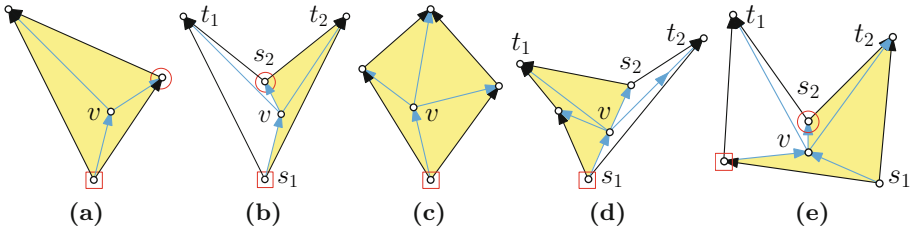


Fig. 2. Distinguished predecessors (enclosed by red squares), distinguished successors (enclosed by red circles), and characteristic cycles (filled yellow). (Color figure online)

Lemma 10. *The vertex v has at most one distinguished predecessor, at most one distinguished successor, and at least one distinguished neighbor.*

We define the *characteristic cycle* $C(v)$ as follows. Let $c_G(v)$ be the subgraph of G induced by the neighbors of v . Since v is simple, the underlying graph of $c_G(v)$ is a cycle. If $c_G(v)$ is an st -cycle, then $C(v) := c_G(v)$; this is always the case if v has degree 3. Otherwise, $c_G(v)$ has two sources s_1 and s_2 and two sinks t_1 and t_2 . Suppose that G contains the edges s_1v and vs_2 , the cases in which it contains the edges s_2v and vs_1 , or t_1v and vt_2 , or t_2v and vt_1 are analogous. Suppose also, w.l.o.g., that s_1, t_1, s_2 , and t_2 appear in this clockwise order along $c_G(v)$. If v has degree 4, then we define $C(v)$ as the st -cycle composed of the edges s_1v, vs_2, s_2t_2 , and s_1t_2 . Otherwise, v has degree 5. Consider the directed path $P_1 = (v_1, v, v_2)$, where v_1 (v_2) is the distinguished predecessor (successor) of v or, if such a node does not exist, the source s_1 (s_2). Then P_1 splits $c_G(v)$ into two paths P_2 and P_3 with length 2 and 3, respectively. Cycle $C(v)$ is defined as the st -cycle composed of P_1 and P_3 . We have the following structural lemma.

Lemma 11. *The characteristic cycle $C(v)$ is an st -cycle which contains all the distinguished neighbors of v . Further, all the vertices of $c_G(v)$ not belonging to $C(v)$ are adjacent to all the distinguished neighbors of v .*

Characteristic cycles are used in order to prove the following.

Lemma 12. *There is a unidirectional upward planar morph $\langle \Gamma, \Gamma' \rangle$, where in Γ' the distinguished neighbors of v are in the kernel of the polygon representing $c_G(v)$.*

Proof sketch. If $C(v)$ is convex in Γ , then by Lemma 11 the distinguished neighbors of v already are in the kernel of the polygon representing $c_G(v)$. Otherwise, we remove the interior of $C(v)$ and use Lemma 9 to make $C(v)$ convex. Then, we suitably reinsert the interior of $C(v)$ to obtain the desired morph. \square

The following concludes our discussion on maximal plane st -graphs.

Theorem 5. *Let Γ_0 and Γ_1 be two upward planar drawings of an n -vertex maximal plane st -graph. There is an $O(n)$ -step upward planar morph from Γ_0 to Γ_1 .*

Proof sketch. If $n = 3$, then the desired morph is constructed as in Lemma 6. If $n > 3$, then by Lemma 2 a simple vertex v exists in G with degree at most 5. By Lemma 10, v has a distinguished neighbor u . By Lemma 12, unidirectional upward planar morphs $\langle \Gamma_0, \Lambda_0 \rangle$ and $\langle \Lambda_1, \Gamma_0 \rangle$ exist, where Λ_0 and Λ_1 are upward planar drawings of G in which u lies in the kernels of the polygons representing $C(v)$. Remove v from G , Λ_0 , and Λ_1 , and insert (suitably oriented) edges between u and the former neighbors of v that are not already adjacent to u , thus obtaining upward planar drawings Γ_0'' and Γ_1'' of an $(n - 1)$ -vertex maximal plane st -graph G'' . Recursively compute an upward planar morph \mathcal{M}'' from Γ_0'' to Γ_1'' . Finally, remove the edges incident to u that are not in G and insert v and its incident edges in \mathcal{M}'' , obtaining an upward planar morph \mathcal{M} from Λ_0 to Λ_1 . This, together with $\langle \Gamma_0, \Lambda_0 \rangle$ and $\langle \Lambda_1, \Gamma_0 \rangle$, provides the desired morph from Γ_0 to Γ_1 . \square

We finally get the following.

Corollary 1. *Let Γ_0 and Γ_1 be two upward planar drawings of an n -vertex plane st -graph. There exists an $O(n)$ -step upward planar morph from Γ_0 to Γ_1 .*

Proof. The statement follows by Lemma 7, Theorem 4, and Theorem 5. \square

5 Upward Plane Graphs

Let G be an upward plane graph, let f be a face of G , and let u , v , and w be three clockwise consecutive switches of f . Also, let v_1 (v_2) be the neighbor of v clockwise preceding (succeeding) v along f , and let u_1 (u_2) be the neighbor of u clockwise preceding (succeeding) u along f . We say that $[u, v, w]$ is a *pocket* for f if $\angle(v_1, v, v_2) = \text{small}$ and $\angle(u_1, u, u_2) = \text{large}$. The following is well-known.

Lemma 13 ([9]). *Let G be an upward plane graph and let f be a face of G that is not an st -face. Then, there exists a pocket $[u, v, w]$ for f .*

Next, we give a lemma that shows how to “simplify” a face of an upward plane graph that is not an st -graph, by removing one of its pockets.

Lemma 14. *Let G be an n -vertex (reduced) upward plane graph, let $[u, v, w]$ be a pocket for a face f of G , and let Γ be an upward planar drawing of G . Suppose that an algorithm \mathcal{A} (\mathcal{A}_R) exists that constructs an $f(r)$ -step ($f_R(r)$ -step) upward planar morph between any two upward planar drawings of an r -vertex (reduced) plane st -graph. Then, there exists an $O(f(n))$ -step ($O(f_R(n))$ -step) upward planar morph from Γ to an upward planar drawing Γ^* of G in which w and u have direct visibility and such that u lies below w , if a directed path exists in f from v to u , and it lies above w , if a directed path exists in f from u to v .*

Proof sketch. Suppose that a directed path p_{vu} exists in f from v to u (see Fig. 3a); the case in which a directed path exists in f from u to v can be treated symmetrically. We first show that there exists an upward planar drawing Γ' of G such that (i) it is possible to add to Γ' an upward planar drawing of two directed paths p' and p'' from u to w that form an st -face (see Fig. 3b), and (ii) there exists an $O(f(n))$ -step ($O(f_R(n))$ -step) upward planar morph \mathcal{M}' from Γ to Γ' . We then show that there exists an upward planar drawing Γ^* of G such that (iii) vertices w and u have direct visibility and u lies below w (see Fig. 3c), and (vi) there exists an $O(f(n))$ -step ($O(f_R(n))$ -step) upward planar morph \mathcal{M}^* from Γ' to Γ^* . Composing \mathcal{M}' and \mathcal{M}^* yields an upward planar morph from Γ to Γ^* . \square

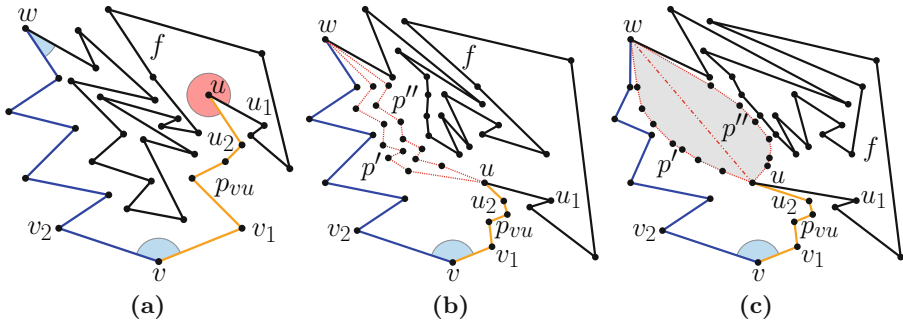


Fig. 3. Illustrations for the proof of Lemma 14.

Theorem 6. *Let Γ_0 and Γ_1 be two upward planar drawings of an n -vertex (reduced) upward plane graph G . Suppose that an algorithm \mathcal{A} (\mathcal{A}_R) exists that constructs an $f(r)$ -step ($f_R(r)$ -step) upward planar morph between any two upward planar drawings of an r -vertex (reduced) plane st -graph. There exists an $O(n \cdot f(n))$ -step (an $O(n \cdot f_R(n))$ -step) upward planar morph from Γ_0 to Γ_1 .*

Proof sketch. By Lemma 7, we can assume that G is biconnected. In order to prove the statement, we show that there exists a $((2\ell(G) + 1) \cdot f(n))$ -step (a $((2\ell(G) + 1) \cdot f_R(n))$ -step) upward planar morph from Γ_0 to Γ_1 , if G is a (reduced) upward plane graph. Since $\ell(G) \in O(n)$, the statement follows.

The proof is by induction on $\ell(G)$. In the base case $\ell(G) = 0$ and thus G is a (reduced) plane st -graph. Hence, by applying algorithm \mathcal{A} (\mathcal{A}_R) to Γ_0 and Γ_1 , we obtain an $f(n)$ -step (an $f_R(n)$ -step) upward planar morph from Γ_0 to Γ_1 .

In the inductive case $\ell(G) > 0$. Then there exists a face f of G that is not an st -face. Thus, by Lemma 13, there exists a pocket $[u, v, w]$ for f . By Lemma 14, we can construct upward planar drawings Γ'_0 and Γ'_1 of G in which u and w have direct visibility and u lies below w (assuming that a directed path exists in f from v to u , the other case being symmetric), and such that there exists an $f(n)$ -step (an $f_R(n)$ -step) upward planar morph \mathcal{M}_{start} from Γ_0 to Γ'_0 and an $f(n)$ -step (an $f_R(n)$ -step) upward planar morph \mathcal{M}_{finish} from Γ'_1 to Γ_1 .

Let G^* be the plane graph obtained from G by splitting f with a directed edge uw . Graph G^* is an upward plane graph whose upward planar embedding is constructed by assigning to each switch in G^* the same label `small` or `large` it has in G . Also, $\ell(G^*) = \ell(G) - 1$, since u is not a switch in G^* . Further, G^* is reduced if G is reduced, since there exists no directed path from u to w in G (due to the fact that $[u, v, w]$ is a pocket of f). Let Γ_0^* and Γ_1^* be the planar straight-line drawings of G^* obtained by drawing the directed edge uw as a straight-line segment connecting u and w in Γ'_0 and in Γ'_1 , respectively. It is easy to see that Γ_0^* and Γ_1^* are upward planar drawings of G^* . Therefore, by the inductive hypothesis and since $V(G^*) = V(G)$, we can construct a $((2\ell(G^*) + 1) \cdot f(n))$ -step (a $((2\ell(G^*) + 1) \cdot f_R(n))$ -step) upward planar morph from Γ_0^* to Γ_1^* . Observe that, since $G \subset G^*$, restricting each drawing in \mathcal{M}^* to G yields a $((2\ell(G) - 1) \cdot f(n))$ -step upward planar morph \mathcal{M}^- of G from Γ'_0 to Γ'_1 . Therefore, by concatenating morphs \mathcal{M}_{start} , \mathcal{M}^- , and \mathcal{M}_{finish} , we obtain a $((2\ell(G) + 1) \cdot f(n))$ -step (a $((2\ell(G) + 1) \cdot f_R(n))$ -step) upward planar morph of G from Γ_0 to Γ_1 . \square

Theorems 2, 4, and 6, imply the following main result.

Theorem 7. *Let Γ_0 and Γ_1 be two upward planar drawings of the same n -vertex (reduced) upward plane graph. There exists an $O(n^2)$ -step (an $O(n)$ -step) upward planar morph from Γ_0 to Γ_1 .*

6 Conclusions and Open Problems

In this paper, we addressed for the first time the problem of morphing upward planar straight-line drawings. We proved that an upward planar morph between any two upward planar drawings of the same upward plane graph always exists. It easy to see that all our algorithms can be implemented in polynomial time.

Several problems remain open. In our opinion the most interesting question is whether an $O(1)$ -step upward planar morph between any two upward planar drawings of the same plane st -graph exists. In case of a positive answer, by Theorem 6, an optimal $O(n)$ -step upward planar morph would exist between

any two upward planar drawings of the same n -vertex upward plane graph. In case of a negative answer, it would be interesting to find broad classes of upward plane graphs that admit upward planar morphs with a sub-linear number of steps. We proved that reduced plane st -graphs have this property and we ask whether the same is true for series-parallel digraphs [8,14].

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