A New Robust Stability Result
for Delayed Neural Networks

Ozlem Faydasicok[1], Cemal Cicek[1], and Sabri Arik[2(✉)]

[1] Department of Mathematics, Istanbul University, Istanbul, Turkey
{kozlem,cicekc}@istanbul.edu.tr
[2] Department of Computer Engineering,
Istanbul University-Cerrahpasa, Istanbul, Turkey
ariks@istanbul.edu.tr

Abstract. This work proposes a further improved global robust stability condition for neural networks involving intervalized network parameters and including single time delay. For the sake of obtaining a new robust stability condition, a new upper bound for the norm of the intervalized interconnection matrices is established. The homeomorphism mapping and Lyapunov stability theorems are employed to derive the proposed stability condition by making use of this upper bound norm. The obtained result is applicable to all nondecreasing slope-bounded activation functions and imposes constraints on parameters of neural network without involving time delay.

Keywords: Delayed systems · Neural networks · Robust stability
Lyapunov theorems

1 Introduction

Analysis of various dynamics of neural networks has recently become an interesting research topic due to their qualitative properties that are employed to solve various practical real-word problems related to combinatorial optimization, image processing and control systems. When solving these types of problems by using neural networks, one needs to establish a neural system possessing a unique and globally asymptotically stable equilibrium point. Thus, one needs to deal with stability of neural networks. The fact that neurons implemented by amplifiers usually have finite switching speeds will result in time delays, which may have undesired affects on the dynamics of neural networks. Another problem is that the parameters of neural systems may involve some uncertainties, which can also have an affect on the equilibria of neural networks. Because of these reasons, for a proper stability analysis, the time delay in the states and uncertainties in the network parameters need to be included in the mathematical model of neural networks. That is to say, the key requirement would be the establishment of robust stability of neural systems which also involve time delay. When reviewing past literature, it can be realized that many researchers published useful

robust stability criteria for delayed neural systems, (see references [1–19]). This paper uses Lyapunov and Homeomorphic mapping theorems to derive a novel condition for global robust asymptotic stability.

Notations: Let $z = (z_1, z_2, ..., z_n)^T$. We will use $|z| = (|z_1|, |z_2|, ..., |z_n|)^T$. For a given matrix $E = (e_{ij})_{n \times n}$, we will use $|E| = (|e_{ij}|)_{n \times n}$, and $\lambda_m(E)$ will represent the minimum eigenvalue of E. If of $E = E^T$, $E > 0$ will show that E is positive definite. $E = (e_{ij})_{n \times n}$ is nonnegative matrix if $e_{ij} \geq 0, \forall i, j$. Assume that $E = (e_{ij})_{n \times n}$ and $F = (f_{ij})_{n \times n}$ are nonnegative matrices. In this case, $E \preceq F$ will denote that $e_{ij} \leq f_{ij}, \forall i, j$. For the vector z, we will use the norm $||z||_2^2 = \sum_{i=1}^{n} z_i^2$, and for E, we use $||E||_2 = [\lambda_{\max}(E^T E)]^{1/2}$.

2   Preliminaries

Consider the neural network of the mathematical form

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t - \tau)) + u_i, \forall i \qquad (1)$$

in this above equation, a_{ij} and b_{ij} are interconnection parameters, c_i are the neurons charging rates, $x_i(t)$ represent state of neuron i, the functions $f_i(\cdot)$ are the nonlinear activation functions, τ represents the time delay, u_i are the inputs.

Neural system (1) can be put into an equivalent system governed by the differential equation:

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t - \tau)) + u \qquad (2)$$

where $C = diag(c_i)$, $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$, $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T$, $u = (u_1, u_2, ..., u_n)^T$, $f(x(\cdot)) = (f_1(x_1(\cdot)), f_2(x_2(\cdot)), ..., f_n(x_n(\cdot)))^T$.

The functions f_i possess the following properties:

$$0 \leq \frac{f_i(x) - f_i(\tilde{x})}{x - \tilde{x}} \leq k_i, \ \forall i, \ \forall x, \tilde{x} \in R, x \neq \tilde{x}$$

with k_i being positive constants. The functions satisfying the above conditions are denoted by $f \in \mathcal{K}$.

The matrices $A = (a_{ij})$, $B = (b_{ij})$ and $C = diag(c_i > 0)$ in (1) are stated by the following intervals:

$$
\begin{aligned}
C_I &:= \{C : 0 \preceq \underline{C} \preceq C \preceq \overline{C}, \ i.e., \ 0 < \underline{c_i} \leq c_i \leq \overline{c_i}\} \\
A_I &:= \{A : \underline{A} \preceq A \preceq \overline{A}, \ i.e., \ \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}\} \qquad (3) \\
B_I &:= \{B : \underline{B} \preceq B \preceq \overline{B}, \ i.e., \ \underline{b}_{ij} \leq b_{ij} \leq \overline{b}_{ij}\}
\end{aligned}
$$

We now introduce the following lemma which is of great importance to obtaining our main result:

Lemma 1: Let D be a positive diagonal matrix with n diagonal entries, x be any real vector having n elements, and consider any real $n \times n$ dimensional matrix $A = (a_{ij})$ with being intervalized as $\underline{A} \preceq A \preceq \overline{A}$. In this case, the following inequality is satisfied:

$$x^T A^T DAx \leq |x^T|[|A^{*T} DA^*| + |A^{*T}|DA_* + A_*^T D|A^*| + A_*^T DA_*]|x|$$

in which $A^* = \frac{1}{2}(\overline{A} + \underline{A})$ and $A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

Proof: If $A \in A_I$, then, a_{ij} can be written as

$$a_{ij} = \frac{1}{2}(\overline{a}_{ij} + \underline{a}_{ij}) + \frac{1}{2}\sigma_{ij}(\overline{a}_{ij} - \underline{a}_{ij}), \quad -1 \leq \sigma_{ij} \leq 1, \forall i, j.$$

Assume that $\tilde{A} = (\tilde{a}_{ij})_{n \times n}$ is a real constant matrix and whose elements are defined as $\tilde{a}_{ij} = \frac{1}{2}\sigma_{ij}(\overline{a}_{ij} - \underline{a}_{ij})$. Then, A can be written as

$$A = \frac{1}{2}(\overline{A} + \underline{A}) + \tilde{A} = A^* + \tilde{A}$$

We can now express the following:

$$\begin{aligned}
x^T A^T DAx &= x^T (A^* + \tilde{A})^T D(A^* + \tilde{A})x \\
&= x^T (A^{*T} DA^* + A^{*T} D\tilde{A} + \tilde{A}^T DA^* + \tilde{A}^T D\tilde{A})x \\
&\leq |x^T||A^{*T} DA^* + A^{*T} D\tilde{A} + \tilde{A}^T DA^* + \tilde{A}^T D\tilde{A}||x| \\
&\leq |x^T||A^{*T} DA^*||x| + |x^T||A^{*T}|D|\tilde{A}||x| \\
&\quad + |x^T||\tilde{A}^T|D|A^*||x| + |x^T||\tilde{A}^T|D|\tilde{A}||x|
\end{aligned}$$

Since $|\tilde{a}_{ij}| \leq \frac{1}{2}(\overline{a}_{ij} - \underline{a}_{ij}), \forall i, j$, it follows that $|\tilde{A}| \preceq A_*$. Then, we obtain

$$\begin{aligned}
x^T A^T DAx &\leq |x^T||A^{*T} DA^*||x| + |x^T||A^{*T}|D|A_*||x| \\
&\quad + |x^T||A_*^T|D|A^*||x| + |x^T||A_*^T|D|A_*||x| \\
&= |x^T|(|A^{*T} DA^*| + |A^{*T}|DA_* + A_*^T D|A^*| + A_*^T DA_*)|x|
\end{aligned}$$

Below are two lemmas and a fact that will be needed in the proofs:

Lemma 2 [1]**:** Let D be a positive diagonal matrix with n diagonal entries, x be any real vector having n elements, and consider any real $n \times n$ dimensional matrix $A = (a_{ij})$ with being intervalized as $\underline{A} \preceq A \preceq \overline{A}$. In this case, the following inequality is satisfied:

$$x^T (DA + A^T D)x \leq |x^T||S||x|$$

where $S = (s_{ij})$ such that $s_{ii} = 2d_i \bar{a}_{ii}$ and $s_{ij} = max(|d_i \bar{a}_{ij} + d_j \bar{a}_{ji}|, |d_i \underline{a}_{ij} + d_j \underline{a}_{ji}|)$ for $i \neq j$.

Lemma 3 [2]: Let the map $H(y) \in C^0$ posses two properties: $H(y) \neq H(z)$, $\forall y \neq z$ and $||H(y)|| \to \infty$ as $||y|| \to \infty$ with $y \in R^n$ and $z \in R^n$. Then, $H(y)$ is said to be homeomorphism of R^n.

Fact 1: If $A = (a_{ij})$ and $B = (b_{ij})$ satisfy (3), then, A and B have bounded norms, i.e., we can find some positive real constants ϵ and ε satisfying the following

$$\|A\|_2 \leq \epsilon \quad \text{and} \quad \|B\|_2 \leq \varepsilon$$

3   Existence and Uniqueness Analysis

The following theorem presents the criterion which ensures that system (1) possesses a unique equilibrium point for each constant input:

Theorem 1: Let neuron activation functions belong $\mathcal{K}$, and assume that the uncertain network elements A, B and C are defined by (3). Then, delayed neural network described by (1) possesses only one equilibrium point, if one can find a matrix $D = diag(d_i > 0)$ satisfying the following condition

$$\Theta = 2\underline{C}DK^{-1} - D - S - Q > 0$$

where $K = diag(k_i > 0)$, $Q = B_*{}^T D|B^*| + B_*{}^T DB_* + |B^{*T} DB^*| + |B^{*T}|DB_*$, $S = (s_{ij})$ is the matrix whose diagonal elements are defined by $s_{ii} = 2d_i \bar{a}_{ii}$ and off-diagonal elements are defined by $s_{ij} = max(|d_i \bar{a}_{ij} + d_j \bar{a}_{ji}|, |d_i \underline{a}_{ij} + d_j \underline{a}_{ji}|)$, the matrix B^* included in Q is defined as $B^* = \frac{1}{2}(\overline{B} + \underline{B})$ and the nonnegative matrix B_* included in Q is defined as $B_* = \frac{1}{2}(\overline{B} - \underline{B})$.

Proof: Consider the associated map $H(x)$ representing neural network (2)

$$H(x) = -Cx + Af(x) + Bf(x) + u \tag{4}$$

For every equilibrium point x^* of (2), by definition of equilibrium equation, the following must be satisfied

$$-Cx^* + Af(x^*) + Bf(x^*) + u = 0$$

Apparently, when a vector x satisfies $H(x) = 0$, it results in the fact that $H(x) = 0$ also corresponds to the equilibrium points representing the solutions of (2). Thus, by the virtue of Lemma 3, one can conclude that neural model (2) possesses only one equilibrium point for the constant u if $H(x)$ fulfills conditions in Lemma 3. For any randomly selected vectors $x \neq y$, using (4), we express

$$H(x) - H(y) = -C(x - y) + A(f(x) - f(y)) + B(f(x) - f(y))$$

Let $H(x, y) = H(x) - H(y)$ and $f(x, y) = f(x) - f(y)$. Then, the previous equation can be put in the form:

$$H(x, y) = -C(x - y) + Af(x, y) + Bf(x, y) \tag{5}$$

Since $f \in \mathcal{K}$, if $x \neq y$ and $f(x) = f(y)$, (5) yields

$$H(x, y) = -C(x - y)$$

in which $C = diag(c_i > 0)$. Therefore, $x - y \neq 0$ will ensure the condition that $H(x, y) \neq 0$. For $x - y \neq 0$, $f(x, y) \neq 0$, and $D = diag(d_i > 0)$, multiplying (5) by the nonzero vector $2f^T(x, y)D$ leads to

$$2f^T(x, y)DH(x, y) = -2f^T(x, y)DC(x - y)$$
$$+ 2f^T(x, y)DAf(x, y)$$
$$+ 2f^T(x, y)DBf(x, y)$$

The following can be written

$$2f^T(x, y)DAf(x, y) = f^T(x, y)(DA + A^T D)f(x, y)$$

Thus, one would obtain

$$2f^T(x, y)DH(x, y) = -2f^T(x, y)DC(x - y)$$
$$+ f^T(x, y)(DA + A^T D)f(x, y)$$
$$+ f^T(x, y)DBf(x, y) \tag{6}$$

For activation functions in $\mathcal{K}$, the following can be derived

$$-2f^T(x, y)DC(x - y) = -\sum_{i=1}^{n} 2d_i c_i (f_i(x_i) - f_i(y_i))(x_i - y_i)$$
$$\leq -2\sum_{i=1}^{n} \frac{d_i c_i}{k_i}(f_i(x_i) - f_i(y_i))^2$$
$$= -2f^T(x, y)\underline{C}DK^{-1}f(x, y) \tag{7}$$

Lemma 2 leads to

$$f^T(x, y)(DA + A^T D)f(x, y) \leq |f^T(x, y)|S|f(x, y)| \tag{8}$$

It is worth noting that

$$2f^T(x, y)DBf(x, y) \leq f^T(x, y)Df(x, y) + f^T(x, y)B^T DBf(x, y)$$

By using Lemma 1, one would get

$$f^T(x, y)B^T DBf(x, y) \leq |f^T(x, y)|Q|f(x, y)|$$

Thus

$$2f^T(x,y)DBf(x,y) \le f^T(x,y)Df(x,y) + |f^T(x,y)|Q|f(x,y)| \tag{9}$$

Using (7)–(9) in (6) will give the following

$$\begin{aligned}
2f^T(x,y)DH(x,y) \le & -2|f^T(x,y)||\underline{C}DK^{-1}|f(x,y)| \\
& + |f^T(x,y)|(S+D+Q)|f(x,y)| \\
= & -|f(x,y)|^T \Theta |f(x,y)|
\end{aligned}$$

Since $\Theta > 0$, one can observe that

$$2f^T(x,y)DH(x,y) \le -\lambda_m(\Theta)||f(x,y)||_2^2 \tag{10}$$

Obviously, $f(x,y) \ne 0$ with Θ being positive definite, that is $\Theta > 0$, (10) leads to

$$2f^T(x,y)DH(x,y) < 0$$

where $f(x,y) \ne 0$ guarantees condition that $H(x) \ne H(y)$ for all $x \ne y$.

Choosing $y = 0$, (10) will directly result in

$$2(f(x) - f(0))^T D(H(x) - H(0)) \le -\lambda_m(\Theta)||f(x) - f(0)||_2^2$$

It follows from the above inequality that

$$|2(f(x) - f(0))^T D(H(x) - H(0))| \ge \lambda_m(\Theta)||f(x) - f(0)||_2^2$$

yielding

$$||H(x) - H(0)||_1 > \frac{\lambda_m(\Theta)||f(x) - f(0)||_2^2}{2||D||_\infty ||f(x) - f(0)||_\infty}$$

Using some basic properties of the vector norms, we can state

$$||H(x)||_1 > \frac{\lambda_m(\Theta)(||f(x)||_2 - ||f(0)||_2 - 2||D||_\infty ||H(0)||_1)}{2||D||_\infty}$$

Knowing that $||H(0)||_1$, $||D||_\infty$, and $||f(0)||_2$ have limited upper bounds will enable us to conclude that $||H(x)|| \to \infty$ if $||x|| \to \infty$. Q.E.D.

4   Stability Analysis

Stability of neural network (1) will be studied in this section. If x^* is defined to denote an equilibrium point of (1), then, by means of $z_i(\cdot) = x_i(\cdot) - x_i^*$, neural system (1) is replaced by a new model whose dynamics is governed by:

$$\dot{z}_i(t) = -c_i z_i(t) + \sum_{j=1}^n a_{ij} g_j(z_j(t)) + \sum_{j=1}^n b_{ij} g_j(z_j(t-\tau)) \tag{11}$$

Note that $g_i(z_i(\cdot)) = f_i(z_i(\cdot) + x_i^*) - f_i(x_i^*)$. We can easily observe that the functions g_i belong to the class $\mathcal{K}$, that is, $g \in \mathcal{K}$ satisfying $g_i(0) = 0$.

The vector-matrix form of neural system (11) is

$$\dot{z}(t) = -Cz(t) + Ag(z(t)) + Bg(z(t - \tau)) \tag{12}$$

In this new system, $z(t) = (z_1(t), z_2(t), ..., z_n(t))^T$, and the new nonlinear output functions state vector is $g(z(\cdot)) = (g_1(z_1(\cdot)), g_2(z_2(\cdot)), ..., g_n(z_n(\cdot)))^T$.

The stability result is given as follows:

Theorem 2: Let neuron activation functions belong $\mathcal{K}$, and assume that the uncertain network elements A, B and C are given by (3). Then, the origin of delayed neural system described by (11) is globally asymptotically stable, if one can find an appropriate matrix $D = diag(d_i > 0)$ satisfying the following condition

$$\Theta = 2\underline{C}DK^{-1} - D - S - Q > 0$$

where $K = diag(k_i > 0)$, $Q = B_*^T D|B^*| + B_*^T DB_* + |B^{*T}DB^*| + |B^{*T}|DB_*$, $S = (s_{ij})$ is the matrix whose diagonal elements are defined by $s_{ii} = 2d_i\bar{a}_{ii}$ and off-diagonal elements are defined by $s_{ij} = max(|d_i\bar{a}_{ij} + d_j\bar{a}_{ji}|, |d_i\underline{a}_{ij} + d_j\underline{a}_{ji}|)$, the matrix B^* included in Q is defined as $B^* = \frac{1}{2}(\overline{B} + \underline{B})$ and the nonnegative matrix B_* included in Q is defined as $B_* = \frac{1}{2}(\overline{B} - \underline{B})$.

Proof: The Lyapunov functional to be exploited for the proof of this theorem is chosen as:

$$V(z(t)) = \sum_{i=1}^{n}(z_i^2(t) + 2\gamma\int_0^{z_i(t)} d_i g_i(s)ds)$$
$$+ \gamma\int_{t-\tau}^{t} |g^T(z(\zeta))|Q|g(z(\zeta))|d\zeta + \xi\int_{t-\tau}^{t} \|g(z(\zeta))\|_2^2 d\zeta$$

where the d_i, γ and ξ are constants. $\dot{V}(z(t))$ is determined to be as

$$\dot{V}(z(t)) = -2z^T(t)Cz(t) + 2z^T(t)Ag(z(t)) + 2z^T(t)Bg(z(t-\tau))$$
$$- 2\gamma g^T(z(t))DCz(t) + 2\gamma g^T(z(t))DAg(z(t))$$
$$+ 2\gamma g^T(z(t))DBg(z(t-\tau))$$
$$+ \gamma|g^T(z(t))|Q|g(z(t))| - \gamma|g^T(z(t-\tau))|Q|g(z(t-\tau))|$$
$$+ \xi\|g(z(t))\|_2^2 - \xi\|g(z(t-\tau))\|_2^2 \tag{13}$$

Let $\alpha = \|A\|_2^2\|C^{-1}\|_2$ and $\beta = \|A\|_2^2\|C^{-1}\|_2$. We now observe the inequalities:

$$2z^T(t)Ag(z(t)) - z^T(t)Cz(t) \leq \alpha\|g(z(t))\|_2^2 \tag{14}$$

$$2z^T(t)Bg(z(t-\tau) - z^T(t)Cz(t) \leq g^T(z(t-\tau))B^TC^{-1}Bg(z(t-\tau))$$
$$\leq \beta\|g(z(t-\tau))\|_2^2 \tag{15}$$

$$2\gamma g^T(z(t))DAg(z(t)) = \gamma g^T(z(t))(DA + A^TD)g(z(t))$$
$$\leq \gamma|g^T(z(t))|S|g(z(t))| \qquad (16)$$

$$2\gamma g^T(z(t))DBg(z(t-\tau)) \leq \gamma g^T(z(t))Dg(z(t))$$
$$+ \gamma g^T(z(t-\tau))B^TDBg(z(t-\tau))$$
$$\leq \gamma g^T(z(t))Dg(z(t))$$
$$+ \gamma|g^T(z(t-\tau))|Q|g(z(t-\tau))| \qquad (17)$$

$$- 2\gamma g^T(z(t))DCz(t) \leq -2\gamma g^T(z(t))DCK^{-1}g(z(t)) \qquad (18)$$

Inserting (14)–(18) into (13) yields

$$\dot{V}(z(t)) \leq \alpha\|g(z(t))\|_2^2 + \beta g^T(z(t))g(z(t))$$
$$- 2\gamma g^T(z(t))P\underline{C}K^{-1}g(z(t)) + \gamma|g^T(z(t))|S|g(z(t))|$$
$$+ \gamma g^T(z(t))Pg(z(t)) + \gamma|g^T(z(t-\tau))|Q|g(z(t-\tau))|$$
$$+ \gamma|g^T(z(t))|Q|g(z(t))| - \gamma|g^T(z(t-\tau))|Q|g(z(t-\tau))|$$
$$+ \xi\|g(z(t))\|_2^2 - \xi\|g(z(t-\tau))\|_2^2$$

Taking $\xi = \beta$ leads to

$$\dot{V}(z(t)) \leq (\beta + \alpha)\|g(z(t))\|_2^2 - 2\gamma g^T(z(t))D\underline{C}K^{-1}g(z(t))$$
$$+ \gamma|g^T(z(t))|S|g(z(t))| + \gamma g^T(z(t))Dg(z(t))$$
$$+ \gamma|g^T(z(t))|Q|g(z(t))|$$
$$= (\beta + \alpha)\|g(z(t))\|_2^2 - \gamma|g^T(z(t))|(D\underline{C}K^{-1} - D - S - Q)|g(z(t))|$$
$$= (\beta + \alpha)\|g(z(t))\|_2^2 - \gamma|g^T(z(t))|\Theta|g(z(t))| \qquad (19)$$

Since $\Theta > 0$, (19) gives

$$\dot{V}(z(t)) \leq - (\gamma\lambda_m(\Theta) - (\beta + \alpha)\|)g(z(t))\|_2^2$$

Thus

$$\gamma > \frac{(\alpha + \beta)}{\lambda_m(\Theta)}$$

guarantees that $\dot{V}(z(t))$ will have negative values $\forall g(z(t)) \neq 0$, or equivalently $\dot{V}(z(t)) < 0$ when $z(t) \neq 0$.

Let $g(z(t)) = 0$. Taking $z(t) \neq 0$ leads to

$$\dot{V}(z(t)) = -2z^T(t)Cz(t) + 2z^T(t)Bg(z(t-\tau))$$
$$- \gamma|g^T(z(t-\tau))|Q|g(z(t-\tau))| - \xi\|g(z(t-\tau))\|_2^2$$

Then

$$-z^T(t)Cz(t) + 2z^T(t)Bg(z(t-\tau)) \leq \xi\|g(z(t-\tau))\|_2^2$$

leads to

$$\dot{V}(z(t)) \leq -z^T(t)Cz(t)$$

in which $\dot{V}(z(t)) < 0 \ \forall z(t) \neq 0$. Finally, $g(z(t)) = z(t) = 0$ leads to

$$\dot{V}(z(t)) \leq -\xi \|g(z(t-\tau))\|_2^2$$

Apparently, $\dot{V}(z(t)) < 0 \ \forall g(z(t-\tau)) \neq 0$. Hence, $\dot{V}(z(t)) = 0$ iff $z(t) = g(z(t)) = g(z(t-\tau)) = 0$. This also means $\dot{V}(z(t)) < 0$ when states and delayed states are not equal to zero. The radially unboundedness the Lyapunov functional is to be easily checked by proving $V(z(t)) \to \infty$ when $\|z(t)\| \to \infty$. Q.E.D.

5   Conclusions

This work proposed a further improved condition for the robustness of neural networks involving intervalized network parameters and including single time delay. For the sake of obtaining a new robust stability condition, a new upper bound for the norm of the intervalized interconnection matrices has been established. The homeomorphism mapping and Lyapunov stability theorems are employed to derive the proposed stability condition by making use of this upper bound norm. The obtained result is applicable to all nondecreasing slope-bounded activation functions and imposes constraints on parameters of neural network without involving time delay.

References

1. Ensari, T., Arik, S.: New results for robust stability of dynamical neural networks with discrete time delays. Expert Syst. Appl. **37**, 5925–5930 (2010)
2. Faydasicok, O., Arik, S.: Robust stability analysis of a class of neural networks with discrete time delays. Neural Netw. **29–30**, 52–59 (2012)
3. Faydasicok, O., Arik, S.: A new upper bound for the norm of interval matrices with application to robust stability analysis of delayed neural networks. Neural Netw. **44**, 64–71 (2013)
4. Shao, J.L., Huang, T.Z., Zhou, S.: Some improved criteria for global robust exponential stability of neural networks with time-varying delays. Commun. Nonlinear Sci. Numer. Simul. **15**, 3782–3794 (2010)
5. Zhu, Q., Cao, J.: Robust exponential stability of markovian jump impulsive stochastic Cohen-Grossberg neural networks with mixed time delays. IEEE Trans. Neural Netw. **21**, 1314–1325 (2010)
6. Huang, T., Li, C., Duan, S., Starzyk, J.A.: Robust exponential stability of uncertain delayed neural networks with stochastic perturbation and impulse effects. IEEE Trans. Neural Netw. Learn. Syst. **23**, 866–875 (2012)
7. Zhou, J., Xu, S., Zhang, B., Zou, Y., Shen, H.: Robust exponential stability of uncertain stochastic neural networks with distributed delays and reaction-diffusions. IEEE Trans. Neural Netw. Learn. Syst. **23**, 1407–1416 (2012)
8. Shen, Y., Wang, J.: Robustness analysis of global exponential stability of recurrent neural networks in the presence of time delays and random disturbances. IEEE Trans. Neural Netw. Learn. Syst. **23**, 87–96 (2012)

9. Song, Q., Yu, Q., Zhao, Z., Liu, Y., Alsaadi, F.E.: Boundedness and global robust stability analysis of delayed complex-valued neural networks with interval parameter uncertainties. Neural Netw. **103**, 55–62 (2018)

10. Li, X., et al.: Extended robust global exponential stability for uncertain switched memristor-based neural networks with time-varying delays. Appl. Math. Comput. **325**, 271–290 (2018)

11. Ali, M.S., Gunasekaran, N., Rani, M.E.: Robust stability of hopfield delayed neural networks via an augmented L-K functional. Neurocomputing **234**, 198–204 (2017)

12. Wang, Z., Wang, J., Wu, Y.: State estimation for recurrent neural networks with unknown delays: a robust analysis approach. Neurocomputing **227**, 20–36 (2017)

13. Wan, Y., Cao, J., Wen, G., Yu, W.: Robust fixed-time synchronization of delayed Cohen-Grossberg neural networks. Neural Netw. **73**, 86–94 (2016)

14. Muthukumar, P., Subramanian, K., Lakshmanan, S.: Robust finite time stabilization analysis for uncertain neural networks with leakage delay and probabilistic time-varying delays. J. Franklin Inst. **353**, 4091–4113 (2016)

15. Gong, W., Liang, J., Kan, X., Nie, X.: Robust state estimation for delayed complex-valued neural networks. Neural Process. Lett. **46**, 1009–1029 (2017)

16. Dong, Y., Liang, S., Guo, L.: Robustly exponential stability analysis for discrete-time stochastic neural networks with interval time-varying delays. Neural Process. Lett. **46**, 135–158 (2017)

17. Arik, S.: New criteria for global robust stability of delayed neural networks with norm-bounded uncertainties. IEEE Trans. Neural Netw. Learn. Syst. **25**, 1045–1052 (2014)

18. Wang, J., Zhang, H., Wang, Z., Shan, Q.: Local synchronization criteria of markovian nonlinearly coupled neural networks with uncertain and partially unknown transition rates. IEEE Trans. Syst. Man Cybern. Syst. **47**, 1953–1964 (2017)

19. Ding, Z., Zeng, Z., Wang, L.: Robust finite-time stabilization of fractional-order neural networks with discontinuous and continuous activation functions under uncertainty. IEEE Trans. Neural Netw. Learn. Syst. **29**, 1477–1490 (2018)