Identifying What Is Critical for Learning 'Rate of Change': Experiences from a Learning Study in Sweden

Robert Gunnarsson, Ulla Runesson, and Per Håkansson

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Abstract Learning study is an adapted version of lesson study developed in Hong Kong and Sweden. It has commonalities with lesson study but is framed within a specific pedagogical learning theory – variation theory. Central in variation theory is the object of learning and what is critical for students' learning. Hence, as with lesson study, it is a collective and iterative work where teachers explore how they can make the object of learning available to students, but what characterises learning study is the use of a specific learning theory. In this process, special attention is paid to the critical aspects of the object of learning. We argue that to identify the aspects that are critical, the aspects need to be verified and refined in classrooms. In this chapter, we demonstrate how teachers gain knowledge about such critical aspects. Particularly, we show how these critical aspects cannot be extracted only from the mathematical content or the students pre-understanding alone, but evolve during the learning study cycles. For this we use a learning study about the mathematical topic of rate of change in grade 9 in Sweden as an illustration. We describe how an analysis of how students solved tasks in pre- and post-test and during the lessons, as well as how the mathematical content was presented in lessons, helped the teachers identify what was critical for learning to understand and express the rate of change for a dynamic situation.

Keywords Learning study · Critical aspects · Rate of change

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1 Introduction

There are a lot of things a teacher must have in mind and take into consideration when planning for teaching a topic to class. One such thing is the students. From previous teaching experience, the teacher is probably aware of difficulties students could have with learning a specific topic. Another is how to choose teaching strategies that, for instance, promote active involvement and meaningful interaction. Furthermore, decisions about how to evaluate the lessons must be made. Although, carefully considering all this, we would argue that there still is one question that must be answered. To teach towards a learning goal, that is, what students are expected to achieve, the answer to the question 'What must be learned to achieve the targeted goal?' must be found. Why? Because the answer to this question fills the gap between teaching content and teaching strategies (what and how to teach) and the learning objectives (the intended results of teaching and learning). Our argument is that a learning objective cannot give the answer to 'what must be learned?', since a learning objective is general for all in a specific course or grade. What must be learned to achieve this goal is specific for every group of learners, however. The question must be answered on a detailed level and cannot be derived from the subject matter alone. Instead, identifying what students need to learn is a transactional process between the specific known (thing) and the specific group of students' knowing of that thing (Dewey and Bentley 1949). This process can provide teachers with 'keys to learning'.

In this chapter we will demonstrate how such 'keys to learning', identified with the aim to enhance student learning regarding the rate of change of linear relationship, are refined in the process of the learning study. One example from a learning study conducted by a group of Swedish mathematics teachers will be used as an example. Before that, we will go more into detail about learning study, variation theory and how a learning study can be executed.

2 The Origin and Purpose of Learning Study

The ultimate aim of lesson study is to improve teaching practice and student learning. Learning study, an adapted version of lesson study, shares this aim, just as it shares the collaborative planning, observing of lessons and student learning. Although the role of a theory in lesson study is often unclear or rarely made explicit (Elliott 2012, p. 114), there are lesson studies where the use of theories is reported (e.g. Clivaz 2015; Martin and Clerc-Georgy 2015; Martin and Towers 2016; Pillay and Adler 2015). Due to a difference in the epistemological and ontological assumptions underpinning the theory, however, the lesson studies will have different foci (Runesson 2016).

Learning study (Cheng and Lo 2013; Marton and Pang 2003; Marton and Runesson 2015) is guided by a specific theory of learning through variation (described below). This variation theory provides a lens to focus on how students learn and develop a specific capability and what is made possible to learn in the lesson.

Learning study was developed in Hong Kong around the year 2000. Inspired by Stigler and Hiebert promoting lesson study in the book *The teaching gap* (Stigler and Hiebert 1999), a research project, aimed at developing teaching to cater for individual differences, was conducted (Lo et al. 2005). Another, and significant, source of inspiration was the development of a research tradition in Sweden – phenomenography – from which the variation theory had been developed (Marton and Booth 1997; Marton and Tsui 2004; Marton 2015). Just as in lesson study, the teachers chose a topic they found difficult for students to learn, drove the process and took part in the analysis of the lesson and students' learning. Variation theory was 'owned' by the teachers and used as a theoretical tool in the process. The term learning study was chosen to distinguish its distinctive character from lesson study, the focus on learning, the object of learning and variation theory as the guiding principles in the process. An evaluation of this, and similar projects, demonstrated the effects of learning study; the learning gap between students decreased (Lo et al. 2005), student learning was sustainable and thus lasted and developed beyond individual lessons (Elliott and Yu 2013). Moreover, when variation theory was applied as a tool for design and analysis, student learning was more in line with the teachers' intentions (Marton and Pang 2006). In the same way, studies showed that when teachers jointly explore and study what is made possible to learn in the lesson in relation to student learning, 'the keys' to learning (Pang and Marton 2017) can be found. Subsequently, the lesson can be designed in a way that promotes student learning. Furthermore, it has been demonstrated that 'results' and insights gained in learning study can be communicated, picked up and developed by other teachers and in other contexts (Kullberg 2012; Runesson and Gustafsson 2012).

Learning study can be conducted by a group consisting of teachers only, or they could be assisted by a researcher or an educator from the university. Although they have different expertise, they have a common object of research, to enhance students' learning by identifying the critical aspects and trying them out in class. Although it might be a challenge to avoid distinguishing between the researcher and teachers (Adamson and Walker 2011), it is intended that the teachers and the researcher should be equal partners in the process.

2.1 The Variation Theory as Guiding Principles for Planning and Evaluating Learning

As stated above, to help learners develop a certain capability (e.g. 'understanding the relation between a graph and its derivative's graph', 'the method to calculate the perimeter of compound rectangles'), it is necessary to find what must be learned to develop that capability – aspects that are critical for learning. Thus, learning, from a variation theoretical perspective, amounts to being able to discern critical aspects of what is learned.

Most capabilities have different constituent aspects. Some learners have acquired some aspects already and have not yet acquired other aspects. Learners must learn what they have not yet learnt. Therefore, what students have not acquired, and thus need to learn, must be explored and identified. Although, candidates for such critical aspects can (and must) come from inside the subject matter itself, from previous teaching experience and from research literature, they cannot be known in advance or be known from a predefined learning objective. Therefore, what is critical for the specific group of learners, or whether there are other aspects these learners have not discerned, would be necessary to identify and test. Subsequently, a significant element in learning study is the analysis of student learning in relation to what is taught in the lesson. What must be taught are the critical aspects.

From a variation theory perspective, discernment stems from experiencing variation (Marton and Pang 2006; Pang and Ki 2016). So, students' attention to the critical aspects can be drawn by the means of variation. If an aspect (e.g. the slope of a linear function) varies and others are kept constant (e.g. the intercept of the line), the varied aspect (here, the slope) is likely to be discerned. Furthermore, each aspect consists of its manifold variations, and to discern an aspect, these variations need to be made visible. In the words of variation theory, this is called to *open up a dimension of variation* (Marton 2015). Hence, first when a dimension of variation of a critical aspect is opened up, the critical aspect can be possible to discern. This principle can be applied when designing the lesson. That, which we want the learners to discern – the critical aspect – should be varied against an invariant background.

The idea of learning is, in variation theory, grounded in distinguishing, thus experiencing differences rather than similarities. Marton (2015) argues that the discernment of a feature requires the experiencing of a difference between (at least) two things or parts of the same thing. To discern a new concept, one needs to experience contrast (variation) between the new concept and another concept and hence how it differs from the other concept. Similarly, Watson and Mason (2006) have demonstrated how the idea that systematic and purposeful use of variation and invariance (i.e. what is varied and what is invariant) that structures students' awareness can be applied when designing exercises and examples. They argue that different kinds of variation in exercises offer different learning possibilities. However, teaching without accomplishing a pattern of variation is hardly possible but might be done almost intuitively and without systematicity (Runesson 2005). In learning study, principles of variation – how patterns of variation can afford and constrain learning – are applied intentionally and systematically when designing and analysing teaching and learning.

2.2 *Identifying the Critical Aspects*

Hence, a central part of a learning study is to identify (with as high precision as possible) the critical aspects. This is done in an iterative cyclic process (see, e.g. Cheng and Lo 2013). The initial planning stage includes choosing the topic, defining the object of learning as well as students' learning problems with this object. Learning problems, and what is assumed to be critical for learning, are initially identified through a diagnostic pre-test or by interviews, together with previous teaching experiences and research literature as a background. On the basis of insights generated at this stage, tentative critical aspects are agreed upon, and the first research lesson is planned. After the first lesson, diagnostic post-test or interview is given. Results from this, together with a close analysis of the lesson, give further insights into what is critical for learning and how the content must be handled to promote learning. Hence, typically the critical aspects are refined. This becomes the basis for the planning of the second lesson in the cycle. This lesson is (usually) taught by a new teacher, and to new students, and again the observed/ recorded lesson and the diagnostic post-tests are analysed. The iteration proceeds until all classes are taught (Cheng and Lo 2013). In this way, the assumed critical aspects are confirmed or rejected, and often new (and previously taken-for-granted) aspects are found to be critical. Most often, however, the initially assumed and tentative critical aspects become revised, refined and specified. The revision of critical aspects is a result of a detailed analysis of data (observation and/or video recordings) from the research lessons together with results from pre- and post-tests. With evidence from lesson and student data, the critical aspects are finally established (ibid). Identifying the critical aspect is thus an emergent process of a deep and systematic reflection and analysis of 'the known', 'the knowing' and the 'knower' (cf. Dewey and Bentley 1949).

3 The Example: A Learning Study About the Rate of Change

We illustrate this search for critical aspects, and how these can emerge and be refined, with a learning study about the mathematical concept of rate of change (RoC). This concept is central in the precalculus stage of the study of functions describing dynamic processes and is closely related to the covariation of variables. The object of learning, *to express the quantitative rate of change of a linear relation*, was explored through a series of three iterative lessons in a learning study about RoC in a Swedish lower secondary school (ages 15–16). In this study, a pre-test (described below) was used to select students to form groups of mixed performance. The students were chosen from three different classes (in total 69 students) and were combined in groups of three and where three such groups constituted a teaching unit class in each lesson. Hence, 27 (i.e. 9 students for each lesson) out of 69 students with average performance were included in the full study.

However, the teachers also used the pre-test to identify what aspects they considered critical for learning rate of change. The tasks in the pre-test (in all essential parts it was also used as post-test) were designed based on the teachers' ideas on students' problems regarding rate of change. The paper-and-pencil pre-test consisted of five tasks, all in which calculating rates were central. Four of the tasks included data points representing rates that were positive, negative or zero or linear segments combining these. In one of the tasks, also a non-linear segment was used. In four of the tasks, the rates were to be calculated from graphical representations, but in one task the two covarying quantities were given numerically.

During the teachers' analysis of the tests, there was no one-to-one correspondence between a certain task and a specific aspect. Thus, the different tasks could help identify (and later, when used as post-test, refine) any of the aspects.

The learning study was conducted by three teachers of which one (third author) works half-time at the school and half-time at the university. He participated in the learning study in the same way as the teachers. The main difference was that as the teacher/researcher, he was responsible for documenting the process and had access to guidance from the university (first and second author).

In each lesson, the activities were alternated between the teacher presenting tasks, the students discussing/solving the tasks in the groups and the whole class discussing the different groups' solutions. The use of three groups, each of three students, was decided upon to enhance the possibilities to capture students' reasoning. With four cameras, the three student groups as a whole class and the work of each group could be separately, and simultaneously, audio- and video-captured. The students' solutions in pre- and post-tests, and their discussions during each of the lessons, were analysed and used by the teachers in the iterative learning study process. This data, capturing the students' understanding of the RoC as well as how the object of learning was enacted during the lessons, helped the teachers identify and refine possible critical aspects and plan for how each dimension of variation could be opened up and thus guide their planning of the coming lesson. Our report is structured to describe one critical aspect at the time, but the lessons were planned to sometimes target the different tentative critical aspects within the same task.

3.1 Three Tentative Critical Aspects

From the analysis of the pre-tests, three different tentative critical aspects (TCA) were identified. The first TCA to be identified concerns the covariational aspect that RoC can be regarded as a relation/ratio between changes in two quantities (i.e. two covarying variables). In the pre-test data, the teachers noticed that some students did not seem to regard the changes in both variables as actual continuous changes. Especially concerning the horizontal variable (time), an interval did not seem to be perceived as a change, but as two values between which a corresponding change in the vertical variable occurs. When asked to find the rate of change of a linear relation in a given interval, some students used segments of the graph instead of the interval as a whole. These students repeatedly associated intervals in the vertical variable with adjacent tick marks on the horizontal axis, forming a 'staircase' function. The students seemed too bound to the tick marks in the graph, the teachers concluded. The teachers then hypothesised that a TCA could be *to discern that RoC is constant in linear relations*. This aspect was conjectured to aid students to freely 'move' along the linear parts of a graph and choose a suitable segment. The teachers acknowledged the use of easily identified, but otherwise arbitrary, segments of the graph to be a powerful strategy with the goal to express the RoC in a linear relation.

From pre-test data, it was concluded that students seemed to implicitly assume that they were dealing with proportional relations. In short, with focus on corresponding values, students tried to calculate the RoC by dividing y-value with x-value, a strategy which works in the case of proportionality. In some solutions, an interval in one variable was combined with a value in the other. The teachers interpreted these observations in terms of a need to discern the two corresponding changes involved in calculating the RoC. Hence, a second TCA was formulated as *to discern that every change in one variable is related to a change in the other*.

One of the pre-test tasks was without graphical representation. The responses to this task indicated that some students divided one change with the other ($\frac{\Delta y}{\Delta x}$), and some students divided the changes the other way around ($\frac{\Delta x}{\Delta y}$), to produce a numerical expression for the RoC. The teachers therefore suggested that in the presence of graphical representations, the question of choosing $\frac{\Delta y}{\Delta x}$ or $\frac{\Delta x}{\Delta y}$ could be the result of adopting a mathematical convention. The general convention is to divide the dependent variable with the independent variable ($\frac{\Delta y}{\Delta x}$). But the teachers decided to avoid the issue of dependency-independency as the object of learning concerned RoC in a 'non-functional' way. They concluded that in the lesson, the dimensions of variation had to be opened up in ways relying on reasoning, rather than on conventions or rules. Hence, a third TCA was formulated as *to discern that two values, inverse to each other, can be used to express the quantitative RoC*. This third TCA was therefore targeting that both ratios contain information about the change in one variable with respect to the other.

Hence, there were three different TCAs that were assumed at the beginning of the learning study. These TCA were then identified based on the results from the pre-test. In the following section, we will illustrate what can trigger an evolution of critical aspects in a learning study by describing how two of these TCAs were refined into critical aspects. In particular, we will describe what the teachers found in the data from pre-test, post-test and observations of students that made them refine the critical aspects during the learning study.

3.2 Refining the Critical Aspects Through the Process of Learning Study

After identifying TCAs, the teachers planned the first lesson. The teachers agreed to use the same framing context in all three lessons: to fill tanks with water in a certain time. The level of water in the tank at different times was given, either as points in a graph or in tabular form. Two variables were used: *water level* (measured in cm) and *time* (measured in min). Each lesson contained a series of tasks within this framing context. In this way, the lessons were refined in each cycle of the learning study, all the time with the goal to open up the right dimensions of variation (based on the critical aspects) and thereby enact the object of learning *to express the quantitative RoC of a linear relation*. At the same time, the critical aspects were not static, but were, themselves, refined in the process.

3.2.1 Refining the Tentative Critical Aspect 'to Discern that the Rate of Change Is Invariant in Linear Relations'

To target the TCA *to discern that RoC is constant in linear relations* in lesson 1, the context was developed into tasks involving different 'observation points' (see Fig. 1).

Initially, the table of data points in Fig. 1a was shown to the students. The teachers had designed an example where the rate of change was calculated based on specific data points. The variation theory was used in the design of the example, particularly focusing on what to keep constant and what to vary. Firstly, data points were compared where time intervals (but not the rate) were kept constant, the time interval between 1 and 6 and between 6 and 11 min, Fig. 1a. Secondly, the data points where water level intervals (but still not the rate) were constant were compared, between 43 to 61 cm and between 61 and 79 cm (difference of 18 cm between each pair of points). Thirdly, a new point (3, 18.4) was inserted (see Fig. 1b), and the two first intervals, where the rate of change was invariant, were compared. The example was used with the variation theory in mind, to make possible to discern that a change in one variable, keeping the other invariant, would change the rate in total. The teacher then emphasised that, in contrast with first comparisons, in which the change in only one variable was altered, different changes in *both* variables could produce the same rate. However, that these different pairs of changes were connected through the same linear relation was not explicitly mentioned, and thereby not offered for discernment, according to the variation theory. Later in the lesson, the same points were used again, now represented as points in a diagram; see Fig. 1c. The teacher then connected all three points with a straight line and asked the students to verify that the RoC over both intervals together (1–6 min) matched the earlier calculated (identical) rates. With this, the teachers' intention was to make it possible to experience that the two ratios were connected through the same linear relationship. However, the results in the post-test (after lesson 1) showed no clear improvement in terms of students' ability to use different intervals to calculate the RoC. In the analysis of the first lesson, the teacher's actions on this task were considered central for the lack of students' improvement (regarding the results from the pre- and post-test). In retrospect, it can be interpreted as that the students were given the opportunity to *verify*, not discern, that the RoC was equal. However, at that time, the group only conjectured that the students did not learn that RoC is constant in linear relations because they were not given the opportunity to discern that different pairs of intervals along a given straight line would produce the same rate of change. Thus, this dimension of variation was not opened up in the lesson, they concluded.

Hence, while the critical aspect was not modified after lesson 1, the task sequence was modified from lesson 1 to lesson 2. In lesson 2, an additional task was

a

Time (min)	Water level (cm)
1	2.0
6	43.0
11	61.0
20	79.0
29	61.0

5 min
5 min
18 cm
18 cm

b

Time (min)	Water level (cm)
1	2.0
3	18.4
6	43.0
11	61.0
20	79.0
29	61.0

8.2 cm/min
8.2 cm/min

c

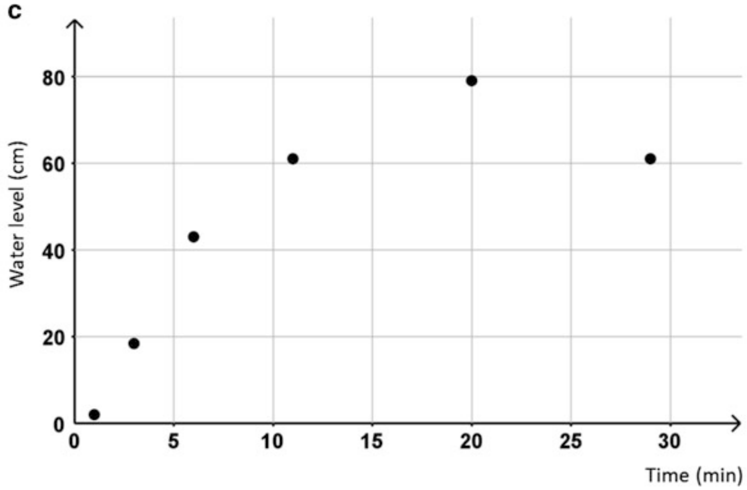

Fig. 1 The example, based on the context of filling a tank of water, used in lessons 1 and 2, here represented as numbers in a table where (**a**) time and water level interval lengths were kept constant, separately, and (**b**) after insertion of an extra point (3, 18.4), where the rate of change was kept constant. (**c**) The graph corresponding to the table in (**b**)

introduced to the students. Starting with a single point (4,12) in a diagram of water level vs time, Fig. 2a, the students were given the task to calculate how fast the tank was filled.

With only one point indicating the water level, many students assumed that the tank was empty from the beginning and calculated the rate as 12 cm/4 min = 3 cm/min. While the origin as the assumed starting point initially was used to vary the next aspect described in this chapter, it also served as part of varying the critical aspect at hand. Some moments later in the lesson, an additional point (1,3) was inserted in the diagram, as shown in Fig. 2b, and students were asked about the RoC between this point and the original point (4,12). This procedure was done to make it possible for the students to discern that the rate is the same between the points; see Fig. 2b. Two out of the three groups then clearly expressed that the rate in this case was the same as they assumed in the first task with the single point. From the video observations, this suggested a critical moment in the lesson, and the observations were connected

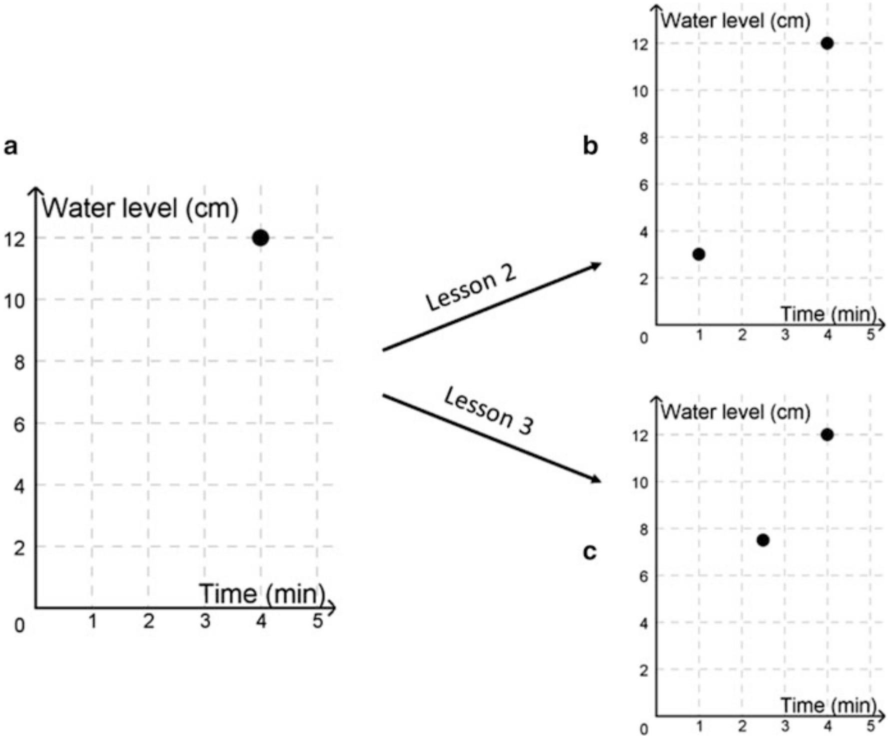

Fig. 2 (**a**) The single point graph used as a starting point to reach the critical aspect in lessons 2 and 3. The same diagram in (**b**) lesson 2 and (**c**) lesson 3, respectively, after inserting a point to open up the dimension of variation that intended to make discernible that the same rate can be achieved by different relations

to a small improvement in the post-test. The teachers then analysed the difference between this event and what happened in the first lesson. They concluded that in both lessons the same mathematical features were varied and kept invariant, respectively. The RoC is constant in linear relations, and hence, different ratios can represent the same RoC. Mathematically the latter follows the first, but for the students, it seemed to be the other way around, the former logically followed the latter, they concluded. Consequently, after lesson 2, the teachers rephrased the critical aspect *to discern that different ratios can represent the same rate*.

In lesson 3, the original set of tasks used in lesson 1 and 2 was abandoned, and the ‘single point task’ was revised by further ‘masking’ the second rate, using the point (2.5,7.5) instead of the point (1,3); see Fig. 2c. The argument was that it should not be too obvious that the two points resulted in the same rate. Video observations of the lesson revealed clearly that all three groups noticed that the rate was the same in the two cases. Two of the three groups in that lesson spontaneously checked their calculations graphically, drawing and examining the graph representing the change of water level over time. The third group first solved the task by examining the point graphically and then also confirmed their answer numerically. In contrast with lesson 2, the teacher in lesson 3 also asked the students for two more examples of how to calculate the rate. Thereby two more ratios (to a total of four) were contrasted with each other, followed by the generalisation of the ratios all producing the same rate. In the post-test, all students in lesson 3 made use of, or suggested to use, multiple pairs of intervals to calculate the RoC in a graphically represented linear relation. Hence, after lesson 3, the formulation of the critical aspect was kept and considered corroborated by the turnover in how it was most efficiently experienced by the students in lesson three.

3.2.2 Refining the Tentative Critical Aspect that 'Every Change in One Variable Is Related to a Change in the Other'

From the pre-test data, the teachers noticed that some students' solutions to calculating the RoC seemed to be based on that they read off two corresponding values in a graph, instead of two corresponding intervals. Therefore, during the planning of lesson 1, it was decided to use the set of data points shown in Fig. 1 to vary the change in one variable, while at the same time, the change in the other variable was kept invariant. (At this time, the teachers often used the words 'changes' and 'intervals' synonymously.) In a concluding task by the end of the lesson, students in all three groups still tended to relate an interval in one variable with a value in the other variable. However, in all such cases, the values the students used were connected to one of the endpoints of the interval. It was also noted that in some cases, students seemed to assume a starting point at the origin, even though they had the full (continuous) graph of the example. This drew the teachers' attention to if and how students experienced the *starting point* of the corresponding interval, especially in relation to the origin. Hence, after the first lesson, the TCA seemed necessary to revise, but it was not formally revised. However, the teachers had an idea that the starting point for an interval where only one point was given was perceived as the origin.

To test their conjecture, the task with one single point present in an otherwise empty diagram space was introduced in lesson 2, as shown in Fig. 2a. The teachers' aim of this manoeuvre was to open a dimension of variation about the starting point from where changes could be measured. When the teacher posed the question 'what does this point mean?', one student replied: 'That after four minutes it has risen twelve centimetres'. Hence, the student already at this point assumed that the tank was empty from the beginning. The teacher then repeated the answer and pointed out that: '...or more accurately, yes, at four minutes, the water level *is* twelve centimetres'. The teacher then posed the question, 'How fast was this tank filled, then?', thus not serving any predetermined interval. Some students immediately solved the task by dividing the y-value of the given point, with its x-value. However, video observations revealed that the presumption that the interval started out from the origin was also challenged by some students, thus opening the desired dimension of variation. Despite disagreements among the students, in their answers the groups stated that the water level rose with a rate of 3 cm/min, possibly expecting that the teacher would not present a task with no unambiguous solution. The teacher then gave additional information: 'You know what? Here comes a secret: There is an observation point, right when the timing began, for this tank'. The point (0,6) was then marked in the diagram, followed by a question to the students if they would like to revise their earlier calculation of the rate. In this way, the position of the starting point for the interval where the rate was to be determined was varied two more times to (2,0) and (1,3), respectively, as shown in Fig. 3a. The data points in this example were designed for this specific critical aspect. However, eventually the points were used also in the example targeting the previously described critical aspect as well.

The teachers now became more aware of the students' strong bias towards the origin as a starting point of functions. Hence, after lesson 2, the critical aspect was rephrased *to discern that the starting point of an interval is not synonymous with the origin*.

In lesson three, this pattern of variation of the starting point was slightly diminished. As the teachers had planned, the students in lesson three discussed whether the starting point was the origin or not, opening a desired dimension of variation. However, in the analysis of lesson 2, it was considered that it was too easy to numerically confuse the RoC with a slope in a graph. Hence, the first 'hidden' starting point was now decided to be (0,7.5) and then exchanging this for a point at (2.5,7.5); see Fig. 3b. The goal was, as in lesson 2, to make a contrast to a starting point at the origin (0,0), which was thought to be assumed by many students. In contrast with the pattern in lesson 2, this was considered to allow for a variation in time to be isolated and highlighted. Anyhow, the changes from lesson 2 to lesson 3 seemed to have an impact on students' learning. In the analysis of this example in the lessons, it was found that two of the three groups in lesson 2 combined one value

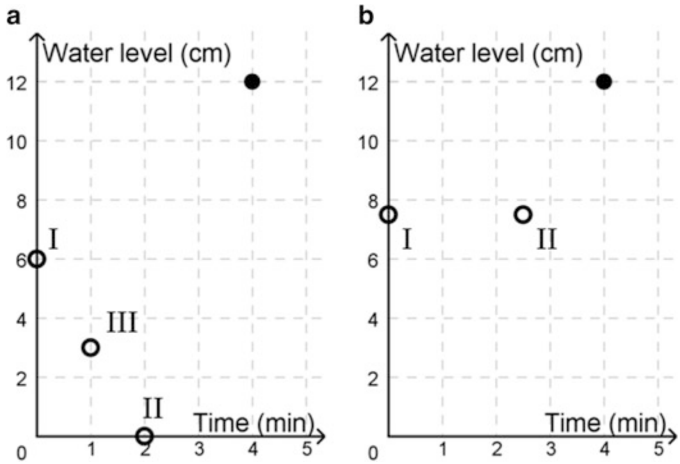

Fig. 3 The different points used to open up the dimension of variation that intervals do not need to start at the origin in (**a**) lesson 2 and (**b**) lesson 3. In both lessons, the alternative points (the empty circles) were initially kept hidden as only the first point (4,12) was shown. These additional points were then shown separately in the order indicated by Roman numerals (I–III)

with one interval, but that in lesson 3, only a single student did so. After lesson 3 the critical aspect was reformulated as *to separate changes starting in the origin from other changes*.

4 Concluding Discussion

We started the chapter by advocating that, when planning for teaching, there is a gap between the learning objective and teaching strategies that must be filled, since learning objectives do not tell what the students must learn to achieve the targeted goal. In the above example, we have shown how the critical aspects, the keys of learning, can be identified but also refined and specified through the process of a learning study.

In Fig. 4, we show how two such tentative critical aspects have evolved during the iterative transactional process. The third tentative critical aspect was transformed in a similar way, not reported on here, however.

In the first round of analysis, the students’ answers to the pre-test and the teachers’ own previous experiences of the topic led them to formulate the tentative critical aspects. The teaching in lesson 1 was planned and enacted accordingly to open up dimensions of variation of these critical aspects. In the following analysis of the teaching of, and students’ responses to, lesson 1, the critical aspects were kept unaltered. We note that for one of the critical aspects (‘Critical aspect 1’, in Fig. 4), it was the analysis of the teacher’s actions that lead the group to change the lesson plan from lesson 1 to lesson 2. Based on students’ responses, the other critical aspect did not seem expressed in a satisfactory way, but it was anyway kept unaltered. However, both critical aspects had now been tested in a teaching situation and could then be considered emerging, rather than tentative. Instead, based on the variation theory, the teachers concluded that in lesson 1, the full dimension of variation was not opened and decided to alter the task for the next lesson.

In the analysis of the second lesson, it was the combination of the teaching and the students’ responses and the mathematical content that attracted attention and resulted in that both critical aspects were revised. Regarding the first critical aspect, it was the insight that students can experience the implication, that linear relations have the same RoC, the other way around (meaning that they saw that students could perceive a constant RoC as a linear relation). Regarding the second critical aspect (‘Critical aspect 2’, in Fig. 4), the teachers started to attend more to the bias of the origin. Anyhow, the third lesson was planned accordingly; patterns of variation were enacted to open up for the students to discern these revised critical aspects. In the analysis of this lesson, it was concluded from students’ responses to the teaching that the first critical aspect was corroborated, whereas from the teachers’ actions (actually, the variation pattern in the example presented by the teacher), and supported by the students’ answers, it was concluded that the second critical aspect had to be revised again.

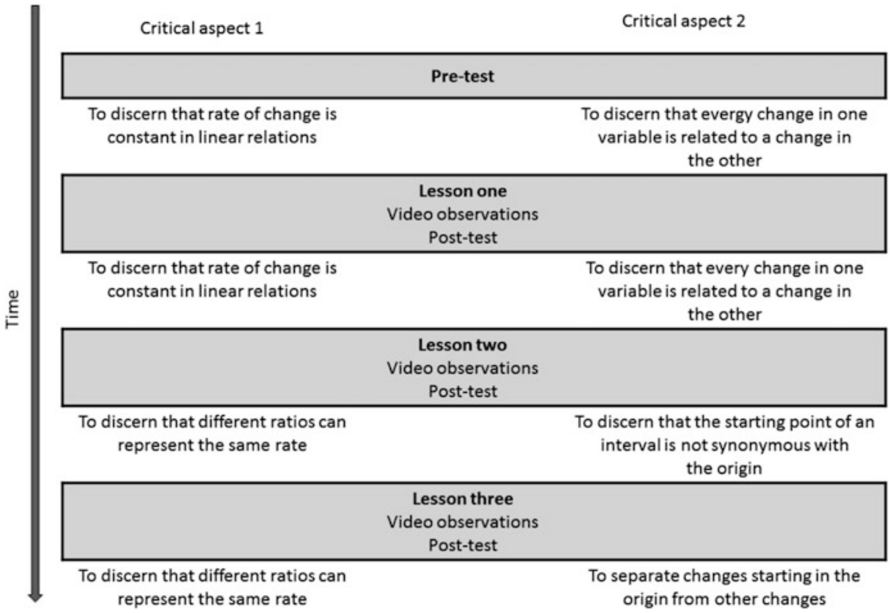

Fig. 4 The evolution (from top to bottom) of two of the critical aspects (the columns to the left and to the right, respectively) during the process of the learning study

Hence, this illustrates that both students' answers and responses to the teaching, as well as teachers' actions and the enactment of the lesson plan, can result in a revision of the critical aspects. Independent of whether it was the students' or the teachers' actions that triggered the revision and specification of the critical aspects, we would suggest that the use of a specific learning theory enabled the teachers to identify these features. Variation theory helped them to focus on the object of learning and students' learning in a relational way. Furthermore, variation theory provided them with a common language to talk about teaching and learning and principles for pedagogical design.

The teachers participating in the process of the learning study took active part in the process of finding and refining the critical aspects. The topic of study stemmed from a problem in their everyday practice, and they planned and analysed the lessons and the student data jointly with the researcher. Actually, they participated to such a high degree that one could claim that they were co-producing the results. This, we would suggest, implies that learning study is more than a development programme. Instead, the teachers are collaboratively acting as researchers, and as such, learning study has the potential to be what Lawrence Stenhouse (1981) suggested: in the research process of generating knowledge for professionals, teachers should be *the* key stakeholders. Therefore, reports on learning studies are not mainly reports on teachers as learners, but on what has been found by the iterative process of investigation. Morris and Hiebert (2011) have suggested lesson study as a system to produce instructional products to be shared and developed. The results from the study reported on here, we think, are an example of such an instructional product. The results are a theoretical description of a lesson design in terms of what was found critical for learning and how this can be manifested in class by means of variation. Hence, a learning study can contribute to the general community with specific pedagogical content by obtaining new information about the keys of learning for a certain topic.

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