Relating an Adaptive Network's Structure to Its Emerging Behaviour for Hebbian Learning

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Abstract. In this paper it is analysed how emerging behaviour of an adaptive network can be related to characteristics of the adaptive network's structure (which includes the adaptation structure). In particular, this is addressed for mental networks based on Hebbian learning. To this end relevant properties of the network and the adaptation that have been identified are discussed. As a result it has been found that in an achieved equilibrium state the value of a connection weight has a functional relation to the values of the connected states.

Keywords: Adaptive network · Hebbian learning · Analysis of behaviour

1 Introduction

A challenging issue for dynamic models is to predict what patterns of behaviour will emerge, and how their emergence depends on the structure of the model, including chosen values for model characteristics or parameters. This applies in particular to network models, where behaviour depends in some way on the network structure, defined by network characteristics such as connections and their weights. It can be an even more challenging issue when adaptive networks are considered, where the network characteristics also change over time, according to certain adaptation principles which themselves depend on certain adaptation characteristics represented by their own particular parameters. It is this latter issue what is the topic of the current paper: how does emerging behaviour of adaptive networks relate to the characteristics of the network and of the adaptation principles used. More in particular, the focus is on adaptive mental networks based on Hebbian learning [1, 3, 4, 6–8]. Hebbian learning is, roughly stated, based on the principle 'neurons that fire together, wire together' from Neuroscience.

To address the issue, as a vehicle the Network-Oriented Modeling approach based on temporal-causal networks [10] will be used. For temporal-causal networks, parameters characterising the network structure are connection weights, combination functions and speed factors. For the type of adaptive networks considered, the connection weights are dynamic based on Hebbian learning, so they are not part of the characteristics of the network structure anymore. Instead, characteristics of Hebbian learning have been identified that play an important role. In this paper, results will be

D. Fagan et al. (Eds.): TPNC 2018, LNCS 11324, pp. 359–373, 2018.
https://doi.org/10.1007/978-3-030-04070-3_28

discussed that have been proven mathematically for this relation between structure and behavior for such adaptive network models, in particular, for the result of Hebbian learning in relation to the connected network states. These results have been proven not for one specific model or function, but for classes of functions that fulfill certain properties. More specifically, it has been found how for the classes of functions considered within an emerging equilibrium state the connection weight and the connected states satisfy a fixed functional relation that can be expressed mathematically.

In this paper, in Sect. 2 the temporal-causal networks that are used as vehicle are briefly introduced. In Sect. 3 the properties of Hebbian learning functions are introduced that define the adaptation principle of the network. Section 4 focuses in particular on the class of functions for which a form of variable separation can be applied, In Sect. 5 a number of examples are discussed. Finally, Sect. 6 is a discussion.

2 Temporal-Causal Networks

For the perspective on networks used in the current paper, the interpretation of connections based on causality and dynamics forms a basis of the structure and semantics of the considered networks. More specifically, the nodes in a network are interpreted here as states (or state variables) that vary over time, and the connections are interpreted as causal relations that define how each state can affect other states over time. This type of network has been called a *temporal-causal network* [10]. A conceptual representation of a temporal-causal network model by a *labeled graph* provides a fundamental basis. Such a conceptual representation includes representing in a declarative manner states and connections between them that represent (causal) impacts of states on each other. This part of a conceptual representation is often depicted in a *conceptual picture* by a graph with nodes and directed connections. However, a *complete conceptual representation* of a temporal-causal network model also includes a number of labels for such a graph. A notion of *strength of a connection* is used as a label for connections, some way to *aggregate multiple causal impacts* on a state is used, and a notion of *speed of change* of a state is used for timing of the processes. These three notions, called connection weight, combination function, and speed factor, make the graph of states and connections a labeled graph (e.g., see Fig. 1), and form the defining structure of a temporal-causal network model in the form of a conceptual representation; see Table 1, first 5 rows.

There are many different approaches possible to address the issue of combining multiple impacts. To provide sufficient flexibility, the Network-Oriented Modelling approach based on temporal-causal networks incorporates for each state a way to specify how multiple causal impacts on this state are aggregated by a combination function. For this aggregation a library with a number of standard combination functions are available as options, but also own-defined functions can be added.

Next, this conceptual interpretation is expressed in a formal-numerical way, thus associating semantics to any temporal-causal network specification in a detailed numerical-mathematically defined manner.

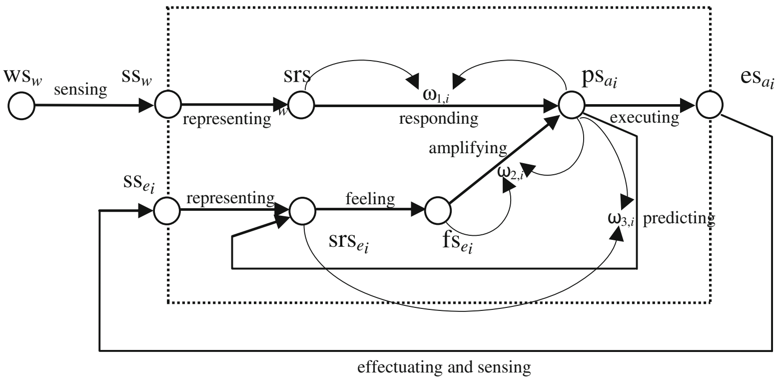

Fig. 1. An adaptive temporal-causal network model for adaptive decision making.

Table 1. Concepts of conceptual and numerical representations of a temporal-causal network.

Concepts	Notation	Explanation
States and connections	X, Y, $X \rightarrow Y$	Describes the nodes and links of a network structure (e.g., in graphical or matrix format)
Connection weight	$\boldsymbol{\omega}_{X,Y}$	The *connection weight* $\boldsymbol{\omega}_{X,Y} \in [-1, 1]$ represents the strength of the causal impact of state X on state Y through connection $X \rightarrow Y$
Aggregating multiple impacts	$\mathbf{c}_Y(..)$	For each state Y (a reference to) a *combination function* $\mathbf{c}_Y(..)$ is chosen to combine the causal impacts of other states on state Y
Timing of the causal effect	$\boldsymbol{\eta}_Y$	For each state Y a *speed factor* $\boldsymbol{\eta}_Y \geq 0$ is used to represent how fast a state is changing upon causal impact
Concepts	Numerical representation	Explanation
State values over time t	$Y(t)$	At each time point t each state Y in the model has a real number value in [0, 1]
Single causal impact	$\mathbf{impact}_{X,Y}(t)$ $= \boldsymbol{\omega}_{X,Y} X(t)$	At t state X with connection to state Y has an impact on Y, using weight $\boldsymbol{\omega}_{X,Y}$
Aggregating multiple impacts	$\mathbf{aggimpact}_Y(t)$ $= \mathbf{c}_Y(\mathbf{impact}_{X_1,Y}(t), \ldots, \mathbf{impact}_{X_k,Y}(t))$ $= \mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t))$	The aggregated causal impact of multiple states X_i on Y at t, is determined using combination function $\mathbf{c}_Y(..)$
Timing of the causal effect	$Y(t + \Delta t) = Y(t) + \boldsymbol{\eta}_Y\, [\mathbf{aggimpact}_Y(t) - Y(t)]\, \Delta t$ $= Y(t) + \boldsymbol{\eta}_Y \left[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y} X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y} X_k(t)) - Y(t)\right] \Delta t$	The causal impact on Y is exerted over time gradually, using speed factor $\boldsymbol{\eta}_Y$

This is done by showing how a conceptual representation based on states and connections enriched with labels for connection weights, combination functions and speed factors, can get an associated numerical representation [10], Ch. 2; see Table 1, last five rows. The difference equations in the last row in Table 1 constitute the numerical representation of the temporal-causal network model and can be used for simulation and mathematical analysis; it can also be written in differential equation format:

$$\begin{aligned} Y(t+\Delta t) &= Y(t) + \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)]\Delta t \\ \mathbf{d}Y(t)/\mathbf{d}(t) &= \boldsymbol{\eta}_Y[\mathbf{c}_Y(\boldsymbol{\omega}_{X_1,Y}X_1(t), \ldots, \boldsymbol{\omega}_{X_k,Y}X_k(t)) - Y(t)] \end{aligned} \tag{1}$$

In adaptive networks connection weights ω are treated in the same way as states, and are defined by combination functions $\mathbf{c}_\omega(\ldots)$ in a similar manner (with suitable arguments refering to relevant states and connection weights):

$$\begin{aligned} \omega(t+\Delta t) &= \omega(t) + \boldsymbol{\eta}_\omega[\mathbf{c}_\omega(\ldots) - \omega(t)]\Delta t \\ \mathbf{d}\omega(t)/\mathbf{d}t &= \boldsymbol{\eta}_\omega[\mathbf{c}_\omega(\ldots) - \omega(t)] \end{aligned} \tag{2}$$

3 Adaptive Networks Based on Hebbian Learning

In this section it is discussed how specific combination functions for Hebbian learning can be defined, and it will be analysed what equilibrium values can emerge for the learnt connections. First a basic definition; see also [2, 5, 9].

Definition 1 (stationary point and equilibrium)
A state Y or connection weight ω has a *stationary point* at t if $\mathbf{d}Y(t)/\mathbf{d}t = 0$ or $\mathbf{d}\omega(t)/\mathbf{d}t$. The network is in *equilibrium* a t if every state Y and connection weight of the model has a stationary point at t. A state Y has is increasing at t if $\mathbf{d}Y(t)/\mathbf{d}t > 0$; it is decreasing at t if $\mathbf{d}Y(t)/\mathbf{d}t < 0$. Similar for adaptive connections based on $\mathbf{d}\omega(t)/\mathbf{d}t$.

Considering the specific type of differential equation for a temporal-causal network model, and assuming a nonzero speed factor, from (1) and (2) more specific criteria can be found:

Lemma 1 (Criteria for a stationary, increasing and decreasing)
Let Y be a state and $X_1, \ldots, X_k$ the states with outgoing connections to state Y. Then

$$\begin{aligned} &Y \text{ has a stationary point at } t &&\Leftrightarrow \quad \mathbf{c}_Y(\omega_{X_1,Y}X_1(t), \ldots, \omega_{X_k,Y}X_k(t)) = Y(t) \\ &Y \text{ is increasing at } t &&\Leftrightarrow \quad \mathbf{c}_Y(\omega_{X_1,Y}X_1(t), \ldots, \omega_{X_k,Y}X_k(t)) > Y(t) \\ &Y \text{ is decreasing at } t &&\Leftrightarrow \quad \mathbf{c}_Y(\omega_{X_1,Y}X_1(t), \ldots, \omega_{X_k,Y}X_k(t)) < Y(t) \end{aligned}$$

Similar criteria are applied to adaptive connection weights:

$$\begin{aligned} &\omega \text{ has a stationary point at } t &&\Leftrightarrow \quad \mathbf{c}_\omega(\ldots) = \omega(t) \\ &\omega \text{ is increasing at } t &&\Leftrightarrow \quad \mathbf{c}_\omega(\ldots) > \omega(t) \\ &\omega \text{ is decreasing at } t &&\Leftrightarrow \quad \mathbf{c}_\omega(\ldots) < \omega(t) \end{aligned}$$

The Hebbian learning principle for the connection between two mental states is sometimes formulated as 'neurons that fire together, wire together'; e.g., [1, 3, 4, 6–8, 11].

This is modelled by using the activation values the two mental states *X(t)* and *Y(t)* have at time *t*. Then the weight $\omega_{X,Y}$ of the connection from *X* to *Y* is changing over time dynamically, depending on these levels *X(t)* and *Y(t)*. As this connection weight is dynamic, following the Network-Oriented Modeling approach outlined in Sect. 2 it is handled as a state with its own combination function $c_{\omega_{X,Y}}(V_1, V_2,)$, and using the standard difference and differential equation format as shown in (2) in Sect. 2

$$\begin{aligned} \omega_{X,Y}(t+\Delta t) &= \omega_{X,Y}(t) + \eta_{\omega_{X,Y}}[c_{\omega_{X,Y}}(X(t), Y(t), \omega_{X,Y}(t)) - \omega_{X,Y}(t)]\Delta t \\ \mathbf{d}\omega_{X,Y}/\mathbf{d}(t) &= \eta_{\omega_{X,Y}}[c_{\omega_{X,Y}}(X, Y, \omega_{X,Y}) - \omega_{X,Y}] \end{aligned} \quad (3)$$

The parameter $\eta_{\omega_{X,Y}}$ is the speed parameter of connection weight $\omega_{X,Y}$, in this case interpreted as learning rate. Note that by the above criteria $\omega_{X,Y}$ increases if and only if $c_{\omega_{X,Y}}(X, Y, \omega_{X,Y}) > \omega_{X,Y}$, and $\omega_{X,Y}$ decreases if and only if $c_{\omega_{X,Y}}(X, Y, \omega_{X,Y}) < \omega_{X,Y}$, and $\omega_{X,Y}$ is stationary if and only if $c_{\omega_{X,Y}}(X, Y, \omega_{X,Y}) = \omega_{X,Y}$.

An example of a mental network model using Hebbian learning is shown in Fig. 1 (adopted from [10], Ch 6, p. 163). It describes adaptive decision making as affected by direct triggering of decision options a_i (via weights $\omega_{1,i}$) in combination with emotion-related valuing of the options by an as-if prediction loop (via weights $\omega_{3,i}$ and $\omega_{2,i}$). For the weights of the adaptive connections the bending arrows show that they are affected by the states they connect. Here ws_w are world states, ss_w sensor states, srs_w and srs_{e_i} sensory representations states for stimulus w and action effect e_i, ps_{a_i} preparation states for a_i, fs_{e_i} feeling states for action effect e_i, and es_{a_i} execution states for a_i. A relatively simple example, also used in [10] in a number of applications (including in Ch 6 for the model shown in Fig. 1) is the following combination function:

$$\begin{aligned} &c_{\omega_{X,Y}}(V_1, V_2, W) = V_1V_2(1 - W) + \mu W \\ \text{or} \quad &c_{\omega_{X,Y}}(X(t), Y(t), \omega_{X,Y}(t)) = X(t)Y(t)(1 - \omega_{X,Y}(t)) + \mu\omega_{X,Y}(t) \end{aligned} \quad (4)$$

Here μ is a persistence parameter. In an emerging equilibrium state it turns out that the equilibrium value for $\omega_{X,Y}$ functionally depends on the equilibrium values of *X* and *Y* according to some formula that has been determined for this case in [10], Ch 12. For some example patterns, see Fig. 2.

It is shown that when the equilibrium values of *X* and *Y* are 1, the equilibrium value for $\omega_{X,Y}$ is 0.83 (top row), when the equilibrium values of *X* and *Y* are 0.6, the equilibrium value for $\omega_{X,Y}$ is 0.64 (middle row), and when the equilibrium values of *X* and *Y* are 0, the equilibrium value for $\omega_{X,Y}$ is 0 (bottom row). This equilibrium value of $\omega_{X,Y}$ is always attracting. The three different rows in Fig. 1 illustrate how the equilibrium value of $\omega_{X,Y}$ varies with the equilibrium values of *X* and *Y*. It is this relation that is analysed in a more general setting in some depth in this paper. In Example 1 in Sect. 5 below, this case is analysed and more precise numbers will be derived for the equilibrium values. In [10], Ch. 12 a mathematical analysis was made for the equilibria of the specific example combination function above. In the current paper a much more general analysis is made which applies to a wide class of functions.

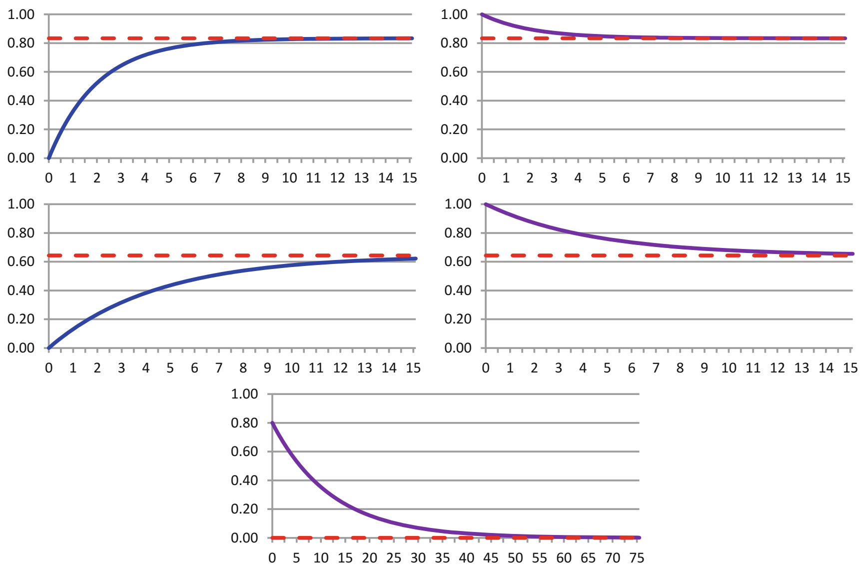

Fig. 2. Hebbian learning $\eta = 0.4$, $\mu = 0.8$, $\Delta t = 0.1$; adopted from [10], pp. 339–340. a) Top row: activation levels $X_1 = 1$ and $X_2 = 1$; equilibrium value 0.83 b) Middle row activation levels $X_1 = 0.6$ and $X_2 = 0.6$; equilibrium value 0.64 c) Bottom row: activation levels $X_1 = X_2 = 0$; equilibrium value 0 (pure extinction)

The following plausible assumptions are made for a Hebbian learning function: one set for fully persistent Hebbian learning and one set for Hebbian learning with extinction described by a persistence parameter μ; here V_1 is the argument of the function $c_{\omega X,Y}(..)$ used for $X(t)$, V_2 for $Y(t)$, and W for $\omega_{X,Y}(t)$.

Definition 2 (Hebbian learning function)
A function c: [0, 1] x [0, 1] x [0, 1] → [0, 1] is called a *fully persistent Hebbian learning function* if the following hold:

(a) $c(V_1, V_2, W)$ is a monotonically increasing function of V_1 and V_2
(b) $c(V_1, V_2, W) - W$ is a monotonically decreasing function of W
(c) $c(V_1, V_2, W) \geq W$
(d) $c(V_1, V_2, W) = W$ if and only if one of V_1 and V_2 is 0 (or both), or $W = 1$

A function c: [0, 1] x [0, 1] x [0, 1] → [0, 1] is called a *Hebbian learning function* with *persistence parameter* μ if the following hold:

(a) $c(V_1, V_2, W)$ is a monotonically increasing function of V_1 and V_2
(b) $c(V_1, V_2, W) - \mu W$ is a monotonically decreasing function of W
(c) $c(V_1, V_2, W) \geq \mu W$
(d) $c(V_1, V_2, W) = \mu W$ if and only if one of V_1 and V_2 is 0 (or both), or $W = 1$

Note that for μ = 1 the function is fully persistent. The following proposition shows that for any Hebbian learning function with persistence parameter μ there exists a monotonically increasing function $f_\mu(V_1, V_2)$ which is implicitly defined for given V_1, V_2

by the equation $c_{\omega_{X,Y}}(V_1, V_2, W) = W$ in W. When applied to an equilibrium state of an adaptive temporal-causal network, the existence of this function $f_\mu(V_1, V_2)$ reveals that in equilibrium states there is a direct and monotonically increasing functional relation of the equilibrium value $\underline{\omega}_{X,Y}$ of $\omega_{X,Y}$ with the equilibrium values $\underline{X}$, $\underline{Y}$ of the states X and Y. This is described in Theorem 1 below. Proposition 1 describes the functional relation needed for that. For proofs of Propositions 1 and 2, see the Appendix.

Proposition 1 (functional relation for *W*)
Suppose that $c(V_1, V_2, W)$ is a Hebbian learning function with persistence parameter μ.
(a) Suppose $\mu < 1$. Then the following hold:
  (i) The function $W \rightarrow c(V_1, V_2, W) - W$ on [0, 1] is strictly monotonically decreasing
  (ii) There is a unique function f_μ: [0, 1] x [0, 1] → [0, 1] such for any V_1, V_2 it holds

$$c(V_1, V_2, f_\mu(V_1, V_2)) = f_\mu(V_1, V_2)$$

This function f_μ is a monotonically increasing function of V_1, V_2, and is implicitly defined by the above equation. Its maximal value is $f_\mu(1, 1)$ and minimum $f_\mu(0, 0) = 0$.
(b) Suppose $\mu = 1$. Then there is a unique function f_1: (0, 1] x (0, 1] → [0, 1], such for any V_1, V_2 it holds

$$c(V_1, V_2, f_1(V_1, V_2)) = f_1(V_1, V_2)$$

This function f_1 is a constant function of V_1, V_2 with $f_1(V_1, V_2) = 1$ for all V_1, $V_2 > 0$ and is implicitly defined on (0, 1] x (0, 1] by the above equation.

If one of V_1, V_2 is 0, then any value of W satisfies the equation $c(V_1, V_2, W) = W$, so no unique function value for $f_1(V_1, V_2)$ can be defined then.

When applied to an equilibrium state of an adaptive temporal-causal network, this proposition entails the following Theorem 1. For $\mu < 1$ this follows from Proposition 1a) applied to the function $c_{\omega_{X,Y}}(.)$. From (a)(i) it follows that the equilibrium value is attracting: suppose $\omega(t) < \underline{\omega}_{X,Y}$, then from $c_{\omega_{X,Y}}(\underline{X}, \underline{Y}, \underline{\omega}_{X,Y}) - \underline{\omega}_{X,Y} = 0$ and the decreasing monotonicity of $W \rightarrow c(V_1, V_2, W) - W$ it follows that $c_{\omega_{X,Y}}(\underline{X}, \underline{Y}, \omega(t)) - \omega(t) > 0$, and therefore by Lemma 1 $\omega(t)$ is increasing. Similarly, when $\omega(t) > \underline{\omega}_{X,Y}$, it is decreasing.

For $\mu = 1$ the statement follows from Proposition 1b) applied to the function $c_{\omega_{X,Y}}(.)$.

Theorem 1 (functional relation for equilibrium values of $\omega_{X,Y}$)
Suppose in a temporal-causal network $c_{\omega_{X,Y}}(V_1, V_2, W)$ is the combination function for connection weight $\omega_{X,Y}$ and is a Hebbian learning function with persistence parameter μ, with f_μ the function defined by Proposition 1. In an achieved equilibrium state the following hold.

(a) Suppose $\mu < 1$. For any equilibrium values $\underline{X}, \underline{Y} \in [0, 1]$ of states X and Y the value $f_\mu(\underline{X}, \underline{Y})$ provides the unique equilibrium value $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$. This $\underline{\omega}_{X,Y}$ monotonically depends on $\underline{X}$, $\underline{Y}$: it is higher when $\underline{X}$, $\underline{Y}$ are higher. The maximal equilibrium value $\underline{\omega}_{X,Y}$ of $\omega_{X,Y}$ is $f_\mu(1, 1)$ and the minimal equilibrium value is 0. Moreover, the equilibrium value $\underline{\omega}_{X,Y}$ is attracting.

(b) Suppose $\mu = 1$. If for the equilibrium values $\underline{X}, \underline{Y} \in [0, 1]$ of states X and Y it holds $\underline{X}, \underline{Y} > 0$, then $\underline{\omega}_{X,Y} = 1$. If one of $\underline{X}, \underline{Y}$ is 0, then $\underline{\omega}_{X,Y}$ can be any value in [0, 1]: it does not depend on $\underline{X}, \underline{Y}$. So, for $\mu = 1$ the maximal value of $\underline{\omega}_{X,Y}$ in an equilibrium state is 1 and the minimal value is 0.

4 Variable Separation for Hebbian Learning Functions

There is a specific subclass of Hebbian learning functions that is often used. Relatively simple functions $c(V_1, V_2, W)$ that satisfy the requirements from Definition 2 are obtained when the arguments V_1 and V_2 and W can be separated as follows.

Definition 3 (variable separation)
The Hebbian learning function $c(V_1, V_2, W)$ with persistence parameter μ *enables variable separation* by functions cs: [0, 1] x [0, 1] → [0, 1] monotonically increasing and cs: [0, 1] → [0, 1] monotonically decreasing if

$$c(V_1, V_2, W) = cs(V_1, V_2)\, cc(W) + \mu W$$

where $cs(V_1, V_2) = 0$ if and only if one of V_1, V_2 is 0, and $cc(1) = 0$ and $cc(W) > 0$ when $W < 1$

Note that the s in cs stands for states and the second c in cc for connection. When variable separation holds, the following proposition can be obtained. For this type of function the indicated functional relation can be defined.

Proposition 2 (functional relation for *W* based on variable separation)
Assume the Hebbian function $c(V_1, V_2, W)$ with persistence parameter μ enables variable separation by the two functions $cs(V_1, V_2)$ monotonically increasing and $cc(W)$ monotonically decreasing:

$$c(V_1, V_2, W) = cs(V_1, V_2)\, cc(W) + \mu W$$

Let $h_\mu(W)$ be the function defined for $W \in [0, 1)$ by

$$h_\mu(W) = (1 - \mu)W / cc(W)$$

Then the following hold.

(a) When $\mu < 1$ the function $h_\mu(W)$ is strictly monotonically increasing, and has a strictly monotonically increasing inverse g_μ on the range $h_\mu([0, 1))$ of h_μ with $W = g_\mu(h_\mu(W))$ for all $W \in [0, 1)$.
(b) When $\mu < 1$ and $c(V_1, V_2, W) = W$, then $g_\mu(cs(V_1, V_2)) < 1$ and $W < 1$, and it holds

$$h_\mu(W) = cs(V_1, V_2)$$
$$W = g_\mu(cs(V_1, V_2))$$

So, in this case the function f_μ from Theorem 1 is the function composition g_μ o cs of cs followed by g_μ; it holds:

$$f_\mu(V_1, V_2) = g_\mu(cs(V_1, V_2))$$

(c) For $\mu = 1$ it holds $c(V_1, V_2, W) = W$ if and only if $V_1 = 0$ or $V_2 = 0$ or $W = 1$.
(d) For $\mu < 1$ the maximal value W with $c(V_1, V_2, W) = W$ is $g_\mu(cs(1, 1))$, and the minimal equilibrium value W is 0. For $\mu = 1$ the maximal value W is 1 (always when $V_1, V_2 > 0$ holds) and the minimal value is 0 (occurs when one of V_1, V_2 is 0).

Note that by Proposition 2 the function $f_\mu(V_1, V_2)$ can be determined by inverting the function $h_\mu(W) = (1 - \mu)W/cc(W)$ to find g_μ and composing the inverse with the function $cs(V_1, V_2)$. This will be shown below for some cases. For the case of an equilibrium state of an adaptive temporal network model Proposition 2 entails Theorem 2.

Theorem 2 (functional relation for equilibrium values of $\omega_{X,Y}$: variable separation)
Assume in a temporal-causal network the Hebbian learning combination function $c_{\omega_{X,Y}}(V_1, V_2, W)$ with persistence parameter μ for $\omega_{X,Y}$ enables variable separation by the two functions $cs_{\omega_{X,Y}}(V_1, V_2)$ monotonically increasing and $cc_{\omega_{X,Y}}(W)$ monotonically decreasing, and the functions f_μ and g_μ are defined as in Propositions 1 and 2. Then the following hold.

(a) When $\mu < 1$ in an achieved equilibrium state with equilibrium values $\underline{X}$, $\underline{Y}$ for states X and Y and $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ it holds

$$\underline{\omega}_{X,Y} = f_\mu(\underline{X}, \underline{Y}) = g_\mu(cs_{\omega_{X,Y}}(\underline{X}, \underline{Y})) < 1$$

(b) For $\mu = 1$ in an equilibrium state with equilibrium values $\underline{X}$, $\underline{Y}$ for states X and Y and $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ it holds $\underline{X} = 0$ or $\underline{Y} = 0$ or $\underline{\omega}_{X,Y} = 1$.
(c) For $\mu < 1$ in an equilibrium state the maximal equilibrium value $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ is $g_\mu(cs_{\omega_{X,Y}}(1, 1)) < 1$, and the minimal equilibrium value $\underline{\omega}_{X,Y}$ is 0. For $\mu = 1$ the maximal value is 1 (always when $\underline{X}$, $\underline{Y} > 0$ holds for the equilibrium values for the states X and Y) and the minimal value is 0 (which occurs when one of $\underline{X}$, $\underline{Y}$ is 0

5 Analysis of Different Cases of Hebbian Learning Functions

In this section some cases are analysed as corollaries of Theorem 2. First the specific class of Hebbian learning functions enabling variable separation with $cc(W) = 1-W$ is considered. Then

$$h_\mu(W) = (1 - \mu)W/cc(W) = (1 - \mu)W/(1 - W) \tag{5}$$

and the inverse $g_\mu(W')$ of $h_\mu(W)$ can be determined from (4) algebraically as follows.

$$\begin{aligned} W' &= (1-\mu)W/(1-W) \\ W'(1-W) &= (1-\mu)W \\ W' - W'W &= (1-\mu)W \\ W' &= (W' + (1-\mu))W \\ W &= W'/[W' + (1-\mu)] \end{aligned}$$

So

$$g_\mu(W') = W'/[W' + (1-\mu)] \tag{6}$$

Substitute $W' = \mathrm{cs}(V_1, V_2)$ in (6) and it is obtained:

$$f_\mu(V_1, V_2) = g_\mu(\mathrm{cs}(V_1, V_2)) = \mathrm{cs}(V_1, V_2)/[(1-\mu) + \mathrm{cs}(V_1, V_2)] \tag{7}$$

and this is less than 1 because $1 - \mu > 0$. From this and Theorem 2b) and c) it follows.

Corollary 1 (cases for function $\mathrm{cc}_{\omega_{X,Y}}(W) = 1 - W$)
Assume in a temporal-causal network the Hebbian learning combination function $c_{\omega_{X,Y}}(V_1, V_2, W)$ for $\omega_{X,Y}$ with persistence parameter μ enables variable separation by the two functions $\mathrm{cs}_{\omega_{X,Y}}(V_1, V_2)$ monotonically increasing and $\mathrm{cc}_{\omega_{X,Y}}(W)$ monotonically decreasing, where $\mathrm{cc}_{\omega_{X,Y}}(W) = 1 - W$. Then the following hold.

(a) When $\mu < 1$ in an equilibrium state with equilibrium values $\underline{X}$, $\underline{Y}$ for states X and Y and $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ it holds

$$\omega_{X,Y} = f_\mu(\mathrm{X}, \mathrm{Y}) = \mathrm{cs}(\mathrm{X}, \mathrm{Y})/[(1-\mu) + \mathrm{cs}(\mathrm{X}, \mathrm{Y})] < 1$$

(b) For $\mu = 1$ in an equilibrium state with equilibrium values $\underline{X}$, $\underline{Y}$ for states X and Y and $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ it holds $\underline{X} = 0$ or $\underline{Y} = 0$ or $\underline{\omega}_{X,Y} = 1$.

(c) For $\mu < 1$ in an equilibrium state the maximal equilibrium value $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ is

$$\mathrm{cs}(1,1)/[(1-\mu) + \mathrm{cs}(1,1)] < 1$$

and the minimal equilibrium value $\underline{\omega}_{X,Y}$ is 0. For $\mu = 1$ the maximal value is 1 (when $\underline{X}$, $\underline{Y} > 0$ holds for the equilibrium values for the states X and Y) and the minimal value is 0 (which occurs when one of $\underline{X}$, $\underline{Y}$ is 0).

Corollary 1 is illustrated in the following three examples.

Example 1. $c(V_1, V_2, W) = V_1V_2(1 - W) + \mu W$

$$\mathrm{cs}(V_1, V_2) = V_1V_2 \quad \mathrm{cc}(W) = 1 - W$$

This is the example shown in Fig. 2

$$f_\mu(V_1, V_2) = cs(V_1, V_2)/[(1-\mu) + cs(V_1, V_2)] \tag{8}$$

Substitute $cs(V_1, V_2) = V_1\ V_2$ in (7) then $f_\mu(V_1, V_2) = V_1\ V_2$ / $[(1-\mu) + V_1\ V_2]$. Maximal W is $W = f_\mu(1, 1) = 1/[2 - \mu]$, which for $\mu = 1$ is 1; minimal W is 0. The equilibrium values shown in Fig. 2 can immediately derived from this (recall $\mu = 0.8$):

Top row $V_1 = 1$, $V_2 = 1$, then $f_\mu(1, 1) = 1/[2 - \mu] = 0.833333$

Middle row $V_1 = 0.6$, $V_2 = 0.6$, then $f_\mu(0.6, 0.6) = 0.36$ / $[(1 - 0.8) +0.36]$ = 0.642857

Bottom row $V_1 = 0$, $V_2 = 0$, then $f_\mu(0, 0) = 0$

Example 2. $c(V_1, V_2, W) = (\sqrt{V_1 V_2})(1-W) + \mu W$

$$cs(V_1, V_2) = \sqrt{V_1 V_2} \quad cc(W) = 1 - W$$

$$f_\mu(V_1, V_2) = cs(V_1, V_2)/[(1-\mu) + cs(V_1, V_2)] \tag{9}$$

Substitute $cs(V_1, V_2) = \sqrt{}(V_1 V_2)$ in (8) to obtain

$$f_\mu(V_1, V_2) = \sqrt{V_1 V_2} / [(1-\mu) + \sqrt{V_1 V_2}] \tag{10}$$

Maximal W is $W = f_\mu(1, 1) = 1/[2 - \mu]$, which for $\mu = 1$ is 1; minimal W is 0.

In a similar case as in Fig. 2, but the using this function the following equilibrium values would be found

Top row $V_1 = 1$, $V_2 = 1$, then $f_\mu(1, 1) = 1/[2 - \mu] = 0.833333$

Middle row $V_1 = 0.6$, $V_2 = 0.6$, then $f_\mu(0.6, 0.6) = 0.6$ / $[(1 - 0.8) + 0.6] = 0.75$

Bottom row $V_1 = 0$, $V_2 = 0$, then $f_\mu(0, 0) = 0$

Example 3. $c_{\omega_{X,Y}}(V_1, V_2, W) = V_1 V_2(V_1 + V_2)(1-W) + \mu W$

$$cs_{\omega_{X,Y}}(V_1, V_2) = V_1 V_2(V_1 + V_2)\ cc_{\omega_{X,Y}}(W) = 1 - W$$

$$f_\mu(V_1, V_2) = cs(V_1, V_2)/[(1-\mu) + cs(V_1, V_2)] \tag{11}$$

Substitute $cs(V_1, V_2) = V_1 V_2(V_1 + V_2)$ in (10) to obtain

$$f_\mu(V_1, V_2) = V_1 V_2(V_1 + V_2)/ [(1-\mu) + V_1 V_2(V_1 + V_2)] \tag{12}$$

Maximal W is $f_\mu(1, 1) = 2/[(1 - \mu) + 2] = 2/[3 - \mu]$, which for $\mu = 1$ is 1; minimal W is 0

In a similar case as in Fig. 2, but the using this function the following equilibrium values would be found

Top row $V_1 = 1$, $V_2 = 1$, then $f_\mu(1, 1) = 2/[3 - \mu] = 0.909090$

Middle row $V_1 = 0.6$, $V_2 = 0.6$, then $f_\mu(0.6, 0.6) = 0.36 * 1.2/ [(1 - 0.8) + 0.36 * 1.2] = 0.632$

Bottom row $V_1 = 0$, $V_2 = 0$, then $f_\mu(0, 0) = 0$

Next the specific class of Hebbian learning functions enabling variable separation with $cc(W) = 1 - W^2$ is considered. Then

$$\mathrm{h}_\mu(W) = (1-\mu)W/\mathrm{cc}(W) = (1-\mu)W/\left(1-W^2\right) \tag{13}$$

and the inverse of h_μ can be determined algebraically as shown in Corollary 2. Inverting $\mathrm{h}_\mu(W)$ to get inverse $\mathrm{g}_\mu(W')$ now can be done as follows:

$$W' = (1-\mu)W/\left(1-W^2\right)$$
$$\left(1-W^2\right)W' = (1-\mu)W$$
$$-W' + (1-\mu)W + W^2W' = 0$$

This is a quadratic equation in W:

$$W'W^2 + (1-\mu)W - W' = 0 \tag{14}$$

As $W \geq 0$ the solution is

$$W = (-(1-\mu) + \sqrt{((1-\mu)^2 + 4W'^2)})/(2W') \tag{15}$$

$$W = -(1-\mu)/2W' + \sqrt{((1-\mu)/2W')^2 + 1}$$

So

$$\mathrm{g}_\mu(W') = -(1-\mu)/2W' + \sqrt{((1-\mu)/2W')^2 + 1} \tag{16}$$

By substituting $W' = \mathrm{cs}(V_1, V_2)$ it follows

$$\begin{aligned} \mathrm{f}_\mu(V_1, V_2) &= \mathrm{g}_\mu(\mathrm{cs}(V_1, V_2)) \\ &= -(1-\mu)/2\,\mathrm{cs}(V_1, V_2) + \sqrt{((1-\mu)/2\,\mathrm{cs}(V_1, V_2))^2 + 1} \end{aligned} \tag{17}$$

All this is summarised in the following:

Corollary 2 (cases for function $\mathrm{cc}_{\omega_{X,Y}}(W) = 1 - W^2$)
Assume in a temporal-causal network the Hebbian learning combination function $\mathrm{c}_{\omega_{X,Y}}(V_1, V_2, W)$ for $\omega_{X,Y}$ with persistence parameter μ enables variable separation by the two functions $\mathrm{cs}_{\omega_{X,Y}}(V_1, V_2)$ monotonically increasing and $\mathrm{cc}_{\omega_{X,Y}}(W)$ monotonically decreasing, where $\mathrm{cc}_{\omega_{X,Y}}(W) = 1 - W^2$. Then the following hold.

(a) When $\mu < 1$ in an equilibrium state with equilibrium values $\underline{\mathrm{X}}$, $\underline{\mathrm{Y}}$ for states X and Y and $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ it holds

$$\omega_{X,Y} = \mathrm{f}_\mu(\underline{\mathrm{X}}, \underline{\mathrm{Y}}) = -(1-\mu)/2\,\mathrm{cs}(\underline{\mathrm{X}}, \underline{\mathrm{Y}}) + \sqrt{((1-\mu)/2\,\mathrm{cs}(\underline{\mathrm{X}}, \underline{\mathrm{Y}}))^2 + 1} < 1$$

(b) For $\mu = 1$ in an equilibrium state with equilibrium values $\underline{\mathrm{X}}$, $\underline{\mathrm{Y}}$ for states X and Y and $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ it holds $\underline{\mathrm{X}} = 0$ or $\underline{\mathrm{Y}} = 0$ or $\underline{\omega}_{X,Y} = 1$.

(c) For $\mu < 1$ in an equilibrium state the maximal equilibrium value $\underline{\omega}_{X,Y}$ for $\omega_{X,Y}$ is

$$-(1-\mu)/2\ \mathrm{cs}(1,1) + \sqrt{((1-\mu)/2\ \mathrm{cs}(1,1))^2 + 1} < 1$$

and the minimal equilibrium value $\underline{\omega}_{X,Y}$ is 0. For $\mu = 1$ the maximal value is 1 (when $\underline{X}$, $\underline{Y} > 0$ holds for the equilibrium values for the states X and Y) and the minimal value is 0 (which occurs when one of $\underline{X}$, $\underline{Y}$ is 0).

Corollary 2 is illustrated in Example 4.

Example 4. $\mathrm{c}_{\omega_{X,Y}}(V_1, V_2, W) = V_1V_2(V_1+V_2)(1-W^2) + \mu W$

$$\mathrm{cs}_{\omega_{X,Y}}(V_1, V_2) = V_1V_2(V_1+V_2)\mathrm{cc}_{\omega_{X,Y}}(W) = 1 - W^2$$

$$\mathrm{f}_\mu(V_1, V_2) = -(1-\mu)/2\ \mathrm{cs}(V_1, V_2) + \sqrt{((1-\mu)/2\ \mathrm{cs}(V_1, V_2))^2 + 1} \quad (18)$$

Substitute $\mathrm{cs}(V_1, V_2) = V_1V_2(V_1 + V_2)$

$$\mathrm{f}_\mu(V_1, V_2) = -(1-\mu)/2V_1V_2(V_1+V_2) + \sqrt{((1-\mu)/2V_1V_2(V_1+V_2))^2 + 1} \quad (19)$$

Maximal W is $W = \mathrm{f}_\mu(1, 1) = -(1-\mu)/4 + \sqrt{(((1-\mu)/4)^2 + 1)} = [-(1-\mu) + \sqrt{((1-\mu)^2 + 16)}]/4 = 4/[(1-\mu) + \sqrt{((1-\mu)^2 + 16)}]$, which for $\mu = 1$ is 1; minimal W is 0. In a similar case as in Fig. 2, using this function the equilibrium values can be found by applying (18).

6 Discussion

In this paper it was analysed how emerging behaviour of an adaptive network can be related to characteristics of network structure and adaptation principles. In particular this was addressed for an adaptive mental network based on Hebbian learning [1, 3, 4, 6–8, 11]. To this end relevant properties of the functions defining the Hebbian adaptation principle have been identified. For different classes of functions emerging equilibrium values for the connection weight have been expressed as a function of the emerging equilibrium values of the connected states. The presented results do not concern results for just one type of network or function, as more often is found, but were formulated and proven at a more general level and therefore can be applied not just to specific networks but to classes of networks satisfying the identified relevant properties of network structure and adaptation characteristics.

Appendix Proofs of Propositions 1 and 2

Proof of Proposition 1. (a) Consider $\mu < 1$. Then by Definition 2 (b) the function $W \rightarrow c(V_1, V_2, W) - \mu W$ is monotonically decreasing in W, and since $\mu - 1 < 0$ the function $W \rightarrow (\mu - 1)W$ is strictly monotonically decreasing in W. Therefore the sum of them is also strictly monotonically decreasing in W. Now this sum is

$$c(V_1, V_2, W) - \mu W + (\mu - 1)W = c(V_1, V_2, W) - W$$

So, the function $W \rightarrow c(V_1, V_2, W) - W$ is strictly monotonically decreasing in W; by Definition 2(d) it holds $c(V_1, V_2, 1) - 1 = \mu - 1 < 0$, and by Definition 2(c) $c(V_1, V_2, 0) - 0 \geq 0$. Therefore $c(V_1, V_2, W) - W$ has exactly 1 point with $c(V_1, V_2, W) - W = 0$; so for each V_1, V_2 the equation $c(V_1, V_2, W) - W = 0$ has exactly one solution W, indicated by $f_\mu(V_1, V_2)$; this provides a unique function f_μ: [0, 1] x [0, 1] $\rightarrow$ [0, 1] implicitly defined by $c(V_1, V_2, f_\mu(V_1, V_2)) = f_\mu(V_1, V_2)$. To prove that f_μ is monotonically increasing, the following. Suppose $V_1 \leq V_1'$ and $V_2 \leq V_2'$, then by monotonicity of $V_1, V_2 \rightarrow c(V_1, V_2, W)$ in Definition 2(a) it holds

$$0 = c(V_1, V_2, f_\mu(V_1, V_2)) - f_\mu(V_1, V_2) \leq c(V_1', V_2', f_\mu(V_1, V_2)) - f_\mu(V_1, V_2)$$

So $c(V_1', V_2', f_\mu(V_1, V_2)) - f_\mu(V_1, V_2) \geq 0$ whereas $c(V_1', V_2', f_\mu(V_1', V_2')) - f_\mu(V_1', V_2') = 0$
and therefore

$$c(V_1', V_2', f_\mu(V_1', V_2')) - f_\mu(V_1', V_2') \leq c(V_1', V_2', f_\mu(V_1, V_2)) - f_\mu(V_1, V_2)$$

By strict decreasing monotonicity of $W \rightarrow c(V_1, V_2, W) - W$ it follows that $f_\mu(V_1, V_2) > f_\mu(V_1', V_2')$ cannot hold, so $f_\mu(V_1, V_2) \leq f_\mu(V_1', V_2')$. This proves that f_μ is monotonically increasing. From this monotonicity of $f_\mu(..)$ it follows that $f_\mu(1, 1)$ is the maximal value and $f_\mu(0, 0)$ the minimal value. Now by Definition 1(d) it follows that $f_\mu(0, 0) = c(0, 0, f_\mu(0, 0)) = \mu\, f_\mu(0, 0)$ so $f_\mu(0, 0) = \mu\, f_\mu(0, 0)$, and as $\mu < 1$ this implies $f_\mu(0, 0) = 0$.

(b) Consider $\mu = 1$. When both V_1, V_2 are > 0, and $c(V_1, V_2, W) = W$, then $W = 1$, by Definition 1(d). This defines a function $f_1(V_1, V_2)$ of $V_1, V_2 \in (0, 1]$, this time $f_1(V_1, V_2) = 1$ for all $V_1, V_2 > 0$. When one of V_1, V_2 is 0 and $\mu = 1$, then also by Definition 1 (d) always $c(V_1, V_2, W) = W$, so in this case multiple solutions for W are possible: every W is a solution, and therefore no unique function value for $f_1(V_1, V_2)$ can be defined then.

Proof of Proposition 2

(a) From $cc(W)$ monotonically decreasing in W it follows that $W \rightarrow 1/cc(W)$ is monotonically increasing on [0, 1). Moreover, the function W is strictly monotonically increasing; therefore for $\mu < 1$ the function $h_\mu(W) = (1 - \mu)W/cc(W)$ is strictly monotonically increasing. Therefore h_μ is injective and has an inverse function g_μ on the range of h_μ: a function g_μ with $g_\mu(h_\mu(W)) = W$ for all $W \in [0, 1)$.

(b) Suppose $\mu < 1$ and $c(V_1, V_2, W) = W$, then from Definition 2(d) it follows that $W = 1$ is excluded, since from both $c(V_1, V_2, W) = W$ and $c(V_1, V_2, W) = \mu W$ it would follow $\mu = 1$, which is not the case. Therefore $W < 1$, and the following hold

$$\begin{aligned} \mathrm{cs}(V_1, V_2)\,\mathrm{cc}(W) + \mu W &= W \\ \mathrm{cs}(V_1, V_2)\,\mathrm{cc}(W) &= (1-\mu)W \\ \mathrm{cs}(V_1, V_2) = (1-\mu)W/\mathrm{cc}(W) &= \mathrm{h}_\mu(W) \end{aligned}$$

So, $\mathrm{h}_\mu(W) = \mathrm{cs}(V_1, V_2)$. Applying the inverse g_μ yields $W = \mathrm{g}_\mu(\mathrm{h}_\mu(W)) = \mathrm{g}_\mu(\mathrm{cs}(V_1, V_2))$.

Therefore in this case for the function f_μ from Theorem 1 it holds:

$$\mathrm{f}_\mu(V_1, V_2) = W = \mathrm{g}_\mu(\mathrm{cs}(V_1, V_2)) < 1$$

so f_μ is the composition of cs(..) followed by g_μ.

(c) For $\mu = 1$ the equation $c(V_1, V_2, W) = W$ becomes $\mathrm{cs}(V_1, V_2)\,\mathrm{cc}(W) = 0$ and this is equivalent to $\mathrm{cs}(V_1, V_2) = 0$ or $\mathrm{cc}(W) = 0$. From the definition of separation of variables it follows that this is equivalent to $V_1 = 0$ or $V_2 = 0$ or $W = 1$.

(d) Suppose $\mu < 1$ and $c(V_1, V_2, W) = W$, then because cs(..) and g_μ are both monotonically increasing, the maximal W is $\mathrm{g}_\mu(\mathrm{cs}(1, 1))$, and the minimal W is $\mathrm{g}_\mu(\mathrm{cs}(0, 0))$. For $\mu = 1$ these values are 1 always when $V_1, V_2 > 0$, and any value in [0, 1] (including 0) when one of V_1, V_2 is 0.

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