Implementing Mathematics Teaching that Promotes Students' Understanding Through Theory-Driven Lesson Study

Rongjin Huang, Zikun Gong, and Xue Han

Contents

Abstract Lesson study (LS) has been practiced in China as an effective way to advance teachers' professional development for decades. This study explores how LS improves teaching that promotes students' understanding. A LS group including didacticians (practice-based teaching research specialist and university-based mathematics educators) and mathematics teachers in China explored and documented how teacher participants shifted their attention to students' learning by incorporating two notions of teaching: learning trajectory (LT) and variation pedagogy (VP). The former describes conjectured routes of children's thinking and learning with pertinent tasks to move toward the learning goals along the route, while the latter

This chapter is built upon an article that formally appeared in ZDM: Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study, ZDM, 48(3), 425–439.

R. Huang (✉)
Middle Tennessee State University, Murfreesboro, TN, USA
e-mail: rhuang@mtsu.edu

Z. Gong
College of Education, Hangzhou Normal University, Hangzhou, Zhejiang, China

X. Han
National College of Education, National Louis University, Chicago, IL, USA
e-mail: xhan2@nl.edu

R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_30

suggests strategies for using systematic tasks progressively. The concepts of LT and VP are used to guide planning, teaching, and debriefing throughout the LS process. Data consist of lesson plans, videotaped lessons, post-lesson discussions, post-lesson quizzes, and teachers' reflection reports. This study reveals that by building on the learning trajectory and by strategically using variation tasks, the lesson has been improved in terms of students' understanding, proficiency, and mathematical reasoning. In addition, the LT was refined through the LS. This study displays how theory-driven LS could help teachers improve their teaching and develop the linkage between theory and practice.

Keywords Lesson study · Learning trajectory · Variation pedagogy · Theory-driven lesson study

1 Introduction

Lesson study (LS) is a form of practice-based professional development originated from Asia and has been widely adopted around the world (Hart et al. 2011; Lewis et al. 2006). One salient feature of LS is to improve teaching and develop teachers' mathematics expertise by focusing on students' learning (Murata 2011). Chinese LS has been practiced for decades (Chen and Yang 2013) and has contributed a great deal to the improvement of teaching and teachers' competence in China (Huang and Bao 2006; Huang and Han 2015). It is found that Chinese teachers have attempted to focus more on polishing teachers' instructional practices in class rather than directly on eliciting student learning during the process of LS (Chen and Yang 2013; Huang and Bao 2006). However, since 2011 the new curriculum standards have expected mathematics teachers to focus on student learning. This study is designed to explore how mathematics teachers develop lessons that promote students' understanding through a theory-driven LS approach. Specifically, the study aims to address the following research questions:

1. How does a LS group improve classroom instruction that promote students' understanding?
2. How do the teachers make these improvements toward students' understanding?
3. How does the LS process inform the refinement of the guided theories of the LS?

2 Literature and Theoretical Framework

This section discusses perspectives of effective mathematics instruction, theories underlying the LS, a model of theory-driven LS, and studies on LS research lessons that focus on division of fractions.

2.1 Effective Mathematics Classroom Instruction

Although there are cross-cultural variations in terms of what constitutes effective mathematics instruction (Cai and Wang 2010), the NCTM (2014) synthesizes eight evidence-based effective mathematics teaching practices, including establishing mathematical goals to focus on student learning, implementing tasks that promote reasoning and problem-solving, using multiple mathematical representations, and using evidence of student thinking. The Chinese curriculum standards (MOE 2011) emphasize that teaching is a process in which teachers and students actively engage in the lesson, interact with each other, and co-develop understanding of mathematics concepts. Mathematics teaching should establish students' self-exploration in problem-solving; guide students in obtaining basic knowledge, skills, thinking methods, and experiences in doing mathematics through practicing, thinking, exploring, and communicating; and continually develop students' abilities in forming, posing, analyzing, and solving problems. Given the shared ideas of effective mathematics instruction in the USA and China, this study emphasizes the following components: set clear mathematics goals, implement mathematical tasks that promote students' learning, develop conceptual understanding, and gather evidence of students' thinking.

2.2 Two Underpinning Theories in the Lesson Study

Based on literature review and mathematics teaching practice in China, the expert team (e.g., university professors and specialists) of the LS adopted two specific notions of learning trajectory (Clements and Sarama 2004) and variation pedagogy (Gu et al. 2004; Marton and Pang 2006) to guide their activities throughout the LS.

Learning trajectories (LT) have been developed and proposed as the foundation for classroom instruction (Simon 1995; Sztajn et al. 2012). In his seminal work, Simon (1995) suggests the hypothetical LT as the pathway on which students might proceed as they advance their learning toward the intended goals. Further research has refined the LT concept to the "descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks" (Clements and Sarama 2004, p. 83). Research shows that the use of LTs can support teachers' knowledge growth and instructional decision-making, allow teachers to focus on students' thinking, and eventually improve students' achievement (Clements et al. 2011).

Variation pedagogy (VP) arises from the Chinese mathematical teaching tradition and focuses on using deliberate and systematic variation in mathematics tasks to help students develop new concepts and problem-solving abilities (Gu et al. 2004). Building on the core notion of variation, researchers have developed a theory of variation (Marton and Pang 2006; Marton and Tsui 2004). According to the theory, learning is to develop new ways of seeing something; specific comparisons allow one to discern critical features of the object being studied and thus learn more about it. For example, contrasting a triangle with "not triangles" (e.g., squares, pentagons, hexagons) allows critical aspects of triangles such as the number of sides to be discerned. Students can then learn other critical features such as the size of angles and the length of sides by looking at different kinds of triangles (e.g., right-angled triangle and isosceles triangle). Furthermore, by examining what remains unchanged as appearances vary (e.g., position and size of triangles), students could further discern the invariant feature of triangles such as the sum of all angles is 180° (Lo and Marton 2012). Thus, in mathematics classroom instruction, it is crucial to create certain patterns of variation: examining what varies against what is invariant (Marton and Pang 2006; Marton and Tsui 2004). Lo and Marton (2012) further argue "when learners need to discern more *than two or more critical features*, the most powerful strategy is to let the learners discern them one at a time, before they encounter simultaneous variation of the features" (p. 11).

To use measurable terms to describe students' learning, researchers have developed frameworks from the theory of variation (Marton and Pang 2006) including object of learning, enacted object of learning, and lived objects of learning. An *object of learning* is "a specific insight, skill, or capability that the students are expected to develop during a lesson or during a limited sequence of lessons" (Marton and Pang 2006, p. 194). The *enacted object of learning* is described as the patterns of variation and invariance co-constructed by the teacher and the students. The patterns of variation consist of the necessary conditions for the appropriation of the enacted object of learning. From the students' answers to the written and oral questions after the lesson, we can characterize the *lived object of learning*, i.e., the object of learning that is experienced by the learners.

Gu et al. (2004) highlight the role of using varying tasks for promoting students' conceptual understanding. This corresponds to Watson and Mason's (2006) argument that varying tasks could be used to develop students' conjectures. Furthermore, Marton and his colleagues (Marton and Pang 2006; Marton and Tsui 2004) have focused on examining *how* lesson study guided by the variation pedagogy impacts teaching and students' learning. In this study, we utilize the lenses of the object of learning and enacted and lived objects of learning to investigate how the theory-driven LS may improve classroom instruction and students' learning.

2.3 A Model of Theory-Driven Lesson Study

Although the concept of LTs suggests the importance of describing a conjectured route of children's thinking and learning with pertinent tasks to move toward the learning goals along the route, how these tasks should be designed and presented to students have not been explicitly addressed. The notion of VP emphasizes specific strategies in using systematic tasks progressively, but it has not paid explicit

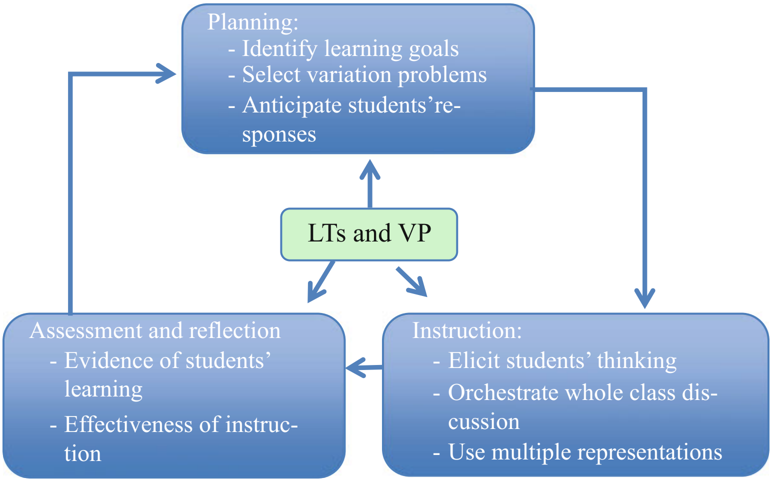

Fig. 1 The lesson study’s cycle

attention to the route of children’s learning. Thus, the incorporation of these two perspectives may provide a useful tool for designing and delivering lessons: VP could help teachers strategically design and implement tasks in line with LT. Centering LTs and VP in lesson design and implementation, we created a model LS cycle as shown in Fig. 1.

When planning lessons, teachers set clear mathematics learning goals, select appropriate tasks based on LTs and VP, and anticipate students’ responses to the tasks. When implementing lessons, teachers encourage students to express their actions and their thoughts while solving tasks, orchestrate whole class discussion of students’ work, and build connections among representations. In the post-lesson reflection, evidence of students’ learning is gathered through classroom observation and assessments. Suggestions for improvement should then be made based on this evidence.

2.4 *Studies on Teaching and Learning of Division of Fractions*

Developing conceptual understanding of the algorithm for division of fractions is not an easy task for either students or teachers (Carpenter et al. 1988). Researchers have developed two general pedagogical approaches for teaching division of fractions (Li 2008). One provides mathematical justifications for the division of fractions algorithm based on the properties of fractions and the meaning of division

(Tirosh 2000). The other uses concrete or visual demonstrations to explain how division of fractions can be computed, such as extending whole number division to division of fractions through a quotitive interpretation (Silvia 1983) or a partitive interpretation (Ott et al. 1991).

Recently, a complementary approach of making sense of division of fractions using quotitive and partitive models and justifying why the algorithm works has been proposed (Sowder et al. 2010). In particular, it is suggested that division of fractions should be taught through solving word problems aligned with the sequence: (1) a unit fraction divided by a non-zero whole number using the partitive model and the relationship between division and multiplication, (2) a whole number divided by a unit fraction using the quotitive model and the relationship between division and multiplication, and (3) a fraction divided by fraction using the quotitive model and the relationship between division and multiplication (CCSSI 2010).

Based on existing research and the framework of equipartitioning (Maloney et al. 2014), the expert team of the LS developed an LT. The LT includes five dimensions: *sequence*, *situation*, *model*, *representations*, and *tasks*.

The *sequence* dimension includes eight levels: (1) a whole number divided by a whole number; (2) a fraction divided by a whole number, where the dividend is a multiplier of divisor; (3) a unit fraction divided by a whole number; (4) a fraction divided by a whole number; (5) a whole number divided by a unit fraction; (6) a whole number divided by a fraction; (7) a unit fraction divided by a unit fraction; and (8) a fraction divided by a fraction. *Situation* includes contextual and mathematical situations. *Model* refers to partitive and quotitive models. *Representations* include visual representations, such as explaining an algorithm using pictures; verbal representations, such as explaining algorithm using verbal language (i.e., how many $^1/_5$s can go to 1 unit?); and symbolic representations (i.e., $1 \div {}^1/_5 = 1 \times {}^5/_1 = 5$ or $1 \div {}^1/_a = 1 \times a$).

Aligned with each level, there are instructions about models and tasks. For example, at level 4, the partitive model is suggested. The following task is offered as an illustration: A $^4/_5$-kg cake is shared with 3 friends. How much does each friend get? In addition, three representations are illustrated.

3 Methods

In this section, we first describe the major components of the LS, including the LS group, instruments, and the procedure of implementing the LS. We then present the methods of data collection and data analysis.

Table 1 Background information of the team members

Code	Name	Title of profession	Highest degree	Experiences
R1	Mr. Kong	University professor	Ph.D. in mathematics education	Twenty-five years in both elementary and university levels
SP1	Mr. Shao	Senior teacher, specialist at district level	Bachelor in elementary education	Twelve years of teaching mathematics and 13 years in serving as a mathematics specialist in elementary school
SP2	Mr. Ren	Senior teacher, specialist at city level	Master in mathematics education	Eleven years of teaching mathematics and 16 years of serving as a mathematics specialist in elementary school
T1	Ms. Tang	Senior teacher	Bachelor in public affair management	Twenty-seven years of teaching mathematics
T2	Ms. Han	Senior teacher	Bachelor in education	Fourteen years of teaching mathematics
DT	Ms. Lu	First level of teacher	Bachelor in information and technology	Five years of teaching mathematics

3.1 *The Setting: The School, Teachers, and the Lesson Study Group*

The LS took place in an elementary school in southeastern China. The school serves around 1500 students in grades 1–6 (45 classes) with 24 mathematics teachers. The LS group consists of three mathematics teachers with various levels of experience and three didacticians (two teaching research specialists and one university mathematics educator). Teaching research specialists are specialists who are employed by various practice-based education divisions and preliminarily work with practicing teachers (see Huang et al. (2014) or Gu and Gu (this book) for details). The backgrounds of these teachers and didacticians (e.g., professor and specialist) are shown in Table 1.

Table 1 shows that the university mathematics educator, Mr. Kong, is experienced in teaching mathematics and mathematics education at elementary and university levels. The two specialists, Mr. Ren and Mr. Shao, are excellent elementary mathematics teachers with experience in teaching research activities. The two more experienced teachers are Ms. Tang and Ms. Han. Ms. Lu has taught research lessons based on the LT suggested by experts. Ms. Lu has a bachelor's degree in information and technology and has 5 years of mathematics teaching experience. She has won several teaching awards at the district and the city levels. With the school-based teaching research group, the two experienced teachers worked with Ms. Lu to develop the initial lesson plans and to watch and improve the lessons.

The expert team was responsible for overseeing the process of LS and developing LT of the division of fractions. The school-based teacher team was responsible for designing and delivering the research lessons. Both experts and teachers participated in observing research lessons and post-lesson debriefings. Although experts provided critical comments on the research lessons, the participating teachers made final decisions about the research lesson revisions.

3.2 The Process of Conducting the Lesson Study

The LS group conducted two consecutive research lessons: a lesson on dividing a fraction by a whole number and a lesson on dividing a fraction by a fraction. A three-phase process (Huang and Bao 2006) was used to develop both research lessons. In phase one, *trial teaching 1*, the teachers collaboratively developed the research lessons, and Ms. Lu delivered them in her class. In phase two, *trial teaching 2*, the group worked to revise the lesson plans based on the first debrief and self-reflection, and Ms. Lu taught the revised lessons to a different group of students. In phase three, the group sought to develop an *exemplary lesson* by teaching the same topic based on the rehearsals and debriefings. During trial teachings 1 and 2, the group observed the teaching, collected post-lesson quizzes from the students and reflections from Ms. Lu, debriefed, and revised the lessons plans. In phase three, only the post-lesson quiz and teacher reflection were administrated. The lesson study was conducted in October 2014. Each of the two research lessons was taught in classes 602 (26 students), 604 (28 students), and 607 (34 students), respectively. The Spring 2014 unified exam passing rates of these three classes were 100%, and their excellent rates were 98% (602), 94.6% (604), and 98.5% (607), respectively. Thus, students had similar mathematical performance in these three classes.

3.3 Data Collection

During the LS, the following data were collected: (a) pre- and post LS versions of the LT, (b) all versions of lesson plans, (c) videotaped research lessons and students' worksheets, (d) videotaped post-research lesson debriefings, (e) videotaped pre-lesson interviews with the teachers, (f) post-lesson quizzes (see Appendix 1), (g) selected student interviews after classes, (h) reflection journals of demonstrating teachers and teaching research specialists, and (i) audio-recorded post-lesson interviews with teachers and teaching research specialists. To address the research questions of this article, the first seven types of data from research lesson 2 were used due to space restriction.

Table 2 Learning trajectory and task (T) sequence in the research lesson

Learning trajectory	Mathematics task	
	Teaching 1	Teaching 2
1. 1 divided by a unit fraction (e.g., 1 ÷ ⅕)	T1: How many ½- liter glasses are there in 2 liters of milk?	T1a: How many ⅕-liter glasses are there in 1 liter of milk?
		T1b: How many ¼-liter glasses are there in 1 liter of milk?
		T1c: How many ⅓-liter glasses are there in 1 liter of milk?
		T1d: How many ⅙-liter glasses are there in 1 liter of milk?
2. 1 divided by a fraction (e.g., 1 ÷ ⅖)	T2: How many ⅖-liter glasses are there in 1 liter of milk?	T2a: How many ⅖-liter glasses are there in 1 liter of milk?
		T2b: How many ²⁄₇-liter glasses are there in 1 liter of milk?
		T2c: How many ¾-liter glasses are there in 1 liter of milk?
		T2d: How many ⅗-liter glasses are there in 1 liter of milk?
3. A whole number divided by a fraction (e.g., 3 ÷ ⅖)	T3: How many ⅖-liter glasses are there in 3 liters of milk?	T3a: How many ⅖-liter glasses are there in 2 liters of milk?
		T3b: How many ⅖-liter glasses are there in 3 liters of milk?
		T3c: How many ⅖-liter glasses are there in 4 liters of milk?
		T3d: How many ⅖-liter glasses are there in 100 liters of milk?
4. A fraction divided by a fraction (e.g., ½ ÷ ⅓)	T4: How many ⅓-liter glasses are there in ½ liters of milk?	T4a: How many ⅖-liter glasses are there in ¾ liters of milk?
		T4b: How many ⅖-liter glasses are there in ⅝ liters of milk?
		T4c: How many ¾-liter glasses are there in ⅘ liters of milk?

Notes: The subtask (T1a, T1b, T1c, etc.) of teaching 2 suggests a sequence of variation intended to highlight the critical features of the learning trajectory

3.4 Data Analysis

The audio-recorded interviews, videotaped lessons, and debriefs were transcribed verbatim in Chinese. Data analysis was performed on the Chinese documents with relevant transcripts translated into English.

To address research question 1 (e.g., improvement of classroom instruction), we examined the data at two levels. At the macro-level, the LT and associated tasks of the research lesson were examined based on lesson plans and video lessons (see Table 2; the relevant LT and associated tasks with the first research lesson can be found in Appendix 2. An essential difference between lesson 1 and lesson 2 is the interpretation model used: a partitive model was used in lesson 1, while a quotitive model was used in lesson 2). At the micro-level, we focused on the transformation of the object of learning, enacted object of learning, and lived object of learning.

The explicitly stated instructional objectives in the lesson plans were synthesized as the object of learning. The videotaped lessons were used to describe the enacted object of learning: dimensions of what varied and what was invariant. For example, after examining $1 \div \frac{1}{5} = 1 \times 5$ with verbal, visual, and arithmetic representations, the teacher presented exploration tasks: $1 \div \frac{1}{3} = 1 \times 3$, $1 \div \frac{1}{4} = 1 \times 4$, and $1 \div \frac{1}{6} = 1 \times 6$. Thus, one dimension of what varied was the divisor, while the form of division remained unchanged as 1 divided by a unit fraction. This was a necessary condition for students to discern the general pattern of 1 divided by a unit fraction equals 1 times the reciprocal of the unit fraction. All dimensions of variation in the first two teachings are displayed in Table 3.

The lived object of learning was examined through students' post-lesson quizzes. On the quiz, students were asked to solve five contextual problems and justify their solutions using a variety of methods. The quiz was rated based on three criteria: correctness, the use of a visual strategy, and the use of proportional reasoning. If a correct answer was reported, one point was credited, and then the strategies utilized were examined. Students received an additional point for their use of a visual strategy or model, and a final point was awarded for an explanation involving proportional reasoning. Thus the scores of correctness, visual strategies, and proportional reasoning each ranged from 0 to 5 points. Mean comparisons for scores and frequencies were conducted among the student cohorts to detect possible differences between the two teachings.

An example of a student's work considering how many 2-/3-liter glasses are there in 3 liters of milk is provided in Fig. 2. The arithmetic expression on the left-hand side shows a correct response and is credited with one point. The accompanying model, which illustrates 3 liters partitioned into thirds and incremented by 2-/3-liter glasses, is also credited with one point. Finally, the student's description of the scenario, "because there are $^3/_2$ glasses in 1 liter, there are $3 \times {}^3/_2$ glasses in 3 liters, namely, $^9/_2$ glasses," was credited with one point for demonstrating proportional reasoning. Thus this response received the full credit of three points.

To address research question 2 on how improvements were made, the post-lesson debriefs were analyzed through constant comparison (Patton 2002). Traditionally, mathematics teachers in China have focused on addressing *three points* (Yang and Ricks 2013) as main themes when designing, implementing, and evaluating a lesson. The three points are *important*, *difficult*, and *critical* content points. The *important point* describes the emphasis the teacher must put on the topic and the essentials that students must grasp. The *difficult point* is the cognitive challenge that students might encounter as they try to learn the mathematical content. The *critical point* is the teacher's consideration of how to help students reach the learning goals while overcoming pitfalls that might arise.

Attention to the *three points* provided the major emphasis for evaluating and improving the research lesson during the debriefing meetings. The emerging ideas

Table 3 Dimensions of variation in the research lesson

	Dimension of variation	
Learning trajectory	Teaching 1	Teaching 2
1. 1 divided by a unit fraction (e.g., $1 \div \frac{1}{5}$)	FV1: multiple ways of computing $2 \div \frac{1}{5}$	SV1: the same variation as the FV1
	Invariant: the same arithmetic equation ($2 \div \frac{1}{5} = 10$)	SV1e: generalization of 1 divided by $^{1}/_{a}$ ($= 1 \times a$)
	Varied: multiple strategies (converting to decimals, using diagrams)	Invariant: the same strategy: how many $^{1}/_{a}$ are there in 1 whole (quotitive model)
		Varied: different divisor unit fractions ($\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{6}$)
2. 1 divided by a fraction (e.g., $1 \div \frac{2}{5}$)	FV2: multiple ways of computing $1 \div \frac{2}{5}$	SV2: the same variation as the FV2
	Invariant: the same arithmetic equation ($1 \div \frac{2}{5} = {}^{5}/_{2}$)	SV2e: generalization of 1 divided by a fraction ($1 \div {}^{b}/_{a} = 1 \times {}^{a}/_{b} = a \div b$)
	Varied: multiple strategies (conjecture, inverse operation of multiplication, diagrams)	Invariant: the same pattern ($1 \div \frac{2}{5} = {}^{5}/_{2}$, $1 \div \frac{2}{7} = \frac{7}{2}$, $1 \div \frac{3}{4} = \frac{4}{3}$)
		Varied: different divisor fractions ($\frac{2}{5}$, $\frac{2}{7}$, $\frac{3}{4}$, $\frac{3}{5}$)
3. A whole number divided by a fraction (e.g., $3 \div \frac{2}{5}$)	FV3: multiple ways of computing $3 \div \frac{2}{5}$	SV3: The same variation as FV3
	Invariant: the same arithmetic equation ($3 \div \frac{2}{5} = 3 \times {}^{5}/_{2}$)	SV3e: generalization of a whole number divided by a fraction ($m \div {}^{b}/_{a} = m \times {}^{a}/_{b}$)
	Varied: multiple strategies (using diagrams and proportional reasoning)	Invariant: the proportional reasoning based on the same stereotype situation of $1 \div \frac{2}{5} = {}^{5}/_{2}$
		Varied: different whole numbers (2, 3, 4, and 100)
4. A fraction divided by a fraction (e.g., $\frac{1}{2} \div \frac{2}{5}$)	FV4: multiple ways of computing $\frac{1}{2} \div \frac{1}{3}$	SV4: generalization of a fraction divided by a fraction $\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times {}^{5}/_{2}$
	Invariant: the same arithmetic equation ($\frac{1}{2} \div \frac{1}{3} = \frac{1}{6}$)	Invariant: proportional reasoning based on the same stereotype situation of $1 \div \frac{2}{5} = {}^{5}/_{2}$
	Varied: multiple strategies (using diagrams and proportional reasoning)	Varied: different dividend fractions ($\frac{3}{4}$, $\frac{5}{8}$, and $\frac{4}{5}$)

Notes: FV#, the order of variation in the first lesson; SV#, the order of variation in the second lesson; SV#e, the extension of variation of SV# during the second lesson

were then classified into four categories: (1) identifying the three content points; (2) strategies for highlighting important content points, emphasizing critical content points, and overcoming difficult content points; (3) dealing with multiple representations; and (4) focusing on students' understanding and thinking. Lesson

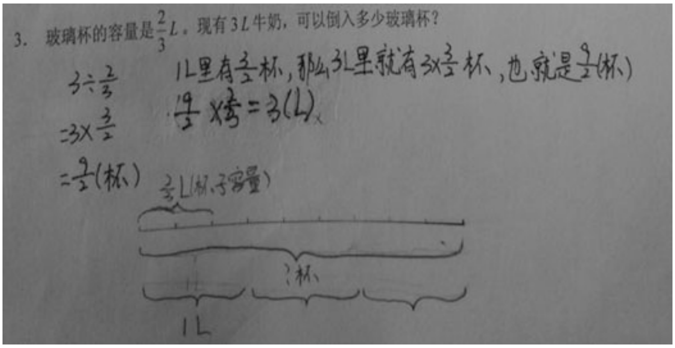

Fig. 2 Student's work demonstrates the number of ⅔-liter glasses in 3 liters of milk

improvements, based on the teachers' reflections and discussions of these ideas, were then summarized. Detailed descriptions of each of these aspects will be presented in the Results section.

Finally, to answer research question 3 on how the LS informed revision of the LT, we looked at the adjustments of the LT throughout the trial teachings. Additionally, the revised LT provided by the lesson group after completion of the project was considered.

4 Results

The results are presented in alignment with the research questions. First, we describe the improvements of the research lessons. Next, we present the factors that led to the changes. Finally, we describe the revisions of the LT.

4.1 Changes in Research Lessons over Repeated Teachings

The two research lessons focused on understanding the meaning of division of fractions and justifying their computation methods. The objects of learning could be synthesized in two areas based on lesson plans. In knowledge and skills, students should understand the meaning of fractions divided by whole numbers or fractions, understand the rationale of the algorithms for these operations, and compute them fluently. Regarding ability and disposition, students should see the application of these algorithms in solving daily life problems, appreciate the connection between mathematics and society, and develop mathematical thinking and reasoning skills related to induction and proportionality.

4.1.1 Learning Trajectory and Associated Mathematics Task

Based on the lesson plans and videotaped lessons, the LT and associated mathematical tasks in the first two teachings are identified as shown in Table 2.

Table 2 reveals that the teacher exactly followed the learning trajectory for the two lessons as suggested by the expert team, but the associated tasks changed dramatically. In teaching 1, all the tasks are related to the same situation of glasses in an amount of milk. However, there were no appropriate scaffoldings between tasks. Task 4 in the first teaching, $1/2 \div 1/3$, presented a huge challenge to students due to the lack of preparation. In contrast, in the second teaching, each level contains several deliberate variation tasks that could help students generalize respective algorithms. In addition, all major tasks (e.g., $2 \div 2/5$, $3 \div 2/5$, $3/4 \div 2/5$) build on a core task of $1 \div 2/5$. This interconnection lays a foundation for developing transformational thinking and proportional reasoning.

4.2 *Enacted Object of Learning*

We present the enacted object of learning by providing a summary of the patterns of variation and comparisons of the two teachings.

4.2.1 Patterns of Variation

The lessons included five stages: (1) introduction of the new topic, (2) exploration of 1 divided by a fraction, (3) exploration of a whole number divided by a fraction, (4) exploration of a fraction divided by a fraction, and (5) practice and summary. When a task was presented, students were asked to work independently for a while and then discuss with their neighbors. After a majority of students raised their hands indicating that they had solutions, the teacher asked some of them to present and explain these solutions. The teacher emphasized and summarized key points after class discussions.

According to the theory of variation, the enacted object of learning is described by the patterns of what varied and what remained the same. In Table 3, the first column presents the LT, and the second and third columns present different dimensions of variation constructed in the first two teachings.

Table 3 demonstrates that in the first teaching, all the dimensions of variation focused on helping students to discern multiple ways of solving a specific problem. For example, it is possible for students to see that they can compute $3 \div 2/5$ using a formula, visual diagram, and proportional reasoning. However, in the second teaching, in addition to the dimensions of variation constructed in the first teaching (FV1, FV2, FV3, and FV4), more extended dimensions of variation were created. Thus, the second teaching created more learning opportunities for students to generalize the algorithms (SV1e, SV2e, SV3e, and SV4e). In particular, the dimensions of variation SV3e and SV4e provided students with opportunities to discern how *proportional reasoning* could be used as a powerful strategy for justifying the algorithm. Based on VP, the second teaching provided much richer learning opportunities for students to justify the algorithm from multiple perspectives.

4.2.2 Other Major Differences Between the Two Teachings

There were three salient changes from the first to second teachings. First, time management became more effective. The first teaching lasted 51 min, while the second teaching lasted 43 min. Moreover, it was found that 1 min was used to introduce the topic in both teachings, and a similar amount of time was used to explore 1 divided by a fraction using visual representations (20 min in first teaching vs. 19 min in second teaching). However, compared to the first teaching, the second teaching spent twice as long on the exploration of a whole number divided by a fraction (5 min vs. 9), laying a sound foundation for developing proportional reasoning. Due to this preparation, the second teaching spent less time exploring a fraction divided by a fraction through proportional reasoning (16 vs. 10). The systematic variation problem exploration at different stages in the second teaching was more efficient (9 vs. 4). Second, although the teacher paid great attention to expose students' thoughts in both teachings, the teacher paid more attention to present and to share students' correct answers during the first lesson and discussed students' errors as learning sources in the second lesson. Third, regarding the use of visual representation, both teachings emphasized the use of arithmetic, verbal, and visual representations simultaneously. But, in the second teaching, the teacher purposefully guided students from the extensive use of visual representation to mental representation and proportional reasoning without visual representation. This assisted students in developing their proportional reasoning skills.

4.3 The Lived Object of Learning

The lived object of learning is described by students' post-lesson quizzes. The mean and standard deviation of the first two teachings are displayed in Table 4.

Table 4 Students' performance based on the post-lesson quizzes

	Overall performance		Visual representation		Proportional reasoning	
	Mean	Std.	Mean	Std.	Mean	Std.
1st ($N = 26$)	4.08	1.65	1.50	1.50	0.15	0.61
2nd ($N = 28$)	4.93	0.38	1.68	1.39	0.96	1.81

Table 4 shows that there are increases of means in performance, visual representation, and proportional reasoning from the first teaching to the second teaching. A t-test further detects that the changes of mean in overall performance ($t = 2.66$, $p = 0.01$) and proportional reasoning ($t = 2.2$, $p = 0.04$) are significant, but the changes in visual representation ($t = 0.45$, $p = 0.65$) are not. The data show that the different teachings resulted in improvement of students' fluency in division of fractions. It is encouraging that in the second teaching, students made improvements in both proportional reasoning and their use of visual representation to justify the algorithm, which reflected the teacher's intentions.

4.4 *Emerging Ideas for Improving Research Lesson*

The analysis of post-lesson debriefings and the teacher's self-reflection on the first teaching provided evidence about the changes teachers made to the second teaching. The major ideas about improving the research lesson that emerged in the post-lesson debriefing include identifying the important, difficult, and critical content points; strategies for highlighting important content points, emphasizing critical content points, and overcoming difficult content points; dealing with multiple representations; and focusing on students' understanding and thinking.

4.4.1 Identifying Three Key Content Points

Ms. Lu originally believed that the important and difficult content point was understanding the algorithm of $\frac{1}{2} \div \frac{1}{3}$ using diagrams, and the critical content point was justifying $3 \div \frac{2}{5} = 3 \times {}^{5}/_{2}$ via the bridge of $1 \div \frac{2}{5} = 1 \times {}^{5}/_{2}$ and proportional reasoning. The didacticians (Mr. Kong and Mr. Ren) helped the teachers clarify the important knowledge point as follows:

> Mr. Kong: I ask you one more question. You implemented the lesson by following the learning trajectory we suggested, namely, five tasks ($2 \div \frac{1}{5}$; $1 \div \frac{2}{5}$; $3 \div \frac{2}{5}$; $\frac{1}{2} \div \frac{1}{3}$; ${}^{3}/_{7} \div \frac{2}{5}$). Which of the five tasks do you think you should spend more time on?
>
> Ms. Lu: I originally thought that I should make great efforts in dealing with $\frac{1}{2} \div \frac{1}{3}$.
>
> Mr. Ren: You said that students were not able to draw diagrams to visualize it.
>
> Ms. Lu: I originally thought that for the first two tasks ($1 \div \frac{1}{5}$, $1 \div \frac{2}{5}$), students should be asked to draw diagrams, but for the third one ($3 \div \frac{2}{5}$) it was not necessary. Here, students should use proportional reasoning to find the result directly.
>
> Mr. Kong: It is acceptable to draw diagrams for $3 \div \frac{2}{5}$, but it is too hard for students to draw diagrams for $\frac{1}{2} \div \frac{1}{3}$. My thought was to emphasize optimal thinking when discussing $\frac{1}{2} \div \frac{1}{3}$ and ${}^{3}/_{7} \div \frac{2}{5}$. It is necessary to use proportional reasoning to justify computation methods when drawing seems unrealistic.
>
> Ms. Lu: Students were not able to draw diagrams for $\frac{1}{2} \div \frac{1}{3}$. I will not ask them to draw these. I want them to understand how many glasses are in 1 liter and then use proportional reasoning to justify.
>
> Mr. Kong: So, it is not appropriate to say $\frac{1}{2} \div \frac{1}{3}$ is the most important point. Actually, $1 \div \frac{2}{5}$ is the most important content point, is not it?

Through extensive discussions, the important, difficult, and critical content points were clarified. The important content point, as well as the critical point, was to help students understand the computation rule for $1 \div \frac{2}{5} = 1 \times {}^{5}/_{2} = {}^{5}/_{2}$ from visual, verbal, and arithmetic perspectives. Another critical content point was to justify why $3 \div \frac{2}{5} = 3 \times {}^{5}/_{2}$ verbally and logically while de-emphasizing visual representation and highlighting proportional reasoning via the bridge $1 \div \frac{2}{5} = 1 \times {}^{5}/_{2} = {}^{5}/_{2}$. The difficult point was to understand why $\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times {}^{5}/_{2}$ from proportional reasoning with the support of visual diagrams. The group decided to replace $\frac{1}{2} \div \frac{1}{3}$ with $\frac{3}{4} \div \frac{2}{5}$ due to its interconnection with previous tasks. A core task on how many glasses of $\frac{2}{5}$ liters are in 1 liter of milk was seen as the hinge linking different tasks throughout the class.

4.4.2 Task Selection Focusing on Three Content Points

After achieving agreement to the three content points, the discussion focused on how to deal with these points strategically by selecting deliberate mathematical tasks.

Strategically Polishing Scaffolding Tasks Exploration of dividing 1 by a unit fraction is relatively simple but fundamental. In the first teaching, the teacher used a task, how many glasses of $\frac{1}{5}$ liters are there in 2 liters of milk, because she intended to help students realize how they can use the result of 1 liter to solve the problem with 2 liters via transformation thinking (i.e., solving a complex problem using the solution to a simpler and solved problem and proportional reasoning). Through discussion, she realized "$1 \div \frac{1}{5}$ can help students visually see there are 5 of $\frac{1}{5}$ in 1 whole." The discussion also revealed "[the] introductory situation of how many glasses of $\frac{1}{5}$-liter are in 1 liter is good. The simple situation would lead students to get the result of $1 \div \frac{1}{5} = 5$ very quickly. Then, the teacher should ask them how you found the result" (Mr. Ren). Moreover, during the discussion, the specialists suggested for the teachers to include several situations of 1 divided by a unit fraction (such as $1 \div \frac{1}{4}$ and $1 \div \frac{1}{6}$) and to have students generalize the pattern.

Digging Deeply into the Important and Critical Points Exploration of 1 divided by a fraction is one of the important points of the lesson. To address the key point of understanding why $1 \div \frac{2}{5} = {}^{5}/_{2} = 2\frac{1}{2}$ from multiple perspectives, a great amount of time was devoted to discussing relevant strategies. Two key ideas occurred. One focused on students' demonstration of a half glass through the use of visual and verbal representations in their explanations. The second idea was the generalization of the pattern through examining several cases of 1 divided by a fraction verbally (e.g., $1 \div \frac{2}{7}$; $1 \div {}^{2}/_{9}$; $1 \div \frac{3}{5}$; $1 \div {}^{5}/_{7}$). The specialist provided the following suggestions:

> Mr. Ren: Yes. You should let students work extensively on this activity. The diagram is only used to verify the result. I think that this lesson should focus on verification rather than discovery because students in their brain already knew the result: multiplying the reciprocal of the divisor. If you ask students the result, they will certainly say the

> reciprocal. So, the key is to verify the relationship rather than to discover it as a new pattern. In this way, the students can see the result of 1 ÷ ⅖ as 2 [glasses] and ½ [glass]. Then, you can ask students about the relationship between the arithmetic expression and the result. Based on previous experience, students should get the result of $^5/_2$. After students understand why 1 ÷ ⅖ is $^5/_2$ visually, then they should explore more variation tasks such as why 1 ÷ ²⁄₇ = $^7/_2$, 1 ÷ ¾ = $^4/_3$, and so on, and finally [you should] encourage students to generalize the pattern of 1 divided by a fraction.

Proportional Reasoning Used to Develop Understanding of a Whole Number Divided by a Fraction and a Fraction Divided by a Fraction Building on the understanding of 1 divided by a fraction, students should be guided to develop a deeper understanding of a whole number divided by a fraction through proportional reasoning. Several cases including 2 ÷ ⅖, 3 ÷ ⅖, and 4 ÷ ⅖ should be strategically explored in order to generalize the pattern of a ÷ ⅖ = $a \times {}^5/_2$. Visual representations and proportional reasoning should be used purposefully to verify 2 ÷ ⅖ and highlight the inconvenience of using diagrams and the effectiveness of using proportional reasoning when verifying 3 ÷ ⅖ and onward. If there are 2 and ½ of ⅖ liter in 1 liter of milk, 3 liters will have 3 times of 2 and ½ of 2/5 liter of milk, which is $3 \times {}^5/_2$. The didacticians and the teachers agreed if students understand the benefit of proportional reasoning, they could apply proportional reasoning to discuss ¾ ÷ ⅖ and so on. Mr. Kong further suggested to the teacher to "use the opportunity to develop optimization thinking by using proportional reasoning; let students realize proportional reasoning is easy [for certain conditions] while drawing diagrams is not convenient." Based on the experience in exploring a whole number divided by a fraction through proportional reasoning, the teachers further designed the lessons to ask the students to verify the pattern about a fraction divided by a fraction using proportional reasoning as Mr. Ren suggested:

> Mr. Ren: . . .After completing these two tasks, we can use the fractions we discussed in the class, such as 1 ÷ ¾, and ask students to explain ⅘ ÷ ¾ [writing beside 1 ÷ ¾ = $^4/_3$]. Students are expected to apply the same reasoning as we discussed with a number divided by ⅖. That is what we discussed about 2 ÷ ⅖ using the result of 1 ÷ ⅖. Thus, we can consider the result of 1 divided by a fraction first, then a divided by the fraction is a times the result. Then, the students are asked to explore how many glasses of ⅖ liter there are in ¾ liters. Students are encouraged to think according to what they did in previous cases. Thus, students can further explore ⅝ ÷ ⅖ and ⅘ ÷ ¾. It is natural to use the result of 1 ÷ ¾.

4.4.3 Dealing with Multiple Representations

Appropriate use of representations was a key element throughout the entire lesson. The discussions on the use of representations focused on the following issues: (1) inappropriate use of visual representations, (2) deliberate use of visual representations, and (3) integrating multiple representations simultaneously.

Inappropriate Use of Visual Representations Based on the experience of drawing diagrams to visualize the result of 1 ÷ ⅕ and 1 ÷ ⅖, students tended to use diagrams to present a fraction divided by a fraction even though the arithmetic is too complicated to visualize. This tendency actually constrained students' thinking as discussed below.

> Ms. Lu: Because of the simplicity of drawing diagrams for visualizing $2 \div \frac{1}{5}$ and $1 \div \frac{2}{5}$, the students' first attempt to explain their results by drawing. However, with the increase in complexity of arithmetic expressions, diagrams become more and more complicated, for example, students were not able to draw an appropriate diagram to visualize $\frac{1}{2} \div \frac{1}{3}$.
>
> Mr. Ren: This is to say, sometimes drawing diagrams may constrain and limit students' mathematical thinking.

Deliberate Use of Visual Representation The failed experience in using a diagram for explaining $\frac{1}{2} \div \frac{1}{3}$ motivated the group to discuss strategies to address this issue. The different ways to treat multiple representations were discussed. The tasks for dealing with important content points ($1 \div \frac{1}{5}$, $1 \div \frac{2}{5}$) should be discussed extensively using multiple representations simultaneously. While the initial variation tasks surrounding the core tasks should be treated using visual and verbal representations, the eventual goal is for students to solely use verbal representations of proportional reasoning in the later tasks. The mathematics education professor, Mr. Kong, provided the following suggestions:

> First, $1 \div \frac{1}{5}$ is the important content point; it should be visualized using diagrams and expressed verbally. Then, when discussing $1 \div \frac{1}{4}$ and $1 \div \frac{1}{3}$, drawing diagrams should be not required; students should be asked to think using their brains. When discussing $1 \div \frac{2}{5}$, the diagram should be used to visualize and justify why it is $^5/_2$. It is very visual. Then, when discussing $1 \div \frac{2}{7}$, students should not be asked to draw a diagram but to think about how many $\frac{2}{7}$'s there are in 1 unit. After students get answers, then the teacher shows a diagram to them. After that, it is not necessary to show diagrams for $1 \div \frac{3}{4}$. The discussions about $1 \div \frac{2}{7}$ and $1 \div \frac{3}{4}$ should progress quickly. When discussing $2 \div \frac{2}{5}$ and then $3 \div \frac{2}{5}$, the emphasis should be on $2 \div \frac{2}{5}$, creating a situation problem, which corresponds to how many $\frac{2}{5}$ are there in 2 (i.e., $2 \div \frac{2}{5}$). Students should be asked to consider the following question: There are $^5/_2$ of $\frac{2}{5}$ liters in 1 liter, how many $\frac{2}{5}$ of a liter are there in 2 liters? Students could use diagrams or use the result ($^5/_2$) of how many $\frac{2}{5}$ in 1 liter. Then, when discussing the result of $3 \div \frac{2}{5}$, students should be guided to think and express using what you did with $1 \div \frac{2}{5}$.

Integration of Using Multiple Representations Simultaneously The difficulty in understanding $\frac{3}{4} \div \frac{2}{5}$ has been discussed and addressed from two aspects. First, it is suggested that proportional reasoning be used. Second, a well-designed diagram given by the teacher should be helpful for students to make sense of the situation.

4.4.4 Focusing on Students' Understanding and Thinking

The discussion also focused on students' learning difficulties and finding ways to overcome these difficulties. As in previous discussions, the teachers found that students had a tendency to draw diagrams to visualize the algorithms, even when the expressions became more and more complex, so the team devised ways to overcome the difficulty by using proportional reasoning. In addition, the use of appropriate language to make sense of the algorithm was an issue. For example, when discussing how many glasses of $2/5$ liters there are in 1 liter, the teacher said 5/2 glasses in the class. However, observers noted that students said two glasses and a half glass. They suggested for the teachers to use a diagram to help students understand why it is 2 glasses and a half glass and then realize the equivalence between $^5/_2$ and $2\frac{1}{2}$.

In addition, during debriefing, issues about questioning skills, instructional language, and board writing were discussed.

4.5 Intended Changes of the Research Lesson

Based on the reflection on the research lesson and its debrief, the teacher explicitly expressed her intended changes. In her reflection report, she summarized five major points she took away from the first teaching and the debrief. First, the process of teaching should reflect a progressively abstracting process. Students should shift their tendency from "drawing" to "thinking and reasoning" when solving problems. Second, the instructional process should reflect an optimization process for thinking (i.e., appropriate use of representations based on the nature of problem to illustrate how the algorithm works). On the one hand, after students extensively explored $1 \div 2/5$, students should be led to explore other purposeful variation tasks. When exploring $1 \div 2/7$, students should be asked to visualize a diagram in their heads. After that, the teacher should show students a diagram to help them verify their ideas. On the other hand, through the recognition of complexity of drawing diagrams as the arithmetic expression becomes more complex, students should be led to realize the need to explore a simpler way to solve the problem, such as proportional reasoning based on the quotitive interpretation of division. Third, it is important to put great effort in breaking through the important and difficult content point of $1 \div 2/5$ and then using it as a tool to solve a series of problems of a whole number divided by $2/5$ (e.g., $2 \div 2/5$, $3 \div 2/5$, $100 \div 2/5$) and transforming to discuss a fraction divided by $2/5$ (e.g., $3/4 \div 2/5$). The discussion with $1/2 \div 1/3$ will be removed completely in the second teaching. Fourth, the board writing needed to improve in order to build connections between different contents and emphasize the core content of $1 \div 2/5$ as a starting point and a bridge linking different parts. Finally, attention must be given to students' feedback, particularly to students who have difficulties.

Overall, the teacher accepted the major ideas and suggestions discussed in the first debrief. In the second teaching, these strategies were implemented appropriately.

4.6 Learning Trajectory Refinement

Table 5 presents the LTs of division of fractions at two stages: pre- and post LS.

Overall, the hypothetical LT developed based on Western literature is applicable in the classroom in China. The three macro-levels fit student learning in the

Table 5 Learning trajectories of division of fractions

Major levels	Pre-lesson study	Post-lesson study
0. Meaning of fractions and division	Division with whole number (4 ÷ 2)	Within a same situation of cake sharing: 1 ÷ 5
1. A fraction divided by a whole number	1a. Numerator of the dividend is the multiplier of divisor (e.g., 4/5 ÷ 2)	The same as pre-lesson study: ⅘ ÷ 2
	1b. The dividend is a unit fraction (e.g., ⅕ ÷ 2)	The same as the pre-lesson study: ⅕ ÷ 2
	1c. The dividend is a proper fraction (e.g., ⅘ ÷ 3)	The same as the pre-lesson study: ⅘ ÷ 3
2. A whole number divided by fraction	2a. The divisor fraction is a unit fraction (2 ÷ ⅕)	Extended to include 1 ÷ ⅕. 1 ÷ ⅓, 1 ÷ ¼,1 ÷ ⅙,. . ., 1 ÷ $^1/_a$
	2b. 1 divided by a proper fraction (1 ÷ ⅖)	Extended to include 1 ÷ ⅖, 1 ÷ 2/7, 1 ÷ ¾, 1 ÷ ⅗.
	2c. A whole number divided by a fraction (3 ÷ ⅖)	Extended to include 2 ÷ ⅖, 3 ÷ ⅖, 4 ÷ ⅖,. . ., 100 ÷ ⅖
3. A fraction divided by a fraction	3a. A unit fraction divided by a unit fraction (e.g., ½ ÷ ⅓)	This sublevel is removed
	3b. A fraction divided by a fraction (¾ ÷ ⅖)	Extended to include ¾ ÷ ⅖, ⅜ ÷ ⅖ ; ⅘ ÷ ¾

classroom in China very well. At the micro-levels, all levels apart from sublevel 3a, a unit fraction divided by a unit fraction, have been implemented in the study and appear to predict students' learning progression well. Importantly, this LS has enriched the LT by incorporating the notion of variation. Due to the variation ideas, students could be guided to develop their ability to make generalizations (2a, 2b, 2c). In addition, by adopting the variation pedagogy, keeping the situation of 1 ÷ ⅖ invariant across multiple tasks and examining varying tasks at level 2c and 3b provided an opportunity to justify the algorithm through proportional reasoning. This scenario demonstrates how developing conceptual understanding and mathematical reasoning could be achieved simultaneously.

5 Discussions and Conclusion

This study describes how a lesson study, guided explicitly by theoretical notions of learning trajectory and variation theory, developed lessons for division of fractions through a cycle of collaborative design, teaching/classroom observation, debriefing, and reteaching. The results show that the lesson has been improved in terms of students' understanding, proficiency, and mathematical reasoning. Meanwhile, the factors that led to these improvements are revealed through analyzing the post-lesson debriefs and the teacher's reflections. The debrief focused on identifying and prioritizing key content points within the overall framework of the LT, exploring effective ways of addressing important content points, strategically overcoming difficult content points and highlighting critical points. Specifically, the debriefing also focused on the purposeful use of multiple representations, addressing students' learning difficulties and errors, and general instructional skills such as questioning, instructional language, and board writing design. Self-reflection, redesign, and reteaching helped the teacher to adopt and to implement those ideas in class, resulting in the improvement of teaching. In addition to improving teaching, this study also evidenced that the LT is refined and enriched through the theory-driven LS. In particular, the variation theory provided strategies for creating scaffolding for progressing students' understanding to higher levels and extending students' experience in generalizing patterns. In the following sections, we highlight some key points.

5.1 Lesson Study as an Effective Way to Implement Standard-Based Teaching

Effectively implementing standard-based mathematics teaching has been a long-standing issue for decades (Woodbury and Gess-Newsome 2002). Although research-based effective mathematics teaching practice (NCTM 2014) may help teachers understand what a standard-based classroom looks like, implementing mathematics teaching practice in classes presents new challenges to most teachers. Research has shown that LS is a promising way to help teachers implement reform-oriented teaching in class (Lee and Lo 2013). This study provides two specific implications for the selection of tasks and the use of multiple representations.

With regard to mathematical tasks, there are a great deal of studies on high cognitive demand tasks (Stein and Lane 1996) and the ways to effectively implement tasks and share students' work (Stein et al. 2008). This study adds additional dimensions to the implementation of tasks. First, the instructional tasks should be embedded in a context with which students are familiar with, such as cake or milk sharing. Subtasks should be intentionally varied in some aspects ($2 \div \frac{2}{5}$, $3 \div \frac{2}{5}$, $4 \div \frac{2}{5}$, $100 \div \frac{2}{5}$, $\frac{3}{4} \div \frac{2}{5}$) while keeping others the same ($1 \div \frac{2}{5}$). By doing so, students' attention could be drawn to important mathematical relations rather than the various contexts. To vary tasks within a context, two strategies could be employed. One is vertical variation that focuses on developing students' understanding from different levels of the LT (such as $1 \div \frac{1}{5}$, $1 \div \frac{2}{5}$, $3 \div \frac{2}{5}$, and $\frac{3}{4} \div \frac{2}{5}$, while the divisor is invariant). The other is horizontal variation (such as $1 \div \frac{2}{5}$, $1 \div \frac{2}{7}$, $1 \div \frac{3}{4}$...) within the same level of the LT, which focuses on developing generalization. Through these two dimensions of variation, conceptual understanding could be developed in depth and in breadth.

Regarding the use of multiple representations, research has shown that US students tend to use concrete or pictorial representations, while Chinese students tend to use general or symbolic representations (Cai and Wang 2006). In addition, US teachers tend to use multiple representations simultaneously, while Chinese teachers tend to use multiple representations selectively (Huang and Cai 2011).

This study further shows how teachers could help students develop their flexibility in using multiple representations through emphasizing multiple representations simultaneously, decreasing the use of visual representations over time, and using verbal and arithmetic representations solely as appropriate to the task. The use of representations in Chinese mathematics classes is purposeful and selective with the intent to develop students' abstract thinking.

5.2 Lesson Study as a Platform for Building Linkage Between Theory and Practice

To address the issue of the gap between theory and practice in education, Kieran et al. (2013) argue that treating teachers as key stakeholders in research is a powerful way for building linkage between theories and practice. Using lesson study in China and Japan as examples, they highlight the critical features of research where the teacher is viewed as a stakeholder. These features include inquiry-based activity, a significant action research component, and dynamic duality of research and professional development. This study includes all three features and demonstrates how a theory-driven lesson study (utilizing learning trajectory and variation pedagogy as guiding principles for lesson design, implementation, and reflection) could improve mathematics teaching practice and, at the same time, refine a learning trajectory. Similarly, Lo and Marton (2012) state that "[lesson] study likewise provides a possible platform where teaching can be cast as an experimental science and a form of action research. We would like to suggest that variation theory offers potential gains to lesson study in the sense that it provides an additional theoretical component to guide decisions about teaching" (p. 21). Based on an extensive discussion of the system of Japanese lesson study (national, district-based, and school-based lesson study), how teachers and researchers learn from crossing boundaries within the lesson study system, and the eventual results of "teaching for understanding in both mathematics and science, successfully spreading some major instructional innovations" (Lewis 2015, p. 58), she argues that lesson study is a type of improvement science (Langley et al. 2009). An improvement science essentially includes the core framework of plan-do-study-act (PDSA) cycle, coupled with three fundamental questions: (a) what are we trying to accomplish? (b) how will we know that a change is an improvement? and (c) what change can we make that will result in improvement? (Lewis 2015, p. 54). The researchers would argue that Chinese lesson study is a form of improvement science as well (Huang and Han 2015), and this study, methodologically, provides a way for researching into improvement science by addressing object of learning, enacted object of learning, and lived object of learning (corresponding to questions (a), (b), and (c), respectively).

5.3 Limitation and Further Studies

Two limitations to the study should be noted. First, this form of LS is not typically utilized in China because it was explicitly guided by specific theoretical perspectives. This special case illustrates the efforts of a group of mathematics educators as they attempt to implement standard-based curriculum through the theory-driven LS. Second, the study presents results from a single, school-based LS group that may not be reflective of the variations of LS that occur at other organizational levels (e.g., district, municipal, or national level LS programs). Thus, the findings in this case cannot be generalized broadly to LS in China but rather present a unique case that offers some insight into the potential of theory-driven lesson study to impact students' mathematical understanding.

Despite these limitations, this study suggests at least four issues worthy of further exploration. First, the study shows the critical role played by knowledgeable others (e.g., university professors, subject specialists, etc.) during the lesson study process. It would be interesting to examine the way these knowledgeable others work with practicing teachers and develop their own professional knowledge and skills through mentoring during LS. Second, throughout this study, the notions of learning trajectory and variation pedagogy have played a critical role in scaffolding students' learning. It would be interesting to look in more detail at the manner in which these constructs relate to specific theoretical perspectives on scaffolding. Third, this case study focuses on the potential of theory-driven LS to impact students' understanding at the classroom level. It would be interesting to explore methods for scaling up this model to teacher professional development in general. Finally, as this study took place in China where LS is embedded in teachers' daily practice, it would be interesting to explore the implementation of similar studies in settings in the West where LS is an innovative initiative.

Acknowledgments We thank anonymous reviewers for their invaluable feedback on the revisions of the paper. We appreciate Dr. Dovie Kimmins and Mr. James Willingham from Middle Tennessee State University for their contribution to the improvement of the article. Our thanks go to participating teachers and didacticians for their commitment to the lesson study and support of data collection.

Appendices

Appendix 1: Post-lesson Assessment

Greetings, class! To understand your learning situation in the class, we designed this questionnaire. Please carefully answer each question according to the instructions given. Just write down what you think. We will not grade your work and compare you answers with others. Thank you for your cooperation.

First, you need to compute arithmetic expressions. Then, justify your computations using as many as methods as possible such as verbal explanation, visual diagrams, or arithmetic expressions. The more details the better.

1. How many glasses of 1/5 liter are there in 3 liters of milk?
2. How many glasses of 2/3 liters are there in 1 liter of milk?
3. How many glasses of 2/3 liter are there in 3 liters of milk?
4. How many glasses of 1/5 liter are there in 1/3 liter of milk?
5. How many glasses of 2/3 liter are there in 4/5 liter of milk?

Appendix 2: Learning Trajectory and Associated Tasks in Research Lesson 1

Learning trajectory	Second teaching
1. Connecting to previous knowledge (divisions with whole numbers) (e.g., $1 \div 5 = 1 \times \frac{1}{5}$)	T1: A 1-kilogram rectangle cake is shared between five friends, how much does each friend get?
2. A fraction divided by a whole number (when the numerator is the multiplier of divisor) (e.g., $\frac{4}{5} \div 2$)	T2: A $\frac{4}{5}$-kg of cake is shared between two friends, how much does each friend get?
3. A unit fraction divided by a whole number (e.g., $\frac{1}{5} \div 2$)	T3: A $\frac{1}{5}$-kg cake is shared between 2 friends, how much does each friend get?
4. A fraction divided by a whole number (when the numerator is not a multiplier of divisor) (e.g.,$\frac{4}{5} \div 3$)	T4: A $\frac{4}{5}$-kg cake is shared between 3 friends, how much does each friend get? Practice: A $^{3}/_{10}$-kg cake is shared between 8 friends, how much does each friend get?
5. 1 divided by a unit fraction (e.g., $1 \div \frac{1}{5}$)	T1a: How many $\frac{1}{5}$-liter glasses are there in 1 liter of milk? T1b: How many $\frac{1}{4}$-liter glasses are there in 1 liter of milk? T1c: How many $\frac{1}{3}$-liter glasses are there in 1 liter of milk? T1d: How many $\frac{1}{6}$-liter glasses are there in 1 liter of milk?
6. 1 divided by a fraction (e.g., $1 \div \frac{2}{5}$)	T2a: How many $\frac{2}{5}$-liter glasses are there in 1 liter of milk? T2b: How many $\frac{2}{7}$-liter glasses are there in 1 liter of milk? T2c: How many $\frac{3}{4}$-liter glasses are there in 1 liter of milk? T2d: How many $\frac{3}{5}$-liter glasses are there in 1 liter of milk?
7. A whole number divided by a fraction (e.g., $3 \div \frac{2}{5}$)	T3a: How many $\frac{2}{5}$-liter glasses are there in 2 liters of milk? T3b: How many $\frac{2}{5}$-liter glasses are there in 3 liters of milk?

(continued)

Learning trajectory	Second teaching
	T3c: How many ⅖-liter glasses are
	there in 4 liters of milk?
	T3d: How many ⅖-liter glasses are there in 100 liters of milk?
8. A fraction divided by a fraction (e.g., ½ ÷ ⅓)	T4a: How many ⅖-liter glasses are there in ¾ liters of milk?
	T4b: How many ⅖-liter glasses are there in ⅝ liters of milk?
	T4c: How many ¾-liter glasses are there in ⅘ liters of milk?

References

Cai, J., & Wang, T. (2006). U.S. and Chinese teachers' conception and construction of representations: A case of teaching ratio concept. *International journal of Science and Mathematics Education, 4*, 145–186.

Cai, J., & Wang, T. (2010). Conception of effective mathematic teaching with a cultural context: Perspectives of teachers from China and the United States. *Journal of Mathematics Teacher Education, 13*, 265–287.

Carpenter, T. C., Lindquist, M. M., Brown, C. A., Kouba, V. L., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Trends and conclusions. *Arithmetic Teacher, 36*(4), 38–41.

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*, 218–236.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Clements, D., Sarama, J., Spitler, M., Lange, A., & Wolfe, C. B. (2011). Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *Journal for Research in Mathematics Education, 42*, 127–166.

Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/Math/Practice

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: An effective way of mathematics teaching in China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–348). Singapore: World Scientific.

Hart, L. C., Alston, A. S., & Murata, A. (2011). *Lesson study research and practice in mathematics education: learning together*. New York: Springer.

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298.

Huang, R., & Cai, J. (2011). Pedagogical representations to teach linear relations in Chinese and U.S. classrooms: Parallel or hierarchical? *The Journal of Mathematical Behavior, 30*(2), 149–165.

Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.

Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese Lesson Study. *ZDM – The International Journal on Mathematics Education, 46*, 239–251.

Kieran, C., Krainer, K., & Shaughnessy, J. M. (2013). Linking research to practice: Teachers as key stakeholders in mathematics education research. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 361–392). New York: Springer.

Langley, G. J., Moen, R. D., Nolan, K. M., Nolan, T. W., Norman, C. L., & Provost, L. P. (2009). *The improvement guide*. San Francisco: Jossey-Bass.

Lee, C. K. E., & Lo, M. L. (2013). The role of lesson study in facilitating curriculum reform. *International journal for lesson and learning studies, 2*, 200–206.

Lewis, C. C. (2015). What is improvement sciences? Do we need it in education? *Educational Researcher, 44*(1), 54–61.

Lewis, C. C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Li, Y. (2008). What do students need to learn about division of fractions? *Mathematics Teaching in the Middle School, 13*, 546–552.

Lo, M. L., & Marton, F. (2012). Toward a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal for Lesson and Learning Studies, 1*, 7–22.

Maloney, A. P., Confrey, J., & Nguyen, K. H. (2014). *Learning over time: Learning trajectories in mathematics*. Charlotte: Informational Age Publishing.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning Science, 15*, 193–220.

Marton, F., & Tsui, A. B. M. (with Chik, P. P. M., Ko, P. Y., Lo, M. L., Mok, I. A. C., Ng, F. P., Pang, M.F., et al.) (Eds.). (2004). *Classroom discourse and the space of learning*. Mahwah: Lawrence Erlbaum.

Ministry of Education, P. R. China. (2011). *Mathematics curriculum standards for compulsory education* (grades 1–9). Beijing: Beijing Normal University Press.

Murata, A. (2011). Introduction: Conceptual overview of lesson study. In C. L. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: learning together* (pp. 1–12). New York: Springer.

National Council of Teachers of Mathematics. (2014). *Principles to action: Ensuring mathematical success for all*. Reston: NCTM.

Ott, J. M., Snook, D. L., & Gibson, D. L. (1991). Understanding partitive division of fractions. *The Arithmetic Teacher, 39*, 7–11.

Patton, M. Q. (2002). *Qualitative research & evaluation methods* (3rd ed.). Thousand Oaks: Sage.

Silvia, E. M. (1983). A look at division with fractions. *The Arithmetic Teacher, 30*, 38–41.

Simon, M. A. (1995). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24*, 233–254.

Sowder, J., Sowder, L., & Nickerson, S. (2010). *Reconceptualizing mathematics for elementary school teachers*. New York: W.H. Freeman & Company.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50–80.

Stein, M., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*, 313–340.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: toward a theory of teaching. *Educational Researcher, 41*, 147–156.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*, 5–25.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91–111.

Woodbury, S., & Gess-Newsome, J. (2002). Overcoming the paradox of change without difference: A model of change in the arena of fundamental school reform. *Educational Policy, 16*, 763–782.

Yang, Y., & Ricks, T. E. (2013). Chinese lesson study: Developing classroom instruction through collaborations in school-based teaching research group activities. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 51–65). New York: Routledge.

Rongjin Huang is a Professor at Middle Tennessee State University. His research interests include mathematics classroom research, mathematics teacher education, and comparative mathematics education. Dr. Huang has published scholarly work extensively. His recently published books include *Teaching and Learning Mathematics Through Variation: Confucian Heritage Meets Western Theories* (Sense, 2017), *How Chinese Acquire and Develop Mathematics Knowledge for Teaching* (Sense, 2018), and *Theory and Practice of Lesson Study in Mathematics: An International Perspective* (Springer, 2019). Dr. Huang has served as a guest editor for *ZDM Mathematics Education and International Journal for Lesson and Learning Studies*.

Zikun Gong is a professor at Hangzhou Normal University. His research interests include teaching and learning procedural knowledge, children' cognitive development such as children' cognitive development of probability and inference, learning trajectories of core math concepts, and teacher professional development. He has edited national compulsory course textbooks, such as primary school mathematics textbooks and middle school mathematics textbooks. Additionally, he also published books, *Teaching and Learning of Procedural Knowledge*, *Mathematics Teachers' Professional Competence*, *Mathematics Pedagogy*, and *Theory and Practice of Chinese "Two Basic" Teaching and Learning*. Dr. Gong has conducted a number of national research projects such as "The adaptability research of national mathematics curriculum standards in compulsory education" and "the children's cognitive development research in probability concepts." Dr. Gong received the second prize in national basic educational teaching achievement in 2014 and presented his research findings at ICME-13 meeting in Germany, 2016.

Xue Han is currently an assistant professor in the College of Education at National Louis University, USA. She received her doctoral degree from Michigan State University with concentration on mathematics teacher learning and professional development. Her research interests include mathematics teacher learning and professional development, elementary and middle school mathematics instruction, and international comparative education.