Advances in Mathematics Education

Rongjin Huang
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Theory and Practice of Lesson Study in Mathematics

An International Perspective

Springer

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ISSN 1869-4918 ISSN 1869-4926 (electronic)
Advances in Mathematics Education
ISBN 978-3-030-04030-7 ISBN 978-3-030-04031-4 (eBook)
https://doi.org/10.1007/978-3-030-04031-4

Library of Congress Control Number: 2019932837

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

If anyone ever questioned the possibilities, the influence, or the global reach of lesson study, this volume quickly puts those questions to rest. In our judgment, no other phenomenon in education has so quickly spread to so many countries and reached so many teachers and researchers in so many different educational settings. There are plenty of good ideas in education, but most of them remain tethered to their original context, encountering too many barriers of cultural differences or simple lack of interest across educational settings to cross local, national, or continental boundaries and impact education worldwide.

In less than one generation, lesson study has found its way from Asia to many countries around the world, making a volume like this one possible, a volume unimaginable just 20 years ago, when we first started learning and writing about lesson study. What might account for lesson study's unparalleled appeal to educators and its unprecedented rapid spread?

In a 2013 article in *The New Yorker*, Atul Gawande tried to answer the same question about the spread of ideas in the clinical practice of medicine. He began his article by asking, "Why do some innovations spread so swiftly and some so slowly?" (p. 36). To answer the question, he compared two breakthroughs in medical practice in the nineteenth century—anesthesia and antiseptics.

Anesthesia, first demonstrated in 1846, spread rapidly. Not surprisingly, both patients and doctors hailed it as a remarkable improvement. Within 1 year after its first appearance, according to Gawande's account, surgical hospitals around the world were using anesthesia. In contrast, the spread of antiseptics, another miracle breakthrough in medicine, was slow and halting. Many analysts would not have predicted this, says Gawande, because infection was the single largest killer of postsurgical patients. An 1867 report in a renowned medical journal reported that antiseptics could drastically reduce infections and save lives. However, it took years for antiseptics and the process of sterilization to become routine, and hospitals still struggle with reducing infections. What was the difference between anesthesia and antiseptics? To oversimplify Gawande's explanation, anesthesia visibly solved problems for both patients and doctors. The benefits of antiseptics were less visible, and mostly for patients.

The story told by Gawande offers some insights into possible reasons for the rapid spread of lesson study. Lesson study, a collaborative endeavor between teachers and researchers, solves problems for both groups in visible ways. With its relentless focus on improving teaching and learning and its inclusion of teachers as essential participants, teachers can quickly see the benefits for themselves and their students. The benefits for researchers might not be as immediately apparent, but they become quickly visible to researchers as soon as they enter the process. We believe it is worth pointing out a few of these benefits, and the accompanying insights they afford, especially if it entices the reader into the richness that awaits them in this volume.

A key benefit of lesson study for teachers is that it moves much of the hard and complex work of teaching outside the classroom, expanding the time and space available to work on improving practice. Lesson study does this by broadening our concept of what teaching is to include not only the teacher's performance in the classroom but also the planning of lessons, in excruciating detail, and then the reflecting on taught lessons using data gathered during the lessons about students' progress toward precisely defined learning goals. This does not make teaching easier, but it offloads much of the challenging intellectual work to outside the classroom, when teachers have time to think through their instructional problems, preferably with other teachers who share the same problems. These collaborations among teachers outside of class provide a less stressful and more thoughtful intellectual setting for learning to make good instructional decisions than is provided by the classroom, with its fast pace, multiple interactions, and demands for individual teachers to make good on-the-fly decisions. As teachers ourselves, we are not surprised that other teachers find these aspects of lesson study extremely beneficial.

The spread of lesson study appears to confirm that teachers around the world have resonated with this shift away from a singular focus on performance in the classroom to a broader focus that includes the study of teaching outside of class. This says to us that almost universally teachers recognize that improving their practice requires a focus on the methods used to teach, not on the personality or charismatic qualities of individuals. This also confirms for us that almost all teachers want to improve their craft if given a chance to do so. Finally, the popularity of lesson study among teachers in very different educational settings means that lesson study is not a formulaic procedure that prescribes a specific routine but rather a set of principles that can be adapted to different settings. The chapters in this book provide a glimpse into how adaptable the process can be.

Although the benefits for researchers might not be as immediately visible, the authors in this book illustrate the wide range of research questions related to improving teaching and learning that can be addressed through lesson study. We mention only a few of them here. Perhaps the greatest gift of lesson study is its ability to lay bare the activity of teaching. By analyzing daily classroom lessons, researchers can look inside the system of teaching, the interactions between all the elements that contribute to shaping the learning opportunities experienced by students. Because the lesson is the smallest unit of teaching that preserves this system, researchers can analyze teaching in great detail without missing critical ingredients.

By laying bare the activity of teaching, researchers working beside teachers also can use lesson study to gain new insights into how teachers think and learn about teaching. What instructional problems do teachers often face? What barriers to solving these problems and improving their practice do teachers confront in their local settings, and what supports can help them surmount these barriers? These are fundamental questions in improving teaching and learning, and researchers gain on-site close-up information through lesson study that can address these questions.

A final benefit we identify for researchers working within the lesson study process, as it spreads around the world, is access to information that can help educators understand cross-cultural differences in teaching and learning at the classroom level. The ways in which lesson study is adapted to fit cultural contexts provides new clues into cultural constraints and affordances in educating students and in systems for improving education. How regions or countries must adapt the lesson study process to make it productive in their environment reveals a great deal about cultural norms, often invisible, that shape the nature of teaching and the paths available to improve it.

Authors of the chapters in this volume illustrate all these benefits, and more. They show what can be gained by analyzing the nature of lesson study as it plays out in different educational settings and for different purposes. Readers will learn what the field is learning about teaching and learning through lesson study, and especially about improving teaching and learning, from reading the fascinating range of cases provided by the authors. Because the list of authors is a who's who of lesson study practice and research, readers will encounter the best we know about how lesson study can benefit both teachers and researchers. That is, readers will learn firsthand why this process has seen a virtual explosion of popularity in the past 20 years. We applaud this rapid spread of lesson study because, like anesthesia in medicine, we believe it has real and visible benefits for teachers and researchers. It is a richly fertile process that changes the way educators at all levels can see teaching and can appreciate what is required to improve teaching. We applaud this volume for documenting so extensively this unique educational phenomenon.

Los Angeles, CA, USA James Hiebert
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Reference

Gawande, A. (2013, December 29). Slow ideas. *The New Yorker*, pp. 36–45.

Acknowledgments

We would like to take this opportunity to thank our 92 contributors from 21 countries or regions for their commitment and contribution. Indeed, the chapters have been revised multiple rounds based on peer reviews over the past 2 years. Our contributors not only have made great efforts to ensure the high quality of their chapters but also volunteered their time to review chapters. We also want to thank external reviewers for their help in reading and commenting on the manuscripts. It would not be possible to have such a high-quality collection of chapters without their constructive inputs. On behalf of all the contributors, we would like to extend our appreciation to the dedicated group of reviewers including Glenda Anthony (Massey University, New Zealand), Mónica Baptista (University of Lisbon, Portugal), Lara Dick (Bucknell University, USA), Sharon Dotger (Syracuse University, USA), Soledad Estrella (Pontifical Catholic University of Valparaiso, Chile), Melanie Haupt (Middle Tennessee State University), Lenuel Hernandez (Wayne State University, USA), Johan Häggström (University of Gothenburg, Sweden), Angelika Kullberg (University of Gothenburg, Sweden), Su Liang (California State University, San Bernardino, USA), Joana Mata-Pereira (University of Lisbon, Portugal), Sarah Smitherman Pratt (University of North Texas, USA), Kyle Prince (Central Magnet School, Tennessee, USA), Candice Quinn (Middle Tennessee State University), Miguel Ribeiro (University State of Campinas (UNICAMP), Brazil), Teresa Schmidt (Middle Tennessee State University, USA), Charlotte Krog Skott (UCC University of Applied Sciences, Denmark), Melissa M. Soto (San Diego State University, USA), Xuhua Sun (University of Macau, China), Hamsa Venkatakrishnan (University of the Witwatersrand, Johannesburg, South Africa), Ke Wang (Texas A&M University, USA), and Trena Wilkerson (Baylor University, USA).

We are grateful to Middle Tennessee State University for various support during the process of editing and publishing the book. We thank the book series editor, Professor Gabriele Kaiser, for her encouragement and support during the process of editing the book. Last but not least, we extend our gratitude to Jennifer Seat and Monica R. Frideczky for their throughout and professional edits of some chapters.

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Part I
Theoretical Perspectives of Lesson Study

Theory and Practice of Lesson Study in Mathematics around the World

Rongjin Huang, Akihiko Takahashi, and João Pedro da Ponte

Contents

Abstract Lesson study, a powerful teacher professional development approach, originating in Asia, has spread globally. Although the positive effects of lesson study on teacher learning and student learning have been widely documented, many challenges and obstacles facing the adaptation of lesson study have been identified. Moreover, theorizing of lesson study and methodologies for researching lesson study have just begun to emerge as research issues. This book is a collaborative attempt to synthesize state-of-the-art research on conceptualization, theorization, and adaptation of lesson study. The structure and major contributions of the book are described.

Keywords Lesson study · Conceptualization · Theorization · Adaptation

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The original version of the chapter has been revised. A correction to this chapter can be found at https://doi.org/10.1007/978-3-030-04031-4_39

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_1

1 Introduction

Continued efforts to improve teaching and to develop teachers have drawn increasing international attention toward *lesson study* (LS hereafter in this book) over recent decades (e.g., Dudley 2014; Hart et al. 2011; Inprasitha et al. 2015; Isoda et al. 2007; Kieran et al. 2013; Lewis 2002). LS is a practice-based, research-oriented, student-focused, collaborative mode of professional development (Fernandez 2002; Lewis and Tsuchida 1998; Murata 2011; Stigler and Hiebert 1999). Originating in Asia (Japan and China) (Chen and Yang 2013; Lewis and Tsuchida 1998; Stigler and Hiebert 1999), it has spread across the globe (Lewis and Lee 2017). For example, more than 1000 participants from around the world attended the annual conference of the World Association of Lesson Studies (WALS) in 2016. In addition, a Special Interest Group of Lesson Study associated with the American Educational Research Association was established in 2017. As a strong endorsement of LS, the International Congress of Mathematicians in 2018 organized a panel discussion on the "use of lesson study to support quality mathematics teaching: practical and theoretical issues raised within the community of mathematics educators and mathematicians." Because teaching is a cultural activity (Stigler and Hiebert 1999), various conceptualizations of LS and associated forms of activity have developed in different countries (e.g., Huang and Bao 2006; Yoshida 2012) both with in-service and pre-service teachers, assuming different purposes and following different formats (da Ponte 2017; Huang and Shimizu 2016; Lewis 2016).

A number of studies have documented that LS contributes to transforming teaching (Chen and Yang 2013; Lewis and Tsuchida 1998; Stigler and Hiebert 1999), promoting teachers' growth (Lewis et al. 2009; Murata et al. 2012), sustaining professional learning communities (Moss et al. 2012), improving students' learning (Lewis and Perry 2017), and building the connections between research and practice (Huang et al. 2016; Kieran et al. 2013; Runesson 2015). Yet, researchers have also identified obstacles and challenges when adapting LS in other counties (da Ponte 2017; Fujii 2014; Huang and Shimizu 2016; Larssen et al. 2018). With in-service teachers, Fujii (2014) indicated six misconceptions of LS such as regarding it as a workshop adhering to the research lesson as explicitly prescribed by the lesson plan. Huang and Shimizu (2016) classified the factors influencing the success of LS into two broad categories: (1) At macro level, these factors include its broad cultural value, the teaching and teacher learning culture, the teacher professional development system, the professional learning community, and the leadership of district leaders and school leaders. (2) At micro level, these factors include appropriate content and pedagogical knowledge of teachers; the development of inquiry stances such as critical lens as researcher, as curriculum developer, and as student; and classroom observation with a focus on student learning, teachers' commitment, and so on.

Regarding LS with pre-service teachers, da Ponte (2017) identified challenges such as defining the aims of LS, establishing the relationships among participants, scaling up LS, and adapting or simplifying LS for the particular purpose of educating future teachers. In addition, Larssen et al. (2018) put forward the challenges in adopting LS with initial teacher education (ITE) programs including how to prepare student teachers to observe, the wide variation in the focus of classroom observation in these lesson studies, and the need for discussion of what is understood by learning to stand at the heart of preparation for LS in ITE. To maximize the benefits of LS and to address challenges facing the implementation of LS both with in-service and pre-service teachers, two major issues emerged: one about conceptualizing LS by consideration of the variation and adaptation of LS (Huang and Shimizu 2016; Tahahashi and McDougal 2016) and another about methodological and theoretical frameworks for researching LS (Quaresma et al. 2018).

A ZDM issue (Huang and Shimizu 2016) was devoted to deepening the understanding of the differences and similarities among different forms of LS with in-service teachers and of the underpinning cultural and/or philosophical rationales. Although the release of this issue has promoted dialogues of relevant questions, many excellent research studies could not be included due to space constraints. Moreover, the release of an ICME 13 monograph on theoretical and methodological issues (Quaresma et al. 2018) calls for the need to extend this line of study.

This book aims to synthesize and extend the current research efforts on adaptation, conceptualization, and theorization of LS by including more than 30 chapters from internationally known researchers to advance the studies on LS and to address the challenges facing the adaptation of LS to different cultures.

2 Structure of This Book

This book includes six parts. Part I provides an introduction and various theoretical perspectives of researching LS. Part II contains the historical and cultural perspectives of LS in China and Japan where LS has been practiced system wide for over a century. Part III focuses on adaptations of LS in selected educational systems. Part IV contains the use of LS for preparing future mathematics teachers. Part V includes studies on key aspects of LS. Part VI, the last part, includes commentary chapters and conclusions. The commentary chapters draw together the research reported in this volume and reflect on what we can learn from this international collaborative publication effort with possible research directions for the future.

Part I includes seven chapters. This introduction chapter provides readers with an overview of the book. The chapter "How Does Lesson Study Work? Toward a Theory of Lesson Study Process and Impact" proposes a theoretical model for explaining the impact of LS on teacher and student outcomes. Lewis and colleagues, building on their studies on LS over two decades, examine all four phases of a LS cycle (study, plan, teach, reflect) and identify major goals, challenges, strategies to overcome challenges, and relevant theoretical perspectives. In the chapter "How Could Cultural-Historical Activity Theory Inspire Lesson Study?," Wei provides a holistic analysis of the structure of LS from the perspective of cultural-history activity theory (CHAT) and illuminates the significance of LS at the ontological, epistemological, methodological, and axiological levels using empirical data collected from his LS with elementary school teachers in China. In the chapter "Developing Teachers' Expertise in Mathematics Instruction as Deliberate Practice Through Chinese Lesson Study," Han and Huang explore how LS developed teachers' expertise in mathematics instruction in China from the perspective of deliberate practice. They concluded that through exploring the deliberate practice of perfecting teaching of division of fractions, teachers developed their expertise with enacting the core practices of revision of mathematical tasks and revision of mathematical representations. In the chapter "Doing and Investigating Lesson Study with the Theory of Didactical Situations," Bahn and Winslow adopt the perspective of the theory of didactical situations (TDS) to examine essential questions such as what is the role of different components of LS, how do they interact, and what are the effects of repeating research lessons. In the chapter "Theorizing Professional Learning Through Lesson Study Using the Interconnected Model of Professional Growth," Wanty and her colleagues use the Interconnected Model of Professional Growth (IMPG) of Clarke and Hollingsworth (2002) to examine the professional learning experiences of individual participants of LS. In the chapter "Teaching for Robust Understanding with Lesson Study," Schoenfeld and his associates describe how an empirically validated framework of the Teaching for Robust Understanding (TRU) can be used to strengthen LS. Using TRB-based LS in mathematics, teachers work together to design, teach, and reflect on a lesson that focuses on key mathematical issues and students' engagement with the TRU framework.

Part II includes five chapters. The chapter "Preface: Historical and Cultural Perspectives on Lesson Study in Japan and China" by Lynn Paine highlights big ideas across the chapters in Part II. Since historical tradition and cultural values shape the goals and enactments of LS, she emphasizes the importance of recognizing the complexity of LS and reminds us of a dilemma of lesson study's role in reinforcing dominant traditions and its potential as an incubator of innovation. In the chapter "The Origin and Development of Lesson Study in Japan," Makinae details the origin and history of LS in Japan. Japanese LS initially is coined using an object lesson approach, evolved through criticism lesson, and, finally, developed as LS. In the chapter "Lesson Study and Textbook Revisions: What Can We Learn from the Japanese Case?," Watanabe examines a critical issue of how LS, besides promoting the implementation of curriculum, impacted textbook improvement in Japan in the 1980s. In the chapter "An Analysis of Chinese Lesson Study from Historical and Cultural Perspectives," Li tracks the origin and development of LS in China from initial demonstrating and critiquing lessons to an institutionalized routine of teaching research activity, to a currently further developed teaching research system. He also explains the cultural value and beliefs related to LS in China. In the chapter "Lesson Study and Its Role in the Implementation of Curriculum Reform in China," Huang and his colleagues provide a holistic picture of the system of Chinese LS and its role in mathematics curriculum reform in China. Through a case study, the authors show how LS can help to implement an innovative idea from curriculum into classroom practice.

Part III includes nine chapters. In the chapter "Preface: Adaption of Lesson Study in Selected Education Systems," Wasyl Cajkler provides insight into each chapter of Part III and highlights two big concerns of guiding theories of LS and knowledgeable others during LS. In the chapter "Using School-Wide Collaborative Lesson Research to Implement Standards and Improve Student Learning: Models and Preliminary Results," Takahashi and McDougal address a critical issue of adapting LS outside Japan without losing authentic features of LS by proposing the Collaborative Lesson Research model. This model has been implemented in the USA and Qatar, and the initial results indicate its usefulness on school-wide LS to implement new curriculum and to improve student learning. In the chapter "Implementing a New Mathematics Curriculum in England: District Research Lesson Study as a Driver for Student Learning, Teacher Learning and Professional Dialogue," Dudley and his colleagues describe a project which harnessed six cycles of Research Lesson Study at school and district level over 2 years to tailor the implementation of a new statutory curriculum in England and report the findings of research carried out regarding the project. In the chapter "A Case of Lesson Study in South Africa," Adler and Alshwaikh present a LS which focuses on how to use examples to promote students' learning during this process. This case shows the power of exemplification when studying and working on mathematics teaching and supports theoretically informed LS in general. In the chapter "How Variance and Invariance Can Inform Teachers' Enactment of Mathematics Lessons," Preciado Babb and his colleagues describe how to use systematic variance and invariance to inform teachers' continuous decision-making during a class as a critical component of LS. They further illustrate a teaching approach consisting of four components developed empirically through a multiple-year project. In the chapter "Capturing Changes and Differences in Teacher Reflection Through Lesson Study: A Comparison of Two Culturally Diverse Malaysian Primary Schools," Kor and colleagues examine the characteristics of post-lesson reflection between different groups of LS. They conclude that at the earlier LS cycles, teacher reflection was mainly at the descriptive story level. Yet, teachers' reflection gradually advanced to a higher dialogic level at the later cycles. In the chapter "Representing Instructional Improvements in Lesson Study Through Principled Analysis of Research Lessons in Singapore: A Case of Equivalent Fractions," Fang and her colleagues develop a principled analysis of research lesson to represent and articulate instructional improvements systematically. They also further informed and improved their own ongoing LS with teachers locally and the LS work globally. In the chapter "What Knowledge Do Teachers Use in Lesson Study? A Focus on Mathematical Knowledge for Teaching and Levels of Teacher Activity," Clivaz and Ni Shuilleabhain examine the knowledge that teachers used at different levels of teacher activity during a cycle of LS from a combination of perspectives of mathematics knowledge for teaching and teacher activity. They found that various dimensions of mathematics knowledge for teaching can be used at varying levels of teacher activity and at all phases of a LS cycle. In the chapter "Identifying What Is Critical for Learning 'Rate of Change': Experiences from a Learning Study in Sweden," Gumarsson and her colleagues examine how teachers developed their knowledge about identifying objects of learning through learning study, which is an adapted version of LS. Guided by variation theory, the aim of learning study is to make the object of learning identified by teachers available to their students. A case study shows how teachers' knowledge about such critical aspect evolves during the learning study cycles.

Part IV includes seven chapters. In the chapter "Preface: Mathematics Teacher Preparation and Lesson Study," Raymond Bjuland provides further research directions in adapting LS in teacher preparation education, building on a critical analysis of each chapter of Part IV and recent research findings. "Developing Learning Communities Through Lesson Study" by Gunnarsdóttir and Pálsdóttir examines how a LS can enhance pre-service teachers' learning community in Iceland. The participants realized that LS can help develop a trustful, collective collegiate relationship and share beliefs about mathematics teaching and learning. Yet, they struggled with anticipating students' response to tasks when planning research lessons. In the chapter "Lesson Study for Preservice Teachers," Lewis examines how LS can build connections between theory and practice for pre-service teachers through a case study in the USA. She concludes that through participating in a LS cycle, pre-service teachers developed an expansive disposition of *mathematical care*, a repertoire of *pedagogical moves* linked to children's learning, and an expanded *sense of the teaching self*. In the chapter "How Lesson Study Helps Student Teachers Learn How to Teach Mathematics Through Problem-Solving: Case Study of a Student Teacher in Japan," Nakamura examines how to help pre-service teachers to teach mathematics through problem-solving through a case study in Japan. He found that the teachers transferred their teaching from lecture-oriented to a students' thinking-driven approach through a 2-week LS process. In the chapter "Lesson Study in a Mathematics Methods Course: Overcoming Cultural Barriers," based on their exploration of how pre-service secondary teachers can develop productive conversation about mathematics and students' thinking about mathematics in a method course in the USA, Peterson and his colleagues share their experience in the iterative revision of courses to overcome the cultural barriers regarding mathematics and mathematics teaching. In the chapter "Improving Prospective Teachers' Lesson Planning Knowledge and Skills Through Lesson Study," Chen and Zhang examine how a modified LS incorporated in a methods course can develop pre-service teachers' lesson planning skills. They found after experiencing the LS process that participants demonstrated significant improvement in thinking about learning objectives, analysis of content and students, anticipating students' solutions, and sequencing of mathematics tasks. "Lesson Study in Mathematics Initial Teacher Education in England," by Baldry and Foster, examines the potential and challenges of incorporating LS in mathematics initial teacher education (ITE) in England and proposes a theoretical model for using LS in mathematics ITE that takes account of contextual issues and offers ways to make the most of the opportunities available.

Part V focuses on studies on several critical aspects of implementation of LS. In the chapter "Preface: Studies on Key Aspects of Lesson Study," Wood Keith provides a framework of teacher learning through LS through which the major ideas of all chapters in Part V are put together, and then he concludes that lesson study, informed by theory and facilitated by knowledgeable others, could promote teacher learning. "Implementing Mathematics Teaching That Promotes Students' Understanding Through Theory-Driven Lesson Study" by Huang and his colleagues examines how theory-informed LS can foster students' understanding and build connections between theory and practice. In the chapter "Learning while Leading Lesson Study," Lewis presents how novice LS facilitators can develop their facilitating skills at a reasonable level after an 18-month learning experience (studying materials, attending conferences, and leading LS), although they have to cope with issues such as teacher resistance and the use of time due to the countercultural bulwark of teacher learning. In the chapter "Characterizing Mathematics Teaching Research Specialists' Mentoring in the Context of Chinese Lesson Study," Gu and Gu examine how experienced LS facilitators mentored practicing teachers during post-lesson debriefs in China. They identify the strengths and weakness of facilitating practice and propose a mode for LS facilitators to improve their professional skills. In the chapter "Designing and Adapting Tasks in Lesson Planning: A Critical Process of Lesson Study," Fujii examines the process and roles of lesson planning in LS based on a multiple-year project in Japan and identifies the key features of planning a research lesson. In the chapter "A Critical Mechanism for Improving Teaching and Promoting Teacher Learning During Chinese Lesson Study: An Analysis of the Dynamics Between Enactment and Reflection," Huang and colleagues examine the dynamics of enactment and reflection during the iterative process of LS. In the chapter "Race to the Top and Lesson Study Implementation in Florida: District Policy and Leadership for Teacher Professional Development," Ahkiba and her colleagues examine how district policy and leadership characteristics are associated with the levels of LS implementation on scale. In the chapter "The Use of Lesson Study to Unpack Learning Trajectories and Deepen Teachers' Horizon Knowledge," Suh and her colleagues examine how a coach-facilitated, vertical LS (including teachers from multiple grade levels) can promote teachers' use of learning trajectory and contribute to the expansion of teachers' horizon content knowledge through investigating teaching of a similar rich task across grades.

The two commentary chapters of Part VI provide insights into understanding chapters and implications of the book from different perspectives. In the chapter "A Western Perspective," building on his long-standing "classroom action research" tradition, John Elliott provides a Western perspective on major themes and issues emerging from the book and discusses fundamental issues concerning the roles of academic experts, teachers in creating a "knowledge platform" and curriculum development, and the use of learning theories to inform LS and methodology issue of studying lesson study globally. In the chapter "An Asian Perspective," building on her development of learning study over a decade, Munling Lo provides insight into the major ideas posed in the volume. She emphasizes the importance of considering context such as school system and culture when adopting LS and developing theories for guiding LS, analyzing research lesson, and dealing with the objects of learning.

3 Contributions and Limitations of the Book

This book makes its unique contributions to the field of LS due to the following features: comprehensiveness, representativeness, and richness. First, it includes all of the following aspects of LS: (1) theorizing LS, (2) researching into LS, (3) origins of LS, (4) use of LS with in-service teachers, and (5) adaptation of LS with pre-service teachers. So, this book will be valuable for both researchers and practitioners. Second, the origins and adaptation of LS are presented systematically. Beyond the well-known Japanese LS, the book for the first time, introduces the origin, development, and cultural roots of LS in China, which may provide an alternative perspective about what LS may look like (essence of LS). For example, what are the benefits and weaknesses if a LS requires repeated teaching and requires knowledgeable others' involvement, with a goal of perfecting a research lesson? In addition, the adaption and/or enrichment of LS in different cultural settings can enrich and broaden the extensions of LS. For example, some questions emerged: Does LS need specific guiding theories? Should LS include pretest and posttest? Should LS focus on developing sharable instructional products? Third, this book particularly places an emphasis on research perspectives of LS from providing theoretical lens to demonstrating research methods and aspects. Any researchers who are interested in doing studies on LS can benefit from reading this book.

However, there are certain limitations of the book. It does not address how the availability of various technologies, particularly online conference systems, may reshape and strengthen LS to make it doable on larger scale. It does not explicitly address how LS may help students in poverty to learn mathematics or how to use LS as a tool to address equity of student learning opportunity.

It is not our intention to cover all possible issues or provide a prescription about how to do research on or how to conduct a LS. Rather, it is our goal to open the kaleidoscope to appreciate the richness and diversity of LS and provide a ground for readers to develop their own research agendas to advance LS as a promising field both theoretically and practically.

References

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*(3), 218–236.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*, 947–967.

da Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6*(2), 169–181.

Dudley, P. (2014). *Lesson study: Professional learning for our time* (pp. 29–58). London/New York: Routledge.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 16*(1), 49–65.

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 2–18.

Hart, L. C., Alston, A. S., & Murata, A. (2011). *Lesson study research and practice in mathematics education: Learning together*. New York: Springer.

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298. Singapore: World Scientific.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher developers, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*(4), 439–587.

Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM Mathematics Education, 48*, 425–439.

Inprasitha, M., Isoda, M., Wang-Iverson, P., & Yeap, B. H. (2015). *Lesson study: Challenges in mathematics education*. Singapore: World Scientific.

Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.). (2007). *Japanese lesson study in mathematics: Its impact, diversity and potential for educational improvement*. Singapore: World Scientific.

Kieran, C., Krainer, K., & Shaughnessy, J. M. (2013). Linking research to practice: Teachers as key stakeholders in mathematics education research. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 361–392). New York: Springer.

Larssen, D. L. S., Cajkler, W., Mosvold, R., Bjuland, R., Helgevold, N., Fauskanger, J., et al. (2018). A literature review of lesson study in initial teacher education. Perspectives about learning and observation. *International Journal for Lesson and Learning Studies, 7*(1), 8–22.

Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: RBS Publishing.

Lewis, C. (2016). How does lesson study improve mathematics instruction? *ZDM Mathematics Education, 48*, 571–580.

Lewis, C., & Lee, C. (2017). The global spread of lesson study: Contextualization and adaptations. In M. Akiba & G. K. Letendre (Eds.), *International handbook of teacher quality and policy* (pp. 185–203). New York: Routledge.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for Research in Mathematics Education, 48*(3), 261–299.

Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly owing river. *American Educator, 22*(4), 12–17.

Lewis, C. C., Perry, R., & Hurd, J. (2009). Improving mathematic instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*, 285–304.

Moss, J., Messina, R., Morley, E., & Tepylo, D. (2012). Building the sustaining professional collaborations: Using Japanese Lesson Study to improve the teaching and learning of mathematics. In J. M. Bay-Williams & W. R. Speer (Eds.), *Professional collaborations in mathematics teaching and learning: Seeking success for all* (pp. 297–309). Reston: National Council of Teachers of Mathematics.

Murata, A. (2011). Conceptual overview of lesson study: Introduction. In L. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 1–12). New York: Springer.

Murata, A., Bofferding, L., Pothen, B., Taylor, M., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics lesson study. *Journal for Research in Mathematics Education, 43*, 616–650.

Quaresma, M., Winsløw, C., Clivaz, S., Ponte, J. P., Ni Shuilleabhain, A., & Takahashi, A. (Eds.). (2018). *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Runesson, U. (2015). Pedagogical and learning theories and the improvement and development of lesson and learning studies. *International Journal for Lesson and Learning Studies, 4*(3), 186–193.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*, 513–526.

Yoshida, M. (2012). Mathematics lesson study in the United States: Current status and ideas for conducting high quality and effective lesson study. *International Journal for Lesson and Learning Studies, 1*, 140–152.

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How Does Lesson Study Work? Toward a Theory of Lesson Study Process and Impact

Catherine Lewis, Shelley Friedkin, Katherine Emerson, Laura Henn, and Lynn Goldsmith

Contents

Abstract This chapter proposes a theoretical model of the impact of lesson study. Outcomes addressed include teacher outcomes (e.g., knowledge and beliefs), professional learning norms and routines, instructional routines and tools, and student learning outcomes. Four theoretical perspectives are used to examine lesson study impact: knowledge integration environment, self-determination theory, self-efficacy theory, and pedagogies of practice. The chapter also examines all four phases of the lesson study cycle – study, plan, teach, and reflect – and for each phase identifies major goals, challenges, strategies to overcome challenges, and relevant theoretical perspectives. In addition, reflection questions for each phase are proposed, which are designed to support educators and researchers to reflect on the effectiveness of their work during each phase. The chapter is based on 20 years of observations of lesson study and is intended to spark further conversation about the process and impact of lesson study.

Keywords Lesson study · Theory-driven lesson study · Knowledge integration environment · Self-determination theory · Self-efficacy theory · Rehearsals · Pedagogies of practice

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_2

1 Introduction

We are often asked “Does Lesson Study work?” – a question that seems a lot like asking “Does teaching work?” (Or, “Does marriage work?”). It all depends on your goals and how you approach them. The goals of lesson study as it is practiced in Japan are much broader than is often appreciated in the West. For example, in Japan, lesson study is expected not only to improve teaching but also to strengthen professional community among teachers (Lewis et al. 2010; Sato 2008), help teachers make sense of changes in national standards (Takahashi and McDougal 2014), build more coherent instruction across classrooms (Matsuzawa Elementary School 2011), and connect individual teachers’ daily instruction to the shared long-term vision for students embraced by the school (Takahashi and McDougal 2016).

This chapter lays out a theoretical model of lesson study that encompasses outcomes for teachers (individually and collectively) and students. We draw on theoretical perspectives from knowledge integration environments (Linn et al. 2004), self-determination theory (Deci and Ryan 1985), self-efficacy theory (Bandura 2001), and pedagogies of enactment (Grossman et al. 2009a, b) to build our theoretical model. Our chapter is organized around the phases of the lesson study cycle, and we examine each phase through the lens of theory to understand the key goals and challenges of each phase and to consider strategies to overcome key challenges. We also propose reflection questions specific to each phase, rooted in the theoretical perspectives on each phase and designed to help lesson study researchers and practitioners gauge the progress of their work.

1.1 What We Mean by “Lesson Study”

Lesson study is a translation of the Japanese term “jugyou kenkyuu,” and it is a professional inquiry approach practiced in more than 90% of schools in Japan (National Education Policy Research Institute 2011). Although lesson study is sometimes misconstrued as focusing primarily on lesson planning, it consists of four stages of cyclical activity, as shown at the left side of Fig. 1. In Japan, lesson study cycles typically take place in the context of school-wide Collaborative Lesson Research, in which lesson study teams throughout a school build and share knowledge around a research theme that captures long-term goals for students and testable ideas about how to reach those goals (Takahashi and McDougal 2016). Some eminent researchers lay out six phases of lesson study (Fujii 2016; Takahashi and McDougal 2016), in order to emphasize goal-setting at the outset and to separate the post-lesson discussion and subsequent reflection on learning, but we have opted for simplicity.

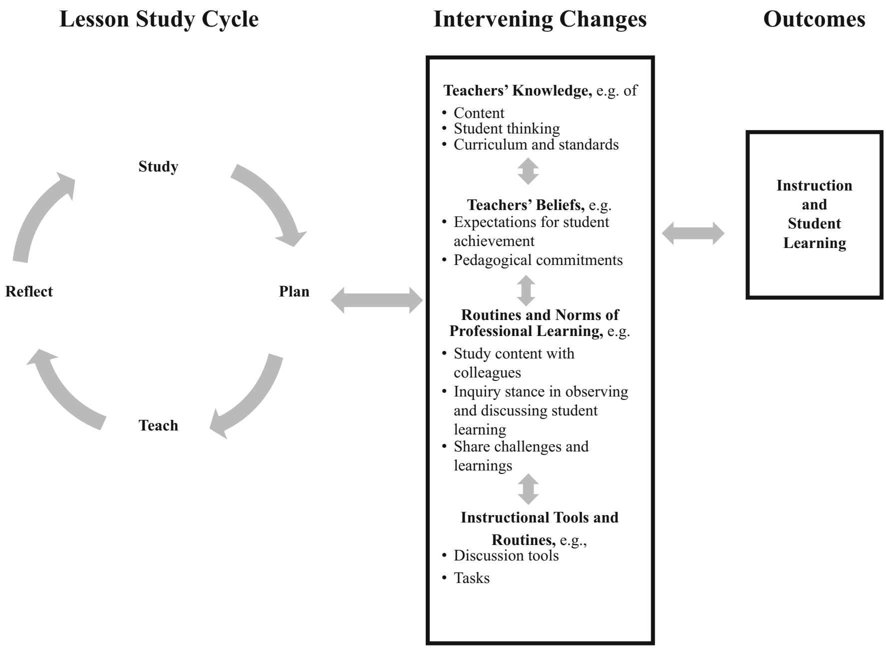

Fig. 1 Theoretical model of lesson study impact

2 Theoretical Model of Lesson Study

Figure 1 posits that lesson study can influence instruction and student learning through intermediate changes in teachers' knowledge and beliefs, professional norms and routines, and instructional materials; in the second part of this section, we provide research-based examples of each type of change. First, we describe four theoretical perspectives that we have found useful in understanding the impact of lesson study. These perspectives focus on four different types of outcomes: knowledge, motivation, self-efficacy, and capacity to enact knowledge (e.g., of content and teaching) in the classroom.

2.1 Knowledge Integration Environment

The knowledge integration environment is a perspective on learning that grew out of Piagetian theory applied to the domain of science learning, and it focuses on the conditions that allow people to develop increasingly sound understandings of complex phenomena. Knowledge integration theory posits that four sequential processes enable development of powerful, integrated knowledge: making one's current ideas visible ("elicit"), encountering new ideas ("add"), developing criteria to compare and distinguish ideas ("distinguish"), and using reflection to solidify and integrate ideas ("consolidate") (Linn et al. 2004, 2013; Linn and Eylon 2011).

2.2 Self-Determination Theory

Self-determination theory posits that human beings have three basic psychological needs – autonomy, competence, and a sense of belonging – and that intrinsic motivation to learn develops to the extent that these needs are met (Deci and Ryan 1985; Ryan and Deci 2000). By extension, lesson study groups that meet teachers' needs for autonomy, competence, and belonging will be valued by teachers and will elicit teachers' intrinsic motivation to engage in the hard, ongoing work of improving instruction.

2.3 Self-Efficacy Theory

Self-efficacy is a belief in one's ability to succeed, and it influences motivation: "It is partly on the basis of efficacy beliefs that people choose what challenges to undertake, how much effort to expend in the endeavor, how long to persevere in the face of obstacles and failures, and whether failures are motivating or demoralizing" (Bandura 2001). In general, people experiencing higher levels of self-efficacy feel more motivated to take on – rather than shy away from – challenging experiences. Both personal experience and indirect experience (observing others) influence efficacy expectations. To the extent that people increase efficacy beliefs through participation in lesson study, they may also increase motivation to take the risks entailed in trying to improve instruction.

2.4 Pedagogies of Practice

Teaching is a practice: Knowledge for teaching is enacted with other people. In their investigation of "pedagogies of practice," Grossman and colleagues identified three pedagogies common to professional training for the "relational" careers they studied (teaching, clergy, clinical psychology): (1) using representations of practice, (2) decomposing practice into components, and (3) engaging with approximations of practice such as rehearsal or microteaching to peers (Grossman et al. 2009a, b). For educators, they note that "Taking clinical practice seriously will require us to add pedagogies of enactment to our existing repertoire of pedagogies of reflection and investigation" (Grossman et al. 2009a, b, p. 274).

2.5 Evidence of Lesson Study Impact on Pathways in Theoretical Model

While a full literature review is beyond the scope of this paper, we briefly note some of the evidence that exists for the pathways of impact shown in Fig. 1. A number of our examples are drawn from a randomized, controlled trial of lesson study with mathematical resource kits on fractions (hereafter "Fractions Lesson Study RCT") (Lewis and Perry 2015, 2017).

Lesson Study Impact on Student Learning The Fractions Lesson Study RCT found a significant impact on students' mathematical proficiency in fractions (Lewis and Perry 2017). A case study of lesson study in language arts in a turnaround school likewise found a significant impact on standardized test scores in language arts (Collet 2017). A multi-year case study of a school practicing school-wide lesson study in mathematics showed an increase in mathematics achievement nearly three times that of the district as a whole (Perry and Lewis 2010) (Lewis et al. 2006).

Lesson Study Impact on Teachers' Knowledge The Fractions Lesson Study RCT also produced a significant increase in teachers' knowledge of fractions (Lewis and Perry 2017). Research has found an impact of lesson study on other aspects of teachers' knowledge including knowledge about tasks (Krystal 2018) and pedagogical content knowledge of mathematics related to students and teaching (Aoibhinn 2016).

Lesson Study Impact on Teachers' Beliefs The Fractions Lesson Study RCT showed a significant impact on teachers' expectations for student achievement (measured by items such as "By trying a different teaching method, I can significantly affect a student's achievement") (Lewis and Perry 2015). Other research has shown impact of teachers' collaborative, lesson-focused work on their beliefs about the value of using errors (Pernilla and Henrik 2018).

Lesson Study Impact on Routines and Norms of Professional Learning The Fractions Lesson Study RCT showed an impact on collegial learning effectiveness (measured by items such as "I have learned a great deal about mathematics teaching from colleagues") (Lewis and Perry 2015). A shift in professional learning routines to consider students' interaction with the content, rather than only the content itself, has also been documented in lesson-focused collaborative professional learning in Sweden (Pernilla and Henrik 2018). Similarly, cross-national lesson study work conducted jointly by Japanese and Iranian educators led the latter to expand their study of curriculum to include consideration of students' interactions with the curriculum (Sarkar Arani 2017).

Lesson Study Impact on Instructional Tools and Routines The Fractions Lesson Study RCT showed an impact on the instructional tasks used to teach fractions, with about half of the participating lesson study teams choosing to adopt a Japanese task they saw in classroom lesson videos; teams that used this task produced an added benefit in student learning over the experimental group as a whole (Lewis and Perry 2017). A case study of Swedish teachers working to improve decimal instruction documented an expansion of their ideas about assessment to include formative assessment and informal observation during lessons (Pernilla and Henrik 2018). There is some evidence that teachers can share instructional tools across countries (Runesson and Gustafsson 2012).

3 Goals and Indicators of Effectiveness for Each Lesson Study Phase

While the research examples cited in the prior section illustrate the potential of lesson study to influence teachers' knowledge and beliefs, instructional tools and routines, and student learning, lesson study does not always achieve these outcomes, even when they are intended. The remainder of this chapter analyzes each step of the lesson study cycle, using the theoretical perspectives introduced in Sect. 1 to illuminate the goals and challenges of each phase and to identify strategies that may support success at that phase. Our thinking about the goals, challenges, and strategies at each phase is heavily informed by our observation of lesson study groups that took part in two randomized trials of fractions lesson study and our ongoing work with 13 US schools involved in Collaborative Lesson Research (Takahashi and McDougal 2016). Figure 2 summarizes the goals and theoretical connections of each phase of the lesson study cycle, which is discussed in more depth in this section.

3.1 Goals and Key Components of Phase 1 (Study)

Phase 1: Study
The Study Phase focuses on two major goals:

- To establish a lesson study team that is valued by its members and has reasonably efficient processes for learning together
- To establish the topical focus of the lesson study cycle and build team members' knowledge about the topic

3.1.1 Study Phase Goal 1: Build a Valued Team with Efficient Processes for Learning

Teachers are busy. For teachers to experience lesson study as a worthwhile use of their time and begin to value it, a lesson study team needs to build effective processes for learning. The processes for learning established during the Study Phase have

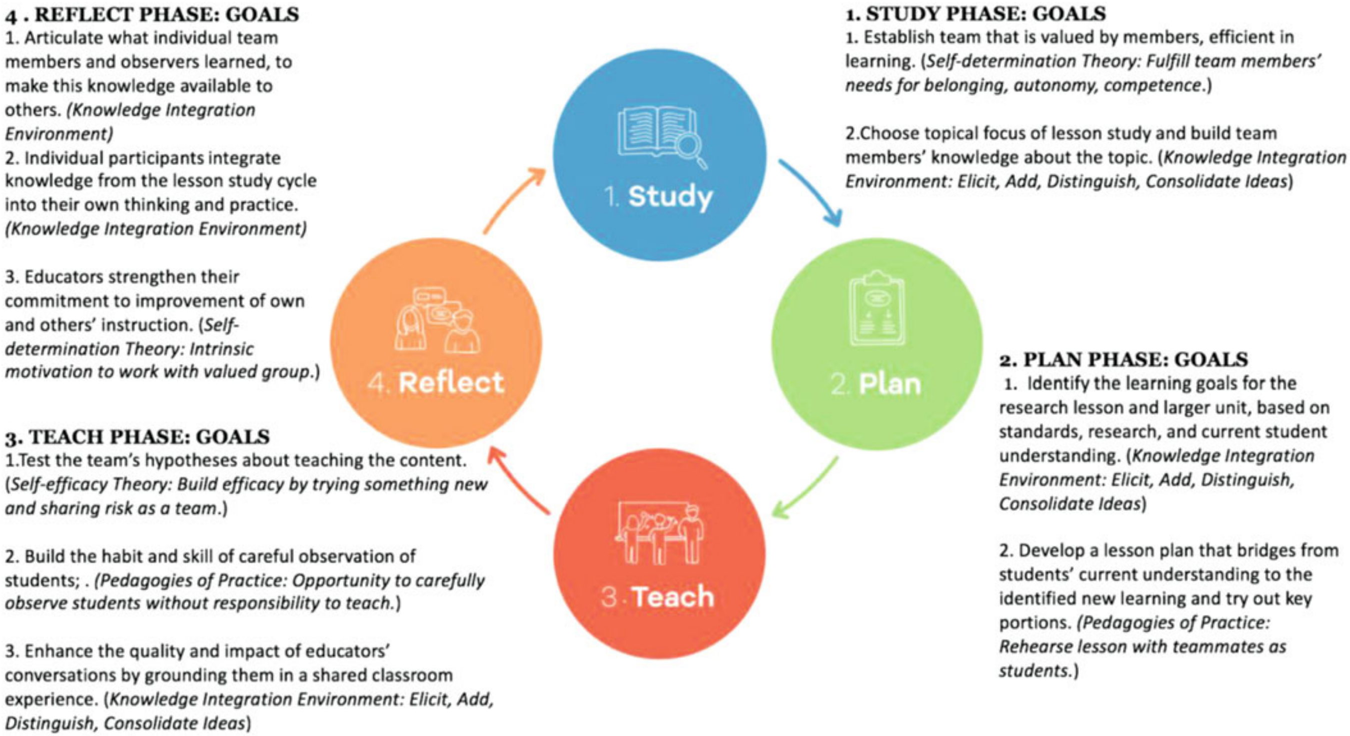

Fig. 2 Goals at each lesson study phase, connection to theory

both short-term and long-term implications. In the short term, these processes allow a lesson study team to work efficiently – for example, to keep track of what they learn and decide at each meeting and to quickly access this information at future meetings, in order to build upon it.

In the long term, the team needs to become a group that is valued by teachers, thus eliciting their commitment to continued learning. Self-determination theory suggests that teams that meet members' fundamental human needs for autonomy, competence, and belonging are best positioned to elicit members' commitment and motivation to do the hard work of improving instruction. Team members who feel supported by the team and committed to its goals will be able to do the hard work of learning – such as admitting gaps in their knowledge and revealing challenges they are facing in the classroom.

In our experience, some key challenges in building valued, efficient teams are:

- Managing time – for example, allocating sufficient time to important conversations and avoiding time-consuming sidetracks
- Managing participation – for example, making sure that all team members feel included and no voice habitually dominates
- Managing learning – for example, making sure that ideas from prior meetings are carried forward and built upon

Some key strategies often used to build valued, efficient teams include the following.

1. *Develop norms and revisit them at each meeting*, to reflect on how the teamwork is going and what you might need to change. Your school or district may already have an established process to do this. If not, you can find processes to do this at http://lessonresearch.net/prepare-step/build-and-practice-norms/.
2. *Adopt meeting tools*, *such as an agenda and meeting notes*, and agreements on how to use these. You may already have systems in place that enable your team to develop an agenda for each meeting in advance, keep and carry forward notes, and update the Teaching-Learning Plan that will inform your research lesson. If not, you can find templates for agendas and notes at http://lessonresearch.net/prepare-step/adopt-an-agenda/ and a Teaching-Learning Plan template at http://lessonresearch.net/study-step/access-tlp/.
3. *Establish roles (*e.g., *notetaker, facilitator) and rotate them each meeting*, so that all team members experience what it's like to manage that facet of the team's work. Many teams start out with a designated facilitator – perhaps someone who brings particular content knowledge or is experienced with the lesson study process. Over time, it may be optimal for teams to rotate responsibility for facilitation, so that all team members gain leadership experience. (Accessing content knowledge does not necessarily need to be joined with the facilitation role – e.g., it can be one standing item on the agenda.) If your team is not familiar with establishing and rotating roles, you can find examples here at https://lessonresearch.net/prepare-step/agree-roles-expectations/.

4. *Develop a shared research theme* that articulates your long-term vision for students. Teachers create it – often as a whole school faculty – by considering the qualities they want their students to have at graduation (or several years down the road), the qualities of their students today, and a gap between those two sets of qualities that they really want to work on as educators. The research theme – whether developed by the whole school or just one lesson study team – helps reconnect educators with the goals that are really vital to them. You can find a suggested process for developing the research theme here at http://lessonresearch.net/study-step/develop-research-theme/.

The research theme provides an agreed-upon long-term vision for the lesson study team's work, and also a starting point for the second goal of the Study Phase, discussed next.

3.1.2 Study Phase Goal 2: Establish a Focus of Study and Build Team Members' Knowledge About It

The research theme establishes a broad, long-term goal, such as building students' perseverance and capacity as problem-solvers. The lesson study team also needs to identify the specific discipline (e.g., mathematics) and topic (e.g., fractions) where they will situate their investigation of the research theme and to study this topic in some depth. This learning is often called "kyouzai kenkyuu" in Japanese – literally, study of teaching materials (Yoshida and Jackson 2011; Watanabe et al. 2008; Takahashi et al. 2005; DosAlmas and Lewis 2017). But, since Japanese curriculum guides provide more information on student thinking and on the content trajectory than counterpart US materials (Lewis et al. 2011), it is probably better to translate this term as "study of curriculum and content." In addition, some important knowledge for this phase (and for the Plan Phase) will come from the educators on the team as they solve tasks themselves, anticipate student thinking, and share solution methods that illuminate different ways of thinking about the mathematics (or other subject matter).

In our experience, challenges that arise as teams identify their focus and study it during the Study Phase may include:

- Premature focus on a single lesson, rather than on the larger unit and trajectory of which it is part
- Failure to go beyond the team's current knowledge
- Study of low-quality resources that do not illuminate the content or what is known about its teaching and learning (e.g., attention-grabbing activities found online that may be side trips from the key mathematical trajectory)
- Choice of a focus that is not of interest to (some) team members

Several strategies may enhance the team's likelihood of success with Goal 2 (establish a focus of study and build knowledge around it).

1. *Provide access to high-quality content resources and frameworks*. High-quality content resources can enable a team to go beyond what team members currently know. For example, the TRU Framework (**t**eaching for **r**obust **u**nderstanding) and TRU Conversation Guide (see chapter "Teaching for Robust Understanding with Lesson Study" this volume for details) can provide important resources as teachers develop their research theme and focus their lesson study work, because they support broad thinking about the multiple determinants of students' mathematical development (Schoenfeld et al. under review). (As discussed below, the TRU materials can also provide important reflection tools during lesson planning and post-lesson discussion.) High-quality resources on the specific topic to be studied (e.g., fractions) and the teaching-learning strategies to support the team's long-term goals (e.g., what is known about building agency) are also essential. Frameworks that illuminate the trajectory of the content over time (e.g., Clements and Sarama 2004), concise summaries of research (e.g., Shifter et al. 2010; Van de Walle et al. 2009), and research-based curriculum materials are all likely to be useful.
2. *Create the environment for knowledge integration*. The knowledge integration perspective reminds us that learning is not just a matter of adding ideas: it also depends on eliciting team members' existing ideas and sparking a need to reexamine and refine ideas. Supportive team dynamics (see Goal 1) will be essential to eliciting ideas, since team members need to feel comfortable revealing their thinking, speaking up when they don't understand or agree, and so forth. Likewise, the right activities are essential to elicit team members' thinking – for example, a mathematics task that teachers can solve and discuss that illuminates new perspectives on the content.
3. *Ask each team member to identify something they would like to learn*. Even if the topic of the lesson (e.g., multiplication of fractions) is not a topic every team member expects to teach, every team member can and should feel a stake in the team's learning. A focus on some dimension of the larger research theme – e.g., examining a strategy to support student agency – will help all team members take an authentic interest in the work. Some lesson study teams ask each team member to develop an inquiry question related to the team's work. For example, a team member may ask how to use reflective mathematics journals to support student agency and integrate this question into the lesson study cycle.

Reflection Questions

1. Did our team develop efficient processes for learning together? For example, were we able to use our time in a worthwhile fashion and carry forward knowledge effectively from one meeting to the next?
2. Did our team develop inclusive processes that value the thinking and learning of all team members?
3. Did we arrive at a research theme and content focus that we are genuinely curious about? Does every team member have something they want to learn?
4. Did we access major research and standards/frameworks to learn about this content and its teaching-learning? Specifically, what did we learn?

5. Did we access knowledge from our own experience, teammates, and students to learn about this content and its teaching-learning? Specifically, what did we learn?

3.2 Goals and Key Components of Phase 2 (Plan)

Phase 2: Plan
The major goals of the Plan Phase are:

- Identify the learning goals for the research lesson and larger unit, based on standards, research, and students' current understanding.
- Develop a lesson plan that bridges from students' current understanding to the identified new learning and think through or try out key portions.

During the Plan Phase, teams continue to investigate the content by studying research, standards, curriculum, student thinking, and tasks (e.g., solving and discussing tasks as adults). This work culminates in a Teaching-Learning Plan (TLP) for a research lesson (explicitly situated within a larger unit and trajectory of learning). The TLP differs from many everyday lesson plans in its focus on providing a rationale for the lesson design and anticipating student responses and how they will be used to build the new learning of the lesson. The Teaching-Learning Plan also specifies the data that will be collected during the research lesson to investigate the students' learning and the team's hypotheses about lesson design. Ideally, the team also conducts a mock-up lesson in which the instructor teaches the lesson to the other team members (who take the role of students). The mock-up lesson allows the team to experience the lesson from the student point of view and the instructor to "rehearse" by trying out the specific teacher questions and moves the team has planned (Lampert et al. 2013). Of all the activities in the Plan Phase, the mock-up lesson most closely approximates the experience of actually teaching the research lesson; it provides an opportunity to fine-tune plans, based on the team's simulation of the lesson they have planned. Mock-up lessons are common in Japan and have recently been encouraged in the US by Akihiko Takahashi, to help teachers specify and test their instructional moves more carefully before the lesson.

3.2.1 Challenges of the Plan Phase

Key challenges of the Plan Phase include:

- Jumping into lesson planning, without first considering the unit design and long-term content trajectory, and clearly identifying the lesson's role within the larger unit and trajectory.

- Failing to identify the new learning that will occur during the lesson – for example, describing the lesson goal as "doing ———," rather than identifying what students will *learn* from the activity.
- Planning the lesson around what the *teacher* will do, rather than around what the students will think, do, feel, and learn.
- Neglecting to incorporate learning from the Study Phase into the Teaching-Learning Plan – for example, neglecting to incorporate what is known about building agency or about learning fractions multiplication.
- Failing to grasp students' current knowledge and to design the lesson based on that knowledge.
- Anticipating student thinking in enough depth, accuracy, and breadth to write a plan that is likely to promote learning.
- Lopsided planning, focused on one element of the lesson (often the launch).
- Focus on logistical elements of the lesson, rather than on the key student experiences that will produce tension or contradiction and drama of breakthrough.
- Team members divide responsibility for writing the Teaching-Learning Plan in a way that fails to build all team members' learning.
- The data collection plan is not well-connected to the lesson goals.
- Gaps in the model of teaching-learning underlying the plan – for example, the plan may anticipate certain important student responses but not plan the teacher questions and moves that will allow other students to grasp these ideas.

Theories of knowledge integration environment and pedagogies of enactment (such as rehearsals) are both important to understanding the Plan Phase. Unit and lesson planning should repeatedly surface team members' knowledge and beliefs about pedagogy and help team members' ideas bump up against each other and against ideas from research. The need to negotiate a joint research lesson plan should spark refinement and reflection on ideas. Enactment of ideas during a mock-up lesson should provide more authentic and "proximate" engagement with core elements of teaching than simply talking about teaching; the mock-up lesson should therefore help team members to elicit, compare, and refine ideas about classroom practices such as teacher questioning, board organization, etc.

3.2.2 Strategies to Support the Plan Phase

1. *Use a template to write the Teaching-Learning Plan*. A well-designed template will remind your team to discuss important elements such as the role of this lesson in the larger unit, the experiences that will lead to "aha's" for students, the connection to the standards and larger trajectory of learning, and so forth. You can access one such template here: http://lessonresearch.net/study-step/access-tlp/.
2. *Investigate your students' thinking*. In addition to learning what researchers know about the topic, it's important to know what your own students know coming into

the lesson and what misconceptions they are likely to struggle with. Interviews of current students (or students who studied the material last year), exploration of student work that you bring to the lesson study group for examination, examination of existing learning studies (see other chapters in this volume), and attention to students' thinking as they experience the prior lessons in the unit can all be valuable sources of information in your lesson planning.

3. *Predict the responses of the specific students who will take part in the lesson.* Our colleague Tad Watanabe notes that if we don't make predictions, we won't be surprised. Predicting how the specific students in the class will each respond to the lesson task gives us the opportunity to check out how well we know individual learners. It also enables us to address, in the Teaching-Learning Plan, how each student will move from their initial response and grasp the key ideas of the lesson.
4. *Revisit the TRU Framework.* The TRU framework provides a useful tool to think about whether the Teaching-Learning Plan has covered important bases – for example, whether the lesson design builds in access for every learner, nurtures student ownership and identity, and allows the teacher to understand and use student thinking.
5. *Ask a knowledgeable outsider to review the Teaching-Learning Plan*. By sharing the Teaching-Learning Plan with a knowledgeable outsider when the lesson task has been chosen – but there is still time to revise the plan – the team creates a valuable opportunity to expand its knowledge. (The team might consult the same knowledgeable other consulted during the Study Phase, just after the topic is selected.) An outsider who has expertise related to the content and its teaching can often pose questions that will help the team sharpen thinking about the lesson goals or suggest small changes that may make big changes in student learning – for example, changes in the choice of number or the presentation of the task or clarification of how and why students are expected to change their thinking.
6. *Conduct a mock-up lesson.* The mock-up lesson, with team members taking the role of students, allows the team to notice things that might only be noticed in a setting that closely approximates classroom practice. For example, the instructor might notice that a question needs to be posed differently, or team members might notice that the organization of ideas on the board is confusing.

Reflection Questions

1. Does the Teaching-Learning Plan consolidate knowledge from the Study Phase – for example, knowledge about the content, student thinking, standards, and what is known about teaching-learning? Does it address the long-term research theme as well as the specific goals of the lesson and unit?
2. Does the Teaching-Learning Plan propose a plausible bridge from students' current knowledge to the desired new learning? Can a reader understand how varied students will respond, and how they will grow during the lesson?
3. Does the Teaching-Learning Plan identify what our team hopes to learn from the research lesson and what data will be collected to inform our learning?

4. Did we try the task ourselves and then discuss our approaches, in order to anticipate student thinking?
5. Did we try a "close approximation to practice," such as a mock-up lesson, to help us think through lesson elements that might not come out in team discussion – for example, specific teacher questions, visual arrangement of the board, time allocation to each phase of the lesson, and lesson summary that will come from students?
6. Did writing the Teaching-Learning Plan help team members develop their thinking about the subject matter and pedagogy? What insights were gained by individual team members? What might we do differently next time in our planning and writing to increase every team member's opportunity to learn something useful to them?

3.3 Goals and Key Components of Phase 3 (Teach)

Phase 3: Teach

The major goals of the Teach Phase are:

- To test the team's hypotheses about teaching the content
- To build the habit and skill of careful observation of students
- To enhance the quality and impact of educators' conversations by grounding them in a shared classroom experience

During Phase 3, the research lesson is taught by one team member to their students, with other team members observing students and collecting data agreed upon by the team. If observers outside the team attend, the research lesson may be immediately preceded by a pre-lesson discussion in which team members present the Teaching-Learning Plan, have observers try out the lesson task, and review observation guidelines. Some teams choose to have two different team members teach the lesson, with sufficient time in between (at least a day) for revision of the lesson based on what was learned during the first teaching.

Though brief, the "teach" phase is the crux of the lesson study cycle, when the team's hypotheses about teaching the particular content are brought to life in the research lesson. Lesson enactment reveals the instructor's and team's theories of learning. For example, is it assumed that students will be able to persevere in solving a challenging, open-ended problem or that they need the problem in small installments? How are individual work, group work, and whole class discussion assumed to contribute to student learning? What models and visual information are hypothesized to be useful, and how are they sequenced and presented? When one student voices an important idea, is it assumed that other students have also grasped that idea? The research lesson also reveals the team's knowledge of the students: How close is the fit between the plan designed by the team and the knowledge, interests, and dispositions students actually bring to the task?

3.3.1 Challenges of the Teach Phase

Akihiko Takahashi notes that "Teaching is easy to talk about but hard to do." The Teach Phase is important (and hard) for precisely that reason – that talking about the lesson and planning for it may not adequately prepare for teaching it. Yet the place where the team's study and planning bump up against reality is a rich site for knowledge integration; a great deal can be learned from unsuccessful, as well as successful, lessons. For example, team members may learn that students did not display key knowledge or dispositions needed to grapple successfully with the task. Team members may discover that the students "successfully" solved the problem but did not learn anything from solving it (Mills College Lesson Study Group 2005). Team members may find that the planned lesson supported learning by several students, but not the vast majority of the class. Observations may also reveal variations in student responses that provide hints for improvement of the lesson in the future; for example, one student's way of seeing and counting a geometric pattern may provide insights that could help classmates solve the problem in a future lesson (Lewis et al. 2009), and the questions posed by a student in one group might constitute valuable prompts for other groups in a future lesson.

Key challenges of the phase include:

1. *Student thinking is not made visible by the research lesson*. During a research lesson, it is typical for at least some of the observers to follow particular students from the beginning to the end of the lesson and to understand how these particular students changed their thinking over the course of the lesson and what catalyzed change (or what acted as a barrier). But what if the student's thinking is never made visible – for example, through their writing, actions, or speech? Ideally, the lesson should be designed to allow a careful observer to discern much about a student's experience; some lesson study groups also allow observers to briefly question students after the lesson is concluded.
2. *The data collection plan misses key elements of teaching-learning*. The data collection points laid out by the lesson study team may miss certain data that is important to understanding the lesson – particularly if the lesson does not unfold as expected.
3. *Observers interfere in student learning*. Observers may forget that they are not supposed to help students or may unwittingly interfere with learning – for example, by blocking a student's view of the board or of a resource such as an anchor chart.
4. *Observers do not collect data on student learning*. Data collection on students is a new experience for many teachers, and they may be reluctant to lean in to hear students' conversations and record the content of their work. In large public research lessons, the venue may be too crowded to permit close observation of

students. In either case, teachers who watch from the periphery will not gain a good understanding of students' experiences of the lesson.

5. *The taught lesson diverges greatly from the planned lesson*. Team members may experience great disappointment when the instructor of the research lesson departs greatly from what the team collaboratively planned. This disappointment, and the conflict it often engenders, is a problem. However, diverging from the lesson plan is not necessarily a problem. Akihiko Takahashi encourages research lesson instructors to study, plan, and collaborate as carefully as possible in order to internalize the ideas behind the lesson plan but to "throw out the lesson plan and teach while watching the students' eyes" once the research lesson commences. If the carefully developed lesson plan does not feel right within the classroom, this provides important feedback to the instructor and team. Perhaps the team had a poor grasp of the students' prior knowledge, in which case the instructor's responsibility is now to teach a lesson that will best support students' learning. Or perhaps the team developed a sound plan but did not adequately support the instructor to understand and practice the specific moves that would enable the lesson to unfold successfully. In either case, these are important learnings for the team.

3.3.2 Strategies to Support the Teach Phase

Several of the strategies below relate closely to the pedagogies of practice framework, since they represent ways to look at the component parts of practice (such as the lesson plan) outside of classroom practice and to rehearse elements of classroom practice (e.g., particular teacher moves or questions) in a mock-up lesson. Other strategies relate to creation of a knowledge integration environment by creating opportunities to study the lesson plan and relate it to the observed lesson and to bring in knowledge from an outside commentator.

1. *Final review of the Teaching-Learning Plan, with a focus on data collection.* At the end of the Plan Phase, as team members finalize the Teaching-Learning Plan, it is important to review it from the perspective of the data to be collected and ask whether the collected data will adequately capture the story of students' learning during the lesson. If not, lesson elements can be adjusted. For example, the team may add a writing prompt or application problem to the end of the lesson that reveals student thinking, add a poll or partner chat that prompts students to share their current thinking at key points in the lesson, or build in a few minutes for observer-student interviews after the close of the lesson.
2. *Mock-up lesson or other "approximation of practice."* The opportunity to try out the lesson plan with team members serving as "students" can support the lesson instructor to try out specific questions and moves that may be unfamiliar but important to making the team's plan work.
3. *Preparation of students.* Students should know in advance that they will be observed and should understand that the observations are intended to help

make lessons better, by carefully studying how the lesson works for students. Students thus enlisted usually enjoy being allies in improving lessons!

4. *Pre-lesson discussion with public review of observation guidelines*. Even if the observers are experienced in lesson study, they should be reminded of the observation guidelines, such as refraining from side conversations and from helping students. During the pre-lesson discussion, the specific data to be collected by the team should also be reviewed.
5. *Monitoring and encouragement of observers during research lesson*. It may be necessary for a designated facilitator to encourage careful observation by participants who are hanging back away from students, to intervene if an observer starts to teach students, or to quiet side conversations. These challenges may also be raised as a general issue at the post-lesson discussion.
6. *Inclusion of an experienced final commentator*. An experienced final commentator typically provides a good model of data collection and knows how to collect data that reveal student learning broadly across the class – even if the team's data collection plan is not fully adequate. Likewise, experienced observers can show by the example of their observation what it means to closely study students.

Reflection Questions

- Did the Teaching-Learning Plan provide good guidance about data collection? Did data collection focus on the key elements our team wants to learn about, as well as a broad picture of student learning during the lesson?
- Did observers understand and take up their role? For example, were they able to closely observe students without interfering with student learning?
- Did observers have a chance to read the Teaching-Learning Plan and understand the team's goals?

3.4 Goals and Key Components of Phase 4 (Reflect)

Phase 4: Reflect

The major goals of the Reflect Phase are:

- To articulate what individual team members and observers learned from the lesson study cycle, so that this knowledge becomes available to others within and outside the team
- For individual participants to integrate knowledge from the lesson study cycle into their own thinking and practice
- For educators to strengthen their commitment to improvement of their own knowledge and practice and that of colleagues

The final phase of the lesson study cycle includes the post-lesson discussion and often an additional meeting to reflect on the entire lesson study cycle. Ideally, the post-lesson discussion occurs just following the research lesson, with a short break between lesson and discussion to allow observers to review the data they collected and make decisions about what to present. The Reflect Phase often creates the most palpable tension between two different theoretical perspectives/goals: lesson study as a place for knowledge integration and lesson study as a place for building community and individual motivation. Participants need to navigate between the extremes of, on the one hand, severe critique that will undermine team members' interest in continued professional learning and, on the other hand, "happy talk" that ignores what needs to be learned from the research lesson and leaves participants feeling they learned very little. Use of a discussion protocol (see below) should support teams to avoid both extremes.

Knowledge integration, self-determination, and self-efficacy theories all provide important theoretical perspectives on the Reflect Phase. From the perspective of knowledge integration, the post-lesson discussion often provides the first in-depth opportunity for the last two stages of knowledge integration: distinguishing and consolidating ideas. During the post-lesson discussion, team members for the first time have data from the classroom that enables them to compare and distinguish ideas that may previously been held independently. To take an example from our Fractions Lesson Study trial, during the Study Phase a teacher argued that pizza is a good model for fractions because it is relevant to students, but also commented favorably on research evidence suggesting that a linear measurement model of fractions helps students see fractions as numbers. These two ideas seemed to be held separately from each other until the negotiation of the lesson plan required one to be chosen. (The team chose to use a linear measure model of fractions.) During the post-lesson discussion, teachers explicitly compared the two models (pizza versus linear measure), evaluating, connecting, and "developing criteria to distinguish" between these two ideas about what constitutes a good model for introducing fractions. They noticed, for example, that students in the research lesson, compared to prior students, had a better understanding of fraction composition (e.g., seeing two-thirds as two one-thirds), and they posited that students might more easily transfer understanding of fraction composition from linear measurement to pizzas than vice versa. The hallmarks of the third stage of knowledge integration – distinguishing – is that criteria for distinguishing between two ideas and evaluating their relative worth are developed. In the final stage of Knowledge Integration, ideas are reflected on and refined so that they fit together. Such integration often seems to occur during the Reflect Phase of the lesson study cycle, after teachers see and discuss the research lesson and contrast it with prior practice. One team member integrated two different criteria for evaluating fraction models (real-world context, consistency), in her reflection at the end of the lesson study cycle:

> "We think we're doing students a favor by relating fractions to food. However, the type of food constantly changes, and we know that a pizza is never really divided into eight equal slices (we've all dove in for the biggest slice at one point or another!). Measurement provides a consistent context in which to imbed the teaching of fractions. Not only would this context make sense, but it would give students more real-world contexts in which to learn. I recently bought a house, and measurement and fractions have come into play several times".

Self-determination theory also provides an important lens on the Reflect Phase, because the post-lesson discussion has great potential to affect team members' experiences of autonomy, competence, and belonging. If data suggest that the instruction supported student learning, team members may deepen their sense of competence and gratitude to colleagues; if data suggest the instruction was problematic, team members' sense of competence and belonging may be threatened. Likewise, self-efficacy theory provides an important lens. The post-lesson discussion can provide a powerful boost – or a powerful threat – to a teacher's sense of efficacy as, for example, a new teaching strategy is tried successfully for the first time, or a familiar strategy is called into question by observers' reports.

3.4.1 Challenges of the Reflect Phase

Major challenges during the Reflect Phase occur within the post-lesson discussion and in the subsequent process of integrating what is learned into daily practice.

1. *The post-lesson discussion does not enable learning*. The post-lesson discussion can fail to build the team's learning for various reasons, which are typically related to the quality of data collected or the quality of the process used to present and discuss it. Data quality may be poor because observers could not see or hear well or because the lesson did not make students' thinking visible. Or observers may not be experienced at noticing and recording the *process* of student learning – imagine the observer who simply notices that a student obtained the correct answer, without noticing the particular counting strategies and materials used to arrive at the answer. Likewise, observers may take an evaluative stance in which they judge student actions, rather than an inquiry stance in which they try to describe and understand student actions. Data collection guidelines provided in the plan may not yield rich data because they are not well-connected to a theory of action; for example, observers may be instructed to notice whether students persevere, rather than to notice the specific resources students use to re-engage with a problem when stuck.

   Even if observers collect valuable data, the post-lesson discussion process may not make good use of observers' data. For example, observers invited to talk about whatever they saw – rather than to focus their observations on key questions related to the team's theory of action – may produce a laundry list of noticings whose significance is unclear. Sometimes observers have difficulty synthesizing what they observed and connecting it back to ideas found in the Teaching-Learning Plan because the team has not clearly laid out the lesson flow – the model of how and why student thinking is expected to progress over the lesson that observers' data can confirm or disconfirm.

   A third source of challenge in post-lesson discussions can stem from the general culture of the post-lesson discussion. Making sense of data and using it to investigate the team's theory of action require an inquiry stance, not simply an evaluative stance toward the lesson. A culture too focused on politeness – or conversely on critique – can undermine inquiry. Thus, lesson study seeks to establish a culture focused on learning.

2. *Team members are not motivated to bring learnings from lesson study back into daily practice at their school or to continue lesson study.* Sometimes lesson study turns into a performance that is disconnected from daily practice and improvement at the school. Teachers may find the lesson study process valuable and integrate what they learn back into their own practice, but not feel motivated to share their learning with other teachers at their school or to initiate subsequent cycles of lesson study.

3.4.2 Strategies to Support the Reflect Phase

1. *Make use of knowledgeable others to review the draft lesson plan (including the data collection plan) and provide final commentary.* A final commentary by an expert in the content and its teaching-learning can add an important dimension to the team's learning by linking the team's work to research and developments outside the school. The final commentator can also provide a model of skilled observation and of how to use classroom data to address the team's questions and goals.
2. *Use a protocol to guide the post-lesson discussion; if observers outside the team will attend the research lesson, consider using a facilitator from outside the team.* Using a discussion protocol increases the likelihood that the post-lesson discussion will focus on presentation and discussion of observers' data, with a focus on the ideas posed by the team. An outside facilitator can keep observers on track with the discussion protocol and redirect any observers who have difficulty deploying an inquiry stance. If there was a disconnection between the team's goals and the lesson that unfolded or the data that was collected, an experienced facilitator may also be able to help bridge that gap – for example, by huddling with the team before the post-lesson discussion to revise the questions the team would like to address during the post-lesson discussion.
3. *Hold an end-of-cycle reflection meeting*. After the post-lesson discussion is concluded, it is useful to have a separate meeting (for the team only) to reflect on the cycle. http://lessonresearch.net/reflect-step/consolidate-your-learning/

   This meeting can help team members think about what they learned at each phase of the cycle, what they want to bring back into their daily practice, and what changes in the lesson study cycle might support their learning in the next cycle. It can also surface questions and noticings that might spark the team's next inquiry and provide an opportunity to revisit the school's vision and needs in light of what was learned during the research lesson.
4. *Open the door on your team's work and connect it to your school's work; connect to an instructional leadership team or school steering committee*. The isolated work of a single lesson study group will never be as powerful as the integrated work of many groups of teachers, particularly if they are working on the same

vision at the same school. For example, when one group of teachers at a school establishes routines that help students explain their thinking, their work can benefit other teachers at the school both by nurturing student capacities that will affect other classrooms (when the students move on) and also by demonstrating that it is possible for students to explain their thinking. One key tenet of self-efficacy theory is that we learn from others' successes and are more willing to try something new when we see others being successful. Organizational studies of schools underline that improvement occurs when the professional learning of teachers connects closely to instruction (which lesson study does) and also connects to school initiatives. Collaborative Lesson Research (school-wide lesson study) provides one structure for ensuring that the work of individual lesson study teams is coordinated, that knowledge is shared across teams, and that the work advances a school-wide vision (Takahashi and McDougal 2016). Connecting your team's work with the important initiatives at your school will enhance the relevance of your team's work, increase its chances of longevity, and decrease the overload teachers often experience from multiple uncoordinated initiatives.

5. *Celebrate!* Self-determination theory reminds us that groups that meet our human needs, such as belonging, are more likely to elicit our commitment. Treating the research lesson teacher to a meal is a Japanese tradition that transfers easily to other countries. A shared celebration of the team's work is always in order – *especially* if the team has encountered many bumps along the way.

Reflection Questions

1. How did the post-lesson discussion add to our learning?
2. What learning do we want to carry forward from this lesson study cycle? For example, what learning about the content, about teaching, and about our students?
3. How will what we learned during this lesson study cycle influence our daily practice as individuals? As a school? How will we share what we learned with the profession more broadly?
4. What questions or ideas were sparked by this cycle that we want to investigate in future lesson study cycles?
5. How might we adjust and improve the lesson study process in the future?
6. To what extent did the lesson study cycle strengthen relationships among team members? For example, did it increase the likelihood we will consult each other informally about problems of practice in the future?
7. Did we allow sufficient time for team members to reflect together at the end of the cycle?

4 Discussion and Conclusion

This chapter lays out what we believe to be the major goals of each phase of the lesson study cycle (study, plan, teach, reflect) and ties them briefly to educational and psychological theories that support both the exploration of teaching and learning (theories relating to knowledge integration and pedagogies of practice) and development of the lesson study team itself (theories relating to self-determination and self-efficacy), along with challenges at each phase, strategies that may help teams overcome these strategies, and reflection questions that may help teams assess and refine their work. We do not see these ideas as final or definitive, since our understanding of lesson study has evolved steadily ever since our first published work on it in 1997. Given the complexity of teaching and the complexity of human learning, even four theoretical lenses are probably insufficient to provide a theoretical framework to understand lesson study impact. We hope that this chapter will stimulate much further conversation.

We hope that the challenges and strategies laid out for each phase can provide a useful framework for other researchers to add their own ideas and critique, so that, as a field, we have a common framework that continues to change and grow as more educators engage in lesson study. We also hope that lesson study teams will use the reflection questions associated with each phase as a way of reflecting on their own work and will add to these questions, creating a practical reflection tool that helps lesson study teams deepen their work.

Acknowledgments This material is based on work supported by three funders:

The Institute for Education Sciences of the US Department of Education under Grant No. R305A150043. Any opinions, findings, conclusions, or recommendations expressed in this article are those of the authors and do not necessarily reflect the views of the Institute for Education Sciences.

The Bill & Melinda Gates Foundation. The views, findings, conclusions, and recommendations expressed herein are those of the authors and do not necessarily express the viewpoint of the foundation.

The National Science Foundation under Grant No. 1503342. Any opinions, findings, and conclusions or recommendations are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Aoibhinn, N. S. (2016). Developing mathematics teachers' pedagogical content knowledge in lesson study: Case study findings. *International Journal for Lesson and Learning Studies, 5* (3), 212–226. https://doi.org/10.1108/IJLLS-11-2015-0036.

Bandura, A. (2001). Social cognitive theory: An agentic perspective. *Annual Review of Psychology, 52*(1), 1–26.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Collet, V. S. (2017). Lesson study in a turnaround school: Local knowledge as a pressure-balanced valve for improved instruction. *Teachers College Record, 119*, 1–58.

Deci, E., & Ryan, R. (1985). *Intrinsic motivation and self-determination in human behavior*. New York: Plenum.

DosAlmas, A., & Lewis, C. (2017). Bowling with walnuts: What we can learn from Kyouzai Kenkyuu (study of teaching materials). *International Journal for Lesson and Learning Studies, 6*(1), 27–31.

Fujii, T. (2016). Designing and adapting tasks in lesson planning: A critical process of Lesson Study. *ZDM Mathematics Education, 48*(4), 411–423.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009a). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*(9), 2055–2100.

Grossman, P., Hammerness, K., & McDonald, M. (2009b). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and Practice, 15*(2), 273–289.

Krystal, B. (2018). Developing teachers' mathematical-task knowledge and practice through lesson study. *International Journal for Lesson and Learning Studies, 7*(2), 136–149.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., et al. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education, 64*(3), 226–243.

Lewis, C., & Perry, R. (2015). A randomized trial of lesson study with mathematical resource kits: Analysis of impact on teachers' beliefs and learning community. In E. J. Cai & Middleton (Eds.), *Design, results, and implications of large-scale studies in mathematics education* (pp. 133–155). New York: Springer.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for Research in Mathematics Education, 48*(3), 261–299.

Lewis, C., Perry, R., Hurd, J., & O'Connell, M. P. (2006). Lesson study comes of age in North America. *Phi Delta Kappan, December 2006*, 273–281.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.

Lewis, C., Akita, K., & Sato, M. (2010). Lesson study as a human science. In W. R. Penuel & K. O'Connor (Eds.), *Learning research as a human science* (National Society for the study of education yearbook) (Vol. 109, pp. 222–237). New York: Teachers College, Columbia University.

Lewis, C., Perry, R., & Friedkin, S. (2011). Using Japanese curriculum materials to support lesson study outside Japan: Toward coherent curriculum. *Educational Studies in Japan: International Yearbook, 6*, 5–19.

Linn, M. C., & Eylon, B.-S. (2011). *Science learning and instruction: Taking advantage of technology to promote knowledge integration*. New York: Routledge.

Linn, M., Eylon, B., & Davis, E. (2004). Internet environments for science education. In M. Linn, E. Davis, & P. Bell (Eds.), *The knowledge integration perspective on learning* (pp. 29–46). Mahwah: Lawrence Erlbaum Associates.

Linn, M. C., Davis, E. A., & Bell, P. (2013). *Internet environments for science education*. New York: Routledge.

Matsuzawa Elementary School. (2011). *School research report by the Matsuzawa elementary school*. Available from http://www.impuls-tgu.org/cms/uploads/File/resource/MatsuzawaLeafletDec12011.pdf

Mills College Lesson Study Group. (2005). *How many seats? Excerpts of a lesson study cycle [DVD]*. Oakland: Mills College Lesson Study Group.

National Education Policy Research Institute, J. K. K. S. K. (2011). *Report of survey research on improvement of teacher quality* [Kyouin no Shitsu no Koujou ni Kansuru Chosa Kenkyuu]. Retrieved from Tokyo.

Pernilla, M., & Henrik, H. (2018). Challenging teachers' ideas about what students need to learn: Teachers' collaborative work in subject didactic groups. *International Journal for Lesson and Learning Studies, 7*(2), 98–110.

Perry, R., & Lewis, C. (2010). Research and practice in education: Building alliances, bridging the divide. In C. E. Coburn & M. K. Stein (Eds.), *Building demand for research through lesson study* (pp. 131–145). Lanham: Rowman & Littlefield Publishers, Inc.

Runesson, U., & Gustafsson, G. (2012). Sharing and developing knowledge products from Learning Study. *International Journal for Lesson and Learning Studies, 1*(3), 245–260.

Ryan, R. M., & Deci, E. L. (2000). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. *American Psychologist, 55*(1), 68.

Sarkar Arani, M. R. (2017). Raising the quality of teaching through Kyouzai Kenkyuu – The study of teaching materials. *International Journal for Lesson and Learning Studies, 6*(1), 10–26.

Sato, M. (2008). Philosophy on the restoration of schools: The vision, principles and activity system of the learning community. *Journal of All India Association for Educational Research, 20* (3&4), 14–26.

Schoenfeld, A. H., Baldinger, E., Disston, J., Donovan, S., Dosalmas, A., Driskill, M., Fink, H., et al. (under review). Learning with and from TRU: Teacher educators and the teaching for robust understanding framework. In K. Beswick (Ed.), *Handbook of mathematics teacher education, volume 4: The mathematics teacher educator as a developing professional*. Rotterdam: Sense Publisher.

Shifter, D., Bastable, V., & Russell, S. J. (2010). *Developing mathematical ideas*. Reston: National Council of Teachers of Mathematics.

Takahashi, A., & McDougal, T. (2014). Implementing a new national curriculum: Case study of a Japanese school's 2-year lesson-study project. In K. Karp (Ed.), *Annual perspectives in mathematics education: Using research to improve instruction 2014*. Reston: National Council of Teachers of Mathematics.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*(4), 513–526.

Takahashi, A., Watanabe, T., Yoshida, M., & Wang-Iverson, P. (2005). Improving content and pedagogical knowledge through kyozaikenkyu. In P. Wang-Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 77–84). Philadelphia: Research for Better Schools.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2009). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). Boston: Allyn & Bacon.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), *Inquiry into mathematics teacher education* (Vol. 5, pp. 131–142). San Diego: Association of Mathematics Teacher Educators.

Yoshida, M., & Jackson, W. C. (2011). Response to part V: Ideas for developing mathematical pedagogical content knowledge through Lesson Study. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education*. New York: Springer.

Catherine Lewis, Ph.D., a developmental psychologist, has directed ten major grants funded by NSF, IES, or private foundations, focused on effective strategies to support instructional improvement. A recent randomized controlled trial (conducted with Rebecca Perry) demonstrated that teams of teachers engaged in lesson study, supported by mathematical resources, can significantly improve students' mathematics learning (Lewis & Perry, Journal for Research in Mathematics Education, 2017, 48:3). Their study has been identified as one of only two mathematics professional learning interventions (of 643 reviewed) to meet What Works Clearinghouse scientific criteria and positively impact students' mathematical proficiency (Gersten, Taylor et al. 2014). Lewis speaks and reads Japanese and has produced video and print materials that introduce lesson study (www.lessonresearch.net).

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Lynn Goldsmith, Ph.D., distinguished researcher at Education Development Center, is a developmental psychologist whose research has focused on varied aspects of teaching and learning. She has studied professional development for mathematics teachers, investigated principals' instructional leadership for mathematics, examined the role that emotions play in learning, explored possible relationships between arts education and mathematical reasoning, and, more recently, has been exploring ways to support elementary teachers in integrating computational thinking into their mathematics and science classes. She has conducted formative and summative research to support curriculum development efforts and has also written about mathematics education for the research community, practitioners, and parents.

How Could Cultural-Historical Activity Theory Inspire Lesson Study?

Ge Wei

Contents

Abstract Lesson study, originating from East Asia, has been widely spread and accepted as a vehicle for teacher continuing professional development. With its features of openness (beginning with a question instead of an answer), involvement (driven by teachers rather than experts), dialogicality (reciprocal learning instead of hierarchical interpersonal relationships), and practicality (linking research and practice), lesson study contributes new energy to educational research. Finding an appropriate theoretical perspective to approach lesson study and glean its advantages is challenging. When cultural-historical activity theory is used to contextualize lesson study, numerous insights are gained by both teachers and researchers. This conceptual chapter introduces the use of cultural-historical activity theory as an analytical lens, followed by a brief overview of the commonalities of various modes of lesson study. Some central tenets of cultural-historical activity theory that echo the essential points of lesson study are elaborated. By collecting data for and analyzing a lesson study case involving an elementary mathematics lesson in Beijing, the author shows that cultural-historical activity theory illuminates the significance of lesson study at the ontological, epistemological, methodological, and axiological levels.

Keywords Lesson study · Cultural-historical activity theory (CHAT) · Teacher professional development

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_3

1 Introduction

Over the past few years, the high performance of East Asian students in international academic assessments, such as the Program for International Student Assessment (PISA) and the Trends in International Math and Science Study (TIMSS), has attracted increasingly wide attention from the Western world. Another global program, the Teaching and Learning International Survey (TALIS), aims to explore how teacher knowledge base impacts student learning outcomes, especially from an international perspective comparing the East and the West (OECD 2017). The sustainable growth of East Asian education is a "learning myth" (Stigler and Hiebert 1999) for educators: what contributes to East Asia's educational success (see also, e.g., Zhao 2014)?

In East Asian societies, especially the Chinese, the teachers' knowledge and competence affect the students' learning achievement more directly than some peripheral factors of teaching and learning activity, such as cultural and institutional factors (Ma 1999). In the process of improving teacher quality, lesson study in China plays an important role in incorporating professional development activities into the teachers' daily tasks. According to Lewis (2009), lesson study challenges the traditional ways of teacher professional development, most of which inform teachers on what to do in a designated manner. However, lesson study activities allow teachers to learn how to teach through collaborative inquiry into the methods of analyzing, designing, developing, implementing, and evaluating a lesson while focusing on student learning (Hart et al. 2011).

Given its essential merits, lesson study as a form of teacher development has been adopted by many schools and districts around the world (Huang and Shimizu 2016). Various perspectives have been employed to interpret the advantages of lesson study for teacher professional learning and development in school-based settings. For example, Pang and Marton (2005, 2013, 2017) adopted "variation theory" to reframe teachers' and researchers' insight, transforming lesson study into learning study; they focused more on students' learning activities instead of teachers' actions. Chen and Yang (2013) used Little's (2003) idea of "decontextualization and recontextualization" to approach teachers' meaning making of curriculum policies and their local actions in school-based lesson study activities. More recently, Fang (2017) employed Lave and Wenger's (1991) conceptualizations of "legitimate peripheral participation" and "transparency" to discern novice teachers' professional learning in teacher research groups, under the discursive circumstances of district curriculum reform. Huang et al. (2016) applied theories of "learning trajectory" and "variation pedagogy" to guide lesson study activities, through which both the teachers' teaching and the students' mathematical understanding were improved as a result. Despite previous inquiry into the subject of lesson study, few scholars have paid intensive attention to cultural-historical activity theory (hereafter CHAT; Engeström 1987/2015) as a robust theoretical underpinning for lesson study.

CHAT, originally inspired by Hegel's dialectics and Marx's historical materialism, effectively integrates various elements in human activities (e.g., subjects, tools, objects, rules, community, division of labor, and outcomes) in a systemic and interacting manner (Wei 2017b). CHAT is a cross-disciplinary theory for understanding human activity and human development in collaborative settings, incorporating the social and historical roots of a specific event into a collaborative inquiry process. Considering lesson study as a collaborative human activity conducted by a group of teachers focusing on student learning and certain subject matters, CHAT could bring about opportunities for researchers and teachers to address the complexity of and dynamics within the holistic process of lesson study. Specially, CHAT is useful for discerning the what and the how of teacher professional learning and development in lesson study activities, which are currently one of the most challenging issues in the research fields of both lesson study and teacher education.

This chapter thus aims to bridge the theoretical perspective of CHAT with lesson study as a practical mode for teacher continuing professional development. The overarching question of this chapter, reflected in the title, is "How could cultural-historical activity theory (CHAT) inspire lesson study?". In order to answer this question, this conceptual chapter first reviews the commonalities and essential characteristics of different modes of lesson study worldwide. Then, CHAT, with its three generations of evolvement, is scrutinized, with the results depicted in representative figures. The central tenets and most innovative conceptualizations in CHAT are discussed. It is argued that the merits of lesson study could be better utilized by incorporating CHAT (e.g., Mosvold and Bjuland 2011; Wake et al. 2016; Wei 2017b). Finally, the author's audio recording of a series of lesson study activities in a mathematics teacher group in an elementary school in Beijing in 2013 is described. This case study demonstrates that CHAT helps to illuminate the broader and deeper significance of the uses of lesson study.

2 Reviewing the Commonalities of Various Lesson Study Modes

"Lesson study" is a translation of the Japanese term "*jugyou kenkyuu*." *Jugyou* means "live instruction" (e.g., a single lesson or many lessons); *kenkyuu* means "research" or "study" (Lewis 2016). Lesson study is a form of practice-based teacher professional development that originated from East Asia; it has been widely adopted around the world due to its benefits for continuing teacher professional learning. The Japanese lesson study has been adapted into different modes mainly in China, the United Kingdom, Hong Kong, and Sweden (Huang and Shimizu 2016).

According to the literature, the essential characteristics of lesson study include being focused on the knowledge of the subject matter, the curriculum, and student learning; being ongoing, inquiry-based, and integrated into the daily tasks of teachers; providing opportunities for teachers to become actively engaged in the meaningful analysis of teaching and learning; and promoting coherence between teachers' professional development and other professional experiences (e.g., Fernandez 2002; Lee and Lo 2013; Lewis 2015, 2016; Murata et al. 2012).

According to Lewis (2002) and Lewis and Hurd (2011), lesson study does not refer to a single practice but a series of practices of teaching experimentation and reflection. With regard to the Chinese models of lesson study, three featured types have been distinguished: (1) public lessons, with their different types conducted by teachers at different stages of their professional development; (2) deliberate practice of teaching the same lesson repeatedly in order to refine its execution; and (3) institutionalized apprenticeship during which novice teachers learn from the "excellent" exemplars of expert teachers (Chen 2017).

Regardless of the location, be it Japan, China, the United Kingdom, or the United States, lesson study has been practiced for decades and has contributed a great deal to the improvement of teacher professionality (e.g., Chen 2017; Huang and Shimizu 2016; Lewis 2015). As a common method of implementing teachers' school-based professional development, lesson study is embedded into the teachers' daily tasks in an integrated way. Lewis (2002, 2015) compared the traditional teacher professional development model with lesson study as the new approach. She argued that lesson study enables openness (beginning with a question instead of an answer), involvement (driven by teachers rather than experts), dialogicality (reciprocal learning instead of hierarchical interpersonal relationships), and practicality (linking research and practice). This leads to an urgent question of how the merits of lesson study may be employed by using an appropriate theoretical lens.

The purpose of this chapter is to review and introduce CHAT and its central tenets to researchers and practitioners who are interested in lesson study. In fact, CHAT is a theoretical perspective within the field of psychology that originated from Lev Seminovich Vygotsky's work in the pre-Soviet era during the 1920s. Since Vygotsky's work, there have been a growing number of European and North American scholars examining the usefulness of CHAT for educational activities (Yamagata-Lynch 2010, p. vii). Some educational researchers (e.g., Mosvold and Bjuland 2011; Tsui and Law 2007; Wake et al. 2016) have incorporated the well-known triangle model of CHAT (see Fig. 1) into their projects and subsequently divided the model into seven elements (i.e., subject, object, artifacts, community, rules, division of labor, and outcomes) using their collected data. Despite the previous literature on this topic, little attention has been paid to the underpinnings and hidden tenets of CHAT, which constrains our deeper understanding of the inner mechanisms of human development.

The following part of this chapter first reviews the historical development of CHAT through its three generations of evolvement. Then, the four central tenets of CHAT, according to the latest interpretations of Yrjö Engeström (2015), a Finnish interventionist researcher, are illustrated theoretically.

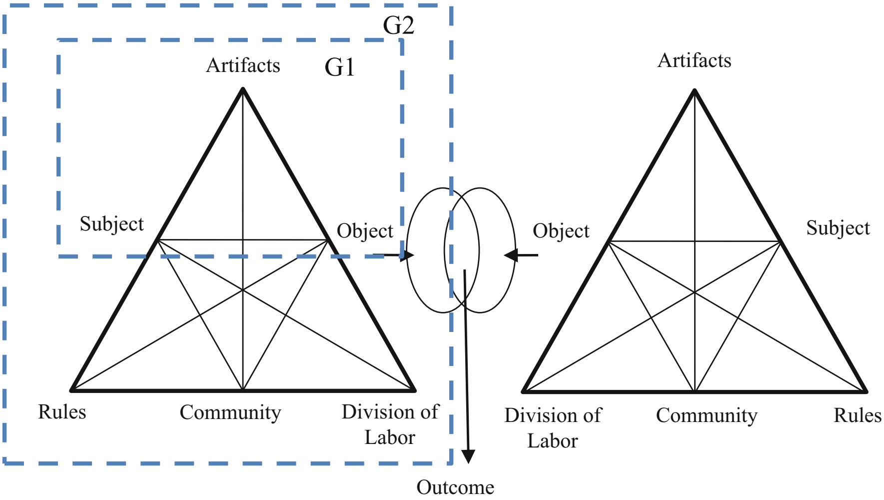

Fig. 1 The theoretical model of CHAT. (Revised from Engeström and Sannino 2010)

3 Scrutinizing Cultural-Historical Activity Theory

According to Engeström (1996), activity theory has evolved through three generations of research. In this section, I focus on the contributions of Vygotsky and the post-Vygotskian scholars who played critical roles in the development of CHAT, using the keywords created by different pioneers as a guide (Engeström 2001).

3.1 Vygotsky and the Mediated Mechanism

Tracing back to Hegel and Marx's philosophical thoughts on dialectics and historical materialism, Vygotsky challenged behaviorism and constructed a social constructivist model of thought to interpret human learning and development (Wei 2017a). As the positivistic paradigm of the social sciences (especially developmental psychology) established in the European continent did not satisfy his inquiry, Vygotsky proposed a new epistemology and methodology to study human development, called the cultural-historical theory (CHT). Different from Pavlov, who developed "stimulation-reflection," Vygotsky introduced *mediation* as a constellation of tools and signs to bridge the subject and the object, which opened a new space for subject-

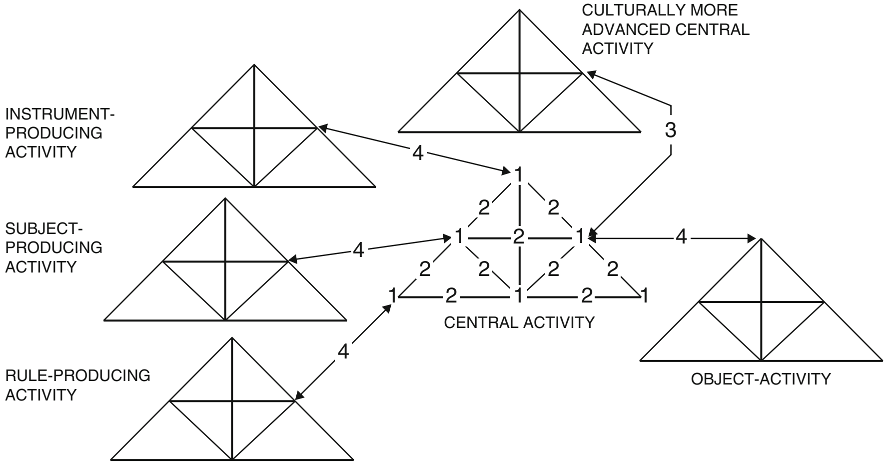

Fig. 2 Four levels of contradictions in human activity. (Engeström 1987/2015)

object relationships (see G1 in the dashed box, the uppermost sub-triangle in Fig. 2). Vygotsky based his psychology on Marxian theory to describe the relationship between individuals and their social environment (Cole 1985; Wertsch 1985). Moreover, he used Marx's political theory regarding collective exchanges and material production to examine the organism and the environment as an organic unit of analysis. Through this reformulation of psychology, Vygotsky attempted to capture the coevolutionary process that individuals encounter in their environment while learning to engage in shared activities (Stetsenko 2005).

Vygotsky further introduced *mediated action* as a concept to explain the relation between thinking and speech that enables human consciousness development through interaction with artifacts and knowledgeable others in a certain environment (Vygotsky 1978). The interactions in which individuals engage allow opportunities for mediated action that contributes to the social formation of their consciousness (Wertsch 1985). In this interaction, individuals are not passive participants waiting for the environment to instigate meaning-making processes for them; rather, through their interactions, individuals make meaning of the world while they modify and create activities that trigger the transformations of artifacts and people in their environment (Scribner 1997).

3.2 Leontiev and the Object-Directed Activity

Contributing to the development of activity theory, Leontiev, a Russian psychologist, identified object-oriented activity, instead of the goal-oriented action, as the unit of analysis. Leontiev (1978, p. 10) defined object-oriented activity as:

> A molar and non-additive unit of a material subject's life. In a narrower and more psychological sense, activity is a unit of life mediated by mental reflection whose real function is to orient the subject to the world of objects. Activity is thus not a reaction or a totality of reactions, but rather a system possessing structure, inner transformations, conversations, and development.

Leontiev provided a clear distinction between object-oriented activity and goal-directed actions. Goal-directed actions are much more temporary in nature and may be a step that subjects take in the process of participating in an object-oriented activity. Goal-directed actions are often individually focused and involve fewer collective endeavors of community improvement (Leontiev 1978). In other words, the object is shared with individuals in a community of practice, with rules and the division of labor (see G2 in the dashed box, the left-side triangle in Fig. 1). Leontiev was the scholar who depicted the fundamental elements of human activity in a systematic way.

3.3 Engeström and the Inter-system Analysis

Based on the scholarship of Marxism and the Vygotskian School, Engeström (1987) challenged traditional theories that considered learning as a process of acquisition or a restructuring of cognition within the individual mind. In contrast, Engeström held a collective and dialectical view of learning and human development based on his work at the University of Helsinki during the past decades (Wei 2017a).

Upon observing the dominant Cartesian views of learning and development, Engeström questioned the legitimacy of cognition abstracted from its contexts. Engeström (1987/2015) further developed analytical methods beyond the single activity system analysis. The third and current generation of activity theory aims to understand dialogues, multiple perspectives, and networks of interacting activity systems. It extends the single activity system into inter-system analysis, considering the complexity and dynamics of human activities (see the whole image in Fig. 1). It specifically addresses both the individual and the sociocultural contexts of the activity in order to move away from former CHAT methods that were too individual focused.

The third generation of CHAT argues that human learning is embedded in object-oriented and artifact-mediated collective activity systems, historically triggered by inner contradictions (Engeström 2015, p. xvi). Under this rationality, Engeström proved that a new type of "learning by expanding" always exists in human history, which implies the existence of potential objects, conscious mastery, and transformative agency (Wei 2017a). Expansive learning takes place in the interaction between two or more activity systems, which is a historically new type of learning moving across collective zones of proximal development, where understanding and changing the world are integrated (Wei 2017b).

Briefly, according to Engeström's (1996, 2001) summary, the three generations of activity theory encompass distinct approaches to understanding human development. Vygotsky's identification of the mediated action triangle refers to the activity theory of the first generation. Second-generation activity theory is attributed to Leontiev's work that emphasized the collective nature of human activity, along with Engeström's own work in 1987 that developed the activity systems model. Finally, based on Engeström's later works, the third generation of activity theory involves explorations of multiple system interactions in developmental research where the investigator often takes a participatory and interventionist role in the participants' activity to help participants experience change.

CHAT helps us to understand educational activities in a systemic way, by analyzing the elements within and between different activity systems (Yamagata-Lynch 2010). In a school context, for example, a mathematics teacher (*subject 1*) would like to improve his/her students' (*subject 2*) academic performance (*object 1*) within an urban school (*community*). He/she intends to introduce a new strategy for teaching (*tool*). Depending on the management structure within the school (*division of labor*), the teacher might be constrained on the basis that the new idea is being interpreted as deviating from implicit norms (*rules*). If so, the teacher would need to negotiate with his/her students about how to employ this new pedagogical method in their teaching and learning activities (*outcome*). This is a simple example of using CHAT as a lens to analyze the day-to-day work of teachers. The use of this theory finds that almost all perceived factors of teachers' work are included and that some interesting issues (e.g., the school's micropolitics) emerge within this context. CHAT also supplies an effective view of lesson study when teachers and researchers reflect on their coworking. In other words, CHAT is not only a lens with which to analyze, conduct research, or *reflect on* teachers' lesson study processes but also a methodological approach to guide teachers' *reflection in* the lesson study process as well.

4 Distilling the Central Tenets of CHAT

The above interpretation exemplifies the benefits of using CHAT as a framework to analyze complicated lesson study activities. Furthermore, it is necessary to delve deeper and determine the core tenets of CHAT emerging from the analysis process.

4.1 Activity System as the Unit of Analysis

Activity system is the fundamental unit of "the whole" (Vygotsky 1978) for analyzing human activity from the perspective of CHAT (Engeström 1987/2015). In the classical triangle model, seven elements constitute the whole system, which describes the complex context of human learning and development in the real world. In the triangle model (see G2 in Fig. 2), the *subject* is the individual or groups of individuals involved in a certain activity. The *artifacts* include tools and signs that can act as resources for the subject during the activity. The *object* is the motive behind the activity. The *community* is the social group to which the subject belongs and in which the subject is engaged in an activity. The *rules* are both formal and informal regulations that can affect how the activity takes place in varying degrees. The *division of labor* refers to how the object is shared among the community. The *outcome* of an activity system is the result of the activity, which is normally unexpected by the subject (Engeström 1987/2015). The seven elements compose a holistic system by which this analysis method enhances an understanding of the human activity situated in a collective context.

4.2 *Contradiction as the Inner Force for Change*

Contradiction is positioned as the driving force for change and transformation (Il'enkov 1977); it enables a new object of activity to be identified and conceptualized. Consequently, the concept of "contradiction" has been a central focal point of a number of studies inspired by CHAT (Engeström 2016). According to Il'enkov (1977, p. 330), contradictions are not just inevitable features of activity. They are the principle of the self-movement of activity and the form in which the development is cast. This means that new stages and forms of activity emerge as solutions to the contradictions of the preceding stages and forms. This in turn takes place in the form of invisible breakthroughs (Wei 2017b). In the analysis of human activity, four levels or layers of contradictions could be discerned. These levels are illustrated in Fig. 2.

Each level of contradiction is labeled with numbers in Fig. 3. The four-level contradictions in the activity systems could be interpreted as:

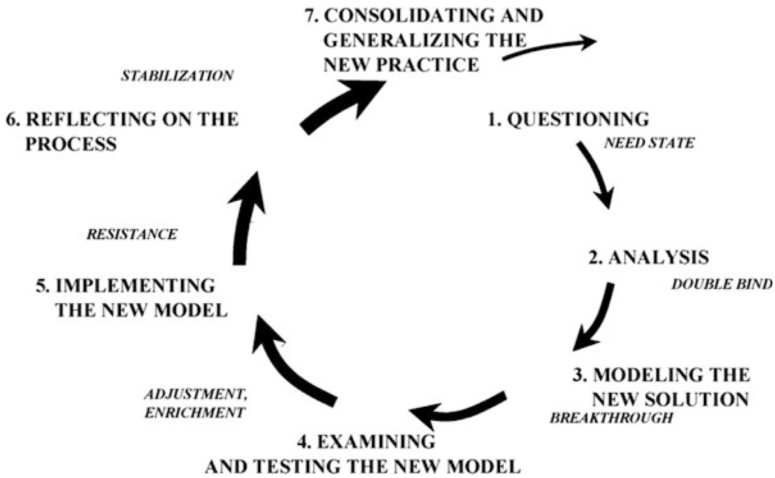

Fig. 3 The cycle of expansive learning. (Engeström and Sannino 2010)

- Level 1: Primary inner contradiction (double nature) within each constituent component of the central activity
- Level 2: Secondary contradictions between the constituents of the central activity
- Level 3: Tertiary contradiction[s] between the object/motive of the dominant form of the central activity and the object/motive of a culturally more advanced form of the central activity
- Level 4: Quaternary contradictions between the central activity and its neighbor [ing] activities (Engeström 1987/2015, p. 71)

More specifically, in CHAT, "contradiction" is the inner force triggering the occurrence of expansive learning. If historically accumulated contradictions are not encountered by the subjects, the crisis would most likely not be solved completely.

4.3 Expansive Learning as an Approach to Transformation

CHAT focuses on the dynamics of human learning. Sfard (1998) suggested two basic metaphors for learning activities: the *metaphor of acquisition* and the *metaphor of participation*. According to the interpretations of Paavola and Hakkarainen (2005), the former emphasizes individual mental processes, and the latter examines the transmission of cultural knowledge and competence from one generation to the next. Neither one of these metaphors appears, however, to examine the knowledge creation that is critical to an advanced knowledge society.

From the point of view of expansive learning, both acquisition-based and participation-based approaches share much of the same conservative bias. Both have little to say about the transformation and creation of culture (Wei 2017b). On the contrary, the theory of expansive learning places importance on the following: communities as learners, the transformation and creation of culture, horizontal movement and hybridization, and the formation of theoretical concepts (Engeström and Sannino 2010). Hence, the theory of expansive learning relies on its own metaphor: *expansion*. In expansive learning, learners learn something that is not yet there (Engeström 2016). In other words, the learners construct a new object and concept for their collective activity and implement this new object and concept in practice. This shift in metaphors has been noted by Paavola et al. (2004), who suggested knowledge creation as the new, third metaphor for theorizing work-based learning.

Together, these actions form an expansive cycle or spiral (see Fig. 3). An ideal-typical sequence of epistemic actions in an expansive learning cycle is described as follows (Engeström and Sannino 2010, p. 7):

- The first action is that of questioning, criticizing, or rejecting some aspects of the accepted practice and existing wisdom. For the sake of simplicity, we will call this action questioning.
- The second action is that of analyzing the situation. Analysis involves mental, discursive, or practical transformation of the situation in order to find out causes

or explanatory mechanisms. Analysis evokes "why?" questions and explanatory principles. One type of analysis is historical-genetic; it seeks to explain the situation by tracing its origins and evolution. Another type of analysis is actual-empirical; it seeks to explain the situation by constructing a picture of its inner systemic relations.

- The third action is modeling the newly found explanatory relationship in some publicly observable and transmittable medium. This means constructing an explicit, simplified model of the new idea that explains and offers a solution to the problematic situation.
- The fourth action is that of examining the model, running, operating, and experimenting on it in order to fully grasp its dynamics, potentials, and limitations.
- The fifth action is that of implementing the model by means of practical applications, enrichments, and conceptual extensions.
- The sixth and seventh actions are those of reflecting on and evaluating the process and consolidating its outcomes into a new stable form of practice.

In other words, expansive learning as an ideal type of human learning describes the inner mechanism and processual ideology of how people learn.

4.4 Formative Intervention as a Stance of Inquiry

From the perspective of CHAT, Engeström developed the methodology of formative intervention used in *Developmental Work Research* (DWR, Engeström 2005). Engeström (2016) noted that an intervention is meant to be not only disruptive but also developmental in relation to the practice in which the intervention takes place. An intervention such as DWR is meant to be deliberative and systematic; to some extent, it should also halt the daily practice of practitioners to allow them to examine their business-as-usual processes by themselves.

CHAT is not a rigorous framework by which researchers may "research" social phenomena. Rather, the positionality of researchers involves maintaining a stance of inquiry, which means that intervention becomes a new and necessary way to conduct research. Two reasons can explain the importance of intervention. First, when we observe, analyze, and interpret social life, we also influence it, whether we want to or not. Thus, it is advisable for us to analyze our own actions and research practices as they interact with those of our subjects. Second, by intervening deliberately and methodically, we generate knowledge about what is possible. Possibility knowledge opens up insights into what may be possible in a human activity and what alternative directions of development and change are available. Possibility knowledge is generated by setting the activity and its subjects into motion, into some form of focused "time travel" that explores the past, the present, and the future in relation to one another (Virkkunen and Newnham 2013, p. xvii).

The claim of the methodology of formative intervention enables participants to do more than simply work on improving their own performance either through action research methods or through participation in researcher-led design experiments. The aim is the development of what Engeström (2007) called "transformative agency" among the practitioners. It is stimulated by the power of the conceptual tools of CHAT in helping participants to analyze how the object of their collective activity is constructed, how rules and a division of labor have emerged historically within a community of practitioners, how artifacts are appropriated by members of that community, and, moreover, how these might be changed for a better future.

5 Shedding Light on Lesson Study Through CHAT

Based on the illustration of CHAT and its featured contributions to educational research, this section, alongside the section detailing the author's research project, describes a lesson study case that took place in an elementary school in China; it provides justification for the use of CHAT to inspire lesson study theoretically. This section maintains that CHAT offers a robust lens for analyzing lesson study process due to its multilayer perspective from the aspects of ontology, epistemology, methodology, and axiology of lesson study itself. In this section, I connect the keywords and central tenets of CHAT with the merits of lesson study emerging from the aforementioned four layers.

5.1 Introducing the Case

The data for this lesson study case were collected by the author, between September 2013 and November 2013, in an elementary school called Youth School in Beijing, China. Youth School has a long tradition of conducting lesson study in teachers' daily work. Lesson study in Youth School has several formats, including research lessons, public lessons, and performing lessons, which place emphasis on school-based teacher professional development. The case in question involved a more ordinary form of lesson study in a mathematics teaching research group that took place during a whole semester in autumn, which aimed to guide novice teacher's growth.

This lesson study was based on a unit in mathematics: "Learning to Locate" (see Fig. 4). A group of teachers who taught Grade 4 formed a lesson study group to research this lesson (see Table 1).

During the semester, six sessions of lesson study were conducted by the group of teachers in their usual way. Teacher Sun, as a novice teacher, designed this lesson independently and taught it in a class for the first time. Subsequently, the other teachers in this lesson study group analyzed Sun's teaching skills and offered some suggestions for improvement. Sun considered her colleagues' feedback and

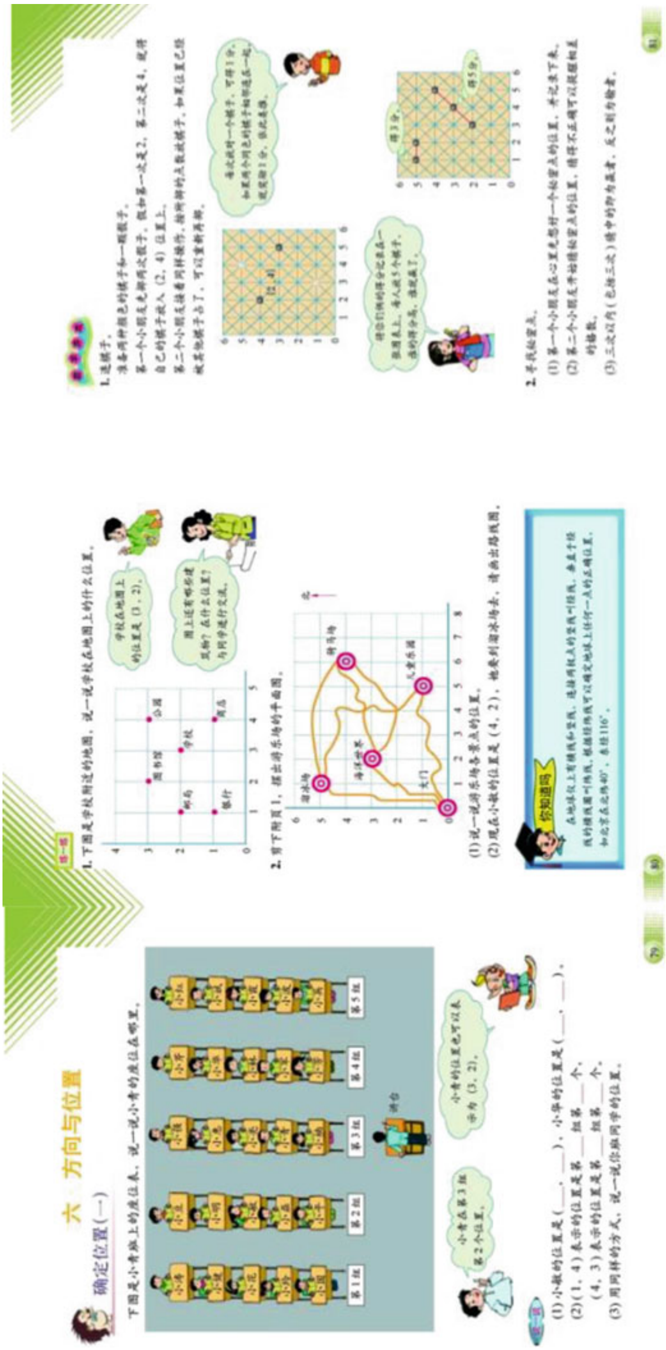

Fig. 4 Textbook pages. (Editorial Team 2016, pp. 63–65)

Table 1 Participants

Teachers' name	Gender	Age	Note
Li	M	45	Group leader
Sun	F	25	Novice teacher
Zhou	F	28	Novice teacher
Zheng	M	35	Experienced teacher
Wu	F	42	School director

redesigned the same lesson again; she then taught the lesson in a new class, with new students. In total, this lesson study cycle, expressed as "teaching→discussion→reflection→reteaching," was implemented six times in six classes. Sun was the one who applied the group members' ideas into practice, while her colleagues played roles as critical friends in shaping Sun's instructional improvements. During this process, the other group members without Sun's visions of teaching and learning were transformed to some extent as well. I audio recorded all of these sessions, including the teachers' performance of teaching activities in the classroom and their group discussions in their office. All of the records of Sun's classroom teaching, group discussion meetings, and teachers' reflection journals were in the form of transcripts taken verbatim.

An interesting question that emerged from this case study could be expressed as "How could the teachers' collective professional learning take place during lesson study?" Although the current study is not an empirical one designed to answer this question, the following sections refer directly to the four aspects of inspiration by which the question could be approached using the insights of CHAT.

5.2 *Ontological Inspiration*

According to CHAT, the minimum unit of analysis is the activity system, instead of the elements. CHAT suggests that we are not isolated individuals interacting with our environment on a purely biological basis; rather, our relationship with the world is mediated by other people and the cultural-historical context in which we live (Yamagata-Lynch 2010). This inspires us to consider lesson study as a collective practice in certain sociocultural contexts. The first essential characteristic of lesson study is its openness, which means that the teaching activity does not focus on a single factor, but a collection of elements that are interconnected with one another. Thus, in this case, the exploration of teacher professional development in lesson study should broaden the view from one of individual teachers to that of the whole group.

Previous research on lesson study has always focused on novice teachers and what they learned in the lesson study process (e.g., Chen and Yang 2013). In fact, as the lesson study involves a community of practice, every member in the lesson study group, even if they are experienced teachers, learns something new during the dialogical process in lesson study. Actions of talking, analyzing, and feedbacking combined all the teachers together toward a same activity object, namely, teaching better and more effectively. The teacher group, as a subject of learning, continually facilitates each member's reflection on his/her teaching style and their respects to each other.

For example, in this case study, the director of Youth School informed me of the following at the end of the lesson study:

> I did not expect to find our novice teachers to have great competence in critical thinking. I should reflect on how to encourage or even protect the novice teachers' own teaching styles. (Teacher Wu)

In other words, the ontology of lesson study, under the discourse of teacher professional development, concerns the community of teachers and their lesson study activity systems, rather than the individuals. Only by adopting a systemic view of lesson study and observing the group activities could we discern the openness of lesson study that enables the improvement of teacher professional learning in a continuous manner.

5.3 Epistemological Inspiration

Lesson study is driven by teachers instead of educational researchers, which means that the authentic voices of teachers should be considered as the resource that allows insight into teacher professional learning. CHAT uses the concept of "multi-voicedness" to present an activity system as a community of multiple points of view, traditions, and interests (Engeström 2001). Different voices are likely to introduce tension, conflicts, and even contradictions. When I took a closer look at the data of this case, I focused on the contradictions and how they drove the development of a teacher community, thereby facilitating teacher learning in the lesson study group.

CHAT views contradictions as the driving force of human development. To enrich the interpretations of the four-level contradiction system (see Fig. 2), the system can be seen as serving as the "skeleton," while the other tenets of CHAT can be seen as functioning as the "flesh and blood." The following dialogues took place during the second cycle of the lesson study. The teacher learning process was found to be embedded in these contradictory dialogues.

> Li: You (Teacher Sun) should tell your students in a simple way to clarify the definitions of "Row" and "Line."
> Zheng: Yes, I agree. Your kids used a lot of different linguistic formats to describe locations. However, which are the ones typically used? [You should say it out loud.]
> Wu: The more you teach, the more your students communicate more ambiguously... By the way, try to add some activities to the beginning of your class.
> Sun: Yep...Eh...Thanks. But I cannot work out how to incorporate your suggestions into my teaching so quickly. I need time to digest.
> Zhou: Maybe we had better respect and listen to her (Teacher Sun's) ideas.

From an epistemological view, analyzing contradictions is an effective way to recognize the inner mechanism of teacher learning in lesson study.

In this episode, Li, Zheng, and Wu suggested that Sun should revise her oral expression of scientific concepts. While Sun's resistance brought a tensional atmosphere into the group, Zhou's feedback guided the tension toward self-reflection for everyone in the group. Reflection indicates the start of learning. In exposing the contradictions that occur within an activity system of lesson study, CHAT aims to help participants to better understand the processes and to identify any necessary actions required to bring about improvements to the practice.

In brief, the CHAT approach enables researchers to consider the contradictions and different motives within a given context. CHAT provides teachers and researchers participating in or examining lesson study with a theoretical framework to analyze the tangible actions and intangible motives of various participants, by stimulating new professional learning outcomes.

5.4 Methodological Inspiration

In this case study, the basic or inner hypothesis was that teacher professional learning exists in the expansive learning process. In other words, the methods for teacher learning are mainly implemented in the workplace and in a dialogical environment, where teachers can generate something that is not yet there (Engeström 2016). In the CHAT framework, expansive learning leads to the formation of a new, expanded object and pattern of activity oriented to the object. This involves the formation of a theoretical concept of the new activity, based on grasping and modeling the initial simple relationship, the "germ cell" (Davydov 1999), which gives rise to the new activity and generates its diverse concrete manifestations. The formation of an expanded object and corresponding new pattern of activity requires and induces collective and distributed agency, which leads to questioning and breaking away from the constraints of the existing activity and embarking on a journey across the uncharted terrain of the zone of proximal development (Engeström 2001).

In this lesson study case, the teachers' learning, as perceived from the interviews, was examined. Each teacher in the group was found to have learned something new, which was not the initial goal of the lesson study activity.

> What is good teaching? Good teaching does not involve activities, nor careful design, but our understanding of the textbooks (Teacher Wu).

> In terms of novice teacher development, we should respect the novice teachers and leave some space for their maneuvers (Teacher Li).

> I am not an experienced teacher. I learned a lot when I observed Teacher Sun's teaching and listened to the comments given by our colleagues (Teacher Zhou).

> We expect to learn something new from our collaboration (Teacher Zheng).

> What a precious opportunity it was for me to receive criticisms from the leaders of our teaching research group! I learned a lot from our democratic communication (Teacher Sun)!

Contextualizing lesson study with CHAT in a methodological way highlights lesson study as a process of expansive learning for teachers. As a dialectical theory, CHAT views human relationships as interwoven with multiple contradictions and conceptualizes learning as a dynamic and nonlinear process. Expansive learning is an ideal type of this kind of learning. Expansive learning is not only a learning theory but also a methodological instrument with which to design and promote teacher professional learning in an extensive manner.

5.5 Axiological Inspiration

The axiology of CHAT stems from its formative intervention, which implicates a possible change and the responsibility of researchers and participants to join together to witness the (successful) process of reforming.

According to Lewis (2015), lesson study is also an improvement science, which guides researchers and teachers in their design, implementation, analysis, and development of conclusions in the lesson study process. It supports a systematic and systemic approach of understanding human activities and interactions in complex, real-world environments.

Formative intervention is the pursuance of the value of CHAT, which is also the basic method to achieve change and transformation. Formative intervention can help researchers and practitioners understand individual activity in relation to its context and how the individual, his/her activities, and the context affect one another. Additionally, it can help document the historical relationships among multiple activities by identifying how the results from a past activity affect new activities. As Engeström (2015) noted, CHAT was developed on an interventionist premise, which means that educational research needs to be actively involved in making the world better. From this perspective, the value of lesson study, different from that of rigorous research, is intervening in the teacher practice and combining theory and practice together.

6 Concluding Remarks

The main advantage of incorporating CHAT into lesson study is that this theoretical lens can help both teachers and researchers make sense of complex contexts in a manageable and meaningful manner. In the lesson study process, CHAT also

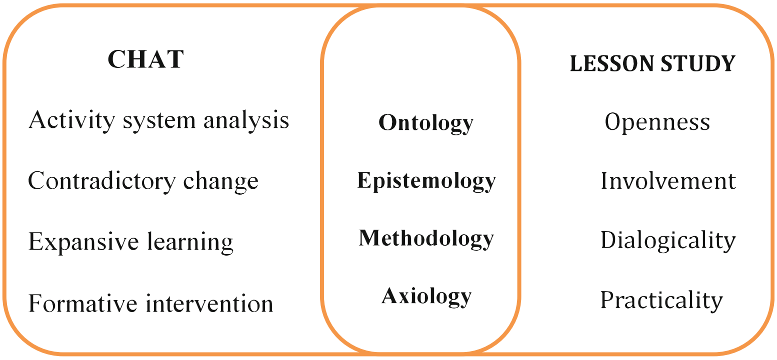

Fig. 5 Intersections between CHAT and lesson study

provides opportunities for teachers and researchers to work with a manageable unit of analysis, to understand systemic contradictions, to discover teacher learning mechanisms, and to transform findings into next-stage changes and practices.

I paired the four tenets of CHAT with the four essential commonalities of various modes of lesson study as their counterparts (see Fig. 5). The analysis results of the case of lesson study practiced in a teachers' group showed that CHAT invites further insight into the ontological, epistemological, methodological, and axiological aspects of lesson study. The CHAT approach, with a focus on systematic analysis, contradictions as the driving force, the learning process as a form of expansion, and the interventionist stance, could inspire and promote the merits of lesson study from various perspectives in the future.

Acknowledgments This work was supported by the Hong Kong Yuen Yuen Foundation (grant no. JY14003). I thank the anonymous reviewers for their invaluable feedback on the revisions of this paper. I would also like to thank Prof. Rongjin Huang from Middle Tennessee State University and Prof. Xiangming Chen from Peking University for their comments which contributed to the improvement of this article.

I would like to thank Anthony Abram for proofreading this manuscript.

References

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal of Lesson and Learning Studies., 6*(4), 283–292.

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*, 218–236.

Cole, M. (1985). The zone of proximal development: Where cultural and cognition create each other. In J. Wertsch (Ed.), *Culture, communication, and cognition* (pp. 146–161). New York: Cambridge University Press.

Davydov, V. V. (1999). The content and unsolved problems of activity theory. In Y. Engeström, R. Miettinent, & R.-L. Punamaki (Eds.), *Perspectives on activity theory* (pp. 39–52). New York: Cambridge University Press.

Editorial Team. (2016). *Mathematics (Grade4-Vol.2)*. Beijing: Beijing Normal University Press.

Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Helsinki: Orienta-Konsultit Oy.

Engeström, Y. (1996). Developmental work research as educational research. *Nordisk Pedagogik: Journal of Nordic Educational Research, 16*(5), 131–143.

Engeström, Y. (2001). Expansive learning at work: Toward an activity theoretical reconceptualization. *Journal of Education and Work, 14*(1), 133–156.

Engeström, Y. (2005). *Developmental work research: Expanding activity theory in practice*. Berlin: Lehmanns Media.

Engeström, Y. (2007). Enriching the theory of expansive learning: Lessons from journeys toward coconfiguration. *Mind, Culture, and Activity, 14*(1–2), 23–39.

Engeström, Y. (2015). *Learning by expanding: An activity-theoretical approach to developmental research* (2nd ed.). Cambridge: Cambridge University Press.

Engeström, Y. (2016). *Studies in expansive learning: Learning what is not yet there*. Cambridge: Cambridge University Press.

Engeström, Y., & Sannino, A. (2010). A. Studies of expansive learning: Foundations, findings and future challenges. *Educational Research Review, 5*(1), 1–24.

Fang, Y. (2017). School-based teaching research and lesson-case study in mediating the second-cycle curriculum reform in Shanghai. *International Journal for Lesson and Learning Studies, 6* (4), 293–305.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 16*(1), 49–65.

Hart, L. C., Alston, A., & Murata, A. (Eds.). (2011). *Lesson study research and practice in mathematics education: Learning together*. Dordrecht: Springer.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher developers, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM-Mathematics Education, 48*, 393–409.

Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM-Mathematics Education, 48*, 425–439.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. London: Cambridge University Press.

Lee, C. K., & Lo, M. L. (2013). The role of lesson study in facilitating curriculum reforms. *International Journal for Lesson and Learning Studies, 2*, 200–206.

Leontiev, A. N. (1978). The problem of activity and psychology. In A. N. Leont'ev (Ed.), *Activity, consciousness, and personality* (pp. 45–74). Englewood Cliffs: Prentice Hall.

Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: Research for Better Schools Inc.

Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research, 17*, 95–110.

Lewis, C. (2015). What is improvement science? Do we need it in education? *Educational Researcher, 44*(1), 54–61.

Lewis, C. (2016). How does lesson study improve mathematics instruction? *ZDM Mathematics Education, 48*, 571–580.

Little, J. W. (2003). Inside teacher community: Representations of classroom practice. *Teacher College Record, 105*(6), 913–945.

Lewis, C., & Hurd, J. (2011). *Lesson study step by step: How teacher learning communities improve instruction*. Portsmouth, NH: Heinneman.

Il'enkov, E. V. (1977). *Dialectical logic: Essays in its history and theory*. Moscow: Progress.

Ma, L. (1999). *Knowing and teaching elementary mathematics*. Mahwah: Lawrence Erlbaum Associates.

Mosvold, R., & Bjuland, R. (2011). An activity theory view on learning studies. *International Journal of Early Childhood, 43*, 261–275.

Murata, A., Bofferding, L., Pothen, B., Taylor, M., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics lesson study. *Journal for Research in Mathematics Education, 43*, 616–650.

OECD. (2017). *Empowering and enabling teachers to improve equity and outcomes for all*. Paris: OECD Publishing.

Paavola, S., & Hakkarainen, K. (2005). The knowledge creation metaphor–An emergent epistemological approach to learning. *Science & Education, 14*(6), 535–557.

Paavola, S., Lipponen, L., & Hakkarainen, K. (2004). Models of innovative knowledge communities and three metaphors of learning. *Review of Educational Research, 74*, 557–576.

Pang, M. F., & Marton, F. (2005). Learning theory as teaching resource: Another example of radical enhancement of students' understanding of economic aspects of the world around them. *Instructional Science, 33*(2), 159–191.

Pang, M. F., & Marton, F. (2013). Interaction between the learners' initial grasp of the object of learning and the learning resource afforded. *Instructional Science, 41*(6), 1065–1082.

Pang, M. F., & Marton, F. (2017). Chinese lesson study, learning study and keys to learning. *International Journal for Lesson and Learning Studies, 6*(4), 336–347.

Scribner, S. (1997). A sociocultural approach to the study of mind). In E. Toback, R. J. Flamagne, M. B. Parlee, L. M. W. Martin, & A. S. Kapelman (Eds.), *Mind and social practice: Selected writings of Sylvia Scribner* (pp. 266–280). New York: Cambridge University Press.

Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*, 4–13.

Stetsenko, A. (2005). Activity as object-related: Resolving the dichotomy of individual and collective planes of activity. *Mind, Culture, and Activity, 12*(1), 70–88.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Tsui, A. B. M., & Law, D. Y. K. (2007). Learning as boundary-crossing in school–university partnership. *Teaching and Teacher Education, 23*(8), 1289–1301.

Virkkunen, J., & Newnham, D. S. (2013). *The change laboratory: A tool for collaborative development of work and education*. Rotterdam: Sense.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wake, G., Swan, M., & Foster, C. (2016). Professional learning through the collaborative design of problem solving lessons. *Journal of Mathematics Teacher Education, 19*, 243–260.

Wei, G. (2017a). Review of learning by expanding: An activity-theoretical approach to developmental research. *Frontiers of Education in China, 12*(1), 130–132.

Wei, G (2017b). *Dynamics of teacher practical knowledge: From cultural-historical activity theoretical perspective* (Unpublished Doctoral Dissertation), Peking University.

Wertsch, J. V. (1985). *Vygotsky and the social formation of mind*. Cambridge, MA: Harvard University Press.

Yamagata-Lynch, L. (2010). *Activity systems analysis methods*. New York: Springer.

Zhao, Y. (2014). *Who's afraid of the big bad dragon? Why China has the best (and worst) education system in the world*. New York: Jossey-Bass.

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Developing Teachers' Expertise in Mathematics Instruction as Deliberate Practice through Chinese Lesson Study

Xue Han and Rongjin Huang

Contents

Abstract This study explored how two lesson study groups developed the teachers' expertise in mathematics instruction in China from the perspective of deliberate practice. The two lesson study groups, respectively, rehearsed and taught six research lessons on the same topic of division with fractions. They collaborated with knowledgeable others to develop research lessons based on a hypothetical learning trajectory. Guided by the learning trajectory, through exploring the deliberate practice of perfecting teaching of division of fraction, teachers developed their expertise with enacting two core practices of teaching mathematics, including revision of mathematical tasks and revision of mathematical representations.

Keywords Teacher development · Deliberate practice · Lesson study · Mathematics instruction

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_4

1 Introduction

In the past three decades, the improvement of student achievements, instructional practices, and school qualities has been closely linked to teachers' professional education and development. Effective professional education and development has been considered the key factor in improving student achievement. As an important approach to educating teachers, Chinese lesson study has been an integrated part of the teachers' professional development system for several decades in China (Huang et al. 2016a, b; Huang et al. 2017a; Wang 2013). Collectively done by a group of teachers who focus on student learning and the subject matter, integrate inquiry stance into their daily practice, and refine their teaching competence through repeated practice, lesson study incorporates many key features of effective professional development programs identified in the literature (Borko 2004; Cochran-Smith and Lytle 1999; Darling-Hammond et al. 2009; Desimone 2009; Desimone and Garet 2015; Franke et al. 2005; Garet et al. 2001; Grossman et al. 2001; Little 2002; McLaughlin and Talbert 2001; Porter et al. 2003; Wilson and Berne 1999).

More studies have emerged in the literature to reveal the influences of lesson study on teaching, teacher learning, and student achievement (Hart et al. 2011; Huang and Han 2015; Huang and Shimizu 2016; Lewis and Perry 2017; Lewis and Hurd 2011; Lewis et al. 2012; Perry and Lewis 2011). These studies revealed the effectiveness of lesson study to improve teaching, teacher learning, and student knowledge. Lewis and her team (Perry and Lewis 2011) designed a longitudinal study to investigate the effects of school-based lesson study on teachers and students. Their studies found changes in teachers' instructional practice and increased standardized test scores of students in mathematics. Further studies done by the team (Perry and Lewis 2011; Lewis and Perry 2017) showed that lesson study improved teachers' and students' mathematical knowledge. Many researchers (Han and Paine 2010; Huang and Bao 2006; Huang et al. 2016a, b; Huang and Han 2015; Huang et al. 2017b) conducted small-scale qualitative studies to understand the influences of Chinese lesson study on teaching practices and teachers' learning. They argued the participant teachers improved the core aspects of instructional practices and identified the changes in their knowledge and beliefs.

Although structurally similar to Japanese lesson study, some unique features of Chinese LS such as emphasizing the repeatedly rehearsals of the same lesson to different groups of students, perfecting an exemplary lesson, and emphasizing knowledgeable others' involvement throughout the entire of lesson study have been identified. Researchers have attempted to interpret the Chinese LS from a cultural perspective (Chen 2017) and perspectives of educational leadership (Cravens and Wang 2017) and community of practice (Fang 2017; Huang and Han 2015). Yet, more empirical studies on how teachers learn from and promote students' learning through Chinese lesson study from various perspectives are needed. Some researchers have argued that Chinese lesson study is a type of deliberate practice which has developed teachers' instructional expertise through repeatedly planning, enacting, and reflecting on a lesson with immediate feedback from knowledgeable others (Han and Paine 2010; Huang et al. 2017b). Interpreting and understanding how Chinese lesson study contributes to mathematics teacher learning will help practitioners and researchers better the mechanism of Chinese lesson study and deepen understanding of the mathematical work of teaching (Ball 2016; Selling et al. 2016). The mathematical work of teaching focuses on what teachers do and acknowledges the deliberate, discursive nature of teaching. Ball (2016) argues that the mathematical work of teaching attends to the practical core of teaching that is to interact, discuss, act, and do. Lesson study in China aimed at improving teaching and learning through examining what teachers do, what teachers should do, and what teachers could do as deliberate practice (Han and Paine 2010; Hu 2005; Huang et al. 2017b). Lesson study as a professional development approach in China, tightly embedded teachers' expertise improvement in practice and focused on the practical core of teaching mathematics, which engaged teachers in rehearsing study lessons to aim for active and effective learning of mathematics.

In this paper we drew on the theoretical lens of deliberate practice to explore how Chinese lesson study developed mathematics teachers' expertise through conducting research lessons as deliberate practice. We attempted at exploring what teaching practice the participant teachers focused on when teaching public lessons as deliberate practice through lesson study—what kind of mathematical work of teaching was acted, analyzed, discussed, and practiced? How did their mathematical work of teaching influence student learning?

2 Theoretical Framework

2.1 The Deliberate, Discursive Nature of Teaching Mathematics

The research on expert performance in some professional fields such as chess, music, medicine, etc. (Ericsson 2003, 2005; Ericsson et al. 1993) revealed that engagement in repetitive special activities was an important factor that resulted in sustained improvement and attainment of expert performance. These special activities were defined as deliberate practice that were developed for individuals to improve and repeatedly pursued by individuals with feedback from experts (Ericsson et al. 1993). Drawing on the wide studies on expert performers in the past 30 years, Ericsson and Pool (2016) argues that a common set of general principles lie at the heart of effective practice for any human practice, including teaching practice. Recently, some researchers adapted the theoretical lens of deliberate practice to investigate teacher learning and redesign teacher education programs to promote teacher learning (Deans for Impact 2016; Han and Paine 2010; Bronkhorst et al. 2011, 2014; van Gog et al. 2005; Lampert and Graziani 2009; Lampert et al. 2013). Han and Paine (2010) argued that teaching public lessons (Chinese lesson study) as deliberate practice in China allowed the teachers to recurrently refine their competence and skills in three representative teaching tasks. Lampert and Graziani (2009) identified a list of more than ten instructional activities that were used to develop novice teachers' competence in the core practice. Ericsson and his collaborators in education (2016) synthesized five key principles of deliberate practice from the science of expertise that are highly relevant to developing teacher skill. The five principles include pushing beyond one's comfort zone; working toward well-defined, specific goals; focusing intently on practice activities; receiving and responding to high-quality feedback; and developing a mental model of expertise.

2.2 Background of the Study

The participant teachers and the university experts on the study team chose to adapt the theories of learning trajectory (Simon 1995) and teaching and learning mathematics with variation (Gu et al. 2017) in designing the lessons. The mathematical topic was division of fractions for sixth graders. Division of fractions had been found to be the least understood topic in elementary school (Carpenter et al. 1988; Fendel 1987; Payne 1976). Some students could not understand the standard algorithm of dividing fractions so that they may invert the dividend instead of the divisor or invert both dividend and divisor (Ashlock 1990; Barash and Klein 1996). Some studies (Ball 1990; Borko et al. 1992; Ma 1999; Son and Crespo 2009; Tirosh 2000) also revealed that teachers usually felt challenged to explain division of fractions for understanding due to their own meager conceptual understanding of the topic.

Learning trajectory proposes sequences of tasks and activities aiming for the progressive development of students' mathematical thinking and skill (Clements and Sarama 2004, 2013; Daro et al. 2011; Simon 1995; Van den Heuvel-Panhuizen 2000). Learning trajectory outlines how student learning might develop when learning a certain mathematical topic. Clements and Sarama (2004) suggested three stages to designing learning trajectory: identifying research-based models to depict students' knowledge construction, selecting and designing key mathematical tasks, and sequencing the tasks to compose the learning trajectory. When composing the learning trajectory, the research team reviewed several versions of the Chinese textbooks, the Common Core State Standards for Mathematics (2010), and research on the topic of division of fractions (Ott et al. 1991; Sowder et al. 2010; Tirosh 2000; Tirosh et al. 1998). Once the research team and the participant teachers agreed upon the learning trajectory, the teachers selected and designed mathematical tasks for the lessons and sequenced those tasks in the lesson plans. In addition, the research team facilitated the teachers to employ the theory of teaching and learning mathematics through variations in selecting mathematical tasks and planning lessons. Teaching and learning through variation asks teachers to select varied mathematics tasks for supporting students' conceptual development and problem-solving (Gu et al. 2004; Lo and Marton 2012; Marton and Pang 2006; Watson and Mason 2006). Appropriate patterns of variation and invariance across different mathematical tasks were crucial for enhancing student learning.

3 Methods

3.1 Research Setting

Four elementary teachers from an Eastern coastal city of China made up a lesson study group on a voluntary basis. Among the four participant teachers, three experienced teachers, Ms. Shao, Ms. Han, and Ms. Tang, had more than 10 years of teaching experience, and Ms. Lu had 5 years of teaching experience. Both Ms. Lu and Ms. Shao volunteered to teach the research lessons on the same topic, but they taught the lessons to different groups of students. The rationale that they both taught the same topic was they used different story contexts in their respective lessons and taught the lessons to different groups of students so that they could bring in different perspectives on teaching the challenging topic. One university professor, Mr. Gong, and two teaching research specialists, Mr. Sao and Mr. Ren, from the district were involved in the lesson study activities as experts. Altogether, the lesson study group designed two lessons by drawing on the learning trajectory of division with fractions. The topic of division of fraction was taught in two lessons. Lesson One was fractions divided by whole numbers, and Lesson Two was whole numbers divided by fractions. Ms. Shao and Mr. Lu, respectively, taught each of the two topics to three different sixth grade classes in the same school. The average class size was about 30 students. Among the three classes for each teacher, one class received the first research lessons, the second class received the second research lessons, and the third class received the final public lesson. Tables 1 and 2 show the timeline and organization of the research lessons.

Table 1 Teacher Shao' timeline and organization of research lessons

Topic	Research lesson 1 Class: 605	Research lesson 2 Class: 603	Final lesson Class: 606
A fraction divided by a whole number	Date: 10-9-2014	Date: 10-10-2014	Date: 10-15-2014
A number divided by a fraction	Date: 10-11-2014	Date: 10-14-2014	Date: 10-17-2014

Table 2 Teacher Lu' timeline and organization of research lessons

Topic	Research lesson 1 Class: 602	Research lesson 2 Class: 604	Final Lesson Class: 607
A fraction divided by a whole number	Date: 10-9-2014	Date: 10-10-2014	Date: 10-15-2014
A number divided by a fraction	Date: 10-11-2014	Date: 10-14-2014	Date: 10-17-2014

3.2 Data Collection

The study was conducted from September to November 2014. The sources for data collection included 12 videotaped research lessons and debriefing meetings, audiotaped interviews with the participant teachers and some students, lesson plans, student classroom work, quizzes, and the participant teachers' reflection journals. The quiz was given to the students right after each research lesson. All completed quizzes were collected. The quiz had five word problems, asking the students to justify their solution methods with words, drawings, and symbols. The one-on-one interviews were conducted with all the participant teachers and ten of the students on a voluntary basis at the end of the study. During the interviews, the students were asked about their understanding of the different solution methods of division with fractions. The participant teachers were interviewed individually once at the end of the study.

3.3 Data Analysis

We analyzed the data set from multiple perspectives. The findings focusing on the interaction between theory and practice, and student learning (Huang et al. 2016a, b) and the findings from the lens of variation theory (Han et al. 2017) were published. This chapter focuses on examining the data set from the perspective of deliberate practice. Reading through all the data sources—the lesson plans, transcribed debriefing meetings, transcribed lesson videos, and transcribed interviews with the teachers and district teaching research specialists—we identified the commonalities and differences among what the two lesson study groups of teachers analyzed, discussed, and practiced. We aimed at understanding the interactive nature of the research lessons in the two groups. Thus we identified the evolving changes in the three research lessons of each group and cross-compared the commonalities and differences that emerged from the data of the two groups. Next we coded the commonalities and differences in changes to identify the themes regarding the interactive nature of the research lessons and the teachers' repeatedly rehearsed practice. We found the following two major themes across the two lesson study groups, including revising mathematical tasks and refining mathematical representations to ultimately redesign and produce the learning trajectory of division with fractions.

4 Results

Deliberate practice produces mental models to guide experts' decision (Ericsson and Pool 2016). In other words, experts in different fields build mental models to improve their performance. The theory-driven lesson study activities did not ask the participant teachers to simply rehearse what they were expected to reproduce. Instead, the teachers employed the theories of learning trajectory and teaching through variation to construct and develop their mental models of how the students should learn the new topics of division with fractions. Their expertise in teaching and learning mathematics was enhanced through revising mathematical tasks, refining mathematical representations, and modifying ways of promoting student thinking. Therefore, we argued that theory-driven lesson study activities focused the participant teachers' mathematical work of teaching on adapting and refining the triangular relationship among task design, student's learning trajectory, and mathematical representation. The three aspects of mathematical teaching practice were intertwined and inseparable.

When the teachers started with their conjectured model of student's learning trajectory in division of fractions, they created and selected a series of tasks in their respective first research lessons. However, after each rehearsed research lesson, the teachers and experts compared their evidence of student learning with the conjectured learning trajectory, they noticed some issues related to student understanding emerged. Thus they revised the mathematical tasks and refined the conjectured learning trajectory, which led to new task design and modified mathematical representations. Across the two lesson study groups, we found that the teachers and experts had two purposes when redesigning the mathematical tasks. One purpose was to use the tasks to scaffold student learning, while another purpose was to select the tasks that provided more opportunities to develop students' cognition which led to deep learning of the conceptual understanding of division with fractions.

4.1 Revision of Review Tasks

It is critical how a teacher activates students' prior knowledge for learning new knowledge in a lesson. The teachers in both lesson study groups redesigned the review tasks aiming more closely to connect the students' prior knowledge with the current critical aspects of the new knowledge in the lessons. In Teacher Lu's research lessons on fractions divided by integers, she and her colleagues—other teachers and the experts— changed the review task from five children equally sharing 5 liters of water to partitioning a cake of 1 kilogram into five equal parts, from $5 \div 5$ to $1 \div 5$. Similarly, in Teacher Shao's research lessons on the same topic, she and her colleagues created two review tasks that were "Two people equally shared 2 liters of juice. How much did each person get?" and "Two people equally shared 1 liter of juice. How much did each person get?". At the debriefing meeting after the first research lesson, Teacher Lu's colleagues commented that the original review task was not highly relevant to the new topic—fractions divided by integers. In addition, the story context of the original review task was not consistent with the following example problem that was to divide $\frac{4}{5}$ of a piece of paper into halves. The revised review tasks more explicitly helped the students discern partition division in the consistent story contexts that was a critical aspect of understanding the standard algorithm of division with fractions. The students were expected to recognize the interpretation of partition division which they had learned with whole numbers in understanding division with fractions. Once making the connection, the students could efficiently transfer their prior knowledge of partition division to the new knowledge in the lessons. The new review tasks effectively paved the way to new learning for the students.

Following the review tasks, both teachers showed the same first example task, $\frac{4}{5} \div 2$, but in different story contexts. Teacher Shao's lesson worked on the task, "How much each person got when two people equally shared $\frac{4}{5}$ of a liter of juice," while Teacher Lu's students solved the problem, "How much each child got when two children equally shared a cake of $\frac{4}{5}$ kg." Throughout the, respectively, following research lessons, the students had no difficulty with discerning the partition division context and setting up a correct division number sentence. To deepen the students' understanding of division with fractions, Teacher Lu and her colleagues added one more task after $\frac{4}{5} \div 2$. Because some students used the special feature that the numerator 4 is divisible by 2, $\frac{4}{5} \div 2 = {}^{(4 \div 2)}/_{5}$, the teachers were not confident the students had understood why they multiplied $\frac{4}{5}$ by the reciprocals of 2 to solve the problem. They decided to have the students do one more example task. The task asked the students to equally distribute a cake of $\frac{1}{5}$ kg between two children. Without the special feature the students needed to rely on their thorough understanding of partition division by 2 as multiplying $\frac{1}{2}$. Dividing a fraction by 2 is the same as looking for $\frac{1}{2}$ of that fraction.

In the six lessons on whole numbers divided by fractions, Teacher Shao revised how to review the students' prior knowledge too. In the first lesson, Teacher Shao reviewed the reciprocals of five numbers. The teachers all noticed that the lesson did not achieve the lesson objectives, and Teacher Shao should revise the review task. Simply reviewing reciprocals of five numbers did not help the students understand the algorithm from the perspective of proportional relationship. In the second research lesson, Teacher Shao reviewed the reciprocals of five numbers and then posed the question of how many $\frac{1}{2}$ hour, $\frac{1}{3}$ hour, $\frac{1}{5}$ hour, and $\frac{1}{10}$ hour are in 1 hour, respectively. Teacher Shao and her colleagues thought those review tasks prepared the students' conceptual understanding of multiplicative relationship in division with fractions. Ms. Han commented at the debriefing meeting that the students would be able to recognize that the distance changed at the same rate as time changed when solving the following example problem. The teachers wanted the students to discern the quantities of distance and the times had the same multiplicative relationship, for example, the time increasing from 2/3 of an hour to 1 hour at the rate of $\frac{3}{2}$, and then the distance also increased at the same rate from 2 km to $\frac{3}{2}$ times as far as 2 km.

After the second research lesson, the teachers felt that reviewing the reciprocals could have given the students an excessive hint so that they purely memorized the standard algorithm of multiplying the reciprocal of a divisor. They decided to remove the review task of reciprocals for the goal that the students would focus on the conceptual understanding of the algorithm from the perspective of proportional relationship. Thus the teachers modified the numbers in the review tasks and added three more questions in the final research lesson. The final review tasks were five similar questions—how many $\frac{1}{3}$ hour in $\frac{2}{3}$ hour, how many $\frac{1}{4}$ hour in $\frac{3}{4}$ hour, how many $\frac{1}{3}$ hour in 1 hour, how many $\frac{1}{5}$ hour in 1 hour, and how many $\frac{1}{10}$ hour in 1 hour. The modified numbers—$\frac{2}{3}$ and $\frac{3}{4}$—were aligned with the numbers in the example problem that followed the review in the final lesson. In addition, Teacher Shao asked the students three new questions when reviewing the prior knowledge: how many times an hour is as much as $\frac{1}{3}$ hour, how many times an hour is as much as $\frac{1}{5}$ hour, and how many times an hour is as much as $\frac{1}{10}$ hour.

T: I can also express in this way- how many times greater is one hour than $\frac{1}{3}$ hour.
S: Three times.
T: Next problem.
S: There are five copies of $\frac{1}{5}$ hour in one hour.
T: How can you express this relationship in a different way?
S: One hour is five times $\frac{1}{5}$ hour.
T: How about $\frac{1}{10}$ hour? How many $\frac{1}{10}$ hour does one hour include?
S: Ten.
T: Express it in another way? Let's say it together.
S: One hour is ten times as much as $\frac{1}{10}$ hour. (Lesson on Oct.17, 2014)

The revised review tasks directly focused on the multiplicative relationship between $\frac{1}{3}$ and $\frac{2}{3}$, $\frac{1}{4}$ and $\frac{3}{4}$, $\frac{1}{3}$ and 1, $\frac{1}{5}$ and 1, and $\frac{1}{10}$ and 1, through which the teachers established a conceptual link to the following example problem. The students' prior knowledge of multiplicative relationship was activated, which got them ready for developing the conceptual understanding of the standard algorithm through proportional relationship. The activated prior knowledge supported the students to make sense of the critical aspect of the standard algorithm in a special partition division context. The problem contexts about unit rates (e.g., speed/hour, price/pound, etc.) represented a special partition division situation. Those contexts of seeking a unit rate are more challenging for students to understand than the measurement interpretation of division that asks how many times a fraction goes into a whole number.

4.2 *Revision of Major Mathematical Tasks*

To better provide scaffolding to the students in understanding dividing whole numbers by fractions, Teacher Lu and her colleagues adjusted the example problems. The original first example task was to figure out how many cups were needed to contain 2 liters of milk if one cup contained $\frac{1}{5}$ of a liter of milk. However, on the test at the end of the first research lesson, there were 13 students out of 28 students who did not solve one problem correctly—how many cups were needed for 1 liter of milk if one cup contained $\frac{2}{5}$ of a liter of milk. The students were required to use a picture and words to show their answers. The test results showed almost half of the class did not understand the quantity relationship between $\frac{2}{5}$ and 1 situated in a story context (Fig. 1).

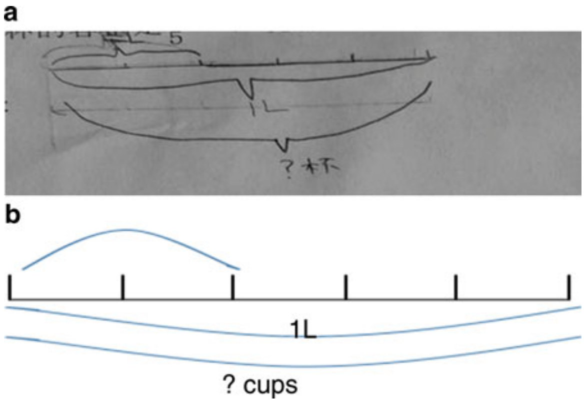


Fig. 1 (**a**) Student's picture of Problem 1 on Test 1(after Research Lesson 1). (**b**) Student's picture of Problem 1 on Test 1(translated)

Teacher Lu reflected that she found the majority of the students could not have used the knowledge she wanted them to use when solving the following problems. Originally, she thought the students understood the idea—how many times a unit fraction goes into one whole—however, in the lesson most of the students did not apply the knowledge to solve problems, which was seen on the test too. Teacher Lu reasoned that she did not do enough to support the students' learning. "I am thinking that $2 \div \frac{2}{5}$ can be solved by drawing on the knowledge of $1 \div \frac{1}{5}$. The answer to $1 \div \frac{1}{5}$ can be figured out in a concrete way. The students will apply the knowledge of $1 \div \frac{1}{5}$ to solve the following problems. So this task is to transit their thinking" (Debriefing meeting on Oct.11, 2014). Teacher Ren followed up, suggesting that Teacher Lu added the similar questions, such as what about one cup holding $\frac{1}{4}$ liter, $\frac{1}{3}$ liter, and $\frac{1}{6}$ liter. They expected the students to discover the pattern from those questions. Once the students answered those questions, Teacher Lu would show next task that was to find how many cups were needed for 1 liter of milk if one cup contained $\frac{2}{5}$ liter. Both Teacher Ren and Mr. Gong commented that the key task was to understand $1 \div \frac{2}{5}$. They thought the students would be able to solve the rest once they correctly understood this task. Therefore, Teacher Lu redesigned the mathematical tasks in the whole lesson while keeping the context unchanged—counting cups for a certain amount of milk. She started the lesson with presenting the first group of tasks $1 \div \frac{2}{5}$, $1 \div \frac{1}{4}$, $1 \div \frac{1}{3}$, and $1 \div \frac{1}{6}$, next showing the second group of tasks $1 \div \frac{2}{5}$ and $1 \div \frac{2}{7}$, and last asking the students to mentally solve the third group of tasks $2 \div \frac{2}{5}$, $3 \div \frac{2}{5}$, and $4 \div \frac{2}{5}$. The test after the second research lesson revealed a different result from the first test (the first and second research lessons were taught to two different classes of students). The students solved a test item—how many cups were needed for 1 liter of milk if one cup contained $\frac{2}{3}$ of a liter of milk—similar to the test problem given to the students during the first research lesson. All 28 students answered the problem correctly with a correct picture (Fig. 2).

Looking across the two lesson study groups, we noticed that both groups of teachers settled on an important mathematical idea to help students make sense of the standard algorithm for whole numbers divided by fractions. The key idea in the learning trajectory of division with fractions was eventually centered on proportional

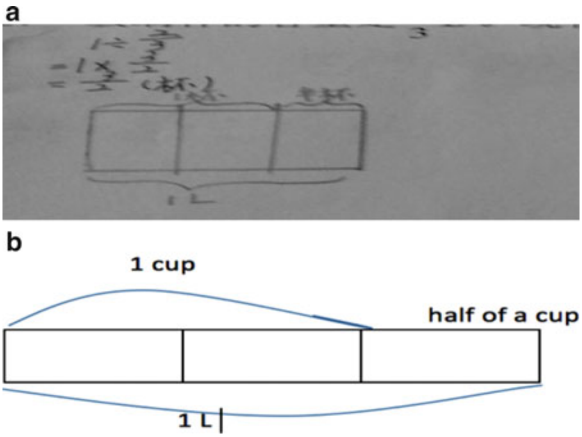


Fig. 2 (**a**) Student's picture of Problem 2 on Test 2 (after Research Lesson 2). (**b**) Student's picture of Problem 2 on Test 2 (translated)

reasoning, drawing on proportional reasoning to conceptually understand the standard algorithm. By sixth grade, the students had learned rate, ratio, and proportion and developed proportional reasoning. The students were expected to understand the algorithm through the proportional relationship among quantities. Teacher Shao and her group redesigned the tasks about distance, time, and speed, while Teacher Lu and her group facilitated the students to use proportional reasoning to understand the relationship between 1 liter and $\frac{2}{5}$ of a liter, 2L and $\frac{2}{5}$ L, 3L and $\frac{2}{5}$ L, etc. For example, when a person walked 5 kilometers within $\frac{2}{3}$ of an hour, finding the speed per hour was a division problem (distance $\div$ time $=$ speed, which was $5 \div \frac{2}{3} =$ speed per hour). Next the students needed to discern a multiplicative relationship among quantities and apply proportional reasoning to understand the standard algorithm. It is a direct proportion between distance and time, such as 5 kilometers within $\frac{2}{3}$ of an hour and the distance within 1 hour.

As we discussed above, in the lessons on whole numbers divided by fractions, Teacher Shao revised the review tasks by adding questions about how many fractions of an hour 1 hour has, e.g., how many $\frac{1}{10}$ of an hour 1 hour includes. Because of the revision the teachers removed the first question in the major mathematical example problem that asked the speed of walking $\frac{3}{4}$ km within 2 hours—$\frac{3}{4} \div 2$. The major example was: *Xiaohua walked 2 kilometers within* $\frac{1}{2}$ *hour. Xiaoming walked 2 kilometers within* $\frac{2}{3}$ *of an hour. Xiaohong walked* $\frac{15}{8}$ *kilometers within* $\frac{3}{4}$ *of an hour. How many kilometers did each of them walk in 1 hour?* The number sentences were $2 \div \frac{1}{2}$, $2 \div \frac{2}{3}$, and $\frac{15}{8} \div \frac{3}{4}$. The central problem the students worked on in the last research lesson had revised numbers: *Xiaohua walked 3 kilometers within* $\frac{1}{2}$ *hour. Xiaoming walked 5 kilometers within* $\frac{2}{3}$ *hours. Xiaohong walked* $\frac{21}{8}$ *kilometers within* $\frac{3}{4}$ *hours. How many kilometers did each of them walk in 1 hour?* The number sentences were $3 \div \frac{1}{2}$, $5 \div \frac{2}{3}$, and $\frac{21}{8} \div \frac{3}{4}$. At the debriefing meeting after the second research lesson, Mr. Gong and two district teaching research specialists pointed out that $2 \div \frac{1}{2}$ and $2 \div \frac{2}{3}$ both had an integer answer, which could be misleading for the students to think about division of whole numbers by fractions. To avoid the possible confusion, the teachers changed the numbers in the major task for the last research lesson. However, more significant revision and discussions among the teachers related to the central mathematical idea of proportional reasoning in the three research lessons were around using appropriate mathematical representations—pictorial representations—which we would analyze in next the section.

Similarly, Teacher Lu and her colleagues also turned to proportional reasoning to help the students understand the standard algorithm after the first unsuccessful lesson. At the debriefing meeting, Mr. Gong, Teacher Ren, and Teacher Han all proposed that Teacher Lu should ask the students to use a new series of tasks (see the discussion above about the third group of tasks) and think about the relationship between $1 \div \frac{2}{5}$ and $2 \div \frac{2}{5}$. They expected the students to realize the direct proportional relation between how many $\frac{2}{5}$ are in two wholes and how many $\frac{2}{5}$ are in one whole. This meant the students needed to first figure out $1 \div \frac{2}{5} = \frac{5}{2}$ and next use the proportional reasoning that 2 liters are two times of 1 liter and thus make sense of $2 \div \frac{2}{5}$ two times as many as $1 \div \frac{2}{5} \rightarrow 2 \div \frac{2}{5} = 2 \times (1 \div \frac{2}{5}) = 2 \times \frac{2}{5}$. "For example, the students first understood $1 \div \frac{2}{5}$, and we ask them what about $2 \div \frac{2}{5}$? Because 2 is 2 times of 1, the result (of $2 \div \frac{2}{5}$) is 2 times (of $1 \div \frac{2}{5}$) too" (Teacher Ren at the debriefing meeting on October 11, 2014). The second research lesson showed that the students incorporated this idea very well.

- T: Let's figure out $2 \div \frac{2}{5}$. I don't want you to tell me the answer quickly. Instead, I want you to pause and think. We have the number sentence now and we just discussed how many $\frac{2}{5}$ of a liter one liter includes. Now we need to figure out how many $\frac{2}{5}$ of a liter two liters include. Can you use what we've discussed to solve this problem? Raise your hands when you are ready. You can try to draw a picture if you need.
- Students are thinking...
- T: Let's hear what you've thought about. How about you?
- S: Based on $1 \div \frac{2}{5}$, I think when the dividend increases twice, the quotient increases twice too. So, 2 divided by $\frac{2}{5}$ means 5 cups. Also $2 \div \frac{2}{5} = 2 \times 5 \div 2$, the answer is 5 too. That means multiplication works too.
- T: OK. He told us how he solved this problem. How many cups can one liter be distributed into?
- Ss: $\frac{5}{2}$ cups.
- T: 2 liters are twice as many as one liter, therefore 2 liters of milk can be distributed into...
- Ss: $\frac{10}{2}$ cups.
- T: That's 5 cups, 2 times of $\frac{5}{2}$, $2 \times \frac{5}{2}$. Now we have a very good explanation to understand $2 \times \frac{5}{2}$. Can you explain the reason in your own words? I want this boy to tell us your explanation.
- S: $2 \div \frac{2}{5}$ equals to $2 \times \frac{5}{2}$, which means a number divided by a fraction is to multiply the reciprocal of the fraction.
- T: You described the calculation process, but you did not tell us why you can multiply the reciprocal. How about that girl?
- S: Because division is the reverse operation of multiplication. A quotient times divisor equals dividend. $5 \times \frac{2}{5}$ equals 2, which tells me this method is correct.

- T: We can think in this way if the problems have simple numbers. For big numbers it may take more time to think in this way. Let's see a picture. When we solve this problem, first we think about 1 liter of milk that can be distributed into...
- Ss: $5/2$ cups.
- T: 2 liters can be distributed into...
- Ss: 2 times of $5/2$ cups.

(The second research lesson of Teacher Lu on October 14, 2014)

Though not every student at this moment can use their own words to clearly explain the proportional relationship between $2 \div 2/5$ and $1 \div 2/5$, they all seemed to understand the direct proportion. When Teacher Lu asked them to work in pairs to solve 3 liters and 4 liters, they all solved the problems by drawing on the proportional relationship, three times of $2/5$ and four times of $2/5$. Teacher Lu called on four students to explain their reasoning, and all four students clearly stated the proportional relationship. Mentally calculating how many cups 100 liters of milk can be distributed into (one cup contained $2/5$ of a liter), all the students answered $100 \times 5/2 = 250$ cups. The lesson demonstrated that the students understood the standard algorithm through proportional reasoning about quantities in the story contexts. Closely related to revision of mathematical tasks, the teachers spent a great deal of time discussing how to select appropriate pictorial representations and support students to draw pictures in making sense of the standard algorithm.

4.3 Revision of Mathematical Representations

Both groups of teachers discussed the issues of drawing diagrams to demonstrate thinking and solving the problems and made tremendous changes in the pictorial representations the teachers should model and the pictures the students were required to draw. The changes were made to go along with the changes in the mathematical tasks, which ultimately produced a refined learning trajectory of division with fractions. Teacher Lu and her group incorporated students' reactions to pictorial representations into the lessons. Teacher Shao and her group faced a more challenging context that was about distance, time, and speed around which they identified the students' difficulties and confusion to change the pictorial models used by the teacher.

In Teacher Lu's second research lesson on fractions divided by whole numbers, the teachers noticed that some students divided the rectangle bar (a cake) made up of five equal parts into halves by dividing each $1/5$ into halves. The more efficient dividing was to simply draw a horizontal line through the middle of the rectangle. In order to help the students understand the two ways of dividing into halves, the teachers decided to ask the students to share their different pictures and discuss the two ways in the last research lesson on fractions divided by numbers (Fig. 3).

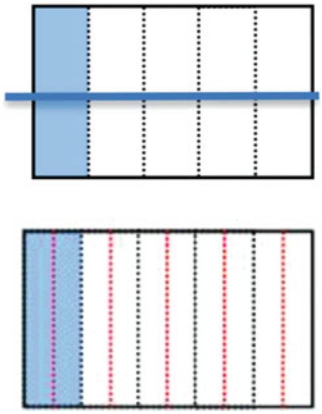


Fig. 3 Two ways to divide a cake into halve

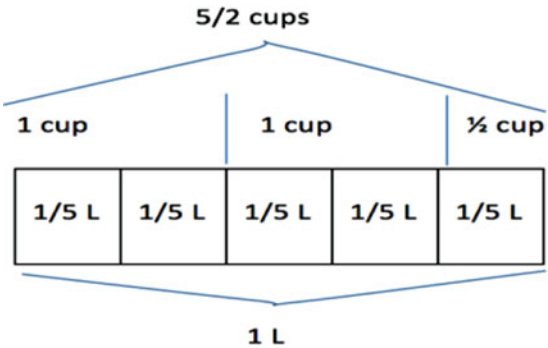


Fig. 4 Teacher Lu's model to explain $\frac{1}{2}$ cup

In Teacher Lu's research lessons on numbers divided by fractions, the students had difficulties in connecting their understanding of the standard algorithm with the pictures they drew. At the debriefing meeting after the second research lesson, one of Teacher Lu's colleagues mentioned that he saw one boy divide one whole into four equal parts and labeled two parts as one cup when solving the problem $1 \div \frac{2}{5}$. He did not know how to show half of a cup—$\frac{1}{5}$ of a liter in the picture. Obviously, the student did not understand the standard algorithm—why $2 \div \frac{2}{5} = \frac{5}{2}$, and how he should use the picture to show his reasoning about the algorithm. In addition, several students drew five equal parts in the whole, divided the whole into halves with a line and then colored four equal parts. Those students could not explain why they divided the whole into halves with a line. Therefore, Teacher Lu and her colleagues decided to invite the students to share their different pictures and emphasized modeling and explaining $\frac{1}{5}$ of a liter as half of a cup (one cup holds $\frac{2}{5}$ of a liter). In the last research lesson, Teacher Lu called on four pairs of students to share their pictures and explained why there was $\frac{1}{2}$ cup. All the four pairs were able to correctly draw a picture and spell out the reason why they got $\frac{1}{2}$ cup for $\frac{1}{5}$ of a liter, though they used different pictures. At the end of the discussions, Teacher Lu showed her model to further elaborate on why the answer was $\frac{5}{2}$ cups and why there was $\frac{1}{2}$ cup (Fig. 4).

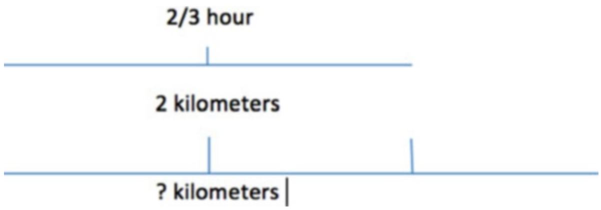


Fig. 5 Double-line diagram

When the numbers became complicated and the students had understood the standard algorithm, Teacher Lu and her group did not require the students to draw pictures anymore when solving the problems. They wanted the students to develop abstract thinking apart away from the concrete pictures, being able to use words and number sentences to explain thinking and solve problems. For example, in the last research lesson on dividing numbers by fractions, Teacher Lu told the students to mentally think about how many cups 3 liters of milk can be distributed into 4 liters without drawing pictures. Once the students solved the two problems mentally, the teacher showed her pictures to demonstrate and justify the students' solutions. It was clear that two new elements were added into the learning trajectory. One was the key to understanding $5/2$ as the answer to $1 \div 2/5$ was to know $1/5$ of a liter indicating half a cup from the picture. The other element was to grow their abstract reasoning out of concrete pictures along the learning trajectory.

It was challenging to explain and understand why the standard algorithm works using pictures, words, and story contexts. The teachers in Teacher Shao's lesson study group took on the challenge to design and revise the pictorial representations in order to develop the students' conceptual understanding. They focused on what visual models Teacher Shao should demonstrate and how to address the students' difficulties in drawing pictures. They made changes between a single-line diagram and double-line diagram and between area model and length model.

Originally, the teachers used one single-line diagram to illustrate the problem in the first research lesson on numbers divided by fractions. At the debriefing meeting, the teachers recognized the difficulty the students showed when using one single line to illustrate the standard algorithm. They decided to adopt double-line diagrams to model the problem in the second research lesson. One line represented the distance the person walked, and the other line represented the speed. From the first lesson, they noticed that the majority of the students did not know how to illustrate the time and the distance on one single line. Many students were not able to represent walking 2 km in $2/3$ hour on one single-line diagram. They stumbled with the one single line representing 2 km or $2/3$ hour. If they labeled the one single line as $2/3$ hour, they did not know where to locate 1 hour on the line. Figure 5 showed a double-line diagram the teachers used in the second research lesson. The top line diagram showed the distance of 2 km walked in $2/3$ hour, and the second line diagram indicated the distance walked in 1 hour.

The teachers thought the double-line diagram could support the students' understanding of the standard algorithm through multiplicative relationships. In the first research lesson, there was only one girl who drew on the idea of multiplicative relationships, while all the other students were struggling with explaining the reason

Fig. 6 Area model in the lesson on fractions divided by whole numbers

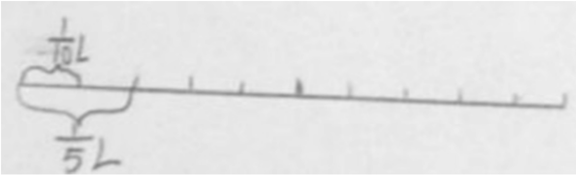
for multiplying by the reciprocal. The girl told her small group if Xiaoming walked 2 km in ⅔ hour, he would walk $\frac{3}{2}$ times in 1 hour as far as 2 km. Thus, Teacher Shao and her colleagues decided they should employ double-line diagrams to focus on scaffolding multiplicative relationships.

In the three lessons on fractions divided by whole numbers, Teacher Shao's group centered their discussions and revision on the critical aspect of understanding the algorithm—elaborating on the relationship between multiplication and division with fractions. At the beginning, the teachers planned to use area models to demonstrate the algorithm. From the first research lesson, the teachers noticed three difficulties the students had in using pictorial models to explain their reasoning. First, they did not know how to equally divide in the picture. Second, they did not partition the whole to show the answer. Last, some students wrote a multiplication number sentence based on the picture without setting up a division equation. Another observation the teacher had about visual representations was that some students correctly drew the picture based on the calculated answer, which indicated that the students did not use visual models as tools to demonstrate their thinking. Instead, they drew pictures as solutions. The teachers expected the students would draw visual models to understand the algorithm. Thus, in the following two research lessons, Teacher Shao encouraged the students to use different visual models including area models and length models for deepening their conceptual understanding of the standard algorithm.

Both models, area models and length models, have their own advantages. The area model can clearly show the relative size of a part to a whole (Cramer et al. 2008). In the lesson the students would easily understand four parts out of five equal parts in a whole rectangle divided into halves—$\frac{4}{5} \div 2$. However, it was challenging for the students to discern the reasoning for the standard algorithm in the area models—why to multiply the reciprocal of the divisor. Solving this problem—how much each person got when two people equally shared $\frac{4}{5}$ of a liter of juice (Fig. 6)—the students could just rely on the visual model to figure out the answer, while not understanding it was the same as finding half of $\frac{4}{5}$ of a liter. The teachers noticed that it was not easy for the students to make this conceptual connection between multiplication and division.

In the second research lesson, Teacher Shao built upon the students' work in instruction that adopted the length model to illustrate the partitive division (Fig. 7). The length model is critical to develop student understanding of fractions and other fraction concepts (Petit et al. 2010; Siegler et al. 2010). After the students discussed and shared their justification of dividing $\frac{4}{5}$ by 2, Teacher Shao elicited different visual models. One student presented his length model and explained, "I divided the line segment into 5 equal parts. I then took 4 equal parts to represent $\frac{4}{5}$. The 4 equal parts, $\frac{4}{5}$, was then split in half. One half of the 4 equal parts thus showed the answer $\frac{2}{5}$" (Lesson on October 14, 2014).

Fig. 7 Length model in the lesson on fractions divided by whole numbers



Solving the second example problem—⅕ of a liter of juice equally shared by two people—some students first divided a line segment into five equal intervals, split ⅕ interval into halves, and then labeled one half of ⅕ as ⅒ to display the answer. In the last research lesson, Teacher Shao gave the students an opportunity to share their area and length models with the whole class.

5 Discussions and Conclusion

In this study we argued that Chinese lesson study engaged mathematics teachers in deliberate practice that developed their expertise through repeatedly refining the two critical elements of constructing learning trajectory. A learning trajectory demonstrated in this study includes two important elements that are mathematical tasks and mathematical representations. Engaging in Chinese lesson study is to intensely, repeatedly practice the two core aspects of teaching mathematics. Chinese lesson study situates teachers in a community of practice with knowledgeable others who provide immediate feedback on their practice. In the learning community, teachers frequently conduct lesson study activities as deliberate practice that focus on the core aspects of teaching mathematics.

The study explored two lesson study groups at the same elementary school that, respectively, investigated the learning trajectory of division with fractions. With the support from the university mathematics experts and district teaching specialists, the teachers from each group designed a learning trajectory and taught six research lessons on the topic of division with fractions. After each research lesson, the lesson study groups had debriefing meetings to discuss student learning and provide feedback on revising the lessons. We found that both of the groups repeatedly redesigned the mathematical tasks and refined the mathematical representations throughout rehearsing the research lessons. Ultimately, they aimed at creating a learning trajectory that best matched with how students learned division with fractions cognitively and developmentally. During the process, we argued that the teachers developed and enhanced their mental models of teaching mathematics that eventually contributed to their expertise development.

Ericsson and Pool (2016) proposed that deliberate practice has a central purpose that is to develop effective mental representations. According to their studies on expertise in several professional fields such as medicine, music, chess, and sports, they argued that better mental representations would be developed when a practitioner adapted and responded to deliberate practice over time, which could lead to possibilities for improved performance. When discussing what mental representations produced from deliberate practice implied for teacher development, Ericsson and his collaborators in Teacher Education (2016) considered three implications, including understanding how students learn; knowing how students make sense of new ideas, keep information, solve problems, and transfer prior knowledge to new situations; and assessing student learning to adjust mental model about how students learn the content. We argued that the deliberate practice the teachers engaged in through rehearsing and teaching research lessons produced and further developed the teachers' mental representations of how students learned the topics of division with fractions.

Mental representations in the study were produced from the teachers' collaborative practice to create a learning trajectory. From the start they used their original mental representations of how students learned division with fractions to design the learning trajectory. Along with the research lessons, collaborative discussions, and immediate feedback, the teachers drew on their observations and understanding of the students' reactions to the lesson and the students' misconceptions and difficulties to modify and redesign mathematical tasks and pictorial representations. When modifying the tasks, the teachers focused on identifying the fundamental mathematical ideas in student learning. There were two fundamental ideas settled on in the study: the relationship between multiplication and division of fractions and proportional reasoning involved in understanding the standard algorithm of division with fractions. Around the fundamental mathematical ideas across the lessons, the teachers centered their practice on creating consistent problem contexts, using appropriate tasks to activate prior knowledge for scaffolding the students' learning of new topics, adjusting the logical chain of tasks to support the students' conceptual understanding of the standard algorithm, and modifying the numbers in the tasks to support student learning. Mathematical tasks rely on proper mathematical representations to fulfill their missions for the purpose of constructing an effective learning trajectory. In the study, the teachers learned that the students had difficulties with linking visual drawings to the conceptual understanding of the algorithm and used different models to illustrate their understanding. Thus, the teachers encouraged the students to express their understanding in various models, addressed the key issues to help them connect conceptual understanding with visual models, and eventually supported them to grow abstract reasoning about division with fractions.

We conclude that the teachers' expertise is related to their ability to form a learning trajectory that depicts how students learn mathematical topics. Such a learning trajectory is based on expert teachers' mental models of what mathematical tasks are compatible with and support student learning cognitively and developmentally and what mathematical representations should be employed along with the tasks to stimulate, elicit, and enhance conceptual understanding (Huang et al. 2016a, b). Better mental model of teaching mathematics is the product of deliberate practice and built upon teachers' profound understanding of fundamental mathematics.

In order to have successful revision of the lessons in lesson study activities, there are some factors in place to make deliberate practice possible. The participant teachers have been working in a professional learning community—school-based teaching research groups where they have been used to conduct public lessons with expectations of receiving collegial feedback from colleagues and district teaching research specialists. The district teaching research specialists regularly visited classrooms, organized and facilitated professional development activities for school teachers, and directly worked with school-based teaching research groups to guide and supervise professional learning of teachers. For example, this study provided an opportunity for the participant teachers to link theory with practice through working with the district teaching research specialists and the university math education experts. They collaborated together to experiment on mapping out the learning trajectory of division with fractions. Meanwhile, the school infrastructure traditionally set aside time for teachers to take part in professional learning each day and each week, as a teacher may teach only two or three lessons a day. The participant teachers did not need to find substitute teachers for being involved in the lesson study activities; instead, it is job-embedded, part of their teaching practice. With the educational reforms in the past two decades, lesson study activities in China paid more attention to student learning through collecting formative assessment data. The participant teachers revised their lessons by drawing on multiple sources of data, such as observation data of student learning collected by their colleagues, end-of-lesson assessment data, and interviews with some students after each lesson. These factors allowed successful revision of each research lesson and made deliberate practice possible for teachers to engage in and develop expertise on their jobs.

References

Ashlock, R. D. (1990). *Error patterns in computation*. New York: Macmillan.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*, 132–144.

Ball, D. L. (2016). Uncovering the special mathematical work of teaching. In *Plenary lecture at the 13th International Congress on Mathematical Education (ICME)*. Germany: Hamburg. Retrieved from https://deborahloewenbergball.com/presentations-intro/#presentations.

Barash, A., & Klein, R. (1996). Seventh grades students' algorithmic, intuitive and formal knowledge of multiplication and division of non negative rational numbers. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 35–42). Valencia: University of Valencia.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher, 33*(8), 3–15.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*, 194–222.

Bronkhorst, L. H., Meijer, P. C., Koster, B., & Vermunt, J. D. (2011). Fostering meaning-oriented learning and deliberate practice in teacher education. *Teaching and Teacher Education, 27*, 1120–1130.

Bronkhorst, L. H., Meijer, P. C., Koster, B., & Vermunt, J. D. (2014). Deliberate practice in teacher education. *European Journal of Teacher Education, 37*(1), 18–34.

Carpenter, T. C., Lindquist, M. M., Brown, C. A., Kouba, V. L., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Trends and conclusions. *Arithmetic Teacher, 36*(4), 38–41.

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 283–292.

Clements, D., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Clements, D., & Sarama, J. (2013). Rethinking early mathematics: What is research-based curriculum for young children? In L. English & J. Mulligan (Eds.), *Reconceptualizing early mathematics learning: Advances in mathematics education*. Dordrecht: Springer.

Cochran-Smith, M., & Lytle, S. (1999). Relationships of knowledge and practice: Teacher learning community. *Review of Research in Education, 24*, 249–305.

Cramer, K., Wyberg, T., & Leavitt, S. (2008). The role of representations in fraction addition and subtraction. *Mathematics Teaching in the Middle School, 13*(8), 490–496.

Cravens, X., & Wang, J. (2017). Learning from the masters: Shanghai's teacher-expertise infusion system. *International Journal for Lesson and Learning Studies, 6*(4), 306–320.

Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession: A status report on teacher development in the United States and abroad*. Oxford: National Staff Development Council.

Daro, P., Mosher, F., & Corcoran, T. (2011). *Learning trajectories in mathematics (Research Report No. 68)*. Madison: Consortium for Policy Research in Education.

Deans for Impact. (2016). Practice with purpose: The Emerging science of teacher expertise. September, 5, 2017, Retrieved from *https://deansforimpact.org/resources/practice-with-purpose/*.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*, 181–200.

Desimone, L. M., & Garet, M. S. (2015). Best practices in teachers' professional development in the United States. *Psychology, Society, and Education, 7*(3), 252–263.

Ericsson, K. A. (2003). Deliberate practice and the acquisition and maintenance of expert performance in medicine and related domains. *Academic Medicine, 79*(10), 70–81.

Ericsson, K. A. (2005). Recent advances in expertise research: A commentary on the contributions to the special issue. *Applied Cognitive Psychology, 19*(2), 233–241.

Ericsson, K. A., & Pool, R. (2016). *Peak*. Boston: Houghton Mifflin Harcourt.

Ericsson, K. A., Ralf, K., & Clemens, T. (1993). The role of deliberate practice in the acquisition of expert performance. *Psychological Review, 100*(3), 363–406.

Fang, Y. (2017). School-based teaching research and Lesson-case study in mediating the second-cycle curriculum reform in Shanghai. *International Journal for Lesson and Learning Studies, 6* (4), 293–305.

Fendel, D. M. (1987). *Understanding the structure of elementary school mathematics*. Newton: Allyn and Bacon.

Franke, M., Kazemi, E., Shih, J., Biagetti, S., & Battey, D. (2005). Changing teachers' professional work in mathematics: One school's journey. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 209–229). Mahwah: Erlbaum.

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38*(4), 915–945.

Grossman, P. L., Wineburg, S. S., & Woolworth, S. (2001). Toward a theory of teacher community. *Teachers College Record, 103*(6), 942–1012.

Gu, F., Huang, R., & Gu, L. (2017). Theory and Development of Teaching through Variation in Mathematics in China. In R. Huang & Y. Li (Eds.), *Teaching and learning through variations* (pp. 13–42). Rotterdam: Sense.

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: An effective way of mathematics teaching in China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–348). Singapore: World Scientific.

Han, X., & Paine, L. (2010). Teaching mathematics as deliberate practice through public lessons. *The Elementary School Journal, 110*(4), 519–541.

Han, X., Gong, Z., & Huang, R. (2017). Teaching mathematical concepts through variation and learning progression: A case study of division of fractions. In R. Huang & Y. Li (Eds.), *Teaching and learning through variations* (pp. 267–293). Rotterdam: Sense.

Hart, L. C., Alston, A. S., & Murata, A. (2011). *Lesson study research and practice in mathematics education: Learning together*. New York: Springer.

Hu, G. (2005). Professional development of secondary EFL teachers: Lesson from China. *Teachers College Record, 107*(4), 654–705.

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298.

Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher developers, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*, 393–409.

Huang, R., Gong, Z., & Han, X. (2016a). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM Mathematics Education, 48*, 425–439.

Huang, R., Ye, L., & Prince, K. (2016b). Professional development system and practices of mathematics teachers in Mainland China. In B. Kaur & K. O. Nam (Eds.), *Professional Development of Mathematics Teachers: An Asian Perspective* (pp. 17–32). New York: Springer.

Huang, R., Fang, Y., & Chen, X. (2017a). Chinese lesson study: An improvement science, a deliberate practice, and a research methodology. *International Journal for Lesson and Learning Studies, 6*(4), 270–282.

Huang, R., Haupt, M., & Barlow, A. (2017b). Developing high-leverage practices as deliberate practice through lesson study. *International Journal for Lesson and Learning Studies, 6*(4), 365–379.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning in and for practice. *Elementary School Journal, 109*(5), 491–509.

Lampert, M., Franke, M., Kazemi, E., Ghousseini, H., Turrou, A., Beasley, H., Cunard, A., & Crowe, K. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education, 64*(3), 226–243.

Lewis, C., & Hurd, J. (2011). *Lesson study step by step: How teacher learning communities improve instruction*. Portsmouth: Heinemann.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial fractions learning. *Journal for Research in Mathematics Education, 48*(3), 261–299.

Lewis, C., Perry, R., Friedkin, S., & Roth, J. (2012). Improving teaching does improve teachers: Evidence from lesson study. *Journal of Teacher Education, 63*, 368–375.

Little, J. W. (2002). Locating learning in teachers' communities of practice: Opening up problems of analysis in records of everyday work. *Teaching and Teacher Education, 18*(8), 917–946.

Lo, M. L., & Marton, F. (2012). Towards a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal for Lesson and Learning Studies, 1*, 7–22.

Ma, L. (1999). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Lawrence Erlbaum.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning Science, 15*, 193–220.

McLaughlin, M. W., & Talbert, J. E. (2001). *Professional communities and the work of high school teaching*. Chicago: University of Chicago Press.

Ott, J. M., Snook, D. L., & Gibson, D. L. (1991). Understanding partitive division of fractions. In *The Arithmetic Teacher* (Vol. 39, pp. 7–11).

Payne, J. N. (1976). Review of research on fractions. In R. Lesh (Ed.), *Number and measurement* (pp. 145–188). Athens: University of Georgia.

Perry, R. R., & Lewis, C. C. (2011). *Improving the mathematical content base of lesson study summary of results*. Oakland: Mills College Lesson Study Group. Retrieved from http://www.lessonresearch.net/IESAbstract10.pdf.

Petit, M., Laird, R. E., & Marsden, E. L. (2010). *A focus on fractions: Bringing research to the classroom*. New York: Taylor & Francis.

Porter, A. C., Garet, M. S., Desimone, L. M., & Birman, B. F. (2003). Providing effective professional development: Lessons from the Eisenhower program. *Science Educator, 12*(1), 23–40.

Selling, S. K., Garcia, N., & Ball, D. L. (2016). What does it take to develop assessments of mathematical knowledge for teaching?: Unpacking the mathematical work of teaching. *The Mathematics Enthusiast, 13*, 35–51.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide (NCEE#2010–4039)*. Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from https://ies.ed.gov/ncee/wwc/pdf/practice_guides/fractions_pg_093010.pdf.

Simon, M. A. (1995). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24*, 233–254.

Son, J. W., & Crespo, S. (2009). Prospective teachers' reasoning and response to a student's nontraditional strategy when dividing fractions. *Journal of Mathematics Teacher Education, 12*, 235–261.

Sowder, J., Sowder, L., & Nickerson, S. (2010). *Reconceptualizing mathematics for elementary school teachers*. New York: W. H. Freeman & Company.

Tirosh, D., Fishbein, E., Graeber, A., & Wilson, J.W. (1998). Prospective Elementary Teachers' Conceptions of Rational Numbers. Retrieved from http://jwilson.coe.uga.edu/texts.folder/tirosh/pros.el.tchrs.html

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*, 5–25.

Van den Heuvel-Panhuizen, M. (2000). *Mathematics education in the Netherlands: A guided tour.* Freudenthal Institute Cd-rom for ICME9. Utrecht: Utrecht University.

van Gog, T., Ericsson, K. A., Rikers, R. M. J. P., & Paas, F. (2005). Instructional design for advanced learners: Establishing connections between the theoretical frameworks of cognitive load and deliberate practice. *Educational Technology Research and Development, 53*(3), 73–81.

Wang, J. (2013). *Mathematics education in China: Tradition and reality*. Singapore: Galeasia Cengage Learning.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense making. *Mathematical Thinking and Learning, 8*(2), 91–111.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of research in education* (Vol. 24, pp. 173–209). Washington, DC: AERA.

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Doing and Investigating Lesson Study with the Theory of Didactical Situations

Jacob Bahn and Carl Winsløw

Contents

Abstract The theory of didactical situations (TDS) is a framework for experimental research into the dynamics of mathematics teaching. At face value it may appear rather distant from lesson study (LS), which is presented (in the West) as a more or less unique format for teachers' professional development. In this chapter, we argue that at least two important connections could further both scholars' and teachers' work with lesson study. First, LS involves sharing and construction of teacher knowledge about the complex dynamics of the mathematics classroom and its dependency on the design of the lesson (materializing in the lesson plan). TDS offers a useful set of vocabulary and models for the researchers' analysis of the lesson's design and dynamics, and with appropriate transposition to the needs of teachers, it might also contribute to the precision of both lesson plan and teacher reflections before and after lessons. Secondly, LS involves a sequence of learning situations for the teachers, with the research lesson at the centre. TDS can be adapted to analyse these learning situations with explicit categories and models. This could contribute towards a more scientific viewpoint on lesson study and, in particular help, answer questions like the following: what is the role of different components of lesson study? How do they interact? What are the effects of repeating research lessons? In this chapter we will elaborate the above theoretical potentials and illustrate them by examples from the doctoral project of the first author.

Keywords Lesson study · Theory of didactical situations · Paradidactical situations

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_5

1 Introduction

Among seasoned practitioners of lesson study (LS), it may seem a somewhat strange idea that theory should be needed in order to study LS, let alone engage in it. To them, it may appear that the name suffices to define roughly what LS is, and then observation and practice do the rest. This could indeed suffice to maintain and develop the practice of LS within a homogenous and strongly connected community of teachers, such as found in Japan. But as seen in this book, practices called LS also appear in many other cultural and institutional landscapes, often as a novelty. There is every reason to doubt that these practices are simply a copy of LS as found in Japan. A unique name does not guarantee a unique object. Even in Japan, LS is used to designate a certain variety of teacher practices (Miyakawa and Winsløw 2018). The main purpose of this paper is to introduce a theoretical machinery which can be used both to conduct LS and to describe more explicitly what LS is.

Indeed, researchers need theoretical models to capture and analyse the variations of LS, which are found across cultural and institutional contexts. The point of theoretical frameworks is not that they are entirely neutral or independent of such contexts but that they support us in being explicit about selected aspects of the phenomena we study – to help us keeping them at analytic distance. Being explicit about the categories and criteria we use to formulate and establish new knowledge is a fundamental gesture of research in any discipline.

The theory of didactical situations (TDS) is a research programme (in the sense of Lakatos 1970) created to investigate and experiment classical teaching-learning situations, occurring in a regular classroom with a teacher and a group of pupils. More details are given in Sect. 2; here, we just note that TDS applies directly and immediately to describe and analyse many important components of mathematics lessons, which in fact make sense and can be found in school mathematics teaching all over the world. The term "lesson" may thus not in itself call for much explanation or theorizing, but the components and internal dynamics of a lesson are quite complex. Even the teacher may be unable to explain everything that happens during his lesson. It is thus a quite natural idea to try to use elements of TDS to enhance the explicitness and precision of teachers' work with lessons in LS and hence to extract tools for doing LS from TDS. We will outline the results of our first experiments with this idea in Sect. 4.

It is less obvious how to use TDS in research on LS, beyond the study lesson itself. To explain and exemplify, this is a main point of the chapter, and is done in Sect. 3. We show how the main phases or episodes of LS can be analysed, in TDS, as a learning situation for the participating teachers, with the various states of the study lesson as what in TDS is called a milieu (we return to this in Sect. 2). As in the case of the lesson itself, this milieu is constructed and developed to achieve such learning. There are also evident differences, of course. One main point of this model is to enable a discussion of how different types of (study) lessons – like more or less teacher-centred lessons – may furnish different learning environments for the teachers participating in the LS.

2 An Outline of the Basics of TDS

TDS is by now a well-established and highly developed research programme in mathematics education research (see Brousseau 1997 for a classical reference). Originating in the aftermaths of the "new math" reforms in France of the 1960s, one of the central aims was to provide researchers with explicit models of mathematical knowledge that are relevant and operational to support experimental research in real classrooms. Unlike the models which prevailed at the time – essentially cognitive psychology and the axiomatic models of mathematics itself – TDS takes into account not only the single student or the fundamental structure of a piece of knowledge to be taught but also the specific dynamics of classroom situations.

In fact, as the name of the theory suggests, the *situation* is the name given to its central model of mathematical knowledge. In Brousseau's (1997 p. 56) terms, a situation consists of two main elements: (a) "the student's game with the adidactical *milieu*, which allow the specification of what the function of the knowledge is after and during the learning" and (b) "the games of the teacher as an organizer of these student's games". Here, the term "game" is used as a metaphor for the actions of teachers and students: (a) the students interact with a problem and some conditions (e.g. resources) and criteria (rules) to solve it, and the students "win the game" if they solve the problem, and (b) the teacher initiates, regulates and evaluates the students' game in ways which are further specified by the types of situations described later in this section. A very simple illustration of this "double game" is given in Fig. 1.

In general, the milieu denotes everything the students interact with in a given situation, in order to learn. In TDS, learning is viewed as the adaptation of students' knowledge to the milieu, but unlike the assumptions in Piagetian theory, the milieu is constructed and managed by a teacher with the explicit aim of promoting learning (Hersant and Perrin-Glorian 2005, p. 116). The function of a milieu depends furthermore on the situation and its characteristics. A situation may in fact be either *didactical* (the teacher interacts with the subsystem of milieu and student, for instance, to modify the former or instruct the latter) or *adidactical* (the teacher does not interact; in terms of Fig. 1, there is no arrow from the T to the student-milieu game). In general, adidactical situations are not simply autonomous work by students on a mathematical problem; instead, such situations are *devolved* (handed over) by the teacher, in didactical situations that are, naturally, called *situations of devolution*.

Adidactical situations come in three fundamentally different states (Brousseau 1997, pp. 60–71):

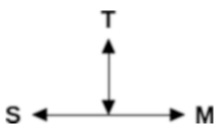


Fig. 1 The basic model of a didactical situation (adapted from Fig. 4 in Brousseau 1997, p. 56)

- Situations of *action*, where the milieu (*of action*) is completely devoid of intention to instruct, such as a problem and everything which the student may draw on to attack it, for instance, doing computations related to special cases of the problem.
- Situations of *formulation*, in which students begin to formulate and share hypotheses about the problem at stake, often based on a previous situation of action. Here, the communication pattern may be more or less structured by the teacher (in a preceding didactical situation of devolution), e.g. asking certain students to work together or to record formulations in a particular manner. A main component of the adidactical milieu is now the students' knowledge about the situation of action, which they seek to formulate.
- Situations of *validation* (or *proof*), where the milieu consists of formulations, typically hypotheses or claims about the problem and possibly other elements of the milieu of action, which may be produced by the students during a situation of formulation or may be devolved by the teacher. Again, the students' game with this *milieu of validation* is devolved by the teacher, which is often necessary to give it an appropriate form, although students may spontaneously engage in such a game.

There is a second type of didactical situations – called *situations of institutionalization*. These consist of the teacher's game with any of the previous situations, in order to establish "the relationships that can be allowed between the students 'free' behaviour or production and the cultural or scientific knowledge and the didactical project" (p. 56). In simpler terms, such situations will find the teacher responsible of relating (parts of) students' work to the target knowledge, its scientific and cultural status, and thus represents a specific form of direct teaching. The status of knowledge is an important notion in the theory; for instance, Hersant and Perrin-Glorian (2005) distinguish *old knowledge* (which may be easily institutionalized, while referring to past situations), *knowledge in progress* (shared by some but not all students) and *new knowledge* (often delicate and difficult to institutionalize).

When talking about a planned or otherwise anticipated situation of any of the above types, one uses the term *adidactical potential* to indicate the degree to which the milieu in the situation is capable of providing feedback to the students, allowing them to produce new knowledge on their own (ibid., 117). An adidactical potential is thus a theoretical feature of the situation; it can be an important research objective for experimentation of the situation to determine the conditions under which this potential is realized.

An event of teaching, such as a lesson, will often comprise several situations as described above, including both didactical and adidactical situations; to be more precise, researchers may plan or analyse the event as such a sequence. Despite the evident importance of this precision in TDS-based work, we will follow common usage and use the term "didactical situation" (then abbreviated DS) also in a more relaxed way, namely, to refer to such an event, which will often span an entire lesson. When needed we can write DS = $(S_1,\ldots,S_n)$ to indicate the situations (in the strict sense) that DS consist of.

Even a brief account of the main ideas of TDS would not be complete without introducing also the notion of *didactical contract*. According to Hersant and Glorian (2005, p. 116) it is "a way of regulating the mutual expectations of the teacher and the students with respect to the mathematical notions at stake". It certainly depends on the mathematical notions, the type of situation involved (in particular if it is adidactical or not) and also on the milieu. Still, it is fundamentally about "expectations", including the responsibilities and actions which teachers and students expect each other to undertake. The teacher, in particular, "wants the student to find the answer entirely by herself, but at the same time she wants – she has the social responsibility of wanting – the student to find the correct answer" (Brousseau 1997, p. 230). In a fundamental way, didactical situations are *paradoxical* – the teacher aims for the students to learn something, yet in many cases, simply telling them what they would know (inappropriate institutionalization) does not suffice to obtain this; the students must, at least to some extent, obtain it by themselves, through interaction with a milieu, as described above. For instance, in a situation of validation, it is of paramount importance that the rationality of students' own reasoning is not ruined by premature institutionalization or "hints" from the teacher (Sierpinska 2007). The students naturally strive to win their adidactical games and to draw, as much as possible, on the milieu, which they know that the teacher has arranged with specific aims in mind. As teachers and students strive to fulfil these somewhat contradictory obligations, the contract evolves and sometimes degenerates in ways which have been described and further characterized by Brousseau (1997, p. 261).

It should be noted that TDS was originally developed to serve the purpose of researchers' construction and experimentation of situations. However, the theory has also been used successfully to analyse "ordinary teaching", that is, situations that were planned and carried out by teachers without using or even knowing about TDS (e.g. Hersant and Perrin-Glorian 2005). Perrin-Glorian (2008, p. 5) notes some issues that such research needs to take into account:

> The first important issue is to identify the target knowledge (it is not always explicit and not always the one expressed by the teacher) and how it appears in the problem to solve. The second one is to identify what could be the milieu: data and all actual givens usable by students without any intervention of the teacher. The third one is to identify prior knowledge of students to foresee actions students may undertake on this milieu and how they could interpret feedback coming from it.

The first and third of these issues are indeed often tackled quite explicitly in LS, unlike what is the case in ordinary teaching. The milieu may be more difficult to identify from the lesson plan and observation records, although they would give valuable information.

A first discussion of how ideas from TDS may be brought to bear on LS was given by Miyakawa and Winsløw (2009) – noting that elements of TDS "could fertilize teachers' practice in lesson study". At the same time, TDS is an international research programme with strong roots in European didactics, and the complete theoretical machinery may not be useful for practitioners (whether in France, Japan or elsewhere). Thus, to that end, choices will have to be made.

To sum up, TDS certainly offers a quite complex set of constructs to describe the dynamics of teaching and learning and comes with a perspective on mathematical knowledge that is in many ways resonant with teachers' own experience of their task to construct and manage a rich learning environment for students to learn it. We now proceed to consider the possible and actual uses of TDS in relation to LS.

3 TDS as a Paradigm for Research on LS

A lesson type DS is evidently the centrepiece of any LS, regardless of how we demarcate and describe the latter. As LS is moreover often concerned with fine details of the lesson which the participating teachers "study", it is quite obvious that one could try to use TDS to dissect the development of a lesson which teachers plan, realize and observe together. This was first exemplified by Miyakawa and Winsløw (2009) for a case of LS in Japan. They also noted some similarity between the professional terminology of Japanese teachers to indicate various parts of the lesson and the TDS characterization of different types of situations which was described in the preceding section. LS has also been described as a kind of scientific research carried out by teachers (e.g. Isoda 2015), and from this perspective, one might even consider TDS as an alternative to the theoretical and methodological framework which is involved in the teachers' scholarship. We return to this point of view in Sect. 4.

However, in this section, we will consider the uses of TDS to do research on LS, as an object of study – or perhaps rather a class of objects. Indeed, it is recognized by most experts that the label LS is commonly used to denote a rather broad variety of teacher practices, in Japan (e.g. Isoda 2015, 84–89; Isoda et al. 2007), China (e.g. Huang et al. 2017) and internationally (e.g. Quaresma et al. 2018). A first motivation to try to have TDS bear on LS as a research object is to enable a clearer distinction between defining and variable features of LS. A second motivation is to investigate the learning processes of teachers engaged in LS, both in order to investigate the mechanisms and sources of their learning in terms of a special kind of situations and milieus and to identify crucial conditions for teachers to engage in and benefit from such learning situations. In this chapter, we focus especially on the last point and the case of teachers who are new to LS, drawing on the PhD dissertation of the first author.

3.1 TDS-Based Framework to Study LS

The central idea in our approach is to consider a sequence, DS, of didactical situations, as illustrated in Fig. 1, as the milieu for the learning of teachers engaged in LS. In other words, LS consists of (various types of) situations where a group ǔE3; of teachers interact with DS; because these situations run in parallel to the DS, we call them *paradidactical*. The milieu is neither didactical nor adidactical, since there is no teacher as in a DS; in fact, there is no devolution and also no pre-established target knowledge which can be institutionalized. We also call such milieus paradidactical. They occur in basically three forms:

1. Anticipated, theoretical didactical situations DS_A, which can materialize as concrete lesson plans and the like.
2. Ongoing, “real time” didactical situations DS_R, as described in the previous section.
3. Observed “past” didactical situations DS_O – this refers to a DS_R that has been observed by ǔE3; – and the observable parts of DS_O are oral and written recollections about DS_R by members of ǔE3;, including the teacher(s) involved in DS_R.

Given the timewise situation of the three types of paradidactical milieus, we denote the corresponding types of paradidactical situations as follows:

1. Predidactical situations (PrS), with DS_A as a milieu, which ǔE3; learns from gaming with, the resources coming mainly from beliefs and experiences rooted in previous observation situations (often not explicit). It is also important to note that other resources such as websites, textbooks or official guidelines may be drawn upon by some or all members of ǔE3;, as media which do inform them on the teaching project (cf. Miyakawa and Winsløw 2018, for more on this notion, which is not standard in TDS).
2. Observation situations (ObS), with DS_R as a milieu which ǔE3; observes without modifying, except naturally for the teacher(s) directly involved in DS_R.
3. Postdidactical situations (PoS) in which DS_O is the milieu for the learning of ǔE3;.

As with didactical situation, we use the abbreviation PS to indicate (a sequence of) paradidactical situations of the above types. As for DS, a more or less explicit problem may be part of paradidactical milieus and to some extent direct the learning which ǔE3; gains from interacting with them.

Notice that any teacher engages to some extent with all kinds of milieu in preparing, delivering and reflecting upon ordinary teaching; but then ǔE3; consists normally of the teacher herself. We consider it a defining feature of LS to involve all the above kinds of situations, with a group of teachers ǔE3; which does not just consist of the teacher of DS_R and which is present in all three kinds of situations related to one and the same situation DS_R. Moreover, we consider it a defining feature that a set of explicit problems concerning teaching is present in the milieus of all the paradidactical situations involved (while it may evolve in time, both within each of the situations and in the PS as a whole). Finally, it is a facultative – but quite common and productive – trait of LS that ǔE3; is sometimes augmented by external facilitators or partners (whose contributions are sometimes less based on teaching experience than on other sources, such as academic research) in any of the paradidactical situations and in particular in postdidactical situations.

A minimal LS can thus be depicted as in Fig. 2: it consists of three paradidactical situations, occurring consecutively as shown. Of course, the notation in Fig. 2 is very compact and does not exhibit the milieus or the group ǔE3;. In particular, ObS

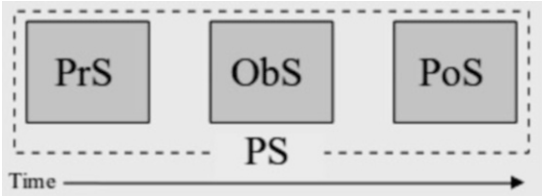


Fig. 2 Lesson study in terms of paradidactical situations

contains DS_R, which may be analysed using the classical notions of TDS, while the interaction of ŭE3; with ObS is passive (except for the teacher involved in DS_R).

3.2 A Specific LS Context

We now proceed to present two cases of how these analytic tools can be deployed by researchers to analyse LS with novice teachers. The case comes from a 1-year-long LS experiment, which was implemented in 2015 by the first author in three schools in Lyngby-Taarbæk Municipality, north of Copenhagen (Denmark). The first author participated in the teacher groups as a facilitator, while the second author took part in some of the postdidactical situations, as an external partner.

The overall purpose of this LS experiment was to investigate conditions that were promoting or hindering the realization of principles and potentials of open-ended approach (OEA, see Nohda 2000) in an experimental lesson on elementary probability. The OEA principles focus on students' autonomous work, their learning of essential knowledge and the teacher's expedient decision-making (Nohda 2000, p. 42). Moreover, OEA lessons typically revolve around one problem with multiple correct solutions (Nohda 2000, p. 40; Miyakawa and Winsløw 2009, p. 208). In terms of TDS, this requires rich milieus which allow for a variety of approaches to action as well as formulation and validation. Concerning the latter, it is a valued feature of OEA lessons to allow pupils to share, compare and modify mathematical ideas (Nohda 1995, 2000; Becker and Shimada 1997). Earlier research on OEA lessons have emphasized the "major difficulty [of] how to give a suitable problem to the students" (Nohda 1995, p. 59) and of how to fruitfully manage students' formulations of solutions.

For details of the methodology, we refer to Bahn (2018).

3.3 A LS on Chance and Circle Sectors: Lesson and Teacher Focus

A group of four teachers, all with more than 10 years of experience, developed a research lesson on geometry and probability for grade 4. The pivotal idea was to investigate the winning odds for a "wheel of fortune" with two colours, red and yellow, distributed on six uneven sectors of alternating colour. The initial problem was to find out what colour had the higher rate of success. The investigation was carried out in groups of two or three pupils, after a brief introduction where the teacher demonstrated the wheel (in a large version) in front of the class, in order to devolve the problem. Besides all groups getting a copy of the wheel (in paper), the milieu further comprised scissors, ruler, protractor and thread which could be used to cut and measure the sectors.

For the researchers' analysis, an a priori analysis of the adidactical situation functioned as a reference for the a posteriori analysis of the actually employed solutions, as is always the case in TDS-based research. The central problem of the lesson concerns plane geometry (area of circle sector, angles and perimeter), proportion (e.g. between angle, area and perimeter of different sectors) and chance (in particular the empirical chance found by spinning the wheel several times and its connection to a theoretical probability found by analysing the geometry of the wheel). At this level, students have at best an intuitive idea of probability as "chance", and in the actual adidactical situation devolved in the lesson, only the geometrical problem was emphasized. In fact, during devolution, the teacher first formulated the problem as: *Which of red and yellow has the highest chance to win?* And then, after the whole-class demonstration in interaction with the students, she reformulated it as: *which colour covers the most of the wheel?* As a part of the institutionalization of the answer, the teacher returned to the chance question and organized an experiment with spinning the wheel several times, to verify the answer. We do not consider this last part of the lesson further here, since it involves no adidacticity.

The a priori analysis of the second of the above problems led to identify nine plausible student methods, which were divided into *comparison* and *measurement*. *Comparison* was again divided into *direct comparison* (e.g. to cut each sector out and compare the pieces of different colours either one by one or somehow "assembled" (see, for instance, Fig. 3)) and *indirect comparison* (e.g. compare the assembled sectors of one colour with a half circle, representing 180°). *Measurement* of one or more sectors at a time is done looking at *angles* (measured by protractor), *arcs* (measured by thread and/or ruler) and *chords* (by thread or ruler). Each approach could be more or less accurate; especially the use of thread led to imprecise answers. In addition, looking at chords is in fact mathematically problematic (the sum of the chords of two sectors is larger than the chord of their union).

The above sketch of the a priori analysis illustrates that the adidactical milieu allows for a variety of actions and formulations and holds, in this sense, a rich adidactical potential for students who are familiar with the tools and notions concerned. This reflects that the LS group had planned the lesson with the principles of OEA in mind. In particular, the teachers chose to provide the students with the different "tools" described above, in order to encourage different methods of measurement and comparison. In the PrS, the teachers primarily focused on facilitating the action state of the adidactical situation; they put much less attention to how the milieu could facilitate students' adidactical formulation and validation of answers. In the actual lesson (ObS), the teacher circulated to observe the students' action and also interacted with individual groups of pupils about what they concluded from it, so that the situation became didactical for the group attended to. Then, it was only the sheer answer that was institutionalized at the end (not the methods). The weak attention to adidactical formulation and validation in the PrS led to a lesson in which students did not get to "share, compare and modify mathematical ideas" beyond individual groups.

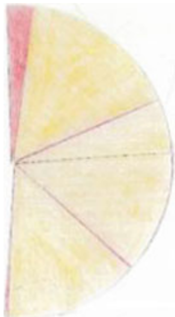
In the a posteriori analysis of the observed lesson, eight different approaches to solve the problem were identified. Five of these are among the ones identified in the priori analysis, one is a variant of one of these, and two were not fully completed. The approaches were all observed in the group's adidactical work (action and formulation). The most explicit validation took place in the individual groups' communication with the teacher, in which they explained their action in the milieu, as exemplified in the following episode (Bahn 2018):

Peter	We put them on top of each other
Karen	To see if they. . .it's because we think it's difficult to see
Teacher	Yes, yes. Good thinking
Karen	It looks as if the red is bigger

The groups' idea is illustrated in Fig. 3 in which the sectors have been cut out, assembled by colour, and finally the yellow sectors cover the darker, red sectors.

A pupil formulated the hypothesis that "red is bigger", and the teacher validated it positively ("good thinking"). All other groups eventually reached this conclusion, which was quickly institutionalized in a (whole-class) didactical situation. However, the validation (reasoning, justification, comparison) of different methods only occurred in individual groups and was mostly done by the teacher. While the groups may have been unable to validate their result on their own, an evident potential for a richer milieu of validation was missed when students did not get to "share, compare and modify" their ideas. In the PoS the teachers also focused on the various strategies actually observed in action situations, while it was pointed out by an external observer that the multiplicity of strategies was only visible to the teacher (and the observers, of course) and was not shared and compared in the class, as is common in OEA lessons.

Fig. 3 Students put cut yellow sectors on top of red ones



In conclusion to our analysis of this LS, we can say that the unrealized adidactical potentials of the situation mainly correspond to aspects of OEA which the teachers did not focus on in their preparation. On the other hand, the relative "success" of the adidactical situation of action can be ascribed to two factors, which were intensively attended to by the teachers in both PrS and PoS. First, the material milieu invites students to try out different ideas to measure and compare the red and yellow areas; in particular the sizes of the six sectors (of alternating colour) were carefully designed so that, on the one hand, the dominance of red is not immediately visible and on the other hand the difference is sufficient to be recognized with a variety of methods. Secondly, the devolution of the problem is clear to the students, and it effectively engages them through the (more or less intuitive) connection to the initial chance problem.

3.4 A "Cyclic" Lesson Study on Subtraction of Natural Numbers

Outside Japan, LS is often described and performed as a cyclic process, in which the DS (materialized in the lesson plan) is revised, retested and re-analysed a number of times. This is not common in Japan (Fujii 2014), but it has been argued that it could be helpful for teachers in other countries, at least when they are new to LS (Winsløw et al. 2018). In such cases, shared experiences from previous PS, and in particular those made explicit in the PoS, become part of the resources for the PrS for the following cycle. In fact, in the case considered here, the PoS of one cycle was held in continuation with the PrS of the following one. But the two situations were clearly distinct, nevertheless: the milieu for the PoS is of type DS_R, while the milieu for the following PrS is of type DS_A (the new iteration to be constructed). Figure 4 illustrates an example of a three-cycled lesson study, where the three different DS have been indicated explicitly.

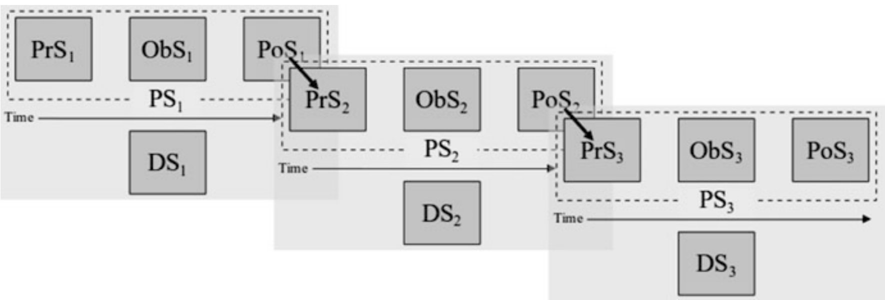


Fig. 4 A (long) paradidactical situation involving the three realized didactical situations (DS_1, DS_2, DS_3) which form the milieu for the three observation situations

The motivating problem for one such three-cycled LS with grade 3 classes came from the teachers' experience that pupils in this grade tend to favour one method for subtraction of natural numbers and stick to it regardless of the subtraction problem. For instance, in the case of 20 minus 18, many pupils would count from 20 down to 2 (18 "steps"), i.e. *take away* 18 from 20 (we refer to this as TA, as in the framework of van den Heuvel-Panhuizen and Treffers 2009). In this case, it is obviously easier to "count" the difference between the two given numbers, i.e. *determine the difference* (DD). During the lesson study, the teachers also distinguished what we might call DD_{down} and DD_{up}, e.g. counting from 20 (down) to 18 or from 18 (up) to 20.

By the end of the first predidactical situation, PrS_1, the following aim was formulated: to teach grade 3 pupils that there are three methods of subtraction by counting and that they have different properties. At the beginning of the cycle of LS, the teaching problem focused on having students realize "when what method would be more expedient", but as we shall see, the focus changed.

The lesson plan included two pivotal activities which were the core objects of the analysis. In the first activity, the teacher would present six sentences related to subtraction (but with no mention of the operation); these were to be discussed with the pupils, one sentence at a time. The sentences are group into two: some about comparison, where the order in which the numbers are mentioned is important (e.g. *how much is 18 larger than 15?*), and some about difference or distance, where the order does not matter (e.g. *how far apart are 16 and 11?*). The broader aim of the first activity was to make pupils aware that difference may be expressed in different ways and that a problem regarding difference can be solved by subtraction ("minus" in the language of the school).

In the second activity, pupils were to choose their own choice of subtraction method and present it to their peers by means of a number line on the board. The numbers of the subtraction tasks given to students were the same as those used in the six sentences of the first activity.

The PrS revolved around anticipated didactical situations, DS_A, related to these two activities. The teachers tried to calibrate the sentences (which are the essential elements of the milieu) so that students could easily find the numbers and would subsequently be able to discover that sentences with *the difference between* are unaffected of the order in which the numbers are given. They discuss what could be left to the students and what could or should be done by the teacher before institutionalizing this fact.

In PrS_1, the teachers expected that the outcome of the first activity would ensure that in the second activity, different children would present and explain different methods (not only TA) for the given subtraction problems. But during PoS_1 a main observation was that the pupils almost exclusively considered TA as a method to solve the presented problems. They also observed that when the teacher, in full alignment with the lesson plan and the underlying didactical idea, tried to make the pupils relate the sentences of the first activity to subtraction, he explicitly used the term *minus*. Also in the last activity, *minus* was explicit since it was written on the board (e.g. as $7-5$) and the problem was presented as: *show how you find 7 minus 5*.

The teachers then formed the hypothesis that *minus* narrowed pupils' minds to think of "take away". Accordingly, they hypothesized that avoiding the use of the word *minus* and the corresponding symbol would enhance the chance that pupils might use other methods in the second activity. When revising the lesson plan in PrS_2, the teachers decided that in the first activity, the teacher should talk about *difference* and only use *minus* when following up on possible student comments using that term. Furthermore, the problems for the last activity should be presented as, for example, *show how you find the difference between 7 and 5*, while the numbers of problem should be displayed as, e.g. 7 and 5 or 4 and 19, i.e. deliberately mixing the orders of the higher and the lower number. This was indeed observed in the second research lesson and noted by the teachers with great satisfaction in PoS_2: all three methods (TA, DD_{down}, DD_{up}) were displayed more or less evenly. In the third cycle, the avoidance of "minus" was retained and emphasized in DS_A, with the same result in DS_O.

This repetition clearly gave the teachers an impression of situation specific, empirically based knowledge which is specific to the situation. The analysis in terms of paradidactical situations gives further insight into how the teachers' knowledge developed during the three-cycled LS. The teachers' initial attempt to solve the teaching problem, using the didactical ideas of the first lesson plan, shares similarities with a situation of action in the sense that no hypotheses were explicitly stated about the specific issue; the teachers acted based on what seems plausible to them, based on experience. Given the "negative" feedback from the milieu (the DS_O of the first cycle), the teachers developed a new hypothesis in PrS_2, which then resembles a situation of formulation. The feedback from DS_O of the second cycle testing the hypothesis led the teachers to consider it as a possible solution to the teaching problem (explicit in a kind of situation of validation, in PoS_2). The last cycle certainly strengthened that assumption among them.

The example presented here also serves to demonstrate a more general phenomenon: gaps between DS_A and DS_O can be crucial to teachers' learning in LS. For researchers (and facilitators), it is also interesting to observe how DS_R sometimes diverges from both DS_A and DS_O. For instance, in the cycle, both DS_A and DS_O differed from the DS_R (which is recorded on video) regarding an activity which was carried out between the two presented here: an intermediate activity had been added between the two mentioned above, where pupils were to construct maths stories based on the six initial sentences. According to what teachers anticipated and reported to observe, the pupils could produce a variety of "meaningful" math stories that would lead them to a deeper understanding of the subtraction. In fact, what most pupils did was to produce a familiar kind of exercise, ending by one of the six sentences from the first activity, such as: *Peter has 42 euros and John has 32 euros. How much is 42 larger than 32?*

While the exact number of cycles may be debatable, it appears that multiple cycles may offer a positive opportunity for teachers to explore the effect of specific didactical choices, as well as an experience such systematic effects may exist, even if they are quite contingent on the situation. In fact this was an explicit learning outcome of the LS in question.

As mentioned above, this particular LS started with a different variant of the teaching problem. Originally the teachers were keen to teach the pupils principles for when to choose what counting method. This resulted in a number of proposals for activities in which pupils were to sort or match subtraction problems with methods. However, what they did not realize at first is that they assumed the students already were familiar with the methods. This realization came as one teacher began to anticipate how pupils would perceive the problem, which possible actions or formulations they could produce and what kind of feedback the milieu could provide. Interestingly, this first anticipation of (elements of) a didactical situation appeared to function as a catalyst for the teachers to enter *kyouzai kenkyuu*, the study of the topic, the learning of it and the resources available (Watanabe et al. 2008). This led to the situation described above and in particular to the first activity. From our analysis of the work of this group of teachers over three 3-cycled lesson studies, we see a clear development of both their focus on and capacity for anticipating pupils' actions and formulations.

4 TDS as a Support to Do LS

As a preparation for the teacher groups involved in the cases we consider here, the first author held a preparatory workshop outlining the basics of TDS (essentially, what is presented in Sect. 2 here). The intention was that at least some of its elements – such as the distinction of didactical and adidactical situation, the role of the milieu and the notion of adidactical potential – would become useful anchor points for the teachers' and facilitator's subsequent work with LS focusing on enabling students' inquiry on open-ended problems (in a relatively loose sense, which will not be detailed here). The background and motivation of this were two independent facts: Danish mathematics teachers do not share a research-based theoretical discourse suitable for didactical design work, and Danish mathematics teaching is dominated by the introduction (by teachers) and training (by students) of techniques for solving standard exercises (Mogensen 2011).

In this section, we shall consider how this intention was or could have been realized in the two cases considered in Sect. 3.

4.1 TDS as a Tool for Teachers

In the data from predidactical and postdidactical situations, we find in fact few explicit instances of teachers making use of TDS tools. Though the teachers had been introduced to TDS, they were probably not sufficiently confident with the notions and models and what they can be used for. Nevertheless, in the course of the three 3-cycled lesson studies that each team conducted, the teachers' knowledge gradually increased, not least because the facilitator and the external observers continued (mainly in PoS) to draw explicitly on TDS.

Two of the ideas that the teachers gradually adopted were *devolution* and *material milieu* (the latter in the sense of "concrete materials and problems" produced by the teachers before the lesson). Devolution is indeed a practical term to refer to that crucial part of the lesson where the teacher articulates the work to be done by the students; there is no common term for this is Danish, but the teachers usually employed an equivalent of "handing (something) over to the students". In practice, the teachers' used the term milieu mainly when discussing the potential of a DS to enable pupils' autonomous actions and to some extend also formulations. A deeper knowledge of TDS might have helped them to consider pupils' potential interplay with different states of the milieu for validation (i.e. the situation of formulation) and how it could be enriched by other pupils' contributions.

The analysis of DS_3, i.e. the last study lesson from the LS presented in Sect. 3.1 (which was the last of that team), provides a very clear example of this. In fact the teachers concluded that the material milieu of this lesson was successfully devolved and that pupils' interactions with the milieu in the adidactical situations of action and formulation realized its potential for generating autonomous student work. But in fact, adidactical situations of validation were not realized since the teacher (in all three lessons) validated the groups' answers individually.

Data from PrS clearly demonstrates how the teachers thoroughly considered details of the devolution and the material milieu, such as the specific words to use (e.g. probability, chance), how to capture pupils' interest (e.g. organizing a game with scores, one wheel or two wheels, the visual design of wheel(s), etc.), the dimensions of sectors on the wheel to investigate (size of each sector, clearly distinguishable colours, etc.), tools for the students' investigations and how to make the spinning of the wheel shareable and solid (on the board). The data also reveals that little attention was given to validation of students' solution. Probably, one of the greatest obstacles to learning from LS is when crucial elements remain "done as usual" with no further reflection. In connection to this, it is interesting to note that as early as in PoS_1, the teacher who had taught the lesson noticed how much feedback about the pupils' learning processes she had received from observing their work (without intervening) and how much that feedback had given her an "overview" (verbatim citation, cf. Bahn 2018) to manage the following situations.

It should also be noted that although the terms *devolution*, *adidactical* and *milieu* are not frequently appearing in the teachers discourse, what they work on may still be interpreted and possibly derive from their exposition to the ideas behind these notions. In PoS_1 of the lesson study presented in Sect. 3.2, the teachers marvelled at how crucial little details in the handing over (i.e. devolution) of the tasks (crucial part of the milieu) can be. Indeed, one of the specific learning outcomes of this lesson study was that the use or non-use of specific words and symbols can have great influence on the pupils' likeliness to interact with the milieu in desired ways. More generally, the teacher teams all went through a clear shift from proposing "activities" to considering (elements of) milieus, the details of how they should be devolved and the effects of both when anticipating pupils' adidactical interaction with the milieu.

The shift away from detached activities was connected to focus in TDS on the interplay between the pupils' knowledge and a milieu with a mathematical problem (in the milieu). The teaching problem for the LS analysed in Sect. 3.2 grew out of the teachers' experience that their pupils had to some extent acquired techniques to solve certain standard tasks of subtraction but displayed no or little deeper knowledge of the mechanisms and meaning behind these. In accordance with that experience, they wished to investigate ways to avoid that pupils would just "solve tasks without achieving any knowledge" (to cite a teacher's expression). Reluctantly, the teachers realized that in order to establish any adidactical potential, they would have to specify the target knowledge explicitly and to anticipate how pupils could realize it through interplay with a carefully designed situation. The increasing awareness and usage of didactical concepts and models indicate a vital dynamics between the evolution of theoretical and practical didactical knowledge and skills, which suggests that a more thorough, continued focus on theoretical training can benefit the teachers' paradidactical work and corresponding learning outcomes. This also indicates that continuous practice with theoretical tools increases the teachers' capacity to anticipate and analyse DS, for instance, using notions related to the didactical contract.

This brings us to the question of how TDS can support also the work of a facilitator of teachers' first experiences with LS.

4.2 TDS as a Tool for Facilitators

As discussed in Sect. 3.2, there are various ways in which one can imagine to structure a LS, and there are no set of rules. For the Danish teachers involved in the present research, it was considered a first "target knowledge" that there is indeed something to be learned about the connection between teaching and pupils' learning. This includes experiencing how different didactical choices may have different or even opposite didactical consequences. Such experiences are inevitably linked to specific mathematical properties in concrete DS, which condition what can be learned (by pupils).

The models and notions of TDS provide the facilitator with tools to facilitate the teachers' interactions with their paradidactical milieu: the notion of teaching problem that has to be an explicit part of the paradidactical situation and the DS as an evolving milieu from which to build knowledge about the problem. Moreover, the facilitation of lesson studies requires the facilitator to analyse potential and realized didactical situations resulting from the teachers' work and to analyse potential and realized interactions between those didactical situations and the teachers.

Since the teachers' learning is assumed to emerge from these interactions, this interplay must also be organized and supported by the facilitator. In this respect, the model illustrates why *kyouzai kenkyuu* is deemed of capital importance to LS: in *kyouzai kenkyuu*, teachers' (old) knowledge becomes explicit and challenged, which alters (their view of) the teaching problem and the possible hypotheses to solve it. Accordingly, the facilitator must ensure that the teachers engage in *kyouzai kenkyuu* to resource the game with the anticipated DS.

In the lesson study of case 3.2, the teachers were initially not aware that “the existence of different methods of subtraction” is not a sufficiently precise target knowledge for the pupils. If the teachers do not have an explicit model of the relevant methods and their properties, how can they design and manage a DS in which pupils can potentially acquire them? Hence, the facilitator worked to establish a (paradidactical) situation in which the teachers realized that they needed such knowledge. This was done through questions like “what are you looking for?”, “what answers would you like to see?”, “what do you want [the pupils] to realize?” or “you say *the methods*, but which methods actually exist?”. In a later LS of that team, the facilitator simply began to examine all possible solutions to a given problem. This inspired one teacher to engage in the examination and the others to consider them thoroughly. The point is that new knowledge about the subject matter is often a key component of teachers’ learning and that teachers do not assume that from the outset.

Another function of the facilitator is to inspire ideas about how to analyse observed DS in a way that promotes learning and leads to improved design. Here, organizing the participation of external observers – other teachers, including teachers who are more experienced with lesson study – can add both exemplars and a felt need for precision, which in the end will bring about the need for more detailed models of the DS itself.

The facilitator can also provide “internal” inspiration in PrS, in the shape of specific ideas. For instance, in early stages of designing the DS of Sect. 3.1, as a solution to the problem of teaching probability by experiments, the facilitator proposed the problem of determining the “best” sack to pick a black marble from, one with three blacks and seven whites or one with two blacks and three whites. The idea was dismissed but eventually led to a pivotal discussion about using fractions or not.

In general, the facilitator can use the model of TDS to decide when and how it will be expedient to regulate the teachers’ interactions with their milieu. Most of all by constantly forcing the teachers to re-examine proposed ideas by asking generic questions like “how could that help the pupils to learn?” or “what options does that provide to the pupils?”

5 Conclusion

In this chapter, we have outlined how LS can be modelled as a set of paradidactical learning situations for teachers, involving three kinds of milieus in the form of anticipated, realized and observed didactical situations. We have analysed two cases where first experiences of LS were organized by a facilitator by using this model explicitly, both as a model of the LS itself and as a model to analyse the DS involved. We have seen how the model can be used to identify crucial obstacles and developments in the teachers' engagement in LS, in close relation to an analysis of the DS they work with. We have also outlined the limitations and potentials of a relatively timid introduction of the teachers to elements of TDS, as a tool for themselves in this work. There is no doubt to us that further development and experiments of TDS as a theoretical framework for research on and practice in LS will help making this (sometimes, rather blurry) form of teacher scholarship more well understood. This could contribute to take it from the occasional alternation between magic and deception to become an accessible and solid interface between didactical research and practice in countries and institutions where it is not as well established as in the fascinating infrastructure of Japanese primary school mathematics.

At the same time, it is also clear that TDS has some limitations: it only models the dynamics of learning situations (lessons, and by the adaptation given here, also LS) but not the cultural and institutional conditions which affect them in many ways. For this, other theoretical frameworks – such as the anthropological theory of the didactic – have more to say (see Miyakawa and Winsløw 2018).

References

Bahn, J. (2018). *Inquiry based mathematics education and lesson study: Teachers' inquiry based learning about inquiry based teaching* (Unpublished PhD dissertation), University of Copenhagen.

Becker, J. P., & Shimada, S. (1997). *The open-ended approach: A new proposal for teaching mathematics*. Reston: National Council of Teachers of Mathematics.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Fujii, H. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 65–83.

Hersant, M., & Perrin-Glorian, M. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactical situations. *Educational Studies in Mathematics, 59*, 113–151.

Huang, R., Fang, Y., & Chen, X. (Eds.). (2017). Theory and practice of Chinese lesson study and its adaption in other countries [Special Issue]. *International Journal for Lesson and Learning Studies, 6*(4).

Isoda, M. (2015). The science of lesson study in the problem solving approach. In M. Imprashita, M. Isoda, P. Wang-Iverson, & B. Har Yeap (Eds.), *Lesson study: Challenges in mathematics education* (pp. 81–108). Singapore: World Scientific.

Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.). (2007). *Japanese lesson study in mathematics: Its impact, diversity and potential for educational development*. Singapore: World Scientific.

Lakatos, I. (1970). Falsification and the methodology of scientific research programmes. In I. Lakatos & A. Musgrave (Eds.), *Criticism and the growth of knowledge* (pp. 91–195). Cambridge: Cambridge University Press.

Miyakawa, T., & Winsløw, C. (2009). Didactical designs for students' proportional reasoning. *Educational Studies in Mathematics, 72*(2), 199–218.

Miyakawa, T., & Winsløw, C. (2018). Paradidactical infrastructure for sharing and documenting mathematics teacher knowledge: A case study of "practice research" in Japan. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-017-9394-y.

Mogensen, A. (2011). *Point-driven mathematics teaching: Studying and intervening in Danish classrooms* (PhD dissertation), Roskilde University. Retrieved from http://milne.ruc.dk/imfufatekster/pdf/484web.pdf. (Feb 5, 2018).

Nohda, N. (1995). Teaching and evaluating using 'Open-ended problems in classroom. *Zentralblatt für Didaktik der Mathematik, 27*(2), 57–60.

Nohda, N. (2000). *Teaching by open-approach method in Japanese mathematics classroom*. Proceeding of the 24th conference of the international group for the psychology of mathematics education (Vol. 1, pp. 39–54). Hiroshima: Hiroshima University.

Perrin-Glorian, M. -J. (2008). *From producing optimal teaching to analysing usual classroom situations. Development of a fundamental concept in the theory of didactical situations: the notion of milieu*. Retrieved from https://www.unige.ch/math/EnsMath/Rome2008/WG5/Papers/PERRIN.pdf. (Feb 5, 2018).

Quaresma, M., Winsløw, C., Clivaz, S., da Ponte, J., Ní Shúilleabháin, A., & Takahashi, A. (Eds.). (2018). *Mathematics lesson study around the world: Theoretical and methodological issues*. New York: Springer.

Sierpinska, A. (2007). *I need the teacher to tell me if I am right or wrong*. Proceedings of the 31st conference of the international group for the psychology of mathematics education, Seoul, South Korea, July 8–13, 2007 (Vol. 1 pp. 45–64).

Van den Heuvel-Panhuizen, M., & Treffers, A. (2009). Mathe-didactical reflections on youngchildren's understanding and application of subtraction-related principles. *Mathematical Thinking and Learning, 11*(1–2), 102–112.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), *Inquiry into mathematics teacher education* (Association of Mathematics Teacher Educators (AMTE) monograph series) (Vol. 5, pp. 131–142). San Diego: Association of Mathematics Teacher Educators.

Winsløw, C., Bahn, J., & Rasmussen, K. (2018). Theorizing lesson study: Two related frameworks and two Danish case studies. In M. Quaresma, C. Winsløw, S. Clivaz, J. da Ponte, A. Ní Shúilleabháin, & A. Takahashi (Eds.), *Mathematics lesson study around the world. ICME-13 monographs*. Cham: Springer.

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Theorizing Professional Learning through Lesson Study Using the Interconnected Model of Professional Growth

Wanty Widjaja, Colleen Vale, Susie Groves, and Brian Doig

Contents

Abstract Lesson study is highly regarded as a model for teacher professional learning. Yet there are few studies that attempt to theorize the learning process for participating teachers. In an earlier paper, we used Clarke and Hollingsworth's (Teach Teach Educ 18:947–967, 2002) *Interconnected Model of Professional Growth* (IMPG) to map the professional growth of a research lesson planning team participating in a lesson study project over two school terms. This chapter uses the IMPG to examine the professional learning experiences of individual participants from the other planning team in the same project. The analysis is based on interviews carried out at the beginning of the lesson study project and after the first and second lesson study research cycles.

Parts of this chapter originally appeared in Widjaja, W., Vale, C., Groves, S. & Doig, B. (2017). Teachers' professional growth through engagement with lesson study, *Journal of Mathematics Teacher Education*, *20*(4): 357–383.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_6

Keywords Lesson study · Mathematics teaching · Interconnected Model of Professional Growth · Enactment · Reflection

1 Introduction

Lesson study is widely regarded as a model for teacher-led professional learning that centers on teachers working collaboratively with their colleagues to plan, observe, and reflect on their practice (Groves et al. 2016; Lewis et al. 2009; Takahashi and McDougal 2016). Features of lesson study that are considered as indicators of high-quality professional development include a focus on student learning, time for reflection and inquiry into practice, a focus on the development of teachers' content knowledge and pedagogical skills grounded in practice, support from school leadership, and the involvement of outside experts (Darling-Hammond and Richardson 2009; Guskey and Yoon 2009).

Despite its long-standing history in Japan and its widespread adaptation in other countries, there have been few studies that investigate the mechanisms through which teacher professional growth occurs through lesson study (Elliott 2012; White et al. 2011; Widjaja et al. 2017). Adaptations of lesson study in other countries have resulted in different "versions" of lesson study, some of which bear little resemblance to Japanese Lesson Study (JLS) and ignore some of its salient features (Fujii 2016). However, Stigler and Hiebert (2016) claim that explicit theories underpinning lesson study would make it possible to "adapt what is essentially a cultural practice or routine to a new setting" (p. 582).

This chapter investigates the professional growth of *individual teachers* from the second planning team participating in the *Implementing structured problem-solving mathematics lessons through lesson study* project. The Clarke and Hollingsworth's (2002) *Interconnected Model of Professional Growth* (IMPG) is used to identify "change sequences and growth networks" (p. 957) in order to explain the mechanisms through which professional growth occurred for this group of teachers.

2 Literature and Theoretical Framework

This section discusses lesson study and Clarke and Hollingsworth's (2002) *Interconnected Model of Professional Growth* (IMPG).

2.1 Lesson Study

Japanese Lesson Study is a professional learning activity, the origins of which can be traced back for over a century. While there are variations in the lesson study process (see, e.g., Fujii 2016; Takahashi and McDougall 2016), in general, the school-based version of lesson study consists of four components: (i) formulation of overarching school goals related to students' learning and long-term development; (ii) group planning of a *research lesson* addressing these goals; (iii) one team member teaching the research lesson, while the planning group, and others, observe in order to gather evidence of student learning; and (iv) the post-lesson discussion where the planning group and other observers (including an "outside expert") discuss and reflect on the evidence gathered during the lesson, using it to improve the lesson, the unit, and teaching more generally (Perry and Lewis 2008 p. 366).

Teachers play a central role in researching classroom practice and exploring ways to improve students' learning in lesson study. The post-lesson discussion are informed by students' work samples and observers' notes collected during the research lessons (Lewis and Tsuchida 1998; Takahashi and Yoshida 2004). The planning process involves setting the goals for the lesson, studying curriculum documents, identifying appropriate teaching resources, and, in mathematics, finding and solving a suitable mathematical problem and anticipating students' solutions. Observers collect evidence of students' learning and document salient moments in the teaching and learning process during the research lesson. Teachers, researchers, and observers then discuss their "evidence" of students' learning and share ideas to improve the teaching and learning process during the post-lesson discussion (Takahashi and Yoshida 2004; Watanabe 2002). It should be noted that the focus of the post-lesson discussion is on the teaching and on students' learning and *not* on the teacher.

2.2 Structured Problem-Solving Mathematics Lessons

The impetus for the widespread interest in Lesson Study originated from the video and print descriptions of the "typical" Japanese Year 8 mathematics lessons captured as part of Stigler and Hiebert's (1997) TIMSS video study. These lessons followed a pattern which Stigler and Hiebert referred to as structured problem-solving. Typically, the research lesson in JLS in mathematics takes the form of a structured problem-solving lesson.

These lessons have a single focus and address a single problem designed to "achieve a single objective in a topic" (Takahashi 2006, p. 4). According to Shimizu

(1999), major characteristics of these structured problem-solving lessons are (i) the *hatsumon*, the thought-provoking question or problem students engage with, which is the key to students' mathematical development; (ii) *kikan-shido*, the *purposeful scanning* that takes place while students are working individually or in groups, which allows teachers not only to monitor students' strategies but also to orchestrate their reports on their solutions in the *neriage* phase of the lesson; (iii) *neriage*, the *kneading* stage of a lesson that allows students to compare, polish, and refine solutions through the teacher's orchestration and probing of student solutions; and (iv) *matome*, the summing up and careful review of students' discussion in order to guide them to higher levels of mathematical sophistication. Critical in the process of planning, a mathematics research lesson is anticipating student responses, which help guide the teacher in selecting students to share their solutions in the *neriage* stage of the lesson.

2.3 The Interconnected Model of Professional Growth

Clarke and Hollingsworth's (2002) *Interconnected Model of Professional Growth* (IMPG) (Fig. 1) has been used in many recent studies of teacher professional growth, including a number on lesson study (e.g., Goldsmith et al. 2014; Schipper et al. 2017; White et al. 2011; Widjaja et al. 2017). The model posits that teacher change happens through the process of enactment and reflection in four domains: the personal domain (teacher knowledge, beliefs, attitudes), the domain of practice

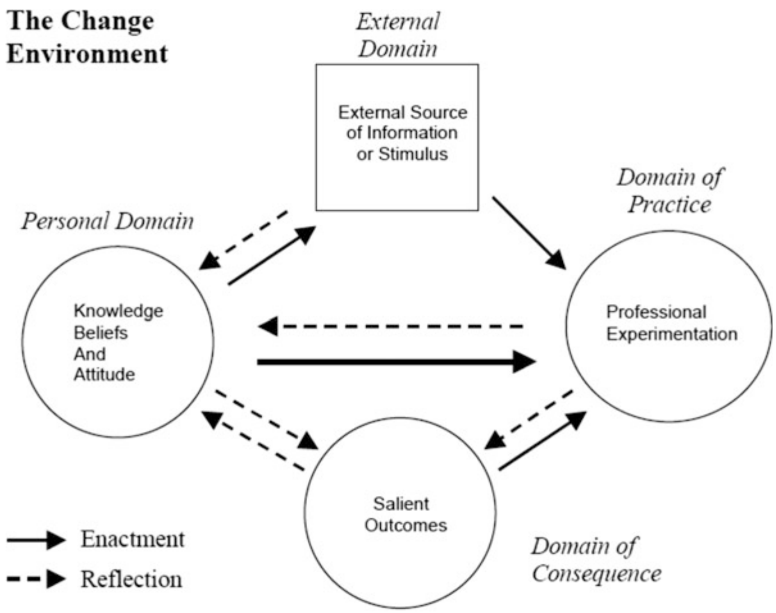


Fig. 1 Interconnected Model of Professional Growth (Clarke and Hollingsworth 2002, p. 951)

(classroom experimentation), the external domain (professional development, external support, and stimuli), and the domain of consequence (salient outcomes).

Drawing on Guskey's (1986) linear model of teacher change, the IMPG model depicts a nonlinear, iterative model that takes into account the "various dynamics at work in social behavior and [how] these interact and combine in different ways" (Clarke and Hollingsworth 2002, p. 378). Clarke and Hollingsworth (2002) argue that multiple "growth pathways" between the domains are possible (p. 950). Opfer and Pedder (2011) argue that the IMPG model is useful in capturing the complexity of teacher learning, particularly in explaining the "cyclic nature of the learning and change process" (p. 385).

3 Method

This section provides an overview of the project and the methods of data collection and analysis.

3.1 Overview of the Project

Three schools from an Australian local school network participated in the *Implementing structured problem-solving mathematics lessons through lesson study* project. A cross-school implementation of JLS was chosen to explore ways in which key elements of JLS could be embedded into mathematics teaching and professional learning for three schools through an interschool collaborative approach. Six teachers, two from each school, the numeracy[1] coach[2] or curriculum coordinator from each school, and the local network numeracy coach participated in the project. Following a professional learning day to introduce participants to JLS processes and JLS-structured problem-solving lessons, participants were divided into two cross-school planning teams, self-named the Matomes and the Bobbies. Each team consisted of three Year 3 or 4 teachers, one from each school, and two coaches. Widjaja et al. (2017) reported on the professional growth of the other team, the Bobbies. In this chapter we explore the professional growth of members of the Matomes team.

[1]Numeracy is the term used to describe the mathematics subject in primary schools in Victoria at the time of the study.

[2]Numeracy coaches were teachers who were nominated by the school principal to support continuing professional learning in mathematics within schools. They were also involved in professional conversations with other coaches in the school networks.

Table 1 Members of the Matomes Planning Team

Name (pseudonym)	School	Role	Teaching experiences (Years)
Trevor	C	Year 3 teacher Teacher of research lesson 1	2
Camilla	B	Year 3–4 teacher Teacher of research lesson 2	5
Sandra	A	Year 4 teacher Numeracy coordinator Curriculum coordinator	12
George	A	Numeracy coach	7
Narah	B	Curriculum coordinator	6

3.2 The Matomes Planning Team

The school, role, and number of years teaching experience of each participant of the Matomes planning team are shown in Table 1. Trevor was the least experienced teacher, with the others having taught in primary schools for at least 5 years. Prior to her 5 years as a primary teacher, Camilla had spent 2 years working as a casual relief teacher. Narah, the curriculum coordinator at School B, had spent 2 years teaching in England. She started as a numeracy coach at School B before taking the role of the curriculum coordinator. George was an experienced teacher who was a numeracy coach with a leadership role in his school. He was studying for a Master's degree in Educational Leadership. Sandra, the most experienced participant, had two leadership roles at her school, coordinator of the school's curriculum program and the school's mathematics program, in addition to teaching a Year 4 class. The second and third authors were also participant-observers in the Matomes planning team.

3.3 The Lesson Study Process

The Matomes and the Bobbies each completed one lesson study cycle during each of two 10-week school terms, planning their own research lesson, based on the same problem provided by the research team. Participants planned each research lesson during four 2-hour sessions. One member of each team taught the research lesson using the problems presented by the researchers (see Appendixes B and C). In the case of the Matomes, Trevor taught the research lesson in Cycle 1 and Camilla in Cycle 2. All members of both planning teams, key staff at each school, together with all interested teachers who could be released from their classes, observed the research lessons and took part in the post-lesson discussions. In all between 20 and 30 people, including members of the leadership teams from other schools, staff from the regional office, mathematics educators, and an outside expert observed each research lesson and took part in the post-lesson discussions. However, due to a change in the date for the research lessons and the subsequent post-lesson discussions, Sandra was unable to attend the research lesson in Cycle 2, although she played a full part in the planning.

3.4 Data Collection and Analysis

Throughout the study, the researchers kept field notes of planning meetings, research lessons, and post-lesson discussions. In addition, all sessions, including the professional learning day, were video-recorded. Planning meeting agenda, together with lesson plans and notes prepared by members of the planning teams were collected, as was students' written work from all research lessons. These data were complemented by individual, audio-recorded, 30-min interviews with participants on three occasions: at the beginning of the project and following the first and second Lesson Study Cycles. Interviews and post-lesson discussions were transcribed.

In Widjaja et al. (2017), we reported on the professional growth of the Bobbies team, using the data collected during planning meetings, interviews, and post-lesson discussions. Due to technical issues with the video-recordings of the Matomes' planning sessions, systematic analysis of data from these meetings was not possible, so this chapter is based on transcripts of the three interviews.

As was the case in our earlier paper, Clarke and Hollingsworth's (2002) IMPG provided the framework for the data analysis. Ethnographic methods involving open coding using a constant comparative method (Corbin and Strauss 2008) were used to analyze the interview data. All four authors jointly coded substantial parts of the interview data and adapted and agreed on a final list of much more detailed codes than those used in the previous analysis for the Bobbies team (see Appendix A for a full list of codes). As the IMPG is a dynamic model, the arrows from one domain to another indicating Enactment and Reflection are a critical part of the model. Therefore, where applicable, interview segments were coded with the Arrows, the Domains, and the detailed codes. As would be expected, there were very few arrows identified in the initial interview transcripts as no Professional Experimentation had yet taken place. However, arrows were prominent in the coding in the later interviews.

This coding was used to construct summaries and Change Environment figures showing the professional growth for each member of the Matomes team.

A brief extract showing part of the summary for Camilla is shown in Fig. 2. This extract shows that Camilla was responding to a question about what were the main things she had learned through her professional experimentations with the lesson study process (JL) in the Domain of Practice (DP). Her answer indicates that her reflection (R) on this aspect resulted in a realization of "just how important it is for the teacher to have a really good understanding of the content" (CK) – a change in her beliefs in the Personal Domain (PD).

In the Change Environment figure for Camilla (Fig. 4), this resulted in a Reflection (R) arrow from the Domain of Practice (DP) to the Personal Domain (PD).

R: DP-JL → PD-CK – main things that you think you've learnt: Just how important it is for the teacher to have a really good understanding of the content

Fig. 2 An extract from the summary for Camilla illustrating the coding process for arrows

A further layer of coding was constructed for the Change Environment figures, designed to show whether the changes observed were an extension (or enhancement) of previous beliefs or practices (coded with an X) or changes that were new understandings (coded with a C). These letters were then numbered to show in which cycles the extensions and changes were noted as occurring – for example, the code C1 is used to indicate *Changing in Cycle 1*.

4 Findings

This section presents the cases for the five members of the Matomes team that were constructed using the process described above.

4.1 Trevor

Trevor enjoyed teaching mathematics and appreciated opportunities to observe other teachers teach and to receive feedback from the literacy and numeracy coaches to enhance his knowledge and practice of teaching. He was aware that not all people are comfortable with receiving feedback about how they could improve their practice:

> I really value observing others but probably a big one for me is getting feedback and giving feedback and taking it on as being constructive, not taking it as a personal attack on what you've taught, but just being able to make you better, make others better in their teaching. (Interview 1, Ln 110–114)

Trevor was therefore looking forward to the project and the opportunity to observe or teach and participate in post-lesson discussions. Planning at Trevor's school involved the Year level team planning the content and sequence and number of lessons for a term. Pairs of teachers, one experienced the other inexperienced, then planned detailed lessons that all teachers would follow. His school (School C) used the Education Department Region's lesson model that involved a warm-up, introduction, student activity and share time. The share time was supposed to be 10–15 min but Trevor found that he often did not have enough time left to conduct this part of the lesson. He recognised that he needed to put more emphasis on "different ways we can solve problems in terms of mathematical thinking" and using the whole class time at the end of the lesson to do this (Interview 1, Ln 719–724). Trevor believed that mathematics teaching needed to be inclusive, that is, to cater for the needs of the individual kids, the ones that need that support and extra guidance, and the case where those ones that need that extension to get to the next level of their learning. (Interview 1, Ln 259–262).

Cycle 1: During the third planning meeting, the Matomes team randomly selected Trevor to teach the first research lesson using the matchstick problem (see Appendix B). The changes implemented and their consequences and impact on Trevor's personal domain and professional practice are displayed in Fig. 3. Trevor trialed

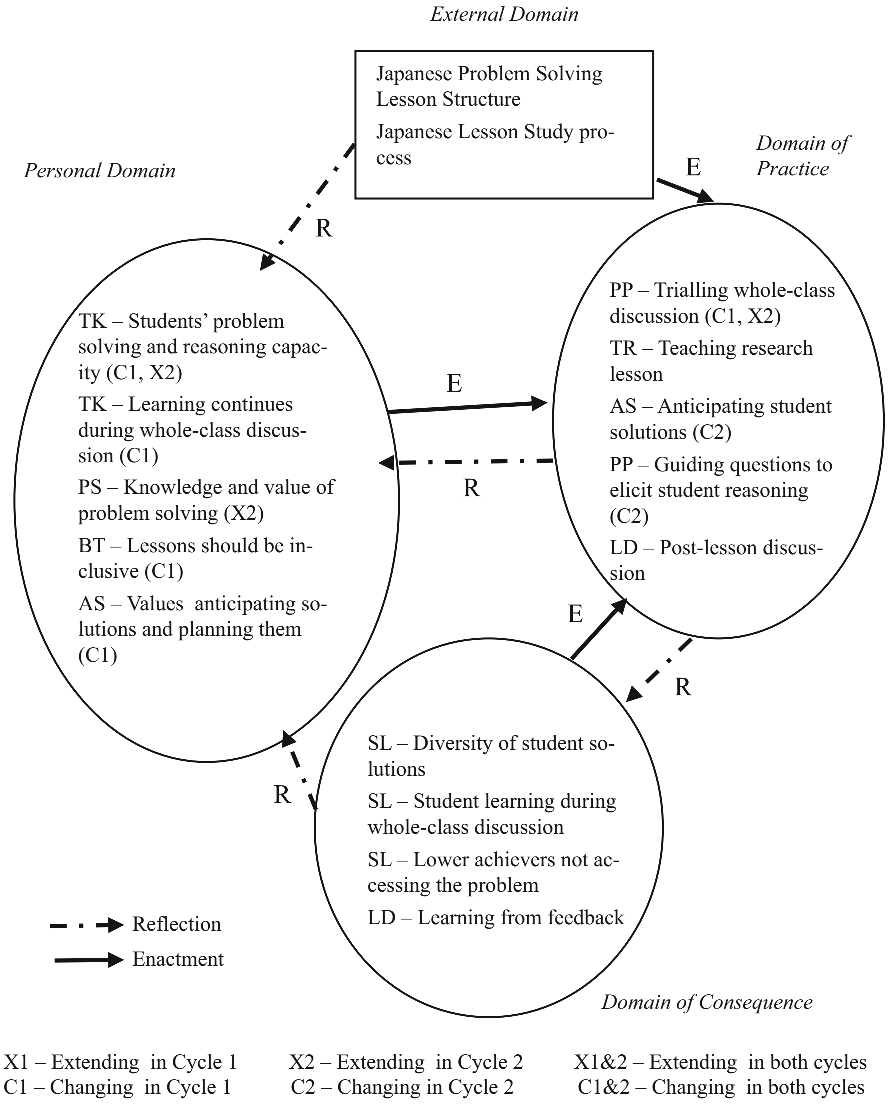


Fig. 3 Trevor's change environment during Cycles 1 and 2

the Matomes' draft lesson plan with another Year 3 grade at the school prior to teaching the research lesson in his class. The numeracy coach at his school (who was a member of the Bobbies LS team rather than the Matomes) worked with him to trial the lesson. Together they reflected on the lesson afterward and confirmed that a larger amount of time devoted to the whole-class discussion would work – that is, students could stay engaged longer than expected. After teaching the research lesson, Trevor reiterated the value of the whole-class discussion for students' learning:

> Actually over the past couple of weeks we've been doing that more and spending more time in the reflection part and it seems to work really, really well, I've seen that. (Interview 2, Ln 39–51) . . . And I think the more times we practice [whole-class discussions] with the kids they'll get better and better at it. (Interview 2, 115–20)

He was surprised by the range of solutions generated by students in his class, and this revealed to him of the value of planning for the possible solutions of students:

> I know looking at this problem we've done today, I would never have that many possible solutions, but it just shows you how kids think differently. (Interview 2, Ln 153–5)

Trevor used the questions and prompts included in the lesson plan to elicit students' reasoning during the whole-class discussion. He concluded that along with anticipating student solutions, planning questions was also important:

> more the questioning and allowing students to answer, giving open questions and allowing them to explain, not just sort of guide, like ask them guiding questions where you know what the answers going to be. Asking open ones where they're explaining and thinking about stuff. (Interview 2, Ln 218–225)

However, Trevor was not convinced that the lesson had worked well for all students in the class especially the lower achievers who used materials and counted by ones and did not see a pattern. He wondered about what enabling prompts he could have used to engage these students:

> But I know there's a couple of kids who got no benefit out of it because I wasn't able to give them any prompting or any guidance to get them going. (Interview 2, Ln 430–434)

The post-lesson discussion was driven by the evidence of students' work and included discussion of teacher actions that could have been used to engage these particular students. The post-lesson discussion confirmed Trevor's beliefs about the value of feedback for his learning.

> I think I got it from the feedback we got from the post lesson discussion, and that I need feedback, especially in teaching, if you don't get any feedback you're not going to improve. (Interview 2, Ln 359–361)

Cycle 2: In the second LS cycle, Trevor contributed to planning the research lesson, observed Lyn from School B teach it, and participated in the post-lesson discussion. As a consequence of planning and observing a second structured problem-solving lesson, Trevor realized that problems did not have to be open-ended to engage all students: "they're open... [but] they're still directed in a certain area as well" (Interview 3, Ln 311–312). They could be designed to achieve a specific learning goal but still allow students to solve them or express their solution in various ways. He also realized that observing a lesson did not just allow him to learn from other teachers but that it provided an opportunity to learn more about students' thinking:

> I just watched like four students. . . . . . I really got to understand how a student thinks. You can make generalisations about how they think but I could never be able to sit there and . . . I really valued that. (Interview 3, Ln 69–74)

Apart from changing his classroom practice to use problems with specific learning goals that provide access for students and various methods of solution increasing the time given to whole-class discussion at the end of the lesson, and using open questions to elicit students' explanations, Trevor stated that he would also press for changes to the way his school planned lessons. He valued the opportunities to collaborate and learn from others:

> I've really enjoyed [the lesson study project], it's given me a lot of opportunities to do things I wouldn't have done before as a – like as a graduate. It's given me the chance to improve my teaching and I've thoroughly enjoyed it. I got to work with a number of different people and listen to the opinions and thoughts and suggestions of you know experts in numeracy and maths. And it's continued my interest in numeracy and makes me want to continue it in the future. (Interview 3, Ln 401–407)

Trevor's reflection on the practices enacted during the lesson study project and in his classroom during the project illuminated salient consequences regarding students' learning, lesson planning, and teacher actions. He acknowledged new understandings of his students and their potential. He continued to develop his practice in conducting whole-class discussions and using questions to elicit children's solutions and reasoning.

4.2 Camilla

At the time of this research, Camilla was teaching Year 3 and 4 composite classes at School B, the same school where Narah was the curriculum coordinator. Camilla was in her seventh year of teaching, five of which had been at these year levels, with the other two being as a casual replacement teacher. According to Camilla, teachers at School B were encouraged to work together, with students in mathematics lessons grouped in hubs based on their needs. Camilla stated that she put a lot of effort into her mathematics teaching "because maths isn't my strong point, I always struggled with maths" (Interview 1, Ln 9–10). She had a bit of intermittent mathematics coaching 2 years earlier and more intensive mathematics coaching in her second year of teaching.

Camilla particularly valued professional learning experiences that were "hands-on. the ones that you get to experience the things yourself" (Interview 1, Ln 645–646). She regarded having experts on hand and understanding and support from leadership, and being trusted to do your job as important elements in supporting her professional learning.

Camilla regarded problem-solving as an important aspect of mathematics "because life is about problem solving . . . if you don't teach the kids how to solve the problems they can't do it" (Interview 1, Ln 575; 579–580). She described a problem-solving session as students working together, coming up with a solution on their own, and then sharing it with the others, and talking about the problem-solving strategies they used – which she saw as a major purpose for discussion. However, Camilla had never specifically focused on anticipating student responses to use when sharing student solutions. She described as one of the challenges she faced as the need to "differentiate very much in my class because I have such a wide scope . . . [with] a lot of children that can't work in small group independently" (Interview 1, Ln 543–546).

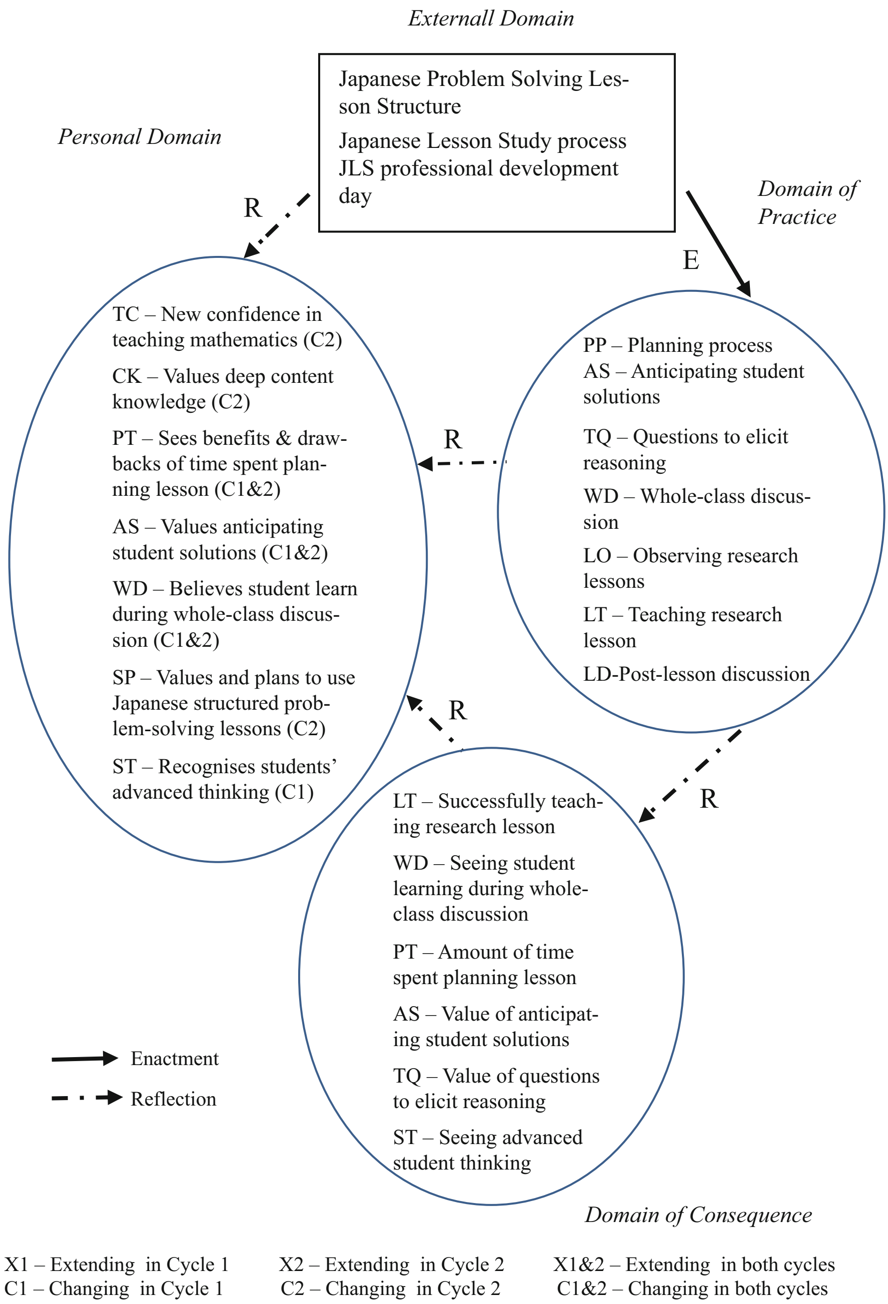


Fig. 4 Camilla's change environment during Cycles 1 and 2

Regarding her assessment practices, Camilla said that she used "lots of anecdotal notes, lots of observations, but pre-tests and post-tests and stuff as well" (Interview 1, Ln 591–592). She reflected on her teaching by thinking about whether the students learned what she intended to teach them and discussing her reflections with other people, rather than writing them down.

Cycle 1: Camilla contributed to planning the Matomes' research lesson and observed both the Matomes' research lesson and the Bobbies' research lesson that used the same task.

While Camilla appreciated the benefits of having such a lot of planning time, she also felt that there were drawbacks, stating that "the amount of time that I've had to spend on this has been a challenge" (Interview 2, Ln 49). However, she saw this investment of time as having paid off in terms of her discovering "how important it is to know where you want to take the lesson . . . finding out what the kids are thinking or knowing what the kids are going to say" (Interview 2, Ln 178–179). Further, Camilla believed that in her own planning, she would be working "backwards a bit more, thinking more 'this is what I want to get out of them through the reflection' and being prepared for any eventuality and any student response" (Interview 2, Ln 214–216).

As an observer of the two research lessons, Camilla found it interesting to see "how other people would have approached the problem and the different ways teachers react to their students and interact with their students" and "watch the students, [when] you knew where they were going" (Interview 2, Ln 225–229). Overall, she believed the most beneficial aspect of observing a research lesson was being able to discuss it in the post-lesson discussion with other people who might notice something that you hadn't noticed. Camilla appreciated the fact that the focus of post-lesson discussion was on the teaching and the learning of the students, and not the teacher, and believed that as a result of participating in these discussions, she would try to "remember to be a little bit more reflective in my lessons" (Interview 2, Ln 406). She also believed that she would be spending more time on thinking about her questioning strategies.

As a result of observing the two research lessons, as well as the trial of earlier problem-solving lessons, Camilla was surprised by how advanced some of her students' thinking was and the way in which some students were able to think algebraically without her realizing it. She saw this as influencing her future teaching with a "lot more of the teaching [being] done during the sharing time . . . [and] helping me think about how I would present future problem solving" (Interview 2, Ln 71–72; 99–100).

Cycle 2: In Cycle 2, as well as participating in the lesson planning and taking part in the post-lesson discussion, Camilla taught the research lesson to her own class. Camilla claimed that, while a lot of what she learned through teaching the research lesson came from:

> the planning and preparation. From the actual specific teaching part of the lesson, at the moment all I can say is I learned that I can do it . . . I wanted to prove to myself and to the other people that I could do it. (Interview 3, Ln 330–332; 311–312)

As further evidence of the increase in her confidence, Camilla stated that she was "hoping to have my own lesson study group or lead a lesson study group next year in the school" (Interview 3, Ln 214–215).

Regarding changes she had made or intended to make to her classroom practice as a result of being part of the project, Camilla said:

> I have already changed my classroom practice in that I am spending a lot more time on the reflection part of it and thinking – trying to get the teaching done. I am making sure that I have a really good understanding . . . of the content of what I'm trying to teach. (Interview 3, Ln 190–194)

Camilla's focus on content was also evident in her response to the question of what changes she intended to make to her planning, where she said she intended to start "spending a little bit more time really getting to know the content really deeply" (Interview 3, Ln 299–300) and later that one of the main things she had learned was "Just how important it is for the teacher to have a really good understanding of the content" (Interview 3, Ln 95–96).

According to Camilla, a main benefit of implementing lesson study was:

> Changing the teachers' thinking . . . Getting the teachers to really understand what it is they're teaching and where they want to go – the point of the lessons, the point of the units. And getting them to do that research before teaching the lesson to sort of think about – why do I want to teach them how to do that. (Interview 3, Ln 582–586)

Regarding the use of the Japanese problem-solving structure, Camilla stated that she intended to use it much more frequently, adding that she had learned that you could

> spend most of your time doing the teaching during the reflection stage . . . And that *neriage* – the way of presenting the answers in the order that leads towards an end point – and summarising it all to the final point . . . [has] been fantastic. (Interview 3, Ln 99–103)

Camilla's views about discussion had also changed; "I always knew that they were really important. But now I can really see how important it is. Just how rich it can be and how good it can be" (Interview 3, Ln 512–514).

However, a major challenge for Camilla was that

> You need to really, really know your kids. You need to really know where you want to go. And you need to . . . have those anticipated responses so that you know which examples to have. So you need to be more prepared prior to the lesson. (Interview 3, Ln 402–405)

This challenge was partly overcome by "having the other teachers have their little practice sessions with the same lesson so that you can gather up the anticipated responses . . . it felt like I'd never been more prepared for a lesson" (Interview 3, Ln 133–137). Camilla believed that in her everyday practice she would "do a lesson with the kids, get their responses and then keep that bank of responses for your anticipated responses for the next time that lesson comes around" (Interview 3, Ln 175–177).

Overall, while Camilla described her experiences in the project and the JLS model as "really worthwhile and really beneficial." One of the problems she had "with it was the amount of time that I was out of the classroom" and "how much extra work was involved" (Interview 3, Ln 572–574; 89–90).

4.3 Sandra

At the beginning of the project, Sandra stated that teaching mathematics “is a passion of mine” (Interview 1, Ln 85–86). She believed that mathematics was all about problem-solving:

> I think [problem solving] underpins what we do in mathematics, and how we use mathematics, we use mathematics to solve everyday problems and people use different strategies to come to an end result and everybody uses a different strategy to get there and in the end it doesn’t really matter how you get there, as long as you understand how you got there and you’re happy with the result. (Interview 1, Ln 332–337)

Sandra believed teaching should engage students in rich tasks with multiple solutions, and that students should be encouraged to explain and justify their solutions. She also focused on developing students’ vocabulary so that they could explain their thinking. She believed that using effective questioning strategies during share time and whole-class discussion was important. Teachers at her school (School A) planned in teams. They were encouraged to engage in professional reading as part of the planning process. They also used formative assessment to plan and sequence lessons. This included pretests, student observations-checklists and anecdotal records, and student work samples. They moderated assessment of student work samples. The school followed the Education Department Region’s lesson model, though Sandra would sometimes conduct the “share time” in the middle of the lesson.

Cycle 1: Sandra contributed to planning the Matomes’ research lesson and observed both the Matomes’ and the Bobbies’ first research lessons that used the same task. Sandra recognized the value of anticipating student solutions when planning the lessons.

> I liked the anticipating the student responses, something that you don’t really do in your everyday teaching ... Certainly thinking about them in the depth that we have... has really helped, particularly with that solution selection at the end...understanding the range of strategies and where they fit in the scheme of, you know, developmentally. (Interview 1, Ln 29–35)

While observing the research lessons, Sandra focused her attention on a group of students in each lesson and noticed their level of engagement – that is, whether they could access the problem or not (Matomes’ Research Lesson) and whether they were challenged to extend their thinking and to generalize (Bobbies’ Research Lesson). She noticed that the students tended to rely on less efficient processes or strategies such as counting all by ones, to solve the problem rather than those anticipated and documented in the lesson plan. She noticed that they also struggled to use appropriate vocabulary.

> The little people at my table were just not trusting what they were doing, and changing their thinking, but not really having the vocabulary or the means to sort of explain their thinking with their drawings or their words, they changed their labels ... you’d think they would, you know, trust the count. (Interview 1, Ln 279–282, 289)

She noted the difference in the introductions to the problem in the two research lessons and the difference in types and number of responses from students in the two grades: “[I’m] wondering . . . how the Grade 4s would have attempted our lesson, compared with the [Grade] 3s” (Interview 2, Ln 270–272).

She valued the “getting other people’s ideas, questioning people’s thinking, questioning your own thinking” (Interview 2, Ln. 159–160) that occurred through the planning, observation, and post-lesson discussion.

Cycle 2: As was the case in Cycle 1, Sandra contributed to planning the Matomes’ research lesson, although she was not able to be present on the day the two research lessons were taught. Following this cycle Sandra noted how students’ engagement and learning, problem selection and representation, and anticipating students’ thinking and solutions were impacting on her planning, teaching, and assessment practice.

> It has definitely been the most valuable professional learning I’ve done this year. And it has changed my focus with my own numeracy lessons so the way I look at the way students work through a problem had definitely become more focussed, more process oriented and more questions to find out well why are they thinking the way that they’re thinking. (Interview 3, Ln 32–37)

At the same time, the lesson study project had also reinforced or refined her beliefs about teaching and learning of mathematics. The impact of the lesson study project on her professional growth is depicted in Fig. 5.

Sandra came to realize the challenge and importance of selecting the right problem and matching it to the learning goal and students’ readiness:

> Probably looking at the problem itself and then trying to match it to specific goals and content area that’s probably been the most challenging. . . . more especially with the last feedback we got. Well why that number? It made perfect sense well yeah why that number when quite easily we could have shown the same thing with simpler number the same sorts of goals that we had so it really identifies the need for picking the right problem for the goal. (Interview 3, Ln 60–68)

She changed the way she conducted whole-class discussion to follow the process used for structured problem-solving lessons in JLS.

> I find I’m . . . spending more time on the share time and the share time for me has always been how have you solved the problem but it’s now more looking at the different types of strategies that students use to solve a problem. (Interview 3, Ln 301–305)

She remained committed to using the “why?” questioning when interacting with students but also came to realize the importance of providing visual representations of students’ work and strategies during whole-class discussions.

Sandra also reported changes to her assessment practices, in particular, in her “in the moment” observations of students. Sandra really valued her experience of observing different lessons that used the same task and focusing her observation on a small group of students.

> I only focussed on a small group or a particular table and you do realise how much you don’t notice about what students are doing when you’re an observer and you just wish you had the eyes all over the place all the time. (Interview 3, Ln 168–171)

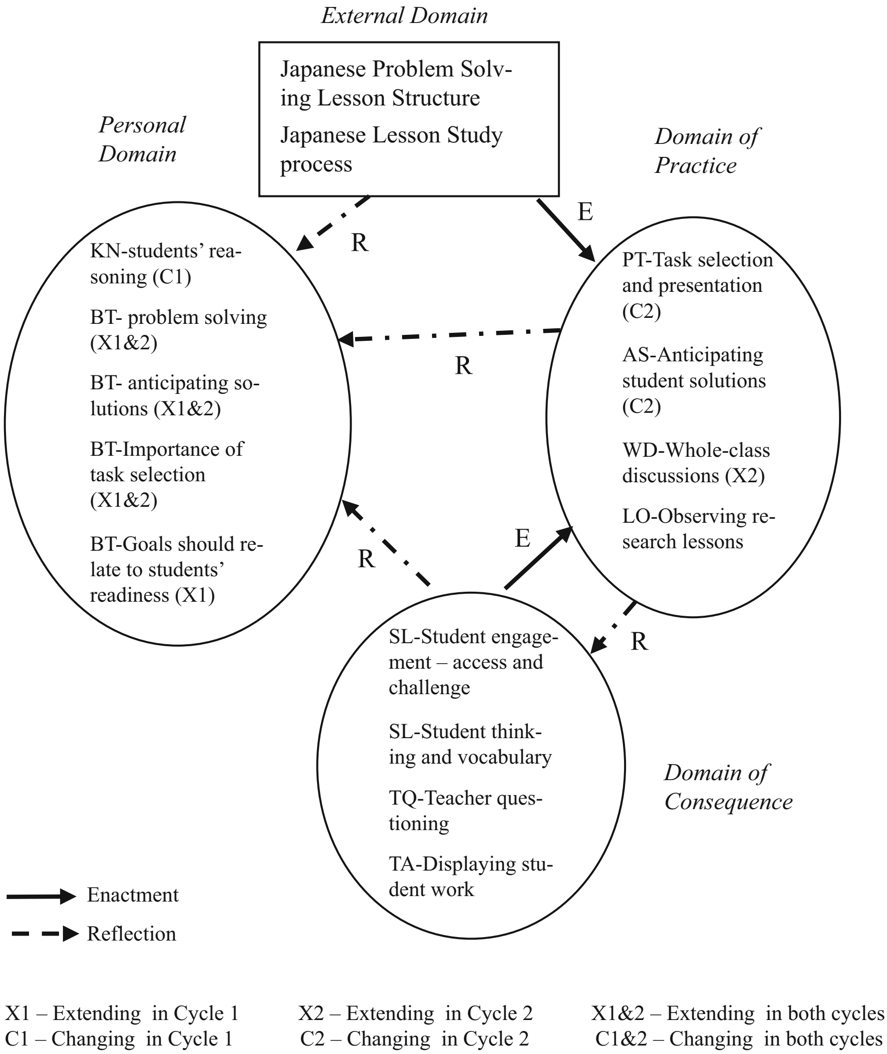


Fig. 5 Sandra's change environment during Cycles 1 and 2

She wants her school to introduce observations of students in other teachers' lessons for teacher professional learning. Sandra reflected on the planning and teaching of the research lessons to identify salient outcomes regarding students' engagement, representation, and launch of the task and the practice of observing students. The experience and consequence consolidated Sandra's beliefs about the value of problem-solving and the importance of teacher questioning during whole-class discussion to support and challenge students when communicating their solutions and reasoning.

4.4 George

George was a numeracy coach and lead teacher at his school, with responsibilities for providing professional development within the school. His focus at the time of the interviews was on the Year 3 and Year 4 teachers, a level at which he had taught for some years. He was "currently undertaking [a] Master[s] [degree] . . . in Numeracy Leadership" (Interview 1, Ln 61–62).

George gave a lot of information about the organization of the classes and the school in general. He was a firm believer in professional learning communities, which he organized for the Year 3 and 4 teachers, and had an emphasis on "student work and assessment" (Interview 1, Ln 84). Under his tutelage the teams planned lessons using the "whole-part-whole" model (Interview 1, Ln 177), with the warm-up "roughly would go for around 5 min . . . then the student activity normally runs for approximately 30 to 40 min . . . then the reflection . . . normally 5 to 10 min" (Interview 1, Ln 188–189). With respect to discussion, George said that he liked "the students to work together for at least the first 10 min of the activity. And then we actually stop as a class and have a class discussion about some strategies . . . then share some different strategies which hopefully allows students . . . to learn from each other" (Interview 1, Ln 221–226). Typical goals for these lessons were for the "students to articulate their thinking . . . show their reasoning and how they actually solve a problem" (Interview 1, Ln 332–334).

Teacher observation of other teachers in their classroom was routinely conducted by George in his role as the numeracy coach as well as by other teachers. He said that "We have collegiate classroom visits . . . that involves a meeting before the lesson to talk about it, the observations and then the meeting after" (Interview 1, Ln 410–411).

George had read about lesson study, as adapted in the United States, and had "taken away and tried to use in [his] team, . . . the idea of research and finding the best way . . . to teach concepts" (Interview 1, Ln 525–528).

Cycle 1: In the second interview, George said that the important things that he had learned during the lesson study cycle were "thinking through the planning and thinking about what student responses might come up" (Interview 2, Ln 22–23). He expressed surprise at how the long planning time was not enough. A further change had occurred in his thinking about the lesson structure: "a really significant shift is how the lesson is structured [with] the share time at the end . . . [and] it's a great increase in the amount of time . . . we are normally used to here" (Interview 2, Ln 26–9). He further explained that

> I've been able to go in and really model the planning process, to really think through what the task's going to evoke from the students and then how I use the students to, sort of, do the teaching through the share time . . . I've really tried to model that since starting this [lesson study] process. (Interview 2, Ln 77–81).

While appreciating the advantages of the JLS model, and the structured problem-solving lesson in particular, he didn't "think that we can jump all the way there, at least to start with" (Interview 2, Ln 145–146). George suggested that teachers would learn a lot about their students' thinking by using the structured problem-solving lesson approach, and that this would benefit the students as later lessons could build on the teacher's knowledge of the students.

When asked what changes he had made in his teaching as a result of his involvement in the lesson study cycle, he said

> the timing of the lesson and how the lesson is structured that minimising the instruction time at the start, letting the students work on the problem, and then having that larger share time at the end where the students are doing the thinking, and learning, and the talking. (Interview 2, Ln 190–194)

In response to a question about changes that are critical to make in lesson planning, he replied that "building the content knowledge [of teachers] through the process, [so that] teachers [were] not just grabbing activities [but were] looking at ... the continuum of skills for [the topic]" (Interview 2, Ln 284–287). As a numeracy coach, George observed many lessons, but thought that the lesson study observation was "great" (Interview 2, Ln 329), because, if "you're the teacher a lot of stuff gets forgotten and missed" (Interview 2, Ln 370). He felt that the benefits for the students of the structured problem-solving lesson was that the students were "spending the time to really ... sinking their teeth into a problem and then also listening to each other, shar[ing] their responses" (Interview 2, Ln 377–379).

The post-lesson discussion aspect of the lesson study cycle had the potential to be difficult, as it might "become an attack on the person who delivered the lesson ... [but] I think, in general, we kept our focus on the team of teachers" (Interview 2, Ln 452–454) and "how overwhelmingly positive it was" (Interview 2, Ln 489). However, summing up his challenges with regard to lesson study, George reiterated a theme running throughout his interview, in that, "we are time poor" (Interview 2, Ln 502–503).

Cycle 2: The main issue for George was the time needed to plan, observe, and discuss, and yet, despite this constraint, he thought that the strengths of lesson study lie in the fact that "it's a very rigorous process ... [and] that it's a collaborative process" (Interview 3, Ln 47–49), all of which takes time. In his role as a numeracy coach, George was able to visit classrooms and see the effect the lesson study experience was having on the teachers.

In his school, some of the staff had already started to implement "orchestrating and ordering, and organizing, the student responses during share time" (Interview 3, Ln 71–72), which he described as a big change in teachers' practice. This was underscored in his comment that "an amazing part of the second [lesson study] cycle was when we started talking about the order that we'd wanted to select students in to share" (Interview 3, Ln 189–190). Clearly, this was an important, and powerful, revelation to both George and his colleagues. A further feature of the whole-class discussion, which had an impact on George, was that he now realized that "teaching can still happen there ... [and] that's been a very powerful lesson from this project" (Interview 3, Ln 85–86) (Fig. 6).

When asked about feedback from the teachers about the lesson study experience, he said that "the idea of the structure of the lesson is something that has [had] a very

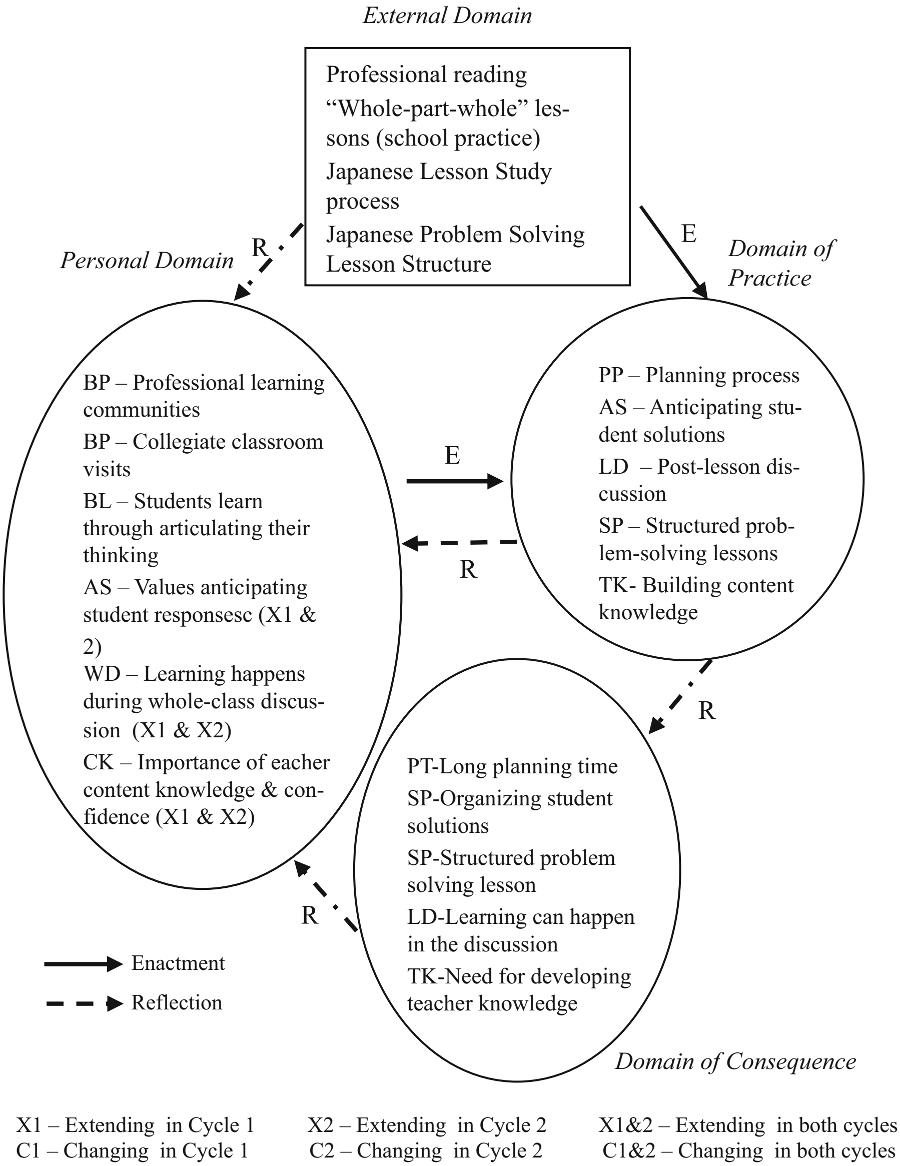


Fig. 6 George's change environment during Cycles 1 and 2

positive comment and it's something we've actually written into our annual implementation plan for next year" (Interview 3, Ln 302–305).

At the end of the research lessons, the observers provided feedback to the planning team, and then a *knowledgeable other*, or outside expert, made concluding remarks aimed at providing an extension to the mathematical topic of the lesson or remarking on a pedagogical issue. What struck George were the comments made by the knowledgeable other on the specific numbers used for the problem set for the lesson. Reflecting on the comments, George admitted that normally:

> You just pick them randomly, whereas that very deliberate thinking through why you want just those numbers, not even a whole problem or question ... I think that was just really, really, good. (Interview 3, Ln 359–362)

In summary, George was very enthusiastic about the lesson study process but worried about the time needed to do it properly. The aspects that he thought were very positive were teacher collaboration, planning, and discussion by both students and teachers. An issue for George was that he thought that teacher knowledge was in need of improvement with regard to student misconceptions. While the structure of the lessons, very different from current practice, was seen in a very positive light.

4.5 Narah

Narah is the curriculum coordinator with 8 years of teaching experience. She moved to School B 18 months ago, and in previous years, she was the mathematics coach at the school. In her recent role as a curriculum coordinator, Narah participated in regular professional development by the regional coach for the network of schools every third week as well as those delivered by the school coach. Narah believed that teaching mathematics should be inquiry-based and underscored the importance of using assessment data to inform planning in order to ensure students' learning progressions and needs being catered for.

> We need to look at what big areas we need to address and then we use a continuum of learning for the children, so we – we need to know where our children are at on that continuum, so based on where they're at that's how we decide our learning intentions. (Interview 1, Ln 355–359)

Her understanding of problem-solving was informed by the regional model problem-solving strategies or heuristics:

> We use the WMR problem solving strategies, so those strategies that are linked to different learning styles, so for example, make a model, make a table, look for a pattern and so on, so those are explicitly taught and used across the school as well, to varying degree of success at this point. (Interview 1, Ln 157–161)

She saw the purpose of discussion in the lesson to "bringing all learning together is happening amongst the children in the discussion at the end." Narah argued that the whole-class discussion should provide students with teaching and learning opportunities beyond just sharing time.

> It's the climax of the lesson, it's the drama, it's that point where its' like wow, but yes that's probably a big weakness in our classes that people see it as a sharing time, not as a learning time. I see that discussion at the end -I almost hesitate to call it reflection as well because it's not, it's – that's when the real learning's happening. (Interview 1, Ln 587–592)

Cycle 1: Narah contributed to planning the Matomes' research lesson and observed both the Matomes' research lesson and the Bobbies' research lesson that used the same task. While Narah thought four planning sessions was "a bit much" at the start, she valued the reflective and collaborative discussion and noticed "a real development in our thinking about, ... the maths that's involved in the lesson and the questioning and really thinking down to every detail." She identified the need for schools to provide time for professional learning and support for teachers "to research and source their own problems" and "resources to support teacher knowledge." She recognized teachers' lack of content knowledge and confidence in teaching mathematics as one of the real challenges and valued the lesson study process in the development of teacher knowledge and teacher questioning skills. She identified the whole-class discussion at the end of the lesson as the most challenging aspect to implementing it in schools.

> The sharing part at the end, sharing strategies, is the crux of the lesson, it's so crucial. So therefore that's the hardest part to implement because if you don't get it right, the lesson's essentially a waste of time... and that all comes back again to teacher knowledge and unpacking the thinking and the progression of thinking and learning within a task and knowing the maths, in the task. (Interview 2, Ln 395–399)

Narah was initially worried that the post-lesson discussion might be a bit shallow. She underscored the importance of establishing the trust between the teams at the start of the project and the thorough planning process to allow for productive post-lesson discussion.

Cycle 2: Narah reiterated the important role of teacher content knowledge and teacher ability to question children and extend their learning as an important driver in teacher professional learning and growth in her Cycle 2 interview. She highlighted the vital role of school leadership to support teachers in terms of structuring time for teachers to collaboratively plan together and embedding this in their professional learning meetings. While acknowledging the importance of collaboration in planning, her view of collaborative planning shifted more toward a community of inquiry perspective.

> I think just the importance of collaboration and a big thing that's dawning on our teachers here, is that collaboration doesn't mean, someone else does the work for you and its less work. It's about really challenging each other's thinking and questioning each other, and that's been a big feature of this. (Interview 3, Ln 140–144)

In her role as the curriculum coordinator, she valued the detailed planning in JLS. In particular, she identified the anticipated student responses as one of her biggest learning in the project that she had implemented in her practice in other subject areas.

> But the big one is, the big one that can be applied in lots of different contexts, be it maths or your unit of enquiry or literacy, is the anticipated responses. That's a huge one and I've used that now, with a lot of teachers, even those who haven't been involved in this project. (Interview 3, Ln 132–135)

Narah acknowledged the role of the knowledgeable other and appreciated his honest and constructive feedback during the post-lesson discussion. She also realized different cultural contexts and reflected on her own experience as the mathematics

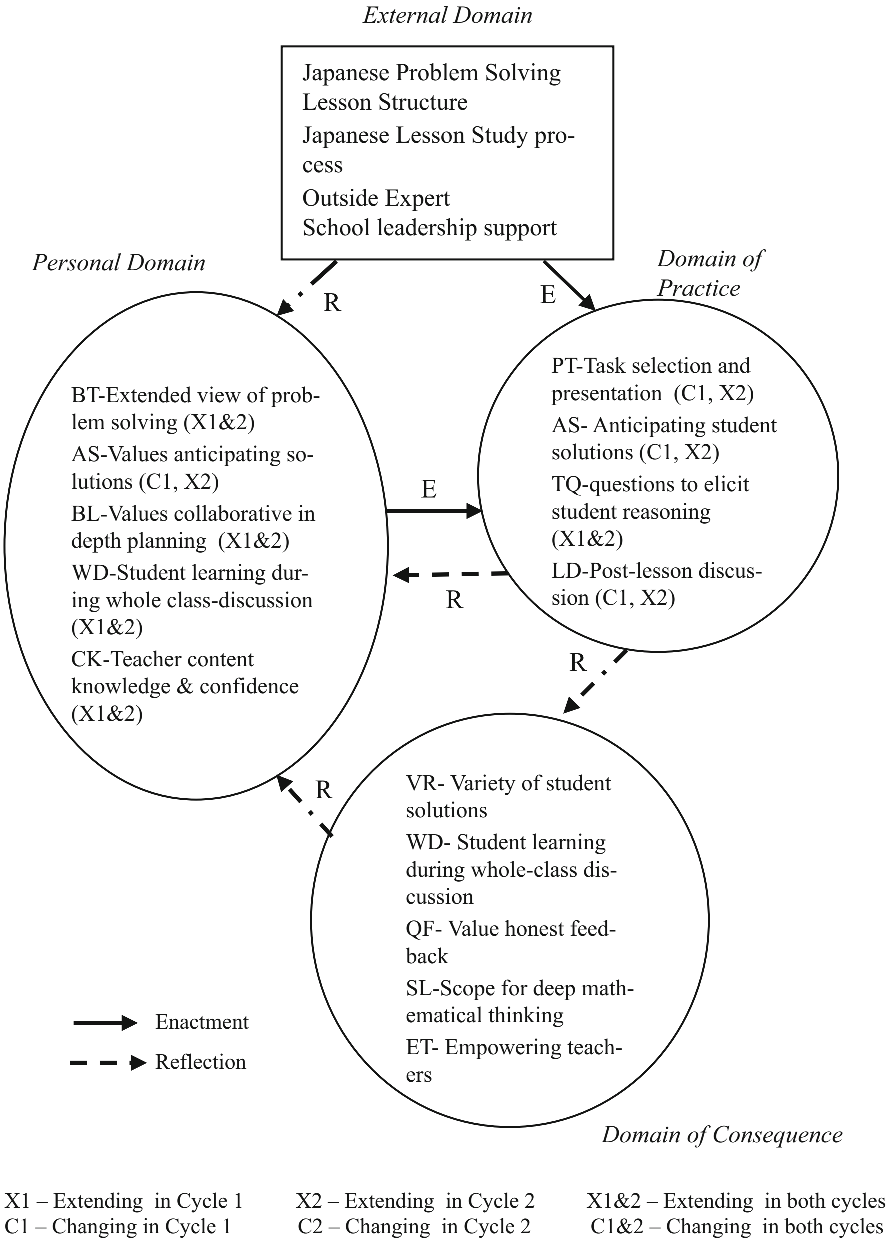


Fig. 7 Narah's change environment during Cycles 1 and 2

coach and the curriculum coordinator in providing feedback and that teachers sometimes take it personally when given honest feedback. She commented that "obviously you've still got to balance the scales, there has to be support, there's also constructive criticism" [Interview 3, Ln101–102] (Fig. 7).

> I just love his bluntness and it's just such a breath of fresh air to have someone come in and say "Hey you didn't do this" or "Why didn't you do this?" or "You got this wrong" or something, we pussyfoot around so much in Australia I think, with worrying about hurting teachers feeling and things. But as we've always said. . . it's not about the teacher, it's about the thinking and the learning. (Interview 3, Ln 84–89)

Narah's views of problem-solving shifted after her participation in the lesson study project. Initially during the first interview, her knowledge of problem-solving seemed to be closely linked to Polya's problem-solving heuristics such as "make a model, act it out, draw a list, work backwards." After participating in lesson study, she noticed that teaching these heuristics led to "structuring [students'] thinking perhaps a little too much." She contrasted this with the structured problem-solving approach in JLS, which encouraged students to use a variety of strategies.

Narah highlighted the importance of teacher content knowledge in providing rich learning opportunities for students during whole-class discussion. She has developed an appreciation of sequencing students' strategies according to increasing levels of sophistication – one salient feature of Japanese problem-solving lesson structure.

5 Discussion

Our previous analysis of the Bobbies planning team (Widjaja et al. 2017) involved the use of the IMPG to analyze the interaction between the experiments occurring in the Domain of Practice and the other domains as it occurred – that is, during the planning meetings, the research lessons, and the post-lesson discussions. The analysis documented the change environment that lesson study provides for teacher professional growth. In this chapter we have relied on participants' reflections on the enactment of each of the JLS processes to identify the nature of the change environment, map the interconnections in this environment, and report on participants' professional growth. Using the three individual interviews with each participant, we have been able to identify the connections between the various domains in the change environment and the mediating processes of enactment and reflection across the two cycles as discussed below.

Figure 8 maps the interconnections between each of the domains over the two study lesson cycles, which we have called *The Interconnected Trajectory of Professional Growth*. PD1 is the personal domain prior to enacting the JLS process, while the personal domain, PD3, represents participants' knowledge, beliefs, and attitudes following the second lesson study cycle.

Different experiences among the members of the Matomes planning team were reflected in the varying pace and degree of growth observed within one cycle and between the two cycles. This is consistent with Clarke and Hollingsworth's (2002) argument that teacher professional learning is nonlinear and happens through an iterative process. At the beginning of the study (PD1), the three teachers in the study who were full-time classroom teachers – Trevor, Camilla, and Sandra – valued "hands-on" professional learning tasks for their professional learning and either

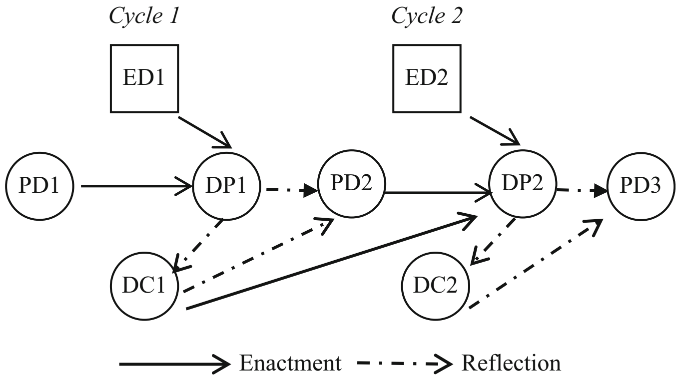


Fig. 8 The *Interconnected Trajectory of Professional Growth* for the Matomes across the two lesson study cycles

feedback from experienced teachers or coaches or the opportunity to share alternate perspectives. All three teachers believed that problem-solving was an important element of the mathematics curriculum and teaching. They believed that small group work was appropriate for problem-solving tasks and that children should share their thinking. Trevor and Sandra had very positive attitudes toward teaching mathematics, whereas Camilla was less confident. In their roles as numeracy coach and curriculum coordinator, George and Narah conducted their coaching and delivered professional development for teachers in their schools. They highlighted the importance of school factors to support teacher professional learning.

Reflecting on the planning and teaching of the first structured problem-solving research lesson (DP1) and the consequences for student engagement, student learning, and teacher actions (DC1), the teachers either extended their knowledge and beliefs about teaching and learning or changed their beliefs (PD2) (See Fig. 8). For example, Trevor and Camilla, the least experienced teachers, developed new understandings of the diversity of students' thinking and solutions and came to believe that learning continues during whole-class discussion when students are encouraged to explain their solutions to others. All three teachers changed their perception of the role of the whole-class discussion and agreed that they needed to develop their students' capacity to discuss their solutions and thinking. Sandra's learning was deeper. She extended her beliefs regarding whole-class discussion and argued that she had learned that students needed to be able to explain and justify their thinking, not just share their solutions. Trevor and Sandra began to change their practice from brief "share time" at end of the mathematics lesson to implement longer whole-class discussions and anticipate student solutions when planning lessons (DP2). All three thought that planning questions to elicit teaching was a practice they should adopt. Sandra also began to think about the way in which the task is represented influences students' thinking. She realized that she needed to pay more attention to the type of problem she selected to achieve the specific learning goal and to ways of representing and posing the problem.

Initially, both George and Narah raised the same concern about the amount of planning time required to plan one research lesson. Following the first JLS cycle both appreciated the value of structured problem-solving lessons in allowing students to take ownership of their learning and realized the need to develop teachers' questioning skills. Narah saw the benefits of anticipating student solutions and embedded this into her practice. While both of them acknowledged the challenges for teachers in conducting productive whole-class discussions, they saw these as a critical part of the lesson. They also valued the opportunity to focus on a few students during research lessons to learn the progression of students' mathematical thinking.

In the Cycle 2, the teachers used their new or extended understandings of the diversity of their students' thinking and the structured problem-solving approach in general (PD2) to plan and teach or observe the second research lesson (DP2). It was during this cycle that Camilla's professional growth was strongest. Camilla taught the second research lesson and successfully teaching this research lesson greatly enhanced her confidence. She deepened her knowledge of her students and appreciated the need to deepen her content knowledge for structured problem-solving mathematics lessons (PD3). During the Cycle 2, Trevor extended his knowledge of students to develop understanding of the nature and range of thinking and strategies students used to solve the problem (PD3). He also began to change his perception of problem-solving tasks to realize that they could be used to achieve specific learning goals. Sandra was the only one of the three teachers to connect her new understandings of students' thinking and the value of observing students closely while they worked on a problem with her assessment practice (PD3). She had begun to record the strategies and reasoning that students were using in her anecdotal notes during her mathematics lessons (DP2). Teaching the research lesson and the post-lesson discussion that followed impacted on their knowledge of students and planning for teacher actions, such as whole-class discussions. Sandra, the most experienced of these teachers, deepen her knowledge and practice of informal formative assessment and task selection through student observation.

At the end of the project, all three teachers aspired to make changes to the planning and evaluation processes at their school (PD3). Each of them recognized the value of research and the use of larger planning teams to bring together a range of experience and knowledge. Sandra was particularly keen to enabling teachers to observe each other's lessons with a particular focus on students' thinking. Camilla wanted to enact lesson study with her colleagues. This largely arose as a consequence of the different teachers' observations and reflections that occurred during the post-lesson discussions (DC1 and DC2), where the focus was on the students' learning and the planned lesson, rather than on the teacher.

Both George and Narah valued collaborative planning in lesson study. Narah perceived the order of comments during the post-lesson discussions – which started with the teacher who taught the lesson, followed by the planning team – as empowering the participating teachers to own the professional learning. In the Cycle 2, their appreciation of the whole-class discussion and the process of anticipating student solutions were strengthened. Both George and Narah also reiterated the value of the outside expert in providing constructive feedback and extending their professional learning. They also emphasized the importance of support from school leadership to provide time for teachers to engage in professional learning through lesson study.

The analysis of the participants' professional growth during the lesson study project reveals the iterative and interconnected nature of their learning. It was not based on a discrete set of events or tasks; rather it involved participants in an ongoing process of research, experimentation, and reflection in their own classrooms, at the same time as they were participating in the formal activities of the lesson study process – the planning meetings, research lessons, and post-lesson discussions that occurred over two school terms.

6 Conclusion

There were some challenges in applying Clarke and Hollingsworth's (2002) *Interconnected Model of Professional Growth* (IMPG) to the data from the interviews of teachers and numeracy coaches. The four domains in the model were to be fleshed out with data drawn from the interview transcripts, but as the transcripts were analyzed, it was discovered that pieces of evidence were being lost in the overarching domain descriptions. Thus, it seemed necessary to define codes that kept the individual pieces of evidence visible, rather than obscured. The number of codes grew as more interviews were analyzed, with the final list being shown in Appendix A.

It was imperative in this research to trace the changes in individual participants' beliefs, understandings, and attitudes as a result of participating in lesson study. Thus, the issue was how to register changes and differences in a static representation, for example, on paper.

Our findings imply that investing in in-depth, quality planning, with a focus on advancing students' thinking, leads to teachers' professional growth (Hargreaves and Fink 2009). It also highlights the importance of establishing support within the school and across the school network, such as providing time for teachers to plan together, observe each other, and learn from each other's teaching. Ongoing collaborative enactment and reflection was vital throughout the two lesson study cycles to facilitate teachers' professional growth through lesson study.

The IMPG depicts the change environment and shows the interconnections between domains. The IMPG model enables us to identify elements in each domain and trace individual teacher' change as a result of their professional experimentation mediated by the processes of enactment and reflection. However, the IMPG model does not allow us to track when the change occurs and whether the change is sustained over a period of time. We posit that the use of the *Interconnected Trajectory of Professional Growth* (Fig. 8) adds the dimension of time so that we can track and map professional growth over time, while acknowledging the interactive elements of the change environment.

Acknowledgments *The Implementing structured problem-solving mathematics lessons through lesson study* project was funded by the Centre for Research in Educational Futures and Innovation, Deakin University. We gratefully acknowledge the contributions of participating teachers, numeracy coaches, students, schools, and the outside experts: Dr. Max Stephens, University of Melbourne, and Prof Toshiakira Fujii, Tokyo Gakugei University, Japan.

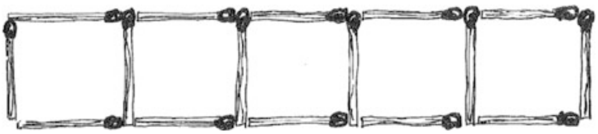
Appendices

Appendix A: List of Codes

Code	Meaning	Code	Meaning
AP	Assessment practice	PL	Professional learning
AS	Anticipated student solutions	PP	Planning process
BD	Belief about discipline	PR	Professional reading
BL	Belief about learning	PS	Problem-solving
BP	Belief about professional learning	PT	Planning time
BS	Belief about support	QF	Constructive quality feedback
BT	Belief about teaching	RC	Resources
CC	Collaborative coaching	RF	Regional factors
CH	Challenges	RP	Reflective practice
CK	Content knowledge (teachers')	SC	Student confidence
DL	Deep learning	SD	Shallow discussion
DP	Domain of practice	SF	School factors
ED	External domain	SG	Small group discussion
EF	Expert feedback	SL	Student learning
ET	Empowering teachers	SM	Student misconceptions
GM	Good models (Japanese teachers)	SO	Salient outcomes
JD	Japanese professional learning day	SP	Structured problem-solving pattern
JL	Japanese lesson study process and model	ST	Articulate student thinking
JP	Japanese lesson study project	TA	Teacher actions
LD	Post-lesson discussion	TC	Teacher confidence teaching mathematics
LG	Learning goal	TI	Teaching/teachers' ideas
LO	Research lesson observation	TK	Teacher knowledge
LT	Research lesson teaching	TP	Teaching practice
LV	Research lesson variations (between two teams)	TQ	Teacher questioning
OE	Outside expert	TR	Trial lessons (and observation of these)
PD	Personal domain	TT	Teachers' thinking time
PE	Professional experience	VS	Variety of student solutions
PI	Practical implementation	WD	Whole-class discussion

Appendix B: The Matchstick Problem (Cycle 1 Research Lessons)

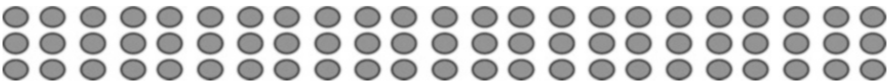
I used some matchsticks to make squares connected side by side as shown below.
How many matchsticks would I need if I was to make eight squares?
What if there were 100?



Appendix C

The 23 × 3 multiplication problem as presented by the researchers for *Cycle 2 Research Lessons* and used by the Bobbies and the Matomes

Here is a diagram of some dots.
Can you work out how many dots are in the diagram without counting them one by one?
Please make sure that you're showing your thinking in the space provided.



References

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*, 947–967.

Corbin, J. M., & Strauss, A. L. (2008). *Basics of qualitative research [electronic resource]: techniques and procedures for developing grounded theory*. Los Angeles/London: SAGE.

Darling-Hammond, L., & Richardson, N. (2009). Teacher learning: what matters. *Educational Leadership, 66*(5), 46–55.

Elliott, J. (2012). Developing a science of teaching through lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 108–125.

Fujii, T. (2016). Designing and adapting tasks in lesson planning: A critical process of lesson study. *ZDM Mathematics Education, 48*(4), 411–423.
Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education, 17*(1), 5–36.
Groves, S., Doig, B., Vale, C., & Widjaja, W. (2016). Critical factors in the adaptation and implementation of Japanese Lesson Study in the Australian context. *ZDM Mathematics Education, 48*(4), 502–512.
Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher, 15*(5), 5–12.
Guskey, T. R., & Yoon, K. S. (2009). What works in professional development. *Phi Delta Kappan, 90*(7), 495–500.
Hargreaves, A., & Fink, D. (2009). What works in professional development. *Phi Delta Kappan, 90* (7), 495–500.
Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.
Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 12*(Winter), 12–17; 50–52.
Opfer, V. D., & Pedder, D. (2011). Conceptualizing teacher professional learning. *Review of Educational Research, 81*(3), 376–407.
Perry, R. R., & Lewis, C. C. (2008). What is successful adaptation of lesson study in the US? *Journal of Educational Change, 10*, 365–391.
Shimizu, Y. (1999). Aspects of mathematics teacher education in Japan: Focusing on teachers' roles. *Journal of Mathematics Teacher Education, 2*(1), 107–116.
Schipper, T., Goei, S. L., de Vries, S., & van Veen, K. (2017). Professional growth in adaptive teaching competence as a result of Lesson Study. *Teaching and Teacher Education, 68*, 289–303.
Stigler, J., & Hiebert, J. (1997). Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study. *Phi Delta Kappan, 79*(1), 14–21.
Stigler, J., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM Mathematics Education, 48*(4), 581–587.
Takahashi, A. (2006). *Characteristics of Japanese mathematics lessons*. Retrieved from http://www.criced.tsukuba.ac.jp/math/sympo_2006/takahashi.pdf
Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*(4), 513–526.
Takahashi, A., & Yoshida, M. (2004). Ideas for establishing lesson-study communities. *Teaching Children Mathematics, 10*(9), 436–443.
Watanabe, T. (2002). Learning from Japanese lesson study. *Educational Leadership, 59*(6), 36–39.
White, P., Wilson, S., & Mitchelmore, M. (2011). Teaching for abstraction: Collaborative teacher learning. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 761–768). Singapore: MERGA.
Widjaja, W., Vale, C., Groves, S., & Doig, B. (2017). Teachers' professional growth through engagement with lesson study. *Journal of Mathematics Teacher Education, 20*(4), 357–383.

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Teaching for Robust Understanding with Lesson Study

Alan Schoenfeld, Angela Dosalmas, Heather Fink, Alyssa Sayavedra, Karen Tran, Anna Weltman, Anna Zarkh, and Sandra Zuniga-Ruiz

Contents

Abstract This chapter describes the synthesis of two powerful approaches to professional development, the Teaching for Robust Understanding (TRU) framework and Lesson Study. The synthesis is known as TRU-Lesson Study.

The TRU framework identifies five essential dimensions of classroom practice: (1) the Content; (2) Cognitive Demand; (3) Equitable Access; (4) Agency, Ownership, and Identity; and (5) Formative Assessment. When classroom practices engage students meaningfully along these five dimensions, students become knowledgeable and resourceful thinkers and problem solvers. In TRU-based professional development, teacher learning communities negotiate their visions of teaching and learning collaboratively by reflecting on artifacts of practice using the TRU framework.

In math-focused Lesson Study (LS), teachers work together to design, teach, and reflect on a lesson that focuses on key mathematical issues and students' engagement with them. Widespread in Japan, Lesson Study is a powerful mechanism for building and sharing understandings of mathematics content, teaching, and learning.

Teachers in the USA typically have little collective time to reflect on teaching practice. TRU-Lesson Study (TRU-LS) supports the growth of teacher learning communities and their engagement with key ideas and practices of TRU and Lesson Study. Like Lesson Study, it profits from teachers' concerted attention to lesson design and reflection on the hypotheses reflected in the design. Like TRU-based

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_7

professional development, it supports teachers to work together explicitly on key dimensions of classroom practice. This paper describes TRU-Lesson Study and provides descriptions of how it plays out in practice.

Keywords Lesson Study · Teaching for Robust Understanding · TRU framework · Professional development

1 Background and Context

1.1 Introduction and Overview

As this volume indicates, Lesson Study (LS) is a worldwide mechanism for professional development. Yet, just as teaching varies from country to country (Stigler and Hiebert 1999), so does Lesson Study, based on the cultural affordances available to teachers (Chen 2017, Perry and Lewis 2009). In the US, teachers typically have little collective time for collaboration; they do not have national curricula supporting rich mathematics instruction; and there has not been a historical tradition of attending to student thinking as students engage with rich mathematics. For Lesson Study to take root and thrive in the US, it must address these realities.

Within mathematics education, Lesson Study's power comes in large part from teachers' collaborative efforts to understand the richness and complexity of mathematical ideas and practices and to support students in engaging with them. When teachers design and reflect on lessons that focus on rich content and student learning of it, their collective negotiations of problems of practice provide an ongoing mechanism to share and develop pedagogical understandings.

The Teaching for Robust Understanding (TRU) framework (Schoenfeld 2013, 2014, 2015, 2017) describes what matters for equitable and robust learning in mathematics classrooms. As elaborated in Sect. 1.3, the TRU framework focuses on five key dimensions of learning environments: (1) the Content (here mathematics); (2) Cognitive Demand; (3) Equitable Access; (4) Agency, Ownership, and Identity; and (5) Formative Assessment. The first dimension concerns the richness and complexity of mathematical ideas and practices essential for powerful mathematics learning. The remaining four dimensions focus on students' experiences of the mathematics and the impacts of those experiences on their mathematical development and understanding.

TRU's model of professional learning engages teacher learning communities (TLCs) in ongoing inquiry cycles grounded in the TRU framework. Rather than specifying any particular approach to teaching, TRU focuses teachers' attention on attending explicitly to aspects of practice that are essential for supporting students' individual and collaborative engagement with rich mathematical ideas and practices. In regularly scheduled meetings, teachers share artifacts from their classrooms that capture evidence of focal problems of practice and are supported to use the TRU framework to inquire more deeply into key practices. Between meetings teachers try new strategies, visit each other's classrooms, and/or collect classroom artifacts to bring to the next meeting.

There is a strong philosophical affinity between TRU and LS. Both attend explicitly to how students experience instruction; both see discourse within TLCs as a venue for teachers' ongoing professional growth; and both value teacher professionalism. In Japan, LS is well established; so are teacher collaboration, peer support for collective growth, and teacher professionalism. In the USA, collaborative support for teacher growth is rare (Horn et al. 2017; Little 2002). Lortie (1975) characterized school buildings as "egg crates," in which each teacher works in his or her room entirely isolated from colleagues. Today, this is still often the case. TLCs in the USA struggle to develop the trust and depth of conversation necessary for inquiring into complex problems of practice in ways that could lead to learning (Horn and Little, 2010; Horn et al. 2017) – a significant obstacle to the implementation of Lesson Study. In addition, teachers in the USA are not necessarily as well prepared as Japanese teachers to focus in detail on student thinking. TRU-LS is designed to help build teacher communities and to focus on student thinking in ways that allow the practices of Lesson Study to take hold, enriching what teachers can learn from focused reflection on designing instruction and student engagement with important mathematical ideas.

1.2 Lesson Study Model[1]

Lesson Study has established a foothold in the USA over the past 15 years (Akiba et al. 2014; Hill 2011). Not surprisingly, there is a range of LS adaptations to the US context. TRU-LS draws on the theoretical model of Lesson Study proposed by Lewis and colleagues, aimed at adapting Lesson Study to the US context (Lewis et al. 2009; Lewis et al. 2006; Perry and Lewis 2009; Takahashi and McDougal 2016; Watanabe et al. 2008). The core aspects of one LS cycle are shown in Fig. 1.

In one LS cycle, a TLC moves together through the four-phase process in Fig. 1. They begin by studying curriculum and standards as well as considering their long-term goals for student learning. In this phase the TLC decides on a research theme and theory of action and selects a mathematical topic on which to focus. It conducts background research, looking at what is known about student understanding of the topic and at relevant instructional materials (possibly including exemplary lessons). With this refined understanding, they *plan* a lesson, identifying precise issues that members of the team want to focus on when the lesson is taught. One or more teachers then *conduct* the research lesson, while the remaining teachers and invited educators observe, gathering evidence of student thinking. Participants then *reflect* on how the lesson played out and discuss their observations with an outside expert. They then move on to their next LS cycle.

[1]Our description is terse, given that this entire volume is devoted to Lesson Study.

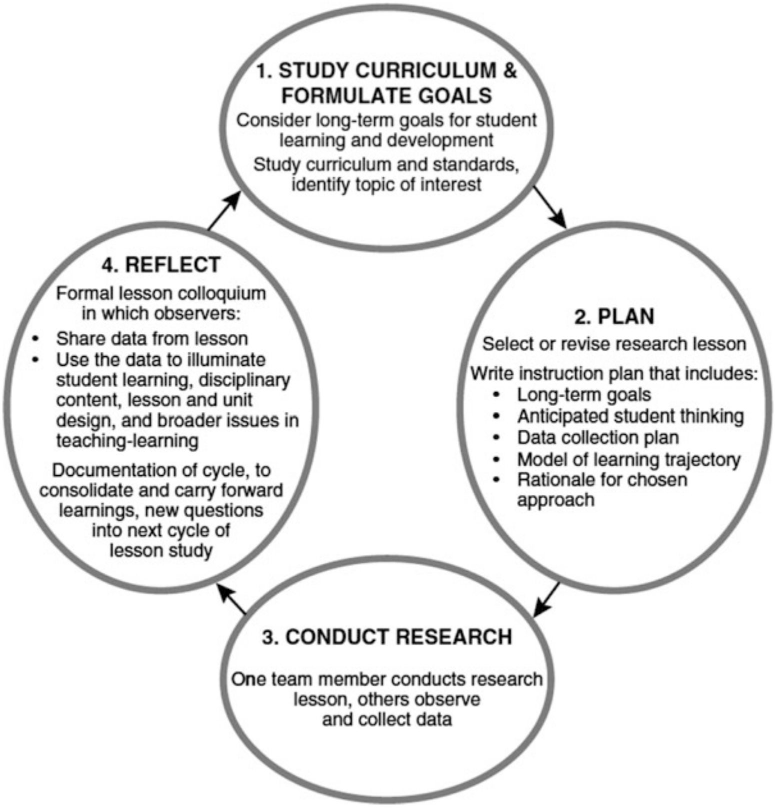

Fig. 1 Lesson Study cycle (From Lewis et al. 2006, p. 4)

The theory of action underlying LS is that such repeated inquiry cycles support the TLC's sense of community and, in an organic way, deepen and modify teachers' beliefs and understandings. This leads to changes in their pedagogical practices and deeper student learning.

1.3 An Introduction to the Teaching for Robust Understanding Framework and PD Structure

The TRU framework provides a theory of powerful learning environments (Schoenfeld 2015). Tools exist for supporting classroom practices in all disciplines (see, e.g., Baldinger et al. 2016, Schoenfeld and the Teaching for Robust Understanding Project 2016). Here we focus on the mathematics version of TRU, TRUmath.

The Five Dimensions of Powerful Mathematics Classrooms

The Mathematics	Cognitive Demand	Equitable Access to Mathematics	Agency, Ownership, and Identity	Formative Assessment
The extent to which classroom activity structures provide opportunities for students to become knowledgeable, flexible, and resourceful mathematical thinkers. Discussions are focused and coherent, providing opportunities to learn mathematical ideas, techniques, and perspectives, make connections, and develop productive mathematical habits of mind.	*The extent to which students have opportunities to grapple with and make sense of important mathematical ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with task difficulty ranging from moderate to demanding. The level of challenge should be conducive to what has been called "productive struggle."*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematical content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.*	*The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations about mathematical ideas, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.*	*The extent to which classroom activities elicit student thinking and subsequent interactions respond to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.*

Fig. 2 The five dimensions of robust mathematics classrooms – the Teaching for Robust Understanding (TRU) framework for mathematics, TRUmath

1.3.1 The Teaching for Robust Understanding Framework

The five dimensions of TRUmath are given in Fig. 2.

As described in detail by Schoenfeld (2013, 2014, 2017), the derivation of the framework included a comprehensive literature search regarding factors affecting student outcomes and the distillation of that list into five categories framed so that each is "actionable" – each can be the focus of professional development. Examinations of videotapes of practice and the use of a scoring rubric (Schoenfeld et al. 2014) documented a positive relationship between scores assigned to classroom practices using the rubric and student performance on measures of mathematical proficiency. Students from classrooms that rate increasingly well along the five dimensions are increasingly knowledgeable and resourceful thinkers and problem solvers.

TRU has several features that afford teacher learning, some of which are enumerated below.

1. The TRU framework provides mechanisms for focusing on what matters in instruction. TRU highlights key aspects of practice for collaborative teacher inquiry, experimentation, and reflection.

Teachers and administrators in the USA have few opportunities to develop shared goals (Grossman and McDonald 2008). Consequently, they can struggle to find common ground when beginning to talk about important problems of practice. This,

Observe the lesson through a student's eyes	
The Mathematics	• What's the big idea in this lesson? • How does it connect to what I already know?
Cognitive Demand	• How long am I given to think, and to make sense of things? • What happens when I get stuck? • Am I invited to explain things, or just give answers?
Equitable Access to Mathematics	• Do I get to participate in meaningful mathematical learning? • Can I hide or be ignored?
Agency, Ownership, and Identity	• Do I get to explain, to present my ideas? Are they built on? • Am I recognized as being capable and able to contribute in meaningful ways?
Formative Assessment	• Do classroom discussions include my thinking? • Does instruction respond to my thinking and help me think more deeply?

Fig. 3 Observing a mathematics lesson from the student perspective

in turn, makes it harder for them to work together to build and achieve shared goals. Even when there is time for collaborative meetings, focus and coherence can be a challenge (Horn and Kane 2015).

The perspective and language offered by TRU serve as a mechanism for boundary-crossing (Akkerman and Bakker 2011; Star and Griesemer 1989) and provide a means of communication among teachers at multiple grade levels and administrators. The five dimensions of TRU support teachers in inquiring into and reflecting on each other's routine practice. The hard work of coming to understand what the TRU dimensions can look like in classrooms helps teachers, administrators, and other stakeholders build shared understandings and goals. This facilitates collective work on problems of practice.

2. In the cultural context of the USA, TRU involves a fundamental shift in perspective from focusing primarily on the activities and/or materials students engage with to a focus on the ways in which students experience mathematics, as supported by engagement with carefully designed activities and materials.

Focusing on the ways students experience instruction is a fundamental point of alignment between TRU and LS. A key question is, "What does math class feel like, from the point of view of the student?" This perspective is represented in Fig. 3, drawn from the TRU Observation Guide (Schoenfeld and the Teaching for Robust Understanding Project 2016).

It can be challenging for teams of teachers in the US to move their conversations beyond scheduling activities, organizing curriculum, and naming topics to be taught. Such conversations are necessary, but they rarely reach the conceptual depth that leads to powerful teacher learning (Horn et al. 2017). TRU shifts conversations away from such activities and materials to what students *perceive and experience*, supporting teachers in exploring goals and orientations toward teaching and learning. Questions such as "What does it feel like for a student to experience these mathematical ideas for the first time? How does this connect to what students already know? What's confusing, interesting, or useful about this?" do not have easy answers. They address what really matters for students and are worthy of extended, collaborative exploration.

3. There are many different ways to design mathematically rich learning environments. Accordingly, the TRU framework does not prescribe specific teaching methods. Professional development aligned with the TRU framework helps teachers problematize their instruction, building on their strengths and refining their practices with the goal of providing their students with increasingly rich experiences along each of the five TRU dimensions.

The TRU framework and TRU tools open up instruction for inquiry and reflection while respecting teacher professionalism and autonomy. Specifically, asking "How are students experiencing the mathematics along this particular dimension? How can those experiences be made richer?" allows teachers the freedom to work within their own styles while building shared understandings of and commitment to a set of goals known to help students become powerful mathematical thinkers. TRU is not laissez-faire: reflecting using the TRU dimensions in a particular context may well reveal that one particular approach in that context is more likely to foster deep student engagement with the content, or provide more equitable access to it, than another approach. But TRU does not tell teachers what to do. It supports and enhances reflection on aspects of instruction that matter.

1.3.2 TRU-Based Professional Development

TRU-based professional development (TRU-PD) leverages the affordances of the TRU framework described above – explicit foci, shift to the student perspective, and the structured problematizing of teaching methods – to help teachers build communities of inquiry around important problems of practice. TRU-PD entails a series of inquiry cycles in which teachers negotiate their visions of teaching and learning mathematics collaboratively, reflect on artifacts of practice (including video), and try out agreed-upon approaches with their students. Each stage of this inquiry cycle is supported by the explicit foci and boundary-crossing language built into the TRU framework.

Once a TLC has been introduced to TRU, it begins the inquiry process with collective reflections (top part of Fig. 4). The TRU framework facilitates the choice of a problem of practice[2] that will be central to their collaborative inquiry. Having

[2]Some examples of problems of practice and the TRU dimensions that support them are, "How can we support students to feel ownership over the mathematics and discuss each other's mathematical ideas and strategies?" (TRU Dimension 4: Agency, Ownership, and Identity) or "How can we support students to struggle productively?" (TRU Dimension 2: Cognitive Demand).

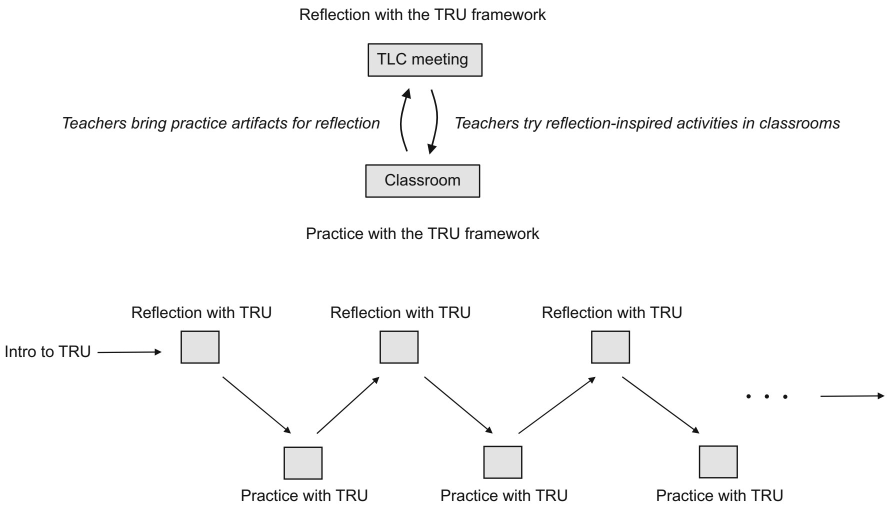

Fig. 4 The TRU inquiry cycle

chosen a focal problem of practice, the teachers identify relevant activities or techniques to try in their classrooms. The framework orients them to what they might try, but they are free to experiment with a variety of methods.

At this point, their inquiry shifts into their classroom practice. Teachers enact their interventions, collecting artifacts that will ground their subsequent collaborative reflections. Those artifacts might include replays or otherwise rich descriptions of what took place (Horn 2005), student work (Kazemi and Franke 2004), or video (Sherin et al. 2009).

In the next TLC meeting, the teachers report and reflect on what happened, using tools such as the *TRU Conversation Guide* (Baldinger et al. 2016) and the *TRU Observation Guide* (Schoenfeld and the Teaching for Robust Understanding Project 2016) to focus their reflections on students' experiences[3]. They then plan another intervention to refine their understanding of their chosen problem of practice, and the cycle again returns to the classroom.

Through the repeated cycles of reflection and practice shown in Fig. 4, TRU-based PD aims to develop teachers' capacity to collaborate on enriching students' experiences of mathematics and mathematics instruction by building routines of inquiring into everyday practice – an explicit goal that extends beyond typical Lesson Study. Working collaboratively with multiple TRU inquiry cycles is aimed at building a sense of teacher community and supporting structured collective reflection on teachers' attempts to create increasingly rich mathematical learning environments.

[3]See http://truframework.org for some of the tools developed to support teachers' planning, observation, and reflection.

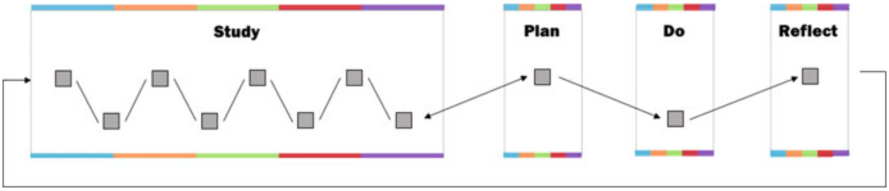

Fig. 5 TRU-LS: A synthesis of TRU and Lesson Study

1.4 The Synthesis of TRU and LS

The goal of synthesizing TRU and LS is to leverage the strengths of both TRU-PD and LS, employing TRU-PD activities and the TRU framework to address the challenges of shifting LS from the Japanese to the US context. Research has highlighted a number of these challenges while concurrently enumerating critical features deemed necessary for Lesson Study to be effective (Fernandez et al. 2003; Perry and Lewis 2009; Takahashi and McDougal 2016; Watanabe et al. 2008). One such critical feature is that a significant amount of time needs to be spent on *kyouzai kenkyuu*[4], literally translated as the "study of materials for teaching." Practically, however, as Wang-Iverson and Yoshida (2005) describe, *kyouzai kenkyuu* focuses not just on teaching materials but also includes investigation of "students' prior knowledge, learning experiences, state of learning, and understanding, which makes it possible for teachers to be able to anticipate students' reactions and solutions" (p. 152). As discussed below, this feature is addressed with TRU's integration into LS and its focus on building expertise about what matters in mathematics teaching and learning. Specifically, TRU-LS does so by providing teachers with opportunities for in-depth explorations of student thinking in their own classrooms, while laying the groundwork for and implementing Lesson Study research cycles occurring on a larger time scale. Two main TRU-based modifications of Lesson Study produce the synthesis and tackle these challenges.

As shown in Fig. 5, TRU-LS incorporates the TRU framework into each stage of the Lesson Study process. TLCs engaged in TRU-LS use the TRU framework to frame each stage of the Lesson Study process. TRU helps teachers select, study, and refine their research theme and theory of action and provides explicit focus for *kyouzai kenkyuu*. TRU observation tools help teachers focus their data collection during the research lesson on what students experience. The TRU framework also guides the final commentary during the post-lesson discussion. In the US context, where students' potential experiences of mathematics are not always central to

[4]Focused attention on the mathematics and to student thinking is essential in the US context. Curricula in the USA vary widely and are very uneven in quality, so teachers in the USA cannot depend on the affordances of curricula that are available in Japan. Moreover, teachers in the USA have not, as a rule, had nearly as much opportunity as Japanese teachers to reflect on the impact of classroom materials and practices on student thinking.

conversations about teaching, the focus that the TRU framework provides is essential. It helps keep the focus on building expertise around student thinking. More generally, the use of TRU in this way makes explicit many of the underlying values and practices that can be tacit in traditional Japanese Lesson Study and are "carried" by cultural tradition (Takahashi 2015).

In Japan, the teaching community is steeped in Lesson Study, so teachers who have not been part of the research lesson planning team can still learn a great deal from the framing questions, enactment of the lesson, and the commentaries. That is not necessarily the case in the USA. Accordingly, the TRU-LS process insures that all members of the department or TLC engage with all aspects of the research lesson before it is taught. As seen in Fig. 5, the TRU inquiry cycles, in which all teachers participate, form the backbone of the study phase of Lesson Study. The research theme and preliminary theory of action for the research lesson are developed in the TLC. All TLC members, including the lesson planning team, engage with the research theme and theory of action through repeated cycles of reflection and practice; relevant literature is referenced, although (given the realities of US teaching) typically not with the intensity and thoroughness of classical *kyouzai kenkyuu*. This ongoing work allows for collective refinement of the research theme and theory of action. Later, as the planning team works to design the research lesson, it brings the core mathematics of the lesson to the larger community for discussion. That conversation includes discussions of the mathematics itself, what the students are likely to know, anticipated student responses, and how mathematical ideas develop throughout the lesson as well as specific design elements – an analog of the work in selecting and analyzing the *hatsumon* in traditional LS. These collective conversations inform the research lesson planning and also raise possibilities for intervention and reflection that the whole department can try out in ongoing inquiry cycles. As shown in Fig. 5, the plan, do, and reflect phases of Lesson Study (cf. Figure 1) interact in ongoing ways with collective study by the TLC.

In TRU-LS, TRU inquiry cycles allow the TLC to leverage the LS process for community-wide development and learning. The TRU cycles also support community building and the collective negotiation of teachers' visions of teaching and learning mathematics, which is essential for effective LS cycles. Finally, they help the learning done in a LS cycle permeate teachers' regular classroom practice.

The changes to Lesson Study described here – engaging the TLC in small inquiry cycles within the larger inquiry cycle that is LS and using the TRU framework to scaffold conversations at each stage of the Lesson Study process – are aimed at bridging the divide between LS as designed for the Japanese context and the reality of teachers' working lives in the USA. They offer the potential to enhance both TRU-PD and LS. To TRU-based PD the changes add the benefit of involvement in larger-scale and more intense design than conducted in TRU inquiry cycles, supporting teachers in delving more deeply into content and student thinking. To LS, the changes add aspects of community building necessary in the USA and the kinds of explicitness and focus on key aspects of instruction that increase the likelihood both of successful lesson design and of collective learning from participating and reflecting on that design.

2 Early Signs of Impact

This section summarizes instances of practice from three school sites to show how early implementations of TRU-LS are taking shape. We selected these examples to highlight consequential differences between TRU-LS and typical LS implementation in the USA.

For the past 3 years, the TRU-LS team has collaborated with high school mathematics departments in the Oakland (California) Unified School District (OUSD) in designing and implementing TRU-LS. OUSD high schools vary substantially in terms of size, schedule, personnel, and priorities; even though there is a common curriculum in the district, implementation of that curriculum varies significantly across sites. Given that TRU-LS is designed to be responsive to local context, the examples demonstrate substantial variation in TRU-LS implementation, even though the implementations are consistent with the underlying design principles of TRU-LS.[5] That said, there is enough curricular consistency to produce overlap in research themes and research lessons.

The study phase of LS is the most extensive and a challenge to implement in the USA – teachers in the USA are not habituated to the deep study of curricula or student thinking and tend to move rapidly to selecting and/or designing tasks for implementation in research lessons. TRU-LS, while still proceeding more rapidly than is typical in Japan, tries to open things up by having a TLC focus on developing a research theme and a supporting theory of action. Here we discuss three examples from the study phase. We also present two examples from the combined plan, do, and reflect stages.

2.1 Establishing a Research Theme and Theory of Action with TRU-LS at Sites A and B

TRU-LS integrates TRU-based inquiry cycles with the study phase of Lesson Study with the goal of helping TLCs create and refine research themes with explicit underlying theories of action. Here, examples from sites A and B illustrate the selection and refinement of research themes resulting from focused attention on students' mathematical experiences. These examples also show how TRU-LS supported teachers' agency and helped them begin to develop routines of collaborating as a department about problems of practice, building the culture essential to LS.

[5]This is an essential, given our wish to respect teacher autonomy and help build local teacher communities.

2.1.1 Site A: Crafting an Initial Research Theme Through Reflection with the TRU Framework

The process began at site A with each teacher sharing one goal they set for the school year. Because this site had studied TRU previously, many of the teachers' goals were already framed by TRU. As teachers talked, it became apparent that three shared themes were emerging – supporting student explanations/arguments, building student ownership of their learning, and finding the "sweet spot" for productive struggle. To narrow the theme, the teachers discussed strengths and areas of growth for their current students. The language of TRU helped keep the discussion focused on the students as learners. Teachers voted on the theme they preferred, settling on the "sweet spot" of productive struggle. Teachers said they saw this idea as connected not only to Cognitive Demand but also to Equitable Access; material has to be "within reach" if all students are to be productively engaged. At the end of the first meeting, teachers agreed to try strategies aimed at supporting productive struggle in their classrooms prior to the next full department PD meeting. (This highlights the use of TRU inquiry cycles as part of the development of the LS research theme.)

With the goal of deepening their collective understanding of productive struggle and deciding which aspects of productive struggle they wanted to focus on in their research lesson, the TLC began its second meeting by watching and reflecting on a video of students in an Algebra 2 class grappling with a challenging problem. The teachers organized their observations using a TRU Cognitive Demand observation tool. They recorded what student behaviors they saw as productive struggle, what factors constrained productive struggle, and what strengths each student brought to the task.

Teachers noted that they saw students in the video trying an alternate strategy when their current strategy was not working as they had expected and asking each other whether a solution or process "makes sense" – productive behaviors that they wanted to support. Then, a description of two students who put their heads down during the task as "disengaged" prompted a discussion of how one might support students to engage productively if their "immediate response" to a math problem is "I don't get it, I'm done." This, in turn, led to discussions of what kinds of classroom environments support productive engagement – students won't persevere and engage in productive struggle if they don't feel safe enough to take risks. These observations led to an enhanced research theme proposal: "How do we set up students so that they are comfortable engaging in productive struggle and how do we arrange what they are grappling with so that they can make meaningful progress toward the learning target and standards?"

This narrative illustrates how collaborative reflection on a classroom video, framed by the shared language of the TRU framework, supported selection and refinement of a research theme during the study phase of TRU-LS. At the initial stages of the TRU-LS study phase, the TRU framework helped the TLC articulate and focus on meaningful problems of practice. Joint reflection on classroom video allowed teachers to deepen their noticing of student experiences. This, in turn, supported refining and negotiating their collective understanding of the research theme. [We note that in Japan, a focus on the students' experience of the mathematics can be taken as a given. That is not necessarily the case in the USA. The TRU framework oriented teachers toward the student experience, and the video provided a grounding for the conversations that kept them from becoming too abstract.]

2.1.2 Site B: Refining a Research Theme and Developing a Theory of Action Through TRU Inquiry Cycles

Similarly, the TLC at site B revised its research theme and developed a theory of action by engaging in TRU-based inquiry cycles of reflection and practice. These cycles helped them develop and sustain the three lenses of *kyouzai kenkyuu* that Fernandez et al. (2003) observed that their Japanese colleagues deem critical – the researcher lens, the curricular lens, and the student lens. Here we describe a department meeting several months into the TRU-LS process in which teachers reflected on artifacts of practice generated when they tried an intervention developed to help them learn about their draft research theme. The TRU framework helped teachers reflect on the artifacts they brought to the meeting and translate their observations into goals for teaching and learning. These resulted in a revised research theme and theory of action.

As had been the case at site A, teachers at site B developed their research theme through reflections on their goals, students' strengths and challenges, and the TRU framework. Their initial research theme was, "Building student perseverance/capacity to struggle productively, together." They connected this theme to the Cognitive Demand and Agency, Ownership, and Identity dimensions of the TRU framework. (Students are likely to persevere only if they have a sense of agency, and such agency is built by having struggled productively – which means that tasks are cognitively demanding but within reach.)

To engage in the practice part of the TRU inquiry cycle, teachers chose to explore a teaching strategy that they believed would help them see their research theme of student perseverance "come to life."[6] They chose a teaching strategy called Three Things: When students are stuck, the teacher would ask them to state three things they know about the problem and three things they are wondering about. Teachers conjectured that this strategy would help students think about what they already know when they got stuck and that it might help students become more independent. They hoped that it would prompt students to explain their thinking and that students might begin using it with each other.

[6]Note that TRU, with its emphasis on the student experience of the content, often leads teachers to consider pedagogical choices as well as content choices – the question being, "how do we enrich the students' experience of the content?"

In subsequent meetings, reflections on Three Things and discussions of classroom artifacts related to the theme of perseverance and productive struggle helped the community take up a student perspective on the research theme. Observations grounded in the shared language of TRU supported the teachers in building on one another's thinking about student perseverance and their efforts to support it. The discussion peaked during their fourth department meeting, slightly more than a month before their first research lesson.

At the beginning of this fourth meeting, the teachers discussed what happened in their classrooms when they tried Three Things. One teacher began the discussion by connecting her reflection to the TRU "equitable access" dimension. This prompted another teacher to reflect that in his experience, Three Things prompted more students to be more open about sharing their ideas verbally. Another teacher said that she had noticed that while Three Things encouraged students who were "borderline engaged" to talk more, it did not seem to support students who were more physically and audibly disengaged from classroom work. These observations stimulated a community discussion of what is meant by equitable access and gave rise to the question: "Does Three Things help us achieve *equitable* access to collaborative productive struggle?" A fourth teacher challenged whether Three Things was actually promoting collaborative productive struggle since the teacher, rather than the students, had provided the strategy.

This challenge led the teachers to examine more deeply each other's definitions of "persistence," a word they had used interchangeably with perseverance and productive struggle – but this time with attention to how persistence manifests for students who experience the challenge of mathematics problems differently. The facilitator encouraged them to try to disentangle their different definitions of persistence using the TRU dimensions. In the discussion that followed, teachers identified specific reasons that students might not persevere, tying their observations to TRU dimensions:

[Students might say] "Why are we doing this? This is so hard" (Cognitive Demand).
"Kids don't feel like they have a toolbox" (Equitable Access).
Fixed mindset – "I'm not good at math" (Agency/Ownership/Identity).

One teacher suggested that when teachers ask students to share three things when they are stuck, students are prevented from drawing on their "toolbox" themselves. From her perspective, students who failed to persevere because they felt as though they did not have the tools to move forward were not supported to persevere by teachers providing those tools for them; those students should be supported to find the tools they need themselves.

This comment led the teachers to wonder how and to what Three Things gives students access. While the strategy might give students more equitable access to the *content* by helping them continue to solve a problem and allowing them to hear strategies developed by other students, it might not give them access to mathematical practices, including perseverance, if they depended on the teacher to move them forward when stuck. If perseverance means individual and collective struggle without teacher intervention, having a teacher ask for three things does not help students develop that capacity.

This articulation of what the teachers really meant by perseverance, tied to their experiences with Three Things and the TRU dimensions, marked a significant step forward in developing a rich research theme tied to student experiences. By the end of this fourth department meeting, the teachers knew that while they cared about supporting perseverance so that students would have equitable access to content, they also cared about supporting perseverance so that students could draw on each other, rather than their teachers, for support during times of need. Moving forward, the teachers oriented toward finding ways to support students to draw on resources such as their peers, notes, or classroom artifacts with less teacher scaffolding.

This is precisely the kind of *problematizing* – inquiring into the whys and wherefores of one's teaching – that TRU is explicitly designed to support. Teachers' experiences at sites A and B demonstrate how TLC-wide TRU-based inquiry cycles, nested within the study phase of a Lesson Study cycle, supported teachers in developing a deeper research theme that was more directly tied to student experiences. The teachers developed a culture of collaborative inquiry into teaching and learning, testing, and discussing conjectures about teaching strategies, in advance of their work crafting and reflecting on research lessons.

2.2 Plan and Study: Interconnectedness Between the Planning Team and an Entire Department's Study Process

One significant difference between TRU-LS and traditional LS in Japan is the interconnection of whole department PD and the research lesson planning team. In TRU-LS, responsibility for planning the research lesson is assigned to a planning team – but that planning team interacts with the whole department in ongoing ways as it works on critical planning issues. Through this interconnection, *kyouzai kenkyuu* is further developed, particularly with regard to the collective development of the curricular and student lenses.

2.2.1 Site A: Refining a Lesson Plan Through a TLC Discussion Grounded in TRU

The TLC at site A, which was focusing on productive struggle (cf. Sect. 2.1.1), decided that its content focus would be on systems of linear equations. The planning team reviewed the relevant progression of Common Core standards from grade 7 through high school, examining tasks they believed would afford rich explorations of the content and also provide students with opportunities to engage in mathematical practices related to productive struggle.

The teacher who had volunteered to teach the research lesson said that his approach to the topic the previous year had been procedural and that he hoped to increase Cognitive Demand this year. Accordingly, the planning team began by working through a sample assignment from the previous year. This task offered possibilities for students to use multiple representations but in a largely procedural way.

The planning team then studied additional tasks that might open up the mathematics and offer additional challenges – but, they were hampered by a narrow interpretation of the Common Core content standards. They read the standards to say that students only needed to learn the slope-intercept form of linear equations and that they only needed to learn to solve pairs of simultaneous linear equations using substitution. This reading constrained their considerations concerning Cognitive Demand. For example, a task they examined was helpful in opening up the space of possible student strategies, but it was rejected from consideration because the equations in the task were presented in standard form.

The content-related question driving the planning team was: "What knowledge do we want students to leave this unit with?" Ultimately, the team decided that it wanted students to understand (1) that the solution set of a pair of linear equations is the set of points where the graphs of the two equations intersect and (2) the conditions under which such a system will have zero, one, or infinitely many solutions. The Formative Assessment Lesson "Classifying Solutions to Systems of Equations" – the lesson can be downloaded from http://map.mathshell.org/lessons.php?unit=8220&collection=8 at no cost – seemed promising in this regard, despite its length (the lesson was designed for 2–3 days) and its use of equations in standard form. The planning team decided to bring the FAL to the TLC so that everyone could do the mathematics and aid the team in adjusting the tasks to hit the "sweet spot" of productive struggle.

Doing the mathematics together at the TLC meeting generated a fruitful discussion that shaped the planning team's subsequent design decisions. TLC members agreed that the level of Cognitive Demand was too high, encouraging the planning team to think more deeply about how to modify the task without "scaffolding away" the most important mathematical ideas. For example, when the planning team proposed changing all the equations in the lesson to slope-intercept form in an effort to make the lesson more accessible, an Algebra 1 teacher challenged, "if you alter all these equations to be in slope-intercept form, then aren't you giving up a lot of the Cognitive Demand of this particular task?" Engaging the whole TLC broadened the collection of possible student strategies for comparing two equations. It expanded the set of connections that might be made, highlighting places where students might engage deeply with the content. Finally, TLC members also suggested changes to the tasks that could make it easier for students to access the central mathematics within the allotted time. The planning team followed up on these ideas in subsequent planning meetings.

This record of teacher conversations demonstrates ways in which coordination between TLC study meetings and planning team meetings, built into TRU-LS, enriches both the study and the planning phases of LS. The larger community's

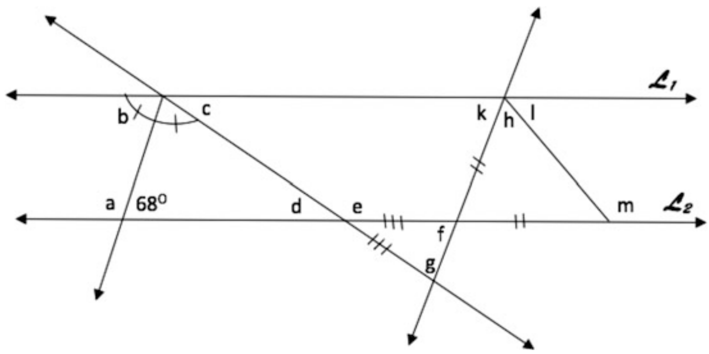

Fig. 6 The main task in the Group Angle Challenge lesson

knowledge of TRU and the Common Core enabled the planning team to think more deeply about the content of the research lesson than their consideration of the standards documents had afforded. At the same time, doing the mathematics together supported all of the teachers in the TLC in engaging more deeply with the content and issues of Cognitive Demand than if they had discussed these issues in the abstract and positioned them to better understand what would transpire in the research lesson. In this way, the synthesis of TRU and LS expanded the opportunities for all TLC members to learn about the mathematical content and practices through a student lens.

2.3 Do and Reflect: Two Different Trajectories Through the Same Lesson Plan at Sites B and C

Sites B and C independently chose to use the same OUSD lesson, the Group Angle Challenge, as the basis for one of their research lessons. The main mathematical activity in the lesson asks students to determine the values of labeled angles in Fig. 6, given that lines L_1 and L_2 are parallel.

The lesson begins with students working in groups to determine each angle and record an explanation of their reasoning in a table. Each group then makes a poster to record its findings. The lesson concludes with students from different groups visiting each other's posters to read and comment. The content focus of the lesson is rather limited, but the lesson does offer possibilities for student discourse and argumentation.

The planning teams at sites B and C chose this lesson for different reasons. Site B teachers hoped that the lesson would help students learn to turn increasingly to classroom resources and each other for help instead of to the teacher. Site C teachers hoped that the lesson would help them support students to justify their thinking and critique the mathematical thinking of others. Thus both sites focused on aspects of dimensions 2 and 4 of TRU, Cognitive Demand and Agency/Ownership/Identity.

Yet their specific research themes and theories of action led them to focus differently in observing and reflecting on their research lessons. Both sites reflected deeply on their students' experiences but very differently.

The examples that follow indicate how the nonprescriptive but focused character of the TRU framework can play out productively in practice. The Group Angle Challenge lesson can be modified in myriad ways, not all of which would support rich opportunities to learn in classroom discourse. Leaving things wide open could lead to unhappy results; but forcing teachers down any particular path – even if the direction taken is useful – runs the risk of engendering resistance and undermining attempts to build teacher community. That teachers at the two sites were able to pursue their specific interests was enfranchising, contributing to teacher agency and the ongoing development of teacher community. At the same time, TRU helped to problematize practice in focused and productive ways.

2.3.1 Site B: What Would Support Students to Turn to Each Other as Resources? A Focus on Agency, Ownership, and Identity

In post-research lesson reflections, site B teachers and their final commentator focused largely on the ways in which students sought help from each other and their teachers and the role teacher interventions played in student perseverance. They commented that few students had drawn on the clearly relevant posters that students had developed over the past week and that teachers expected students to use as resources. Although some students quickly referred to their notes or to posters, many did not do so – seemingly at a loss for how to proceed without heavy scaffolding from a teacher. These observations challenged the teachers' theory of action. The resources and questions that they had developed had not supported as many students to struggle productively together as they had hoped.

Site B teachers did not explicitly categorize their observations into the TRU dimensions, but the Cognitive Demand, Equitable Access, and Agency/Ownership/Identity (AOI) dimensions featured in their conversations.[7] The final commentator encouraged them to focus on AOI, especially with respect to the ways that teacher interventions can both enhance and undermine student agency during group work. She highlighted observations in which teachers had described having to scaffold student thinking more than they felt was appropriate for their goals.

She also discussed a specific observation that she felt had implications for student agency. She had heard a student discuss an idea with confidence and certainty when the teacher wasn't present. But when the teacher dropped by to check in, the student had asked a question about that idea as if she hadn't already thought of it. The teacher, unknowingly, went over some of the ideas the student had already reasoned

[7]Although organizing observations according to the TRU dimensions can be useful, it is certainly not necessary. Indeed, the goal is for the TRU language and perspective to become internalized, so that they serve as a natural framing for discussions.

through. When the teacher left, the student attempted to reproduce the teacher's phrasing and approach, seemingly losing track of the way she had dealt with it herself. This kind of interaction has the potential to undercut student agency.

These observations deepened the group's conversation about AOI. The teachers reflected on the implications of their interventions during group work, noting that high school students' mathematical agency might be fragile despite a teacher's best efforts. The final commentator also encouraged the teachers to think of ways that the resourceful mathematical work being done by some groups might be made public to the class, so that students could learn about using resources from each other rather than their teachers.

In subsequent TRU-LS cycles, site B teachers focused further on helping their students develop mathematical resources that they could draw on during group work. They also attended more carefully to the ways in which students were supported to engage with the big mathematical ideas in their lessons. They observed that students would be better positioned for collaborative productive struggle if the mathematical ideas present in a lesson were worth the struggle and if available resources engaged students with those ideas.

2.3.2 Site C: Reflection with the TRU Framework Supports Shifts in Theory of Action in a Subsequent TRU-LS Cycle

In their post-research lesson discussion, site C teachers noted that although students had justified their reasoning to each other while working in small groups on the Group Angle Challenge, much of that mathematical talk did not make it onto their posters or in comments to their peers. This observation ran against their theory of action. The teacher leading the research lesson had modeled "what critiquing reasoning looks like" and the lesson format provided students opportunities to critique each other's work. However, students' critiques contained neither the mathematical thinking the planning team had expected nor the mathematical thinking that had been observed in the small group work that preceded the critiques.

The teachers recorded these observations in the context of the Mathematics and Agency/Ownership/Identity dimensions. Under Mathematics they wrote, "Students knew how to justify <u>verbally</u>, writing didn't match." They also suggested "Maybe tell them what kind of info to include in the justifications," thinking that students' mathematical writing might be richer if the teachers were explicit about the criteria they had in mind for richness. Under AOI they wrote "Student critiques were vague," referencing the AOI dimension's attention to students' critiquing and building on each other's thinking. They added "Students did a good job of holding each other accountable <u>while solving</u>," but "This thinking didn't make it onto the poster."

The final commentator started by touching briefly on each of the five TRU dimensions, highlighting some of the observations that had been made by the teachers. She then turned to formative assessment (TRU dimension 5), which had not received much attention during the teachers' lesson planning or reflection. After discussing the lesson through the lens of formative assessment, she turned to next steps, suggesting that the team revisit their theory of action in light of some of the teachers' planning notes.[8] She drew the teachers' attention to where they had written, "Our goal is for students to be able to (1) think (2) conjecture (3) critique and (4) revise... 'Revise' allows students to know that it's okay to make mindful mistakes." She prompted the teachers to think about how, if at all, this final step of "Revise" showed up in their research lesson. She drew from the TRU Observation Guide, highlighting two Student look-fors ("Provides specific and accurate feedback to fellow students" and "Makes use of feedback in revising work") and one Teacher look-for ("Flexibly adjust content and process, providing students opportunities for re-engagement and revision"). She suggested that if the students had been given the chance to revise their work, they would have better understood the purpose of giving precise, constructive feedback.

As they moved into their second Lesson Study cycle, teachers at site C decided to keep the research theme from their first LS cycle, "building students' mathematical reasoning through student justification." However, reflections on first research lesson led to significant changes in their theory of action. The teachers had selected the Group Angle Challenge for their first research lesson because they felt it would provide students opportunities to justify their answers. They were now more aware of the limitations of the original curricular task: because there was only one correct answer for each angle measure and only one or two ways to explain each answer, students' justifications were not as central to the task as they could have been. One teacher suggested that "more rigorous open-ended tasks" would offer more opportunities for students to share a variety of justifications and critiques.

As a result of these reflections, teachers adjusted their theory of action for their second LS cycle to incorporate the four-step justification process highlighted in the final commentary. They chose a richer and more challenging set of tasks for the core content of the lesson, and they used a TRU Formative Assessment lens when considering instructional strategies, students' learning experiences, and task design in preparation for their second research lesson.

This sequence of events illustrates how TRU and Lesson Study work in synergy. TRU helped frame teacher reflection, making it more productive and powerful. Furthermore, the use of an external commentator versed in TRU broadened and deepened the set of focal issues the TLC took into account.

[8]Planning for the research lesson was supported by and recorded in a TRU-Lesson Study Discussion Guide, which yields a lesson plan similar in detail to those developed in traditional Lesson Study.

3 Discussion

To take hold and flourish, any mechanism for teachers' professional growth must mesh with the context in which it is implemented. In Japan, Lesson Study is deeply embedded in supportive school contexts and structures. Teachers are provided time and space intended explicitly for collaboration; they are supported in working together; there is a decades-long tradition of attending to content and practices in instruction and professional development; and curricula provide a rich and stable base for curricular inquiry. Under such circumstances, Lesson Study can function as intended, supporting the continuous development of Japanese teachers' knowledge and practices.

The US context for professional development differs substantially. Although conditions vary, teachers in the USA typically have few opportunities for collaboration. Efforts to support professional growth and collaboration are typically undertheorized, if they are theoretically grounded at all. Teaching practice is largely considered private; when external observers walk into a classroom, the purpose is often evaluative. Curricula vary substantially in the affordances they provide both for content and practices, and there is much less of a tradition of attending to student thinking than in Japan. TRU-based professional development was designed in and for the US context. Explicit attention is given to community building among teachers. TRU offers a structure and tools that help teachers focus on key aspects of instruction while not being prescriptive in ways that would undermine teacher agency.

A primary affordance of Lesson Study is the learning that comes from teachers' extended study of important mathematical ideas and the ways that students make sense of them. A primary affordance of TRU is the explicitness with which it highlights productive aspects of the learning environment, with a focus on the students' experience of the mathematics. The five dimensions of the TRU framework provide a clear framing that highlights essential aspects of classroom practice. This approach has been useful across the USA, providing grounding for professional development efforts that might otherwise be scattershot (Schoenfeld et al. press). In addition, the TRU framework helps to make explicit the goals and theories of action for Lesson Study research lessons and observations of them – something essential in the USA but also helpful in Japan, where key aspects of practice are largely tacit and passed on by tradition. The use of TRU inquiry cycles, which bring aspects of research lesson design to an entire teacher learning community, helps to prepare all members of that TLC to profit from observing the research lesson and hearing the lesson commentaries. It also supports the development of teacher community. In these ways, TRU-LS provides a mechanism for the functional enrichment and adaptation of LS to the US context.

TRU-Lesson Study has unique potential for broad implementation across the USA and other nations with similar cultures of teaching. As this volume demonstrates, Lesson Study is a powerful mechanism for professional development, when the culture supports it. The TRU part of TRU-LS, which is powerful on its own – see

Schoenfeld et al. (press) for evidence of its impact – helps to establish fertile ground in which the Lesson Study part of TRU-LS can take hold and flourish. Admittedly the TRU-LS work is in its early stages, and the evidence presented in Sect. 2 is preliminary. This is as it must be: it takes time to build supportive contexts, and the development of teacher knowledge and expertise is, like all expertise, a process that takes thousands of hours of concerted practice and reflection. However, the evidence thus far suggests that there is some reason for optimism.

Acknowledgment This paper was produced with support from the National Science Foundation grant 1503454, "TRUmath and Lesson Study: Supporting Fundamental and Sustainable Improvement in High School Mathematics Teaching," a partnership between the Oakland Unified School District, Mills College, the SERP Institute, and the University of California at Berkeley. We greatly appreciate the support of the SERP team, including Matt Ellinger and Suzanne Donovan; the Oakland Unified School District mathematics team, including Warren Currie, Geetha Lakshminarayanan, Courtney Ortega, Mary Reed, Barbara Shreve, and Phil Tucher; and Catherine Lewis of Mills College. The authors, and not the National Science Foundation, are responsible for the contents.

References

Akiba, M., Ramp, L., & Wilkinson, B. (2014). *Lesson study policy and practice in Florida: Findings from a statewide district survey*. Tallahassee: Florida State University.

Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169.

Baldinger, E., Louie, N., & The Algebra Teaching Study and Mathematics Assessment Project. (2016). *TRU conversation guide: A tool for teacher learning and growth*. Berkeley/E. Lansing: Graduate School of Education, University of California, Berkeley/College of Education, Michigan State University Retrieved from: http://ats.berkeley.edu/tools.html and/or http://map.mathshell.org/materials/pd.php.

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 83–292.

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A US–Japan lesson study collaboration reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*(2), 171–185.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184–205.

Hill, H. C. (2011). The nature and effects of middle school mathematics teacher learning experiences. *Teachers College Record, 113*(1), 205–234.

Horn, I. S. (2005). Learning on the job: A situated account of teacher learning in high school mathematics departments. *Cognition and Instruction, 23*(2), 207–236.

Horn, I. S., & Kane, B. D. (2015). Opportunities for professional learning in mathematics teacher workgroup conversations: Relationships to instructional expertise. *Journal of the Learning Sciences, 24*(3), 373–418.

Horn, I. S., & Little, J. W. (2010). Attending to problems of practice: Routines and resources for professional learning in teachers' workplace interactions. *American Educational Research Journal, 47*(1), 181–217.

Horn, I. S., Garner, B., Kane, B. D., & Brasel, J. (2017). A taxonomy of instructional learning opportunities in teachers' workgroup conversations. *Journal of Teacher Education, 68*(1), 41–54.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203–235.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and north American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.

Little, J. W. (2002). Locating learning in teachers' communities of practice: Opening up problems of analysis in records of everyday work. *Teaching and Teacher Education, 18*(8), 917–946.

Lortie, D. (1975). *School teacher: A sociological study*. Chicago: University of Chicago press.

Perry, R., & Lewis, C. (2009). What is successful adaptation of lesson study in the US? *Journal of Educational Change, 10*(4). https://doi.org/10.1007/s10833-008-9069-7.

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM, The International Journal on Mathematics Education, 45*, 607–621. https://doi.org/10.1007/s11858-012-0483-1.

Schoenfeld, A. H. (2014, November). What makes for powerful classrooms, and how can we support teachers in creating them? *Educational Researcher, 43*(8), 404–412. https://doi.org/10.3102/0013189X1455.

Schoenfeld, A. H. (2015). Thoughts on scale. *ZDM, The International Journal on Mathematics Education, 47*, 161–169. https://doi.org/10.1007/s11858-014-0662-3.

Schoenfeld, A. H. (2017). Teaching for robust understanding of essential mathematics. In T. McDougal (Ed.), *Essential mathematics for the next generation: What and how students should learn* (pp. 104–129). Tokyo: Tokyo Gagukei University.

Schoenfeld, A. H., & The Teaching for Robust Understanding Project. (2016). *The Teaching for Robust Understanding (TRU) observation guide: A tool for teachers, coaches, administrators, and professional learning communities*. Berkeley: Graduate School of Education, University of California, Berkeley Retrieved from: http://TRU.berkeley.edu or http://map.mathshell.org/ or http://ats.berkeley.edu/.

Schoenfeld, A. H., Floden, R. E., & The Algebra Teaching Study and Mathematics Assessment Project. (2014). *The TRU Math Scoring Rubric*. Berkeley/E. Lansing: Graduate School of Education, University of California, Berkeley and College of Education, Michigan State University Retrieved from http://ats.berkeley.edu/tools.html.

Schoenfeld, A. H., Floden, R. B., & The Algebra Teaching Study and Mathematics Assessment Project. (2018). On Classroom Observations. Manuscript submitted for publication.

Schoenfeld, A. H., Baldinger, E., Disston, J., Donovan, S., Dosalmas, A., Driskill, M., Fink, H., Foster, D., Haumersen, R., Lewis, C., Louie, N., Mertens, A., Murray, E., Narasimhan, L., Ortega, C., Reed, M., Ruiz, S., Sayavedra, A., Sola, T., Tran, K., Weltman, A., Wilson, D., & Zarkh, A. (in press). Learning with and from TRU: Teacher educators and the teaching for robust understanding framework. In K. Beswick (Ed.), *Handbook of mathematics teacher education, volume 4: The mathematics teacher educator as a developing professional*. Rotterdam: Sense Publishers.

Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student thinking. *Journal of Teacher Education, 60*(3), 213–230.

Star, S. L., & Griesemer, J. R. (1989). Institutional ecology, translations, and boundary objects: Amateurs and professionals in Berkeley's Museum of Vertebrate Zoology, 1907-39. *Social Studies of Science, 19*(3), 387–420.

Stigler, J., & Hiebert, J. (1999). *The teaching gap*. New York: Free Press.

Takahashi, A. (2015). Personal communication, July 1, 2015.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM, 48*(4), 513–526.

Wang-Iverson, P., & Yoshida, M. (2005). *Building our understanding of lesson study*. Philadelphia: Research for Better Schools.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond [AMTE monograph 5]. In *Inquiry into Mathematics Teacher Education* (pp. 131–142). San Diego: Association of Mathematics Teacher Educators.

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Part II
Historical and Cultural Perspectives in Japan and China

Preface: Historical and Cultural Perspectives on Lesson Study in Japan and China

Lynn Paine

Contents

Lesson study offers a powerful set of lenses that can sharpen our gaze on teaching. It invites the teachers engaged in conducting a research lesson, observers of the lesson, or even those studying lesson study as a practice to consider small moments and movements in a single lesson and fine details of teaching with the goal of understanding and improving teaching and learning. Yet anyone engaged in lesson study would remind us that it is not only a lens that magnifies the current moment, the act of teaching, but lesson study also gains its power from its ability to focus the educator on connections between the present and future. By considering goals for learning in guiding a lesson plan, looking closely at the enactment of that plan in an instance of teaching, and revising and reflecting on the plan's effect, teachers and others engaged in the practice of lesson study move between big ideas and micro-moments. Lesson study thus naturally weaves back and forth from future to present back to future.

The chapters in this section on "historical and cultural perspectives in Japan and China" offer a reminder that lesson study is neither solely determined by the present moment nor some future goal. Instead, its practice is shaped by traditions constructed over time and in particular contexts. These chapters, considered together, argue for the need to understand lesson study in context, both temporal and spatial. Cultural history and curricular developments and policies toward teachers are dynamic; each affects how lesson study is enacted, what role it plays, and, more profoundly, how teaching is defined. Lesson study also affects teachers differently at different stages of their careers. Whether in terms of an educational system's history or the biography of a single teacher, time and timing are relevant to lesson study. This section demonstrates the value of a historical perspective for understanding what lesson study is and what it achieves (and what its limits are) at

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_8

any one point in time. In addition, lesson study is inherently a social practice. It connects teachers as they plan together and as they observe, discuss, reflect on, and refine a lesson. These chapters show how, depending on the version of lesson study in play, it may bring together a small group of teachers from a single school or link educators across schools, a district, or even nationally or internationally.

The research presented in this section provides insights into lesson study practices in Japan and China. The chapters remind us of the broad range of practices that fall under the umbrella of the term "lesson study," as well as the need to think more precisely about what that term means at any one moment in time and in any one locale. The four essays outline details about the development and practice of lesson study in Japan and China. There are clear similarities: focusing on "unit" of a lesson as the means of improvement of teaching, relying on making teaching public and open to shared reflection as a way to bring new insights and deeper understandings, invoking, in different contexts, opportunities to model exemplars or chances for critique during the process of exploring and polishing new approaches, developing artifacts and resources for a community of practice and its shared knowledge base, and serving as a means to implement curriculum reform. Yet there are subtle differences worth exploring: to what extent is the focus on the lesson or on student learning? How much is lesson study used to enact reform or to shape it? How is "jointness" defined and collaboration enacted and with what purposes? Each chapter offers us a window on some of these. By considering the essays together, we are left with a richer and more complex understanding of the range of practices, over time and place, that we call lesson study.

Clearly, lesson study has a long history. Today, it is a practice gaining popularity around the world, acting as a kind of "traveling policy" (Thompson 2013) endorsed by educational reformers in many countries.[1] Often, literature refers to "Japanese lesson study" and treats this practice as originating in Japan. Intriguingly, as Naomichi Makinae (this volume) makes clear, the roots of lesson study themselves reflect the ongoing history of movement of ideas about education across national boundaries, in particular between Japan and the West. Indeed, Makinae argues, lesson study was developed to respond to the need to help teachers develop reform-minded practices informed by Pestalozzian and other ideas about teaching imported from afar; it served as a mechanism to spread a type of teaching. Chapters here by both Makinae and Li provide information about how the development of lesson study in Japan and China, at different moments, has been sparked by outside reforms. In both countries, lesson study has a long history and has gone through different periods as some elements of its form and goals have shifted. These accounts persuade us to be wary of treating lesson study as a monolithic or single practice, even within one country's tradition. And today, as the attention to lesson study has grown internationally, there is even greater value in recognizing the ways in which lesson study creates a dialogue not only among practitioners (as they plan, observe,

[1]The creation and growth of the World Association of Lesson Studies and its journal (the *International Journal for Lesson and Learning Studies*) reflect this rapid global reach.

and reflect on a lesson) but among its proponents, advocates, and scholars. Global interest in lesson study has created an international dialogue, not only about how to develop lesson study in a school or district but also about how to support teaching, teachers, curriculum reform, and student learning.

The focus on history also points to the powerful ways cultural assumptions about teaching and learning shape the goals and enactment of lesson study. Traditions of teaching that recognize the power of models allow lesson study practices to hue toward that purpose. Traditions that see teaching as a collective practice afford a community engagement in lesson study that is quite different from lesson study development in contexts in which teaching has traditionally been highly individualized practice (Fernandez 2002). China's lengthy history of teaching research groups (Paine and Ma 1993; Huang et al. this volume) and the cultural roots of Chinese lesson study (Chen 2017) mean that ideas and assumptions, as well as an infrastructure, have long been in place to support the particular embodiments of lesson study—varied as they are—that are present in China today. Li (this volume) and Chen (2017, p. 284) point to the importance, for example, of an epistemology that sees the "unity of knowing and doing," a pragmatic valuing of learning from concrete experience, and the importance of emulating models. These elements nurture approaches to working in and learning through lesson study that uniquely informs the development of the practice and its evolution in China. As Huang et al. (this volume) suggest, envisioning lesson study in other settings requires that we consider the cultural resources of that context: "when adapting the Chinese LS into other educational systems, the cultural value and professional development system should be given careful consideration." While the particular cultural resources drawn in Japan's lesson study practices are not identical to these in China, both contexts remind us of the need to recognize culture as a resource and to acknowledge that, despite the spread of lesson study, such resources do not automatically accompany the traveling practice/policy. Instead, these chapters might invite readers to reflect on what aspects of local culture can be exploited to support hybrid practices that combine imported ideas and local resources.

Lesson study advocates are quick to point to its power to support growth over time—for an individual teacher, a group of teachers, as well as for a system (Lewis and Tsuchida 1998). There is strong evidence in both Japan and China for the ways lesson study has allowed teachers to develop as more accomplished educators. In some Chinese settings, for example, teachers working collaboratively to plan and observe lessons are at the core of the induction of new teachers (Paine et al. 2003). As such, lesson study can be a form of socialization into a community of practice. Huang et al.'s (this volume) detailed analysis of one series of lesson study activities of a teacher group reveals how an array of educators, in the school and the district, support not only the improvement of a lesson but also the development of a teacher with stronger understandings of teaching, deeper content knowledge, and new insight about pedagogy. Lesson study thus can help develop teaching and, as several of these chapters argue, can assist teachers in making their teaching to be consistent with new curriculum. The lesson study systems in Japan and China historically have supported larger system-level reform efforts.

As much as lesson study helps form teachers in both countries, however, Watanabe (this volume) also raises the question of how lesson study can shape broader discourses of teaching. That is, the flow of information through lesson study is not necessarily from center or expert to others. It can, as Watanabe's analysis of artifacts of Japanese mathematics lesson study suggests, provide a feedback mechanism from frontline teachers to curriculum decision-makers. Globally, there is a long-standing challenge in creating responsive teaching systems informed by the wisdom of practice of experienced teachers. Lesson study may offer a distinctive, promising means for that sharing of information.

Perhaps one of the rich contributions of lesson study is the ways in which it supports movement within an individual, a community of teachers, and—in the most ambitious possibilities (as Watanabe outlines)—a large educational system. The meticulous tracking of iterations of changes of a lesson, and the actors engaged in that process, in one mathematics lesson study project (Huang et al. this volume) points to the ways lesson study offers wide-ranging participation with the possibility of rounds of learning across time and space/location/actors.

The set of chapters invite questioning about the complexity of the relationships inherent in lesson study, and how relationships that are hierarchical or position some teachers as learning from (other) experts afford learning that is similar and different from lesson study conceived as fundamentally collaborative, with teacher learning from and through joint work. Each offers possibilities, but not the same. Intriguingly, the experience over decades in both China and Japan offer examples of variation, within a country, in how relationships in lesson study are assumed to serve teacher learning. Similarly, these chapters remind us of a dilemma of lesson study's role in reinforcing and reproducing dominant traditions and its potential as an incubator of innovative challenges to status quo teaching. How elements of cultural tradition inform how educators view the role(s) of lesson study, the engagement of participants, and the ultimate goals of the practice appear pivotal.

Finally, these chapters do not only help us recognize the complexity of lesson study and call us to avoid treating it as some known or uniform practice, but they also model the rich variation possible in approaches to studying it—from historical analysis of documents and careful tracing of artifacts in relation to discourses (of textbooks, policy, and so on) to observation, interview, and video analysis of the process itself. As we seek to unpack a practice as diverse and rich as lesson study, our questions deserve equally rich forms of inquiry.

References

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 283–292.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 53*(5), 393–405.

Lewis, C., & Tsuchida, I. (1998). A river is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator*. (Winter), *12–17*, 50–52.

Paine, L., & Ma, L. (1993). Teachers working together: A dialogue on organizational and cultural perspectives of Chinese teachers. *International Journal of Educational Research, 19*(8), 675–697.

Paine, L., Fang, Y., & Wilson, S. (2003). Entering a culture of teaching: Teacher induction in Shanghai. In E. D. Britton, L. Paine, & S. Raizen (Eds.), *Comprehensive teacher induction: Systems for early career learning* (pp. 20–83). Dordrecht: Kluwer.

Thompson, P. (2013). Learner-centered education and "cultural translation". *International Journal of Educational Development, 33*, 48–58.

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The Origin and Development of Lesson Study in Japan

Naomichi Makinae

Contents

Abstract This study aims to clarify the history and issues in the origin of lesson study in Japan. In 1872, a new school system was started by the Meiji government. In that period, the object lesson approach was introduced in the normal school [teacher training college] as an updated teaching method for the new elementary schools. The object lesson approach method is based on Pestalozzian theory, according to which teaching in school should start from observation of objects that helps children recognize concepts through their intuition. This method introduced a significant change in teaching. The normal school undertook the responsibility of spreading the new teaching method through training teachers as well as editing and publishing instruction manuals and textbooks. In the process, the criticism lesson was introduced as a teacher training method. In this method, the normal school students presented a lesson to the class, while other students observed and discussed it. For observation and discussion, four points of criticism were used: matter, method, teacher, and children. These are also useful for today's lesson study. Along with the criticism lesson, the object lesson approach was spread to the

This chapter is built upon an article that formally appeared in Proceedings of 5th East Asia Regional Conference on Mathematics Education: Makinae, N. (2010). The Origin of Lesson Study in Japan, Proceedings of 5th East Asia Regional Conference on Mathematics Education, Vol. 2, 140-147.

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whole country through normal school products such as instruction manuals, new textbooks, and the teachers who graduated. This form of lesson study shows us its origin and its principles. Later, the criticism lesson expanded its role from preservice teacher training to in-service professional development. This describes how lesson study originated in Japan.

Keywords Lesson study · Object lesson approach · Normal school · Criticism lesson

1 Introduction

Japanese lesson study has been a popular professional development approach in recent years (Stigler and Hiebert 1999). In this method, teachers collaborate to study teaching content and instructions by observing lessons and discussing them. The careful development of research lesson, as well as observation and discussion, is an important component for lesson study. Through such an approach, teachers try to improve the quality of their teaching.

The origins of this type of lesson study can be traced back to the early 1900s when study meetings about new teaching methods took place in attached schools and private schools (Nakatome 1984). Attached schools were schools that carried pre-primary education, primary education, and secondary education attached to the normal school. In those schools, advanced education and educational experiments were carried. Curriculum development and the development of new textbooks and teaching methods were conducted in collaboration with teachers of normal schools. The study meetings were places for presentation of their studies. The attached schools had the role of spreading the research results of the normal school to the ordinary schools. After World War II, lesson study was well established as a strategy of in-service teacher training by the middle of the 1960s (Fernandez and Yoshida 2004). This paper attempts to determine the origin of lesson study prior to the early 1900s. The primary characteristic of lesson study involves teachers careful observing each other presenting research lessons followed by critical discussion, but the origin of this style and the background of its acceptance are not well known. How was this professional development approach built? In this paper, this question will be explored through a historical methodology.

2 The Beginning of the New School System

During the early Meiji era (1868–1912) in Japan, the Meiji government introduced the Western social system for modernization of the country's resources, including the school system. Before the Meiji era, Japan had adopted isolation policies in a feudal society, because contact with Western civilization was not convenient for the control of the feudal society. Equality and political participation of the people in Western civilization were incompatible with a feudal society. But what the isolation policy brought was not only isolation from social thought but also isolation from the progress of science and technology. The policy deprived Japan of opportunities to absorb progress in the scientific revolution and military technologies of Western countries. Japan was delayed from Western civilization.

The nineteenth century was the era when the Western countries spread colonies. Developing countries were being turned into colonies of Western countries. The Meiji government was established by breaking the feudal system in order to respond to such an international situation. The Meiji government aimed to adopt Western civilization to avoid the colonization of Japan.

The Meiji government, established in 1868, quickly promoted social institutional reforms and adopted Western civilization. It soon established an educational system. In 1872, a new school system named “Gaku-sei” [School System] was started by the government. Previously, only “Terakoya” [primary school for general public] and “Hankou” [regional government school for feudal clans] were available, which were not managed by the national government. Therefore, we can say that the modern school system in Japan began with “Gaku-sei.”

In “Gaku-sei,” the schools consisted of “Dai-gaku” [university], “Tyu-gaku” [secondary school], and “Syo-gaku” [elementary school]. The Meiji government endeavored to spread “Syo-gaku” to the whole nation and to build “Dai-gaku” in big cities. The government developed the school system from elementary education to higher education. Elementary education was available to all students, while higher education was considered to be the elite education of talented people. In this era, the matter of urgency was to introduce Western civilization and technology and to build a modern country similar to Western countries. The schools undertook the responsibility to provide the country with appropriately trained human resources.

For mathematics education, the Meiji government directed schools to introduce Western arithmetic instead of Japanese arithmetic, because they needed to import Western technology. They needed to understand Western mathematics in order to read Western books and understand the theories and mathematical formulas written in them. The subject “San-yo” [Arithmetic] was set up for elementary mathematics education. “San-yo” used Arabic numbers and written calculations based on decimal notation. This was totally different from Japanese arithmetic during that era. In addition, the subject name was later changed from “San-yo” to “San-jutsu”, both having same meaning of Arithmetic.

3 Normal School and the Object Lesson Approach

To achieve the new school system, “Gaku-sei,” teacher training was also needed. In 1872, the Meiji government established a normal school in Tokyo, to ensure that the preservice teachers learned the new teaching methods. American teachers, having experience in teacher training, were invited to teach in the normal school. One of these was Marion Scott, who introduced the classroom teaching method in which one teacher teaches all the students together by using the blackboard or teaching charts and a specific teaching approach, the object lesson approach.

The object lesson approach is one of the teaching methods which was famous in American normal schools in that era. The aim was to apply Pestalozzian theory to elementary teaching. According to Pestalozzian theory, all cognition is based on one's intuition, and intuition is absolutely essential for human cognition. We recognize things by intuition and then form a concept. This was seen as a natural order of mental development. The starting point of intuition is the image we receive through the senses. Therefore, teaching should not begin by reading books but instead from the observation of a familiar object. Teachers then expect their students to learn intuitively (Pestalozzi 1801).

From this standpoint, children learn from their direct experience with the world by using natural objects. They recognize things and form concepts through concrete manipulation and observation of objects. This is founded upon the idea that children already have the ability themselves to learn and to grow. Education is regarded as a developmental activity in terms of promoting voluntary growth and an organic activity in terms of emphasizing the child's inner personality. Teaching must be shifted to emphasize the child's internal cognition through the use of natural objects.

In arithmetic, a teacher has students count real objects, such as fingers or small stones. The student can recognize numbers intuitively through counting the real objects. In the case of $3 + 4 = 7$, for example, the teacher should not start from the abstract equation and operation. She should give her students three stones and four stones and then have them gather all the stones and count the sum of them. This activity will provide students a recognition of the process of addition and an intuition of the meaning of $3 + 4 = 7$.

The object lesson approach was introduced as an improvement over teaching by lecture, since that had been the previous method. In lecture, the teacher imparted the correct knowledge, and the students received it. Although the object lesson approach was an unfamiliar method when first introduced at that time in Japan, the normal school teachers were challenged to use this new education method. In normal school, new teachers were trained to understand the object lesson approach and to introduce it to all the schools in the country. Graduates of the normal school were expected to play a role in spreading new teaching methods at each school. Some were assigned as teachers and others as instructors at regional teacher schools in a leading position in schools across Japan.

4 Arithmetic Teaching by the Object Lesson Approach

The role of normal schools was not only to train teachers but also to propose new rules for instruction through editing and publishing instruction manuals and textbooks. At the beginning of "Gaku-sei," the only books available were translations of

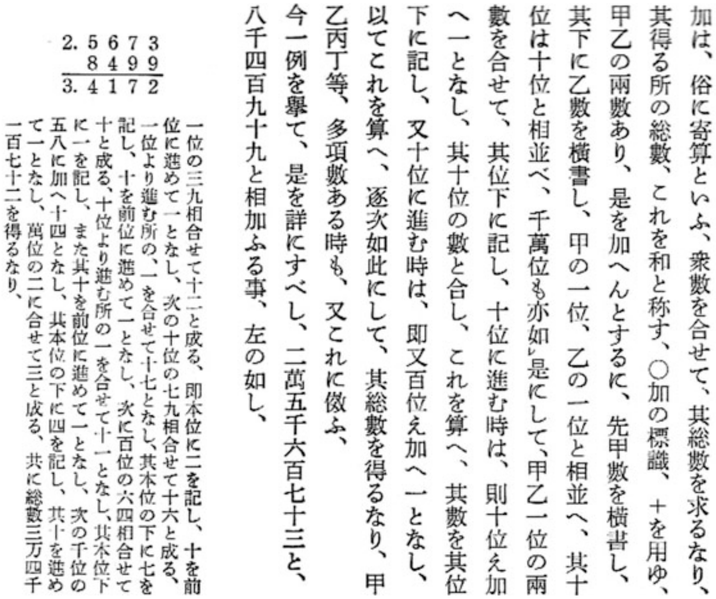
Fig. 1 The cover of *Hisan Kunmou* [*Enlightenment on Written Calculation*] (Tsukamoto 1869)

foreign texts, but these were not useful for elementary students because of the high level of information. Suitable textbooks for students were in strong demand by teachers. Responding to the demands, the normal school edited textbooks for each subject, using foreign textbooks as a guideline. These textbooks became the foundation of modern textbooks in Japanese schools. In arithmetic textbooks, we can see significant changes influenced by the object lesson approach.

4.1 Textbook for Introducing Written Calculation

Hisan Kunmou [*Enlightenment on Written Calculation*] was the designated textbook for elementary schools in "Gaku-sei" (Fig. 1). The textbook was published during the Meiji era for new arithmetic education, but it did not introduce the object lesson approach. Instead, it aimed to introduce written calculation, as performed in Western arithmetic, in Japan. In this textbook, the example 25673+ 8499 = 34172 was used to introduce addition (Fig. 2). Using these numbers, the algorithm of written addition was explained. This was a general and abstract explanation for the algorithm. In addition, the introductory example was not suitable for elementary students. After this introduction, practice drills were given. At the beginning of "Gaku-sei,", such a textbook was commonly used.

The abstract of the translation of Fig. 2 is as follows:



加法

加は、俗に寄算といふ、衆數を合せて、其総數を求るなり、其得る所の総數、これを和と称す、○加の標識、十を用ゆ、甲乙の兩數あり、是を加へんとするに、先甲數を横書し、其下に乙數を横書し、甲の一位、乙の一位と相並へ、其十位は十位と相並べ、千萬位も亦如レ是にして、甲乙一位の兩數を合せて、其位下に記し、十位に進む時は、則十位え加へ一となし、其十位の數と合し、これを算へ、其數を其位下に記し、又十位に進む時は、即又百位え加へ一となし、以てこれを算へ、逐次如此にして、其総數を得るなり、甲乙丙丁等、多項數ある時も、又これに傚ふ、

今一例を擧て、是を詳にすべし、二萬五千六百七十三と、八千四百九十九と相加ふる事、左の如し、

2.5673
　8499
3.4172

一位の三九相合せて十二と成る、即本位に二を記し、十を前位に進めて一となし、次の十位の七九相合せて十六と成る、一位より進む所の、一を合せて十七となし、其本位の下に七を記し、十を前位に進めて一となし、次に百位の六四相合せて十と成る、十位より進む所の一を合せて十一となし、其本位下に一を記し、また其十を前位に進めて一となし、次の千位の五八に加へ十四となし、其本位の下に四を記し、其十を進めて一となし、萬位の二に合せて三と成る、共に総數三万四千一百七十二を得るなり、

Fig. 2 Addition in *Hisan Kunmou* [*Enlightenment on Written Calculation*] (Tsukamoto 1869)

Addition

The calculation putting numbers together is called addition. You add all the numbers and find the total number. The total number is called the "sum." The sign of addition is "+."

When you add two numbers A and B, first write number A horizontally and then write number B under it. Write numbers according to the positional notation, ones and ones and tens and tens, and other places in same way. Add numbers for each position and write the sum under the line beneath the number B. If there is a carry forward, add 1 to the sum in the next place.

Explanations of the positional notation and the algorithm in general:

$$\begin{array}{r} 25673 \\ 8499 \\ 34172 \end{array}$$

$3 + 9 = 12$. Write 2 on ones, $7 + 9 = 16$, $16 + 1 = 17$. Write 7 on tens, $6 + 4 = 10$, $10 + 1 = 11$. Write 1 on hundreds, $5 + 8 = 13$, $13 + 1 = 14$. Write 4 on thousands, $2 + 1 = 3$. Write 3 on ten thousands. Therefore the sum is 34172.

Fig. 3 The cover of "Sho-gaku Sanjutsu-syo" [Elementary Arithmetic Textbook] (Monbusyo 1873)

4.2 New Arithmetic Textbook for Object Lesson Approach

In 1873, *Sho-gaku Sanjutsu-syo* [*Elementary Arithmetic Textbook*] edited by the normal school was published (Fig. 3). This was not only a new arithmetic textbook but also a representative of the object lesson approach in that era. In this textbook, addition was introduced through pictures (Fig. 4). By being shown objects, students were expected to understand the meaning of addition by their intuition. No equations were initially given, as these were intended to be derived from students' observations. Initially, small numbers, such as 1 and 1, were used. Such an approach seems very natural today, but compared with the earlier-used textbooks, this was a major change in student learning. After this introduction, the addition of numbers proceeded step by step, up to 10 + 9.

The translated text of Fig. 4 is as follows:

Section 5 Addition

(1) One flag added to one flag makes two flags.
(2) Two peaches added to one peach makes three peaches.

The differences between these textbooks are clear, namely, the foundational principles of teaching. The former focused on teaching an algorithm for an abstract operation, using difficult numbers to explain this general operation. The latter showed addition and the results of this operation through pictures. No explanations

Fig. 4 Addition in "Sho-gaku Sanjutsu-syo" [Elementary Arithmetic Textbook] (Monbusyo 1873)

were provided for the algorithm, and equations were absent in the initial parts of the textbook. The students were expected to recognize addition through their intuition by observing objects. The numbers given were simple and easy to count, gradually being increased. It is clear that the latter is more suitable as an introduction to elementary arithmetic.

5 Spreading the Object Lesson Approach

To use the new textbook based on the object lesson approach, the teachers in the nation needed to be well trained. To spread the object lesson approach, a new teacher training program was developed in the normal school. The program started for preservice teacher training.

5.1 The Use of Criticism Lessons and Model Lessons as a Teacher Training Method

The normal school teacher referenced some books on the introduction of the object lesson approach. These were imported from America. One of the books was written by Edward Sheldon, who was the principal of New York State Normal School in Oswego. He introduced Pestalozzian theory to elementary education, experimented with it on attached schools and public schools in Oswego, and developed the teacher training method to help teachers practice this theory. In his book, "A Manual of Elementary Instruction for the Use of Public and Private Schools and Normal Classes; Containing a Graduated Course of Object Lesson approach for Training the Senses and Developing the Faculties of Children," he introduced criticism lesson and model lessons as the methods to be used in normal schools (Sheldon 1871).

The criticism lesson was a lesson presented by a student in normal school. Other students and the teacher observed the lesson and then expressed their opinions on the various points of the lesson in which they thought the teacher had succeeded or failed.

The model lesson was a lesson given by a well-trained teacher. The normal school students would take notes on the lesson, observing the teacher's ideas and plans. It is important to observe a sufficient number of model lessons, as well as criticism lessons, to develop a well-trained teacher.

Teacher training and the object lesson approach cannot be separated. To succeed at spreading the object lesson approach throughout the whole country, well-trained teachers were needed. On the other hand, to train teachers, the object lesson approach itself is very useful, because the teachers required teaching skills in order to be successful at the object lesson approach. The object lesson approach provided a suitable learning experience for teachers.

5.2 Criticism Lesson and Points of Criticism

Sheldon (1871) defined specific points of criticism in his criticism lesson. These four categories included matter, method, teacher, and children. Regarding subject matter, three points were awarded on the basis of whether it was suitable for children. In the method category, seven were awarded on the basis of whether the teacher clearly comprehended the distinction between information that had to be imparted by the teacher and information that had to be developed by the child. For the teacher category, four points were awarded, one each for classroom management, the standing position in the classroom, the manner of the teacher, and the language of the teacher. In the children category, two points were awarded, one for attitude toward the children and one for showing an understanding of the students. What exactly these described were as follows:

Points of Criticism

I. Matter

1. Whether suitable to children; whether exercising observation, conception, reason, or all these.
2. Lesson – whether bearing on one point; into what heads divided.
3. Whether, in a Scripture or moral lesson, an application be made; whether the right one. In a lesson on an animal, whether the children are led to see the wisdom and goodness of God in the adaptation of parts to mode of life, and whether human feelings are cultivated.

II. Method

1. Whether the teacher clearly apprehends the distinction between what must be told and what must be given.
2. Whether she distinguishes the various mental faculties one from another; knows which should be, and how exercised.
3. Whether good illustrations are used; the specimens large enough and sufficient for distribution; whether diagrams were drawn when required.
4. Whether appropriate questions were used when general answers are wanted. Leading questions only to obtain an admission, on which another question is based.
5. Whether the board was sufficiently used – new terms written on it; also titles and heads of lessons; also, with elder children, definitions and statements.
6. Summary, of what kind; whether of the kind most appropriate to the children and the lesson.
7. Whether proper use was made of "hands out" and S. R. (Summary Report)

III. Teacher

1. Whether capable of swaying the class according to her will and of awakening sympathy.
2. Whether attending to all, of carrying on the lesson with a few forward children; whether taking the right standing position.
3. Manner – whether appropriate – bustling and excited – slow and languid – cheerful and energetic; whether, if Scriptural lesson, reverential tone of voice.
4. Language – whether appropriate; syntax and correct pronunciation.

IV. Children

1. Whether respectful, attentive; whether interested; if so, to what interest is owing.
2. Whether likely to carry the lesson away as a whole; if a Scripture or moral lesson, whether their hearts were touched.
(Sheldon 1871, 24–25)

To conduct a criticism properly, Sheldon mentioned that it was necessary to have a presiding critic and to develop a summary of the opinion at the end. This is similar to what is currently done. The commentator should be present at the lesson study. In the criticism lesson, two principal objectives should always be achieved: to acquaint future teachers with the principles of education that are based on the nature of children and to utilize the principles of teaching. By comparing the objectives with the outcomes, the criticism lesson can succeed in training teachers. These approaches are also appropriate in today's lesson study. The research class was carefully prepared according to the teaching objectives. The results of the lesson should be evaluated with reference to the objectives.

These points of criticism are exactly the same as those introduced in teacher training classes at normal schools. In the instruction manual "Kaisei Kyojujutsu"

[Revision of Teaching Methods], written by normal school teachers in 1883, the points of criticism were introduced as viewpoints to use while observing other teachers' classes (Wakabayashi and Shirai 1883). This method was also emphasized for use by normal school students. In those days, normal schools were established in local regions of Japan. The object lesson was spread from the Tokyo normal school to the local normal schools through manuals, the new textbooks mentioned previously, and the teachers who graduated from Tokyo. This dissemination then spread from the local normal schools to the entire country.

5.3 Development of Lesson Study

Inagaki (1995) described the criticism lesson as being practiced around the late 1890s. These lessons were conducted in elementary schools attached to normal schools. In this case, although the teaching methods were changing from the object lesson to Herbert's five phases of teaching, the criticism lesson remained a useful method to develop and train new teachers. Herbert's five phases of teaching were introduced as formalization of the teaching method in Japan. In the first phase, the teacher prepared a topic for imparting new knowledge to the children. He then actually presented the topic to the children and questioned them about it. Third, the children compared the new knowledge with what they already knew. Fourth, the children organized both new and old ideas into one system. Finally, the children reviewed the systematized knowledge and applied it to daily life. These five phases of teaching were the prototype of the problem-solving approach in Japan.

In the early 1900s, many local boards of education held conferences for teachers in order to develop new teaching methods. The criticism lesson was accepted as a method to be used during teachers' conferences. The conferences were called "Jugyo-hihyo-kai" [criticism lesson conference] or "Jugyo-kenkyu-kai" [Lesson study conference]. This can be seen in the educational magazines published in that era; some records were found about the use of lesson study not only in normal schools but also in local elementary schools (Figs. 5 and 6).

As stated earlier, the criticism lesson was originally established as part of lesson study, later spreading to local normal schools for teacher training. The criticism lesson later expanded its role from preservice teacher training to in-service professional development. From the historical development of the criticism lesson, we can see the origin of lesson study in spreading the object lesson and teacher training during the early Meiji era in Japan.

Fig. 5 The cover of "Kyouiku Kenkyu" [Educational Research] (Shotou-kyouiku Kenkyukai 1904)

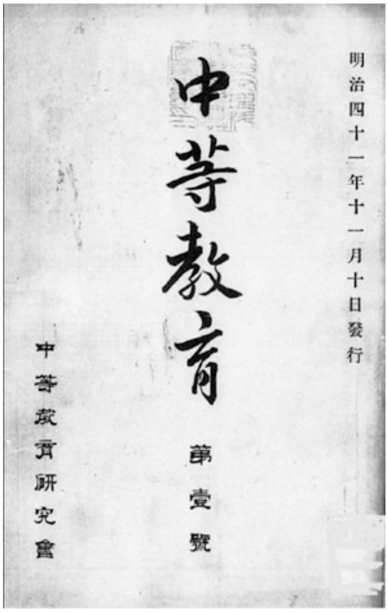


Fig. 6 The cover of "Tyutou Kyouiku" [Secondary Education] (Tyutou-kyouiku Kenkyukai 1908)

6 Conclusion

The origin of Japanese lesson study can be seen in the early Meiji era. During this time, the object lesson approach was introduced as a new teaching method. To spread the method, teacher training became an important issue. Preservice teachers in normal schools would practice the object lesson approach by using the criticism lesson. This demonstrates the origin of lesson study and its principles. The criticism lesson later expanded its role from preservice teacher training to in-service professional development.

References

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Mahwah: Lawrence Erlbaum Associates, Inc.

Inagaki, T. (1995). *Meiji kyouju rironshi kenkyu* [A historical research on teaching theory in meiji-era]. Tokyo: Hyuuron-Sya. [in Japanese].

Monbusyo. (1873). *Sho-gaku sanjutsu-syo* [Elementary arithmetic textbook]. Tokyo: Shihan-gakkou. [in Japanese].

Nakatome, T. (Ed.) (1984). *Kounai kensyu wo tsukuru: Nihonn no kounai kensyu keiei no sougouteki kenkyu* [Developing teacher training in school: A comprehensive study of management of teacher training in Japanese school]. Tokyo: Eideru Kenkyusyo. [in Japanese].

Pestalozzi, J. (1801). *Wie gertrud ihre kinder lehrt* [transl. by Nagao T. and Fukuda H. in 1976, Tokyo: Meiji-tosyo]. [in Japanese].

Sheldon, E. (1871). *A manual of elementary instruction for the use of public and private schools and normal classes: Containing a graduated course of object lessons for training the senses and developing the faculties of children* (6th ed.). New York: Charles Scribner and Co.

Shotou-kyouiku Kenkyukai. (1904). *Kyouiku Kenkyu* [Journal of Educational Research] vol. 1. [in Japanese].

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Tsukamoto, A. (1869). *Hisan kunmou* [Enlightenment on written calculation]. Tokyo: Numazu Gakkou. [in Japanese].

Tyutou-kyouiku Kenkyukai. (1908). *Tyutou Kyouiku* [Journal of Secondary Education] vol. 1. [in Japanese].

Wakabayashi, T., & Shirai, T. (eds.) (1883). *Kaisei kyojujutsu* [Revision of teaching methods]. Tokyo: Fukyu-Sya. [in Japanese].

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Lesson Study and Textbook Revisions: What Can We Learn from the Japanese Case?

Tad Watanabe

Contents

Abstract According to Japanese mathematics educators, Lesson Study is not only a professional development activity for teachers but also a part of a feedback mechanism to inform curricular revisions. The learning from Lesson Study informs the revision of the National Course of Study (NCOS), which takes place about every 10 years. Lesson Study also informs the revision of commercially produced textbook series, which are revised much more frequently than the NCOS. In an earlier study, Watanabe (Transformation of Japanese elementary mathematics textbooks: 1958–2012. In: Li Y, Silver EA, Li S (eds), Transforming mathematics instruction: multiple approaches and practices, 199–216. Springer, Cham, 2014) examined how Japanese elementary mathematics textbooks have changed over the years as Japanese mathematics teachers transformed mathematics teaching from teacher-centered instruction to what Stigler and Hiebert (*Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom*. The Free Press, New York, 1999) called structured problem-solving style instruction. That study revealed that Japanese textbooks indeed changed over the years to reflect more closely the structured problem-solving style lessons. However, the previous study could not attribute those changes to Lesson Study. In this follow-up study, we examined Lesson Study artifacts from the 1980s to see how Lesson Study might have affected the changes in Japanese textbooks. Although we cannot definitively conclude that

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_10

Lesson Study led to specific changes in textbooks, there are some evidences that suggest its influence. We speculate that certain features of the Japanese education system may contribute to making such influences possible.

Keywords Lesson study · Textbooks · Elementary school mathematics · Curriculum revision

1 Introduction

The publication of *Teaching Gap: Best Ideas from the World's Teachers for Improving Education in the Classroom* (Stigler and Hiebert 1999) piqued the interests of mathematics educators throughout the world about Japanese mathematics teaching and learning for a couple of reasons. This book was based on the video study conducted in conjunction with the Third International Mathematics and Science Study (TIMSS, now known as Trends in International Mathematics and Science Study), and it described mathematics classrooms from Germany, Japan, and the United States. Based on the analysis of video data, the authors described the characteristics of mathematics lessons in these three countries.

One of the reasons that this book brought the attention of mathematics educators to Japanese classrooms was because Japanese mathematics teaching, what Stigler and Hiebert labeled "structured problem-solving" (Stigler and Hiebert 1999 p. 27), was so different from how they described mathematics teaching in Germany and in the United States. Moreover, Japanese mathematics lessons "emphasized student thinking and problem solving, multiple solution methods, and the kinds of discourse described in U.S. reform documents to a greater extent than U. S. lesson did" (p. 106). In other words, Japanese teachers appeared to practice the type of mathematics teaching that was elusive to US teachers.

However, *Teaching Gap* not only described high-quality mathematics teaching in Japan, but also it offered the description of a mechanism through which Japanese teachers gained their teaching proficiency. In chapter "Teaching for Robust Understanding with Lesson Study" of the book, which was based on Makoto Yoshida's dissertation (Yoshida 1999), Stigler and Hiebert described the professional development practice called Lesson Study.[1] They concluded that through Lesson Study, Japanese teachers gradually, yet successfully, improved their teaching over the years. As a result, many mathematics educators throughout the world implemented lesson study with varying degree of success.

[1]Since the publication of *Teaching Gap*, lesson study has been attempted throughout the world. However, in part because of the limitation of the information about lesson study as practiced in Japan, what actually implemented varied drastically. Thus, in this chapter "Lesson Study" is used to refer to lesson study as practiced by Japanese educators, while "lesson study" will be used to describe the activity more generally.

Although it is true that Lesson Study is the primary form of school-based professional development activity in Japan, some scholars also note that Lesson Study is a part of a feedback mechanism to inform curriculum revisions (e.g., Lewis and Tsuchida 1997; Murata and Takahashi 2002). Reys et al. (2004) suggest that variations in mathematics curricula can partially explain the differences in student achievement on international studies. Kilpatrick et al. (2001) argued that "what is actually taught in classrooms is strongly influenced by the available textbooks" (p. 36). Thus, if Lesson Study influences curriculum revisions in Japan, understanding how it happens may be important for lesson study practitioners outside of Japan. However, there is very little known about the role of Lesson Study in curriculum revisions. Therefore, a study was conducted to investigate how Lesson Study may have influenced curriculum revisions in Japan. In particular, various Lesson Study artifacts were examined to see if there is any evidence that learning from Lesson Study may have affected elementary school mathematics textbooks. This chapter presents the findings from the study.

1.1 Lesson Study in Japan

Lesson Study has been practiced in Japan since the late nineteenth century (see chapter "How Does Lesson Study Work? Toward a Theory of Lesson Study Process and Impact" of this book for a more detailed discussion on the history of Lesson Study in Japan). Although Lesson Study in its origin was not tied to a specific subject area, Lesson Study has soon become an important tool for improving mathematics teaching and learning. Nakano (1939) discussed how Lesson Study in mathematics should be conducted, along with several examples of research lessons. Nakano argued that teachers must carefully design a research lesson after in-depth examination of teaching materials (*kyozai* in Japanese) and careful assessment of their students' current understanding. He also noted that observers must carefully study the lesson plan and understand the rationale behind the lesson design. Moreover, while observing a research lesson, Nakano suggested that observers pay attention to the interplays among the teacher, the students, and the content of the lesson. These recommendations are in alignment with contemporary recommendations for conducting lesson study more effectively (e.g., Takahashi and McDougal 2016; Watanabe et al. 2008).

Teaching Gap describes Lesson Study that was conducted as a part of school-based professional development activity. However, in Japan, there are different types of Lesson Study. Tsubota (2004) lists five different types of Lesson Study:

1. School-based Lesson Study
2. District-wide Lesson Study
3. Lesson Study organized by a local education agency
4. Lesson Study by a university affiliated school
5. Lesson Study organized by special interest groups

APEC Human Resources Development Working Group (n.d.) discussed different purposes for different types of Lesson Study, school-based, district-wide, and regional or nationwide. While the main purposes of school-based and district-wide Lesson Study focus on improving instruction and learning in a school or in a school district as a whole, the cross-district Lesson Study focuses on investigating curriculum sequences and contents as well as developing curriculum.

Lewis and Tsuchida (1997) discuss how Lesson Study influenced the shift in Japanese elementary science instruction, moving from teacher-centered lecture style teaching to student-centered instruction. Lewis and Tsuchida (1998) describe public research lessons and how they influence attending teachers. Both of these papers also discuss how Lesson Study plays a key role in implementing changes in the Japanese National Course of Studies, which are revised about every 10 years. However, perhaps because Lesson Study has been a routine part of Japanese educators' professional lives, not much has been written about the effects of Lesson Study on Japanese education by Japanese scholars. However, some Japanese scholars have expressed their concerns for the way Lesson Study is being conducted in Japan. For example, Yokosuka (1990) points out that, at some schools, teachers might be simply going through the motion of Lesson Study. Tsubota (2004) encourages Japanese teachers to re-examine how they engage in Lesson Study in light of the international attention Lesson Study has attained in recent years.

1.2 Problem-Solving in Japanese Mathematics Classrooms

As noted above, Stigler and Hiebert (1999) characterized mathematics teaching as structured problem-solving. In a typical lesson, there are four main stages: (1) teacher poses the problem, (2) students work on the problem independently, (3) the whole class discusses students' solutions, carefully orchestrated by teacher, and (4) teacher and students jointly summarize their learning. Oftentimes, the flow of a research lesson in Lesson Study is structured around these four stages of a lesson. According to a survey conducted by the Japan Society of Mathematical Education (JSME), virtually all respondents agreed that this approach to teaching mathematics was effective (JSME 2001). Although problem-solving has always been a major focus in Japanese mathematics education, teaching mathematics through problem-solving is something Japanese teachers developed and continue to develop through lesson study (McDougal and Takahashi 2014).

Nagasaki and Senuma (1986) examined the research trends on problem-solving by examining the programs of the JSME annual meetings. A vast majority of presentations at the JSME annual meeting were based on practitioners' research, that is, Lesson Study. They noted that "problem-solving" as a session strand was offered every year since 1955 except in 1973 at the elementary school level, but not quite as often at lower secondary school level. Moreover, they noted that in 1982 and 1983, there was a symposium focused on problem-solving at the annual meeting, and the records of these symposia were published in the official JSME journals.

Nagasaki and Senuma concluded that these symposia show a heightened awareness among the JSME leaders of the need for further research on problem-solving.

Hino (2007) examined the trends in problem-solving in Japanese mathematics classrooms. She noted that the 1980 publication of *Agenda for Action* by the National Council of Teachers of Mathematics (NCTM) in the United States had significant impacts on Japanese mathematics educators' thinking. In particular, NCTM (1980) first recommendation that problem-solving should be the focus of school mathematics in the 1980s resonated with Japanese mathematics educators. It is reasonable to assume that this publication was one of the reasons for the 1982 and 1983 symposia at the JSME annual meetings. Hino also noted that the emphasis on problem-solving research in the 1980s and beyond followed by years of research focuses on mathematical thinking. In fact, developing students' mathematical thinking remains a major focus of Japanese mathematics educators. Nagasaki and Senuma (1986) also noted that the research trends on problem-solving in Japan must be understood in the context of research on mathematical thinking. Hino (2007) posited that Japanese mathematics educators considered problem-solving as important partly because they have been investigating ways to help students develop the abilities to think mathematically.

1.3 Japanese Curriculum

Travers and Westbury (1989) distinguished three types of curriculum: intended curriculum, implemented curriculum, and attained curriculum. Curriculum materials such as textbooks play an important role of bridging between the intended curriculum and the implemented curriculum. In Japan, the intended curriculum is defined by the Ministry of Education, Culture, Sports, Science and Technology (MOE) through the National Course of Study (NCOS). The NCOS determines the content of each subject area, including mathematics, in each grade for elementary and lower secondary schools (Grades 1 through 9) and each course for the upper secondary schools (Grades 10 through 12). The NCOS is revised periodically, about once every 10 years, and the most recent revision of elementary and lower secondary school NCOS was released in 2017, with full implementation to start in the 2020 school year for elementary schools and in the 2021 school year for lower secondary schools.

Although the MOE specifies the contents in each subject area through the NCOS, textbooks are published by private publishers, who carefully interpret the NCOS and accompanying elaboration document, also published by the MOE. In the case of elementary mathematics, there are currently six different series published by six publishers. Whenever textbooks are revised, textbook publishers must submit their draft versions to the MOE for an official review. The MOE review focuses primarily on the content alignment with the NCOS. Note that although the NCOS specifies the grade-level placement of topics, it does not specify how a topic is treated or how topics within a grade level are sequenced and organized.

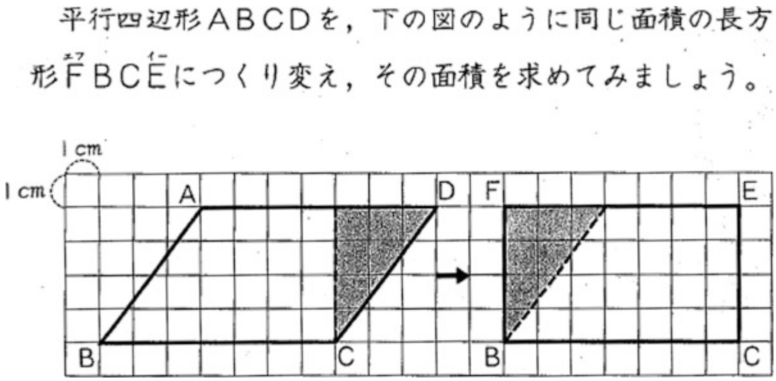

Fig. 1 The introductory question on the area of parallelogram in the 1977 edition (Tokyo Shoseki n.d., Grade 5, p. 75)

Watanabe (2014) examined how six editions of one of the most widely used elementary school mathematics series, published by Tokyo Shoseki (n.d.), transformed since the late 1950s to today. His analysis focused on area of triangles and quadrilaterals in Grade 5 and multiplication of fractions in Grades 5 and 6. Watanabe's analysis showed that the series indeed transformed over five decades to include features that are more supportive of the "structured problem-solving" (Stigler and Hiebert 1999) style of mathematics teaching.

For example, in the edition that followed the 1977 NCOS, the section on the area of parallelogram starts with a statement, "Let's think about ways to calculate the area of parallelogram ABCD," just as they do in the later editions. However, in the 1977 edition[2], the textbook immediately asks students to determine the area of the parallelogram by transforming it to a rectangle as shown in Fig. 1.

In contrast, the same section in the 1989 edition does not suggest a specific method. Rather, students are asked to actually figure out ways to calculate the area of the parallelogram – with a hint to think about ways to transform it to a rectangle as shown in Fig. 2.

In addition, in the 1989 and later editions, the textbook often included multiple-solution approaches and asked students to explain how each method worked. For example, Fig. 3 shows two ways hypothetical students transformed a parallelogram in the 1989 edition, and Fig. 4 shows two ways hypothetical students calculated $\frac{4}{5} \times \frac{2}{3}$ in the 2012 edition.

Another change was the type of representations and their placements within a unit. For example, in the unit on multiplication of fractions, the 1977 edition used a "double-sided" number line as shown in Fig. 5 to represent the following problem situation:

[2]Because some of the old editions obtained for the analysis did not include the publication years, these editions are referenced in this chapter by the corresponding NCOS years.

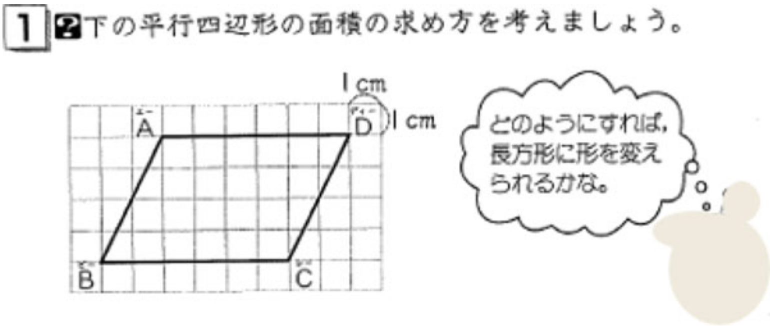

Fig. 2 The introductory question on the area of parallelogram in the 1989 edition (Tokyo Shoseki n.d., Grade 5). An avatar says, “How can we change this into a rectangle?” (p. 69)

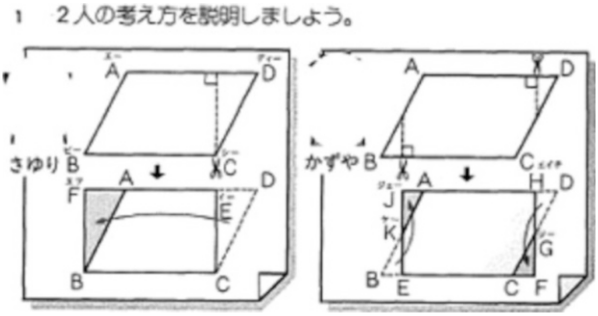

Fig. 3 The 1989 edition (Tokyo Shoseki n.d. Grade 5) shows two ways hypothetical students transformed a parallelogram into a rectangle to determine its area (p. 70)

1 m of cloth costs 360 yen. (1) What will be the price of this cloth when you buy $1\frac{1}{3}$ m? (2) $\frac{2}{3}$ m?

In this diagram, a single number line is labeled differently – the labels above the line are for the prices (yen), while the labels below are for the length (m). For a similar problem in the 1989 edition (the price of 1 m is 240 yen), the textbook used a double number line diagram as shown below, making it clearer that two distinct quantities are involved (Fig. 6).

Furthermore, in the 1977 edition, this “double-sided” representation was used after students have already learned the procedure to multiply fractions. It was used for the first time with the particular problem as they investigated the relationship between the size of the multiplier and the product. On the other hand, in the 1989 edition, a double number line diagram was included in the opening problem of the unit (Fig. 7).

The diagram was intended to support students’ understandings of the problem situation with a fraction multiplier. This intention became even clearer in the 2012 edition as one of the hypothetical students used the diagram to explain why she thinks the problem can be solved by $\frac{4}{5} \times \frac{2}{3}$ for the same opening problem in the unit (Fig. 8).

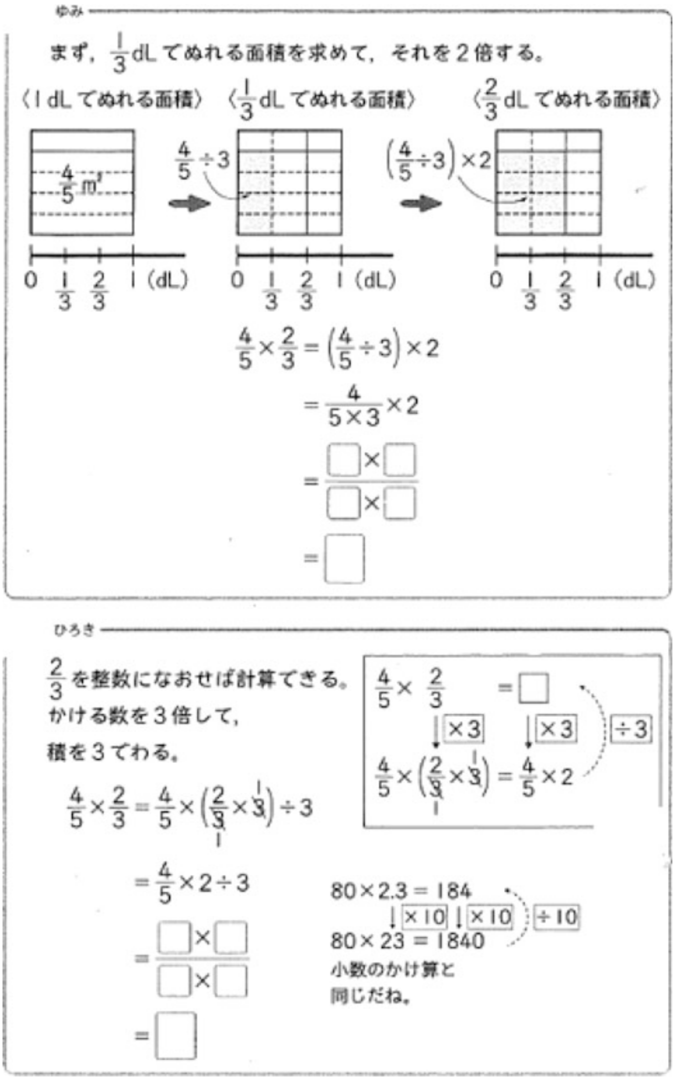

Fig. 4 The 2012 edition (Tokyo Shoseki n.d., Grade 6) includes two ways hypothetical students calculated $\frac{4}{5} \times \frac{2}{3}$ (p. 25)

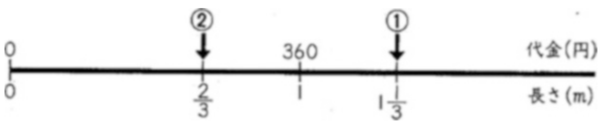

Fig. 5 “Double-sided” number line used in the 1977 edition (Tokyo Shoseki n.d., Grade 6, p. 9)

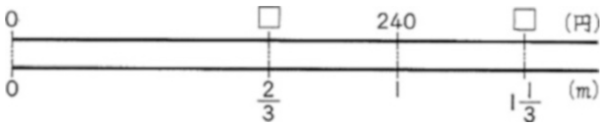

Fig. 6 A double number line diagram used in the 1989 edition (Tokyo Shoseki n.d., Grade 6, p. 11)

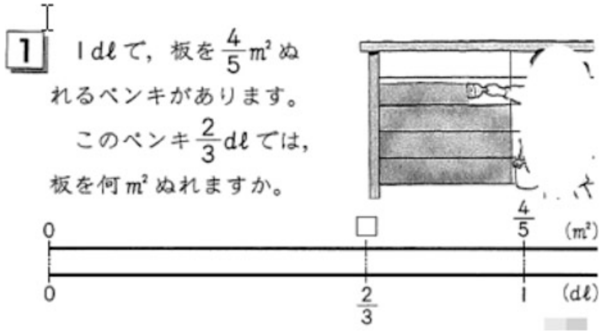

Fig. 7 The opening problem in the multiplication of fraction unit in the 1989 edition (Tokyo Shoseki n.d., Grade 6) includes a double number line diagram: "With 1 *dl* of paint you can paint $\frac{4}{5}$ m^2 of boards. How many m^2 can you paint with $\frac{2}{3}$ *dl* of paint?" (p. 7)

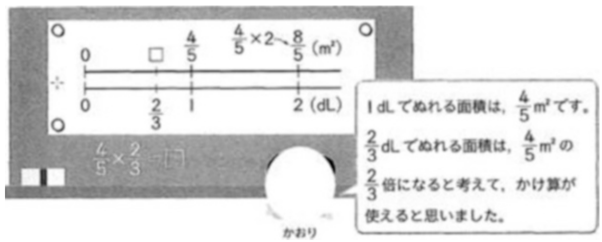

Fig. 8 A hypothetical student explains why she thought the problem can be solved by $\frac{4}{5} \times \frac{2}{3}$ in the 2012 edition of the textbook (Tokyo Shoseki n.d., Grade 6, p. 24)

The 2000 edition of Tokyo Shoseki series included a new feature entitled "Let's write learning reflection," in Grades 4 and above. This feature was then expanded in the 2012 edition to "Record of Learning: My Math Notes," starting in Grade 3. This feature encouraged students to keep a record of their learning in their notebooks so that they can always look back. In particular, students are encouraged to record lesson date, problem, own idea, other students' ideas, and summary in their notebook. In addition, students are asked to reflect on their own learning and write a brief journal entry.

In summary, the major changes identified by Watanabe's (2014) analysis of Tokyo Shoseki's textbooks are (1) inclusion of multiple-solution approaches and (2) types of representations and ways they are used. In addition, in the two most recent editions, the series include features that encourage students to develop good notebooks. These changes are consistent with the structured problem-solving (Stigler and Hiebert 1999) approach to mathematics teaching. Watanabe (2014) also noted that the changes between two successive editions were generally small except between the 1977 and 1989 editions.

2 Research Question

The brief literature review on problem-solving in Japanese mathematics education and the transformation of Japanese elementary school mathematics curriculum clearly showed there have been significant changes in Japanese mathematics education in the 1980s. Takahashi (personal communication) noted that it was during the 1980s when Japanese mathematics teachers developed "structured problem-solving" (Stigler and Hiebert 1999) style of mathematics teaching. Although the analysis of research trends, in particular practitioner research, suggests that Lesson Study played an important role in transforming Japanese mathematics instruction in the 1980s, if or how Lesson Study impacted the transformation of Japanese curriculum materials is not quite as clear. Thus, a study was conducted to examine various Lesson Study artifacts to identify any potential evidence that suggests Lesson Study may have impacted changes in Japanese elementary mathematics textbooks. Because problem-solving became a major focus among Japanese mathematics educators in the 1980s, and also because the prior study of elementary mathematics textbooks suggested that there were major differences in the textbooks from the 1977 edition to later editions (Watanabe 2014), it was decided to focus the study on Lesson Study artifacts from the 1980s. The specific research question investigated is as follows:

> What evidences are there in various Lesson Study artifacts from the 1980's that may suggest the impacts of Lesson Study on Japanese elementary school mathematics textbooks?

3 Methodology

Addressing the research question will require a multifaceted approach, including, perhaps, interviews with textbook editors and authors. However, identifying and interviewing people who were involved in textbook revisions 30 years ago was not feasible for a number of reasons. In particular, the 1989 edition listed over 30 names as members of the authoring team, and it was not possible to identify who actually wrote the units that were analyzed in Watanabe (2014). Therefore, as an initial step of the study, we decided to conduct a document analysis (Bowen 2009) of Lesson Study artifacts from the 1980s. Moreover, we decided to analyze those artifacts through the lenses of the three major changes identified by prior analysis of the textbook series: (1) use of multiple-solution approaches, (2) types and roles of representations, and (3) student notebook and journal writing.

The selection of documents to be included in this study was somewhat limited. Again, obtaining artifacts from Lesson Study activities over 30 years ago proved to be difficult. As a result, the documents included in the study were convenient samples. However, they were different types of documents, providing a range of data sources.

The first set of documents were annual *Kenkyuu Kiyou* (research reports) produced by Setagaya Elementary School attached to Tokyo Gakugei University from

1986 through 1989. These reports were produced at the end of each academic year (1985–1986 through 1988–1989), and they contained an overview of the school-based research and subject-specific research reports. Each subject report would typically include one or more research lesson plans and the summary record of the lessons. For the purpose of this study, only the school research overview and research reports by the mathematics group were analyzed.

The second document was a book published by a public elementary school in Tokyo (Sakamoto Elementary School 1983). This book is the final report of the Lesson Study conducted by this elementary school in the 1981–1982 and 1982–1983 school years, with a research grant from the MOE. The research theme was "Developing materials and instructional approach that foster students' problem solving capacity." Fourteen teachers and a school nurse, along with both the principal and vice principal, were involved in the writing of this book. The book consisted of four sections: (1) importance of instruction based on problem-solving, (2) key ideas for instruction based on problem-solving, (3) summaries of 12 research lessons, and (4) instruction based on problem-solving and students' reasoning.

The third document set obtained was the journal *Atarashii Sansuu Kenkyuu* (Elementary Mathematics Teaching Today, hereafter *ASK*) published by Society of Elementary Mathematics Education. This publication included a regular feature entitled "Lessons of the month." An article in this feature typically included a research lesson proposal similar to what is typically prepared for a research lesson, summary of the lesson implementation, and either commentary by an invited expert or a report of a post-lesson panel discussion including an invited guest (or guests). Each issue of the journal usually included one article for each of the elementary grades (Grades 1 through 6). For this study, we selected Grades 5 and 6 articles in the 1982 issues. Altogether, there were 21 articles (11 Grade 5 and 10 Grade 6) that were reports of a research lesson. One Grade 6 article was a record of a group of 13 Grade 6 teachers collaboratively planning a research lesson.

4 Findings

The findings of the analysis are organized according to the three specific changes identified in the earlier study.

4.1 Use of Multiple-Solution Approaches

Of the 21 research lesson reports from *ASK*, 10 lessons followed the pattern of "structured problem-solving" by having students share multiple-solution approaches. For example, in a Grade 5 lesson on multiplication of decimal numbers, the teacher asked students to determine the price of 2.8 m of ribbon when the price of

1 *m* of the same ribbon is 150 yen (Enokizono 1982). The report showed that the following five different approaches were shared and discussed.

1. 2.8 *m is* 2 *m and* 0.8 *m*
   $150 \times 2 = 300$
   $150 \div 10 = 15$ $15 \times 8 = 120$
   $300 + 120 = 420$
2. 2.8 *m is* 2 *m and* 80 *cm*
   $150 \times 2 = 300$
   $150 \div 100 = 1.5$ $1.5 \times 80 = 120$
   $300 + 120 = 420$
3. $150 \times 280 \div 100 = 420$
4. $2.8 \times 10 = 28$ $150 \times 28 = 4200$
   $4200 \div 10 = 420$
5. $150 \div 10 = 15$
   $15 \times 28 = 420$

(Enokizono 1982, p. 54. Translation by Watanabe)

During the whole class discussion, another student shared an idea based on the fact that 2.8 *m* is 0.2 *m* less than 3 *m*. Commenting on this lesson, Nakajima (1982) noted that discussing various ideas in a lesson is educationally worthwhile. However, he noted, we should recognize that orchestrating students' ideas to approach the goal for the lesson was very challenging.

Of the remaining 11 lessons reported in *ASK*, 9 of them followed the basic structured problem-solving lesson framework. However, instead of sharing a variety of solution approaches and then comparing and contrasting them, the classroom teacher orchestrated the whole-class discussion, bringing out different ideas from students through carefully timed questions. For example, in a Grade 5 research lesson on dividing a fraction by a whole number, students were asked to think about the question, "How many *l* of milk will each person get if $\frac{1}{3}$ *l* is shared fairly between 2 people?" The teacher first asked students if anyone can solve the problem by using diagrams like an area model or number line. After students shared how they could solve using those models, the teacher wondered aloud, "I wonder if we can figure out $\frac{1}{3} \div 2 = \frac{1}{6}$ using mathematical expressions and equations." In the previous lesson, students have dealt with the case where the numerator of the dividend is divisible by the divisor, and at the beginning of the research lesson, they noted that the difference between the prior lesson and the research lesson was that the numerator of the dividend is not divisible by the whole number. Students then realized that they could utilize equivalent fractions to make the numerator of the dividend divisible by the divisor, and the lesson eventually concluded with the formula $\frac{a}{b} \div c = \frac{a}{b \times c}$.

Sakamoto Elementary School (1983) discussed seven key ideas for instruction based on problem-solving. One key idea they discussed is allowing students to think and try freely. Further elaborating on this idea, the authors stated that "freely" means allowing students to use their own ideas as they see fit without any constraint. All of the 12 research lessons reported in the book followed the structured problem-solving style instruction with multiple-solution approaches.

In the research reports by Setagaya Elementary School, there were seven research lesson reports altogether. All seven lessons follow the structured problem-solving framework, with students sharing multiple-solution approaches. The research focus of this elementary school in the 1985–1989 school years were on lessons in which students can enlighten each other. They argued that in order to support students to enlighten each other, lessons needed to involve conflicting ideas coming out from students. The mathematics group at the school identified that one of the key features of a "good" task was that there are a variety of possible solution approaches, thus making it likely to bring up conflicting ideas during a lesson. Moreover, they consider characteristics of ideal students to include the ability to clearly articulate the similarities and differences of their own ideas and those of their classmates and the disposition to incorporate good ideas even if they are different from their own ideas. Thus, it is not surprising that the group's research lessons would involve multiple-solution approaches and extended discussion on those approaches.

These findings suggest that Japanese teachers in the 1980s considered discussing multiple-solution approaches during a lesson, whether they were generated by students during the independent problem-solving time or through whole-class discussion, as a productive approach to mathematics teaching. However, as Nakajima (1982) pointed out, this way of teaching is challenging, thus requiring research through Lesson Study. What was not clear is whether or not this idea originated in Lesson Study because this approach was already in use in the early 1980s. In order to identify the origin of this idea, we need to examine Lesson Study artifacts before 1980.

4.2 Types and Roles of Representation

Because the specific representations on which we are focusing, double-sided number line and double number line diagram, are typically used in multiplication and division of decimal numbers and fractions, we focused our analysis of research lesson reports on those topics. There were six articles in *ASK*, one research lesson report in Sakamoto Elementary School (1983), and one research lesson report in Setagaya Elementary School (1986) on these topics.

One of the six research lessons in *ASK* was on dividing a fraction by a whole number when the numerator of the dividend is not divisible by the divisor (Takayanagi 1982). In this particular lesson, even though the problem in the lesson involved quantities from two different measure fields, only a single number line was used to represent the dividend. The class then equally divided the segment representing the dividend into the number of equal parts indicated by the divisor. Thus, neither double-sided number line nor double number line diagram was used in the lesson. Of the remaining seven research lesson reports on these topics, four lessons in *ASK* and one by Setagaya Elementary School (1986) used only double-sided number lines in their lessons. One research lesson reported in Sakamoto Elementary School (1983) used only double number line diagrams. Finally, one article, which was the record of research lesson planning by 13 Grade 6 teachers, included both types of representations.

Overall, of the eight research lessons, six focused on helping understand what it means to multiply or divide by decimal numbers or fractions. Those lessons were positioned at the very beginning of the respective unit, usually the first or the second lesson of the unit. The other two lessons focused on helping students develop ways to calculate the indicated operation, and those lessons followed immediately after the initial lesson focusing on the meaning of operations. Thus, all eight lessons used diagrams, single number line, double-sided number line, or double number line diagram, in the very early part of a unit. This is a stark contrast to the way the double-sided number line was used in the 1977 edition of the textbook. However, Kikuchi (1982) questioned the usefulness of number lines and their variations in helping students understand why division by a fraction was the appropriate operation. Kikuchi's comment indicated that the idea of using either double-sided number line or double number line diagram as a tool to help students understand multiplication and division by decimal numbers or fractions was still in the germination stage in the early 1980s. However, it is likely that Japanese teachers continue to investigate the effectiveness of number line based model in Lesson Study throughout the 1980s, which may have resulted in its inclusion in the early part of a unit in the 1989 edition of the textbook.

4.3 Student Notebook and Journal Writing

The only Lesson Study artifacts that explicitly discussed student notebook and journal writing were the research reports by Setagaya Elementary School (1987, 1988, 1989). Starting in the 1987 research report, the mathematics group at the elementary school began to discuss the use of students' journals. In the 1987 report, the group listed the three purposes for having students write reflective journals at the end of a lesson:

1. Assessing students' understanding of the problem posed by the teacher
2. Assessing the degree of mutual enlightenment among students
3. Assessing students' understanding of the lesson goals

Then, in the group's writing in 1987 through 1989, they discussed what they learned from examining students' reflective journals. Thus, it appeared that the initial interest in having students write reflective journals was as a tool for an effective assessment. However, they also noted that students' reflections were useful for students to clarify their own ideas as well as realizing values in other students' ideas. In the 1989 report, the group reported two ways to assign students to write reflections: (1) as an activity to pursue mathematically valuable ideas to solve the assigned task and (2) as a self-assessment activity through reflecting on problem solutions.

As mentioned above, although student notebook and reflective journal writing was a prominent feature in the research report by Setagaya Elementary School, none of the other Lesson Study artifacts included any discussion on this theme. Thus, whether or not Lesson Study influenced the creation of a new feature in the 2000 and 2012 editions of the textbook series could not be answered from the current analysis.

5 Discussion

The analysis of the three focal points resulted in three distinct patterns. The use of multiple solutions in virtually all research lessons discussed in the Lesson Study artifacts suggests that there was a consensus among Japanese mathematics educators on the effectiveness of this approach. Whether or not Lesson Study influenced textbook revisions cannot be concluded definitively from the current analysis. However, in the Tokyo Shoseki textbook series, multiple solutions did not appear until the 1989 edition. Thus, it is possible to argue that Japanese textbooks began to incorporate multiple-solution approaches to reflect the consensus of Japanese teachers.

The use of double-sided number line or double number line diagram at the beginning of multiplication and division of decimal numbers or fractions units was a different story. As described, in the early 1980s there was still some debate whether or not those representations were useful in helping students understand the meaning of multiplication or division by decimal numbers and fractions (Kikuchi 1982). However, Lesson Study groups throughout Japan continued to explore the idea during the 1980s. Perhaps such persistent experimentations eventually led to the consensus among Japanese educators, which resulted in the changes in textbooks as double number line diagrams are commonly seen in more recent editions. Further investigation is necessary to clearly answer this question.

Finally, the treatment of student notebook and journal writing did not appear to be a common or widespread research question for Lesson Study teams in the mid- to late 1980s. However, because the only Lesson Study artifacts that were available for the period was from one elementary school's research reports, further investigation is needed. We believe it is reasonable to conjecture that the mid- to late 1980s was the germination, or perhaps the incubation, period for this idea among Japanese teachers, just like the use of double-sided number line and double number line diagram to support students' understanding of the meaning of multiplication and division by rational numbers. Further analysis of Lesson Study artifacts from the 1990s might show whether or not Lesson Study was an impetus for Tokyo Shoseki to start the new feature in the 2000 edition of their textbook.

Although we cannot definitively say that Lesson Study influenced textbook revisions in Japan based on the current analysis alone, as Murata and Takahashi (2002) noted, teachers are hired to write and revise textbooks in Japan. It is also likely that those teachers who take leadership roles in various Lesson Study groups are more likely to be those teachers who would be selected by publishers. Thus, it is possible for individual teachers to try to incorporate their own ideas, or ideas from their Lesson Study groups, in their writing. However, they still need to be accepted by the editorial group before they could be adopted into a textbook.

Use of Lesson Study practitioners as members of textbook authoring team is one structural feature of Japanese education system that may allow Lesson Study to influence textbook revisions. Other factors to consider are threefold. The first feature is simply the fact that the National Course of Study, i.e., the intended curriculum, is revised regularly. In the United States, there is no plan or system for revising the Common Core State Standards (CCSS). In fact, some states have begun to make their own revisions and thus weakening the "common" part of the CCSS. Before the CCSS, each state developed its own standards, but there was no regular timetable for their revisions. Under such condition, it may be difficult to be motivated to think about ideas that might contribute to an improvement of curriculum materials.

Another important feature seems to be the research grants the MOE awards to schools to examine different ideas. Sakamoto Elementary School (1983) was a result of 2 years of Lesson Study funded by such a grant. The MOE continues to provide such grants even today. Of course, not all schools publish their findings in a book, but virtually all schools will hold a public research open house toward the end of the grant period to disseminate their learning in the form of several research lessons. Such open houses will provide not only teachers but also textbook editorial staffs the opportunity to learn about new ideas developed by Lesson Study groups.

Finally, another important feature in the Japanese education system that might promote Lesson Study's influence on curriculum materials is the existence of university-attached schools, called *fuzoku* schools, like Setagaya Elementary School attached to Tokyo Gakugei University. Every Japanese national university with teacher education programs has a collection of *fuzoku* schools. The three missions of *fuzoku* schools focus on (1) education of their students, (2) support of teacher education programs of the affiliated university, and (3) leadership in professional development of teachers in the region. At a *fuzoku* school, teachers engage in their own school-based Lesson Study where research lessons are for their own faculty. However, they also hold a public research open house once a year where teachers from the region, and sometimes outside of their own regions, come to participate in research lessons and discussion. Many *fuzoku* schools also hold content-specific research open houses. Murata and Takahashi (2002) described that one purpose of Lesson Study at these cross-district Lesson Study events is to promote new ideas. It is possible that the idea of student notebook and reflective journal writing was a new idea one such *fuzoku* school was promoting in the mid- to late 1980s, which eventually spreads to other Lesson Study groups and finally influenced the curriculum materials.

6 Limitations and Future Research

Although Lesson Study artifacts examined in this study seem to suggest that textbook revisions might have been influenced, or at least informed, by various Lesson Study activities, the conclusion is only speculative. The evidences are rather circumstantial, and additional investigation is clearly needed to more definitively link Lesson Study and curriculum revisions in Japan. Even if interviewing the people who were involved in textbook writing since 1980 is not feasible, we can certainly gather additional Lesson Study artifacts. Additional articles from *ASK* in the late 1980s and 1990s might be helpful to determine whether or not the idea of making better use of student notebook and reflective journal writing has become more widespread. Research reports from other *fuzoku* schools and other schools who received MOE grants will be useful. However, collecting these documents is rather difficult for scholars outside of Japan. Therefore, we are seeking collaboration from Japanese scholars for a more in-depth examination.

References

APEC Human Resources Development Working Group (n.d.). *Lesson study overview*. Retrieved March 15, 2016, from http://hrd/apec.org/index.php/Lesson_Study_Overview. (no longer available).

Bowen, G. A. (2009). Document analysis as a qualitative research method. *Qualitative Research Journal, 9*(2), 27–40.

Enokizono, K. (1982). Shousuu no kakezan (multiplication of decimal numbers). *Atarashii Sansuu Kenkyuu, 134*, 53–55.

Hino, K. (2007). Toward the problem-centered classroom: Trends in mathematical problem solving in Japan. *ZDM, The International Journal of Mathematics Education, 39*, 503–514.

Japan Society of Mathematical Education. (2001). Sansu jugyou no houhou ni kansuru chousa no kekka (Results of the survey on mathematics teaching approaches). *Arithmetic Education, 83* (2), 31–42.

Kikuchi, H. (1982). Koment: Bunsuu no warizan (comment: Division of fractions). *Atarashii Sansuu Kenkyuu, 137*, 57.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Lewis, C., & Tsuchida, I. (1997). Planned educational change in Japan: The case of elementary science instruction. *Journal of Education Policy, 12*, 313–331.

Lewis, C. & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: how research lessons improve Japanese education. *American Educator, winter*,14-17 & 50-52.

McDougal, T., & Takahashi, A. (2014). *Teaching mathematics through problem solving*. Retrieved from http://www.nais.org/MagazinesNewsletters/ITMagazine/Pages/Teaching-Mathematics-Through-ProblemSolving.aspx

Murata, A. & Takahashi, A. (2002). Vehicle to connect theory, research and practice: How teacher thinking changes in district-level lesson study in Japan. In Editors (Eds.), *Proceedings of the twenty-fourth annual meeting of North American chapter of the international group of the Psychology of Mathematics Education* (pp. 1879–1887). Athens: USA.

Nagasaki, E., & Senuma, H. (1986). *Suugaku kyouiku ni okeru mondai kaiketu ni tuiteno kenkyuu no doukou (1) (Research trends on problem solving in mathematics education)*. Tokyo: National Institute for Educational Research.

Nakajima, K. (1982). Koment: Shousuu no kakezan (comment: Multiplication of decimal numbers). *Atarashii Sansuu Kenkyuu, 134*, 56.

Nakano, K. (1939). *Sanjutsu no jugyou kenkyu (lesson study in elementary school mathematics)*. Tokyo: Kenbunkan.

National Council of Teachers of Mathematics. (1980). *An agenda for action*. Reston: NCTM.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership, 61*(5), 61–66.

Sakamoto Elementary School. (1983). *Sansuuka mondai kaiketu nouryoku no ikusei (fostering students' problem solving capacity in elementary school mathematics)*. Tokyo: Meiji Tosho.

Setagaya Elementary School Attached to Tokyo Gakugei University (1986) *Kenkyuu Kiyou* (vol. 26). Tokyo.

Setagaya Elementary School Attached to Tokyo Gakugei University (1987) *Kenkyuu Kiyou* (vol. 26). Tokyo.

Setagaya Elementary School Attached to Tokyo Gakugei University (1988) *Kenkyuu Kiyou* (vol. 26). Tokyo.

Setagaya Elementary School Attached to Tokyo Gakugei University (1989) *Kenkyuu Kiyou* (vol. 26). Tokyo.

Stigler, J. W., & Hiebert, J. (1999). *Teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*, 513–526. https://doi.org/10.1007/s11858-015-0752-x.

Takayanagi, M. (1982). Gakushuuji no hyouka: Bunsuu no warizan (assessment during a lesson: Division of fractions). *Atarashii Sansuu Kenkyuu, 134*, 54–56.

Tokyo Shoseki (n.d.) *Atarashii sansu* (New elementary school mathematics). Tokyo: Tokyo Shoseki Publishing.

Travers, K. J., & Westbury, I. (1989). *The IEA study of mathematics I: Analysis of mathematics curricula*. New York: Pergamon Press.

Tsubota, K. (2004). *Sansu jugyo kenkyu saikou (re-thinking lesson study in elementary school mathematics)*. Tokyo: Toyokan Shuppan.

Watanabe, T. (2014). Transformation of Japanese elementary mathematics textbooks: 1958–2012. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 199–216). Cham: Springer.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), *Inquiry into Mathematics Teacher Education. Association of Mathematics Teacher Educators (AMTE) Monograph Series* (Vol. 5, pp. 131–142). San Diego: AMTE.

Yokosuka, K. (1990). *Jugyo kenkyu yougo jiten (dictionary of lesson study terms)*. Tokyo: Kyouiku Shuppan.

Yoshida, M. (1999). *Lesson study: An ethnographic investigation of school-based teacher development in Japan*. Doctoral dissertation, University of Chicago.

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An Analysis of Chinese Lesson Study from Historical and Cultural Perspectives

Xuhui Li

Contents

Abstract This chapter first reviews the origin and development of Lesson Study in China from the late nineteenth century to the present and identifies three main historical periods: (1) 1896–1949 when the initial forms of Lesson Study activities were first introduced and practiced, (2) 1949–1999 when Lesson Study became an institutionalized routine practice among prospective and practicing teachers, and (3) the year 2000 and beyond when Lesson Study in China started to take many more forms and developed its own distinguishing characteristics. Along the cultural dimension, this chapter introduces three sets of traditional beliefs and principles regarding teaching and learning that could help to make sense of Chinese Lesson Study: (1) It is important to learn exemplary knowledge and practice from masters and experts; (2) effective teaching and learning must blend profound theories with cycles of deliberate practice and refinement; and (3) learning also takes place among learner peers through mutual observations and discussions. Some of the challenges of implementing Lesson Study in China during the reform era and in international contexts are discussed. Topics for potential future research are outlined.

Keywords Chinese Lesson Study · Historical development · Cultural roots

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_11

1 Introduction

Lesson Study has been a fundamental and essential form of teaching research activity in Chinese schools for over a century (Chen and Yang 2013). It originated at the beginning of the twentieth century as a component of the newly adopted normal school education (i.e., prospective teacher preparation) system and soon expanded to the professional development of practicing teachers in elementary and secondary schools. The simplest form of Lesson Study involves a lesson carefully planned and implemented by an instructor and observed by supervisors and fellow teachers, then followed by the instructor's explanations and supervisors' and fellow teachers' questions, comments, and suggestions. Although the Chinese phrase for Lesson Study,课例研究, did not come into existence until around the year 2000, various forms of Lesson Study activities have taken place in China since the 1900s under broader names such as teaching research (教學研究), pedagogical research (教學法研究), and instructors' research (教授研究) or organized by the more precise name of teaching critique or lesson critique (授業批評). From the 1950s on, it has become a core routine activity of teaching research groups (教研組) in every school in China, supported by a nationwide, multitier teaching research network (Wang 2013a).

More recently, Lesson Study has taken more diverse forms and played crucial roles in implementing curriculum reforms, transforming teaching practice, preparing and developing teachers, and promoting student learning (Gu and Wang 2003a, b; Huang et al. 2016; Yang 2009; Yuan and Li 2015). Researchers have summarized the unique characteristics as well as some major differences between Lesson Study activities in China and those in Japan (Huang et al. 2017a, b; Huang and Han 2015). To better understand those distinguishing characteristics and major differences, it is crucial to examine the historical development and cultural foundations Chinese Lesson Study. Historical reviews (such as Chen and Yang (2013) and Yang and Ricks (2013)) are made as part of the background or introduction of research and hence are very sketchy and typically focus on the modern period (the 1950s and after) only. Very recently researchers have started to explore the underlying cultural roots (Chen 2017) and theoretical interpretations (Huang et al. 2017a, b) of Lesson Study in China. New research in this direction that makes more reference to the historical development of Lesson Study is desired.

Based on findings from the aforementioned research and other existing studies, this chapter aims to more systematically address two fundamental research questions:

1. How did Lesson Study begin and develop into an integral part of Chinese teachers' professional practice?
2. What cultural beliefs and practices in China best explain the characteristics of Chinese Lesson Study?

Results from this study could help educational researchers and practitioners better understand the multifaceted context in which Lesson Study originated, adapted, and evolved in Chinese school systems and make better sense of its popularity, characteristics, and influences.

2 Historical Development

This part addresses the first research question: How did Lesson Study begin and develop into an integral part of Chinese teachers' professional practice?

The author reviewed and cross-referenced literature in the following categories:

1. Research publications specifically discussing the historical development of Lesson Study and, more broadly, teaching research activities in elementary and secondary schools in China. The majority of these publications focus on Lesson Study activities in the era of the People's Republic of China (since 1949).
2. Research publications and archives of policy documents on the history of teacher education in China, including teacher preparation and professional development.

The result of such a review process is a set of three main time points: 1896, 1949, and 1999. As explained later in this chapter, each of these is a major turning point in the history of (teacher) education and also a milestone for the development of Lesson Study in China. These three time points divide history into four periods (before 1896, 1896–1949, 1949–1999, and beyond 1999). Below is a brief introduction of the history before 1896, followed by detailed portraits of the backgrounds and major events related Lesson Study during each of the three modern periods.

Historically, there were no teacher education institutions in China before 1896. Teachers were educated individuals who went through several years of private tutoring or were self-taught (Wang 1997). Many students and scholars spent years studying classic works and trying to achieve high marks on high-stakes civic service examinations (科舉考試). Consequently, their paths to administrative positions and to teaching careers often overlapped: many current or retired government officials were themselves scholars and/or educators of varying capacities (Wang 1997). The qualification and reputation of a teacher correlated strongly with his administrative rank, seniority, and expertise; moral and academic achievements; and depth of his knowledge and understanding of truths, laws, doctrines, principles, and reason. Since traditional teaching and learning were mostly accomplished through lectures, and occasionally heuristics and discussion, a novice teacher would primarily acquire instructional techniques based on his own prior experiences as a learner, through observing and imitating his teachers' instructional procedures and behaviors, and through reflecting on his teachers', and later his own, teaching practices.

Modern school systems, including teacher preparation institutions and professional development programs, did not come into existence in China until the end of the nineteenth century when the imperial Qing Dynasty (清朝, 1636–1912) underwent dramatic political reforms and social, cultural, and technological revolutions. A major influence came from Japan where the Meiji Restoration (明治維新) in 1868 brought forth successful modernization, the earliest teacher training institutions were established, and the first set of regulations on normal schools were promulgated between 1872 and 1880 (Ferguson 1985; Makinae 2010). The modern educational system, institutions, and regulations established in China between the late nineteenth century and the 1910s more or less resembled their Japanese counterparts (Abe 1987; Li 1998).

Primitive forms of Lesson Study activities were practiced as an integral component of prospective teacher preparation programs and soon expanded to professional development programs for practicing teachers. The major phases of Lesson Study development are discussed below, along with an analysis of the historical context for each phase.

2.1 1896–1949: The Introduction and Early Development of Lesson Study

2.1.1 The Official Adoption of Basic Lesson Study Practice

The year 1896 witnessed the foundation of the Nanyang Public Institute (南洋公學) in Shanghai following an imperial edict (Wang 1997). It included a Normal School (師範院) for teacher preparation and an affiliated elementary school as the designated venue for prospective teachers' on-site teaching practice. This was the very first teacher preparation school in China and marked the beginning of institutionalized teacher education.

In 1904, the imperial government issued the *Presented School Regulations* (奏定學堂章程), the first set of educational laws formally adopted and implemented nationwide in the modern history of China (Ju et al. 1994; Shu 1981). It included several sets of regulations for teacher education and outlined the structures of normal schools at elementary and advanced levels. These normal schools were expected to create affiliated elementary and secondary schools that would serve multiple purposes: (1) as model public schools, (2) as a venue for conducting research related to teaching and learning, and, most importantly, (3) for prospective teachers from the normal schools to fulfill their teaching practice (教育實習in traditional Chinese characters and 教育实习in simplified Chinese characters) requirements (Ju et al. 1994; Shu 1981).

The regulations on Elementary Normal Schools state: "During their teaching practice at the elementary school affiliated with an elementary normal school, student teachers get to practice principles and strategies regarding how to educate young children…they take turns teaching children. Faculty members from the normal school and the principal and teachers from the elementary school must facilitate and monitor prospective teachers' teaching, commenting on whether it is appropriate. If necessary, they themselves should teach lessons to demonstrate model teaching in front of prospective teachers" (Shu 1981, p.671. The original text was in Chinese, translated into English by the author of this chapter).

This is the earliest educational regulation that explicitly addressed two primitive types of Lesson Study activities during student teachers' teaching practice: (1) experienced teachers observe and provide feedback on lessons taught by student teachers and (2) experienced teachers demonstrate to student teachers how to teach. These two types and their variations have been practiced in schools in China until the present day.

Between 1896 and 1903, fourteen normal schools were founded across China (Ruan 2012). After the introduction of the *Presented School Regulations* in 1904, the country saw a boom in normal school openings. By 1909, a total of 415 normal schools were in operation across all 22 provinces and 4 administrative districts in China, with 28,572 students enrolled (Chen 1969).

In early 1912, imperial China was formally brought to its end. The Ministry of Education of the newly established Republic of China released a series of laws on normal education, upgrading teacher education institutions and mandating the creation of affiliated elementary and secondary schools to promote public education and support prospective teachers' teaching practice (a.k.a. student teaching) (Shu 1981).

The next three decades witnessed a few more rounds of reform and revision of school education and teacher education policies (Li 1998; Wang 1997; Yan 2003). A modern teacher education system was established, comprised of normal schools, normal colleges, teacher education departments or colleges in comprehensive universities, and designated teachers' universities. Along with the steady increase in the number of students pursuing teacher education credentials, many textbooks from developed countries were translated into Chinese and adopted by schools, colleges, and universities in China. Popular pedagogical methods, both general and subject-specific, were also widely adapted and implemented. These imposed greater demands on teacher preparation and professional development programs which needed to help prospective and practicing teachers quickly adapt to the changes and effectively implement new theories, curricula, and pedagogical practices. Faculty councils and other teacher professional organizations at local, provincial, and national levels also shared responsibilities. As one type of school-based teaching research activity, Lesson Study became an important platform on which teachers could collectively demonstrate or observe teaching; experiment, discuss, and reflect on various ideas and strategies; and revisit and revise their own beliefs and practices.

2.1.2 Lesson Study in Written Policies and Published Guidelines

In some rigorous teacher education programs, Lesson Study activities were explicitly regulated by written policies on student teaching. For example, Beijing Normal University, the first public university dedicated to teacher education in China, issued *Teaching Practice Regulations* in 1924 to guide prospective teachers' teaching practice at its affiliated secondary school, including the lesson critique (授業批評) meetings: "During teaching practice, student teachers must consult with practicing teachers who teach the same subject at the secondary school regarding the beginning and ending of the lessons in the textbook, and discuss teaching methods. The practicing teachers have the obligation to advise and support the student teachers. Before teaching a lesson, a student teacher needs to make a lesson plan and submit it to the Lead Teacher who oversees the subject at the secondary school for review and advice. After teaching, the student teacher, fellow student teachers, faculty supervisor from the normal university, and the Lead Teacher of the subject from the secondary school should hold a critique meeting. Each subject group should hold general critique meetings on a regular basis to summarize results from the individual critique meetings during the preceding period" (Wang 2013b, p. 90).

Some scholars summarized general principles and procedures for prospective teachers' teaching practice, providing guidelines for student teaching and lesson critique meetings. For example, Hu (1932) described two basic types of lesson critique:

1. Practice teaching critique meetings (批評試教會). The student teacher who is going to teach the lesson and be observed and critiqued prepares copies of his/her lesson plan in advance. These are distributed at the meeting so participants know what to expect, can compare with what actually happens, and can make comments.
2. Weekly discussion meetings (每周討論會). On a specific day of each week, the Lead Teacher of each grade at the affiliated school calls a meeting among all student teachers. The student teachers reflect on their teaching practice during the past week; the practicing teachers then make comments on the strengths and weaknesses of the lessons, with "honest attitude and solid points of view" (p. 206).

2.1.3 Lesson Study Activities Among Prospective Teachers

Students enrolled in teacher education programs were required to take a sequence of courses in the field of education. Teaching practice (a.k.a., student teaching) (教育實習, sometimes also called實事授業or 實事練習) typically took place in the last year of the program at the elementary or secondary school affiliated to the normal school, college, or university the prospective teachers were attending. During teaching practice, prospective teachers participated in two basic forms of Lesson Study activities: (1) observing and critiquing lessons taught by their fellow student teachers and (2) teaching lessons in their own specialized fields/subjects and then being critiqued by fellow student teachers, practicing teachers, supervisors, and administrators. Both forms were often enacted as a basic activity of various teaching research councils and their routine meetings.

Below is an example of a complete Lesson Study activity for prospective teachers (Deng and Li 1995). On April 25, 1914, the Pedagogical Research Council (教授法研究會) of the Elementary School Affiliated to Beijing Women's Normal School held its commencement ceremony as well as its first meeting. In the opening speech, the Chairman, Mr. Jiefan Zhou, discussed the mission of the council and the purposes of its meetings: "The discussions should focus on whether the pedagogy is effective. Pedagogy could stay the same or change over time. After the discussion is done, the effective aspects of the pedagogy should be maintained, whereas the less effective aspects need be modified" (Deng and Li 1995, p. 247).

Mr. Zhou then outlined a standard procedure for critiquing lessons taught by a student teacher: (1) the student teacher comments on his or her own teaching during the lessons; (2) the participating teachers ask questions about the lesson; the student teacher responds. The following groups and individuals then take turns sharing their critiques: (1) fellow student teachers, (2) practicing teachers at the elementary school, (3) supervisors and teachers at the normal school, (4) the Lead Teacher at the elementary school, (5) the Director of the elementary school, and (6) the Chairman of the meeting.

Further, Mr. Zhou emphasized the intention and principles of critiquing: "The so-called critiques are merely ways to make comments and provide guidance. They are irrelevant to personal feelings. Whenever there is a doubt, questions should be raised. If questioning is not sufficient, discussions should follow. If questions and discussions do not involve any personal affection or disgust, advantage or disadvantage, then all we want from the critiques is to know what worked well or not so well. This is what we call research. Only by organizing the research meetings in this way, should we expect lasting effects without having to worry about seeing no reward. Let us all try our best" (Deng and Li 1995, p. 247).

Ms. Wenru Lin, a prospective teacher at the normal school, then took the stage and taught a mathematics lesson on division in a 4th grade arithmetic class. Specifically, the lesson focused on how to divide four-digit whole numbers by two-digit whole numbers with three-digit quotients. According to the lesson plan on record, the lesson included three main episodes:

1. *Preparation*. Ms. Lin first had the class review topics from previous lessons by practicing mental calculations on six division problems as well as solving two word problems. She then introduced the topic of the lesson: dividing four-digit numbers by two-digit numbers, where the tens place of the quotient is 0.
2. *Teaching next content*. First, Ms. Lin demonstrated two examples, $6479 \div 31$ and $5125 \div 25$, and then assigned five practicing problems for the class to work on. She asked pupils to read and interpret these problems before solving. Next, after walking around the class monitoring the pupils' progress, she asked selected pupils to show work on the blackboard and then asked the whole class to compare their own work with the work on the board. All mistakes were corrected.
3. *Application*. Ms. Lin showed two real-world problems: (1) 7112 rolls of cloth are equally distributed among 14 people, how many rolls does each get? (2) 8360 sheets of paper are equally split among 40 people, how many sheets does each get? She once again asked pupils to read and interpret the problems before solving, then asked selected pupils to show work on the blackboard, and had the whole class compare their own work with the work on the board, correcting any mistakes.

During the post-lesson critiques, Ms. Lin reflected on her own teaching of the lesson, feeling it was quite mechanical and in many ways did not follow basic principles for effective teaching. Student teachers and supervisors from the normal school and practicing teachers from the elementary school who were in attendance during the lesson then made comments on and discussed various aspects of Ms. Lin's teaching, such as the number of examples and exercises, the relationship between the pacing and student learning outcomes, whether the teacher should explain the word problems first, whether the content written on the board should be erased, how to motivate pupils, the need to pay attention to pupils' questions and provide feedback, time distribution across the lesson, etc. The participants also pointed out some of areas in Ms. Lin's teaching that needed improvement.

Toward the end of the meeting, the Chairman, Mr. Zhou, made some concluding remarks regarding how to help pupils develop mental calculation skills and how to better explain the examples and exercises.

2.1.4 Lesson Study Among Practicing Teachers

From early on, education administrations and organizations encouraged practicing teachers to engage in various kinds of research related to school teaching and learning. In July 1906, the Ministry of Education of the Qing government issued regulations on provincial Education Societies, their local branches, and disciplinary branches (各省教育會章程). Educational Research Societies were one of the main branches. Since then, these societies and branches had held annual meetings, promoting school-based research and sharing research findings. Such a tradition was continued and reinforced later in the Republic era. Teaching research council meetings were occasionally held as part of the annual events, where practicing teachers demonstrated new teaching methods and observed and commented on by the audience (e.g., the Sciences Teaching Research Council of the Education Society of Jiangsu Province organized a Lesson Study on the teaching of radios).

At the school level, the Association of Provincial Education Societies (各省教育總會聯合會) recommended in 1911 that teachers in each type and level of school form associations to conduct collaborative research (Ju et al. 1994). The Ministry of Education of the Republic of China issued two regulations in 1938 and 1941, requiring board members of each secondary school to chair the teaching research committee in their own specialty areas (各科教授研究會). They were to organize routine meetings among teachers in each subject to discuss research activities and results related to curriculum standards, teaching methods, textbooks, semester pacing schedules, extracurricular activities, teachers' continuing development, writing and discussing book and journal reports, etc.

Lesson Study was a special type of teaching research and was practiced in very organized ways. For example, all public schools in Zhejiang Province in the 1940s were required to have an On-site Teaching Critique Council (實地授業批評會). According to the regulations of the council (Wang 2013b), all teachers in each school were members of the council and were expected to participate in a monthly teaching critique meeting. For each meeting, one teacher was to be elected as the instructor, having other members visit his/her classroom on the meeting day. The instructor was required to distribute a detailed lesson plan to all members the day before the class as a reference for the classroom observation and teaching critique. The critique meeting would begin right after the lesson ended.

The regulations also outline a general procedure of a critique meeting: it starts with (1) the instructor's own summary and reflection of the lesson, followed by (2) the participants' questions, and (3) the instructor's responses. Then (4) all participants further critique the lesson by making comments and suggestions. The meeting ends with (5) the Chairman's critique and summary.

Besides these steps, the instructor had opportunities to respond to any participant's critique after the critique was completed. Each participant wrote notes about the main points of the critiques he/she made and submitted them to the Chairman after the meeting. The Chairman then compiles and edits all notes into a report, distributing it to the participants as a reference.

2.2 1949–1999: Lesson Study as a Main Form of Teaching Research Activities

During World War II, the education system in China was severely damaged due to the Japanese invasion and occupation from 1937 to 1945. After that it continued to suffer from the Civil War between 1945 and 1949. When the Communist government took over the mainland and established the People's Republic of China in 1949, it became imperative to reconstruct the entire country, including the education system. Increasing the quantity and improving the qualifications of practicing teachers throughout the country was one of the most urgent demands of reconstruction.

2.2.1 Adapting the Teaching Research System from the Soviet Union

Due to the success of the former Soviet Union in science, technology, engineering, and mathematics, as well as the long-time ally relationship between the Communist Parties in China and the Soviet Union, the new government of China decided to adapt the Soviet Union's educational infrastructures and major regulations regarding school education and teacher education. Most notable was the notion of a Pedagogy Group, which was introduced to all schools in the Soviet Union in 1938 and has since become a prevailing teachers' organization for collaborative research on teaching. It was adapted into the teaching research group in the Chinese system.

In March 1952, the Ministry of Education of China issued provisional regulations for schools across the country, requiring all secondary schools to set up subject-specific teaching research groups (學科教學研究組織, in short, 教研組), with the purpose of studying and improving teaching (Li 2014). Each group consisted of all teachers in the same subject group, and they held research meetings every other week to discuss the content, methods, and pacing schedule of teaching. In the meantime, elementary schools were required to hold Teaching and Training Research Meetings (教導研究會議) biweekly among all teachers to discuss teaching. If there were enough teachers in a school, they could organize teaching research meetings by subject-specific groups.

Meetings of the teaching research groups focused on three main themes (Li 2014):

1. In-depth analysis of textbooks, other instructional materials, and pedagogy (鉆研教材教法)
2. Collective lesson planning by teachers in the same group (集體備課)
3. Various basic forms of Lesson Study: observing an exemplary lesson taught by expert teachers or an experimental lesson involving new teaching strategies (觀摩課); teaching or observing a public lesson, followed by the audience commenting on the lesson (公開課); and mutually observing lessons taught by other teachers in the same group and providing feedback (互相聽課、評課)

An upgraded and detailed set of regulations on teaching research groups was issued in 1957. These regulations facilitated the rapid growth of a teaching research national infrastructure and personnel network which was coordinated at the provincial level and reached down to municipal, country, and school levels. By the early 1960s, teaching research in schools in China had developed into its own system, gradually moving away from Russian influences. Such development provided a strong foundation and support for teaching research groups and activities, including Lesson Study, and made them more professional, widespread, and long-lasting (Li 2014).

2.2.2 Examples of Group Lesson Planning and Lesson Study

During the 1952–1953 school year, the Mathematics Teaching Research Group at Jiading Secondary School (1953) in Jiangsu Province collectively conducted mathematics lesson planning according to the following procedure:

1. *Become knowledgeable of the curriculum content*. First, read carefully through a chapter or unit; figure out the major concepts and their connections. Then analyze each section and determine the pacing schedule. Work on all exercises in the textbook; select representative problems for teaching and/or homework assignment.
2. *Analyze the textbook*. Determine the objectives and expected learning outcomes. Determine the key points for teaching, including basic definitions, theorems, and formulas, laws of operations and critical points for problem-solving, and connections between the sections studied earlier and the one to be studied.
3. *Look for necessary reference materials*. To employ diverse teaching strategies and make the content relevant to real life, teachers look for more examples and teaching methods that may make it easier for the students to understand the content.
4. *Determine instructional procedure*. To achieve this, group members typically go through six steps:

    1. Ask probing questions to review prior content and prepare students for new content.

2. Streamline the instructional materials, choosing the best examples and representative, special, or difficult problems, making the various pieces coherent and logically connected.
3. Highlight key points in concepts and problem-solving.
4. Integrate moral and ideological education in the lesson.
5. Select and analyze homework assignments; an experienced teacher explains the main strategies for solving homework problems, expected student solutions, and estimated time needed
6. Prepare tools, models, and manipulatives for demonstrating.

5. *Exchange ideas*. Teachers teaching the same subject and using the same textbook share thoughts and experiences.
6. *Design lesson plans.* This step is based on the results from the first five steps and needs to be completed in compliance with the standard lesson plan form (pp. 37–40).

Another good example of mathematics Lesson Study was shared by the Mathematics Teaching Research Group at Anyang No.1 Secondary School in Henan Province (Suo 1953). In the fall semester of 1952, the group organized a 3-hour public lesson activity in a high school classroom. The topic was the relationship among the sides and angles of a triangle. Before the lesson, all teachers in the group discussed the lesson plan for the public lesson and made suggestions to the teacher who was chosen to teach the public lesson for maximizing teaching effectiveness.

After the lesson, students who participated in the lesson first discussed the aspects of the lesson that suited their needs, as well as the strengths and weaknesses of the teaching. Representatives of these students then joined the commentary and discussion meeting (評議會) along with all members of the Mathematics Teaching Research Group, administrators from the same school, and mathematics teachers from other schools.

The meeting started with the instructor’s self-critique, discussing how well his actual teaching was aligned with the lesson plan, the successes and failures of his teaching, and determining reasons for those successes or failures. Next, the student representatives summarized and reported opinions from all students in the earlier discussion. Then mathematics teachers shared their comments. The meeting ended after a brief summary by the organizer. By attending this meeting, participants had a chance to listen to different perspectives, and mathematics teachers all felt they benefited from this experience.

2.2.3 Revitalization and New Development of Lesson Study Between the Late 1970s and 1990s

The Cultural Revolution was launched in China in 1966. At the peak of this unprecedented political movement, schools, colleges, and universities were temporarily shut down nationwide. Even though they were later partially reopened, students spent much time engaging in social and political activities. Teacher preparation and professional development programs, as well as teaching research groups and activities, virtually came to a halt.

After the Cultural Revolution ended in October 1976, schools gradually shifted back to focus on academic teaching and learning. University entrance exams were administered again nationwide starting in 1977. The Ministry of Education revised regulations on elementary and secondary school education, reemphasizing the importance of teachers being effective in examining textbooks, designing or adopting instructional materials, and employing various teaching methods. Classroom teaching and learning were encouraged to focus on the two basics: basic knowledge and basic skills. Subject-specific teaching research groups were expected to play vital roles in improving the quality of teaching and learning (Li 2014).

By the early 1980s, collective lesson planning once again became a dominant form of teachers' teaching research activities, and master-apprentice peers arose as a new, special form of teaching research group. Experienced teacher mentors helped novice teacher mentees develop a more profound understanding of instructional materials, learn new teaching methods, and become more familiar with student learning characteristics. These mentor-mentee interactions were also combined with their observations of each other's classroom teaching and Lesson Study style comments and discussions (Li 2014). Such dynamics occurred not only among practicing teachers but also between mentor teachers and prospective teacher mentees during their teaching practice.

School education in China underwent a series of reforms in the late 1980s and 1990s after the milestone *Compulsory Education Law of the People's Republic of China* was promulgated (National People's Congress 1986). There were strong calls for replacing test scores and college admission rates with students' comprehensive qualifications and balanced development as the new focus of school education. Classroom teaching, learning, and teaching research group activities started to shift learners to the center stage of classrooms, aiming to stimulate learners' active learning, spontaneity, and creativity. Correspondingly, Lesson Study activities during this period emphasized more designing and teaching lessons based on student learning and understanding (Li 2014).

2.2.4 Lesson Study, Teaching Research, and Teachers' Professional Recognition and Ranking

Since the 1950s, various honorary titles have been created to reward outstanding teachers. Teaching public lessons have been tied to the selection and competition processes. For instance, the so-called *Backbone Teacher* (骨干教师) title emerged in 1962 for early- and middle-career teachers who demonstrated accomplishments and promise and had taken or were expected to take leadership roles (Jiang 2008). There were also Excellent Teacher (优秀教师) and Prominent Teacher (名师) titles for recognizing expert and master teachers for their expertise, achievements, and contributions. Basic criteria for earning one of these titles varied but typically included (1) winning top awards in teaching contests at municipal and higher levels or (2) having been selected to teach a public lesson or observation lesson at municipal or higher levels (Li et al. 2011).

Lesson Study played an even more crucial role after China established a nationwide contract-based professional title and appointment system for all elementary and secondary school teachers (Ministry of Education of China 1986a, b). To be eligible for renewing a current appointment or moving up to a higher rank, teachers are required to routinely participate in Lesson Study and other types of teaching research activities. They also need to demonstrate appropriate levels of instructional expertise through designing and teaching public lessons while being observed and evaluated by review committee members and fellow teachers (Huang et al. 2016).

Teachers are also required to participate in teaching contests to boost their credentials and become eligible for higher ranks (National Research Council, 2010). Such contests may be held at school, district, municipal, provincial, or national levels, organized by educational administrations or organizations. Participating teachers carefully design a lesson and execute it in front of an expert panel. They may teach to their own students or a class selected by the contest organizers and hosting institutions. Their teaching performance is typically judged on content accuracy, instructional coherence, the appropriateness of instructional goals and how well they are achieved, interaction with students, the use of technology, expressiveness and charisma, etc. Winners of the contests are recognized and awarded and may be qualified to advance to a contest at a higher level (Li and Li 2009).

Lesson Study and other teaching research activities also help teachers fulfill another requirement for retention and promotion: conducting research and publications on teaching. Many teachers write research reports based on their observations of lessons taught by other teachers or reflections on their own teaching.

Teachers at senior ranks are expected to organize teaching research activities and mentor teachers at junior and middle ranks. Lesson Study is a natural and productive platform to fulfill such responsibilities (Huang et al. 2016).

2.3 Teaching Research and Lesson Study in the Twenty-First Century

Entering the new millennium, the Chinese government launched a new round of education reform and issued the first set of curriculum standards in mathematics and other subject areas. It has been promoting a school-based (以校為本) teaching research system as a vehicle to support the implementation of the new standards and reform measures. The school-based teaching research focuses primarily on issues arising within individual campuses, attempting to resolve challenges that teachers and students encounter in their own classrooms.

As an important form of teaching research activity, Lesson Study has taken on more forms with increased depth, requiring participating teachers' reflective practice. Besides the traditional forms of collective lesson planning, master-apprentice partnerships, demonstration lessons, public lessons, lesson narratives, and lesson commentaries and critiques, there also emerged micro-teaching (微格教學), teacher-focused study (師本研究), same lesson different construct (同課異構), same lesson multiple rounds (一课多轮), etc. Most of these forms have been performed among both practicing teachers and prospective teachers (Li 2014).

Many of the Lesson Study activities are organized around a specially designed classroom environment and chosen topic, rather than a lesson naturally happening in a regular teaching schedule. This occurs in two different settings:

1. An expert teacher or a teacher engaging in a teaching contest demonstrates exemplary teaching skills in an artificial classroom where the topic of the lesson may not be exactly what the students are learning in their regular mathematics classes. In this case, the research and investigative aspect of Lesson Study has increased.
2. Prospective teachers in their student teaching phase practicing simulated teaching or rehearsal teaching (模擬教學) where fellow prospective teachers in the same group act as students when there are no real students present. This case is mostly for training and practicing purposes.

These started to happen at the same time as Japanese Lesson Study (授業研究) was being introduced to the western world around 2000 and gained international recognition (Fernandez 2002; Fernandez and Yoshida 2004; Lewis 2000; Lewis et al. 2004; Stigler and Hiebert 1999). When Chinese and other teacher education researchers reflected on the traditional practice of teaching research activities in China, they began to scrutinize, characterize, and experiment with the Chinese style Lesson Study (Huang and Bao 2006; Huang and Li 2009; Yang 2008, 2009). Supported by other popular research theories and practice, such as communities of practitioners, peer coaching, case methods of teaching, and action research, some educational researchers and practitioners went further and established new types of Lesson Study norms. The most recognized work was done by the Action Research Project team at Qingpu County in Shanghai (Gu and Wang 2003a, b; Gu and Yang 2003). Based on over 20 years of teaching experiments and empirical research in mathematics classrooms, the project team formulated an Action Education model for teacher professional development that incorporated Lesson Study as the main platform for teacher learning, planning, teaching, reflections, and new behaviors (Gu and Gu 2016; Huang and Bao 2006).

Supported by internet technology and online platforms, many web portals have been developed in the new millennium and are devoted to distributing online demonstration lessons and excellent-quality lessons (优质课、精品课) originally taught by expert teachers or renowned teachers (名师). This new source of model lessons shares the same underlying philosophy of Lesson Study and has also been utilized in similar ways as a vehicle for teaching research for both prospective and practicing teachers.

2.4 Characterizations of the Chinese Style Lesson Study

A century after it was adopted and implemented, Lesson Study has evolved into an essential form of learning and development activity among Chinese teachers and schools with its own unique characteristics that distinguish itself from the Japanese style Lesson Study (Huang and Han 2015). As an experienced practitioner, Cai (2008) proposed a model which portrays a teacher's path to excellence as a trilogy of three types (or stages) of lesson design and enactment:

1. Simulation lessons (模课): a teacher observes expert teachers' teaching, simulates the essence of their lesson designs and implementations, and yet makes adaptations when designing lessons and teaching his/her own classes.
2. Polishing lessons (磨课): a teacher drafts an initial version of a lesson plan and teaches it (or gives a narrative) in front of his/her teaching research group; fellow teachers in the group collaborate and go through cycles of commentary, discussion, revision or redesign, re-implementation, and experimentation.
3. Reflection lessons (悟课): a teacher reflects on his/her behaviors and strategies in teaching the previous two types of lessons; develops a deeper understanding of theories, curriculum, students, and instructional processes and techniques; and gradually establishes his/her own effective teaching style and expertise.

From these three basic forms, one can identify some key features of Chinese Lesson Study, such as learning and getting feedback from master teachers and purposefully going through cycles of practice, discussion, revision, and reflection to achieve a more profound understanding of the lessons and to improve teaching.

Based on recent developments in Chinese Lesson Study and findings in related research, Huang et al. (2017a, b) theorized Chinese Lesson Study as a design-enactment-analysis-revision cycle that has three main characteristics:

1. It is a deliberate practice of "special activities developed for and repeatedly pursued by individual teachers with feedback from experts" (p. 272). It involves a task with a well-defined goal, motivations for improvement, and opportunities for feedback and repetition, which eventually lead to refinement of participating teachers' teaching behaviors and expertise.
2. It is a research methodology that blends research and practice in cycles of design, enactment, analysis, and redesign. It starts with a teaching and research question and a theoretical stance and aims to answer the question to a certain extent after the cycles of Lesson Study activities.
3. It is an improvement science that shares two core features: "the 'plan-do-study-act (PDSA) cycles' and organizational conditions that support such continuous improvement across levels systematically" (p. 274).

Next, this chapter turns to analyzing the cultural roots of Lesson Study in China, which is beneficial for us to better understand both the historical path as well as the theoretical features of the Chinese Lesson Study.

3 Cultural Roots

This section addresses the second research question: What cultural beliefs and practices in China best explain the characteristics of Chinese Lesson Study?

Lesson Study is a dual process that involves both learning and teaching (in a broad sense). On the one hand, it is a venue for teachers to learn about different aspects of teaching (educational theories, curriculum standards and instructional materials, subject matter content, student characteristics, lesson design and teaching strategies, assessment, etc.) from expert and fellow teachers as well as through practice and reflection. On the other hand, it is a process for teachers with more expertise, experience, or new ideas and strategies to guide or inspire fellow teachers.

Hence, in answering the research question, the author extensively searched Chinese classic works to identify those well-established fundamental principles and practices regarding teaching and learning which would (1) reflect the above due process, (2) echo the historical development of Lesson Study in China, and (3) best explain the main characteristics of Chinese Lesson Study.

The result includes three main principles and practices: (1) respecting and learning from masters and experts, (2) teaching and learning by integrating profound theory and deliberate practice, and (3) consolidating teaching and learning into one complete process. These themes provide guidance to the roles and behaviors of all parties involved in Lesson Study activities: mentors, experts, experienced and novice teachers, and fellow teacher participants.

3.1 Learning from Masters and Experts with Exemplary Knowledge and Skills

As observed from history and the present, Chinese teachers have been engaged in two basic forms of Lesson Study activities:

1. Observation lessons (觀摩课) and demonstration lessons (示範課): Prospective or novice teachers observe lessons taught by experienced or expert practicing teachers, then have conversations with them, and/or write reports afterward. There are also excellent-quality lessons (优质课、精品课) that all teachers could observe and learn from regardless of level of experience.
2. Critique lessons or commentary lessons (評課) and report lessons (匯報課): Prospective or novice teachers carefully prepare and teach a lesson; expert teachers and mentors observe this lesson, asking questions and providing comments or advice afterward.

At the core, both forms share the fundamental principle of learning from experts, and they highlight the authoritarian status that experienced teachers (including teacher educators, mentors, supervisors, administrators, etc.) have over prospective and novice teachers (Chen 2017), which is a special version of the hierarchical relationship between teachers and students or the traditional master-apprentice relationship. Master teachers and expert teachers must have valuable ideas, skills, and experiences that are worth being conveyed to novice teachers who are learning to teach and work with students. Learning from masters and experts by directly observing their teaching practice and then discussing the rationale behind lesson designs and implementation is an essential process for the growth of novice teachers. Historically, such master-apprentice relationships have been recognized as unconditional and unquestionable. In a more profound way, they reflect the social status that teachers have had and the crucial roles they have played historically in Chinese society. It also explains why teaching public lessons to a large audience for feedback and winning teaching competitions that are judged by experts are both criteria required of teachers applying for senior ranks or honorary titles.

Confucian ideologies became the dominant educational and cultural doctrines in China during the Han Dynasty (220 B.C.–200 A.D.). Confucius (551–479 B.C.) himself has since been recognized in China and many oriental societies as the earliest master teacher and greatest role model for all later generations. The classic works of Confucian schools stressed the nobleness and honor of the teaching profession and its doctrines (師道尊严), as well as the importance of respecting teachers and valuing education (尊師重教), both appearing in *The Classic of Rites* (禮記). Teachers must be rigorous in order for truths and laws to be respected by students, which would in turn inspire students to learn diligently. Confucian theories also systematically addressed moral and intellectual standards of the teaching profession. These have helped build a long-standing social superiority of teachers. The concept of teachers goes beyond those who hold teaching as their career and extends to everyone who is knowledgeable and virtuous enough for others to learn from him. According to Confucianism, the main purpose of education is to cultivate future generations of gentlemen, whereas teachers' crucial responsibility is to instill both knowledge and morality within students (Gu 2014; Reagan 2000; Zhu 1992). To achieve these goals, teachers themselves should first become role models to their students, both in intellect and in virtue.

The notion of teachers as models can be traced back to some of the earliest official historical books in China. For instance, the phrase 師表 was used by Sima Qian (司馬遷) in *The Records of the Grand Historian* (史記, written around 91 B.C.) to describe government officials with exemplary knowledge and capabilities and being deemed role models. It was also used in *The Records of the Three Kingdoms* (三國志) to show tremendous respect to Confucius as the holy master teacher.

The phrase 師範was first used in *The Book of the Late Han* (後漢書) (published around 289 A.D.) to characterize the teaching profession which holds high standards in knowledge and morality. Here 師means (1) a teacher or an expert, as a noun, and (2) to learn from or to imitate, as a verb. The word 範also has dual denotations as a noun: (1) norms, rules (which corresponds to the original meaning of the word "normal" as in "normal education") and (2) models, exemplifications (which highlight the fundamental roles of teachers). Since then, this phrase has been often used in both China and Japan. By the 1890s, 師範was widely used by Chinese politicians, entrepreneurs, and educators to promote normal education (Wang 1997). When the

Chinese government adapted educational policies and system settings from Japan in 1904, 師範was officially adopted for various teacher education institutions (normal schools, colleges, and universities) (Gu 2003; Li 1998).

The Chinese cultural tradition encourages people to not only show respect to teachers, experts, masters, and gentlemen but to also wholeheartedly learn directly from them. *The Confucian Analects* (論語) (written during 480–350 B.C.) quotes Confucius stating, "When we see men of worth, we should think of equaling them (见賢思齐)." This provides a fundamental norm of expectation on one's attitude toward experts and learning: always be ready to learn from others who have high morality and capability, with the goal to at least achieve their levels. It is a basic principle underlying Lesson Study activities and teacher learning: novice teachers should always be eager to learn knowledge and skills from experienced and master teachers.

In the new millennium, a large amount of demonstration lesson and excellent-quality lesson (优质课、精品课) videos have been created and distributed by major online educational portals. These lessons are typically associated with authors who are expert teachers or renowned teachers (名師). This implicitly reflects the cultural tradition that master teachers are respected and trusted for their expertise. Master teachers are both capable of and responsible for sharing their expertise so that all other teachers can benefit. These online model lessons became a new type of resource for teachers with varying levels of experience to engage in Lesson Study and other teaching research activities.

3.2 *Teaching and Learning by Integrating Profound Theory and Deliberate Practice*

There are several long-lasting cultural beliefs that stress the importance of unifying words and actions, linking learning, knowing, thinking, and understanding with practice.

3.2.1 Teaching Through Combining Verbal Explanations and Exemplary Actions

One of the most influential principles for teaching appeared in the Taoist classic *Zhuangzi* (庄子) (written between 350 B.C. and 250 B.C.) which maintains that a teacher should not only teach his students through verbal explanations but also demonstrate ideas and model skills through exemplary actions (言傳身教). Since the word 身 in Chinese carries multiple meanings (body, physical, in person, action, behavioral, etc.), in later history this principle has been interpreted and practiced broadly in two ways: (1) A teacher should not only teach important thoughts and good virtues in words, but also apply them in his own daily routines and practice.

This also turned out to be very relevant as a principle for parenting: parents should not merely preach to their children what is right but also become role models to their children by demonstrating exemplary behaviors. (2) A teacher should not only teach theoretical perspectives but also enact them in teaching and learning practice so that students can have a better understanding of the theories. These interpretations once again emphasize teachers' status as role models and have implications for teacher education. Teacher educators and mentors help novice teachers learn teaching skills not only by instilling theories and perspectives but also through modeling effective teaching strategies, behaviors, and skills in person.

Hence, when it comes specifically to teaching practice, experienced teachers are naturally considered the masters and mentors who could demonstrate their knowledge, visions, skills, experiences, etc. through actual teaching in the classrooms, which should and could all be modeled by novice teachers later. In the meantime, experienced teachers can help novice teachers reflect on, analyze, and improve their own teaching skills. Lesson Study activities perfectly integrated words and actions in teacher learning.

This principle also aligns with the traditional concept that teaching is a heuristic-persuasion process for masters of knowledge to tell and explain concepts, ideas, and methods and to also demonstrate the actual procedures (Hsieh et al. 2018). Teachers need to maintain a balance between telling and querying. In order to earn the recognition of school administrators, experts, and fellow teachers, a novice teacher must demonstrate in front of them, through public lessons and other forms of Lesson Study, that he/she has acquired practical teaching skills. In other words, teaching skills are observable, measurable, and comparable. Lesson Study provides a template for assessing teaching effectiveness.

3.2.2 Understanding Through Linking Learning with Practice

The traditional concept of learning is subtle. The Chinese phrase for learning, 學習, incorporates the dual processes of 學 (study) and 習 (review, drill, practice, and experiment). Similarly, the Chinese notion of knowledge (知識) also interweaves learning and knowing (知) and profound understanding based on experiences and practice (識). In *The Analects*, Confucius made his famous quote: "Often review and practice while studying, isn't it fun? (學而時習之, 不亦乐乎?)" (Legge 1869).

Regarding the best ways to achieve knowing and understanding, one set of epistemological principles involves learning and knowing (知) versus practice (行) or inner knowledge versus actions. Classic works such as *The Confucian Analects*, *The Book of Documents* (尚書), and the Mohism collection *Mozi* (墨子), as well as the great Confucian philosopher Zhu Xi (朱熹) of the Song Dynasty, all emphasized the dual importance of learning and practice. But they might be obscure or differ in terms of which of the two is harder or should precede the other. Wang Shouren (王守仁) (1472–1259), who was a follower of Confucianism and the founder of the Yangming School (阳明學派), famously proposed the unity of learning and practice (知行合一). It could also be interpreted as (1) integrating thinking and actions and

(2) acquiring factual knowledge, truths, and principles while experimenting with or implementing them practically. Depending on who is learning what, the order of learning and practice processes may vary, but the two processes must mutually reinforce each other. The final outcome is a unified body of knowledge and skills developed from both learning and practice.

Sheng Xuanhuai (盛宣懷), the founder of the Nanyang Public Institute and the first normal school in China, wrote in 1896 the following in his proposal for fundraising for the Institute and its normal school: "…we plan to establish a Normal School in the Institute… By imitating the Japanese system, we also plan to build an affiliated elementary school, enroll 120 smart children aged between 10 to 17 or 18, …let prospective teachers teach those children. During a whole year, students in the normal school would be studying while practicing teaching (且學且誨), benefiting from making progress in both learning and practice (頗得知行并进之益)" (Wang 1997, p. 6).

As Sheng obviously already realized, the 知行合一principle especially makes sense for acquiring sophisticated professional skills, including teaching skills. Through Lesson Study activities, novice teachers are able to observe how expert teachers apply various types of professional knowledge and teaching techniques in an actual classroom. They can also experiment with what they have learned in teacher education classes through teaching public lessons or critique lessons and receiving immediate feedback from mentors and fellow teachers. Lesson Study is an ideal venue where the 知行合一principle can be demonstrated and observed.

This principle also resonates with the aforementioned teaching principle: teaching through both words and actions, teaching theories, and implementing/demonstrating them in practice (言傳身教).

3.2.3 Improving and Perfecting Skills Through Deliberate Repetitions

When Lesson Study becomes an institutionalized and routine activity for teachers, teachers often spend a considerable amount of time repeatedly designing, implementing, discussing, and refining episodes of a lesson or multiple lessons, just to maximize the effectiveness of the lesson(s) and strengthen their mastery teaching skills. This reflects another cultural tradition associated with skill learning: to master a certain set of skills, one may have to undergo deliberate and repeated practice over an extended period of time, combined with constant reflection and revision (Chen 2017).

For instance, "practice makes perfect" (熟能生巧) is a traditional belief that has been firmly held and widely practiced throughout Chinese history (Dai 2003; Li 1999; Li and Dai 2009). It is partially rooted in the traditional Chinese agrarian economy and agrarian culture which emphasize hard work, diligence, immersion, and fine skills. A good work ethic and solid basic skills are widely considered the cornerstone for success, not only for physical labor but also for learning and teaching (Hsieh et al. 2018; Li et al. 2015).

Such a belief could also be traced to the profound influences from the meritocratic ideology, the lasting existence of the imperial civic service examination mechanism, and the Confucian educational philosophies. Learners are expected to tirelessly study and practice, focus, and immerse themselves in the subject matter. Learners are likely to benefit from repetitive but purposeful practice which could help them internalize the learning principles and routines (Li et al. 2015).

The aforementioned notions of polishing lessons (磨课) and excellent-quality lessons (优质课、精品课) also reflect this set of beliefs: an individual teacher and fellow teachers immerse themselves in cycles of lesson design, enactment, commentary, and discussion and gradually polish (磨) the lesson plan and make it closer and closer to perfect. Teaching skills can also be improved during this process.

3.2.4 Consolidating Teaching and Learning into One Complete Process

In the Chinese language, both the single character 教and the phrase 教學denote "teach" or "teaching," hence are often used interchangeably. However, 教學 carries richer meaning because學 means "learning or study." The popular use of this phrase reveals the traditional belief that teaching cannot be independent from learning. A teacher's teaching process indeed encompasses student learning as an important component, and they are mutually beneficial.

Such a belief was discussed and analyzed in the chapter "On Learning" in *The Classic of Rites:* "When he learns, one knows his own deficiencies; When he teaches, he knows the difficulties of learning. After knowing his deficiencies, one is able to turn around and examine himself; after knowing the difficulties, he is able to stimulate himself to effort. Hence it is said, "Teaching and learning help each other" (學然後知不足，教然後知困。知不足，然後能自反也；知困，然後能自強也。故曰：教學相長也). Since then 教學相長 has become a common belief and principle for school education in China.

3.3 Mutual Learning Among Peers and Through Collaboration

The principles discussed above are mostly concerned with the hierarchical master-apprentice, expert-novice, and teacher-student types of dynamics. Teaching and learning are observed as occurring between two parties with considerably different levels of knowledge, skill, and/or morality. Expertise is shared unidirectionally from mentors and experts to novice teachers. In reality, Lesson Study activities also involve mutual learning among fellow teacher participants with similar levels of experience, such as a group of prospective teachers in the middle of their student teaching. Even expert and experienced teachers could potentially learn good ideas and strategies from novice teachers in Lesson Study or other teaching research activities. In a broad sense, all teachers involved in Lesson Study could be considered members of a learning community with equal status yet varying goals and learning needs (Yuan and Li 2015).

Students can learn not only from experts but also from each other. This modest vision and attitude are reflected in Confucius' famous quote: "Whenever I engage in practice with several other fellows, I can always identify at least one of them that could be considered my teacher" (三人行必有我師). With such an attitude, participants in Lesson Studies would have more opportunities to learn and improve. The popular practice of group lesson planning (集體備課) in Lesson Study is a direct application of this philosophy.

The chapter "On Learning" in *The Classic of Rites* includes a statement of four principles for effective teaching at advanced levels, one of which is indeed about group learning and collaborative work: students learn and improve by observing each other and having discussions (相觀而善之謂摩). Based on this principle, the two verbs 观 (watch, observe) and 摩 (discuss, scrutinize) were later merged to the new phrase观摩 which describes the joint actions of observation and discussion. Now it appears to make sense that one of the basic and primary forms of Lesson Study activities is called 观摩课 – since the very origin and nature of Lesson Study is for teachers to observe each other's lessons, have discussions, and improve teaching effectiveness together.

More often than not, group learning through mutual 观摩 activities occurred in history in a scholastic research type of private school, 書院 (college or academy). Teaching and learning occur based on a Confucian heuristic tradition (Hsieh et al. 2018). Teacher and students interact with one another, having dialogues and conversations, asking questions and figuring out the answers, and reaching consensus, with an ultimate goal of resolving the original problem. This reflects the collaborative nature of traditional Chinese teaching and learning.

4 Concluding Remarks

This chapter first reviews the origin and development of Lesson Study in China over the last 120 years. Three main historical periods are identified:

1. The first half of the twentieth century when the initial forms of Lesson Study activities were first introduced as a component of the newly established normal schools and colleges and later expanded to practicing teachers. Lesson Study activities at the time often followed established procedures.
2. The second half of the twentieth century when Lesson Study became a routine practice among both prospective and practicing teachers. The nationwide teaching research network and the teaching research groups established at each school made Lesson Study institutionalized and systematic. Lesson Study became an important platform not only for teachers to learn, share, and improve teaching

expertise but also for teachers to demonstrate their capabilities during teaching contests or in fulfilling requirements for retention and promotion.

3. The first two decades of the twenty-first century when Lesson Study in China started to take many more forms along with modern school education reforms. When the Japanese Lesson Study became internationally recognized and studied, researchers examined the nature and impact of Chinese style Lesson Study, theorized its main characteristics, and compared it with the Japanese Lesson Study.

This chapter is the latest attempt in providing a comprehensive narrative of the history of Lesson Study in China, which is intended to help researchers and practitioners make sense of the status and main characteristics of Chinese Lesson Study in its historical context. It especially fills a vacuum in the research on the early history of Lesson Study in the first half of the twentieth century. Due to dramatic and frequent political movements, foreign invasions and interventions, civil wars, and government changes, many education-related first-hand materials between 1900 and 1950 were lost or scattered. This chapter portrays a more detailed picture of that period based on the extensive review of compilations of historical records and existing research literature.

Historically speaking, Lesson Study as a unique form of teacher preparation and professional development activity did not originate naturally in China. Instead, it was introduced into China twice, half a century apart, both as a government mandate and more or less as a foreign implant:

1. Basic Lesson Study activities (e.g., lesson observations and commentaries) were brought to existence between the 1890s and 1910s along with the normal education system which in many ways resembled the Japanese normal education infrastructure and practice.
2. Lesson Study and some other activities for teachers got a new look when the Chinese government formally adopted the teaching research network system in the 1950s from the Soviet Union.

It was one of the measures China took for restoration and reconstruction after the World War II and the following Civil War. During both periods, engaging in Lesson Study activities helped to relieve the high demand for improving the quality of teachers and their teaching in a relatively narrow time frame.

Lesson Study and teaching research returned to schools in China as a routine practice for teachers after taking a break during the Cultural Revolution (1966–1976). They played important roles during the educational revitalizations and reforms in the 1980s as well as in the 2000s. Lesson Study has become an integral component of the school and teacher education systems in China and developed distinguishing styles and features. To fully make sense of this development, the author turned to educational and cultural traditions for insights.

Along the cultural dimension, this chapter brings to light three main themes (traditional beliefs about and principles for teaching and learning) that could help to explain why and how Lesson Study in China is being practiced in its own ways:

(1) It is highly important to learn exemplary knowledge and practice from masters and experts; (2) effective teaching and learning must blend profound theories with cycles of deliberate practice and refinement; (3) learning also takes place among learner peers through mutual observation and discussion. The first two themes overlap with and yet are elaborated on more broadly than existing theories of Lesson Study in China (Chen 2017). The third theme has not been explicitly discussed in research in Chinese Lesson Study, and it could help to justify the collaborative nature of Chinese Lesson Study.

Cultural perspectives could help us better understand not only the nature and features of Lesson Study in China, but also some of its downsides or unintended consequences. For instance, expert teachers are more respected and trusted when they are involved in Lesson Study activities; online video clips of lessons use “excellent-quality lessons” or “renowned teachers” in their titles to appeal to visitors. Being invited to teach public lessons or demonstration lessons or winning public lesson teaching contests at the municipal or higher levels became criteria for reappointment and promotion to senior rank positions. Some teachers have taken a utilitarian or opportunist approach, participating in Lesson Study activities mainly for the sake of winning a contest or promotion, i.e., fame, social status, and financial incentives. All these phenomena can be attributed to the same cultural inclination: experts are highly respected and worth learning from. Their experience and insights are almost always valuable.

Respecting and pursuing exemplary or best knowledge and practice may encounter dilemmas during current Lesson Study activities. Chen (2017) pointed out the occasional difficulty in generating a commonly agreed-upon criteria for “good” lessons during Lesson Study amid ongoing curricular reforms. It would require both willingness and effort of experienced teachers to clearly explain to novice teachers what constitutes a “good” lesson.

Similarly, there have been discussions regarding what a “standard” Lesson Study process should encompass (Wang and Gao 2012a, b), what focal points a Lesson Study should have (An 2008), whether Lesson Study should include a few essential stages, or if it is acceptable to focus on only one stage at a time (e.g., planning or enactment stage alone), etc. (Yang 2008). Future research and practice on Lesson Study could shed light on how to find a balance between the norms and flexibilities so that Lesson Study activities are not reduced to a checklist or a sequence of programmed actions, without also missing any characteristic step or action.

China and Japan have shared many similar origins and features of cultural traditions, as well as in the origin and historical development of Lesson Study. Nonetheless Lesson Study in the two countries does have major differences. It would be an interesting topic for future research to trace such differences in Lesson Study to the subtle distinctions between the social and cultural traditions in the two countries.

In an even broader international context, would Lesson Study work equally or be sufficiently effective in an education system where historical traditions and cultural norms are somewhat different from those in China and Japan? If not, what should or could be done for Lesson Study to work out where? In the case of the USA, researchers (Chokshi and Fernadez 2004) have detailed a number of cultural and logistical barriers to implementing Lesson Study in the USA more widely and effectively and proposed strategies for overcoming challenges originating from cultural and systemic factors. Lewis and Perry (2006) summarized the main progress made in implementing Lesson Study among US teachers, as well as some emerging evidence of success. In a few more recent projects, researchers adapted the Chinese Lesson Study model to schools and districts in the USA (Cravens and Drake 2017; Huang et al. 2017a, b) and in Italy (Bartolini Bussi et al. 2017). Results show improved instructional practice, changed views about student learning, and strengthened pedagogical knowledge and content knowledge among school teachers after they have engaged in well-designed Chinese Lesson Study activities. For such adaptations to be successful, it is necessary to modify the structure, components, and process of Lesson Study or to develop new tools in order to overcome the conflicts between the Chinese model and local school cultures and needs (Huang et al. 2017a, b).

Acknowledgment The author would like to thank Dr. Hongyu Su of South China Normal University and Professor Qin Dai of Neimenggu Normal University of China for sharing information regarding historical resources of mathematics education in China and thank the editor of this handbook, Professor Rongjin Huang, for many insightful comments and suggestions. The author would like to also thank the anonymous reviewers for their invaluable feedback.

References

Abe, H. (1987). Borrowing from Japan: China's first modern educational system. In R. Hayhoe & M. Bastid (Eds.), *China's education and the industrialized world* (pp. 57–80). Armonk: M. E. Sharpe.

An, G. (2008). The connotations and values of lesson study. *Global Education, 37*(7), 15–19 (in Chinese).

Bartolini bussi, M. G., Bertolini, C., Ramploud, A., & Sun, X. (2017). Cultural transpositions of Chinese Lesson Study to Italy: An exploratory study on fraction in a fourth grade classroom. *International Journal for Lesson and Learning Studies, 6*(4), 380–395.

Cai, Y. (2008). Simulation lessons, polishing lessons, and reflection lessons. *Secondary and Elementary Teacher Education, 7*, 41–42 (in Chinese).

Chen, Q. (1969). *History of education in modern China*. Taipei: Zhonghua Publishing House (in Chinese).

Chen, X. (2017). Theorizing Chinese Lesson Study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 283–292.

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*(3), 218–236.

Chokshi, S., & Fernadez, C. (2004). Challenges to importing Japanese Lesson Study: Concerns, misconceptions, and nuances. *Phi Delta Kappan, March*, 520–525.

Cravens, X., & Drake, T. (2017). From Shanghai to Tennessee: Developing instructional leadership through teacher peer excellence groups. *International Journal for Lesson and Learning Studies, 6*(4), 348–364.

Dai, Q. (2003). *Confucian thoughts and traditional mathematics in China*. Beijing: Commerce Publishing House.

Deng, J., & Li, C. (1995). *Resources for modern history of elementary education in Beijing*. Beijing: Beijing Education Press.

Ferguson, P. (1985). Teacher education in Japan: An historical and comparative perspective. *Journal of Teacher Education, September–October*, 211–224.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 53*, 393–405.

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Mahwah: Lawrence Erlbaum Associates, Publishers.

Gu, M. (2003). Traditions and evolutions of teacher education. *Teacher Education Research, 15*(3), 1–6 (in Chinese).

Gu, M. (2014). *Cultural foundations of Chinese education*. Leiden: Koninklijke Brill NV.

Gu, F., & Gu, L. (2016). Characterizing mathematics teaching research mentoring in the context of Chinese Lesson Study. *ZDM Mathematics Education, 48*(4), 441–454.

Gu, L., & Wang, J. (2003a). Teachers' growths in educational actions: A study on the model of teacher education based on curriculum (Part 1). *Curriculum, Textbook, Pedagogy, 1*, 9–15 (in Chinese).

Gu, L., & Wang, J. (2003b). Teachers' growths in educational actions: A study on the model of teacher education based on curriculum (Part 2). *Curriculum, Textbook, Pedagogy, 2*, 14–19 (in Chinese).

Gu, L., & Yang, Y. (2003). School-based research on teachers' professional growths. *Research on Educational Development, 6*, 1–7 (in Chinese).

Hsieh, F., Lu, S., Hsieh, C., Tang, S., & Wang, T. (2018). The conception of mathematics teachers' literacy for teaching from a historical perspective. In Y. Li & R. Huang (Eds.), *How Chinese acquire and improve mathematics knowledge for teaching* (pp. 37–56). Leiden: Koninklijke Brill NV.

Hu, S. (1932). *Guidelines for teaching practice in New China*. Beijing: Zhonghua Publishing House (in Chinese).

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298.

Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.

Huang, R., & Li, Y. (2009). Pursuing excellence in mathematics classroom instruction through exemplary lesson development in China: A case study. *ZDM Mathematics Education, 41*, 297–309.

Huang, R., Ye, L., & Prince, K. (2016). Professional development system and practices of mathematics teachers in Mainland China. In B. Kaur & K. O. Nam (Eds.), *Professional development of mathematics teachers: An Asian perspective* (pp. 17–32). New York: Springer.

Huang, R., Fang, Y., & Chen, X. (2017a). Chinese Lesson Study: An improvement science, a deliberate practice, and a research methodology. *International Journal for Lesson and Learning Studies, 6*(4), 270–282.

Huang, R., Barlow, A. T., & Haupt, M. E. (2017b). Improving core instructional practice in mathematics teaching through lesson study. *International Journal for Lesson and Learning Studies, 6*(4), 365–379.

Jiang, X. (2008). The evolution of the "Backbone Teacher" training system since 1949 and its implications. *Contemporary Educational Science, 14*, 10–12 (in Chinese).

Ju, X., Tong, F., & Zhang, S. (1994). *A compilation of materials for the history of modern education in China*. Shanghai: Shanghai Education Press (in Chinese).

Legge, J. (1869). *The Chinese classics: Translated into English with preliminary essays and explanatory notes by James Legge. Vol. 1. The life and teachings of Confucius.* Second Edition. London: N. Trübner. Retrieved on June 4, 2018 at http://oll.libertyfund.org/titles/2270

Lewis, C. (2000). *Lesson study: The core of Japanese professional development*. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.

Lewis, C., & Perry, R. (2006). Professional development through lesson study: Progress and challenges in the U.S. *Tsukuba Journal of Educational Study in Mathematics, 25*, 89–106.

Lewis, C., Perry, R., & Hurd, J. (2004). A deeper look at lesson study. *Educational Leadership, 61* (5), 18–23.

Li, J. (1998). The development of a Chinese style normal education in modern China. *Teacher Education Research, 56*, 46–54 (in Chinese).

Li, S. (1999). Does practice make perfect? *For the Learning of Mathematics, 19*(3), 33–35.

Li, S. (2014). Teaching research in secondary and elementary schools in the past 60 years: Reflections and prospects. *Contemporary Educational Science, 17*, 17–21 (in Chinese).

Li, S., & Dai, Q. (2009). Chinese traditional culture and mathematics education. In J. Wang (Ed.), *Mathematics education in China: Traditions and reality* (pp. 1–30). Nanjing: Jiangsu Education Publishing House (in Chinese).

Li, Y., & Li, J. (2009). Mathematics classroom instruction excellence through the platform of teaching contests. *ZDM Mathematics Education, 41*(3), 263–277.

Li, Y., Huang, R., Bao, J., & Fan, Y. (2011). Facilitating mathematics teachers' professional development through ranking and promotion practices in the Chinese mainland. In N. Bednarz, D. Fiorentini, & R. Huang (Eds.), *International approaches to professional development of mathematics teachers* (pp. 72–87). Ottawa: Ottawa University Press.

Li, X., Li, S., & Zhang, D. (2015). Cultural roots, traditions, and characteristics of contemporary mathematics education in China. In B. Sriraman et al. (Eds.), *The first sourcebook on Asian research in mathematics education* (pp. 67–88). Charlotte: Information Age Publishing.

Makinae, N. (2010). *The origin of lesson study in Japan*. Proceedings of EARCOME5, Japan Society of Mathematics Education.

Mathematics Teaching Research Group at Jiading Secondary School. (1953). A report on improving the mathematics teaching at Jiading Secondary School. *Bulletin of Mathematics, March–April*, 37–42 (in Chinese).

Ministry of Education of the People's Republic of China. (1986a). *Tentative regulations on secondary school teacher professional titles*.

Ministry of Education of the People's Republic of China. (1986b). *Tentative regulations on elementary school teacher professional titles*.

National People's Congress. (1986). *Compulsory education law of the People's Republic of China.* Retrieved on June 3, 2018 from http://www.lawinfochina.com/display.aspx?lib=law&id=1166&CGid

National Research Council. (2010). *The teacher development continuum in the United States and China: Summary of a workshop*. Washington, DC: The National Academies Press.

Reagan, T. (2000). *Non-Western educational traditions: Alternative approaches to educational thought and practice*. Mahwah: Lawrence Erlbaum Associates, Publishers.

Ruan, C. (2012). An investigation of teacher education in the late Qing Dynasty around the promulgation of the Presented School Regulations. *Guangdong Social Sciences, 4*, 133–139.

Shu, X. (1981). *Resources for modern history of education in China*. Beijing: People's Education Press (in Chinese).

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Suo, Q. (1953). The experiences of the mathematics teaching research group at Anyang no.1 secondary school in organizing demonstration teaching. *Bulletin of Mathematics, March–April*, 37–42 (in Chinese).

Wang, B. (1997). Teacher preparation and teacher education in China: Commemorating the 100th anniversary of teacher education in China. *Teacher Education Research, 54*, 3–9 (in Chinese).

Wang, J. (2013a). *Mathematics education in China: Tradition and reality*. Singapore: Galeasia Cengage Learning.

Wang, M. (2013b). A preliminary examination of teachers' teaching research during the Republic of China era. *Journal of East China Normal University (Educational Sciences), 31*(1), 89–95 (in Chinese).

Wang, R., & Gao, J. (2012a). Lesson study: Domestic experiences and multiple forms (Part 1). *Research on Educational Development, 8*, 31–36 (in Chinese).

Wang, R., & Gao, J. (2012b). Lesson study: Domestic experiences and multiple forms (Part 2). *Research on Educational Development, 10*, 44–49 (in Chinese).

Yan, G. (2003). An analysis of teacher education in modern Chinese compulsory education. *Teacher Education Research, 15*(1), 61–66 (in Chinese).

Yang, Y. (2008). How should teachers conduct lesson study. *Research on Educational Development, 8*, 72–82 (in Chinese).

Yang, Y. (2009). How a Chinese teacher improved classroom teaching in teaching research group: A case study on the Pythagorean Theorem teaching in Shanghai. *ZDM Mathematics Education, 41*, 279–296.

Yang, Y., & Ricks, T. E. (2013). Chinese Lesson Study: Developing classroom instruction through collaborations in school-based teaching research group activities. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 51–65). New York: Routledge.

Yuan, Z., & Li, X. (2015). "Same content different lesson constructs" activities and their impact on prospective mathematics teachers' professional development – A case study of Nadine. In L. Fan, N. Wong, J. Cai, & S. Li (Eds.), *How Chinese teach mathematics – Perspectives from insiders* (pp. 565–588). Singapore: World Scientific.

Zhu, W. (1992). Confucius and traditional Chinese education: An assessment. In R. Hayhoe (Ed.), *Education and modernization: The Chinese experience* (pp. 3–22). New York: Pergamon Press.

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Lesson Study and Its Role in the Implementation of Curriculum Reform in China

Xingfeng Huang, Rongjin Huang, Yan Huang, Chenqi Wu, and Cecilia Anne Wanner

Contents

Abstract This chapter aims to provide a holistic picture of the system of lesson study (LS) and its role in mathematics curriculum reform in China. By means of official documents (policy) and a review of literature, we begin by examining features of the teaching research system that is part of the infrastructure of LS in China. Following this introduction, the chapter presents a case of LS at district and school levels to examine how a curriculum reform idea could be implemented in classrooms through teaching a specific topic through the LS approach. Following typical Chinese LS phases, a LS team including knowledgeable others and school teachers set the goals of understanding the concepts of milliliter and liter and developing problem-solving skills, then carried out eight iterations of rehearsing and reflecting upon research lessons, and, finally, taught an exemplary lesson. The data collected include videotaped research lessons, lesson planning and debriefing sessions, post-lesson teachers' reflections, experts' comments on the final exemplary

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_12

lesson, and post-LS teachers' and teaching research specialists' interviews. An analysis of this data showed that the research lesson was substantially improved in its alignment with the goals of LS. In addition, the practicing teacher further developed her own understanding of teaching mathematics through experiment, and the specialist developed her professional expertise through mentoring the LS.

Keywords Chinese lesson study · Curriculum reform · Teaching research system · Teaching research specialist · Mathematics experiment · Community of practice · Encounter objects

1 Introduction

Lesson study (LS) in China has been in place for over a century (Chen and Yang 2013; Yang 2009). While structurally similar to Japanese LS, Chinese LS is distinct in various ways. For example, Chinese LS emphasizes repeated rehearsals of research lessons, the product (exemplary lessons) and process (dynamic between reflection and enactment) of LS, and involvement of knowledgeable others throughout the LS process (Huang and Han 2015). Studies have documented the critical roles of Chinese LS in implementing reform curriculum (Chen and Yang 2013; Fang 2017), providing professional development for teachers (Huang and Li 2009; Yang 2009), and promoting students' learning (Gu and Wang 2003; Huang et al. 2016b). Furthermore, researchers have explored the success of Chinese LS from various perspectives such as culture (Chen 2017), community of practice (Fang 2017), and educational leadership (Cravens and Wang 2017). In particular, researchers have attempted to portray how innovative ideas from curriculum could be translated and implemented in classrooms through LS within a multiple-tiered teaching research system (Cravens and Wang 2017; Fang 2017). However, the process of how Chinese LS contributes to these translations and implementations of new curriculum is largely unknown. This chapter aims to unveil the process of translating curriculum ideas into classroom implementation through a case study of one particular LS. We begin with a brief introduction of education and curriculum systems, including relevant research findings.

2 Context of Curriculum Reform and Background of LS

In this section, we first provide an overview of the curriculum reform system in China. We then describe the teaching research system in detail. We follow this with a review of the relationship between curriculum implementation and LS. Finally, we present the framework of this study.

2.1 Curriculum Reform in China

In 2011, a revised version of mathematics curriculum standards was released in China (Ministry of Education [MoE] 2001, 2011). Many significant changes regarding curriculum goals, structure, content, teaching and learning, and assessment were recommended in the revisions. For a number of decades before 2000, there was only one set of unified national curriculum standards and one unified textbook in China (Liu and Li 2010). More recently, as part of a decentralized curriculum reform effort, several textbook publishers have been authorized to develop their textbooks, which are required to align with the national standards (Wang 2013). To assist teachers in the implementation of the new curriculum, various teacher-training programs are currently in place (Huang et al. 2016a). All teachers are required to attend at least 40 hours of training before using the new textbooks. A "cascade model" has been adopted to train teachers to use the reform-oriented curriculum. This model begins by "seeding" the new ideas and strategies of the reform-oriented curriculum and textbooks through the training of trainers and key teachers at the national level. The national trainers then instruct local trainers, who in turn facilitate the training of classroom teachers (Huang et al. 2016a). To promote these translations from the intended curriculum to an enacted curriculum, the hierarchical teaching research system provides a supportive infrastructure, and LS serves as an effective vehicle for conducting teaching research activities (Cravens and Wang 2017; Huang et al. 2017a).

2.2 Teaching Research System

A teaching research system consisting of hierarchical Teaching Research Offices (TROs, hereafter) from national to district levels is one of the most important infrastructures for supporting teacher professional development and implementing curriculum reform in China (Huang et al. 2016a; Wang 2013). "Teaching research activities" refer to various types of professional development organized by the TROs. Mathematics teaching research specialists (TRSs, hereafter) employed in TROs are officially responsible for organizing teaching research activities at different levels (Secondary Mathematics Instruction Council (SMIC), China Education Association 2012). Based on official documents and requirements of TRSs in China, Huang et al. (2012) held that Chinese TRSs should have expertise in conducting effective teaching, doing teaching research, effectively organizing school-based teaching activities, and evaluating teachers' teaching and students' learning. Moreover, SMIC, China Education Association 2012 explicitly emphasized the role of TRSs in discovering and developing master teachers, as well as in serving as consultants for educational authorities. Several studies have tried to conceptualize the core competencies of TRSs in China (e.g., He 2013), which include general competence as a teacher and specific competence as a TRS. The former refers to teaching, evaluating lessons, the assessing of student learning, negotiating and communication, and language expression, and the latter consists of teaching research, educational research, and implementation of policy, administration, and creativity.

2.3 Curriculum Reform and LS

Lee and Lo (2013) contended that teachers play crucial roles in bringing the intended curriculum to life in class and that LS provides a mechanism through which the intended, enacted, and lived curricula can be better synthesized. They claimed that "it is only through such collaborative discourse among teachers supported by 'knowledgeable others' that reform ideas can take root in classrooms and bring about lasting change" (p. 200). Lewis and Takahashi (2013) described how an interlocking LS system in Japan successfully supported the implementation of curriculum reform over decades. Moreover, Lewis (2016) described four types of LS, as practiced by school, district, university, and professional association in Japan. The features of different types of LS vary somewhat. For example, the school- and district-based groups focus primarily on meeting the needs of their local students, while the LS practiced by laboratory schools (affiliated with university or professional associations) focus primarily on developing innovations in curriculum and instruction. Teachers in school-based LS groups can benefit from materials and approaches developed by district-based and university-based teams. In contrast, educators from laboratory schools are often invited to serve as advisors to local LS efforts or as commentators on research lessons. This opportunity to observe local LS efforts allows laboratory school educators an opportunity to investigate which innovations are most successful, using this analysis to further expand the innovation development process.

Regarding LS in China, several studies show how Chinese LS has consistently supported teachers' implementation of innovative teaching strategies (Huang et al. 2014; Yang 2009). These studies address both newly added content and traditionally difficult topics (Huang and Li 2009; Huang et al. 2014). Fang's (2017) study presented three schools' different responses to Shanghai's recent curriculum reform. In all three responses, the familiar structures and processes of school-based teaching research and Chinese LS have made curriculum reform transparent for teachers. From a perspective of distributed leadership, Cravens and Wang (2017) found that a multiple-tiered promotion system (Huang et al. 2016a) in Shanghai served as the main mechanism for subject leaders at district or city levels to take a leadership role in carrying out school-based teaching research activities. These leaders helped teachers interpret the curriculum standards, demonstrate their reform-oriented teaching, and mentor other teachers. It seems that LS could effectively promote implementation of curriculum reform nationwide in both China and Japan.

2.4 A Theoretical Framework of This Study

We adopted the framework of community of practice (Wenger 1998) to analyze the process of translating innovations from curriculum into actual teaching through LS. According to Wenger (1998), learning is defined as the process of participating in the practices of communities and constructing meanings and identities in relation to these communities. Within a district-based LS, knowledgeable others (i.e., experts), including the district TRS and master teachers from other schools, form one community of research, while the participating teachers of a LS team form another community of practice. The boundaries between different communities make it difficult to exchange knowledge across communities. Akkerman and Baker (2011) noted that boundaries both divide and connect communities in ways that problematize knowledge. Within the research community, knowledge involves competence in the design of a lesson or task. Within the teaching community, knowledge involves competence in promoting student learning, given a specific classroom. When specialists and teachers work together during LS, the LS creates a space in which these two communities come together. This space is conceptualized as a "boundary encounter." Participants from separate communities who are involved in boundary encounters negotiate meaning of effective teaching of a particular topic and/or an idea both "across the border" with another community and within their original community. By creating these boundary encounters, LS provides an opportunity for members from separate communities to communicate about, collaborate around, and potentially transform practice. Thus, LS as a boundary encounter allows for the negotiation of meanings of effective teaching and learning of mathematics between and within both the research and the teaching communities.

Both specialists and mathematics teachers who come together during LS are "boundary brokers": community members who introduce elements of practice from one community into another (Wenger 1998). Boundary brokers support connections across communities and create opportunities for new meanings to merge in different communities. Boundary encounters are organized around brokers who learn together and from each other and then return to their own communities with new practices that potentially transform that community.

When brokers from two communities come together in a boundary encounter that generates boundary practices, they work around representations of knowledge that convey meanings across multiple communities. These representations are "boundary objects," which inhabit intersecting worlds and satisfy particular requirements from each of them. During LS, research lessons with students' learning artifacts represent boundary objects that manifest the meaning of mathematics teaching and learning across research and teaching communities. They allow for the emergence of shared boundary practices. Researchers and mathematics teachers work together using the research lesson to make sense of curriculum ideas and enact these ideas in the classroom for student learning.

2.5 Research Questions

The purpose of this study was to provide a case study showing how innovative ideas from curriculum standards could be translated into the classroom environment in Shanghai through a LS approach. Within the community of practice framework, the development of research lessons is a fundamental feature of conceptualizing LS as boundary encounters. In this study, we aimed to answer the following research questions: (1) What major changes can be identified across the multiple enactments of the research lesson? (2) What is the teachers' perceived learning regarding the understanding of goals, design, and implementation of the research lesson? (3) How did the research specialist mentor the LS? (4) What are the research specialist's perceived gains derived from the LS?

3 Methodology

3.1 Setting

Shanghai is one of the five major cities directly under the central government of China. It is an international metropolis highly developed in economy, science, technology, and culture. With 16 districts in an area of 6340 square kilometers, Shanghai has a population of about 24 million permanent residents in 2017. There is a 9-year compulsory education system (5 years of elementary school and 4 years of junior high school). According to statistics from 2017, there are 741 elementary schools in the city with a total of 785,000 students, as well as 818 secondary schools enrolling 571,000 students. The compulsory education enrollment rate always remains at 99%. Like other provinces and cities, Shanghai has built a three-level teaching research system at the municipal, district, and school levels (Ji 2016).

The Shanghai Education Commission directs a TRO, where specialists in each subject are responsible for the implementation of curricula to guide teaching and educational research. Each district has an Institute of Education, where again specialists are responsible for each subject at all stages. According to the plan of the Shanghai TRO and each district's specific conditions, these specialists guide the schools and teachers in the district to implement and fulfill the objects and requirements of the curriculum, as well as conduct research on teaching. At the same time, each school also sets up a Teaching Research Group to explore the teaching innovation found in each subject area and to carry out classroom teaching research. Unlike other provinces and cities, Shanghai has its own curriculum standards and textbooks. LS has become the main acting paradigm for elementary and secondary schools to carry out teaching research activities (Gu and Wang 2003).

3.2 Goals of This LS

The Shanghai Mathematics Curriculum Standards (SHEC 2014) indicate three categories of goals for mathematics teaching and learning, namely, (1) knowledge and skills, (2) process and method, and (3) emotion, attitude, and value. In practice, most teachers simply list these three goals in their lesson plans without explicitly implementing them in their classrooms (Jiang and Liu 2017; Yu 2017). The national mathematics curriculum standards (MoE 2001, 2011) emphasize developing students' number sense as one of the main objectives of mathematics teaching. Although number and quantities are two inseparable concepts, early national curriculum standards described number sense without explicitly connecting to quantity (MoE 2001). However, in the national curriculum standards issued in 2011, quantities such as length, area, volume, weight, time, and speed were included, and students needed to make sense of them. Neither the Shanghai nor the national curriculum standards provide specific ways in which to develop students' quantity sense, yet in elementary school, students learn content related to quantities (i.e., measurements) (MoE 2001, 2011; SHEC 2014). The teaching guide for Shanghai elementary school mathematics (Huang 2017b) also clearly stated goals of developing students' conceptual understanding of quantities (including various measurements, such as length, area, volume, capacity, mass, weight, and time) in relevant units. How to cultivate students' quantity sense in elementary school has drawn great attention in mathematics education research in China (Dong 2015; Fei 2014; Rui 2015; Shen and Tan 2014; Shi and Tan 2014; Zhou 2014). Dong (2015) argued that mathematical experiments could improve students' number sense by providing opportunities to learn mathematics through hands-on activities and offering a rich learning environment for students to explore and discover. Mathematical experiments, according to Yu and Dong (2016), provide an active way through which students understand mathematical knowledge and discover and verify mathematical concepts and skills with the help of relevant tools. They further claimed that mathematics experiments could promote the development of students' cognition and affection harmoniously, fulfilling goals of developing students' quantity sense as stated in the mathematics curriculum standards. However, how to promote students' quantity sense through doing mathematics experiments in elementary schools is largely unexplored. Moreover, developing quantity sense is closely related to the broader goals of process and methods suggested in the curriculum standards. Thus, this LS sets its overarching goal as the implementation of the process and methods goals in classrooms. To do this, a topic of making sense of quantities by means of doing mathematical experiments in elementary schools was selected.

3.3 Process of Developing the Exemplary Lesson

This research project, entitled "Learning mathematics through doing mathematical experiments in elementary schools," is a district-wide key research project funded by the district authority. With the support of this funding, a group of teachers formed a

LS research team that included seven members. Miss Hong (all names given are pseudonyms), a district elementary school mathematics TRS, was the initiator and team leader of the LS. She had over 10 years of experience teaching elementary school mathematics in addition to over 10 years of experience as a TRS. Miss Ye, the team member who taught the research lesson, had 5 years of experience teaching elementary mathematics. Five other teachers from different schools in the district acted as lesson observers.

The research team aimed to develop a series of exemplary mathematic lessons, which could be used to demonstrate how mathematics could be taught through mathematics experiments. To achieve the overarching goals of the LS, the research team focused on the topic of *Knowing Milliliter and Liter* in fourth grade in a Shanghai elementary school. The team decided to use mathematics experiments to promote students' quantity sense of milliliter and liter.

The specialist asked Miss Ye to develop an exemplary lesson of *Knowing Milliliter and Liter*. The team planned to spend 1 year in developing an exemplary lesson that would then be disseminated to mathematics teachers across different districts in Shanghai. Miss Hong proposed ideas on how to enact the lesson, and Miss Ye designed the initial lesson plan. Together, they developed an initial lesson plan for rehearsal. Through iterative cycles of LS, the lesson plan was revised a total of nine times. At the conclusion of this process, the research lesson was taught and observed by more than 300 elementary mathematics teachers from different districts in Shanghai.

The development of the exemplary lesson included various stages. After the initial lesson plan was designed, Miss Ye implemented the lesson according to the lesson plan, while Miss Hong observed the lesson on site. Five teacher observers carefully recorded students' behaviors in class, collecting data for analysis in the post-lesson discussion. Miss Hong facilitated the post-lesson discussion, where the team shared their observations regarding the strengths and weaknesses of the lesson. The team discussed how to revise the lesson plan. Miss Ye modified the lesson plan based on comments from the post-lesson debrief and a consultation with Miss Hong. Then Miss Ye taught the lesson to another class in the same school. This cycle was repeated multiple times in order to develop an exemplary lesson. During the process, Miss Hong sometimes invited other experts to participate in lesson observation and post-lesson discussion. These experts included master teachers and specialists from the Teacher Research Office of the Shanghai Education Commission and mathematics educators from Shanghai Normal University.

3.4 Data Collection and Analysis

The data collected in this study included: (1) nine versions of the lesson plan; (2) videotaped fourth, sixth, seventh, and eighth lessons; (3) written comments from two experts on the eighth lesson; and (4) audio recordings of semi-structured interviews with Miss Ye, Miss Hong, and, one observer, Mr. Liu. The interviews with Miss Ye and Mr. Liu aimed to elicit their interpretations of the research lessons, the process of revision, and the perceived learning. The interview with Miss Hong aimed to elicit her opinions about and perception of the goals of this LS, the process of revision, the mentoring role of the TRS, the challenges of conducting this LS, and the perceived gains from the LS.

The original Chinese data was analyzed. Only necessary transcripts were translated by the first author to support relevant statements. To answer the first research question regarding major changes across the multiple implementations of the research lesson, we analyzed the lesson plans and lesson videos of *Knowing Milliliter and Liter*. The major phases and characteristics of each phase were juxtaposed in tables, and changes were identified through comparison. To feature the final exemplary lesson, both the video lesson and experts' comments on the lesson were used to capture the major characteristics.

To address the remaining research questions, both the teacher's and the TRS's interviews were analyzed using constant analysis to identify the major themes (Corbin and Strauss 2008). Specially, the interview with Miss Ye was analyzed to identify the teacher's perceived learning regarding the goals, design, and implementation of the research lesson. Miss Hong's responses during the interview process were analyzed to identify the TRS's role in mentoring LS and her perceived gains in learning.

4 Results

The findings are presented in three sections. The first section pertains to the revision process that led to the final exemplary lesson. The second section focuses on teacher-perceived learning. The final section describes how the TRS mentored LS and what she gained from the mentoring.

4.1 Strengthening the Research Lesson

The research lesson was rehearsed eight times before the final presentation of the exemplar lesson. Four critical changes were identified and explained. These include (1) adding mathematics experiments to the tasks originally presented in the textbook, (2) removing unnecessary mathematics experiments due to time constraints, (3) focusing on the mathematical ideas of estimation, and (4) focusing on problem-solving by means of experiment and multiplicative reasoning. After discussing each of these categories of revision, we will highlight features of the final exemplary lesson and include comments from experts regarding this lesson.

4.1.1 Adding Mathematics Experiments

The Shanghai mathematics textbook includes the topic of *Knowing Milliliter and Liter* broken down into three parts. Part 1 uses two activities to teach the concept of milliliter. The first activity asks students to compare the capacities of two different water kettles by measuring how many cups are in each. This activity teaches the importance of using the same referent unit in measurement. The second activity applies the unit of milliliter to quantities of food. Part 2 teaches the concept of liter through an activity involving the connection of this quantity with familiar items (soft drinks, oil, milk). Part 3 consists solely of exercise problems (Huang 2017a). According to the definition of mathematics experiment (Yu and Dong 2016), the activities in the textbook are not mathematics experiments. Rather, they can be considered as illustrative experiments where experimental tools are used to make measurements and record the experimental results. This provides students with the opportunity to perceive relevant concepts concretely instead of focusing on the prompting of student understanding through discovering and justifying mathematical knowledge.

To supplement this topic, Miss Hong provided Miss Ye with additional situation problems as a resource for the mathematics experiments (Table 1). The two problems have a common feature: Students are required to solve the contextual problems based on experiments. The experiments aim at helping students establish the general sense of the quantity of milliliters and liters, as well as the quantitative relationship between the two quantities.

In the initial lesson plan, Miss Ye designed five experiments to help children establish the sense of quantity for milliliter and liter. In Experiment 1 (EDP), students established the relationship between the number of drops and 1 mL and then use computational research (multiplicative reasoning) to solve the problem. Experiment 2 used cups and spoons to help students make sense of the quantity of 10 mL. Experiment 3 sought to develop students' sense of the quantity of 100 mL. Through Experiment 4 (WP), students established the number of milliliters in one cup of water and then determined how many cups would be needed for 1400 mL of water. Experiment 5 used a pouring demonstration to help students perceive the quantity of 1 L. The goal of these five experiments was to help students understand the quantitative relationship between 1 mL, 10 mL, 100 mL, and 1000 mL, thus

Table 1 Two mathematics experiment programs

Eye drops problem (EDP)	We started swimming lessons in school this semester. After swimming, we need to use eye drops to protect our eyes. If each student needs 4 drops of eye drops and there are 50 students in my class, how many 10 mL bottles of eye drops do I need?
Water problem (WP)	Scientists claim that on average, a person drinks somewhere between 2000 and 3000 mL a day to maintain body balance and ensure good health. An average person should drink about 1400 mL of water daily. Using the cups provided, how many cups of water do you need to drink each day?

building a preliminary sense of quantity. Experiments 2, 3, and 5 are, like activities in the textbook, illustrative experiments and not mathematics experiments (Yu and Dong 2016).

Agreeing with the substance of Miss Ye's lesson plan, Miss Hong proposed two amendments. First, mathematics experiments should allow for student conjecture and inquiry. In Experiment 4, Miss Ye planned to let the students begin by estimating how many milliliters were contained in a glass of water. Students would then find the quantitative relationship between 100 mL and cups through experimentation and finally determine how many cups of water a person should drink daily. Miss Hong felt that this teaching process was too "closed" and that the teacher should let students determine the experimental steps and implement them according to their own methods. Groups could then compare different methods used to solve the problem. Thus, in Experiment 1, the teacher should first ask students to estimate how many milliliters are contained in a bottle, then give students time to discuss their experimental methods in small groups, and finally have students actually drop 1 mL into a bottle.

Miss Hong's second recommendation was to allow the students to pour 1400 mL of water instead of exactly 1 L in Experiment 5, because this is more coherent with Experiment 4. In addition, using 1400 mL of water not only helps students conceptualize the relationship between 1000 mL and 1 L, but it also smoothly transitions to the next lesson involving the process of decomposing 1400 mL into liters and milliliters. Miss Ye revised the teaching plan based on Miss Hong's suggestions.

4.1.2 Removing Unnecessary Experiments

When Miss Ye taught the lesson the second time, Miss Hong invited Mr. Cao, a master teacher in another district in Shanghai, to observe. The lesson, which would typically take 35 min, lasted almost an hour and a half. Mr. Cao analyzed the features and value of the mathematics experiments and suggested that two illustrative experiments (Experiments 2 and 3) be removed, leaving only the mathematics experiments of EDP and WP and the illustrative experiment of pouring 1400 mL of water. The remaining three experiments still provide students with an opportunity to concretely experience the quantities of milliliter and liter. The team concluded that such a lesson design would continue to allow students to meet the goals of the lesson while keeping within a 35-min duration, making the lesson more feasible.

4.1.3 Addressing Students' Learning Difficulty in Estimation

When asked to identify from his observations students' learning difficulties in class, Mr. Liu replied that students always had trouble estimating the liquid capacity in different boxes, bottles, and buckets, especially for quantities like 750 mL or 5 L. But he also remarked that it seemed as if the students performed better when Miss Ye gave the students opportunities to first perceive 1 mL, 10 mL, 100 mL, 100 mL, or

1000 mL. The observers found that most of the students had difficulty measuring with referent units. The lesson plan evolved in order to address these difficulties. In the first lesson plan, although Miss Ye presented many referents, such as a 1 cm^3 block to represent 1 mL and a bottle of eye drops to represent 10 mL, the term "referent unit" was not yet found in the lesson plan. It was not until the fourth version of the lesson plan that Miss Ye wrote: "Through the experiment just now, we learned 1 mL and 10 mL. Please regard yourself as a little gauge, and estimate how many milliliters are in the bottle of liquid. What do you think?. .. This student's method is very good. She uses the block and the eye-drops to estimate, which we call finding a referent." This indicates that the teacher has begun to guide her students to make estimates using referents, although estimation is not yet a major focus.

In the seventh lesson plan, students were explicitly provided with two opportunities to develop the sense of quantity through estimation activities. The first activity involved guessing and checking. The teacher first asked students to estimate certain quantities of milk, water, and soda and then showed the actual capacity. In this manner, students were provided with different referents. The second activity consisted of estimating 100 mL of milk, 1 L of milk, and 5 L of oil in groups. After this activity, the teacher asked students to explain their estimation and reasoning. In the eighth lesson plan, the teacher not only required students to estimate the capacity of the goods but also explicitly helped students learn how to use referents to estimate quantity.

4.1.4 Shifting Focus from Experimental Procedure to Actual Experiments and Mathematical Reasoning

Originally, students spent much time on writing an experimental report, which they were unable to complete well. Miss Ye and Mr. Liu both mentioned this difficulty in their interviews. To address this problem, the team decided to simplify the expectations. Beginning with the fourth lesson plan, students were no longer required to record their experimental procedures. Instead, they orally reported the process after completing the experiment. The team decided that a lesson involving a mathematics experiment should put more emphasis on actually doing the experiment, rather than spending an excessive amount of time on components such as written reports (Yu and Dong 2016).

When solving the EDP, many students rushed to do the experiment without first considering the type of problem-solving strategy to employ. In the sixth rehearsal, two observers gave feedback that students did not know how to begin the experiment. They suggested that teachers fully present how to solve the problem from the experiment. Miss Ye commented in her interview, "After teaching this lesson three times, we found that few students would use the method of connecting the experiment with computation to solve the problem, so we decided to emphasize the method in the eye-drops problem. You can find that in the seventh lesson plan, we list this point as one of the teaching aims of this lesson. We didn't clearly note that before."

In the final exemplary lesson, before conducting the experiment, Miss Ye asked students to discuss methods of solving the problem. Then, after they solved the problem through experiments, students shared their solutions and reasoning (see Table 2). Using this format, almost all of the groups made the connection between the experiment and computation.

Table 2 Lesson episode in EDP (the eighth lesson plan)

1	T	Well. Think about it. How can you use these materials to justify your idea? Please discuss in pairs and write down your steps... Any volunteers to share your ideas?
2	S	We used a small metering tank. First [we] dropped 1 mL and found how many drops are there [20 drops]. And then [the 20 drops in 1 mL] times 10, resulting in 200 drops
3	T	Do you understand? Are there any different methods?
4	S2	I think that we should first drop 10 mL since the eye drops are 10 mL total. Then we finish dropping all of it, counting the number of drops, and compare the result with 200 drops
		(Students execute the experiment)
5	T	Ok, please be seated. Who wants to share your experiment results with us?
6	S3	Here is our thinking. We first dropped 1 mL, counting the number of its drops, 20. 10 mL times 20 is 200 drops. We found that 200 drops is exactly 10 mL. So one bottle of eye drops is enough
7	T	What other groups also measured 1 mL? Hands up, show me, please! How many drops did you measure in 1 mL?
8	S4	We measured that there were about 25 drops in 1 mL
9	S5	We measured that there were about 22 drops in 1 mL
10	S6	We measured that there were about 23 drops in 1 mL
11	T	Well, you all first measured how many drops there were in 1 mL and then calculated how many drops there were in 10 mL. Are there any other different ways? I saw that a group had measured 2 mL of eye drops. Please talk about your way
12	S7	We counted that there were 44 drops in 2 mL of eye drops, and then we multiplied 44 by 5. Last, we compared the result 220–200. So we need to take more than one bottle of eye drops
13	T	Well, this group compared 220–200. Can other students understand this way? Are there other different ways?
14	S8	We first measured how many drops there were in 10 mL, and we counted that there were about 220 drops in 10 mL. 220 drops are more than 200 drops, so we need to take more than one bottle
15	T	Oh, there were so many ways. Some measured 1 mL, others measured 2 mL, and still others measured the whole bottle of eye drops. Which way is your favorite between the three ways? Give me your reasons
16	S9	I prefer the first way, because we simply need to measure 1 mL. We don't have to measure the whole bottle. So it's easier than other ways
17	T	Do you agree? We can measure how many drops there are in 1 mL by experiment, and we can know how many drops there are in 10 mL by computation. Through experiments and computation, the problem can be more efficiently solved

Note: T stands for the teacher, while S# presents student #

Fig. 1 Students experimenting in groups

4.1.5 Features of the Final Exemplary Lesson

Miss Ye taught the lesson *on Knowing Milliliter and Liter* as a public exemplar lesson in her school. More than 300 teachers from different districts observed the lesson and participated in the post-lesson discussion. At the same time, Miss Hong also invited two specialists from different districts to observe and comment on the lesson. The class consisted of 30 fourth-grade students. The students were divided into five equal groups and seated at five tables (see Fig. 1). The lesson structure, lesson segments, lesson activities, and duration are given in Table 3.

The major goal of this lesson is to encourage students to solve problems using estimation and multiplicative reasoning. The following interaction documents how this occurs (Table 4). During the interaction, Student 13 first determined the unit of measurement by weighing and then used a bottle of eye drops as a referent to estimate. When Students 14 and 15 measured the capacity of a bottle of milk and Students 18 and 19 estimated the capacity of a barrel of oil, they chose different referents. The performance of these students shows that they have established an initial sense of quantity for milliliter and liter and have learned to use the referent and reasoning to make estimations.

4.1.6 Expert Comments on the Final Exemplary Lesson

In their written comments on the exemplary lesson, two experts (Miss Yu and Miss Chen) highlighted two aspects of the lesson: (1) mathematics experiment is beneficial to students' experiencing mathematical thinking and (2) mathematics experiment is beneficial to developing students' sense of quantity.

Table 3 Structure and activities in the exemplary lesson

Lesson structure	Lesson segment	Lesson activity	Duration (min)
Introduction	Knowing milliliter	The teachers set up the context and show students a bottle of eye drops. Students learn about the quantity of mL in whole class interaction	1
Exploration	Perceiving 1 mL	The teacher introduces the EDP, requires students to share their different conjectures with the whole class, discusses with each group its problem-solving plan, and communicates the different plans with the entire class. Then students carry out the experiment and solve the problem, sharing their solutions with the entire class	14
	Perceiving 1000 mL	The teacher introduces the WP, requires students to share their different conjectures with the whole class, and discusses with each group its problem-solving plan. Then students carry out the experiment and solve the problem, sharing their solutions with the entire class	10
	Knowing liter	Students pour 1 L of water with the provided cup, perceiving how many cups are contained in 1 L of water	2
	Converting between milliliter and liter	In the teacher-student interaction, students learn the conversion between milliliter and liter: 1400 mL =? L and? mL; 2500 mL =? L and? mL; 4 L + 700 mL =? L	2.5
	Estimating capacity	The teacher shows different everyday items to the students, requiring them to estimate the capacities of the items using units of milliliter or liter	3
	Estimating with referent	The teacher requires students to estimate the capacities of a bottle of Yakult, a bottle of milk, and a barrel of oil and then shares their methods with the whole class	7
Summary		The teacher and students summarize what they learned in the lesson	1
Homework		1. Find the capacities of everyday items 2. Estimate how many glasses of water are in 1400 mL, and then check your answer using the measuring cylinder	1

Mathematics Experiment Is Beneficial to Students' Experiencing Mathematical Thinking Miss Yu reflected on how students were led to experience conjecturing, verifying, and drawing conclusions. She believed that in the process of mathematics experiment, the students experienced four steps that are similar to Polya's problem-solving, namely, understanding the mathematics problem, formulating an experimental plan, implementing the experimental plan, and reviewing and

Table 4 Lesson episode of interaction after student estimation (eighth lesson plan)

1	T	You have learned many capacities of different items. Now I ask you to do estimation. Please estimate how much liquid there is inside these three items. Please write down the answer on your paper. And the group leaders put the paper on the desk. OK. Let's reveal the answer. Please tell us the answer
2	S13	100 mL
3	T	If you agree, please raise your hand. How did you get the answer?
4	S14	We weighed it by hand. Because Yakult is much smaller, we used milliliters
5	T	What did you compare it with to get the answer? Sit down, please. Thank you. Next, please
6	S14	Because eye drops are 10 mL and Yakult is 10 times as much as the eye drops, it is 100 mL
7	T	What did she compare it with? Eye drops. She thought the bottle of Yakult is 10 times as much as the eye drops. Then she got 100 mL. The method is good. Now, let's reveal the second answer. Milk. 1 L. If you have the correct answer, please raise your hand. How did you get the answer?
8	S15	I compared it with Yakult directly. I thought that the bottle of milk is 10 times as much as Yakult
9	T	Great! What else?
10	S16	I compared it with a 1 L glass. The weight of a 1 L glass, which is full of water, is the same as the bottle of milk. So I think the capacity of the bottle is 1 L.
11	T	Do you agree? He compared the bottle of milk with a glass of water and weighed them. The method is good. Last, let's turn to oil. Tell us the answer
12	S17	5 L
13	T	If you agree, please raise your hand. How did you get the answer?
14	S18	I think we need five 1 Liter graduated cylinders to contain the oil. So I think it is 5 L
15	T	What else?
16	S19	We can also compare the oil with milk

summarizing the process. Miss Chen also expressed similar opinions in her commentary. She believed that many students experienced experiments in the classroom that inspired mathematical thinking. Students' reasoning was developed by the comparison of different quantities. Both Miss Yu and Miss Chen believe that the mathematics experiments provided students with a valuable environment for the exploration necessary to enhance mathematical thinking.

Mathematics Experiment Is Beneficial to Developing Students' Sense of Quantity Miss Chen mentioned the role of mathematics experiments in making sense of quantity at two levels. First, students' sense of quantity is established through the direct experience provided by mathematics experiments. As Miss Chen commented, "Learning like this through experimental operations is more intuitive and profound." Second, students' sense of quantity is developed by using the established sense of quantity to estimate capacities in daily life.

Along with her approval of the exemplary lesson, Miss Chen also provided suggestions for improvement of the lesson. First, teachers should provide students with more opportunities to experience quantity in multiple ways, such as dropping

1 mL into a cup. Since liquid is without fixed shape and changes with the shape of the container, this would give students further insights into the quantity of 1 mL. Also, teachers should consciously elicit students to form their own referents in their mind and then can use these referents to do estimation at different quantitative levels, based on their established sense of quantity.

In summary, the exemplary lesson demonstrated critical features of teaching mathematics such as developing problem-solving skills through experimentation and mathematical reasoning. These features were established but may be further enhanced.

4.2 Teachers' Perceived Learning in the LS

Although there are many theoretical discussions about teaching mathematics through mathematics experiments (Sun et al. 2016; Tan and Zhu 2016; Zhao and Zhang 2016), there is a lack of illustrative examples of implementing mathematics experiments in the classroom. The EDP and WP were purposefully designed to make sense of quantity and to experience mathematical problem-solving. During the first two rehearsals, Miss Ye focused on details of doing experiments such as how to operate equipment, how to read the measurements, and how to write the experiment report. In the third rehearsal, while still attending to the operational details of the experiment, she began to focus more on mathematical reasoning. Three more rehearsals were needed to fully develop the strategies of integrating experiment (making conjectures and verification) and mathematical reasoning.

The team as a whole gradually developed their own understanding of quantity through the LS process. Initially, all members seemed to think that illustrative experiments were most beneficial in developing students' quantitative sense. After several rehearsal teachings, team members realized how mathematics experiments served to help students establish quantitative relationships, use multiplicative reasoning to solve problems, and use referent units to make estimations.

4.3 Translating Innovative Ideas from Standards into Classrooms: A Specialist's Perspective

An analysis of the interview with the TRS, Miss Hong, reveals themes regarding the process of translating ideas into classroom practice, the role of mentoring LS, and how the TRS can learn from LS.

4.3.1 General Process of Translating Ideas into Practice Through LS

In her interview, Miss Hong stated that one major role of the TRS is to "understand the curriculum standards and implement the innovative ideas [from standards] in classrooms." To do this, the TRS may form a project team. A project team consists of several voluntarily participating schools and volunteer teachers from these schools. With the mentoring of a TRS, project teams explore how to implement innovative ideas using standards from their classrooms. Once the team identifies feasible strategies, the TRS organizes a district-wide teaching research activity to demonstrate the exemplary lesson. Other teachers may be attracted and want to join the project. For example, Miss Hong commented that "through. .. public lessons, teachers can see how those innovative ideas are implemented [in actual classrooms], how students are actively engaged in learning and excited about sharing their thinking, then they want to imitate." In one semester, a school-based LS group could conduct one or two research lessons to be implemented through mathematics experiment. Another school-based LS group could repeatedly explore the same topic based on previous LS. After several enactments and reflections, if the LS group feels comfortable with their research lesson, the group's experience and accomplishment could be disseminated through district-wide public lessons and online learning communities.

4.3.2 The Mentoring Role of a TRS

In addition to general administrative roles, the TRS also plays specific roles in mentoring LS. Miss Hong described her roles in this specific LS as follows: (1) creating an authentic learning situation and (2) encouraging teachers to be reflective learners.

Creating an Authentic Learning Situation Miss Hong emphasized the effect of creating an interesting and mathematically worthwhile problem situation. She commented, "I pay attention to the design problem situation to see if a teacher designs a very attractive and motivating student [learning] situation. If he/she creates such a [learning] situation, then students will be engaged [in learning]. It will be easy to process the activity because [the students] are motivated to do experiments. This is very important."

Encouraging Teachers to be Reflective Learners Miss Hong stressed the need to adjust the class based on questions raised by students, developing the teachers' reflective ability. She noted, "I emphasize how teachers handle student-raised questions or incidentally occurred issues. I ask them to write reflection notes after each rehearsal teaching. Through this process, it is hoped to develop teachers' instructional expertise."

4.3.3 Co-learning Through Mentoring LS

In addition to teacher learning that occurs through the LS process, Miss Hong also articulated her own learning through mentoring LS. First, she deepened her understanding of the mathematical ideas embedded in the experiment. In her interview, Miss Hong acknowledged, "I have grown through the process [of LS]. For example, I realized the importance of mathematical [multiplicative] reasoning when doing a mathematics experiment. I initially intended to develop student problem solving ability, but I have realized that developing [multiplicative] reasoning ability is very important." In addition to deepening her mathematical content knowledge, Miss Hong also realized that developing student ability is more important than mastering knowledge. She stated:

> Through the entire process, I became more and more convinced that in a lesson with mathematics experiment, developing ability is more salient.. .. Someone may think that there are weaknesses in knowledge and skill-training in this lesson. There are fewer practicing problems. But I feel that in this lesson, compared to the ability development [multiplicative thinking], the [basic skill] practicing is not so important.

5 Discussion

This study reveals how a LS group (a team of teachers and a TRS, with support from knowledgeable others) collaboratively worked together to implement the goals of process and methods of mathematics learning through teaching the topic of *Knowing Milliliters and Liter* by mathematics experiments. Through multiple iterations of design, teaching, debriefing, reflecting, and revision, the research lesson better met the overall goal of the LS, to make sense of quantity and develop problem-solving ability (experiencing process and methods in particular). The chapter documented the struggles the team faced during the process: selecting appropriate tasks and experiments, doing experiments, the relationship between experiments and mathematical reasoning, big mathematical ideas embedded in the experiments (estimation), and strategies of problem-solving (using a referent unit to estimate). This study portrays how the team explored the translation of ideas from standards and textbooks into classrooms and what they learned through the development of the research lesson. From the perspective of community of practice (Wenger 1998) and characteristics of Chinese LS, we highlight the following features.

5.1 Chinese LS Promotes the Translation of Curriculum Reform Ideas into Classrooms and Develops Teacher Instructional Expertise

Chinese LS is a type of deliberate practice (Han and Paine 2010; Huang and Han 2015; Huang et al. 2017a). In this study, the LS team, with specific goals (making sense of quantity and developing problem-solving ability) in mind, repeatedly conducted the research lessons with immediate feedback from knowledgeable others (specialists). Through implementation and reflection, the research lesson was improved to better align with the intended goals of instruction. This process echoes the effect of deliberate practice (Ericsson et al. 1993) in continuously refining performance. Meanwhile, the participating teacher developed a deeper understanding of how to help students make sense of quantity through making use of mathematics experiments to teach the concepts of milliliters and liters. This finding concurs with Widjaja et al.'s (2017) observation that LS could promote teachers' professional growth through interplay (enactments and reflection) between different domains.

5.2 Chinese LS Promotes the Development of the TRS

The role of experts in facilitating LS has been recognized widely (e.g., Takahashi 2014). Yet the development of experts has not yet been explored. This study shows the roles of experts (the TRS, in particular) in brokering teacher learning through the encounter objects of research lesson. The TRS not only provided the initial tasks and general principles for teaching a topic through experiments but also gave specific suggestions on tailoring tasks and sharpening the focus of this lesson. During the facilitating process, the TRS also developed her own professional knowledge and vision. She established a deeper understanding of how to make sense of quantity, how to integrate experiments and mathematical reasoning, and how to develop big ideas and methods embedded in the activities. In addition, she established a vision that developing student ability is more important than students' mastering of knowledge when doing experimentation. This finding that the growth of knowledgeable others through mentoring LS in China is supported by Huang and Han (2015) and Huang et al. (2017b). Complimentary to studies that have explored the development of facilitators through specific coursework (Campbell and Malkus 2014) and workshops (Borko et al. 2015), this study provides an action research approach of professional development for knowledgeable others in China.

5.3 Chinese LS Promotes Implementation of Curriculum Reform

In Japan, LS has played a critical role in implementing curriculum reform and transforming classroom teaching from telling to exploring (Lewis 2015; Lewis and Takahashi 2013). Other studies have indicated that several mechanisms promote implementation of a new curriculum in China (Cravens and Wang 2017; Fang 2017; Gu and Wang 2003). This provides a detailed and concrete illustration about how Chinese LS appears to function as an improvement science (Bryk et al. 2016). A LS group launched their exploratory journey through repeated cycles of plan-do-study-action. Within a school-based LS group, the iterative cycle of plan-do-study-action, with the involvement of knowledgeable others, seems to be a deliberate practice that helps to constantly improve the research lesson. The development of research lessons for the same topic across different school-based LS groups will enrich the repertoire for teaching the same topic to different students. The well-developed annotated lesson plans and videotaped lessons may be shared within a district through public lessons and the online learning community. The exemplary lessons developed through LS in a district could be shared at city or national levels (Huang and Li 2009). The various exemplary lessons with relevant documents developed through various lesson studies may provide valuable resources for teachers to implement this curriculum in their own classrooms.

6 Conclusion

This chapter provides an analysis of a supportive system for conducting LS nationwide in China. The case study shows how LS assists in the development of exemplary lessons, demonstrating how to implement reform ideas in actual classrooms. The participating teacher and TRS furthered their professional expertise through the development of encounter objects between the community of teaching (teachers) and community of teaching research (specialists). LS in China seems to be a deliberate practice when it is school-based, while it may be viewed as an improvement science when it is district-wide or beyond. The networked LS system could help teachers implement curriculum in their own classrooms and develop their instructional expertise through conducting LS. However, when adapting the Chinese LS into other educational systems, the cultural value and professional development system should be given careful consideration (Bartolini et al. 2017; Cravens and Drake 2017).

References

Akkerman, S., & Baker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169.

Bartolini Bussi, M. G., Bertolini, C., Ramploud, A., & Sun, X. (2017). Cultural transpositions of Chinese lesson study to Italy: An exploratory study on fraction in a fourth grade classroom. *International Journal for Lesson and Learning Studies, 6*(4), 380–395.

Borko, H., Jacobs, J., Koellner, K., & Swackhamer, L. (2015). *Mathematics professional development: Improving teaching using the problem-solving cycle and leadership preparation models*. New York: Teachers College Press.

Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. G. (2016). *Learning to improve: How America's school can get better at getting better*. Cambridge, MA: Harvard Education Press.

Campbell, P. F., & Malkus, N. N. (2014). The mathematical knowledge and beliefs of elementary mathematics specialist-coaches. *ZDM Mathematics Education, 46*(2), 213–225.

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 283–292.

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*(3), 218–236.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research* (3rd ed.). Los Angeles: Sage.
Cravens, X., & Wang, J. (2017). Learning from the masters: Shanghai's teacher-expertise infusion system. *International Journal for Lesson and Learning Studies, 6*(4), 306–320.
Cravens, X., & Drake, T. (2017). From Shanghai to Tennessee: Developing instructional leadership through teacher peer excellence groups. *International Journal for Lesson and Learning Studies, 6*(4), 348–364.
Dong, L. W. (2015). Developing sense of quantity in mathematics experiment. *China Mathematics Education, 11*, 2–6 (in Chinese).
Ericsson, K. A., Krampe, R., & Tesch-Romer, C. (1993). The role of deliberate practice in the acquisition of expert performance. *Psychological Review, 100*(3), 363–406.
Fang, Y. (2017). School-based teaching research and lesson-case study in mediating the second-cycle curriculum reform in Shanghai. *International Journal for Lesson and Learning Studies, 6* (4), 293–305.
Fei, L. F. (2014). Sense of quantity as an important content of teaching quantitative unit: A case of knowing millimeter. *Elementary Teaching Design, 8*, 4–5 (in Chinese).
Gu, L. Y., & Wang, J. (2003). Teachers' growth in educational action: A study on the model of teacher education based on curriculum. *Global Education, 185*(1), 44–49 (in Chinese).
Han, X., & Paine, L. (2010). Teaching mathematics as deliberate practice through public lessons. *The Elementary School Journal, 110*, 519–541.
He, X. B. (2013). The components, and characteristics and development of core capacity of teaching research specialists. *Journal for Introduction to Education, 10*(1), 38–40 (in Chinese).
Huang, J. H. (2017a). *Shanghai elementary mathematics textbook, grade 4(1)*. Shanghai: Shanghai Education Publisher (in Chinese).
Huang, J. H. (2017b). *Teaching reference materials, grade 4(1)*. Shanghai: Shanghai Education Publisher (in Chinese).
Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.
Huang, R., & Li, Y. (2009). Pursuing excellence in mathematics classroom instruction through exemplary lesson development in China: A case study. *ZDM Mathematics Education, 41*(3), 297–309.
Huang, R., Xu, S., & Su, H. (2012). *Teaching researchers in China: Hybrid functions of researching, mentoring and consulting*. Paper presented at 12th International conference on Mathematics education, Seoul, July 8–15.
Huang, R., Fang, Y., & Chen, X. (2017a). Chinese lesson study: An improvement science, a deliberate practice, and a research methodology. *International Journal for Lesson and Learning Studies, 6*(4), 270–282.
Huang, R., Gong, Z., & Han, X. (2016b). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM Mathematics Education, 48* (4), 425–439.
Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese lesson study. *ZDM Mathematics Education, 46*(4), 239–251.
Huang, R., Ye, L., & Prince, K. (2016a). Professional development system and practices of mathematics teachers in mainland China. In B. Kaur & K. O. Nam (Eds.), *Professional development of mathematics teachers: An Asian perspective* (pp. 17–32). New York: Springer.
Huang, R., Zhang, J., Mok, I., Zhou, Y., Wu, Z., & Zhao, W. (2017b). The competence of teaching research specialists and their development in China. *International Journal for Lesson and Learning Studies, Special issue, 6*(4), 321–335.
Ji, M. (2016). How to realize inheritance and innovation in Shanghai teaching and research. *People Education, 20*, 20–23.
Jiang, Y. G., & Liu, G. B. (2017). How to achieve the teaching goals effectively: A case study of elementary mathematics teaching. *Education and Teaching Research, 31*(6), 85–90 (in Chinese).

Lee, C. K., & Lo, M. L. (2013). The role of lesson study in facilitating curriculum reforms. *International Journal for Lesson and Learning Studies, 2*, 200–206.
Lewis, C. (2016). How does lesson study improve mathematics instruction? *ZDM Mathematics Education, 48*(4), 571–580.
Lewis, C. C. (2015). What is improvement sciences? Do we need it in education? *Educational Researcher, 44*(1), 54–61.
Lewis, C., & Takahashi, A. (2013). Facilitating curriculum reforms through lesson study. *International Journal for Lesson and Learning Studies, 2*, 207–217.
Liu, J., & Li, Y. (2010). Mathematics curriculum reform in the Chinese mainland: Changes and challenges. In F. K. S. Leung & Y. Li (Eds.), Reforms and issues in school mathematics in East Asia. Rotterdam: Sense Publisher
Ministry of Education (MoE). (2001). *Mathematics curriculum standards for compulsory education (trial)*. Beijing: Beijing Normal University Publisher (in Chinese).
Ministry of Education (MoE). (2011). *Mathematics curriculum standards for compulsory education*. Beijing: Beijing Normal University Publisher (in Chinese).
Rui, J. F. (2015). The practical strategy of accumulating the experience on sense of quantity in mathematical experiments. *Basic Education Research, 5*, 49–51 (in Chinese).
Secondary Mathematical Instruction Council (SMIC), China Education Association. (2012). *Guidance for secondary school teaching research activity at city and county levels*. September 4, 2012. Retrievable at: htpp://www.wendangku.net/doc/7547a01dff00bed5b9f31da6.html
Shanghai Education Committee [SHEC]. (2014). *Shanghai elementary and secondary school mathematics curriculum*. Shanghai: Shanghai Education Publisher (in Chinese).
Shen, X. J., & Tan, N. J. (2014). The cultivation of sense of quantity is inseparable from real experience: The lesson record and comments on knowing millimeter. *Hunan Education, 6*, 52–57 (in Chinese).
Shi, M. M., & Tan, N. J. (2014). *Improvement of sense of quantity: Only slow work can make a fine work: Lesson record and comment on grams and kilograms* (Vol. 9, pp. 50–55). Hunan Education (in Chinese).
Sun, C. R., Dong, L. W., & Zhu, G. F. (2016). Measurement framework construction and strategy analysis of mathematics experiment conception in middle school. *Curriculum, Teaching Materials, and Methods, 36*(7), 90–95 (in Chinese).
Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 4–12.
Tan, D. L., & Zhu, J. M. (2016). Teaching evaluation for middle school mathematics experiment. *Curriculum, Teaching Materials, and Methods, 36*(8), 108–113 (in Chinese).
Wang, J. (2013). *Mathematics education in China: Tradition and reality*. Singapore: Galeasia Cengage Learning.
Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. New York: Cambridge University Press.
Widjaja, W., Vale, C., Groves, S., & Doig, B. (2017). Teachers' professional growth through engagement with lesson study. *Journal of Mathematics Teacher Education, 20*, 357–383.
Yang, Y. (2009). How a Chinese teacher improved classroom teaching in teaching research group: A case study on Pythagoras theorem teaching in Shanghai. *ZDM Mathematics Education, 41*(3), 279–296.
Yu, P., & Dong, L. W. (2016). Analysis of nature of mathematics experiment in middle school. *Curriculum, Teaching Materials, and Methods, 36*(8), 89–95 (in Chinese).
Yu, Z. Q. (2017). What is more important than the goal of mathematics classroom teaching in elementary school? *People's Education, 27*, 55–58 (in Chinese).
Zhao, W. K., & Zhang, J. Y. (2016). Teaching design of mathematics experiment in middle school. *Curriculum, Teaching Materials, and Methods, 36*(8), 102–107 (in Chinese).
Zhou, G. R. (2014). *Survey and analysis of Grade 6 students' sense of quantity*. Unpublished master thesis. Chongqing: Southwest University (in Chinese).

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Part III
Adaption Lesson Study in Selected Education Systems

Preface: Adaptation of Lesson Study in Selected Education Systems

Wasyl Cajkler

Contents

Abstract The eight chapters in this section demonstrate how lesson and learning study have been adapted and applied in a variety of educational contexts. The authors provide an informative series of research accounts which clearly demonstrates the extent to which there is now significant variation in the way that lesson and learning study are used in mainstream education systems. This preface briefly introduces the chapters and their detailed reports of how researchers, teachers and teacher educators enhance the quality of teaching and learning through lesson or learning study, highlighting the power, flexibility and versatility of both lesson study and learning study.

1 Introduction

Since the publication of *The Teaching Gap* (Stigler and Hiebert 1999), the adaptation of lesson study to educational contexts across the globe has gathered pace. There is now significant variation in the way that lesson study is adapted and applied in all phases of education. The chapters in this section provide detailed accounts of how researchers, teachers and teacher educators enhance the quality of teaching and learning through lesson study in a number of jurisdictions. These chapters provide

The original version of the chapter has been revised. A correction to this chapter can be found at https://doi.org/10.1007/978-3-030-04031-4_39

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_13

evidence of the versatility of both lesson study and learning study, allowing them to be adapted across the globe.

2 Introduction to the Chapters

Takahashi and McDougal open this section with their evaluation of an adapted form of lesson study, Collaborative Lesson Research, henceforth CLR, designed to address the demands of new curriculum mathematics in the USA. However, they begin by claiming that key aspects of Japanese lesson study may be lost in translation, leading to projects with limited impact. Drawing on Takahashi's extensive experience, the chapter serves as a reminder to all of us, when adapting lesson study, of the essential qualities required to secure its effective implementation.

Five characteristics are defined by the writers as essential to the effective use of lesson study, namely, the focus on the refinement of pedagogic expertise through a clearly defined research purpose, not just the refinement of an individual lesson as in many projects outside Japan; secondly, lesson study is a school- or district-wide initiative, not individual; thirdly, time for 'kyouzai kenkyuu' (the study of teaching materials) is essential; fourthly, a lesson study cycle, with live observation and detailed post-lesson evaluation, is conducted over several weeks, not in a few hours; and, finally, knowledgeable others play a very important part at all points in the lesson study process, but their contributions to post-lesson evaluation are particularly highlighted. The following chapters echo this claim about the vital contribution of knowledgeable others.

CLR has been implemented in eight US city schools and in a project in Qatar. The five essential conditions are incorporated in CLR, the key guiding question being: 'How can we design a lesson so that students learn a certain concept or skill better than they have in the past?' This is allied to a general research theme, agreed and shared by the school community, e.g. 'fostering students' ability for problem-solving and reasoning by using teaching through problem-solving'.

Takahashi and McDougal argue that conducting multiple cycles of CLR leads to a situation in which the school becomes 'a place for teachers to improve mathematics teaching and learning, and implementing the new standards will be a collaborative effort'. Here, there are distinct echoes of Stigler and Hiebert's plea for professional development initiatives to be firmly focused on improving teaching as a whole, not just individual teachers.

Readers will find a detailed replicable practical explanation of how to do CLR. Impacts from the projects are also described, notably the growth of teacher understanding of children's mathematical conceptions and misconceptions. When used school-wide, CLR 'can help teachers meet the challenge of changing both what they teach and how they teach', an aspiration that should be at the heart of thinking behind any lesson study adaptation.

In chapter "Implementing a New Mathematics Curriculum in England: District Research Lesson Study as a Driver for Student Learning, Teacher Learning and Professional Dialogue", Dudley and colleagues describe a London-based project of

6 cycles of research lesson study (RLS) in 96 schools, which sought improvements in students' mathematics learning. RLS differs from Japanese lesson study and CLR, discussed above, because it observes a small number of case pupils and uses post-lesson interviews to explore pupils' perspectives on learning. This iterative 'design study approach' affords opportunities for reconnaissance (Elliott 1991) and tentative 'field testing of hypotheses' in research lessons (Dudley 2013). At the planning stage, teachers engage in deliberative predictive visualisation of what is expected to happen (imagining) and how case pupils are likely to act at each stage of the lesson. Thus, imagining and reflective discussion are essential in the process.

The writers situate RLS in the action research tradition, drawing on Stenhouse (1981) and Elliott (1991), pioneers in promoting collaborative action research in the UK in the 1970s and 1980s. Dudley et al. demonstrate how policy failures and patchy use of action research in the past had left the field open for a lesson study adaptation to be accepted. RLS was thus a timely counterweight to increasingly top-down mandated forms of professional development which risked robbing teachers of their pedagogic agency. Anyone interested in establishing district-wide research lesson study (RLS) could benefit from reading this chapter.

While different to CLR, RLS allows teachers, through processes of hypothesising and rehearsing, to share experience, understanding and knowledge that are often tacit and difficult to access. Space limitations forbid detailed commentary on the multiple approaches to data analysis, but the researchers' data came from video recordings of planning and evaluation sessions, teachers' lesson study workbooks, survey data and pupil interviews. Studies of teachers' descriptive learning processes (DLP) and interpretive learning processes (ILP) feature strongly. Another distinctive feature of this research was the analysis of teacher workbooks for evidence of impact on both teaching and learning.

The Wits Maths Connect Secondary Project described in chapter "A Case of Lesson Study in South Africa" (Adler and Alshwaikh) was conducted in South Africa between 2013 and 2016 in three low-income disadvantaged school clusters, with a particular focus on exemplification in the teaching of mathematics. The project draws on Mathematical Discourse in Instruction (MDI) (Adler and Ronda 2015), which focuses on four aspects of a mathematics lesson: the object of learning, exemplification, explanatory communication and learner participation. MDI was organised into a teaching framework, the Mathematical Teaching Framework (MTF). In contrast to the expectations expressed by Takahashi and McDougal above, time limitations play a significant part in Adler and Alshwaikh's adaptation of lesson study in South Africa, with face-to-face collaboration restricted to six after-school hours over three consecutive weeks. Teachers planned the first lesson in advance of or during the first meeting on an agreed topic; this was taught in the second week in after-school time, following which there was a review meeting, which also prepared the second lesson for teaching after-school in the third week. This is an example of how lesson study is enacted in time-constrained settings where the Japanese model would not be possible. Some might consider this adaptation inappropriate, while others would applaud the flexibility and versatility of lesson study.

The research described in the chapter focuses on work of four teachers and the simplification of algebraic expressions using brackets in different positions, exploring two questions about changes in the example set across lesson plans and how these changes occur. This becomes a study of how teacher perspectives and plans change during reflection on research lessons. The researchers show how changes were collectively agreed as a result of the attention teachers gave to learner activity in the research lessons. Consequently, Adler and Alshwaikh are able to make a strong case for the inclusion of an explicit focus on ‘exemplification and example sets in lesson study’ given the critical contribution of examples in the teaching of mathematics. Like Takahashi and McDougal, this chapter highlights the importance of knowledgeable others but also stresses the essential driving agency of teachers and the importance of ‘theoretically informed observation and reflection’ in lesson study, in this case under the theoretical umbrella of MDI/MTF.

Chapter “How Variance and Invariance Can Inform Teachers’ Enactment of Mathematics Lessons” (Preciado Babb, Metz and Davis) is a study of variance and invariance, through a partial application of lesson/learning study, in the Maths Mind Initiative in Canada, using variation theory (Marton 2015) and ‘Chinese perspectives on variation’, along with a strong focus on continuous formative assessment in the classroom. Impressive gains are reported through the Maths Mind Initiative in three elementary schools, which enabled the research team to identify effective teaching strategies in order to inform the development of an observation protocol. Their application of variation theory is described in detail; beginning with the identification of necessary discernments, they discuss the intended, enacted and lived objects of learning.

Four critical elements of Maths Mind teaching, related to critical discernment of the object of learning, are described and elaborated: ravelling, prompting, interpreting and deciding. Two classroom examples are used to show how the four elements of ravelling, prompting, interpreting and deciding are combined to support more effective teaching and learning of (a) how to represent numbers up to 20 and (b) partitive and ‘quotitive division’. The key to effective learning lies in supporting learners to ‘notice and integrate critical features of the targeted learning outcomes’.

We learn how learning study can be used to enable teachers to interpret children’s understanding of number and division. Clear development of teacher thinking is illustrated in discussion of how the research lessons were crafted and amended through teachers’ decision-making (deciding). In the Maths Mind approach, explicit guidance from the teacher is considered important as are the use of formative assessment and feedback:

> While our approach may be seen as offering explicit guidance, such guidance involves creating conditions for students to make critical discernments of the targeted object of learning.

In chapter “Capturing Changes and Differences in Teacher Reflection through Lesson Study: A Comparison of Two Culturally Diverse Malaysian Primary Schools”, Kor, Tan and Lim, in a comparative study, explore the experience of two culturally diverse schools in Malaysia, one Chinese and one Malay, drawing on the research tradition associated with teacher reflection, including reference to Dewey (1933). For their analytical framework, the writers settle on Hatton and Smith’s (1995) four-level model of reflection as their critical lens: (i) descriptive writing, (ii) descriptive reflection, (iii) dialogic reflection and (iv) critical reflection.

The lesson study cycle followed the model proposed by Lewis (2009), with teacher learning evaluated in relation to Lave and Wenger's (1991) Situated Learning Theory. The role of knowledgeable others in promoting deeper reflection was considered critical, echoing the advice of Takahashi and McDougal in chapter "Using School-Wide Collaborative Lesson Research to Implement Standards and Improve Student Learning: Models and Preliminary Results".

Data analysis, from five reflection sessions, revealed that reflection was often judged to be at descriptive levels (descriptive storying and reflection), with only gradual progress to dialogic reflection and hardly any evidence for critical reflection, mirroring other studies that have highlighted shallow levels of reflection (e.g. Myers 2012; Parks 2009).

Fang, Wang and Kim-Eng in their chapter "Representing Instructional Improvement in Lesson Study Through Principled Analysis of Research Lessons: A Case of Equivalent Fractions" explore how discourse analysis can be used to analyse and document improvements in teaching through a 2006–2007 lesson study project. The study was framed by Bruner's CPA model (Concrete-Pictorial-Abstract), in which novice teachers evaluated how the CPA approach could enable learners in their lesson study classrooms to understand and work with equivalent fractions. In Singapore, lesson study has been officially recommended since 2009 as an approach that contributes to the building of professional learning communities in schools. In their adaptation, they explore using lesson study as a vehicle for mentoring novice teachers, in relation to the teaching of equivalent fractions, following diagnostic tests of third to fifth graders. Wells' (1999) discourse levels (move, exchange, sequence and episode) were used to analyse classroom discourse and levels achieved by their learners in a detailed iterative study which demonstrates the power of lesson study to support the growth of expertise in novice teachers.

Clivaz and Ni Shuilleabhain (chapter "What Knowledge Do Teachers Use in Lesson Study? A Focus on Mathematical Knowledge for Teaching and Levels of Teacher Activity"), through a case study involving eight grade 3–4 primary teachers in Switzerland, explore the types of knowledge, both subject and pedagogical, that teachers used in four lesson study cycles over 2 years. Analysis of the first lesson study cycle, teaching integer number and place value, is the focus of the research reported here. Two frameworks were used in combination, very clearly and informatively, to shape the research: Mathematical Knowledge for Teaching (Ball et al. 2008) and Levels of Teacher Activity (Margolinas et al. 2005). They were chosen to enable researchers to delve into the 'multi-layered knowledge required of teachers during various stages of teaching', a theoretical combination neatly captured in Figure 1 of their chapter.

One of the principal insights lies in the identification of the power of lesson study to afford opportunities for the explicit articulation of 'all levels of activity, from the values and conceptions about learning and teaching to seeing mathematics through the eyes of the student'. Consequently, teachers sought to visualise more (like the imagining reported in chapter "Implementing a New Mathematics Curriculum in England: District Research Lesson Study as a Driver for Student Learning, Teacher Learning and Professional Dialogue") as they prepared for teaching. Clivaz and Ni

Shuilleabhain make important contributions to our understanding of teacher learning, not least its nonsequential iterative nature in all phases of the lesson study process. Their adaptation did not include the presence of knowledgeable others, and they acknowledge that this absence may have contributed to the relative lack of 'horizon content knowledge' in teacher discussions in their case study. The authors concluded by identifying further avenues for research to deepen our understanding of teacher engagement in and development through lesson study.

The final chapter by Gunnarsson, Runesson and Håkansson reports a learning study project from Sweden conducted by three mathematics teachers, focusing on how students understand rate of change (water level and time), in a lower secondary school (27 students aged 15–16). The chapter contains another thorough but different discussion of variation theory and its application, to complement that in chapter "How Variance and Invariance Can Inform Teachers' Enactment of Mathematics Lessons", particularly in relation to the precise identification of critical aspects. The discussion of the research takes this further, guiding the reader through the identification of tentative critical aspects to their refinement into critical aspects. The steps of the learning study are very clearly outlined, including graphs and tables of data points used to vary the task sequence during the research lessons, providing a replicable account of how the sequence of lessons progressed and how students refined their understanding of the rate of change. In their conclusions, the authors argue that variation theory not only provided teachers with a common language to discuss pedagogy but critically helped them 'to focus on the object learning and students' learning in a relational way'.

> It presents the key question which must be asked as:
>
> To teach towards a learning goal, that is, what students are expected to achieve, the answer to the question "What must be learned to achieve the targeted goal?" must be found. Why? Because the answer to this question fills the gap between teaching content and teaching strategies (what and how to teach), and the learning objectives (the intended results of teaching and learning).

Gunnarsson, Runesson and Håkansson helpfully demonstrate how 'keys to learning' 'are refined in the process of the learning study', showing very clearly how a learning study can be realised. When teachers find the keys to learning, these are used to inform the planning of lessons, so that they are responsive to students' needs and promote successful learning. Despite variation from approaches recommended by Takahashi and McDougal, this focus on keys to learning returns us to their question in chapter "Using School-Wide Collaborative Lesson Research to Implement Stan dards and Improve Student Learning: Models and Preliminary Results": How can we design a lesson so that students learn a certain concept or skill better than they have in the past?' That is at heart of lesson study in all its adaptations.

3 Reflections Concerning Lesson Study Adaptations

The chapters contribute to our understanding of how lesson study is adapted and applied in a variety of settings, not only highlighting the complexity of learning and teaching but also demonstrating how lesson study can be adapted to open up that complexity for scrutiny and allow teachers to explore the pedagogic black box in order to afford opportunities for reflection on and improvement of practice. A range of theoretical frameworks is used in the studies reported, variation theory for learning studies, but a number of others for lesson study projects, which tended to evaluate impact on teacher development more explicitly than learner development.

The role of the knowledgeable other is highlighted as critical, although their presence was not possible in all the projects described. Increasingly, lesson study is becoming positioned as a significant contributor to teacher development at a range of levels, and this section provides several examples that should serve as both guide and inspiration to other lesson study users. These chapters provide convincing evidence that when teachers collaboratively explore what challenges their learners, then pedagogy is enriched.

References

Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education, 19*(3), 237–254.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: D. C. Heath.

Dudley, P. (2013). Teacher learning in lesson study: What interaction-level discourse analysis revealed about how teachers utilised imagination, tacit knowledge of teaching and fresh evidence of pupils learning, to develop practice knowledge and so enhance their pupils' learning. *Teaching and Teacher Education, 34*, 107–121.

Elliott, J. (1991). *Action research for educational change*. Buckingham: Open University Press.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: University of Cambridge Press.

Hatton, N., & Smith, D. (1995). Reflection in teacher education: Towards definition and implementation. *Teaching and Teacher Education, 11*(1), 33–49.

Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research, 17*(1), 95–110.

Margolinas, C., Coulange, L., & Bessot, A. (2005). What can the teacher learn in the classroom? *Educational Studies in Mathematics, 59*, 205–234.

Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.

Myers, J. (2012). Lesson study as a means for facilitating preservice teacher reflectivity. *International Journal for the Scholarship of Teaching and Learning, 6*(1), Article 15. Retrieved from https://doi.org/10.20429/ijsotl.2012.060115

Parks, A. N. (2009). Collaborating about what? An instructor's look at preservice lesson study. *Teacher Education Quarterly, 36*(4), 81–97.

Stenhouse, L. (1981). What counts as research? *British Journal of Educational Studies, 29*(2), 103–114.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap; best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. New York: Cambridge University Press.

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Using School-Wide Collaborative Lesson Research to Implement Standards and Improve Student Learning: Models and Preliminary Results

Akihiko Takahashi and Thomas McDougal

Contents

Abstract Many schools in America have been working to implement the new common standards for mathematics. In Japan, Lesson Study is used to incorporate revised national standards in the classroom. While there are many projects related to Lesson Study outside of Japan, they have met with varying degrees of success, often because they diverge from authentic Japanese Lesson Study. This chapter is built upon an article we wrote that appeared in *ZDM* in 2016. We developed Collaborative Lesson Research (CLR) for classrooms outside of Japan, based on Japanese Lesson Study. We have been piloting CLR projects in American and Qatari schools. Some schools had teachers who had at least some experience with Lesson Study, while other schools had no prior experience with Lesson Study. We are developing models to introduce CLR to both kinds of schools. We believe that the initial success of these projects shows that CLR may be used on a school-wide scale to implement new standards and improve student learning.

Keywords Lesson Study · CCSS-M · Collaborative Lesson Research

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Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_14

1 Introduction

In 2010, new national standards for K–12 mathematics, called the Common Core State Standards for Mathematics (CCSS-M), were introduced in the USA to replace the individual state standards. Forty-two states chose to adopt them (Common Core State Standards Initiative 2010); however, implementing them has been a challenge. CCSS-M comprises two parts: one part is content standards, which dictate which specific mathematical topics students should learn in each of grade K–8, and the other part, the Standards for Mathematics Practice, describes eight general practices that students should learn in all grade levels. These practice standards are largely based on the process standards from *Principles and Standards for School Mathematics* (NCTM 2000) and include mathematical problem-solving, reasoning, and communication. However, some of these standards can be said to go back several decades. For example, a focus on problem-solving in school mathematics was advocated in 1980 in *An Agenda for Action* (NCTM 1980). Problem-solving was also a theme in the *Curriculum and Evaluation Standards for School Mathematics* (NCTM 2000). However, problem-solving, reasoning, and communication remain stubbornly absent from many US classrooms. NCTM (2014) argues that we must move from “pockets of excellence” to “systemic excellence.”

Mathematics instruction in the USA has changed very little since the early 1900s (National Research Council 2001; Stigler and Hiebert 2009). Changing *what* is taught at each grade can probably be addressed by changing the textbooks. But history suggests that changing *how* mathematics is taught is more difficult. Given the lack of progress in American education in changing teaching practices, it is worth considering other models of professional development than what is currently being used.

We are currently testing, in eight urban US schools, a model that we call Collaborative Lesson Research (CLR), which is based on Japanese Lesson Study. We are also using what we have learned from those schools to run a CLR project in Qatar. The results from the beginning stages of these projects show that CLR, used on a school-wide basis, can help teachers meet the challenges of changing both what they teach and how they teach.

2 *Jyugyou kenkyuu* versus Lesson Study

Jyugyou kenkyuu has been the primary form of professional development for educators in Japan for over a hundred years. It was introduced outside of Japan in the late 1990s and translated as “Lesson Study” (NCTM 2000). Early research articles were based on studies of Japanese schools (National Council of Teachers of Mathematics 1989), but did not explain the rationale behind the teachers’ actions. These articles also did not determine which parts of the process are essential and which parts could be modified for a non-Japanese classroom.

2.1 Early Attempts to Implement Lesson Study outside Japan

Educators around the world have attempted to use Lesson Study to shift from traditional teacher-centered instruction to student-centered learning focused on mathematical thinking and problem-solving (e.g., NCTM 2014; Stigler and Hiebert 1999). However, almost none of these educators had any prior experience with *jyugyou kenkyuu*. While some of these projects faithfully followed the descriptions of Japanese Lesson Study, most adapted the process to fit the limited time frames and resources that their schools offered.

As a result, the effectiveness of these projects was uneven. There are only a few cases that show strong evidence of the impact of Lesson Study on teaching and learning (NCTM 2000). For example, the Lesson Study Group at Mills College conducted a randomized, controlled trial of Lesson Study using mathematical resource kits which resulted in a significant impact on both teachers' and students' mathematical knowledge (Catherine Lewis and Perry 2017; C. Lewis and Perry 2014). A recent review, which used a process modeled on *What Works Clearinghouse* guidelines, examined 643 studies on mathematics professional development (Gersten et al. 2014b). Out of those 643 studies, only the abovementioned 2014 Lewis and Perry study and one other study actually met scientific criteria and showed substantial impact on student learning.

In Japan, Lesson Study plays a critical role in the effective implementation of new curricula (e.g., NCTM 2014; Stigler and Hiebert 1999). We believe that Lesson Study can support the implementation of CCSS-M here in the USA and local standards abroad. However, Lesson Study has been less impactful outside of Japan. We believe that this is because important aspects of Japanese Lesson Study are simply getting "lost in translation." We introduced Collaborative Lesson Research to capture the important aspects of Lesson Study.

2.2 Understanding Lesson Study

Lesson Study is an integral part of teaching for Japanese teachers. One teacher describes it as "like the air" (Fujii 2014). This is in part why it has been hard for outsiders to understand Lesson Study. However, both recent studies of Japanese Lesson Study and flawed attempts to use Lesson Study outside of Japan have helped to highlight and clarify the practice as a whole.

2.2.1 Lost in Translation

Takahashi, the main author of this article, practiced Lesson Study as a teacher in Japan. He has nearly 20 years of experience observing activities referred to as "Lesson Study" that looked very different from what he knows as Japanese Lesson

Study. Many Lesson Study projects outside of Japan omit some of the crucial elements, which hinder their success. For example, Fujii (NCTM 1989) examined Lesson Study in some African countries and noted that many aspects of Japanese Lesson Study are left out. The same occurs in America; many projects omit the first crucial phase of Lesson Study, *kyouzai kenkyuu*[1], the "study of teaching materials," which helps teachers gain knowledge and insight into mathematics and student thinking (e.g., NCTM 2014; Stigler and Hiebert 1999). Leaving out important steps of Japanese Lesson Study limits the effectiveness of these projects.

Takahashi has seen that the time frame is also often severely compressed for many Lesson Study projects outside of Japan. Some school districts tried to fit an entire Lesson Study cycle into a single day. In the morning, a team of teachers came together and spent 30 minutes planning a lesson. They taught this lesson to students and recorded what they observed. That afternoon, they spent 30 minutes modifying the lesson plan and then taught a revised version of the same lesson. On the surface, this 1-day process includes all the components of Lesson Study as it is described in most journal articles and resources. However, the typical duration of a Lesson Study cycle in a Japanese elementary school is at least 5 weeks (Murata & Takahashi, 2002). It is certainly never done in just a single day.

Takahashi has also observed that there are also often misunderstandings about the purpose of Lesson Study. One team of educators conducted multiple research lessons on the same topic, based on the misconception that the purpose was to create a perfect lesson plan. They did Lesson Study six times on the same lesson. As a result, they felt afterward that they had not learned much from the last lesson. This is unsurprising, as a major purpose of Japanese Lesson Study is to gain new knowledge for teaching and learning, not to perfect a specific lesson plan. In fact, reteaching is generally never done in Japan (Fujii 2014).

Another pitfall that Takahashi has witnessed is that many teachers and educators misconstrue Lesson Study as showcasing the best practices for teaching mathematics. For example, they may invite an experienced teacher to teach an unfamiliar group of students in an auditorium in front of a large audience. This may demonstrate innovative teaching strategies, but it does not necessarily help teachers implement those ideas into their classroom. Japanese teachers and educators refer to these kinds of demonstration lessons as s*hihan jyugyou* to distinguish them from research lessons, *kenkyuu jyugyou*.

2.2.2 Investigating Japanese Lesson Study

Researchers have been investigating how and why Japanese teachers use Lesson Study and how it helps them improve their mathematics teaching and learning. This research helps both Japanese and non-Japanese educators better understand the conditions necessary to conduct effective Lesson Study (e.g., Lewis et al. 2006;

[1]*Kyouzai kenkyuu* is discussed in more detail in Sect. 3.2.

Murata & Takahashi 2002; Watanabe 2002; NCTM 2014; Stigler and Hiebert 1999). What this research revealed is that Japanese Lesson Study is typically a highly structured, school-wide project, involving all or nearly all of the staff, aimed at addressing a common teaching-learning challenge (e.g., NCTM 2014; Stigler and Hiebert 1999). This contrasts with most Lesson Study projects outside of Japan, which are done by small groups of volunteer teachers independent of their school's official professional development programs.

Another important finding from this research concerns the role of supporting professionals, *koshi*, often referred to in English as "knowledgeable others." A knowledgeable other is someone from outside the school's planning team who has content expertise, often has teaching expertise, and is well-versed in Lesson Study. Many Lesson Study projects in the USA are done by teachers under the guidance of a facilitator who knows the Lesson Study process, but this is not the same as having the support of a knowledgeable other. Japanese Lesson Study almost always includes a knowledgeable other who provides final comments during the post-lesson discussion. Sometimes a second knowledgeable other is invited to join the planning phase; their role is to draw attention to key issues (Makinae 2010). Takahashi conducted a case study, based on an earlier study by Watanabe (e.g., C. Lewis and Tsuchida 1998; Yoshida 1999; Stigler and Hiebert 1999), that looked at three experienced knowledgeable others in Japan. The purpose of this study was to better understand the roles of knowledgeable others. Takahashi noted several ways in which their final comments helped participants connect the lesson with larger mathematics and pedagogy issues. Lewis argues that knowledgeable others from outside the planning team may be critical to scaling up successful school-based Lesson Study in the USA (e.g., Hart et al. 2011).

2.2.3 Important Elements of Japanese Lesson Study

Lesson Study has been the primary mechanism of professional development for both prospective and practicing teachers since the Japanese public education system started (e.g., C. Lewis et al. 2006). The most common form of Lesson Study takes place within a single school as an official school-wide professional development program (C. Lewis and Perry 2014). A very common purpose of school-based Lesson Study is to seek practical ideas for the effective implementation of the Japanese national curriculum (Gersten et al. 2014a).

Japanese teachers begin Lesson Study by carefully reading the curriculum and studying other relevant materials in a process called *kyouzai kenkyuu*, or "study of teaching materials" (Takahashi 2014b). Based on their *kyouzai kenkyuu*, they then design a lesson focused on addressing certain issues that relate to a broader teaching and learning research theme. This lesson, known as a "research lesson," is taught by a teacher from the planning team, while the other team members and educators who are not on the planning team observe. The planning team and observers then conduct a post-lesson discussion about how students responded to the lesson. This enables them to gain insights into the teaching-learning process and into how the curriculum should be implemented (Fujii 2014).

We examined the "Lesson Study" projects that so clearly deviated from Japanese Lesson Study and the research on Japanese Lesson Study. We realized that the following features are key for effective Lesson Study:

1. Participants engage in Lesson Study to build expertise and learn something new, not simply to refine a certain lesson.
2. The Lesson Study is part of a highly structured, school-wide, or district-wide process.
3. It includes significant time spent on *kyouzai kenkyuu*, the "study of teaching materials."
4. It is done over several weeks, rather than just a few hours.
5. Knowledgeable others contribute insights during the post-lesson discussion and during planning as well.

We applied these principles to develop the idea of CLR. School-wide support is crucial. In this chapter, we will show how we are using CLR in American districts to implement CCSS-M as well as in Qatar to implement local standards.

3 Collaborative Lesson Research (CLR): A Powerful Form of Lesson Study

We developed and proposed CLR as a program to adapt Lesson Study to classrooms outside of Japan (2014). The term is drawn from Catherine Lewis's original translation of *jyugyou kenkyuu* as "Lesson Research," which we revive in order to emphasize the research aspect (Takahashi et al. 2005; C. Lewis et al. 2011). We developed this program based on our experience and research on Lesson Study in Japan (Murata and Takahashi 2002), as well as our experiences working with schools outside of Japan. CLR is an investigation undertaken by a group of educators, usually teachers, using live lessons to answer shared questions about teaching and learning. CLR is defined as having the following components:

1. A clear research purpose
2. *Kyouzai kenkyuu*, the "study of teaching materials"
3. A written research proposal
4. A live research lesson and discussion
5. Knowledgeable others
6. Sharing of results

Each component will be elaborated upon in the following sections. If any component is missing, then the activity cannot be called CLR. Lessons learned from one CLR cycle can lead to revised theories about how to address the research theme or adjustments in the theme itself, which can lead to another cycle on a different topic perhaps taught at a different grade level (Fig. 1).

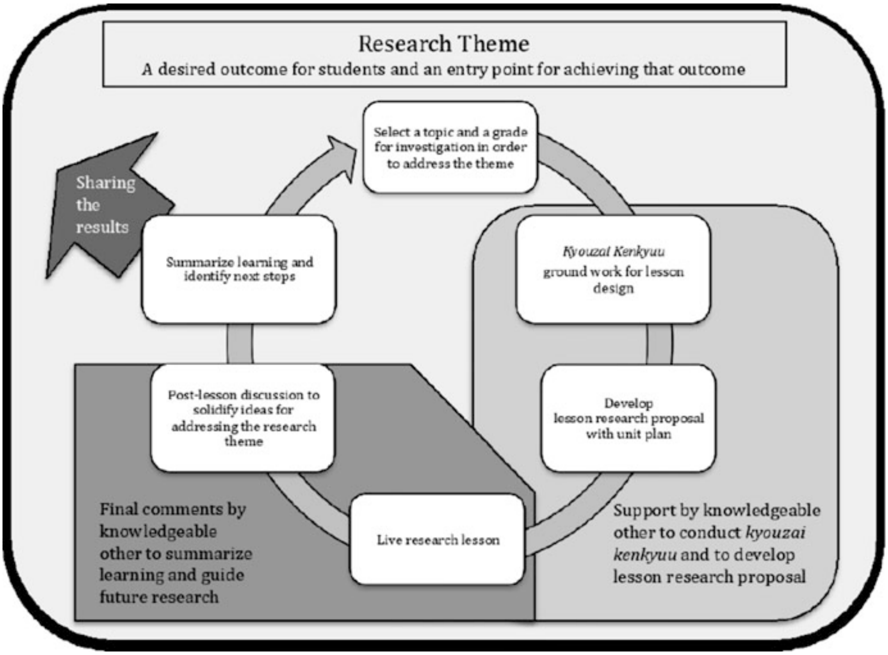

Fig. 1 Collaborative Lesson Research (Takahashi and McDougal 2016)

3.1 A Clear Research Purpose

The research focus of CLR usually has two layers. One is the teaching of specific content: how can we design a lesson so that students learn a certain concept or skill better than they have in the past? Therefore, the topic of the research lesson is typically presented as a challenge for students or teachers. The second layer involves a broad teaching-learning goal shared by the CLR community that goes beyond any particular topic or grade level and may even be cross-disciplinary. This second layer is referred to as the "research theme."

There is no widely shared definition of "research theme," but in our view, a research theme describes (a) a desired outcome for students and (b) an entry point for achieving that outcome. For example, say a group of teachers want to improve their students' ability to give a viable argument and critique the reasoning of others. The entry point for this goal could be teaching students to use journals to record their own and others' ideas. The research theme statement should be short enough to be memorable. In this example, the statement could be, "For students to be able to clearly explain their thinking and consider the ideas of others through the support of their own journals."

We have seen in both America and in Qatar how a compelling research theme is an important motivator. As teachers become conscious of a gap between the outcomes they desire for their students and what they have been able to achieve, they become eager for the opportunity to work together to close that gap. When the research theme is relevant to all grade levels, teachers see how they can benefit by observing research lessons with students older or younger than their own. This can be seen especially at the schools in our Qatar project, where they chose one common theme for all the schools, “Fostering students’ ability for problem-solving and reasoning by using teaching through problem solving.” This research theme is relevant to all the schools because they all share the same concern in implementing their national standards. By working together on this compelling theme, they are learning from each other about the best ways to improve teaching and learning to reflect the standards.

3.2 *Kyouzai Kenkyuu: The Study of Teaching Materials*

Kyouzai kenkyuu, the careful study of academic content and teaching materials, is integral to Japanese Lesson Study (e.g., C. Lewis et al. 2006; Murata and Takahashi 2002; Watanabe 2002; Fernandez and Yoshida 2004; Shimizu 2002). It is analogous to a literature review in scientific research. It involves an investigation of the intended learning trajectory related to the topic from lower to higher grades, through a review of the standards and curriculum, and research into teaching and learning issues such as typical misunderstandings students may have about the topic. *Kyouzai kenkyuu* also includes consideration of possible tasks, tools, manipulatives, or materials that may be used. Thorough *kyouzai kenkyuu* helps avoid “reinventing the wheel,” making it more likely that CLR will contribute new knowledge to the education community.

One of the obstacles the teachers at the American schools in our projects faced was the quality of available materials to support *kyouzai kenkyuu*. Generally speaking, there are many more materials available to Japanese teachers (Takahashi 2014b; Takahashi and McDougal 2014; Yoshida 1999). Most American textbooks are designed to support didactic instruction, which does not help students become independent problem solvers. At our American schools, no curricula were available that aligned well with the new standards. Therefore, the teachers at these schools had to rely on a translation of a Japanese textbook series to conduct *kyouzai kenkyuu* (Fujii and Iitaka 2012).

3.3 *A Written Research Proposal*

A CLR planning team creates a written document, called the *lesson research proposal*, to communicate what the team learned from their *kyouzai kenkyuu* and to explain their instructional thinking. It includes learning goals for a unit, an overview of the unit, a detailed teaching-learning plan for one particular lesson within the unit (the *research lesson*), the rationale for the design of the unit and research lesson, and a clear statement about how the research lesson aims to address the research theme and learning goals. In our experience, a thorough lesson research proposal may be nine pages long. The authors developed a template document to help CLR teams organize their Lesson Study work and write their lesson research proposals.[2]

The following represents the components of a typical lesson research proposal:

- Title of the lesson.
- Brief description of the lesson.
- Research theme: Broad goals of the CLR cycle.
- Goals of the unit.
- Goals of the lesson.
- Relationship of the unit to the standards: How this unit relates to the standards from prior grades and those for the current or later grades.
- Background and rationale: Justification for the choice of theme and topic, usually expressed in terms of the contrast between the current state of students (or students in previous years) and what you and your colleagues want to accomplish.
- *Kyouzai kenkyu*: Findings from various curricula and other resources, as well as the rationale behind the tasks, the manipulatives, and the unit and lesson design
- Unit plan: How this lesson fits into the larger unit. Briefly describes lessons before and after the research lesson.
- Unit and lesson design: Briefly describes the unit and lesson design and the rationale for those choices.
- Research lesson plan: May use a structure such as the one seen in Fig. 2 to describe the lesson plan. It is best to include anticipated students' responses, including possible misunderstandings and incorrect answers, as well as questions the teacher can use to help guide them.
- Evaluation: Often includes questions that the planning team hopes to explore through the lesson and post-lesson discussion.
- Board plan: A diagram showing how work will be organized on the blackboard. It is best to run a simulation before finalizing the board writing plan. Have some teachers act as students and the teacher who will teach the research lesson practice organizing the work on the board. It is also helpful to include a photo of the board taken after the simulation.
- Reflection: Written after the lesson, this typically includes what the team had hoped to observe during the lesson, what was actually observed during the lesson, major points raised during the post-lesson discussion and the team's own opinions, points made by the knowledgeable other, and ideas for future study. This reflection may be a few paragraphs in length and makes the final document much more valuable to an outside audience.

[2] http://lsalliance.org/public_docs/lesson_research_proposal_template.doc

Flow: Main activities	***Support from Teacher:*** Actions, questions, or statements that the teacher may need to make to help students	***Points of Evaluation:*** What the teacher and observers should look for to determine whether each segment of the lesson is having the intended effect.
1. Introduction: Review ideas from a prior lesson or discuss a simple problem designed to prepare students for work on the main problem.		
2. Posing the Task: How the problem or task will be presented to students. Exact phrasing of the key question(s) and the specific numbers used.	Indicate here whether the problem will be written on the board, posted, handed out as a worksheet, or glued into student notebooks.	
3. Anticipated Student Responses: How students might respond to the problem, including incorrect solutions and places where they might get stuck. It can be helpful to tag different responses such as "R1" for "Response 1," etc. R1: 2 + (3 * 5) [correct] R2: 3 x 5 = 15; 2 + 15 = 17	Can describe how the teacher will handle different student responses, especially incorrect solutions, students who get stuck, or students who finish early.	
4. Comparing and Discussing: Which student solution methods should be shared and in what order, or how to handle the discussion.	What are the ideas to focus on during the discussion?	
(If needed, repeat steps 2, 3, and 4)		
5. Summing up: How the teacher will summarize the main ideas of the lesson. May also include an assessment activity.		

Fig. 2 Typical outline of a research lesson plan

3.4 A Live Research Lesson and Post-Lesson Discussion

One member of the team teaches the research lesson, observed by the entire planning team and additional members of the CLR community. Observers collect data on how the lesson impacts the students, relative to the research theme and the learning goals. Ideally, the research lesson should be taught by the students' usual teacher in their regular classroom. A video recording of the lesson can be useful. However, CLR requires observations of student learning process from multiple viewpoints, so video cannot substitute for actual live observation.

As soon as practical after the research lesson, observers share their data and discuss what they mean, especially with respect to the learning goals of the lesson and the research theme. We call this activity the "post-lesson discussion." This is a cumbersome term compared to the more popular "debrief." However, "debrief" implies a simple report without discussion, and discussion is exactly the point we wish to emphasize. The primary goal of this discussion is to gain insights into teaching and learning and to inform the design of future lessons, *not* to revise the lesson plan. These discussions generally benefit from a moderator, someone not on the planning team or acting as a knowledgeable other, who helps focus the discussion on important issues and keeps the conversation grounded in data.

3.5 Knowledgeable Others

"Knowledgeable others" are persons with both extensive knowledge of the lesson topics and CLR. They are invited by the team to provide valuable expert advice. Ideally, a CLR community would have two knowledgeable others: one to help with the development of the research lesson proposal and another to provide final comments at the end of the post-lesson discussion.

During planning, a knowledgeable other may share relevant instructional examples, articles, results from other CLR projects, and feedback on the research lesson proposal. Besides having vast knowledge of the subject matter and lesson topic, the knowledgeable other should also be familiar with the school's curriculum and students. An experienced teacher or a content coach who often works at the school may play the role of this kind of knowledgeable other. In one of the schools in an urban American district that is participating in our CLR project, this role is played primarily by the math and science coordinator. At our other schools in the district, one or both of us join some of the Lesson Study meetings to act as the knowledge other. As local teachers gain experience with Lesson Study, we expect that they will also be able to provide this service to each other in the future.

The second knowledgeable other is needed during the research lesson. At the end of the post-lesson discussion, they are expected to highlight important events from the research lesson that were not discussed and to make connections between the lesson and new knowledge from research and standards. This knowledgeable other also provides suggestions to the CLR community of possible future steps they could take toward accomplishing their research theme (Watanabe and Wang-Iverson 2005).

3.6 Sharing of Results

CLR should also contribute to the education community at large by including a structure or process for disseminating what was learned from each research lesson. Simply inviting people from outside the planning team to observe and discuss the research lesson is one valuable way that CLR teams can do this. The team can also distribute their research lesson proposal, which can be useful to other educators as it encapsulates their research and instructional ideas. This document is made more powerful through the addition of a written reflection by the team, which describes what they learned from the live lesson observation and post-lesson discussion about their research theme, research lesson proposal, mathematics, student thinking, teaching, etc. At the schools we are working with in an urban American district, the school administrators have found ways to enable teachers who are not on the planning team to participate in observing and discussing the research lessons and have often invited teachers from other schools to observe them as well.

4 The Importance of School-Wide Support for Lesson Study

Takahashi and his colleagues have been working in a large American urban school district at over 30 different schools to help teachers improve mathematics teaching and learning using Lesson Study since 2002. However, almost all these projects were done by small teams of enthusiastic volunteers, disconnected from school initiatives. The teachers on each team often came from different schools. Takahashi has many years of experience with Lesson Study as a teacher in Japan, so we are confident that the work done by these teachers captured the most important aspects of Lesson Study. Unfortunately, these schools discontinued Lesson Study after just a few years as teachers moved away, schools changed administrators, or teachers just grew tired of trying to practice Lesson Study without adequate time or support from administrators and colleagues. Lesson Study could not thrive without school-wide support, which is why it is a crucial element of CLR.

When CCSS-M was adopted in 2010, with full implementation to begin in the fall of 2014, the professional development climate in this district changed. Many teachers and administrators recognized the magnitude of the changes that they needed to make, both in terms of content and instruction. Furthermore, they recognized that the professional development they were getting was not adequate, although they weren't always sure what was missing. A study of professional development in three large US cities found no consistent impact on teacher growth (2014a). Most of that professional development in those cities focused on building teachers' knowledge of teaching. However, teachers need to develop their teaching expertise and their ability to teach problem-solving, reasoning, and communication skills as called for by the new standards.

We are now focusing on public elementary schools where the administrations not only support CLR but want to make it a routine component of professional development for all teachers. These schools, which serve students from kindergarten through grade 8, are using CLR as their major form of professional development for mathematics. All of these schools are high-poverty schools whose students face many challenges. Expanding Lesson Study to school-wide CLR can be challenging. From over 16 years of experience in this district, we find that the following elements seem to be both necessary and sufficient to ensure a successful school-wide implementation of Lesson Study:

1. Enthusiasm for Lesson Study from the school principal, clearly communicated to the faculty
2. A persistent Lesson Study advocate in addition to the principal
3. A compelling school-wide goal for teaching and learning
4. Commitment on the part of the school administration to provide time for Lesson Study through use of funds, staff, and district-mandated professional development time

The rest of this chapter will discuss how schools can use school-wide CLR to drive durable, long-term change in teaching and learning and meet the expectations of new standards.

5 Employing School-Wide CLR to Help Implement New Standards

Lesson Study in Japan is commonly used to help implement revisions to the national standards and is usually conducted as a major part of school-wide professional development. In the same vein, CLR on a school-wide level is ideal to help schools fully implement new standards, especially those that relate to teaching problem-solving skills. The best way to establish school-wide CLR varies depending on the school and the teachers' previous professional development experience. We are currently testing two pathways for establishing school-wide CLR, one in schools which have had some previous experience with Lesson Study and another project in which the schools had no prior experience with Lesson Study.

5.1 Schools Which Have Some Experience with Lesson Study

In January 2015, we began piloting school-wide CLR in an urban American school district to implement CCSS-M. Our goal is to develop a model for establishing school-wide CLR in schools that already have a small group of teachers experienced with Lesson Study. The teachers in this study had already voluntarily participated in annual Chicago Lesson Study conferences and at the Lesson Study Alliance Summer Institute. We built upon their knowledge and experience to expand their small-scale volunteer Lesson Study projects to school-wide CLR. This path can be broken down into three major phases, which we will discuss in depth. We believe that these phases can be adopted by other schools where some teachers are already familiar with Lesson Study to help those schools make CLR a school-wide professional development program.

5.1.1 The First Phase

Some ground work needs to be done before the school year and the full implementation of school-wide CLR begins. Both a school research steering committee and research theme must be established. This can be done either during the summer or at the end of the previous school year.

The school research steering committee typically includes teachers from various grade levels and a teacher leader. The committee leads the school's CLR efforts, ensuring that what is learned from each research lesson is disseminated to the rest of the school and that cohesiveness of ideas across the grades is maintained (C. Lewis et al. 2006). The research steering committee is responsible for:

1. Developing a master plan for the school's research
2. Scheduling and leading meetings to find strategies to address the school's research theme
3. Planning, editing, and publishing school research reports, including reports on a research lesson open house
4. Arranging for knowledgeable others to give lectures, teach demonstration lessons, and share advice about the research lessons

One of the first tasks of the research steering committee is to create a schedule of research lessons for the following year.

Another important step in the first phase is to establish a research theme, which will focus teachers' efforts on implementing the new standards. This research theme should reflect a gap between the school's educational goals and the actual state of the students. A research theme can be drafted by the research steering committee, but it should be approved by a consensus of all the mathematics teachers.

To accommodate developmental differences in students, it is necessary to create at least three "CLR teams," which break up the grades into smaller subgroups or grade bands. For example, one team could be from kindergarten to second grade, another could be third to fifth grade, and another from sixth to eighth grade. These teams are responsible for planning their own research lessons, and each team should create an ideal student profile of characteristics in terms of the research theme. One of our schools chose the research theme, "Supporting students' ability to explain their mathematical thinking and the thinking of their peers using evidence based strategies commonly used in Literacy." Their ideal student profiles are as follows:

- *Lower Grade Band (K–2):* Students will be able to verbally explain how and why they solved a problem without being prompted and progress to a written explanation by the end of the year.
- *Middle Grade Band (3–5):* Students will be able to provide a written explanation of how and why they solved a problem and begin to respond to their peers with clarifying questions, adding relevant opinions, and disagreeing by identifying errors in the mathematical process with prompting from the teacher.
- *Upper Grade Band (6–8):* Students will be able to use Middle Grade Band to discussion strategies to facilitate class discussions about how and why they solved a problem. In addition, students will be able argue multiple solutions through use of their math journals.

Creating ideal student profiles provides observable behaviors or outcomes by which success can be measured. Each team should also come up with concrete steps they can apply in their everyday lessons that will help students develop the characteristics of the ideal profile.

5.1.2 The Second Phase

The second phase of school-wide CLR consists of multiple cycles of CLR (Fig. 1). Each CLR team conducts two cycles during the year to test and refine their ideas about how to overcome the issues they identified during the first phase. Since there are at least three teams, each school will have at least six CLR cycles that year. Teachers will participate in their team's two CLR cycles, planning and conducting the research lessons. They are also encouraged to participate in all the research lessons and post-lesson discussions of other CLR teams. This means that each teacher will engage in discussions about the school research theme and research lessons almost every month of the school year. In this way, schools will become a place for teachers to improve mathematics teaching and learning, and implementing the new standards will be a collaborative effort.

At the end of a CLR cycle, the team summarizes their learning from the research lesson. They then submit a report to the research steering committee. When all the teams have submitted their reports, the research steering committee reviews, consolidates, and shares the results with the teams.

5.1.3 The Third Phase

The third phase of school-wide CLR takes place during the following school year. Just as in the second phase, multiple cycles are CLR are conducted, but the purpose of these cycles in this third phase is to refine and consolidate findings on the research theme. The school then publishes a report on their work, which includes:

1. How the school came up with the research theme
2. All research lesson plans from the 2 years with summaries of the post-lesson discussions and key learnings
3. A summary of a wrap-up discussion at the end of the school-based CLR project

It may be a good idea for a school to continue the CLR project until they see real impacts on student learning in regard to the research theme. Typical Japanese CLR projects take several years.

5.1.4 Initial Changes Observed Regarding Teacher and Student Learning

Many of the components of CLR were successfully in place a few months into the first phase of this project. Both teachers and administrators expressed great satisfaction. The following teacher statements are some examples:

> Lesson Study has enabled me to reflect upon my own teaching, not only individually, but within a supportive team. It has helped push me to be more introspective and ask myself the question “why” within my planning and instructional decisions.
>
> Lesson Study has completely changed my approach to mathematics instruction… My math pedagogical knowledge has increased tremendously. It is through Lesson Study that I’ve been able to truly reflect and gain insights into children’s mathematical understandings/misunderstandings.
>
> We are loving Lesson Study. This could be transformative for our school. Analyzing research, creating lessons, and discussing student performance with other teachers is clearly the most productive professional development for a teacher. We clarified each other’s misunderstandings as we read the material on fractions and discussed how ideas could be utilized in our classrooms. As teachers, we enriched our own understanding of fraction content and student perceptions…This Lesson Study has profoundly affected the activities I use.
>
> As I watched the lesson unfold I saw how, with good intentions, we teachers stop the thinking of our students by providing too much scaffolding…I saw students working themselves from an incorrect answer to recognizing the answer was wrong, puzzling over how to correct it, only to have a teacher ask “yes-no” questions that stopped their problem solving and led [the students] to the correct answer. I recognize this trait in myself and have committed myself to allowing the students time to struggle and…an opportunity to learn from mistakes. This will impact all of my instruction, not just fraction work.

Even in the early stages of the first phase, the teachers felt that CLR is having a significant impact on teaching and learning.

As for the CLR components not yet in place, teachers and administrators are beginning to recognize the need for them. For example, one principal asked about ways to share valuable insights from a post-lesson discussion with teachers who were not present. This would be the “sharing of results” component from our list of the six necessary components that define CLR. Thus, we are optimistic that CLR fits the needs of the teachers and may become a primary form of professional development.

We also observed changes in the students. At one of the schools where developing students' note-taking skills was emphasized in their school-wide research theme, students are writing their ideas for solving problems, including diagrams, mathematical expressions, and reasoning.

5.2 Schools Which Had No Prior Experience with Lesson Study

The existing culture in primary schools in Qatar does not encourage teachers to study mathematics content and pedagogy to improve their classroom instruction. To address this issue, a project at Tokyo Gakugei University called the International Math-teacher Professionalization Using Lesson Study (IMPULS[3]) and the College of Education at Qatar University (QU) started a CLR project in 2014 to establish a professional development program in Qatari independent schools. The goal of this QU-IMPULS Lesson Study project is to teach primary school students the problem-solving skills they need to be ready for secondary education, as required by their national standards.

5.2.1 The First Year of the Project: Phase 1

The first year of the project, or Phase 1, focused on nurturing professional development specialists (PDSs) from the National Center for Educational Development at Qatar University to become CLR leaders. These PDSs had special knowledge about teaching, but as CLR leaders, they needed to develop expertise in observing student learning, designing lessons based on those observations, and evaluating the student learning process to ensure that students are learning the necessary reasoning and problem-solving skills.

The PDSs did not have any Lesson Study experience. Therefore, the goal of Phase 1 was for them to deepen their own knowledge of teaching mathematics and to learn how to conduct CLR. Although becoming a CLR leader typically requires several years, the PDSs received intensive support so they could finish Phase 1 as CLR leaders.

During Phase 1, they observed a total of 18 research lessons, 2 of them taught by Japanese experienced teachers at a Japanese school in Doha. The PDSs also themselves planned and taught 16 research lessons at a QU-affiliated independent school. Project IMPULS personnel led professional development sessions, including live research lesson observations and post-lesson discussions. As a result of this intensive professional development, the PDSs reported that their view of teaching and learning mathematics shifted by deepening their understanding of problem-solving,

[3]International Math-teacher Professionalization Using Lesson Study.

mathematical reasoning, and teaching through problem-solving. Also, they reported readiness to lead the teachers of the QU partner independent schools in order to conduct Lesson Study.

5.2.2 Initial Changes Observed Regarding Teacher and Student Learning

For the first two research lessons, most of the PDSs' comments during the post-lesson discussion were mainly advice about what the teacher should have said, rather than how to support the students to think mathematically. As the PDSs observed more mathematics lessons designed to teach problem-solving and mathematical reasoning, they began to comment more about how to nurture students' mathematical thinking:

> My main learning point of today is that all students have potential and are creative, and they can think and communicate mathematically if we give them the opportunity and enough time to think and answer.
>
> The discussion that we have after the lesson made me think deeply about the way that we look as educators to the students. Even when no students reach any answer, some of the comments were very optimistic about the "trying" to find any solutions. I learned that it is very important for the teachers to have skills of building on that very little amount of results obtained from students' attempts. Instead of presenting the solutions, the teacher should have the prompting skill. Leading questions are very important in running the class successfully.

The PDSs, professors, and teachers already knew key terms that came from the national standards, such as "problem-solving" and "reasoning," but simply knowing these terms was not enough to help their students develop these abilities. That only came after observing and teaching several research lessons.

The PDSs designed a series of four lessons for seventh grade students. Each PDS taught the four lessons as a mini unit. All the PDSs and IMPULS professors observed and discussed the lessons. This intensive CLR experience taught PDSs to focus more on the student learning process:

> We have to care about the way the students learn and think, not just about the final solution of the problem.
>
> The students during the discussion show a good level of their thinking and discussion which also shows me the benefit of changing the lesson plan.
>
> The way which the teacher followed to collect information from the students and built on it, then asked them to help each other during the discussion, helped them to observe their progress and gave them self-confidence.
>
> We should work with a belief in our ability to make a change in the way of teaching math. We could see the potential that students have to learn through using problem solving. We saw that potential because we focused our observation on students' performance. Usually, when observing classes, the focus is on the teacher performance, and the observer provides comments on the teacher's effort without looking carefully to the students abilities. I learned in this program to look carefully and observe the details of the students' work to

> collect data. All the statistics that we had after observing the lesson made a difference in deciding what is the next step.

School-wide CLR gave them the hands-on experience they needed to realize they must understand how their students think mathematically in order to improve their teaching methods.

Teachers need an appropriate professional development program to develop necessary teaching expertise. Participants in our project felt that CLR had helped them:

> Any changing or modification in the lesson plan occurred because of a reason that supported by evidences. Those evidences will not be collected if we keep observing only the teacher performance as in the traditional way. This close observation for the students' work that allow us to collect the data that help a lot in taking decisions to modify based on real needs for the students. That is called professional development.

This first-hand experience helped PDSs see how CLR can help implement national standards. This project provides the necessary experience for educators to become PDSs, leaders of CLR. With the proper support of PDSs, even schools with no prior experience of Lesson Study could start conducting cycles of CLR.

6 The Future

In the summer of 2002, a joint US/Japan seminar titled "The professionalization of teachers through Lesson Study" was held in Utah. One of the major goals of the seminar was to clarify the mechanisms and operating principles of Lesson Study. However, at that time, Japanese mathematics education researchers and teachers did not have clear definitions to distinguish authentic Lesson Study from Lesson Study-like activities. Now, more than a decade after Lesson Study was introduced outside of Japan, the important mechanisms and operating principles of effective Lesson Study are becoming clear. We have coined the term CLR and have been studying its implementation in schools in different countries.

Our CLR projects have resulted in clear changes in teacher practice, and we expect they will yield clearly identifiable improvements in student learning. They also appear sustainable. School-wide Lesson Study has been conducted at other US schools for periods of 5 years or more and has shown impact on student achievement (C. Lewis 2000; C. Lewis and Tsuchida 1998; Murata and Takahashi 2002; Takahashi 2000; Takahashi and Yoshida 2004; Makinae 2010; Yoshida 1999). Accounts of these school-wide Lesson Study efforts (Yoshida 1999) and videos of key elements of the process[4] illustrate the power of teachers developing a shared research theme and conducting Lesson Study cycles to investigate and improve instruction. Although it is premature to assess the full impact on student learning at

[4] http://lessonresearch.net/resources2.html

our projects in American and Qatari schools, in the future we hope to use the data we collect to assess the viability and effectiveness of implementing school-wide CLR.

Acknowledgments This publication is based in part on a project funded by the Bill & Melinda Gates Foundation. The findings and conclusions contained within are those of the authors and do not necessarily reflect positions or policies of the Bill & Melinda Gates Foundation.

This publication is based in part on a project funded by the Qatar Petrochemical Company (QAPCO). The findings and conclusions contained within are those of the authors and do not necessarily reflect positions or policies of QAPCO.

References

Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. http://www.corestandards.org/the-standards/mathematics.

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. New York: Routledge.

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 65–83.

Fujii, T., & Iitaka, S. (2012). *Mathematics international (grade 1-grade 9)* Retrieved from www.globaledresources.com

Gersten, R., Taylor, M. J., Keys, T. D., Rolfhus, E., & Newman-Gonchar, R. (2014a). *Summary of research on the effectiveness of math professional development approaches*. Washington, DC: US Department of Education, Institute of Education Sciences, National Center for Educational Evaluation and Regional Assistance, Regional Educational Laboratory Southeast.

Gersten, R., Taylor, M. J., Keys, T. D., Rolfhus, E., & Newman-Gonchar, R. (2014b). *Summary of research on the effectiveness of math professional development approaches. (REL 2014–010)*. Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southeast.

Hart, L. C., Alston, A., & Murata, A. (Eds.). (2011). *Lesson study research and practice in mathematics education*. New York: Springer.

Lewis, C. (2000). Lesson Study: The core of Japanese professional development. In *AERA annual meeting*, April 2000.

Lewis, C., & Perry, R. (2014). Lesson study with mathematical resources: A sustainable model for locally-led teacher professional learning. *Mathematics Teacher Education and Development, 16* (1), 22–42.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for Research in Mathematics Education, 48*(3), 261–299.

Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 22*(4), 12–17, 50–52.

Lewis, C., Perry, R., Hurd, J., & O'Connell, M. P. (2006). Lesson study comes of age in North America. *Phi Delta Kappan, 88*(04), 273–281.

Lewis, C., Perry, R., & Friedkin, S. (2011). Using Japanese curriculum materials to support lesson study outside Japan: Toward coherent curriculum. *Educational studies in Japan: International yearbook: ESJ, 6*(Classrooms and Schools in Japan), pp. 5–19.

Makinae, N. (2010). The origin of lesson study in Japan. In Y. Shimizu, Y. Sekiguchi, & K. Hino (Eds.), *The 5th East Asia regional conference on mathematics education: In search of excellence in mathematics education* (Vol. 2, pp. 140–147). Tokyo: Japan Society of Mathematics Education.

Murata, A., & Takahashi, A. (2002). Vehicle to connect theory, research, and practice: How teacher thinking changes in district-level lesson study in Japan. In D. L. Haury (Ed.), *Twenty-fourth annual meeting of North American chapter of the international* group *of the* Psychology *of Mathematics Education, Columbus, OH, 2002b* (Vol. 1–4, pp. 1879–1888): ERIC/CSMEE Publications.

National Council of Teachers of Mathematics. (1989). *Curriculm and evaluation standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston: Author.

NCTM. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston: National Council of Teachers of Mathematics.

NCTM. (2000). *Principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Shimizu, Y. (2002). Sharing a new approach to teaching mathematics with the teachers from outside the school: The role of lesson study at "fuzoku" schools. In *US-Japan cross cultural seminar on the professionalization of teachers through lesson study, Park City, Utah*, July 2002.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Stigler, J., & Hiebert, J. (2009). Closing the teaching gap. *Phi Delta Kappan, 91*(3), 32–37.

Takahashi, A. (2000). Current trends and issues in lesson study in Japan and the United States. *Journal of Japan Society of Mathematical Education, 82*(12), 15–21.

Takahashi, A. (2014a). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 4–21.

Takahashi, A. (2014b). Supporting the effective implementation of a new mathematics curriculum: A case study of school-based lesson study at a Japanese public elementary school. In I. Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 417–441). New York: Springer.

Takahashi, A., & McDougal, T. (2014). Implementing a new national curriculum: A Japanese public school's two-year lesson-study project. In A. R. McDuffie, & K. S. Karp (Eds.), *Annual perspectives in mathematics education (APME): Using research to improve instruction* (pp. 13–21). National Council of Teachers of Mathematics.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*, 513–526.

Takahashi, A., & Yoshida, M. (2004). How can we start lesson study?: Ideas for establishing lesson study communities. *Teaching Children Mathematics, 10*(9), 436–443.

Takahashi, A., Watanabe, T., Yoshida, M., & Wand-Iverson, P. (2005). Improving content and pedagogical knowledge through kyozaikenkyu. In P. Wang-Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 101–110). Philadelphia: Research for Better Schools.

Watanabe, T. (2002). The role of outside experts in lesson study. In C. Lewis (Ed.), *Lesson study: A handbook of teacher-led instuctional imporvement*. Philadelphia: Research for Better Schools.

Watanabe, T., & Wang-Iverson, P. (2005). The role of knowledgeable others. In P. Wang-Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 85–91). Philadelphia: Research for Better Schools.

Yoshida, M. (1999). *Lesson study: A case study of a Japanese approach to improving instruction through school-based teacher development*. Dissertation, University of Chicago, Chicago.

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Implementing a New Mathematics Curriculum in England: District Research Lesson Study as a Driver for Student Learning, Teacher Learning and Professional Dialogue

Peter Dudley, Paul Warwick, Maria Vrikki, Jan D. Vermunt, Neil Mercer, Nicolette van Halem, and Anne Mette Færøyvik Karlsen

Contents

Abstract Against a backdrop of a transformation in teacher professional development and learning and state school organisation in England this century, this chapter describes the evolution of Research Lesson Study, a model that optimises dialogic teacher learning, its effects in a project which harnessed six cycles of research lesson study at school and district level over 2 years to tailor the implementation of a new statutory curriculum in England to address the professional development needs of teachers and classroom learning needs of London students. It then reports the findings of research carried out during the project into how these teachers learned and developed this new curricular expertise and practice knowledge through lesson study dialogues that supported student learning. It concludes by proposing future directions for teacher professional learning research and practice.

Keywords Lesson study · School collaboration · Dialogue · Teacher learning

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Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_15

1 Introduction

This chapter introduces a popular form of lesson study (LS) in the UK and reports on its use in supporting 96 primary and secondary schools in London as they reviewed and rewrote their mathematics curricula in order to improve the mathematics learning of their students. The background context to this work was the lead-up to changes made to the statutory mathematics curriculum in England (DfE 2013).

In the first section, we will set the scene by placing this development in teacher learning in the context of transformational changes in teacher professional development taking place over the past 25 years in England. We will then focus on the model of 'research lesson study' used in the project. We will describe the features it incorporated of both Japanese lesson study (JLS) and district level lesson study into a deliberate, interleaved process and focus on two particular RLS elements 'case pupils' and 'pupil interviews' – which helped to promote the modes of teacher learning that we discuss in depth in Sect. 3.

The second section will describe the context of the 2-year Camden[1] New Curriculum Mathematics Lesson Study Development and Research Project: its aims, scale and methods. It will present evidence from the project evaluation, of impact on teachers' development of pedagogical content knowledge in areas of mathematics that they had identified as being the hardest to teach and the hardest for pupils to learn. It will also present evidence of the impact this work had on pupil learning in the new curricula that were developed.

The third section focuses on research undertaken in this project into mechanisms of teacher learning within LS planning and reflection discussions. It examines the forms of interaction that were productive for teacher learning; the nature of teacher learning processes in these discussions and their relationship to dialogue moves by teachers; the learning patterns evident as the teachers engaged in these discussions over time and their relationship to ideas about teacher identity; and, finally, how teachers incorporated pupil perspectives into these discussions.

The fourth and concluding section considers the next steps for practice and research.

2 A Background to the Development and Adoption of Lesson Study in the UK

2.1 A Prelude to the Late Twentieth Century 'Neo-liberal' Reforms

Figures such as Lawrence Stenhouse and John Elliot positioned 'critical' teacher development and agency alongside classroom action research as essential

[1] A municipality and school district in central London

components of successful curriculum development and sustainable improvement of classroom learning (Stenhouse 1981; Elliott 1991). However, in reality typical experiences of teachers of later twentieth century 'In Service Education and Training' (INSET[2]) in the UK varied, depending the degree to which the school (or local authority) valued, promoted and funded such activity.

The first statutory English National Curriculum and assessment processes introduced in the early 1990s formalised an annual entitlement to 5-day school closure for INSET. But the same legislation transformed state schools into quasi-independent, competitive businesses regulated by a national inspectorate, 'Ofsted',[3] ranked annually by examination results and inspection ratings. Teachers became subject to statutory lesson observation-based performance management.

A revolution in INSET followed as schools realised that if they were to gain the positive inspection grades and examination ranking needed to attract pupils and funding, they were going to need good results.

2.2 Lone Practice- and Centre-Based Professional Development

Despite these changes, over half of English 11-year-olds 'failed' the first national tests in 1995. The 1997 Blair government responded with the 'National Strategies' (NS) (Department for Education and Skills 1997) aiming to increase attainment in literacy and mathematics with training from 450 local authority-based consultants and 900 school-based 'leading teachers'.

However, most of this training continued to take place *away* from classrooms and students despite evidence to the contrary (Lave and Wenger 1991; Scribner 1999; Puttnam and Borko 2000).

2.3 Harnessing Teacher Agency and Classroom Led Reform

An independent review of NS by the Ontario Institute for Studies in Education found that while initial gains had been impressive, continued top-down delivery of centralised policies was diminishing returns and risked de-professionalising teachers (Earl et al. 2003). It recommended encouraging 'disciplined innovation' (Hargreaves 1999) of improvement in schools, led by practitioners.

The door was opened for LS in England.

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[2]INSET is used here to distinguish 'in service' teacher education from 'pre-service'.

[3]Oftsed is the abbreviation of 'Office for Standards in Education' England's national schools inspectorate.

2.4 The Introduction of Research Lesson Study

First piloted from 2003 to 2005, it was 2008 before research lesson study (RLS – see Fig. 2 below) found its way into mainstream policy (Dudley 2005, 2008). In 2008–2009 teachers of 11-year-olds in half of England's underperforming, 'coasting' primary schools were supported by NS 'leading teachers' using RLS to develop pedagogical content knowledge and practice knowledge in literacy and mathematics. In both years, national test result improvements for the RLS-supported schools were double those of the other schools supported through traditional approaches and higher still than the national increases (Hadfield et al. 2011[4]).

Although only 2% of teachers reported experiencing professional learning taking place over time or involving in-class, experimental collaboration by teachers (Opfer and Pedder 2010), evidence that this has most impact on student learning was mounting (Cordingley et al. 2004; James et al. (2007). Meta-studies by Robinson et al. (2009)) confirm the primacy that leading such systematic school-wide teacher-enquiry into how to improve *learning* (not teaching) has on improved student outcomes.

2.5 Austerity and the Renaissance of School-Led Development

Government-sponsored Continuing Professional Development (CPD) largely ceased in England after 2010; supporting materials were archived. Schools were encouraged instead to develop enquiry-informed, self-improvement processes (McKinsey 2007; DfE 2010; Mourshed et al. 2010.) Four hundred 'teaching schools' were created to support these activities locally. 'Teachers as researchers' blossomed in the form of social media groups actively encouraged by a national policy environment promoting innovation, enterprise and competition.

Lesson study thrived as school partnerships developed their own in-school enquiry approaches to professional learning. By 2012 it was estimated that up to 10% of English schools had used lesson study (Dudley 2011).

[4]The RLS model has been subject to two further large-scale trials of 800 schools and 280 schools, respectively (DfE 2016; Murphy et al. 2017), demonstrating improvements in pupil outcomes and receiving positive reactions from professionals in both. While neither trial showed an overall effect size, results in the 2016 study led to a recommendation that RLS be considered as one of only two interventions by schools for narrowing attainment gaps.

3 The New Mathematics Curriculum Development Lesson Study Project

It was then that the Greater London Authority (GLA) funded the London Borough of Camden in partnership with the University of Cambridge, to run a 2-year development and research programme (2013–2015) studying how 96 London schools used a design-based, district level LS approach to develop and implement the new mathematics national curriculum. The new curriculum aimed to raise standards in mathematics at 11 and 16. It expected 11-year-olds to be competent at mathematics levels previously expected of 12.5-year-olds.

This project concentrated efforts on students aged 10–13 years completing their primary schooling and commencing secondary education. Teachers in the project were encouraged to concentrate on areas of the new curriculum they believed would be hardest for them to teach and for students to learn. Teaching areas in which at least a quarter of the teachers identified they had little or no confidence included:

- Place value for decimal notation of fractions
- Written methods of division involving decimals
- Finding 100% given a percentage part
- Algebraic distinctions and terminology, providing explanations and solving simultaneous equations algebraically
- Identifying formulae and models for algebraic equations, direct proportion and graphical representations
- Explaining the distinctions between types of numbers and illustrating particular forms of number
- Calculations involving fractions
- Dividing a quantity in a given ratio

(Source: Bufton, N. Analysis of teacher confidence survey, New Curriculum Mathematics LS project, 2013)

3.1 How LS Helps Teachers Learn Mathematics Pedagogical Content and Practice Knowledge

Part of this project focused on investigating how modes of teacher learning in lesson study identified by Vermunt and Endedijk (2011) and Dudley (2013) operate at scale and how they might be modified to improve the learning of teachers in ways that improve the mathematical learning of pupils.

Inspired by Japanese processes of curriculum renewal such as those described by Kuno (2015), the project attempted to create a forum for primary and secondary schools to reflect on the kinds of mathematicians they wished to develop, the means by which this might be achieved, and the curriculum and pedagogical approaches that might best achieve these outcomes. In the next sections we consider the RLS model in comparison with others and further adaptations made for the Camden project.

3.2 The Model of Lesson Study Used for the Programme

Different models of lesson study exist in Japan, but most models used outside Japan include the following components (Takahashi and McDougal 2016): (a) curriculum study (progression before and after the unit); (b) lesson design (critical features of the lesson or unit design); (c) progress related to the school's research theme; and (d) next steps for future teaching. They also summarised principles contributing to effective Japanese lesson study stating that: its purpose is to collaboratively build expertise (not perfect lessons); it is part of a highly structured, school-wide, even district-wide process; it includes significant curriculum review, student and progression study (*kyouzai kenkyuu*); it takes weeks (not hours); and planning and post-lesson discussions are supported by 'knowledgeable other' experts.

Catherine Lewis (2016) has reported on the different forms of professional learning and development that tend to take place at each of these stages.

3.3 How RLS Differs from Other Forms and Why

Research lesson study (RLS) was first developed in England in a national pilot (Dudley 2005, 2011) (see Fig. 2 below). We will briefly explain the rationale for and genesis of the development of its key distinguishing features: three research lessons, case students, student interviews and a focus on dialogic learning.

3.3.1 A Note on Learning Theoretical Underpinnings of RLS

When RLS was first piloted in 2003, literature on LS in English was scarce. Little was known about Chinese lesson study. Dominant models were Japanese LS and its US implementation, which was mainly in mathematics education (Lewis 1998; Fernandez et al. 2003; Yoshida 2002). England's national pilot developed LS in a range of curriculum areas (Dudley 2011). It was of course developed with teachers who, for the first time, experienced LS's ability to help them see their students' learning with new eyes and to access their tacit stores of practice knowledge (Dudley 2013). Teachers in England were often deeply wary of situations where another teacher was with them in their classroom. Lone practice, the tyranny of inspections and performance management processes had discouraged risk-taking or 'disclosure' of professional learning needs. Therefore the RLS pilot developed safe spaces in which teachers learned not only about improving their student's learning but also how they themselves can learn collaboratively with fellow teachers; that it is acceptable not to know everything (nobody does!); and that it is both rewarding and a professional imperative to develop practice knowledge together and to pass it on to others. For these reasons, post-research lesson discussions were also conducted iteratively within RLS's three research lesson (RL) cycle, before what was learned was made public and open to the critique of commentators and peers.

The overall theoretical underpinning for both teacher and student learning in RLS is broadly sociocultural, drawing on Vygotskian principles. However, in the 'study phase' teachers are also encouraged to draw upon well-evidenced curricular and pedagogical developments in subject teaching and to use RLS to incorporate these into their classrooms. With the benefit of increased LS literature and the work of organisations such as the World Association of Lesson Studies, it is now clear that there are strong parallels between RLS and Chinese lesson study. Another LS form found in England is learning study. Developed in Hong Kong, Sweden and Indonesia, it combines Chinese and Japanese LS approaches with Marton's 'variation' theory of learning (Lo 2016). Its use in the UK is most common in schools for children with learning difficulties. It was not used in this project.

3.3.2 Why Three Research Lessons?

Lewis' 'study' phase (Fig. 1), during which teachers review their existing practices and research alternatives, is carried out before RL planning begins. However, in RLS the 'study' phase is revisited between the three RLs. This is because when the RLS model was first co-evolved (2003–2005), teachers in England found it helpful to develop and test hypotheses over *several* RLs. This is common in some Chinese forms of LS. Teachers then were not used to close observation of student learning – they were used to 'observing *teaching*'. This iterative 'design study' RLS approach thus afforded opportunities for 'reconnaissance' (Elliott 1991) and field testing of hypotheses and their subsequent development in later RLs (Dudley 2013).

To develop this point, the first research lesson is often 'revelatory' reconnaissance for teachers. They discover, through close observation of learning, stark differences between what they had imagined to be true about their pupils as learners and what they actually observed to be the case. As a result, assumptions made in the study phase needed adjusting and further testing in a second research lesson. Hypotheses developed in the initial study phase and refined as a result of the second RL, often need to be further refined, or simply tested for replicability in the third.

The three-cycle model increases threefold the collaborative imagining and discussion of what might happen (when planning) or what might have happened differently (in a post-lesson discussion), influencing how teaching can be changed next time. Action and reflection processes are thus *interleaved* in this model. Dudley (2013) reports that such deliberate, interleaved processes are associated with the formation of close bonds of joint endeavour between LS group members. These help

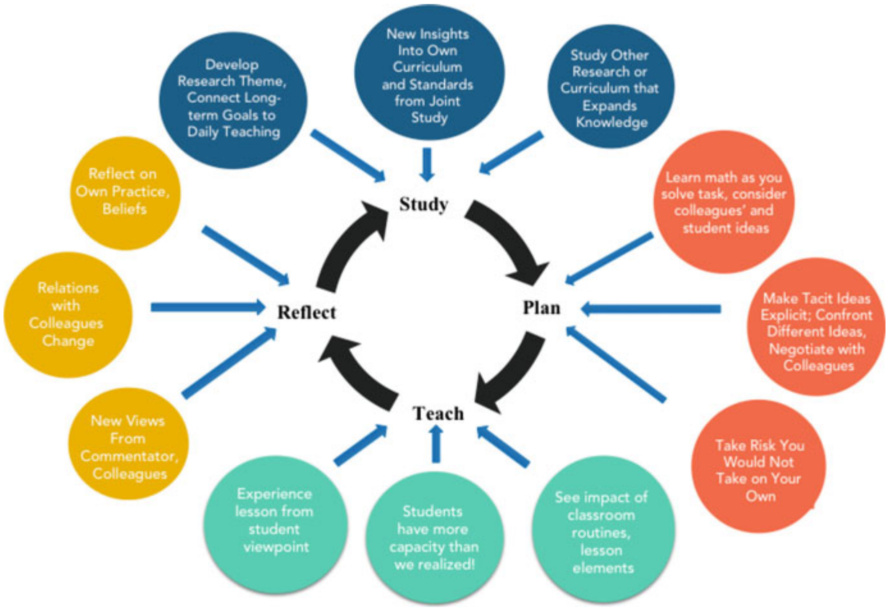

Fig. 1 What learning occurs at each stage of the lesson study cycle. (© Lewis (2016) Reproduced with kind permission)

to create conditions of high intrinsic motivation (to help the students overcome identified barriers), high trust and high social capital. These not only enable teachers to take more risks (c.f. also Lewis 2016 in Fig. 1) but also enable them to access each other's normally invisible tacit knowledge. Accessing tacit knowledge of teaching and changing beliefs about practice (see Fig. 2) are notoriously difficult to achieve (Brown and McIntyre 1993; Eraut 1994). The occurrence of these phenomena in RLS is most closely associated with discursive processes of (i) hypothesising (what might happen or might have happened) and (ii) 'rehearsing' hypothetical sections of the research lesson before and after it takes place and doing so *in role* as teachers, using their 'teacher' voices to collectively conjure up the imaginary class before them (Dudley 2013) as they 'interthink' (Mercer 2000).

A further affordance of the three-cycle RLS model is that it can be used to create two distinct forms of research lesson, either developing an object of learning (such as a skill) over a sequence of three different lessons with the same students or the development and refinement of what is essentially the same lesson but with three different classes (a common feature of Chinese LS). The latter is more easily carried out in larger schools with three or more parallel classes (typically secondary schools (11–18) in England). The former is attractive to smaller schools.

There are two further distinguishing features of the RLS model: its use of 'case pupils' to help teachers focus on student learning in research lessons and the use of 'pupil interviews' to explore the students' own perceptions of their learning in these lessons.

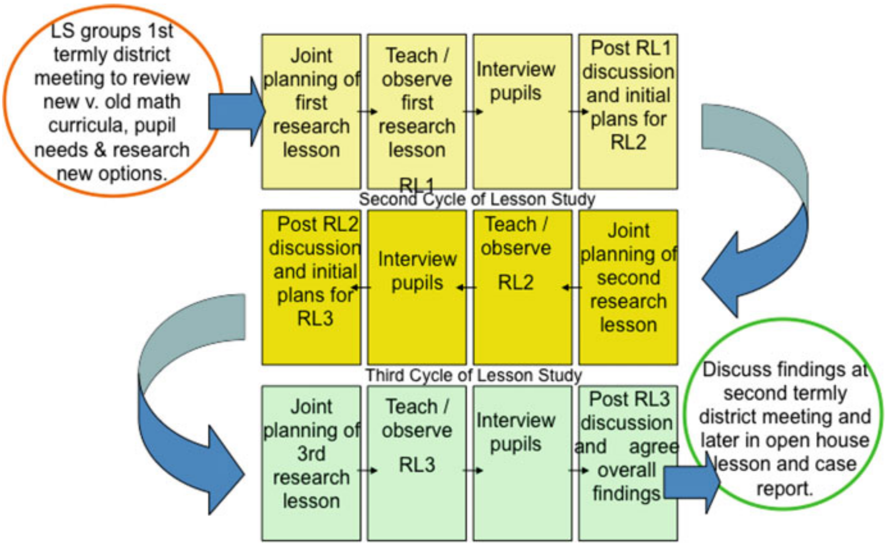

Fig. 2 Three-cycle research lesson study model used in this project

3.3.3 Case Pupils

Collaborative dialogic learning is not only encouraged between teachers. It is used in RLS to promote student learning through exploratory talk (Barnes and Todd 1977) and also, so that by eavesdropping on this talk student thought processes become 'visible'. However, an additional strategy was incorporated into RLS.

In order to help teachers see the students in front of us (rather than the constructs of students we carry in our minds), three or so *case pupils* are selected for specific attention during the RLS process. They are identified during the 'study phase'. In contexts such as that of this project, where whole curriculum review and renewal is taking place, case pupils will typify learner groups in the class – perhaps one who is confident in mathematics and one who may require additional support. The third may be drawn from those falling between the two. At key points during the planning of the RLs, the LS group will imagine the learning that is to take place and predict what these case students are each likely to be doing, saying, drawing or writing if they are making expected progress and are ready to move on to the next lesson stage. These predictions help to set the agenda for the RL observers who check what case pupils are actually doing, noting their observations down against the group's earlier predictions. Following the student interviews (see below), the post-lesson discussion begins by discussing the actual behaviours observed in comparison with the predictions. This process helps focus discussion less on teaching and *more* on students' learning, throwing differences between predicted and observed behaviours into sharp relief.

3.3.4 Post-research Lesson Pupil Interviews

Immediately following the RL, a few students are interviewed by one or two members of the lesson study group. Case pupils may or may not be included. The aim of the interview is to discover the views and insights that the students themselves hold about their learning in the RL. Questioning might begin with:

> *If we were teaching this lesson again tomorrow to a class like yours, how could we change the lesson to help the students learn more easily?*

Evidence from lesson studies using this approach (Dudley 2011; Warwick et al. 2018) suggests that students frequently have insights (about what helped or hindered their learning, aided understanding or 'lightbulb' moments) that no member of the LS group had imagined. It has proven a useful additional process for eliciting evidence of student learning and of what students value in teaching and lessons.

3.3.5 How Lewis's Forms of Learning Map onto the RLS Model

The forms of learning identified by Lewis (2016) in Fig. 1 above have been reported by Dudley as occurring in the RLS model (2013). There are notable differences between the stages at which these forms of learning fall in the RLS model. The RLS model focuses three RLs on developing practice knowledge that might be the focus of only one RL in JLS. School leaders nevertheless bring in facilitators to support groups where such expertise is not available in school and where LS is networked across schools; commentators will distil what has been learned further at district LS meetings.

In Fig. 3, RLS's three consecutive research lessons are depicted as concentric circles each embodying elements of the review, plan, teach and reflect phases. Red dots depict the recursive opportunities for teachers to test out and develop ideas, observe their effects on pupil learning and adjust their teaching. Lewis's forms of learning are mapped onto this depiction of RLS, using information about teacher learning in RLS reported by Dudley (2013).

Evidence from studies of RLS in England involving teachers 'interthinking' (Mercer 2000) (including those reported in this chapter) suggests that deliberate processes of collective review, hypothesising; imagining, predicting and observing student learning; and finally negotiating an agreed interpretation, afford LS groups opportunities to collectively solve teaching problems they could not have solved alone (Dudley 2013). This mirrors processes of 'interthinking' observed by Mercer (1995) amongst younger learners and later amongst studies of professional learners (Mercer 2014).

The recursive nature of the three RL cycle allows LS group hypotheses to be re-formed and tested out again. Usually, by the close of the overarching 'meta-discussion' drawing conclusions from all three RLs, the LS group has developed practice knowledge from which they feel others would also benefit.

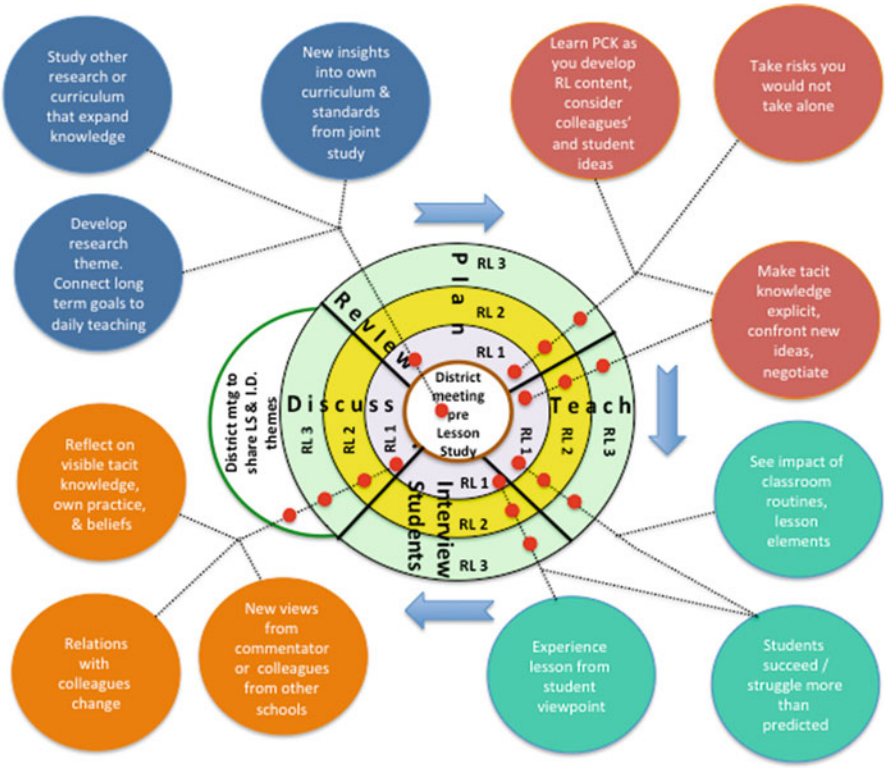

Fig. 3 Forms of learning in research lesson study in the new mathematics curriculum lesson study project (Adapted from Lewis 2016)

At that stage the RLS model re-joins the JLS model described above. Teachers share their newly discovered practice knowledge with colleagues though open house lessons, presentations or case reports. There is evidence (Dudley 2013) that teachers who pass on their practice knowledge to others are more likely to adopt it long term.

3.3.6 The LS Model Used in This Project

The three-cycle RLS model was further adapted for this London project. National Strategies RLS involved subject-expert teachers exploring with classroom teachers how national mathematics and literacy materials could improve pupil learning. While the London RLS district approach contained these elements, there were important differences, as follows:

- The core curriculum materials were the framework for the new national mathematics curriculum (Department for Education 2013), but the purpose of the project was to develop schemes of learning and core lessons to help schools to tailor the new curriculum to the needs of their largely bilingual and often disadvantaged pupils.

- LS teams met with national experts in mathematics, curriculum development and knowledge-creation through lesson study twice each term to facilitate learning and collective review (see Fig. 5 below).
- School LS teams conducted lesson studies that bridged this district level knowledge-creation with the needs of their own students. (They also worked across the primary and secondary divide to build progression).

3.3.7 The Roles of the District Level LS Meetings and the District Support Teams

While additional London schools joined the project's second year bringing the total to 96, we will focus mainly on the 22 Camden primary, secondary and special schools that were involved for the full 2 years (Fig. 4).

Two district level meetings (1 × 1 day and 1 × 0.5 day) attended by school LS groups were held each term. The district team provided expert input on mathematics subject and pedagogical content knowledge– curriculum development and collaborative professional learning – with a focus on LS group dialogue. It comprised experts in mathematics education (including a national adviser and former chief inspector); teacher learning in lesson study (including, in the second year, a Chinese district lesson study leader/researcher); and (termly) the Cambridge University team (which included international experts in 'interthinking' and dialogic learning).

The first district meeting each term began the collective 'study' phase. Mathematics experts external and internal to the group worked with the teachers to elicit their conceptions of 'hard to learn' areas of mathematics, to probe their understandings of (and knowledge gaps in) how they might be more effectively learned. As well as this extended guided review and study phase, LS groups also began to scope their specific lesson studies, exploring their ideas with the district team.

After the RLS cycle in schools was complete, a second (5-hour) district meeting shared lesson studies and collectively reflected upon findings. A final stage in the termly process involved members of the district team – including the Cambridge

Fig. 4 The timeline of the project

University research team – taking away the LS 'workbooks' (containing teachers' lesson study data), as well as videorecordings of the LS groups planning and evaluating their research lessons together. Cross-case examination of these data revealed themes not evident at the sharing meeting. So at the first event of the following term, these common themes were presented by the district team and discussed amongst the school groups to further inform the lesson studies for the coming term.

These iteratively informed, district level meetings connected elements of the design cycle across groups of schools, helping to form common lines of enquiry and knowledge building in these 'challenging-to-teach' areas of mathematics. Figure 5 illustrates how these common experiences developed into co-evolving enquiry themes. Schools shared their findings with each other, enabling the development of knowledge capable of use for curriculum development across schools more broadly.

Each term, a number of schools held 'open house lessons' in which teachers and school leaders from all schools were invited to participate and where invited subject experts and commentators also shared their perspectives and advice on the lessons and curricula observed.

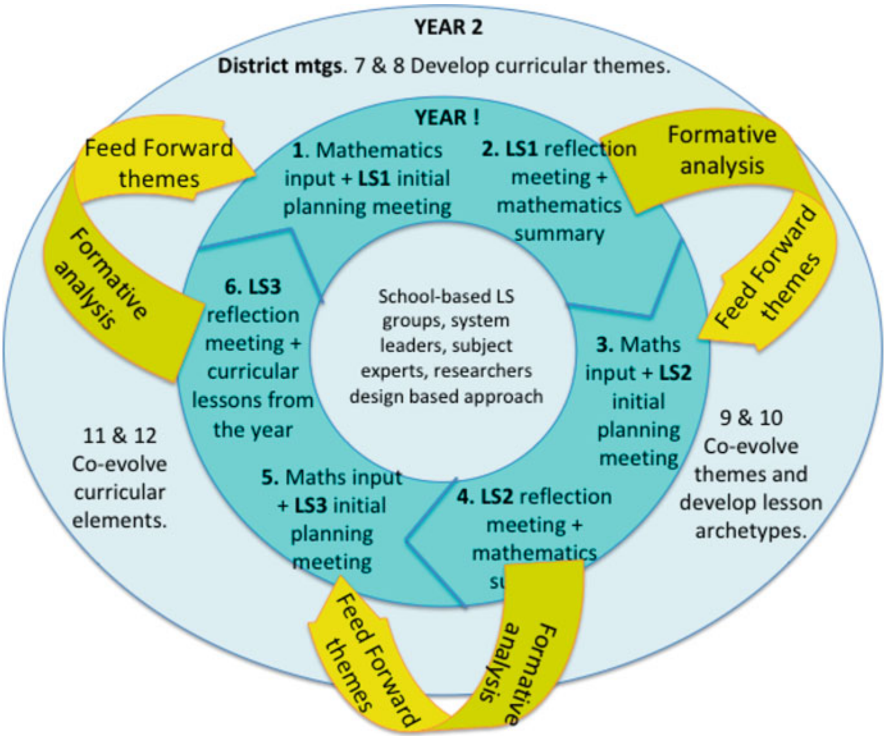

Fig. 5 District level lesson study system 2-year cycle in the London new curriculum mathematics lesson study project

3.3.8 Project Development Outcomes and Learning Outcomes

One hundred thirty-two research lessons were conducted by the 22 schools over the 2-year period. All teachers, primary and secondary, reported greater confidence in teaching the 'hard to teach and learn' areas of mathematics and across all 22 new curriculum mathematics areas in project surveys (Ylonen et al. 2015). Teachers also reported gains in pedagogical content knowledge, increased understanding of student misconceptions and knowledge of a wider range of misconceptions. Schools developed their new mathematics curricula based on a shared understanding of what progression in the new curriculum should 'look like' for their students and frequently built around key lessons developed from project lesson studies.

Evidence from 70% of LS workbooks revealed that by the end of the six terms, teachers were dedicating much more time than before to consolidating pupil learning and to strengthening pupils' long-term memory and recall of key information, underlying methods and procedures in mathematics. This helps secure recent mathematical learning before progressing to more challenging areas.

Teacher comments included:

> We were struck by how important the pace of a lesson is. . .we noticed that we sometimes moved on too quickly or assumed understanding. This realisation tied in with our appreciation of the New Curriculum focus on mastery.

> It is crucial that key concepts are secured and consolidated at each stage.

> Real need is to focus on basics, like place value, before you can move on to more complex areas, as without a solid base, everything else just gets lost. It is not wrong to go over these things again and again in a variety of different ways.

The importance of language, vocabulary and verbalisation in mathematics teaching and learning was highlighted in over a third of workbooks. Structured use of dialogue enabled students to give their reasoning voice. Expressing what they already knew helped students themselves to identify areas in which they were less secure.

> We have found that the children value the opportunity to talk and discuss what they are learning, including the strategies they are using. . .having the opportunity to talk through their understanding and reasoning.

The use of 'talk partners' (a technique supporting focused paired discussion) was highlighted in 40 per cent of the LS workbooks – not only for strengthening students' understanding but also teachers' ability to understand *how* their students understood the mathematics.

Sharing pedagogical approaches supported teachers to access practices developed in other lesson studies that they subsequently applied to their own teaching. Examples included:

- Scaffolding the language of reasoning so students could more accurately express their ideas and give explanations and reasons.
- Concept-checking students' development of understanding and reasoning that leads to deeper learning and a greater repertoire. Early lesson studies revealed that

it was common for children to be able to ‘carry out’ mathematical processes without understanding why the process worked.
- Allowing students sufficient time to talk about the mathematics with peers during the learning processes.
- Developing a whole-school, long-term, strategic view of planning of mathematics.
- Sharing examples of what can be achieved and processes for how to get there proved more useful than simply identifying good or bad practices.

The most profound learning, echoed by all involved, was the need to slow down initial concept introduction and to ensure students had thoroughly grounded their understanding for application. This led to the catch-phrase ‘*Go slow and grow’*. Some schools used their research lessons as archetypal components of the new curricula they developed serving as reference points on the curricular landscape, for example, for inducting new teachers.

3.3.9 Differences in Attainment for the Pupils in the Project Schools

Student attainment was first tested for 11-year-olds in the new national mathematics tests of 2016 (a year after the project ended). Because the standards demanded of 11-year-olds were raised in the new tests, the only constant that project school results could be compared with was the national average result for England. Importantly, the results of the primary schools in Camden that had *not participated* in the project were at a higher point than those of the schools in the project at its outset in 2013 (so the schools involved in the project were those with initially lower mathematics attainment scores than others in the district). The scores of project schools rose against the national average between 2013 and 2016 by two percentage points, while the attainment of district schools not involved in the project fell against the national average by two percentage points – a 4% difference in total. Both quantitative and qualitative data suggest that teachers had been developing their own practices in response to their LS learning experiences in ways that subsequently improved the learning of their pupils.

We will now turn our attention to what was found about these teachers’ learning processes from analysis of LS groups planning and post-lesson discussions.

4 Researching Teacher Learning in LS in the Context of Curriculum Innovation and Development

Within the wider context of adaptations of LS to the UK context reported above, we now report outcomes of the University of Cambridge team’s research undertaken within this project. The team analysed teachers’ LS planning and post-lesson discussions in order to:

(i) Understand how and why LS discussions[5] may impact upon teacher learning patterns and processes.
(ii) Identify more or less powerful components of these discussions.
(iii) Understand the contribution of pupil voice in LS discussions, relating this to subsequent mathematics teaching intentions.
(iv) Consider the longitudinal development of mathematics teacher identity in relation to LS.

In addition to these foci, the research team also sought to contribute to theory development in understanding and improving teacher learning in the context of educational innovation. Methodologically, it was our intention to contribute to the field though the development of analysis tools for the categorisation of dialogue within LS discussions and a reliable coding scheme incorporating teacher learning. Here we draw on all elements of the research, both theoretical and methodological, to provide an overview of our findings.

The importance of professional communities of mathematics teachers was central to the research and development endeavour, since creating and sustaining such communities can be one of the most effective mechanisms for enhancing professional learning and sustained professional development (Lieberman and Miller 2008; McLaughlin and Talbert 2006; Stole et al. 2006; Webster-Wright 2009). Indeed, schools with strong teacher communities seem to have higher student achievement (Horn and Kane 2015; Bryk et al. 2010). To uncover the mechanisms of teacher learning in communities that are inspired and sustained by a shared interest in LS in mathematics teaching and learning, we focused our research on the role of dialogue and its influence on learning in LS discussions. In so doing, we considered the intersections of three different theoretical perspectives for our study (Fig. 6).

Fig. 6 Considering the intersections of three theoretical perspectives

[5]In the remainder of the chapter, the term 'LS discussions' refers specifically to teacher planning and discussions conducted in relation to LS research lessons.

Here, it is useful to briefly review how we understand teacher learning and dialogue as concepts informing the research.[6]

4.1 Teacher Learning in the Context of LS Discussions

There is a pre-existing line of research that considers the structure or form of professional development programmes and the impact this has on outcomes for teachers and for students such as Desimone's (2009) linear, intervention-outcome model of teacher learning. Such models conceptualise teacher learning as features of interventions leading to teacher learning outcomes (i.e. knowledge, skills, attitudes), which can then lead to changes in teacher behaviour in the classroom. Others (Clarke and Hollingsworth 2002) have suggested a non-linear model involving interconnected change sequences and growth networks. Others still have explored effects of classroom complexity (Opfer and Pedder 2011) or cross-institutional models (Kazemi and Hubbard 2008) on teacher learning.

In many studies the intention is not necessarily to explore teacher learning processes or patterns, although interesting insights often arise though analysis of outcomes. Borko et al. (2010) intentionally focused on exploring teacher learning processes occurring in the 'black box' between stimulus (intervention) and response (learning outcomes) (Vermunt 2013). Our research fits within this latter frame, although the form provided through the recursive, reflexive nature of RLS is clearly the enabling context.

In considering the exploration of teacher learning through LS, the field of student learning is potentially of interest. For example, Marton and Säljö (1984) identified two distinct learning approaches that students tend to adopt in the context of their normal studies: a deep approach and a surface approach. A deep approach is characterised by the student's intention to understand, while a surface approach is characterised by the student's intention to remember the material. Others have built on these findings by elaborating and extending their conceptualisations. Vermunt and Vermetten (2004) used the term 'learning pattern' as an encompassing concept in which the cognitive processing of subject matter, the metacognitive regulation of learning, conceptions of learning and learning orientations are united. In a series of studies, they consistently found four such learning patterns: undirected, reproduction directed, meaning directed and application directed. Further work (Bakkenes et al. 2010), revealed six categories of learning activities, the four most frequently occurring categories of which were (i) experimenting; (ii) considering one's own practice; (iii) getting ideas from others; and (iv) experiencing 'friction'. Subsequent work by Vermunt and Endedijk (2011), considering patterns in teacher learning, inspired us to view LS as a 'contextual factor' that may lead to the creation of positive learning

[6]We do not review LS here, given the overall focus of this volume and the fact that LS in the UK context has already been considered in this chapter.

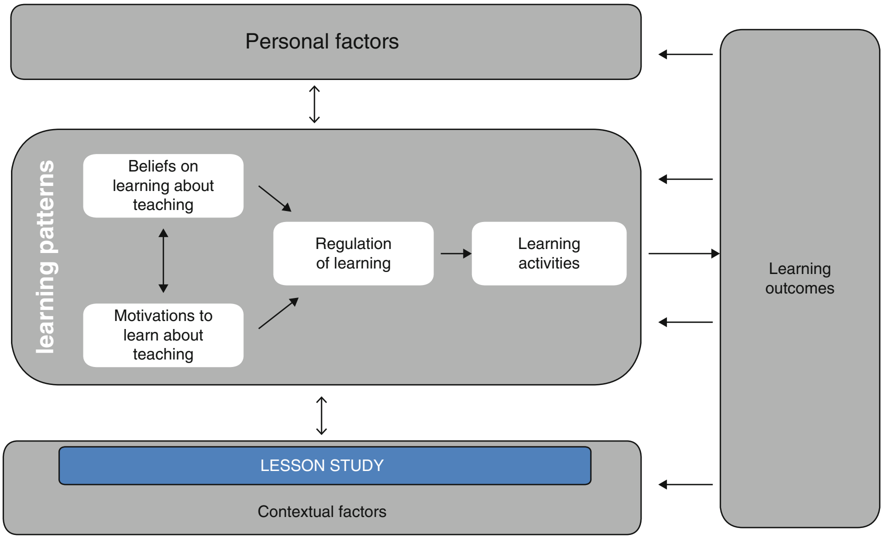

Fig. 7 A model of teacher learning in the context of lesson study

patterns and processes resulting in teacher learning. We therefore broadened Vermunt and Endedijk's model of teacher learning in the context of LS (Fig. 7).

We were interested to see how the content and character of LS discussions influenced the 'what' and 'how' of teacher learning, hence our interest in the next element of our intersecting theoretical perspectives – dialogue.

4.2 *Dialogue in the Context of LS Discussions*

A focus on dialogue reflected our wish to understand the processes by which teachers exchange information and build professional knowledge in collaborative groups. In everyday use, dialogue means talking together. But within the sociocultural framework of understanding that underpinned our work (Vygotsky 1962, 1978; Bakhtin 1981), dialogue refers to 'any kind of human sense-making, semiotic practice, interaction, thinking and communication, as long as these phenomena are 'dialogically' (or 'dialogistically') understood' (Linell 2009, p. 990). Dialogue can be thought of as a very specific kind of talk, designed to enable people to come to an understanding of one another's knowledge and perspectives through 'interthinking' (Mercer 2000).

Collaborative interaction is intimately bound with talk, and research into collaborative working, in professional and educational contexts, suggests that 'groups seem to achieve some of the best, and some of the worst, outcomes' (Littleton and Mercer 2013, p. 24). The quality of communication within a group can provide a clue as to why some groups achieve productive outcomes while others flounder. It seems that group discussion is far from reliable in generating positive outcomes, since the implicit social norms of discussion in *some* groups may be used to strictly control how knowledge exchange and knowledge building, and particularly dissent around ideas, are handled (Littleton and Mercer 2013; Mercer 2000). Labelled 'groupthink' (Esser 1998; Hart 1994; Janis 1982), this phenomenon has been blamed for some catastrophic political decisions.

In contrast, professional groups that regularly produce good solutions to problems seem to do so because of 'ground rules' that generate positive conditions for interaction and knowledge building through dialogue; they seem to act in what 'may be imagined as a socially distributed cognitive process' (Måseide 2003, p. 363). The talk and interaction in such professional groups mirrors that found in research on effective group interaction amongst school students (Mercer and Littleton 2007; Dudley 2013). From this, it seems that both the features of group dialogue and the environment that enables such dialogue are of crucial importance.

In our work on LS in mathematics, we focused explicitly on how shared information about student performance can act as a mediating artefact for the development of common knowledge amongst teachers (Edwards 2012; Edwards and Mercer 1987; Engstrom 2003). We were interested in the teacher learning processes evident in the LS discussions and the professional learning outcomes taking the form of developed pedagogic intentions that arise from teachers using 'knowledge that combines knowing about students and knowing about mathematics' (Ball et al. 2008, p. 401). And we were further interested in the learning patterns that might be apparent as teachers engaged in sequences of RLs.

Taking our extended model from Vermunt and Endedijk (2011) one stage further, we hypothesised that one or more perspectives in LS discussions may be held in tension as a 'dialogic space' is created in which teachers can 'engage with each other and, in a sense, learn to see the task [of understanding students' mathematical learning] through each other's eyes' (Wegerif 2007, p. 4). This idea of 'thinking ... in dialogues' (Wegerif 2007, p. 116) seems highly pertinent in the context of LS discussions and is presented in Fig. 8.

4.3 Data Collection and Analysis Approaches

Data for the research element of this project came from three main sources:

1. Videorecordings of the LS group planning and reflective sessions conducted in each school, ranging from 20 minutes to 1 hour in length. The video data were accompanied by LS workbooks to assist teacher discussions; these had been completed by the teachers during their LS meetings.
2. Survey data, collected at three time points during the project, to examine teacher learning over time.
3. Post-RL pupil interview data.

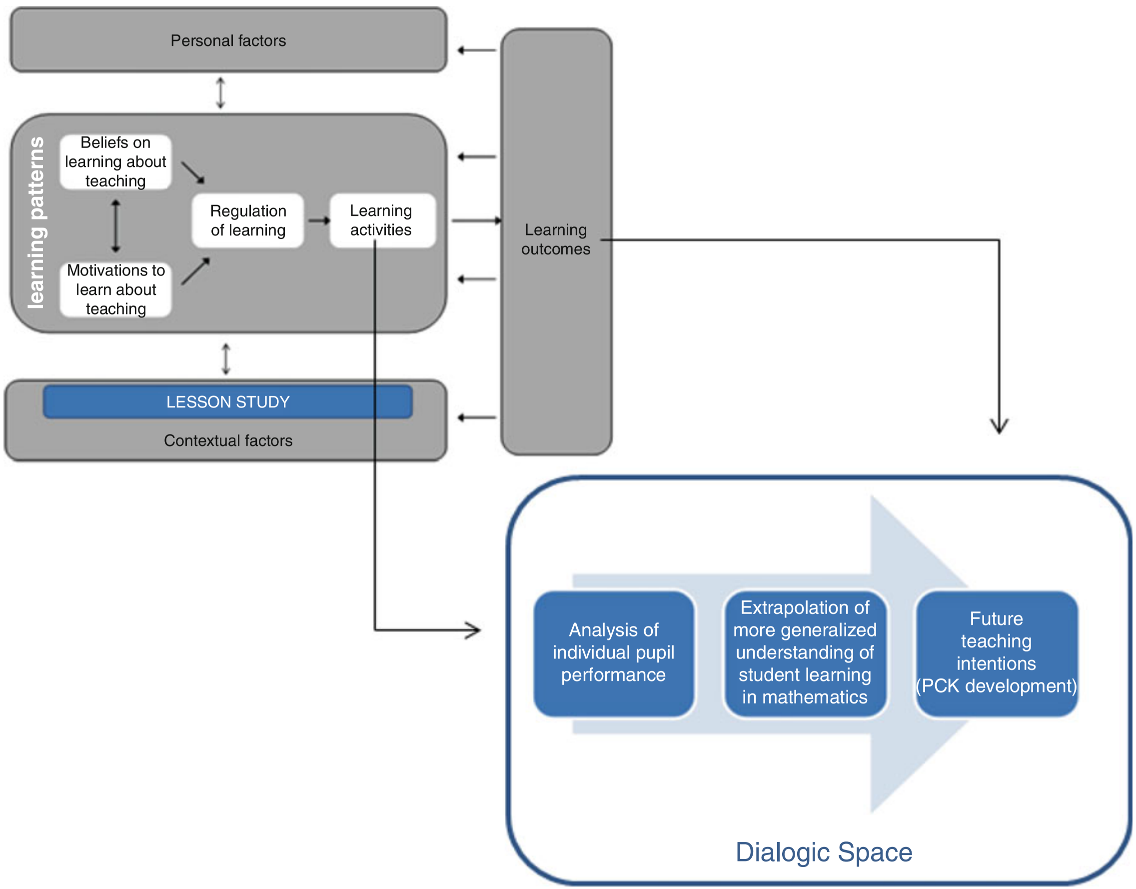

Fig. 8 Conceptualising a dialogic space for pedagogical development through LS

Several approaches were taken in the analysis of the data, in line with research questions of the strands we explored. In considering the video data, we developed an initial coding protocol, building on studies that detail the use of digital video as a tool for capturing the complexity of teaching and learning processes (Powell et al. 2003; Sorenson et al. 2006). The protocol comprised three sections, with features arrived at through methods associated with sociocultural discourse analysis (SDCA) (Mercer 2004; Mercer et al. 2004). This focuses on 'the use of language as a social mode of thinking – a tool for teaching-and-learning, constructing knowledge, creating joint understanding and tackling problems collaboratively' (p. 137). After building and testing the protocol by revisiting videorecordings of numerous LS discussions, the version of the protocol used for the initial analysis was eventually agreed. This consisted of three major categories: (1) *discourse-related features* detailed important elements of talk that contributed to a collaborative and productive learning environment; (2) *content-related features* pointed to teachers' mental processes in relation to specific content; and (3) *learning points* (Dudley 2013) identified types of teacher learning from LS discussions, with direct reference to teaching mathematics, understanding students' learning and learning about the subject of mathematics. At this stage particular emphasis was given to dialogic moves that could be seen to reliably influence teacher learning processes in project LS groups – termed descriptive learning processes (DLP) and interpretative learning processes (ILP) (Warwick et al. 2016). These terms are explained more fully in the next section.

Further work (Vrikki et al. 2017) developed reliability of protocol coding which was then used to examine the interrelationships between teacher learning and dialogue and between differing teacher learning processes. Thus, the relationships between DLP and ILP were examined at group and individual levels, and their relationship with dialogue moves was re-examined in light of the revised protocol.

Parallel work (Vermunt et al. 2017) examined the role of dialogue in teacher learning and identity formation within the LS groups. Here, and in other work (Vermunt et al. 2015), patterns in teacher learning were the focus, rather than teacher learning processes. Using the project survey data (described below), links between learning patterns and three identified types of teacher identity (Beijaard et al. 2004) were explored and developed in the context of mathematics teaching and learning.

Finally, transcriptions of teachers' reflective LS discussions were analysed to discover the extent to which and how pupil voice data was used by teachers within their post-lesson discussions generally and for articulating intentions for future action in particular (Warwick et al. 2018, in press). The focus once again was on the relationship between teacher learning processes and evidence, with the evidence in question being RLS student interview data and what students had said during lessons (as reported by the teachers).

4.4 Findings

We do not attempt here to provide detail of the methodological complexity of the analyses undertaken for this work; this detail can be found in the papers cited above. Rather we hope to give a 'flavour' of our findings, illustrating their complexity and throwing light on the variables in play in professional development and teacher learning in the context of LS. We present this work by noting the findings of each study in turn.

4.5 Study 1: Features of Teacher Talk Promoting Teacher Learning

We were initially concerned to examine what in the structure and content of the video's teacher talk, seemed to drive the learning conversation forward (Warwick et al. 2016). Nine episodes from post-lesson, teacher discussions were analysed. We identified as significant for teacher learning, those parts of the dialogue that began with an analysis of student learning strategies (or, in some cases, outcomes) and concluded with agreement on future pedagogical intentions. While it is true to say that these episodes formed only a small part of the total content of the teachers' discussions, they were highly significant in moving pedagogic thinking forward.

Where the discussions were more dialogic in character, teachers tended to encourage co-construction of understanding within the group by requesting information, giving reasons, providing evidence, making supportive comments and articulating shared ideas. Less dialogic discussions (in which some might take a dominant role preventing others from expressing their own perspectives or feeling a part of a collective solution) were not identified as driving learning forward. What seemed most significant was the power of particular 'dialogue moves' to effect progress in the dialogue. Questioning (including negotiating meaning), building on each other's ideas, coming to some agreement and providing evidence or reasoning and challenging seemed to be dialogue moves that drove forward productive professional discussion.

In addition, two levels of dialogic features were found in the data; we called these 'dialogic moves' and 'supportive moves'. Dialogic moves were accepted within groups because they were accompanied by supportive interactional cues that seem crucial for the creation of a supportive dialogic space of 'reciprocity, mutuality (allowing) the continual (re)negotiation of meaning' (Mercer and Littleton 2007, p. 25). It is interaction of the dialogic moves and the supportive moves that seems to create a productive learning environment.

It is worth noting that links between vocabulary and conceptual understanding were a strong focus of the teacher LS dialogues. This is hardly surprising, given that '[mathematical] language is crucial to reasoning – to the construction of mathematical knowledge – for it provides the medium in which claims are developed, made and justified' (Ball and Bass 2000, p. 205).

4.5.1 Study 2: Relationships Between DLP, ILP and Teacher Learning

Working with data from two of the project phases (Vrikki et al. 2017), we worked to produce a valid, reliable protocol that other researchers might be able to adopt. The protocol was reduced from 54 codes to 7 (see Fig. 9).

Three dialogue moves that proved particularly significant in teacher LS discussions appear in this final version of the protocol. This is not to say that other dialogue moves and supportive moves are not important; it is simply that *requesting* information, opinion or clarification, *building* on ideas and *providing* evidence or reasoning are those with the strongest, most reliable association with the learning processes identified in our analysis: descriptive learning processes (DLP) and interpretative learning processes (ILP). When engaged in DLP, teachers explicitly co-construct knowledge at the level of representing what is known. 'What is known' can include information from lesson plans or activities, expectations agreed for students and teaching, observations of student learning and teaching and information gathered from the students, such as interview data. ILP take place when teachers attempt to go beyond what is given, unpicking this information, for example, by evaluating teaching or student learning, diagnosing student errors or misconceptions and drawing on insights from teaching experience in considering 'next steps' for individuals or groups. While this distinction between types of learning was inspired by literature proposing such processes such as deep and

DIALOGIC MOVES	SCOPE OF DISCUSSION	LEARNING PROCESSES
[DM1] Requesting information, opinion or clarification	**[S1]** Groups of pupils	**[DLP]** Descriptive processes
[DM2] Building on ideas	**[S2]** Particular pupils	**[ILP]** Interpretative processes
[DM3] Providing evidence or reasoning		

Fig. 9 The final, reliable version of the coding protocol

surface learning, there is no evidence that either descriptive or interpretive learning is more or less important than the other.

Multilevel modelling revealed a range of complex findings, revealing differential predictors of these two learning processes. When a LS group engaged collectively in DLP, in building on one another's ideas and in talking about individual students, then individual teachers showed evidence of descriptive learning. When individual teachers engaged in reasoning to provide support for their statements, they are also likely to show evidence of descriptive learning. However, when a LS group engages collectively in ILP and focuses on groups of students, then individual teachers show evidence of interpretative learning.

4.5.2 Study 3: Learning Processes and Development of Learning Patterns and Professional Identity

Here we approached the data from a slightly different perspective, using survey data to analyse changes in teacher learning patterns over time. While teacher learning processes affect the establishment of learning patterns and identity, the *interweaving* of developing learning patterns and professional identity formation was our concern here. Being dynamic, identity constantly shifts according to the influence of internal and external factors; as Danielsson and Warwick (2015, p. 73) put it, 'identity is a constant becoming'. We adopted Beijaard et al.'s (2000) conceptualisation of three forms of professional identity – pedagogical expertise (related to student development), didactic expertise (related to teaching) and subject knowledge expertise.

Our analysis revealed three types of learning pattern: meaning oriented, application oriented and problematic.

> Meaning-oriented learning was characterised by learning activities such . . . trying to understand how students learn, and reflecting on one's own teaching practices. . . . Application-oriented learning was typified by wanting to know which teaching methods work. . . and learning best when trying out new ideas in practice. Struggling with new ways of teaching, not knowing how to teach one's subject in another way than one is used to. . . and only wanting to learn things that can be used immediately in one's teaching, were some defining elements of problematic learning. (Vermunt et al. 2017, p. 148)

A longitudinal comparison showed an increase in meaning-oriented teacher learning and a decrease in problematic learning during the time the teachers worked with RLS. The learning patterns were strongly associated with the three forms of professional identity previously examined by Beijaard et al. (2000). The meaning-oriented learning pattern was significantly and strongly associated with all three of these professional identities. Application-oriented learning was also significantly, but to a somewhat lesser degree, associated with all three identities. Problematic learning showed negative but non-significant correlations with all three identities. These results, and the further detail provided elsewhere (Vermunt et al. 2017), suggest in RLS contexts a significant, positive correlation between what might be termed 'good-quality learning patterns' – namely, meaning-oriented learning and application-oriented learning – and all three identity variables. Finally, examination of the data suggested that dialogue in collaborative professional groups plays a significant role in teacher identity formation.

4.5.3 Study 4. Student Perspectives and Teacher Learning in RLS

In the final study reported here, we considered the place of ‘pupil voice’ in the reflective LS discussions of the teachers. Here (Warwick et al., under review), we were concerned with the ways in which teachers incorporated the views of students into their post-lesson discussions. We were particularly interested in whether the use of ‘pupil voice data’ was associated with descriptive or interpretative learning patterns and whether it was used to inform stated intentions for subsequent RLs. Since student interviews have been a feature of RLS since its inception, this seemed a legitimate focus of study. In Dudley’s earlier work, student views were observed to contribute perspectives and insights that more traditional approaches and measures did not replicate. There was evidence of (i) pupil perspectives revealing interpretations and insights of which teachers had been unaware and (ii) these perspectives directly informing the design of subsequent RLs and practice. Teachers overwhelmingly reported that pupil interviews were valuable for the same reasons (Dudley 2013).

Using data from schools that could provide a full dataset of one LS cycle, including videorecordings of pupil interviews, we show that teachers used both descriptive learning processes and interpretative learning processes in integrating student interview data into their discussions. Twelve episodes featured 15 instances of a direct link being made between the reporting category in our coding representing teachers’ descriptive learning processes and the interpreting category in our coding representing teachers’ interpretative learning processes, evidenced in the interpretation of observations, student interview responses, broader lesson evaluations and resultant lesson planning. Two episodes featured a ‘jump’ to interpretative learning without evidence of prior descriptive learning in the teacher discussions (because the teacher speaking does not feel a need to report before interpreting). Four episodes showed evidence of descriptive learning without subsequent interpretative learning. Interestingly, all four of these latter instances led directly to wider lesson evaluation and/or the linking of this learning to subsequent planning. This adds strength to the assertion by Vrikki et al. (2017) that, while descriptive learning processes may influence interpretative learning processes, they should be seen as of equivalent value to the professional learner.

4.6 Looking Forward

The blend of school and district level lesson study work undertaken in this project continues to develop in London. The Camden Primary Teaching School Alliance has developed a popular LS-based approach called ‘connecting classrooms’, and some schools are utilising Shanghai ‘mathematics mastery’ LS methods. ‘Learning hubs’ have been established: groups of schools researching at classroom, school and cluster level to develop curricular and practice knowledge (across disciplines) for all Camden schools. A cadre of lead mathematics lesson study facilitators is enabling two additional London areas to continue with their LS practice as 'London Subject Knowledge hubs' https://www.london.gov.uk/what-we-do/education-and-youth/improving-standards-schools-and-teaching/subject-knowledge-hubs-2. Largely self-funded and operated, their longevity suggests that this school-hub blend and the RLS model work together effectively enough for schools to self-sustain such work over a period of years, even at a time of falling budgets.

Further work therefore needs to build on the links reported in this chapter between dialogue-based teacher learning in RLS and improved student learning, in order to study the effects of this over time on learning processes and outcomes for students and teachers at both school and district levels.

References

Barnes, D., & Todd, F. (1977). *Communication and learning in small groups*. London: Routledge & Kegan Paul Ltd..

Bakhtin, M. M. (1981). In M. Holquist (Ed.), *The dialogic imagination: Four essays*. Austin/London: University of Texas Press.

Bakkenes, I., Vermunt, J. D., & Wubbels, T. (2010). Teacher learning in the context of educational innovation: Learning activities and learning outcomes of experienced teachers. *Learning and Instruction, 20*(6), 533–548.

Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the national society for the study of education, constructivism in education* (pp. 193–224). Chicago: University of Chicago Press.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Beijaard, D., Verloop, N., & Vermunt, J. D. (2000). Teachers' perceptions of professional identity: An exploratory study from a personal knowledge perspective. *Teaching and Teacher Education, 16*, 749–764.

Beijaard, D., Meijer, P. C., & Verloop, N. (2004). Reconsidering research on teachers' professional identity. *Teaching and Teacher Education, 20*, 107–128.

Borko, H., Jacobs, J., & Koellner, K. (2010). Contemporary approaches to teacher professional development. In E. Baker, B. McGaw, & P. Peterson (Eds.), *International encyclopaedia of education (part 7)* (3rd ed., pp. 548–555). Oxford: Elsevier Scientific Publishers.

Brown, S., & McIntyre, D. (1993). *Making sense of teaching*. Buckingham: Open University Press.

Bryk, A. S., Sebring, P. B., Allensworth, E., Luppescu, S., & Easton, J. Q. (2010). *Organizing schools for improvement: Lessons from Chicago*. Chicago: University of Chicago Press.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947–967.

Cordingley, P., Bell, M., Rundell, B., Evans, D., & Curtis, A. (2004). *How do collaborative and sustained CPD and sustained but not collaborative CPD affect teaching and learning?* London: EPPI-Centre, Institute of Education.

Danielsson, A., & Warwick, P. (2015). Identity and discourse: Gee's discourse analysis as a way of approaching the constitution of primary science teacher identities. In L. Avraamidou & W. M.

Roth (Eds.), *Studying science teacher identity: Theoretical perspectives, methodological approaches and empirical findings* (pp. 71–88). Rotterdam: Sense Publishers.

Department for Education. (2010). *The importance of teaching: The schools white paper 2010*. London: DfE.

Department for Education. (2013). *The national curriculum for England to be taught in all local-authority-maintained schools.* Retrieved from https://www.gov.uk/government/collections/national-curriculum

Department for Education and Skills. (1997). *Excellence in schools*. London: HMSO.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*(3), 181–199.

DfE. (2016). Closing the gap test and learn: School based research testing teaching practices using experimental methods, National College of Teaching and Leadership. https://www.gov.uk/government/publications/closing-thegap-test-and-learn.

Dudley, P. (2005). *Getting started with research lesson study*. Nottingham: National College for School Leadership.

Dudley, P. (2008). *Improving practice and progression through lesson study: A handbook for headteachers, leading teachers and subject leaders*. London: DCSF.

Dudley, P. (2011). Lesson study development in England: From school networks to national policy. *International Journal for Lesson and Learning Studies, 1*(1), 85–100.

Dudley, P. (2013). Teacher learning in lesson study: What interaction-level discourse analysis revealed about how teachers utilised imagination, tacit knowledge of teaching and fresh evidence of students learning, to develop practice knowledge and so enhance their students' learning. *Teaching and Teacher Education, 34*, 107–121.

Earl, L., Fullan, M., Leithwood, K., Watson, N., Jantzi, D., Levin, B., & Torrance, N. (2003). *Watching and learning 3: Final report of the external evaluation of England's national literacy and numeracy strategies*. Toronto: Ontario Institute for Studies in Education.

Edwards, A. (2012). The role of common knowledge in achieving collaboration across practices. *Learning, Culture and Social Interaction, 1*(1), 22–32.

Edwards, D., & Mercer, N. (1987). *Common knowledge: The development of understanding*. London: Methuen.

Elliott, J. (1991). *Action research for educational change*. Buckingham: Open University Press.

Engstrom, Y. (2003). Activity theory and individual and social transformation. In Y. Engstrom, R. Miettinen, & R. L. Punamaki (Eds.), *Perspectives on activity theory*. Cambridge: Cambridge University Press.

Eraut, M. (1994). *Developing professional knowledge and competence*. London: Falmer Press.

Esser, J. K. (1998). Alive and well after 25 years: A review of groupthink research. *Organisational Behaviour and Human Decision Processes, 73*(2–3), 116–141.

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A US-Japan lesson study collaboration reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*(2), 171–185.

Hadfield, M., Jopling, M., & Emira, M. (2011). *Evaluation of the national strategies' primary leading teachers programme*. London: Department for Education.

Hargreaves, D. (1999). The knowledge creating school. *British Journal of Education Studies, 47*(2), 122–144.

Hart, P. (1994). *Government: A study of small groups and policy failure*. Baltimore: The Johns Hopkins University Press.

Horn, I. S., & Kane, B. D. (2015). Opportunities for professional learning in mathematics teacher workgroup conversations: Relationships to instructional expertise. *The Journal of the Learning Sciences, 24*(3), 1–46.

James, M., McCormick, R., Black, P., Carmichael, P., Drummond, M. J., Fox, A., MacBeath, J., Marshall, B., Pedder, D., Proctor, R., Swaffield, S., Swann, J., & Wiliam, D. (2007). *Improving learning how to learn: Classrooms, schools and networks*, *TLRP improving learning series*. London: Routledge.

Janis, I. (1982). *Groupthink: Psychological studies of policy decisions and fiascos* (2nd ed.). New York: Houghton Mifflin.
Kazemi, E., & Hubbard, A. (2008). New directions for the design of professional development: Attending to the coevolution of teachers' participation across contexts. *Journal of Teacher Education, 59*(5), 428–441.
Kuno, H. (2015). Evolving the curriculum through lesson study. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 104–117). London: Routledge.
Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.
Lewis, C. (1998). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, (Winter), 22*, 12–17 & 50–51.
Lewis, C. (2016). *What learning occurs at each stage of the lesson study process*. Presentation given at the World association of lesson studies annual conference, 3rd Sept 2016, University of Exeter.
Lieberman, A., & Miller, L. (2008). *Teacher in professional communities*. New York: Teachers College Press.
Linell, P. (2009). *Rethinking language, mind, and world dialogically: Interactional and contextual theories of human sense-making*. Charlotte: Information Age Publishing.
Littleton, K., & Mercer, N. (2013). *Interthinking: Putting talk to work*. Abingdon: Routledge.
Lo, M. L. (2016). You can only see what you have chosen to see: Overcoming the limitations inherent in our theoretical lenses. *International Journal for Lesson and Learning Studies, 5*(3), 170–179.
Marton, F., & Säljö, R. (1984). Approaches to learning. In F. Marton, D. Hounsell, & N. Entwistle (Eds.), *The experience of learning* (pp. 39–58). Edinburgh: Scottish Academic Press.
Måseide, P. (2003). Medical talk and moral order: Social interaction and collaborative clinical work. *Text, 2*(3), 369–403.
McKinsey. (2007). *How the world's best-performing school systems come out on top*. London: McKinsey & company.
McLaughlin, M., & Talbert, J. (2006). *Building school-based teacher learning communities: Professional strategies to improve student achievement*. New York: Teachers College Press.
Mercer, N. (1995). *The guided construction of knowledge: Talk amongst teachers and learners*. Clevedon: Multilingual Matters.
Mercer, N. (2000). *Words and minds: How we use language to think together*. London: Routledge.
Mercer, N. (2004). Sociocultural discourse analysis: Analysing classroom talk as a social mode of thinking. *Journal of Applied linguistics and Professional Practice, 1*(2), 137–168.
Mercer, N. (2014). The social brain, language, and goal-directed collective thinking: A social conception of cognition and its implications for understanding how we think, teach, and learn. *Educational Psychologist, 48*(3), 148–168.
Mercer, N., & Littleton, K. (2007). *Dialogue and the development of children's thinking: A sociocultural approach*. London: Routledge.
Mercer, N., Littleton, K., & Wegerif, R. (2004). Methods for studying the processes of interaction and collaborative activity in computer-based educational activities. *Technology, Pedagogy and Education, 13*(2), 193–209.
Mourshed, M., Chijioke, C., & Barber, M. (2010). *How the world's most improved school systems keep getting better*. London: McKinsey and Company.
Murphy, R., Weinhardt, F., Wyness, G. and Rolfe, H. (2017). *Lesson study: Evaluation report and executive summary*. Education Endowment Foundation. Available at: https://educationendowmentfoundation.org.uk/public/files/Projects/Evaluation_Reports/Lesson_Study.pdf. Accessed 20 Feb 2017.
Opfer, V. D., & Pedder, D. (2010). Benefits, status and effectiveness of continuous professional development for teachers in England. *The Curriculum Journal, 21*(4), 413–431.
Opfer, V. D., & Pedder, D. (2011). Conceptualizing teacher professional learning. *Review of Educational Research, 81*(3), 376–407.

Powell, A., Francisco, J., & Maher, C. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behaviour, 22*(4), 405–435.

Puttnam, R., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1), 4–15.

Robinson, V., Hohepa, M., & Lloyd, C. (2009). *School leadership and student outcomes: Identifying what works and why best evidence synthesis*. Auckland: New Zealand Ministry of Education.

Scribner, S. (1999). Knowledge at work. In R. McCormick & C. Paechter (Eds.), *Learning and knowledge* (pp. 103–111). London: Paul Chapman.

Sorenson, P. D., Newton, L. R., & Harrison, C. (2006). *The professional development of teachers through interaction with digital video*. Paper presented at the BERA annual conference 2006, Sept 2006. Location.

Stenhouse, L. (1981). What counts as research? *British Journal of Educational Studies, 29*(2), 103–113.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximising the impact of lesson study. *ZDM Mathematics Education, 48*(4), 513–526.

Vermunt, J. D. (2013, 22 May). *Teacher learning and student learning: are they related?* Inaugural lecture given at the University of Cambridge, Faculty of Education.

Vermunt, J. D., & Endedijk, M. D. (2011). Patterns in teacher learning in different phases of the professional career. *Learning and Individual Differences, 21*(3), 294–302.

Vermunt, J. D., & Vermetten, Y. J. (2004). Patterns in student learning: Relationships between learning strategies, conceptions of learning, and learning orientations. *Educational Psychology Review, 16*(4), 359–384.

Vermunt, J. D., Vrikki, M., Mercer, N., & Warwick, P. (2015). *UK teachers' perceptions of Lesson Study and its effects on teacher learning: a survey study.* Paper presented at the 16th conference of the European association of research on learning and instruction, Limassol, Cyprus.

Vermunt, J. D., Vrikki, M., Warwick, P., & Mercer, N. (2017). Connecting teacher identity formation to patterns in teacher learning. In D. J. Clandinin & J. Husu (Eds.), *The SAGE handbook of research on teacher education* (pp. 143–159). London: SAGE.

Vrikki, M., Warwick, P., Vermunt, J. D., Mercer, N., & Van Halem, N. (2017). Teacher learning in the context of lesson study: A video-based analysis of teacher discussions. *Teaching and Teacher Education, 61*, 211–224.

Vygotsky, L. (1962). *Thought and language*. Cambridge, MA: MIT.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Warwick, P., Vrikki, M., Vermunt, J. D., Mercer, N., & Van Halem, N. (2016). Connecting observations of student and teacher learning: An examination of dialogic processes in lesson study discussions in mathematics. *ZDM Mathematics Education, 48*, 555–569.

Warwick, P., Vrikki, M., Færøyvik Karlsen, A-M., Dudley, P., & Vermunt, J. D. (2018). '*Instead of numbers, I would make it decimals': The role of pupil voice in teacher learning in Lesson Study*. Under review.

Webster-Wright, A. (2009). Reframing professional development through understanding authentic professional learning. *Review of Educational Research, 79*(2), 702–739.

Wegerif, R. (2007). *Dialogic education and technology: Expanding the space for learning*. New York: Springer.

Ylonen, A., Dudley, P., & Lang, J. (2015). *London schools excellence fund new curriculum higher order mathematics lesson study programme final report*. London: Greater London Authority.

Yoshida, M. (2002). *Lesson study: An introduction*. Madison: Global Education Resources.

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A Case of Lesson Study in South Africa

Jill Adler and Jehad Alshwaikh

Contents

Abstract The Wits Maths Connect Secondary Project, a research-linked professional development project, included Lesson Study with teachers in school clusters. When two teachers shared their experiences of doing Lesson Study, they spoke spontaneously about how much they had learned about choosing and using examples in their teaching. Exemplification is a key element of the mathematics teaching framework developed in the project to support planning and reflection in our Lesson Study work. The teachers' reflection suggested that working on examples had been enabling and empowering. We zoom in on one Lesson Study cycle with the same teachers to give meaning to what, how and why we work with examples in the way that we do and how our LS practices provide a supportive context for this work.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_16

Through this we build a case for a focus on exemplification when studying and working on mathematics teaching and for support at a more general level for theoretically informed Lesson Study.

Keywords Lesson Study · Mathematics · Secondary · Exemplification · South Africa

1 Introduction

> What I have learned from these Lesson studies, it is not only about my teaching and the strategies I use, but it is about giving them [learners] the mathematics that is behind each and every topic so that they understand what we are doing. The reflection has helped a lot. We can reflect on what language are the learners using … **And most of all looking at the examples, that is one where we have grown a lot, what examples we use and why we use such examples**. (Thembi Ndlovu,[1] teacher participant, Lesson Study seminar, June 2016, our emphasis)

The quote above was Thembi's opening reflection on her experience with Lesson Study. She was talking in a seminar for mathematics teachers and teacher educators visiting the Wits Maths Connect Secondary project (WMCS). Located at the University of the Witwatersrand in Johannesburg, WMCS is a research-linked professional development project aimed at improving the learning and teaching of mathematics in previously disadvantaged secondary schools in one province in South Africa. The purpose of the seminar was to share the WMCS approach to Lesson Study. Thembi voiced the importance of exemplification as a focus in the Lesson Study work of the project. In this chapter, and our presentation of a case of Lesson Study in South Africa, we focus on the possibilities opened up for learning when exemplification is explicitly attended to in Lesson Study (LS).

While reflective of the widely held view of professional learning as a collaborative practice, and with some similarities to the Japanese model, the LS model developed in the WMCS project has two distinct features. Firstly, it is framed by a theoretically driven and empirically elaborated framework for mathematics teaching that includes an emphasis on exemplification. Secondly, the practice of LS in the project has been shaped by conditions of learning and teaching in the WMCS project schools and thus by similar constraints experienced in the implementation of LS elsewhere in South Africa and other low-income schooling contexts (Adler 2017).

In 2016, we embarked on systematic study of Lesson Study in the WMCS, investigating the opportunities for learning opened up through a case of theoretically driven and time-constrained Lesson Study. In this chapter we focus on one LS cycle from the most active and sustained cluster. It vividly illustrates what

[1]This is a pseudonym, as are the names of other participants in the paper except the authors of this chapter.

and how exemplification, as an explicit object of attention for planning and reflection, opened opportunities for learning about mathematics teaching in the LS group.

2 Literature Review

2.1 Lesson Study

Lesson Study, a long-standing practice in Japan (Fernandez 2002) and China (Gu and Gu 2016; Huang and Shimizu 2016), has become widely used in professional development globally and differently interpreted across contexts (Huang and Shimizu 2016; Huang et al. 2017; Quaresma et al. 2018). Different models of LS have different foci. In Japanese LS there is a focus on problem-solving and student thinking with considerable time given to initial deliberations, discussion and planning with ranging curriculum resources. Particular attention is given to developing a key task for the lesson, as well as to the significant role of knowledgeable others in the LS group (Fujii 2016; Takahashi and McDougal 2016). In Chinese LS, there is a focus on exemplary lessons. Huang and Shimizu (2016) reviewed the main characteristics of Chinese LS such as phases of teaching practice and phases of reflection and the Chinese three-point framework (important knowledge point, difficult point and critical point), which is rooted in a culture of achieving effective teaching.

Much of the research writing on LS focuses on how it functions (e.g. Yoshida 2012) and its impact on teacher education and advocates its use for mathematics professional learning and development (da Ponte 2017; Huang and Shimizu 2016). Papers typically highlight how lessons or tasks in them evolve over a Lesson Study cycle and what is learned from this (Fujii 2016; Pang 2016) and/or what and how teachers learn about their learners, their teaching and mathematics content (Dudley 2013; Murata et al. 2012; Shuilleabhain 2016). There has also been attention to how facilitators in LS learn to do this work (Lewis 2016).

In addition to the doing of, and learning from LS, questions have been raised about the pedagogy and learning theory that supports Lesson Study work (Elliot 2012). Swedish Learning Study is a model of LS developed in Sweden and Hong Kong with an explicit theory of learning: variation theory (e.g. Marton and Tsui 2004). For Marton and Tsui, learning is always about learning something, an *object of learning*, and is a function of discerning difference through variation (Runesson 2006). Wood (2017) has interpreted a focus on difference as one of the common aspects across Japanese LS, Chinese LS and Swedish Learning Study, in addition to collaborative learning in professional communities.

The idea of professional learning communities working to improve mathematics teaching and learning would, as expected, capture attention, particularly of mathematics teacher educators and researchers around the world. Adaptations of Japanese LS have taken root in the USA, the UK, and Australia, for example. In these contexts LS projects, while neither systemic nor rooted in a professional learning culture, also focus on problem-solving and learner thinking. As expected, questions have emerged about the wider applicability of LS and its expansion and sustainability in these contexts (e.g. Wood 2017).

The range of active pursuance of LS in mathematics teacher professional development spreads from the countries mentioned through to Asia and Africa, including South Africa, with most influenced by the Japanese model. Recent reviews reflect that adaptations are apparent (Huang et al. 2017; Larssen et al. 2018; Wood 2018). Challenges and successes in conducting LS have been illuminated, particularly cultural fit (e.g. with the Japanese model), the time investment and so resistance from teachers, the perceived limitations of focusing on one lesson and whether and how researchers or knowledgeable others are appropriately participative (Lewis 2016). Fujii (2014) argued that cultural practices in Japanese schools have not been "transferred" together with doing LS elsewhere and that there were "misconceptions" about the Japanese LS such as LS being carried out as a workshop rather than a process of cycles of planning, teaching and reflection. He was reporting on a project where JICA promoted LS in countries such as Malawi and Uganda.

There are a number of cases of LS in South Africa influenced by collaboration between universities and provincial education departments in SA and universities and JICA from Japan (e.g. Jita et al. 2008; Ono and Ferreira 2010). In South Africa, while there is recognition of the benefits, Jita et al. (2008) state many challenges. For example, where the official education department was involved, a sense of ownership by teachers was thwarted as teachers were selected and assigned by the department of education. Another challenge was the available time for running the training and enacting the process (planning, teaching and reflection). It was not practical to ask teachers to do extra work during or after teaching in or out the school. A third challenge was that participant teachers crossed grades. Teachers' communication was shaped by power relations which made peer teacher learning difficult. Finally, the role of knowledgeable others (Japanese experts or SA university educators) in conducting training and development for teachers was entangled by official policy. A cascade model was effected "first to the curriculum implementers who then presented it, in turn, to the cluster leaders. The cluster leaders then cascaded the training further down to the teachers at the school level" (p. 480). The difference between this enactment of LS and practices in Japan is apparent, particularly in relation to LS being an in-school practice, where experienced and knowledgeable others are part of the systemic culture.

Similar challenges and constraints have been reported on LS in Kenya and Zambia where through JICA, LS has been adapted and implemented systemically. Baba and Nakai (2011) described challenges that teacher education in developing countries face such as a focus on quantity over quality and where there are underqualified teachers. In Zambia, for example, the department of education has worked with JICA to implement LS as professional development for Zambian teachers. Inevitably, a form of cascading evolved for training participants. One consequence was insufficient focus by teachers on developing a common understanding about the target lesson to be achieved. Confirming these kinds of challenges, Benson et al. (2014), pointed out that "a school with high pupil-teacher ratio, and a school with heavy teachers' workload appeared to be negatively related to CPD implementation" (p. 476).

These reports point to the significant contextual challenges in LS work. Limited human and material resources open up critical reflection on the benefits that accrue to participants and on what is actually possible. It is precisely these conditions that inspired the WMCS to (re)design Lesson Study in ways that both supported the goals of the wider project (see below) and the conditions of teachers' work in South Africa. The key design features of the WMCS model mentioned earlier thus make sense. In particular the model takes constraints on time into account and it focuses on current teaching practices like exemplification. Before describing LS in WMCS, we look briefly at the literature base on examples in mathematics education.

2.2 Exemplification and Variation

There is an extensive literature on the significance of exemplification in mathematics teaching, e.g. on the value of multiple examples in a lesson and on the importance of the role of examples in building generalisation and recognising structure (e.g. Antonini et al. 2011; Watson and Mason 2006; Sinclair et al. 2011; Kullberg et al. 2017). In referring to examples as the "raw material for generalization", Bills and Watson (2008) and Sinclair et al. (op cit) point to the work of teaching in their use. Not only is deliberate and careful selection and sequencing of examples needed, but teaching needs to bring learners' attention to connections between selected examples and to the underlying patterns of variance amidst invariance that enable generalisation and/or appreciation of structure. While the importance of variation in mathematics teaching dates back to the work of Dienes (1960), it has re-emerged in more recent years through the work of Watson and Mason and their vivid illustration of variance amidst invariance as a tool for engaging with generality and with mathematical structure (2006), as well as the links made between their work and those working with variation theory mentioned above. For Kullberg et al. (op cit), the application of variation theory in teaching enables critical features of the object of learning to come into focus. This linked work on examples and variation in and for mathematics teaching has inspired and influenced our attention to exemplification in our research and professional development work (Adler and Venkat 2014; Adler and Ronda 2015).

In her review of research on the roles and use of examples in mathematical learning and thinking, Zaslavsky (in press) distinguished between three settings of example-use, spontaneous example-use, evoked example-use and responsive example-use to a provided example and thus between settings where the source of examples is the learner (the first two) and where the source is the teacher/researcher. In this latter setting, as she notes: "The provided example-use requires scaffolding of students' attention to relevant features of the example". Zaslavsky's research has focused on teachers' awareness of their example use in their teaching (Zodik and Zaslavsky 2008) and students' use of examples in their thinking (Ellis et al. in press). Teachers' were not necessarily aware of how they were using examples, and students' use was more or less productive. Teachers' thus can be supported in this work.

Kullberg et al. (op cit) made the important observation that multiple examples are not simply cumulative. Their ordering, simultaneous presentation and the teacher drawing attention to similarities and differences are critical. We agree, but we have also argued previously that research related to a lesson needs to attend to the accumulating example space, clearly in relation to the object of learning.

> The research on examples, [however,] while illuminating of what teachers do and why, does not enable a view of whether and how examples accumulate to bring the object of learning into focus for learners, and whether there is movement towards generality. (Adler and Ronda 2017a, p. 68)

This reinforces the notion above of examples as "raw materials", and while an example is always "an example of something" (Goldenberg and Mason 2008), learners' attention needs to be drawn to the general case of which it is an instance, and this is a function of a set of examples and their mediation in teaching. This encouraged us to work with teachers on their use of examples. In order to bring the object of learning into focus and enable learners to generalise through the use of examples, we have highlighted two necessary aspects for a sequence of examples, and the accumulating example set in a lesson. There needs to be *similarity* from one example to the next – where a feature is kept *invariant* while other features vary. In addition, there need to be *contrasting* examples that can draw attention to difference between key features. The key influences here are Watson and Mason and the variation theorists op cit. With a focus on variance amidst invariance, and contrast, it is possible to build generality and create possibilities for appreciating the critical features and structure of particular mathematical objects.

A recent Swedish Learning Study, drawing on many of the above considerations, is salient here as the object of learning in the study was the distributive law – the focus of the LS cycle in this chapter. Olteanu (2017) reviewed the literature on "misconceptions" related to the distributive law, including overgeneralisation, and the importance of learners being able to apply the distributive law in its direct ($a(b+c) = ab+ac$) and indirect ($ab+ac = a(b+c)$) forms. She argued that:

> . . . students (need) to . . . discern the meaning of the whole i.e. $2(3 + 4) = 2 \cdot 3 + 2 \cdot 4$ . . . by knowing the meaning of the simple parts (i.e. the numbers, factors and terms), the semantic significance of a finite number of syntactic modes of composition (i.e. operations and parenthesis), and how the whole is built up out of simple parts. (p. 63)

In his paper on "Tunja sequences", Mason (2001) also pointed to the importance of both syntax and semantics in learning and teaching algebra that are relevant to our LS cycle. He described work with learners where their attention towards generality was provoked by a key starting set of examples. Learners had to continue the pattern and build their own examples in response to "what comes next"?

$$
\begin{aligned}
&1 \times 6 + 6 = 3 \times 4\\
&2 \times 7 + 6 = 4 \times 5\\
&3 \times 8 + 6 = 5 \times 6\\
&4 \ldots.\\
&x
\end{aligned}
$$

This is an exploratory task where, as Mason notes, it is a relatively easy step to $* \times (*+5)+6 = (*+2)(*+3)$ and thus an expression of generality (pp. 165–166). Our interest here is the related algebraic work learners were to do. They had to apply the distributive law, do and undo operations (or inverses), and thus focus on the syntax of the equations, relating this to their meaning. Of course, it is possible to follow a pattern without knowing why it works. However, Mason argued that by expressing generality, learners can come to appreciate rules for expansion of brackets and ultimately the distributive property of ordinary arithmetic.[2]

Together Olteanu and Mason provide support for our work on exemplification where we work with teachers and their learners on syntax and semantics. Both are crucial to algebraic fluency. We elaborate these aspects of algebra below when we discuss the unfolding of the Lesson Study and its focus on "the position of brackets in algebraic expressions".

Of course, examples are always embedded in a task. While examples are selected as particular instances of the general case, and for drawing attention to relevant features, generality and/or structure, tasks are designed to bring particular capabilities to the fore (Marton and Tsui 2004). For example, expanding a(b + c) and factoring ab + ac are different tasks. They are also inverses, and noticing or working on the relation between these, or doing and undoing in Mason's terms, yet a further capability. Different tasks require different actions, at different levels of complexity, and so make available different opportunities for mathematics learning. In our LS work, constructing an example set across a lesson includes attention to tasks, and as we will show, this was the collective work of the LS group – teachers and researchers.

3 The WMCS, Framework of Mathematics Teaching and Model of Lesson Study

3.1 The Wits Maths Connect Secondary Project and Its Mathematics Teaching Framework

WMCS, a research-linked professional development project, has aimed at improving mathematics teaching in disadvantaged schools in one province in South Africa. In

[2]Mason makes this point in an earlier version of the 2001 paper with the same title and available on his website.

our early research work, we examined practices in these schools. With a sociocultural orientation to learning as mediated, and to mathematics as a network of scientific, connected and hierarchic concepts (Vygotsky 1978), we zoomed in on exemplification and explanatory talk in the lesson, viewing these as key mediational means in instruction. These were common across lessons yet diverse in terms of the quality of mathematics offered for learning. We explored coherence and connections between examples and the explanatory talk that elaborated these within an episode (Venkat and Adler 2012) and later as these unfolded across episodes in a lesson (Adler and Venkat 2014). These initial studies pointed to diversity across teachers but with an undercurrent of incoherence and fragmentation of mathematics, hence the emphasis in WMCS on mathematical coherence in our work with teachers.

The cornerstone of WMCS development work is a year-long 16-day[3] *mathematics for teaching* course, offering participating teachers an opportunity to strengthen their own relationship with mathematics and with mathematically coherent teaching. In line with the overall sociocultural orientation to learning and to mathematics, we developed an analytic framework for describing and interpreting shifts in mathematics made available in teaching (Adler and Ronda 2015). Called Mathematical Discourse in Instruction (MDI), the framework provides a set of analytic tools to look closely at mathematics teaching and to analyse the mathematics offered to learners and how learners participate in their own learning and is detailed in Adler and Ronda (2015).

MDI focuses on four key elements of a mathematics lesson: first is the *object of learning*, the "what" of learning or lesson goal and what students are expected to know and be able to do at the end of a lesson. The object of learning could be a concept, a procedure or mathematical practice, together with the relevant capability. This leads to *exemplification* – to the sequence of examples, with their associated tasks and representations that can be used to bring the object of learning into focus with learners. As reflected in our discussion on exemplification, attention is on how to work with variance amidst invariance, to build generality and illuminate structure. We note the third and fourth elements of MDI, though they are not in focus in this chapter. *Explanatory communication* includes attention to naming/word use (what is said and what is written) and substantiations of mathematics as specialised knowledge (what counts as mathematical knowledge). *Learner participation* focuses on what learners do and say with regard to the mathematics they are learning. We consider whether and how learners' talk moves beyond the prevalent but limited chorusing of single words, towards responding to and asking questions and to more open dialogue with others (the teacher and other learners). In Adler and Ronda (2015), we discuss how we view *exemplification* and *explanatory communication* as mediational means and explain why and how the elements of the framework and constructs within each of these are both theoretically and empirically informed. The overarching importance of the framework is the coherence of a lesson and thus how

[3]In some papers, this is described as a 20-day course, 16 days on campus and 4 days in school work.

Lesson goal:		
Exemplification Examples, tasks and representations	**Learner Participation** Doing maths and talking maths	**Explanatory communication** Word use and justifications
Coherence and connections to the lesson goal		

Fig. 1 WMCS mathematics teaching framework

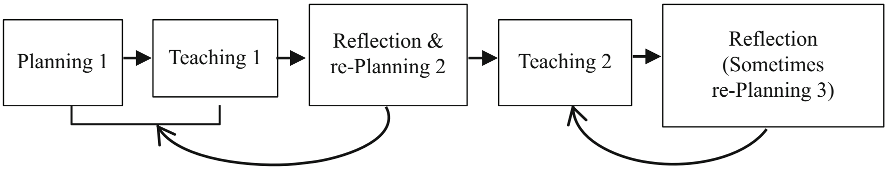

Fig. 2 One cycle of the LS in WMCS

all elements interact, link back to the object of learning and open opportunities to learn mathematics. We have reported on the relationship between our analysis of teaching using MDI and on teaching informed by this work (Adler and Ronda 2017a, b).

Between 2013 and 2016, we explored a model of LS in the project. We worked in a school after school hours in small collaborating groups of teachers from clusters of schools. We used MDI as a structuring device to guide lesson planning and reflection. In order to communicate MDI with teachers, the project team organised MDI into a teaching framework (referred to in the project as a Mathematical Teaching Framework – MTF), illustrated in Fig. 1. MDI/MTF has thus functioned as a boundary object, moving between being a research tool and a tool for teaching (Adler 2017). It has framed the research work of the project (how lesson analysis is done) and informed PD practice. As shown in Fig. 1, the MTF includes all the components of the MDI framework and is intended to assist teachers in planning and then as an observation/reflection tool on the “quality” of mathematics offered in their teaching in LS. In this chapter we focus on exemplification in the MTF framework in our LS work.

3.2 *Our Lesson Study Model*

Figure 2 shows one LS cycle in the WMCS. In some cases, like the cycle we discuss in this chapter, the team works up a final plan (not to be implemented) if further changes were recommended.

While the cycle follows mainstream practice of LS, *differences*, *particularly in relation to time and place*, are a function of resource constraints. Our LS cycle takes place one afternoon a week *after school hours* for 3 consecutive weeks and involves *teachers from a cluster of neighbouring schools*. In week 1, the LS group plan a 1-h lesson on a topic agreed by the teachers in advance of or during this first meeting. The plan usually includes a short pretest to be given at the start and then again as a post-test at the end of the lesson. In week 2, one teacher teaches the lesson to a class of learners who agree to remain after school for the lesson. After the learners leave, the LS group spends the following hour reflecting on the enacted lesson and planning the second lesson which is taught in the third week, to a different class, but the same grade. *A Lesson Study cycle thus involves 6 hours* of face-to-face collaboration, an evolving lesson plan and reflection on both taught lessons. Of course, other planning goes into the LS. The teacher who is teaching plans the actual lesson more carefully before teaching. The project team prepares and brings relevant instructional resources to the sessions. We refer readers back to our review of LS where we discussed constraints on time and place. In our model, a LS cycle takes place over a relatively short time. It takes place outside of normal class teaching, and for some of the participating teachers', this is not their own school and/or classroom.

4 The Study: Data and Method

While we carried out LS in three clusters of schools, in 2016 we undertook a systematic study of our most participative and sustained LS cluster. Elsewhere we have reported aspects of this research. We have shown how LS opened up opportunities for teachers and researchers together to learn about teaching and how the tensions and dilemmas we faced were simultaneously opportunities for strengthening the coherence of the community in doing LS (Alshwaikh and Adler 2017a, b). A specific question in the study, and the focus of this chapter, was on the evolution of the example set or what others have called the instructional example space (Goldenburg and Mason op cit) in a LS cycle. The questions we aim to answer in this chapter are:

1. What changes occur in the example set across the lesson plans over a cycle?
2. How do these changes evolve?

We have selected one of the three LS cycles in 2016, where four teachers (Thembi, Lerato, Thabi and Sipho[4]) from three different schools in the cluster and four researchers (Frank, Linda, Jehad and Jill) from the project met in May for planning the first teaching.[5]

[4]Sipho was present but not critically active in the group.

[5]There are practical reasons for the unusual almost one-to-one relation between researchers and teachers. The second author was relatively new in the project, learning about LS, and so supported by the first author, the project director and another graduate student who had participated in a previous cycle.

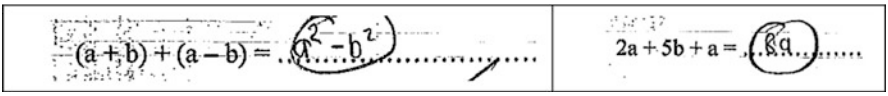

Fig. 3 Thembi's students' answers to an assessment test (grades 11 and 12)

The teachers chose to work on "simplifying algebraic expressions with brackets in different positions" in Grade 10. The teachers justified their selection based primarily on their experience with learners applying the distributive law and the persistent errors made in their manipulation of algebraic expressions when these included brackets in various positions. This issue was raised during the first planning meeting when we analysed the curriculum and the mathematical content in relation to the suggested object of learning (position of the brackets and distributive law). Thembi showed the group the results of a preliminary assessment she did with her students (grades 11 and 12) and highlighted some common errors such as $(a+b)+(a-b)=a^2-b^2$ (as shown in Fig. 3), confusion between $3-(2x+5)$ and $-3(2x+5)$ and where conjoining still appeared in learner scripts (see Fig. 3).

Our experience and study of learning gains in the WMCS project (see Pournara et al. 2015) confirmed that learner error was prevalent in our schools. While these errors are similar to those identified in previous research (Olteanu op cit), in South Africa these persist even as learners progress to higher grades. In particular, we were aware of the importance of bringing learners' attention to the position of the bracket in an expression – to the operation with the term preceding or following the bracket – and so to the syntax of the expressions. The group also commented on common teacher talk as a source of some errors, such as the instruction to "do what is in the bracket first" or "bracket first" when learning about the order of operations (brackets, orders (exponents), division, multiplication, addition, subtraction – BODMAS), references to "invisible brackets" around integers and why/when an operation sign is needed between brackets and its meaning.

In order to systematically research our co-learning, we collected all relevant information: written lesson plans; pre- and post-'tests'; audio and/or video recordings of all sessions, i.e. planning week 1; and lesson and reflection discussion in the second and third weeks. We also recorded the two teachers' reflections on their experience in a seminar setting; and all the teachers provided feedback in a brief written questionnaire at the end of 2016. All audio and video recordings were transcribed. We draw from this data, particularly the lesson plans, post-lesson reflective discussions and teacher reflections to engage with our research questions.

Our analysis proceeded with a description of the exemplification in the first lesson plan in terms of the MTF framework, and so our interpretation of variance amidst invariance, and similarity and contrast in the accumulating example set. We identified examples that were removed/added, where this occurred and how this linked with changing representations and tasks in Plans 2 and 3. Our analysis of the reflective discussions after each lesson included the identification of "example change moments" – moments in the reflective discussion where there was specific attention to exemplification – *and* the discussion led to some change. We described by whom and how the discussion was initiated and how it evolved. We paid particular attention to the moves made to change examples/representations/tasks and by whom. The changes were then interpreted and explained in relation to the theoretical framing of the LS by MDI.

Before examining the plans and reflection sessions, we briefly capture Thembi and Lerato's reflections in the seminar and their responses to the questionnaire. In particular, we identify talk related to exemplification.

5 Teachers' General Reflections on and in Lesson Study

We opened the paper in Thembi's voice in a seminar about what LS meant to her. She spontaneously emphasised learning from and about choosing and using examples, and this was reinforced by Lerato who also linked this to learner thinking.

> we also think about what are learners going to do – what do we expect them to do with the examples . . . their misconceptions, what are we going to try to correct and how. (Lerato Sello, teacher participant, WMCS Lesson Study seminar, June 2016)

Thembi gave more mathematical substance to her general appreciation of learning about examples when she described her "key take-away" from the LS work in her written survey response. She pointed to how examples can operate to build generality and illuminate structure:

> Linking concepts and generalisations after each example . . . directing learners' attention to the structure of examples as this help them to connect to different strategies for solving problems. (Thembi survey, Nov 2016)

Learning from and about exemplification stands out for the teachers in focus in this paper. This was our experience across the LS cycles in this cluster and other clusters of schools/teachers. Indeed, this is the case for teachers in the wider WMCS project (e.g. Ntow and Adler 2017; Adler and Ronda in preparation).

6 The Lesson Plans and Evolving Example Sets

In Fig. 4 below, we present the cumulative example set in each lesson plan produced during the cycle: the initial plan for Lesson 1 (Plan 1, P1), the revised plan for Lesson 2 (P2) and then a re-revised plan that completed the LS cycle (P3). This is to keep our focus on exemplification and our particular research questions. Each plan had additional information related to explanatory talk and learner participation that, as noted, are backgrounded here.

With respect to the examples, the LS group agreed to introduce the lesson with numerical expressions. This representation of expressions with brackets would have meaning for learners and provide a semantic basis for moving on to algebraic

P1	P2	P3
Pre-test assessment	Pre-test assessment	Pre-test assessment
Introduction: Calculate the following: a) $4 + 3(4 + 5)=$ b) $(4 + 3)4 + 5 =$ c) $(4 + 3)(4 + 5)$	Introduction: simplify the following expressions: a) $4 + 3(4 + 5)=$ b) $(4 + 3)4 + 5 =$ c) $(4 + 3)(4 + 5)=$	Introduction: simplify the following expressions: a) $4 + 3(4 + 5) =$ b) $(4 + 3)4 + 5 =$ c) $(4 + 3)(4 + 5) =$ d) $(4 + 3) - (4 + 5) =$ Insert brackets in the left side to result in the given sum e) $4 - 3 + 5 = -4$ f) $4 + 3 - 5 = -35$
Activity 1: Simplify the following $x + 3(x + 5)=$ $4 + 3(4 + 5)=$ $(x + 3)x + 5 =$ $(4 + 3)4 + 5 =$	Activity 1: Simplify the following $x + 3(x + 5)=$ $(x + 3)x + 5 =$	Activity 1: Simplify the following 1. $x + 3(x + 5) =$ 2. $(x + 3)x + 5 =$ 3. $x - 3(x + 5) =$ 4. $-(x + 3)x - 5 =$
Activity 2: Simplify $(x + 3)(x + 5) =$ $(x + 3) + (x + 5) =$	Activity 2: Simplify $(x + 3)(x + 5) =$ $(x + 3) - (x + 5) =$	Activity 2: Simplify 5. $(x + 3)(x + 5) =$ 6. $(x + 3) - (x + 5) =$ 7. $(x + 3)(-x + 5) =$
Activity 3: Simplify a) $(x - 3x) + 5 =$ b) $(x - 3)x + 5 =$ c) $x(-3x + 5) =$ d) $x - (3x + 5)$	Activity 3: insert bracket(s) in the expressions on the left side so that the two sides are equal a) $x - 3x + 5 = -3x^2 + 5x$ b) $x - 3x + 5 = -2x - 5$ c) $x - 3x + 5 = x^2 - 3x + 5$	Activity 3: insert bracket(s) in the expressions on the left side so that the two sides are equal 8. $x - 3x + 5 = -3x^2 + 5x$ 9. $x - 3x + 5 = -2x - 5$ 10. $x - 3x + 5 = x^2 - 3x + 5$
Activity 4: Simplify a) $x - 8(x + 6) =$ b) $(x - 8)x + 6 =$ c) $(x - 3)(x + 3) =$ d) $(x - 3) - (x + 3) =$	Activity 4: Simplify a) $x - 8(x + 6) =$ b) $(x - 8)x + 6 =$ c) $(x - 3)(x + 3) =$ d) $(x - 3) - (x + 3) =$	Activity 4: Simplify a. $x - 8(x + 6) =$ b. $(x - 8)x + 6 =$ c. $(x - 3)(x + 3) =$ d. $(x - 3) - (x + 3) =$
Activity 5 (Post-Test): Simplify a) $2p - (4 + p) =$ b) $2p(-4 + p) =$ c) $(2 + p) + (-4 + p) =$	Activity 5 (Post-Test): Simplify a) $2p - (4 + p) =$ b) $2p(-4 + p) =$ c) $(2 + p) + (-4 + p) =$	Activity 5 (Post-Test): Simplify a) $2p - (4 + p) =$ b) $2p(-4 + p) =$ c) $(2 + p) + (-4 + p) =$

Fig. 4 Examples in the three plans (Lesson goal: Learners can simplify expressions with brackets when these are in different positions)

expressions and so more abstract symbolic forms. As visible in the introductory examples, the numbers 4, 3, 4 and 5 and the + operation in each expression were invariant. Only the position of the brackets changed from one example to the next. The task was for learners to calculate the value of these expressions. Communication following the completion of the calculations by learners was to focus on "what was the same and what was different" across the examples, thus bringing into focus the impact of the changing position of the brackets. The group also discussed the underlying application of the distributive law and that as these were numerical examples, learners might apply BODMAS, adding the terms within the brackets first. The teacher was to encourage learners who did this to look at each expression and do the calculation applying the distributive law.

The examples of algebraic expressions in Activity 1 and 2 were then similarly structured with x, 3, x and 5 and the + operation invariant while the position of the brackets changed with each subsequent example. These two activities on simplifying expressions formed the lesson presentation and were to be done through teacher-led whole class interaction. At the end of each activity, the teacher drew learners' attention to what was similar and different in the expressions. Activity 3 which included the operation of subtraction/negative terms and, if time, Activity 4 were then to be done by learners, independently of the teacher. Learners would complete the post-test in the last minutes of the lesson.

The accumulating example set reflects the groups' interpretation of variation for this lesson where similarity first within each activity example set and then across these build generality in relation to the application of the distributive law and the more visible form of the changing position of brackets the expression. Contrast was introduced in Activity 2 where two binomials were now added and thus potentially bringing attention to the operation between brackets.

6.1 The Changes Made to the Examples in Plan for Lesson 2 and the Final Plan 3

The highlights in P2 and P3 in Fig. 4 identify the changes to the example sets in the lesson plans following reflection on Lessons 1 and 2, respectively. Changes from P1 to P2 can be seen in Activities 2 and 3. The operation of subtraction (inclusion of the " – " sign) and so contrasting + and – signs now appear earlier in the lesson, in Activity 2, and again potentially drawing attention to the operation between brackets. In Activity 3 in P2, there are now equations, with the expressions on the left retaining the same numbers and variables as in P1 but without brackets. Brackets now need to be inserted to produce an identity. The task for learners thus changed. They have to decide where to place brackets on one side of the equation so that the application of the distributive law produces the expression on the other. In Mason's terms, doing and undoing was now incorporated and important for attaching meaning to algebraic syntax.

Further changes appear from P2 to P3. There are additional examples of numerical expressions in the introductory activity and of algebraic expressions in Activities 1 and 2. The " − " sign now appears in a numerical expression, as does the task of inserting brackets. It appears that the change in Activity 3 in P2 led to a change in the introductory examples in P3.

Across the three plans, we see the overall example sets expanded. The + and − signs were both included in the introduction. This increases dimensions of variation at the outset and could perhaps detract from the focus on the position of the brackets. This was accompanied by a movement between the numerical and algebraic representations of expressions and changes in task demand. Even at face value, the changes in the example sets across the three plans reflects attention to variation and the interconnectedness of the selection and sequencing of examples, the tasks in which they were embedded and their representational forms. How then, did these changes in exemplification come about? What were the reflective processes in the LS group that provoked and produced these?

7 How Did the Change in Plans Evolve?

As discussed above, we identified key "example change moments" in the reflective discussion where there was specific attention to exemplification, *and* the ensuing discussion led to a change in the lesson plan. Changes were mainly focused on additional expressions, both numeric and algebraic, that brought more attention to the position of the bracket and application of the distributive law, through a change to a negative sign (operation) or through the change in the task and so insertion of brackets (Activity 3). The two vignettes below, from the reflection after Lesson 1 and Lesson 2, respectively, take us into the discussion, allowing us to see when the selected change moment occurred, who initiated it and how it was taken up by others and then evolved into a change in the example set. Following the vignettes we discuss the changes, how these relate to the theoretical framing of the LS by the MDI/MTF and to the wider literature on LS.

7.1 Vignette 1: When the Exemplification Is "Not Enough"

Thembi taught Lesson 1 and closely followed the joint plan (P1). The hour long reflective discussion began, as was the practice in the WMCS LS, with the teacher herself reflecting on the lesson. In the first 20 minutes, the group discussed the persistence of conjoining errors made by some of the learners. In agreement with Thembi, the LS group agreed the lesson goal was achieved, particularly when they looked at the pre- and post-tests where there was evident improvement.[6] At this point Frank (WMCS teacher educator/researcher) asked the group to focus on the examples across the lesson. Lerato commented that “the examples were enough”, and Jehad probed further suggesting that following the enactment there should be further reflection. It was in response to this that Thembi voiced a key concern with her learners’ responses to Activity 3.

Frank: Can we . . . (look) at the examples that were used? Do you think that it helped . . . to achieve the object of learning? Did it bring it into focus?

Lerato: I think . . . because we had the same numbers and the brackets were just shifting . . . I think any learner must realise the effect of the bracket faster . . . I think the examples were enough . . . . Unlike previous [teaching] we’ve changed the numbers . . . So I think the examples here helped in terms of them seeing that the brackets are having an effect.

Jehad: But do we need to change . . . now we saw them in action?

Attention shifted back to the pre- and post-tests for a few minutes and their role – and whether and how they informed the next lesson. Thembi asserted that this was not simple as the next lesson was with a different class and turned to respond to Frank and Jehad.

Thembi: Remember . . . activity three . . . they all answered “yes it will make a difference” but then I didn’t know how . . . that’s why I had to ask them for solutions in the activity. . . . I know when we planned we said that the main aim is to investigate the bracket. But already they picked up from activity one and two that . . . if the bracket is put in a different place it changes the solution. . . . [after unclear interactional exchange]. . . They were able to answer that as a group . . . they said “no the answers will not be the same because the brackets are in different positions”. But for me . . . that was not enough . . .

It was not immediately apparent to others what Thembi meant by “not enough”. Some thought she felt that Activity 3 was “too easy”. Following further discussion on how to “change” Activity 3, Linda (WMCS) suggested changing the form of the expression to $(3x^2)$ or $(3+x)^2$. The teachers rejected this suggestion with Thabi pointing out that “this changes too much”. He was referring to the goal of the lesson saying it will be changed, as “we will have to deal with exponents”. Discussion continued until Linda made another suggestion that caught Thembi’s attention (“that’s brilliant”) and resulted in a change to the example set:

Linda: I had another suggestion for Activity 4. Instead of just cancelling it, what if you, if you almost like repeated Activity 3 but you gave them the answer and then said: where must I put the brackets to get this answer?

Thembi: (I think that could) be the Activity 3, where you give them § minus three § plus five but then give them the solution and ask them where should we put the brackets? That will be, yes I think that’s brilliant ja. . . . .

And a little later:

[6]The pre- and post-test data are not sufficiently rigorous to make claims about learners’ learning. The pre- and post-tests were used for teachers to see whether they had enabled learners to attend to the object of learning. In this particular case, it was evident across the scripts that most learners were able to correctly expand more expressions than in the pretest.

Thembi: But this one we can also discuss after they've put the bracket then we can now discuss what is the bracket now with respect to the solution? Brilliant!

Attention moved to collectively generating the example set that appears in Activity 3, P2 above. Staying with the expressions in b, c and d in P1, adjustments were made to the sequence, the expressions were expanded, and the brackets from the original expression removed. The task now was to "insert brackets on the left expression "so that the two sides are equal". Thabi expressed concern that this would be too much of a "jump" for learners, but the group, spurred by Thembi's enthusiasm, agreed that while this might be a jump, Thabi (who was to teach the next lesson) could give more time to the activity and then perhaps not complete all of the other activities in the lesson.

This vignette evidences that the moment of explicit attention to examples was initiated by Frank and Jehad (WMCS) and thus by the project interest supporting teachers to use the MTF to frame and steer reflection. Significantly, it was Thembi (teacher) who responded to this initiative with a teaching/learning concern in her enactment that unsettled her. In this way, Activity 3 came into focus. It took some time for the group to understand Thembi's concern and to offer productive suggestions. Finally, the suggestion from Linda (WMCS) satisfied Thembi, and despite Thabi's concerns, it was accepted and the example set changed.

7.2 Vignette 2: When Representational Form Matters

Thabi taught Lesson 2 and followed P2. Here too the hour long reflective discussion began with Thabi's reflection on the lesson. There was much enthusiasm for the lesson and how Thabi encouraged and responded to learner thinking. Again a scan through the pre- and post-tests suggested improvements. It was obvious in the lesson that the learners had difficulty with Activity 3. After 24 minutes of general discussion Jill (WMCS) brought explicit attention to the example set and particularly Activity 3.

Jill: … in retrospect then Thabi, this (Activity 3) was an unfamiliar example, and it's interesting that they felt like they didn't know how to start … you needed to show them what to do. If you were doing the lesson again would you keep them in? Or would you change them?

Thabi: hmm… I will still keep them in but I will have to have a, a kind of a clue on how to approach it. Because definitely I could see they were stuck and some told me they can't start, so I have to think on how can I help them to start. So it's a good exercise because definitely if you, if you can put those expressions, the brackets … you will need to simplify before … in your head you're thinking how can I have this number, you're simplifying already. … so you just need to give them a clue to approach the problem.

Discussion[7] followed on what could be the "clue" or "approach". Jill suggested more direct attention to variation and so "what changed and stayed the same" in the

[7]Elsewhere (Alshwaikh and Adler 2017b) we have discussed Thabi's spontaneous example offered in the lesson. This incident introduced an error and much reflection by the LS group on what and how to deal with a teacher's error during a lesson and our joint responsibility in this.

previous examples, and so attention to how the position of the brackets changed the final expression. This could then be a "clue" or scaffold to look at the final expression that now asked where should the brackets be placed. This shifted discussion onto a difference with Lesson 1 where Thembi continually drew learners' attention to variance amidst invariance and so a general focus on the position of the brackets. In Lesson 2, Thabi's focus was on applying the distributive law correctly. Jill noted that these different emphases in the lessons pointed to different objects of learning in focus for the teacher. Soon after, Frank asked Thabi a question that led to example set change:

Frank: Do you . . . think focusing on the arithmetic . . . was helpful in transitioning to the algebra? Because in this case the arithmetic was very smooth even for us to realise in Activity 2 like where they're trying to generalise . . .

Thabi: Oh, you know, I . . ., when I plan my lesson I try to find a way where I can get arithmetical problems that are related to this thing that I'm going to teach . . . that the arithmetic is more easier for everyone. So if I find a way of bringing arithmetic problems, then I believe learners just working with numbers if they can solve that quickly . . . (and so) grasp whatever concept I'm bringing in now.

There is a break in the recording at this point, with the tape picking up discussion on additional numerical examples in the Introduction as the bridge for Activity 3.

Linda: So you could have it the other way round, but then you say four, maybe four subtract three plus five equals . . . [unclear . . . negative four?]

Thembi: Yeah it's fine, it's fine.

Linda: So where must I put the brackets to get that. So you said four. . . Around four plus three . . .

Thembi: So there you have the brackets and then you can still go back to your arithmetic and introduce activity three.

Finalising the example set took a further 30 minutes at which point there was a return to the importance of including negative signs and to the numerical activities in the introduction.

Thembi: I think activity one I will also add two, I think it's um because in activity one we don't have a multiplier that's negative. So we introduce a multiplier that is negative.

Jehad: For activity one?

Thembi: Yes in activity one for future use.

Frank: Everything's positive, everything's plus, plus, plus, plus.

Lerato: Yes, everything

Thabi: Oh yes, activity one use negative. And also activity two?

In Fig. 5, we present the collective record of the reflective deliberations. What becomes visible is that consideration of Thabi's comment resulted in numerical examples being added to the introductory section – including an example that brought in a negative sign/operation, as well as examples requiring the insertion of brackets to make the equation true.

This vignette evidences that the moment of explicit attention to examples was again initiated by the project team (Jill) and here too in response to learner actions in

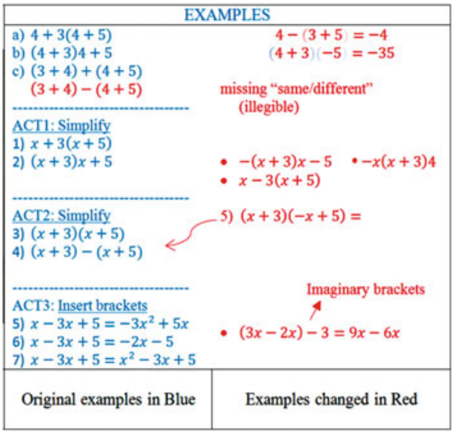

Fig. 5 WMCS LS's reflective deliberations after Lesson 2

the lesson enactment. While the question was directed at Thabi, it was his response and his teaching/learning approach that directed attention to addressing the bridge needed to Activity 3 through numerical examples, as well as attention to keeping the numbers in use in the introduction invariant (3, 4 and 5) and so drawing attention to the position of brackets in the expressions. At the same time, the addition of examples of expressions with negative terms brought in contrasting examples that could bring attention to the operation between brackets.

8 Discussion

In the data presented, we have shown that the example set changed over the lesson cycle. Through the two vignettes, we have illustrated how changes to the example set were initiated through "example change moments". It is interesting that in both the vignettes, the initiation was by a project member/researcher and reflective of the structuring of the reflection by the elements of the MTF framework, in particular by attention to exemplification. However, it was the teachers' engagement with these initiations that directed the discussion in relation to their learners and their teaching/learning concerns. In each case, it was possible to trace the mathematical substance of these discussions, as we elaborate below. Together this leads us to posit three themes that emerge from our case study and illuminate our case for exemplification as an explicit focus in our (and others') LS and as powerful for teachers.

8.1 Theme 1: Changes Are a Collective Accomplishment of Teachers' Attention to Their Learners' Activity in the Lesson Enactment and the Resources Researchers Bring

Through the two vignettes, we have illustrated how the initiation and evolution of change to the example set across the lesson, and particularly in the Introduction and Activity 3, were a *collective accomplishment* of both the teachers and the WMCS teacher educators/researchers. While a researcher initiated an explicit focus on the example set, in each case the change that materialised was evidently a function of the driving concern of the teacher and the suggestions and additional instructional resources offered by teachers and particularly researchers together. Our evidence here, and our experience in other LS cycles, reinforces the emphasis on the role of "knowledgeable others" in other models of LS (cf. Takahashi 2014; Lewis 2016). We suggest that this role is crucial and has significant implications for any attempt to "cascade" the implementation of LS in contexts where experience with professional collaboration on the one hand, and focused lesson engagement on the other, is new. Such was the case for most of the teachers in our project. The reports on LS implementation elsewhere in Africa are also indicative here, as these have tended to rely on cascading down, and inevitably, the dilution of such roles.

The important yet troubling implication for the practice of LS in "less resourced" contexts is that the human resources of knowledgeable others should not be side-stepped nor undervalued. This is a learned role and perhaps too a function of teacher education and research knowledge and practice. It is not a role that officials in a system can necessarily carry out without themselves being inducted into LS activity over time (cf. Jita et al. 2008).

8.2 Theme 2: Changes Are a Function of the MTF Framework and So Theoretically Informed

The vignettes also show how discussion following a change moment is shaped by attention being drawn to the MTF framework and particularly the example set. The example set, being informed and shaped in the initial planning by principles of variation, was subsequently changed, again with similar principles in mind. Here we see also the deep interrelation between theory and practice constituted in LS reflection. In both vignettes it was the teachers' practice-based concerns that influenced the ultimate joint decisions. In Thembi's case, it was her concern for what one could say was evidence that her learners were moving beyond the surface features of the brackets in the expressions, their syntax. This pushed the group to producing a task that could probe the semantics of the expressions and so in her terms her learners' understanding of the underlying distributive law. It was also in this vignette that we see the teachers' appreciation of coherence with the lesson goal or object of learning. Some of the examples suggested would veer the lesson off its focus and, as Thabi clearly understood, have too many things to concentrate on and so too much variance. As in Kullberg's et al. (2017) learning study, the role of variation as a theoretical resource in the LS work was also in evidence in Vignette 2. The additional examples in the example set kept the numbers and/or terms in all the examples invariant while the signs were changed or the task was shifted from expansion to insertion of brackets.

8.3 Theme 3: Each Change Is an Opportunity for Learning Mathematics Teaching

As indicated when the cycle began, while the focus of attention with learners was on the position of brackets in algebraic expressions, the underlying mathematics was the application of the distributive law. Vignette 1 illustrates how the group, and Thembi in particular, grappled with learners being able to operate on symbolic forms, particularly when there were patterns in past examples to follow, and how this related to their understanding of these forms. The tension between syntax and semantics, or operational and structural thinking in algebra as discussed earlier, was a substantive part of the work of the LS, as was how fluency with the application of the distributive law could be supported by tasks that required "undoing" and so the insertion rather than "removal" of brackets to produce true equations.

While we have not given sufficient attention to all the issues that arose with the numerical examples in the introduction, it is apparent in Vignette 2 that the LS provided the group opportunity to reflect on the relationship between arithmetic and algebra and whether and how number is always a suitable bridge into algebra. We intimated earlier, but did not elaborate, that the numerical examples did not necessarily require the distributive law in the first instance. This can exacerbate the tendency for learners to conjoin and so move from, for example, $3+4(5+3)$ as $3+32$ to $x+4(5+x)$ as $x+20x$, particularly if there is a strong discourse related to "doing what is in the brackets first" as following BODMAS.

9 Concluding Remarks

In this chapter, we reported a LS cycle on "simplifying algebraic expressions with brackets in different positions" with Grade 10 learners in the context of a professional development project in South Africa. Structured by a framework developed in the project to support lesson planning and reflection, we described a key change moment in the reflection following the first enactment of the plan where Thembi expressed her concern with the example set in Activity 3 and how this shaped the subsequent lesson plan, its enactment and the reflection following. Through this, and a second change moment in the reflection on Lesson 2, we have given empirical substance to the claims of the teachers in their seminar presentation. We have shown opportunities for learning from and about choosing and using examples and how these support the overall goals of the project for providing opportunities for teachers to learn mathematics teaching. In this particular cycle, opportunities arose for discussion and reflection on the relationship between arithmetic and algebra, and possibilities for building algebraic symbol sense and the meaning of the distributive law through numerical examples; and at the same time how to change the task in the particular example set and its attention to variation amidst invariance so as to bring generality, structure and strategy into focus with learners.

We have made a case for explicit focus on exemplification and example sets in LS. Examples are critical for mathematics teaching and learning, and so deliberate attention to how they are represented, selected and sequenced and embedded in appropriate tasks is important in professional activity like LS. We built our case by zooming in on one LS cycle with one LS cluster, indeed our most sustained and participative cluster. This was to illustrate the possible, which of course then could have broader purchase.

A related concern here could be that our focus on example sets and what they offer for teacher learning might be relevant to SA given prevalent teaching practices but not extend to lessons that are more problem based. We would suggest (following Wood 2018) that different solutions or strategies to a set of problems are themselves examples, and how and what is variant and invariant in these would shape what is possible to learn. And so attention in LS reflection to the different examples of possible solutions would hold similar possibilities and particularly if framed by principles of variation.

Giving such attention to these changes in example sets happened in the lesson plans and reflection on enactment. Within those discussions we highlighted teachers' concerns about their learners' learning as experienced in the teaching of the lesson. This gives meaning to teachers' voices about improvement in their teaching simply through discussing and reflecting on exemplification in Lesson Study.

Furthermore, changes to the example set, and so exemplification, the additions, deletions, changing representations and changing task demands, evolved as a function of the collective enterprise of the LS group – teachers and teachers educators or knowledgeable others. To repeat, the role of knowledgeable others is critical and presents a significant challenge in situations of constrained resources, where if in our case, for example, we were to expand our LS activity, the building up of that expertise would need to be a part of the LS programme overall. The implications for this in LS across contexts are serious, as there are significant cost implications if knowledgeable others are to be included in any expanded form. Without this, however, LS will not function optimally, whatever the context.

Finally, we have pointed to the impact and the value of structured and theoretically informed observation and reflection. Planning, teaching and reflection were structured and shaped by the WMCS theoretical approach, the MDI/MTF tool used with teachers in our LS. We thus close by suggesting that our illustrative case study of LS in South Africa has added impetus to those who argue for theoretically informed LS and hope through our work we stimulate further research along these lines.

Acknowledgements This work is based on the research supported by the South African Research Chairs Initiative of the Department of Science and Technology and National Research Foundation (Grant No. 71218). Any opinion, finding and conclusion or recommendation expressed in this material is that of the author(s), and the NRF does not accept any liability in this regard.

References

Adler, J. (2017). *Mathematics discourse in instruction (MDI): A discursive resource as boundary object across practices*. Paper presented at the 13th international congress on mathematical education, ICME-13, Hamburg.

Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education, 19*(3), 237–254. https://doi.org/10.1080/10288457.2015.1089677.

Adler, J., & Ronda, E. (2017a). Mathematical discourse in instruction matters. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 64–81). Abingdon: Routledge.

Adler, J., & Ronda, E. (2017b). A lesson to learn from: From research insights to teaching a lesson. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 133–143). Abingdon: Routledge.

Adler, J., & Ronda, E. (in preparation). *What changes in the quality of mathematics in instruction following teachers' participation in subject matter focused professional development*.

Adler, J., & Venkat, H. (2014). Teachers' mathematical discourse in instruction: Focus on examples and explanations. In H. Venkat, M. Rollnick, J. Loughran, & M. Askew (Eds.), *Exploring mathematics and science teachers' knowledge: Windows into teacher thinking* (pp. 132–146). Abingdon: Routledge.

Alshwaikh, J., & Adler, J. (2017a). Researchers and teachers as learners in lesson study. In M. K. Mhlolo, S. N. Matoti, & B. Fredericks (Eds.), *SAARMSTE book of long papers* (pp. 2–14). Free State: Central University of Technology.

Alshwaikh, J., & Adler, J. (2017b). Tensions and dilemmas as source of coherence. In A. Chronaki (Ed.), *Mathematics education and life at times of crisis: Proceedings of the ninth international mathematics education and society conference (V2)* (pp. 370–381). Volos: University of Thessaly.

Antonini, S., Presmeg, N., Mariotti, M. A., & Zaslavsky, O. (2011). On examples in mathematical thinking and learning. *ZDM Mathematics Education, 43*(2), 191–194. https://doi.org/10.1007/s11858-011-0334-5.

Baba, T., & Nakai, K. (2011). *Teachers' institution and participation in a lesson Study Project in Zambia: Implication and Possibilities. Paper presented at the Africa-Asia University Dialogue for Educational Development*. Report of the international experience sharing seminar (2). Actual Status and Issues of Teacher Professional Development. Retrieved from http://aadcice.hiroshima-u.ac.jp/e/publications/sosho4_2-06.pdf.

Benson, B., Mudenda, V., Tindi, E., & Nakai, K. (2014). Lesson study practice of science teachers in Zambia: Its effects, enhancing and hindering factors. In Pixel (Ed.), *Conference proceedings: New perspectives in science education* (pp. 473–477). Florence: libreriauniveritaria.it.

Bills, L., & Watson, A. (2008). Editorial introduction. *Educational Studies in Mathematics, 69*(2), 77–79. https://doi.org/10.1007/s10649-008-9147-z.

da Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6*(2), 169–181. https://doi.org/10.1108/IJLLS-08-2016-0021.

Dienes, Z. (1960). *Building up mathematics*. London: Hutchinson Educational Ltd.

Dudley, P. (2013). Teacher learning in lesson study: What interaction-level discourse analysis revealed about how teachers utilised imagination, tacit knowledge of teaching and fresh evidence of pupils learning, to develop practice knowledge and so enhance their pupils' learning. *Teaching and Teacher Education, 34*, 107–121. https://doi.org/10.1016/j.tate.2013.04.006.

Elliot, J. (2012). Developing a science of teaching through lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 108–125. https://doi.org/10.1108/20468251211224163.

Ellis, A. B., Ozgur, Z., Vinsonhaler, R., Dogan, M. F., Carolan, T., Lockwood, E., Lynch, A., Sabouri, P., Knuth, E., & Zaslavsky, O. (in press). Student thinking with examples: The criteria-affordances-purposes-strategies framework. *Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2017.06.003.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 53*(5), 393–405. https://doi.org/10.1177/002248702237394.

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 65–83 Retrieved from https://eric.ed.gov/?id=EJ1046666.

Fujii, T. (2016). Designing and adapting tasks in lesson planning: A critical process of lesson study. *ZDM Mathematics Education, 48*(4), 411–423. https://doi.org/10.1007/s11858-016-0770-3.

Goldenberg, P., & Mason, J. (2008). Spreading light on and with example spaces. *Educational Studies in Mathematics, 69*(2), 183–194.

Gu, F., & Gu, L. (2016). Characterizing mathematics teaching research specialists' mentoring in the context of Chinese lesson study. *ZDM Mathematics Education, 48*(4), 441–454. https://doi.org/10.1007/s11858-016-0756-1.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*(4), 393–409. https://doi.org/10.1007/s11858-016-0795-7.

Huang, R., Fang, Y., & Chen, X. (2017). Chinese lesson study: A deliberate practice, a research methodology, and an improvement science. *International Journal for Lesson and Learning Studies, 6*(4), 270–282. https://doi.org/10.1108/IJLLS-08-2017-0037.

Jita, L. C., Maree, J. G., & Ndlalane, T. C. (2008). Lesson study (Jyugyo Kenkyu) from Japan to South Africa: A science and mathematics intervention program for secondary school teachers. In B. Atweh et al. (Eds.), *Internationalisation and globalisation in mathematics and science education* (pp. 465–486). Dordrecht: Springer.

Kullberg, A., Runesson, K., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education, 49*, 559–569. https://doi.org/10.1007/s11858-017-0858-4.

Larssen, D. L. S., Cajkler, W., Mosvold, R., Bjuland, R., Helgevold, N., Fauskanger, J., & Norton, J. (2018). A literature review of lesson study in initial teacher education: Perspectives about learning and observation. *International Journal for Lesson and Learning Studies, 7*(1), 8–22. https://doi.org/10.1108/IJLLS-06-2017-0030.

Lewis, J. M. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM Mathematics Education, 48*(4), 527–540. https://doi.org/10.1007/s11858-015-0753-9.

Marton, F., & Tsui, A. B. M. (Eds.). (2004). *Classroom discourse and the space of learning*. London: Lawrence Erlbaum.
Mason, J. (2001). Tunja sequences as examples of employing students' powers to generalize. *The Mathematics Teacher, 94*(3), 164–168.
Murata, A., Bofferding, L., Pothen, B. E., Taylor, M. W., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics lesson study. *Journal for Research in Mathematics Education, 43*(5), 616–650. https://doi.org/10.5951/jresematheduc.43.5.0616.
Ntow, F., & Adler, J. (2017). An exploration into teachers' take up of professional development teaching resources. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 313–320). Singapore: PME.
Olteanu, L. (2017). Distributive law as object of learning through direct and inverse tasks. *International Journal for Lesson and Learning Studies, 6*(1), 56–65. https://doi.org/10.1108/IJLLS-05-2016-0014.
Ono, Y., & Ferreira, J. (2010). A case study of continuing teacher professional development through lesson study in South Africa. *South African Journal of Education, 30*(1), 59–74.
Pang, J. (2016). Improving mathematics instruction and supporting teacher learning in Korea through lesson study using five practices. *ZDM Mathematics Education, 48*(4), 471–483. https://doi.org/10.1007/s11858-016-0768-x.
Pournara, C., Hodgen, J., Adler, J., & Pillay, V. (2015). Can improving teachers' knowledge of mathematics lead to gains in learners' attainment in mathematics? *South African Journal of Education, 35*(3), 10. https://doi.org/10.15700/saje.v35n3a1083.
Quaresma, M., Winsløw, C., Clivaz, S., da Ponte, J. P., Ní Shúilleabháin, A., & Takahashi, A. (Eds.). (2018). *Mathematics lesson study around the world: Theoretical and methodological issues* (ICMI-13 monographs). Dordrecht: Springer.
Runesson, U. (2006). What is it possible to learn? On variation as a necessary condition for learning. *Scandinavian Journal of Educational Research, 50*(4), 397–410. https://doi.org/10.1080/00313830600823753.
Shuilleabhain, A. N. (2016). Developing mathematics teachers' pedagogical content knowledge in lesson study: Case study findings. *International Journal for Lesson and Learning Studies, 5*(3), 212–226. https://doi.org/10.1108/IJLLS-11-2015-0036.
Sinclair, N., Watson, A., Zazkis, R., & Mason, J. (2011). The structuring of personal example spaces. *The Journal of Mathematical Behavior, 30*(4), 291–303. https://doi.org/10.1016/j.jmathb.2011.04.001.
Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 4–12.
Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*(4), 513–526. https://doi.org/10.1007/s11858-015-0752-x.
Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction. *Pythagoras, 33*(3), 25–32. https://doi.org/10.4102/pythagoras.v33i3.188.
Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91–111. https://doi.org/10.1207/s15327833mtl0802_1.
Wood, K. (2017). Is there really any difference between lesson and learning study? Both focus on neriage. *International Journal for Lesson and Learning Studies, 6*(2), 118–123. https://doi.org/10.1108/IJLLS-02-2017-0008.
Wood, K. (2018). The many faces of lesson study and learning study. *International Journal for Lesson and Learning Studies, 7*(1), 2–7. https://doi.org/10.1108/IJLLS-10-2017-0047.

Yoshida, M. (2012). Mathematics lesson study in the United States: Current status and ideas for conducting high quality and effective lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 140–152. https://doi.org/10.1108/20468251211224181.

Zaslavsky, O. (in press). There is more to examples than meets the eye: Thinking with and through mathematical examples in different settings. *Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2017.10.001.

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*(2), 165–182. https://doi.org/10.1007/s10649-008-9140-6.

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How Variance and Invariance Can Inform Teachers' Enactment of Mathematics Lessons

Armando Paulino Preciado Babb, Martina Metz, and Brent Davis

Contents

Abstract The use of systematic variance and invariance has been identified as a critical aspect for the design of mathematics lessons in many countries where different forms of lesson study and learning study are common. However, a focus on specific teaching strategies is less frequent in the literature. In particular, the use of systematic variation to inform teachers' continuous decision-making during class is uncommon. In this chapter, we report on the use of variation theory in the Math Minds Initiative, a project focused on improving mathematics learning at the elementary level. We describe how variation theory is embedded in a teaching approach consisting of four components developed empirically through the longitudinal analysis of more than 5 years of observations of mathematics lessons and students' performance in mathematics. We also discuss the pivotal role of the particular teaching resource used in the initiative. To illustrate, we offer an analysis of our work with a Grade 1 lesson on understanding tens and ones and a Grade 5 lesson on distinguishing partitive and quotitive division.

Keywords Variation pedagogy · Math Minds · Elementary · Formative assessment · Division · Place value

The original version of the chapter has been revised. A correction to this chapter can be found at https://doi.org/10.1007/978-3-030-04031-4_39

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_17

1 Introduction

Systematic variation has been identified as an important element of lesson study and learning study (e.g., Huang and Li 2017), which have become popular models for teacher professional development around the world (Hart et al. 2011; Huang and Shimizu 2016). Generally, lesson study involves the design of a lesson by a team of teachers and teacher educators; the implementation of the lesson in the classroom, observed by the team; the refinement of the lesson; a second implementation; and a report of the results (sometimes with further iterations). Learning study is similar to lesson study but includes testing students before and after the lesson as well as the use of variation theory as a theoretical framework (Huang and Li 2017; Marton et al. 2004). Pang et al. (2017) explained that two complementary perspectives on "variation pedagogy" (Watson 2017) have developed, first in parallel and later through extensive interaction between researchers from each tradition: "Bianshi Jiaoxue" was developed in China, and pedagogical approaches based on the variation theory of learning proposed by Marton (2015) and colleagues were first developed in Sweden. Both perspectives are prominent in the literature regarding learning study (Huang and Li 2017; Marton and Häggström 2017; Marton et al. 2004) and have been implemented in lesson study (cf. Bruce et al. 2016; Han et al. 2017). This chapter addresses the model of teacher professional development developed by the Math Minds Initiative, which includes elements of lesson study enacted in a particular way and builds on both Marton's approach to variation and Chinese perspectives on variation.

While different forms of lesson study have been implemented in Canada (Bruce and Ladky 2011; Bruce et al. 2016; Preciado-Babb and Liljedahl 2012; Tepylo and Moss 2011), there have been other forms of collaboration in the design of teaching artifacts as a means of professional development. For instance, Preciado and Liljedahl elaborated on different modes of "teachers' collaborative design" (p. 23), which involve elements of lesson study such as codesign, enactment of a lesson, debriefing of results, and refinement of the lesson. Local constraints such as timetables and lack of supports for substitute teachers have made it difficult to implement lesson study in Canada. In our work with the Math Minds Initiative, we also found that having few or no participating teachers teaching the same grade level made it very difficult for teachers to teach, redesign, and reteach a lesson during the same year. Nonetheless, we have included key elements of lesson study in our work, as we describe in this chapter.

A significant feature of our approach to lesson study has been the use of systematic variation for both lesson adaptation and for directly informing in-the-moment teaching decisions. In this chapter, we describe a lesson observation protocol that we use as a tool for analyzing and refining lessons with teachers. It was developed through the analysis of classroom observations and longitudinal data on students' performance in mathematics. Systematic variation is key to each aspect of the protocol.

This protocol represents a theoretical contribution that has the potential to inform different modes of teachers' collaborative design, including lesson and learning study. While a focus on systematic variation has been widely reported in the literature on learning study as critical for task design and research analysis of mathematics lessons, we assert that such focus can also have a significant impact on how the tasks are enacted in the classroom. Like Kullberg et al. (2014), who reported how the same task can be enacted in different ways and offer different possibilities to learn, we have observed (Preciado-Babb et al. 2016a) how the use of the same lesson plans by different teachers yielded contrasting results in terms of students' learning (as measured by standardized tests and as observed in terms of student engagement in lessons). In this chapter, we stress the importance of both continuous assessment during the enactment of a lesson and teachers' appropriate responses to student feedback. We posit that systematic variation can inform both. This approach emphasizes not only lesson design (which can be anticipated) but also emergent situations that require teachers to improvise during class. The chapter elaborates on the methodology and key findings of the Math Minds Initiative, describes how variation has informed the development of the observation protocol, and presents two examples of how the protocol has been used in the project.

2 The Math Minds Initiative

The Math Minds Initiative started in 2012 with the intent of improving mathematics instruction at the elementary level through design-based research (Cobb et al. 2003) that involved the collaboration of several organizations, including the University of Calgary, Calgary Catholic School District, Golden Hills School Division, JUMP Math (2018), and Suncor Energy Foundation (the latter as sponsor). In Phase 1 (2012–2017), the initiative focused on an elementary school with a highly diverse demography and a long history of low performance in mathematics. Although the initiative started in 2012, the project was not fully implemented until September 2013. A second elementary school was added in 2014 and a third school from another school district in 2016. In total, the initiative has included professional development and weekly mathematics lesson observations for 31 participant teachers and the video recording of about 300 lessons. The study also included a longitudinal analysis of student performance in mathematics, as measured by the Canadian Test for Basic Skills (CTBS; Nelson 2018), and 44 teacher interviews and 228 student interviews. We are currently involved in a second phase of the project that aims to design a model for teacher professional development that is based on the results from the first phase.

In Phase 1 of the initiative, we observed consistent improvement in student performance in mathematics. Scores from the mathematics component of the CTBS (Grades 2 to 6) were collected each year. We used a linear mixed model (LMM) to accommodate this *unbalanced* study design. In this way, "not all individuals need to have the same number of observations and not all individuals need to

Table 1 LMM estimates of average t-scores for CTBS

	n	2013	2014	2015	2016	2017
School 1	147	43.582	45.691	47.511	49.654	–
School 2	131	–	47.113	49.373	53.202	52.09
School 3	85	–	–	–	46.881	49.398
All schools	363	44.054	45.943	47.725	50.052	50.93

All estimates are significant at $p < 0.001$

be measured at the exact same time points" (West 2009, p. 212). CTBS scores were converted to t-scores and normalized before conducting the analysis. Table 1 summarizes the results of the longitudinal analysis per school and for the whole intervention. The analysis showed a significant improvement in student performance in mathematics from Year 2 to Year 5 of the project (the study was fully implemented in its second year), with national percentile rankings rising from 27 to 55, which is equivalent to a rise in t-scores from 44.054 to 50.93 (see Table 1).

The results were not the same for all groups (Preciado-Babb et al. 2016a). By contrasting results and observations from different classrooms, we were able to identify teaching approaches associated with higher rates of improvement in CTBS scores, particularly conceptual understanding, and higher levels of student engagement in mathematical activities, as observed during classroom visits. These teaching approaches informed the observation protocol described in this chapter.

Lesson analysis has been an essential component of our work with teachers in the Math Minds Initiative. Because we worked with small schools, there were limited opportunities for regular cycles of lesson study in the same year. Often, there was only one participant teacher teaching a particular grade level. The initiative's approach to lesson analysis, debriefing, redesign, and re-implementation was therefore based mostly on modification from 1 year to the next and between closely related lessons within and across grade levels. We also considered longitudinal results of students' performance to inform lesson redesign.

The interaction of teachers and researchers and the use of a shared resource provided by JUMP Math (2018) have been pivotal to efforts to revise lessons and compare results of different implementations. JUMP Math resources include students' *assessment-and-practice* books, teacher guides, and predesigned SMART® Notebook slides for use with interactive whiteboards. In turn, this work informed professional development sessions for teachers. Thus, we have been able to test not only the improvement of lesson plans but also the teaching approaches used during the lessons. Thus, the Math Minds approach emerged iteratively with a close focus on individual lessons and themes that became sites of attention that encompassed teachers from different grades, over multiple years, and over multiple research sites. Over the course of the project, our work has grown in ways that distinguish it from common approaches often situated as opposite in North America; it does not neatly align with either traditional (commonly associated with direct teaching and rote memorization) or progressive (often associated with inquiry-based learning) teaching approaches (cf. Metz et al. 2016).

It is important to stress that many classrooms in Canada have considerable diversity regarding achievement levels in mathematics. The classrooms we observed all had such gaps, spanning up to three grade levels. The strategies we describe were developed to support all students in such diverse classrooms.

3 Approach to Variation

Marton's (2015) theory is based on the conjecture that "novel meanings are acquired through contrast and not through induction" (Marton and Häggström 2017, p. 391). Close attention to what varies and what remains the same in the examples and tasks that students are exposed to is essential. Contrast supports learners in discerning critical features of the intended object of learning. The term *critical discernment* is used here to indicate a critical feature that students need to discern in order to learn something.

We emphasize that a mathematical concept involves multiple discernments *woven* together into a coherent, powerful idea. There is a cumulative aspect to these discernments: As concepts are developed, they become elements of contrast needed for discerning new concepts. In other words, learning a concept involves (a) becoming familiar with the discernments that comprise it and (b) appropriately integrating those discernments with one another. For instance, when learning to represent numbers from 9 to 20 in the decimal system, it is important for students to integrate two critical discernments: (a) grouping by tens and (b) assigning value with respect to place (e.g., there are one ten and three units in 13).

Teaching, thus, involves formatting a concept in a manner that supports learning, starting with the identification of necessary discernments. Taken together, these discernments and the relationships between them comprise the *intended object of learning* (Marton 2015). We note that the manner in which discernments are woven together must take into account the limits of learners' abilities to hold multiple discernments in mind. The variation theory of learning can inform a systematic sequencing of activities and examples offered to students; such sequencing becomes the *enacted object of learning*. It is important that the teacher assesses all students' understanding every time a new discernment is presented and as students respond to prompts intended to support their integration into broader mathematical structures; when done well, doing so can offer insight into what Marton described as the *lived object of learning*. Further, this assessment can inform how the teacher decides to proceed with the lesson.

3.1 The Teaching Approach

The teaching approach developed in Math Minds consists of four elements enacted recursively during class. These now form the basis of our framework for lesson design and classroom observation. The elements are (a) raveling, (b) prompting, (c) interpreting, and (d) deciding, which we now elaborate.

Raveling refers to the identification of a critical discernment within a coherent sequence of discernments. One part of *raveling* happens prior to teaching and involves long-range planning on the part of teacher and resource. The teacher and/or the resource thoroughly decompose a concept (to identify fine-grained discernments that are embedded within) and recompose that concept, drawing attention to how discernments may be knitted back together and to how the concept fits within a broader network of mathematical understanding; we refer to this as *macro*-raveling. *Micro*-raveling occurs at the level of an individual lesson. Ideally, students are continuously prompted to critical discernments woven together into a meaningfully connected big idea that stays focused on the heart of lesson. Each new idea is anchored to previously developed ideas that have been summarized to carry forward (e.g., a quick summary, a key word, an image); there is attention to transforming the new into something familiar and to anticipating next steps. We see strong connections between micro-raveling and Gu, Huang, and Marton's (Gu et al. 2004) descriptions of conceptual and procedural variation in the Bianshi tradition.

Prompting is about engaging students in ways that (a) channel their attention to each critical discernment, through the use effective patterns of variation to draw attention to each discernment, and (b) require students to *make those distinctions*. For instance, the teacher might offer relevant contrasts that are clearly juxtaposed, highlighted, and appropriately sequenced followed by a task in which students are asked to make the intended distinctions.

Interpreting involves getting a read from each learner on the sense being made of the critical discernment. Essential to effective interpreting are student responses that quickly and clearly inform the teacher as to whether all have made the intended discernment or noticed the intended connection.

Deciding is the way the teacher chooses between stepping back, lingering, or pressing on, based on learners' demonstrated understandings. The teacher's decisions might involve clarification of prompts, modification of tasks, and/or adjustment of the pace of the lesson in ways that benefit all learners. An important feature of deciding is the way students are offered opportunities to extend significant ideas during class.

In addition to the variation theory of learning, we have also attended to the extensive research showing the limitations of working memory (Engle et al. 1992): Humans have limited capacity to attend simultaneously to multiple pieces of information. Working memory has been widely studied in the context of mathematics learning (e.g., Berg 2008). A detailed review of the literature on this subject is beyond the scope of the chapter; rather, we stress the relevance of considering its limitations when designing for learning. For instance, visually juxtaposing two well-crafted examples can support students' discernment of key distinctions.

We note both tensions and complementarities imposed by the manner in which human perception is drawn to *difference* while simultaneously having strict limits on how much can be attended to simultaneously. Contrast requires at least two elements. We have seen many cases where attempts to reduce demands on working memory have eliminated the very contrast needed to draw attention to a particular discernment. We differentiate our approach to working memory from those that focus primarily on explicit guidance: Clark et al. (2012) claimed that "Teachers are more effective when they provide explicit guidance accompanied by practice and feedback, not when they require students to discover many aspects of what they must learn" (p. 6). While our approach may be seen as offering explicit guidance, such guidance involves creating conditions for students to make critical discernments of the targeted object of learning.

We also contrast our teaching practices with problem-based and project-based approaches to teaching mathematics that offer limited guidance. Lessons in these approaches often open with a *rich* question that offers investigative spaces where students collaborate to develop deep and connected understanding of big mathematical ideas. However, some students may lack both the skills to effectively structure their own inquiries and sufficient background knowledge to attend to multiple ideas at once, thereby overwhelming their working memories. We have found that the teaching strategies identified in Math Minds can support students in developing the conceptual understanding and the skills necessary to address more open explorations. The focus is on supporting students to engage in activities that help them notice and integrate critical features of the targeted learning outcomes.

Continuous assessment that informs teachers' immediate decisions during class is critical to the teaching approach and the observation protocol that we describe in this chapter. This is consistent with recent work in formative assessment (Chappuis 2015; Wiliam 2011; Wiliam and Leahy 2015). Wylie and Wiliam (2007) identified the need for *hinge* questions to support teachers who struggled with continuously assessing all students during class and with providing appropriate immediate responses. Hinge questions can be answered in less than a minute, allowing the teacher to assess students' understanding and make informed decisions about how the lesson might proceed. Although there has been important work in the last decade on the development of such questions, particularly in mathematics (Wiliam 2011; Wiliam and Leahy 2015; Wylie and Wiliam 2007), most of the advice to teachers has been limited to re-explaining or reteaching, with little emphasis on *how* teaching might be effectively modified.

Ongoing decisions about teaching that are based on continuous assessment support both learners who struggle and those who quickly meet targeted learning goals. Preciado-Babb et al. (2017) reported how a focus on variation has resulted in a better way to support students in making critical discernments during class based on "on-the-fly" teacher's decisions. Clear juxtaposition and systematic variation can support students in both making discernments and proceeding from a familiar question or task to further challenges as they meet the learning goals during class.

3.2 Teaching Resources

We have found it important to consider how responsibility for the four elements of the Math Minds model might be effectively shared between a teacher and a teaching resource. While a resource can identify and sequence discernments in a manner that supports long-term coherence, a teacher can be responsible for attending to the needs and development of particular learners as they engage with key ideas.

At Math Minds we have found that, in contrast to other commonly available classroom resources designed to support Alberta's Program of Studies for Mathematics, JUMP Math (2018) offers a carefully measured presentation of information and emphasizes assessment of each key idea. We hypothesize that this approach respects limitations on students' working memories (Engle et al. 1992) and makes key ideas explicit. However, we have found that modifications to the JUMP Math materials based on systematic variation have often resulted in more effective ways of *drawing students' attention to critical features* or *connecting* key mathematical ideas (Metz et al. 2017).

The JUMP Math (2018) resource is consistent with the idea of constantly assessing students and responding accordingly during class. The teacher guides suggest "stepping back" to a place where students can reengage in the lesson when they have trouble in class (Mighton et al. 2010a). The guides also recommend offering bonus questions that challenge students who move quickly through assigned tasks. We have observed that both suggestions are often difficult for teachers (Preciado-Babb et al. 2016b): Step back to where? How should they create effective bonus questions? We have found that systematic variation can support both "stepping back" and "bonusing." Effective adaptations, however, often have more to do with effective variation than with smaller steps or bigger numbers. While the JUMP Math resource has carefully identified critical discernments to be noticed by learners and has sequenced topics coherently, we have observed that adapting the resource using more clearly structured variation (Marton 2015; Pang et al. 2016; Watson 2017) and remaining mindful of broader learning targets can provide better opportunities for students' learning. Doing so has opened pathways that both supported the weakest students and challenged even the most capable students (Metz et al. 2017; Preciado-Babb et al. 2017).

4 Examples from the Classroom

We offer two examples to show how the four elements of teaching described here have been used for both lesson analysis and redesign. The first example also shows the elements of lesson study enacted in the initiative in which lessons have been analyzed, redesigned, and implemented in subsequent years; excerpts from one classroom are included to show instances of interpreting and deciding. The second example illustrates how a focus on raveling and prompting was used to analyze a lesson plan and support teacher professional learning; we also include suggestions for further adaptations to the lesson.

4.1 Example 1: Representing Numbers Up to 20

Early in the Math Minds Initiative, we observed a Grade 1 lesson in which the teacher made numerous attempts to support students in representing numbers to 20 using base ten blocks. In addition to providing several opportunities for each child to respond to questions, the teacher paid regular attention to one student, Heidi, who struggled far more than her classmates. The teacher's careful use of assessment, persistence in seeking an effective response, and effective adaptation of the given variation allowed everyone to reach understanding by the end of the lesson. The episode also included examples of extensions for students who met the expected outcomes during class. Insights gleaned during this lesson informed other lessons as well as new implementations in subsequent years of the project.

Representing numbers in the decimal system requires understanding and combining (raveling) the notions of (a) grouping by powers of ten and (b) identifying the value associated with the position of each digit in a numeral. Base ten blocks are commonly used for representing numbers and were used to help prompt students' attention to these critical discernments.

4.1.1 Analysis of the Lesson

The teacher started the lesson with a series of questions and took answers either in chorus or from individual students. Some students gave wrong answers. In particular, there seemed to be confusion about the number of ten-blocks and one- blocks used to represent a particular number. The instructional sequence involved using blocks to represent 11, 12, 13, 14, 15, and 18; students were asked to identify the number of ten-blocks and one-blocks and then write the number. Note that in this sequence, the number of units varied, while the number of tens remained constant (always 1 ten).

After offering the first item in the sequence, the teacher checked on each student individually. She then offered explanations for 10, 11, 12, and 13. At this point, she asked the class to represent 14 and checked each student's response. Heidi was struggling, so the teacher gave her 4 one-blocks and 1 ten-block. Then she requested the attention of all students and provided an explanation on the board. The teacher then asked students to show blocks for 18. She walked through the class checking on every student, offering assistance and posing further challenges as needed (19, 17). These actions illustrate the deciding step in the observation protocol. The actions gave her some time to work individually with some students. Only two or three students, including Heidi, still had trouble representing the numbers.

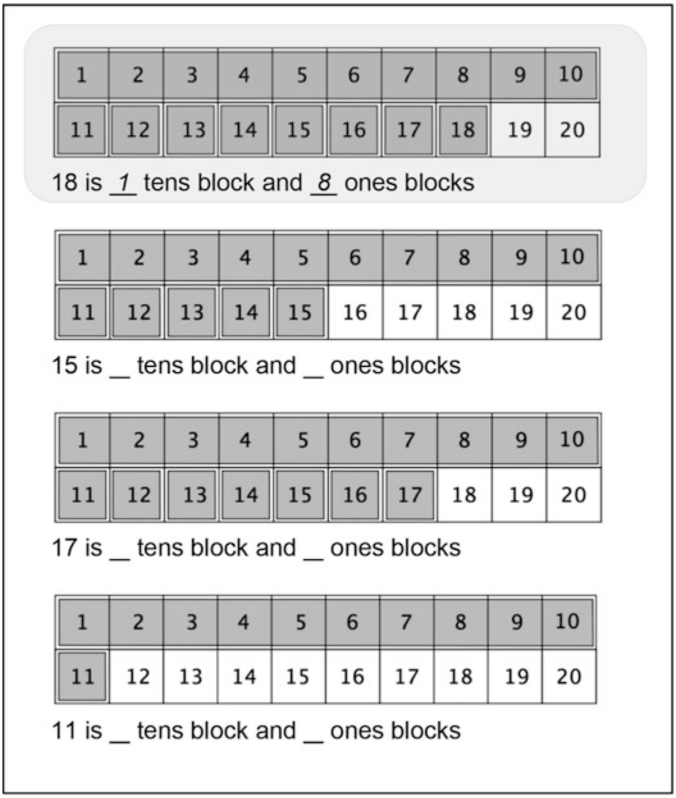


Fig. 1 Student's task from *Assessment and Practice* book. (Image based on Mighton et al. 2009a, p. 48)

The class moved to the next part of the lesson, in which the students worked in their *Assessment and Practice* books (see Fig. 1). The teacher introduced this work by using a document camera to show the example solved in the students' book for the case of 18. She then asked individual students to solve the rest of the exercises in front of the class.

When the teacher asked how many ten-blocks were in 15, some students answered in chorus: "10." The teacher then explained that there was only 1 ten-block. She gave every student 1 ten-block, asking "How many tens blocks did I give you?" until every student answered "one."

Subsequent questions required students to identify tens and ones without the support of colored 20 charts: They were now required to place blocks on a given 20 charts to represent the given number and then complete a sentence similar to those in Fig. 1. For instance, the first task was:

14 is ___ tens block and ___ ones blocks.

One student showed the answer on the board, and the teacher re-explained the use of one-blocks and ten-blocks. Subsequent tasks in the practice book asked students to consider 19, 11, 13, 12, and 20. Note that so far, all but one of the examples focused on a single ten paired with varying numbers of units. The exception comes at the very end, where it is unlikely to serve as a useful contrast.

As students worked individually, the teacher walked through the class checking on each of them. At least three students, including Heidi, still had trouble differentiating the ten-block from the one-block. At this point, the teacher started giving these students more ten-blocks. She started with Heidi, showing 3 ten-blocks and counting, "One, two three." For the first time, the pattern of variation now included a contrast in the number of ten-blocks.

The teacher went to another student, who had apparently written a 5 in the space for ten-blocks; she indicated that in the case of 15, there were not 5 ten- blocks, but only 1 ten-block. This student seemed to understand and corrected his work.

Another student had written 15 in the space for ten-blocks. The teacher showed her 15 ten-blocks while saying "Here are 15 ten-blocks" and asking "How many ten-blocks are there [in the given example]?" The teacher gave 1 ten-block to the student, asking: "How many blocks did I give you?" The following dialogue ensued:

Student: One.
[Teacher gave another]
Teacher: Now how many blocks?
Student: Two.
[Teacher took away one block]
Teacher: OK? How many ten-blocks did I give you right now?
Student: One
Teacher: Then write one.

The class finished, and the teacher asked Heidi to stay; she spent 4 min working with her. The teacher held up 10 ten-blocks.

Teacher: This is 10 ten-blocks.
[Teacher gave the blocks to Heidi to hold in her hand]
Teacher: See ten blocks?

Heidi started giving the blocks back, one by one to the teacher. The teacher counted, while the student gave the blocks back.

Teacher: One, two, three tens. Can you show me 5 ten-blocks?
Heidi: One, two, three, four, five [moving the tens-blocks one by one]
Teacher: So that's 5 tens blocks. Now show me 1 ten-block. Just one.
[Heidi moved one block.]
Teacher: Now, how many ten-blocks do you have here?
Heidi: One.
Teacher: So, write the number one. How many one-blocks? How many ones?

In this episode, we stress that the teacher led group discussions in which one student or students in chorus responded to her questions, instead of using an all-student response system. Nonetheless, it was clear to her that there was confusion about the distinction between 1 ten-block and 10 one-blocks. The teacher prompted attention to this difference several times during the lesson, which included giving 1 ten-block to each student and asking, “How many blocks did I give you?” Still, some students were confused. By the end of the lesson, the teacher started to vary the number of ten-blocks in the examples and questions posed to students. This was more effective for the students who were still confused.

4.1.2 Adaptations to Other Lessons

We discussed this issue with other teachers in the project, suggesting to vary the tens digit to draw attention to the meaning of the tens place in similar lessons: They did so in different ways. One teacher prompted attention to the meaning of the tens digit by including numbers beyond 19. She simply kept a handful of ten-blocks at her side as she worked with her students. When the predicted difficulty with 1 ten-block-vs. 10 one-blocks came up, she was prepared with several ten-blocks to show the difference.

Another adaptation based on this suggestion included an explicit prompt to numbers with and without ten-blocks. The teacher started the lesson by offering a series of slides in which students were supposed to identify the number of dots, preferably by counting on from a full ten frame (Fig. 2).

Many students struggled to identify the number of dots in the slides. After a few more tries, the teacher recognized that continuing with the planned lesson was unlikely to be productive. She decided to switch to a task in which students had to identify a number based on the number of dots she held up. She presented a yellow card with ten dots and a blue card with anywhere from zero to nine dots (see Fig. 3).

For each question (prompt), she held up a blue card *with or without the yellow ten-card*, and the students' job was to name the number. Note that although multiple tens were not involved, as in the previous case, the presence of the *zero-or-one-ten* option helped to draw students' attention to role of the ten's digit. The students quickly became adept at recognizing the numbers she held up.

In another case, a second-grade lesson that already included variation in the number of tens was adapted to more clearly highlight this variation. Initially, the four examples shown on the left in Fig. 4 were presented separately. By clearly juxtaposing them (as on the right), students found it easier to attend to key contrasts in 28 as (a) 28 ones, (b) 1 ten and 18 ones, and (c) 2 tens and 8 ones (something many struggled with during their first attempt with this lesson).

These adaptations resulted from the teacher's awareness that students were struggling. She requested a meeting with a lead teacher and a researcher after class

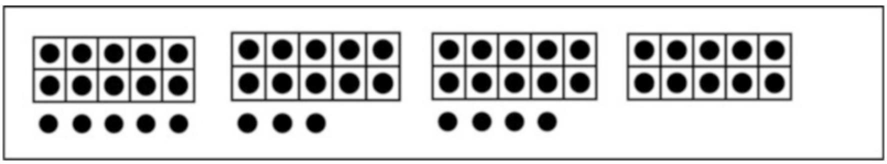


Fig. 2 Modification of the JUMP Math slides to juxtapose examples. (Mighton et al. 2009a)

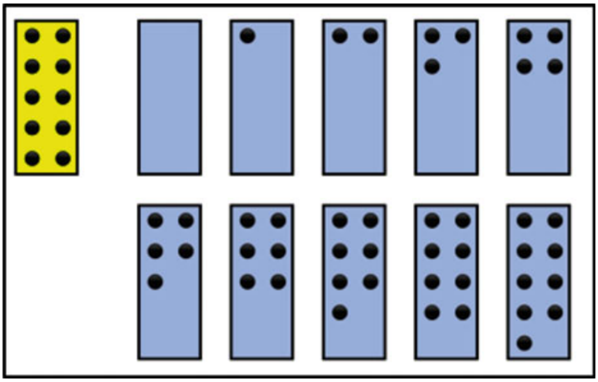


Fig. 3 Slide showing variations in the use of ten or no ten for representing numbers

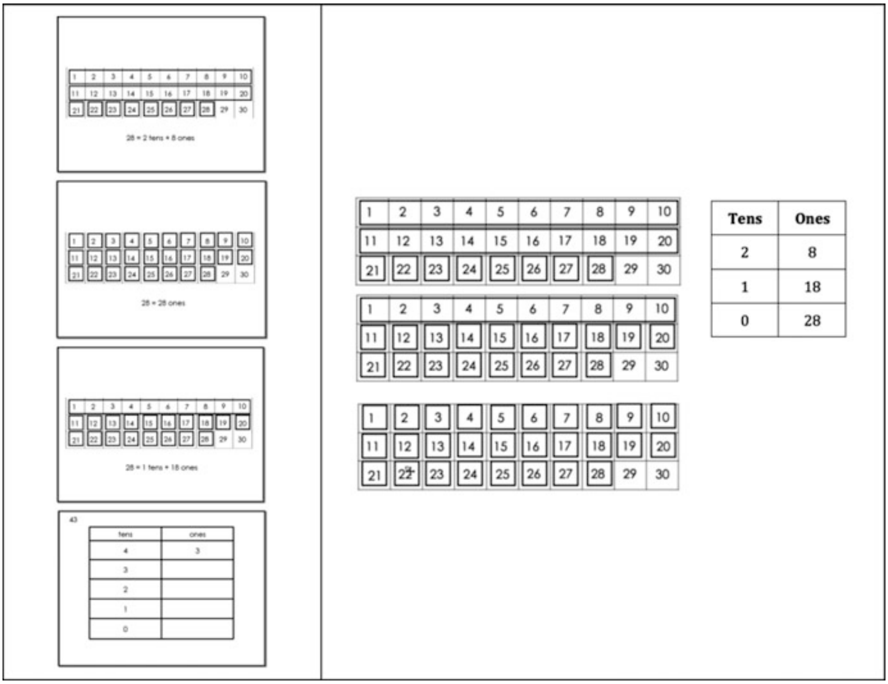


Fig. 4 On the left, original sequence of slides shown to students; on the right, modified version with a single slide. (Mighton et al. 2010a, p. B-58)

to adapt the lesson, with the intent to reteach it the following day. It was then that they decided to juxtapose the first three images and the T-Table on a single slide. Further, students were to identify what was the same and different in the three representations of 28. Following this, students were asked to identify possible numbers of tens and ones in other numbers, supported by an erasable hundred chart taped to a whiteboard (see Fig. 5); the use of the hundred chart was also an

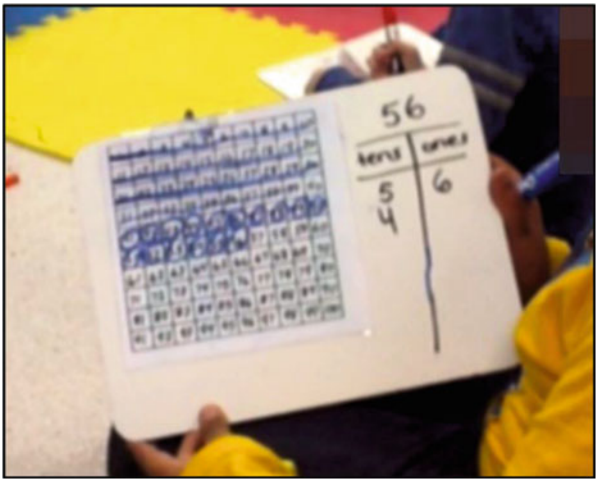


Fig. 5 Individual whiteboard and hundred chart used to represent numbers

adaptation from the original lesson, during which students used base ten blocks that they found hard to keep track of. Here, the teacher indicated that a ten-block must be shown by a continuous line through an entire row, while a one-block must be shown with a slash mark through a single number, thus further highlighting the contrast between a ten-block and a one-block. The total numbers of each were recorded in a T-Table positioned directly beside the hundred chart on the whiteboard. Students experienced greater success with this approach.

The decision to use the whiteboards and the hundred charts also facilitated the process of interpreting students' understanding: It allowed the teacher to quickly check individual understanding of all students during this part class.

Note that in contrast to previous lessons where the number being represented varied, here, the number remained the same but was represented by varying numbers of ten and one-blocks. Including the T-Table on the same slide allowed further contrast in terms of representation before moving to other numbers (as in the fourth slide on the left in Fig. 4).

4.2 *Example 2: Partitive vs. Quotitive Division*

The following example focuses on the distinction between partitive and quotitive division. Learners' and teachers' difficulties distinguishing partitive and quotitive division have been widely reported in the literature. These include limitations in correctly solving word problems related to division (e.g., Fischbein et al. 1985; Graeber et al. 1989; Tirosh 2000). The JUMP Math resource (Mighton et al. 2010b) includes a Grade 5 lesson, "Two Ways of Sharing" (p. 81) intended to draw attention to manners of grouping that are consistent with partitive and quotitive division. This lesson precedes other lessons on knowing whether to multiply or divide in particular contexts and understanding the long division algorithm. In Math Minds, we have

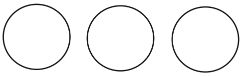
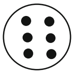
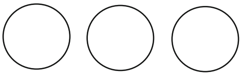
Original Contrast **(Mighton et al. 2009b, p. 80)**	**Modified Contrast**
Tory has 18 cookies. There are two ways she can share or divide her cookies equally: 1) She can decide how many sets (or groups) of cookies she wants to make: Tory wants to make 3 sets of cookies. She draws 3 circles. She then puts one cookie at a time into the circles until has placed 18 cookies.  2) She can decide how many cookies she wants to put in each set: Tory wants to put 6 cookies in each set. She counts out 6 cookies. She counts out sets of 6 cookies until she has placed 18 cookies in sets. 	Tory has 18 cookies. There are two ways she can share or divide her cookies equally: 1) She can decide how many sets (or groups) of cookies she wants to make: Tory wants to make 3 sets of cookies. She draws 3 circles. She then puts one cookie at a time into the circles until has placed 18 cookies. We can describe her action as 18 divided into 3 sets, or 18 ÷ 3.  2) She can decide how many cookies she wants to put in each set: Tory wants to put 3 cookies in each set. She counts out 3 cookies. She counts out sets of 3 cookies until she has placed 18 cookies We can describe her action as 18 divided into sets of 3, or 18 ÷ 3. 

Fig. 6 Two tasks intended to prompt attention to the difference between partitive and quotitive division

observed that neither students nor teachers who engaged with this lesson made the desired distinction in subsequent lessons. For instance, teachers made no reference to the two types of sharing (division) and explained the division algorithm in terms of quotitive language (e.g., "how many times does 3 goes into . . .") even though the resource refers explicitly to images of partitive division (e.g., splitting 600 blocks into 3 equal groups). We decided to make this lesson the focus of a group discussion with participant teachers, including one who was teaching Grade 5 at that time.

During the session with teachers, we analyzed ways to support learners' understanding of the difference between the two types of division. Then, we asked teachers to compare the tasks in Fig. 6 and to discuss which version would better draw attention to the difference between partitive and quotitive division. The first task is included JUMP Math (Mighton et al. 2009b) for the selected lesson; the second was a modification of this task.

Note that in the original (Fig. 6, left) version of the task, the circle/dot diagrams both show 3 sets of 6 cookies, with no reference to division notation; had this been included, one would have had a divisor of 6 and the other a divisor of 3. Here, the actions performed in forming the groups (based on what information is given in the problem) change, while the resulting groups are identical. In the modified version (Fig. 6, right), the circle/dot diagrams correspond to two different actions, both of which can be represented by $18 \div 3$; in one case, the resulting diagram shows 3 sets of 6 cookies; the other shows 6 sets of 3 cookies. In this case, the only thing that varies is the type of division. This distinction prompted several "a-ha's" among the teachers; it was here that the point of the partitive/quotitive distinction became more clearly apparent to them. After this discussion, we proceeded to analyze the rest of the lesson plan.

4.2.1 Analysis of the Original Lesson Plan: Two Ways of Sharing

The "Two Ways of Sharing" lesson offered in JUMP opens with a number of contrasts. The first is between two situations in which 12 learners are "numbered off" to form 2 teams:

(a) 1, 1, 2, 1, 1, 3, 1, 1, 4, 1, 1, 1
(b) 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4

Here, what varies is whether or not the groups are equal; the point of emphasis is equal-sized groups. Following a's uneven distribution, students are invited to make the teams fair by reassigning numbers to some students.

Next, students are offered containers labeled 1, 2, 3, and 4 and invited to distribute 12 blocks into the 4 containers. Compared with the previous task, this offers a variation in the manner of representing the 12 students being divided.

Then, the resource invites learners to consider sharing 12 cookies among 4 people. This offers a contrast in context; there are still 12 objects being split into 4 groups, but the particular objects change. Suggested questions in the teacher's guide draw attention to the number of containers and counters needed and to what these represent (cookies and people). Learners are then invited to consider circles for containers and dots for cookies, and the teacher models placing 1 cookie in the first circle, then 1 in the second, etc. until the 12 cookies have been distributed. Here, we see another contrast between this example and the previous one: representation with containers and counters vs. representation with circles and dots. Students have an opportunity to practice this by dividing 12 students into 4 cars. Again, the total remains constant, as does the number of groups into which the total must be divided; the only thing that changes is the context.

Note that so far, the variations in the lesson have drawn attention to the need for equal-sized groups, to how objects and groups might be represented (by people standing together, by counters in containers, or by dots in circles), and to the idea that even if contexts change, the numbers remain constant (though this is not explicitly mentioned).

Following this, a set of practice examples varies the number of cookies and people (12 cookies with 3 people, 15 cookies with 5 people, 10 cookies with 2 people), each of which produces a different number in each group.

At this point, it is important to note that, based on our previous observations to the enactment of this lesson, all of the examples used so far could easily be visualized by most students as four groups of three; they did not need to divide (or partition their counters) to find the answer. Students responded to questions of this nature with reference to a multiplication sentence; for example, when asked how they knew there were 3 students on a team, students responded: "Because $4 \times 3 = 12$."

The resource then announces, to the teacher, a shift in intention: "So far, questions have told students how many piles (or sets) to make. Now questions will tell students how many to put in each pile (or in each set) and they will need to determine the number of sets" (Mighton et al. 2010b, p. 82). However, there is no suggestion that the teacher should flag this shift for students, nor are there direct contrasts between the quotitive contexts that follow and the partitive ones with which they have already worked.

The first question students are invited to consider in this portion of the lesson is a person with 30 apples who wants to give five apples to each friend. Returning to counters and containers, the teacher is supposed to ask students what the containers and counters represent and to ask students if they know how many containers are needed and if they know how many counters should go in each container. If students can remember back to their earlier work with containers, this offers a potential contrast between known and unknown number of containers, but this contrast is not highlighted, and the two are not directly juxtaposed. Nonetheless, this is first time that partitive vs. quotitive grouping has been opened as a dimension of variation.

As the lesson continues, the direct contrast between partitive and quotitive fades from view: Examples that contrast different quotitive contexts become the focus of the lesson. The next contrast involves modeling the apple problem a second time, this time with circles and dots (another contrast in representation with no contrast regarding what is known/unknown). Once the 30 apples have been drawn in groups of five, students are asked to consider 30 apples in groups of three; this might draw attention to the fact that when you use smaller groups you get more groups. Practice examples then invite students to divide varying numbers of dots into sets of known size: 9, 6, and 12 dots into sets of 3; 6 and 12 dots into sets of 2; 15 dots into sets of 5; 12 dots into sets of 4; and 16 dots into sets of 2. Further practice involves varying numbers of apples being placed into boxes of varying sizes (10 apples, 2 per box; 12 apples, 3 per box; 18 apples, 6 per box). Finally, practice shifts to situations in which students are required to highlight "important information," which includes total and group size and to find the number in each group. In each of these cases, set sizes and totals vary, but the manner of grouping does not.

At this point, the distinction between the two "methods of sharing" is made explicit to students, and they are invited to make this distinction.

a) 15 children with 5 in each canoe, b) 15 children with 5 canoes, and c) a girl with 40 stickers who gives 8 stickers to each of her friends.

Fig. 7 Slide 19 in the "Two Ways of Sharing" lesson

a) There are 24 strings on 4 guitars. b) There are 3 hands on each clock. There are 15 hands altogether. c) There are 18 holes in 6 sheets of paper. d) There are 15 rings on 5 binders. e) 15 people sit on 5 couches.

Fig. 8 Slide 21 in "Two Ways of Sharing" lesson

g) There are 15 people. 5 people fit on each couch. h) There are 4 cans of tennis balls. There are 3 tennis balls in each can. i) There are 20 chairs in 4 rows. j) There are 3 rows of chairs. There are 9 chairs altogether

Fig. 9 Slide 22 in the "Two Ways of Sharing" lesson

> Sometimes a problem provides the number of sets, and sometimes it provides the number of objects in each set. Have your students tell you if they are provided with the number of sets or the number of objects in each set for the following problems (Mighton et al. 2010a, b, p. 83).

The situations offered include a slide showing three cases simultaneously (see Fig. 7). The only thing that changes between the two canoe cases is how the 15 is divided into 5: Is it divided into 5 groups or groups of 5? This is the first time that students are invited to make this distinction. Note, however, that students are not asked to represent either situation with a division sentence. We also observe that the third example on the slide could in fact distract from the clear contrast in the first two, as it might seem to suggest three independent examples.

In the slides that follow, various contextual situations are offered that require students to identify (a) what is being shared, (b) how many sets, and (c) how many are in each set (using a question mark to denote the unknown element). The distinction, then, is phrased in terms of what is known vs. what is not known. These distinctions must be discerned against a background of changing context, numbers, and manner of grouping. Nevertheless, none of these examples are presented as pairs in which only the manner of grouping changes (as in the canoe example). Here, total objects, number of sets, and number in each set all change from example to example, with a couple of exceptions. There is only one pair that forms a clear partitive/quotitive contrast (clear in that only the manner of grouping changes), but it is split between two slides, as shown in Figs. 8 and 9. Note the contrast between Slide 21 (e) and Slide 22 (g).

	What has been shared or divided into sets?	How many sets?	How many in each set?
a) Kathy has 30 stickers. She put 6 stickers in each box.	30 stickers	?	6
b) 24 children are in 6 vans.	24 children	6	?
c) Andy has 14 apples. He gives them to 7 friends.			
d) Manju has 24 comic books. She puts 3 in each bin.			
e) 35 children sit at 7 tables.			
f) 24 people are in 2 boats.			
g) 12 books are shared among 4 children.			
h) 10 flowers are shared in 2 rows.			
i) 8 hamsters are in 4 cages.			

Fig. 10 Task of the "Two Ways of Sharing" lesson corresponding to *Assessment and Practice* student book. (Mighton et al. 2009a, b, p. 81)

We also observed that the distinction between known and unknown can be difficult to discern when the numbers are such that both are immediately obvious due to students' knowledge of basic multiplication facts; in other words, if both are easily known, the known/unknown distinction becomes hazy, and this is critical to distinguishing partitive and quotitive division.

The practice examples in the *Assessment and Practice* book (Mighton et al. 2009b) follow a pattern similar to the lesson; there is a set of practice involving partitive contexts followed by a set of practice involving quotitive contexts. Finally, there is a chart describing numerous groupings in which students are to identify total, number of sets (if possible), and number in each set (if possible); the distinction between known and unknown is key (see Fig. 10). The first two cases are solved for students as examples. Notice that among the nine statements that students are offered, only two are quotitive: (a) and (d).

In summary, we have highlighted three features that may prevent students from noticing key critical features of the partitive and quotitive division. First, there were few opportunities to contrast the two ways of sharing in which only the meaning of sharing was changed (e.g., the canoe situation in Fig. 7). Second, these two ways of sharing were not connected explicitly to division. This connection is not explicit in subsequent lessons, either. Finally, the reference to known and unknown may be ambiguous for students who know multiplication facts. For instance, if a student knows the multiplication fact $30 = 5 \times 6$ and connects it with $30 \div 6$, then the result of the division, 5, is known by the student.

4.2.2 Adaptations and Suggestions for Implementation

We considered that the contrast between quotitive and partitive division should be made more explicit in the lesson. For a revised version of the lesson, we encouraged the teacher to juxtapose partitive and quotitive examples from the beginning of the lesson. We also emphasized the importance of prompting students' attention to what changes and what stays the same from example to example. For instance, the teacher lesson used a modified version of Slide 19 (Fig. 7) with only the first two examples. In a further implementation of this part of the lesson, we observed that all students were able to discern the difference between the two types of division when the modified slide was used.

A relevant discernment for this lesson is the distinction between number of sets and number of objects in each set. This is an example of *raveling*, as a critical discernment was identified and connected to other discernments in the lesson. The chart in the *Assessment and Practice* book (Fig. 10) was modified to include (a) direct contrasts between partitive and quotitive contexts (in which only the

	What has been shared or grouped into sets?	How many sets?	How many in each set?	Draw the sets.	Division Sentence	P or Q?
Kathy has 30 stickers. She put 6 stickers in each box.	30 stickers	?	6		30 ÷ 6 = 5	
Kathy has 30 stickers. She has 6 boxes.					30 ÷ 6 = 5	
24 children are in 6 vans.						
There are 24 children with 6 in each van.						
Andy has 14 apples. He gives them to 7 friends.						
Andy has 14 apples. He gives 7 apples to each friend.						
Manju has 24 comics. She puts 3 in each bin.						
Manju has 24 comics. She puts them in 3 bins.						

Fig. 11 Modified chart

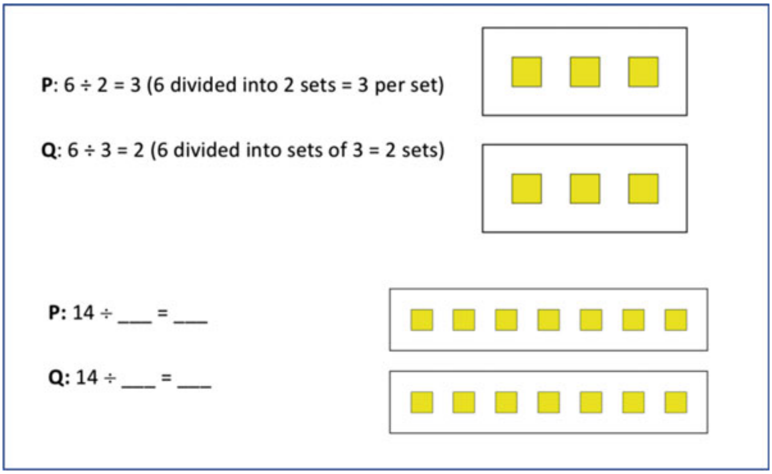


Fig. 12 Examples of two different ways of division corresponding to the same image

manner of grouping changed), (b) space to sketch the groupings, and (c) space for a division sentence (see Fig. 11). This adaptation may also be considered part of the *raveling* approach to teaching, as key discernments are both separated and connected. The adapted chart includes an explicit reference to division, which supports connections to future lessons (*macro-raveling*). The chart is also an example of *prompting*, as the examples are presented in contrasting pairs that support students distinguishing between the two types of division.

We also suggest to further emphasize the significance of *known quantity/ unknown quantities*: Is there a *known number of sets* or a *known number in each set*? Although the unknown factor is flagged with a question mark, we observed in the classroom that the relation between the position of the question mark and the type of division remained a point of confusion for some students. Nonetheless, most students successfully identified the known and unknown elements and found an appropriate solution, and several invented their own pairs of examples as a bonus (an example of *deciding* that supported advanced students).

Finally, students who complete the modified chart might be offered images showing numbers split into sets and asked to identify the mathematical expressions that indicate partitive and quotitive groupings (Fig. 12). This would allow a broader contrast involving the same groupings with different number sentences.

5 Conclusion

The four teaching strategies outlined in this chapter – raveling, prompting, interpreting, and deciding – are informed by Marton's (2015) variation theory of learning. While *raveling* refers to the identification of critical discernments, and their connections to other mathematical ideas and concepts, *prompting* refers to the sequence of examples and tasks offered during class that require students to make distinctions relating to each critical discernment. Earlier, we noted that the selection of what should vary and what should remain invariant is not obvious; consistent with Marton, we have observed that varying the critical feature often works better to prompt learners' attention. While raveling and prompting can be supported by teaching resources such as textbooks, lesson plans, and teaching guides, teachers must be aware of the need for potential modifications, which may include clearer patterns of variation and more direct juxtaposition of tasks and examples that contrast key features of the intended object of learning.

The role of the teacher is more significant in interpreting students' understanding during class, as well as in deciding how to proceed. Systematic variation can inform the questions or tasks posed to students as assessment, as well as teachers' decisions regarding how best to support students who are struggling or those who require additional challenges to extend their learning.

Although the Math Minds Initiative has involved only selected aspects of lesson study, the focus on variation theory makes it relevant for the literature on lesson study and learning study. In particular, we suggest paying attention not only to lesson plans but also to the decision-making process during lesson implementation. Systematic variation can inform teachers' decisions that can immediately address the needs of all students in class rather than waiting for another iteration of the lesson to be developed (though this is also important). On another level of iteration, the four elements of the Math Minds protocol are also impacting the redesign of the JUMP Math (2018) materials, as suggestions for adapting lessons are informing future editions of the teaching materials.

The approach to teaching described here has been developed from empirical evidence in the Math Minds Initiative. In Phase 2 of the project, we continue to refine lessons plans and to more clearly articulate the four strategies presented in this paper so as to better support teachers in understanding and enacting them in class.

References

Berg, D. H. (2008). Working memory and arithmetic calculation in children: The contributory roles of processing speed, short-term memory, and reading. *Journal of Experimental Child Psychology, 99*(4), 288–308. https://doi.org/10.1016/j.jecp.2007.12.002.

Bruce, C. D., & Ladky, M. S. (2011). What's going on backstage? Revealing the work of lesson study with mathematics teachers. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 243–249). New York: Springer.

Bruce, C. D., Flynn, T. C., & Bennett, S. (2016). A focus on exploratory tasks in lesson study: The Canadian 'Math for young children' project. *ZDM Mathematics Education, 48*(4), 541–554.

Chappuis, J. (2015). *Seven strategies of assessment for learning* (2nd ed.). Hobeken: Pearson.

Clark, R. E., Kirschner, P. A., & Sweller, J. (2012). Putting students on the path to learning: The case for fully guided instruction. *American Educator, 36*(1), 6–11.

Cobb, P., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.
Engle, R. W., Cantor, J., & Carullo, J. J. (1992). Individual differences in working memory and comprehension: A test of four hypotheses. *Journal of Experimental Psychology: Learning, Memory, & Cognition, 18*, 972–992.
Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3–17. https://doi.org/10.2307/748969.
Graeber, A. O., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 20*(1), 95–102.
Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–347). Hackensack: World Scientific.
Han, X., Gong, Z., & Huang, R. (2017). Teaching and learning mathematics through variation in lesson study. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 267–293). Boston: Sense Publishers.
Hart, L. C., Alston, A., & Murata, A. (Eds.). (2011). *Lesson study research and practice in mathematics education: Learning together*. New York: Springer.
Huang, R., & Li, Y. (Eds.). (2017). *Teaching and learning mathematics through variation: Confucian heritage meets western theories*. Boston: Sense Publishers.
Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*, 393–409.
JUMP Math. (2018). *JUMP Math: Multiplying potential*. https://jumpmath.org/jump/en/
Kullberg, A., Runesson, U., & Mårtensson, P. (2014). Different possibilities to learn from the same task. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 8(4), pp. 139–150). Vancouver: PME.
Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.
Marton, F., & Häggström. (2017). Teaching through variation: A European perspective. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 389–406). Boston: Sense Publishers.
Marton, F., Tsui, A. B., Chik, P. P., Ko, P. Y., & Lo, M. L. (2004). *Classroom discourse and the space of learning*. Mahwah: Routledge.
Metz, M., Preciado-Babb, P., Sabbaghan, S., Davis, B., Pinchbeck, G., & Aljarrah, A. (2016). Transcending traditional/ reform dichotomies in mathematics education. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 1252–1258). Tucson: The University of Arizona.
Metz, M., Preciado-Babb, P., Sabbaghan, S., Davis, B., & Ashebir, A. (2017). Using variation to critique and adapt mathematical tasks. In P. Preciado Babb, L. Yeworiew, & S. Sabbaghan (Eds.), *Selected proceedings of the IDEAS conference: Leading educational change* (pp. 169–178). Calgary: Werklund School of Education, University of Calgary.
Mighton, J., Sabourin, S., & Klebanov, A. (2009a). *JUMP Math 1.1: Assessment & practice*. Toronto: JUMP Math.
Mighton, J., Sabourin, S., & Klebanov, A. (2009b). *JUMP Math 5.1: Assessment & practice*. Toronto: JUMP Math.
Mighton, J., Sabourin, S., & Klebanov, A. (2010a). *JUMP Math: Teacher's guide: Workbook 2*. Toronto: JUMP Math.
Mighton, J., Sabourin, S., & Klebanov, A. (2010b). *JUMP Math: Teacher's guide: Workbook 5*. Toronto: JUMP Math.
Nelson. (2018). *Canadian tests of basic skills (CTBS).* Nelson. www.assess.nelson.com/

Pang, M. F., Marton, F., Bao, J., & Ki, W. W. (2016). Teaching to add three-digit numbers in Hong Kong and Shanghai: Illustration of differences in the systematic use of variation and invariance. *ZDM Mathematics Education, 48*(4), 455–470.
Pang, M. F., Bao, J., & Ki, W. W. (2017). 'Bianshi' and the variation theory of learning. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 43–67). Boston: Sense Publishers.
Preciado-Babb, A. P., & Liljedahl, P. (2012). Three cases of teachers' collaborative design: Perspectives from those involved. *Canadian Journal of Science, Mathematics and Technology Education, 12*(1), 22–35.
Preciado-Babb, A. P., Metz, M., Sabbaghan, S., & Davis, B. (2016a). Fine-grained, continuous assessment for the diverse classroom: A key factor to increase performance in mathematics. In *Proceedings of the American Education Research Association Annual Meeting 2016* (pp. 1–20). http://www.aera.net/Publications/Online-Paper-Repository/AERA-Online-Paper-Repository/Owner/973698
Preciado-Babb, A. P., Aljarrah, A., Sabbaghan, S., Metz, M., Pinchbeck, G., & Davis, B. (2016b). Teachers' perceived difficulties for creating mathematical extensions at the border of students' discernments. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 514–517). Tucson: The University of Arizona.
Preciado-Babb, A. P., Metz, M., Sabbaghan, S., & Davis, B. (2017). The role of continuous assessment and effective teacher response in engaging all students. In R. Hunter, M. Civil, N. P. Herbel-Eisenmann, & D. Wagner (Eds.), *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp. 101–120). Rotterdam: Sense Publishers.
Tepylo, D. H., & Moss, J. (2011). Examining change in teacher mathematical knowledge through lesson study. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 59–77). New York: Springer.
Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*(1), 5–25.
Watson, A. (2017). Pedagogy of variations: Synthesis of various notions of variation pedagogy. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 85–103). Boston: Sense Publishers.
West, B. T. (2009). Analyzing longitudinal data with the linear mixed models procedure in SPSS. *Evaluation & the Health Professions, 32*(3), 207–228.
Wiliam, D. (2011). *Embedded formative assessment*. Bloomington: Solution Tree Press.
Wiliam, D., & Leahy, S. (2015). *Embedding formative assessment: Practical techniques for K – 12 classrooms*. West Palm Beach: Learning Sciences International.
Wylie, C., & Wiliam, D. (2007). *Analyzing diagnostic items: What makes a student response interpretable?* Paper presented at Annual Meeting of the National Council on Measurement in Education (NCME), April 2006, Chicago. http://www.dylanwiliam.org/Dylan_Wiliams_website/Papers.html

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Capturing Changes and Differences in Teacher Reflection through Lesson Study: A Comparison of Two Culturally Diverse Malaysian Primary Schools

Liew Kee Kor, Saw Fen Tan, and Chap Sam Lim

Contents

Abstract This chapter discusses the changes and differences in the levels of teacher reflection as they engaged in the lesson study (LS) process. Two culturally diverse primary schools were selected to form a LS group each. These schools were culturally different in terms of medium of instruction and ethnicity. Six mathematics teachers from School A and three mathematics teachers from School B participated in this study. Both groups conducted five LS cycles. The research lesson taught in each cycle varies in topic and grade level. Data were collected through video recording of lesson observation, reflection sessions, and interviews with the teachers and selected pupils. In this study, Hatton and Smith's four-level reflection framework (descriptive story, descriptive reflection, dialogic reflection, and critical reflection) was used as a basic structure to capture the changes in teacher reflection across the five LS cycles. Using the same framework, the differences in reflection between the two LS groups were also compared. Analysis of the data revealed that there were some slight and gradual changes in the levels of teacher reflection as they progressed

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_18

from the first to the fifth reflection session. At the earlier LS cycles, teacher reflection was mainly at descriptive story level. However, their reflection gradually advanced to a higher level, the dialogic reflection at the later cycles. Although both groups were culturally different, there were not much observable differences in their levels of reflection after completing five LS cycles. This study suggests that LS as a teacher professional development program is able to nurture and foster the quality of teacher reflection.

Keywords Levels of reflection · Primary mathematics teachers · Lesson study · Cultural diversity

1 Introduction

Lesson study (LS) is a form of teacher professional development originated in Japan. It comprises four sequential steps: study, plan, do research lesson, and reflect. Teachers actively carry out LS to enrich instructional practice and to enhance students' learning. Tapping into its potential, researchers worldwide continually explore and expedite LS to achieve greater pedagogical goals. For instance, they have widen the scope of their studies to include both teacher learning (Meyer and Wilkerson 2011; Suh and Seshaiyer 2014) and student learning (Lasut 2013; Mak 2016). Other researchers engaged LS to promote teachers' questioning skills (Ong 2010), explore learners' observation skills (Myers 2012b), and for teacher reflection (Suratno and Iskandar 2010).

Since 2011, LS has attracted the attention of the Ministry of Education (MOE) of Malaysia. The Teacher Education Division was entrusted to promote professional development communities using LS as the core approach to improve teaching and learning. Consequently, the number of participating schools nationwide has multiplied considerably from the original 289 to more than 600 schools to date. In addition, review on these LS projects gave encouraging feedback. Among few, LS helps teachers in advancing their content knowledge and pedagogical content knowledge (Chiew and Jong 2009; Goh et al. 2007), enhances their teaching practices (Chia 2014; Ong 2010), strengthens teacher collaboration (Goh et al. 2007; Lim 2006), secured teachers' attention on student learning (Chiew and Jong 2009; Lim et al. 2005), improves student performance (e.g., Mak 2016), and promotes teacher reflection (Chiew 2009). However, study on Malaysian teachers' reflection is still scarce, and past studies did not examine details of reflection in-depth.

In LS, reflection occurs at the planning, teaching/observing, and reflecting stages (Suratno 2012; Suratno and Iskandar 2010). This study focused solely on teachers' reflection at the reflecting stage. It explored and studied the depth of teachers' reflection attained at four different levels. In the process, changes in reflection across five LS cycles from two LS groups based in two culturally diverse primary schools were documented. Results were compared and contrasted to screen for observable changes at different levels.

2 Literature and Theoretical Framework

This section delineates definition of reflection, types of reflection, and theories underlying the reflection in the LS.

2.1 Definition of Reflection

The notion of reflection was initiated by John Dewey (1859–1952). In his 1933 work, Dewey defined reflection as the "active, persistent, and careful consideration of any belief or supposed form of knowledge in light of the grounds that support it and the further conclusions to which it tends" (1933, p. 9). Likewise, many other definitions were used by present-day researchers to describe reflection. Closely related to the aim of this study is Barnett and O'Mahony's (2006) definition that reflection is "a learning process examining current or past practices, behaviors, or thoughts in order to make conscious choices about future actions" (p. 501). In essence it suggests that reflection is a learning process which includes hindsight, insight, and foresight. In this study, we adopted Barnett and O'Mahony's (2006) definition of reflection, and we viewed reflection as "an intellectual activity carried out by a group of participants. They look back at the pupils' learning during the research lesson. Thereafter, they identify the reasons for the occurring incidents and explore alternatives to improve pupils' learning." This view was informed by Hatton and Smith's (1995) three natures of reflection (Descriptive, Dialogic, and Critical) for reflection on action (see Sect. 2.2.1 for more details).

2.2 Types and Levels of Teacher Reflection

In order to assess teachers' reflection, researchers have developed several models to classify the different types or levels or dimensions of reflection. The earliest attempt to categorize the level of reflection was through Van Manen (1977) who staged a three-level reflection, namely, technical rationality, practical action, and critical reflection. Hatton and Smith (1995) expanded Van Manen's three categories to form a four-category analytical framework consisting of descriptive writing, descriptive reflection, dialogic reflection, and critical reflection. Valli (1997) examined "what" and "how" do teachers think about in her literature review along with several teacher education programs and concluded that there are five types of reflection: technical reflection, reflection-in- and reflection-on-action, deliberative reflection, personalistic reflection, and critical reflection. Jay and Johnson (2002) proposed a typology showcasing three dimensions of reflection, namely, descriptive reflection, comparative reflection, and critical reflection. Ward and McCotter (2004) analyzed the reflective texts written by 13 pre-service teachers through grounded theory approach. They developed a rubric consisting of four levels of reflection, namely, routine, technical, dialogic, and transformative. Each level was described from the dimensions of focus, inquiry, and change. Moreover, an in-depth analysis of the levels of reflection shows that teachers who reflect at higher level tend to look at the incidents from a number of perspectives prior to making judgment and decision for further action (Hatton and Smith 1995; Jay and Johnson 2002). Consequently, they gain insight and change or even improve their practice (Lee 2005; Ward and McCotter 2004).

Table 1 displays a summary of the types of reflection advocated by the researchers mentioned above and the respective year of their publication. Although the list is by no means exhaustive, it is sufficient to support the research framework of this study.

Of the many frameworks developed, Hatton and Smith's (1995) model was found helpful in identifying the levels of reflection and is popular in studies conducted in secondary schools and institutions of higher learning. Among a few studies that engage this model is, for instance, Myers (2012a) who examined 20 participating pre-service American teachers enrolled in a mathematics method course. Analysis of individual reflections and group reports revealed that the pre-service teachers' reflections were at the lowest levels: descriptive writing and descriptive reflection without any critical reflection reported. Most recently, Nurfaidah et al. (2017) investigated the development of levels of reflection encapsulated in reflective teaching practice of four Indonesian EFL pre-service teachers. They found that the

Table 1 Types of teacher reflection

Researcher (year of publication)	Types of reflection
Van Manen (1977)	Technical rationality
	Practical action
	Critical reflection
Hatton and Smith (1995)	Descriptive writing
	Descriptive reflection
	Dialogic reflection
	Critical reflection
Valli (1997)	Technical reflection
	Reflection-in and on-action
	Deliberative reflection
	Personalistic reflection
	Critical reflection
Jay and Johnson (2002)	Descriptive reflection
	Comparative reflection
	Critical reflection
Ward and McCotter (2004)	Routine
	Technical
	Dialogic
	Transformative

teachers' level of reflectivity resided at dialogic reflection and dialogic reflection. Similar to Myers' (2012a), no one attained the quality of critical reflection. Likewise, Iksan and Rahim's (2017) study on 17 Malaysian secondary teachers' reflection on teaching and learning mathematics through lesson study and video critique found that a majority of these teachers reflected at dialogue reflective stage. Some of these teachers reached descriptive reflective stage, and only few teachers advanced to critical reflective level.

In this study, we investigated teacher reflection using Hatton and Smith's (1995) four-category analytical framework consisting of descriptive writing, descriptive reflection, dialogic reflection, and critical reflection.

2.2.1 Hatton and Smith's Four-Level Reflection

Hatton and Smith's (1995) work in categorizing the levels of reflection was an extended development of Van Manen's three categories of reflection. In their study conducted at the University of Sydney focusing on reflective teaching, they investigated the types of reflection in 60 pre-service teachers' writing. Based on the analysis of the reports written by these teachers, Hatton and Smith categorized four types of writing: descriptive writing, descriptive reflection, dialogic reflection, and critical reflection. They elaborated that critical reflection is at the highest level in which practitioners become increasingly aware of the problematic nature of their actions and begin to search and examine why things occur the way they do. It is subjected to the historical and sociopolitical context of an event or action and is more demanding as it requires knowledge and experience to develop. In descriptive reflection, teachers attempt to provide the reason and justification besides reporting the incidents. Dialogic reflection is analytical in nature. It requires the use of judgments and other possible alternatives of explaining and hypothesizing to explore the experience, events, and actions. Lastly the descriptive writing is nonreflective as it merely describes events without supporting reasons and justification.

Descriptive reflection, dialogic reflection, and critical reflection are all reflective. The highest level is the critical reflection. The lowest level is the nonreflective descriptive story. Figure 1 shows the four levels arranged according to developmental sequence.

In this study, we stress on the word "level" to underline the hierarchical nature of the four types of reflection. We replaced the original term "descriptive writing" to "descriptive story" which we adopted from Suratno and Iskandar (2010). Suratno and Iskandar used the term "story" to evaluate teachers' written reflection and oral reflection which is similar to the forms of data collected in this study. For this study, we drew on Hatton and Smith's (1995) four- level reflection as the research framework and the analytical tool.

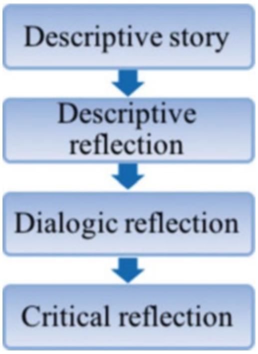

Fig. 1 The hierarchical four-level reflection model

2.2.2 The Language Factor

It should be noted that there are also contextual aspects that need to be considered when conducting reflection sessions. For example, the issue of language and communication skills may actually hamper some participants from engaging in a thorough discussion. Posthuma (2012) asserts that language is a contextual factor that possibly influences participants' reflective practice. In her study she found that the participating teachers struggled to express themselves when they were asked to reflect in English, which was not their home language. In this respect, we sought to minimize the effect of language incompetency in teachers' reflection in this study by allowing the participating teachers to use their mother tongue when they were reflecting.

2.3 The Theoretical Framework

Figure 2 shows the theoretical framework of this study. Situated learning theory (Lave and Wenger 1991) theorizes that there are novice and expert in a community of practice. The community of practice in this study refers to the lesson study groups set up by the teachers and researchers (who played the role of knowledgeable others). The novices were teachers who were new to the lesson study and reflection in lesson study. Meanwhile, the experts were teachers or knowledgeable others who were familiar with lesson study or were able to reflect at higher level.

In Fig. 2, lesson study cycle is the practice of the community. There are four sequential steps in a lesson study cycle (Lewis 2009), namely, (1) Study, (2) Plan, (3) Do, and (4) Reflect. Suratno and Iskandar (2010) claim that reflection is the heart of lesson study as it is found not only in the Step 4 but in all the steps. They believe that every lesson study group applies different kinds of analysis in reflection. As Fig. 2 shows, prospective analysis is used when doing a study and planning. Situated analysis is applied during the research lesson in Step 3. Finally, retrospective analysis is applied when teachers are reflecting in Step 4.

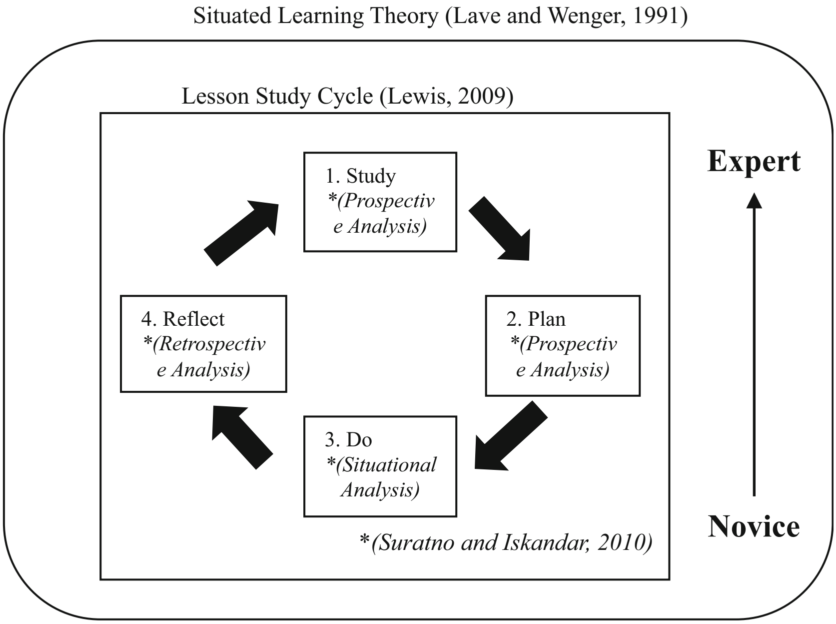

Fig. 2 The theoretical framework

This study examined and discussed reflection at Step 4 only as it was at this instance the participating teachers and researchers came together to contemplate retrospective practice or working through reflection-on-action (Hatton and Smith 1995; Schön 1983). As the teachers participated in the lesson study cycles, they collaborate and interact with other novices and experts in the community of practices. The novice teachers internalize the culture and belief of the community of practice and eventually mold to become experts.

3 Methods

In this section, we describe the background of the participants of the study. Besides, we also discuss the process of lesson study conducted by the two lesson study groups. Lastly, methods of data collection and data analysis are presented.

3.1 The Setting: The Participating School, the Lesson Study Groups, and the Teacher Participants

There are three mainstream primary schools in Malaysia. They are the national school, the national type Chinese school, and the national type Tamil school.

Although these three types of school adhere to the same curriculum, they are different in the medium of instruction and the ethnicity of the pupils. In national school, most of the pupils and teachers are Malay. The medium of instruction is the Malay language. Meanwhile, the majority of the pupils and teachers in national type Chinese school are Chinese. They use Mandarin as their medium of instruction. Likewise, in the national type Tamil school, most of the pupils and teachers are Indian, and they use Tamil as their instructional medium.

This chapter discusses LS conducted in national school and national type Chinese school. LS in Tamil school was not discussed because more time is needed to decipher the Tamil language.

3.1.1 Lesson Study Group A

Lesson Study Group A was set up in a national type Chinese school. This school was a small school located in a rural area. It has a headmaster, 10 teachers, and 138 pupils in the school. There was only one class in each grade. The school daily session started at 7:40 am and ended at 1:10 pm, operating from Monday to Friday. The school had just moved into its new building when the study was initiated. The new building was well equipped with computer laboratory, science laboratory, mathematics laboratory, library, music room, and audiovisual room. Interactive whiteboards were installed in some of the classrooms.

Lesson Study Group A was initiated to all four mathematics teachers in the school, known as TA1, TA2, TA3, and TA4. Table 2 displays the background information of the participating teachers. In this chapter, all the participating teachers were given codes starting from "T," whereas the researchers or knowledgeable others were given codes starting from "R." "TA" refers to teachers from Lesson Study Group A, while "TB" indicates teachers from Lesson Study Group B.

Lesson Study Group A conducted five LS cycles. However, only TA1, TA2, and TA4 participated in all the five LS cycles. TA3 joined the first three LS cycles, and he was transferred to other schools after the third LS cycle. TA5 and TA6 were new mathematics teachers who joined the school after TA3 left the school. Both of them were invited to be part of the LS group. So, both of them only conducted the fourth and fifth LS cycles. Table 3 displays the overall participation of the six participating teachers in the five LS cycles.

Table 2 Background information of the participating teachers in LS Group A

Participant	Gender	Teaching experience (year)	Mathematics teaching (level)
TA1	Male	1	Year 4
TA2	Female	5	Year 5
TA3	Male	4	(Year 1–6) tutoring classes
TA4	Female	7	Year 1 and 6
TA5	Female	10	Year 2
TA6	Female	6	Year 3

Table 3 Attendance of participants in Lesson Study Group A

Participant	LS1			LS2		LS3		LS4		LS5	
	Setting goal 2 February 2012	Refining lesson plan 15 June 2012	Research lesson and reflection 29 June 2018	Refining lesson plan 22 October 2012	Research lesson and reflection 29 October 2012	Refining lesson plan 9 May 2013	Research lesson and reflection 11 June 2013	Refining lesson plan 15 January 2014	Research lesson and reflection 22 January 2014	Refining lesson plan 25 February 2014	Research lesson and reflection 20 March 2014
TA1	√	√	√*	√	√	√	√	√	√	√	√
TA2	√	√	√	√	√*		√*	√	√	√	√
TA3	√	√	√	√		√	√				
TA4	√	√	√	√	√	√	√	√	√*	√	√
TA5								√	√	√	√
TA6								√	√	√	√*
RL	√	√	√	√	√	√	√				
RO					√		√				
RF					√	√	√	√	√	√	√
Key											
√	Teacher who attended the session										
√*	Teacher who taught the research lesson										

All the six teachers were mathematics teachers in the school. They were teaching mathematics at different grades of the school. Their teaching experience ranged from 1 year to 10 years. Other than these six teachers, there were three knowledgeable others involved in the LS process, who were RL, RO, and RF.

RL and RO were lecturers from a local higher institution. Both of them had 30 over years of teaching mathematics experience. RF was a postgraduate student who had about 2 years of teaching experience.

LS cycles were conducted based on the teachers' availability, and convenience were put on hold when the teachers were busy with school events, like school opening, examination, and sport's day. Lesson Study Group A spent about 2 years to complete five lesson study cycles. They started in February 2012 and ended in March 2014. They conducted two cycles in the year 2012, one cycle in the year 2013, and two cycles in the year 2014. The teachers started with the preparation of lesson plan. The researchers joined in thereafter to refine the lesson plan during their visit to school. The teachers then conducted the research lesson and held reflection session immediately after the lesson. The duration of the sessions between refining lesson plan and research lesson and reflection throughout the LS cycles lasted from a week to a month (see Table 3).

3.1.2 Lesson Study Group B

Lesson Study Group B was set up in a national school. This school was a small school located in the rural area. It has a headmaster, 18 teachers, and 172 pupils. The school's daily session started at 7:45 am and ended at 2:15 pm. There was only one class in each grade.

Lesson Study Group B consisted of three teachers. Table 4 presents the background information of these three participating teachers. They were the mathematics teachers in their school, teaching mathematics at different grades. Three of them conducted five lesson study cycles together as exhibited in Table 5.

There were six knowledgeable others who participated in Lesson Study Group B, who were RL, RC, RS, RK, RF, and RM, as listed in Table 5. RL, RC, RS, and RK were lecturers from higher institution who had about 30 years of teaching mathematics experience, while RF and RM were postgraduate students. RM had about 5 years of teaching mathematics experience, and RF had 2 years of teaching mathematics experience when they participated in the study.

As shown in Table 5, Lesson Study Group B conducted five lesson study cycles in about two and a half years. They conducted the research lesson 1 or 2 weeks after the refining lesson plan session. Like Lesson Study Group A, they carried out the reflection session immediately after each research lesson.

Lesson Study Group B was participated by six knowledgeable others, while Lesson Study Group A had only three. Since not all the six took part in observing the lesson study cycles at the same time, there is no obvious impact in the level of reflection from the participating teachers.

Table 4 Background information of the participating teachers in Lesson Study Group B

Participants	Gender	Teaching experience (year)	Mathematics teaching (level)
TB1	Male	14	Year 2, 4, and 6
TB2	Female	23	Year 5
TB3	Male	6	Year 1

3.2 The Process of Conducting the Lesson Study

As mentioned in the previous section, both Lesson Study Groups A and B carried out five LS cycles. The LS cycles were adapted from Chiew (2009) who followed the sequential procedure: (1) identify and formulate goals; (2) plan lesson plan collaboratively; (3) teach/observe research lesson; and (4) reflect and refine lesson plan.

Both lesson study groups delivered their research lessons to different groups and levels of pupils. The topics of the research lessons were also different, as shown in Table 6. Topics chosen by the teachers were those they found difficult to teach or learned by the pupils. They did not reteach the research lesson after refining the lesson plan because there was only one class in each grade in both schools.

3.3 Data Collection

Qualitative data was collected through participatory observation, reflection sessions, and collection of artifacts.

(i) *Participatory observation*

The researchers played the roles as participant as well as observer in this study. They participated in the steps: (2) plan lesson plan collaboratively; (3) teach/observe research lesson; and (4) reflect and refine lesson plan. The researchers observed the teachers' activities, behavior, commitment, as well as interaction among them. They recorded their observation by writing field note.

As the participants, the researchers assumed the roles of the knowledgeable others. They guided the teachers the process of LS, like the preparation of detailed lesson plan and the way of observing the research lessons. During the reflection sessions, the knowledgeable others gave final comments after all the teachers have given their reflections.

(ii) *Reflection sessions*

Both Lesson Study Groups A and B conducted five lesson study cycles; therefore, there were a total of five reflection sessions in both groups, respectively. All the reflection sessions began with the teacher who taught the research lesson reflected on his/her own research lesson. Then, other observing teachers took turns

Table 5 Attendance of participants in Lesson Study Group B

	LS1			LS2		LS3		LS4		LS5	
Participants	Setting goal 14 November 2011	Refining lesson plan 30 April 2012	Research lesson and reflection 10 May 2012	Refining lesson plan 26 June 2012	Research lesson and reflection 4 July 2012	Refining lesson plan 25 February 2013	Research lesson and reflection 11 March 2013	Refining lesson plan 11 July 2013	Research lesson and reflection 25 July 2013	Refining lesson plan 24 February 2014	Research lesson and reflection 3 March 2014
TB1	√	√	√*	√	√	√	√	√	√	√	√*
TB2	√	√	√	√	√*	√	√	√	√	√	√
TB3	√	√	√	√	√	√	√*	√	√*	√	√
RL	√		√		√		√		√		
RC	√	√	√	√	√	√	√				
RS	√		√								
RK							√				
RF								√	√	√	√
RM									√		√
Key											
√	Teacher who attended the session										
√*	Teacher who taught the research lesson										

Table 6 Topics and grade levels of research lessons

Lesson Study Group A		
Lesson study cycle	Topic of research lesson	Grade level
LS1	Conversion of time and calculation of interval of time	4
LS2	Calculation of the volume of cubes and cuboids	4
LS3	Calculation of volume	4
LS4	Proper fraction and equivalent fraction	3
LS5	Improper fraction and mixed number	4
Lesson Study Group B		
Lesson study cycle	Topic of research lesson	Grade level
LS1	Duration	6
LS2	Mass	5
LS3	Whole number	6
LS4	Measurement in length	4
LS5	Fraction	3

to reflect on the research lesson. Lastly the knowledgeable others gave the final comments. All the reflection sessions were video recorded with permission for data analysis purpose.

(iii) *Collection of artifacts*

The artifacts collected included the lesson plans, observation sheets, and worksheets or handouts given by the teachers. During the research lessons, all the observing teachers and knowledgeable others were given an observation sheet. The observation sheet guided the observing teachers and knowledgeable others what to observe during the research lesson. They filled in their observation in the observation sheets when they were observing the research lesson. Then, they reflected based on the observation sheets they have filled in during the reflection sessions.

3.4 Data Analysis

All the videos of reflection sessions were transcribed in verbatim in Chinese (for Lesson Study Group A) and Malay (for Lesson Study Group B). Analysis of the transcripts was performed based on the original language. First, the transcripts were divided into segments. A segment means a part of the transcript which was related to a topic of theme of reflection. It ended when the topic of reflection changed. The length of the segment could be as short as a sentence from a participant, for instance, "for those students who are weak [academically], we should give them more attention" (TB2, LS1). Or it could be several utterances from different participants. An utterance refers to an uninterrupted chain of spoken language. For example, when RL and TB2 discussed ways to relate the lesson with the pupils' real life during the first reflection session:

RL: Actually, can also conduct very simple activity, like, when [the pupils] are wearing shoes, going to wash their hands, what is the duration…
TB2: Sometimes they use the concept, but they don't know.
RL: Yes.
TB2: They applied [in their real life] but they are not aware of.

Then, all the segments were coded to the four levels of reflection as advocated by Hatton and Smith (1995), which comprised of descriptive story, descriptive reflection, dialogic reflection, and critical reflection. Triangulation of the reflection transcripts, observation sheets, lesson plans, worksheets, and field notes was also conducted. The percentages of utterances at each level of reflection were tabulated using NVivo software.

After analyzing the levels of reflection of each reflection sessions, the levels of reflection were compared across the five reflection sessions to explore the changes of levels of reflection across the five reflection sessions. Lastly, the results from both Lesson Study Groups A and B were compared to study the similarities and differences between the two groups in terms of the levels of reflection.

4 Results

This section reports the levels of teachers' reflection in both Lesson Study Groups A and B, separately. Next, the levels of teachers' reflection in Lesson Study Group A will be compared with those of Lesson Study Group B to explore the similarities or differences between the two groups.

4.1 Teachers' Levels of Reflection in Lesson Study Group A

Figure 3 displays the percentage of utterances of the four levels of reflection in Lesson Study Group A across the five lesson study cycles. During the first reflection session, more than half of the utterances (54.62%) were at descriptive story level. The participants, especially the teachers, merely described the pupils' learning, teacher's time management, and teaching strategies used during the research lesson, without further analysis or intention to improve the lesson. For instance, TB3 described that "at the beginning, [the pupils] were shy, not engaged yet, [but] during the activity of creating question, [they] were more daring to talk, voice out actively, contribute [ideas to the group]" (LS1).

More a quarter of the utterances of the first reflection session (27.69%) were at dialogic reflection level. The utterances at this level were contributed by both the teachers and knowledgeable others. They did not analyze the incidents from several perspectives, but they gave suggestions to improve the lesson. For example, the

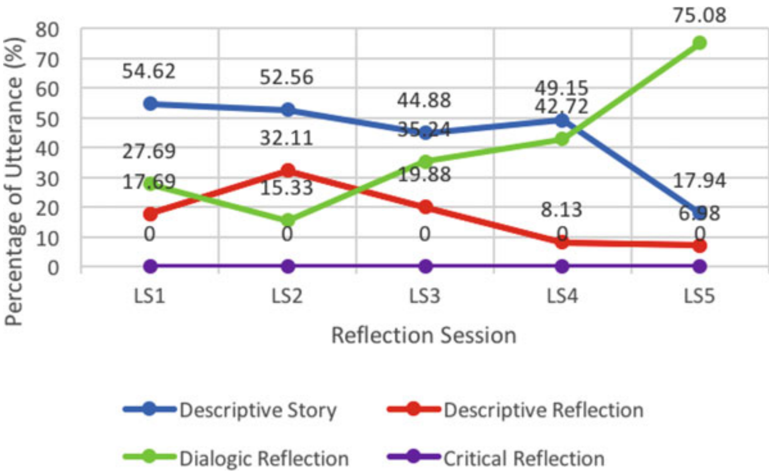

Fig. 3 The percentage of utterances of the four reflection levels across the five reflection sessions in Lesson Study Group A

teachers and knowledgeable others found that the pupils did not understand the concept of "Golden Hour" which was the concept taught in the first research lesson. Thus, they proposed some ways to refine the lesson. TB3 suggested adding a context of disaster, as he mentioned: "explain what is going to happen, [some victims] injured, buried under the soil, nothing to eat, no drink, so you think they can stay for how long?" (LS1). RL also gave similar suggestion, "add [some episodes where] some victims were injured. . . some people were injured and some people were buried [in the video]" (LS1).

The second level of reflection is descriptive reflection. Only 17.69% of the utterances were at this level during the first reflection session. Out of the 17.69%, 7.06% of the utterances came from TA1, the teacher who taught the first research lesson. He gave justifications when other participants commented on the research lesson. For instance, TA2 commented that the teacher should give more opportunities for the pupils to involve or express in the class. Regarding this comment, TA1 justified that he planned to get the pupils to explain their answers; however, he eliminated this section because "not enough time. [Besides], the pupils were not good in explaining. They know the way of calculating, but, they don't know how to explain" (LS1).

As Lesson Study Group A progressed from the first to the fifth lesson study cycles, the percentages of utterances at each levels of reflection gradually changed. The percentages of utterances at the first level, descriptive story, gradually dropped to 17.94% during the last reflection session.

On the other hand, the percentage of utterances at dialogic reflection level gradually increased to 75.08% during the fifth reflection session. As displayed in Table 7, the teachers' (TA1, TA2, and TA4, who participated in all the five reflection

Table 7 Percentage of utterances of each participant at dialogic reflection level across the five reflection sessions

Participants	Reflection Session				
	LS1	LS2	LS3	LS4	LS5
TA1	3.06	2.79	0.31	3.21	8.90
TA2	13.55	5.12	11.06	18.04	27.32
TA3	6.64	–	3.26		
TA4	0.22	1.30	4.75	5.42	7.62
TA5				4.57	3.36
TA6				5.41	21.81
RL	4.22	3.10	5.30		
RO		3.01	9.87		
RF			0.69	6.07	6.07
	27.69	15.33	35.24	42.72	75.08

sessions) utterance at dialogic level gradually increased from the first to the fifth reflection session. It was because the participating teachers analyzed or viewed the incidents from several perspectives. For examples, during the fifth reflection session, the teachers found the pupils' misconceptions in determining the denominator of the improper fraction. The pupils added up all the portions in the diagrams to get the denominator. The teachers analyzed this problem first from the perspectives of pupils' prior knowledge. TA2 suspected that "the pupils have not mastered the basic concept of the fraction" (LS5). However, her comment was rejected by TA4 and TA6. TA6 linked the pupils' misconception with the previous lesson, where the pupils learned about the basic concept of proper fraction, "could it be because the pupils were confused about the concept taught in the previous lesson? Because in that lesson, we taught them to count all the portions" (LS5). Nonetheless, the statement was rejected again by the group members. At the end of the discussion, all the teachers agreed that the misconception was caused by the instructional content delivered in that particular research lesson. TA4 concluded that "the teacher did not emphasize that there are many pieces, but you should not count all the portions, you only count the number of portions in one piece" (LS5).

Other than analyzing the problem from several perspectives, the high percentage of utterances at dialogic reflection level was also due to the teachers, and knowledgeable others gave more suggestions to improve the lesson. Their suggestions were more concrete, specific, and detailed.

Next, the percentages of utterances at the descriptive reflection level in all the five reflection sessions were not high. The teachers and researchers seldom justified the reasons of the incidents happened in the research lesson. Teachers who taught the research lesson would justify the reason of his/her teaching when commented by about their teaching.

Critical reflection was the highest level of reflection among the four levels. Nevertheless, none of the teachers and researchers reflected at this level across the five reflection sessions.

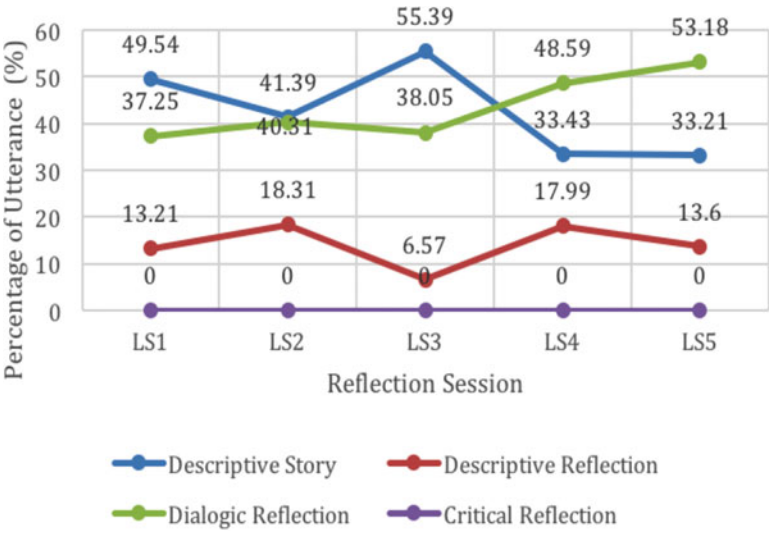

Fig. 4 The percentage of utterances of the four reflection levels across the five reflection sessions in Lesson Study Group B

4.2 Teachers' Levels of Reflection in Lesson Study Group B

This section discusses about the reflection of teachers and knowledgeable others from Lesson Study Group B. Figure 4 displays the percentage of utterances at the four levels of reflection across the five reflection sessions.

As shown in Fig. 4, the percentage of utterances at the descriptive story was high at the early stages of lesson study, which were 49.54%, 41.39%, and 55.39% in the first, second, and third reflection sessions, respectively. The participating teachers and researchers merely described the incidents in the research lesson, like pupils' learning, pupils' behavior, teaching strategy, and mathematical task. For instance,

> The pupils were not sure how to use the timelines of hour and minute (TB1, LS1).
> The time taken for discussing the pupils' answers was a bit longer [than expected] (TB3, LS3).

The second level of reflection is descriptive reflection. The percentages of utterances at this level were relatively low if compared with those of descriptive story and dialogic reflection, ranging from 6.57% to 18.31%. Most of the utterances at this level were contributed by the teachers who taught the research lessons. They justified their reasons when the other participating teachers or knowledgeable others commented on their research lesson. For example, during the second refelction, RC commented that TB2 who taught the research lesson did not get the pupils to present their answers as indicated in lesson plan, showed all the correct answers instead. Based on the comment, TB2 justified that she changed the plan because "I was lacking of time" (LS2).

The third level of reflection is dialogic reflection. As shown in the Fig. 4, the percentage of utterances at dialogic reflection was quite high across the five reflection sessions. However, most of the utterances at this level during the first three reflection sessions were contributed by the knowledgeable others. As shown in Table 8, 9.30%, 14.16%, and 3.7% out of the 37.25% of the utterance during the first reflection session were contributed by the knowledgeable others RL, RC, and RS, respectively. Similarly, during the second reflection session, RL and RC contributed 15.98% and 8.24% out of the 40.31% of utterances, which were at dialogic reflection level. It was because the knowledgeable others gave a lot of suggestions to improve the lesson during the reflection session.

As the teachers progressed to the fourth and fifth reflection sessions, they started to reflect at dialogic reflection level. The percentages of their utterances at dialogic reflection level increased. For instance, the percentage of TB1's utterances at this level were 14.31% and 10.98%, respectively, during the fourth and fifth reflection sessions, respectively. The participating teachers gave lots of suggestions to refine the lesson, for instance, "prepare two sets of questions in the envelope, one as given, for the high performers, I should prepare another [set of] questions, maybe with diagrams only, no numbers. So, the pupils find out the equivalent fractions based on the diagram provided, for the low performers" (TB1, LS5). Their suggestions were more specific, and they elaborated the benefits or importance of the new ideas.

Lastly, critical reflection is the highest level of reflection. However, none of the participating teachers and knowledgeable others reflected at this level in the five reflection sessions.

Table 8 Percentage of utterances of each participant at dialogic reflection level across the five reflection sessions

	Reflection session				
Participants	LS1	LS2	LS3	LS4	LS5
TB1	4.94	10.07	7.39	14.31	10.98
TB2	3.88	4.12	5.27	8.19	14.56
TB3	1.27	1.90	0.99	5.84	11.78
RL	9.30	15.98	14.42	16.62	
RC	14.16	8.24	2.62		
RS	3.70				
RK			7.35		
RF				3.64	13.75
RM					2.12
	37.25	40.31	38.05	48.59	53.18

4.3 Comparison Between the Levels of Reflection of Participants from Lesson Study Groups A and B

In both lesson study groups, the percentages of utterances at descriptive story levels were high at the beginning stages. It was because most of the participants, especially the participating teachers, merely described the incidents which happened during the research lessons. However, the percentages utterances at this level decreased as the participants progressed to the later stages of lesson study.

The percentages of utterances at dialogic reflection level were the second highest. This trend was shown in both Lesson Study Groups A and B. In Lesson Study Group B, at the beginning stages, most of utterances at this level were contributed by the knowledgeable others. It was because the teachers did not articulate much during the reflection sessions. Although the participating teachers also gave comments in improving the lesson, their suggestions were short, simple, and general. On the other hand, even though percentages of utterance at this level in Lesson Study Group A were low at the beginning stages, those utterances were contributed by both the participating teachers and researchers.

Although the trends of percentage of utterance at dialogic reflection level were a bit different between the two LS groups, both groups showed an increase in the percentage of utterances at the later stages of lesson study. The participating teachers started to analyze the lesson from several perspectives. Besides giving more suggestions to improve the lesson, their suggestions were more specific and elaborated.

Descriptive reflection is the second level of reflection. The percentages of utterances at this level were relatively low if compared with those of descriptive story and dialogic reflection, in both lesson study groups. It was because most of the utterances at this level were contributed by the teachers who taught the research lessons. They justified their reasons when other teachers or knowledgeable others commented on their lessons.

In both cases, none of the participating teachers and knowledgeable others reflect at the highest level, critical reflection level.

5 Discussions and Conclusion

This chapter discusses the changes and differences in the levels of teacher reflection as they engaged in the LS process. Two culturally diverse primary schools were selected to form a LS group each. Although we compared two lesson study groups which were set up at two schools which were culturally different, we found not much observable differences in the levels of reflection after completing five LS cycles. In both lesson study groups, most of the recorded utterances of teacher reflection at the beginning were mainly at descriptive story level. This finding is consistent with Myers' (2012a) finding. In her study, the teachers conducted one lesson study cycle, and their reflection was found predominantly at the descriptive story level. So, the high percentage of utterances at descriptive story level at the beginning stages in the present study could be attributed to teachers unfamiliar with LS and had yet to acquire the reflective skills.

As the teachers progressed to the later stages of LS, their reflection was noted to shift to a higher level, the dialogic reflection. This finding aligns with the situated learning theory developed by Lave and Wenger (1991). The theory posits that learning is embedded within activity, context, and culture. Applying the theory, we explain that as the participating teachers reflected together with their peers and knowledgeable others in the LS group, they observed and learned the tactic of dialogic reflection. At the end they emulated successfully the dialogic reflection after taking part in several reflection sessions.

However, an important point to note in this study is none of the teachers and knowledgeable others in both LS groups reflected at critical reflection level, which is the highest level of reflection. In the future, researchers may consider introducing workshop on reflection in order to foster reflective skills. Teachers and knowledgeable others can be trained to reflect at critical reflection level.

Results of comparison in levels of reflection between the two LS groups in two cultural diverse primary schools showed that neither culture nor ethnicity of the participants has affected their reflection. Instead, it was the knowledgeable others and the interaction within the group that augmented the quality of reflection. In both cases, the knowledgeable others in the Lesson Study Group B were capable in reflecting at the dialogic reflection level even at the early stages of LS. Modeling the knowledgeable group, the rest of the participating teachers in the same group were able to gradually reflect at dialogic reflection level as they progressed to the fourth and fifth reflection sessions.

The findings of this study also showed that after five reflection sessions, there were some noticeable positive changes in the content as well as the level of teacher reflection. These observed changes included (i) improvement in the depth of reflection when reflecting on pupils' learning, (ii) development of the awareness about all possible misconceptions and confusions underlying pupils' learning, (iii) reflection with critical lenses, and (iv) improvement in teachers' reflection at the descriptive story level at the beginning stages to the dialogic reflection level at the later stages of the lesson study. These findings are in agreement to Suratno and Iskandar (2010) and Chiew (2009) that to some extent lesson study is feasible in promoting the reflection among teachers.

Research literature suggests that knowledgeable others play crucial roles in facilitating discussion (Huang and Shimizu 2016; Takahashi 2014). As such, we recommend highly that future study may continue to examine the nature and pattern of reflection from two groups: teacher vs. knowledgeable other. Nonetheless, we uphold Takahashi's view that more reflection sessions in lesson study are needed to observe more visible changes in teacher reflection. We conclude that LS as a teacher professional development program is able to nurture and foster the quality of teacher reflection.

References

Barnett, B. G., & O'Mahony, G. R. (2006). Developing a culture of reflection: Implications for school improvement. *Reflective Practice, 7*(4), 499–523.

Chia, H. M. (2014). *Exploring the pedagogical flow of Chinese primary school mathematics lessons before, during and after lesson study.* Unpublished master's thesis, Universiti Sains Malaysia, Malaysia.

Chiew, C. M. (2009). *Implementation of lesson study as an innovative professional development model among mathematics teachers.* Unpublished doctoral dissertation, Universiti Sains Malaysia, Malaysia.

Chiew, C. M., & Jong, C. M. (2009, July). *Perceptions of lesson study: An exploratory case study among pre-service mathematics teachers.* Paper presented at the Seminar Penyelidikan Pendidikan Institut Pendidikan Guru Malaysia Zon Utara 2009, Hotel Copthorne Orchid, Penang.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: D. C. Heath.

Goh, S. C., Tan, K. A., & Lim, C. S. (2007). *Engaging in lesson study: Our experience.* In C. S. Lim et al. (Eds.), Proceedings of the 4th East Asia Regional Conference on Mathematics Education [EARCOME4] (pp. 574–579). Penang: Universiti Sains Malaysia..

Hatton, N., & Smith, D. (1995). Reflection in teacher education: Towards definition and implementation. *Teaching and Teacher Education, 11*(1), 33–49.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*(4), 393–409.

Iksan, Z. H., & Rahim, M. B. (2017). Reflection on teaching and learning of mathematics through lesson study and video critique. *Advances in Social Sciences Research Journal, 4*(1), 50–63.

Jay, J. K., & Johnson, K. L. (2002). Capturing complexity: A typology of reflective practice for teacher education. *Teaching and Teacher Education, 18*, 73–85.

Lasut, M. (2013). Effect of implementation lesson study to improve students' learning achievement in Calculus I of mathematics department. *Journal of Education and Practice, 4*(20), 182–188.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: University of Cambridge Press.

Lee, H.-J. (2005). Understanding and assessing preservice teachers' reflective thinking. *Teaching and Teacher Education, 21*(6), 699–715.

Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research, 17*(1), 95–110.

Lim, C. S. (2006). *Promoting peer collaboration among pre-service mathematics teachers through lesson study process*. In Yoong Suan et al. (Eds.), Proceedings of XII IOSTE symposium: Science and technology in the service of mankind (pp. 590–593). Penang: School of Educational Studies, Universiti Sains Malaysia.

Lim, C. S., White, A. L., & Chiew, C. M. (2005). *Promoting mathematics teacher collaboration through lesson study: What can we learn from two countries' experience?* Paper presented at the 8th International Conference of The Mathematics Education into the 21st Century Project, Johor Bahru.

Mak, W. F. (2016). *Effect of lesson study incorporating phase-based instruction on Form one students' achievement and learning motivation in Geometry.* Unpublished master's thesis, Universiti Sains Malaysia, Malaysia.

Meyer, R. D., & Wilkerson, T. L. (2011). Lesson study: The impact on teachers' knowledge for teaching mathematics. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 15–26). Dordrecht: Springer.

Myers, J. (2012a). Lesson study as a means for facilitating preservice teacher reflectivity. *International Journal for the Scholarship of Teaching and Learning*, *6*(1), Article 15. https://doi.org/10.20429/ijsotl.2012.060115

Myers, J. (2012b). The effects of lesson study on classroom observations and perceptions of lesson effectiveness. *The Journal of Effective Teaching, 12*(3), 94–104.

Nurfaidah, S., Lengkanawati, N. S., & Sukyadi, D. (2017). Levels of reflection in EFL pre-service teachers' teaching journal. *Indonesian Journal of Applied Linguistics, 7*(1), 80–92.

Ong, E. G. (2010). *Changes in mathematics teachers' questioning techniques through the lesson study process.* Unpublished doctoral dissertation, Universiti Sains Malaysia, Malaysia.

Posthuma, B. (2012). Mathematics teachers' reflective practice within the context of adapted lesson study. *Pythagoras*, *33*(3). https://doi.org/10.4102/pythagoras.v33i3.140

Schön, D. (1983). *The reflective practitioner: How professionals think in action*. London: Temple Smith.

Suh, J., & Seshaiyer, P. (2014). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. *Journal of Mathematics Teacher Education*, 1–23.

Suratno, T. (2012). Lesson study in Indonesia: An Indonesia University of Education experience. *International Journal for Lesson and Learning Studies, 1*(3), 196–215.

Suratno, T., & Iskandar, S. (2010). Teacher reflection in Indonesia: Lessons learnt from a lesson study program. *US-China Education Review, 7*(12), 39–48.

Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 4–21.

Valli, L. (1997). Listening to other voices: A description of teacher reflection in the United States. *Peabody Journal of Education, 72*(1), 67–88.

Van Manen, M. (1977). Linking ways of knowing with ways of being practical. *Curriculum Inquiry, 6*(3), 205–228.

Ward, J. R., & McCotter, S. S. (2004). Reflection as a visible outcome for preservice teachers. *Teaching and Teacher Education, 20*(3), 243–257.

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Representing Instructional Improvement in Lesson Study through Principled Analysis of Research Lessons in Singapore: A Case of Equivalent Fractions

Yanping Fang, Xiong Wang, and Christine Lee Kim-Eng

Contents

Abstract This chapter reports our effort in developing a principled way of representing instructional improvement in lesson study in the form of a process analysis, based on a thorough analysis of classroom discourse of four research lessons on equivalent fractions, a third-grade topic in Singapore, which we conducted at the early stage of our lesson study work in 2006–2007. Our social-cultural view guided a systematic effort in making consistent and iterative improvements within an instructional system mediated dynamically at multilevels – the content, discourse, activity, tasks, and tools level. Each level of the analysis was conceptualized to build coding schemes. The coded patterns were quantitatively and qualitatively presented to indicate how a balance in mathematical representation was achieved and how construction of the meaning-making was escalated across the research lessons. This principled way of representation has not only enabled us to represent and articulate the instructional improvements systematically but also further informed and improved our own ongoing lesson study with teachers locally and the lesson study work globally.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_19

Keywords Lesson study · Instructional improvements · Discourse analysis

1 Introduction

The use of instructional resources, such as classroom discourse, teaching and learning tasks, and tools, which are explicitly used in classroom instruction, could impact the effectiveness of classroom teaching and learning within any instructional systems (Cohen et al. 2003). However, resource use as technology in improving classroom teaching and learning has been taken for granted (Lave and Wenger 1991; Little 1999). Lesson study, a practice originated from Japan and China, offered an ideal site to pursue high-quality mathematics education in which the three levels of resource use (discourse, tasks, and tools) in mathematics instruction are well embodied through continuous improvements (Paine 1990; Stigler and Hiebert 1999). This chapter aims to capture those improvements in research lessons in Singapore's early lesson study experience in which researchers had worked closely with an elementary school for 2 years from 2006 to 2007. More specifically, we focused on four research lessons (two preliminary research lessons and their improved and retaught versions) on the topic of equivalent fractions, a third-grade topic (Fong et al. 2003), which the team conducted in two cycles in August 2006 and 2007, respectively. The four lessons provided us a wonderful opportunity of not only studying an emerging instructional system more systematically but also capturing how the abovementioned instructional resources were used to bring about the instructional improvements and more engaged student learning. Our two major research questions are as follows: (1) What constitute instructional improvements in teaching equivalent fractions through lesson study and how to represent them? (2) What can we learn about the nature and power of lesson study for curriculum and instructional resource development as well as professional learning of teachers?

Guided by our conceptual framework, a thorough coding analysis of the research lessons, see details in the third section later, has revealed differences in the instructional quality, particularly, how the classroom discourse mediated improved students' engagement in the designed mathematical experience across the four research lessons along five major dimensions: (1) balancing instructional representations; (2) escalating mathematical meaning construction; (3) transforming the social-mathematical culture of the classroom; (4) problematizing the lesson tasks; and (5) honing mathematical tools as learning support. In this chapter, we focus on the first two dimensions as they were identified most from our coding analysis in enabling us to represent the instructional improvements and illustrate how such improvements or lack of them were captured. We hope our effort in more systematically representing improvements is able to assist teachers and researchers in observing and assessing the research lessons and articulating the desired improvements in the quality of their teaching practice.

2 Context of Study

2.1 Lesson Study

Originated in Japan and China, lesson study represents organic institutional structures and work cultures in the two educational systems in which teachers work together to develop curriculum, instructional, and learning resources to improve classroom teaching and learning, often in response to reform (Chen 2017; Lewis and Tzuchida 1998). In lesson study, it is the "research lessons" which are carefully designed and taught in live classrooms and observed by a team of colleagues who gather evidences on student learning that inform deliberate and continuous improvement effort and bring to life instructional goals (Lewis and Tsuchida 1998). Teachers learn to teach and improve teaching practice while engaged in doing lesson study (Lewis and Tsuchida 1998; Stigler and Hiebert 1999; Yang and Ricks 2011). Over the past two decades, successful cases of adaption as well as challenges in trying out lesson study have been reported in the English-speaking world (e.g., Fang et al. 2011; Fernandez 2002; Perry and Lewis 2009; White and Lim 2008). Even though lesson study features data analysis at the heart of the professional development experience (Fernandez et al. 2003; NEA Foundation for the Improvement of Education 2003), it is not until quite recently that analysis of teacher discourse and tool development, particularly viewed from classroom improvement perspective, has been reported, such as in Dudley (2011) and Goh and Fang (2017).

2.2 Singaporean Instructional Practice and Context of Our Research

While many believe that Singapore might not lack good classroom pedagogy given its high student performance internationally (e.g., TIMSS and PISA), instructional practice in Singaporean classrooms fell short of what educational policies had targeted (Hogan and Gopinathan 2008). Moreover, classroom teaching was characterized by reproduction of procedural knowledge without much interpretation, application, and generalization of new knowledge and was mostly conducted as whole-class teaching and individual seatwork (Hogan et al. 2012). In recent years, key strategies to improve the intellectual quality of knowledge work have been proposed, such as the improvement of the quality of the instructional tasks teachers set for students; the proper alignment between learning goals, instructional tasks, and assessment; and more recently, the emphasis on assessment tasks and use of technology to improve instructional quality (Koh 2011). In addition, building professional learning communities in schools was launched officially by MOE of Singapore in 2009 with lesson study recommended as one of the major avenues (Hairon and Dimmock 2012; Hogan and Gopinathan 2008).

The research lessons reported in this chapter took place in a local government school[1] over two cycles in the second semester of 2006 and 2007, respectively. At a time with many teachers at their retiring age and an influx of novice teachers joining the school, in the first cycle in 2006, seven teachers were involved, of which four were novice teachers with less than 1 year of teaching experience. To support the novice teachers in learning to teach effectively, two of the novice teachers were given the opportunity to teach the research lessons (namely, RL1 and RL2), and thus a more traditional, direct teaching was tried out to facilitate their learning the fundamentals of teaching. In the second cycle in 2007, the team aimed toward a more interactive way of involving students in experiencing mathematical thinking to enhance their understanding of the meaning of EF. The same novice teacher who taught RL2 in the first cycle taught the first research lesson (RL3) and an experienced female teacher taught the revised lesson (RL4). It is in this sense that our lesson study teams became sites for mentoring the novice teachers in learning to teach (Lewis and Tsuchida 1998; Paine 1990) in Singapore (Fang et al. 2009).

2.3 *The Significance of and Challenges in Teaching and Learning Equivalent Fractions*

Much research has indicated that a solid understanding of equivalent fractions is a stepping-stone for learning more advanced topics (Jigyel and Afamasaga-Fuata'I 2007; Wong and Evans 2007) such as fractions and rational function, but fraction equivalence has posed a major stumbling block to children (Hunting 1984; Kamii and Clark 1995; Wong and Evans 2007). Equivalent fractions are complex (Dossey et al. 1988; Jigyel and Afamasaga-Fuata'I 2007) because of the multiplicative property (Hunting 1984; Kamii and Clark 1995) and the different semantic meanings of rational numbers embodied in the different representations (Ball 1993; Ni 2001). Unfortunately, many teachers teach the concept of equivalent fractions by engaging students in symbolic exercises to practice algorithmic strategies in converting fractions to equivalent fractions (Frobisher et al. 1999). Very few researchers and practitioners have made an effort to help students understand the meaning of equivalent fractions (Moss and Case 1999). More recently, computer technology has assisted students' understanding of the meaning of the EF concept, such as the slow motion animation (Kervin 2007) and dynamic virtual objects and movable visual representations (Westenskow and Moyer-Packenham 2016).

Singapore's mathematics textbook (Fong et al. 2003) has paid attention to the significance of and difficulty in teaching and learning this concept. Seven lessons altogether are allocated to teaching the topic in the third grade, together with examples of manipulatives for teacher use, which is considerably more than those found in Japan's (Hironaka and Sugiyama 2006) and Shanghai's textbooks (Chen

[1]The Project was funded (2006–2008) by the Center for Research in Pedagogy and Practice (CRPP), National Institute of Education (NIE), Nanyang Technological University.

and Zhou 2005). Our diagnostic tests given to students from third to fifth grades in the school found, however, that they lacked conceptual understanding of equivalent fractions. For example, many fourth and fifth grade students failed to understand fractions with different denominators and about half of them took subtracting a fraction from a mixed fraction only as a procedural algorithm. In addition, most of them took "equivalent" as "like." Taking this situation into account, our lesson study team decided to focus the research lessons on understanding the meaning of equivalent fractions as a foundational step to address these issues.

2.4 The CPA Model

In our lesson study, we also worked on the concrete-pictorial-abstract model (Bruner 1966) which was advocated by Singapore's Primary Mathematics Syllabus (Ministry of Education 2007). We encouraged teachers to facilitate students' mathematical learning through manipulating concrete learning tools and form a pictorial mental image of the concept before gradually associating the image with the abstract representations of symbols and algorithms. Although the teachers were aware of the CPA model, their packed teaching schedule discouraged them from ever trying it out in their regular lessons. Therefore, the lesson study teams designed, implemented, and improved research lessons by learning how to create various problem situations and sufficient time for students to think about equivalent fractions using various representational tools (Kamii and Clark 1995; Moss and Case 1999). Meanwhile, the "area" model and the "set" model were taken as manipulative contexts to contribute to the development of the students' conceptual understanding of EF concept (Jigyel and Afamasaga-Fuata'I 2007).

Through the process of conducting the research lessons, teachers, particularly the novice teachers, who taught the research lessons and those involved throughout a complete cycle of lesson study, not only had experienced how the CPA model could be used in teaching and how to help students understand equivalent fractions but also developed a deeper appreciation of its value (Fang et al. 2008).

3 The Development of an Analytical Framework and the Coding Scheme

The above background was set against an era when researchers, practitioners, and policymakers have all paid close attention to improving the quality of mathematics instruction (Cai 2007; Cai et al. 2009; Clarke et al. 2007). Research has identified the role of the teacher, classroom discourse, and mathematical tasks as three key dimensions that determine the quality of mathematics instruction (Kunter et al. 2013; Leon et al. 2017; Munter 2014).We developed a more systemic analytical framework that enabled us to look into and beyond these three dimensions. In this section, we provide the detailed components of the analytical framework and the coding scheme developed based on the framework.

3.1 Levels of Mediation in Instructional Systems

From social-cultural perspective, classroom teaching is viewed as tool-, artifact-, tasks-, and language-mediated instructional systems (Cobb 1995) from specific to more general functions: concrete tools (such as manipulates) to mediate thinking to derive representations and relationships that lead to meaning and understanding at the acting/doing and manipulating level in knowledge construction (Wells 1999, 2002); artifact mediation, such as through CPA; tasks as activities through embodiment of language, representation, and meaning to assist instruction; and language and discourse in mediating teaching, learning, and understanding (Cazden 2001) such as, among those mentioned later, in the inductive process of classroom teaching (Schoenfeld 2002; Truxaw 2005), language mediation of thinking through spiral upper movement in building representations and relationships (Bruner 1966; Cobb 1995).

3.2 Discourse Function in Mathematics Classroom

Classroom instruction was typically seen as teacher and student interaction over content in a learning environment (Cohen et al. 2003). The traditional I (teacher initiation), R (student response), and E (teacher evaluation) or F (teacher feedback to the student's response) pattern (Cazden 2001) features the teacher as the source of truth or authoritative knowledge and the student, the receiver of that knowledge, with little construction of knowledge or intellectual meaning-making going on, which is seen in many classrooms, including those of Singapore (Hogan Gopinathan 2008).

When children learn in an environment where they could have a clearer sense of the application purpose (Noss and Hoyles 1996), they make better sense of mathematical meaning. This requires teachers to go beyond knowledge delivery and use instructional language to help with students' knowledge construction. Meanwhile, classrooms should foster the social interactions that engage students in doing mathematics, exchanging ideas, comparing solution strategies, and discussing arguments so as to allow interaction and collaboration to mobilize reflection (Corte et al. 1996; Steinbring 2005).

This knowledge constructionist view takes all culture resources as mediational approaches/tools to achieve the goals of human activities (Cobb 1995; Wells 1999). However, to enable the mediational function, "they must be in the hands of a person who understands their meaning and mode of functioning in relation to the goals of the activity they mediate" (Cole 1994, as cited by Wells 1999, p. 138). In that sense, a teacher's primary mission is to help a learner to "discover – in action – when, where and how to use the culture's most important tools" (Wells 1999, p. 138) to know "their semiotic significance" (Wells 1999, p. 139).

Language, mediating the development of higher-order processes (Cole et al. 1978), has two actions in the classroom in mediating teaching and student thinking: "acting" and "understanding" (Wells 1999, p. 139). Acting mediates actions in negotiating goals and means, monitoring nonverbal behavior, and managing the interpersonal relationships involved, while understanding can be achieved by using language to reflect upon experiences. Mathematics can therefore "be viewed as a dynamic process of doing and communicating" (Hucker 1994, p. 10; Nassaji and Wells 2000).

This dual function of classroom language requests discourse analysis to capture two dimensions: how the classroom discourse functions in moving successfully or unsuccessfully toward instructional objectives and how the meaning construction progresses in each meaningful exchange with the advancement of the learning activities. The two dimensions together would yield more sophisticated patterns of whether and how the discourse succeeded in developing the students' understanding by giving meaning to their mathematical experience in an activity. We operationalized our analysis using Well's (1999) discourse levels (a move, an exchange, a sequence, and an episode) in building our coding schemes along the move level (the smallest building block of discourse) and sequence level (least meaningful exchanges) to quantify and describe the lesson representation of content and lesson orchestration of the mathematical meaning construction.

3.3 The Epistemological Model of Mathematical Knowledge

The mathematical signs or symbols, determined by conventions, do not have a meaning of their own unless when the epistemological subject establishes thought to a suitable reference context (Steinbring 2006). Therefore, the thought between symbol and object is "defined by the mathematical rules of connection – as precisely as possible," while the definition of the object rests on "social negotiation" in the development of mathematical knowledge (Steinbring 1998, p. 173). The epistemological triangle in Fig. 1 indicates such a relationship.

In the above figure, if a sign and referent constitute a mathematical concept, then there must be a productive mediation between symbol and object beyond thought/reference. If the relationship between symbol and referent refers to a mathematical concept, this concept is not exhausted by definitions or subjective notions. The mathematical concept is not identical with thought/reference but requires a theory; concepts reflect new *relationships* and are no mere images of representation (Steinbring 1998, p. 173). Therefore, the object or reference context is potential to represent the manipulative stage of CPA model; thus we named *object* in coding the manipulative stage of CPA model in classroom and the *symbol* to represent and code the abstract stage of CPA model. In between, ideally, the pictorial stage acts as a stepping-stone for the theory building required in the mathematical concept

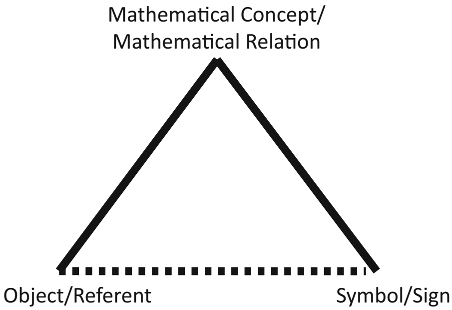

Fig. 1 The epistemological triangle of mathematical concept. (Adapted from Steinbring 1998, p. 174)

Table 1 The code at the move level of the classroom discourse

The first level	The second level	The third level	Definition
Teacher/student talk	Question/explanation	Object	The question or explanation of the teacher or a student (s) demonstrates the manipulative stage, such as folding the rectangle paper strips, shading the rectangle paper strips, etc.
		Symbol	The question or explanation of the teacher or a student (s) demonstrates the abstract or formalized stage, such as the expression of ½ = $^2/_4$
		Relationship	The question or explanation of the teacher or a student (s) demonstrates the process of abstracting from the manipulative to the abstract stage or concretizing from the abstract to the manipulative stage. For example, abstracting and expressing the fraction ½ from the manipulation of shading half of the rectangle paper strip or understanding the meaning of ½ in a shaded rectangle paper strip

construction which is termed *relationship* to code the pictorial stage of CPA model in classroom discourse.

Putting together, we took the smallest unit, "move," to capture the content representation in order to render a precise description of what was being talked in the classroom. At the move level of the discourse analysis, we used the codes of *object*, *relationship*, and *symbol* to represent the concrete manipulative, pictorial, and abstract stage, respectively, to capture the productive mediation of language, thought, and concept in building relationships to embody potential mathematical meaning. The coding scheme at move level is illustrated below (see Table 1).

3.4 The Inductive Teaching Model

To trace the sequences making up the episodes in which the learning tasks of the research lessons were accomplished, we adopted the inductive teaching model

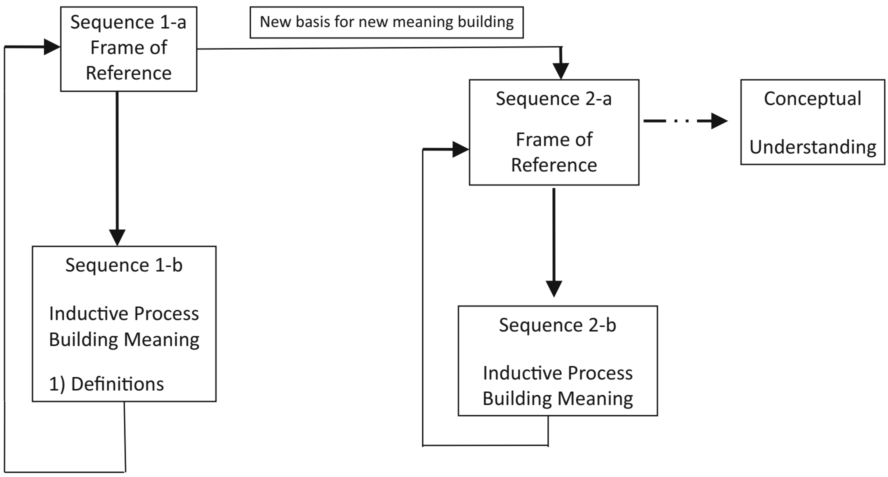

Fig. 2 Inductive teaching model

(Truxaw 2005) to capture the progressive sequential meaning in the discourse analysis. Take our research lesson four as an example. In a learning episode consisting of four sequences, a problem was introduced in Sequence 1, "what is the meaning of one-half?" (indicated as Sequence 1-a in Fig. 2). Building on this reference frame, Sequences 2 and 3 developed a common understanding of this key idea through definitions, explorations, and hypotheses of denominator and numerator, which was indicated as Sequence1-b. In Sequence 4, the problem was investigated in small groups, and solutions were presented by some students which achieved one-step advancement of mathematical meaning construction toward the lesson target and provide a new upscale basis for meaning-making, indicated as Sequence 2-a. Also in Sequence 4, through manipulating and discussion, the class achieved the consensus that the meaning of one-half was one shaded part out of two equal parts (as indicated in Sequence 2-b). This consensus, in turn, provided a new basis for meaning-making in the upcoming sequences. For instance, the meaning of one-half became the basis for the meaning of equivalence that fractions shared the same value but different graphical representations. The recursive nature of the discourse sequences demonstrated that the common understanding was achieved through teacher-guided exploring, conjecturing, testing, and revising hypotheses step by step or stage by stage in one lesson or across a unit of lessons, leading the students to achieve conceptual understanding of mathematical concept(s).This process is shown in Fig. 2.

Based on the rationale of the inductive teaching sequence, an evaluating norm was established according to the advancement of mathematics meaning construction in each sequence; see Table 2. In doing so, we were able to analyze the attributes of each discourse sequence and reason with the advance, stagnancy, or retrogression of meaning-making within each sequence and across sequences to elucidate the

Table 2 The norm of evaluating each sequence in classroom discourse

Value	Conditions
0/1(base value)	The discourse doesn't directly relate to the intended process of mathematical meaning (MM) construction, such as when it is about disciplining students or laying down the rule for using manipulatives and so on, the value of the sequence is defined as 0; otherwise, the value of the sequence is 1
+1/−1 (increased or decreased in value)	When the MM construction in a sequence is built on or superior to that in the previous or a most related sequence, the value of MM is plus 1; otherwise, if there is no MM construction or it belongs to that of the previous or the most related sequence, the value of MM is minus 1
+/−0 (stagnant in value)	When the constructed MM in a sequence is equivalent to that of the previous or a most related sequence, the value of MM remains the same as that of the previous or the most related sequence

differences in the progression of the mathematical meaning construction of the four research lessons and the ways in which the meaning was built.

4 Data Analysis

We drew on three major sources of data from our intervention, namely, video footage and transcriptions of the research lessons, the post-lesson discussions, and the planning sessions for cross-referencing and triangulation. The four research lessons (RL1 and RL2 in 2006 and RL3 and RL4 in 2007) were encoded using the above-mentioned coding scheme to identify valid evidences in terms of what improvements were made and how and why they were made. Both the quantitative patterns and the qualitative instances were captured to represent the improvements through constant comparison between and across the four research lessons.

NVivo 8.0 software not only assisted in building coding scheme but also in coding and interpretation. The advantage of the video footage allowed the coding to capture the nonverbal behaviors and language cues adding nuances to the interpretation. NVivo also helped with defining the sequence via its *set* function.

To ensure coding reliability, the research team remained mindful of the various potential emerging discrepancies in the course of coding. Two team members individually coded approximately one-fourth of one transcribed lesson via Nvivo8.0 to measure the agreement coefficient between their coding. To raise the agreement coefficient which was close to 60% in the first round, a third member from our research team joined the coding as a monitor and went over every disagreeing coding outcome to clear up discrepancies and built up a revised agreement based on enhanced mutual understanding. Then the first two members completed the coding of part of each of the four research lessons individually according to the revised agreement to have eventually raised the agreement coefficient to 85%. The coding reliability was thus ensured.

5 Results

A thorough coding analysis of the research lessons has revealed important differences in the quality of mathematical discourse and its mediation in students' mathematical experience and thinking. These differences demonstrated critical instructional improvements. In this section, we present, with both quantitative representations and qualitative examples, how a balance in mathematical meaning representation was achieved and how construction of the meaning of EF escalated through the classroom discourse that mediated the teacher-student interactions over content. Even though we cannot demonstrate fully how these improvements were established, we could show what those in place during classroom interactions meant. From the remarkable differences between research lessons within each cycle and across the two cycles, we could show how the improvements impacted on students' engagement in the conceptual thinking.

5.1 *The Balancing of Instructional Representations at the "Move" Level*

Coding of the "move" level of the first research lesson (RL1) and its improved lesson (RL2) in the first cycle indicated that in RL1, the teacher and the students worked more at the object level (65.9%) and less at the relationship (6.9%) and least at the symbol (1.0%) level; see Table 3.

RL2 had a higher relationship level (46.0%) and symbol level (6.6%) and a lower object level (18.9%). This suggests that RL2 remarkably shifted away from the overemphasis on the manipulative stage performed in RL1 and succeeded in moving to the pictorial stage by building mathematical relationship between the concrete manipulating and the mental images. However, the low symbol level suggests that the teacher did not have enough time, in the final stage of the lesson, to adequately guide students in abstracting the mathematical meaning.

In cycle two in 2007, the classroom discourse of the initial research lesson (RL3) and its improved research lesson (RL4) fell mainly on the object and relationship levels. For example, in RL3, 21.0% of classroom talk was at the object level, and 29.4% was at the relationship level. In RL4, approximately a quarter of classroom discourse was at both the object and the relationship levels, respectively. This

Table 3 Classroom discourse on the move level in the research lessons

	Cycle 1 (July to Sep 2006)		Cycle 2 (July to Sep 2007)	
"Move" level codes	RL1	RL2	RL3	RL4
Object	65.9%	18.9%	21.0%	25.5%
Relationship	6.9%	46.0%	29.4%	25.5%
Symbol	1.0%	6.6%	3.8%	11.6%

Table 4 Teacher talk on the "move" level in the research lessons

"Move" level codes	Teacher questions				Teacher explanations			
	RL1	RL2	RL3	RL4	RL1	RL2	RL3	RL4
Object	24.3%	7.8%	4.8%	9.3%	33.9%	9.1%	12.1%	13.1%
Relationship	1.4%	27.3%	16.2%	13.3%	5.0%	17.4%	9.8%	8.5%
Symbol	0.7%	4.7%	0.8%	4.3%	0.1%	1.2%	1.5%	4.0%

indicates that the discourse mainly focused on the manipulating and building pictorial mental representations. At the symbol level, however, RL4 markedly improved, moving to 11.6% from 3.8% in RL3. Therefore, RL4 witnessed greater effort to engage the students at all three levels of representations – concrete, pictorial, and abstract – showing more balance among the three stages and thus giving the students more opportunities to reflect on and construct the abstract meaning from their manipulations. Further analysis indicated that the balanced representations were attributed to the RL4 teacher's relationship questions and explanations.

5.1.1 Teacher's Relationship Questions

The higher incidence of teacher questions at the relationship level in RL2, RL3, and RL4 as indicated in Table 4 reveals that effort was made effort to improve the lessons and enable students to abstract mathematical meaning embedded in the manipulations to promote students' conceptual thinking.

Nevertheless, higher amount of teacher questions at the relationship level would not guarantee students' conceptual understanding because it also relied on what kinds of questions were asked and how they were asked. For example, in RL3, to encourage students to reflect on their sorting, the teacher asked questions at the relationship level as much as possible (16. 2%), such as "Okay tell me why. Why did you sort it this way?" "Why did you, why did you do this?" "Why, why did you put these three together?" Such questions were repeatedly asked for 34 times but no further scaffolding of students' thinking was provided when no students could answer them. In fact, answering such questions required the students to articulate the sorting process and generalize and abstract the mathematical meaning of the process, which demanded a higher level of cognition and thinking. Without enough scaffolding, these higher cognitive questions failed to promote students' conceptual thinking.

In RL2 and RL4, a higher amount of relationship questions were more successfully asked albeit in different ways to gear the students toward conceptual thinking. In RL2, for example, when the students got stuck at a relationship-level question, the teacher strategically changed the way he posed the question while keeping it at the relationship level so that the concrete manipulating and the abstract concepts could be linked. While in RL4, the teacher strategically utilized questions to match with the students' cognitive levels to facilitate and enhance their thinking. For example,

Table 5 Examples of the teacher's questions in RL4

Turn	Questions	Instruction	Coding
356	Look at the paper which had been pasted on your table, how many groups do you see? (followed by the students answering "two groups")	The first step of grouping	Object
361	What are you supposed to do with the circles that have been given to you? (followed by the students answering "split the circles into groups")	The second step of grouping	Object
375	Put the circles into groups. How many groups? (followed by the students answering "two groups")	The third step of grouping	Object
383	What are you looking at when you group the circles? (followed by the students answering "the shaded amount")	The fourth step of grouping	Relationship

Table 6 Teacher questions interwoven in student answers in RL4

Turns	Speaker	Utterance	Coding
867	Teacher	Why you have cropped the circles this way?	Relationship
884	Student A	The shaded ones	Object
886	Teacher	Now the shaded parts are?	Object
888	Student A	Two	Object
890	Teacher	Two. Where? Which one?	Object
894	Student A	These (pointing at his grouping)	Object
896	Teacher	These two, and then?	Object
898	Student A	And then, count the total	Object
900	Teacher	We count the total, which is?	Object
902	Student A	Three	Object
904	Teacher	Three, so why do you put this thing here then?	Relationship

before the students began to group different circles, the teacher posed a series of questions found in Table 5.

In giving the grouping instructions, the teacher focused the first three questions on the object manipulation. When she posed the fourth question, the students already moved forward from the concrete starting point of thinking to a relationship question by building it on a series of related object-level questions. In addition, in the same lesson, the relationship-level questions were also interwoven in students' talk. In the following interaction found in Table 6, the teacher facilitated a student to articulate his reasons behind his grouping by asking questions geared towards relationship building.

The above example shows that the teacher facilitated the student in articulating his grouping through each concrete step by building her questions upon the student's answers and strategically repeating the key word(s) in the student's responses and adding "new information" in her feedback as a way to acknowledge the student's knowledge. For instance, the teacher repeated the keyword "shaded" in the student's response, "the shaded ones" (Turn 884), and added the word "parts" before questioning again, "the shaded parts are?" (Turn 886). Such patterns in her questioning occurred in Turns 894 and 896 and Turns 898 and 900. Through this chain of questioning and answering, the teacher not only facilitated the student to articulate how he grouped the circles by focusing on the shaded parts (the central focus of the whole manipulating activity at this stage) but also led him to notice the discrepancies in his grouping. In this way, the teacher was able to lead the student up to the relationship level (which demanded a higher cognitive level) by going through the object level.

5.1.2 Teacher's Explanations on the Relationships

We also found that the incidence of teacher's explanation at the relationship level was high in RL2 compared with the other three lessons (see Table 4). The relationship level explanation is potential to promoting the students' conceptual thinking, particularly through extensive explaining. For example, in the construction of EF concept, the teacher's explaining was to further the students' thinking about the mathematical meaning behind a student's answer, such as 1. "Yes, okay very good, she says the strips are the same when you erase out the lines dividing the strips. 2. That means they are all representing the same whole. Moreover, when I asked you to shade half of each strip, it is actually half of the same whole, Right?" In part 1 the teacher repeated the student's answer and made the answer clearer for the other students and part 2 furthered the student answer by demonstrating the mathematical meaning implicated in the student answer by turning the object-level answer into a relationship-level one. In fact, from the manipulative perspective (the object level), the student answer was correct. Moreover, the teacher offered a positive comment on it with an encouraging tone. This was commendable for a novice teacher to achieve with the team's, particularly the mathematics teacher educator's careful mentoring in planning the revised lesson (RL2) based on lessons learned from RL1, which was also planned by the team (Fang et al. 2008).

5.2 The Escalating Mathematical Meaning Construction at the "Sequence" Level

We numbered all the sequences in order of which research lesson they belonged to. For example, in RL4, the third sequence was coded 4S3. The improved mathematical meaning construction occurred in both cycles.

5.2.1 Improved Mathematical Meaning Construction in the First Cycle

In RL1, see Fig. 3, the coding analysis of the sequences showed that two-thirds of the sequences stayed at low or extremely low value of meaning construction. The value even dropped from the preceding ones, such as in code 1S30, 1S34, 1S37, 1S39,

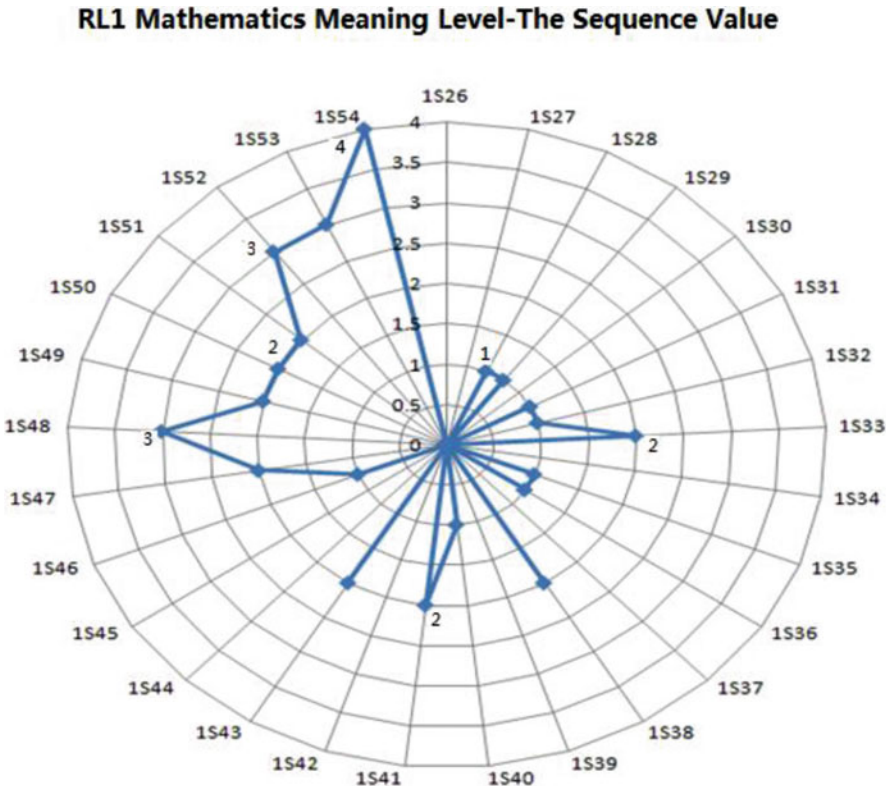

Fig. 3 Values of sequences in constructing the meaning of EF in research lesson 1 (RL1)

1S42, 1S44, 1S45, and 1S49; except for 1S49, the rest dropped to 0. In addition, in the final stage of abstraction, several sequences were constant in value, such as in 1S50, 1S51, and 1S53, which suggests a failed attempt to assist students in abstracting the meaning from the lengthy manipulating. The low level or drop in mathematical meaning construction was attributed to limited productive interaction between the teacher and the students and the disconnected manipulation activities caused by the cumbersome shading of different color strips designed by the lesson study team.

Taking Sequence 1S37 as an example, the interaction was initiated when students were asked to draw a line to show half in a strip folded three times in the following instruction: "Now draw a line using pencil and ruler to show half again. To show half." Yet some students did not understand the teacher's instruction clearly and manipulated wrongly by cutting the strip. "Teacher, ( ) go and cut then cut wrongly already," uttered one student. "Cut? Why you cut the paper? I never ask you to cut," responded the teacher. He then emphasized again "no need to cut the paper. Just fold it. I give you another one. Why you cut the paper?" and closed the interaction of this sequence. Consequently, most of the interactions did not build or advance mathematical meaning. In addition, a fundamental understanding of the concept of equivalent fractions entails part-whole conception of a fraction and the visual cognition that the shaded regions of equivalent fractions are equal. In RL1, however, these two related properties were disconnected in the lesson's manipulation activity. For example, naming fractions (1S49–1S50) occurred after the manipulations of folding, shading, and aligning of the rectangular strips and comparing the features of the manipulations (1S26–1S48). Therefore, the meaning construction of fractions lagged behind the manipulative process.

In RL2 (see Fig. 4), the meaning construction of equivalent fractions concept occurred from sequences 2S33 to 2S52. The deepening understanding of the EF concept occurred from sequences 2S53 to 2S55. The structure of the learning activities in RL2 was similar to that of RL1; however, changes were made after learning from RL1: reviewing fraction concept earlier and eliminating the distracting folding activity by providing children with three pre-divided strips of the same color which children could align under one whole strip as a reference framework. More importantly, a discussion about the conflicting notion of the sameness of shaded regions and differences in the shaded parts was conducted to highlight the meaning

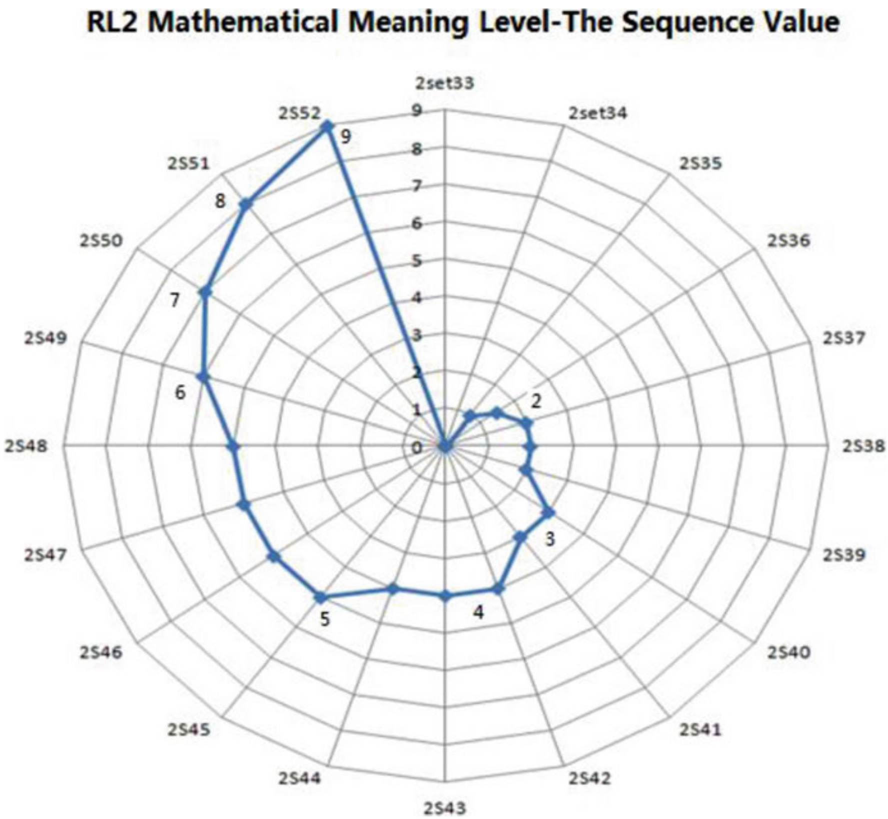

Fig. 4 Values of sequences in constructing the meaning of EF in research lesson 2 (RL2)

of equivalent fractions. Although the values of sequences 2S38 and 2S39; 2S42, 2S43, and 2S44; and 2S46, 2S47, and 2S48 were constant, mathematical meaning increased spirally in RL2, which suggests that the advancement of the mathematical meaning construction escalated in a structured manner in RL2.

Students' understanding of the part-whole concept was scaffolded by the sequential actions such as review, articulation, shading, and observation. The shading of the three strips proceeded coherently with the second (2S38) and the third strip (2S39) built on that of the first one (2S34) through analogical thinking. Following the shading, the three strips were immediately pasted under one whole strip as the reference frame for comparison and visual cognition. Understanding of fraction concept was fused into the shading stage itself, such as naming two-quarters followed the shading acts in 2S40 and 2S41, and discussion followed based on each shading process. These sequential moves avoided the confusion that the students experienced in RL1 during folding and aligning the strips.

More importantly, the comparison and articulation based on the aligned strips could have led the students to center on what was the same and what was different in the shaded areas: the sizes of the shaded parts were all the same (i.e., equivalent to ½), but the number of the shaded parts were different, so the fractional numbers were represented differently: 1/2, 2/4, and 4/8. The teacher then led the students to understand that the sameness can be mathematically represented in the following expression, 1/2 = 2/4, 1/2 = 4/8, and 2/4 = 4/8, which not only laid the mathematical representation of equivalent fractions but also implied a basic algebraic idea – transitive law enabling the three separate expressions to be expressed in a chain "1/2 = 2/4 = 4/8" (in 2S52). This surfaced the understanding that the three fractions were the same sizes, which more conspicuously expressed the essential meaning of equivalence. It also became the point of entry to introduce the concept of equivalence.

5.2.2 Improved Mathematical Meaning Construction in the Second Cycle

In RL3, as shown in Fig. 5, the construction of the conceptual meaning of equivalent fractions occurred from sequences 3S10 to 3S79 (see Fig. 5) and advanced in four stages: naming fractions (3S10–3S30); identifying the pattern, sorting, and articulating the grouping rationale (3S31–3S64); and further manipulating – folding of the circles (3S65–3S75). In the last stage, the folding was followed with more students' articulating (3S76–3S79). The value of mathematical meaning went up stage by stage, but the progress was not smooth.

In fact, despite the fact that there was a fair amount of time and space in RL3 for the students to manipulate and articulate, due to the ineffective questioning, as mentioned earlier, articulation was not effective most of the time. In addition, the students responded, "the shaded parts are halves" in 3S45 and 3S61 and "the shaded parts are the same" in 3S48, which indicated that they had identified the pattern of the EF concept through grouping – the shaded parts were the same sizes. However,

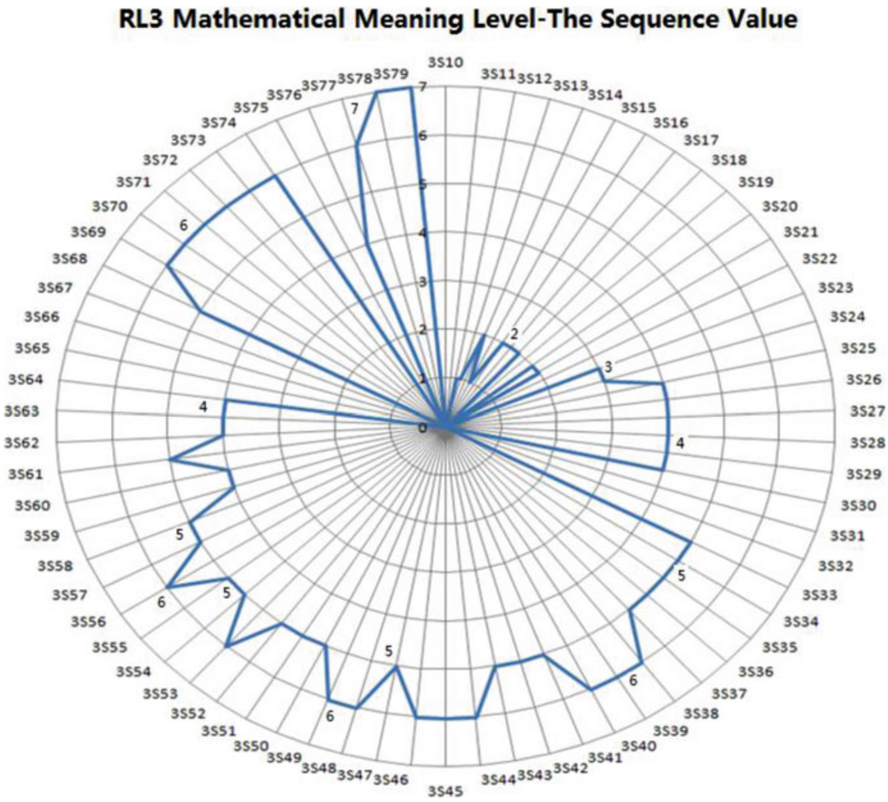

Fig. 5 Values of sequences in constructing the meaning of EF in research lesson 3 (RL3)

such initial understanding was not utilized to link the pictorial images to the abstract symbols and relationships. Frustrated at student's difficulty in articulating their sorting idea, the teacher gave up on the sorting experience and led them to compare the sizes of the shaded parts and non-shaded parts (3S65–3S75) to confirm whether the shaded parts were equal to the non-shaded parts. Students' confusion at the sudden shift in the orientation of their manipulation lowered the value of mathematical meaning construction involved, as indicated in 3S57–3S64.

In RL4, the meaning construction of equivalent fractions developed in two stages, as represented in Fig. 6, the abstraction of the concept from the manipulating and grouping (4S15–4S62) and the deepening of the concept of equivalence (4S63–4S67). Similar to RL3, the "sequence" advancement happened in four stages: students' manipulations (4S15–4S37), demonstrating and sharing their grouping on the whiteboard (4S38–4S52), articulating their perspectives based on the whiteboard demonstrations (4S53–4S62), and further deepening the understanding through comparison and discussion (4S63–4S67). Compared with RL3, the mathematical meaning construction in RL4 went with more progressive escalation stage by stage, which can be attributed to the effective articulation created in RL4.

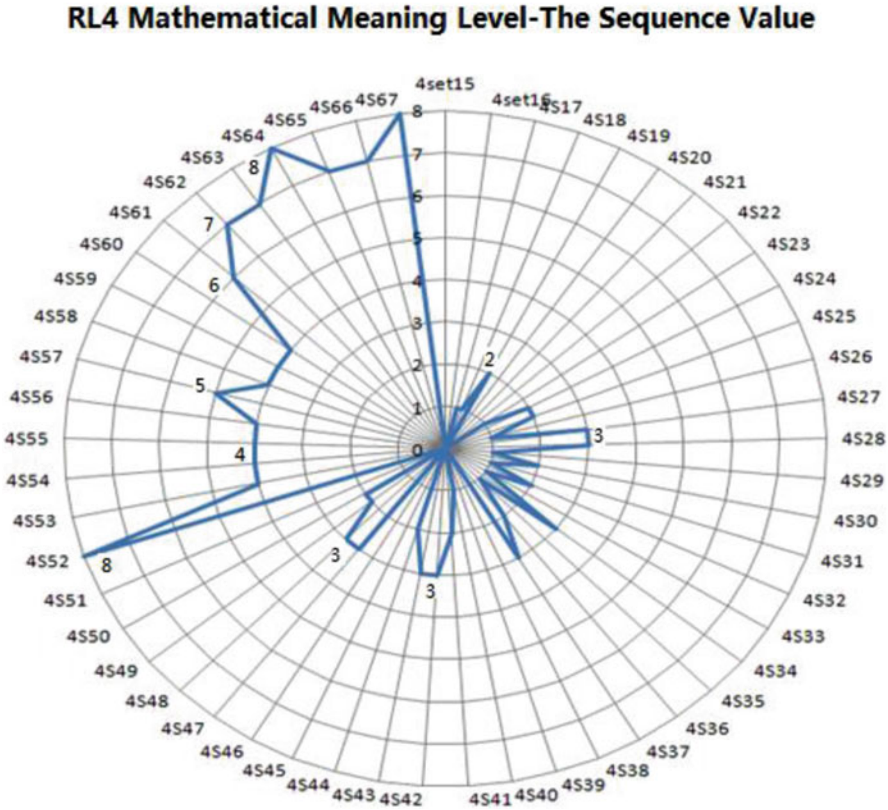

Fig. 6 Values of sequences in constructing the meaning of EF in research lesson 4 (RL4)

Learning from RL3, the lesson study team carefully designed RL4 to build up students' sharing. For example, students would share their grouping on the whiteboard immediately after the manipulation stage. During the manipulation, the teacher let the students know that they would be invited to demonstrate on the whiteboard, which triggered them to think about their manipulating process while doing the grouping. In sequences 4S38 to 4S52, three students were willing to demonstrate their sorting solutions on the whiteboard.

As depicted in Fig. 7, the first student's sorting strategy was incomplete because the second group consisted of equivalent fractions (3/6, 4/8, 5/10, and 6/12) and the first group consisted of nonequivalent fractions (1/2, 2/3, 3/4, 3/5). The second student succeeded in identifying the nonequivalent fractions (2/3 and 3/5) from the one-half (see Fig. 8). The third student offered a solution which was unanticipated by the teacher and even the lesson study team members and researchers. He grouped the one-half together and separated the two-thirds from the three-fifths (see Fig. 9) because the two are not equivalent. Even though the third grouping did not account for the teacher's earlier idea of identifying the nonequivalent fractions from the

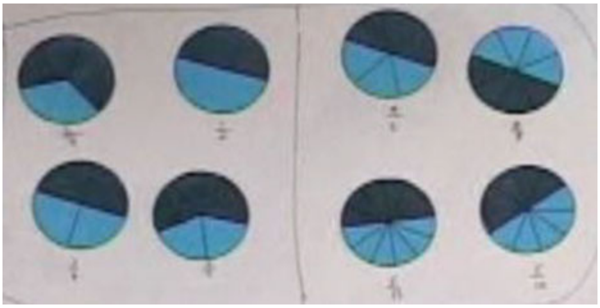

Fig. 7 The first student's demonstration

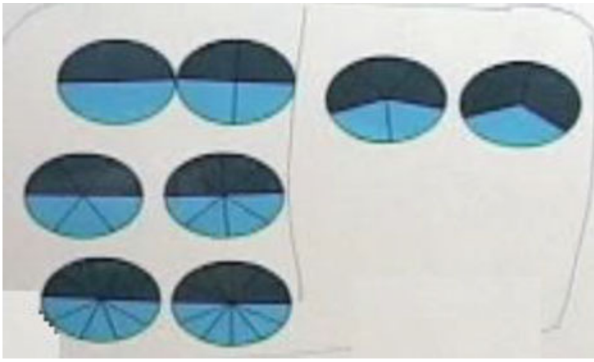

Fig. 8 The second student's demonstration

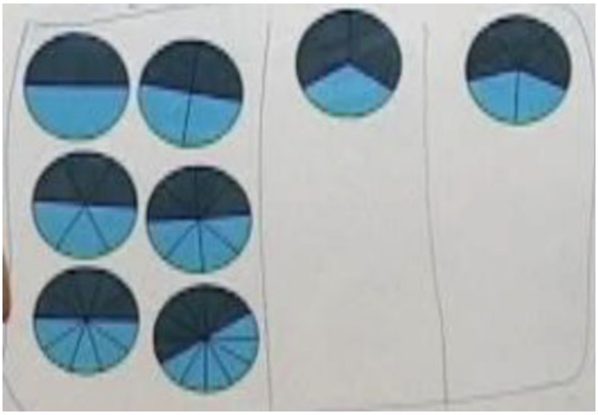

Fig. 9 The third student's demonstration

equivalent ones, the student hit the nail on the head. His grouping strategy pointed out what the teachers had overlooked – that two-thirds was not equivalent to three-fifths. This not only revealed a context in which the conceptual understanding of the concept of equivalence (4S52) occurred but also provided the teacher with an opportunity to learn from the student. In addition, the clearer board demonstrations built the fundamental platform for students' articulations (4S53 to 4S62).

The teacher then urged the whole class to discuss the three different sorting solutions and articulated their perspectives on the grouping. The visual support from the board demonstration reduced the difficulty and built students' confidence in articulating and interpreting the abstract meaning of each grouping leading them toward more conceptual understanding of EF. Overall, building on the students' contributions, the teacher facilitated the class discussion by opening up space to enhance students' conceptual understanding. Our analysis further demonstrated the role played by proper manipulation and timely articulation in making effective transition from concrete level to formal system of mathematics and thus contributing to conceptual understanding of equivalent fractions.

6 Discussion

Our effort in building a principled way to represent instructional improvements in lesson study has allowed us to not only answer our research questions but also see them in new light and ask new questions. What we highlighted in this chapter are representations of two major instructional improvements. First, by adapting Steinbring's (1998) epistemological triangle of mathematical concept to support the CPA model, we systematically coded the discourse of four research lessons at the "move" level: major improvements in revised lessons occurred in both cycles when the concrete manipulation, the forming of pictorial mental image, and the abstracting of mathematical meaning became more balanced. This means that the more balanced the instructional representation levels in building mathematical relationships, the more opportunities children would have to reflect on and construct the abstract meaning from their manipulations. This finding further reinforces that children should be provided with sufficient time and various representational tools to think about, struggle with, and invent equivalent fractions in various situations (Kamii and Clark 1995; Moss and Case 1999; Streefland 1991). To achieve the balance, teachers' asking relationship questions and providing relationship-oriented explanations to guide students along were again proven fundamental (Campbell and Rowan 1997).

Our use of Truxaw's (2005) inductive teaching model in guiding our systematic coding on the "sequence" level of the discourse uncovered the other major instructional improvement trend taking place wherever the discourse of the revised lessons escalated and maintained progression in meaning construction. Representing the instructional improvements in this way has allowed us to reason with and articulate the advance, stagnancy, or retrogression of meaning-making within each sequence and across sequences of the discourse and the ways in which the meaning was built.

Our resource-mediated social-cultural perspective has led our attention to the what and how of resource use which is aligned with "the tools and processes" that the improvement science also calls for research attention (Lewis 2015). Burrowing deep into the smallest grain size of meaningful discourse to search and map out how mathematical relationships and meanings were built revealed dynamic interactive movement and “the extent and causes of variation within a system” (p. 55) of instruction (Lewis 2015). Those that most intimately impacted on teaching and learning should be targeted for resource allocation to bring forth improvement and innovation (Cohen et al. 2003). The discourse analysis model we built in this chapter contributes to lesson study’s tool kit as a useful resource to guide evidence-based decision-making in planning, revising, and post-lesson reflection. In fact, lesson study practices themselves can be regarded as a set of established tools and processes aimed to build understanding of problems of curriculum, teaching, and learning and conceptions of innovation (Huang et al. 2017).

Most importantly, these improvements were made by teachers involved in the lesson study work through repeated teaching and reflection by engaging in deliberate practice to create variations in order to understand the meanings behind them. The opportunity to meticulously reexamine the discourse patterns in their deliberation across the four research lessons confirms that through making these improvements, the teachers involved eventually learned to grasp “the meaning and mode” of the mediational function of language with the CPA model “in relation to the goals of the activity they mediate” (Wells 1999, p. 138).

The process of building this analytical model has once again impressed us on how much novice teachers learned to teach effectively and expediently with the support of a lesson study community (Goh and Fang 2017). This capacity to improve not only the practice but also all those involved in the process of making the learning happen resides in the fact that making mistakes is legitimatized as natural and fundamental to making improvements collectively, which is common in improvement science in the industries (Lewis 2015). The stumbling in RL1 and RL3, although regretful, motivated the lesson study teams to improve and strengthen their commitment to students’ learning when they felt empowered by the possibility to make change through team effort (Lewis et al. 2006), for example, when they learned how to use CPA model by doing it and turning it from a textbook version into reality (Leong et al. 2015). Against a strong kiasu culture (afraid of losing), lesson study holds real promise to build tolerance for mistakes and errors in enabling teacher development and learning to meet a fast-paced change environment (Goh and Fang 2017).

7 Conclusion

Our lesson study work with Singapore teachers was an effort to explore an alternative to the traditional teacher-dominated direct teaching, particularly with teaching elementary mathematics, which is still found in most East Asian classrooms (Hogan et al. 2012). In fact, facing quiet but generally attentive students, it is quite challenging for teachers in these fast-paced classrooms to find and provide enough space for students to experience the manipulative process and construct mathematical meaning which is essential for achieving the transitional stage in the CPA model.

Nevertheless, East Asian students' strong mathematical performance in international studies was often attributed to "extended discourse" (Schleppenbach et al. 2007, p. 381) and teachers' better understanding of the subject knowledge (Ma 1999) in their mathematics classrooms. This might be owning to teachers' appropriate use of language in facilitating mathematics learning and its "power to help children organize and link their partial understandings as they integrate and develop mathematical concepts" (Campbell and Rowan 1997, p. 64). When we geared toward students' conceptual understanding, critical thinking, and creativity, however, we found such classroom discourse and interactive patterns quite inadequate (Hogan et al. 2012; Paine and Fang 2006). In our lesson study work, we constantly find the need to adapt teaching for understanding to better fit into East Asian pedagogy. For instance, by strategically combining manipulation and articulation as key components of the learning activities with careful teacher questioning and explaining in direct teaching tradition, both teachers and students would be more comfortable in experiencing mathematical thinking through lesson study. Therefore, the demonstrated pedagogy of research lessons accommodated perspectives of both the East and the West as a hybrid model (Paine and Fang 2006).

References

Ball, D. (1993). Halves, pieces and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49–84). Hillsdale: Lawrence Erlbaum Associates.

Bruner, J. (1966). *Toward a theory of instruction*. London: Harvard University Press.

Cai, J. (2007). What is effective mathematics teaching? A study of teachers from Australia, Mainland China, Hong Kong SAR, and the United States. *ZDM, 39*(4), 265–270.

Cai, J., Kaiser, G., Perry, G., & Wong, N. Y. (2009). *Effective mathematics teaching from teachers' perspectives*. Rotterdam: Sense Publishers.

Campbell, P. F., & Rowan, T. E. (1997). Teacher questions + student language + diversity =mathematical power. In M. J. Kenney (Ed.), *Multicultural and gender equity in the mathematics classroom: The gift of diversity* (pp. 60–70). Reston: National Council of Teachers of Mathematics.

Cazden, C. (2001). *Classroom discourse: The language of teaching and learning*. Portsmouth: Heinemann.

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 283–292.

C. Chen, & Z. Zhou (Eds.) (2005). Mathematics for 4th grade, textbooks for nine-year compulsory education. Shanghai: Shanghai Education Press.

Clarke, D. J., Keitel, C., & Shimizu, Y. (2006). *Mathematics classrooms in twelve countries: The insider's perspective*. Rotterdam: Sense Publishers.

Cobb, P. (1995). Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education, 26*(4), 362–385.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resource, instruction, and research. *Education Policy Analysis Archives, 25*(2), 119–142.

Cole, M., John-Steiner, V., Scribner, S., & Souberman, E. (Eds.). (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Corte, E. D., Greer, B., & Verschaffel, L. (1996). Mathematics teaching and learning. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 491–549). New York: Macmillan.

Dossey, J. A., Mullis, I. V. S., Lindquist, M. M., & Chambers, D. L. (1988). *The mathematics report card: Are we measuring up? Trends and achievement based on the 1986 National Assessment*. Princeton: Educational Testing Service.

Dudley, P. (2011). Lesson study development in England: From school networks to national policy. *International Journal for Lesson and Learning Studies, 1*(1), 85–100.

Fang, Y., Lee, C. K. E., & Haron, S. T. S. (2008). Lesson study in mathematics: Three cases for Singapore. In K. Y. Wong, P. Y. Lee, B. Kaur, P. Y. Foong, & S. F. Ng (Eds.), *Mathematics education: The Singapore journey* (pp. 104–129). Singapore: World Scientific.

Fang, Y. P., Lee, K. E. C., & Haron, S. T. (2009). Lesson study in mathematics: Three cases from Singapore. In K. Y. Wong, P. Y. Lee, B. Kaur, P. Y. Foong, & S. F. Ng (Eds.), *Mathematics education - the Singapore journey* (pp. 104–129). Singapore: World Scientific.

Fang, Y. P., Lee, K. E. C., & Yang, Y. (2011). Developing video cases from research lessons as curriculum and pedagogical support for teacher learning – A case of long division. *International Journal of Lesson and Learning Studies, 1*(1), 65–84.

Fernandez, C. (2002). Learning from Japanese approaches to professional development. *Journal of Teacher Education, 53*, 393–405.

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A U.S.–Japan lesson study collaborative reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*, 171–185.

Fong, H., Ramakrishnan, C., & Choo, M. (2003). *My pals are here!* Singapore: Federal Publications.

Frobisher, L., Monaghan, J., Orton, A., Orton, J., Roper, T., & Threlfall, J. (1999). *Learning to teach number*. Cheltenham: Stanley Thornes.

Goh, R., & Fang, Y. (2017). Improving English language teaching through lesson study: Case study of teacher learning in a Singapore primary school grade level team. *International Journal for Lesson and Learning Studies, 6*(2), 135–150.

Hairon, S., & Dimmock, C. (2012). Singapore schools and professional learning communities: Teacher professional development and school leadership in an Asian hierarchical system. *Educational Review, 64*(4), 405–424.

Hironaka, H., & Sugiyama, Y. (Eds.) (2006). *Mathematics 4A for elementary school.* Translated into English by Yoshida et al. Tokyo Shoseki, Co., Ltd.

Hogan, D., & Gopinathan, S. (2008). Knowledge management, sustainable innovation, and pre-service teacher education in Singapore. *Teachers and Teaching, 14*(4), 369–384.

Hogan, D., Chan, M., Rahim, R., Towndrow, P., & Kwek, D. (2012). Understanding classroom talk in secondary 3 mathematics classes in Singapore. In B. Kaur (Ed.), *Connections, reasoning and communication: New directions in mathematics education*. Singapore: Springer.

Huang, R., Fang, Y., & Chen, X. (2017). Chinese lesson study: A deliberate practice, a research methodology, and an improvement science. *International Journal of Lesson and Learning Studies, 6*(4), 270–282.

Hucker, J. (1994). *Creating paths to mathematical literacy: A.S.B./A.P.P.A.travelling fellowship 1994 report*. Auckland: Author.

Hunting, R. (1984). Understanding equivalent fractions. *Journal of Science and Mathematics Educations in S. E. Asia, 7*(1), 26–33.

Jigyel, K., & Afamasaga-Fuata'I, K. (2007). Students' conceptions of models of fractions and equivalence. *Australian Mathematics Teacher, 63*(4), 17–25.

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior, 14*, 365–378.

Kervin, K. (2007). Exploring the use of slow motion animation (Slowmation) as a teaching strategy to develop year 4 students' understandings of equivalent. Fractions. *Contemporary Issues in Technology and Teacher Education, 7*(2), 100–106.

Koh, K. H. (2011). Improving teachers' assessment literacy through professional development. *Teaching Education, 22*(3), 255–276.

Kunter, M., Klusmann, U., Baumert, J., Richter, D., Voss, T., & Hachfeld, A. (2013). Professional competence of teachers: Effects on instructional quality and student development. *Journal of Educational Psychology, 105*(3), 805–820.

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge: Cambridge University Press.

Leon, J., Medina-Garrido, E., & Núñez, J. L. (2017). Teaching quality in math class: The development of a scale and the analysis of its relationship with engagement and achievement. *Frontiers in Psychology, 8*, 1–14.

Leong, Y., Ho, W., & Cheng, L. (2015). Concrete-pictorial-abstract: Surveying its origins and charting its future. *The Mathematics Educator, 16*(1), 1–19.

Lewis, C. (2015). What is improvement sciences? Do we need it in education? *Educational Researcher, 44*(1), 54–61.

Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: Research lessons and the improvement of Japanese education. *American Educator (Winter), 14–17*, 50–52.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Little, J. W. (1999). Organizing schools for teacher learning. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 233–262). San Francisco: Jossey-Bass.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Lawrence Erlbaum.

Ministry of Education. (2007). *Primary mathematics syllabus*. Singapore: Author.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122–147.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584–635.

Nassaji, H., & Wells, G. (2000). What's the use of 'triadic dialogue'?: An investigation of teacher-student interaction. *Applied Linguistics, 21*, 376–406.

NEA Foundation for the Improvement of Education. (2003). *Using data about classroom practice and student work to improve professional development for educators*. Washington, DC: Author.

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology, 26*, 400–417.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht: Kluwer.

Paine, L. W. (1990). The teacher as virtuoso: A Chinese model for teaching. *Teachers College Record, 92*(1), 49–81.

Paine, L. W., & Fang, Y. P. (2006). Reform as hybrid model of teaching and teacher development in China. *International Journal for Education Research, 45*(4–5), 279–289.

Perry, R. R., & Lewis, C. C. (2009). What is successful adaptation of lesson study in the U.S.? *Journal of Educational Change, 10*(4), 365–391.

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and U.S. mathematics classrooms. *Elementary School Journal, 108*, 131–147.

Schoenfeld, A. H. (2002). A highly interactive discourse structure. *Social Constructivist Teaching, 9*, 131–169.

Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1*(2), 157–189.

Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction: An epistemological perspective*. New York: Springer.

Steinbring, H. (2006). What makes a sign a mathematical sign?–An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics, 61*(1–2), 133–162.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap; best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht: Kluwer.
Truxaw, M. P. (2005). *Orchestrating whole group discourse to mediate mathematical meaning.* DigitalCommons@UConn.
Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. New York: Cambridge University Press.
Wells, G. (2002). The role of dialogue in activity theory. *Mind, Culture, and Activity, 9*(1), 43–66.
Westenskow, A., & Moyer-Packenham, P. (2016). Using an iceberg intervention model to understand equivalent fraction learning when students with mathematical learning difficulties using different manipulatives. *International Journal for Technology in Mathematics Education, 23*(2), 45–62.
White, A. L., & Lim, C. S. (2008). Lesson study in Asia Pacific classrooms: Local responses to a global movement. *ZDM – The International Journal on Mathematics Education, 40*(6), 915–925.
Wong, M., & Evans, D. (2007). Students' conceptual understanding of equivalent fractions. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Group of Australasia) (pp. 824–833). Adelaide: MERGA.
Yang, Y., & Ricks, T. E. (2011). How crucial incidents analysis support Chinese lesson study. *International Journal for Lesson and Learning Studies, 1*(1), 41–48.

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What Knowledge Do Teachers Use in Lesson Study? A Focus on Mathematical Knowledge for Teaching and Levels of Teacher Activity

Stéphane Clivaz and Aoibhinn Ni Shuilleabhain

Contents

Abstract This chapter combines the frameworks of Ball, Thames, and Phelps (J Teach Educ 59:389–407, 2008) and Margolinas, Coulange, and Bessot (Educ Stud Math 59:205–234, 2005) to demonstrate the elements of subject and pedagogical content knowledge utilized at varying levels of teacher activity in a cycle of lesson study. Qualitative data generated in a mathematics-based lesson study, conducted with eight primary school teachers in Switzerland, is analyzed and visualizations of the knowledge occurring at each phase of lesson study are provided. This fine-grained analysis of the mathematical knowledge incorporated by teachers in lesson study demonstrates that all forms of Mathematical Knowledge for Teaching, at each level of teacher activity, can occur across a cycle. In addition, the paper provides evidence that phases of lesson study do not necessarily occur in succession but can rather take place in a confluence of teachers' work across a full cycle.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_20

Keywords Lesson study · Professional development · Mathematical Knowledge for Teaching · Level of teacher activity

1 Introduction

Since the introduction of lesson study in Stigler and Hiebert's seminal book *The Teaching Gap* (1999), it has grown in popularity around the world as a model of professional learning for mathematics teachers. Originating in Japan and China and practiced as part of educational cultural routines in those countries (Stigler and Hiebert 2016), teachers from Shanghai to Switzerland and from Detroit to Dublin now participate in this practice-based and research-oriented form of professional development (Huang and Shimizu 2016). However, while lesson study may be "like the air" (Fujii 2018, p. 1) in how familiar and fundamental it is in some countries, it is difficult to transport such cultural routines to new educational systems where the core points and central processes may be unrecognized or misunderstood (Fujii 2014; Stigler and Hiebert 2016; Takahashi and McDougal 2016). Therefore, with the increased popularity of lesson study around the world, there have been calls to deepen our understandings of how it contributes to teacher learning (Lewis et al. 2009; Widjaja et al. 2017). While studies have demonstrated the potential of lesson study to develop teacher community and enhance teacher knowledge (e.g., Lewis and Perry 2017; Ni Shuilleabhain 2016; Warwick et al. 2016), research has also highlighted the need to explore the theoretical underpinnings of mathematics teacher learning in lesson study (Clivaz 2015b; Miyakawa and Winsløw 2009; Xu and Pedder 2015).

In this chapter, in the context of further understanding teacher learning in lesson study, we investigate the types of knowledge employed by teachers in lesson study. Based on a case study conducted with eight primary school teachers in Switzerland (Ni Shuilleabhain and Clivaz 2017), we detail their participation across one cycle of lesson study utilizing a combination of the cognitive and situated theoretical frameworks of Ball et al. (2008) and Margolinas et al. (2005). This fine-grained analysis demonstrates the constituents of both subject and pedagogical content knowledge employed by teachers, at varying levels of pedagogical activity, for each phase of the lesson study cycle. In this case study, teachers' pedagogical content knowledge, particularly related to their consideration of content, was the most utilized form of knowledge incorporated in their lesson study work. Teachers' values and considerations about teaching and learning were also apparent throughout their planning and reflection of the research lesson. These findings provide a detailed representation of the types of knowledge employed by teachers across the phases of lesson study and contribute to our understanding of how and what teachers may learn in their participation in lesson study. In addition, our analysis demonstrates that each phase of lesson study need not necessarily take place in succession but rather occur in a confluence of teachers' conversations over one full cycle.

2 Theoretical Framework

There are a multitude of theoretical frameworks which particularize the knowledge and practices required to teach mathematics. While there is agreement within the research literature that both content knowledge and pedagogical content knowledge (PCK) are requirements in the teaching of mathematics (Hill 2010; Krauss et al. 2008; Rowland et al. 2005; Schoenfeld 2010; Speer et al. 2015), there is, at present, a separation between the cognitive and situated models of teacher knowledge (Rowland et al. 2011). Furthermore, differences exist between traditions in their considerations of the subject-related knowledge required to teach mathematics and the ways through which such knowledge can be developed in Anglo-American, continental European, and East Asian research (Depaepe et al. 2013).

A number of studies have explored mathematics teacher learning in lesson study and investigated the types of knowledge employed by teachers in their lesson study work. Tepylo and Moss (2011) detail the development of teachers' content and pedagogical content knowledge over a number of cycles of lesson study based on fractions. Research by Suh and Seshaiyer demonstrates the development of teachers' understanding of their students' mathematical learning progression through their participation in multiple cycles of lesson study (Suh and Seshaiyer 2015). Murata and colleagues' research shows how teachers developed new mathematical knowledge for teaching based on their lesson study conversations connecting student thinking and mathematical content (Murata et al. 2012). Furthermore, Lewis et al. (2009) demonstrate the development of teachers' content knowledge through their participation in the planning phases of lesson study, and similar findings are confirmed by Lewis and Perry (2017) who demonstrate the development of teachers' content knowledge when participating in lesson study using specially designed mathematics resource kits. Additional research by Dudley (2013) and Ni Shuilleabhain (2016) demonstrate the development of teachers' pedagogical content knowledge due to their participation in lesson study. However, in order to further understand the relationship between teacher knowledge and student learning in lesson study, more fine-grained analysis will be required to capture the complex and active mix of knowledge used by mathematics teachers in their practices (Davis and Renert 2013).

We therefore conduct a particularized analysis of the knowledge incorporated by mathematics teachers during their participation in lesson study, utilizing a combination of frameworks of teachers' mathematical knowledge (Ni Shuilleabhain and Clivaz 2017). The theoretical frameworks of Mathematical Knowledge for Teaching (Ball et al. 2008) and levels of teacher activity (Margolinas et al. 2005) combine differing perspectives and traditions of teacher knowledge and allow for a more complete perspective on the type of knowledge included by teachers in cycles of lesson study. We refer to "combining" these frameworks in the sense of Prediger, Bikner-Ahsbahs, and Arzarello (2008). Clivaz (2015a) previously studied the condition and benefit of combining these two frameworks, an action which does not require "the complementarity or even the complete coherence of the theoretical approaches in view," since "theories with conflicting basic assumptions can be combined" (Prediger et al. 2008, p. 173). In a similar vein, we undertake this combination in order to gain further, multifaceted insight into the knowledge incorporated by mathematics teachers in their participation in lesson study.

2.1 Mathematical Knowledge for Teaching

Ball and her colleagues (2008) developed a practice-based theory of the knowledge required "to carry out the work of teaching mathematics," presented as a framework of Mathematical Knowledge for Teaching (MKT) (p. 395). This model built on Shulman's theoretical suggestion of PCK as a specific type of knowledge unique to teachers and distinguished it from subject matter or content knowledge (1986, 1987). In their model, Ball and her colleagues highlighted particular categories of knowledge within the subject matter delineations and PCK delineations:
Subject Matter Knowledge

– Common Content Knowledge (CCK)
– Horizon Content Knowledge (HCK)
– Specialized Content Knowledge (SCK)

Pedagogical Content Knowledge

– Knowledge of Content and Teaching (KCT)
– Knowledge of Content and Students (KCS)
– Knowledge of Content and Curriculum (KCC)

In their review of the conceptualization and evidence of PCK in mathematics education research, Depaepe et al. (2013) noted the MKT model as "probably the most influential re-conceptualizations of teacher PCK within mathematics education" (p. 13). However, Steinbring (1998) and Margolinas (2004) suggest that in Shulman's proposed framework of teacher knowledge (1986), on which the MKT framework is modelled, fixed categories of teacher knowledge are "not a good model for teacher's activity, which is more complicated" (Margolinas et al. 2005, p. 207). Indeed, Ball and her collaborators acknowledge that these categorizations of teacher knowledge can be interpreted as static and distinct (2008) and therefore difficult to incorporate within the active practices of teaching. Others have critiqued such consideration of teachers' knowledge for teaching as solely applied within a context of instruction, without consideration of the complex, social environment of a mathematics classroom (Hodgen 2011; Putnam and Borko 2000).

Davis and Renert (2013) suggest that understanding the relationship between teacher knowledge and student learning "will require more fine-grained analyses than large-scale assessments" (p. 264) in order to fully capture the amalgam of knowledge utilized by mathematics teachers. The theory of didactical situations (Brousseau 1997) can provide researchers with a tool to conduct such qualitative, fine-grained, and mobile analyses (Winsløw et al. 2018) and is described further below.

2.2 Levels of Teacher Activity

In the 1970s, Brousseau's theory of didactical situations (1997) first modelled a learning situation by focusing on student learning without explicitly incorporating the role of the teacher (Bloch 2005). However, in analyzing student learning from the 1990s, the importance of the teacher's role became increasingly evident in the theorization of classroom situations (Bloch 1999; Dorier 2012; Roditi 2011). This new lens provided opportunity to introduce a situated theory embedded in the context of the classroom, through which the various levels of practices, skills, and knowledge required of mathematics teachers could be analyzed.

The concept of *milieu*[1] is central to Brousseau's theory of didactical situations. The *milieu* is defined by "all of the pertinent features of the student's surroundings, including the space, the teacher, the materials and the presence or absence of other students" (Warfield 2014, p. 66). Based on this theory of didactical situations, Margolinas (2002) developed a model of a mathematics teacher's milieu to describe the teacher's activities, both in and outside of the classroom. This model was designed to acknowledge the complexity of teachers' actions, while also capturing the broad range of activities contained in teaching and learning (Margolinas et al. 2005). Centering on the action of the classroom, the model depicts the various levels at which a teacher must situate themselves within their pedagogical practices. In this model (Fig. 1) level + 3 refers to teachers' values and conceptions about learning and teaching, level + 2 concerns teachers' actions and discourses about the global didactic project, level + 1 pertains to the local didactic project, level 0 is the didactic action, and level − 1 deals with the observation of pupils' activity. The teacher's point of view can be related to his or her considerations and reflections at different levels of generality. Observing students' work (including noticing student talk) relates to a more deliberate focus of the teacher on individual students and, hence, relates to level − 1. Planning the local didactic project (about the lesson) refers to the preparation and sequencing of content within the lesson and, hence, is placed at level + 1. At level + 2 the teacher considers the didactic project in a more holistic or global sense (e.g., teaching a particular element of a topic as one lesson in a series of lessons or throughout a term). While at level + 3 the teacher draws upon their beliefs about the teaching and learning of mathematics, which can be related to how the didactic projects may be constructed and to how the teacher will engage with individual students. The model is not intended as a linear interpretation of teachers' work but rather identifies the multidimensional tensions involved in teaching (Margolinas et al. 2005). At every level the teacher not only has to deal with the current, most prescient, level of activity but also with the levels directly before and after and, in some instances, with levels extending beyond.

As a situated model of teacher knowledge, based within the context of teaching and learning, Margolinas et al. (2005) proposed this model as a way of delineating

[1]Milieu is the usual translation for Brousseau's French term "milieu." However, in French it refers not only to the sociological milieu, but it is also used in biology or in reference to Piaget's work. A more accurate translation would be "environment."

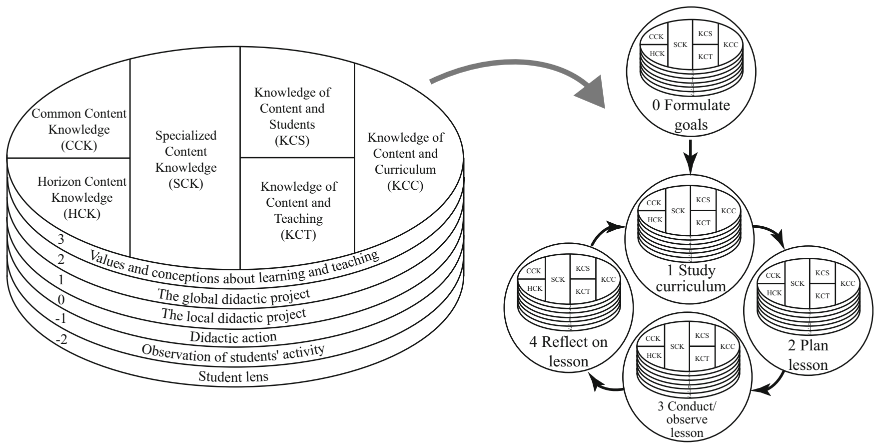

Fig. 1 MKT and levels of teacher activity in a cycle of lesson study

the multilayered knowledge required of teachers during varied stages of teaching – from the overarching pedagogical values underpinning a lesson to the interactions between teachers and students in the classroom. However, while this model incorporates teacher's beliefs and acknowledges the pedagogical skills required to notice and interpret student thinking, it does not make explicit how a teacher's specific content or pedagogical content knowledge may be encompassed in such activities.

2.3 A Proposed Theoretical Framework

Domains of MKT (Ball et al. 2008) have been shown to be utilized and developed through teachers' participation in lesson study, at both in-service and pre-service levels (Leavy and Hourigan 2016; Ni Shuilleabhain 2016; Tepylo and Moss 2011). However, considering the many forms and levels of knowledge and practices incorporated within each phase of lesson study – studying the curriculum; planning; conducting or observing; and reflecting on a mathematics lesson – the MKT framework does not wholly capture the incorporation of teachers' beliefs nor the considerations involved in structuring content for a research lesson. Ni Shuilleabhain (2015) utilized the MKT framework to investigate mathematics teacher learning in lesson study over a number of iterative cycles in one academic year. In an attempt to capture the knowledge incorporated by teachers in their planning and reflection conversations, she combined this with the idea of a critical lens relevant to student thinking (as suggested by Fernandez et al. 2003) and suggested this form of knowledge an additional layer of analysis of teacher learning in lesson study. This concept of a "student lens" relates to the PCK a teacher utilizes in seeing mathematics "through the eyes of their students" (Fernandez et al. 2003, p. 179) but is separate to an action of the teacher noticing students' mathematical work in teaching

(Jacobs et al. 2010). In our proposed theoretical framework, this “student lens” is added to Margolinas et al.’s (2005) model as a level − 2. This layered model relates to work by Clivaz (2014, 2017), who used the situated activity model (Margolinas et al. 2005) to observe teacher classroom practice and, in an effort to detail both the mathematical knowledge for and in teaching, aligned it with the cognitive framework of MKT.

In our research (Ni Shuilleabhain and Clivaz 2017), we proposed a combination (Prediger et al. 2008) of these two existing frameworks of Mathematical Knowledge for Teaching (Ball et al. 2008) and levels of teacher activity (Margolinas et al. 2005) to analyze the knowledge incorporated by teachers in two case studies. The graphical representation of this framework (Fig. 1) shows that the categorization of knowledge lies in one plane (“the egg”), while the levels of activity are characterized in a contrasting cross-sectional plane (“the cake”). In this chapter, we develop this work and employ the framework as a tool to further detail and analyze mathematics teachers’ knowledge in various phases of planning, conducting or observing, and reflecting on teaching in one case study cycle of lesson study (see Fig. 1).

3 Context and Methodology

Eight generalist grade 3–4 primary school teachers from the Lausanne region, a French-speaking part of Switzerland, were introduced to lesson study and conducted four lesson study cycles over two school years. The group was facilitated by two university teacher educators, one specialist in teaching and learning and one specialist in mathematics didactics (first author of this chapter). All meetings (37 of an average of 90 min duration) and research lessons (8) were transcribed and coded in a qualitative analysis software (NVivo). Student work, teachers observations during lessons, and lesson plans were also recorded and coded (Clivaz 2016).

In this chapter, our case study refers to the first lesson study cycle (Lewis et al. 2006) of the group, where teachers decided to focus on the topic of integer number and place value. Within this cycle of lesson study, teachers met on nine occasions with the research lesson being taught, discussed, redesigned, and taught in the redesigned form. Generated data, segmented into conversational utterances or section of note-taking, was coded according to its classification within each of the following categories: first with the form of MKT attributed to the speaker, then with the lesson study phase, and finally with the level of teacher activity. A detailed coding structure was developed, with indicators formed and revised through iterative phases of analysis (Ni Shuilleabhain and Clivaz 2017) to attribute the codes and sub-codes (see full coding map in Appendix). The proportion of data coded with MKT forms and then with phases and levels, within each lesson study meeting and research lesson, varied from approximately 10% to 65%. All teachers’ and students’ names referred to throughout the chapter are pseudonyms. The lesson study facilitators, Anne and Stéphane (first author of this chapter), retain their real names.

4 Analysis and Findings

In this reporting of our analysis, we detail the different forms of Mathematical Knowledge for Teaching (Ball et al. 2008) and levels of teacher activity (Margolinas et al. 2005) that occur over each phase of the lesson study cycle. Through the use of quotes and graphical data we explicate teachers' knowledge recorded in their participation in lesson study. While the data was originally generated and coded in French, translations included here were translated collaboratively by the two authors of this chapter, with an explicit effort made to keep the nuance and color of the spoken language as pertinent to teacher dialogue.

4.1 Levels of Teacher Activity Incorporated in One Cycle

In an attempt to identify the levels of activity at which teachers operated over the course of the lesson study cycle, we first focus specifically on the levels of teacher activity (Margolinas et al. 2005) accounted for in our case study. As might be anticipated, phase 2 (planning the research lesson) was mostly attributed to teachers' activities at level + 1, focusing on the local didactic project within the research lesson (see Fig. 2). Similarly, phase 3 (conducting and observing the research lesson) was mostly related to level 0 of teacher activity, i.e. the didactic action within the classroom.

However, as the following extract shows, during the initial phases of 0 and 1, where teachers formulate goals and study the curriculum, teachers articulated their values and conceptions about teaching and learning (level + 3) while also considering the global didactic project relative to particular class group of students (level + 2):

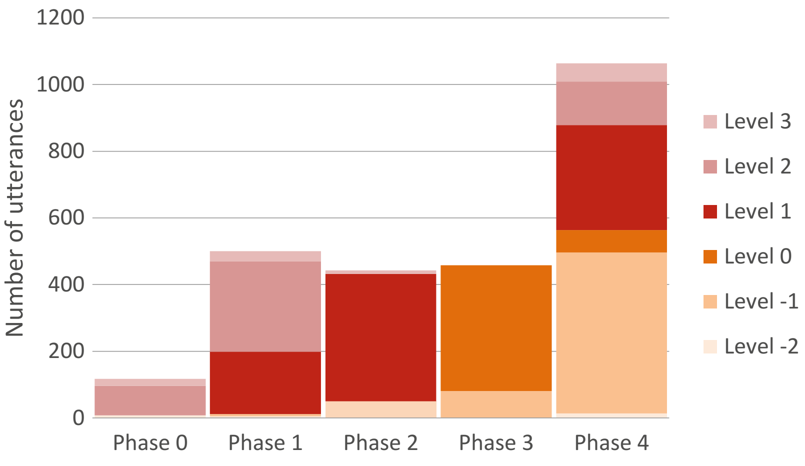

Fig. 2 Levels of teacher activity incorporated in one cycle

Edith (T)	Our habits of doing things. Even having discussions with other people and hearing: "Bah, I don't like that!" or "I don't do it that way"… It makes me think: "Oh, I didn't think about that!" […] Because, to be honest, some stuff, you know, frankly…
Stéphane (F)	Of course.
Edith (T)	I mean, these cubes. I don't like these manipulatives at all. To be honest, we just lose those little cubes!
Stéphane (F)	Yeah.
Edith (T)	And oh my God, that drives me crazy. I mean it. It drives me crazy. I don't use them, I'll do it another way. But, then… Sometimes it's good to get a kick in the ass and say: "Well, give it a go." Since, yeah, really, we are sometimes a little bit selfish! It annoys me, but if it is for the good of my students, well yeah, let's see. Maybe it works and maybe it doesn't!
Stéphane (F)	Hmm hmm.
Edith (T)	Exactly! It is true. In one's teaching we are a little bit selfish. I mean, we cut our cloth to how it suits us. Because there are so many things, we make choices that are not… I mean, we suit ourselves.
Stéphane (F)	That's important too!
Edith (T)	So, here we are! I'll try it in another way and see.

Teachers' values and conceptions impact their approaches to teaching and learning (Ni Shuilleabhain and Seery 2017) and, in this context, teachers had the opportunity to make their implicit practices explicit in their planning of the research lesson (Fujii 2018). In this case, Edith thought that manipulatives were too difficult to use but she decided to use them because they were helpful for her students' learning, thereby explicitly linking her values (level + 3) and her global didactic project.

As demonstrated in Fig. 2, teachers also had the opportunity to see the mathematics from the perspectives of the students in their planning of the lesson. As part of phase 2 of lesson study, teachers are encouraged to complete the tasks which might be incorporated within the research lesson. In our case study teachers played a game which they intended as a key learning activity within the lesson and, in doing so, took part in the activity as if they themselves were the students.

Caroline (T)	For me, when I see that, my first reaction is: I can't!
Anne (F)	But you have not...
Caroline (T)	So, I did the exchange.
Anne (F)	And then?
Caroline (T)	Yes, but it's because I saw the ten, that's why. But if I see a ten, do I always think to make an exchange to ten units?
Anne (F)	But that's what you just did. After that, why didn't you do it?
Caroline (T)	I did it then. But after that, I don't know.
Anne (F)	What should you do, ideally?

Caroline (T) I think. . .
Valentine (T) We need to start with the hundred.

This teacher activity, utilizing a student lens at level − 2, provided teachers with further insight into student thinking and afforded them greater insight in deciding how to conduct the game within the research lesson.

In addition to seeing the mathematics through the eyes of the student, our analysis also reveals a blend of levels of activity over the phases of the lesson study cycle. Teachers' participation in phases 0 to 2 focused largely on the local and global didactic projects (levels +2 and + 1), while the work of phases 3 and 4 was mostly concerned with the action within the research lesson and focusing on students' thinking (levels 0 and − 1). Teachers' values and conceptions of teaching and learning of mathematics (level + 3) played a part in both their planning and reflection conversations (C. Lewis et al. 2009), as did teachers' actions in seeing the mathematics through the eyes of the student (level − 2). It is worthy to note that the post-lesson discussions included all levels of teacher activity (see Fig. 2). Such a finding demonstrates the significance of the reflection phase of lesson study, where teachers are provided with opportunity to articulate their pedagogical practices at all levels.

4.2 MKT Expressed in One Cycle of Lesson Study

Analyzing teachers' participation in lesson study according to the MKT framework (Ball et al. 2008), all categories of MKT were found to be expressed across each of the phases of the lesson study cycle (see Fig. 3). It is notable that all forms of MKT were incorporated in phase 1, studying the curriculum.

In phase 2, planning the research lesson, teachers regularly drew upon their KCT in designing the mathematical instruction. In this collaborative work, teachers were obliged to deliberate and agree on the aims of the learning activity. Teachers

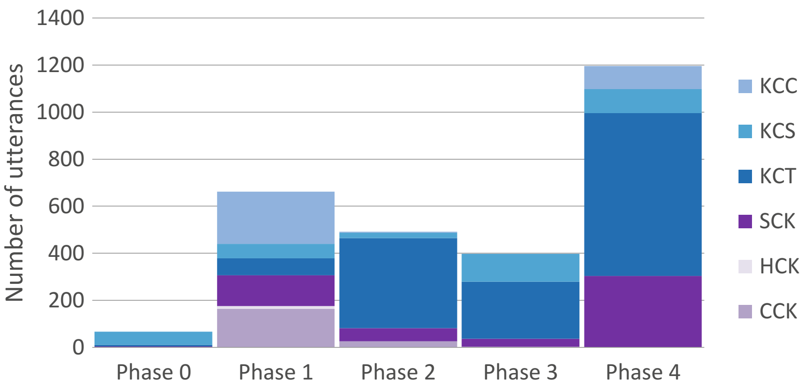

Fig. 3 MKT expressed in one cycle of lesson study

considered whether the goal of the lesson should be for students to discover the idea of exchange or simply practice the exchange of hundreds, tens, and units.

Vanessa (T) We thought about that, hmm. In fact, we are reflecting on the objectives of the task. Does the learning also lie in being familiar with the idea of exchange? I think so, but I don't know. . .

Edith (T) Then, either we decide we don't want them [the students] to come to it by themselves and the goal of the lesson is really that they practice doing the exchanges. Because, if that's the case, we can explain the concept and after that it's OK!

Valentine (T) Yeah.

Edith (T) Or we decide that the goal is really to cause them to think, so that they find this solution of exchange and, in that case, we must really define what our lesson goal is.

These discussions provided opportunity for teachers to select appropriate models, representations, and examples which would support students' mathematical understanding (indicator for KCT) of exchanging units, tens, and hundreds.

Phase 3 included occurrences of KCT, KCS, and SCK as relevant to the work of the conducting teacher and to the notes recorded by observers during the research lesson. For example, in Fig. 4, the observer noted a student's suggestion to change one one-hundred unit into nine tens and ten units. This observation note demonstrates the teacher's noticing of student thinking (level − 1) and the incorporation of the teacher's interpretation of the mathematical meaning associated with student's responses (indicator for KCS).

In our analysis the occurrence of KCT in teachers' conversations was predominant across all of the phases, supporting the suggestion that participating in lesson study can develop teachers' PCK (Ni Shuilleabhain 2016; Tepylo and Moss 2011). As an example of a less frequently occurring knowledge from the data, in their study of the curriculum during phase 1 teachers had the opportunity to utilize their CCK (indicator: performing mathematical task) while testing various tasks to potentially include within the research lesson. Teachers undertook a task named "Hit" from the

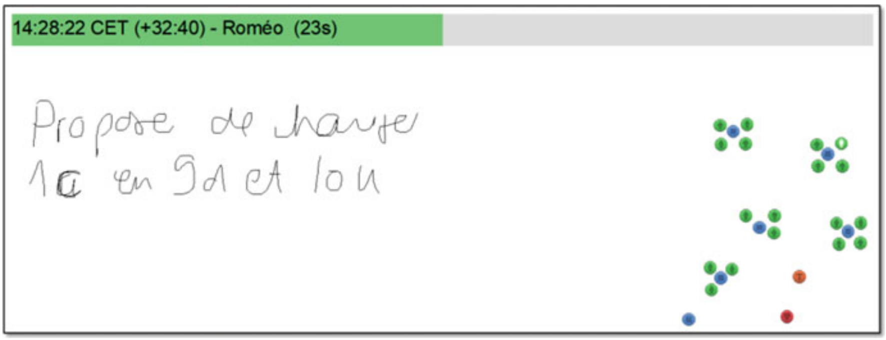

Fig. 4 Teacher's observation, recorded on Lesson Note (Lesson Study Alliance 2012)

textbook, which had the objective of computing 387 on a calculator utilizing only 0 and 3. Drawing on their mathematical knowledge for common calculations, teachers made decisions on the appropriateness of the task for students.

Marius (T)	"Hit," on page 97.
Océane (T)	"Hit"? Oh yeah, it's with the calculator. [. . .] They just have to do addition or. . .?
Marius (T)	They must do those. . .
Océane (T)	Okay yeah, agreed. Three and zero. Yeah, then. . .If they can do thirty times thirty. . . Thirty times three. . .
Marius (T)	It would make nine hundred divided into...
Océane (T)	No, three times. . .it's thirty and then it makes ninety.
Marius (T)	Plus thirty times three.
Océane (T)	Yeah, yeah [. . .]
Marius (T)	Subtract three. They have to see that already and see how we get to three hundred and how we get to ninety.

While the group thought the game could be useful, they decided it was not the most appropriate task for students and did not include it within the research lesson.

Participating in the lesson study cycle required teachers to draw on their various forms of MKT in their discussions and decisions around planning, conducting or observing, and reflecting on the research lesson. While the content of phase 0 work mostly related to students' learning (KCS), phase 4 required a far richer breadth of teacher knowledge related to teacher learning (KCT, SCK, KCS, and KCC). These varying forms of knowledge were distributed across all of the phases of lesson study. Furthermore, our analysis demonstrates that teachers' PCK was particularly evident in their participation, an important feature to consider in detailing the work that teachers do as part of lesson study.

4.3 From Phases to Meetings

The visualizations often used to depict lesson study suggest a chronological order of phases, where the planning of the research lesson (phase 2) proceeds the study of curriculum (phase 1) and the reflection of the research lesson (phase 4) follows the conduction of that research lesson (phase 3). However, our analysis of the data demonstrates that phases of lesson study do not always occur in sequential order but rather arise throughout the cycle as interconnected and related phases during teachers' collaborative work (see Fig. 5).

As demonstrated by our analysis, the planning of a research lesson might also include reflections of teaching, as depicted in meeting 7 (Fig. 5). In the same manner, a post-lesson discussion might also contain deliberations on the goal of the lesson study. Such conversations are depicted in meeting 9 (see Fig. 5), where the most recent research lesson was discussed alongside the previous research lesson and goals of the cycle. Our findings suggest that phases are not necessarily consecutive within lesson study. Such fluidity in the chronology of phases may be useful to

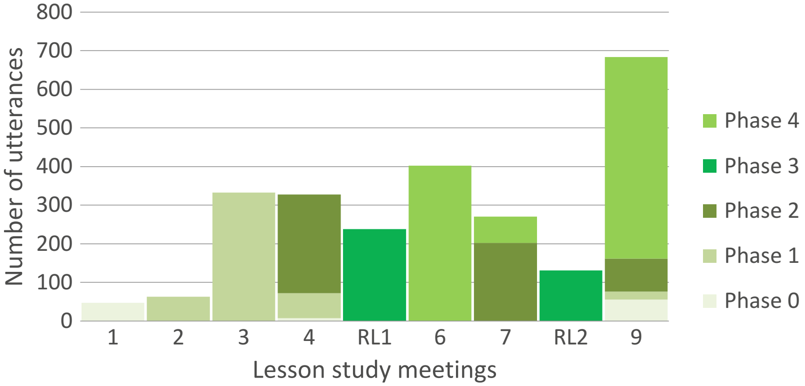

Fig. 5 Lesson study phases and meetings

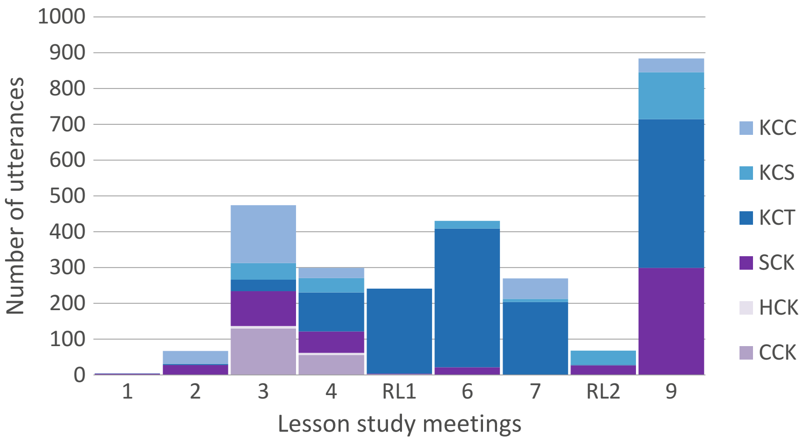

Fig. 6 MKT expressed across all meetings

highlight to participating teachers and facilitators. It may be particularly relevant to make teachers aware that their work continuously evolves across a lesson study cycle in, for example, refining goals or considering elements of student thinking through each phase.

Analyzing the MKT across the meetings (see Fig. 6), it is evident that teachers had more of a focus on the mathematical content of the lesson, as relative to student learning, after the first research lesson. This analysis across the lesson study cycle provides us with insight into the increased occurrences of KCT, KCS, and SCK following the conduction and observation of research lessons, where teachers had increased focus on students' thinking in their lesson study work. This finding may be relevant when considering whether a research lesson should be redesigned and taught in a new form, since teachers may have increased focus on content as relevant to student learning.

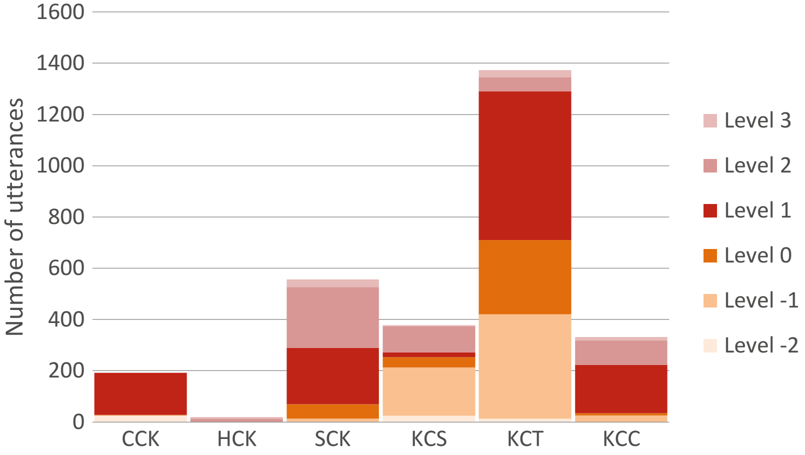

Fig. 7 MKT expressed according to levels of teacher activity over one cycle of lesson study

4.4 MKT and Levels

Taking the lesson study cycle as a unit of investigation, we examined the occurrences both of MKT and levels of teacher activity throughout (see Fig. 7). This analysis provides a holistic picture of the type of knowledge, at a particular level of teacher activity, which was evident in the case study cycle.

From Fig. 7 it is clear that KCT was the most prevalent type of knowledge in teachers' collective participation in lesson study. It is interesting to observe that this type of knowledge was utilized at almost every level of teacher activity across the lesson study phases. As might be expected, teachers' noticing of student thinking (level -1) was prevalent in their utilization of both KCT and KCS during the cycle. It is also interesting to illustrate that teachers saw mathematics through the eyes of their students (level -2) when drawing on their KCS.

During their participation in lesson study, teachers reflected (unprompted) on how their participation in this collaborative form of professional development impacted on their pedagogical practices outside of lesson study. In the example below, Valentine, in phase 1, reflected on linking mathematical knowledge to a specific task (indicator for KCC) and suggested that she was employing changes to her choice of activities in her teaching practices outside of lesson study.

Valentine (T)	Maybe participating in this lesson study cycle, maybe it has changed my… I mean it has modified some approaches in my teaching, on reflection, in the subject. I'm thinking of other things.
Other teacher	Hum. Like what?
Valentine (T)	Well, for example, it's a bit like what Edith said, I think of different perspectives. I do extra activities. For example, "in Pieces," which is an activity in the book, I did it a second time. I'm doing my usual things, but I'm also trying to visualize more. I am aware… well… I

> am more attentive to some, ah, to some of the difficulties that I wouldn't have noticed before.

It is also interesting to note the lack of evidence of horizon content knowledge (HCK) in teachers' lesson study conversations in this case study (Fig. 7). This may be due to the fact that within this cycle of lesson study, there was no knowledgeable other (Takahashi 2014) distinct to that of a facilitator, who joined in the lesson study cycle. The knowledgeable other often articulates future pathways of students' learning and guides teachers' thinking beyond that of the research lesson (Takahashi 2014), thereby potentially highlighting future pathways of students' mathematical knowledge and incorporating HCK in the post-lesson discussion. Further research may be required on the prevalence of HCK in the contributions of the knowledgeable other and in lesson study conversations in countries such as Japan and China.

5 Discussion and Conclusion

While lesson study is increasingly practiced around the world (Huang and Shimizu 2016), there have been calls within the mathematics education and lesson study research communities to develop the theoretical underpinnings of mathematics teacher learning in lesson study (Clivaz 2015b; Miyakawa and Winsløw 2009; Xu and Pedder 2015). In this chapter, drawing on a case study of eight generalist primary teachers participating in a cycle of lesson study, we have attempted to provide a detailed account of the knowledge explicitly incorporated by teachers in their lesson study work. Utilizing a framework combining Mathematical Knowledge for Teaching (Ball et al. 2008) and levels of teacher activity (Margolinas et al. 2005), we have analyzed teachers' participation in one cycle of lesson study through their lesson study conversations and related notes. This research builds on our previous work (Ni Shuilleabhain and Clivaz 2017) and provides further fine-grained analysis of the knowledge incorporated by teachers in their participation in lesson study. Furthermore, this work contributes to the literature outlining and defining the knowledge utilized and developed by teachers in lesson study.

Several graphical representations in this chapter demonstrate the repartition of each of the components of MKT and levels of teacher activity across a cycle of lesson study. Our research demonstrates that all levels of teacher activity, from the values and conceptions about learning and teaching to seeing mathematics through the eyes of the student, are afforded opportunities of articulation in teachers' participation in the collaborative work of lesson study (see Fig. 2), making implicit teacher knowledge explicit (Fujii 2018; Warwick et al. 2016). Analysis of the data also evidences the presence of all categories of MKT across the phases of the cycle, particularly those of KCS and KCT (types of pedagogical content knowledge) and SCK (a form of subject matter knowledge) (see Fig. 3). Combining both frameworks, our data shows a prevalence of KCT within the lesson study cycle (see Fig. 7), which may support other research findings which have demonstrated changes to teachers' classroom practices as a result of their participation in lesson study (e.g., Batteau 2017; Goldsmith et al. 2014; Ni Shuilleabhain and Seery 2017; Takahashi and McDougal 2018). Furthermore, Figs. 2, 3, and 7 depict the benefit of participation in lesson study, where teachers have the opportunity to utilize almost all elements of their MKT across each phase of the cycle and across all levels of teacher activity. An advantage of participating in lesson study may be the fact that it encourages teachers to incorporate, draw on, and potentially develop their knowledge at various levels by openly articulating their knowledge through active participation across each of the phases.

Within our findings it is interesting to note the distinct lack of occurrences of teachers' horizon content knowledge (HCK) in our case study data (see Fig. 7). Research from Suh and Seshaiyer (2015), however, references a development of teachers' understanding of students' learning progressions (i.e. HCK) in lesson study and suggests this development of knowledge was due to the fact that teachers from multiple grade levels participated in this collaborative work. However, in their study, the development of teachers' knowledge of students' vertical knowledge progression was a key focus for the professional development initiative and was a deliberate focus for the lesson study facilitation team (Suh and Seshaiyer 2015). The findings in our research may further demonstrate the key role played by the facilitators in guiding the focus and conversations of lesson study groups (J. M. Lewis 2016). Further research may be necessary to investigate the prevalence of this type of knowledge in lesson study, particularly in settings where there are both a facilitator and knowledgeable other involved in a cycle. However, it may be also worth considering the suggestion of Speer et al. (2015) to deliberate on HCK as a distinct element of MKT.

In analyzing the types of knowledge occurring across each of the lesson study meetings, our research demonstrates that the phases of lesson study do not necessarily occur in a strict chronological or sequential order but rather take place at varied points throughout the cycle. This may be an important finding in facilitating and analyzing lesson study, where teachers can articulate goals, student learning, or subject topics at all points throughout their lesson study conversations.

This articulation of teacher knowledge in lesson study, and the way that teachers are propelled and compelled teachers to express this professional knowledge, is a key element of this model of professional development. Further investigation and theorization of teacher knowledge and teacher learning are required in order to sustain lesson study as a professional situation where teachers can develop their knowledge in and for teaching.

Acknowledgments This research was supported, in part, by a Charlemont Award from the Royal Irish Academy and by the Movetia Swiss National Agency.

Appendix

The indicators are *in italics*. The codes are in color, corresponding to Figs. 2, 3, 5, and 7

MKT

SMK	CCK	*Performing mathematical task*
		Use of notations and vocabulary
		Determining if a solution, a definition, a representation… is correct
	HCK	*Considering other uses of a mathematical knowledge*
		Considering later purpose of a mathematical knowledge
	SCK	*Looking for patterns in student errors*
		Sizing up whether a nonstandard approach would work in general
		Unpacking of mathematics
		Understanding different interpretations of a concept/techniques appreciating the differences
		Talking explicitly about how mathematical language is used
		Choosing, making and using mathematical representations effectively
		Explaining and justifying mathematical ideas
		Analyzing/building examples having mathematical characteristics
		Determining if a mathematical concept or rule is a convention or a mathematical necessity
PCK	KCT	*Sequencing mathematical content*
		Identifying or developing learning activities
		Selecting models, representations, examples, and procedures that support the development of mathematical understanding
		Anticipating/analyzing teacher's reaction to students' response or difficulties
		Anticipating/analyzing teacher's actions in relation to mathematical content
		Sharing or comparing representations and procedures in teaching
		Selecting appropriate mathematical language, analogies and metaphors
	KCS	*Identifying students' knowledge or learning*
		Identifying students' difficulties or misconceptions
		Anticipating students' mathematical responses
		Noticing and interpreting the mathematical meaning associated with students' responses
		Choosing an example that students will find interesting and motivating
		Selecting questions and tasks that seek out the presence of misconceptions
		Noticing and interpreting the mathematical meaning associated with students' responses
		Choosing an example that students will find interesting and motivating
		Selecting questions and tasks that seek out the presence of misconceptions
	KCC	*Linking mathematical knowledge to the syllabus (maybe implicit)*
		Linking mathematical knowledge to a specific task available (in textbook or...)
		Lateral curriculum knowledge
		Vertical curriculum knowledge

LS phase

0 Consider issues and formulate general goals	*In/for student learning and development*	
	In/for teaching	
	In/for teacher's professional knowledge	
1 Study curriculum and formulate content specific goals	Consider learning of the topic	*Identify topic of interest*
		Formulate goals for student learning specific to the topic
		Discuss a learning trajectory related to the topic through grades
	Identify/analyze specific difficulties.	*In teaching*
		In student knowledge or learning
		In content
	Study curriculum, standards and material	*Study course of study, standards...*
		Study textbook, specific task, manipulative...
		Link topic to other topics
		Read and reference research literature
2 Plan	Select (or revise) content	*Select (or revise) research lesson*
		Select (or revise) sequence of lessons
	Consider elements of the research lesson	*Long term goals*
		Learning objectives
		Model of learning trajectory
		Rationale for chosen approach
	Detail the conduction of the lesson	*Anticipated student thinking*
		Anticipated teacher actions
		Incorporating resources (tasks, material)
		Data collection plan
3 Do research lesson	*Conduct research lesson*	
	Observe and collect data	
4 Reflect	Use the data to illuminate	*Student actions*
		Student learning
		Teacher actions
		Disciplinary content
		Lesson and unit design
		Reflect on curriculum
		Broader issue in teaching-learning
	Documentation of cycle	*Consolidate and carry forward learning*
		New questions
	Reflect about other teachings of the research lessons	

Level of teacher activity

3 Values and conceptions about learning and teaching	*Educational project: educational values, conceptions of learning, conceptions of teaching*
2 The global didactic project	*The global didactic project, of which the planned sequence of lessons is a part: notions to study and knowledge to acquire*
1 The local didactic project	*The specific didactic project in the planned sequence of lessons: objectives, organization of work*
0 Didactic action and observation	*Observation of teacher's didactic action*
	Interactions with pupils, decisions during action
-1 Observation of pupils' activity	*Perception of pupils' activity, regulation of pupils' work*
-2 Student critical lens	*Articulation of teaching and learning from a student's perspective*

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407.

Batteau, V. (2017). Using lesson study in mathematics to develop primary school teachers' practices: A case study. *Quadrante, 26*, 127–157.

Bloch, I. (1999). L'articulation du travail mathématique du professeur et de l'élève dans l'enseignement de l'analyse en première scientifique. Détermination d'un milieu – connaissances et savoirs. *Recherches en Didactique des Mathématiques, 19*, 135–194.

Bloch, I. (2005). Quelques apports de la théorie des situations à la didactique des mathématiques dans l'enseignement secondaire et supérieur : contribution à l'étude et à l'évolution de quelques concepts issus de la théorie des situations didactiques en didactique des mathématiques. HDR, Paris 7.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Clivaz, S. (2014). *Des mathématiques pour enseigner? Quelle influence les connaissances mathématiques des enseignants ont-elles sur leur enseignement à l'école primaire?* La Pensée Sauvage: Grenoble.

Clivaz, S. (2015a). Des mathématiques pour enseigner? Quelques réflexions à partir d'un cas de combinaison de cadres théoriques. In L. Bacon, D. Benoit, C. Lajoie, & I. Oliveira (Eds.), *Croisements variés de concepts, d'approches et de théories: les enjeux de la création en recherche en didactique des mathématiques. Colloque du groupe de didactique des mathématiques du Québec 2014*. Montréal: UQAM.

Clivaz, S. (2015b). French Didactique des Mathématiques and Lesson Study: A profitable dialogue? *International Journal for Lesson and Learning Studies, 4*, 245–260.

Clivaz, S. (2016). Lesson study: From professional development to research in mathematics education. *Quadrante, XXV*, 97–112.

Clivaz, S. (2017). Teaching multidigit multiplication: Combining multiple frameworks to analyse a class episode. *Educational Studies in Mathematics, 96*(3), 305–325.

Davis, B., & Renert, M. (2013). Profound understanding of emergent mathematics: Broadening the construct of teachers' disciplinary knowledge. *Educational Studies in Mathematics, 82*, 245–265.

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education, 34*, 12–25.

Dorier, J.-L. (2012). La démarche d'investigation en classe de mathématiques : quel renouveau pour le questionnement didactique ? In B. Calmettes (Ed.), *Démarches d'investigation. Références, représentations, pratiques et formation*. Paris: L'Harmattan.

Dudley, P. (2013). Teacher learning in lesson study: What interaction-level discourse analysis revealed about how teachers utilised imagination, tacit knowledge of teaching and fresh evidence of pupils learning, to develop practice knowledge and so enhance their pupils' learning. *Teaching and Teacher Education, 34*, 107–121.

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A US–Japan lesson study collaboration reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*, 171–185.

Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*, 65–83.

Fujii, T. (2018). Lesson study and teaching mathematics trough problem solving: The two wheels of a cart. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Da Ponte, A. Ni Shuilleabhain, A. Takahashi, & T. Fujii (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education, 17*, 5–36.

Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 41*, 513–545.

Hodgen, J. (2011). Knowledge and identity: A situated theory of mathematics knowlege in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching*. London: Springer.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*, 393–409.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169–202.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM Mathematics Education, 40*, 873–892.

Leavy, A., & Hourigan, M. (2016). Using lesson study to support knowledge development in initial teacher education: Insights from early number classrooms. *Teaching and Teacher Education, 57*, 161–175.

Lesson Study Alliance. (2012). *LessonNote*. Chicago.

Lewis, J. M. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM Mathematics Education, 48*, 1–14.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for Research in Mathematics Education, 48*, 261–299.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*, 3–14.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*, 285–304.

Margolinas, C. (2002). Situations, milieux, connaissances: Analyse de l'activité du professeur. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la 11e école d'été de didactique des mathématiques*. Grenoble: La Pensée Sauvage.

Margolinas, C. (2004). Modeling the teacher's situation in the classroom. In H. Fujita, Y. Hashimoto, B. Hodgson, P. Lee, S. Lerman, & T. Sawada (Eds.), *Proceedings of the ninth international congress on mathematical education*. Dordrecht: Springer Netherlands.

Margolinas, C., Coulange, L., & Bessot, A. (2005). What can the teacher learn in the classroom? *Educational Studies in Mathematics, 59*, 205–234.

Miyakawa, T., & Winsløw, C. (2009). Didactical designs for students' proportional reasoning: An "open approach" lesson and a "fundamental situation". *Educational Studies in Mathematics, 72*, 199–218.

Murata, A., Bofferding, L., Pothen, B. E., Taylor, M. W., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics lesson study. *Journal for Research in Mathematics Education, 43*, 616–650.

Ni Shuilleabhain, A. (2015). *Developing mathematics teachers' pedagogical content knowledge through lesson study: A multiple case study at a time of curriculum change.* Doctor of Philosophy Ph.D, Trinity College Dublin.

Ni Shuilleabhain, A. (2016). Developing mathematics teachers' pedagogical content knowledge in lesson study: Case study findings. *International Journal for Lesson and Learning Studies, 5*, 212–226.

Ni Shuilleabhain, A., & Clivaz, S. (2017). Analyzing teacher learning in lesson study: Mathematical knowledge for teaching and levels of teacher activity. *Quadrante, 26*, 99–125.

Ni Shuilleabhain, A., & Seery, A. (2017). Enacting curriculum reform through lesson study: A case study of mathematics teacher learning. *Professional Development in Education, 44*, 1–15.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*, 165–178.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*, 4–15.

Roditi, E. (2011). *Recherches sur les pratiques enseignantes en mathématiques: apports d'une intégration de diverses approches et perspectives*. Paris V: Université René Descartes.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). *Developing mathematics teaching: The genesis of the knowledge quartet*. Gent: Rijksuniversiteit Gent.

Rowland, T., Ruthven, K., & Hodgen, J. (2011). Knowing and identity: A situated theory of mathematics knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching*. Dordrecht: Springer Netherlands.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4–14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Revue, 57*, 1–22.

Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education, 18*, 105–122.

Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1*, 157–189.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the worlds teachers for improving education in the classroom*. New York: The Free Press.

Stigler, J. W., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM Mathematics Education, 48*, 1–7.

Suh, J., & Seshaiyer, P. (2015). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. *Journal of Mathematics Teacher Education, 18*, 207–229.

Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examination of comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*, 4–21.

Takahashi, A., & Mcdougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*, 1–14.

Takahashi, A., & Mcdougal, T. (2018). Collaborative lesson research (CLR). In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Da Ponte, A. Ni Shuilleabhain, A. Takahashi, & T. Fujii (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Tepylo, D. H., & Moss, J. (2011). Examining change in teacher mathematical knowledge through lesson study. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education*. Dordrecht: Springer Netherlands.

Warfield, V. (2014). *Invitation to didactique*. New York: Springer.

Warwick, P., Vrikki, M., Vermunt, J. D., Mercer, N., & Van Halem, N. (2016). Connecting observations of student and teacher learning: An examination of dialogic processes in lesson study discussions in mathematics. *ZDM Mathematics Education, 44*, 1–15.

Widjaja, W., Vale, C., Groves, S., & Doig, B. (2017). Teachers' professional growth through engagement with lesson study. *Journal of Mathematics Teacher Education, 20*, 357–383.

Winsløw, C., Bahn, J., & Rasmussen, K. (2018). Theorizing lesson study: Two related frameworks and two Danish case-studies. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Da Ponte, A. Ni Shuilleabhain, A. Takahashi, & T. Fujii (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Xu, H., & Pedder, D. (2015). Lesson study: An international review of the research. In P. Dudley (Ed.), *Lesson study: Professional learning for our time*. London/New York: Routledge.

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Identifying What Is Critical for Learning 'Rate of Change': Experiences from a Learning Study in Sweden

Robert Gunnarsson, Ulla Runesson, and Per Håkansson

Contents

Abstract Learning study is an adapted version of lesson study developed in Hong Kong and Sweden. It has commonalities with lesson study but is framed within a specific pedagogical learning theory – variation theory. Central in variation theory is the object of learning and what is critical for students' learning. Hence, as with lesson study, it is a collective and iterative work where teachers explore how they can make the object of learning available to students, but what characterises learning study is the use of a specific learning theory. In this process, special attention is paid to the critical aspects of the object of learning. We argue that to identify the aspects that are critical, the aspects need to be verified and refined in classrooms. In this chapter, we demonstrate how teachers gain knowledge about such critical aspects. Particularly, we show how these critical aspects cannot be extracted only from the mathematical content or the students pre-understanding alone, but evolve during the learning study cycles. For this we use a learning study about the mathematical topic of rate of change in grade 9 in Sweden as an illustration. We describe how an analysis of how students solved tasks in pre- and post-test and during the lessons, as well as how the mathematical content was presented in lessons, helped the teachers identify what was critical for learning to understand and express the rate of change for a dynamic situation.

Keywords Learning study · Critical aspects · Rate of change

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_21

1 Introduction

There are a lot of things a teacher must have in mind and take into consideration when planning for teaching a topic to class. One such thing is the students. From previous teaching experience, the teacher is probably aware of difficulties students could have with learning a specific topic. Another is how to choose teaching strategies that, for instance, promote active involvement and meaningful interaction. Furthermore, decisions about how to evaluate the lessons must be made. Although, carefully considering all this, we would argue that there still is one question that must be answered. To teach towards a learning goal, that is, what students are expected to achieve, the answer to the question 'What must be learned to achieve the targeted goal?' must be found. Why? Because the answer to this question fills the gap between teaching content and teaching strategies (what and how to teach) and the learning objectives (the intended results of teaching and learning). Our argument is that a learning objective cannot give the answer to 'what must be learned?', since a learning objective is general for all in a specific course or grade. What must be learned to achieve this goal is specific for every group of learners, however. The question must be answered on a detailed level and cannot be derived from the subject matter alone. Instead, identifying what students need to learn is a transactional process between the specific known (thing) and the specific group of students' knowing of that thing (Dewey and Bentley 1949). This process can provide teachers with 'keys to learning'.

In this chapter we will demonstrate how such 'keys to learning', identified with the aim to enhance student learning regarding the rate of change of linear relationship, are refined in the process of the learning study. One example from a learning study conducted by a group of Swedish mathematics teachers will be used as an example. Before that, we will go more into detail about learning study, variation theory and how a learning study can be executed.

2 The Origin and Purpose of Learning Study

The ultimate aim of lesson study is to improve teaching practice and student learning. Learning study, an adapted version of lesson study, shares this aim, just as it shares the collaborative planning, observing of lessons and student learning. Although the role of a theory in lesson study is often unclear or rarely made explicit (Elliott 2012, p. 114), there are lesson studies where the use of theories is reported (e.g. Clivaz 2015; Martin and Clerc-Georgy 2015; Martin and Towers 2016; Pillay and Adler 2015). Due to a difference in the epistemological and ontological assumptions underpinning the theory, however, the lesson studies will have different foci (Runesson 2016).

Learning study (Cheng and Lo 2013; Marton and Pang 2003; Marton and Runesson 2015) is guided by a specific theory of learning through variation (described below). This variation theory provides a lens to focus on how students learn and develop a specific capability and what is made possible to learn in the lesson.

Learning study was developed in Hong Kong around the year 2000. Inspired by Stigler and Hiebert promoting lesson study in the book *The teaching gap* (Stigler and Hiebert 1999), a research project, aimed at developing teaching to cater for individual differences, was conducted (Lo et al. 2005). Another, and significant, source of inspiration was the development of a research tradition in Sweden – phenomenography – from which the variation theory had been developed (Marton and Booth 1997; Marton and Tsui 2004; Marton 2015). Just as in lesson study, the teachers chose a topic they found difficult for students to learn, drove the process and took part in the analysis of the lesson and students' learning. Variation theory was 'owned' by the teachers and used as a theoretical tool in the process. The term learning study was chosen to distinguish its distinctive character from lesson study, the focus on learning, the object of learning and variation theory as the guiding principles in the process. An evaluation of this, and similar projects, demonstrated the effects of learning study; the learning gap between students decreased (Lo et al. 2005), student learning was sustainable and thus lasted and developed beyond individual lessons (Elliott and Yu 2013). Moreover, when variation theory was applied as a tool for design and analysis, student learning was more in line with the teachers' intentions (Marton and Pang 2006). In the same way, studies showed that when teachers jointly explore and study what is made possible to learn in the lesson in relation to student learning, 'the keys' to learning (Pang and Marton 2017) can be found. Subsequently, the lesson can be designed in a way that promotes student learning. Furthermore, it has been demonstrated that 'results' and insights gained in learning study can be communicated, picked up and developed by other teachers and in other contexts (Kullberg 2012; Runesson and Gustafsson 2012).

Learning study can be conducted by a group consisting of teachers only, or they could be assisted by a researcher or an educator from the university. Although they have different expertise, they have a common object of research, to enhance students' learning by identifying the critical aspects and trying them out in class. Although it might be a challenge to avoid distinguishing between the researcher and teachers (Adamson and Walker 2011), it is intended that the teachers and the researcher should be equal partners in the process.

2.1 The Variation Theory as Guiding Principles for Planning and Evaluating Learning

As stated above, to help learners develop a certain capability (e.g. 'understanding the relation between a graph and its derivative's graph', 'the method to calculate the perimeter of compound rectangles'), it is necessary to find what must be learned to develop that capability – aspects that are critical for learning. Thus, learning, from a variation theoretical perspective, amounts to being able to discern critical aspects of what is learned.

Most capabilities have different constituent aspects. Some learners have acquired some aspects already and have not yet acquired other aspects. Learners must learn what they have not yet learnt. Therefore, what students have not acquired, and thus need to learn, must be explored and identified. Although, candidates for such critical aspects can (and must) come from inside the subject matter itself, from previous teaching experience and from research literature, they cannot be known in advance or be known from a predefined learning objective. Therefore, what is critical for the specific group of learners, or whether there are other aspects these learners have not discerned, would be necessary to identify and test. Subsequently, a significant element in learning study is the analysis of student learning in relation to what is taught in the lesson. What must be taught are the critical aspects.

From a variation theory perspective, discernment stems from experiencing variation (Marton and Pang 2006; Pang and Ki 2016). So, students' attention to the critical aspects can be drawn by the means of variation. If an aspect (e.g. the slope of a linear function) varies and others are kept constant (e.g. the intercept of the line), the varied aspect (here, the slope) is likely to be discerned. Furthermore, each aspect consists of its manifold variations, and to discern an aspect, these variations need to be made visible. In the words of variation theory, this is called to *open up a dimension of variation* (Marton 2015). Hence, first when a dimension of variation of a critical aspect is opened up, the critical aspect can be possible to discern. This principle can be applied when designing the lesson. That, which we want the learners to discern – the critical aspect – should be varied against an invariant background.

The idea of learning is, in variation theory, grounded in distinguishing, thus experiencing differences rather than similarities. Marton (2015) argues that the discernment of a feature requires the experiencing of a difference between (at least) two things or parts of the same thing. To discern a new concept, one needs to experience contrast (variation) between the new concept and another concept and hence how it differs from the other concept. Similarly, Watson and Mason (2006) have demonstrated how the idea that systematic and purposeful use of variation and invariance (i.e. what is varied and what is invariant) that structures students' awareness can be applied when designing exercises and examples. They argue that different kinds of variation in exercises offer different learning possibilities. However, teaching without accomplishing a pattern of variation is hardly possible but might be done almost intuitively and without systematicity (Runesson 2005). In learning study, principles of variation – how patterns of variation can afford and constrain learning – are applied intentionally and systematically when designing and analysing teaching and learning.

2.2 Identifying the Critical Aspects

Hence, a central part of a learning study is to identify (with as high precision as possible) the critical aspects. This is done in an iterative cyclic process (see, e.g. Cheng and Lo 2013). The initial planning stage includes choosing the topic, defining the object of learning as well as students' learning problems with this object. Learning problems, and what is assumed to be critical for learning, are initially identified through a diagnostic pre-test or by interviews, together with previous teaching experiences and research literature as a background. On the basis of insights generated at this stage, tentative critical aspects are agreed upon, and the first research lesson is planned. After the first lesson, diagnostic post-test or interview is given. Results from this, together with a close analysis of the lesson, give further insights into what is critical for learning and how the content must be handled to promote learning. Hence, typically the critical aspects are refined. This becomes the basis for the planning of the second lesson in the cycle. This lesson is (usually) taught by a new teacher, and to new students, and again the observed/recorded lesson and the diagnostic post-tests are analysed. The iteration proceeds until all classes are taught (Cheng and Lo 2013). In this way, the assumed critical aspects are confirmed or rejected, and often new (and previously taken-for-granted) aspects are found to be critical. Most often, however, the initially assumed and tentative critical aspects become revised, refined and specified. The revision of critical aspects is a result of a detailed analysis of data (observation and/or video recordings) from the research lessons together with results from pre- and post-tests. With evidence from lesson and student data, the critical aspects are finally established (ibid). Identifying the critical aspect is thus an emergent process of a deep and systematic reflection and analysis of ‘the known’, ‘the knowing’ and the ‘knower’ (cf. Dewey and Bentley 1949).

3 The Example: A Learning Study About the Rate of Change

We illustrate this search for critical aspects, and how these can emerge and be refined, with a learning study about the mathematical concept of rate of change (RoC). This concept is central in the precalculus stage of the study of functions describing dynamic processes and is closely related to the covariation of variables. The object of learning, *to express the quantitative rate of change of a linear relation*, was explored through a series of three iterative lessons in a learning study about RoC in a Swedish lower secondary school (ages 15–16). In this study, a pre-test (described below) was used to select students to form groups of mixed performance. The students were chosen from three different classes (in total 69 students) and were combined in groups of three and where three such groups constituted a teaching unit class in each lesson. Hence, 27 (i.e. 9 students for each lesson) out of 69 students with average performance were included in the full study.

However, the teachers also used the pre-test to identify what aspects they considered critical for learning rate of change. The tasks in the pre-test (in all essential parts it was also used as post-test) were designed based on the teachers’ ideas on students’ problems regarding rate of change. The paper-and-pencil pre-test consisted of five tasks, all in which calculating rates were central. Four of the tasks included data points representing rates that were positive, negative or zero or linear segments combining these. In one of the tasks, also a non-linear segment was used. In four of the tasks, the rates were to be calculated from graphical representations, but in one task the two covarying quantities were given numerically.

During the teachers' analysis of the tests, there was no one-to-one correspondence between a certain task and a specific aspect. Thus, the different tasks could help identify (and later, when used as post-test, refine) any of the aspects.

The learning study was conducted by three teachers of which one (third author) works half-time at the school and half-time at the university. He participated in the learning study in the same way as the teachers. The main difference was that as the teacher/researcher, he was responsible for documenting the process and had access to guidance from the university (first and second author).

In each lesson, the activities were alternated between the teacher presenting tasks, the students discussing/solving the tasks in the groups and the whole class discussing the different groups' solutions. The use of three groups, each of three students, was decided upon to enhance the possibilities to capture students' reasoning. With four cameras, the three student groups as a whole class and the work of each group could be separately, and simultaneously, audio- and video-captured. The students' solutions in pre- and post-tests, and their discussions during each of the lessons, were analysed and used by the teachers in the iterative learning study process. This data, capturing the students' understanding of the RoC as well as how the object of learning was enacted during the lessons, helped the teachers identify and refine possible critical aspects and plan for how each dimension of variation could be opened up and thus guide their planning of the coming lesson. Our report is structured to describe one critical aspect at the time, but the lessons were planned to sometimes target the different tentative critical aspects within the same task.

3.1 Three Tentative Critical Aspects

From the analysis of the pre-tests, three different tentative critical aspects (TCA) were identified. The first TCA to be identified concerns the covariational aspect that RoC can be regarded as a relation/ratio between changes in two quantities (i.e. two covarying variables). In the pre-test data, the teachers noticed that some students did not seem to regard the changes in both variables as actual continuous changes. Especially concerning the horizontal variable (time), an interval did not seem to be perceived as a change, but as two values between which a corresponding change in the vertical variable occurs. When asked to find the rate of change of a linear relation in a given interval, some students used segments of the graph instead of the interval as a whole. These students repeatedly associated intervals in the vertical variable with adjacent tick marks on the horizontal axis, forming a 'staircase' function. The students seemed too bound to the tick marks in the graph, the teachers concluded. The teachers then hypothesised that a TCA could be *to discern that RoC is constant in linear relations*. This aspect was conjectured to aid students to freely 'move' along the linear parts of a graph and choose a suitable segment. The teachers acknowledged the use of easily identified, but otherwise arbitrary, segments of the graph to be a powerful strategy with the goal to express the RoC in a linear relation.

From pre-test data, it was concluded that students seemed to implicitly assume that they were dealing with proportional relations. In short, with focus on corresponding values, students tried to calculate the RoC by dividing y-value with x-value, a strategy which works in the case of proportionality. In some solutions, an interval in one variable was combined with a value in the other. The teachers interpreted these observations in terms of a need to discern the two corresponding changes involved in calculating the RoC. Hence, a second TCA was formulated as *to discern that every change in one variable is related to a change in the other*.

One of the pre-test tasks was without graphical representation. The responses to this task indicated that some students divided one change with the other ($\frac{\Delta y}{\Delta x}$), and some students divided the changes the other way around ($\frac{\Delta x}{\Delta y}$), to produce a numerical expression for the RoC. The teachers therefore suggested that in the presence of graphical representations, the question of choosing $\frac{\Delta y}{\Delta x}$ or $\frac{\Delta x}{\Delta y}$ could be the result of adopting a mathematical convention. The general convention is to divide the dependent variable with the independent variable ($\frac{\Delta y}{\Delta x}$). But the teachers decided to avoid the issue of dependency-independency as the object of learning concerned RoC in a ‘non-functional’ way. They concluded that in the lesson, the dimensions of variation had to be opened up in ways relying on reasoning, rather than on conventions or rules. Hence, a third TCA was formulated as *to discern that two values, inverse to each other, can be used to express the quantitative RoC.* This third TCA was therefore targeting that both ratios contain information about the change in one variable with respect to the other.

Hence, there were three different TCAs that were assumed at the beginning of the learning study. These TCA were then identified based on the results from the pre-test. In the following section, we will illustrate what can trigger an evolution of critical aspects in a learning study by describing how two of these TCAs were refined into critical aspects. In particular, we will describe what the teachers found in the data from pre-test, post-test and observations of students that made them refine the critical aspects during the learning study.

3.2 Refining the Critical Aspects Through the Process of Learning Study

After identifying TCAs, the teachers planned the first lesson. The teachers agreed to use the same framing context in all three lessons: to fill tanks with water in a certain time. The level of water in the tank at different times was given, either as points in a graph or in tabular form. Two variables were used: *water level* (measured in cm) and *time* (measured in min). Each lesson contained a series of tasks within this framing context. In this way, the lessons were refined in each cycle of the learning study, all the time with the goal to open up the right dimensions of variation (based on the critical aspects) and thereby enact the object of learning *to express the quantitative RoC of a linear relation*. At the same time, the critical aspects were not static, but were, themselves, refined in the process.

3.2.1 Refining the Tentative Critical Aspect 'to Discern that the Rate of Change Is Invariant in Linear Relations'

To target the TCA *to discern that RoC is constant in linear relations* in lesson 1, the context was developed into tasks involving different 'observation points' (see Fig. 1).

Initially, the table of data points in Fig. 1a was shown to the students. The teachers had designed an example where the rate of change was calculated based on specific data points. The variation theory was used in the design of the example, particularly focusing on what to keep constant and what to vary. Firstly, data points were compared where time intervals (but not the rate) were kept constant, the time interval between 1 and 6 and between 6 and 11 min, Fig. 1a. Secondly, the data points where water level intervals (but still not the rate) were constant were compared, between 43 to 61 cm and between 61 and 79 cm (difference of 18 cm between each pair of points). Thirdly, a new point (3, 18.4) was inserted (see Fig. 1b), and the two first intervals, where the rate of change was invariant, were compared. The example was used with the variation theory in mind, to make possible to discern that a change in one variable, keeping the other invariant, would change the rate in total. The teacher then emphasised that, in contrast with first comparisons, in which the change in only one variable was altered, different changes in *both* variables could produce the same rate. However, that these different pairs of changes were connected through the same linear relation was not explicitly mentioned, and thereby not offered for discernment, according to the variation theory. Later in the lesson, the same points were used again, now represented as points in a diagram; see Fig. 1c. The teacher then connected all three points with a straight line and asked the students to verify that the RoC over both intervals together (1–6 min) matched the earlier calculated (identical) rates. With this, the teachers' intention was to make it possible to experience that the two ratios were connected through the same linear relationship. However, the results in the post-test (after lesson 1) showed no clear improvement in terms of students' ability to use different intervals to calculate the RoC. In the analysis of the first lesson, the teacher's actions on this task were considered central for the lack of students' improvement (regarding the results from the pre- and post-test). In retrospect, it can be interpreted as that the students were given the opportunity to *verify*, not discern, that the RoC was equal. However, at that time, the group only conjectured that the students did not learn that RoC is constant in linear relations because they were not given the opportunity to discern that different pairs of intervals along a given straight line would produce the same rate of change. Thus, this dimension of variation was not opened up in the lesson, they concluded.

Hence, while the critical aspect was not modified after lesson 1, the task sequence was modified from lesson 1 to lesson 2. In lesson 2, an additional task was

a

Time (min)	Water level (cm)
1	2.0
6	43.0
11	61.0
20	79.0
29	61.0

5 min; 5 min; 18 cm; 18 cm

b

Time (min)	Water level (cm)
1	2.0
3	18.4
6	43.0
11	61.0
20	79.0
29	61.0

8.2 cm/min; 8.2 cm/min

c

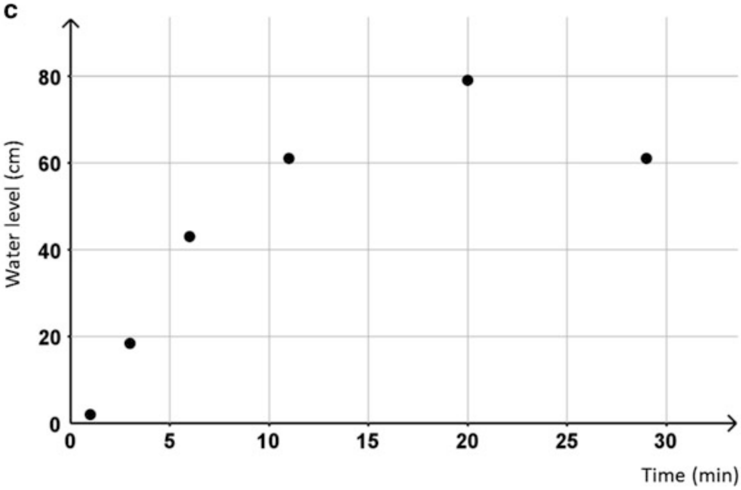

Fig. 1 The example, based on the context of filling a tank of water, used in lessons 1 and 2, here represented as numbers in a table where (**a**) time and water level interval lengths were kept constant, separately, and (**b**) after insertion of an extra point (3, 18.4), where the rate of change was kept constant. (**c**) The graph corresponding to the table in (**b**)

introduced to the students. Starting with a single point (4,12) in a diagram of water level vs time, Fig. 2a, the students were given the task to calculate how fast the tank was filled.

With only one point indicating the water level, many students assumed that the tank was empty from the beginning and calculated the rate as 12 cm/4 min = 3 cm/min. While the origin as the assumed starting point initially was used to vary the next aspect described in this chapter, it also served as part of varying the critical aspect at hand. Some moments later in the lesson, an additional point (1,3) was inserted in the diagram, as shown in Fig. 2b, and students were asked about the RoC between this point and the original point (4,12). This procedure was done to make it possible for the students to discern that the rate is the same between the points; see Fig. 2b. Two out of the three groups then clearly expressed that the rate in this case was the same as they assumed in the first task with the single point. From the video observations, this suggested a critical moment in the lesson, and the observations were connected

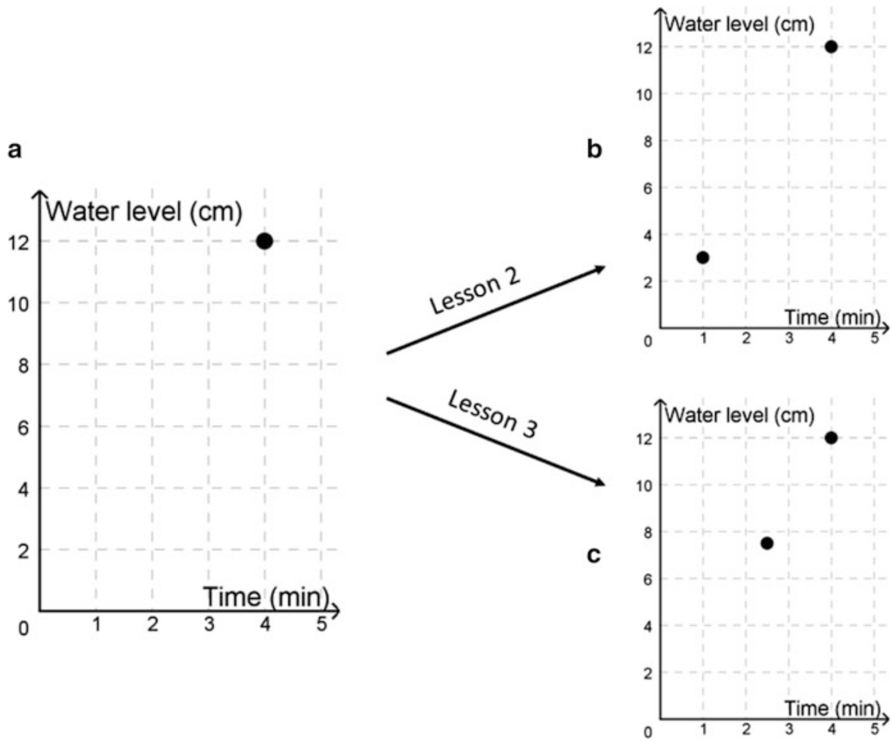

Fig. 2 (**a**) The single point graph used as a starting point to reach the critical aspect in lessons 2 and 3. The same diagram in (**b**) lesson 2 and (**c**) lesson 3, respectively, after inserting a point to open up the dimension of variation that intended to make discernible that the same rate can be achieved by different relations

to a small improvement in the post-test. The teachers then analysed the difference between this event and what happened in the first lesson. They concluded that in both lessons the same mathematical features were varied and kept invariant, respectively. The RoC is constant in linear relations, and hence, different ratios can represent the same RoC. Mathematically the latter follows the first, but for the students, it seemed to be the other way around, the former logically followed the latter, they concluded. Consequently, after lesson 2, the teachers rephrased the critical aspect *to discern that different ratios can represent the same rate*.

In lesson 3, the original set of tasks used in lesson 1 and 2 was abandoned, and the 'single point task' was revised by further 'masking' the second rate, using the point (2.5,7.5) instead of the point (1,3); see Fig. 2c. The argument was that it should not be too obvious that the two points resulted in the same rate. Video observations of the lesson revealed clearly that all three groups noticed that the rate was the same in the two cases. Two of the three groups in that lesson spontaneously checked their calculations graphically, drawing and examining the graph representing the change of water level over time. The third group first solved the task by examining the point graphically and then also confirmed their answer numerically. In contrast with lesson 2, the teacher in lesson 3 also asked the students for two more examples of how to calculate the rate. Thereby two more ratios (to a total of four) were contrasted with each other, followed by the generalisation of the ratios all producing the same rate. In the post-test, all students in lesson 3 made use of, or suggested to use, multiple pairs of intervals to calculate the RoC in a graphically represented linear relation. Hence, after lesson 3, the formulation of the critical aspect was kept and considered corroborated by the turnover in how it was most efficiently experienced by the students in lesson three.

3.2.2 Refining the Tentative Critical Aspect that 'Every Change in One Variable Is Related to a Change in the Other'

From the pre-test data, the teachers noticed that some students' solutions to calculating the RoC seemed to be based on that they read off two corresponding values in a graph, instead of two corresponding intervals. Therefore, during the planning of lesson 1, it was decided to use the set of data points shown in Fig. 1 to vary the change in one variable, while at the same time, the change in the other variable was kept invariant. (At this time, the teachers often used the words 'changes' and 'intervals' synonymously.) In a concluding task by the end of the lesson, students in all three groups still tended to relate an interval in one variable with a value in the other variable. However, in all such cases, the values the students used were connected to one of the endpoints of the interval. It was also noted that in some cases, students seemed to assume a starting point at the origin, even though they had the full (continuous) graph of the example. This drew the teachers' attention to if and how students experienced the *starting point* of the corresponding interval, especially in relation to the origin. Hence, after the first lesson, the TCA seemed necessary to revise, but it was not formally revised. However, the teachers had an idea that the starting point for an interval where only one point was given was perceived as the origin.

To test their conjecture, the task with one single point present in an otherwise empty diagram space was introduced in lesson 2, as shown in Fig. 2a. The teachers' aim of this manoeuvre was to open a dimension of variation about the starting point from where changes could be measured. When the teacher posed the question 'what does this point mean?', one student replied: 'That after four minutes it has risen twelve centimetres'. Hence, the student already at this point assumed that the tank was empty from the beginning. The teacher then repeated the answer and pointed out that: '...or more accurately, yes, at four minutes, the water level *is* twelve centimetres'. The teacher then posed the question, 'How fast was this tank filled, then?', thus not serving any predetermined interval. Some students immediately solved the task by dividing the y-value of the given point, with its x-value. However, video observations revealed that the presumption that the interval started out from the origin was also challenged by some students, thus opening the desired dimension of variation. Despite disagreements among the students, in their answers the groups stated that the water level rose with a rate of 3 cm/min, possibly expecting that the teacher would not present a task with no unambiguous solution. The teacher then gave additional information: ‘You know what? Here comes a secret: There is an observation point, right when the timing began, for this tank’. The point (0,6) was then marked in the diagram, followed by a question to the students if they would like to revise their earlier calculation of the rate. In this way, the position of the starting point for the interval where the rate was to be determined was varied two more times to (2,0) and (1,3), respectively, as shown in Fig. 3a. The data points in this example were designed for this specific critical aspect. However, eventually the points were used also in the example targeting the previously described critical aspect as well.

The teachers now became more aware of the students’ strong bias towards the origin as a starting point of functions. Hence, after lesson 2, the critical aspect was rephrased *to discern that the starting point of an interval is not synonymous with the origin*.

In lesson three, this pattern of variation of the starting point was slightly diminished. As the teachers had planned, the students in lesson three discussed whether the starting point was the origin or not, opening a desired dimension of variation. However, in the analysis of lesson 2, it was considered that it was too easy to numerically confuse the RoC with a slope in a graph. Hence, the first ‘hidden’ starting point was now decided to be (0,7.5) and then exchanging this for a point at (2.5,7.5); see Fig. 3b. The goal was, as in lesson 2, to make a contrast to a starting point at the origin (0,0), which was thought to be assumed by many students. In contrast with the pattern in lesson 2, this was considered to allow for a variation in time to be isolated and highlighted. Anyhow, the changes from lesson 2 to lesson 3 seemed to have an impact on students’ learning. In the analysis of this example in the lessons, it was found that two of the three groups in lesson 2 combined one value

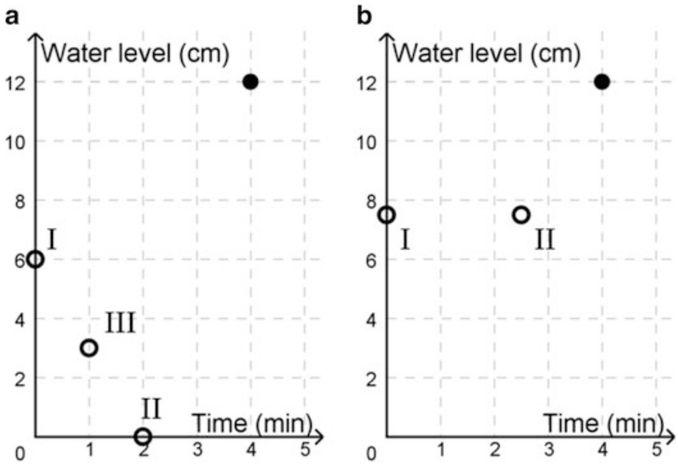

Fig. 3 The different points used to open up the dimension of variation that intervals do not need to start at the origin in (**a**) lesson 2 and (**b**) lesson 3. In both lessons, the alternative points (the empty circles) were initially kept hidden as only the first point (4,12) was shown. These additional points were then shown separately in the order indicated by Roman numerals (I–III)

with one interval, but that in lesson 3, only a single student did so. After lesson 3 the critical aspect was reformulated as *to separate changes starting in the origin from other changes*.

4 Concluding Discussion

We started the chapter by advocating that, when planning for teaching, there is a gap between the learning objective and teaching strategies that must be filled, since learning objectives do not tell what the students must learn to achieve the targeted goal. In the above example, we have shown how the critical aspects, the keys of learning, can be identified but also refined and specified through the process of a learning study.

In Fig. 4, we show how two such tentative critical aspects have evolved during the iterative transactional process. The third tentative critical aspect was transformed in a similar way, not reported on here, however.

In the first round of analysis, the students' answers to the pre-test and the teachers' own previous experiences of the topic led them to formulate the tentative critical aspects. The teaching in lesson 1 was planned and enacted accordingly to open up dimensions of variation of these critical aspects. In the following analysis of the teaching of, and students' responses to, lesson 1, the critical aspects were kept unaltered. We note that for one of the critical aspects ('Critical aspect 1', in Fig. 4), it was the analysis of the teacher's actions that lead the group to change the lesson plan from lesson 1 to lesson 2. Based on students' responses, the other critical aspect did not seem expressed in a satisfactory way, but it was anyway kept unaltered. However, both critical aspects had now been tested in a teaching situation and could then be considered emerging, rather than tentative. Instead, based on the variation theory, the teachers concluded that in lesson 1, the full dimension of variation was not opened and decided to alter the task for the next lesson.

In the analysis of the second lesson, it was the combination of the teaching and the students' responses and the mathematical content that attracted attention and resulted in that both critical aspects were revised. Regarding the first critical aspect, it was the insight that students can experience the implication, that linear relations have the same RoC, the other way around (meaning that they saw that students could perceive a constant RoC as a linear relation). Regarding the second critical aspect ('Critical aspect 2', in Fig. 4), the teachers started to attend more to the bias of the origin. Anyhow, the third lesson was planned accordingly; patterns of variation were enacted to open up for the students to discern these revised critical aspects. In the analysis of this lesson, it was concluded from students' responses to the teaching that the first critical aspect was corroborated, whereas from the teachers' actions (actually, the variation pattern in the example presented by the teacher), and supported by the students' answers, it was concluded that the second critical aspect had to be revised again.

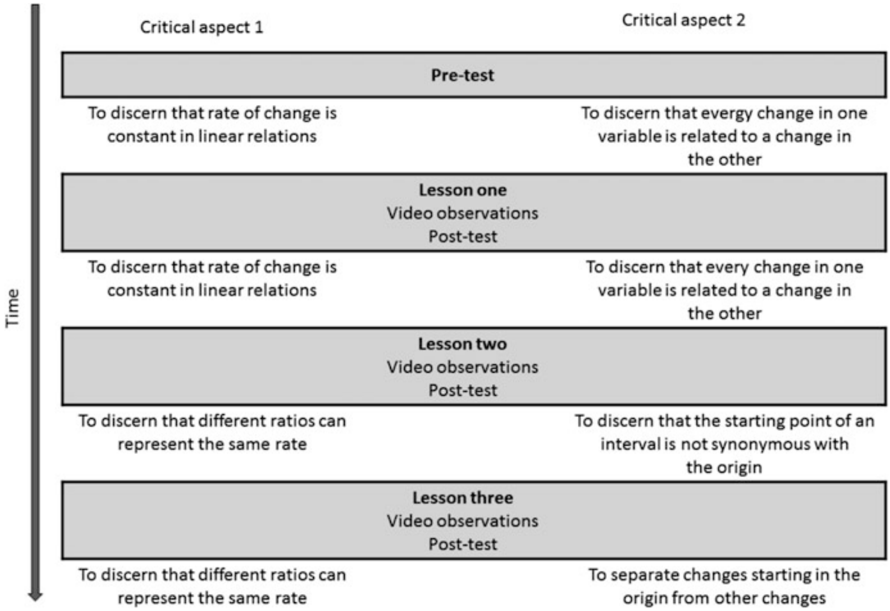

Fig. 4 The evolution (from top to bottom) of two of the critical aspects (the columns to the left and to the right, respectively) during the process of the learning study

Hence, this illustrates that both students' answers and responses to the teaching, as well as teachers' actions and the enactment of the lesson plan, can result in a revision of the critical aspects. Independent of whether it was the students' or the teachers' actions that triggered the revision and specification of the critical aspects, we would suggest that the use of a specific learning theory enabled the teachers to identify these features. Variation theory helped them to focus on the object of learning and students' learning in a relational way. Furthermore, variation theory provided them with a common language to talk about teaching and learning and principles for pedagogical design.

The teachers participating in the process of the learning study took active part in the process of finding and refining the critical aspects. The topic of study stemmed from a problem in their everyday practice, and they planned and analysed the lessons and the student data jointly with the researcher. Actually, they participated to such a high degree that one could claim that they were co-producing the results. This, we would suggest, implies that learning study is more than a development programme. Instead, the teachers are collaboratively acting as researchers, and as such, learning study has the potential to be what Lawrence Stenhouse (1981) suggested: in the research process of generating knowledge for professionals, teachers should be *the* key stakeholders. Therefore, reports on learning studies are not mainly reports on teachers as learners, but on what has been found by the iterative process of investigation. Morris and Hiebert (2011) have suggested lesson study as a system to produce instructional products to be shared and developed. The results from the study reported on here, we think, are an example of such an instructional product. The results are a theoretical description of a lesson design in terms of what was found critical for learning and how this can be manifested in class by means of variation. Hence, a learning study can contribute to the general community with specific pedagogical content by obtaining new information about the keys of learning for a certain topic.

References

Adamson, B., & Walker, E. (2011). Messy collaboration: Learning from a learning study. *Teaching and Teacher Education, 27*(1), 29–36.

Cheng, E., & Lo, M. (2013). *Learning study: Its origins, operationalisation, and implications* (OECD education working papers, No. 94). Paris: OECD Publishing.

Clivaz, S. (2015). French Didactique des Mathématiques and Lesson Study: a profitable dialogue? *International Journal for Lesson and Learning Studies, 4*(3), 245–260.

Dewey, J., & Bentley, A. (1949). *Knowing and the known. The later works, 1949–1952* (Vol. 16, pp. 1–280). Boston: Beacon Press.

Elliott, J. (2012). Developing a science of teaching through lesson study. *International Journal of Lesson and Learning Studies, 1*(2), 108–125.

Elliott, J., & Yu, C. (2013). Learning studies in Hong Kong schools: A summary evaluation. Report on the 'variation for the improvement of teaching and learning' (VITAL) project. *Education & Didactique, 7*(2), 147–163.

Kullberg, A. (2012). Can findings from learning studies be shared by others? *International Journal for Lesson and Learning Studies, 1*(3), 232–244.

Lo, M. L., Pong, W. Y., & Chik, P. (Eds.). (2005). *For each and every one. Catering for individual differences through learning study*. Hong Kong: Hong Kong University Press.

Martin, D., & Clerc-Georgy, A. (2015). Use of theoretical concepts in lesson study: An example from teacher training. *International Journal for Lesson and Learning Studies, 4*(3), 261–273.

Martin, L., & Towers, J. (2016). Folding back and growing mathematical understanding: A longitudinal study of learning. *International Journal for Lesson and Learning Studies, 5*(4), 281–294.

Marton, F. (2015). *Necessary conditions of learning*. New York: Routledge.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah: Erlbaum.

Marton, F., & Pang, M. F. (2003). Beyond "lesson study": Comparing two ways of facilitating the grasp of economic concepts. *Instructional Science, 31*(3), 175–194.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *Journal of the Learning Sciences, 15*(2), 193–220.

Marton, F., & Runesson, U. (2015). The idea and practice of learning study. In K. Wood & S. Sithampram (Eds.), *Realising learning. Teachers' professional development through lesson study and learning study* (pp. 103–121). New York: Routledge.

Marton, F., & Tsui, A. B. M. (2004). *Classroom discourse and the space of learning*. Mahwah: Lawrence Erlbaum.

Morris, A. K., & Hiebert, J. (2011). Creating shared instructional products: An alternative approach to improving teaching. *Educational Researcher, 40*, 5), 5–5),14.

Pang, M. F., & Ki, W. W. (2016). Revisiting the idea of "critical aspects" *Scandinavian Journal of Educational Research, 60*(3), 323–336.

Pang, M. F., & Marton, F. (2017). Chinese lesson study, Learning study and keys to learning. *International Journal for Lesson and Learning Studies, 6*(4), 336–347.

Pillay, V., & Adler, J. (2015). Evaluation as key to describing the enacted object of learning. *International Journal for Lesson and Learning Studies, 4*(3), 224–244.
Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *The Cambridge Journal of Education, 35*(1), 69–87.
Runesson, U. (2016). Pedagogical learning theories in lesson and learning studies – Revisited. *International Journal for Lesson and Learning Studies, 5*(4), 295–299.
Runesson, U., & Gustafsson, G. (2012). Sharing and developing knowledge products from learning study. *International Journal for Lesson and Learning Studies, 1*(3), 245–260.
Stenhouse, L. (1981). What counts as research? *British Journal of Educational Studies, 29*(2), 103–114.
Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
Watson, A., & Mason, J. (2006). Seeing exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Teaching and Learning, 8*(2), 91–111.

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Part IV
Mathematics Teacher Preparation and Lesson Study

Preface: Mathematics Teacher Preparation and Lesson Study

Raymond Bjuland

Contents

Lesson study has been increasingly incorporated in global teacher education programs as a professional learning approach. Yet, there has been a call for further detailing the use and adaptation of the lesson study structures that support prospective teachers' learning (e.g., Larssen et al. 2018; Ponte 2017). Research has revealed that there are significant differences in educational cultures and systems (e.g., Stiegler and Hiebert 2016), and it is necessary to consider organizational and logistical variation when incorporating lesson study in teacher education programs (e.g., Davies and Dunnill 2008). The chapters in this section address some of these issues, reporting from experiences with incorporating lesson study in teacher education programs from many parts of the world.

Based on 15 years of developmental work from experiences and research on lesson study within teacher education in Iceland, Guðný Helga Gunnarsdóttir and Guðbjörg Pálsdóttir report in their chapter from a recent study in which prospective mathematics teachers ($n = 21$) participated in a yearlong teacher education program where lesson study was incorporated as a professional learning strategy. One important finding reveals that the prospective teachers were clearly influenced by resources given by their teacher educators in the courses at the study program. The prospective teachers were eager to discuss and try out ideas in practice, and they become more aware of the complexities of teaching during this structured and inquiry-oriented approach to teaching and learning. This study emphasizes the important role of the teacher educators as the knowledgeable others since the prospective teachers had the opportunity to discuss the progress of their work and get immediate feedback from the teacher educators during the lesson study process.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
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These authors have found that a crucial driver for learning to take place for prospective mathematics teachers within lesson study is the significance of establishing and participating in learning communities of practice.

Jennifer Lewis presents another study carried out in a methods class for elementary preservice teachers in the United States, showing that lesson study can serve as a bridge between theory and practice. Lewis argues that the research lesson is the main component of the lesson study model in bridging the theory-practice divide since it is the actual enactment of a lesson with careful planning of a real-time lesson and reflection and revising afterward. This author points to the challenges of incorporating lesson study with preservice (prospective) teachers since it differs from settings in schools with collaborative and experienced teachers with shared leadership. In Lewis' study, there were little contributions from the preservice teachers in planning of the research lesson. Lewis was the instructor with a "top-down" mandate that, together with a team of university instructors, did most of the design of the research lesson. However, with a more fine-grained examination of the lesson study process, Lewis found that the preservice teachers interacted more equally with the university teachers in the post-lesson discussions based on observations of the enacted lesson. The value of this study lies in its illustration of the challenges and affordances of lesson study with preservice teachers. It shows the important role of the university instructor as the knowledgeable other, particularly in the design of a research lesson in which the university instructors have the expertise to find and pose possible tasks which the preservice teachers do not yet possess. In accordance with previous studies (e.g., Stiegler and Hiebert 2016), Lewis suggests that lesson study with preservice teachers requires extensive adaptations, but this case study indicates that lesson study provides preservice teachers with important experience to cultivate mathematical care for children's learning of mathematics in a fourth-grade classroom.

In contrast to the approaches of the studies reported on from cultural settings in Iceland and the United States, Koichi Nakamura takes on the task of making visible to readers how Japanese prospective teachers learn to teach mathematics through problem solving through a lesson study approach. This author attempts to capture the process of improving lessons, focusing on one prospective teacher's activities throughout the process of planning and enacting research lesson, and reflecting on the lesson over 3 weeks of teaching practice. In Japan, it is common that prospective teachers experience three steps of a lesson study cycle. They write a lesson plan with learning goals and main questions and anticipate student solutions to the math problem given to the class (step 1). The prospective teachers teach their revised lesson plan, while other prospective teachers and the supporting teacher make observations and collect data (step 2). The prospective teacher participates in a post-lesson discussion with those who observed the lesson (step 3). As readers, we learn that a typical Japanese lesson consists of four main activities: "the teacher presents the problem, students try to solve the problem on their own, the class discusses solution methods, and the teacher gives a wrap-up summary." Nakamura concludes that the supporting teacher as a qualified highly skilled knowledgeable other plays an important role in order to facilitate prospective teachers to learn teaching mathematics through problem solving. This finding resonates with findings from the studies conducted by Lewis and by Gunnarsdóttir and Pálsdóttir in which university teachers are knowledgeable others, helping prospective teachers to refine their understanding of what their students need to learn.

Reflecting on 15 years of using lesson study in a mathematics methods course in the United States, Blake E. Peterson, Dawn Teuscher, and Thomas E. Ricks describe that they have struggled to help secondary preservice teachers (PSTs) to develop a rich conceptual understanding of mathematics. These authors have made an effort to encourage PSTs to have rich conversations about the mathematics of their lessons and be aware of how students think about this mathematics. Two crucial US cultural views prevented the PSTs from engaging in rich mathematical conversations. The PSTs' views of mathematics were mostly procedural, and they were also more concerned with their own teaching performance rather than paying attention to student learning. This finding resonates with findings from studies in Western Europe (e.g., Bjuland and Mosvold 2015). From the reflections on the revisions and changes made to their methods course, addressing the two US cultural barriers that made it difficult for the PSTs to engage in rich mathematical conversations, the authors noticed that these adaptations have much in common with authentic Japanese lesson study (JLS). Critical facets of JLS include (1) *the knowledgeable other*, (2) *structured reflection meetings*, (3) *PSTs working in groups*, and (4) *the iterative process*. The value of this study "highlights the importance of consistent, multi-cycle lesson study implementations by educational researchers, with continual refinement through multiple iterations, to account for unforeseen issues that will naturally arise."

There is a danger in lesson study research that many studies report from descriptions of stories without including a theoretical framework for analyzing data collected from lesson study cycles and without relating discussions of findings to theories (e.g., Bjuland and Mosvold 2015; Larssen et al. 2018). This is not the case with the chapter written by Suanrong Chen and Bo Zhang. These authors report from a research study, aiming at improving prospective teachers' (PTs) lesson planning knowledge and skills through an adapted lesson study cycle. Thirty-nine prospective teachers (PTs), enrolled in a methods course at a teacher preparation program in China, participated in a program of developing the work of planning a research lesson. The framework of mathematical knowledge for teaching (MKT) developed by Ball et al. (2008) was used to make an in-depth analysis of the PTs' lesson plans in order to capture the PTs' existing knowledge and the progress they made through revised lesson plans. One important finding from this study illustrates that the PTs had difficulties in combining both theory and content into practice in their individual lesson plans, but after experiencing the lesson study process, "the participants demonstrated significant improvements in thinking about learning objectives, analysis of content and students, anticipating students' solutions and sequencing mathematical tasks."

In the last chapter of this section, Fay Baldry and Colin Foster take us back to the importance of establishing productive partnerships between university and school in a UK context when exploring the potential of incorporating lesson study in initial teacher education (ITE). The authors highlight the need for mathematics preservice teachers to develop pedagogical learning based on collaborative lesson planning and observation. In line with Bjuland and Mosvold (2015), Baldry and Foster emphasize the importance of formulating a clear research question that relates to student learning of mathematics. The authors propose a theoretical model that illustrates key features of a lesson study for mathematics preservice teachers in ITE. They also emphasize the important role played by knowledgeable other(s) in England. There are two facets of the role: one focusing on the lesson study process itself and one being more concerned with mathematics pedagogy. Baldry and Foster make it clear that there are considerable challenges when trying to modify and adapt Japanese lesson study in a new context or country. But according to these authors, “lesson study has been successfully operationalized in our country by making numerous adaptations and compromises.”

The chapters in this section have addressed the important role played by a knowledgeable other (e.g., university educators) when prospective teachers are presented for lesson study within teacher education programs. In a recent study, Bjuland and Helgevold (2018) have explored a more detailed analysis of facilitation in which lesson study was used as a context for establishing a dialogic learning community, investigating dialogic processes in prospective teachers’ mentoring conversations in field practice. In line with Warwick et al. (2016), these authors have applied an analytic framework involving five dialogic moves that have the potential to illustrate dialogic chains of utterances produced by a mentor teacher (the knowledgeable other) and a group of four prospective science teachers. By applying a fine-grained dialogic framework, it is possible to identity how a dialogic space of interthinking is created in the mentoring conversations where the mentor teacher and the prospective teachers act together (interact) and think together (interthink) (Warwick et al. 2016). From such a framework, it is possible to identify how the mentor teacher challenges the prospective teachers to make predictions of chosen activities for the research lesson and bring the reflections about student learning forward.

In a systematic literature review of 200 recent English language studies, focusing on the exploration of lesson study with in-service teachers beyond Japan, Seleznyov (2018) confirms the complexity and challenges of translating critical components of Japanese lesson study in new contexts or countries. From another systematic and structured literature review on lesson study in initial teacher education, we learn how the concept of learning is represented and discussed and how observation is used to capture evidence for student learning from the lesson study cycle (Larssen et al. 2018). This review indicated that most of these studies did not have structured observations, illustrating no universally held understanding of the observation process. The review also indicated a lack of clarity how learning was defined in order to use learning theory to support observations. The chapters of this section, with emphasis on incorporating lesson study in teacher education programs, are in line with these literature reviews, illustrating the complexity of challenging traditional approaches to learning how to teach. They provide a significant and added value since these studies illustrate the demanding process for prospective teachers to design good tasks and to make structured observations for the lesson study cycle, paying more attention to student learning rather than being concerned with their own teaching performance. Following Larssen et al. (2018), these studies show how research on the use of lesson study in initial teacher education can be rigorous in the ways they incorporate and discuss critical components of lesson study in teacher education programs. Future studies must keep this direction, assuring great clarity of how learning is defined and to be more explicit about the use of conducting structured observation with emphasis on student learning.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407.

Bjuland, R., & Helgevold, N. (2018). Dialogic processes that enable student teachers' learning about pupil learning in mentoring conversations in a lesson study field practice. *Teaching and Teacher Education, 70*, 246–254.

Bjuland, R., & Mosvold, R. (2015). Lesson study in teacher education: Learning from a challenging case. *Teaching and Teacher Education, 52*, 83–90.

Davies, P., & Dunnill, R. (2008). Learning study as a model of collaborative practice in initial teacher education. *Journal of Education for Teaching, 34*(1), 3–16.

Larssen, D. L. S., Cajkler, W., Mosvold, R., Bjuland, R., Helgevold, N., Fauskanger, J., Wood, P., Baldry, F., Jakobsen, A., Bugge, H. E., Næsheim-Bjørkvik, G., & Norton, J. (2018). A literature review of lesson study in initial teacher education. Perspectives about learning and observations. *International Journal for Lesson and Learning Studies, 7*(1), 8–22.

Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6*(2), 169–181.

Seleznyov, S. (2018). Lesson study: An exploration of its translation beyond Japan. *International Journal for Lesson and Learning Studies, 7*(3), 217–229.

Stiegler, J. W., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM – The International Journal of Mathematics Education, 48*(4), 581–587.

Warwick, P., Vrikki, M., Vermunt, J. D., Mercer, N., & van Halem, N. (2016). Connecting observations of student and teacher learning: An examination of dialogic processes in lesson study discussions in mathematics. *ZDM – The International Journal of Mathematics Education, 48*(4), 555–569.

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Developing Learning Communities through Lesson Study

Guðný Helga Gunnarsdóttir and Guðbjörg Pálsdóttir

Contents

Abstract In this chapter, we present our experiences and research on Lesson study in mathematics teacher education in Iceland, based on 15 years of developmental work with teachers and prospective teachers in mathematics.

Recently we conducted a research study with teachers specialising in mathematics teaching and learning. In the study programme, emphasis was put on the development of learning communities by introducing Lesson study. The focus of the research study was to analyse whether Lesson study was useful for the participants to connect new knowledge to their practice and how it influenced their learning communities. The participants formed five Lesson study groups. We gathered data from their preparations and evaluations as well as their lesson plans. Four themes emerged when analysing the data. The themes are preparation of the lesson, lesson plans, post-lesson discussions and evaluation of the lesson study process. They followed the timeline of the Lesson study process.

Our main findings are that the participants formed learning communities built on trust, collective learning and shared beliefs and practice that developed further during the Lesson study process. They also found Lesson study a fruitful

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professional learning strategy to improve their teaching and learning. The main challenge for the participants was to anticipate students' responses to tasks. These findings are in accordance with our previous studies, but in the present study, the participants were teaching whilst studying, and hence the connection between theory and practice is stronger.

Keywords Lesson study · Teacher education · Learning communities · Professional development · Lesson plans

1 Introduction

Teacher education programmes have seen considerable development in recent years with a growing body of research focusing on the structure and content of such programmes. In these studies, emphasis is put on the connection between theory and practice, in which teaching practice plays an important role. Careful preparation of lessons in conjunction with discussions and reflections of the teaching experiences are critical features of teaching practice. Lesson study is an approach that integrates these elements and has been used with both teachers and prospective teachers. Lesson study is grounded in the idea that a fruitful way to learn to teach is to work together with colleagues on lesson planning and by collaboratively observing and reflecting on students' learning and each other's teaching (Dudley 2015a; Grevholm 2006; Hiebert et al. 2003).

We became familiar with Lesson study for the first time in connection with the TIMSS 95 video study, when Stigler and Hiebert (1999) drew attention to the practice in their book *The Teaching Gap*. In 2002, we began introducing Lesson study as a professional development strategy to our prospective teachers specialising in mathematics teaching. Through the years we have developed our approach further and adopted it to different groups of prospective mathematics teachers and to our teacher education programme that has gone through many changes in the last 15 years.

We have studied our use of Lesson study with various groups of teachers and prospective teachers (Gunnarsdóttir and Pálsdóttir, 2010, 2011, 2013, 2016). It is our belief that participation in learning communities creates favourable conditions for professional development, and therefore it is important for prospective teachers to learn and try out ways to be active members of learning communities. It is our understanding that prospective teachers start their professional development at the onset of their teacher education. It is only the starting point in their career-long learning process in which they have to take responsibility for as active participants.

In 2017 we conducted a research study with prospective mathematics teachers. The focus of the research study was to analyse whether Lesson study was useful for the participants to connect new knowledge to their practice and how it influenced their learning communities. Our research questions were:

- How does Lesson study support the participants in connecting theory and practice?
- Do the participants find Lesson study useful in developing their learning communities and as a professional development strategy in a wider context?

2 Learning Communities and Lesson Study

Many researchers have worked on defining a knowledge base or important competencies for prospective mathematics teachers (Grevholm 2006; Hiebert et al. 2003; Niss 2003). Competency for professional development is considered important, as well as learning how to create professional learning communities during formal teacher education. It is assumed that prospective teachers have to learn to work together and establish learning communities that can support them in acquiring knowledge as they enter the teaching profession.

One of the things we have put an emphasis on in planning courses for prospective mathematics teachers is the creation of learning communities. Hord and Sommers (2008) define professional learning communities as "communities of professionals caring for and working to improve student learning together, by engaging in continuous collective learning of their own". We refer to that definition but use the concept learning community for a group of professionals working and learning together. Characteristics of learning communities are:

- Shared beliefs, values and visions
- Shared and supportive leadership
- Collective learning and its application
- Supportive conditions
- Shared personal practice

(Hord and Sommers 2008; p. 9)

Research indicates that learning communities can play an important role in supporting teachers on their continuous path of improving their teaching and sustaining their professional knowledge (Loucks-Horsley et al. 2010; Wei et al. 2009). Learning communities, where teachers share their understanding of the nature of good teaching and work together on improving their practice, seem to create fruitful conditions for teachers' professional development (Borko et al. 2011; Desimone 2009; Hammerness et al. 2005). In learning communities, competencies of its members can develop in several regards such as professional communication and collaboration (Jaworski 2007; Loucks-Horsley et al. 2010).

Our aim is also to introduce professional learning strategies to our prospective teachers that they can use when they enter the teaching profession. In professional development projects, teachers are now more than ever encouraged to analyse and deal with factors in their own working environment, with regard to both the learning environment in the classroom and the role of the teacher in a wider sense (Fernandez 2002; Loucks-Horsley et al. 2010; Wei et al. 2009)

Lesson study is often referred to as an example of professional development strategy. Through Lesson study, teachers create a learning community in which they learn and cultivate their profession in cooperation with their colleagues (Lewis et al. 2009). According to Loucks-Horsley et al. (2010), there are four factors that characterise effective professional development. These four factors are:

- Increasing teacher knowledge
- Developing good teaching
- Increasing leadership abilities
- Creating learning communities

Lesson study is an example of a professional development strategy that meets all four criteria.

3 Adaptation of Lesson Study

Recently, Lesson study and variants like Learning studies have become more noticed and used in the United States and in European countries. The first European research meeting on Mathematics Education and Lesson study was held in Dublin in March 2018. Many researchers and users of Lesson study are concerned with whether a professional development activity like Lesson study with a century-old history in Japan can be replicated in other cultures (Doig and Groves 2011; Dudley 2015a; Perry and Lewis 2009). Cultural conditions, which support Japanese Lesson study, are according to Lewis and Tsuchida (1997) a shared curriculum, collaboration amongst teachers, critical self-reflection and stability in educational policy.

Lesson study is the main model for professional development in Japan. In Japan an emphasis is put on preparing the research lesson by studying the mathematics and student thinking, writing up a detailed lesson proposals, teaching and observing the research lesson, post-lesson discussions and deep reflection about the context and what was learned. Some of these elements, like preparation, are often superficial in the cases outside Japan (Fujii 2016; Takahashi and McDougal 2016). In Japan there is a strong culture for teachers researching teaching in their own school, and teaching is considered as a public activity (Doig and Groves 2011).

This calls for an adaptation of Lesson study in other cultural settings (Doig and Groves 2011; Perry and Lewis 2009). The development of a professional learning strategy like Lesson study requires time for collaboration as well as for developing the practice to understand its main components. It is not about developing a good lesson plan that can be used by others; it is about capturing students' learning and development, teachers' own learning and deepening both content knowledge and pedagogical content knowledge in collaboration with others. This calls for an outside expert who has an extensive knowledge of the topic, its teaching and learning and insight into important features of Lesson study.

4 Lesson Study in Prospective Teacher Education

During the last decade, there has been growing interest for the use of Lesson study in prospective teacher education. Cajkler and Wood (2015) reviewed research on Lesson study in prospective teacher education and found that the majority of the studies have focused on the teaching of mathematics. There are many variations in how Lesson study is incorporated into prospective teacher education. Many of the studies closely follow the formal Japanese Lesson study approach, where the teaching of the research lesson is carried out during teaching practice. There are some differences in the ways that teacher educators and mentoring teachers engage with the Lesson study process and whether they emphasise the role of the specialist in teaching the subject (Cajkler and Wood 2015; Murata and Pothen 2011). Access to external expertise can lead to more critical evaluations and further development of teaching. There are also examples of prospective teachers using lesson plans designed by others (Corcoran and Pepperell 2011; Jansen and Spitzer 2009).

In the Lesson study model developed and promoted by Dudley (2015b) in the United Kingdom, research lessons are planned with three particular types of students in mind. For example, there can be a case where teachers are concerned about particular students, or there might be one high achiever, one low achiever and one that is representative of the middle range of achievers. The intention focusing on three cases is to make it easier for the prospective teachers to plan and teach the research lesson whilst keeping in mind the diversity in the class and to provide the prospective teacher with an opportunity to observe how the teaching is affecting different students in different ways. Other versions of Lesson study in teacher education include microteaching to peers, planning lesson and discussing the plan with mentors and university teachers and studying and evaluating lesson plans (Cajkler and Wood 2015).

Several studies have been conducted on Lesson study showing positive results when used with prospective teachers (Burroughs and Luebeck 2010; Cajkler and Wood 2015; Fernandez 2005; Potari 2011; Tsui and Law 2007). Lesson study seems to be an effective way to bridge theory and practice, to develop knowledge and encourage reflection about student thinking and to develop knowledge about mathematics and its teaching in collaboration with peers and knowledgeable others (Cajkler and Wood 2015; Potari 2011). According to Tsui and Law (2007), the cooperation of prospective teachers and their teacher educators changed from being a community in which the emphasis was on teaching the prospective teachers, to a community where everyone (prospective teachers, mentor teachers, teacher educators) was learning. Burroughs and Luebeck (2010) studied the learning community of prospective teachers and practicing teachers who collectively took part in Lesson study. Following their participation in Lesson study, the prospective teachers became more acutely aware of the implications entailed in building up learning processes as well as the importance of cooperation between teachers at different levels. It also made them more critical in their discussions and encouraged them to think more about their role as teachers. However, both the teachers and the prospective teachers struggled with anticipating the extent and content of the students' knowledge and assessing the appropriate solution strategies. There is evidence that the development of reflective skills is one of the main challenges facing practicing teachers and prospective teachers taking part in Lesson study (Bjuland and Mosvold 2015; Fernandez et al. 2003).

5 Our Previous Research

In our previous research (Gunnarsdóttir and Pálsdóttir, 2011, 2016), we studied what kind of learning community was created through use of Lesson study with prospective teachers. The learning community was characterised by the development of professional language, collaborative competence, focus on student learning and focus on mathematical content. According to our findings, the prospective teachers developed their professional identity during the Lesson study process. They became more aware of the complexities of teaching, but they also became more eager and stronger in dealing with the issues arising from these complexities. The discussion about what is important to know and understand about a mathematical content area became a part of the work itself, and furthermore, it became obvious that the prospective teachers' understanding of the content influenced their ideas on what they considered important. The prospective teachers had a common knowledge base in mathematics teaching and learning through their studies. Through the Lesson study process, they used it in their discussions and planning and got an opportunity to connect theory and practice.

Lesson study also gave us as teacher educators an opportunity to enter the practice field in the role of a partner rather than an evaluator, since the emphasis was placed on increasing the students' learning opportunities rather than assessing the prospective teachers' performance. Lesson study has a potential for shifting the focus in the teaching practice and place the collaboration between the teacher educators, prospective teachers and mentor teachers into a different context. But it also has to be noted that we also took on the role of a knowledgeable other in the whole Lesson study process.

In a second study (Gunnarsdóttir and Pálsdóttir 2013), we interviewed teachers who had taken part in Lesson study during their teacher education and had been teaching 1–2 years. The study revealed that participation in collaborative practices like Lesson study influenced their ideas about professional development. They all found it very important and supportive to be a part of a learning community. They see their professional development as closely linked to practice. Even though the teachers take part in collaboration and team teaching in their school, they would also like to get opportunities to form learning communities outside their school to enable in-depth discussions about mathematical content and teaching practices. Due to their specialisation, they all had a leading role in mathematics teaching at their school in spite of limited experience. Therefore, they felt a need to be a part of a bigger professional community of mathematics teachers. Schools in Iceland are often small with only one or two teachers who have some specialisation in mathematics teaching and learning.

Because of our positive experiences with the use of Lesson study in prospective teacher education, we want to develop our practice with different groups and within different settings in teacher education. An important part of continuous development is to be aware of changes and study their effects.

6 The Context of the Present Study

In the school year 2017–2018, two groups of prospective mathematics teachers participated in a yearlong teacher education programme for specialising in mathematics education. All the participants were practicing teachers, but they had little or no specialisation in mathematics teaching previously. We therefore consider them to be prospective mathematics teachers. The programme consisted of three 5 ECTS courses which were taken one at a time throughout the school year. One group focused on teaching mathematics in grades 1–4 (12 participants) and the other in mathematics teaching in grades 5–7 (9 participants). Half of the participants where distance learners who took part in the teaching sessions via their computers.

In the first course, the content was numbers and number-talks with didactical focus on discussions. Emphasis was also put on creating a learning community amongst the participants. The second course focused on geometry, inquiry-based learning, creativity and the use of manipulatives and other resources. The last course was on operations with whole numbers and fractions where emphasis was put on teaching and learning processes and professional development. The book series, *Making Sense of Mathematics for Teaching* (Nolan et al. 2016), was used in all courses. In the book series, the readers are encouraged to try out ideas in collaborative settings in order to deepen their understanding of mathematics and improve their mathematics instructions.

Lesson study was introduced in the first course, and the participants read articles/ book chapters and discussed Lesson study as a professional development strategy with focus on learning communities. During the school year, the participants' communication and collaboration developed, and learning communities were formed. The participants read about and tried out new ideas on mathematics teaching in the courses which they experimented with in their own practice. The groups discussed and reflected on their experiences. In course assignments the participants planned lessons together, taught them, reflected on them and then delivered a written report.

The prospective mathematics teachers first tried out a formal Lesson study process in the third course. They formed five groups with 3–5 participants (three groups teaching grades 1–4 and two teaching grades 5–7). In some cases the groups were geographically spread so they communicated via Internet and observed lessons via Skype or by watching video recordings of lessons.

The groups planned and taught one lesson, discussed and revised the lesson plan based on their experiences and taught the lesson again in a different class in a different school. The teacher educators played the role of the knowledgeable other and participated in some group discussions during the preparation. The Lesson study process was in progress during the whole course, for 6 weeks. Each group used 3–6 hours per week for their work with Lesson Study. The goal settings and planning process was given 2 weeks. All the groups used a *four-column lesson plan* form from Matthews et al. (2009). In the lesson plan, the overall goal and the steps of the lesson are presented. The form also requires predicting student responses, preparation of appropriate teacher responses and assessment of student learning. The research lesson was taught during a period of 2 weeks. Participants in each group observed the teaching of the lesson with focus on student learning.

After the first round of teaching, the whole process of the lesson was discussed within each Lesson study group, and the lesson plan was revised on the basis of these experiences before the lesson was taught again in a different class. After the second round of teaching, each of the two groups (1–4 grade group and the 5–7 grade group) met with their teacher educators. There, the whole process was analysed, and the use of Lesson study to develop mathematics teaching and as a professional development strategy was discussed. All groups submitted a written report of their work where they described their journey, how they managed the process and gave insight in their discussions and thoughts.

7 Method and Data

In the study we wanted to gain insight into the whole Lesson study process conducted by the five Lesson study groups. The focus was to analyse their experiences when participating in a Lesson study process. Our interest was to find out how they benefitted from this way of collaborating around the planning and teaching of research lessons. Focus was put on the learning communities the participants formed and the opportunities Lesson study created for collaboration and communication. We wanted to look into where the ideas they decided to try out in practice came from, how their ideas developed during the planning process and how they argued for their choices of topics and instructional approaches. We also wanted to analyse their reflections and discussions during and after the teaching of the lessons and how their ideas about the content, students learning and the learning community developed.

Data was gathered during the whole Lesson study process. The data consisted of audiotapes from meetings when the participants prepared and evaluated the lessons, lesson plans and written reports from each of the five Lesson study groups. We listened carefully to the audiotapes and read the written reports with our questions in mind looking for important elements in each phase of the Lesson study process. We organised the data in a timeline, where we first look at the preparation phase and then lesson plans, post-discussions and the teachers' view on their participation in the

Lesson study process and the learning community created. When presenting the results, we have given the groups the names LS1, LS2, LS3 (teaching grades 1–4), LS4 and LS5 (teaching grades 5–7). All groups were able to meet and observe lesson together except for group LS5. In LS5 two participants were present during the teaching of the lesson, and the other two observed via Skype. It has to be noted that there were less than ten students in the classes being observed. All participants are prospective mathematics teachers, but for simplification we refer to them as teachers when presenting the results and in the discussion.

8 Results

Results are presented following the timeline of the Lesson study process. First we present results regarding the preparation of the lessons. Then we analyse the lesson plans and how they reflect the teachers' ideas about the chosen topic and students' reactions to the problems posed. In the post-lesson discussions, we look into how the teachers talk about the progress of the lessons and what they find important to discuss and change before the lesson is taught again. Finally we give insight into the teachers' experiences and views on Lesson study as a professional development strategy.

8.1 Preparation of the Lessons

In the preparation phase, the teachers discussed openly various ideas they wanted to work with, both regarding the content and mathematical processes. They were clearly influenced by the topics and readings in the courses they had attended during their participation in this study programme. They had common ground and explored and experimented with new ideas they had encountered in the courses. They wanted to challenge themselves in how to approach a topic and how to lead and take part in classroom discussions. This was apparent in all groups. As an example, teachers in grades 1–4 had read about and explored how children's understanding of the equal sign develops. Two of the groups (LS2 and LS3) wanted to build their lessons on this knowledge and try to find out their own students' understanding.

> We built our lesson plan on the four benchmarks to work toward as children's conception of equal sign evolves, that are presented in the book *Thinking mathematically*. (Carpenter et al. 2003) (LS3)

A group from grades 5–7 was preoccupied with students' understanding of fractions and how to relate them to real-life situations. This group had been working with number talks on fractions and wanted to encourage their students to talk about their learning experiences.

> In the article *Digesting student-authored story problems* by Alexander and Ambrose (2010) it is discussed that it is a good way to let students make word problems with fractions if you want to find out about their understanding and thinking about fractions. (LS4)

The teachers had individually read different articles about number sense and operations. When discussing a topic for a lesson, a teacher in LS1 drew attention to an idea she had read about, and the group decided to put that idea into practice. The teacher had read about *Division Quilts – A Measurement Model of Division* (Pratt et al. 2015). She found it interesting to experiment with this model because the authors claimed it to be a fruitful way for students to see a reminder clearly. The group bought the idea and based their lesson plan on the use of this model.

In the discussions in the Lesson study groups, the teachers share with each other their visions about mathematics teaching and voice ideas and ask questions about suitable tasks and approaches. They are learning and developing their ideas together and strengthening their learning community.

> We decided to investigate the meaning of the equal sign after we saw and discussed an interesting video where a teacher is supporting students to build up an understanding of the equal sign. (LS2)

In LS4 the teachers discussed that their students found it difficult to work with fractions and word problems. They decide to combine ideas from two readings about fractions and word problems and build their lesson plan with the aim to find out students' understanding and thinking about the task.

These examples show that the resources that were introduced in the courses influenced the teachers. Common experiences and the learning community they had created shaped their ideas and discussions.

8.2 Lesson Plans

From the lesson plans, it is clear that the teachers were trying to put into practice new ideas and approaches they had been introduced to in their studies as indicated in the previous section. Four out of five lesson plans refer directly to book chapters or articles. and from them ideas for assignments and instructional approaches were developed. The participants choose well-defined problems to work with and put them in an instructional sequence where communication between students and students and teachers is central. It is evident in the plans that they were preoccupied with improving their ability to lead classroom discussions.

In all lesson plans, the four-column setup presented by Matthews et al. (2009) was used.

Steps of the lesson: Learning activities and key questions	Expected student reactions or responses	Teacher's response to student reactions/things to remember	Goals and method(s) of evaluation

The groups looked at other ideas to organise a lesson plan, but they all choose the same form. When, for instance, LS3 argues for their choice of the four-column lesson plan, they refer to the experiences of the teachers quoted in the article by Matthews et al. (2009).

> Teachers that have used the four-column lesson plan feel that this way of organising a lesson improves their instruction and makes the lesson more students centred. (LS3)

For them (LS3) this way of organising lesson seemed helpful in preparing carefully for a lesson by foreseeing how students would react to problem and register their possible solution strategies.

When analysing the plans, it was clear that teachers think of the lesson in steps and that the flow of the lesson is based on the communication between teacher and students. In the lesson plans, a problem for the students to solve and discuss is the focus of the lesson, and the students are supposed to argue for their solutions. When the teachers try to anticipate the students' responses, they focus more on right or wrong answers than the main mathematical ideas in the lessons. They give examples of possible answers but not ideas about student thinking. LS2 worked with the task $8 + 4 = \_\_ + 3$. Here the teachers anticipated students' answers like 12, 15 and 9 but did not elaborate on students thinking and misconceptions leading to these answers. LS3 worked with the same type of task and encountered similar problems when recording possible student responses even though they thought the four-column lesson plan would be helpful in this respect.

The third column in the plan, where the teachers anticipate teachers' reactions to students' responses, is supposed to prepare the teachers to be flexible and to be able to react to the responses (Matthews et al. 2009). From the lesson plans, it is evident that often the teachers were prepared for different responses and that they might have to take a different approach to keep the focus of the lesson. Examples of that are:

> The teacher invites the student to work with another student. (LS1)

> The teacher invites the students to use manipulatives. (LS3)

> The teacher uses lower numbers or higher numbers in a problem. (LS2)

Sometimes, the teachers have anticipated only one way to solve a problem. For example, in a lesson plan from LS2 where the focus is on the meaning of the equal sign, the teacher is supposed to ask:

> Is this right or wrong? $7 = 3 + 4$ (LS2)

In this lesson plan, it is clear that the students are expected to argue for their right or wrong answer. At the same time, the teacher plans to show the students in the end that the way to solve the problem is to add the numbers 3 and 4 and get 7 on both sides of the equal sign. There is no mentioning of other ways to work with the problem, for instance, by using manipulatives.

From the fourth column, it can be seen that the teachers want to put emphasis on seeing and noticing students' reactions and learning. In the lesson plans, the teachers write, for instance, that they intend to do that by listening to the students' discussions and by evaluating their understanding of the mathematical concepts in focus. In the fourth column, remarks like the following can be seen:

> Listen carefully and try to find out if a student is understanding the concept and able to connect it to daily experiences. (LS1)

> Let students work in pairs and explain their solutions. (LS3)

The teachers are also preoccupied with the students' participation in the lesson. From all lesson plans, it is evident that students' involvement in the activities is an important indicator of a successful lesson. In order to evaluate students' involvement, LS4 divided students in small-mixed ability groups, and one observer followed each group and noted the role and participation of the members.

8.3 Post-lesson Discussions

During the post-lesson discussions, the groups followed a procedure often recommended in Lesson study. It starts by giving the person that taught the lesson the opportunity to express his/her first impressions. Then all the other members of the group talk about their observations during the lesson, and the group discusses their experiences. On this basis the lesson plan is revised.

When observing, the teachers used different ways to see and notice what was happening in the classroom. For example, LS1 and LS3 used focal questions from Lewis (2002) which were arranged in the following categories: Academic learning, motivation and engagement, social behaviour, students' attitude towards learning, instructional features and information requested by the instructor. This clearly influences their post-lesson discussions. They take into consideration many elements and try to connect them and analyse the outcome of the lesson, for instance, assessing student learning by observing gestures and expressions

> It was pleasant to see what happened when the boys found out they did not need to calculate to find the missing number – their face lighted up. (LS3)

Both groups (LS2 and LS3) discuss the importance of peer support and notice how students help each other by discussing and explaining solution strategies. They also express that their new insight into how children develop their understanding of the equal sign helped them to assess the students' learning and progression.

> By having benchmarks and study the development of children's mathematical thinking we can more quickly find out where the child is and support its further development. (LS3)

LS2 focuses mostly on the teacher and his/her actions and whether he/she succeeds to follow the lesson plan and reach the goals of the lesson. The other groups focused mostly on the students, their interest and reactions during the lesson and whether they seemed to be understanding and learning.

In the post-lesson discussions after the first round of teaching, the teachers are preoccupied to improve the lesson in order to reach the main goals and make the whole teaching process better. They have a tendency to simplify the problems, explain the main concepts and give more direct instructions on how to approach the problems.

> In the teaching of the second lesson we felt everything went better because we had fixed the lesson plan and skipped what we had found in the first round to be too difficult for the students. We had less constrains and the students didn't have to use both addition and subtraction (of fractions) in the same word problem. (LS4)

This is an example of a change where a task is simplified without losing focus of the main goal of the lesson which was to use certain fractions and make two word problems, one with the answer less than one and the other bigger than one. The teachers had also given the students both circles and squares as models to work with. It confused the students, and in the second round of teaching, only circles were used.

Another example is that in Icelandic the same word, *afgangur*, is used for a reminder in division and for what you have left when you have paid a certain amount with money. In the first lesson plan of LS1 the teachers start to discuss the concept from what you have left when you have bought some sweets. They realise when observing the teaching that this is not helpful for understanding what a reminder is in division.

> After our redesign of the lesson plan the teaching of this strategy for dividing went much better. The teacher showed a problem on a screen. She/he used manipulatives in solving the problem with the students' participation and then discussed with them the concept reminder. The teacher guided the class through the solution before they started to work in pairs. (LS1)

In the lesson plan, the students are to use a new strategy that makes the reminder clearly visible. In the second round of teaching, it was decided to begin the lesson with explaining and demonstrating this strategy and then put focus on the reminder.

This shows that the teachers had a need to discuss with each other mathematical concepts and refine and deepen their own understanding of them. This is also evident in LS3. They had to discuss which words could be used when working with meanings of the "equal sign". They decided on the basis of their observations and discussions that they needed to change the beginning of the lesson in order to direct the students' focus on the main goals. The teachers felt that they needed to add a first step where the main concept, the equal sign, is discussed and explored with the students.

In the post-lesson discussions, the groups discussed the structure of the lessons, how to begin a lesson, the meaning and which words to use for mathematical concepts and students engagement and understanding.

8.4 Evaluation of the Lesson Study Process

Our data – written reports from each group and evaluation meetings of the grade-level groups – show that participation in the Lesson study was a positive experience for the teachers. They found it easy and valuable to follow such a detailed process as

Lesson study. It was new for them to observe teaching and students learning in such a systematic way and get the opportunity to reflect and discuss their observations with fellow members of a learning community.

> We found it very educative to follow the Lesson study process in mathematics. It was demanding and interesting to work within a certain framework and have to anticipate students responses. (LS2)

All participants were surprised how much they learned from observations of lessons they had taken part in planning. Members of LS5 who observed the lesson through Skype had the same experience. All the groups gave positive feedback on their observations.

> To observe other teacher teach, analyse the teaching and evaluate it is very valuable for us as professionals. (LS3)

The lesson plan was made in close collaboration, and all group members took ownership of it. The careful planning had resulted in a common understanding and responsibility for the process of the lesson and that made it easier for them to discuss critically how to refine it.

> Because we made the lesson plan together and discussed it a great deal we mean that we all understood the plan the same way. (LS4)

This common understanding also made it easier for the teachers to focus on the students and their learning and how instructional changes affected how students approach their tasks. By observing and discussing the lessons with other teachers, they got the opportunity to improve the lessons and learn from each other, for example, from how other teachers communicate with students and respond to their questions.

The use of the four-column lesson plan was demanding, and it was challenging for the teachers to foresee the students' reactions and prepare teacher's responses to them. The teachers found the plan useful in order to make the lessons more focused and student centred.

> By going thoroughly through the teaching process, following a detailed plan and discussing and evaluating it after the teaching of the first lesson we got better results and the students in the second round of teaching developed better understanding. (LS1)

In their evaluation of the Lesson study process, LS3 used a list of items designed to guide reflection after Lesson study from Lewis (2002). This helped them to focus on the goal of the lesson and the learning and behaviour of the students. The teachers also discuss issues regarding their own content knowledge and pedagogical content knowledge. Their discussion became deeper and put their experiences in a wider context than the other groups.

All groups found Lesson study an effective and professional development strategy they would like to take part in on a regular basis, for instance, once every semester. It is clear that being a part of a supportive learning community that fosters collaboration between professionals is valuable for the teachers and it encourages them to take risks and try out new ideas. Their common knowledge base and shared visions established during their joint studies created opportunities for the teachers to develop professionally.

In their final remarks about their participation in the Lesson study process, the teachers also focus on the teacher's role in lessons and how participation in the whole process could contribute to their professional development. They expressed a wish to participate in a learning community and believe it is a fruitful way for them to improve their teaching. The teachers find Lesson study a helpful and well-structured professional development strategy.

9 Discussion

The use of Lesson study as a final assignment in this study programme for prospective mathematics teachers was fruitful both for the participants and us the teacher educators. The teachers had developed a good and productive learning community. Through the courses they had participated in rich discussions and learning environment and had gotten new ideas from readings and communications within the group. This is what research has pointed as good conditions for teachers' professional development (Cajkler and Wood 2015; Desimone 2009).

The teachers tried out ideas they learned about in the courses and developed them to use in their practice. They concentrated on discussions between teachers and students and tried to develop good teaching practices with student learning in mind. Their choices of topics were strongly related to the content of their courses and of their wish to challenge themselves by trying out new instructional approaches. They didn't feel they were restricted by curriculum or school culture/traditions. Being a part of a learning community such as Lesson study motivates and encourages the teachers to challenge themselves as has been pointed out by Loucks-Horsley et al. (2010) and Wei et al. (2009).

It surprised the teachers how valuable it was to observe lessons taught by others. It is not a common practice in Iceland, and research has shown the teachers do not get much professional guidance in their schools except from the support they get by working and planning together with teachers at the same grade level (Ólafsson 2014). The teachers both found they learned a lot from observing the teacher and by focusing on students' learning. LS4 built on the idea from Dudley (2015b) to focus on specific students when observing the lesson. That helped them to focus on students' learning but not on the teachers' actions. The teachers in the countryside are technologically minded, and they find ways to observe lessons through different media. The development in ICT is constantly making communication and cooperation easier. This opens up for building learning communities across schools which is important for schools in Iceland which often are small and geographically spread. Based on our experiences working with distance learners in teacher education, it is possible to build effective learning communities especially when you follow a structured approach like Lesson study (Gunnarsdóttir and Pálsdóttir 2013).

In the reports and final discussions it was evident that the learning communities created were built on trust, and the teachers are not afraid to voice their ideas and concerns. The support they got within the study programme, both from each other and the teacher educators, created conditions for collective learning and its application in their practice. This indicates that they have built learning communities that have characteristics of learning communities according to Hord and Sommers (2008).

Burroughs and Luebeck's (2010) research on teachers in mathematics Lesson study has shown that both teachers and prospective teachers have problems with anticipating the students' reactions. Bjuland and Mosvold (2015) and Fernandez et al. (2003) came to similar results. This was also the case in our study. The teachers had a tendency to talk about right or wrong answers that are often associated with traditional view of mathematics teaching and learning. It can also be seen that the teachers were not used to talk about mathematics and mathematics teaching. They did not seem to have the vocabulary to talk about mathematics teaching and students learning in details. Our previous studies with prospective mathematics teachers have shown progress in the use of professional language (Gunnarsdóttir and Pálsdóttir 2011, 2016). The group in the present study had more experience and knowledge of teaching in general but were not familiar with professional literature in mathematics education. The Lesson study process helped them to develop their professional language quickly and through that and observations of lessons to link theory and practice.

10 Concluding Remarks

There is criticism that some of the elements of Lesson study, like the preparation, are often superficial in the cases outside Japan (Takahashi and McDougal 2016). In this Lesson study process, shorter time was used for preparation phase than in our previous work with Lesson study (Gunnarsdóttir and Pálsdóttir 2011, 2016). The teachers were in a learning community and had been gathering and discussing ideas they were eager to try out in practice. They did not need a very long time studying the topics and planning. It was also an advantage when it came to the Lesson study process that the teacher educators had built up a relationship with the teachers and had knowledge of their strength and weaknesses that was useful during the whole process. During the preparation and the teaching of the lesson, the teachers were attending the course and had the opportunity to discuss the progress of the work and get support from the teacher educators.

One of the aims of this study was to examine how the teachers managed to use the resources from their studies in developing their practice. The use of Lesson study in a study programme like this gave us as teacher educators valuable insight into how the teachers in communication about their practice used theory and ideas presented in the courses. It was, for example, interesting to see that all groups chose to work with operations and classroom discussions but none with geometry. Little emphasis was also put on working with manipulatives or creating learning experiences in none traditional settings.

Even though we have been going through many Lesson study processes over the years, every Lesson study gives us new experience and insight into how important the learning community is both for the teachers/prospective teachers and for us as teacher educators. We find our role as the knowledgeable other important, but at the same time, it is important to be aware of that it has to be carried out differently depending on the group and situation. In this Lesson study process, we gave the teachers space to work independently in groups but made sure we were available to them anytime. Because the learning communities had been established, our main role was to begin the process and to lead the final discussions where connections between theory and practice were made and the learning experiences were put into a wider perspective. The experience of taking part in this Lesson study process and conducting this study has strengthened our belief that a collaboration with peers and knowledgeable others is a productive way in developing mathematics teaching and learning.

References

Alexander, C. M., & Ambrose, R. C. (2010). Digesting student-authored story problems. *Mathematics Teaching in the Middle School, 16*(1), 27–33.

Bjuland, R., & Mosvold, R. (2015). Lesson study in teacher education: Learning from a challenging case. *Teaching and Teacher Education, 52*, 83–90.

Borko, H., Koellner, K., Jacobs, J., & Seago, N. (2011). Using video representations of teaching in practice based professional development programs. *ZDM Mathematics Education, 43*(1), 175–187.

Burroughs, E. A., & Luebeck, J. L. (2010). Pre-service teachers in mathematics lesson study. *The Montana Mathematics Enthusiast, 7*(2–3), 391–400.

Cajkler, W., & Wood, P. (2015). Lesson study in initial teacher education. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 105–127). Oxon: Routledge.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Corcoran, D., & Pepperell, S. (2011). Learning to teach mathematics using lesson study. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (pp. 213–250). Dordrecht: Springer.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*(3), 181–199.

Doig, B., & Groves, S. (2011). Japanese lesson study: Teacher professional development through communities of inquiry. *Mathematics Teacher Education and Development, 13*(1), 77–93.

Dudley, P. (Ed.). (2015a). *Lesson study: Professional learning for our time*. Oxon: Routledge.

Dudley, P. (2015b). How lesson study works and why it creates excellent learning and teaching. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 1–28). Oxon: Routledge.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 53*(5), 393–405.

Fernandez, C. (2005). Lesson study: A means for elementary teachers to develop the knowledge of mathematics needed for reform-minded teaching? *Mathematical Thinking and Learning, 7*(4), 265–289.

Fernandez, C., Cannon, J., & Chokski, S. (2003). A US–Japan lesson study collaboration reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*, 171–185.

Fujii, T. (2016). Designing and adapting tasks in lesson planning: A critical process of lesson study. *ZDM Mathematics Education, 48*, 411–423.

Grevholm, B. (2006). Matematikdidaktikens möjligheter i en forskningsbaserat lärarutbildning. In S. Ongstad (Ed.), *Fag og didaktik i lærerutdanning. Kunnskap i grenseland* (pp. 183–206). Oslo: Universitetsforlaget.

Gunnarsdóttir, G. H., & Pálsdóttir, G. (2010). Mathematics teacher education at Iceland University of Education. In B. Sriraman, C. Bergsten, S. Goodchild, G. Pálsdóttir, B. D. Søndergaard, & L. Haapasalo (Eds.), *The first sourcebook on Nordic research in mathematics education* (pp. 467–477). Charlotte: Information Age Publishing.

Gunnarsdóttir, G. H., & Pálsdóttir, G. (2011). *Lesson study in teacher education: A tool to establish a learning community*. In M. Pytlak, E. Swoboda, & T. Rowland (Eds.), Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education (pp. 2660–2669). Rzeszów: University of Rzeszów.

Gunnarsdóttir G. H., & Pálsdóttir G. (2013). *New teachers´ ideas on professional development*. In B. Ubuz, C. Haser, & M. A. Mariotti. (Eds.), CERME8 Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education (pp. 3085–3094). Ankara: Middle East Technical University.

Gunnarsdóttir, G. H., & Pálsdóttir G. (2016). Lesson study in mathematics teacher education. In T. E. Rangnes & H. Alrö (Eds.), *Matematiklæring for framtida. Festskrift til Marit Johnsen-Höines* (bls. 61–85). Bergen: Caspar Forlag AS.

Hammerness, K., Darling-Hammond, L., & Bransford, J. (2005). How teachers learn and develop. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 358–389). San Francisco: Jossey Bass.

Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An "experiment" model for teaching and teacher preparations in mathematics. *Journal of Mathematics Teacher Education, 6*(3), 201–222.

Hord, S. M., & Sommers, W. A. (2008). *Leading professional learning communities. Voices from research and practice*. Thousand Oaks: Corwin Press.

Jansen, A., & Spitzer, S. M. (2009). Prospective middle school mathematics teachers' reflective thinking skills: Descriptions of their students' thinking and interpretations of their teaching. *Journal of Mathematics Teacher Education, 12*(2), 133–151.

Jaworski, B. (2007). Introducing LCM – learning communities in mathematics. In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild, & B. Grevholm (Eds.), *Læringsfællesskap i matematikk: Learning communities in mathematics* (pp. 13–25). Bergen: Caspar Forlag AS.

Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: RBS Research for Better Schools, Inc.

Lewis, C., & Tsuchida, I. (1997). The case elementary science instruction. *Journal of Educational Policy, 12*(5), 313–331.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*, 285–304.

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Hewson, P. W., & Love, N. (2010). *Designing professional development for teachers of science and mathematics* (3rd ed.). Thousand Oaks: Corwin.

Matthews, M. E., Hlas, C. S., & Finken, T. M. (2009). Using lesson study and four-column lesson planning with pre-service teachers. *Mathematics Teacher, 102*(7), 504–508.

Murata, A., & Pothen, B. (2011). Lesson study in preservice elementary mathematics courses: Connecting emerging practice and understanding. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 103–116). Dordrecht: Springer.

Niss, M. (2003). The Danish "KOM" project and possible consequenses for teacher education. In R. Strässer, G. Brandell, B. Grevholm, & O. Helenius (Eds.), *Educating for the future:*

Proceedings of an international symposium on mathematics teacher education. Stockholm: The Royal Swedish Academy of Sciences.
Nolan, E. C., Dixon, J. K., Roy, G. J., & Andreasen, J. B. (2016). *Making sense of mathematics for teaching-grades 3–5*. Bloomington: Solution Tree Press.
Ólafsson, R.F. (2014). *TALIS 2013: Starfsaðstæður, viðhorf og kennsluhættir kennara og skólastjóra á Íslandi í alþjóðlegum samanburði.* [OECD Teaching and Learning International Survey]. Reykjavík: Námsmatsstofnun. Available at https://mms.is/sites/mms.is/files/talis_skyrsla_2014.pdf
Perry, R., & Lewis, C. (2009). What is successful adaptation of lesson study in the US? *Journal of Educational Change, 10*(4), 365–391.
Potari, D. (2011). Response to part II: Emerging issues from lesson study approaches in prospective mathematics teacher education. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 127–132). Dordrecht: Springer.
Pratt, S. S., Lupton, T. M., & Richardson, K. (2015). Division quilts: A measurement model of division. *Teaching Children Mathematics, 22*(2), 102–109.
Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*(4), 513–526.
Tsui, A. B. M., & Law, D. Y. K. (2007). Learning as boundary-crossing in school-university partnership. *Teaching and Teacher Education, 23*, 1289–1301.
Wei, R. C., Darling-Hammond, L., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession.* A status report on teacher professional development in the United States and abroad. Technical report. Dallas: National Staff Development Council.

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Lesson Study for Preservice Teachers

Jennifer M. Lewis

Contents

Abstract The research lesson sets lesson study apart from most other forms of teacher learning. The observed interactive, real-time lesson puts theories about learning, expressed as a lesson plan, to the test of practice, in the company of one's professional peers. In a cycle of lesson study, the research lesson sits between the planning that teachers do in preparation for interactive work and the analysis that teachers do after a lesson with students. As such, lesson study holds particular promise for preservice teacher education because it can serve as a bridge between theory and practice, a divide that has confounded teacher education for decades. Typically, preservice coursework provides opportunities for planning before lessons and analysis of artifacts coming out of lessons, but the actual interactive work with children is distant—and it is precisely this interactive work that most concerns preservice teachers. Thus, lesson study, with its inclusion of the research lesson, provides a promising model for preservice teacher learning. This chapter presents a case of lesson study carried out in a methods class for elementary preservice teachers. Analyses of data in this case study show that preservice teachers developed an expansive disposition of *mathematical care*, a repertoire of *pedagogical*

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_24

moves linked to research and children's learning, and an expanded *sense of the teaching self* from their participation in a cycle of lesson study.

Keywords Lesson study · Teacher preparation · Preservice teacher education

1 Introduction

One of the enduring challenges of preservice teacher education is the perceived disconnect between teacher preparation coursework and the realities of classroom life (Lortie 1975; Clift and Brady 2005). Lesson study is a promising addition to common activities in teacher preparation, because it links study and investigation (Grossman and McDonald 2008) to an actual lesson taught in real time and observed in the company of one's colleagues. But lesson study is typically conducted with inservice teachers or a mix of preservice and inservice teachers. Although lesson study was not developed for preservice teachers *per se*, this form of professional development addresses some of the perennial challenges facing preservice teacher education. This chapter begins with an enumeration of some of those perennial challenges and how lesson study might address them directly. The chapter then presents a preservice lesson study experience, describing the ways in which a cycle of lesson study was modified for use with preservice teachers. The chapter then presents the outcomes from a case study (Yin 2017) that examined this lesson study cycle for preservice teachers to determine what preservice teachers stand to learn from their participation in a modified form of lesson study.

2 Challenges of Preservice Teacher Education in the USA

Preservice teacher education in the USA is regarded by its participants and the schools that employ them as inadequate, "too theoretical" on the one hand, and too easy on the other. The coursework is often critiqued as "too ivory tower" and remote from the needs of the classroom; teachers report that most of what they learned came from their field experiences and not from their teacher training program, which is often regarded more as a rite of passage than a true professional preparation. The standards of entry are quite low—teachers are taken from the bottom quartile of college graduates (Lortie 1975). Teacher education has been critiqued as distant from clinical practice (Alter and Coggshall 2009; Grossman et al. 2009). Whether these perceptions of teacher preparation are true or not is beside the point; teacher education is not viewed as a rigorous, worthwhile experience that prepares its participants with practical know-how or the intellectual hardware to help children achieve (Lortie 1975). Perception is reality.

Against this backdrop, lesson study offers features that address some of the challenges of preservice teacher education enumerated above. Here I want to argue that it is the research lesson that distinguishes lesson study from other forms of teacher education. The research lesson provides the sense of time and urgency that is otherwise absent in preservice education (Zeichner 2010). Lesson study's apex is the research lesson, the actual enactment of a lesson with real children in a chosen classroom. The planning beforehand is the lead, and the reflection and revision afterward its denouement. That all elements build toward a publicly viewed live lesson anchors lesson study in practice. Other instructional designs in teacher education may encompass the planning phase of teaching or the defining of a problem of practice. They may include reflection on a videotaped excerpt of classroom teaching and suggested revisions. But the centerpiece of lesson study that lends time and urgency so integral to practice is the research lesson. This is where the fruits of reflection, deliberation, and practical judgment are reintroduced into the test of real-time work. In this sense, lesson study inquiry is driven by practice, and the standards to which its outcomes are judged are practice-based (Stigler and Hiebert 1999).

Lesson study is typically conducted with inservice teachers or predominantly inservice teachers with a few preservice teachers mixed in. Because teachers drive so much of the lesson study process, preservice teachers' relative lack of experience with children and their thin knowledge of curriculum are limiting factors. Also, preservice teachers are not always cohorted long enough to constitute a stable community of practice that can build knowledge and experience together over time, a hallmark of lesson study. Yet some of lesson study's virtues are especially relevant for preservice teachers. The research lesson and the fact that the collaborative planning beforehand and the analysis afterward are anchored to an actual lesson with real children in real time lend a verisimilitude to "real teaching" that captures the urgency of teaching work and the essential feature of teaching—that all one's cases are present simultaneously. These features of lesson study make its constituent activities more credible for preservice teachers than other practices conducted at a remove from classroom life. And lesson study intervenes on the notion that teaching is a private affair. Teaching is, in lesson study, very much a public enterprise. Planning for a lesson is collaborative, and the actual teaching of the lesson is conducted in the presence of a roomful of colleagues. This is not trivial in a culture where the norm is for teaching to proceed behind closed doors (Lortie 1975).

3 Conducting Lesson Study in a Preservice Methods Class

A single cycle of lesson study was conducted within the confines of a semester-long preservice mathematics methods class at a large public university. All preservice teachers in this class were undergraduates nearing the end of their teacher preparation program; none had any formal experience as teachers although they may have had informal experiences volunteering in schools, working in summer camps, and tutoring. There are special logistical and substantive challenges to manage lesson study in the context of a university course:

1. Coherence with methods curriculum: a university preservice math methods class has its own curricular demands, so it is not self-evident how lesson study can be woven into the fabric of such a class and whether it adds intrinsic value to students' experience.
2. Access to a classroom of children: Preservice methods classes that meet at the university must find a willing classroom of children, arrange ways to get there, and become sufficiently informed about the learning needs of a class virtually unknown to them.
3. Knowledge base of preservice teachers: Preservice teachers by definition have relatively little knowledge or experience with children and curriculum. Because participants typically drive so much of the lesson study goals and lesson design, for preservice participants, accommodations are necessary. This involves some modifications to the canonical form of lesson study that some might say constitute deviations from its essential features.
4. Time. In a course that is already short on time for covering the content preservice teachers need, carving out time for lesson study is a challenge. This is especially acute because lesson study is not the completion of a single lesson study cycle; the power of lesson study is typically realized in its continuing conduct through multiple cycles. The timeline for the lesson study cycle is shown in Fig. 1.

In this lesson study cycle, preservice teachers were invited to prepare an introductory lesson on decimal numbers, with an emphasis on conceptual understanding, by the teacher of a local fourth grade class. There was almost no contact between the school, classroom teacher, instructor, and students in preparation for the research lesson. All 23 preservice teachers in the methods class participated in all aspects of the lesson study cycle described here.

To plan a lesson for the fourth grade class, preservice teachers studied the children's textbook materials introducing decimal numbers in *Everyday Mathematics* as well as sections of other textbooks on the same topic, including *Mathematics* published by Scott Foresman–Addison Wesley. The preservice teachers also examined where this lesson fell in the textbooks' scope and sequence of curriculum in Fourth Grade *Everyday Mathematics: Teacher's Manual and Lesson Guide, Volume A*, to see its introductory presentation of decimals. To better understand the topic, preservice teachers also studied a relevant chapter in their course textbook (Van de Walle et al. 2004) as a resource.

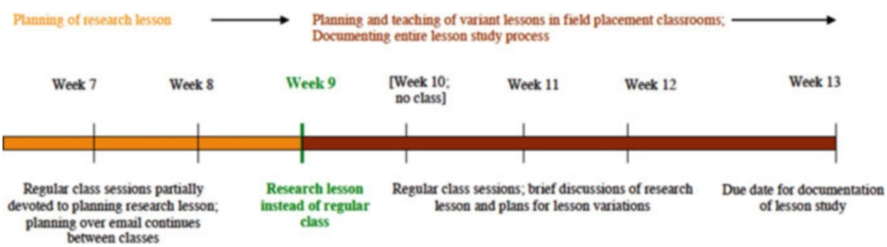

Fig. 1 Timeline for modified lesson study in preservice mathematics methods class

In contrast to the shared goal setting that inservice teachers determine as part of standard lesson study practice, in this case the classroom teacher set the goals for the lesson: fourth graders “will gain a conceptual understanding of part-whole idea in decimal numbers.” The preservice teachers then spent about two hours in each of three class sessions talking about the mathematical topic of decimal numbers and considered various possible contexts for teaching this idea. The preservice teachers worked in small groups to generate possible decimal representations or story contexts and as a class explored each one in depth. For example, one preservice teacher suggested that times in sporting events are reported in tenths of a second, and this context was thought to be an engaging one for fourth graders. An athlete’s achievement in a race, for example, is reported in seconds: 10.7. This was a context in which children would understand that the small differences in time represented by the decimal numbers matter. Another part of the reasoning here was that children like sports and this would be a familiar context. On the other hand, preservice teachers noted that a possible source of confusion could be that running times can be expressed in minutes and seconds (fractions of one-sixtieth), as in the following: 35:58, which represents 38 min and 58 s, not tenths or hundredths of minutes. Another suggestion was to have children create a situation that would generate statistics with decimal numbers in them—this suggestion involved launching toy cars from ramps of different heights to see the time and distance traveled. These would be measured in decimal numbers as well. This was also thought to be an engaging context for children and familiar to them as a game. The physics of the activity and how well the cars could be controlled and measured were presented as possible drawbacks. There were many such possible contexts explored. Here one preservice teacher recalls the planning phase:

> Before deciding on these types of problems, we contemplated several other ideas. One was to use money. We knew from our interviews in project one that fourth graders were able to “understand value and purpose with converting dollars into cents.” While money is an easy topic to use in story problems, money would be difficult to translate into thousandths. The language difference between decimals and money may also have to be explained. For example, $4.46 would be read differently than 4.46. These are good points to raise about the use of money. Another idea was to use a number line. We thought that students could use a number line to find where certain decimal numbers fit in relation to one another. The difficulty with using a number line is that it doesn’t lend itself easily to story problems that would be engaging to fourth graders. Another was to use a stopwatch to measure time. We were unsure to which place stop watches measure time, so we were hesitant to use it in our story problems. (Lara[1])

Three sections of this mathematics methods course were being offered during the same semester; all sections followed a common curriculum with instructors from all three sections collaborating a great deal, although only this one section included a cycle of lesson study. The instructors from the other two sections attended a single class session to contribute to the discussion of lesson planning that focused on learning the terrain of the topic. Later, a group of mathematicians and mathematics

[1] All names are pseudonyms.

educators in a university study group in mathematics education provided feedback to lesson plan ideas via email as the preservice teachers developed the research lesson.

Transcripts from the class' planning sessions document the preservice teachers' proposed approaches to teaching decimals. Mostly they were searching for a real-world context in which decimals appear. In one brief class discussion, the following possible contexts were mentioned: finish times for race cars, elapsed times for Olympic sporting events, money, bank balance sheets, digital thermometers, linear measurement using meters and centimeters, and odometer displays of mileage. For each context, the preservice teachers would consider its appeal to children, the complexity of the representation, the display of decimals, and feasibility. So, for example, the children's textbook, *Everyday Mathematics*, employed the odometer as one context where decimals are found. The preservice teachers in this discussion raised the following questions: "How many fourth graders have been behind a steering wheel? Does every child have a car in their family?" As one preservice teacher observed, "This just doesn't seem very 'everyday' for a fourth grader." There were similar deliberations around bank balance sheets: "You'd have to understand this format. How many kids would be familiar with a bank balance sheet?" Another added: "The context [of a bank statement] plus the decimals seems overwhelming." There was a very interesting exchange around how race times are displayed and whether this would be confusing or helpful: "There are multiple ways that race times are shown on a stopwatch. There's zero, zero, zero, and the decimal point. We'd have to do something that wouldn't go into minutes so that you'd have decimals." "The display is zero zero colon zero zero." "No, the display goes colon zero zero minutes colon seconds point decimal points." In reading over this transcript, the preservice teachers were concerned about the authenticity of the contexts and the representations of mathematical ideas. Does a context appeal to children and make use of their life experiences? Are decimals displayed in a clear and comprehensible way? In contrast, the comments of the mathematicians and mathematics educators who came to visit the methods class brought the preservice teachers back to mathematical concerns that were more pedagogical in nature. One asked whether the representation of base-ten blocks in *Everyday Mathematics* would confuse children about part and whole when the unit switches between a unit cube and a larger cube representing 1000. Another instructor asked a parallel question regarding measurement with centimeters and meters: What would count as one whole? These kinds of questions had not been raised by preservice teachers. The questions raised by the mathematicians showed how robust mathematical knowledge can make possible the consideration of pedagogy in new ways. If this single joint meeting with the preservice teachers and the mathematicians and mathematics educators is representative, it was clear that knowledge of teaching techniques is limited absent what Ma has called "profound understanding of fundamental mathematics" (Ma 1999).

The preservice teachers continued to weigh different approaches in the design of the research lesson. As the research lesson date approached, there was a flurry of email between class sessions as participants were still deciding what tasks to try and in what sequence. In the end, the instructor chose the "broken calculator" idea that is

Mrs. Hayward's calculator is broken. The problem is that the 6 key doesn't work. I
you type 6, it doesn't show anything in the display.
Is her calculator useless now, or can she figure out a way to use it to do any problems she wants?

Can she still display any number she wants into her calculator?
We are going to explore this and decide.
My prediction and reason:

My conclusion and reason:

Can she get her calculator to show these numbers? Try each one, and write down the steps you think she can take.

406	4.06
.65	1.065
3.640	6.000
5.62	6.66

Fig. 2 Research lesson handout

shown in Fig. 2 and formulated a few "warm-up" exercises to precede that central task.

On the day of the research lesson, the entire class met at a local elementary school instead of the university classroom. The university instructor taught the lesson designed by the class; preservice teachers and other university instructors observed and took field notes. The research lesson was videotaped and transcribed; all children's work was collected and scanned so that all preservice teachers could analyze the children's work. As seen in Fig. 1, for 2 weeks following the research lesson, homework assignments included reflections on the research lesson including examination of all records of practice, and class meetings devoted an hour each week to sharing analyses of the research lesson.

4 Participation Structures in Lesson Study

Figure 3 notes the ways in which participants' roles were modified for the conduct of lesson study with preservice teachers. In lesson study with inservice teachers, facilitators or teacher leaders endeavor to cede authority to the teacher participants (for more on this, see Lewis, J., Chapter "Learning While Leading Lesson Study," this volume). Teachers are meant to lead all aspects of lesson study: select the learning goals for students, drive the study toward that goal and devise a research lesson plan, teach the lesson in front of peers, and collect and analyze the data from the research lesson. In a preservice context, these roles are modified by necessity. The instructor plays a more dominant role, leading with a stronger hand than a facilitator might with more experienced teachers. The instructor's knowledge and experience with children, curriculum, and teaching dictates that the preservice participants lead less and observe more, although all are able to participate to varying degrees. The exception to this rule is in the post-lesson analysis and reflection, when data from this cycle of lesson studied showed that preservice teachers and their instructor were able to contribute nearly equally. And in fact, because the instructor taught the research lesson and the preservice teachers observed and collected data, in many ways their reflections and analyses were stronger and more evidence-driven than the instructor's. This inversion of the typical asymmetry between instructor and students in a university class is a noteworthy feature of lesson study; we will return to this later in the chapter.

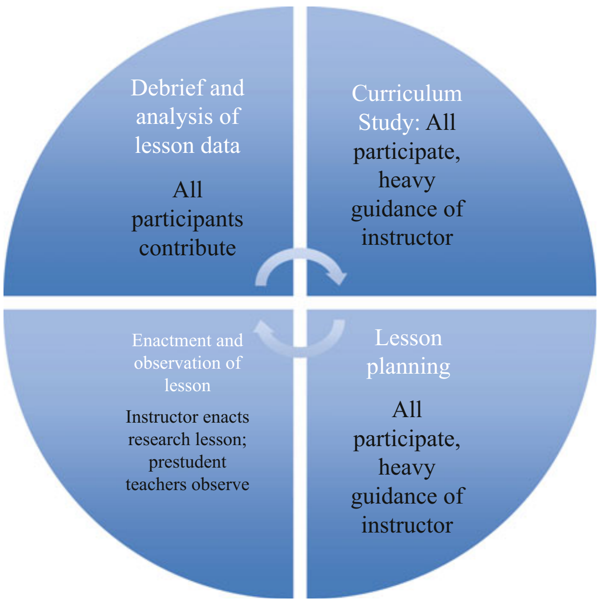

Fig. 3 Roles for inservice versus preservice participants in a cycle of lesson study

5 Method and Data Sources

This study followed the tradition of teacher research as "systematic self-critical enquiry" (Stenhouse 1985, p. 8) to generate knowledge *for practice*. For this specific study, the intent was to generate knowledge for teacher educators and for preservice teachers, to be used in practice (Lampert 1985). Brown (1992) described how her work as a researcher-teacher allowed her to develop "a theoretical model of learning and instruction rooted in a firm empirical base" (p. 143). Brown's work demonstrated the ways in which a researcher-teacher could create or design teaching and learning situations that she may not find in the field or laboratory. This kind of research is problematic, Brown says, because one cannot control variables, but it is useful because it captures the realities of classroom life, and it projects what is possible in the classroom with all its complexity and variability and unpredictability.

For this research, the primary data sources would be the preservice teachers' work in multiple forms and other artifacts of classroom practice. These data sources include transcriptions of class videotapes, lesson plans, and field notes, instructor's field notes following each class meeting and the field notes of observers, and the written notes and assignments of the preservice teachers in this cycle of lesson study. Children's written work from the research lesson was also collected. Preservice teachers in this university methods class kept a class journal that contained responses to specific writing prompts assigned during each class session. In addition, an extensive assignment that required the preservice teachers to document their experience of the lesson study cycle was also analyzed.

Following Glaser and Strauss (2017), the purpose of data collection and analysis for this research was to develop theory that would contribute to our understanding of preservice teacher learning in lesson study. It is important to note that preservice teachers learn from many experiences in and out of their coursework, and it is impossible to attribute their learning exclusively to lesson study. By comparing preservice teachers' spoken and written artifacts from their methods course before, during, and after the lesson study cycle, we are able to generate some hypotheses about how lesson study may have contributed to their evolving views about teaching, learning, and mathematics.

All data sources were analyzed using QSR NVivo software for qualitative data analysis, with more careful inspection of data from four focal preservice teachers. Preservice teachers' written work and all relevant transcribed talk were coded at the sentence level or greater, labeled with themes that stood out for each passage. The qualitative analysis software allowed for open coding without any hierarchy or order at first; codes were later grouped and reordered as needed. Passages of written work or transcribed talk could be coded such that they appeared in multiple categories. Several rounds of qualitative coding were conducted to analyze the data: first, using open coding (Corbin and Strauss 2008), one researcher coded all the data. Open coding produced 107 codes at the outset. These were grouped into five focused codes. These five focused codes were identified through inspection of the coded data, naming emerging themes and noticing categories and patterns (Corbin and Strauss 2008). Following open coding, the researcher recoded all the data with the focused codes.

6 Results

Five broad categories of preservice teacher learning are apparent in the data from this study: mathematical care, general pedagogical concerns, views of mathematical pedagogy, beliefs about instructional design, and development of the teacher identity. In this chapter, we will examine three of these themes: the disposition of *mathematical care* evident in preservice teachers' discussions of the lesson and in their planning for other lessons; the refinement and expansion of possible *pedagogical moves*, where mathematics, children, and pedagogic design are seen to compose an interdependent and dynamic system; and the emergence of *teacher identity*, a teaching self that is clearly developing over time. Each represents a domain of teaching practice. *Mathematical care* is a disposition in teaching, a way of approaching the work, a cognitive state that organizes how a teacher thinks about what comes at her. *Pedagogical moves* are acts that teachers perform; they are driven by dispositions perhaps but visible as actions. The language of "move" recalls chess moves or dramatic moves that are contingent, conscious responses to earlier events and that occasion events to follow. *Development of teacher identity* is located in the domain of self as instrument in instruction and is an emerging awareness and cultivation of the self in a professional role; it is a form of embodied knowing (Belenkey et al. 1986). Taken together, these domains constitute a whole of practice by including dispositions, ways to act, and knowledge of the self. The table below shows some of the categories of preservice teacher learning revealed in the data and discussed in this chapter (Table 1):

Table 1 Selected categories of preservice teacher learning in lesson study

Mathematical care (*dispositions*)
Specific numeric considerations
Analysis of math processes
Mathematically worthwhile
Math taught to self
Children's perceptions of math
Pedagogical moves (*actions*)
Meticulous use of language
Teaching for understanding
Precision in representations
Linking classroom management to content learning
Attention to time allocation
Teacher identity (*self-knowledge*)
Own math knowledge
Personal needs vs student needs
Emotions in teaching
Confidence and vulnerability

6.1 Mathematical Care

"Mathematical care" is the name for an umbrella category encompassing 20 different subcategories coded under mathematical capacities, such as "the use of actual number examples," "mathematical analysis," "mathematically worthwhile," or "taught self this math." Preservice teachers' writing and discussion surrounding lesson study was remarkably full of mathematical concerns expressed in such subcategories—this constituted 40% of the text units in the preservice teachers' writing, up from 27% in their previous work in the methods class. And the level of detail was striking. In the passage below, Ariana describes her teaching a lesson in her field placement classroom following our research lesson. She designed a lesson also on place value but modified for first grade. Note her careful consideration of numeric choices in the lesson:

> For the first number, 12, almost everybody used one long and two cubes.[2]
>
> Many may have "borrowed" this idea from their neighbor, but I think that's ok.
>
> The important thing was that everybody could count the long as ten and add eleven, twelve for the cubes.
>
> For the next number, 21, answers varied a bit more. One student demonstrated two longs and a cube at the overhead and another showed one long and eleven cubes. Both counted longs as ten and cubes as singles.
>
> The final number they made was thirty-four and it was wonderful to see one of my "struggling" math students put three longs on her mat, count ten, twenty, thirty, and then add four cubes! The student who was picked to demonstrate her answer on the overhead, however, showed she wasn't quite getting it. She put down two longs and four cubes. When asked to count her number, she reverted back to one-to-one correspondence and counted twenty-four. We asked, "You've got twenty-four, how can you make it thirty-four?" She added a cube instead of a long. Because she is not yet counting by tens, it is not surprising that she did not automatically add ten. She counted her total again and got twenty-five. (Ariana)

There are a number of points to note in this passage. One is the mathematical detail that she provides in her description. Ariana references actual numbers, repeatedly, and she provides specific detail about how children composed and represented those numbers: "The final number they made was thirty-four and it was wonderful to see one of my 'struggling' math students put three longs on her mat, count ten, twenty, thirty, and then add four cubes!"

Why does this matter? How is this different, and more helpful in teacher education, than simply asserting that one child, who usually struggles in math, seemed to understand? This preservice teacher's discussion of mathematical details matters for a number of reasons. Reflections on teaching rarely include the kind of detail about content, and about children's encounter with content, that allows for teacher growth. The recounting of precise details about how the content can be represented—longs and cubes here—and how children think about them, "ten, twenty, thirty, and then add four cubes" here, provides a window onto the minutiae of practice that so often

[2]A "long" is a rod of 10 units in length; a "cube" is one unit.

fly by in practice and therefore elude consideration outside of a structure like lesson study, where observation of instruction in the company of one's colleagues is the norm.

"Mathematical care" was supported by preservice teachers' use of academic resources on and about teaching, which otherwise seem so remote from the "real work" that teachers do. Lesson study builds a credible context in which these academic resources on teaching suddenly strike preservice teachers as valuable. A classic example is the struggle in teacher education for students to find relevance in the readings and assigned exercises they do in coursework to their work vis-à-vis their fieldwork. This lesson study cycle prompted participants to draw upon readings from their methods course to explicate what they saw in the field. In papers the preservice teachers wrote about the lesson study cycle, readings were referenced extensively, without any prompting. This contributed to their growing "mathematical care." Excerpts from two preservice teachers' papers below show this:

> The course readings have given me a strong idea of what children are expected to learn in place value. According to one reading, students should develop a full understanding of number meanings from their transition through K to 5, as well as begin to experience some number sense for large numbers. Ideally, students will be able to perform and understand the following:
>
> 1. Perceive sets of ten (and tens of tens) as single entities. These sets can then be used to describe how many. This is the main principle of base ten numeration.
> 2. The positions of digits in numbers determine what they represent – which size group they count. This is the main principle of place-value numeration.
> 3. There are patterns to the way numbers are formed.
> 4. The groupings of ones, tens, and hundreds can be taken apart in different ways.
> 5. "Really big" numbers are best understood in terms of familiar real-world referents. (Van de Walle et al. 2004). (Alissa)

> "A full understanding of place value includes a complex array of ideas and relationships that develop over the K-6 grade span." (Van de Walle et al. 2004, p. 149)
>
> Place value concepts build on earlier number ideas (Ibid. 150). While children in kindergarten begin to count beyond the primary numbers and up to 100, it is counting by ones, based strictly on one-to-one correspondence with no conception of place-value ideas. They can neither sort by tens for counting purposes, nor explain the value of the tens place (Composite Picture of What Children Know About Place Value, handout, ED 518, October 10, 2002). It is a "pre-base ten understanding" of numbers referred to by some researchers as "unitary." (Van de Walle et al. 2004, p. 150) (Malorie)

"Care" in the term "mathematical care" combines commitment to student learning with appreciation for mathematics, along the lines of what Bruner has called "intellectual honesty" (1960, p. 33): "Any subject can be taught effectively in some intellectually honest form to any child at any stage of development." It combines attention to the discipline with attention to the learner. Thus, mathematical care is comprised of both pedagogical attunements and mathematical attunements. Both are needed.

The following passage from the debriefing of the research lesson illustrates the intent focus on aspects of mathematical import. The preservice teachers are discussing how decimal numbers should be read and how much to press for conventions and accuracy when reading such numbers aloud:

PT[3]	Oh I have a question about Carolyn, and she was up there [in front of the class] with a calculator and then she was explaining how, what her thinking was and the reason that she was going to, she had the one point zero two five and that she was going to add four. And I wondered if that was going to get her in trouble and then I, you didn't talk about it at all or anything and it worked [inaudible] out but I wondered and everyone seemed to understand what she was talking about, I did too. But I –it's just one of those things that can be misinterpreted in a way.
I	I've found their oral language often [inaudible] will trip them up and they were trying to figure out what to call something that was like one point zero six five, how do you say that six five? You know we went through having them say them in a lot of different ways but in terms of how to quickly, explicitly say that number and use correct mathematical terms that wasn't something that the kids were really comfortable with oral language.
PT	Sometimes they said point six hundred and forty and that was [inaudible] is that, do you think that that's acceptable because I know you didn't, you said oh that's another way but you didn't make a note that you wouldn't want to say it that way.
I	What did I do, who knows what I did with those, there were a bunch of them?
PT	You'd say it, you'd say okay one hundred, or one point two [inaudible] thousandths should've corrected in saying it again but without bringing attention [inaudible].

In this exchange, the preservice teachers are wondering aloud about the accuracy of children's oral productions. They question how much adult correction is needed to make public utterances comprehensible and how this would make the child feel in public. There are at least three levels of possible intervention under consideration: (1) the teacher faithfully revoices the student productions, right or wrong; (2) the teacher restates, in correct conventional form, but without commenting on the change; and (3) the teacher restates correctly but makes public the nature of the change made, so as to make that an object of instruction.

6.2 Pedagogical Moves: Ways to Act

Preservice teachers showed an expanded facility with nuanced pedagogical moves through their work in lesson study. The data show an increased ability to notice and consider the manipulation of fine-grained teacher actions to occasion learning, and participants were more likely to approach instructional planning with an experimental stance. Prior to the lesson study, there was little mention of what a teacher would have to do to make a task work with children—that was taken for granted and therefore left invisible. It was not until the research lesson that the preservice teachers on their own would initiate conversations about what a teacher would have to do, in its most considered detail, and how such subtle teacher's moves could shape instruction differently. They wondered aloud about trying out a variety of moves and thought about the respective outcomes for children's learning. This heightened sensitivity, I should note, was the ability to consider or analyze what a

[3]PT = preservice teacher; I = instructor.

teacher would do or had done, either by observation or in an imagined future lesson; it is important to distinguish this from their ability to execute such actions in the flow of instruction. The preservice teachers' growing ability to notice, discuss, and dissect fine-grained teaching moves can be seen across a number of subjects: meticulousness regarding teaching language, commitment to conceptual understanding, design of instruction, and concern for management issues. Below I consider each of these four in turn.

Meticulousness Regarding Teaching Language From the research lesson forward, the preservice teachers demonstrated a newfound meticulousness regarding teaching language. For example, in the post hoc analysis of the research lesson, one of the preservice teachers began to play with alternative wordings of pivotal discussion questions in the lesson for fourth graders. Carly, one of the preservice teachers, asked the following in the post-lesson discussion:

Carly:	I wrote down, I heard you saying a lot of, "What do other people think?" to get other people involved and get their opinion. I was wondering if it helps or doesn't help to say, "Is anyone thinking something different?" Cause like I felt like when you said, "What do other people think?" they kind of just went along with what the first person said rather than finding other answers.

The conversation continued, with the instructor and the preservice teachers considering how different phrasings of this might play out. What would different wordings elicit from children? How would the substance of children's ideas be changed by alternative formulations of this prompt? This indicated a careful attention to nuances in lexical choice and inflection and how these choices shape what is learned. Again, what is remarkable here is the degree to which choices of language were understood to affect the mathematics and what children would learn, as well as how they would feel. This playing with alternative wordings does not appear prior to the research lesson.

Luke, another of the preservice students, picked up this same thread a few minutes later in the debriefing, following on Carly. Luke: "You could ask questions that distinguish between 'Does everyone agree?' and 'Does anyone think differently?'" He then continues, voicing the implications that may follow from these alternative wordings: "Kids will learn to read whether the teacher is signaling correctness. You would need to learn to question equally. What questions would you ask of right and wrong answers so that the kids don't guess from the way you ask if something is right or wrong?" The preservice teachers were experimenting with different possible teaching "moves" in the debriefing session that followed the observation lesson.

Emphasis on Conceptual Understanding The preservice teachers returned again and again to the idea that children should attain conceptual understanding of mathematics, and this appeared in their views of children and of instructional design. This is related to mathematical care but is expressed in a set of actions and thus is included as a pedagogical move rather than a disposition.

Often this commitment to conceptual understanding is stated as a kind of policy orientation regarding mathematics teaching, as Kayla has done below. She links what a child is learning now to that child's mathematical horizon.

> We were able to see that without a conceptual understanding or at least the beginnings of a conceptual understanding, students' difficulty in understanding Place Value appeared to increase as the grade level increased. For example, an area that strikes me as one that students have difficulty with is the idea of "applying" the concept of Place Value to math work. A student might be able to point to the ones or the tens place in a number, but he or she might not be able to transfer this knowledge to the organization of an operations problem, and might therefore come up with an incorrect answer or come up with an answer that he or she doesn't understand. A false sense of understanding will not be useful as the math being taught in school becomes more complex, and demands a conceptual understanding of Place Value. (Kayla)

Classroom Management Linked to Content Learning Like most preservice students, these students were greatly concerned with issues of classroom management. They worry about how they will lead children to do a particular task, pay attention to the learning at hand, follow directions, and maintain order. These are not simple demands for new teachers, though once overcome they are practiced with little self-awareness. These concerns occupy little of the teacher education curriculum at the university, and they are viewed as incidental and even a bit beneath the lofty goals of teaching content of intellectual heft. What was noticeable in the lesson study cycle is that the preservice teachers talked about management concerns in ways linked directly to ambitious learning goals for their pupils. There are numerous examples of this from the preservice teacher work; here is one:

> After watching the lesson taught at North Bluff Elementary, I immediately wanted to redesign the lesson to appropriately teach my Kindergarten class of 22. I felt that the lesson's overall goal of Place Value was very fitting to the learning needs of the Kindergarteners with which I work. Through the several lessons on Place Value that I have taught to this class, and through the information that I gathered as a result of [an earlier assignment], it has become clear to me that my students need as much exposure to Place Value as possible. As I thought about how to redesign this lesson for my class, I knew that I too, wanted to use a type of manipulative, and thought that I might also be able to include a part in the lesson similar to the work that was done with the "Broken Calculator." I chose to use Unifix cubes for the manipulatives, and I had made enough sticks of 10 for each student in the class. I decided to use Unifix cubes because they are easy to handle, brightly colored, and the students have worked with them before in a mathematical context. I chose to make sticks of 10 for them ahead of time not only to save time, but also to emphasize the importance of learning about groups of 10. By this I mean that I was able to make a big deal of the fact that I had made these special sticks which I think helped to get their attention focused on what we were doing. (Kayla)

Kayla has management concerns: she wants materials that are "easy to handle, brightly colored, and the students have worked with them before." But these management concerns are linked to mathematical learning goals she has for her kindergarteners: "I chose to make sticks of 10 for them ahead of time not only to save time, but also to emphasize the importance of learning about groups of ten . . .[this] helped to get their attention focused on what we were doing."

Awareness of Time The preservice teachers gave much attention to the issue of time in instruction. They were concerned about how much time tasks will take, how time can slip away, and how time is the enemy of understanding. Their lesson plans after the lesson study experience were filled with careful consideration of timing and of instructional sequence. This attention to time was not found in their prior work, and unlike the earlier issues that were on the table throughout the semester (children, mathematics), time was not explicitly asked about in course assignments or in my directions.

> Time is also a factor in my thinking these days. There is never enough of it when teaching a mathematics lesson, which basically means, that if you are going to be pressed for time, you need to be very certain that you are conducting a review and not presenting new information. Exploring new concepts in mathematics is something that requires a good deal of both time and patience. Being patient, seems almost impossible to me when I am constantly watching the clock. (Leah)

Related to time, the preservice teachers are very conscious of sequencing in lessons. A number of preservice teachers raise the issue of sequencing ideas in a lesson. Here is an interesting example below. Note the repeated use of the term "next step" and the mathematical details contained in this participant's comment from our post-research lesson discussion:

PT	The issue with the money, do you think the next step for that class would be that in order to add decimals you've got to be talking about the same type of unit? So that that way they'd know that to have cents that they would have to add one hundred and five cents to their one penny, one cent, that they would have to change just like they change fractions, that might be a next step to have to add similar units when you're talking about decimals.

6.3 Attention to the [Teaching] Self: Embodied Professional Knowing

The preservice teacher work in the lesson study shows a good deal of attention focused on the self as a developing teacher. In their writings more than in discussion, the preservice students talk about themselves in the guise of a teaching persona, and this was an unexpected finding. This is frequently cast in terms of an awareness of their own mathematical understandings:

> This lack of conceptual understanding might seem alarming, but, in fact, we have found in our own graduate-level course work that we college students have some of the same misconceptions. We could throw our hands up in frustration and continue to teach the same way that we were taught, but I believe it is this lack of conceptual understanding that allows even our most procedurally fluent students to make mistakes. As a new teacher, I have to tackle my own misconceptions, and use what I learn to help my students. (Ariana)

The preservice teachers make frequent reference to their insecurities about their mathematical knowledge. Interestingly, Leah ties this to what this means for instruction, as many of the preservice teachers did. Seeing the teaching self in this way is challenging and takes courage. There were repeated comments in the preservice students' papers about the need to learn more mathematics so that they could teach well. There were also candid and critical portrayals of their missteps in teaching. See, for example, Leah:

> Unfortunately, I have to take some (or most) of the credit for my students' confusion. As noted in my lesson plan, I was supposed to start out by asking students to create some examples of their own, in regard to the thousandths place. I was then intended to ask how they knew it was the thousandths place. I was hoping for a very brief discussion that would allow me to assess where some of my students were in their abilities with place value in regard to decimals. I had also hoped that my questions would get them thinking about decimals, by activating their prior knowledge. (My cooperating teacher assured me that this would be a review for the majority of students; this was absolutely not the case.) Regardless of that setback, I blew it when I launched right into the "All-Star Runner" problem. I was so nervous about not having enough time, that I did not follow my lesson plan. Consequently, students did not have time to warm up to the problems I was about to present. Equally problematic was the fact that I did not clarify the objective for them. They were likely unprepared to thoughtfully participate as I said, "We will be working with decimals today" and then launched immediately into the worksheet. When we got to the part about ordering decimal numbers, I did not explain what that really meant. I should have given them a clear illustration, some examples to back up what I was trying to convey and then asked their thoughts on how to correctly order decimals. . . . So did I teach a lesson? Yes, I did. Did I "teach" my students? Unfortunately, the answer to that question is a resounding no. Additionally, I may have confused them in the process. Although this is very disappointing to me, it did cause me to learn a great many things about myself and the way that I need to "teach" mathematics in the future. (Leah)

Kayla, upon reflection, comes to see how her own needs for order may have cut off a child's opportunity to think out loud. She realizes this only upon listening to an audiotape of her lesson:

> I found it very difficult to handle students calling things out and was not happy with the way that I handled all the outbursts. For example, at one point, the boy who had been continually calling out shouted out an incorrect answer. In the split second that it happened, I reminded him that he needed to raise his hand. However, when I went back to listen to the tape, I realized that his calling out might have been an interesting teaching moment if I had let him expand upon his thinking. I felt badly listening to the tape when I realized this, but feel torn as to how to handle those situations. (Kayla)

The preservice teachers show a great deal of self-awareness throughout their written work. They even had an occasional positive thing to say about their teaching!

> I think because of my excitement about the use of a visual representation I have never had such energy or anticipation before a lesson. I could not wait to start. This was the first time I had ever had the feeling that I had something so great that everyone was going to learn. I learned about myself that day that if I design lessons I feel really good about my energy level and desire to teach skyrocket. I was so excited to be focused on the fact that I felt students were really going to learn today and I am sure that excitement came through in my teaching. My body language was more animated. My voice was not monotone and I was having fun; so were the students. We had some banter about how funny it would be to see me run. Students were laughing and having fun. This is how I imagined teaching could be. Maybe not everyday, but a great deal of the time. Throughout this semester this is what I had been trying to improve. In most of my lessons I felt boring and flat. I wanted to have more

> interactions with the students, but I could not engage them. Now I know why, I was not engaged in what I was teaching. (Thomas)

What is clear from the many examples in the preservice teachers' writing is a sense of themselves physically present in their work and the kind of knowing that devolves from their presence in the classroom and in the work. This produces knowledge that is inaccessible to them otherwise. To emphasize this point, I use the term "embodied knowing" (Belenky et al. 1986). Dispositions are essentially cognitive even when they imply action, and dispositions can be present outside the physical reality of the classroom. Pedagogical moves are about doing things interactively with children and materials. Moves follow from dispositions or may lead to the formulation of dispositions. Knowledge of the self can be present when doing pedagogical moves, but it entails an awareness and use of the self in role. This trio of domains functions as an interesting set that can help us organize our thinking about teaching practice and points to areas where interventions in teacher education may be fruitful.

7 Challenges and Affordances of Lesson Study with Preservice Teachers

Lesson study is typically practiced in inservice settings, with experienced teachers. Lesson study is meant to be collaboratively led and participation voluntary. Neither of these features was present in this lesson study: the composition of the group and its leadership was preordained. The students in the preservice methods class were required to participate, and the leadership was not shared as much as assumed by me as the instructor. Catherine Lewis writes: "Top-down mandates and high-stakes assessment have well-known disadvantages, and many common forms of professional development appear to have little impact on instruction. Lesson study provides a collaborative process for teachers to make sense of educational goals and standards and to bring them to life in the classroom" (2002, p. 7). Lewis implies here that voluntary participation in a lesson study group is one essential ingredient for improving practice. Typically, a school or school district plans a number of professional development inservice days that teachers are required to attend. Teachers experience these as a random collection of workshops taught by consultants that may or may not be relevant for instruction. They have little choice but to participate and exercise no control over the content or format of most professional development opportunities. Lesson study, along this dimension, is a stark alternative. Teachers choose to participate, and Lewis suggests that the absence of a "top-down" mandate is key.

In a required course in a preservice teacher education program, of course, participation was not voluntary. It was "top-down" in the sense that the instructor required participation, and it may even qualify as involving "high-stakes assessment," since participation in the lesson study was graded. Is the voluntary, collaborative nature of lesson study one of its essential qualities, or would their absence undermine the possibility of instructional improvement?

Lesson study in an inservice setting presupposes a level of experience and knowledge that preservice teachers are unlikely to possess. While a lesson study group commonly includes outside experts—mathematicians, psychologists, and mathematics educators—the teacher participants are assumed to have a repertoire of teaching moves, knowledge of curriculum, experience with the subject matter, and a sense of the children for whom the research lesson is being designed sufficient to draw from in the design of the lesson. In a preservice methods class, this background knowledge simply could not be taken for granted. Typically, participants join a lesson study group out of interest and desire—again, decidedly not the case for a required project in a required course. Nothing about this lesson study was voluntary for participants. No one came to the work driven by a burning question from practice or seeking opportunities to work in a collaborative environment. Would this process then have little impact on instruction, as Catherine Lewis suggests?

It is important to emphasize here the extent to which this version of lesson study was not driven by the participants but rather by the instructor and by the classroom teacher. This was especially palpable in the planning phase of the research lesson. Although the planning of the research lesson is meant to be a collaborative effort where all participants contribute, in this version the participants, the preservice teachers in this class, contributed very little to the design of the research lesson. There are gestures made at soliciting their ideas, but ultimately the lesson was designed by a team of university instructors, choosing contexts and numerical examples to populate the full lesson plan. This contradicts a fundamental tenet of lesson study, even if it was a logical adaptation given the background of the preservice teachers.

A more fine-grained examination, though, reveals a range of collaborative structures across this lesson study. There were degrees of collaboration across this lesson study, ranging from the minimal collaboration between the instructor and the classroom teacher to the multiple and complex forms of collaboration between instructor and preservice teachers and among the preservice teachers themselves. The methods course instructor, in this instance, also played the roles of lesson study facilitator and the teacher of the research lesson. This entailed collaboration with the fourth grade classroom teacher, mostly via shorthand conveyed between teachers who typically have little time for conversations about practice but who have many shared understandings of teaching. The overt expressions of collaboration were minimal.

What of the collaborations between the preservice teachers and their instructor? The contributions to the design of the lesson study were differential, as mentioned earlier. The instructor provided background materials and ultimately chose the tasks for the lesson. But the preservice teachers were strong partners in considering carefully each possible task, doing analytical work together that is a nice example of the kind of collaboration that lesson study promotes. Similarly, the post hoc analysis of the lesson was highly collaborative in nature. These discussions, anchored in facets of the mathematical tasks, how children would and did perceive them, and what a teacher might and did say, show no distinction between instructor and the preservice teachers. The preservice teachers interacted as equals with the university instructor in these discussions. Certain tasks in the process promoted a kind of work together that erased differences in experience and formal knowledge base, while others were less symmetrical in nature. Finding and posing possible tasks for the research lesson drew more heavily on an expertise that preservice teachers do not yet possess; analyzing how the lesson itself played out and what children did as they learned allowed everyone to participate as more equal participants. This leads me to suggest that the facets of lesson study that require knowledge of curriculum, of child development, and of pedagogical method cast the role of instructor as a more "knowledgeable other." The facets of lesson study that drew upon observation of the enacted lesson and its analysis allowed all of us to participate on a more equal footing. This was, after all, a class of fourth graders unknown to all participants. This contrasts with the typical work between supervising teacher and student teacher, where the supervising teacher is clearly more knowledgeable in nearly every aspect of the relationship.

The preservice teachers also had an opportunity to watch the collaboration between methods instructors, who came to help design the tasks for the enacted research lesson. The instructors batted around several candidates for tasks and talked through how each would play out. This was an example of distributed cognition in teaching work (see, e.g., Hutchins 1996): the group of instructors was able to do work that individually they might not have accomplished. Similarly, the post hoc analysis of the research lesson could in no way have been as rich or multifaceted without collaboration. Such work depends on careful observation from many angles and perspectives and *of* many subjects. Viewing teaching, in all its complexity, is greatly enhanced by the company of others, and even with 23 observers, one has the sense that not all was noted. Collaboration in this context is not for its own sake; it is essential to seeing teaching and learning with clarity. One strong message that participants could glean from this practice in lesson study is that teaching in general benefits from collaboration: teachers see more and better, they design thoughtfully, and they observe and analyze multiple perspectives that only others can bring.

8 Conclusion

Lesson study revolves around the performance of teaching a lesson. And lesson study manipulates the perception of time in instruction. The research lesson plays out in real time, pretending to have been present in the longer flow of instruction that teaching entails. Time is also suspended in the cycle of a lesson study: long study sessions lead up to a single research lesson, and records of the research lesson are kept and referred back to, stretching one lesson over time for the lesson study participants. Lesson study also makes human agency visible in practice. There are a number of facets to lesson study that engender agency: the collaboration among participants, along with the kind of apprenticeship that is afforded in this particular version of lesson study between instructor and preservice teachers. The cycle of planning a lesson, observing it, and then refashioning it promotes a kind of agency. I emphasize here in particular the opportunity to refashion or revise what has been done before as an opening for agency: one is not simply left to repeat mistakes or mimic a form.

Lesson study with preservice teachers requires extensive adaptations and in this case study produced worthwhile gains for participants. The research into lesson study with preservice teachers shared in this chapter indicates that lesson study cultivates mathematical care, provides teachers tools to weigh possible pedagogical moves, and contributes to a developing sense of teacher self among participating preservice teachers. Although the lesson study experience steals precious hours away from the normal routines and curricular demands of a preservice mathematics methods class, these outcomes present a compelling case for including lesson study in preservice experiences.

References

Alter, J., & Coggshall, J. G. (2009). Teaching as a clinical practice profession: Implications for teacher preparation and state policy. Issue Brief. *National Comprehensive Center for Teacher Quality*.

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice, and mind* (Vol. 15). New York: Basic Books.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences, 2*(2), 141–178.

Bruner, J. (1990). *Acts of meaning*. Cambridge, MA: Harvard University Press.

Clift, R. T., & Brady, P. (2005). In M. Cochran-Smith & K. Zeichner (Eds.)., *Studying teacher education Research on methods courses and field experiences* (pp. 309–424). Mahwah: Lawrence Erlbaum.

Corbin, J. M., & Strauss, A. L. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks: Sage.

Glaser, B. G., & Strauss, A. L. (2017). *Discovery of grounded theory: Strategies for qualitative research*. New York: Routledge.

Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and Practice, 15*(2), 273–289.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184–205.

Hutchins, E. (1996). Learning to navigate). In S. Chaiklen & J. Lave (Eds.), *Understanding practice: Perspectives on activity and context* (pp. 35–63). Cambridge, UK: Cambridge University Press.

Lewis, C. C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: Research for Better Schools.

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review, 55*(2), 178–194.

Lortie, D. (1975). *Schoolteacher: A sociological analysis*. Chicago: University of Chicago Press.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Lawrence Erlbaum.

Stenhouse, L. (1985). *Research as a basis for teaching: Readings from the work of Lawrence Stenhouse*. London/Portsmouth: Heinemann Educational Books.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Summit.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2004). *Elementary and middle school mathematics*. Boston: Allyn and Bacon.

Yin, R. K. (2017). *Case study research and applications: Design and methods*. Thousand Oaks: Sage.

Zeichner, K. (2010). Rethinking the connections between campus courses and field experiences in college and university-based teacher education. *Journal of Teacher Education, 89*(11), 89–99.

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How Lesson Study Helps Student Teachers Learn How to Teach Mathematics through Problem-Solving: Case Study of a Student Teacher in Japan

Koichi Nakamura

Contents

Abstract In Japan, one of the goals of student teaching is for prospective teachers to learn how to use problem-solving to teach mathematics. Japanese student teachers conduct a type of professional development called Lesson Study in order to reflect upon and improve their teaching. To help prospective teachers get the most out of their student teaching experience, I examined how Lesson Study helps student teachers learn how to teach mathematics through problem-solving. I investigated whether or not a prospective teacher was able to improve her teaching and how her teaching methods changed throughout her student teaching experience. I collected data from the student teacher's Lesson Study experience by recording her pre-lesson discussions, lessons, and post-lesson debriefs. I analyzed the verbal interactions between her and her students as well as the verbal interactions between her and her supporting teacher. I specifically focused on classroom discussions, the cornerstone of teaching mathematics through problem-solving. My analysis showed that in the beginning, the student teacher was only able to lecture about mathematical facts, but by the end of 3 weeks, she was also able to use her students' mathematical thought processes to make connections between their solutions. I also show how this shift in teaching methods reflects what the student teacher learned about students'

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_25

thought processes during her pre-lesson and post-lesson discussions. In this way, Lesson Study was able to help the student teacher learn how to teach mathematics through problem-solving.

Keywords Lesson Study · Prospective teachers · Problem-solving

1 Introduction

In general, teachers are expected to improve their lessons through reflecting on their daily practices themselves. In Japan, teaching practices are conducted in only 3 weeks. In teaching practice, prospective teachers experience the process of improving their lessons through Lesson Study. Research on prospective teachers generally focuses on pedagogical content knowledge and skills (Karp 2010). For example, Jacobs et al. (2010) focused on examining children's mathematical thinking, and Star and Strickland (2008) used video to help prospective teachers improve their ability to understand how children think mathematically. My research in this case study attempts to capture the process of improving lessons, so we focus on one prospective teacher's activities in lessons, planning, and reflecting in a 3-week teaching practice.

In Japan, math is taught by presenting appropriate problems for the students to solve on their own. Therefore, prospective teachers must learn how children approach solving mathematical problems (Hashimoto 1987; Fujii 2017; Sawada 1999). Rather than simply lecture, teachers give students problems to solve on their own, which fosters their mathematical understanding and thinking. Stigler and Hiebert (1999) refer to this Japanese teaching method as "structured problem solving," as there is a definite structure to how problems are presented, tackled, and discussed. Classes generally spend a whole class period on one problem. Each student tries to solve it on their own first. In the second part of class, the students explain their solutions and discuss them. Wrong and incomplete solutions are also shared so students can learn valuable thinking processes to overcome their wrong and incomplete solutions. Teaching through problem-solving not only teaches students mathematical concepts but also how to think mathematically; but it is not easy for teachers to learn how to conduct teaching through problem-solving.

Lesson Study is the main form of professional development for teachers in Japan. In Lesson Study, teachers will prepare and conduct a lesson to which their colleagues are invited to observe, and afterward the teacher will discuss the lesson with them. Reflection and improvement are indispensable skills for teachers (Shön 1983). Prospective teachers will continue to rely on these skills in their future careers.

In this case study, I examined how a prospective teacher's teaching methods changed over the course of 3 weeks, to identify how and if her teaching improved. I videotaped her mathematics lessons, pre-lesson discussions, and post-lesson discussions. I used these videotapes to analyze the interactions between her and students during lessons as well as the interactions between her and her supporting teacher during pre-lesson and post-lesson discussions. My analysis shows that the subject of my case study's Lesson Study experience improved her ability to teach mathematics through problem-solving.

2 Student Teaching in Japan

2.1 Preservice Teacher Education

To become an elementary school teacher in Japan, one must have at least 67 credits from an accredited 4-year institution to obtain a teaching license and earn a bachelor's degree. Each credit corresponds to fifteen 90-minute lectures. There are four categories of courses students take, elementary subject courses for 8 credits (such as mathematics, Japanese language and literature, social studies education, and physical Education), education courses for 41 credits (such as education concepts and history, elementary mathematics pedagogy, elementary education curriculum and methods), subject or education courses for 10 credits (such as elementary mathematics teaching materials, mathematics curriculum theory), and others for 8 credits (such as Foreign language communication and ICT).

Students at Tokyo Gakugei University obtain more than the ministry of education-mandated minimum number of education credits. They will complete at least 129 credits. Student teaching is part of the education courses. Mathematics major students at Tokyo Gakugei University must complete 32 credits of mathematics courses, such as calculus, algebra, statistics, and others; those are counted in 129 credits. The subject of this case study is a student teacher from Tokyo Gakugei University, specializing in mathematics.

2.2 Student Teaching and Lesson Study

Education majors at Tokyo Gakugei University student teach for 3 weeks during both their junior and senior years. The juniors teach at schools affiliated with the University, and the seniors teach at public schools. Juniors only conduct four or five lessons during these 3 weeks. Through student teaching, one supporting teacher is responsible for a group of five or six student teachers. This group of student teachers mainly teaches classes to students in a single homeroom. The student teachers write lesson plans, observe each other's lessons, and reflect on those lessons together. A prospective elementary school teacher who specializes in mathematics would conduct three mathematics lessons, one Japanese lesson, and one science lesson.

Juniors only conduct four or five lessons during these 3 weeks, but they engage deeply in lesson discussions and reflections as part of Lesson Study. While a full cycle of Lesson Study is comprised of five steps (Fig. 1), prospective teachers generally only experience three of these five steps: lesson planning, research lessons,

Fig. 1 Lesson Study cycle (Fujii 2014, p. 113; Fujii 2016, p. 412)

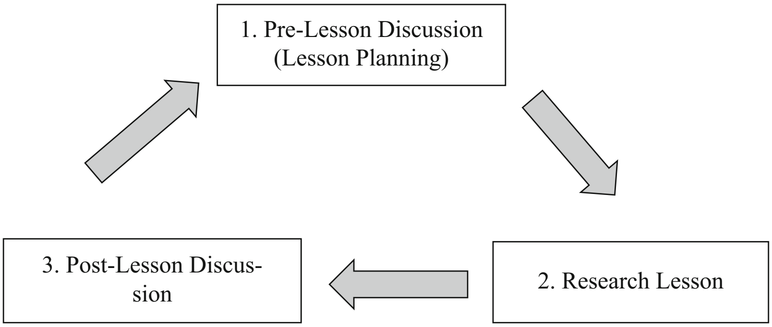


Fig. 2 The three steps of the Lesson Study cycle conducted during student teaching

and post-lesson discussions (Fig. 2). A full cycle of Lesson Study corresponds to school-based lesson study. The first step, Goal Setting, includes identifying gaps between long-term goals and current reality. For prospective teachers, it is clear that their goal of Lesson Study is how to use problem-solving to teach mathematics. The way of setting a goal by prospective teachers is different from that of school-based Lesson Study. We need not consider the first step and also the fifth step, Reflection, for the same reason in student teaching.

The first step of Lesson Study that student teachers experience is lesson planning. They write their own lesson plans. They must write a unit plan which includes the learning goals of that unit and an outline of six or seven lessons. They consult textbooks, the curriculum, and elementary mathematics education books. Each lesson includes the learning goals, main questions, the math problem the students will tackle, anticipated student solutions, anticipated discussion points, and a wrap-up summary. The unit and lesson plans are examined and reviewed closely during pre-lesson discussions with a supporting teacher and other prospective teachers.

The lesson plans are implemented during the next step of Lesson Study, the research lesson. Student teachers teach their revised lesson plan, while other student teachers and the supporting teacher observe and collect data. Afterward, the student teacher who conducted the research lesson discusses it with those who observed, a supporting teacher, other prospective teachers.

This post-lesson discussion is the third and final step of the student teacher Lesson Study cycle. They discuss whether the goals of the lesson were met, the students' responses and learning process, and whether the math problem used to teach the lesson was effective. The supporting teacher gives final comments on the lesson and post-lesson discussion itself. Student teachers use data that was collected by observers during the research lesson and shared during the post-lesson discussion to reflect on their lesson and help them revise their next lesson plan. My case subject taught three mathematics lessons and went through the three Lesson Study steps for each lesson.

3 Structure of a Typical Japanese Mathematics Lesson

In order to analyze how Japanese teachers learn how to teach mathematics through problem-solving, I first need to identify the structure of a typical Japanese mathematics lesson. Japanese teachers aim to teach not just mathematical knowledge and skills but also to how to think mathematically (Souma 1983). This results in a unique lesson structure.

Several researchers have broken down the structure of a typical Japanese mathematics lesson into a consistent sequence of activities. Becker et al. (1990) break down the lesson into seven potential parts: reviewing the previous day's problems or introducing a problem-solving topic, understanding the topic, problem-solving by students, working in pairs or small groups, comparing and discussing, summing up by the teacher, and assigning exercises. Out of these there are four main activities that appear in any lesson: the teacher presents the problem, students try to solve the problem on their own, the class discusses solution methods, and the teacher gives a wrap-up summary. Researchers, such as Stigler and Hiebert (1999), Stigler and Perry (1988), and Miwa (1992), have also all discussed the Japanese mathematics classroom as consisting of these four main activities. Therefore, in this case study, I will also discuss the classroom as consisting of these four main activities.

These four main activities closely relate to each other and must be consistently applied in every lesson. Stigler and Hiebert discuss this:

> A great deal of attention is given to development. They [lessons] are planned as complete experiences- as stories with a beginning, a middle, and end. Their meaning is found in the connections between the parts. If you stay for only the beginning, or leave before the end, you miss the point. If the lesson like this are going to succeed, they must be coherent. The pieces must relate to one another in clear way. (Stigler and Hiebert 1999, pp. 95–96)

The flow of the lesson is crucial. Teaching through problem-solving would not be successful without the third activity, discussion. Discussions explain and reinforce a mathematical concept and are central to every lesson. Miyazaki et al. describe this in depth:

> In this learning system, in order to emphasize variety of students' thinking and realize creative learning activities, discussion among students is an essential activity. Discussion among students is core of lesson. After students solve the presented problem individually, students discuss their solutions in order to understand mathematical idea and get the sophisticated solution. (Miyazaki et al. 1969, p. 185)

Discussion is key to understanding mathematics and developing mathematical thinking. In order to successfully teach mathematics through problem-solving, a teacher must skillfully make connections between students' solutions during discussion. In this case study, I focused on how my subject Koyama connected various student solutions during discussion. By examining how she navigated discussions, I show how she improved.

4 Data

From September 24 to October 10, I collected data on my case study subject, Koyama[1]. The subject of this case study is a student teacher from Tokyo Gakugei University, specializing in mathematics. Koyama is one of the students who understand mathematics education well. She taught fourth graders in a group with five other prospective teachers and one experienced supporting teacher, Aoki. Aoki has 6 years of experience teaching in elementary public schools and 6 years of experience teaching at a university-affiliated elementary school. He specialized in elementary mathematics in these 7 years. Teachers who belong to a university-affiliated elementary school have their strong subject and deeply understand the Lesson Study. They teach five or six prospective teachers every year.

I observed Koyama's full 3 weeks of student teaching. There were five other student teachers in her group. This group of five student teachers observed each other's lessons and participated in every pre-lesson and post-lesson discussion. They taught a collective total of six consecutive mathematics lessons. Koyama taught three of these six mathematics lessons. Her full schedule is shown in Appendix. She observed a total of 64 elementary school lessons, including 15 mathematics lessons.

I collected data from this group of prospective teachers' for 3 weeks of student teaching. As a group, they conducted six consecutive lessons that covered the introduction of the concept of area, formulating the area of squares and rectangles, and finding the area of composite figures. I will refer to these six lessons as lessons #1, #2, #3, #4, #5, and #6, respectively (Table 1). Koyama taught three of these six

[1]All names of teachers and students have been changed to protect their privacy.

Table 1 Titles of lessons

Number of lessons	Title of lessons
#1	Which territories has more space, and how much more?
#2	Which territories has more space, and how much more?: Discussion
#3	Formula for the area of rectangles and squares
#4	What is the area of our classroom?
#5	Units for large areas: 1 a, 1 ha, 1 km^2
#6	Finding the area of composite figure

Table 2 Number of utterances made during Koyama's lessons

		Number of utterances	
	Duration (minutes:seconds)	Teacher	Students
First lesson	45:29	104	95
Second lesson	45:55	121	131
Third lesson	46:38	148	174

lessons, specifically lessons #2, #4, and #6. I videotaped her lessons as well as every pre-lesson and post-lesson discussion in which she participated, for a total of 11 discussions (Appendix).

In order to show how Koyama's teaching method changed, I first analyzed the quantitative data I collected from my video recordings of classroom activities. I measured how much the teacher and students spoke by tracking the number of utterances each made during the lessons. I also measured how long they spoke by counting the number of syllables.

I used my recordings of the classroom to examine the interactions between the prospective teacher and students during discussions. I also examined the videotape data I recorded during pre-lesson and post-lesson discussions to analyze the interactions between the prospective teacher and the supporting teacher. I then used these analyses to make connections between the discussions and lessons to show how Koyama's experience with Lesson Study helped her learn how to teach using problem-solving.

5 Quantitative Data Analysis

Table 2 shows the number of utterances made during Koyama's three mathematics lessons. Both her and the students' number of utterances increase during the second lesson and increase again during the third lesson. In the first lesson, the teacher spoke more than the students. In the third lesson, the students spoke more than the teacher, in a 12:10 ratio.

Table 3 shows the length of utterances made during the first lesson and during the third lesson. I counted the length of utterances by counting the number of syllables spoken in Japanese.[2] I counted the length of utterances during the verbal activities: when the teacher presented the problem, during the discussion of solution methods, and when the teacher summarized the lesson. In the first lesson, the student teacher spoke almost 80 percent of the time. This ratio shifted by the third lesson, in which she only spoke about 60 percent of the time. This marked increase in student speaking can be seen in each verbal activity of the third lesson. Furthermore, in both the first and third lesson, the ratio of the length of utterances made during "discussing solution methods" is similar to the overall total ratio. This underscores how discussions are core to teaching using problem-solving.

6 Interpretive Analysis

6.1 Lesson #2, the Student Teacher's First Math Lesson

Discussion of student solutions is crucial to the success of a Japanese math lesson, but navigating this discussion can be very difficult for student teachers. They must respond in real time to their students' reactions and help them make connections between the solutions on their own, without resorting to pure lecture. This is what teaches children to think mathematically. To analyze if and how Koyama improved her teaching methods, I compared the classroom discussion of her first math lesson, #2, to the one in her third and final lesson, #6. I also recorded Koyama's and her supporting teacher Aoki's thoughts on these lessons during the pre-lesson and post-lesson discussions for these lessons.

Table 3 Length of utterances made during Koyama's lessons

	Koyama's first lesson			Koyama's third lesson		
Activity	Teacher	Students	Total	Teacher	Students	Total
Presenting the problem	247 (89.5%)	29 (10.5%)	276	1464 (59.3%)	1006 (40.7%)	2470
Discussing solution methods	4147 (77.6%)	1200 (22.4%)	5347	2281 (58.2%)	1639 (41.8%)	3920
Summarizing	940 (82.3%)	202 (17.7%)	1142	310 (65.1%)	166 (34.9%)	476
Total	5334 (78.8%)	1431 (21.2%)	6765	4055 (59.1%)	2811 (40.9%)	6866

[2] The length of utterances is easy to measure with counting the number of Hiragana, Japanese cursive syllabary.

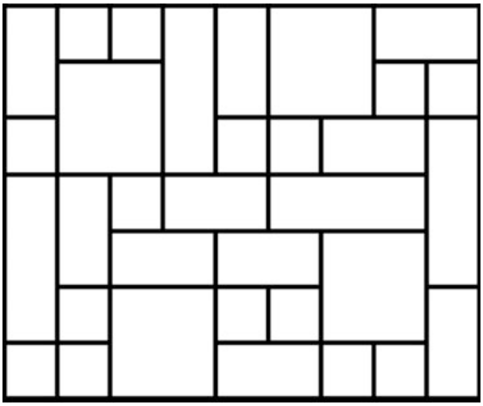


Fig. 3 Problem situation in the first lesson: a territory battle game (Fujii 2015, p. 13)

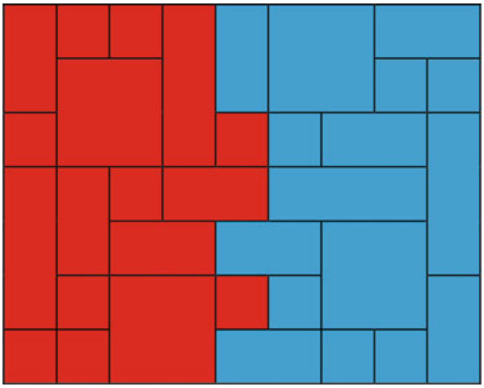


Fig. 4 The results of a territory battle game students played in lesson #1

6.1.1 Lesson #2 Pre-lesson Discussion

In the first lesson, one prospective teacher presented the problem, a territory battle game, shown in Fig. 3. Koyama planned to begin her first lesson, #2, by presenting the results of a territory battle game students played in lesson #1, shown in Fig. 4:

> Play Rock-Paper-Scissors. The person who wins will shade in a rectangle and it becomes his or her territory. The first time you win, you can pick any rectangle. After that, you can only capture a rectangle that is adjacent to your territory. The person whose territory is the largest wins the game.

She would ask students to determine if red or blue had the greater area. During her pre-lesson discussion with her supporting teacher Aoki, she shared her ideas about how to present the students' solutions:

> Koyama: Ok first, I'll have them share each way they compared them [the areas]. After they've shared and explained their solutions, then they can share how they feel about them. . .They are all going to be presenting at once so they can line them [the solutions] up on the board. And then finally I'll ask them as a class to decide which way is the best way to compare areas. (Koyama, pre-lesson discussion for lesson #2, September 25th)

Her idea was to put all the solutions up on the board and rank them. However, the supporting teacher, Aoki, warned her about the potential issues of presenting student solutions in this way:

> Aoki: If you start out showing a direct comparison solution and then show solutions one-by-one until you get to one that uses units. . .It'll be like the children whose solution you show first, the ones who cut out and overlapped the shapes [direct comparison], it'll be like they don't exist. . .Each solution is going to feel separate from each other and they [the students] are probably going to guess that you'll ask, "Which idea do you think is the best?" at the end. (Aoki, pre-lesson discussion of lesson #2, September 25th)

He pointed out that presenting students' solutions in order from naïve to sophisticated is not an effective way to help them make connections. The goal of the lesson is not to recognize which solution is the "best" but to understand the underlying mathematical thought process that leads to a solution.

Aoki tried to help Koyama think of a better way to make connections between student solutions. He helped her see what the naïve and sophisticated solutions had in common:

> Aoki: If you overlap them [use direct comparison], what can you understand about them?
> Koyama: How different they are in size.
> Aoki: Right? So. . .If you convert that part into a number, you can tell exactly what the difference in area is, right?. . .So even these students' ideas, as long as you convert the part that sticks out into a number. . .they [the solutions] have that in common, the same shape is made up of many parts that they can count. . .you don't have to think of them as separate ideas. The children can see how they are connected.
> Koyama: Ok, so I'll put them all up at once, these four ideas and say, "We came up with a lot of ideas, didn't we?" I won't ask what's good about the ones that use direct comparison or what's good about the ones that use indirect comparison, and what's bad about them, I won't say any one is bad. (Pre-lesson discussion of lesson #2, September 25th)

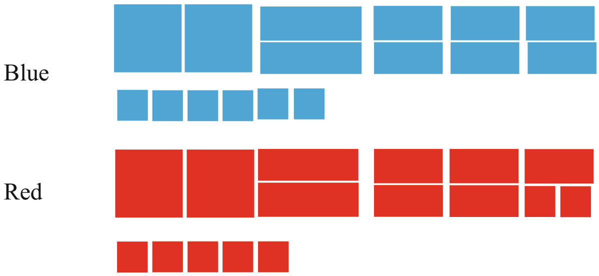
The supporting teacher connected direct comparison and indirect comparison by focusing on converting the remainder of the direct comparison method to a numerical value. This underscores the mathematical thinking the different solution types have in common. Koyama realizes that she needs to focus on this, and not what is "good" or "bad." This approach includes all the students in the discussion, allowing them to participate and make the mathematical connection for themselves.

6.1.2 Lesson #2 Classroom Discussion

During the classroom discussion in lesson #2 on September 30, five student solutions were presented. One student, Shimoda, arranged pieces of paper by lining up the shapes in the red and blue groups to make it clear how many of each there were (Fig. 5). Shimoda showed how two small squares equal one short rectangle, making it clear that the blue group has the greater area.

Even though Koyama and Aoki had earlier conversed in depth about how to run this discussion, this first lesson still proved to be a challenge for Koyama. Instead of

Fig. 5 Student Shimoda's solution, shown during lesson #2



letting the students relate the solutions to each other based on mathematical thinking, she lectured about individual solutions based on a mathematical fact, the unit. An example can be seen in her reaction to student Shimoda's explanation of her solution:

Shimoda: This [blue] has 1-unit blocks [small squares], um, six 1-unit blocks, and this [red] has seven 1-unit blocks

Koyama: Yes, six and seven!

Shimoda: And also this, this is the same as this [red and blue's two large squares, two long rectangles, and five short rectangles]. So put this here [puts two small red squares under a short red rectangle]. So now this one [blue] has six [small squares] and this one [red] has five.

Koyama: Yes, this one [blue] has six [small squares] and this one [red] has five, so you can see that the blue group has more 1-unit blocks [inaudible]. You used units to compare them. So if you do this. . .you get six here and five here, so the blue group has more small units, more 1-unit blocks, so blue wins. So yes, this idea, you had the idea that these units, the small units together are the same size as a 2-unit block. You showed us how we can put the small units together to make 2-unit blocks and then compare the number of each kind of unit in each group. (Lesson #2, September 30th)

Koyama explained the meaning of the student's solution immediately after they gave it. She repeated the student's explanation while drawing attention to the difference in number of the small squares, emphasizing the word "unit." She also responded to four other students' explanations in the same way during this lesson, emphasizing measuring with units. Furthermore, not only did Koyama explain Shimoda's solution herself, she also made connections between Shimoda's and the other students' solutions herself:

> She [Shimoda] put two of the smallest unit blocks together to make a 2-unit block. . . So this [other] idea here is very similar. . .they put two 2-unit blocks together. This is just like how Shimoda put two together to make a 2-unit block. . .If you look at the shapes, all of them can be made up of 1-unit blocks. (Koyama, Lesson #2, September 30th)

She focused on the mathematical fact of single units to lecture about the student solutions. She was unable to guide them to realize this for themselves.

6.1.3 Lesson #2 Post-lesson Discussion

During the classroom discussion in lesson #2, Koyama focused on mathematical facts in students' solutions instead of her students' thought processes. As a result, she created an environment in which students were reluctant to share all their ideas. She laments this in her post-lesson discussion:

> Koyama: I had wanted to share a lot more ideas because there were so many. But I messed up . . .I asked them, "Did anyone cut out the images?" Which might've been okay, except I also meant to ask, "Did anyone compare them without cutting them out?" But instead I said, "Did anyone think of them as pieces, as unit-blocks?" Which I think just made it hard for the other students to then talk about their [direct comparison] solutions. So after that they were really aware of the unit issue and I think it made them think, "Oh, I wish I had used units." I wanted them to think about units so I forced that idea on them myself. I messed up my question to the class. Students stopped raising their hands as much. I think it didn't go well because I forced the idea that all the units are made up of single units. By the end of the class I had written everything I had wanted to up on the board, but I think the flow of the discussion was problematic. (Koyama, post-lesson discussion of lesson #2, September 30th)

Even though Koyama and Aoki discussed how to guide students during lesson #2 in depth during their pre-lesson discussion, it was still difficult for Koyama to execute these ideas in real time during the actual lesson. By bringing up the concept of unit blocks before the students did, she made them shy about sharing all their solutions. By focusing on mathematical fact, rather than students' thought processes, she limited the discussion. Because she lectured them about mathematical knowledge, she was unable to help her students connect their solutions by realizing what mathematical thought processes they had in common. Her post-lesson discussion helped her realize and reflect upon this issue.

6.2 Observations of Other Lessons

Observations of other lessons also helped Koyama understand how children think about mathematics. For example, on October 2, she observed a fellow student teacher's math lesson and participated in the post-lesson discussion. She shared how impressed she was that a student could relate their solution to another student's solution:

> Koyama: One of the good things about this lesson, when the two solutions were up on the board, vertical length times horizontal length and horizontal length times vertical length, Takayama [a student] said that we could do two times eight or eight times two, because eight rows of two was equal to two rows of eight, which I thought was really good! After that comment, another student. . .said that the only thing that was different was just the way of counting them. I thought that was so amazing, the teacher didn't even have to prompt them. I was really impressed that they were able to look at the mathematical expressions and share those ideas. (Koyama, post-lesson discussion of lesson #3, October 2nd)

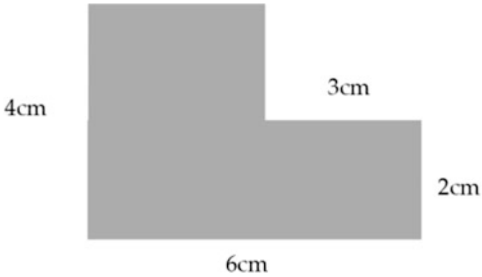


Fig. 6 L-shape composite figure presented in lesson #6

Fig. 7 Solution image (1) from lesson #6

Students were not able to make connections in this way during her first lesson, #2. Observations such as the one above let her see the students' potential. Observing and discussing this class helped her understand how students think and influenced her future lessons for the better.

6.3 *Lesson #6, the Student Teacher's Third and Final Math Lesson*

6.3.1 Classroom Discussion

Koyama taught her third lesson, #6, on measuring the area of an L-shape composite figure (Fig. 6). Students had to figure out the length of the edges themselves. One common solution she anticipated was that students would combine the L-shapes into a square (Fig. 7).

Instead of explaining their solutions for them, Koyama asked five students to share their solutions by writing the mathematical expressions they used on the board and then asked other students to explain the meanings. A student named Watanabe wrote "$6 \times 6 = 36$, $36 \div 2 = 18$" on the board and an another student, Takayama, explained it:

Takayama: Um, four plus two is, um, this is four centimeters and this is two centimeters so that makes six.

Koyama: I had better put this up on the board, how are they put together? They are put together like this [shows Fig. 6]. . .

Takayama: So then the area of this and this is thirty-six because six times six is thirty-six. And there are two areas in that thirty-six so half of that is eighteen centimeters.

Koyama: Yes! Great! Great job explaining Watanabe's solution! This is Watanabe's idea. Let's leave it up here [on the board] okay, six times six. Which area is six times six?. . .

Ishikawa: The whole area made up of the two L-shapes put together.

Koyama: That's right, six times six makes this whole area, six times six makes thirty-six. Six times six, and then why do I divide it in half?. . .

Konishi:

Because there are two uneven shapes [L-shapes], and you want the area for just one of them. You have the area of them both together and then you divide it by two so you get the area for just one of them.

Koyama: Konishi mentioned dividing by two. Two of them together make this shape, so half of that is the area of one of them, and that's the answer.

Suzuki: The area of two L-shapes combined is thirty-six.

Koyama: Ok, let's move on for a second, why did they put this like this over here?

Arita:

Um, they make a square.

Koyama: They make a square, if you put the opposite sides together [of the L-shapes] they make a square.. . .

Koyama: Let's think about it some more, why do I want to make a square?. . .

Tsuchida: Because it's easy to find, it's easy to find the area of the two rectangle-square-shapes [L-shapes] that are drawn in it.

Koyama: If I make it a square, I can calculate the area of that square easily, and then I can find the area that's drawn in it, like Tsuchida said. Or rather, I can find the area of the two shapes that are put together. It's a "put together" type of solution. (Lesson #6, October 9th)

Koyama asked students to explain their classmates' solutions instead of lecturing about the results herself. Koyama also gave opportunities to other students to explain the two main steps: making a square and dividing by two. One student, Ishikawa, pointed out that the combined area was six multiplied by six. Konishi then explained why you then divide by two. When Koyama asked why you put the two L-shapes together, the student, Arita, answered, "They make a square." When Koyama asked why would you want to make a square, Tsuchida explained, "It's easy to find the area of the two rectangle-square-shapes [L-shapes] that are drawn in it." By directing questions to the students in this way, she was asking them to think mathematically in their solutions.

Koyama asked students to explain the reasoning behind their ideas, instead of immediately offering an explanation herself. Another example of this in the classroom discussion of Fig. 8 is:

StudentA: This upper part is three times two. This lower part, this side is two centimeters and this side is six centimeters, so three times two is six and two times six is twelve. And six plus twelve is eighteen.

Koyama: Yes, this is also a good idea. This part is three times two and this area is two times six, which is twelve, then six plus twelve is eighteen.

StudentB: The length of that side is two centimeters. . .So you just add the two shapes together, like we did before.

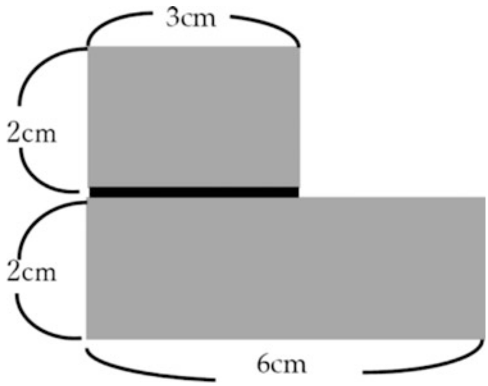


Fig. 8 Solution image (2) from lesson #6

Koyama: That's right, other students gave this idea too...
StudentC: In that solution [an earlier solution], they divided it [the L-shape] with a vertical line...in this solution, they divided it with a horizontal line.
Koyama: That's right, before we divided the L-shape with a vertical line and made two rectangles. This time, we divided the L-shape with a horizontal line and made two rectangles. We found rectangles again.
StudentD: Oooh! In all the solutions we made a square or rectangle.
Koyama: Yes!
StudentE: Because it's easy to calculate if you make a rectangle.
Koyama: That's right! (Lesson #6, October 9th)

Koyama guided the students to make connections among the solutions on their own, such as "like we did before," "they divided it [the L-shape] with a vertical line...in this solution, they divided it with a horizontal line," and "In all the solutions we made a square or rectangle." Furthermore, a student also remarked, "It's easy to calculate if you make a rectangle," identifying the mathematical thought process key to solving the problem. By letting the students make connections between the solutions on their own, they learn how to think mathematically and become more skilled at solving problems on their own.

Koyama's engagement with the students in the third lesson stands in marked contrast to her first lesson. In the first lesson, she explained students' solutions by herself and focused on the meaning of the word "unit." Her teaching method was a lecture on mathematical knowledge. However, in her third and final lesson, she was able to guide students to find connections between the solutions based on mathematical thinking. Her experience with Lesson Study helped her reflect upon and improve her teaching methods.

6.3.2 Post-lesson Discussion

Koyama's discussion activity in lesson #6 was more successful than the one in her first lesson. In the post-lesson discussion for this lesson, she remarks on how she tried to guide students to make connections between the solutions based on mathematical thinking and knowledge:

> For the introduction, I slowly pulled out an L-shape from an envelope to help them think of squares and rectangles. But when I asked them why they thought of a square or rectangle, they said because they are square-shaped or rectangle-shaped. . .but I thought they were going to say because rectangles and squares are shapes we know how to find the area for. I was going to use that as a launching point for them to start solving the problem on their own once they said that. But. . .they were really focused on qualities of the shapes themselves, like, "It's ninety degrees so it's probably a rectangle.". . .I wanted to connect it to the fact that it's a shape they know how to find the area of, so I changed the main question I asked the class. I asked them, "I want to find the area of this shape [L-shape], how do you think we should think about it?" Was it a good idea to change my question?. . .When they fixed them [the L-shapes] into a rectangle, everyone used the word "rectangle." When I asked them why, they said, "it's easy to find the area". . .As they explained their solutions I engaged with them to focus on what I had brought up earlier, that we can find the area of rectangles and squares easily. I do feel like it's possible that what we discussed during the introduction led to these results. I also feel like maybe I had to make them realize that it was good to use rectangles and squares. But when we wrapped up the lesson, the students just did it on their own, it went smoothly. (Koyama, post-lesson discussion of lesson #6, October 9th)

Koyama was better able to identify and react to her students' mathematical thought processes in this third lesson, compared to her first lesson. She noticed that the students were so focused on the rectangles and squares being recognizable shapes that they were not thinking about how they were shapes for which they know how to measure the area. To engage them without lecturing, she responded in real time to ask, "I want to find the area of this shape [L-shape], how do you think we should think about it?" As a result, when she asked them why they have arranged the L-shapes into a rectangle, they said it is because that made it easier to find the area. By guiding the students in this way, they were able to relate the solutions to each other based on this common mathematical thought process.

7 Conclusions

Teaching through problem-solving is more difficult than lecturing. Student teachers rely on Lesson Study to learn this nuanced teaching method. The difficulty of this approach is reflected in Koyama's first lesson. She struggled during her first pre-lesson discussion. Her first instinct was to line up all the student solutions and compare them herself. Her supporting teacher had to help her realize that ranking solutions do not create a productive classroom discussion. Koyama put this advice into practice during her subsequent lessons; she did not ask students which one was "best." Still, she continued to struggle. Instead of guiding students to make the connections between their solutions, she lectured on units, the objective of the lesson. As a result, not all of the students' solutions related to each other, because the naïve solutions didn't use units. The post-lesson discussion helped her reflect on her problematic lecture.

By the end of her 3 weeks of student teaching, Koyama was better able to understand how her students think and was able to guide them through her lesson.

Her observations of other lessons helped her realize the students' potential. She saw them make their own connections to each other's solutions and learn how to think mathematically. This helped her craft her future lessons. She became more able to respond in real time, even changing her main question to the class. With her new understanding of students' mathematical thinking processes, she was able to guide them to make the key connections between their solutions. She reflected upon and internalized this experience in her post-lesson discussion. The processes of reflection and improvement that characterize Lesson Study have taught her how to teach mathematics through problem-solving.

8 Looking Forward

The key features of Lesson Study for student teachers are observation, a supporting teacher, and discussion. These features contributed to the success of this case study. If we are to understand how student teachers learn from Lesson Study, we should examine these features.

Observation techniques are essential to successful Lesson Study because teachers rely on observations made in the classroom to imagine their students' thought processes. Observing students' thought processes during these lessons is an important but difficult skill. Prospective teachers record student-teacher interactions and sometimes individual students' activities. During post-lesson discussions, prospective teachers talk about what the students did in class and try to imagine why. Improving observation techniques, such as using different recording devices, will improve the quality of teachers' observations and understanding of their students.

The role of a supporting teacher is important for prospective teachers to learn how to teach mathematics through problem-solving. Supporting teachers create opportunities to think about and imagine students' learning process. They focus their comments not on criticizing the student teacher but on explaining how children think about mathematics in the classroom. Supporting teachers emphasize to understand the underlying mathematical thought process that leads to a solution. And they also emphasize mathematical thought process that connects one student's solution to another solution. It is important for supporting teachers to focus on students' thinking process and mathematical thought process in Lesson Study process in student teaching. A qualified supporting teacher is key to a successful cycle of Lesson Study. We need to make sure that student teachers have enough opportunities to meet with highly skilled supporting teachers.

Selection of appropriate teaching materials is also important. In this case study, the student teacher relied on the L-shape problem they found in a textbook. Teachers need effective problems such as the L-problem on which to build their lessons. We need to be able to provide them with enough appropriate teaching material.

Acknowledgments I wish to acknowledge the assistance of the following people who contributed to the writing of this paper. I am grateful to the prospective teacher and the supporting teacher who allowed me to video record their activities in student teaching. I also want to provide thanks to Katherine Beckwitt for editing an earlier version of this paper.

Appendix: Case Study Subject Koyama's Student Teaching Schedule

1st week	Sept. 25th Mon.	Sept. 24th Tues.	Sept. 25th Wed.	Sept. 26th Thurs.	Sept. 27th Fri.
Homeroom	National holiday (no school)				
1		Lecture by staff	(Music)		(Japanese) HR
2		Lecture by staff	(Music) HR	(Math) HR	#1 (Math) HR
3		(Library) HR	Lecture by staff	(Music) HR	(Science) HR
4		(Special Homeroom) HR	(Math) HR	(Math) HR	(Special Homeroom) HR
5		Lecture by staff		(Social Studies) HR	(Art) HR
6		Lecture by staff		(After school club activities)	(Art) HR
7		(Special Homeroom) HR		Lecture by staff	
			Pre-lesson discussion for #2		Post-lesson discussion for #1
					Pre-lesson discussion for #2

2nd week	Sept. 30th Mon.	Oct. 1st Tues.	Oct. 2nd Wed.	Oct. 3rd Thu.	Oct. 4th Fri.
Homeroom		(Social Studies) HR		(Special Homeroom)	
1	KOYAMA #2 Math HR	KOYAMA Japanese HR	#3 (Math) HR	(P.E.)	(Math)
2	(Japanese) HR	(Special Homeroom) HR	(Music) HR	(P.E.)	(Math)
3	(Science) HR	(Library) HR	(Handwriting) HR	(Math)	(Japanese)
4	(Special Homeroom) HR	(Math) HR	(Japanese) HR	(Math)	(Math)
5	(P.E.)	(Social Studies) HR	(Special Homeroom) HR	(Social Studies) HR	KOYAMA #4 (Math) HR
6	(P.E.)	(Math) HR		(Art)	(Science) HR
	Post-lesson discussion for #2		Post-lesson discussion for #3	Pre-lesson discussion for #4	Post-lesson discussion for #4

3rd week	Oct. 7th Mon.	Oct. 8th Tue.	Oct. 9th Wed.	Oct. 10th Thu.
Homeroom	(English)	(Social Studies) HR		
1	(Math) HR	(Japanese) HR		Lecture by staff
2	(Science) HR	(Science) HR		(Social Studies) HR
3	(Art) HR	(Library) HR		(P.E.)
4	(Art) HR	#5 (Math) HR	KOYAMA #6 Math HR	(P.E.)
5	(Math)	(P.E.)	(P.E.)	Farewell party with students
6	(P.E.)	(P.E.)	(P.E.)	(After-school club activities)
	Pre-lesson discussion for #6	Pre-lesson discussion for #6	Post-lesson discussion for #6	
		Post-lesson discussion #5		

Note: All classroom subjects in parentheses, such as "(Math)", indicate lessons Koyama observed.

Classes taught by the supporting teacher guiding Koyama's group of student teachers are highlighted in yellow.

Classes Koyama taught are highlighted in orange and marked KOYAMA.

Classes taught by other student teachers are highlighted in blue.

The student teacher-led math lessons are labeled #1, #2, #3, #4, #5, and #6, respectively.

Boxes marked "HR" refer to the main homeroom to which Koyama was assigned. Boxes not marked "HR" refer to other homerooms.

Koyama was scheduled to teach the science class on October 7 but was suddenly unable to do so, and the supporting teacher taught the class instead.

References

Becker, J. P., Silver, E. A., Kantowski, M. G., Travers, K. J., & Wilson, J. W. (1990). Some observations of mathematics teaching in Japanese elementary and junior high schools. *Arithmetic Teacher, 38*(2), 12–21.

Fujii, T. (2014). Theorizing Lesson Study in mathematics education as emerging research area: Identifying components and its structure of Lesson Study. In Japan Society of Science Education (JSSE) (Eds.), *Proceedings of second annual spring conference of Japan Society of Mathematical Education* (Vol. X, pp. 111–118). (in Japanese).

Fujii, T. (2015). Mathematics 4B, Tokyo Shoseki (in Japanese).

Fujii, T. (2016). Designing and adapting tasks lesson planning: A critical process of Lesson Study. *ZDM Mathematics Education, 48*, 411–423.

Fujii, T. (2017). Unifying lesson study with teaching mathematics through problem solving. In International math-teacher professionalization using lesson study. In *Essential mathematics for the next generation* (pp. 85–103). Tokyo: Tokyo Gakugei University Press.

Hashimoto, Y. (1987). *Classroom practice of problem solving in Japanese elementary school*. In J. P. Becker & T. Miwa (Eds.), *Proceedings of the U.S.-Japan seminar of mathematical problem solving* (pp. 94–101). Board of Trustees of Southern Illinois University.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*(2), 169–202.

Karp, A. (2010). Analyzing and attempting to overcome prospective teachers' difficulties during problem-solving instruction. *Journal of Teacher Education, 13*, 121–139.

Miwa, T. (1992). *The teaching mathematical problem solving in Japan and the US*. Tokyo: Toyokanshuppan (in Japanese).

Miyazaki, K, Kumazawa, A., Odaka, T., & Okamoto, K. (1969). Teaching methods in Modernization of mathematics education, *Kindaishinsho*. (in Japanese).

Sawada, D. (1999). Mathematics as problem solving: A Japanese way. *Teaching Children Mathematics, 6*(1), 54–58.

Shön, D. A. (1983). *The reflective practitioner*. New York: Basic Books.

Souma, K. (1983). Problem-solving approach: Mathematics education and problem-solving. *Journal of Japan Society of Mathematical Education, 65*(9), 2–11 in Japanese.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: using video to improve preservice mathematics teachers' ability to notice. *Journal of Teacher Education, 11*, 107–125.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Stigler, J., & Perry, M. (1988). Cross cultural studies of mathematics teaching and learning recent findings and new direction. In D. A. Grouws, T. J. Cooney, & D. Jones (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 194–223). Reston: The National Council of Teacher of Mathematics.

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Lesson Study in a Mathematics Methods Course: Overcoming Cultural Barriers

Blake E. Peterson, Dawn Teuscher, and Thomas E. Ricks

Contents

Abstract In order to support the development of deeper mathematical understanding of preservice teachers, we have two main goals in our university mathematics methods course. They are for secondary preservice teachers to have rich conversations about (1) the mathematics of their lessons and (2) how students think about that mathematics. This paper describes our application of modified Japanese lesson study to meet these goals and how the US cultural views of teaching and mathematics were barriers to achieving those goals. Reflecting on 15 years of using lesson study to meet our methods course goals, we describe specific course revisions and changes necessary to overcome these cultural barriers and, thus, further improve the quality and quantity of preservice teacher candidates' conversations about the mathematics of their lessons and how students think about that mathematics.

Keywords Secondary preservice · Mathematics teaching · Lesson study · Cultural scripts · Methods class

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_26

1 Introduction

As we have worked with secondary preservice teachers (PSTs) in our mathematics methods course, we have faced the struggle of helping them deepen their mathematical understanding. When asked to describe the fundamental mathematics concept of a lesson, PSTs tend to describe the steps of a procedure rather than the underlying concepts. This tendency was consistent with Ball's claim that PSTs often have weak mathematical understanding because US primary and secondary students do not develop that understanding (1990). We also observed PSTs struggle to handle student mathematical thinking that differed from their own when they taught practice lessons in the methods course which further highlighted that PSTs lacked a robust mathematical understanding (Ma 1999; Sim and Walsh 2009) and steps need to be taken to better support their development of that understanding. The steps we took to deepen our PSTs' mathematical understanding were captured in our two main methods course goals which were for secondary PSTs to have rich conversations about (1) the mathematics of their lessons and (2) how students think about that mathematics.

In the search for ways to support the PSTs' development of deeper mathematical understanding in the methods course, the first author noted a description of *lesson study* in *The Teaching Gap* (Stigler and Hiebert 1999). Japanese lesson study (JLS) is one form of collective professional development existing within East Asian professional development practices that encourages teachers to examine their pedagogy in the presence of peers. JLS is a compact, coherent, and connected professional experience that encourages collaborative reflection among participating teachers. Over many months, a local group of subject-specific teachers design, test, refine, and reteach a single lesson to their students—the lesson study process. In their explanation of lesson study, Stigler and Hiebert (1999) provide a description of a group of Japanese mathematics teachers who spent significant time discussing whether to use the subtraction problem 13–9 or 15–8. This mathematical discussion seemed very different from the discussions that occurred in the first author's methods course yet something he desired for his own PSTs. Seeing that lesson study was a productive tool to impact teacher practice and yield rich mathematical discussions (Stigler and Hiebert 1999), implementing lesson study in our mathematics methods course seemed a natural way to support PSTs' development of robust mathematical understanding. In addition, providing the PSTs with opportunities to talk about student mathematical thinking while preparing lessons in a lesson study context would better prepare them to respond to student thinking as they implemented the lessons they prepared (Parks 2008).

In an effort to encourage our PSTs to have rich conversations about the mathematics of their lessons and how students think about that mathematics, a model of lesson study was implemented into a mathematics methods course for secondary PSTs at Brigham Young University (BYU). Over the last decade, many researchers have reported on their efforts to implement lesson study in preservice teacher education (e.g., Griffiths 2016; Larssen et al. 2018), but most research has implemented lesson study in a student teaching setting (Cajkler and Wood 2015) rather than in a methods course (Parks 2008). In order to understand the landscape of research done on lesson study in preservice settings and how our study uniquely contributes, we define terms to describe characteristics that we use to compare these studies. A lesson planning *cycle* is one plan-teach-reflect-revise sequence. In some cases, a lesson planning cycle includes a second teaching of the revised lesson. Lesson study *implementation* is the duration of time used to complete all lesson study cycles in the research study. The variation we saw is that in some studies, there were three lesson planning cycles in a 1-month implementation and in other studies there was one cycle over a full semester implementation.

In general, we saw that most research studies reported at most three lesson planning cycles, during a lesson study implementation that lasted at most one semester in duration (e.g., Bjuland and Mosvold 2015; Fernandez and Zilliox 2011). We only found one case when a research study was repeated after making adjustments to the lesson study implementation (Sim and Walsh 2009).

There are few research studies that have used the results and observations from one implementation to make adjustments for a subsequent implementation. Thus, studies of multiple lesson study planning cycles and lesson study implementations in methods courses are needed for proper refinement, testing, evaluation, and appropriate cultural modification. Studying repeated implementations of lesson study could help address Takahashi and McDougal's (2016) observation about cultural modifications that "…it will be important to understand why lesson study has been less consistently impactful outside of Japan—whether there are important aspects of lesson study as practiced in Japan that are getting 'lost in translation' and can be fixed, or whether the problem is due to cultural difference that cannot be fixed" (p. 514). Our study contributes to this nascent domain by reporting on 15 years of iterating lesson study implementation in a secondary mathematics methods course with at least seven lesson study planning cycles per semester. Thus, we have come to better understand the cultural views of teaching and learning mathematics in the USA that were barriers for our PSTs to reach the two desired goals of the methods course and required several revisions and changes to the way we implemented lesson study. As a result of these efforts, elements of lesson study have permeated BYU's methods course. The unique feature of the work reported here is the duration and the multiple revisions and changes we were able to make over these 15 years.

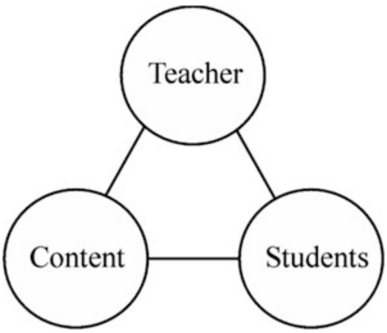
This chapter details our reflection of the following question: What cultural adjustments to our mathematics methods course were needed for lesson study to be an effective tool to support our goals of focusing PSTs' conversations on mathematical concepts and student mathematical thinking? In this chapter, we describe the model of lesson study that was implemented along with revisions and changes that were made to overcome barriers to effective implementation. We also describe the components of lesson study that supported our goals which, in turn, became a valuable support in their development of mathematical content knowledge.

2 Framing Our Goals

We use the instructional triangle in Fig. 1 (e.g., Kilpatrick et al. 2001) to describe the interaction of our two main goals of improving our PSTs' conversations about mathematics and student thinking of that mathematics. A broader teaching culture largely shapes individual interpretations of the parts of the instructional triangle—although they can be modified through proper education and continual reflection on practice (e.g., Kilpatrick et al. 2006). In the USA, teachers commonly interact with students through a performance of mathematical procedures and then monitor students' ability to enact these procedures. The role of students is to dutifully remember and mimic the teacher's demonstration of the mathematics; thus, students interact with mathematics by "learning terms and practicing procedures" (Stigler and Hiebert 1999, p. 27). This student consumption and reproduction of teacher-performed mathematical procedures in scaffolded situations put the focus squarely on the teacher and the upper part of the instructional triangle. This view of mathematics instruction is held by many in the US mathematics teacher culture as well as the broader US culture.

We—as mathematics teacher educators—have a different view of the instructional triangle with mathematical contents' conceptual nature as a vital foundation for understanding conventional procedures (Kilpatrick et al. 2001). Therefore, teachers should teach mathematical concepts and procedures for understanding (e.g., Kilpatrick et al. 2003). We view student thinking as an important aspect of the learning process (e.g., Kazemi and Franke 2004), and as such, we view students as active generators of their own mathematical understandings as mathematics meaning is developed during social interactions. The teacher's role becomes a facilitator of student thinking through orchestrating productive mathematical discussions about important mathematical concepts (Stein et al. 2008). This view of the teacher's role as a facilitator of student mathematical discussions is illustrated by the instructional triangle on the right of Fig. 2 and could be referred to as a "reform" view, while the view of mathematics instruction described in the previous paragraph is illustrated by the instructional triangle on the left of Fig. 2 and could be referred to as a "traditional" view of mathematics teaching.

Fig. 1 The instructional triangle



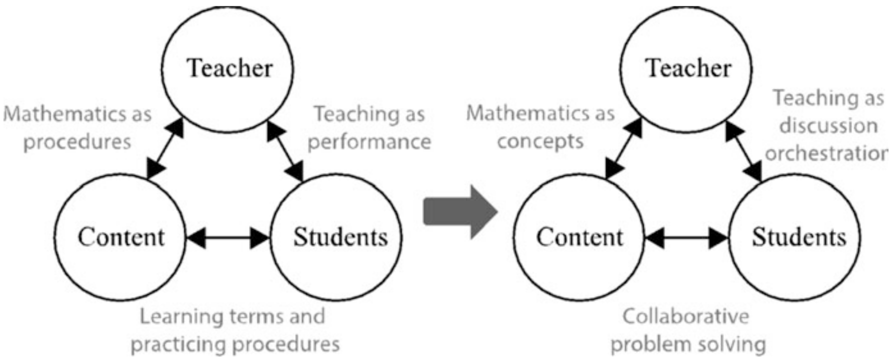


Fig. 2 Methods course goals

As methods instructors we encourage PSTs to consider alternative conceptions of teaching mathematics beyond the US cultural norm of demonstrating procedures to students. We want our PSTs to focus more strongly on the interaction between their students and conceptual mathematics content during lesson planning and implementation and less on the teacher actions as did the PSTs in Chassels and Melville (2009). In particular, students learn mathematics with understanding through discussions of their sensemaking during productive problem-solving. We wanted to challenge the strong tendency for our PSTs to continue to emphasize their performance of presenting mathematical procedures and instead structure their pedagogy around facilitating discussions by eliciting, supporting, and extending students' mathematical conceptual thinking (Fraivillig et al. 2002). For this end, we wanted our PSTs to self-modify their internal interpretations of the instructional triangle toward our view of the instructional triangle on the right of Fig. 2. We believed that as PSTs engaged in conversations with other PSTs about the mathematics of their lessons and how students (might) think about this mathematics, especially as each PST encountered their peers' varied ways of thinking about mathematics (e.g., Hiebert et al. 2005), they would develop a richer conceptual understanding of mathematics.

We view the edges of the instructional triangle connecting the teacher-student-content nodes as linkages in the ways the nodes might interact with one another. In particular, our first methods course goal (to increase PSTs' conversations about mathematics) was to improve the linkage between teacher and content, and our second goal (to increase PSTs' conversations about how students might think about that mathematics) was to improve the linkage between the teacher and students. We hoped improvement along these two linkages would provide for better student learning of mathematics (the linkage between students and content) when the PSTs taught lessons.

3 Context

To better understand the evolution of the course and the implementation of lesson study, we first describe the structure of the course prior to lesson study implementation. We then describe the structure of the course when lesson study was first implemented.

3.1 *Before Lesson Study*

Prior to the implementation of lesson study, the Methods for Teaching Mathematics course at BYU consisted primarily of students having microteaching-type experiences in that they had a "scaled-down teaching encounter" (Cruickshank and Metcalf 1993, p. 87). It was different than what Cruickshank and Metcalf (1993) describe in that our PSTs taught about 15 peers instead of only 3–5 and focused on pulling many teaching abilities together instead of performing only "one of several desirable teacher abilities" (p. 87). PSTs were assigned sections out of secondary mathematics textbooks from which they would prepare and teach lessons (25 min long) to all of their peers in the methods class. Throughout the rest of the paper, we will refer to the lessons prepared from assigned textbook sections as *methods lessons*. Given the 25-min time limit, the PSTs had latitude to select the portion of the textbook section they felt was most relevant given the sections that surrounded it. The textbook chapters were selected so that the PSTs taught several lessons in sequence from a chapter in algebra, then several in geometry, and so forth. A little less than half of the textbooks from which these chapters were drawn had a more traditional (procedural) approach to the mathematics, and the rest of the textbooks were from a reform perspective (e.g., Lappan et al. 1998; Fendel et al. 1997). The reform textbooks from which lessons were prepared were often based on worthwhile mathematical tasks and used less familiar approaches to the mathematics and the teaching of it. This encouraged the PSTs to dig deeper to understand the intended mathematics of the tasks, which led to a better focus on the underlying mathematical concepts instead of emphasizing procedures. During the semester, each PST individually planned and taught two methods lessons—one in the first half of the semester and one in the second half. For each methods lesson, there were 3–4 PSTs who individually wrote a lesson plan on the same textbook section but *did not* meet together, and only one PST taught the lesson. By giving several PSTs an opportunity to individually prepare a lesson on a textbook section and then participate as a student when the lesson was taught, they were privately able to contrast their individual conceptualization of the lesson with the one enacted by their peer. As is common in microteaching settings (Cruickshank and Metcalf 1993), each 25-min lesson, prepared and taught by an individual PST, was videotaped, so the teaching PST could watch and reflect on their lesson. A 10-min debriefing was held after each 25-min lesson.

Every written lesson plan required the identification and articulation of a fundamental mathematics concept (FMC): the key underlying mathematical idea the lesson was designed to help the students understand. Class time was spent discussing the fact that the description of the FMC was not just a description of a procedure but rather a description of the concepts that underlie the procedure. The focus of the course grading was on the written lesson plans with particular attention on the FMC descriptions being clearly articulated.

3.2 Lesson Study Implementation

When lesson study was implemented in 2002, PSTs worked in groups of 3–4 using the lesson study process to prepare a single lesson that was taught in the public schools at the end of the semester. We will refer to the lesson prepared as part of the lesson study process as the *research lesson*. These lesson study groups were paired with a secondary mathematics teacher who they observed throughout the semester. Early in the semester, the PSTs met with their assigned teacher and set a goal for the research lesson (Lewis 2002) to be taught in the secondary classroom at the end of the semester. In this capacity, the secondary teacher functioned as a "knowledgeable other" (Parks 2008) because they provided feedback to the PSTs about the lesson they prepared. In JLS, a knowledgeable other is a more experienced teacher outside of the lesson study group who functions as a mentor during the research lesson planning process. Takahashi (2014) found that the responsibility of the knowledgeable other was "(1) bringing new knowledge from research and the curriculum; (2) showing the connection between the theory and the practice; and (3) helping others learn how to reflect on teaching and learning" (p. 10).

Each group of PSTs then prepared a lesson that would be taught in the secondary classroom at the end of the semester. Since all lessons were to be taught from a reform perspective, they almost always were based on a task. For the 25-min time slot in the methods course, the groups would identify a portion of the lesson to teach which usually involved the main mathematical task of the lesson. A member of the lesson study group would teach that portion to their peers in the middle of the semester. The other members of the group would observe the lesson focusing on the thinking of those participating in the lesson. The observing PSTs were repeatedly reminded that they were not to focus on the teacher but on how those participating in the lesson thought about and reacted to the questions and tasks of the lesson.

Following the initial teaching of the research lesson, a discussion was held similar to those outlined in Lewis (2002). From this point forward, we will refer to such discussions as *reflection meetings* because that is what they are commonly called in Japan. These reflection meetings had a more formal structure (Lewis 2002) than the debriefings held after the methods lessons. In this reflection meeting, the instructor of the methods course functioned as a knowledgeable other (Bjuland and Helgevold 2018). Based on the feedback the PSTs received from their peers and the course instructor about their research lesson, the group revised the lesson in anticipation of teaching it again in the secondary classroom at the end of the semester. The research lesson taught to the peers in the class was also videotaped for additional reflection during revision.

After all groups had taught their research lessons in secondary classrooms at the end of the semester, a discussion was held in the methods course focusing on broader pedagogical principles gleaned from their varied yet similar experiences. Finally, each PST wrote an individual report on the evolution of their group's research lesson, what parts of the lesson were changed and why, and how the lesson could be further modified if it were to be taught again. In the end, the groups had opportunities to plan-teach-reflect-revise the research lesson twice—a beneficial iterative process inherent in the lesson study structure.

Over the course of our efforts to modify JLS for our specific US context, three graduate students conducted studies that investigated different aspects of the conversations that occurred as PSTs planned their research lessons: nature of reflection in lesson study conversations (Ricks 2003), steps students took during lesson preparation conversations to avoid revealing weaknesses in mathematical knowledge (Stafford-Plummer 2002), and how PSTs anticipated student thinking for incorporation in whole class discussions (Webb 2006). Beyond these studies, the authors continued an informal teaching experiment by constantly comparing (Glaser 1965) the PSTs' behaviors against the underlying goals for the course—PSTs' ability to have conversations that focused on mathematical concepts and on the student thinking about those concepts. In each instantiation of the methods course, we compared the PSTs' behavior during the lesson study experience against these underlying course goals. Although our 15-year experiment was not a formal study, we have enacted efforts to address many of these challenges other researchers have identified but not addressed due to their lack of duration (Larssen et al. 2018). Thus, we are able to report on the effectiveness of these modifications.

4 Revisions Over Time

We initially thought that JLS's inherent structure where teachers talk together about lesson design would encourage PSTs to have conversations focused on mathematical concepts and student thinking about those concepts. We noticed, however, that two powerful US cultural views prevented our PSTs from having the conversational focus we had hoped. First, similar to what Bjuland and Mosvold (2015) found, PSTs thought about teaching as what the teacher does with little attention to how the students would respond to teacher actions. They saw teaching as a performance where the spotlight was on the teacher and how they presented the material, rather than on how students thought about the mathematics or how they could facilitate a discussion among students about their mathematical thinking. Webb (2006) found that PSTs would anticipate student thinking for the purpose of creating a smooth-running lesson—a focus on performance—and not on how that anticipated thinking could be used to orchestrate a productive discussion. This emphasis by PSTs to focus on their own teaching performance at the expense of student learning—even during lesson study—is an issue widely reported by preservice lesson study research (e.g., Bjuland and Mosvold 2015; Larssen et al. 2018).

Second, the PSTs' views of mathematics were mostly procedural. Most PSTs had school experiences where mathematics was seen as a set of procedures and were very good at using these procedures efficiently. However, their understanding of the mathematical concepts upon which these procedures were built was limited. Plummer and Peterson (2009) found that once this lack of conceptual understanding became apparent to one PST in our methods course, she subconsciously used evasive techniques to avoid revealing these weaknesses in her mathematical understanding. Because the PSTs generally thought that mathematics was easy due to their procedural knowledge, they spent little time discussing the conceptual mathematics in their conversations.

Because these cultural views prevented our PSTs from reaching our two goals, we refer to them as *cultural barriers*. We found these cultural barriers to be very powerful and so deeply embedded that PSTs embraced them before being formally integrated into the teaching culture, something observed by researchers of US preservice lesson study (Parks 2008). Whereas, because lesson study is part of the Japanese teaching culture, it is a natural part of what Japanese teachers talk about and often a critical part of student teaching (Corey et al. 2010); therefore, the Japanese culture itself inherently supports the enactment of lesson study as intended. In the US teaching culture, however, lesson study is not common, and the US PSTs did not have similar cultural machinery to enact lesson study.

Over time we made modifications to our lesson study implementation to address these two cultural barriers that impeded our PSTs' ability to have conversations that focused on mathematical concepts and on the student thinking about those concepts. In the following sections, we describe the revisions and changes made to our methods course and lesson study implementation that were designed to address these two cultural barriers.

4.1 Addressing the Cultural Barrier of Performance

To address the barrier that teaching is focused on the performance of the teacher, we made multiple revisions over the course of several years to the methods course that we hypothesized would de-emphasize the PSTs' focus on teacher performance and emphasize conversations about the mathematics of the lesson and student mathematical thinking. First, we realized that having all PSTs in the lesson study group take *ownership of the lesson* regardless of who was assigned to teach it was a vehicle to encourage PSTs to engage in productive conversations about mathematics and student mathematical thinking. We share three major revisions we made to the methods course that encouraged PSTs to take ownership in the planning of all lessons (the methods lessons as well as the research lesson) for which they were assigned to write lesson plans regardless of whether they actually taught the lessons.

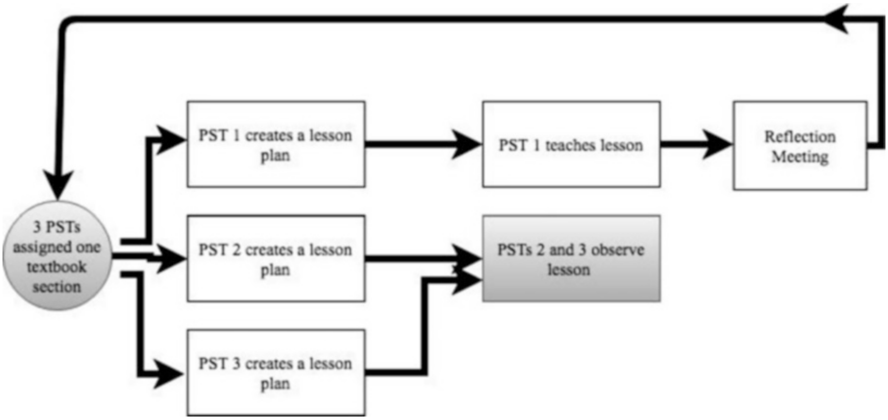


Fig. 3 First revision of six methods lesson planning cycles

First Revision The motivation for the first revision of the methods course was the PSTs were somewhat resistant to planning a single research lesson as a group because they claimed "it is unreasonable for us to come up with a single lesson because we all have different styles of teaching." Griffiths (2016) found similar comments: "I feel that teaching is largely an independent practice, therefore teaching in teams has not been useful" (p. 235). This common comment made by multiple PSTs in different semesters highlighted that PSTs were not focused on students' understanding of the mathematics but rather on their performance of the lesson.

To address PSTs' resistance to planning their research lesson as a group, we hypothesized that having PSTs work more often in groups would encourage them to have more in-depth discussions about the mathematics of their lessons and student thinking as related to the mathematics. Thus, we revised the methods course (see Fig. 3), so PSTs worked in groups to plan their six *methods lessons* similar to the research lesson using the lesson study process.

Figure 3 displays the methods lesson planning cycle that PSTs participated in for each of the six methods lessons during the first revision. The shaded areas indicate the revisions that were made to the lesson planning process to address the cultural barrier of teacher performance. Rather than each PST in the group *independently* planning and writing lesson plans about a specific textbook section, PSTs were assigned to meet in their groups and plan the lesson together but still write individual lesson plans. As before one PST was assigned to teach the methods lesson to the peers in the course.

Another modification was that the PSTs who wrote lesson plans but did not teach them became observers when the methods lesson was taught to their peers. This meant that they no longer participated as students in the lesson but could quietly walk around the classroom to observe peers' work and thinking. We anticipated that this would provide PSTs with more opportunities to see different ways that students think about mathematics as well as encourage them to focus on students' thinking, rather than on the teacher's performance. This "observer" role was what the PSTs were asked to do for the teaching of the research lesson, so doing it for the six methods lessons gave them more experience for this type of observation—noticing student thinking.

Second Revision After we implemented the first revision illustrated in Fig. 3, we continued to see evidence that the PSTs were still focusing on the teaching as a performance. We now share some of that evidence that precipitated our second revision. We found although our PSTs were meeting in a group multiple times to plan lessons, they were still bumping up against the cultural barrier of their performance and having a smooth-running lesson. Webb (2006) noted, “when someone felt inclined to reflect more deeply on the mathematics, others were worried about task details” (p. 100) which points toward this teacher performance emphasis. This conversational emphasis was seen in the PSTs continued focus on teacher actions rather than on student thinking. For example, the observing PSTs would stand in the back of the methods classroom and watch their group member present the main task rather than move around the classroom to observe others’ mathematical thinking for the given task as they had been encouraged to do. Similarly, Cajkler and Wood (2015) found that PSTs in their study had a difficult time observing student learning because they were focused on the teacher’s actions. Related to this struggle to observe student thinking is the nature of the comments made by the observing PSTs during the reflection meeting after the lesson when they were asked to stand as a group at the front of the class with the teaching PST. The observing PSTs would often turn toward the teaching PST and say things like “I liked the way she did this” or “I liked the way she wrote on the board.” These comments were made as if these observing PSTs had no part in the preparation of the lesson and were only focused on the teacher’s performance.

A second piece of evidence of the cultural barrier of performance was the graded lesson plans. Although PSTs were assigned groups to plan and discuss the lesson, we found, similar to Fernandez and Zilliox (2011), that the PSTs were more focused on planning a lesson that they were comfortable teaching—a lesson that would fit their personal teaching style. Because the teaching PST wanted to make sure he/she knew all the answers and could explain the mathematics clearly, his/her lesson plan included more detail than their peers’ lesson plans on the same lesson. Since the observing PSTs did not actually teach the lesson and did not see the mathematics as problematic, they did not see any reason to describe the mathematics more than at a procedural level in their lesson plans. They only needed to understand the mathematics from their own perspective and not from others’ perspectives because their teaching performance was not going to be on display.

A third piece of evidence that highlighted PSTs’ continued focus on their performance was their discomfort in teaching their peers and the reflection meeting that followed. The PSTs felt that a successful teacher performance meant they were able to successfully answer all of the students’ (peers’) questions during the lesson. Thus, they wanted to make sure their explanations of the procedures were clear, so students could apply them without mistakes. An example of this was that typically PSTs designed their tasks in such a way that students would not bump up against

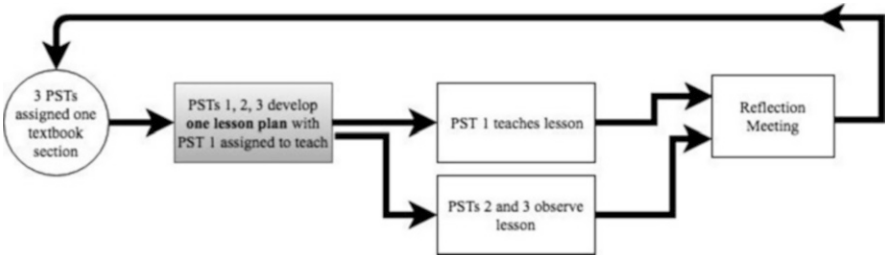


Fig. 4 Second revision of the six methods lesson planning cycles

misconceptions; they wanted the task to be straightforward and clear, so all students would get the correct answers (Fernandez and Zilliox 2011; Webb 2006). Similar to Sim and Walsh (2009), our PSTs were also nervous about the reflection meeting after they taught a lesson because they felt the peers' or the instructor critiques were about them rather than the lesson.

With the implemented first revision of having PSTs work in groups to plan the methods lessons, we hypothesized that it would redirect PSTs away from a focus on teacher performance and encourage an in-depth conversation about the mathematics and student mathematical thinking. Yet, as we have discussed, the deep mathematical conversations did not happen like we had hoped. This led to our second revision where we took further steps to move the focus of the lesson planning away from the teacher performance. This revision to the methods course is displayed in Fig. 4 where the shaded area indicates the revision addressing the cultural barrier made to the methods lesson planning cycle.

First, each group of PSTs submitted a group lesson plan for each of the methods lessons, rather than each PST submitting their own lesson plan. The reason for this revision was that if the group of PSTs were working together to plan one lesson, it would redirect them away from conversations about how the lesson should be performed, because they all had different views of that performance and toward a joint lesson about student understanding of mathematics. We hypothesized that this would require PSTs coming to a consensus of the FMC and certain pedagogical moves to reach that FMC. PSTs were also forced to reconcile their personal understanding of the mathematics with the understandings shared by their peers. By having the PSTs work together to develop a group lesson plan, individual PST's understanding became publicized, meaning that the PSTs in the group had to make sense of each other's understanding of the mathematics, which in turn, gave them a window into the diversity of thinking among their peers and thus their future students in general.

This revision, similar to the first revision, brought the planning of the methods lesson to be more in line with the Japanese lesson study process. We realized that overcoming this cultural barrier could not be done by working on only one lesson but the key elements of lesson study needed to be practiced over and over again to begin to have an impact on the PSTs.

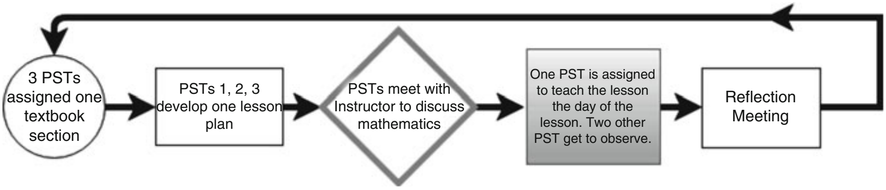


Fig. 5 Third revision of the six methods lesson planning cycles

Third Revision Although we saw progress after the first and second revisions and found that the discussions they had among their group were more focused on understanding the mathematics of the lesson, which redirected more of the PSTs to take ownership in the lesson, we continued to see evidence of PSTs disengaging from the lesson planning process. We found that because the PSTs knew who was teaching the lesson, finishing touches on the lesson plan were left to the teaching PST the night before the lesson was taught. The other PSTs in the group had no problem with these modifications because the lesson was not about their performance but only the performance of the teaching PST. Burroughs and Luebeck (2010) reported similar findings that when they chose the teacher who was to reteach the lesson too early, the other teachers did not take ownership over the lesson. Thus, the third revision of the methods course was to not tell the PSTs who would teach the methods lessons or the research lesson until the moment the lesson was to be taught. This forced the group of PSTs to work together, and all contribute to the lesson plan because any member of the group could be asked to teach the lesson. This revision is displayed in Fig. 5 where the shaded area indicates the aspect of the methods lesson planning cycle that was revised to address the cultural barrier.

With this revision, all members of the group were interested in discussing the mathematics of the lesson because they all needed to understand it at a deeper level in order to be prepared to teach it. This led to all PSTs taking greater ownership of the lesson during the entire lesson planning process. This revision redirected the PSTs away from the performance of the lesson and toward developing a lesson that would help students understand the mathematics. This, in turn, led to a greater desire to understand the ways students might think about the mathematics.

Prior to this revision, it was common for PSTs to look at the lesson task but not actually work through it. They tended to think it was sufficient to simply read the questions of the task and not do it. Therefore, this revision encouraged all of the PSTs to complete the mathematical task of the lesson and then discuss with each other how they had solved and thought about the task. By having multiple PSTs complete the task on their own, their conversations during the planning meetings began to focus more on identifying and understanding the underlying mathematics of the task (Mathis 2018). It also provided a natural connection to anticipating student thinking, because the PSTs generally approached the task differently. The subsequent conversations were naturally more rich as they sought to reconcile the different approaches to the task and discussed why students may do or think about the mathematics differently.

Although we saw our PSTs begin to take ownership of the lesson and their discussions were focused more on the mathematics and student mathematical thinking, we found that their conversations were limited by shallow conceptual knowledge. While they might see different procedural approaches in the tasks, the PSTs had a limited capacity to discuss the conceptual underpinnings of these mathematical procedures. Therefore, to help PSTs have deeper mathematical conversations, they needed someone who could ask questions that encouraged them to see the mathematics conceptually and to think about common misconceptions. This additional need of more involvement of a knowledgeable other is why the third shape in Fig. 5 is bolded. In the next section on the cultural barrier of viewing mathematics as a set of procedures, we will discuss the importance of a knowledgeable other (Chichibu 2016; Mathis 2018) to aid the PSTs to have deeper conversations about mathematics and student mathematical thinking.

4.2 Addressing the Cultural Barrier of Mathematics as a Set of Procedures

All of the revisions discussed while addressing the cultural barrier of seeing teaching as a performance occurred chronologically. The changes made to address the cultural barrier of seeing mathematics as a set of procedures did not happen in a similar chronological order. In fact, some of these changes happened at the same time as the revisions previously discussed and others happened at times independent of those revisions. Because the changes we are about to discuss are best understood by what they were intended to address rather than in the details of their chronology, we use the term “change” as opposed to “revision” in the previous section. It suffices to say that these changes had some overlap with but did not coincide with the revisions described above.

Although the research done by Stafford-Plummer (2002), Ricks (2003, 2011), and Webb (2006) indicated that PSTs were talking about mathematics, there was still a tendency for PSTs to view the mathematics as a set of simple procedures, which was not problematic for them (Ball 1990), and thus their conversations about conceptual mathematics was superficial. We now discuss changes that were made in the implementation of the methods lessons that helped increase the depth of conversions and nudged the PSTs toward more conceptual views of mathematics. Some of these changes were made directly to the lesson study portion of the course, and other changes were made to the process of preparing the methods lessons which, over time, began to look more and more like the lesson study process. We will discuss changes made to the textbooks, the post-lesson reflection meeting, and the role of the course instructor—a knowledgeable other.

First Change As mentioned previously some of the mathematics textbooks used in the course were traditional in nature. As PSTs prepared lessons based on these curricula, the description of the lesson’s FMC tended to be very simplistic and procedural. In some cases the FMC that students wrote was merely the title of the textbook section like "Solving Two-Step Equations" or a simple description of the associated procedure. By contrast the tasks in the "reform" curricula were less familiar to the PSTs and forced them to have conversations about the underlying mathematics or the mathematics they hoped students would learn by engaging in the task. Because the expectations associated with the reform curricula were for the PSTs to work through the task to identify the underlying mathematics, it lent itself to a greater focus on mathematics concepts as opposed to procedures. Thus, we abandoned our use of the traditional curricula and selected all of the methods lessons to be based on reform curricula.

Second Change After 2 years of implementing lesson study in the methods course, the first author spent 2 months studying student teaching in Japan (Corey et al. 2010; Peterson 2010). Since there are many elements of Japanese student teaching that mirror elements of lesson study, several new ideas were added to the preparation of and reflection on the methods lessons as well as the research lesson. One of those ideas was a clear structure to the post-lesson reflection meeting. Early on, the reflection meetings would often float away from the mathematical concepts for which the lesson was designed or the student thinking related to that mathematics. The encouragement for the observing PSTs to focus on student thinking was not sufficient to keep the reflection meetings centered on mathematics. They tended to only see the correct or incorrect execution of procedures and not the underlying conceptual mathematical understandings. This was seen when PSTs proffered compliments about the presentation of the teaching PST's lesson and not the mathematics they had observed. The PSTs would say things like "I liked how you spoke in a clear, loud voice" or "your management of that student disturbance was excellent." Thus, steps were taken to change the reflection meeting structure (outlined below) to improve the focus on conceptual mathematics.

The teaching PST was asked to respond to the following questions one at a time:

- What was the FMC of your lesson?
- How were the tasks designed to meet that FMC?
- How did the lesson play out?

The peers in the class then asked questions about the lesson choices made during the design and implementation of the lesson. These questions typically took up the bulk of the time in the reflection meeting. The peers were then given the opportunity to provide feedback to the group of PSTs who planned the lesson. The last portion of the reflection meeting was reserved for the methods course instructor to ask questions and make comments.

By starting the reflection meeting with the teaching PST sharing their FMC and describing the intended purpose of the task, it discouraged comments about the performance and focused the rest of the conversation on the mathematics and student mathematics thinking associated with that FMC and task. By having the PSTs in the class ask questions before they made comments allowed the teaching PST to clarify the intentions and decisions of the group before making comments about what happened in the lesson. To encourage PSTs' participation in the reflection meetings, the methods course instructor withheld many questions until the end of the reflection meeting. In these reflection meetings, the instructor acted as a knowledgeable other by modeling the type of questions on which the PSTs should focus. By iteratively experiencing this modeling, the PSTs began to see those important things in the later methods lessons and the research lesson. Near the end of the semester, PSTs typically asked most of the questions that the instructor had planninstructional triangle on the right in Figed to address in the reflection meeting, similar to Chassels and Melville's (2009) findings.

Third Change Another change made to the methods course (see the bolded third shape in Fig. 5), intended to help the PSTs prepare methods lessons which in turn influenced the preparation of the research lesson, was having the PSTs meet with the methods instructor as part of the lesson preparation process. In Japan, the student teachers prepare a draft lesson plan and then meet with the cooperating teacher to discuss what they had created. After receiving feedback on the lesson plan, they revise the lesson and meet again. They meet 2–3 times before receiving a literal "stamp of approval" to teach that lesson in the cooperating teacher's classroom. In our methods course, we simply required the PSTs to work together in their groups to create an outline of the lesson and then meet with the course instructor to discuss that outline prior to teaching the lesson. Based on this feedback, the PSTs would create their lesson plan before teaching it in the methods course.

Once this process of having groups of PSTs meet with the course instructor was implemented, it became clear that the most significant area of struggle for the PSTs as they prepared these lessons was having a clear conceptual understanding of the mathematics they were expected to teach. Potari (2011) reported that "the knowledgeable other played an important role in helping prospective secondary teachers to develop their understanding of mathematics teaching and learning" (p. 130). Therefore, the meeting with the course instructor became an important resource for the PSTs because time was spent discussing the mathematics the PSTs intended to teach and how that mathematics would emerge from the task they had selected. Because of this need to have conversations about and solidify the mathematics of the lesson in the minds of the PSTs, the meeting did not move on to anticipating student thinking. Thus, the student responses anticipated by the PST tended to be either correct thinking or incorrect procedures and not on the strategies or reasoning the students might use or how that thinking might be used when facilitating a class discussion (Webb 2006).

In order to better solidify the PSTs' understanding of the mathematics of the lesson and have some time to discuss anticipated student thinking, another slight adjustment was made to the process of the PSTs meeting with the course instructor. Prior to the meeting, the PSTs were required to submit the lesson FMC they had developed. This forced PSTs to spend time as a group thinking about and discussing the conceptual underpinnings of the mathematics of the lesson before meeting with the instructor. By encouraging the PSTs to significantly engage with the FMC prior to meeting with the instructor, the conversation in the meetings moved beyond just talking about the mathematics toward conversations about anticipating student responses. Recently, Mathis (2018) studied PSTs' conversations prior to and after the instructor meeting and found that the PSTs' conversations about mathematical meanings after the instructor meeting were significantly more common.

5 Reflections

Given our perspective that deeper mathematical understanding for students can be developed through their having rich conversations about mathematics and how students think about that mathematics, we attempted to improve the quality and quantity of such conversations by adopting modified JLS in our methods course. We learned that continual revision was necessary to account for subtle and unforeseen manifestations of traditional US teaching culture that were barriers to effective implementation. We also recognized that these revisions and changes aligned with authentic lesson study principles.

5.1 Continual Revision

We initially hoped that jointly planning and having conversations about conceptually based lessons (focused on student solutions to challenging tasks) in a JLS setting would support our PSTs to deepen their conceptual understandings as they tried to make sense of their peers' diverse thinking. Having the PSTs recognize that their conceptual understanding was enhanced by these conversations would also suggest to them a similar pedagogy where students would learn conceptually by discussing each other's mathematical thinking.

During the 15 years of iteratively implementing lesson study, however, we were surprised at the tenacity of the tacit cultural norms held by our PSTs. This traditional US teaching culture is more deeply ingrained in our PSTs than we had initially realized. Because teachers self-select their career—opting to work in the very environment they spent so much time in during childhood—their comfortableness with traditional school culture is deeply rooted and affects their perceptions of pedagogy and content. Stigler and Hiebert (1999) recognized the power of these "cultural scripts" on teacher action (p. 87).

The PSTs' cultural views of (a) teaching as a performance and (b) mathematics as procedural were two major barriers to the goals we had for the methods course and to the effective implementation of lesson study that we believed would help in meeting those goals. Subtle aspects of these two cultural barriers persistently prevented PSTs from operating in the ways we had anticipated. As soon as we made revisions to address these, different aspects of the cultural barriers were manifest. Even when the PSTs engaged in the reform-oriented activities we designed, the pervasiveness of traditional US educational culture manifests itself. We recommend that future reform considerations of teacher preparation emphasize more the importance of cultural forces on PSTs' actions. Reform recommendations that do not address such cultural forces in a consistent and careful manner risk rapid derailment by entrenched—but often hidden—cultural habits. We better understand now why teachers often revert back into traditional modes of instruction despite continued experiences with reform professional development.

We emphasize the role that challenging traditional US teaching culture should play in teacher education. Although we feel that engaging PSTs in reform activities is necessary for their professional development, it is not sufficient to "cure" them of undesirable pedagogical perceptions and habits. Addressing PSTs' traditional culture and encouraging them to adopt more reform approaches required our constant assessment of the ways in which our revisions and changes were or were not working as we had intended. We found that continual reflection on our PSTs' experiences helped us better design our interventions to reach our methods course goals.

5.2 Key Aspects of Japanese Lesson Study

As we reflected on the revisions and changes we made to our methods course in order to better meet our goals, we noticed that these adaptations also brought us more in line with authentic JLS; we now understand JLS better ourselves. We identified four aspects of JLS that are critical facets for our PSTs to engage in the rich conversations that deepened their understanding of the mathematics and student mathematical thinking. They were (1) *the knowledgeable other*, (2) *structured reflection meetings*, (3) *PSTs working in groups*, and (4) *the iterative process*.

The *knowledgeable other* was a key aspect in the methods course in many ways. They met with PSTs to have conversations that would redirect them to think about the mathematics in ways they had not typically been taught, which allowed PSTs to learn how to talk about and teach mathematics conceptually. They modeled types of questions to be asked in the reflection meetings to keep PSTs' conversations focused on the mathematics of the lesson and student thinking.

The *structured reflection meetings* allowed all PSTs to be involved in a discussion about the mathematics and student mathematical thinking. As PSTs heard specific ideas and feedback, they used those as they worked in groups to plan subsequent lessons. As PSTs moved away from a focus on their performance and moved toward a focus on students interacting with the mathematics, they were more receptive to the conversations and feedback during the reflection meetings because they realized that the feedback was not about their performance.

As *PSTs met in different groups* for each of their methods lessons and for their research lesson, they became more aware of different ways that their peers (and thus their future students) would think about a mathematical task. They were forced to move beyond a discussion of mathematical procedures and dig into the underlying concept to make sense of the fact that members of the group thought about the same piece of mathematics differently. These different ways of thinking about the mathematics were further explored as PST groups met with the knowledgeable other. PSTs were pushed by the knowledgeable other beyond where they were comfortable which increased the depth and focus of their conversations. Although the instructor meetings with the different PST groups were quite time-consuming, we found the benefit to the PSTs made the time investment worthwhile (Mathis 2018).

We found *the iterative process* manifested itself in many ways in the evolution of the methods course as well as in the methods course activities for the PSTs. The continual revision of the course was an iterative process. The plan-teach-reflect-revise process done twice in the lesson study process is clearly iterative but so were the iterations of planning methods lessons in groups, meeting with knowledgeable others, participating in reflection meetings, and watching videos of the lesson enactment. Each iteration of these activities was essential to the growth of the PSTs. The first methods lesson was particularly useful for the PSTs' growth because they learned what they do not know about teaching mathematics. They typically realized that, while planning, they had not discussed the mathematics at a depth needed for them to successfully teach the lesson. They also realized that they had not adequately anticipated student thinking. Because a peer thought about a mathematical question differently than the PST had anticipated, it caused the enacted lesson to deviate substantially from the lesson plan (Lamb 2015). Some may ask how many times do PSTs have to go through the iterative process, and our answer is as many times as possible (Sim and Walsh 2009; Takahashi and McDougal 2016). Even at the end of the semester, PSTs are still learning, which is what we want them to realize, that they will have good and bad lessons, but if they keep thinking about and talking about the mathematics and student mathematical thinking, their ability to plan lessons will improve.

6 Conclusions

Our two main goals of our university mathematics methods course were for secondary PSTs to engage in rich conversations about (1) the mathematics of their lessons and (2) how students think about that mathematics. These goals exist, in part, because we believe that mathematical understanding is deepened by such conversations. In order for lesson study to help us better meet these goals, we found that several adjustments needed to be made to help the PSTs overcome the cultural barriers of viewing teaching as a performance and viewing mathematics as a set of procedures. In short, we had to take steps for PSTs to view teaching as facilitating the interaction between the students and the mathematics—the horizontal linkage in the instructional triangle on the right in Fig. 2. We found that one key vehicle in helping the PSTs to change their view was to get them to take ownership of every lesson they prepared, whether they taught it or not. This, in turn, forced them to have authentic, rich conversations about the mathematics of the lesson and ways students think about the mathematics because they had to reconcile the differing ways the PSTs in their groups thought about the mathematics. We found that the steps taken to overcome cultural barriers in order to meet our course goals also brought us more in line with authentic JLS as we have come to understand it. We identified components of JLS that we found to be particularly valuable in meeting our goals and found that our continual effort is necessary to prevent the US cultural views of teaching and mathematics from undermining the effectiveness of these components. Our study helps contribute to the growing theoretical understanding of implementing lesson study in international settings, something called for by past research (Xu and Pedder 2014).

Additionally, we believe beginning lesson study experiences prior to student teaching to be an important component in successful lesson study implementation during student teaching and beyond. Our study raises awareness about the importance of long-term enculturation by participants into the lesson study processes, because these processes often clash with non-East Asian educational practices (Bjuland and Helgevold 2018; Bjuland and Mosvold 2015).

As reported earlier, most preservice lesson study research reports at most three lesson study planning cycles of one semester lesson study implementation duration with no additional iterations. Our study, however, highlights the importance of consistent, multi-cycle lesson study implementations by educational researchers, with continual refinement through multiple iterations, to account for unforeseen issues that will naturally arise. Our methods course revisions and changes consisted of making the overall methods lesson planning process more and more like authentic JLS processes, something that with hindsight seems obvious to us now but required many years of continual improvement for us to come to realize.

References

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The elementary school journal, 90*(4), 449–466.

Bjuland, R., & Helgevold, N. (2018). Dialogic processes that enable student teachers' learning about pupil learning in mentoring conversations in a Lesson Study field practice. *Teaching and Teacher Education, 70*, 246–254.

Bjuland, R., & Mosvold, R. (2015). Lesson study in teacher education: Learning from a challenging case. *Teaching and Teacher Education, 52*, 83–90.

Burroughs, E. A., & Luebeck, J. L. (2010). Pre-service teachers in mathematics lesson study. *The Montana Mathematics Enthusiast, 7*(2/3), 391–400.

Cajkler, W., & Wood, P. (2015). Lesson study in initial teacher education. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 105–127). Oxon: Routledge.

Chassels, C., & Melville, W. (2009). Collaborative, reflective, and iterative Japanese Lesson Study in an Initial Teacher Education program: Benefits and challenges. *Canadian Journal of Education, 32*(4), 734–763.

Chichibu, T. (2016). Impact on lesson study for initial teacher training in Japan: Focus on mentor roles and *kyouzai-kenkyuu*. *International Journal for Lesson and Learning Studies, 5*(2), 155–168.

Corey, D. L., Peterson, B. E., Lewis, B. M., & Bukarau, J. (2010). Are there any places that students use their head? Principles of high-quality Japanese mathematics instruction. *Journal for Research in Mathematics Education, 41*(5), 434–478.
Cruickshank, D. R., & Metcalf, K. K. (1993). Improving preservice teacher assessment through on-campus laboratory experiences. *Theory Into Practice, 32*(2), 86–92.
Fendel, D., Resek, D., Alper, L., & Fraser, S. (1997). *Interactive mathematics program: Integrated high school mathematics: Year 1*. Berkeley: Key Curriculum Press.
Fernandez, M. L., & Zilliox, J. (2011). Investigating approaches to lesson study in prospective mathematics teacher education. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 85–102). Dordrecht: Springer.
Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (2002). Chapter 16: Advancing children's mathematical thinking. In J. Sowder & B. Schappelle (Eds.), *Lessons learned from research* (pp. 135–142). Reston: The National Council of Teachers of Mathematics.
Glaser, B. J. (1965). The constant comparative method of qualitative analysis. *Social Problems, 12* (4), 436–445.
Griffiths, J. (2016). Bridging the school placement gap with peer micro-teaching lesson study. *International Journal for Lesson and Learning Studies, 5*(3), 227–238.
Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., et al. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 Video Study. *Educational Evaluation and Policy Analysis, 27*(2), 111–132.
Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203–235.
Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.
Kilpatrick, J., Martin, W. G., & Schifter, D. (Eds.). (2003). *A research companion to principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics.
Kilpatrick, J., Mesa, V., & Sloane, F. (2006, November). *US algebra teaching and learning viewed internationally*. In: Paper presented at the 2nd IEA international research conference, Brookings Institution, Washington, DC.
Lamb, P. (2015). Peer-learning between pre-service teachers: embracing Lesson Study. *International Journal for Lesson and Learning Studies, 4*(4), 343–361.
Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (1998). *Variables and patterns: Introducing algebra–teacher's guide*. Boston: Dale Seymour Publications.
Larssen, D. L. S., Cajkler, W., Mosvold, R., Bjuland, R., Helgevold, N., Fauskanger, J., et al. (2018). A literature review of lesson study in initial teacher education: Perspectives about learning and observation. *International Journal for Lesson and Learning Studies, 7*(1), 8–22.
Lewis, C. C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: Research for Better Schools.
Ma, L. (1999). *Knowing and teaching elementary mathematics*. Hillsdale: Erlbaum.
Mathis. (2018). *Secondary preservice mathematics teachers' curricular reasoning: What influences their decisions.* Unpublished master's thesis. Brigham Young University, Provo, Utah.
Parks, A. N. (2008). Messy learning: Preservice teachers' lesson-study conversations about mathematics and students. *Teaching and Teacher Education, 24*, 1200–1216.
Peterson, B. E. (2010). Mathematics student teaching in Japan: Where's the management. In G. Anthony & B. Grevhold (Eds.), *Teachers of mathematics: Recruitment and retention, professional development and identity* (pp. 135–144). Kristiansand: Writings from Swedish Society for Research in Mathematics Education No. 8.
Plummer, J. S., & Peterson, B. E. (2009). A preservice secondary teacher's moves to protect her view of herself as a mathematics expert. *School Science and Mathematics, 109*(5), 247–257.
Potari, D. (2011). Response to part II: Emerging issues from lesson study approaches in prospective mathematics teacher education. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 127–132). Dordrecht: Springer.

Ricks, T. E. (2003). *An investigation of reflective processes during lesson study by mathematics preservice teachers.* Unpublished master's thesis. Brigham Young University, Provo, Utah.

Ricks, T. E. (2011). Process reflection during Japanese lesson study experiences by prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education, 14*, 251–267.

Sim, L., & Walsh, D. (2009). Lesson Study with preservice teachers: Lessons from lessons. *Teaching and Teacher Education, 25*, 724–733.

Stafford-Plummer, J. (2002). *An analysis of the influence of lesson study on preservice secondary mathematics teachers' view of self-as mathematics expert*. Unpublished master's thesis. Brigham Young University, Provo, Utah. http://scholarsarchive.byu.edu/etd/62

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*, 313–340.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap*. New York: Free Press.

Takahashi, A. (2014). The Role of the Knowledgeable Other in Lesson Study: Examining the Final Comments of Experienced Lesson Study Practitioners. *Mathematics Teacher Education and Development, 16*(1), *1–17*.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: maximizing the impact of lesson study. *ZDM, 48*(4), 513–526.

Webb, M. M. (2006). *What does it mean to preservice mathematics teachers to anticipate student responses?* Unpublished master's thesis. Brigham Young University, Provo, Utah. http://scholarsarchive.byu.edu/etd/1122

Xu, H., & Pedder, D. (2014). Lesson Study. An international review of research. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 29–58). London: Routledge.

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Improving Prospective Teachers' Lesson Planning Knowledge and Skills through Lesson Study

Suanrong Chen and Bo Zhang

Contents

Abstract Improving prospective mathematical teachers' lesson planning skills is a very crucial task in teacher preparation programs. The purpose of this study is to examine how a lesson study approach implemented in a method course could develop prospective mathematical teachers' lesson planning skills from the perspective of Mathematical Knowledge for Teaching (MKT). Thirty-nine prospective teachers (PTs) enrolled in the course "Instructional Skills of Mathematics" participated in the program of developing lesson planning skills at a public university. Participants experienced an adapted lesson study cycle concentrating on lesson planning. Data included 39 PTs' lesson plans, 8 groups' revised lesson plans, and group reflections. The framework of MKT was used to capture the weakness of PTs' existing knowledge and the progress they made from the initial to the revised lesson plans. The analysis of the initial 39 lesson plans showed 18 different types of problems regarding planning a lesson. About 75% of the problems fell in the domain of pedagogical content knowledge. After experiencing the lesson study process, the participants demonstrated significant improvement in thinking about learning objectives, analysis of content and students, anticipating students' solutions, and sequencing mathematics tasks. Their reflections further confirmed their gains from the process. This study indicated that lesson study is an effective way to enhance PTs' teaching cognition and lesson planning skills.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_27

Keywords Prospective teachers · Lesson planning · Skills and knowledge · Adapted lesson study

1 Background

Teaching is becoming more complex along with the rapidly changing environment of the world, but some novice teachers hold the view that teaching is an easy task (Grossman et al. 2009). In addition, many pre-service teachers do not think it is necessary to receive training because teaching is perceived as a natural talent (Ball and Cohen 1999; Munby et al. 2001). Novice and prospective teachers (PTs) learn how stressful teaching is once they are placed in the position (Britzman 2003). PTs "had not yet developed the pedagogical content knowledge to feel confident making curricular decisions on their own" (Grossman and Thompson 2004, p. 5). Hence, developing habits of continued professional learning is part of the mission of teacher education programs (Mostofo 2014). Lesson planning, as a vital process of teaching, is a central focus in learning how to teach (Norman 2011). However, when novices confront the complexities of planning, they feel the need for more curricular guidance and structured assistance (Kauffman et al. 2002). That is why lesson planning is identified in university teacher preparation standards, as well as teacher certification standards and other related standards for teaching (Norman 2011). Thus, it is crucial to develop PTs' lesson planning skills through method course in teacher preparation programs.

In China, the development of teacher preparation systems has been in place for decades (Huang et al. 2010). In this system, the traditional teacher preparation programs for training pre-service secondary teachers are specialized and discipline-based, in which the proportion of credits on educational courses is far less than content courses (the ratio between both varies from 1:9 to 3:7) (Ding et al. 2014). From the above structure of courses, we see that the programs emphasize building PTs' strong content knowledge base and problem-solving skills, while the development of their pedagogical knowledge and practical experience are overlooked (Li et al. 2008; Li and Huang 2009; Yang et al. 2012). Research findings from Fan (2013) and Yan (2014) indicate that PTs may not have gained enough pedagogical content knowledge for teaching mathematics from colleges and university preparation. Based on the above situation, Li and Huang (2018) pointed out that we need to rethink how to nurture secondary mathematics teachers to equip them with adequate knowledge for teaching. This may lead the teacher educators to think about the ways of helping PTs acquire enough knowledge for teaching within their methods courses.

In recent years, developing mathematical competences in classrooms has become urgent due to a new round of revisions of mathematics curriculum standards in China (CMERGCS 2017). Here, Mathematical competences refer to mathematical knowledge and skill base, mathematical ability, mathematical consciousness, and mathematical emotions, attitudes, and values (Wu and Zheng 2012). This requires preparing competent mathematical teachers who are able to teach for developing students' competences. However, most of the pre-service teachers have learned mathematics in a typical traditional way which is teacher-centered and test-driven; they are not focused on developing their mathematical competence and practical knowledge. In fact, the newly recruited teachers who have recently graduated from universities cannot quickly adapt to school settings that demand high qualifications (Huang 2016). One possible reason is that many novices are weak in making sense of the complexity of teaching and they do not take it seriously (Grossman et al. 2009); another possible reason is that the current teacher education programs create a gap between the methods courses and the field experiences (Darling-Hammond 2006; Hou 2016; Huang et al. 2013).

What may be an effective way for teacher educators to help PTs to develop the understanding of the complexity of teaching and narrow the gap? PTs must be placed in a practice position to engage in learning how to teach, and then, they must experience the difficulty of teaching. More and more researchers consider using lesson study (LS) as a professional development (PD) context for training PTs (Ponte 2017). LS originated from Asia is now widely applicable for PD around the world (Hart et al. 2011; Huang et al. 2013; Lewis et al. 2006). This is where the idea came from for this study: using a LS-based process for improving PTs' lesson planning knowledge and skills within mathematical methods course. The purpose of this study is to develop prospective mathematics teachers' lesson planning knowledge and skills based on the framework of Mathematical Knowledge for Teaching (MKT) (Ball et al. 2008) through an adapted LS cycle. Specifically, the study aims to answer the following research questions:

1. Does the lesson study build connections between theory and practice in lesson planning? And how?
2. Does the adapted LS improve PTs' skills in planning a lesson? And how?

2 Literature Review and Theoretical Framework

2.1 Literature Review

Placing the focus on developing highly qualified teachers to make educational progress, teacher education programs confront the increasing responsibility to prepare new teachers (Sandholtz 2011). After reviewing the research in pre-service teacher education in the areas of science, mathematics, and technology, Mecoli (2013) stated that "beginning teachers provided with excellent teacher education developed more substantial pedagogical content knowledge than novices without the coursework" (p. 21). It was further suggested that "neither pedagogical nor content knowledge will suffice, but rather, that finding better ways to help novice teachers learn and use pedagogical content knowledge will help to realize this goal (all the knowledge and skills they will need for their teaching careers)" (p. 26). LS adapted for PTs is one potential way for helping them reach that goal as demonstrated by

Meng and Sam (2013), Rasmussen (2016), and Peterson (2005). The LS process involves lesson planning and can be modified to focusing on developing the knowledge and skills of learning how to design a good lesson plan.

2.1.1 Lesson Study

LS is a practice-based professional learning process, usually using a systematic method of refining lessons through collaborative lesson planning, implementing the plan with students, reflecting on testing the plan, and revising the plan (Fernandez and Yoshida 2004; Huang et al. 2013; McMahon and Hines 2008; Stigler and Hiebert 1999). LS can also be considered as a "research lesson," involving a group of teachers or a combined group of teachers and teacher educators/researchers exploring how a lesson can be best implemented for students' learning (Ponte 2017). One prominent feature of LS is promoting teachers' teaching competence and increasing efficiency of teaching by concentrating on students' learning (Murata 2011). In order to achieve excellence of teaching and learning, LS may involve several cycles of planning, teaching, observing, and revising while having parallel classes allowed to implement the revised plan (Hart et al. 2011; Robinson and Leikin 2012). LS has been playing significant roles in teachers' professional training around the world (Yildiz and Baltaci 2017). Chinese LS has been suggested as an effective way to enhance teachers' competence and students' learning for decades (Chen and Yang 2013; Huang and Bao 2006; Huang and Han 2015; Huang et al. 2016). In Japan, LS has become a common format utilized in teachers' PD, showing the effectiveness in promoting teachers' teaching and students' learning (e.g., Peterson 2005; Fernandez and Yoshida 2004). Outside of these two countries, LS is also a popular format of PD for improving teachers' teaching (e.g., Pang 2016; Mon et al. 2016; Takahashi and McDougal 2016; Groves et al. 2016; Verhoef et al. 2014; Unal and Jakubowski 2007).

Can lesson studies be adapted to support PTs' professional development? Many teacher educators/researchers have explored the appropriate manners of LS into specific context (Rasmussen 2016). Elipane (2012) elaborated investigations on the cultural and institutional conditions that make it possible for LS to be effectively integrated into the Japanese prospective mathematics teacher education. Rasmussen (2016) transposed LS to Danish educational and cultural context for enhancing pre-service teachers' learning when participating in a teacher education program. In his study, the post-lesson reflection was the most important element of lesson study. Murata and Pothen (2011) gave details on how LS was implemented in an American context to help PTs build connections between teachers' teaching and students' learning. Meng and Sam (2013) described how LS was adopted in a Malaysian context and showed a significant difference in secondary teachers' technological pedagogical content knowledge (TPACK) for teaching mathematics with GSP (Geometer's Sketchpad) before and after conducting LS.

Although activities carried out in LS can build connections between theory and practice, the use of LS in pre-service teacher education may face some unique challenges: "given the nature of the national policies of the future, and the usual scarcity of time and resources available" (Ponte 2017, p. 169; Rasmussen 2016). After reviewing many studies regarding the use of LS in prospective teacher education, Ponte (2017) identified several pending issues such as identifying the aims, establishing the relationships among participants, scaling, and adapting lesson studies for the specific purpose of educating future teachers. In this study, we particularly focus on improving PTs' lesson planning knowledge and skills. When incorporating the elements of LS into a method course, we made adaptions to the standard lesson study format, especially in the planning stages. Before practicing the lesson plans and performing group discussions, we asked PTs to plan a lesson individually and then asked them to revise their plans after receiving feedback from their initial plans from the teacher educator, who will also provide an exemplary plan analysis and a hands-on microteaching session in the class. The purpose of this adaption is to push PTs to get some sense of the process of lesson planning, what the lesson plan should look like, and how to make an effective lesson plan. After these stages, we then suggested PTs practicing the plan by using microteaching within groups and revising the plans further after discussing the teaching experience with peers.

2.1.2 Lesson Planning

A lesson plan is a written description of how teachers direct students' learning activities to attain specific objectives, which also serves as a framework to guide their students to the certain learning destinations (Farrell 2002; Vdovina and Gaibisso 2013). Basically, a lesson plan contains the illustrations of learning goals, content, and sequence, as well as activity procedure, implementation, and assessment (Jacobs et al. 2008). In China, learning objectives and difficulties are also required to be specified in a lesson plan (Wang 2008). Effective teaching requires teachers to complete "three types of transformations" during their lesson planning: transforming the knowledge and methods presented in the textbook and other resources into teachers' own knowledge and methods, transforming the curriculum expectations into teachers' guiding ideology of implementing teaching activities, and transforming the comprehension of the content and students as well as the interrelationships between both into the instructional strategies mastered by teachers (Wang 2008). When planning a lesson, teachers should meet the following needs: understand the "big ideas" and set clear mathematical learning goals according to mathematics curriculum standards; specify learning objectives and learning focuses and difficulties through analyzing the content and students; select appropriate tasks based on learning trajectories (Simon 1995; Sztajn et al. 2012) and variation pedagogy (Gu et al. 2004); and anticipate the possible responses to the tasks from students (Huang et al. 2016; Wang 2008). Thus, lesson planning involves teachers' mastery of content and skills of the subject, as well as their mastery of pedagogy knowledge. Mecoli (2013) demonstrated that "beginning teachers with excellent teacher education developed more substantial pedagogical content knowledge than novices without the course work after reviewing the research in pre-service science, math, and technology teacher education" (p. 21). For PTs, learning how to plan a lesson is one of the crucial skills needed to be gained during their teacher education (Sahin-Taskin 2017).

However, research has shown that PTs feel it difficult to plan their lessons (Tashevska 2008). They often need more time than experienced teachers to plan lessons (Richards 1998; Senior 2006). The key issue is the disequilibrium of theory and practice (Latham and Vogt 2007; Parsons and Stephenson 2005). Too much theoretical learning and too little practice are problematic in university methods courses. Teaching is an activity that requires more practical experience (Basturk 2016), so does planning. Since non-novice teachers have had lots of experiences in planning a lesson and some experiences come from teaching a topic multiple times, the planning process is probably internalized in their minds. For PTs, the planning process needs to be taught; their thoughts may be different from experienced teachers' on how to plan a lesson (Sahin-Taskin 2017). Lesson planning requires teachers to build a connection among the requirements of curriculum and textbooks and what will be presented in their classes (Li et al. 2009). Comprehending curriculum and textbooks is a crucial stage of mastery of the content and setting learning goals (Graff 2011). PTs may learn the content knowledge in a certain area quickly but may not achieve the depth or flexibility of mastery that they desire (Larkin 2013). It is essential to help them conduct the content analysis regarding curriculum. Moreover, PTs lack field experiences with students and have the difficulties of understanding how students approach learning content (Basturk 2016). This results in a diminished sense of what strategies should be selected and how learning activities should be implemented. Schmidt (2010) explored the value of peer teaching, early field experience, and student teaching for enriching PTs' knowledge of pedagogy. Thus, more curricular guidance and structured assistance (such as organizing group discussions of learning difficulties and creating an environment for testing the plan) are needed for PTs (Kauffman et al. 2002).

2.2 *Theoretical Framework*

The work of planning a lesson is an important part of teaching a lesson. In order to carry out the work of teaching mathematics effectively, teachers must have Mathematical Knowledge for Teaching (MKT), the concept developed by Ball et al. (2008). The domains of MKT are described in Fig. 1. Since Ball, Thames, and Phelps developed the MKT theory built on Shulman's (1986) notion of pedagogical content knowledge in 2008, the frame of MKT is effectively used in many professional development programs to develop pre- and in-service mathematics teachers' teaching competence (e.g., Beswick et al. 2012; Corkin et al. 2015; Jacob et al. 2017; Karagöz-Akar 2016; Lannin et al. 2013; Livy et al. 2016; Miheso-O'Connor Khakasa and Berger 2016; Wikie 2016; Youchu 2016).

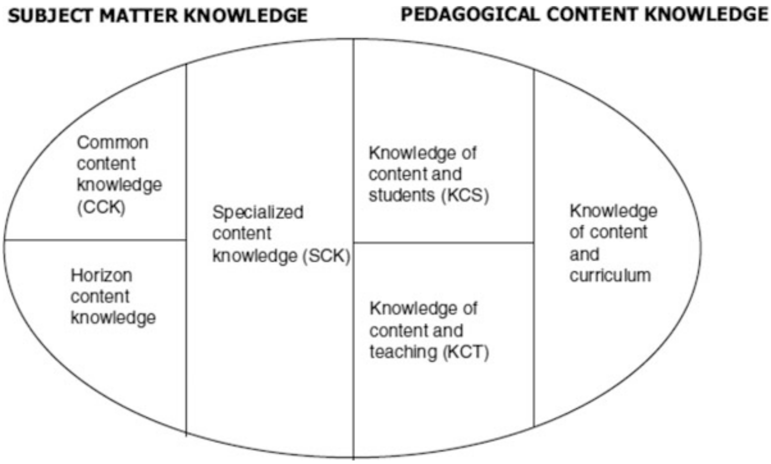

Fig. 1 Domains of mathematical knowledge for teaching (Ball et al. 2008)

There are two major subdomains in MKT: subject matter knowledge (SMK) and pedagogical content knowledge (PCK). SMK refers to the knowledge dealing with common mathematical knowledge of math, special mathematical knowledge of teaching math, and the interrelationship of mathematical knowledge in the curriculum; PCK refers to the content knowledge dealing with the teaching process and running the learning activities (Ball et al. 2008; Shulman 1986). Table 1 describes the meaning of each category in the two major subdomains (Ball et al. 2008). We provide concrete examples to illustrate each domain.

Doing an in-depth analysis of pre-service teachers' lesson plans by utilizing MKT theory can provide teacher educators with a sense of what teachers know and need to learn (Loughran et al. 2008; Nilsson 2008). Reflecting on the provided feedback of what domain of MKT the pre-service teachers lack through the specific lesson plan, they have the directions to revise and improve the lesson plan. Moreover, they get to know what areas need to be strengthened in integrating theory and practice for future acquisition of PCK.

3 Methods

3.1 Setting

This study was conducted in a secondary mathematics methods classroom (course title: Instructional Skills of Mathematics) at a public university in Southern China. A collaborating research group was established before the class began. Two associate professors of mathematics education have formed the collaborative group. In addition, 17 graduate students majoring in mathematics education were invited to assist

Table 1 Description of six categories of MKT

Domains of MKT	Definition	Example
Common Content Knowledge (CCK)	Mathematical knowledge and skill used in the settings and not special to the work of teaching	Knowing 12 is the correct answer of 79 subtracting from 91
Specialized Content Knowledge (SCK)	Mathematical knowledge and skill distinctive to teachers' teaching	Knowing that 91 minus 79 can be solved by the ways of "11–9 + 80–70," "91–80 + 1," or "1 + 90–80 + 1"
Horizon Content Knowledge (HCK)	The knowledge of knowing about how mathematical topics are related to the span of mathematics included in the curriculum	Knowing how teaching of two-digit subtraction like "91–79" relates to one digit subtracting from two-digit and/or multiple-digit subtraction
Knowledge of Content and Students (KCS)	The knowledge of knowing about what and how students think about and interpret mathematics	Anticipating students' different responses and difficulties of doing subtraction with "91–79"
Knowledge of Content and Teaching (KCT)	Knowledge of knowing about what and how teachers sequence mathematics and learning activities	Deciding which way of solving "91–79" should be discussed first in class and what will be placed next
Knowledge of Content and Curriculum (KCC)	Knowledge of mathematics relating to the requirements of curriculum	The overall expectation of two-digit subtraction in the curriculum

in the assessment of lesson plans. The methods class is a 16-week course consisting of two 45-minute class sessions per week taught by one of the two associate professors. The participants were junior undergraduate PTs in a secondary education program majoring in mathematics. These participants also took another theory course, Curriculum and Pedagogy, a 16-week course consisting of three 45-minute class sessions per week offered by another associate professor who is the co-researcher of this study. Two teacher educators met every week and discussed how these two courses complement each other and apply theory and content into practice. This study was about a designed coursework taken in the methods classroom setting. PTs were educated in the theory of lesson plans and the corresponding requirements in the course "Curriculum and Pedagogy"; then they were taught the process of lesson planning, the frame of a lesson plan, and the specific guiding questions of each aspect regarding the frame in the course "Instructional Skills of Mathematics." When they had the foundation knowledge of lesson planning, they were asked to plan a lesson and gain improvement of the plan through an adapted LS process.

The general process of lesson planning before implementing the lesson should include five steps according to Wang (2008). Stage One is getting the larger picture about overall expectations set by the curriculum; Stage Two is comprehending inside the content and its structure of forward and backward knowledge; Stage Three is considering the students' knowledge background, learning experiences and interests, and characteristics; Stage Four is analyzing both content and students to set up the learning focuses and difficulties; Stage Five is planning the actual lesson and creating a document for the plan. In order to realize self-improvement in teaching, the entire process of lesson planning should also include reflection on the intended plan, its implementation, and refinement of the lesson plan as a reference lesson plan for the second-round teaching of the same lesson. This reflection serves as a meaningful Stage Six in the adapted LS process.

The frame of creating a lesson plan includes the title of the topic, version of the textbook being used, learning objectives, learning focuses, learning difficulties, content analysis, student analysis, and instructional process and activities. More specifically, the instructional process and activities are required to describe the teaching stages, the instructor's activities, the application of technology, and students' activities in each stage. This frame is commonly used in school settings in China. For non-novice teachers, descriptions of content analysis and student analysis are not required to be given in the plans in some schools. For novice or PTs, it is necessary to do so because the teacher educators need to examine their SMK and PCK through their description of content analysis and student analysis. In order to give PTs a good start, a lesson plan template was provided as below (Table 2), which was created by the teacher educator according to Wang (2008) about the principles of instructional design.

3.2 Participants

There were 39 junior PTs in the methods class. All of them voluntarily chose to participate in the study. They were grouped into eight teams. Each team has four or five candidates which includes both males and females, along with both outgoing and endocentric styles. In this study, they were required to do independent work and group work in terms of the specific need for the research. They played roles of teachers or students when implementing their lesson plans through microteaching; they discussed and reflected on their plans and implementation within a group.

3.3 Data Sources

The lesson topic selected for the research LS was that of irrational equations. This topic occurs in the eighth-grade mathematics curriculum for compulsory education in Shanghai, China. We chose this topic because students have finished the content of rational equations previously, and irrational equations are a relatively hard content to learn for students. We want to see how these PTs help students build the connections among all algebraic equations. Participants were asked to design a lesson for the first section of this topic. The content includes the concept of irrational equation, the structure of algebra equations, and solving simple irrational equations including equations with one radical expression. Three sources of qualitative data

Table 2 Lesson plan template

Topic		Name		Student ID	
The version of textbook					
Objectives					
Learning focuses					
Learning difficulties					
Content analysis					
Student analysis					
Instructional process and activities					
Teaching stages	Instructor's activities		The application of technology		Students' activities
Stage One:					
Stage Two:					
...					

about this topic were collected through the LS process. One source was the pre-service teachers' initial lesson plans. This data was examined by a team of evaluators to identify the problems that occurred in their lesson plans in terms of MKT framework. The other two sources were the revised lesson plans and reflections conducted by groups. The revised lesson plans were used to examine their progress after several processes in LS. Their reflections were used to uncover how they made progress in designing a specific lesson.

3.4 Process of Improving Lesson Plan Through an Adapted Lesson Study Process

Enhancing the efficacy of pre-service teachers, lesson study was an effective approach to the collaborative nature of the process, the teaching practice opportunities, and the observation of others' teaching (Mostofo 2014). The typical LS process is collaboratively planning the lesson plan, seeing the lesson plan in action, discussing the lesson plan, revising the lesson plan, teaching the new version of the lesson, and sharing reflections about the new version of the lesson (Meng and Sam 2013). We think the above process should be adapted for PTs, especially focusing on improving their lesson planning knowledge and skills in this study. Unlike experienced teachers, novices or pre-service teachers need help to develop individual lessons because they lack well-developed schema for imagining lessons (Norman 2011). Figure 2 depicts the process that was adapted for PTs to use in the methods class.

The whole process was described in the six boxes along the bigger circle in Fig. 2. There was extra work required for both teacher educators and PTs between any two

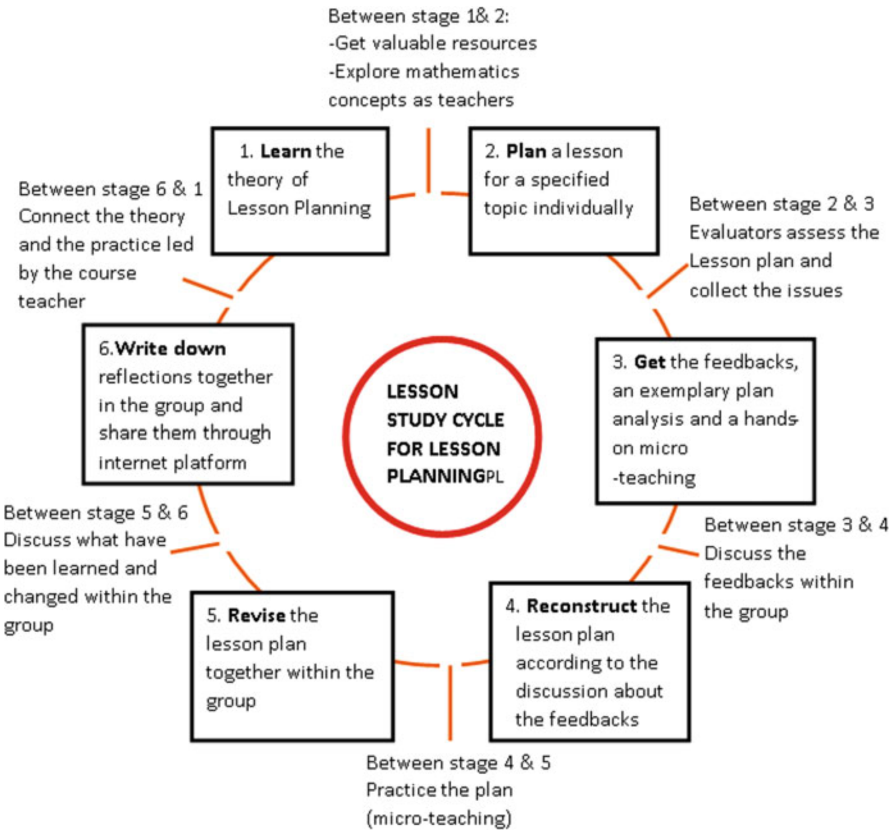

Fig. 2 The adapted lesson study cycle

consecutive stages. We decided that the adapted LS process should start from learning the theory of lesson planning because foundational knowledge is very important for PTs as they begin their careers. It is also essential to let PTs go through the planning process independently after gaining the foundational knowledge. Planning a lesson individually was the second stage. Of course, PTs need to get valuable resources and explore mathematical concepts between Stage One and Stage Two. After Stage Two, PTs received feedback from the teacher educator after the team of lesson plan evaluators analyzed the planned lessons according to the MKT framework. An exemplary lesson plan was also presented to the PTs along with the feedback. The exemplary lesson plan was created by the teacher educator and used for PTs to discuss what and how the MKT framework embodied in the plan, how the teacher educator thought about the content and students, and how they reflect on their own plans. The teacher educator emphasized that individual lessons can be structured differently, but it must meet the overall expectations set by the curriculum. After this class discussion, PTs were instructed to rethink and reconstruct their plans based on the received feedback. In order to test whether the reconstructed plan was effective, the teacher educator selected one prospective teacher from each group to conduct microteaching for the introduction part of the plan and form the concept of irrational equations. The observers (the teacher educator and other prospective teachers) gave feedback immediately to the selected presenters in class. This was designed as a hands-on activity for guiding how microteaching aided in reflection on the lesson plan. Next, the teacher educator encouraged all PTs to implement the reconstructed plans within their groups through microteaching outside of the classroom so that they could find out what part of the plan did not embody the plan intention. Each student was allowed to have about 20 min to present the lesson. Then, they had a group discussion about aspects of the lesson that were effective as well as ineffective. This process helped them to rethink about and revise the plans together. The quality of lessons can be improved after pre-service teachers receive feedback and revise their plans (Chassels and Melville 2009; Ganesh and Matteson 2010). The final stage was writing reflections together on what they have learned and improved during the LS process and then sharing the reflections on the learning internet platform. Finally, the teacher provided a summary on connecting the theory and practice with the students in the class.

In a traditional methods course, teacher educators introduced the theory of lesson planning and then assigned PTs to write a lesson plan and evaluate each plan to provide some feedback to peers. This simple learning process may help them learn more about lesson planning, but it cannot help them explore their struggles and find the ways to improve their work. We think it very crucial to create a collaborative environment to support and develop their first experience of planning. They can be led to go through a completed lesson planning process and a set of improvements of activities by being given appropriate support.

3.5 Data Analysis

An action research model was implemented in this study. In this model, teacher-researchers conduct a systematic inquiry by gathering information about how well the learners learn based on an innovation. The adapted LS was the innovation for improving learners' learning outcomes. MKT framework was used to analyze PTs' lesson planning competence. Qualitative data was collected as the study progressed. Data from participants' initial lesson plans were analyzed by a specified collaborative team including 2 teacher educators (experts in lesson planning) and 17 graduate students majoring in mathematics education who were taking the course "Case Analysis of Instructional Design." These graduate students were guided to read through the curriculum standards and the textbook content, especially the irrational equation section, and then were trained how to analyze the lesson plan under the guidance of the principles of instructional design. They separately analyzed the plans and presented what specific problems were found. The teacher educators gathered all the information and invited the graduate students to discuss and identify the problems shown in each plan. Data from reflections and revised lesson plans on the process of LS were analyzed by the two teacher educators. The analysis of lesson plans was based on the initial frame of lesson plan in which the identified problems were categorized according to the domains of MKT framework. The analysis of reflections was open-coded and categorized separately by the two teacher educators, and then, the teacher educators made an agreement on the hierarchical classification of group reflections and what specific categorization each reflection point belongs to.

4 Results

All PTs who participated in this study were able to accomplish their course work before the deadline set by the teacher educator. They had passion to work on their individual plans, as well as the revised plans and reflections during the LS process. The report of the results including the problems found in initial plans, the final plans, and group reflections is detailed below.

4.1 Problems Found in PTs' Initial Lesson Plans

After examining PTs' initial plans, we found a total of 18 types of problems from 39 initial lesson plans, which were shown in the 4 major components of lesson plans: instructional objectives, content analysis, student analysis, and instructional process and activities (see Table 3). Each problem was aligned with a category in the domains of MKT. "Ratios" showed the percentage of the plans from the total 39 plans and indicated the type of problem present. For example, 17/39 plans (43.59%) demonstrated the percentage of the problem that the objectives of learning process or emotions and attitudes were missing. Obviously, some problems have very high frequencies. For example, 29 PTs' lesson plans were found to be a lack of specialized content knowledge in dealing with the balance of procedural knowledge and the theory of problem-solving; 28 PTs' lesson plans were short of horizon content knowledge in revealing the deep connections among all the concepts in the specified knowledge structure in the curriculum. More details about the percentages of each problem shown in the initial plans can be seen in Table 3.

Table 4 provides a summary of problems classified by using the MKT framework. Among 18 problems, 5 problems fell in the subdomain of SMK (28.8%), and 13 problems fell in the subdomain of PCK (72.2%). This means that PTs' performance on subject matter knowledge was significantly better than their pedagogical content knowledge, which supports the finding from Pang (2011), Fan (2013), and Yan (2014). This indicates PCK can hardly be learned from a textbook or a short course. PTs really need more experiences in putting pedagogical knowledge into practice either in planning or microteaching. When they received the feedback from the teacher educator, they could learn from their misapplication. Although the amount of problems within the SMK domain is less than the PCK domain, the top

Table 3 Results of initial lesson plans

Aspects	Type of problems	MKT	Frequency
Instructional objectives	P1: The objectives about learning process or emotions and attitudes are missing	KCC	17/39 (43.59%)
Content analysis	P2: The content analysis does not reveal the deep connections among all the concepts in the specified knowledge structure	HCK	28/39 (71.79%)
	P3: The content analysis includes some misconception of mathematics	CCK	2/39 (5.13%)
Student analysis	P4: The obstacles of students' learning process are not specified	KCS	5/39 (12.82%)
	P5: The analysis of knowledge difficulties is not very accurate	KCS	8/39 (20.51%)
Instructional process and activities	P6: Students' responses to math problems are anticipated in lesson plan based on solutions that the teacher expected, but there are some different solutions	KCS	9/39 (23.08%)
	P7: The instructional process does not match the objectives.	KCC	8/39 (20.51%)
	P8: The learning importance and difficulties in lesson plan are pointed out but are not embodied in the teaching process	KCT	4/39 (10.26%)
	P9: There is a lack of the coherence from one learning process to the next	KCT	10/39 (25.64%)
	P10: The lesson plan focuses more on the instructor's activity but less on the students'	KCT	7/39 (17.95%)
	P11: The transformation of mathematical thought is not demonstrated in lesson plan	KCT	14/39 (35.90%)
	P12: Understanding of the concept is not strengthened through clarifying the different types of equations	KCT	4/39 (10.26%)
	P13: There are mathematical errors in instructional process	CCK	4/39 (10.26%)
	P14: The knowledge that should be taught later in the curriculum is misplaced to be taught earlier	HCK	12/39 (30.77%)
	P15: The time distribution for the teaching process is not reasonable	KCC	7/39 (17.95%)
	P16: The lesson plan only emphasizes on the procedural knowledge but not on the theory of problem-solving	SCK	29/39 (74.36%)
	P17: The application of learning context is not clear	KCT	5/39 (12.82%)
	P18: Examples and problems shown in the lesson plan do not embody the levels of difficulty from low to high	KCT	13/39 (33.33%)

two ratios of problems, P2 and P16, both belong to the subdomain of SMK according to Table 3. There is no doubt that many PTs lack the mastery of the content beyond the surface knowledge presented in the textbook even if they had more than 3 years of college mathematics professional learning. This indicates that

Table 4 Summary of classified problems according to MKT

MKT		Type of problems	Ratios (total 18 problems)
SMK	CCK	P3, P13	2/18 (11.11%)
	SCK	P16	1/18 (5.56%)
	HCK	P2, P14	2/18 (11.11%)
PCK	KCS	P4, P5, P6	3/18 (16.67%)
	KCT	P8, P9, P10, P11, P12, P17, P18	7/18 (38.89%)
	KCC	P1, P7, P15	3/18 (16.67%)

PTs need to spend more time in making sense of planning even if they have gained the foundational knowledge of mathematics and pedagogy. When they are put into the practice, they feel lost.

4.2 Problems Found in the Final Lesson Plans

After going through several activities of lesson study such as getting feedback from the teacher educator, engaging in an exemplary lesson plan analysis, discussing the sources of feedback within the group, reconstructing the lesson plan based on the discussion, and conducting microteaching within the groups, each group (total eight groups) produced a revised plan based on their reconstructed individual plans. Among the previous 18 types of problems, 5 of these 18 types of problems were not solved thoroughly in the 8 revised plans again; no new problems were discovered (see details in Table 5). The two most common problems (P16 and P2) in initial plans were still the first two most common ones in the revised plans. That tells us that PTs need to practice more on how to improve their horizon content knowledge as described by P2 and specialized content knowledge as described by P16. Two groups completely solved all the previous problems in their lesson plans. What may be the reason behind the fact that some groups still had unsolved problems? The teacher educator did not give the individual feedbacks to each prospective teacher instead of telling them 18 types of problems found in their plans overall. The reason for this is that the teacher educator wanted PTs to examine their own problems according to these 18 types of problems. This is a designed process for PTs to analyze their plans individually and collaboratively in later activities.

More details about the improvement of each group were specified in the following tables from Group 1 (G1) to Group 8 (G8). We have mentioned that the teacher educator did not show the exact problems that each plan had in order to allow PTs to conduct self-assessment according to the given 18 types of problems found in 39 lesson plans. Thus, some of the problems might not be discovered by them; or when they combined each plan to make one revised plan, they neglected some problems they should paid attention to. Although some groups have one or two problems unsolved, the revised plan had progressed greatly compared to their

Table 5 Summary of problems in the final plan

Problems unsolved in the final plan	Frequency
P2: The content analysis does not reveal the deep connections among all the concepts in the specified knowledge structure	2/8 (25%)
P7: The instructional process does not match the objectives	1/8 (12.5%)
P10: The lesson plan focuses more on the instructor's activity but less on the students'	1/8 (12.5%)
P16: The lesson plan only emphasizes on the procedural knowledge but not on the theory of problem-solving	3/8 (37.5%)
P18: Examples and problems shown in the lesson plan do not embody the levels of difficulty from low to high	1/8 (12.5%)

initial plans. Overall, PTs' planning performances were greatly improved on this specific topic after going through the lesson study activities (Table 6).

4.3 *Reflections on What PTs Have Gained in LS Process*

According to the adapted lesson study cycle, PTs were asked to conduct a reflection on their lesson study activities and lesson planning process about what they have been benefited in general and what changes have been made specifically.

4.3.1 Reflections on General Benefits

What have PTs benefited from their experiences of the whole lesson study cycle? Evidence provided below demonstrates the general benefits they received through the adapted lesson study process, especially in their improved cognition with lesson planning. PTs recognized that planning is a complex process and involves many duties. G1 wrote that "Through learning, we know that teaching design is not a single act, it is the whole cycle of preparation, including practice, reflection and modification, and many factors should be taken into consideration." They also realized the importance of collaboration during their professional learning. G2 wrote that "By classroom learning and group discussion, what impressed us most was the power of cooperation." Moreover, they realized that planning should consider more about students' thinking and understanding rather than teachers' one-sided wishes. This can be supported by G8's statement that "When we first constructed our teaching design, we began to consider organizing the teaching activities just relying on our own expectation, without fully taking the academic conditions into consideration. But now we rebuilt the teaching activities from a student's perspective." Thus, their reflections support that teaching is not an easy or natural thing (Ball and Cohen 1999; Munby et al. 2001; Grossman et al. 2009).

Table 6 The comparison problems between the initial and final plans

Group	PTs	Problems in initial plans	Problems in the final plans
G1	PT140701207	P1, P2, P4, P6, P7, P8, P9, P11, P18	P18: Examples and problems shown in the lesson plan do not embody the different levels of difficulty from low to high
	PT140701215	P1, P3, P16	
	PT140701220	P11, P16, P18	
	PT142203132	P1, P2, P4, P11, P13, P16	
	PT142506104	P3, P4, P16	
G2	PT140701208	P1, P2, P5, P11, P17, P18	P16: The lesson plan only emphasizes on the procedural knowledge but not on the theory of problem-solving
	PT140701214	P1, P2, P6, P16	
	PT140701225	P5, P9, P11, P15, P16	
	PT140701230	P1, P2, P7, P10, P11, P12, P16	
	PT140701231	P2, P5	
G3	PT140701206	P4, P8, P9, P10, P18	P2: The content analysis does not reveal the deep connections among all the concepts in the specified knowledge structure P16: (see the above)
	PT140701210	P2, P7, P12, P16	
	PT141602132	P1, P2, P3, P5, P7, P11, P14, P15, P16	
	PT142202120	P1, P5, P9, P11, P12, P15, P16, P17	
G4	PT140701203	P2, P7, P9, P12, P15, P16	P2: (see the above)
	PT140701204	P1, P2, P14, P11, P16	
	PT140701227	P2, P6, P11, P13, P16	
	PT140804314	P2, P4, P14, P16, P18	
	PT142402228	P2, P14	
G5	PT140701205	P5, P6, P7, P16, P17	P7: The instructional process does not match the objectives P16 (see the above)
	PT140701211	P1, P2, P11, P16, P18	
	PT140701212	P2, P14, P15,	
	PT140703102	P1, P2, P6, P7, P15, P16	
	PT140703129	P1, P6, p14, P18	
G6	PT140701201	P2, P14, P16, P18	None
	PT140701202	P2, P14, P18	
	PT140701219	P1, P2, P11, P16	
	PT140701228	P2, P5, P9, P16, P18	
	PT140702130	P1, P4, P9, P10, P13, P16	

(continued)

Table 6 (continued)

Group	PTs	Problems in initial plans	Problems in the final plans
G7	PT140701216	P1, P7, P8, P10, P16, P18	None
	PT140701226	P1, P2, P9, P15, P16	
	PT140701229	P2, P6, P11, P14, P16	
	PT140701232	P6	
	PT140703101	P2, P7, P9, P16, P18	
G8	PT140701218	P2, P5, P9, P10, P13, P14	P10: The lesson plan focuses more on the instructor's activity but less on the students'
	PT140701222	P2, P6, P14, P15, P16	
	PT140701223	P1, P2, P8, P10, P11, P16, P18	
	PT140701224	P2, P5, P10, P16, P17	
	PT140701233	P2, P14, P16, P 17	

4.3.2 Reflections on What Problems They Had Before and What Changes They Made After

Reflections on what PTs have changed were categorized into several focuses such as instructional objectives, introduction of the lesson, designed examples and exercises, learning focuses and difficulties, students' learning and knowledge sequencing, sense of design, subject matter knowledge, teaching cohesion, time distribution, and conclusion of the lesson.

Through PTs' reflections on instructional objectives (IO), we see that they re-thought about the instructional objectives in order to reflect on the actual teaching and learning process and that they made the objectives more concrete and added some dimensional objectives which were missed but required by the curriculum. For example, G2 wrote that "The objectives about learning process are missing. Most of the group members' learning objectives did not have a description about the process, which were all about 'understanding', 'grasping' rather than 'by means of', 'going through'." G3 reflected that "The modified instructional objectives were more concrete and added some like 'develop students' ability to explore independently', which made the whole instructional objectives more complete and meet the requirements of quality education." PTs made these changes because they found problems on the identified instructional objectives in their initial plans after they discussed the requirements of the curriculum and the feedback about the instructional objectives from their teacher educator.

Six out of eight groups reflected on the introduction of the lesson. They reconsidered using what questions or mathematics problems could work better for conducting the introduction part. G4 reflected that "We had many types to make an introduction and most of the students chose to set a context, using the same example as that in the book directly. . . .. . .It must have a direct purpose and stick to the topic to make a context introduction." They also thought about their original wrong anticipation with students' responses and wanted to take full considerations of students' thoughts. For instance, G5 wrote that "In the first teaching design, especially in the introduction section, we anticipated the results we wanted rather than taking students' level of knowledge and skills into consideration." G5 then reflected that "Meanwhile, we should think from students' perspectives." They appreciated the process of reconstructing the lesson plan and doing microteaching to test the plan. For example, G7 wrote that "In the process of the second reconstructing and micro-teaching, (especially in the introduction section) we used to anticipate what the students thought about and enhance the understanding the concept for students,. . . ." This helped them improve the effectiveness of implementing the introduction of the lesson.

The reflection focused on designed examples and exercises was mainly about how to present the exercises and homework problems in an orderly fashion from low- to high-level difficulty. PTs assigned the problems with a random level of difficulty in their initial plans. For example, G2 reflected that "The problem came that we directly assign the homework rather than further considering the layers." They made changes because they realized this would negatively impact students' learning. G6 wrote that "Sometimes they are too difficult to go beyond students' cognition." So, they tried to reorganize the order of example and exercise problems.

From PTs' reflections on learning focuses and difficulties, we see that many groups realized they did not embody the learning focus and difficulties of the teaching and learning processes. The reflections of G6, G8, and G2 mentioned about this issue. For instance, G2 wrote that "We pointed out the key points, but our design did not stress or highlight the key points." And then, G2 also wrote that "We detailed the key and difficult points." Hence, they made changes for highlighting the key points in different ways.

How to sequence the students' learning and knowledge was a topic that engaged many of the participants. PTs reflected on students' learning and knowledge sequencing to promote their recognition of this question. After examining their initial plans and going through the LS process, the PTs realized that they should consider the dynamic learning process and the student-centered classroom because the real setting is complex and might be as straightforward as expected. As G1 described in their reflections, "If you had preset your ideal students' behaviors, teaching accidents may happen during the actual teaching process." So, they realized that "It was more important to design the interactive section than presetting students' behavior in the teaching design."

Although the reflection focuses on the sense of design, we see that some of PTs had nearly no sense of planning a lesson and merely copied contents from the textbook. G3 reflected that "The lesson plan should have a sense of design." And further stated that "It should not be an accumulation of the contents in the book." This resulted in PTs not thinking about the connection between teaching and learning activities nor considering the necessary activity for mastering the key concept in a lesson. After they analyzed these problems in their initial plans, they made better sense of design. For example, G2 wrote that “In this rebuilding, with the discrimination of the concepts added, the key points were obviously highlighted and made the whole design look more exquisite.”

The focus of PTs’ refection also covered many other aspects, such as subject matter knowledge, teaching cohesion, time distribution, and conclusion of the lesson. For instances, G6 wrote that “The lack of the subject-matter knowledge of mathematics not only resulted in the deviation, omission, presetting in every link design, but it also led in the lack of grasping of contents and academic analysis.” G7 addressed the teaching cohesion issue that “In addition, the teaching link was not a single part and should be linked together smoothly and fluently.” G5 reflected on the class time distribution that “The time used in the actual practice should be taken into consideration and we should allocate the time in a good way.” G1 pointed out that “When finishing the session, we should concisely grasp the key points and draw the conclusion of the whole lesson, rather than being prolix or verbose.” We see that PTs examined the problems of lacking subject matter knowledge, not considering the cohesion among teaching and learning activities, allocating unreasonable teaching and learning activity time, and overlooking the importance of making a conclusion of the lesson in their initial plans. The improvements that they made in these aspects are supported by the results of their final lesson plans.

5 Discussion and Conclusion

5.1 Discussion

We found that PTs had difficulties in putting both theory and content into practice on their own through the analysis of their initial plans. This supports the findings that new teachers struggle to prepare content and materials when they were put into the real-life setting (Kauffman et al. 2002). After we implemented the adapted LS, a set of LS activities functioned in connecting theory and practice. For instance, in their final plans, PTs made sense of how to set the three-dimensional objectives (knowledge and skills, process and methods, emotions and attitudes) according to the requirements of the curriculum after they discussed the exemplary lesson plan with the teacher educator. However, in designing their initial plans, they had no idea of setting the objectives which were supposed to be achieved by experiencing the learning process and methods. In order to make PTs think about what and why they misunderstood the theory or content and how they could make it better, they were asked to conduct self-assessment with their individual initial plans according to the feedback from the teacher educators about the overall assessment of their initial plans. Later, they were also asked to conduct microteaching to test whether what they planned could achieve the instructional objectives. The above activities helped PTs to improve their lesson planning knowledge and skills from both the results of the comparison between the initial and final lesson plans and the analysis of group reflections. This supports the findings from Murata and Pothen (2011), Elipane (2012), Meng and Sam (2013), and Rasmussen (2016) that LS is an effective way of helping PTs enhance their Mathematical Knowledge for Teaching. Overall, the adapted LS cycle (see Fig. 2) for the specific purpose of concentrating on enhancing lesson planning skills in this study is a vital contribution for educating future teachers. This successfully solved the pending issue of adapting LS that Ponte (2017) pointed out.

There are possible concerns with implementing LS. Some of the professional learning activities were carried out by eight groups outside of the classroom. For example, PTs were required to discuss the feedbacks within their group in the activity between Stages Three and Four, while they were not monitored or guided by the teacher educator. The effectiveness of this activity highly depended on PTs' self-consciousness and the power of the group leader. We should mention that the teacher educator did extra work for training the group leaders in order to enhance the quality of group activities. The training included the following: recording their group discussion as resources for the teacher educator to assess group participation, reporting group members' contributions in the activity to the teacher educator in the coming class, organizing members to evaluate their learning mutually in the activity, and conducting hands-on microteaching activity in the class. These strategies could contribute to promoting the effectiveness of independent group activities. However, it should be a regret that the teacher educator could not observe PTs' microteaching when they conducted this action out of the classroom. There was a practical difficulty for the teacher educator to observe 39 PTs' microteaching. Hence, training the group leader is essential. Exploring the effectiveness of training group leaders may become a meaningful research project in the future. Additionally, in order to maximize the effectiveness of LS, the teacher educator may create more opportunities to place PTs in a real classroom context to test the plan. Future study will focus on exploring how LS contributes to improving PTs' instructional skills through the collaboration with in-service teachers by placing them in a real context.

5.2 Conclusion

First, PTs made great progress in lesson planning through participating in the LS process. We found that 18 types of problems present in initial plans were reduced to 5 in final plans; there are 2 out of 8 groups that created final plans with no problems. More specifically, they gained improvements on identifying the instructional objectives about students' learning process and attitudes, analyzing students' learning background and pointing out their learning difficulties, anticipating students' responses of solving math problems, embodying the learning importance and difficulties in instructional process and learning activities, creating activities for discriminating mathematical concepts, allocating the instructional time, presenting the appropriate mathematical knowledge, and determining the appropriate position of the new knowledge in curriculum. They also made large progress in analyzing the content, abstracting mathematical thoughts after dealing with the procedural knowledge, sequencing the math problems according to the levels of difficulty, balancing the instructors' and students' activities, and implementing the objectives into the learning process. Overall, PTs did enhance their subject matter knowledge and pedagogical content knowledge after they experienced the series of LS activities.

Second, multiple collaborations and interactions among the teacher educators, graduate students, and PTs through the lesson study process made stronger connections between theory and practice in this study. Collaborative and interactive activities were embodied in the cooperation between the two teacher educators, between the teacher educator and the graduate students who acted the roles of lesson evaluators under the guidance of their teacher educator, and among the participating PTs who conducted microteaching within a group or discussed in a class. The two educators cooperated to teach on how to plan and think about how to adapt the typical lesson study process for promoting PTs' lesson planning skills. They discussed what should be taught and how they made a supplementary teaching on the theory of planning based on their own strong, professional background. They led the team of lesson planning evaluators to analyze the problems identified in initial plans and diagnosed PTs' weakness of Mathematical Knowledge for Teaching. This process actually stimulated the graduate students to gain a deeper understanding of an effective lesson plan which is mutually beneficial for their teacher and themselves. The multi-interaction among the teacher educators and PTs assured the effectiveness of PTs reflecting on the feedbacks which contributed to their promoted cognition of theory and practice. The group reflection on their professional learning activities was a collaborative way to help achieve the professional learning goals. All collaborations in this research were to lead PTs to build a strong connection between theory and practice.

Third, from the results of the final plans and the analysis of group reflections, we could draw a conclusion that the adapted lesson study cycle concentrating on the improvement of lesson planning did work effectively in this study. From Stage One to Stage Six, PTs experienced the activities of learning foundational knowledge of planning: planning a specific lesson, gaining feedback and more knowledge of planning a lesson from an exemplary analysis, reconstructing individual plans in terms of their group discussion about the feedbacks, revising a group plan based on their reconstructed individual plans, and reflecting on their performance and cognition of lesson planning. Every stage showed how to make an effective plan. The activities designed between any two stages of the LS model were vital for assuring the quality of the next stage's activity. Although this model took PTs a large amount of time, commitments, and energies to create and ameliorate a lesson plan, they did make a significant improvement when compared with their initial plans. The group reflections described what specific changes were made and how these changes were made within the progress. Therefore, the LS cycle promoted the effectiveness of PTs' professional learning and the depth of their cognition of lesson planning, which supports what Chassels and Melville (2009) and Ganesh and Matteson (2010) found.

Acknowledgments This project is made possible with the funding from the grant of Innovation and Entrepreneurship Project (numbered with yzucx2016-1B) supported by Yangzhou University and the grant of Jiangsu Brand Professional Development Project in Universities (numbered with PPZY2015B109). The authors would like to thank the graduate students in 2016 Mathematical Master of Education program who conducted the assessment of prospective teachers' initial lesson plans with the two teacher educators and the prospective teachers who voluntarily participated in this study.

References

Ball, D., & Cohen, D. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice*. San Francisco: Jossey-Bass.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education, 59*(5), 389–407.

Basturk, S. (2016). Investigating the effectiveness of micro-teaching in mathematics of primary pre-service teachers. *Journal of Education and Training Studies, 4*(5), 239–249.

Beswick, K., Callingham, R., & Watson, J. (2012). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education, 15*(2), 131–157.

Britzman, D. P. (2003). *Practice makes practice: A critical study of learning to teach* (rev. ed.). Albany: State University of New York Press.

Chassels, C., & Melville, W. (2009). Collaborative, reflective, and iterative Japanese lesson study in an initial teacher education program: Benefits and challenges. *Canadian Journal of Education, 32*(4), 734–763.

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*, 218–236.

China's Ministry of Education Review Group of Curriculum Standard (CMERGCS). (2017). *A list of core competences for each subject of general high schools* [EB/OL]. Retrieved from: https://wenku.baidu.com/view/a8da25359e3143323868936b.html

Corkin, D. M., Ekmekci, A., & Papakonstantinou, A. (2015). Antecedents of teachers' educational beliefs about mathematics and mathematical knowledge for teaching among in-service teachers in high poverty urban schools. *Australian Journal of Teacher Education, 40*(9), 31–62.

Darling-Hammond, L. (2006). *Powerful teacher education*. San Francisco: Jossey-Bass.

Ding, G., Li, M., Sun, M., Li, Y., Chen, L., Yang, F., et al. (2014). *National survey and policy analysis on the pre-service teacher education of student teachers in normal universities and colleges*. Shanghai: East China University Press (in Chinese).

Elipane, L. E. (2012) *Integrating the essential elements of lesson study in pre-service mathematics teacher education*. IND Skriftserie: Copenhagen University.

Fan, Q. (2013). *Survey on pedagogical content knowledge among pre-service mathematics teachers from normal universities and colleges* (Unpublished master thesis). Xian: Shaanxi Normal University (in Chinese).

Farrell, T. S. C. (2002). Lesson planning. In J. C. Richards & W. A. Renandya (Eds.), *Methodology in language teaching: An anthology of current practice* (pp. 30–39). New York: Cambridge University Press.

Femandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. London: Routledge.

Ganesh, B., & Matteson, S. M. (2010). The benefits of reteaching lessons in preservice methods classes. *Action in Teacher Education, 32*(4), 52–60.

Graff, N. (2011). “An effective and agonizing way to learn”: Backwards design and new teachers’ preparation for planning curriculum. *Teacher Education Quarterly, 38*(3), 151–168. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23479622

Grossman, P., Compton, C., Lgra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teaching College Record, 111*(9), 2055–2100.

Grossman, P., & Thompson, C. (2004). *Curriculum materials: Scaffolds for new teacher learning*. Stanford, CA: Center for the Study of Teaching and Policy. Retrieved from http://depts.washington.edu/ctpmail/Reports.html

Groves, S., Doig, B., Vale, C., & Widjaja, W. (2016). Critical factors in the adaptation and implementation of Japanese lesson study in the Australian context. *ZDM: The International Journal on Mathematics Education, 48*(4), 501–512.

Gu, L., Huang, R., & Marton, F. (2004). In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.),. How Chinese learn mathematics: Perspectives from insiders *Teaching with variation: An effective way of mathematics teaching in China* (pp. 309–348). Singapore: World Scientific.

Hart, L. C., Alston, A. S., & Murata, A. (2011). *Lesson study research and practice in mathematics education: Learning together*. New York: Springer.

Hou, D. (2016). An empirical study of improving students’ teaching skill under the background of colleges and universities transformation. *Journal of Northwest Normal University (Social Sciences), 53*(6), 109–114.

Huang, R., & Bao, J. (2006). Towards a model for teacher’s professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298.

Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students’ understanding through theory-driven lesson study. *ZDM Mathematics Education, 48* (4), 425–439.

Huang, R., & Han, X. (2015). Developing mathematics teachers’ competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.

Huang, R., Peng, S., Wang, L., & Li, Y. (2010). Secondary mathematics teacher professional development in China. In F. K. S. Leung & Y. Li (Eds.), *Reforms and issues in school mathematics in East Asia* (pp. 129–152). Rotterdam: Sense Publishers.

Huang, X., Sun, L., & Wu, S. (2013). To enhance pre-service teachers’ professional accomplishment in education of science. *Teacher Education Research, 25*(5), 56–61.

Huang, Y. (2016). The missing and improvement of pre-service teachers’ practical knowledge. *Teacher Education Research, 28*(5), 85–90 (in Chinese).

Jacobs, C., Martin, S. N., & Otieno, T. C. (2008). A science lesson plan analysis instrument for formative and summative program evaluation of a teacher education program. *Science Education, 92*, 1096–1126.

Jacob, R., Hill, H., & Corey, D. (2017). The impact of a professional development program on teachers’ mathematical knowledge for teaching, instruction, and student achievement. *Journal of Research on Educational Effectiveness, 10*(2), 379–407.

Karagöz-Akar, G. (2016). Prospective secondary mathematics teachers' perspectives and mathematical knowledge for teaching. *EURASIA Journal of Mathematics*., *Science & Technology Education, 12*(1), 3–24.

Kauffman, D., Johnson, S. M., Kardos, S. M., Liu, E., & Peske, H. G. (2002). Lost at sea: New teachers’ experiences with curriculum and assessment. *Teachers College Record, 104*(2), 273–230.

Lannin, J. K., Webb, M., Chval, K., Arbaugh, F., Hicks, S., Taylor, C., & Bruton, R. (2013). The development of beginning mathematics teacher pedagogical content knowledge. *Journal of Mathematics Teacher Education, 16*(6), 403–426.

Larkin, D. (2013). 10 things to know about mentoring student teachers. *The Phi Delta Kappan, 94* (7), 38–43. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23611699

Latham, N. I., & Vogt, W. P. (2007). Do professional development schools reduce teacher attrition? Evidence from a longitudinal study of 1000 graduates. *Journal of Teacher Education, 58*(2), 153–167. https://doi.org/10.1177/0022487106297840.

Lewis, C. C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Li, S., Huang, R., & Shin, Y. (2008). Discipline knowledge preparation for prospective secondary mathematics teachers: An East Asian perspective. In P. Sullivan & T. Wood (Eds.), *Knowledge and beliefs in mathematics teaching and teaching development* (pp. 63–86). Rotterdam: Sense Publishers.

Livy, S. L., Vale, C., & Herbert, S. (2016). Developing primary pre-service teachers' mathematical content knowledge during practicum teaching. *Australian Journal of Teacher Education, 41*(2), 152–173.

Li, Y., Chen, X., & Khum, G. (2009). Mathematics teachers' practices and thinking in lesson plan development: A case of teaching fraction division. *ZDM Mathematics Education, 41*, 717–731.

Li, Y., & Huang, R. (2009). Examining and understanding prospective mathematics teacher preparation in China from an international perspective. *Journal of Zhejiang Education Institutes, 1*, 37–44 (in Chinese).

Li, Y., & Huang, R. (2018). *How Chinese acquire and improve mathematics knowledge for teaching*. Boston: Sense Publishers.

Loughran, J., Mulhall, P., & Berry, A. (2008). Exploring pedagogical content knowledge in science teacher education. *International Journal of Science Education, 30*(10), 1301–1320.

McMahon, M. T., & Hines, E. (2008). Lesson study with preservice teachers. *Mathematics Teacher, 102*(3), 186–191.

Mecoli, S. (2013). The influence of the pedagogical content knowledge theoretical framework on research on preservice teacher education. *The Journal of Education*, *193*(3), 21–27. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/24636918

Meng, C. C., & Sam, L. C. (2013). Developing pre-service teachers' technological pedagogical content knowledge for teaching mathematics with the Geometer's sketchpad through lesson study. *Journal of Education and Learning, 2*(1), 1–8. https://doi.org/10.5539/jel.v2n1p1.

Miheso-O'Connor Khakasa, M., & Berger, M. (2016). Status of teachers' proficiency in mathematical knowledge for teaching at secondary school level in Kenya. *International Journal of Science and Mathematics Education, 14*, 419–435.

Mon, C. C., Dali, M. H., & Sam, L. C. (2016). Implementation of lesson study as an innovative professional development model among Malaysian school teachers. *Malaysian Journal of Learning and Instruction, 13*(1), 83–111.

Mostofo, J. (2014). The impact of using lesson study with pre-service mathematics teachers. *Journal of Instructional Research*, 355–363.

Munby, H., Russell, T., & Martin, A. K. (2001). Teachers'knowledge and how it develops. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 877–905). Washington, D.C.: American Educational Research Association.

Murata, A. (2011). Introduction: Conceptual overview of lesson study. In C. L. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 1–12). New York: Springer.

Murata, A., & Pothen, B. E. (2011). Lesson study in preservice elementary mathematics methods courses: Connecting emerging practice and understanding. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 103–116). New York: Springer.

Nilsson, P. (2008). Teaching for understanding: The complex nature of pedagogical content knowledge in pre-service education. *International Journal of Science Education, 30*(10), 1281–1299.

Norman, P. (2011). Planning for what kind of teaching? Supporting cooperating teachers as teachers of planning. *Teacher Education Quarterly*, *38*(3), 49–68. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23479617

Pang, J. (2016). Improving mathematics instruction and supporting teacher learning in Korea through lesson study using five practices. *ZDM: The International Journal on Mathematics Education, 48*(4), 471–483.

Pang, Y. (2011). *A study of prospective teachers' mathematical knowledge for teaching and how to develop it* (Unpublished master thesis). Shanghai: East China Normal University (in Chinese).

Parsons, M., & Stephenson, M. (2005). Developing reflective practice in student teachers: Collaboration and critical partnerships. *Teachers and Teaching: Theory and Practice, 11*(1), 95–116. https://doi.org/10.1080/1354060042000337110.

Peterson, B. E. (2005). Student teaching in Japan: The lesson. *Journal of Mathematics Teacher Education, 8*(1), 61–74.

Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6*(2), 169–181.

Rasmussen, K. (2016). Lesson study in prospective mathematics teacher education: Didactic and Para-didactic technology in the post-lesson reflection. *Journal of Mathematics Teacher Education, 19*(4), 301–324.

Richards, J. C. (1998). *Beyond training: Perspectives on language teacher education*. New York: Cambridge University Press.

Robinson, N., & Leikin, R. (2012). One teacher, two lessons: The lesson study process. *International Journal of Science and Mathematics Education, 10*(1), 139–161.

Sahin-Taskin, C. (2017). Exploring pre-service teachers' perceptions of lesson planning in primary education. *Journal of Education and Practice, 8*(12), 57–63.

Sandholtz, J. (2011). Preservice teachers' conceptions of effective and ineffective teaching practices. *Teacher Education Quarterly*, *38*(3), 27–47. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23479616

Schmidt, M. (2010). Learning from teaching experience: Dewey's theory and preservice teachers' learning. *Journal of Research in Music Education*, *58*(2), 131–146. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/40666239

Senior, R. (2006). *The experience of language teaching*. New York: Cambridge University Press.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Simon, M. A. (1995). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24*, 233–254.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*, 147–156.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM: The International Journal on Mathematics Education, 48*(4), 513–526.

Tashevska, S. (2008). *Some lesson planning problems for new teachers of English*. CELTA Syllabus and Assessment Guidelines. Retrieved from http://www.cambridge.efl.org_teaching

Unal, H., & Jakubowski, E. (2007). Middle and secondary preservice mathematics teachers' comparative analysis of TIMSS videotape lesson study. *Turkish Online Journal of Educational Technology – TOJET, 6*(3), 61–69.

Verhoef, N., Tall, D., Coenders, F., & van Smaalen, D. (2014). The complexities of a lesson study in a Dutch situation: Mathematics teacher learning. *International Journal of Science and Mathematics Education, 12*(4), 859–881.

Vdovina, E., & Gaibisso, L. C. (2013). Developing critical thinking in the English language classroom: A lesson plan. *ELTA Journal, 1*(1), 54–68.

Wang, Q. (2008). *Instructional skills of mathematics*. Shanghai: East China Normal University Press (in Chinese).

Wilkie, K. J. (2016). Learning to teach upper primary school algebra: Changes to teachers' mathematical knowledge for teaching functional thinking. *Mathematics Education Research Journal, 28*(2), 245–275.

Wu, X., & Zheng, Y. (2012). Exploring students' mathematical competences based on the new curriculum. *China Teaching Journal, 4*, 52–55 (in Chinese).

Yan, L. (2014). *Research on pre-service mathematics teachers' ability to analyze and deal with students' errors* (Unpublished master thesis). Xian: Shanxi Normal university (in Chinese).
Yang, Y., Li, J., Gao, H., & Xu, Q. (2012). Teacher education and the professional development of mathematics teachers. In J. P. Wang (Ed.), *Mathematics education in China: Tradition and reality* (pp. 205–238). Singapore: Cengage Learning Asia.
Yildiz, A., & Baltaci, S. (2017). Reflections from the lesson study for the development of techno-pedagogical competencies in teaching fractal geometry. *European Journal of Educational Research, 6*(1), 41–50.
Youchu, H. (2016). A qualitative study on the development of pre-service teachers' mathematical knowledge for teaching in a history-based course. *EURASIA Journal of Mathematics, Science & Technology Education, 12*(9), 2599–2616.

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Lesson Study in Mathematics Initial Teacher Education in England

Fay Baldry and Colin Foster

Contents

Abstract Drawing on research in mathematics initial teacher education (ITE) in England, this chapter explores the potential of lesson study to support pre-service mathematics teachers' pedagogical development. We draw attention to ways in which lesson study can be highly transformative of beginning mathematics teachers' pedagogical learning. We consider the current ITE context in England and analyse potential opportunities and challenges for incorporating lesson study into mathematics ITE courses. We conclude by proposing a theoretical model for using lesson study in mathematics ITE that takes account of these contextual issues and offers ways to make the most of the opportunities available. We particularly highlight the importance of a productive partnership between university and school, the need to focus both collaborative lesson planning and observation on a clear research question that relates to student learning of mathematics and the necessity of a tight protocol for post-lesson discussions.

Keywords Initial teacher education · Lesson study · Mathematics education · School partnerships

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_28

1 Introduction

In recent years, several variants of lesson study (Fernandez and Yoshida 2012) have been deployed in mathematics initial teacher education (ITE) courses in England (e.g. Archer 2016). The adoption of lesson study outside of Japan has brought a diversification of approaches, but all generally involve a cycle of collaborative planning, an observed research lesson and some form of post-lesson discussion, similar to the model described by Cajkler and Wood (2016a; see Figure 1). The patterns followed in mathematics lesson study in ITE in England have been shaped by the particular features of the ITE context here, which present challenges to the implementation of lesson study but also potentially offer considerable opportunities to support pre-service teachers' pedagogical development. In this chapter, we survey key aspects of approaches to mathematics ITE lesson study that could increase the likelihood of its success. We summarise these in a proposed theoretical model for using mathematics lesson study with pre-service teachers.

Currently, ITE provision in England is heavily fragmented, with trends moving away from university-led provision and towards an emphasis on school-led pathways. Pre-service teachers now choose from a range of different routes, which are typically a 1-year postgraduate programme but with variations in their levels of higher education institution (HEI) involvement. The shift towards school-led routes has had wider effects on the power relationships between schools and HEIs, meaning that expertise in teacher education may be seen to reside primarily in the school, or in the HEI, or be distributed across some combination of the two. This has implications for mathematics lesson study in ITE, where the school may take a central or a more peripheral role in the process. For example, if a 'knowledgeable other' (KO) role is included in the lesson-study model, then this might be provided by the HEI, if that is where the relevant expertise is considered to lie. Alternatively, this role could be taken by a senior leader in the school or by another individual regarded as having the appropriate pedagogical expertise, either within the school or from another school.

In this chapter, we outline what we see as key features of mathematics lesson study in ITE that should be carefully considered within the context in England. We draw these together to offer a research-informed theoretical model for lesson study in mathematics ITE (Fig. 2), based on ways in which lesson study has been shown to be successful. We argue that this model is realistic and has considerable potential to support pre-service teachers' pedagogical development.

2 Lesson Study in Mathematics ITE

Lesson study is a powerful model for professional development. In Japan, pre-service mathematics teachers typically attend multiple lesson-study events during their ITE and may also sometimes teach research lesson themselves. However, when lesson study has been adapted for use in ITE in other parts of the world, it has become common for pre-service teachers both to observe and to teach the research lessons, and this may indeed be seen as central to the process (Larssen et al. 2018). In England, while the composition of the lesson-study group varies, it is normal for pre-service teachers to be involved in some teaching as well as observing.

Drawing on many accounts of mathematics lesson study in ITE (e.g. Corcoran and Pepperell 2011; Elipane 2011; Fernandez and Zilliox 2011; Gunnarsdóttir and Pálsdóttir 2011; Rasmussen 2016), we now address three aspects of implementing and sustaining mathematics lesson study in ITE which have been found to be particularly important in the English context.

2.1 Productive Partnership Between HEI and School

Successful lesson study in ITE depends on a productive partnership between the university-based teacher educator and the host school (Baldry and Foster in press). All pre-service teachers undertake at least 24 weeks of teaching placements, where they are assigned an experienced teacher as their mentor to oversee and assess their progress. If pre-service teachers undertake lesson study, it is usually conducted during the placements. In these circumstances, the cooperation, if not the active involvement, of the school-based mentor is required. Pre-service teachers occupy a position in between HEI and school and may sometimes need to mediate aspects of the process between their university tutor and their school-based mentor (see Baldry and Foster in press). This 'boundary crossing' can prove a challenge, as reported by Tsui and Law (2007) in a study of trainee teachers of English in Hong Kong.

The current ITE landscape in England is currently experiencing a shift from HEI-based routes to school-based ones. Political rhetoric emphasises the practical knowledge that practising teachers possess and identifies schools as the best place to learn to teach; conversely, university ITE courses are portrayed as being dominated by ideology and dogma and as taking negative stances on politically promoted programmes and emphases. However, this negative view of what universities have to offer is not universally accepted, and many pre-service teachers and schools are highly supportive of the role of university education departments and the experience and expertise that mathematics education tutors bring. Nevertheless, it is important to consider where power relationships among university tutors, in-service teachers (such as school-based mentors) and pre-service teachers may place constraints on how lesson study may be enacted in ITE. One particularly important aspect in this regard is the role of the 'knowledgeable other' (KO), which we discuss in the next section.

It is usual for the idea of embarking on mathematics lesson study in ITE to originate with the HEI, and thus there can be an issue of ownership, in which lesson study may appear to be 'done to' the pre-service teacher and school-based mentor by the HEI. Ideally, lesson study would be seen as a supportive vehicle for the pre-service teacher's development and one to which the mentor can make a powerful contribution. However, in some models there is an expectation that the mentors teach a research lesson, a process that mentors may find uncomfortable, particularly if they link observation to teacher evaluation. Moreover, there are likely to be constraints imposed by the HEI concerning when and how the lesson study is carried out, resulting in a danger that the school, and perhaps pre-service teachers, complies in a reluctant or uninterested way, merely in order to meet course requirements. When schools are reluctant to engage, it is likely that the process will be of far less value to all parties and may indeed be ‘lesson study’ only by name. In this case, the process may often default to more familiar and accepted forms of mutual lesson observation and evaluative feedback, which focus on ‘how the teacher did’ and ‘what they could do to improve’.

Power imbalances between a pre-service teacher and a school-based mentor can also be a hindrance to effective lesson study. In this case, we see the lesson-study process itself as holding a possible solution. Archer (2016), writing about power imbalances between pre-service teachers and mentors, found that lesson study:

> could break barriers established by positions of power and afford trainee teachers an opportunity to see themselves as experts whose power comes from their knowledge and reflections on their plans and from having been able to observe and reflect on children’s work (p. 17)

Although carrying out mathematics lesson study in ITE may be complicated and ‘messy’ (Parks 2008), the structure of lesson study, when done in an appropriate way, can be effective in focusing all participants’ attention onto school students’ learning. This allows everyone’s contribution – whether university tutor, school-based mentor or pre-service teacher – to be valued. Although pre-service teachers will naturally have more limited pedagogical knowledge than other participants, they may be good observers of what takes place in the classroom. Indeed, they may be able to approach this task with greater neutrality and objectivity, having less prior experience of students and classrooms to colour their perceptions. Moreover, the pre-service teachers may have had a good amount of induction into lesson study from their university tutors and therefore can bring that expertise into the school setting, increasing what they may be perceived to have to offer to the process. Burroughs and Luebeck (2010) found that pre-service teachers could engage productively in lesson study alongside in-service teachers. Although they found tasks such as anticipating student responses to mathematics tasks difficult, the authors report that in-service teachers also did, and the authors concluded that this should not be seen as a serious limitation.

HEI-based routes for ITE, in which pre-service teachers spend a considerable portion of the year in block school placements, devote considerable attention to seeking to bring theory and practice together productively for the pre-service teachers. Fernández (2005) found that microteaching lesson study helped pre-service mathematics teachers to make links between theory and practice, and lesson study, with its emphasis on a ‘research’ lesson, can be a powerful way of linking theory to practice for pre-service teachers (Gunnarsdóttir and Pálsdóttir 2011). However, Bjuland and Mosvold (2015) found that a mathematics lesson study in Norway with pre-service teachers ran into difficulties when the pre-service teachers did not focus in their observations on their students' learning. Moreover, reporting on lesson study undertaken by pre-service teachers in Spain, Angelini and Álvarez (2018) argued that the evaluation of student learning needed further development. Therefore, if the lesson-study process is to be as valuable as possible, we would underline the centrality of having a clear focus on student learning.

Theoretical perspectives play an integral part in planning and interpreting student learning, and HEIs are well placed to support pre-service teachers with these. This could be a way of levelling power relations, because, for lesson study to be effective, planning and post-lesson discussion must involve more than drawing on existing professional experience. When collaboratively planning the research lesson, some participants may be better placed to predict student learning for each stage of the lesson, but such predictions are shared by all members of the planning group. With every participant focused on capturing student responses, about which none of them will have detailed prior knowledge, the privilege of prior experience is reduced.

2.2 The Locus of Pedagogical Knowledge

A central aspect of mathematics lesson study as traditionally practised in Japan is the involvement of a 'knowledgeable other' (KO), an experienced in-service teacher or mathematics educator from outside the immediate setting who provides extensive expert commentary following the research lesson. Prestigious KOs draw large crowds of teachers to high-profile lesson study events in Japan, and the teachers' collective learning through the expert comments is frequently seen as the highlight of the entire process (Wake et al. 2016). In the model of mathematics ITE lesson study under consideration here, where there are only a small number of pre-service teachers (perhaps just one), together with their school-based mentor or university tutor, it is not practicable to bring in an external KO. This raises the question of how best to address the issue of pedagogical knowledge and how it might be developed within ITE lesson study.

Although unnecessary in Japan, since teachers are already very familiar with lesson study as a way of working, outside Japan a KO will also need to bring some expertise about the mechanics of orchestrating a lesson study. To this end, the KO may additionally be involved in the lesson planning stage, helping in-service teachers to collaborate and to raise the level of their planning beyond what they may be accustomed to. This can be helpful in facilitating a more beneficial research lesson for everyone and increasing the likelihood of positive early experiences of lesson study that all participants are then more likely to want to repeat (Wake et al. 2016). This adaptation of the traditional role of the KO seems warranted in these cases by the need in countries less familiar with lesson study than Japan for teachers to be inducted into this model of professional development.

For practical reasons, many models of lesson study in England do not attempt to incorporate a KO at all, since it can be too difficult and expensive to acquire the services of a suitable individual. However, if groups of teachers are not merely to recirculate their prior views and perspectives and are to gain new pedagogical insights, it seems necessary to consider how this might take place. Within ITE settings, it could be that the university tutor could offer this alternative perspective, even if not with the same level of expertise as in Japan. Often, in-service teachers will have limited knowledge of lesson study, whereas for pre-service teachers this will probably be addressed by sessions at the university, which outline how the process should operate. Written materials can be passed to the school, and training sessions may be run to support mentors' and other participating in-service teachers' understanding of the purposes and practices of lesson study. It is clear that in mathematics ITE lesson study, the participating in-service teacher, such as a school-based mentor, and/or the HEI tutor, brings considerable mathematics pedagogical expertise. However, it is much more difficult to address the mathematics pedagogical knowledge needed for fruitful post-lesson discussion (Foster et al. 2014).

The pre-existing knowledge and habits of in-service teachers may not easily lend themselves to a lesson-study approach, resulting in them falling back on models of lesson observation that focus on attempting to evaluate the quality of the teaching and offering advice for future lessons. Additionally, if lesson study is undertaken by a pre-service teacher and their school-based mentor, evaluative approaches may well have become the normal mode of operation, and it will be difficult for both of them to suspend this for the research lesson and resist treating the research lesson like every other lesson (Cajkler and Wood 2016b). We recognise that this entails a major shift in behaviour on the part of both mentor and pre-service teacher. This can be particularly difficult if in the mentor's other school roles (e.g. as mathematics department subject leader or member of senior leadership team), highly evaluative lesson observations are expected. Nevertheless, there is great potential in the pre-service teacher and mentor working together to modify their practice in this way (Cajkler and Wood 2015). For lesson study undertaken with university tutors working on ITE courses, there may be similar issues, as making evaluative judgements of the teaching of pre-service teachers is an assessment requirement.

We believe that there is no 'quick fix' for the role of mathematics pedagogical knowledge in ITE lesson study, but we see considerable benefits in a rigorous observation focus on the students, rather than the teacher, in which all participants are encouraged to make detailed notes concerning what the students actually did and said, perhaps supplemented by photographs of the students' writing. In this way, the post-lesson discussion can be dominated by accounts of what students did and said, rather than more subjective, high-inference *accounts for* (Mason 2001) what was happening, especially when this is construed in terms of the teacher's actions.

In their study, Bjuland and Mosvold (2015) found that some pre-service teachers in their second year of a 4-year undergraduate teacher education course experienced difficulty in the implementation of lesson study, which was attributed to their inexperience of mathematics pedagogy. However, experience alone appears not to be the critical factor; Amador and Weiland (2015) found that pre-service teachers' observations in lessons were more likely to include features of students' mathematical thinking than was the case for more experienced in-service teachers and university tutors. Therefore, if pre-service teachers can be successfully inducted into lesson-study practices during their ITE course, then more constructive observation habits may be engendered that could persist into their teaching career.

2.3 Lesson-Study Cycle

Some variant of the lesson-study cycle presented in Fig. 1 is typically used to characterise the chief features of the lesson-study approach (e.g. Cajkler and Wood 2016a; Dudley 2014a). In the normal course of their school placements, pre-service teachers in England are expected to write lesson plans and evaluations for each lesson that they teach, all of which is overseen by their school-based mentor. Thus, to pre-service teachers and their school-based mentors, the lesson-study cycle may sound superficially similar to their normal practice. While it has been estimated that 10% of schools in England are involved in lesson-study projects (Dudley 2014b), it is still not common practice for most teachers. So, there are limited points of reference to contrast lesson study with established practice and thereby discern critical features of the process. However, when implemented as intended, lesson study has important differences (Cajkler and Wood 2015), as we now consider.

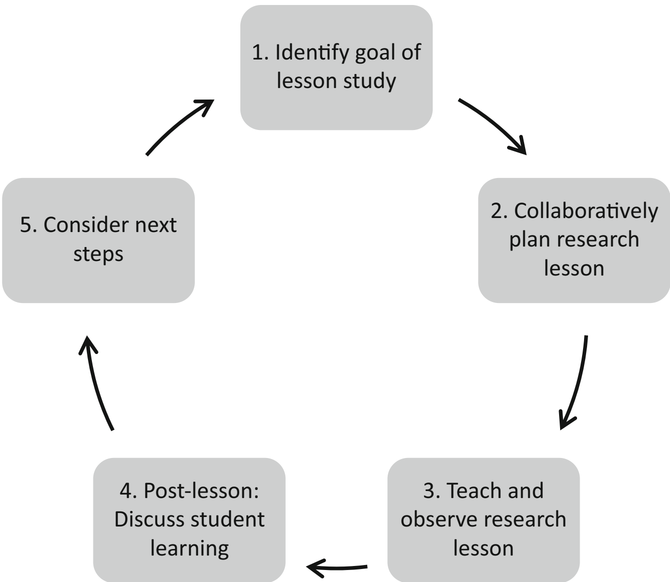

Fig. 1 A cycle of lesson study (Adapted from Cajkler and Wood 2016a, p. 2)

2.3.1 Planning the Research Lesson

The adoption of lesson study in England appears to have brought with it a considerable compression in the time and scope of the planning phase compared with normal practice in Japan. Traditional Japanese lesson study usually takes place over a cycle of up to 2 years, with an overarching research theme derived from wider institutional priorities and shared beyond individual teams (Fujii 2014). In contrast, within ITE programmes in England, timescales are far shorter – often just a few weeks – and lesson topics are usually selected pragmatically, based on those that the teachers were due to teach anyway. *Kyouzai kenkyuu*, the interrogation of relevant research and in-depth study of teaching materials, has been highlighted as one of the features of lesson study that has been 'lost in translation' when lesson study has been adopted outside of Japan (Takahashi and McDougal 2016, p. 514). There is some evidence that theoretical frameworks that underpin lesson plans are less well articulated in English settings, if present at all (Cajkler and Wood 2016a; Larssen et al. 2018), leading to collaborative planning being more reliant on existing professional knowledge than drawing on wider research. In addition, typical English classroom resources (like those used in the USA) are often of low pedagogical and mathematical quality, and their design may impede the planning process in the context of lesson study (Takahashi and McDougal 2016). However, with the growing awareness of how lesson study can be adapted in different cultural contexts (e.g. Fujii 2014), ITE courses should be better situated to draw together the strands of research, resources and mathematics pedagogy into the planning phase of effective lesson-study projects.

In England, the tradition is for school departments to write curriculum plans ('schemes of work' or 'schemes of learning') that outline the topics that are to be taught, and when, but teachers individually plan the lessons for their own classes, and extensive periods of collaborative planning are relatively uncommon. In 2016, as one mechanism for reducing teachers' workload, the UK government published guidance to schools outlining the expectation that collaborative planning in schools should become standard practice (DfE 2016). Consequently, as with other aspects of lesson study, engagement in collaborative planning may be both a challenge and an opportunity for pre-service teachers. They may enter school environments where collaborative planning is not established practice, and in-service teachers may struggle to commit sufficient time to the process. However, with appropriate support, pre-service teachers could have both the time and resources to develop collaborative practices, which would both enhance their professional development and provide an opportunity to bring 'expertise' to lesson-study projects in schools. Lesson study scaffolds a collaborative, interactive approach which may enhance a school's approach to teaching mathematics.

2.3.2 Teaching the Research Lesson

Where mathematics pre-service teachers in England are involved in some kind of lesson study, it is common for the pre-service teacher to both teach and observe research lessons. The composition of lesson-study teams within ITE varies, with almost all combinations of pre-service teachers, university tutors and in-service teachers (who may or may not be the school-based mentor) occurring and with varying levels of involvement. For example, a pre-service teacher and school-based mentor may work as a pair, or several pre-service teachers may work together, with the in-service teachers taking on different roles, from observer or teacher of the lesson to a fully integrated member of the lesson-study team. Within these diverse approaches, how participants consider ownership of the research lesson will vary. For example, Archer (2016) found that pre-service teachers who planned the research lessons felt entitled to hold the in-service teachers to account when the in-service teacher taught the research lesson but deviated from the lesson plan. Developing a shared responsibility for designing and understanding the lesson plans can be enhanced if the decision of who teaches the research lesson is made *after* the planning is completed and is perhaps decided at random (Forsythe and Baldry 2017). The notion of repeated cycles of lesson study has been advocated by some in the UK (e.g. Dudley 2014a); if featured in ITE lesson study, then the timescale for this tends to be condensed to as little as a week, due to the structural constraints of courses. The notion of 'reteaching' the 'same' lesson to a different class (e.g. Cajkler and Wood 2016a), which does not appear to be central to Japanese lesson study, can be problematic if it leads participants to infer that the main goal of lesson study is to 'perfect' a lesson plan, rather than to develop a deeper understanding of teaching and learning (Takahashi and McDougal 2016).

2.3.3 Observation of the Research Lesson

The observation of several case-study students, chosen in advance, rather than the whole class or the teacher, is an integral part of lesson study and yet something that pre-service teachers, and often their school-based mentors, find difficult. Transmission teaching approaches (Askew et al. 1997), which are common in mathematics lessons in England, may make deep learning hard to discern. Problem-solving combined with in-depth classroom discussion, a central feature of mathematics teaching in Japan, is not a routine part of many mathematics lessons in schools in England. More commonly here, the teacher models a single method, followed by individual students' practice of very similar questions. In observations of such lessons, observers may be looking for clear explanations from the teacher and evidence of students completing large numbers of questions successfully. The usual focus for lesson observations in England is a judgement of the teacher's performance, and it can be very difficult to move away from this (Archer 2016). Consequently, clear prior direction is required to ensure that sufficiently detailed appropriate data is collected from the research lesson to enable productive conversations in the post-lesson discussion.

Both pre-service and in-service teachers require considerable coaching in how to make such observations about students' behaviours, rather than generalised evaluative judgements, such as 'they were really engaged', 'they got on really well with their work' or 'they had a really good discussion'. Highlighting beforehand that such comments are likely to be less useful can be helpful in enabling participants to raise the level of their observations, but this still takes time to achieve. A widely used pro forma for lesson-study observations is the annotation of a lesson plan that includes anticipated student responses (Dudley 2014a), though, in this case, the quality of observations may be limited by the quality of the planning. We have found that making detailed notes about observed behaviours is a skill that pre-service teachers can develop quickly and early in ITE courses. This can allow them to capture sufficient data for productive discussions about the interpretation of those behaviours in terms of learning at a later stage. One additional source of information advocated by Dudley (2014a) is to interview the case-study students. While taking additional time, and making further demands on the students and teacher, this can provide further insights into students' learning and emphasise the central nature of the students to the lesson-study process.

2.3.4 Post-lesson Discussion

The post-lesson discussion is the culmination of Japanese lesson study, a central role being played by a knowledgeable other (KO), who offers an expert commentary on what has taken place during the research lesson and summarises what may be learned from that (Takahashi and McDougal 2016). In England, this is another area where teachers' and pre-service teachers' day-to-day experiences may detract from the purposes of lesson study. In normal practice in England, discussions about an observed lesson tend to be highly evaluative judgements about the teacher's performance rather than focused on student learning (Noyes 2013). Moreover, the observer is unlikely to have the experience of both pedagogy and lesson study that would be normal in Japan, meaning that there is a danger that the post-lesson discussion can fail to interrogate the available data about student learning to the full benefit of the participants.

When the post-lesson discussion includes experienced in-service teachers and university tutors, matters may be improved, but this is not necessarily the case, as there are some indications that these participants may also be likely to shift the focus away from student learning (Amador and Weiland 2015). When there are repeated cycles of lesson study, the post-lesson discussion can sometimes focus unhelpfully on proposed changes rather than on developing an understanding of teaching and learning from the observed lesson. However, case studies have also demonstrated that pre-service teachers can observe in sufficient detail to start discussing what the various behaviours observed may indicate about student reasoning (Forsythe and Baldry 2017). Forsythe and Baldry (2017) found that while discussions often drifted towards classroom organisation and teaching, tight discussion protocols sometimes allowed productive discussions to be extended. With teacher performance so central in English schools and ITE courses, mathematics lesson study is both difficult to enact successfully here but also has great potential to bring new perspectives to both in-service and pre-service teachers.

3 Key Features of Mathematics Lesson Study for ITE in England

Now that lesson study has been adopted and adapted in several countries, recent research has looked at how the approach can be most effectively deployed in different cultural contexts (e.g. Takahashi and McDougal 2016). Drawing on the research literature discussed in Sect. 2 and our own experiences of lesson study in ITE, here we highlight how key features of mathematics lesson study may be enacted successfully in English ITE contexts.

3.1 Partnership

Productive relationships need to be developed between all parties if lesson study is to be effective (Baldry and Foster in press). University tutors are often best placed to introduce lesson study into ITE courses, with the understanding that the schools' contexts are, however, critical. For example, we have seen that highlighting the fact that in Japan all teachers engage and expect to learn from the process has facilitated greater participation by school-based mentors and in-service teachers in England, some of whom had expressed unease about an unfamiliar process.

3.2 Knowledgeable Others

In England there are two facets to the role of a KO: one related to the lesson study process itself and one related to mathematics pedagogy. As lesson study is not an established practice in England, the university tutor usually takes responsibility to act as the 'expert' in lesson study, even though they themselves may not have participated in lesson study in schools. Within mathematics ITE lesson study, if university tutors do not work directly with schools, their design of introductory materials and support processes still allows them to fulfil this function of a KO. Mathematics pedagogy is more complex, as university tutors are likely to bring mathematics education research expertise, while in-service teachers tend to contribute situated and professional knowledge, and these may be seen as being in tension, with neither party typically having the expertise that the designation of KO would bring in Japan.

However, the lesson-study KO role provides a powerful mechanism for considering mathematics pedagogy in more detail. For example, in our local lesson-study projects, the use of wider research in the planning stage is promoted through university tutors working directly with pre-service teachers and by providing access to research findings for school-based mentors. As discussed below, at each stage of the lesson-study cycle, structures and processes can be instigated that allow expertise from outside the immediate lesson-study group to be drawn on, and indeed this may be from more than one person (Fig. 2).

We have also seen the professional capital of school-based mentors enhanced through participation in lesson study, which gradually leads to the growth of KOs in schools. For example, one group of local schools participated in lesson study with pre-service teachers for 4 years; over that time some school-based mentors extended the use of lesson study to their own department's practice. Use of lesson study in ITE thus contributes to a stronger focus on pedagogy that can lead to wider impact on in-service teachers.

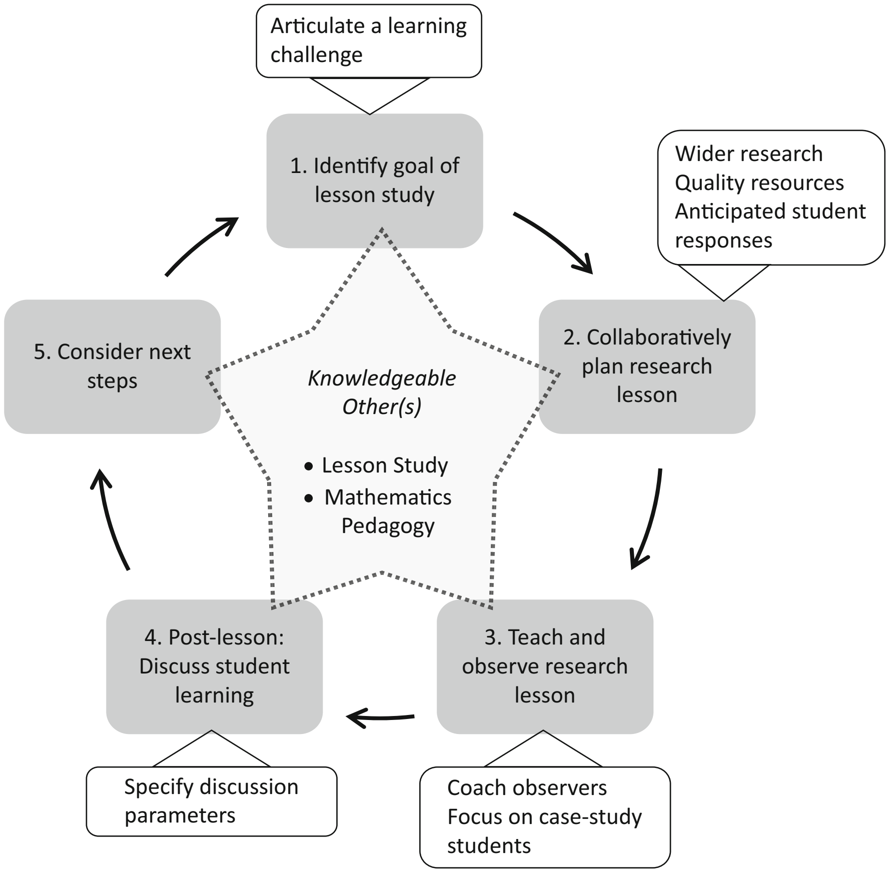

Fig. 2 Key features of a lesson study for mathematics ITE

3.3 Structure

If there is more than one pre-service teacher involved in the lesson study, then engagement with planning may be more productive if the choice of who will teach the research lesson takes place *after* the planning is completed and perhaps randomly. We have found that this encourages all participants to be fully invested in the planning process, since each person needs to be comfortable with what is to be taught and how.

The notion that the research lesson needs to be repeated has some traction in England, and so it is worth clearly articulating that this is not a requirement. If more than one research lesson is going to be undertaken, then subsequent lessons do not necessarily need to use the same plan; if the next lesson is based on the same topic being taught to a different class, then care is needed to ensure that the notion of 'perfecting' a lesson plan is not taken to be the central purpose, as this can detract from a focus on students' learning of mathematics.

Following a research lesson, a document may be produced outlining the key learning points from the lesson for the participants, and this could be more widely distributed among other pre-service teachers on the ITE course and also among the rest of the mathematics department in the school.

3.4 Research Question

The goal of mathematics lesson study is to develop greater understanding of teaching and learning mathematics. Consequently, when planning the research lesson, specifying a clear research question can help to focus the planning and the observation on one small aspect of students' learning of mathematics. In this way, participants may be able to avoid vague, general remarks that do not advance the interrogation of student learning. We have found that it can take a number of iterations before pre-service and in-service teachers are able to articulate a specific research question that moves beyond matters of 'engagement' or 'discussion', and providing examples of productive research questions has been very helpful. Additionally, mathematics lessons with dual goals that include ways of working as well as content can be richer and particularly useful for research lessons; for instance, we have offered 'for students to make and test conjectures' as well as 'for students to develop an understanding of the gradient and intercept parameters in straight lines' as examples. A topic that is considered 'hard to teach' may benefit most from the intense scrutiny that is possible in lesson study; we have used the term 'learning challenge' as a prompt for this and as a reminder to focus on learning and not engagement or behaviour (Fig. 2, Sect. 1).

3.5 Collaborative Planning

Pre-service teachers generally need considerable support in completing detailed lesson plans that go beyond the procedural aspects of what the students are going to do. And, indeed, all participants may be unfamiliar with collaborative planning, including appreciating the amount of time needed. Therefore, providing access to wider research and well-considered resources should be facilitated – a role usually performed by the university tutor (Fig. 2, Sect. 2). We have found it highly beneficial to specify that all those involved in planning should complete all the mathematical tasks for themselves, considering possible alternative solution strategies and representations and seeking to anticipate how students may respond to the task. This has not only helped participants to build an understanding of how learning may be 'visible' as students engage in the tasks, it also provided a learning focus for discussion in collaborative planning meetings. Carrying this out 'as a student might' can be of considerable benefit, allowing teachers to consider likely errors and misconceptions and how the teacher will address these (Fig. 2, Sect. 2). This articulation of anticipated student responses – and planning for how to respond to them – is a critical part of assessing and developing understanding of student reasoning, as well as providing a focus for the lesson observation (Wake et al. 2016).

3.6 Research Lessons

We have often found it necessary to reinforce the message that the observation focus is to be students and their engagement in mathematics tasks, rather than the teacher's 'performance'. In order to be in a position to collect sufficiently detailed information, a maximum of three case-study students may be selected per observer, and, as detailed observation notes are required, all observers should be inducted into what and how to observe (Fig. 2, Sect. 3). As with the research questions, it can take time for participants to move away from general statements about student engagement and evaluative judgements. Modelling the process through the use of either a live lesson or a video recording of students engaged in mathematical activities can be effective. Dudley (2014c) advocates the use of lesson plans with space next to anticipated student responses as an observation pro forma; this can be an effective way of drawing the attention of the observer to the specific actions of the students. If case-study students are interviewed, stimulated recall (where students are asked about how and why they completed particular tasks) may produce more useful information about learning than more general interviews.

3.7 Post-lesson Discussions

In our experience, clear protocols are needed in order to steer participants towards sharing detailed descriptions of particular incidents, followed by the interpretation of these descriptions in terms of those individual students' mathematical reasoning (Fig. 2, Sect. 4). The inclusion of a 'stop' protocol if classroom organisation, evaluation of teaching or deficit models of student reasoning appear in the discussion can be particularly useful. As suggested above, it may be helpful to list in advance examples of the kinds of statements which are likely to be less helpful, such as 'They really got on well with the task' and why these kinds of comments are unlikely to contribute to a productive post-lesson discussion. When student errors were discussed, we have found it powerful to have reminders that any comments about students 'not understanding...' could be reversed to articulate how the students might be reasoning; the idea that students are attempting to make sense of mathematics appears to shift some discussions away from student deficit models and towards learning. Time can be a limiting factor for some participants; we have seen that giving permission for part of the lesson to be discussed in detail, rather than attempting to cover the whole lesson, has made better use of limited time.

4 Conclusion

The challenge of modifying Japanese lesson study so that it both adapts to a different cultural context and at the same time challenges aspects of that context is considerable. Introducing lesson study in a new context or country means challenging some cultural traditions, which takes time and can be difficult (Stigler and Hiebert 1999; Wake et al. 2016). Features of the educational landscape in England present considerable challenges to the kind of approaches adopted in Japan. The constraints are tighter still for mathematics and ITE, and yet the potential benefits of even a highly constrained form of lesson study are considerable, both for pre-service teachers and for the schools in which they are placed for teaching practice.

Lesson study has been successfully operationalised in our context by making numerous adaptations and compromises. Important among these have been the acceptance of a shorter time scale and the absence of a well-defined knowledgeable other. Typical lesson observation practices in schools in England need considerable modification for productive lesson study, as do approaches to lesson planning. However, these challenges are highly productive in terms of developing pre-service teachers' practices and have enormous potential to support their pedagogical development, as we have described. Inducting future teachers into lesson-study practices, and facilitating first experiences of lesson study which are positive and clearly beneficial to all, may prepare pre-service teachers for future careers in which they value such ways of working and perhaps bring them into their future schools.

Where productive partnerships between university and school can be engineered, where time can be found for collaborative lesson planning that focuses on a clear research question that also forms the basis for detailed lesson observation relating to student learning of mathematics and where a tight protocol for post-lesson discussions is in place, the chances for productive mathematics lesson study in ITE are maximised.

Acknowledgement We would like to thank Professor Wasyl Cajkler for very helpful comments on an early draft of this chapter and Dr. Sue Forsythe for her insights from her work with mathematics ITE pre-service teachers. We would also like to thank the anonymous reviewers for their comments on this chapter.

References

Amador, J., & Weiland, I. (2015). What preservice teachers and knowledgeable others professionally notice during lesson study. *The Teacher Educator, 50*(2), 109–126.

Angelini, M. L., & Álvarez, N. (2018). Spreading lesson study in pre-service teacher instruction. *International Journal for Lesson and Learning Studies, 7*(1), 23–36.

Archer, R. (2016). *Lesson study in initial teacher education: Students' positioning analysed through the lens of figured worlds.* Paper presented at the British Society for Research into Learning Mathematics.

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy: Report of a study carried out for the teacher training agency*. London: King's College, University of London.

Baldry, F., & Foster, C. (in press). Lesson study partnerships in initial teacher education. In P. Wood & D. Larssen (Eds.), *Lesson study in initial teacher education. A critical perspective*. London: Emerald.

Bjuland, R., & Mosvold, R. (2015). Lesson study in teacher education: Learning from a challenging case. *Teaching and Teacher Education, 52*, 83–90.

Burroughs, E. A., & Luebeck, J. L. (2010). Pre-service teachers in mathematics lesson study. *The Mathematics Enthusiast, 7*(2), 391–400.

Cajkler, W., & Wood, P. (2015). Lesson study in initial teacher education. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (1st ed.). London: Taylor & Francis Group.

Cajkler, W., & Wood, P. (2016a). Adapting 'lesson study' to investigate classroom pedagogy in initial teacher education: What student-teachers think. *Cambridge Journal of Education, 46*(1), 1–18.

Cajkler, W., & Wood, P. (2016b). Mentors and student-teachers "lesson studying" in initial teacher education. *International Journal for Lesson and Learning Studies, 5*(2), 84–98. https://doi.org/10.1108/IJLLS-04-2015-0015.

Corcoran, D., & Pepperell, S. (2011). Learning to teach mathematics using lesson study. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (Vol. 50, pp. 213–230). Dordrecht: Springer.

DfE. (2016). *Eliminating unnecessary workload around planning and teaching resources*. London: Crown Retrieved from https://www.gov.uk/government/publications/reducing-teacher-workload-planning-and-resources-group-report

Dudley, P. (2014a). How Lesson Study works and why it creates excellent learning and teaching. In P. Dudley (Ed.), *Lesson Study* (pp. 1–28). London: Routledge.

Dudley, P. (2014b). *Lesson Study: Professional Learning for our time' a resume*. Retrieved from http://lessonstudy.co.uk/2014/09/lesson-study-professional-learning-for-our-time

Dudley, P. (2014c). Lesson study handbook (revised).
Elipane, L. (2011). Incorporating lesson study in pre-service mathematics teachers education. *Proceedings of the 35th Conference of the International Group of the Psychology of Mathematics Education, 1*, 305–312.
Fernández, M. L. (2005). *Exploring "lesson study" in teacher preparation.* Paper presented at the Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education.
Fernandez, C., & Yoshida, M. (2012). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. New York: Routledge.
Fernandez, M. L., & Zilliox, J. (2011). Investigating approaches to lesson study in prospective mathematics teacher education. In *Lesson study research and practice in mathematics education* (pp. 85–102). Dordrecht: Springer.
Forsythe, S., & Baldry, F. (2017). *Developing effective Learning Circles: An account of how student teachers participated in a shared dialogue of a team teaching experience.* Paper presented at the European Conference on Educational Research 2017, Copenhagen.
Foster, C., Wake, G., & Swan, M. (2014). Mathematical knowledge for teaching problem solving: Lessons from lesson study. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PMENA 36* (Vol. 3, pp. 97–104). Vancouver: PME.
Fujii, T. (2014). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education & Development, 16*(1), 4–21.
Gunnarsdóttir, G. H., & Pálsdóttir, G. (2011). *Lesson study in teacher education: A tool to establish a learning community.* Paper presented at the Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education.
Larssen, D. L. S., Cajkler, W., Mosvold, R., Bjuland, R., Helgevold, N., Fauskanger, J., et al. (2018). A literature review of lesson study in initial teacher education: Perspectives about learning and observation. *International Journal for Lesson and Learning Studies, 7*(1), 8–22.
Mason, J. (2001). *Researching your own practice: The discipline of noticing*. London: Routledge.
Noyes, A. (2013). The effective mathematics department: Adding value and increasing participation? *School Effectiveness and School Improvement, 24*(1), 1–17.
Parks, A. N. (2008). Messy learning: Preservice teachers' lesson-study conversations about mathematics and students. *Teaching and Teacher Education, 24*(5), 1200–1216.
Rasmussen, K. (2016). Lesson study in prospective mathematics teacher education: Didactic and paradidactic technology in the post-lesson reflection. *Journal of Mathematics Teacher Education, 19*(4), 301–324.
Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.
Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*(4), 513–526.
Tsui, A. B., & Law, D. Y. (2007). Learning as boundary-crossing in school–university partnership. *Teaching and Teacher Education, 23*(8), 1289–1301.
Wake, G., Swan, M., & Foster, C. (2016). Professional learning through the collaborative design of problem-solving lessons. *Journal of Mathematics Teacher Education, 19*(2–3), 243–260.

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Part V
Studies on Key Aspects of Lesson Study

Preface: Studies on Key Aspects of Lesson Study

Keith Wood

Contents

All of the papers in this section focus on aspects that are critical to the success of lesson study. In the first chapter, Rongjin Huang, Zikun Gong, and Xue Han explore the relationship between learning theory and lesson study (LS). They describe a path of learning (Marton 2015) for students identified by a teacher group supported by facilitators as they design and refine a research lesson on division of fractions guided by variation pedagogy. This represents a turn in Chinese lesson study shifting the focus from teaching practice to students' learning. Figure 1 shows the sequence of patterns of variation that characterizes the path of learning developed by the teachers engaged in the LS.

The pattern of variation prepared the students to see by induction that $1 \div a/b = b/a$; $2 \div a/b = 2 \times b/a$; $n \div a/b = n \times b/a$; and $c/d \div a/b = c/d \times b/a$.

The authors discuss issues arising from theory-guided LS including the following: What is the role of facilitators or knowledgeable others in LS? What role could theory-guided LS play in teachers' professional development? They observe that:

> . . . as this study took place in China where LS is embedded in teachers' daily practice, it would be interesting to explore the implementation of similar studies in settings in the West where LS is an innovative initiative. (Huang et al. this volume)

An international dialogue might ensue from this observation since teachers in the West are engaging with this form of theory-guided LS and making observations such as:

> If variation is a prerequisite for students' learning, we can assume that this is the case also for teachers' learning and professional development . . . through the pre- and post-tests, and by analyzing the differences in students' learning as well as by watching our colleagues'

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_29

numerator	denominator	quotient
invariant (1)	**v**ariant (2/5, 2/7, 3/4, …)	**v**ariant (5/2, 7/2, 4/3, …)
v (1,2, 3 … 3/4)	**i** (2/5)	**v** (5/2, 2x5/2, 3x5/2, … 3/4x5/2)
v (1,2, 3 … 3/4)	**v** (2/5, 2/7, 3/4, …)	**v** (5/2 … 2x7/2 … 3x4/3 … 3/4x4/3 …)

Fig. 1 The path of students' learning division of fractions

Object of learning (what the teachers see as critical aspects of the object of learning for their students)	**Teaching design**	
v	**i**	**contrast** – when, through participation in LS, teachers observe differences in the critical aspects of the object of learning (OL) for their students.
i	**v**	**generalization** – when teachers (and facilitators working with teachers) vary the teaching (change the design) so that teachers come to see the effect of their new design on their students' experience of the OL. If the design is ineffective, the teachers may, in the light of the evidence, reconfigure the OL for their students and redesign and teach the lesson again.
v	**v**	**fusion** – when teachers see that, to be effective, teaching must be focused on evidence of the OL for their students (i.e. they can, through this LS process, develop their 'pedagogical content knowledge') This is achieved through cycles of LS as the teachers experience contiguously variation in the OL in relation to variation in the design of teaching.

Fig. 2 The path of teachers' learning through LS (Wood 2018)

> different ways of handling the same object of learning, we became able to discern the critical aspects [what the students need to learn to achieve the intended outcomes], and realize what created the differences. (Vikström et al. 2013, p. 39)

Figure 2 shows the sequence of patterns of variation that can be used to characterize the *teachers'* path of learning in a lesson study. The two aspects – the object of learning (from the perspective of the teachers) and the teaching design – must be separated. Separation is of two kinds: contrast and generalization. Contrast reveals a necessary aspect – the object of learning – and generalization shows how teaching and the object of learning may differ from each other. Both aspects are brought back together again by fusion.

Jennifer Lewis explores the role of facilitators in this section. She recorded the progress of two "novice teacher developers" destined to become facilitators of LS groups in the United States. They found that facilitating LS is challenging:

> Lesson study includes the close [study] of student activity during classroom lessons, methodical study of multiple curriculum materials, the joint construction of highly detailed lesson plans, and evidence-based reflections on research lessons that are observed in person. All these aspects of lesson study diverge from current forms of professional development in the US that feature direct instruction dissemination of information from expert presenters.

The LS is successful when the teachers in the group take ownership of the study. However, helping teachers to explore what their students need to learn – the object of learning – can be challenging if this has not been their way of planning teaching before the study. Lewis compares the change required with subscribing to an emergent curriculum.

> . . . the core idea is that the curriculum follows children's concerns. Lesson study is the analog in teacher learning. To be able to conduct this kind of professional development entails a kind of agility that is rare.

One of the facilitators in Lewis's study identified a moment where the LS group made a breakthrough:

> In their lesson on systems of equations, high school teachers saw in the research lesson, and talked about in the post-lesson analysis session, the fact that students struggled to give words to describe the visual representations they had generated; that students had few tools for justifying mathematical assertions; that visualization might have helped students in advance of graphing and describing their thinking.

In this extract, the focus of discussion has shifted to the object of learning – contrast – on the path of teachers' learning. The facilitators had to interact with teachers in ways that led to growth.

Feishi Gu and Lingyuan Gu explored a different type of facilitation. They examined how mathematics teaching research specialists (TRS) mentored practicing teachers during post-lesson debriefs of a lesson study in China. How different is this to the type of facilitation and the focus on the object of learning that is described in the previous chapters? Gu and Gu explain:

> Chinese TRS in mathematics pay greater attention to developing teachers' practical knowledge, rather than mathematical content or pedagogical knowledge . . . Chinese TRS spent one-third of their mentoring time on instructional task design and around half of the time on improving instructional behaviors, <u>but they rarely spent time on setting learning goals and formative assessment</u> <u>of student learning</u> . . . the weaknesses of Chinese mathematics TRS' mentoring are obvious. <u>Their discussions about task design and improvement of instructional behaviors were mainly based on their past experiences rather than evidence of student learning in class</u> . . . This type of mentoring may lead teachers to focus on their teaching strategies and skills rather than soliciting student thinking and learning . . . (my emphasis)

The value of this study lies in its contrast with the previous accounts of LS. It is through such a contrast that we can come to see what the critical aspects of LS are. Teacher ownership, the discovery of the object of learning from the students, and the design of teaching to achieve this object appear to be sidelined by the TRS. The LS appears to begin and end with a focus on ways to vary teaching. However, Gu and Gu have developed a framework for analyzing the TRS' mentoring strategies that reveal their strengths and weaknesses and suggest ways in which the TRS' activities might be improved.

In contrast to the approach of the Chinese TRS, Toshiakira Fujii takes on the task of making visible to readers how Japanese teachers develop the lesson plan through engagement in lesson study. In doing so, Fujii makes it clear to the reader that he is not focusing on "the impact of lesson planning on teachers' mathematical and pedagogical knowledge . . . [and] how the lesson planning process exposes teachers' beliefs" and, therefore, not focusing on the path of teachers' learning through Japanese LS. Instead the focus is on the flow of the lesson for the learners.

In Japanese LS the problem-solving task receives considerable attention. It is new to the students but they are expected to be able to solve it albeit in different ways. There is an assumption that, at the *neriage* phase of the problem-solving lesson, the students' attention will be drawn to the variation in their different methods of solving the problem. At this stage of the lesson, the experience of variation in their correct solutions should be enough for the students to see that the "ideas involved may not have equal value."

> The teacher at school S clearly stated, "Although each strategy is sure to get the correct answer, we should not end there . . . I want the students to know that getting the answer is not the final goal".
>
> In the case of school M, teachers wanted students to compare two word problems, for partitive and quotitive division, through the use of multiplication sentences to model situations. The lead teacher of the research steering committee posed the question, "What should we ask to elicit a multiplication maths sentence?"
>
> At school T, teachers talked about which point or theme for discussion would be best: the number of groups of quadrilaterals, where the teacher might say "this student made two groups and the other student made three groups, what made these difference? What were the thoughts behind these categorizations?" or how to characterize each group, for example "This student made two groups. Can you see the common characteristics of the quadrilaterals in each group?" One teacher asked, "Which is the higher level of thinking?" to which another teacher responded, "Probably the number of groups is higher. This point is proposed in the lesson plan". So they decided to ask students to discuss how many groups there were and reasons behind this in the neriage phase.

The object of learning appears to be the elevation of *students' mathematical thinking*. It is not enough that the students can solve the problems. They must see that the solutions are different in terms of the degree of mathematical thinking involved. Fujii explains that "From the point of view of mathematical value, the lesson should clarify the relative value of the different solutions, generally by contrasting these." In the final, *matome*, stage of the lesson, it falls to the teacher to identify which problem-solving strategy may be the "most sophisticated." This suggests that the experience of variation may not be sufficient for the students to experience this contrast without input from the teacher. The lessons are very carefully scripted, learners are considered to be fast or slow, and knowledgeable others facilitate the development of the script based on their prior experience. Variation and contrast seem to be central to the design of the lesson, but the theory of learning guiding the lesson planning remains implicit unlike the design based explicitly on variation pedagogy described by Rongjin et al. When it comes to facilitating teacher groups, the knowledgeable others, drawing on a century or more of accumulated knowledge of teaching mathematics through LS, must surely experience a sense of déjà vu. Fujii reveals much that is fascinating about how Japanese teachers develop the lesson plan in LS, but some of the mystique surrounding Japanese lesson study as a transferable form of teachers' professional development remains.

Fujii makes it clear that his study was not focused on the impact of LS on teachers' pedagogical content knowledge and on their conceptions of teaching. However, Rongjin Huang, Dovie Kimmins, and Jeremy Winters *are* focused on this in their analysis of the dynamics between enactment and reflection as a mechanism for improving teaching and promoting teacher learning through Chinese lesson study occurring in the United States. Their findings appear to provide support for the path of learning for teachers shown in Fig. 2. The authors report:

> This study in general demonstrated how Chinese lesson study . . . can improve teaching that promotes students' conceptual understanding and participating teachers' learning as well. . . . [It] showed that during the debriefing sessions, both knowledgeable others and participating teachers collaboratively worked together in critically analyzing the observed research lessons and constructively providing suggestions on improving them. . . . The study also showed that the ideas emerging from debriefing sessions need to be verified and refined through re-enactments. . . . This study revealed that both the knowledgeable others' facilitation and the interactions of enactment and debrief are crucial for maximizing effects of LS in improving teaching and promoting teachers' learning.

These findings resonate with findings from studies in Western Europe in subjects other than mathematics such as the one quoted above (Vikström et al. 2013). Teachers' analysis of successive cycles of research lessons with contributions from a facilitator – a knowledgeable other – allows them to refine their understanding of what their students need to learn and how their teaching must change to bring this learning about. In the study by Vikström et al. (2013), an example of contrast emerged when the teachers, with reference to the curriculum, wanted the concept of solubility in different contexts to be part of the object of learning for their science students but discovered from the pretest results that it was necessary to limit the object of learning to focus on fewer critical aspects to achieve "a dynamic and particulate view of matter" on the part of the students. Few of the students made reference to a particle model in the test. After the first lesson, the posttest revealed that the macroscopic view still dominated. The teachers realized that they had to vary their teaching to focus on the molecular level. By the end of the third cycle, they reported that "the most critical aspect was the notion of 'empty space,' which points out the particulate character of matter."

Huang and his colleagues chose to refer to this process as deliberate practice "which emphasizes the necessity of repeated teaching and immediate input of knowledgeable others, and highlights the unique impact of the dynamic between enactments and reflection on teachers' learning."

Motoko Akiba, Aki Murata, Cassie Howard, Bryan Wilkinson, and Judith Fabrega report on their study of the introduction of LS as a top-down initiative in Florida, USA. They explain that:

> districts that implemented lesson study district-wide first internalized lesson study through communicating and funding a district-wide expectation of job-embedded, inquiry-based professional development. Following this internalization, the district leaders institutionalized it by supporting school ownership and leadership in organizing and embedding lesson study into the school organizational structures and routines.

Initially the districts provided "training, templates, and coaching and promoted schools' decision making in organizing and supporting lesson study," As these were withdrawn, there was a danger that local adaptation would dilute core principles of LS that lead to the improvement of teaching and student learning.

Akiba and her colleagues collected some insights into the success of the initiative from interviews with three professional development directors. One explained that all schools in her district were expected to carry out one cycle of lesson study per semester.

> That's considered best practice. . . .the way the process works here is professional learning communities are supposed to look at areas of student achievement that need to be supported. So, usually when they decided on an area that needs support, then they do their research, but usually it comes down to classroom practices need to be changed. So that's where lesson study comes in, because they located a problem; they researched that problem; they looked through the resources available to them, and then they take it to the classroom level and work on perfecting materials, perfecting the lesson, that would support the area of academic concern.

Already this sounds like an adaptation of lesson study with one cycle unlikely to lead teachers to fusion through a path of learning (Fig. 2) – also unlikely to establish the dynamic between enactments and reflection to which Huang, Kimmins, and Winters refer. It cannot go unnoticed that the teachers' voices are absent from this study.

Jennifer Suh, Sara Birkhead, Terrie Galanti, Rachelle Farmer, and Padmanabhan Seshaiyer take us back to the important role played by the knowledgeable other in a Japanese-style LS with an account of a US study in which the teachers explored the question: What would kindergarten to sixth grade students do with a typical fourth grade equal-sharing task involving fractions, i.e., sharing six sandwiches with eight students? The teachers were engaged in the planning and preparation, classroom teaching, review, and second iteration stages of the lesson study. Importantly, they were introduced to the idea of a mathematics learning trajectory from kindergarten to sixth grade by a knowledgeable other to provide a connection with mathematics curriculum standards. The authors proposed that lesson study using a "learning trajectory-based professional development routine" would develop "horizon content knowledge" (i.e., an awareness of the mathematical landscape in which mathematics instruction is located). They report that it did.

> [The] Lesson Study served as an effective school-based coaching tool that deepened the articulation of standards as well as helped teachers to analyze student work using the learning trajectories. It is important to note the role of the coach and mathematics educators who brought the equipartitioning learning trajectory to the lesson study meeting. In our study, the coach and the mathematics educator facilitated teacher learning by providing teachers with the research-based learning trajectories. Bridging the connection between curricular learning standards for multiple grade levels and LTs allows teachers to better attend to students who may have gaps in their learning.

The discovery of gaps in their students' learning – what they need to learn to achieve the intended outcomes of the curriculum – can provide contrast to initiate the teachers' path of learning (Fig. 2). It follows that teachers would go on to redesign teaching through cycles of LS so that the gaps are filled.

From reading the chapters in this section, it is possible to see how lesson study, informed by theory and facilitated by knowledgeable others, could promote teacher learning. Bringing variation in the object of learning, as it appears to be experienced by their students, to the attention of teachers, prompts them to reflect on their teaching through the separation of teaching and learning in their awareness. Changing their teaching to achieve this varying object of learning through iterative cycles of LS can change the way the teachers are aware of teaching and learning. These become fused in a new, more powerful relationship.

References

Marton, F. (2015). *Necessary conditions of learning*. London/New York: Routledge.

Vikström, A., Billström, A., Fazeli, P., Holm, M., Jonsson, K., Karlsson, G., & Rydström, P. (2013). Teachers' solutions: A learning study about solution chemistry in Grade 8. *International Journal for Lesson and Learning Studies, 2*(1), 26–40.

Wood, K. (2018). *What and how do teachers learn by taking part in lesson study?* Presentation to European Association for Research on Learning and Instruction (EARLI) Special Interest Group 9: Phenomenography and Variation Theory, Birmingham University, UK, 16–18 September.

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Implementing Mathematics Teaching that Promotes Students' Understanding Through Theory-Driven Lesson Study

Rongjin Huang, Zikun Gong, and Xue Han

Contents

Abstract Lesson study (LS) has been practiced in China as an effective way to advance teachers' professional development for decades. This study explores how LS improves teaching that promotes students' understanding. A LS group including didacticians (practice-based teaching research specialist and university-based mathematics educators) and mathematics teachers in China explored and documented how teacher participants shifted their attention to students' learning by incorporating two notions of teaching: learning trajectory (LT) and variation pedagogy (VP). The former describes conjectured routes of children's thinking and learning with pertinent tasks to move toward the learning goals along the route, while the latter

This chapter is built upon an article that formally appeared in ZDM: Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study, ZDM, 48(3), 425–439.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_30

suggests strategies for using systematic tasks progressively. The concepts of LT and VP are used to guide planning, teaching, and debriefing throughout the LS process. Data consist of lesson plans, videotaped lessons, post-lesson discussions, post-lesson quizzes, and teachers' reflection reports. This study reveals that by building on the learning trajectory and by strategically using variation tasks, the lesson has been improved in terms of students' understanding, proficiency, and mathematical reasoning. In addition, the LT was refined through the LS. This study displays how theory-driven LS could help teachers improve their teaching and develop the linkage between theory and practice.

Keywords Lesson study · Learning trajectory · Variation pedagogy · Theory-driven lesson study

1 Introduction

Lesson study (LS) is a form of practice-based professional development originated from Asia and has been widely adopted around the world (Hart et al. 2011; Lewis et al. 2006). One salient feature of LS is to improve teaching and develop teachers' mathematics expertise by focusing on students' learning (Murata 2011). Chinese LS has been practiced for decades (Chen and Yang 2013) and has contributed a great deal to the improvement of teaching and teachers' competence in China (Huang and Bao 2006; Huang and Han 2015). It is found that Chinese teachers have attempted to focus more on polishing teachers' instructional practices in class rather than directly on eliciting student learning during the process of LS (Chen and Yang 2013; Huang and Bao 2006). However, since 2011 the new curriculum standards have expected mathematics teachers to focus on student learning. This study is designed to explore how mathematics teachers develop lessons that promote students' understanding through a theory-driven LS approach. Specifically, the study aims to address the following research questions:

1. How does a LS group improve classroom instruction that promote students' understanding?
2. How do the teachers make these improvements toward students' understanding?
3. How does the LS process inform the refinement of the guided theories of the LS?

2 Literature and Theoretical Framework

This section discusses perspectives of effective mathematics instruction, theories underlying the LS, a model of theory-driven LS, and studies on LS research lessons that focus on division of fractions.

2.1 Effective Mathematics Classroom Instruction

Although there are cross-cultural variations in terms of what constitutes effective mathematics instruction (Cai and Wang 2010), the NCTM (2014) synthesizes eight evidence-based effective mathematics teaching practices, including establishing mathematical goals to focus on student learning, implementing tasks that promote reasoning and problem-solving, using multiple mathematical representations, and using evidence of student thinking. The Chinese curriculum standards (MOE 2011) emphasize that teaching is a process in which teachers and students actively engage in the lesson, interact with each other, and co-develop understanding of mathematics concepts. Mathematics teaching should establish students' self-exploration in problem-solving; guide students in obtaining basic knowledge, skills, thinking methods, and experiences in doing mathematics through practicing, thinking, exploring, and communicating; and continually develop students' abilities in forming, posing, analyzing, and solving problems. Given the shared ideas of effective mathematics instruction in the USA and China, this study emphasizes the following components: set clear mathematics goals, implement mathematical tasks that promote students' learning, develop conceptual understanding, and gather evidence of students' thinking.

2.2 Two Underpinning Theories in the Lesson Study

Based on literature review and mathematics teaching practice in China, the expert team (e.g., university professors and specialists) of the LS adopted two specific notions of learning trajectory (Clements and Sarama 2004) and variation pedagogy (Gu et al. 2004; Marton and Pang 2006) to guide their activities throughout the LS.

Learning trajectories (LT) have been developed and proposed as the foundation for classroom instruction (Simon 1995; Sztajn et al. 2012). In his seminal work, Simon (1995) suggests the hypothetical LT as the pathway on which students might proceed as they advance their learning toward the intended goals. Further research has refined the LT concept to the "descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks" (Clements and Sarama 2004, p. 83). Research shows that the use of LTs can support teachers' knowledge growth and instructional decision-making, allow teachers to focus on students' thinking, and eventually improve students' achievement (Clements et al. 2011).

Variation pedagogy (VP) arises from the Chinese mathematical teaching tradition and focuses on using deliberate and systematic variation in mathematics tasks to help students develop new concepts and problem-solving abilities (Gu et al. 2004). Building on the core notion of variation, researchers have developed a theory of variation (Marton and Pang 2006; Marton and Tsui 2004). According to the theory, learning is to develop new ways of seeing something; specific comparisons allow one to discern critical features of the object being studied and thus learn more about it. For example, contrasting a triangle with "not triangles" (e.g., squares, pentagons, hexagons) allows critical aspects of triangles such as the number of sides to be discerned. Students can then learn other critical features such as the size of angles and the length of sides by looking at different kinds of triangles (e.g., right-angled triangle and isosceles triangle). Furthermore, by examining what remains unchanged as appearances vary (e.g., position and size of triangles), students could further discern the invariant feature of triangles such as the sum of all angles is 180° (Lo and Marton 2012). Thus, in mathematics classroom instruction, it is crucial to create certain patterns of variation: examining what varies against what is invariant (Marton and Pang 2006; Marton and Tsui 2004). Lo and Marton (2012) further argue "when learners need to discern more *than two or more critical features*, the most powerful strategy is to let the learners discern them one at a time, before they encounter simultaneous variation of the features" (p. 11).

To use measurable terms to describe students' learning, researchers have developed frameworks from the theory of variation (Marton and Pang 2006) including object of learning, enacted object of learning, and lived objects of learning. An *object of learning* is "a specific insight, skill, or capability that the students are expected to develop during a lesson or during a limited sequence of lessons" (Marton and Pang 2006, p. 194). The *enacted object of learning* is described as the patterns of variation and invariance co-constructed by the teacher and the students. The patterns of variation consist of the necessary conditions for the appropriation of the enacted object of learning. From the students' answers to the written and oral questions after the lesson, we can characterize the *lived object of learning*, i.e., the object of learning that is experienced by the learners.

Gu et al. (2004) highlight the role of using varying tasks for promoting students' conceptual understanding. This corresponds to Watson and Mason's (2006) argument that varying tasks could be used to develop students' conjectures. Furthermore, Marton and his colleagues (Marton and Pang 2006; Marton and Tsui 2004) have focused on examining *how* lesson study guided by the variation pedagogy impacts teaching and students' learning. In this study, we utilize the lenses of the object of learning and enacted and lived objects of learning to investigate how the theory-driven LS may improve classroom instruction and students' learning.

2.3 A Model of Theory-Driven Lesson Study

Although the concept of LTs suggests the importance of describing a conjectured route of children's thinking and learning with pertinent tasks to move toward the learning goals along the route, how these tasks should be designed and presented to students have not been explicitly addressed. The notion of VP emphasizes specific strategies in using systematic tasks progressively, but it has not paid explicit

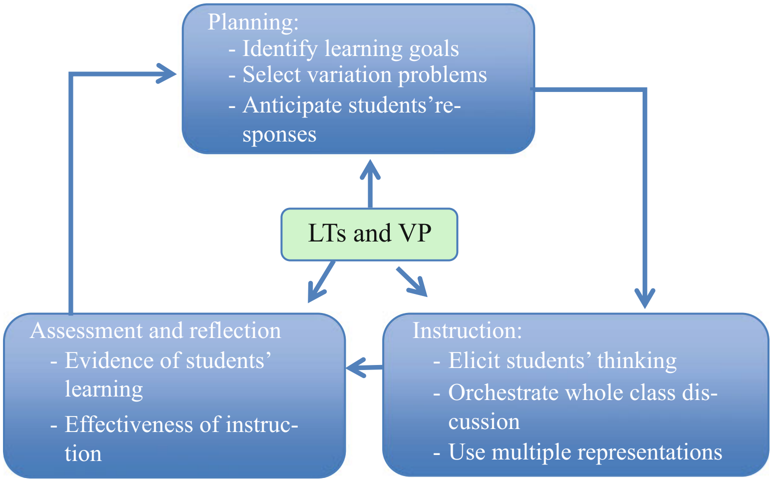

Fig. 1 The lesson study's cycle

attention to the route of children's learning. Thus, the incorporation of these two perspectives may provide a useful tool for designing and delivering lessons: VP could help teachers strategically design and implement tasks in line with LT. Centering LTs and VP in lesson design and implementation, we created a model LS cycle as shown in Fig. 1.

When planning lessons, teachers set clear mathematics learning goals, select appropriate tasks based on LTs and VP, and anticipate students' responses to the tasks. When implementing lessons, teachers encourage students to express their actions and their thoughts while solving tasks, orchestrate whole class discussion of students' work, and build connections among representations. In the post-lesson reflection, evidence of students' learning is gathered through classroom observation and assessments. Suggestions for improvement should then be made based on this evidence.

2.4 *Studies on Teaching and Learning of Division of Fractions*

Developing conceptual understanding of the algorithm for division of fractions is not an easy task for either students or teachers (Carpenter et al. 1988). Researchers have developed two general pedagogical approaches for teaching division of fractions (Li 2008). One provides mathematical justifications for the division of fractions algorithm based on the properties of fractions and the meaning of division

(Tirosh 2000). The other uses concrete or visual demonstrations to explain how division of fractions can be computed, such as extending whole number division to division of fractions through a quotitive interpretation (Silvia 1983) or a partitive interpretation (Ott et al. 1991).

Recently, a complementary approach of making sense of division of fractions using quotitive and partitive models and justifying why the algorithm works has been proposed (Sowder et al. 2010). In particular, it is suggested that division of fractions should be taught through solving word problems aligned with the sequence: (1) a unit fraction divided by a non-zero whole number using the partitive model and the relationship between division and multiplication, (2) a whole number divided by a unit fraction using the quotitive model and the relationship between division and multiplication, and (3) a fraction divided by fraction using the quotitive model and the relationship between division and multiplication (CCSSI 2010).

Based on existing research and the framework of equipartitioning (Maloney et al. 2014), the expert team of the LS developed an LT. The LT includes five dimensions: *sequence*, *situation*, *model*, *representations*, and *tasks*.

The *sequence* dimension includes eight levels: (1) a whole number divided by a whole number; (2) a fraction divided by a whole number, where the dividend is a multiplier of divisor; (3) a unit fraction divided by a whole number; (4) a fraction divided by a whole number; (5) a whole number divided by a unit fraction; (6) a whole number divided by a fraction; (7) a unit fraction divided by a unit fraction; and (8) a fraction divided by a fraction. *Situation* includes contextual and mathematical situations. *Model* refers to partitive and quotitive models. *Representations* include visual representations, such as explaining an algorithm using pictures; verbal representations, such as explaining algorithm using verbal language (i.e., how many $^1/_5$s can go to 1 unit?); and symbolic representations (i.e., $1 \div {}^1/_5 = 1 \times {}^5/_1 = 5$ or $1 \div {}^1/_a = 1 \times a$).

Aligned with each level, there are instructions about models and tasks. For example, at level 4, the partitive model is suggested. The following task is offered as an illustration: A $^4/_5$-kg cake is shared with 3 friends. How much does each friend get? In addition, three representations are illustrated.

3 Methods

In this section, we first describe the major components of the LS, including the LS group, instruments, and the procedure of implementing the LS. We then present the methods of data collection and data analysis.

Table 1 Background information of the team members

Code	Name	Title of profession	Highest degree	Experiences
R1	Mr. Kong	University professor	Ph.D. in mathematics education	Twenty-five years in both elementary and university levels
SP1	Mr. Shao	Senior teacher, specialist at district level	Bachelor in elementary education	Twelve years of teaching mathematics and 13 years in serving as a mathematics specialist in elementary school
SP2	Mr. Ren	Senior teacher, specialist at city level	Master in mathematics education	Eleven years of teaching mathematics and 16 years of serving as a mathematics specialist in elementary school
T1	Ms. Tang	Senior teacher	Bachelor in public affair management	Twenty-seven years of teaching mathematics
T2	Ms. Han	Senior teacher	Bachelor in education	Fourteen years of teaching mathematics
DT	Ms. Lu	First level of teacher	Bachelor in information and technology	Five years of teaching mathematics

3.1 The Setting: The School, Teachers, and the Lesson Study Group

The LS took place in an elementary school in southeastern China. The school serves around 1500 students in grades 1–6 (45 classes) with 24 mathematics teachers. The LS group consists of three mathematics teachers with various levels of experience and three didacticians (two teaching research specialists and one university mathematics educator). Teaching research specialists are specialists who are employed by various practice-based education divisions and preliminarily work with practicing teachers (see Huang et al. (2014) or Gu and Gu (this book) for details). The backgrounds of these teachers and didacticians (e.g., professor and specialist) are shown in Table 1.

Table 1 shows that the university mathematics educator, Mr. Kong, is experienced in teaching mathematics and mathematics education at elementary and university levels. The two specialists, Mr. Ren and Mr. Shao, are excellent elementary mathematics teachers with experience in teaching research activities. The two more experienced teachers are Ms. Tang and Ms. Han. Ms. Lu has taught research lessons based on the LT suggested by experts. Ms. Lu has a bachelor's degree in information and technology and has 5 years of mathematics teaching experience. She has won several teaching awards at the district and the city levels. With the school-based teaching research group, the two experienced teachers worked with Ms. Lu to develop the initial lesson plans and to watch and improve the lessons.

The expert team was responsible for overseeing the process of LS and developing LT of the division of fractions. The school-based teacher team was responsible for designing and delivering the research lessons. Both experts and teachers participated in observing research lessons and post-lesson debriefings. Although experts provided critical comments on the research lessons, the participating teachers made final decisions about the research lesson revisions.

3.2 The Process of Conducting the Lesson Study

The LS group conducted two consecutive research lessons: a lesson on dividing a fraction by a whole number and a lesson on dividing a fraction by a fraction. A three-phase process (Huang and Bao 2006) was used to develop both research lessons. In phase one, *trial teaching 1*, the teachers collaboratively developed the research lessons, and Ms. Lu delivered them in her class. In phase two, *trial teaching 2*, the group worked to revise the lesson plans based on the first debrief and self-reflection, and Ms. Lu taught the revised lessons to a different group of students. In phase three, the group sought to develop an *exemplary lesson* by teaching the same topic based on the rehearsals and debriefings. During trial teachings 1 and 2, the group observed the teaching, collected post-lesson quizzes from the students and reflections from Ms. Lu, debriefed, and revised the lessons plans. In phase three, only the post-lesson quiz and teacher reflection were administrated. The lesson study was conducted in October 2014. Each of the two research lessons was taught in classes 602 (26 students), 604 (28 students), and 607 (34 students), respectively. The Spring 2014 unified exam passing rates of these three classes were 100%, and their excellent rates were 98% (602), 94.6% (604), and 98.5% (607), respectively. Thus, students had similar mathematical performance in these three classes.

3.3 Data Collection

During the LS, the following data were collected: (a) pre- and post LS versions of the LT, (b) all versions of lesson plans, (c) videotaped research lessons and students' worksheets, (d) videotaped post-research lesson debriefings, (e) videotaped pre-lesson interviews with the teachers, (f) post-lesson quizzes (see Appendix 1), (g) selected student interviews after classes, (h) reflection journals of demonstrating teachers and teaching research specialists, and (i) audio-recorded post-lesson interviews with teachers and teaching research specialists. To address the research questions of this article, the first seven types of data from research lesson 2 were used due to space restriction.

Table 2 Learning trajectory and task (T) sequence in the research lesson

	Mathematics task	
Learning trajectory	Teaching 1	Teaching 2
1. 1 divided by a unit fraction (e.g., $1 \div \frac{1}{5}$)	T1: How many $\frac{1}{2}$- liter glasses are there in 2 liters of milk?	T1a: How many $\frac{1}{5}$-liter glasses are there in 1 liter of milk?
		T1b: How many $\frac{1}{4}$-liter glasses are there in 1 liter of milk?
		T1c: How many $\frac{1}{3}$-liter glasses are there in 1 liter of milk?
		T1d: How many $\frac{1}{6}$-liter glasses are there in 1 liter of milk?
2. 1 divided by a fraction (e.g., $1 \div \frac{2}{5}$)	T2: How many $\frac{2}{5}$-liter glasses are there in 1 liter of milk?	T2a: How many $\frac{2}{5}$-liter glasses are there in 1 liter of milk?
		T2b: How many $\frac{2}{7}$-liter glasses are there in 1 liter of milk?
		T2c: How many $\frac{3}{4}$-liter glasses are there in 1 liter of milk?
		T2d: How many $\frac{3}{5}$-liter glasses are there in 1 liter of milk?
3. A whole number divided by a fraction (e.g., $3 \div \frac{2}{5}$)	T3: How many $\frac{2}{5}$-liter glasses are there in 3 liters of milk?	T3a: How many $\frac{2}{5}$-liter glasses are there in 2 liters of milk?
		T3b: How many $\frac{2}{5}$-liter glasses are there in 3 liters of milk?
		T3c: How many $\frac{2}{5}$-liter glasses are there in 4 liters of milk?
		T3d: How many $\frac{2}{5}$-liter glasses are there in 100 liters of milk?
4. A fraction divided by a fraction (e.g., $\frac{1}{2} \div \frac{1}{3}$)	T4: How many $\frac{1}{3}$-liter glasses are there in $\frac{1}{2}$ liters of milk?	T4a: How many $\frac{2}{5}$-liter glasses are there in $\frac{3}{4}$ liters of milk?
		T4b: How many $\frac{2}{5}$-liter glasses are there in $\frac{5}{8}$ liters of milk?
		T4c: How many $\frac{3}{4}$-liter glasses are there in $\frac{4}{5}$ liters of milk?

Notes: The subtask (T1a, T1b, T1c, etc.) of teaching 2 suggests a sequence of variation intended to highlight the critical features of the learning trajectory

3.4 Data Analysis

The audio-recorded interviews, videotaped lessons, and debriefs were transcribed verbatim in Chinese. Data analysis was performed on the Chinese documents with relevant transcripts translated into English.

To address research question 1 (e.g., improvement of classroom instruction), we examined the data at two levels. At the macro-level, the LT and associated tasks of the research lesson were examined based on lesson plans and video lessons (see Table 2; the relevant LT and associated tasks with the first research lesson can be found in Appendix 2. An essential difference between lesson 1 and lesson 2 is the interpretation model used: a partitive model was used in lesson 1, while a quotitive model was used in lesson 2). At the micro-level, we focused on the transformation of the object of learning, enacted object of learning, and lived object of learning.

The explicitly stated instructional objectives in the lesson plans were synthesized as the object of learning. The videotaped lessons were used to describe the enacted object of learning: dimensions of what varied and what was invariant. For example, after examining $1 \div 1/5 = 1 \times 5$ with verbal, visual, and arithmetic representations, the teacher presented exploration tasks: $1 \div 1/3 = 1 \times 3$, $1 \div 1/4 = 1 \times 4$, and $1 \div 1/6 = 1 \times 6$. Thus, one dimension of what varied was the divisor, while the form of division remained unchanged as 1 divided by a unit fraction. This was a necessary condition for students to discern the general pattern of 1 divided by a unit fraction equals 1 times the reciprocal of the unit fraction. All dimensions of variation in the first two teachings are displayed in Table 3.

The lived object of learning was examined through students' post-lesson quizzes. On the quiz, students were asked to solve five contextual problems and justify their solutions using a variety of methods. The quiz was rated based on three criteria: correctness, the use of a visual strategy, and the use of proportional reasoning. If a correct answer was reported, one point was credited, and then the strategies utilized were examined. Students received an additional point for their use of a visual strategy or model, and a final point was awarded for an explanation involving proportional reasoning. Thus the scores of correctness, visual strategies, and proportional reasoning each ranged from 0 to 5 points. Mean comparisons for scores and frequencies were conducted among the student cohorts to detect possible differences between the two teachings.

An example of a student's work considering how many 2-/3-liter glasses are there in 3 liters of milk is provided in Fig. 2. The arithmetic expression on the left-hand side shows a correct response and is credited with one point. The accompanying model, which illustrates 3 liters partitioned into thirds and incremented by 2-/3-liter glasses, is also credited with one point. Finally, the student's description of the scenario, "because there are $^3/_2$ glasses in 1 liter, there are $3 \times {}^3/_2$ glasses in 3 liters, namely, $^9/_2$ glasses," was credited with one point for demonstrating proportional reasoning. Thus this response received the full credit of three points.

To address research question 2 on how improvements were made, the post-lesson debriefs were analyzed through constant comparison (Patton 2002). Traditionally, mathematics teachers in China have focused on addressing *three points* (Yang and Ricks 2013) as main themes when designing, implementing, and evaluating a lesson. The three points are *important*, *difficult*, and *critical* content points. The *important point* describes the emphasis the teacher must put on the topic and the essentials that students must grasp. The *difficult point* is the cognitive challenge that students might encounter as they try to learn the mathematical content. The *critical point* is the teacher's consideration of how to help students reach the learning goals while overcoming pitfalls that might arise.

Attention to the *three points* provided the major emphasis for evaluating and improving the research lesson during the debriefing meetings. The emerging ideas

Table 3 Dimensions of variation in the research lesson

	Dimension of variation	
Learning trajectory	Teaching 1	Teaching 2
1. 1 divided by a unit fraction (e.g., $1 \div \frac{1}{5}$)	FV1: multiple ways of computing $2 \div \frac{1}{5}$	SV1: the same variation as the FV1
	Invariant: the same arithmetic equation ($2 \div \frac{1}{5} = 10$)	SV1e: generalization of 1 divided by $^{1}/_{a}$ ($= 1 \times a$)
	Varied: multiple strategies (converting to decimals, using diagrams)	Invariant: the same strategy: how many $^{1}/_{a}$ are there in 1 whole (quotitive model)
		Varied: different divisor unit fractions ($\frac{1}{5}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{6}$)
2. 1 divided by a fraction (e.g., $1 \div \frac{2}{5}$)	FV2: multiple ways of computing $1 \div \frac{2}{5}$	SV2: the same variation as the FV2
	Invariant: the same arithmetic equation ($1 \div \frac{2}{5} = {}^{5}/_{2}$)	SV2e: generalization of 1 divided by a fraction ($1 \div {}^{b}/_{a} = 1 \times {}^{a}/_{b} = a \div b$)
	Varied: multiple strategies (conjecture, inverse operation of multiplication, diagrams)	Invariant: the same pattern ($1 \div \frac{2}{5} = {}^{5}/_{2}$, $1 \div \frac{2}{7} = \frac{7}{2}$, $1 \div \frac{3}{4} = \frac{4}{3}$)
		Varied: different divisor fractions ($\frac{2}{5}$, $\frac{2}{7}$, $\frac{3}{4}$, $\frac{3}{5}$)
3. A whole number divided by a fraction (e.g., $3 \div \frac{2}{5}$)	FV3: multiple ways of computing $3 \div \frac{2}{5}$	SV3: The same variation as FV3
	Invariant: the same arithmetic equation ($3 \div \frac{2}{5} = 3 \times {}^{5}/_{2}$)	SV3e: generalization of a whole number divided by a fraction ($m \div {}^{b}/_{a} = m \times {}^{a}/_{b}$)
	Varied: multiple strategies (using diagrams and proportional reasoning)	Invariant: the proportional reasoning based on the same stereotype situation of $1 \div \frac{2}{5} = {}^{5}/_{2}$
		Varied: different whole numbers (2, 3, 4, and 100)
4. A fraction divided by a fraction (e.g., $\frac{1}{2} \div \frac{2}{5}$)	FV4: multiple ways of computing $\frac{1}{2} \div \frac{1}{3}$	SV4: generalization of a fraction divided by a fraction $\frac{3}{4} \div \frac{2}{5} = \frac{3}{4} \times {}^{5}/_{2}$
	Invariant: the same arithmetic equation ($\frac{1}{2} \div \frac{1}{3} = \frac{1}{6}$)	Invariant: proportional reasoning based on the same stereotype situation of $1 \div \frac{2}{5} = {}^{5}/_{2}$
	Varied: multiple strategies (using diagrams and proportional reasoning)	Varied: different dividend fractions ($\frac{3}{4}$, $\frac{5}{8}$, and $\frac{4}{5}$)

Notes: FV#, the order of variation in the first lesson; SV#, the order of variation in the second lesson; SV#e, the extension of variation of SV# during the second lesson

were then classified into four categories: (1) identifying the three content points; (2) strategies for highlighting important content points, emphasizing critical content points, and overcoming difficult content points; (3) dealing with multiple representations; and (4) focusing on students' understanding and thinking. Lesson

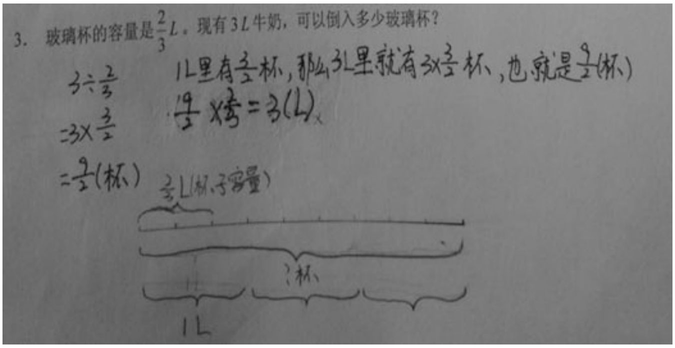

Fig. 2 Student's work demonstrates the number of ⅔-liter glasses in 3 liters of milk

improvements, based on the teachers' reflections and discussions of these ideas, were then summarized. Detailed descriptions of each of these aspects will be presented in the Results section.

Finally, to answer research question 3 on how the LS informed revision of the LT, we looked at the adjustments of the LT throughout the trial teachings. Additionally, the revised LT provided by the lesson group after completion of the project was considered.

4 Results

The results are presented in alignment with the research questions. First, we describe the improvements of the research lessons. Next, we present the factors that led to the changes. Finally, we describe the revisions of the LT.

4.1 Changes in Research Lessons over Repeated Teachings

The two research lessons focused on understanding the meaning of division of fractions and justifying their computation methods. The objects of learning could be synthesized in two areas based on lesson plans. In knowledge and skills, students should understand the meaning of fractions divided by whole numbers or fractions, understand the rationale of the algorithms for these operations, and compute them fluently. Regarding ability and disposition, students should see the application of these algorithms in solving daily life problems, appreciate the connection between mathematics and society, and develop mathematical thinking and reasoning skills related to induction and proportionality.

4.1.1 Learning Trajectory and Associated Mathematics Task

Based on the lesson plans and videotaped lessons, the LT and associated mathematical tasks in the first two teachings are identified as shown in Table 2.

Table 2 reveals that the teacher exactly followed the learning trajectory for the two lessons as suggested by the expert team, but the associated tasks changed dramatically. In teaching 1, all the tasks are related to the same situation of glasses in an amount of milk. However, there were no appropriate scaffoldings between tasks. Task 4 in the first teaching, $\frac{1}{2} \div \frac{1}{3}$, presented a huge challenge to students due to the lack of preparation. In contrast, in the second teaching, each level contains several deliberate variation tasks that could help students generalize respective algorithms. In addition, all major tasks (e.g., $2 \div \frac{2}{5}$, $3 \div \frac{2}{5}$, $\frac{3}{4} \div \frac{2}{5}$) build on a core task of $1 \div \frac{2}{5}$. This interconnection lays a foundation for developing transformational thinking and proportional reasoning.

4.2 *Enacted Object of Learning*

We present the enacted object of learning by providing a summary of the patterns of variation and comparisons of the two teachings.

4.2.1 Patterns of Variation

The lessons included five stages: (1) introduction of the new topic, (2) exploration of 1 divided by a fraction, (3) exploration of a whole number divided by a fraction, (4) exploration of a fraction divided by a fraction, and (5) practice and summary. When a task was presented, students were asked to work independently for a while and then discuss with their neighbors. After a majority of students raised their hands indicating that they had solutions, the teacher asked some of them to present and explain these solutions. The teacher emphasized and summarized key points after class discussions.

According to the theory of variation, the enacted object of learning is described by the patterns of what varied and what remained the same. In Table 3, the first column presents the LT, and the second and third columns present different dimensions of variation constructed in the first two teachings.

Table 3 demonstrates that in the first teaching, all the dimensions of variation focused on helping students to discern multiple ways of solving a specific problem. For example, it is possible for students to see that they can compute $3 \div \frac{2}{5}$ using a formula, visual diagram, and proportional reasoning. However, in the second teaching, in addition to the dimensions of variation constructed in the first teaching (FV1, FV2, FV3, and FV4), more extended dimensions of variation were created. Thus, the second teaching created more learning opportunities for students to generalize the algorithms (SV1e, SV2e, SV3e, and SV4e). In particular, the dimensions of variation SV3e and SV4e provided students with opportunities to discern how *proportional reasoning* could be used as a powerful strategy for justifying the algorithm. Based on VP, the second teaching provided much richer learning opportunities for students to justify the algorithm from multiple perspectives.

4.2.2 Other Major Differences Between the Two Teachings

There were three salient changes from the first to second teachings. First, time management became more effective. The first teaching lasted 51 min, while the second teaching lasted 43 min. Moreover, it was found that 1 min was used to introduce the topic in both teachings, and a similar amount of time was used to explore 1 divided by a fraction using visual representations (20 min in first teaching vs. 19 min in second teaching). However, compared to the first teaching, the second teaching spent twice as long on the exploration of a whole number divided by a fraction (5 min vs. 9), laying a sound foundation for developing proportional reasoning. Due to this preparation, the second teaching spent less time exploring a fraction divided by a fraction through proportional reasoning (16 vs. 10). The systematic variation problem exploration at different stages in the second teaching was more efficient (9 vs. 4). Second, although the teacher paid great attention to expose students' thoughts in both teachings, the teacher paid more attention to present and to share students' correct answers during the first lesson and discussed students' errors as learning sources in the second lesson. Third, regarding the use of visual representation, both teachings emphasized the use of arithmetic, verbal, and visual representations simultaneously. But, in the second teaching, the teacher purposefully guided students from the extensive use of visual representation to mental representation and proportional reasoning without visual representation. This assisted students in developing their proportional reasoning skills.

4.3 The Lived Object of Learning

The lived object of learning is described by students' post-lesson quizzes. The mean and standard deviation of the first two teachings are displayed in Table 4.

Table 4 Students' performance based on the post-lesson quizzes

	Overall performance		Visual representation		Proportional reasoning	
	Mean	Std.	Mean	Std.	Mean	Std.
1st ($N = 26$)	4.08	1.65	1.50	1.50	0.15	0.61
2nd ($N = 28$)	4.93	0.38	1.68	1.39	0.96	1.81

Table 4 shows that there are increases of means in performance, visual representation, and proportional reasoning from the first teaching to the second teaching. A t-test further detects that the changes of mean in overall performance ($t = 2.66$, $p = 0.01$) and proportional reasoning ($t = 2.2$, $p = 0.04$) are significant, but the changes in visual representation ($t = 0.45$, $p = 0.65$) are not. The data show that the different teachings resulted in improvement of students' fluency in division of fractions. It is encouraging that in the second teaching, students made improvements in both proportional reasoning and their use of visual representation to justify the algorithm, which reflected the teacher's intentions.

4.4 Emerging Ideas for Improving Research Lesson

The analysis of post-lesson debriefings and the teacher's self-reflection on the first teaching provided evidence about the changes teachers made to the second teaching. The major ideas about improving the research lesson that emerged in the post-lesson debriefing include identifying the important, difficult, and critical content points; strategies for highlighting important content points, emphasizing critical content points, and overcoming difficult content points; dealing with multiple representations; and focusing on students' understanding and thinking.

4.4.1 Identifying Three Key Content Points

Ms. Lu originally believed that the important and difficult content point was understanding the algorithm of $\frac{1}{2} \div \frac{1}{3}$ using diagrams, and the critical content point was justifying $3 \div \frac{2}{5} = 3 \times {}^{5}/_{2}$ via the bridge of $1 \div \frac{2}{5} = 1 \times {}^{5}/_{2}$ and proportional reasoning. The didacticians (Mr. Kong and Mr. Ren) helped the teachers clarify the important knowledge point as follows:

Mr. Kong: I ask you one more question. You implemented the lesson by following the learning trajectory we suggested, namely, five tasks ($2 \div \frac{1}{5}$; $1 \div \frac{2}{5}$; $3 \div \frac{2}{5}$; $\frac{1}{2} \div \frac{1}{3}$; ${}^{3}/_{7} \div \frac{2}{5}$). Which of the five tasks do you think you should spend more time on?

Ms. Lu: I originally thought that I should make great efforts in dealing with $\frac{1}{2} \div \frac{1}{3}$.

Mr. Ren: You said that students were not able to draw diagrams to visualize it.

Ms. Lu: I originally thought that for the first two tasks ($1 \div \frac{1}{5}$, $1 \div \frac{2}{5}$), students should be asked to draw diagrams, but for the third one ($3 \div \frac{2}{5}$) it was not necessary. Here, students should use proportional reasoning to find the result directly.

Mr. Kong: It is acceptable to draw diagrams for $3 \div \frac{2}{5}$, but it is too hard for students to draw diagrams for $\frac{1}{2} \div \frac{1}{3}$. My thought was to emphasize optimal thinking when discussing $\frac{1}{2} \div \frac{1}{3}$ and ${}^{3}/_{7} \div \frac{2}{5}$. It is necessary to use proportional reasoning to justify computation methods when drawing seems unrealistic.

Ms. Lu: Students were not able to draw diagrams for $\frac{1}{2} \div \frac{1}{3}$. I will not ask them to draw these. I want them to understand how many glasses are in 1 liter and then use proportional reasoning to justify.

Mr. Kong: So, it is not appropriate to say $\frac{1}{2} \div \frac{1}{3}$ is the most important point. Actually, $1 \div \frac{2}{5}$ is the most important content point, is not it?

Through extensive discussions, the important, difficult, and critical content points were clarified. The important content point, as well as the critical point, was to help students understand the computation rule for $1 \div 2/5 = 1 \times 5/2 = 5/2$ from visual, verbal, and arithmetic perspectives. Another critical content point was to justify why $3 \div 2/5 = 3 \times 5/2$ verbally and logically while de-emphasizing visual representation and highlighting proportional reasoning via the bridge $1 \div 2/5 = 1 \times 5/2 = 5/2$. The difficult point was to understand why $3/4 \div 2/5 = 3/4 \times 5/2$ from proportional reasoning with the support of visual diagrams. The group decided to replace $1/2 \div 1/3$ with $3/4 \div 2/5$ due to its interconnection with previous tasks. A core task on how many glasses of $2/5$ liters are in 1 liter of milk was seen as the hinge linking different tasks throughout the class.

4.4.2 Task Selection Focusing on Three Content Points

After achieving agreement to the three content points, the discussion focused on how to deal with these points strategically by selecting deliberate mathematical tasks.

Strategically Polishing Scaffolding Tasks Exploration of dividing 1 by a unit fraction is relatively simple but fundamental. In the first teaching, the teacher used a task, how many glasses of $1/5$ liters are there in 2 liters of milk, because she intended to help students realize how they can use the result of 1 liter to solve the problem with 2 liters via transformation thinking (i.e., solving a complex problem using the solution to a simpler and solved problem and proportional reasoning). Through discussion, she realized "$1 \div 1/5$ can help students visually see there are 5 of $1/5$ in 1 whole." The discussion also revealed "[the] introductory situation of how many glasses of $1/5$-liter are in 1 liter is good. The simple situation would lead students to get the result of $1 \div 1/5 = 5$ very quickly. Then, the teacher should ask them how you found the result" (Mr. Ren). Moreover, during the discussion, the specialists suggested for the teachers to include several situations of 1 divided by a unit fraction (such as $1 \div 1/4$ and $1 \div 1/6$) and to have students generalize the pattern.

Digging Deeply into the Important and Critical Points Exploration of 1 divided by a fraction is one of the important points of the lesson. To address the key point of understanding why $1 \div 2/5 = 5/2 = 2\frac{1}{2}$ from multiple perspectives, a great amount of time was devoted to discussing relevant strategies. Two key ideas occurred. One focused on students' demonstration of a half glass through the use of visual and verbal representations in their explanations. The second idea was the generalization of the pattern through examining several cases of 1 divided by a fraction verbally (e.g., $1 \div 2/7$; $1 \div 2/9$; $1 \div 3/5$; $1 \div 5/7$). The specialist provided the following suggestions:

> Mr. Ren: Yes. You should let students work extensively on this activity. The diagram is only used to verify the result. I think that this lesson should focus on verification rather than discovery because students in their brain already knew the result: multiplying the reciprocal of the divisor. If you ask students the result, they will certainly say the

> reciprocal. So, the key is to verify the relationship rather than to discover it as a new pattern. In this way, the students can see the result of 1 ÷ ⅖ as 2 [glasses] and ½ [glass]. Then, you can ask students about the relationship between the arithmetic expression and the result. Based on previous experience, students should get the result of $^5/_2$. After students understand why 1 ÷ ⅖ is $^5/_2$ visually, then they should explore more variation tasks such as why 1 ÷ ²⁄₇ = $^7/_2$, 1 ÷ ¾ = $^4/_3$, and so on, and finally [you should] encourage students to generalize the pattern of 1 divided by a fraction.

Proportional Reasoning Used to Develop Understanding of a Whole Number Divided by a Fraction and a Fraction Divided by a Fraction Building on the understanding of 1 divided by a fraction, students should be guided to develop a deeper understanding of a whole number divided by a fraction through proportional reasoning. Several cases including 2 ÷ ⅖, 3 ÷ ⅖, and 4 ÷ ⅖ should be strategically explored in order to generalize the pattern of a ÷ ⅖ = $a \times {}^5/_2$. Visual representations and proportional reasoning should be used purposefully to verify 2 ÷ ⅖ and highlight the inconvenience of using diagrams and the effectiveness of using proportional reasoning when verifying 3 ÷ ⅖ and onward. If there are 2 and ½ of ⅖ liter in 1 liter of milk, 3 liters will have 3 times of 2 and ½ of 2/5 liter of milk, which is $3 \times {}^5/_2$. The didacticians and the teachers agreed if students understand the benefit of proportional reasoning, they could apply proportional reasoning to discuss ¾ ÷ ⅖ and so on. Mr. Kong further suggested to the teacher to "use the opportunity to develop optimization thinking by using proportional reasoning; let students realize proportional reasoning is easy [for certain conditions] while drawing diagrams is not convenient." Based on the experience in exploring a whole number divided by a fraction through proportional reasoning, the teachers further designed the lessons to ask the students to verify the pattern about a fraction divided by a fraction using proportional reasoning as Mr. Ren suggested:

> Mr. Ren: . . .After completing these two tasks, we can use the fractions we discussed in the class, such as 1 ÷ ¾, and ask students to explain ⅘ ÷ ¾ [writing beside 1 ÷ ¾ = $^4/_3$]. Students are expected to apply the same reasoning as we discussed with a number divided by ⅖. That is what we discussed about 2 ÷ ⅖ using the result of 1 ÷ ⅖. Thus, we can consider the result of 1 divided by a fraction first, then a divided by the fraction is a times the result. Then, the students are asked to explore how many glasses of ⅖ liter there are in ¾ liters. Students are encouraged to think according to what they did in previous cases. Thus, students can further explore ⅝ ÷ ⅖ and ⅘ ÷ ¾. It is natural to use the result of 1 ÷ ¾.

4.4.3 Dealing with Multiple Representations

Appropriate use of representations was a key element throughout the entire lesson. The discussions on the use of representations focused on the following issues: (1) inappropriate use of visual representations, (2) deliberate use of visual representations, and (3) integrating multiple representations simultaneously.

Inappropriate Use of Visual Representations Based on the experience of drawing diagrams to visualize the result of 1 ÷ ⅕ and 1 ÷ ⅖, students tended to use diagrams to present a fraction divided by a fraction even though the arithmetic is too complicated to visualize. This tendency actually constrained students' thinking as discussed below.

Ms. Lu: Because of the simplicity of drawing diagrams for visualizing $2 \div \frac{1}{5}$ and $1 \div \frac{2}{5}$, the students' first attempt to explain their results by drawing. However, with the increase in complexity of arithmetic expressions, diagrams become more and more complicated, for example, students were not able to draw an appropriate diagram to visualize $\frac{1}{2} \div \frac{1}{3}$.

Mr. Ren: This is to say, sometimes drawing diagrams may constrain and limit students' mathematical thinking.

Deliberate Use of Visual Representation The failed experience in using a diagram for explaining $\frac{1}{2} \div \frac{1}{3}$ motivated the group to discuss strategies to address this issue. The different ways to treat multiple representations were discussed. The tasks for dealing with important content points ($1 \div \frac{1}{5}$, $1 \div \frac{2}{5}$) should be discussed extensively using multiple representations simultaneously. While the initial variation tasks surrounding the core tasks should be treated using visual and verbal representations, the eventual goal is for students to solely use verbal representations of proportional reasoning in the later tasks. The mathematics education professor, Mr. Kong, provided the following suggestions:

> First, $1 \div \frac{1}{5}$ is the important content point; it should be visualized using diagrams and expressed verbally. Then, when discussing $1 \div \frac{1}{4}$ and $1 \div \frac{1}{3}$, drawing diagrams should be not required; students should be asked to think using their brains. When discussing $1 \div \frac{2}{5}$, the diagram should be used to visualize and justify why it is $^5/_2$. It is very visual. Then, when discussing $1 \div \frac{2}{7}$, students should not be asked to draw a diagram but to think about how many $\frac{2}{7}$'s there are in 1 unit. After students get answers, then the teacher shows a diagram to them. After that, it is not necessary to show diagrams for $1 \div \frac{3}{4}$. The discussions about $1 \div \frac{2}{7}$ and $1 \div \frac{3}{4}$ should progress quickly. When discussing $2 \div \frac{2}{5}$ and then $3 \div \frac{2}{5}$, the emphasis should be on $2 \div \frac{2}{5}$, creating a situation problem, which corresponds to how many $\frac{2}{5}$ are there in 2 (i.e., $2 \div \frac{2}{5}$). Students should be asked to consider the following question: There are $^5/_2$ of $\frac{2}{5}$ liters in 1 liter, how many $\frac{2}{5}$ of a liter are there in 2 liters? Students could use diagrams or use the result ($^5/_2$) of how many $\frac{2}{5}$ in 1 liter. Then, when discussing the result of $3 \div \frac{2}{5}$, students should be guided to think and express using what you did with $1 \div \frac{2}{5}$.

Integration of Using Multiple Representations Simultaneously The difficulty in understanding $\frac{3}{4} \div \frac{2}{5}$ has been discussed and addressed from two aspects. First, it is suggested that proportional reasoning be used. Second, a well-designed diagram given by the teacher should be helpful for students to make sense of the situation.

4.4.4 Focusing on Students' Understanding and Thinking

The discussion also focused on students' learning difficulties and finding ways to overcome these difficulties. As in previous discussions, the teachers found that students had a tendency to draw diagrams to visualize the algorithms, even when the expressions became more and more complex, so the team devised ways to overcome the difficulty by using proportional reasoning. In addition, the use of appropriate language to make sense of the algorithm was an issue. For example, when discussing how many glasses of $2/5$ liters there are in 1 liter, the teacher said 5/2 glasses in the class. However, observers noted that students said two glasses and a half glass. They suggested for the teachers to use a diagram to help students understand why it is 2 glasses and a half glass and then realize the equivalence between $^5/_2$ and $2\frac{1}{2}$.

In addition, during debriefing, issues about questioning skills, instructional language, and board writing were discussed.

4.5 Intended Changes of the Research Lesson

Based on the reflection on the research lesson and its debrief, the teacher explicitly expressed her intended changes. In her reflection report, she summarized five major points she took away from the first teaching and the debrief. First, the process of teaching should reflect a progressively abstracting process. Students should shift their tendency from "drawing" to "thinking and reasoning" when solving problems. Second, the instructional process should reflect an optimization process for thinking (i.e., appropriate use of representations based on the nature of problem to illustrate how the algorithm works). On the one hand, after students extensively explored $1 \div 2/5$, students should be led to explore other purposeful variation tasks. When exploring $1 \div 2/7$, students should be asked to visualize a diagram in their heads. After that, the teacher should show students a diagram to help them verify their ideas. On the other hand, through the recognition of complexity of drawing diagrams as the arithmetic expression becomes more complex, students should be led to realize the need to explore a simpler way to solve the problem, such as proportional reasoning based on the quotitive interpretation of division. Third, it is important to put great effort in breaking through the important and difficult content point of $1 \div 2/5$ and then using it as a tool to solve a series of problems of a whole number divided by $2/5$ (e.g., $2 \div 2/5$, $3 \div 2/5$, $100 \div 2/5$) and transforming to discuss a fraction divided by $2/5$ (e.g., $3/4 \div 2/5$). The discussion with $1/2 \div 1/3$ will be removed completely in the second teaching. Fourth, the board writing needed to improve in order to build connections between different contents and emphasize the core content of $1 \div 2/5$ as a starting point and a bridge linking different parts. Finally, attention must be given to students' feedback, particularly to students who have difficulties.

Overall, the teacher accepted the major ideas and suggestions discussed in the first debrief. In the second teaching, these strategies were implemented appropriately.

4.6 Learning Trajectory Refinement

Table 5 presents the LTs of division of fractions at two stages: pre- and post LS.

Overall, the hypothetical LT developed based on Western literature is applicable in the classroom in China. The three macro-levels fit student learning in the

Table 5 Learning trajectories of division of fractions

Major levels	Pre-lesson study	Post-lesson study
0. Meaning of fractions and division	Division with whole number (4 ÷ 2)	Within a same situation of cake sharing: 1 ÷ 5
1. A fraction divided by a whole number	1a. Numerator of the dividend is the multiplier of divisor (e.g., 4/5 ÷ 2)	The same as pre-lesson study: ⅘ ÷ 2
	1b. The dividend is a unit fraction (e.g., ⅕ ÷ 2)	The same as the pre-lesson study: ⅕ ÷ 2
	1c. The dividend is a proper fraction (e.g., ⅘ ÷ 3)	The same as the pre-lesson study: ⅘ ÷ 3
2. A whole number divided by fraction	2a. The divisor fraction is a unit fraction (2 ÷ ⅕)	Extended to include 1 ÷ ⅕. 1 ÷ ⅓, 1 ÷ ¼,1 ÷ ⅙,. . ., 1 ÷ $^1/_a$
	2b. 1 divided by a proper fraction (1 ÷ ⅖)	Extended to include 1 ÷ ⅖, 1 ÷ $^2/_7$, 1 ÷ ¾, 1 ÷ ⅗.
	2c. A whole number divided by a fraction (3 ÷ ⅖)	Extended to include 2 ÷ ⅖, 3 ÷ ⅖, 4 ÷ ⅖,. . ., 100 ÷ ⅖
3. A fraction divided by a fraction	3a. A unit fraction divided by a unit fraction (e.g., ½ ÷ ⅓)	This sublevel is removed
	3b. A fraction divided by a fraction (¾ ÷ ⅖)	Extended to include ¾ ÷ ⅖, ⅜ ÷ ⅖ ; ⅘ ÷ ¾

classroom in China very well. At the micro-levels, all levels apart from sublevel 3a, a unit fraction divided by a unit fraction, have been implemented in the study and appear to predict students' learning progression well. Importantly, this LS has enriched the LT by incorporating the notion of variation. Due to the variation ideas, students could be guided to develop their ability to make generalizations (2a, 2b, 2c). In addition, by adopting the variation pedagogy, keeping the situation of 1 ÷ ⅖ invariant across multiple tasks and examining varying tasks at level 2c and 3b provided an opportunity to justify the algorithm through proportional reasoning. This scenario demonstrates how developing conceptual understanding and mathematical reasoning could be achieved simultaneously.

5 Discussions and Conclusion

This study describes how a lesson study, guided explicitly by theoretical notions of learning trajectory and variation theory, developed lessons for division of fractions through a cycle of collaborative design, teaching/classroom observation, debriefing, and reteaching. The results show that the lesson has been improved in terms of students' understanding, proficiency, and mathematical reasoning. Meanwhile, the factors that led to these improvements are revealed through analyzing the post-lesson debriefs and the teacher's reflections. The debrief focused on identifying and prioritizing key content points within the overall framework of the LT, exploring effective ways of addressing important content points, strategically overcoming difficult content points and highlighting critical points. Specifically, the debriefing also focused on the purposeful use of multiple representations, addressing students' learning difficulties and errors, and general instructional skills such as questioning, instructional language, and board writing design. Self-reflection, redesign, and reteaching helped the teacher to adopt and to implement those ideas in class, resulting in the improvement of teaching. In addition to improving teaching, this study also evidenced that the LT is refined and enriched through the theory-driven LS. In particular, the variation theory provided strategies for creating scaffolding for progressing students' understanding to higher levels and extending students' experience in generalizing patterns. In the following sections, we highlight some key points.

5.1 *Lesson Study as an Effective Way to Implement Standard-Based Teaching*

Effectively implementing standard-based mathematics teaching has been a long-standing issue for decades (Woodbury and Gess-Newsome 2002). Although research-based effective mathematics teaching practice (NCTM 2014) may help teachers understand what a standard-based classroom looks like, implementing mathematics teaching practice in classes presents new challenges to most teachers. Research has shown that LS is a promising way to help teachers implement reform-oriented teaching in class (Lee and Lo 2013). This study provides two specific implications for the selection of tasks and the use of multiple representations.

With regard to mathematical tasks, there are a great deal of studies on high cognitive demand tasks (Stein and Lane 1996) and the ways to effectively implement tasks and share students' work (Stein et al. 2008). This study adds additional dimensions to the implementation of tasks. First, the instructional tasks should be embedded in a context with which students are familiar with, such as cake or milk sharing. Subtasks should be intentionally varied in some aspects ($2 \div \frac{2}{5}$, $3 \div \frac{2}{5}$, $4 \div \frac{2}{5}$, $100 \div \frac{2}{5}$, $\frac{3}{4} \div \frac{2}{5}$) while keeping others the same ($1 \div \frac{2}{5}$). By doing so, students' attention could be drawn to important mathematical relations rather than the various contexts. To vary tasks within a context, two strategies could be employed. One is vertical variation that focuses on developing students' understanding from different levels of the LT (such as $1 \div \frac{1}{5}$, $1 \div \frac{2}{5}$, $3 \div \frac{2}{5}$, and $\frac{3}{4} \div \frac{2}{5}$, while the divisor is invariant). The other is horizontal variation (such as $1 \div \frac{2}{5}$, $1 \div \frac{2}{7}$, $1 \div \frac{3}{4}$...) within the same level of the LT, which focuses on developing generalization. Through these two dimensions of variation, conceptual understanding could be developed in depth and in breadth.

Regarding the use of multiple representations, research has shown that US students tend to use concrete or pictorial representations, while Chinese students tend to use general or symbolic representations (Cai and Wang 2006). In addition, US teachers tend to use multiple representations simultaneously, while Chinese teachers tend to use multiple representations selectively (Huang and Cai 2011).

This study further shows how teachers could help students develop their flexibility in using multiple representations through emphasizing multiple representations simultaneously, decreasing the use of visual representations over time, and using verbal and arithmetic representations solely as appropriate to the task. The use of representations in Chinese mathematics classes is purposeful and selective with the intent to develop students' abstract thinking.

5.2 Lesson Study as a Platform for Building Linkage Between Theory and Practice

To address the issue of the gap between theory and practice in education, Kieran et al. (2013) argue that treating teachers as key stakeholders in research is a powerful way for building linkage between theories and practice. Using lesson study in China and Japan as examples, they highlight the critical features of research where the teacher is viewed as a stakeholder. These features include inquiry-based activity, a significant action research component, and dynamic duality of research and professional development. This study includes all three features and demonstrates how a theory-driven lesson study (utilizing learning trajectory and variation pedagogy as guiding principles for lesson design, implementation, and reflection) could improve mathematics teaching practice and, at the same time, refine a learning trajectory. Similarly, Lo and Marton (2012) state that "[lesson] study likewise provides a possible platform where teaching can be cast as an experimental science and a form of action research. We would like to suggest that variation theory offers potential gains to lesson study in the sense that it provides an additional theoretical component to guide decisions about teaching" (p. 21). Based on an extensive discussion of the system of Japanese lesson study (national, district-based, and school-based lesson study), how teachers and researchers learn from crossing boundaries within the lesson study system, and the eventual results of "teaching for understanding in both mathematics and science, successfully spreading some major instructional innovations" (Lewis 2015, p. 58), she argues that lesson study is a type of improvement science (Langley et al. 2009). An improvement science essentially includes the core framework of plan-do-study-act (PDSA) cycle, coupled with three fundamental questions: (a) what are we trying to accomplish? (b) how will we know that a change is an improvement? and (c) what change can we make that will result in improvement? (Lewis 2015, p. 54). The researchers would argue that Chinese lesson study is a form of improvement science as well (Huang and Han 2015), and this study, methodologically, provides a way for researching into improvement science by addressing object of learning, enacted object of learning, and lived object of learning (corresponding to questions (a), (b), and (c), respectively).

5.3 Limitation and Further Studies

Two limitations to the study should be noted. First, this form of LS is not typically utilized in China because it was explicitly guided by specific theoretical perspectives. This special case illustrates the efforts of a group of mathematics educators as they attempt to implement standard-based curriculum through the theory-driven LS. Second, the study presents results from a single, school-based LS group that may not be reflective of the variations of LS that occur at other organizational levels (e.g., district, municipal, or national level LS programs). Thus, the findings in this case cannot be generalized broadly to LS in China but rather present a unique case that offers some insight into the potential of theory-driven lesson study to impact students' mathematical understanding.

Despite these limitations, this study suggests at least four issues worthy of further exploration. First, the study shows the critical role played by knowledgeable others (e.g., university professors, subject specialists, etc.) during the lesson study process. It would be interesting to examine the way these knowledgeable others work with practicing teachers and develop their own professional knowledge and skills through mentoring during LS. Second, throughout this study, the notions of learning trajectory and variation pedagogy have played a critical role in scaffolding students' learning. It would be interesting to look in more detail at the manner in which these constructs relate to specific theoretical perspectives on scaffolding. Third, this case study focuses on the potential of theory-driven LS to impact students' understanding at the classroom level. It would be interesting to explore methods for scaling up this model to teacher professional development in general. Finally, as this study took place in China where LS is embedded in teachers' daily practice, it would be interesting to explore the implementation of similar studies in settings in the West where LS is an innovative initiative.

Acknowledgments We thank anonymous reviewers for their invaluable feedback on the revisions of the paper. We appreciate Dr. Dovie Kimmins and Mr. James Willingham from Middle Tennessee State University for their contribution to the improvement of the article. Our thanks go to participating teachers and didacticians for their commitment to the lesson study and support of data collection.

Appendices

Appendix 1: Post-lesson Assessment

Greetings, class! To understand your learning situation in the class, we designed this questionnaire. Please carefully answer each question according to the instructions given. Just write down what you think. We will not grade your work and compare you answers with others. Thank you for your cooperation.

First, you need to compute arithmetic expressions. Then, justify your computations using as many as methods as possible such as verbal explanation, visual diagrams, or arithmetic expressions. The more details the better.

1. How many glasses of 1/5 liter are there in 3 liters of milk?
2. How many glasses of 2/3 liters are there in 1 liter of milk?
3. How many glasses of 2/3 liter are there in 3 liters of milk?
4. How many glasses of 1/5 liter are there in 1/3 liter of milk?
5. How many glasses of 2/3 liter are there in 4/5 liter of milk?

Appendix 2: Learning Trajectory and Associated Tasks in Research Lesson 1

Learning trajectory	Second teaching
1. Connecting to previous knowledge (divisions with whole numbers) (e.g., $1 \div 5 = 1 \times \frac{1}{5}$)	T1: A 1-kilogram rectangle cake is shared between five friends, how much does each friend get?
2. A fraction divided by a whole number (when the numerator is the multiplier of divisor) (e.g., $\frac{4}{5} \div 2$)	T2: A $\frac{4}{5}$-kg of cake is shared between two friends, how much does each friend get?
3. A unit fraction divided by a whole number (e.g., $\frac{1}{5} \div 2$)	T3: A $\frac{1}{5}$-kg cake is shared between 2 friends, how much does each friend get?
4. A fraction divided by a whole number (when the numerator is not a multiplier of divisor) (e.g.,$\frac{4}{5} \div 3$)	T4: A $\frac{4}{5}$-kg cake is shared between 3 friends, how much does each friend get? Practice: A $^{3}/_{10}$-kg cake is shared between 8 friends, how much does each friend get?
5. 1 divided by a unit fraction (e.g., $1 \div \frac{1}{5}$)	T1a: How many $\frac{1}{5}$-liter glasses are there in 1 liter of milk?
	T1b: How many $\frac{1}{4}$-liter glasses are there in 1 liter of milk?
	T1c: How many $\frac{1}{3}$-liter glasses are there in 1 liter of milk?
	T1d: How many $\frac{1}{6}$-liter glasses are there in 1 liter of milk?
6. 1 divided by a fraction (e.g., $1 \div \frac{2}{5}$)	T2a: How many $\frac{2}{5}$-liter glasses are there in 1 liter of milk?
	T2b: How many $\frac{2}{7}$-liter glasses are there in 1 liter of milk?
	T2c: How many $\frac{3}{4}$-liter glasses are there in 1 liter of milk?
	T2d: How many $\frac{3}{5}$-liter glasses are there in 1 liter of milk?
7. A whole number divided by a fraction (e.g., $3 \div \frac{2}{5}$)	T3a: How many $\frac{2}{5}$-liter glasses are there in 2 liters of milk?
	T3b: How many $\frac{2}{5}$-liter glasses are there in 3 liters of milk?

(continued)

Learning trajectory	Second teaching
	T3c: How many ⅖-liter glasses are
	there in 4 liters of milk?
	T3d: How many ⅖-liter glasses are there in 100 liters of milk?
8. A fraction divided by a fraction (e.g., ½ ÷ ⅓)	T4a: How many ⅖-liter glasses are there in ¾ liters of milk?
	T4b: How many ⅖-liter glasses are there in ⅝ liters of milk?
	T4c: How many ¾-liter glasses are there in ⅘ liters of milk?

References

Cai, J., & Wang, T. (2006). U.S. and Chinese teachers' conception and construction of representations: A case of teaching ratio concept. *International journal of Science and Mathematics Education, 4*, 145–186.

Cai, J., & Wang, T. (2010). Conception of effective mathematic teaching with a cultural context: Perspectives of teachers from China and the United States. *Journal of Mathematics Teacher Education, 13*, 265–287.

Carpenter, T. C., Lindquist, M. M., Brown, C. A., Kouba, V. L., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Trends and conclusions. *Arithmetic Teacher, 36*(4), 38–41.

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*, 218–236.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Clements, D., Sarama, J., Spitler, M., Lange, A., & Wolfe, C. B. (2011). Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *Journal for Research in Mathematics Education, 42*, 127–166.

Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/Math/Practice

Gu, L., Huang, R., & Marton, F. (2004). Teaching with variation: An effective way of mathematics teaching in China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–348). Singapore: World Scientific.

Hart, L. C., Alston, A. S., & Murata, A. (2011). *Lesson study research and practice in mathematics education: learning together*. New York: Springer.

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298.

Huang, R., & Cai, J. (2011). Pedagogical representations to teach linear relations in Chinese and U.S. classrooms: Parallel or hierarchical? *The Journal of Mathematical Behavior, 30*(2), 149–165.

Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.

Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese Lesson Study. *ZDM – The International Journal on Mathematics Education, 46*, 239–251.

Kieran, C., Krainer, K., & Shaughnessy, J. M. (2013). Linking research to practice: Teachers as key stakeholders in mathematics education research. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 361–392). New York: Springer.

Langley, G. J., Moen, R. D., Nolan, K. M., Nolan, T. W., Norman, C. L., & Provost, L. P. (2009). *The improvement guide*. San Francisco: Jossey-Bass.

Lee, C. K. E., & Lo, M. L. (2013). The role of lesson study in facilitating curriculum reform. *International journal for lesson and learning studies, 2*, 200–206.

Lewis, C. C. (2015). What is improvement sciences? Do we need it in education? *Educational Researcher, 44*(1), 54–61.

Lewis, C. C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Li, Y. (2008). What do students need to learn about division of fractions? *Mathematics Teaching in the Middle School, 13*, 546–552.

Lo, M. L., & Marton, F. (2012). Toward a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal for Lesson and Learning Studies, 1*, 7–22.

Maloney, A. P., Confrey, J., & Nguyen, K. H. (2014). *Learning over time: Learning trajectories in mathematics*. Charlotte: Informational Age Publishing.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning Science, 15*, 193–220.

Marton, F., & Tsui, A. B. M. (with Chik, P. P. M., Ko, P. Y., Lo, M. L., Mok, I. A. C., Ng, F. P., Pang, M.F., et al.) (Eds.). (2004). *Classroom discourse and the space of learning*. Mahwah: Lawrence Erlbaum.

Ministry of Education, P. R. China. (2011). *Mathematics curriculum standards for compulsory education* (grades 1–9). Beijing: Beijing Normal University Press.

Murata, A. (2011). Introduction: Conceptual overview of lesson study. In C. L. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: learning together* (pp. 1–12). New York: Springer.

National Council of Teachers of Mathematics. (2014). *Principles to action: Ensuring mathematical success for all*. Reston: NCTM.

Ott, J. M., Snook, D. L., & Gibson, D. L. (1991). Understanding partitive division of fractions. *The Arithmetic Teacher, 39*, 7–11.

Patton, M. Q. (2002). *Qualitative research & evaluation methods* (3rd ed.). Thousand Oaks: Sage.

Silvia, E. M. (1983). A look at division with fractions. *The Arithmetic Teacher, 30*, 38–41.

Simon, M. A. (1995). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24*, 233–254.

Sowder, J., Sowder, L., & Nickerson, S. (2010). *Reconceptualizing mathematics for elementary school teachers*. New York: W.H. Freeman & Company.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50–80.

Stein, M., Engle, R., Smith, M., & Hughes, E. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*, 313–340.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: toward a theory of teaching. *Educational Researcher, 41*, 147–156.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*, 5–25.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning, 8*(2), 91–111.

Woodbury, S., & Gess-Newsome, J. (2002). Overcoming the paradox of change without difference: A model of change in the arena of fundamental school reform. *Educational Policy, 16*, 763–782.

Yang, Y., & Ricks, T. E. (2013). Chinese lesson study: Developing classroom instruction through collaborations in school-based teaching research group activities. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 51–65). New York: Routledge.

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Learning While Leading Lesson Study

Jennifer M. Lewis

Contents

Abstract This chapter presents research on how teacher developers in the United States learn to conduct lesson study. In contexts such as the United States where this form of professional development is relatively novel, few teachers have participated in lesson study, so leaders of lesson study groups do not have that prior experience to draw upon for facilitation. To investigate how facilitators learn to lead a practice that is new to them, two novice teacher developers were followed for a period of 18 months, from their first exposure to the literature on lesson study, through their participation in lesson study conferences, apprenticeship with an experienced lesson study leader, and into their independent conduct of lesson study groups. Data show that the facilitators learned to contend with such issues as teacher resistance, the use of time, and the shifting imperatives of directing teachers' work versus stepping back to give teachers autonomy in determining their collective work. The chapter concludes by suggesting that lesson study functions as a countercultural bulwark in the field of teacher learning by promoting a participant-driven, time-intensive form of

This chapter is based on an article that appeared in a special issue on lesson study in the journal *ZDM*: Lewis, J. M. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM Mathematics Education, 48* (4): 527–540.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_31

professional development and that despite its complexity, teacher developers who are new to lesson study become reasonably skillful facilitators in a surprisingly short period of time if they have strong mathematical and pedagogical backgrounds.

Keywords Lesson study · Facilitation · Professional development

1 Introduction

As the demand for lesson study has expanded across districts and schools across the United States, the need for high-quality facilitators of lesson study has grown apace. In contexts such as the United States where this form of professional development is relatively new, few teacher developers have experienced lesson study as teachers themselves, so many are beginning to lead this complex practice more or less de novo. Even experienced professional developers may have little to draw on in leading lesson study, since it is meant to be teacher-driven and therefore unlike other forms of professional development more commonly practiced. Lesson study represents a significant departure from the modal kind of professional development in the United States that positions teachers as receivers of expert knowledge imparted from teacher developers and as such presents challenges for teacher developers who may never have experienced this kind of teacher learning themselves.

This chapter examines how leaders of lesson study learn to lead this complex form of professional development at the same time that they are learning to participate in it.

2 Background

Research from the last decade has provided images of lesson study in Asia as a model for the improvement of mathematics teaching and learning (Isoda et al. 2007; Lewis et al. 2006, 2009; Puchner and Taylor 2006; Stigler and Hiebert 1999). Lesson study has been shown to transform mathematics and science instruction in Japan (Lewis and Tsuchida 1998; Stigler and Hiebert 1999) and raise achievement of Japanese students on international assessments. A number of studies have demonstrated its efficacy in the United States (see, e.g., Isoda et al. 2007; Lewis et al. 2006, 2009; 2013; Puchner and Taylor 2006). In fact, lesson study was one of the only two programs of professional development in mathematics (out of 643) that led to statistically significant positive gains for students in the United States, according to the criteria of the US Department of Education Institute of Education Sciences

(Gersten et al. 2014). Accordingly, the need for articulating the work of facilitation is growing as the practice of lesson study expands in the United States and elsewhere.

Lesson study could be thought of as a form of "professional learning community," or PLC. Professional learning communities are characterized by "shared values and vision; shared and supportive leadership; collective learning and its applications; supportive conditions for the maintenance of the community; and shared personal practice" (Hord and Sommers, 2008, p. 9). PLCs are frequently mandated in school district policies across the United States, although there is little specification about how they are to be conducted and what teachers learn in PLCs. Note that the definition of professional learning community above is somewhat silent about subject matter teaching and learning, so lesson study's strong focus on children's thinking and on mathematical content would render lesson study a very particular form of PLC.

Despite its ubiquity in policy documents, it is likely that US teachers and teacher educators have never taken part in the kind of teacher-driven, content-rich professional development that the PLC literature calls for. Contrast this with the experience of Chinese teachers, for example, where lesson study is the norm: "First, [teachers] have experienced the development process as novice teachers, then as experienced teachers, and finally as expert teachers. Thus, they can draw on their own experiences of professional development as teachers to mentor others at a range of professional phases" (Gu and Gu 2016). That makes the launch of lesson study a challenge, with facilitators having to lead a practice that they may have never witnessed or experienced themselves. This article studies the experience of two novice lesson study leaders in the United States to understand some of these challenges.

3 Theoretical Framework

3.1 Linking Theory and Practice

Lesson study is especially promising as a form of professional development because it links theory and practice in a way that has eluded many efforts in the field. Feiman-Nemser wrote that "when teachers talk about their professional learning they rarely mention formal preservice or inservice courses. Instead, they talk about the experience of teaching itself, and the chance to observe and talk with other teachers" (Feiman-Nemser 1983, p. 151). This underscores the tension between the formal theory that teachers learn in preservice coursework and professional development and the experience of practice that they often find more vivid and compelling. *Practice* in education is often defined in contrast to *theory*: practice is the interactive, experiential dimension of teaching work. To better understand teaching *practice*, we turn to Andrew Pickering (1995), who has written about scientific practice to afford a consideration of practice as a generic construct. One definition Pickering provides is the notion that "'practice' is the generic one around which all that follows is organized—practice as the work of cultural extension and transformation in time" (p. 4). In science, Pickering writes that "'practice' relates to specific, repeatable sequences of activities on which scientists rely in their daily work" (p. 4). Pickering's definitions provide contours for thinking about practice in general and apply to the practice of teaching in specific.

Bourdieu points out that one essential aspect of practice is its integral location *in time* and that any analysis of practice or distance from it violates this essential feature. He writes:

> Practice unfolds in time and it has all the correlative properties, such as irreversibility, that synchronization destroys. Its temporal structure, that is, its rhythm, its tempo, and above all its directionality, is constitutive of its meaning. As with music, any manipulation of this structure, even a simple change in tempo, either acceleration or slowing down, subjects it to a destructuration that is irreducible to a simple change in an axis of reference. In short, because it is entirely immersed in the current of time, practice is inseparable from temporality, not only because it is played out in time, but also because it plays strategically with time and especially with tempo. (Bourdieu 1990, p. 81)

He goes on to describe the aspect of urgency that is integral to practice:

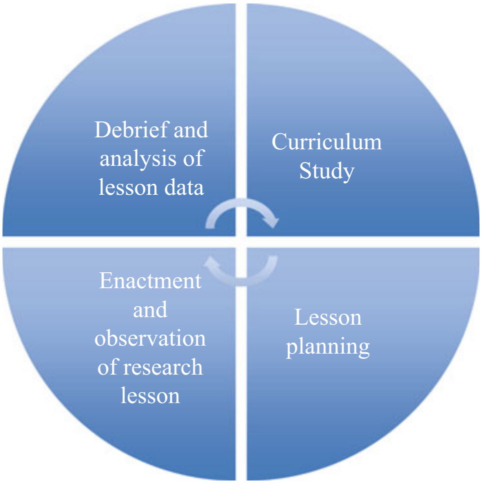
> Urgency, which is rightly seen as one of the essential properties of practice, is the product of playing in the game and the presence in the future that it implies. One only has to stand outside the game, as the observer does, in order to sweep away the urgency, the appeals, the threats, the steps to be taken, which make up the real, really lived-in, world. (p. 82)

Nothing better captures one dimension of teaching practice than this description of urgency. Facing a classroom full of expectant, impulse-driven, curious children, one can be overcome by the sense of urgency that demands action in the moment. To be a teacher means to be required to respond to immediate and sometimes conflicting demands *right now*. In contrast, much of teacher education is conducted outside such demands, and this gives it a seeming lack of authenticity. The suspension of time and press for immediate action is what lends teacher education its sense of remove from the "real work" that teachers must do. Videotape, discussions of practice from afar, literature discussions—indeed, most forms of professional development for teachers—fail to include this sense of urgency. That distance from the press of time allows for the kind of deliberative reflection that is difficult to achieve in the presence of a room full of children. If the urgency that is a function of time is indeed what Bourdieu called "an essential property of practice," then professional development experiences, at least some of them, need to include that sense of urgency.

We use this theoretical framework because lesson study can in some measure provide this sense of time and urgency so lacking in many other forms of professional development.

Figure 1 shows the lesson study cycle. Following what Pickering called the "specific, repeatable sequences of activities" (1995, p. 4) of lesson study shown in this cycle, lesson study's apex, for the purposes of linking theory and practice, would be the research lesson, the actual teaching with real children in a chosen classroom. This is where the fruits of reflection, deliberation, and practical judgment are reintroduced into the test of real-time work. In this sense, lesson study inquiry is driven by practice and the standards to which its outcomes are judged are practice-

Fig. 1 Lesson study cycle



based. It serves, potentially, as a bridge between theory and practice: teachers design a lesson based on their theoretical hunches about what will be productive with students, and they then bring that lesson to the test of practice.

3.2 *Lesson Study: Framed and Driven by Problems of Practice*

The lesson study process begins with teachers formulating a problem to study drawn from their own practices. Teachers' planning of a lesson is likewise anchored in the specifics of teaching work, blending teachers' knowledge and experience with theory from relevant academic and professional disciplines. Child psychology, research mathematics, philosophy, and other disciplines are drawn upon in the planning of the lesson, but unlike other professional development efforts, these disciplines do not drive the lesson, but rather serve as resources for teaching and learning in the context of a single lesson designed around a real problem of practice. The lesson study group continues its investigation by trying out a lesson with children in the presence of other teachers. The teachers reflect on the outcomes of that lesson using evidence of learning as the criteria of reference. The iterative steps that follow—reflection, redesign, teaching, and reflecting again—are also anchored in the practice of teaching as reference points, not some theoretical framework, policy document, or disciplinary lens, although all these may be woven into the teachers' appraisal and understanding of the lesson. Teacher education often approaches the improvement of teaching piecemeal, where a single dimension of teaching work, or one relevant discipline, is addressed. So, for example, in preservice teacher education, students typically take courses that focus on social science disciplines meant to have applications in teaching generally. Professional development days are often comprised of single-session workshops treating either a new policy affecting teachers (e.g., suicide prevention, as currently proposed by Michigan legislators) or a new technique (reciprocal teaching) or a new resource (a new mathematics textbook). These workshops are divorced from practice in their presentation and leave it to the teacher to apply to their work. Even long-term, discipline-rich professional development interventions such as that described by Grossman, Wineburg, and Woolworth (2001) do not take practice as a point of departure. In the professional development effort described in this article, teachers gathered to read together in the fields of English and history over a period of years. How teachers were to make use of this experience in practice, however, was left unspecified. Like other teacher education efforts, teachers' translations of such inputs into their practices are left for teachers to figure out on their own.

Lesson study provides opportunities for teachers to make use of educational research in the crucible of teaching practice, but also to generate knowledge about teaching and to share that knowledge through various pathways of dissemination. This constructs a role for teachers to be both consumers and producers of knowledge about teaching and casts different kinds of educational research as relevant and useful.

The very nature of the lesson study process presents challenges for facilitation. The fact that lesson study incorporates live practice means that facilitators cannot script complete sessions in advance. Much of lesson study is driven by teachers, so the preparation for sessions is relatively indeterminate. Since teachers set the agenda based on their own assessments of their students' needs and their own sense of their teaching challenges, facilitators have to follow the lead of teachers in the moment, sometimes as it is being formed. Contrast this with the kinds of professional development that most facilitators experienced themselves: preset stacks of slides, activities designed in advance, and structured discussion topics with a specified end in sight. Thus, because so much of lesson study is entwined in experience, and because teachers' interests and needs drive the process, facilitators and teachers have to co-construct their competencies as their time together unfolds. These competencies for facilitation are unlikely to have been developed in other forms of professional development that novice lesson study facilitators have experienced themselves. Yet the rapid increase in demand for lesson study around the United States has meant that facilitators inexperienced in this form of professional development are being called upon to lead lesson study groups. This study follows two novice lesson study facilitators to better understand what such novice facilitators have to contend with and how best to support them.

4 Data and Methods

This chapter presents a case study of two facilitators learning to conduct lesson study, which was, for them and for the teacher participants, a novel form of professional development. For the purposes of this investigation, a multiple-case

case study (Yin 2009) was preferred. Studying how novice facilitators conceptualize and carry out the work of lesson study involved close readings of mostly qualitative data collected over time. Two facilitators were followed over a period of a year for this study.

The two facilitators who are the objects of this multiple-case case study are experienced teachers and both new to lesson study. Both have a solid background in mathematics. Karl[1] is a lecturer in a college mathematics department and coordinates the mathematics education classes for teachers. He has worked for 18 years in a summer math program for middle school and high school students. Karl has a BA and an MA in mathematics. Louis was a high school mathematics teacher for 12 years and taught middle school mathematics for 1 year. He currently works as the mathematics coordinator in a student resource center at a public university. Both began their studies interested in other fields but gravitated toward mathematics. Both first learned about lesson study as doctoral students in a course on teacher learning; Karl attended a national lesson study conference where he observed a very established lesson study team's research lessons followed by post-observation panels that included commentary from renowned lesson study scholars from around the world. Both were apprentices to an experienced lesson study leader, attending several days of lesson study with the groups of teachers that they would eventually lead themselves. Those lesson study days focused primarily on curriculum study (*kyozai kenkyu* in Japanese); only Louis was able to attend the research lesson for a middle school group. Both Karl and Louis were novice facilitators in lesson study although both had some experience working with teachers in professional development forms of different kinds. Louis had served as the mathematics department chair at his high school and in this capacity led teacher learning in different formats. Karl leads a cohort of instructors in an intensive summer math program for adolescents and in that role is responsible for ongoing informal professional development during the summer program. Both work in a university where the student population is 36% minority and academically underprepared relative to undergraduates in peer institutions across the state.

Data for this study were collected in three waves. The first wave of data collection occurred as facilitators were apprenticing with an experienced lesson study facilitator. This was for a middle school mathematics department's first exposure to lesson study, and they completed a full cycle. The facilitators' observation notes, transcriptions from meetings, and annotations from readings about lesson study were collected during this phase.

The second wave of data collection occurred as facilitators were together leading middle school and high school lesson study groups through a cycle of lesson study, the second one for the middle school teachers. Facilitators' session plans, transcriptions from meetings, photographic records of inscriptions during sessions, participants' evaluations, and interviews with the facilitators were collected. The third wave of data collection occurred during the facilitators' third cycle of lesson study.

[1] All names in this article are pseudonyms.

By "cycle," we mean a complete round of curriculum study, lesson planning, research lesson, and debriefing phases of lesson study, as shown in Fig. 1 earlier.

It is important to note that the middle school and high school teams of teachers were composed of entire mathematics departments in a single school district, and these teachers were new to lesson study as well. Teachers on these teams did not choose to participate in lesson study; they were mandated by their administrators to attend as part of their required professional development.[2]

Data sources included verbatim transcriptions of interviews, photographs of inscriptions during lesson study sessions, scanned feedback forms from teacher participants following lesson study sessions, verbatim transcriptions of facilitator meetings, and audiotaped recordings of meetings. Multiple perspectives on individual lesson study sessions were solicited from participating teachers, from observers of lesson study sessions, and from the facilitators themselves as a way to triangulate the data. All data were uploaded into atlas.ti, a software program for qualitative data analysis. These data were coded in atlas.ti using a cross-case synthesis method (Yin 2009) where trends and patterns were noted and labels assigned to clusters of teacher developer actions and thoughts. The first coding was done using open coding in an iterative process (Corbin and Strauss 2008); as patterns emerged, codes were combined and subsumed into thematic codes. Focused coding was then conducted, re-examining the data using these thematic codes.

Below in Table 1 is an excerpt from an observer's field notes recording a conversation of apprentice facilitators' observation of a lesson study session. Alongside those notes, the original open codes are recorded.

Analytic memos were written as the data were coded. Following this process, the themes of facilitator learning were identified, and these are shared in the Results section below. To better understand how novice facilitators of lesson study learn to conduct this form of professional development, we report their experiences in their own words to preserve their standpoint as much as possible. These excerpts of facilitators' thoughts are offered against the backdrop of records of practice (lesson plans, teachers' comments) to provide another angle on facilitator learning.

5 Results

A number of themes emerged as issues to contend with as these two leaders learned to lead lesson study. We group these themes into three broad categories: teachers as learners, facilitators as learners, and the nature of the lesson study process itself. For the most part, we report these findings in the voices of the facilitators themselves, so that we can understand how they learned to conduct lesson study through their eyes.

[2]Some lesson study advocates recommend that participants volunteer to take part to ensure buy-in; in Asian countries it is common that an entire faculty or department participate in a lesson study group.

Table 1 Excerpt of data with open codes

Apprentice facilitators' discussion, transcribed	Open codes
Experienced facilitator: What did you see teachers working on during this session?	
Sharon: I don't understand the issue that they're finding their students talking. Is that an issue with the students or an issue with the teachers? My question here at the [facilitators'] table, Could an obstacle be the kind of lesson plans they have to turn in? We were persistent about getting them to interject open questions. Tina needs a clearer idea as to what her focus should be. Because what they're faced with is simple targets and not these deeper questions. That seems to be where they are. In order for us to understand to help them see, we need to understand what their obstacles are	Teacher buy-in Immediate- vs long-term External conflicts
Principal: Targets are investigative approach and vocabulary	District support Mismatch
Sharon: They saw content as interfering with practice	Mismatch Teacher buy-in Math content
Karl: Teachers kept hammering away at dependent and independent variable	Math content Aims
Mira: Joanie asked them about that. Corey said this is a foundational concept. [We need to] speak to aims and targets	Math content Aims Teachers' struggles
Sharon: Why is it important? It allows for the same picture	Math content
Mira: It is important to know that the number of customers on the price	Math content
Sharon: I was upset that I didn't suggest to Corey to give a piece of the lesson as a "do now"	Withholding/ stepping back
Mira: Should we have spent more time at the front identifying the content and the practice goals in a more careful way? One, it's a good practice. It might have helped us guide them a little more. What is it we want to get out of this lesson?	Math content Aims

5.1 Teachers as Learners

5.1.1 An Overview of the Lesson Study Cycles

In the three cycles of lesson study conducted by these novice facilitators, teachers chose to work in grade level and topic teams. They spent almost a full day determining the mathematical topics that they wanted to pursue and ended up dividing into three teams to pursue different mathematical topics for study; these included exponential growth and decay, systems of equations, and properties of similar triangles. All three teams chose to work the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers 2010), Standard for Mathematical Practice #1: "Make sense of

8:00 AM	• Introduction and gather • Review the process of Lesson Study • Determine two objectives ○ Overarching math practice to study (student discussions) ○ Mathematical concept (exponential growth)
8:30 AM 9:00 AM	• Gather resources on mathematical concept • Explore other textbooks on approaches of presenting mathematical concept • Determine positives and negatives of different approaches • Break
9:30 AM 10:00 AM 10:30 AM 11:00 AM	• Begin creating lesson plan ○ What is the focus of the lesson? ○ What do you want students to gain from the lesson? • Focus on the Launch ○ How will you engage students? ○ What materials will you use?
11:30 AM 12:00 PM	Lunch Break
12:30 PM 1:00 PM 1:30 PM 2:00 PM 2:30 PM	• Continue working on lesson plan ○ Write three questions that will provoke student thinking. ▪ Come up with ways students will answer each question ▪ How will you respond to their answers? • How will you assess if the students achieved the expectations of the mathematical concept? • How will you determine if the lesson achieved the overarching objective of the study?

Fig. 2 Facilitator's plan for a lesson study session

problems and persevere in solving them," and studied methods for developing students' capacity to hold rich mathematical discussions as a way of working on this practice. The middle school team was additionally interested in developing real-life contexts to support the textbook lessons they were using.

Half-day and full-day sessions were then spent studying the mathematical topics by examining textbook and supplementary curriculum materials on the topics chosen; often the teachers did the mathematics problems themselves and talked through how they completed them or what they anticipated would be difficult for students. Lesson plans were constructed based on adaptations of existing textbook materials, and these adaptations were especially aimed at supporting rich mathematical discussions of the textbook lessons chosen. Research lessons were taught when each team determined their plans were finished enough to be enacted; the number of days for planning varied by team. Figure 2 shows a facilitator's schedule for one lesson study day.

In Fig. 2, we can see how the facilitator tried to lead by structuring the day and the function of different time segments of the lesson study day while at the same time leaving substantive mathematical and pedagogical questions to teachers to determine. For example, at 1:00, the facilitator directs teachers this way: "Write three questions that will provoke student thinking. Come up with ways students will answer each question. How will you respond to their answers?"

5.1.2 Teacher Resistance

The facilitators frequently reference the issue of teacher "buy-in" versus resistance. Little of value can be done with participants who do not perceive the time spent in lesson study as worthwhile; because lesson study departs in significant ways from teachers' other experiences in professional development, resistance is a common concern and not easily managed or dismissed:

> Buy-in is a huge issue. I feel like I should meet [the teachers] where they are instead of trying to persuade them. I'm feeling better about managing that and breaking down some resistance that was there in the fall. I backed off a lot then because I could tell they weren't welcoming. (Karl, Wave 3)

The novice facilitators had few tools to manage resistance. As Grossman et al. (2007) have documented, teachers have a surprisingly thin understanding of resistance and a correspondingly paltry skillset for working with resistant students. This is in contrast with therapists who, during their training, rehearse a carefully defined set of strategies for handling client resistance.

5.1.3 Comfort and Discomfort

Related to resistance, these novice facilitators identified areas where teachers exhibited "comfort" and "discomfort." For the high school lesson study group, doing mathematics together was a comfortable exercise where teachers felt confident and authoritative. As a group, the teachers found doing mathematics for themselves to be enjoyable and non-threatening.

For the middle school teachers group, doing mathematics problems together was not uniformly comfortable for all participants. The middle school teachers include two special education teachers who work closely with the mathematics teachers, embedded in certain classes and providing external support for other classes. For this group, doing math was positive for some and off-putting for others:

> This session went a lot better, it went well. The teachers liked the math problem a lot. A couple of the special education teachers seemed less engaged by it but one new one was really into it in small groups. The "How many dots?" problem generated a lot of discussion. It was simple enough but you couldn't just count it easily. Finding different ways to count the number of dots—the problem was good, not sure what appealed to them: math, comfort zone. Maybe what I was taking as them not into it in the last session was actually their discomfort. (Louis, Wave 2)

> Exactly what we did this last cycle, especially the high school teachers, they see the value in it and willing to push it, new mathematics, into changing their teaching to look for ways to improve their teaching. The middle [school team], it was better than it was, but I'd be kind of surprised if they wanted to do it again. The bulk of the discomfort I mentioned early came from their lack of knowledge came from the mathematics. A lot of the reform ideas are very difficult for the special education kids. (Karl, Interview)

This last quote touches on a number of issues that appear frequently for these facilitators. Teachers' content knowledge in mathematics varies across grade levels, individuals, and certification areas. The middle school lesson study team for this school is comprised of mathematics and special education teachers, so the special education teachers' background in mathematics is sometimes less than that of their subject area counterparts. But in his comments above, Karl is also pointing to the fact that special education teachers in his team often lobbied for more skills-based goals for their students and were less committed to working on mathematical practices for complex, abstract, nonroutine problem-solving that drives much of the agenda in lesson study. The facilitators did not challenge this notion directly. This remains a delicate issue for facilitators and teachers alike to navigate and one that the field of mathematics education has perhaps not addressed sufficiently.

Doing mathematics together affords teachers the opportunity to enhance their own content knowledge, and it also can provide a shared experience of teaching and learning that the group can refer to in their ongoing work. Facilitators also used these opportunities to model mathematical teaching practices that they thought were relevant for the teachers in these groups. Doing mathematics together can also occlude the focus on student learning, and invite a retreat from work on teaching, especially for teachers who can easily get absorbed in the pleasure of doing mathematics for themselves without necessarily linking that mathematical work to what students need. Facilitators vacillate between those conflicting goals.

5.1.4 Teachers' Content Knowledge

Related to the doing of mathematics together as one of the lesson study activities, facilitators gave attention to teachers' content knowledge in a number of different ways. One approach was to bolster content knowledge for some teachers through the doing of math problems together; conversely the facilitators struggled with how much time to devote to this aspect of their sessions:

> Session 4 felt short in terms of their time expectations; discussion of dot figure [math problem] took up too much time, should have used time more towards the lesson planning. (Louis, Cycle 2)

5.1.5 Teachers' Goals

The novice facilitators frequently mention teachers' aims, both for the immediate lesson study lesson plans and in mathematics instruction in general:

> [I'd like] for teachers to think more about what they're doing in their lesson plans that will help their students to do better, rather than just doing problems to get through them. (Louis, Wave 3)

Facilitators expressed their desire for teachers to set their sights on student learning beyond immediate activities and textbook pages. Because so much of the lesson study sessions were spent in lesson planning, this issue was a constant for the facilitators:

> A big problem is the product versus the process. A lot of teachers feel like they just want the product—the lesson plan—and that the process is only to achieve that perfect lesson plan versus what is that lesson plan doing in their classroom? How is that affecting the students within the classroom? There's still that challenge—the more that we do that, the more the teachers see that it is about what we're doing with the lesson plan, not the lesson plan itself. (Louis, Interview)

5.2 Facilitators as Learners

5.2.1 Time

Facilitators address the number of issues related to time: time management, time shortage, scheduling, and use of time across all the academic year. It is evident that time is a primary issue for teachers and facilitators in many different contexts. In the quote below, we see a facilitator wrestling with time at the intersection of logistics and teacher engagement: is a whole day of lesson study work more productive or a half-day when students are released for the afternoon?

> In the first cycle I felt there was a lot of teacher pushback, then that was gone by the second cycle.
>
> Interviewer: What was that about?
>
> Some logistical stuff that we didn't have any control of or knowledge about. The second time around they were more willing. Timing issue was very big– after a full day they were not in great shape. Teachers are getting comfortable with us, trusting us. (Louis, Wave 3)

> Teachers feel like the bigger block of time lets them be more productive instead of start-stop. They like working a full day instead of half day when they've been teaching in the morning. Compared to their old [professional learning community format] it's better to have a longer block of time to work. A whole day keeps us focused. (Karl, Wave 3)

Scheduling time for teachers to work together in lesson study is not a trivial issue. US teachers spend more time with students than their international counterparts (Organisation for Economic Co-operation and Development 2014); this leaves scheduling time for professional development a challenge. Louis and Karl were working in a district that was unusually generous in granting time for professional development, but the schedule was not established in advance or a norm for the entire district; thus, for Louis and Karl, carving time out for teachers to attend lesson study sessions was an ongoing concern.

Time plays out in a number of different ways in lesson study. Facilitators noticed that teachers are, in earlier cycles, dismayed at the idea of spending hours planning a single lesson and then later on disappointed that they don't have more time to plan:

> Teachers felt the process was rushed, in their prebriefs they felt they didn't have enough time to plan so didn't want to be judged. (Louis, Wave 2)

5.2.2 Teachers Noticing the Use of Time

Facilitators found that teachers often noticed how time was used during the lessons they designed. These excerpts from a meeting with both facilitators following a lesson study cycle include multiple mentions of teachers' noticing of time use:

> The teachers didn't get through an ambitious worksheet and the creation of the manipulatives took a very long time and that took a long time, that took 13 minutes.
>
> Good stuff came from using the manipulatives. That chunk dominated the instructional time.
>
> The discussion of similarities took only 5 minutes. This led to interesting questions about how to handle this.
>
> The other group also ran out of time in their lesson. We should have seen this as a two-day activity.
>
> The algebra group planned something if they had too much time and if they didn't have enough time!
>
> They only got to the first exit ticket, and they didn't feel they had enough time for student discussion. (Karl and Louis, Wave 3)

5.2.3 Progress Over Time

As must be clear from the quotes across this article, the facilitators sense improvement from cycle to cycle: teachers seem increasingly engaged; the facilitators find their footing and trust themselves more. By the third cycle of lesson study, facilitators felt more comfortable presenting less, expressed more confidence in their ability to respond to teachers in the moment, and received positive evaluations from teachers and administrators:

> In the beginning I didn't, I was treating it like a direct instruction situation. In that first cycle with [the high school team] I found myself talking much more, I felt like I had to describe the process, going through this and seeing that it wasn't a zero in terms of what they learned but it wasn't where they needed to be. In the next cycle that first general info was out there already but by them doing it and me being in those smaller groups and just facilitate instead I thought that they learned more and they did as well. It was weird, because I haven't done those professional development things I just went to a place where it was more traditional and then later it became a bit more social, learning by doing, taking myself out of it, monitoring it. (Karl, Interview)

I feel more confident.

Interviewer: What's changed?

Going through another cycle, seeing that it gets better. It felt like a real cycle instead of showing them what a cycle looks like. Being more comfortable with the cycle itself, with the pieces of the cycle, the teachers have been through the cycle at least once. It's like teaching a class for the second time, you don't know where the trouble spots are going to be but then you start seeing that. I knew where important spots were in the process for me. Part of the discussion in the algebra group was about the script and they didn't have set questions that they wanted to pose. The second time around I had more of that script and I had questions I wanted to use to get them thinking. You know what will be trouble for them and you know how to direct them there. (Karl, Wave 3)

5.2.4 Supporting Teachers' Disclosive Acts

The facilitators noted particular turning points in the lesson study process that seemed to portend increased teacher engagement and positive disposition toward the work together. These included teachers' disclosive acts, especially around issues that are socially risky. Karl reported that one of the most respected high school teachers shared that he learned something new in the mathematics of the lesson plan; another high school teacher said that she had never fully understood the concept of geometric mean until the research lesson:

Pam said, "I never knew what geometric mean meant." The algebra team was working on exponential growth and decay. They created a worksheet for the students to have. Laura had never thought through how the formula came from doubling repeatedly. (Karl, Wave 3)

These disclosures signal a new level of safety, engagement, and potential for authentic community (Grossman et al. 2001), and they arose consistently during the post-lesson debriefing sessions. Teachers are traditionally supposed to be experts in subject matter knowledge, so sharing what one doesn't know as opposed to trumpeting what one does know is a delicate act. Teachers talked about the mathematics they didn't know well, and the pedagogical skills they needed to develop, and these admissions clear the way for serious work on instructional improvement. This demands an unusual form of expertise from facilitators. At times, facilitators have to themselves convey a sense of humility to make a safe space for teachers to express doubts and their lack of knowledge; at other times facilitators have to demonstrate deep knowledge of mathematics and of pedagogy to be credible leaders. Miller and Stiver (2015) have described how disclosure fosters a feeling of connectedness; in the context of teacher learning, it also opens possibilities for teachers to work on aspects of practice that need improvement. Perhaps because lesson study includes real-time co-located teaching experiences, disclosive acts are more likely to surface than in other forms of professional development.

This is related to our next category, withholding or stepping back.

5.2.5 Withholding or Stepping Back

A central question for facilitators surrounds how much to direct teachers in lesson study:

> I found it hard to find the balance between observing and interjecting. I thought it was extremely difficult, I really felt tension, didn't feel comfortable suggesting things. I have to work on that for myself. (Louis, Wave 1)

As they led subsequent cycles of lesson study, the facilitators continued to be concerned about when to be directive and when to allow teachers to set the direction more, but they found ways of managing this tension more productively over time. For example, Karl explains that he grew in this with time, and he developed analytic categories for determining when to steer the conversation:

> Once we were in smaller groups then it felt like I could contribute more naturally. I put as much as I could into it. I didn't totally take myself out and do their thing. I really felt like I put myself in the team as one of them, that felt much better and much more natural. They responded to that with much more enthusiasm than I'm telling you in this process. In terms of the mathematics I didn't hold anything back, that was pretty clear to hold out and correct, but in terms of how to do something mathematically I would let them go with it. (Karl, Interview)

> In lesson study, it is taking a step back—letting the teachers do the talking, letting the teachers figure things out on their own. With the professional developments that I've gone through, they don't really let us figure it out. They let us work through the worksheets and all that and then they would say "This is what you thought," and it wasn't really letting the teachers/us figure it out for themselves. For the lesson study, we lead the teachers but let them go where they need to go and it was always hard when you see that the teachers are going into the wrong direction, to tell them that they're doing that—you kind of let them go to where they want to/need to. We try to refocus them but having to tell them "We need to do it this way..." is difficult. (Louis, Interview)

This challenge is directly related to the fact that lesson study positions teachers as leading the agenda and driving much of the content. Facilitators struggle to define for themselves a form of leadership that is, on the one hand, credible and valued, and on the other, respectful of teachers' choices in directing the process.

5.2.6 Sources of Learning About Facilitation

When asked how they learned to conduct lesson study, the facilitators mentioned books and online resources about lesson study specifically; they did not originally volunteer the observation or apprenticeship experiences as sources to draw from nor did they mention anything at first about their own participation in professional development experiences prior to this. During the first cycle, Karl asked if there was a manual he could follow for conducting lesson study. By the third cycle, he noted that teachers had developed foci for observing research lessons, something that he wished to learn from a manual in the first cycle:

> I did read on how to facilitate, there were some good texts on how this should be done. Catherine Lewis' book, another one was a math facilitators' guide for lesson study. Those texts as well were suggesting that the real value is going to come from them, taking ownership, leading it. Learning from their mistakes, seeing what didn't work, was much more powerful. (Karl, Interview)

5.2.7 Learning from Experience

The quote above underscores the extent to which these facilitators learned from experience as much as they learned from manuals, observation, etc. For example, in planning for the second cycle of lesson study, the experienced facilitator suggested that Karl and Louis have to do less talking and frontal presentation of video than they had originally planned, but they were reluctant to leave their planned presentations. Their post-hoc analyses of those presentations, however, convinced both Louis and Karl that it was more productive to work with teachers in curriculum study and lesson planning, rather than talking about those things to them:

> How closely we worked with the two groups this time made me learn more, in it the whole time. As opposed to the first time where I was just telling people about it, not getting a chance to get my hands in. Doing more observation in the first cycle, it was a lot more of lecture, do it, lecture, do it, lecture, do it. This one we just got down to it, being hands on. (Louis, Wave 3)

> [The teachers] were better when they were doing the work and not just hearing about it. (Karl, Wave 2)

The notion of learning from facilitation experiences, then, operates here on two levels: the facilitators are convinced that teachers learn more from their hands-on, experiential work in lesson study, and we also see that the facilitators themselves draw conclusions from their own experiences in conducting lesson study sessions as opposed to being told by a more veteran facilitator.

5.2.8 Need for Resources

Facilitators voiced the need for locating appropriate resources for team members to study as part of the collaborative curriculum study work with their teams. Louis notes that this is especially true in attempting to move from facilitator-driven modes of delivery to more teacher-driven work, where the collaborative work in lesson planning depends on the presence of rich materials to consider:

> We didn't do a lot of the front-end lecturing. What I did start with was taking a look at those other textbooks. We need a lot more curriculum resources to share, draw from, compare. (Louis, Cycle 3)

These facilitators are currently based at a university; this means that they teach and are immersed in mathematics content—but it also means that they are somewhat removed from school resources, practices, and policy. On the other hand, both of these facilitators have extensive recent experience teaching middle and high school mathematics classes. Textbooks, standards, accountability structures, and the like are the currencies teachers trade in, and these two facilitators have to search out such resources and be familiar with them:

> We then showed a lesson plan from the Chicago lesson study group. It'd be great to have a lesson plan with a video of it. They were intimidated by it, it's pretty top-heavy with lots of rationale. Rationale was important though because they need to step back from getting through activities and looking instead at learning goals. They were a little taken aback, but that's what's we were trying for. The Rubicon Atlas lessons from Oakland County [provide materials for] lesson planning. (Karl, Cycle 2)

Similarly, Karl mentioned that he is now on the lookout for "good math problems" to use with the teachers in his lesson study group.

5.3 The Nature of the Lesson Study Process

5.3.1 Collaboration

Above we noted that the facilitators devoted the bulk of their time to collaborative lesson planning and found that the more they talked, the less the teachers were engaged. Both Karl and Louis said that time and again they learned to have teachers "doing" rather than "talking about" lesson study:

> Out of both sessions what stands out the most is what teachers seems to value most is collaborating with each other rather than us leading it and us spoon-feeding it. (Louis, Wave 2)

Despite the fact that teachers originally could not imagine spending days planning a single lesson, they came to value this part of lesson study a great deal. Many teachers commented in their post-session evaluations that creating lesson plans together and analyzing the research lesson together were the most valuable aspects of lesson study.

This collaborative planning also became a significant site for facilitator learning. To give one example, we look to the lesson plan created by a high school team working on systems of equations. The team chose to adapt an existing lesson, and make explicit some of the moves for expanding rich mathematical discussions, one of their stated goals as a team. To do so, the team showed a video to the class about the prices of tacos and drinks as planned in the original curriculum materials, but the team then spelled out some of the supports for sustaining mathematical discourse in the lesson:

> Teacher will direct students to brainstorm answers as to what they think one taco and one drink costs. Teachers will then write the data on the board about the possible answers the students come up with. Each group is required to get at least three answers and to share one of them. Students will graph the points on the board. (From High School Systems of Equations Research Lesson Plan, Wave 3)

> The lesson continues, with students considering a second scenario with different prices of tacos and drinks. Again, the team designed this to mirror the above plan: watch a video, talk about it, and then have students share possible prices for combinations of tacos and drinks.
>
> The team then designed questions for students to talk about in small groups:
>
> What do you notice about the data?
>
> What do you notice about the graphs? Compare and contrast. Points of intersection?
>
> How much does one taco and one drink cost? (From High School Systems of Equations Research Lesson Plan, Wave 3)

These were simple additions and edits to the lesson, but significant for the group of teachers who were beginning to work on incorporating more student-driven discussion in their mathematics lessons. Although Karl, the facilitator, had some concerns about the plan, he left it for teachers to discover those concerns for themselves during the research lesson and post-lesson analysis.

5.3.2 Centrality of the Research Lesson

Karl felt that he had not fully apprenticed to lesson study without having observed the research lesson that grew from the team's planning. He also felt that the research lesson experience propelled the rigor of lesson study for the high school teachers in a way that other phases of the lesson study cycle had not:

> Those debriefs felt really good, their observations were dead on, what was lacking [in the lesson plan] came out with the discussion. (Karl, Wave 3)

> In their lesson on systems of equations, high school teachers saw in the research lesson, and talked about in the post-lesson analysis session, the fact that students struggled to give words to describe the visual representations they had generated; that students had few tools for justifying mathematical assertions; that visualization might have helped students in advance of graphing and describing their thinking. (From Karl's notes, High School Lesson on Systems of Equations, Wave 3)

These were all concerns that Karl anticipated, but teachers saw them with their own eyes through the experience of the research lesson.

5.3.3 Administrator Support

Louis and Karl found that administrators were supportive of the lesson study work. Principals, an assistant superintendent, and the curriculum specialist attended sessions, sometimes for long periods, and often contributed substantive comments. The structure of lesson study makes this kind of cross-occupational participation possible, allowing for all to collaborate around the specifics of instruction:

> [The assistant superintendent] and [principal] came for parts of the research lesson. [The principal] was active with the high school group, added to their lesson plan. He also came for the hybrid group and did the same thing, gave suggestions, more management things than math-specific. [The assistant superintendent] participated too, has been to every session, participated in video discussions. (Karl, Wave 2)

> [The curriculum specialist] has been a support person there, very helpful. Her goals mesh with ours.... That lesson didn't have an explicit learning goal, and people then started to ask, What were we going for? She kept hammering away at that. She was hammering on a single immediate learning goal of the day/lesson. (Karl, Wave 3)

6 Discussion

This chapter began with the proposition that lesson study might provide a bridge between theory and practice in a way that could support teacher learning. Here we return to this question, examining the findings to see whether and how facilitators and teachers wove together theory and practice in the context of lesson study.

In an article by Arthur Bolster (1983), he noted that "most research, especially that emanating from top-ranked schools of education, construes teaching from a theoretical perspective that is incompatible with the perspective teachers must employ in thinking about their work" (p. 295). This same idea has been voiced repeatedly and over the decades, not only by researchers and journalists but by teachers as well. Lesson study has the potential to draw upon theory to infuse practice, but in a way that seems relevant to practice in the moment.

In the three cycles of lesson study analyzed in this study, teachers and facilitators explored a number of mathematical and pedagogical themes in great depth, and these explorations were in service of the research lessons that were being planned and eventually taught to students. One of the high school groups developed a lesson on comparing the altitudes of similar triangles, and in the planning phases of the lesson study cycle, the group worked on how students could use physical models to develop the ideas in the lesson, what questions the teacher would pose to elicit high-level conversation among students, and how the notion of geometric mean could come up in discussion. These are the kind of "theoretical" tropes that, in another professional development context, may feel forced or stunted or simply irrelevant for what teachers feel they need. Instead, the team observed the research lesson and saw, immediately, how their plans pan out. This constitutes, perhaps, the kind of linkages between theory and practice that for so long have evaded teacher education: teachers planned a lesson based on theories about the use of manipulatives, student discourse, and co-construction of mathematical ideas and then saw how they fared in practice. Lampert and Clark (1990) claim that "the way in which teachers acquire and use knowledge is contextual, interactive, and speculative" (1990, p. 21), and they suggest that knowledge about teaching is best acquired situated in practice. Lesson study can step into this breach, but through the kind of facilitation that allows teachers to shape the work, to choose the topics to research and design. Thus, understanding how this kind of facilitation can be cultivated is important.

A significant number of the recurring themes for the facilitators in learning to lead lesson study arise in the interstices of theory and practice: the use of time, emotional comfort and discomfort, stepping back to allow teachers to lead, addressing teachers' buy-in, and disclosure. The interactive, real-time nature of lesson study brings these issues to the fore; these issues may be less salient in forms of professional development that remain more in the realm of theoretical explorations of teaching.

Data from this study point to the various types of knowledge, skills, and dispositions that facilitators of lesson study must possess to implement this structure successfully. The facilitators have to be comfortable and conversant in many facets of school life, which is a world unto itself. They have to be familiar with the curriculum and understand the demands and affordances of classroom teaching. The facilitators had to interact with teachers in ways that led to growth, but also work with administrators and content specialists who are instrumental in scheduling and supporting the work of lesson study. The facilitators must have deep mathematical knowledge and wide-ranging pedagogical know-how, both for teaching teachers and teaching students; that is, they have to know how to work with adult learners but also know a great deal about how children learn. Unlike forms of professional development that are more didactic in nature, lesson study facilitators cannot predict and plan in advance for multiple aspects of the sessions, since teachers drive the agenda and co-design the learning opportunities in real time. Yet facilitators must come to sessions prepared on a number of levels: they bring resources such as textbooks, readings, math problems, and videos to stimulate and extend teachers' thinking and to respond to expressed interests in previous sessions. Facilitators in lesson study follow a kind of "emergent curriculum" (Edwards 1993), an idea well-known in early childhood education, for example, but less so in higher levels of education. "Emergent curriculum" is an approach to education where the curriculum planning grows from children's interests. In this way of teaching, teachers pay close attention to children's interests expressed in play and work and build lessons based on what they see children care about. Teachers may guide children toward materials and topics that they know are important, but the core idea is that the curriculum follows children's concerns. Lesson study is the analog in teacher learning. To be able to conduct this kind of professional development entails a kind of agility that is rare.

The list of attributes needed by a lesson study facilitator seems daunting when viewed this way. But both Louis and Karl were able to get up to speed as facilitators in very short order. By their third cycle of lesson study, their confidence as leaders had increased and teachers rated the sessions highly. We are currently investigating whether external measures of teacher learning correlate with teachers' and facilitators' self-reports regarding these lesson study sessions.

One is struck by the challenges and pushbacks that dogged the facilitators and teachers in learning to enact the lesson study process, and this is revealed in analyzing novices' articulation of what they are learning to contend with. These challenges speak to how profoundly countercultural lesson study is in the US context. Take, for example, the use of time in lesson study. Teacher learning in lesson study is incremental, ongoing, and unfolds over the span of one's career. This stands in stark contrast to the professional development model dominant in the United States, where the expectation is that teachers will learn a complex pedagogical skill in a session or two. Teaching for deep understanding involves endemic challenges that are not easily put to rest in short bursts of professional development, and lesson study gives form to the abstract notion that serious headway can be made on these challenges by working on them together over many years. In this sense, the instances of resistance in lesson study reveal flashpoints where lesson study actually constitutes a countercultural bulwark to existing modal practices for teacher learning in the United States. Lesson study includes the close reading of student activity during classroom lessons, methodical study of multiple curriculum materials, the joint construction of highly detailed lesson plans, and evidence-based reflections on research lessons that are observed in person. All these aspects of lesson study diverge from current forms of professional development in the United States that feature direct instruction dissemination of information from expert presenters, "professional learning community" meetings that take the form of data inspection (usually aggregate scores on students' standardized tests), and discussion of instruction that is distant from shared experiences in classrooms. Lesson study emphasizes incremental improvement of complex practice over time, fed by a focus on observed student learning in shared classroom experiences. Modal professional development emphasizes information transfer; in contrast, "lesson study focuses on the direct improvement of teaching in context" (Stigler and Hiebert 1999, p. 122).

This analysis of facilitation in this article is relevant for settings such as the United States where lesson study is not commonly practiced and where leaders have to jump-start a process that in other contexts is self-perpetuating. Still, the categories of novice facilitator efforts documented in this study point to areas that likely occupy the minds of many lesson study facilitators. As lesson study expands in the United States, these categories provide a useful road map for developing the thoughtful practice of this form of professional development for novices and veterans alike.

References

Bolster, A. S. (1983). Toward a more effective model of research on teaching. *Harvard Educational Review, 53*(3), 294–308.

Bourdieu, P. (1990). *The logic of practice* (R. Nice, Trans.). Stanford: Stanford University Press.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks: Sage.

Edwards, C. (1993). *The hundred languages of children: The Reggio Emilia approach to early childhood education*. Norwood: Ablex Publishing Corporation.

Feiman Nemser, S. (1983). Learning to teach. In L. Shulman & G. Sykes (Eds.), *Handbook of teaching and policy* (pp. 150–170). New York: Longman.

Gersten, R., Taylor, M. J., Keys, T. D., Rolfhus, E., & Newman-Gonchar, R. (2014). *Summary of research on the effectiveness of math professional development approaches*. Washington, DC: US Department of Education, Institute of Education Sciences, National Center for Educational Evaluation and Regional Assistance, Regional Educational Laboratory Southeast.

Grossman, P., Wineburg, S., & Woolworth, S. (2001). Toward a theory of teacher community. *The Teachers College Record, 103*(6), 942–1012.

Grossman, P., Compton, C., Shahan, E., Ronfeldt, M., Igra, D., & Shaing, J. (2007). Preparing practitioners to respond to resistance: A cross-professional view. *Teachers and Teaching: Theory and Practice, 13*(2), 109–123.

Gu, F., & Gu, G. (2016). Characterizing mathematics teaching research specialists' mentoring in the context of Chinese lesson study. *ZDM Mathematics Education, 48*(4), 441–454.

Hord, S. M., & Sommers, W. A. (Eds.). (2008). *Leading professional learning communities: Voices from research and practice*. Thousand Oaks: Corwin Press.

Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.). (2007). *Japanese lesson study in mathematics: Its impact, diversity and potential for educational improvement*. Singapore: World Scientific Publishing.

Lampert, M., & Clark, C. M. (1990). Expert knowledge and expert thinking in teaching: A response to Floden and Klinzig. *Educational Researcher, 19*(5), 21–21.

Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. American Educator, Winter, 12–17 & 50–52.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Lewis, C. C., Perry, R. R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.

Lewis, J. M., Fischman, D., Riggs, I., & Wasserman, K. (2013). Teachers learning in lesson study. *The Mathematics Enthusiast, 3*(10), 583–619.

Miller, J. B., & Stiver, I. (2015). *The healing connection: How women form relationships in therapy and in life*. Boston: Beacon Press.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards: Standards for mathematical process*. Washington, DC: Author.

Organisation for Economic Co-operation and Development. (2014). *Education at a glance 2014: OECD Indicators*. Paris: OECD Publishing. https://doi.org/10.1787/eag-2014-en.

Pickering, A. (1995). *The mangle of practice*. Chicago: University of Chicago.

Puchner, L. D., & Taylor, A. R. (2006). Lesson study, collaboration and teacher efficacy: Stories from two school-based math lesson study groups. *Teaching and Teacher Education, 22*(7), 922–934.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.

Yin, R. K. (2009). *Case study research: Design and methods*. Thousand Oaks: Sage.

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Characterizing Mathematics Teaching Research Specialists' Mentoring in the Context of Chinese Lesson Study

Feishi Gu and Lingyuan Gu

Contents

Abstract This study examined how mathematics teaching research specialists mentored practicing teachers during post-lesson debriefs of a lesson study in China. Based on a systematic, fine-grained analysis of 107 h of videotaped mentoring meetings of 20 groups of teachers and teaching research specialists from different elementary schools, this study revealed that the Chinese teaching research specialists paid a great deal of attention to *practical knowledge* which consists of setting student learning goals, designing instructional tasks, formative assessment of student learning, and improving instructional behaviors. Less attention was paid to mathematics content knowledge and pedagogical knowledge in general. Meanwhile, the teaching research specialists tended to comment on lessons in general and addressed anticipated problems based on their previous experience, with less attention to addressing issues raised by the teachers or engaging in dynamic dialogue with them. On the basis of the data analysis, a framework for analyzing mentoring activities emerged. The strengths and weaknesses of the teaching research

This chapter is built upon an article that formally appeared in ZDM: Gu, F., Gu, L. (2016). Charachterizing mathematics teaching research specialists' mentoring in the context of Chinese lesson study. ZDM, 48(4), 441–454.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_32

specialists' mentoring strategies are identified through the framework, and suggestions for the improvement of teaching research specialists' mentoring strategies are discussed.

Keywords Teaching research specialists (TRSs) · Teaching research system · Chinese lesson study · Mentoring strategies

1 Introduction

Teaching research specialists (TRSs, hereafter) in China, a form of didactician in mathematics (Jaworski and Huang 2014) who are employed by various practice-based education divisions and preliminarily work with practicing teachers, have played a crucial role in improving teachers' professional growth (Huang et al. 2014). The interactions between teachers and TRS have contributed to the improvement of classroom teaching in mathematics (Huang et al. 2010; Yang 2009), plausibly resulting in students' success in mathematics assessments such as the Programme for International Student Assessment (PISA) (OECD 2013; Tang 2014). Due to the outstanding performance of Shanghai students on PISA, a better understanding of mathematics teacher professional development in China in general, and the role of mathematics TRS in particular may contribute to the literature on the improvement of mathematics teaching and teacher education globally. However, few studies have been devoted to didacticians in mathematics regarding their working modes and strategies with practicing teachers (Barlow et al. 2014; Even 2014; Huang et al. 2014; Mudzimiri et al. 2014). This research was designed to explore how TRSs help mathematics teachers enhance their abilities in instruction by means of Chinese lesson study (Huang and Han 2015). Specifically, this study aims to address the following questions:

1. What types of knowledge do teaching research specialists (TRSs) pay attention to during post-lesson debriefing within the context of lesson study in mathematics in China?
2. What professional dynamics between teaching research specialists (TRSs) and teachers can be observed during post-lesson debriefing?

2 Background

In this section, the researchers will briefly introduce the professional development system of teaching research in China and describe the recent development of Chinese lesson study called *action education*. These ideas lay the foundation for understanding mathematics teacher professional development in China.

2.1 Multitiered Teaching Research System

A teaching research system, which has played a fundamental role in improving classroom teaching and implementing curriculum reforms (Huang et al. 2012; Yang 2009), has been in place in China since 1950. There are at least three levels in the teaching research system: school-based teaching research groups, district-level teaching research units, and teaching research institutes at the city or province level. This well-established, multitiered system through which teachers and TRS work together to design, deliver, and revise lessons to promote high-quality student learning has supported mathematics teachers in China in continuing professional development (Huang et al. 2010; Ma 1999). The functions of the teaching research system include four aspects: implementing national curriculum; mentoring teachers' activities in teaching; organizing public lessons including preparing, observing, and evaluating the lessons; and designing and administering various standardized examinations. As a result, TRSs have taken authoritative leadership without being publicly recognized in grades 1–12 education (Cong 2011). The effectiveness of the TRS' work depends on their capacity in transforming intended curricula into enacted curricula in classrooms (Zhao 2008). Although collaborative preparation, observation, and evaluation of lessons have been a routine practice in China, critical issues that relate to the mechanism have not been investigated appropriately (Chen 2006).

2.2 An Innovative Teacher Professional Development Model

One of the most influential teacher professional initiatives is *Action Education Project* (Gu and Wang 2003). To address the challenges facing the implementation of new curriculum reforms since 2001 (MOE 2001, 2011), building on literature and a survey on excellent teachers' learning from their practice, a group of researchers in China developed a model known as *Action Education* for in-service teacher training (Fig. 1).

Action Education is a form of school-based professional development that aims at updating ideas of teaching and learning, designing new learning situations, and improving classroom practice through *Keli*. *Keli*, a fundamental feature of *Action Education*, is a community-mediated process of developing an exemplary lesson, including the lesson planning, delivery, debrief, revision, and reteaching. The *Keli* group is a professional learning community consisting of teachers and TRS who are concerned with issues related to the implementation of the new curricula.

The model of *Action Education*, rooted in a long-standing tradition of teaching research groups (Wang 2013), emphasizes experts' professional guidance, peer coaching, enactment of classroom teaching, and reflection on teaching effect. It is a type of deliberate practice (Ericsson et al. 1993), which includes three phases of teaching practice (existing, designing, and implementing behaviors) and two phases of reflection on rehearsal teachings (Gu and Wang 2003; Huang and Bao 2006; Qingpu Research Institute 2012). The model focuses on the transformation of

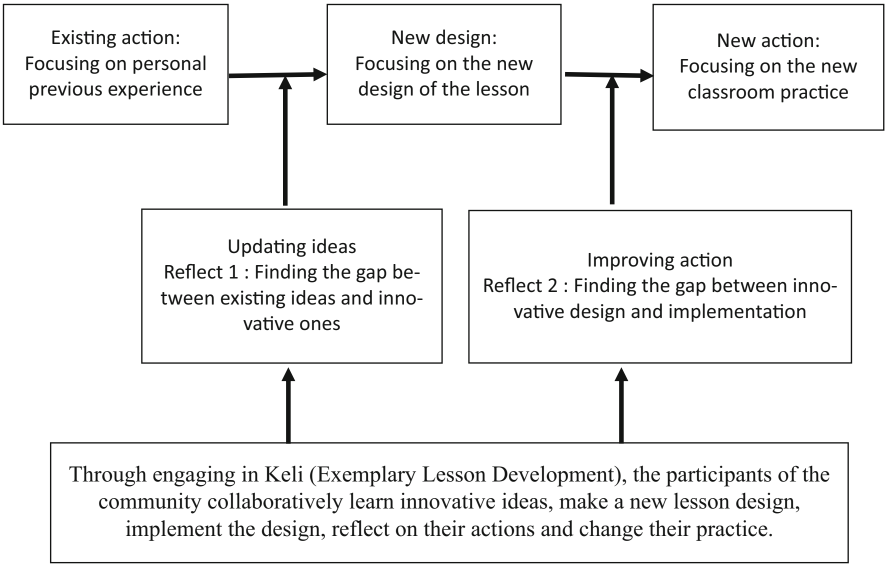


Fig. 1 *Action Education* model of teacher professional development

teaching practice from existing status via standard-based design to standard-based teaching practice through two phases of reflection. The first reflection focuses on identifying the differences between existing practice and innovative ideas. The second aims at identifying the gap between innovative designs and implementation of the designs. From 2003 to 2007, the model of *Action Education* was widely spread around the nation through a project to establish school-based teaching research institutions (Huang et al. 2010). This model reflects two essential Chinese ideas about teacher learning: one, the recognition of the *wisdom of action* and, two, the *unity of knowing and acting* (Ruth 2004). *Wisdom of action* refers to the practical knowledge that integrates subject knowledge with pedagogical knowledge in the context of purposefully improving action. The unity of knowing and acting is an epistemological theory put forward by an Ancient philosopher, Wang Shouren (B.C. 1472–1529). It emphasizes that knowing guides acting, while acting is the effect of knowing. Knowing and acting should go hand in hand. It criticizes the arbitrary acting without understanding or boundless thinking without acting ground. In addition, some scholars argue that the culture of teaching as a public activity and the infrastructure of multitiered teaching research systems support productive interactions between teachers and mathematics TRS (Stigler et al. 2012).

The model of *Action Education* has become a key component of teaching research activities recently (known as Chinese lesson study) (Yang and Ricks 2013), and it has further evolved into parallel lesson study (Huang and Han 2015). Some studies on the Chinese lesson study model have investigated its capacities for improving teaching (Huang and Bao 2006; Huang and Li 2009; Yang 2009) and developing teachers' competences in teaching (Huang et al. 2011; Huang and Han

2015), but it is largely unknown how the model works for improving teaching. In particular, the nature of the interaction between teachers and TRS during post-lesson debriefs in Chinese lesson study remains largely unexplored. To this end, this research examines the nature of TRS' work in mentoring practicing teachers. It can help us better understand mathematics teacher professional development in China and provide an alternative way for educators in other cultures to reflect on effective teacher professional development approaches.

3 Theoretical Background

In this section, relevant literature on teachers' knowledge for teaching dynamics between teachers and didacticians when working together will be reviewed, and based on Chinese practices, the research team will propose a framework for conceptualizing the post-lesson debriefing within the context of lesson study in China.

3.1 Mathematical Knowledge for Teaching

To describe the knowledge of effective teachers, Shulman (1986) proposed a framework, which included seven types of teacher knowledge. These are knowledge of content, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge (PCK), knowledge of students, knowledge of educational contexts, and knowledge of educational ends, purpose, and values. Among the seven types of knowledge, PCK is the most important component to differentiate the subject-based education experts from the pedagogical experts and expert teachers from novice teachers.

Building on Shulman's (1986) seminal work, Ball and her colleagues further developed a framework known as mathematical knowledge for teaching (MKT) (Ball et al. 2008; Hill et al. 2008), classifying MKT into six categories including common content knowledge, horizon content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum (Ball et al. 2008). According to this model, subject matter knowledge is divided into two categories: *common content knowledge* (CCK), which can be developed in anyone who has had school mathematics education, and *specialized content knowledge* (SCK), which is used mainly by teachers. Meanwhile, the model makes a distinction between two main categories within pedagogical content knowledge: *knowledge of content and students* (KCS) and *knowledge of content and teaching* (KCT). This model highlights the kind of mathematical content knowledge that is the specialty of teachers and recognizes that knowledge of mathematics for teaching is partially the product of content knowledge interacting with students in their learning processes and with teachers in their teaching practices.

In addition, the Teacher Education Development Study in Mathematics (TEDS-M) (Tatto et al. 2008) identified two components of MKT including mathematics content knowledge and mathematical pedagogical content knowledge. Specially, the mathematical pedagogical content knowledge is defined as mathematics curricular knowledge, knowledge of planning, and knowledge for enacting mathematics.

3.2 Three Categories of Teacher Knowledge

Although multiple ways have been used to describe mathematics teachers' knowledge for teaching (Ball et al. 2008; Tatto et al. 2008), all of them recognized that teachers need to have knowledge of mathematics, didactics, and pedagogy (Jaworski and Huang 2014). The relationships between these three components of knowledge and the practices with which they are associated have been emphasized. Rooted in educational practice in China, some scholars have developed a similar framework that consists of three components: subject matter knowledge, conditional knowledge, and *practical knowledge* (Lin et al. 1999). Subject matter knowledge refers to the knowledge of the subject that teachers need to teach; conditional knowledge refers to pedagogical and psychological knowledge in general that teachers need to know; and *practical knowledge* refers to knowledge that is situated in teaching practices, related to particular teaching behaviors when carrying out a purposeful action. This third type of knowledge has been accumulated and developed through teaching practice and experience and is essentially important for TRS. It is closely related to knowledge in practice and knowledge of practice (Cochran-Smith and Lytle 1999). According to Cochran-Smith and Lytle (1999), *Knowledge in practice* refers to knowledge constructed as the teaching community works toward the goal of improving practice at an individual and local level. It is "embedded in practice and in teachers' reflections on practice" (Cochran-Smith and Lytle 1999, p. 250). *Knowledge of practice* is generated "when teachers treat their own classrooms and schools as sites for intentional investigation at the same time as they treat the knowledge and theory produced by others as generative material for interrogation and interpretation" (Cochran-Smith and Lytle 1999, p. 250). The process of systematic, critical, and collaborative inquiry into teaching, learning, and schooling places teachers at the center of educational change and generates local knowledge with a democratic agenda. Thus, *practical knowledge* hereafter means both *knowledge in practice and knowledge of practice (*Cochran-Smith and Lytle 1999).

Based on a literature review and Chinese practice, the researchers adopted three categories of knowledge – mathematics knowledge, pedagogical knowledge, and practical knowledge (i.e., wisdom of action) – as the base of mathematics teachers' professional knowledge. With regard to practical knowledge, Simon's (1996) ideas provide insight into developing core elements of practical knowledge in the context of China. Simon argued that science focuses on the phenomena of nature, investigating what exactly things are, while professions that focus on human activities and products explore how these things should be improved. Professions such as architecture, business, law, medicine, and education require researchers to find more effective ways to achieve particular goals through the cycle of designing, implementing, and evaluating.

In the case of professional development for teachers, it is necessary to identify instructional goals and then continuously evaluate to what extent those goals have been achieved. Chinese TRSs work with practicing teachers toward this end by designing and implementing mathematics tasks and improving instructional practices. This features the core work of Chinese TRS in mathematics.

3.3 Dynamics Between Didacticians and Practicing Teachers

Different terminologies have been utilized to coin the mechanisms through which didacticians and practicing teachers work together to improve class teaching and develop teachers. For example, coaching and mentoring are commonly used in Western countries, while teaching research activity or Chinese lesson study has been used in China. For another example, lesson study is commonly used to feature this type of mechanism (Hart et al. 2011; Lewis and Hurd 2011). Several studies on coaching revealed knowledge structure of coaches (Sutton et al. 2011) and models of working with teachers and strategies of facilitating during post-lesson debriefing meeting (Barlow et al. 2014; Mudzimiri et al. 2014). Barlow et al. (2014) identified six coaching practices that are shared among three commonly used models of mathematics coaching. The six practices are focusing the coaching discussion on mathematics, attending to student learning, redirecting teachers' questions, providing positive feedback, using questioning to engage teachers in reflecting, and facilitating the coaching session. Mudzimiri et al. (2014) further found that during one-to-one sessions with teachers, coaches focus on seven aspects: mathematics content and pedagogy, reflection on instruction, student classroom management and curriculum, curriculum issues, resources and professional development, feedback, and goals-setting. Coaches also use a combination of directive and collaborative approaches when working with teachers and serve as external experts within a hierarchical context. Regarding post-lesson debriefing, the knowledgeable others in Japanese lesson study focus on (1) bringing new knowledge from research and the curriculum (subject content, new ideas), (2) showing the connection between the theory and the practice, and (3) helping others learn how to reflect on teaching and learning (Takahashi 2014). In summary, these studies suggest that didacticians during post-lesson debriefing should focus on bringing new knowledge from research and curriculum, discussing about mathematics and pedagogy, reflecting on teaching and learning, providing constructive feedback, attending to student learning, and facilitating productive discussions.

3.4 A Framework for This Current Study

This study aims to conceptualize the features of post-lesson debriefing within the context of Chinese lesson study by focusing on two dimensions: What types of teachers' knowledge TRS pay attention to and the dynamics between TRS and practicing teachers. The former includes three types of knowledge: mathematical knowledge, pedagogical knowledge, and practical knowledge. The latter includes the ways that TRSs give comments in terms of the nature from directive to collaborative conversation.

4 Methodology

This study has been carried out through three phases: initial, intermediate, and final codes. The researchers describe the process of conducting the research, data collection, and data analysis, respectively. A qualitative research paradigm (Corbin and Strauss 2008) has been utilized in this study. Based on observation of TRS' mentoring and videotaped mentoring meetings, the researchers analyzed the mentoring activities through systematic codes and categorization. Inductively, the researchers established an analytical framework to describe the ways of TRS' mentoring strategies. The process of developing these multitiered coding systems is described as follows.

4.1 Process of Data Collection and Developing Initial Categories

A research team consisting of two university mathematics educators and three TRSs was established in February 2011. The two mathematics educators were from the mathematics department at East China Normal University (ECNU) (the second author was the director of this project) and the School of Education at Hangzhou Normal University and three TRSs serving at elementary, middle, and high school levels. The team held multiple meetings and developed a research proposal including research questions and methodology, recruited participating teachers and TRS, and set an implementation schedule. Teaching research specialist participants were from provincial and district education departments, expert teacher training programs, and school-based teaching research groups. At this initial stage, the research team recruited two practicing teachers from a primary school in Yiwu, a city in Zhejiang, and four TRSs from three levels of the teaching research system. One practicing teacher was a beginning teacher, and the other was an experienced teacher. The coordinator of the teaching research specialist team was an exceptional teacher

(Li et al. 2011), an honorary title of the secondary teacher professional in China. Additionally, he was a teaching research specialist in Hangzhou city, the provincial capital of the province of Zhejiang. The other three TRSs in this team were from Yiwu, with a senior professional title, the highest professional rank in China. This teaching research specialist team set the schedule, including topics to be taught, who would teach and observe and who would be responsible for videotaping.

In the initial stage, the research team invited two practicing teachers and four TRSs to conduct a lesson study. In later stages of the study, more than 30 mathematics educators from various universities, more than 20 practicing teachers from urban and suburban schools, and more than 50 TRSs participated in the project. The practicing teachers were from two cities of Yiwu and Hangzhou in Zhejiang province, Suzhou city in Jiangsu province, and three districts of Qingpu, Songjiang, and Jingan in Shanghai. The TRSs were mainly from these areas, with several from other districts in these three provinces.

Next, the research team collected videos of lessons and post-lesson debriefing sessions, audio-recorded interviews with participating TRS, and collected field notes. In total, the research team collected 107 h of videotaped debriefing meetings from 20 teams of teachers and TRS. The videotaped debriefing meetings were transcribed verbatim. Additionally, the research team interviewed TRS about their perceptions of their mentoring activities, asking them to explain what they were doing, why they did so, and how they achieved their goals. The interviews were recorded and transcribed verbatim. Based on a careful reading of these documents using constant comparisons (Corbin and Strauss 2008), the research team developed and revised initial codes to establish a preliminary categorization of what TRSs normally do when mentoring teachers. The 12 main categories of TRS' activities are shown in Table 1. Most of the categories are self-explanatory; for example, the category "theories of education" refers to the TRS mentioning specific theories which could be used to guide teaching, such as saying, "We suggest you set learning goals based on Vygotsky's zone of proximal development."

4.2 Developing an Overall Analytical Framework

The initial study on mentoring activities began in Yiwu, in May 2011, and extended to other locations in East China including Qingpu, a district in Shanghai. The mentoring activities took place in an elementary experimental school in Yiwu. Two teachers were selected to teach "subtraction with two-digit numbers." These teachers developed the lesson through a typical three cycles of teaching/observing, debriefing, and reteaching. Four TRSs mentored the entire cycle of lesson study. Pre- and posttests were given to students immediately before and after each lesson, and all the lessons and debriefs were videotaped. After each lesson, interviews with the teachers and selected students were audio-recorded.

Table 1 Major types of work that TRS often do

Category	Explanation	Example
Advanced mathematics knowledge	Mathematics knowledge that teachers should teach in their classes including knowledge structures and relevant advanced knowledge	When we teach irrational numbers, we should know the real number theory
Curriculum standards	Interpretations of mathematics standards released by the ministry of education in China	When we focus on arithmetic, we should know the primary school curriculum standard requirement
Mathematics teaching materials	Understanding of the textbook used in class	The part of algorithm in irrational numbers in the textbook helps students to understand the irrational number much better
Students' situation	The students' previous learning experiences can impact their learning in class	Some students know how to add two-digit numbers, so you do not need to teach so slowly
Related information from magazines or literature	Knowledge of articles about mathematics teaching	When we are preparing the class on sectorial area, we suggest you look over an article about the computational formula for sectorial area from the last issue of *Shanghai Research on Education*
Theories of education	The theories can be used in informing mathematics education	We suggest you set learning goals based on Vygotsky's zone of proximal development
Mathematics thinking methods	How to understand mathematical theories and content, such as classification thinking and transformation methods	In double-digit addition, you can use the thinking methods of single-digit addition
Studies on different designs of the same content	Some other designs of teaching the same content which could provide alternatives for teaching	There is another way of teaching elliptic curves by using compression deformation
Assessment	Assessing the effects of teaching	Your questions in class are too simple and too many; this prevents students from thinking about mathematics knowledge in depth
Recommendation of mathematical worthwhile tasks	Suggestions of using some exercises or tasks that could provide a better result	We have tasks for you to use in class that could help students understand more clearly
Recommendation of topics for further research	Some research themes based on the lesson	You can do some interviews of students after class to make sure students understand what they were taught
Others resources	Some other resources can be used in class, such as those found on the Internet or from parents	There is a video clip of Caesar online, which can be used in teaching functions

Table 2 Explanation of these four types of mentoring strategies

Type of conversation	Explanation
General comments	Comments on what teachers should know and do in classroom teaching in general; nothing specifically related to what the teacher taught
Comments on anticipated problems	Comments on the anticipated problems that practicing teachers might encounter, providing relevant advice for the practicing teachers to deal with problems
Responses to teachers' questions	Responses to the questions that practicing teachers raised related to the issues in the class taught
Dialogues with teachers	Teaching research specialists and practicing teachers dynamically and dialogically discuss and share their own opinions about the problems that occurred in the class

The research team organized more than 20 TRSs (including those who mentored the practicing teachers) to watch the videotaped mentoring meetings and try to explain the mentoring activities in terms of their purposes, actions, intentions, and effects. The discussions about the nature of videotaped debriefing meetings were also videotaped. Based on the discussions, the 12 categories of activities identified in the previous stage were illustrated and enriched. These categories are shown in Table 1.

Through extensive weekly discussions over a 6-month period, the research team (two university mathematics educators and three TRSs) came to agreement on the classification of the three categories of *mathematical knowledge*, *pedagogical knowledge,* and *practical knowledge*. In particular, specific coding instructions were developed. Based on the purpose of the TRS' activities, the researchers classified the 12 types of activities into 3 types of knowledge. For example, the activity of related information from journals or other literature could belong to all categories of knowledge because it depends on the purpose of the action. If a teaching research specialist introduces an article on general pedagogical theory, then it is classified as pedagogical knowledge. If a teaching research specialist introduces an article about mathematical knowledge, then it is classified as mathematical knowledge.

Similarly, the researchers examined how the TRS mentored debriefing meetings through comparing different cases. Four types of mentoring strategies were identified: general comments, comments on anticipated problems, responses to teachers' questions, and dialogues with teachers. These four types of mentoring correspond to the nature of conversations between TRS and teachers from more authoritative to more dialogic (see Table 2).

Thus, a two-dimensional overall framework for analyzing debriefing meetings was established: type of knowledge (mathematical knowledge, pedagogical knowledge, and practical knowledge) and type of mentoring strategies (comments in general, comments on anticipated problems, responses to teachers' questions, and dialogues with teachers).

4.3 *Refinement of Practical Knowledge*

After establishment of the overall analytical framework, further data analysis was focused on identifying the “core elements” of practical knowledge and logical relationships of these elements and refining the overall analytical framework for characterizing TRS’ mentoring activities.

To refine the overall analytical framework for TRS’ mentoring, the research team examined core elements, practical knowledge, and critical relationships of these elements. More than 50 TRSs from three provinces in East China were involved in the process of development of subcategories of practical knowledge. Finally, over the span of a year, through joint effort, the research team finally identified and described the core elements of practical knowledge and the relationship among these elements as shown in Fig. 2. The sections that follow provide the details of conceptualization of practical knowledge.

4.3.1 The Model for Conceptualizing Practical Knowledge

Using the videotaped mentoring meetings, the research team further analyzed core elements of practical knowledge. It was found that there were two major types of activities: instructional design before the class and improving instructional behavior after class. If teachers and TRS organize a subsequent lesson on the same topic, then instructional design and instruction behavior improvement could be mutually informed. The instructional improvement of the previous lesson would be the basis for a new instructional design of the next lesson and further improvement of that lesson. Thus, instructional design is normally based on experience of previous designs. However, the evaluation of instructional implementation is based solely on in-class observation. Teaching research specialists in China usually organize public lessons

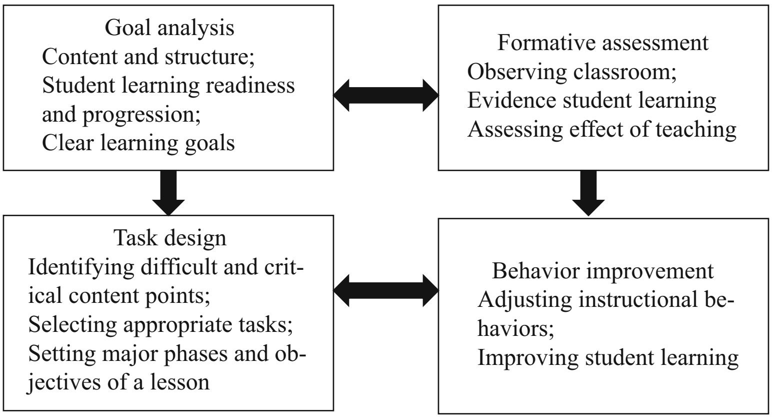


Fig. 2 An analytical model for conceptualizing practical knowledge

and are involved in the whole process from instructional design to implementation and evaluation of the instruction (Han and Paine 2010). They help practicing teachers improve their professional skills in preparing, observing, and assessing public lessons. They provide concrete examples of how to design a lesson and how to evaluate a lesson in order to improve practicing teachers’ practical knowledge.

A four-core component model including goal analysis, task designing, formative assessment, and behavior improvement has been developed and illustrated (Gu and Zhu 2012), shown in Fig. 2. *Goal analysis* refers to setting clear student learning goals based on a deep analysis of mathematical knowledge structure and student learning readiness. *Tasks design* means designing mathematical tasks in terms of learning goals and learning progression (identification of difficult, important, and critical content points) (see Huang et al. this volume). Formative assessment refers to collecting evidence of student learning, assessing student learning results, and providing feedback for improving teaching. *Behavior improvement* means improving instructional behaviors by adjusting classroom behaviors to meet student learning situations and improve student learning.

Furthermore, the relationships among these four components were illustrated as follows. The *task design* should be built on *goal analysis*, and *informative assessment* should provide evidence for *behavior improvement*. Additionally, the mutually beneficial relationship between *task design* and *behavior improvement* is highlighted. Meanwhile, the *goal analysis* and *formative assessment* should be used to inform each other. According to the model, *goal analysis* is the first step of lesson planning. Based on the goal analysis, teachers can design mathematical tasks to help students implement the instructional tasks and achieve the learning goal. During a class or after class, teachers can get evidence of student learning and compare teaching effect and learning goals in order to assess the effectiveness of design and implementation of the lesson. This formative assessment could provide evidence for identifying the differences between design and implementation of a lesson and further for teachers’ improvement of their instructional behaviors in class. These efforts provide opportunities for teachers to further improve their designs and implementations of the next lesson.

4.3.2 An Analytical Tool for Conceptualizing Mentoring Activity

Incorporating knowledge structure with mentoring strategies, the research team further developed an analytical tool for conceptualizing mentoring activity as shown in Fig. 3. This tool was then adopted to systematically analyze all of the 107 hours of mentoring data.

According to this framework, each element of evidence during debriefing sessions is associated with two dimensions of knowledge and mentoring strategies. For example, the cell of the row of *task design* and the *column of comments on anticipated problem* presents the conversation between TRS and teachers about

Mentoring content		Mentoring mode			
		General comments	Comments on anticipated problems	Responses to teachers' questions	Dialogue
Mathematical knowledge					
Pedagogical knowledge					
Practical knowledge	Goal analysis				
	Task design				
	Formative assessment				
	Behavior improvement				

Fig. 3 An analytical tool for conceptualizing mentoring activity

how to design a task to address a problem which is anticipated by the teaching research specialist. Figure 4 presents a representative episode of coding mentoring knowledge.

In the column of *mentoring content* in Fig. 4, codes 1, 2, 3, 4, 5, and 6 represent subject matter knowledge, pedagogical knowledge, goal analysis, task analysis, formative assessment, and behavior improvement, respectively. R1, R2, R3, and R4 stand for teaching research specialist numbers one through four, and T1 stands for teacher number one.

The research team first invited mathematics educators (researchers) and TRS to watch the videos of the mentoring meetings and read the transcripts and then to discuss how to code the videos. Based on extensive discussions, the research team found that the six codes of knowledge could be applied appropriately to coding the mentoring videos. Then, two researchers in the team were asked to code 10 h of video independently. The inter-rater agreement for this sample was about 78%, with disagreements discussed and resolved among the researchers. Afterward, one researcher coded the remaining videos, and the other researcher checked the results with disagreements discussed and resolved.

5 Results

In alignment with the research questions, the results are reported in two sections. First, the findings on the major aspects that teaching research focused on during debriefing are presented. Secondly, the results on dynamics between TRS and practicing teachers are described.

Series number	Mentoring content	Speaker	Transcript	Starting time	Time (Sec.)
58	1	R2	There is a minor problem when you said there is 12 on the unit place.	17.50	40
59	1	R2	In base-ten system, the biggest digit is 9. It is impossible to have 12 on the unit place. But you can say there 12 unit 1 (or 1 ten and 2 unit)		
60	1	R1	We hope teachers get a very clear understanding of mathematics concepts and structure when preparing lessons	18.30	50
61	1	R1	What is the mathematical property? The properties regarding base-ten system are introduced in the courses in undergraduate education.		
62	1	R1	If you always teach the students in low grades (1 and 2) , you may forget the mathematics term.		
63	1	R4	It is for practical use rather than mathematical rigor		
64	1	R1	Yes, I feel that we should not use too many mathematical terms		
65	1	R1	But we, as teachers, should have a clear understanding of mathematical concepts and terminologies		
66	1	R1	Let's look at the textbook. Beijing Normal University publisher publishes it. From the entire system [of the textbook], what contents introduced in previous chapters are closely related to the knowledge to be taught in this class?	19.20	10
67	3	R1	When preparing the class, have you considered students' readiness, namely, whether students understand prerequisite knowledge or not?	19.30	
68	3	T1	Mental calculation of subtraction with numbers less than 20 should be fluent. Subtraction with two-digit numbers without regrouping should be fluent also.		
69	1	R1	What are the relationships between the knowledge you are going to teach and subtraction with two-digit numbers?	20.00	20
70	1	R4	Which knowledge points are closely related?		
71	1	T1	Subtraction with numbers less than 20.		
72	1	R1	Absolutely.		
73	3	R1	When did the students learn the knowledge?	20.20	10
74	3	T1	Last semester.		
75	3	R1	What do you think about whether they still remember it or not?	20.30	
76	3	T1	They should remember it. (Laughing)		
77	3	R1	How about low-achieving students?		

Fig. 4 A coded segment of debriefing meeting

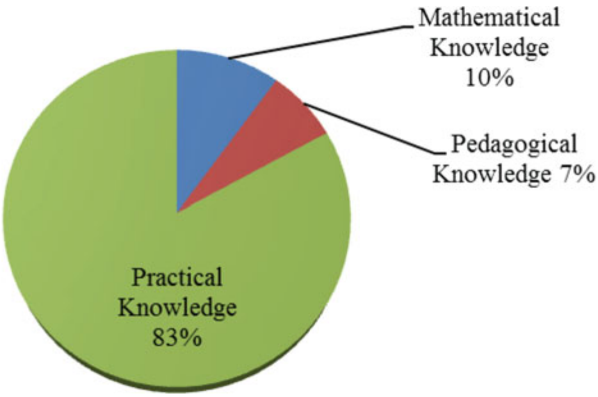
5.1 Major Aspects Teaching Research Specialists Pay Attention to During Post-lesson Debriefing

First, the percentages of time distributed in *mathematical*, *pedagogical*, *and practical knowledge* are reported. Then, within *practical knowledge*, the time devoted to *goal analysis*, *task design, formative assessment*, and *behavior improvement* is identified. As a result, the advantages and disadvantages of Chinese TRS' mentoring are synthesized.

5.1.1 Major Focused Knowledge When Mentoring Practicing Teachers

The percentages of the three categories of knowledge entailed during mentoring activities are displayed in Fig. 5. The figure indicates that TRS spent about 80% of

Fig. 5 Distribution of three types of knowledge when mentoring teachers



the mentoring time on discussing *practical knowledge*. Only around one-tenth of time was focused on discussing mathematical knowledge, and less than one-tenth of duration was devoted to discussing general pedagogical knowledge. Based on these findings, it appears that Chinese TRSs mainly focus on discussing *practical knowledge* when mentoring practicing teachers during post-lesson debriefing.

As Fig. 5 indicated, Chinese TRS did not tend to discuss theories and knowledge at a general and abstract level. Yet, they tried to help teachers understand mathematical knowledge and pedagogical knowledge through analyzing concrete instructional cases that embrace mathematical and pedagogical ideas (which belong to practical knowledge). A great deal of vivid mentoring cases demonstrated the strengths of this type of mentoring (see Gu and Gu in press). Based on the criteria of recruiting TRS such as a strong background in mathematics, excellent skills in teaching mathematics, strong capacities in conducting research and writing articles, and strong abilities in personal communication and administration (Huang et al. 2012), Chinese TRSs have a varied and rich professional training background including three key experiences. First, they have experienced the development process as novice teachers, then as experienced teachers, and finally as expert teachers. Thus, they can draw on their own experiences of professional development as teachers to mentor others at a range of professional phases. Second, TRSs have excellent leadership abilities and skills in teaching and in management. Finally, they are good at mentoring teachers using concrete lesson cases and instructional events. Thus, TRS can provide not only pertinent points or key ideas in context, but they can also communicate with teachers using teachers' language and considering teachers' viewpoints. Overall, Chinese TRSs possess strong practical knowledge in addition to solid subject knowledge (Huang et al. 2012; Ma 1999). These strengths of TRSs play unique roles in mentoring teachers.

5.1.2 Core Components of Practicing Knowledge

Four subcategories of practical knowledge were identified, and the occurrence of each subcategory is shown in Fig. 6.

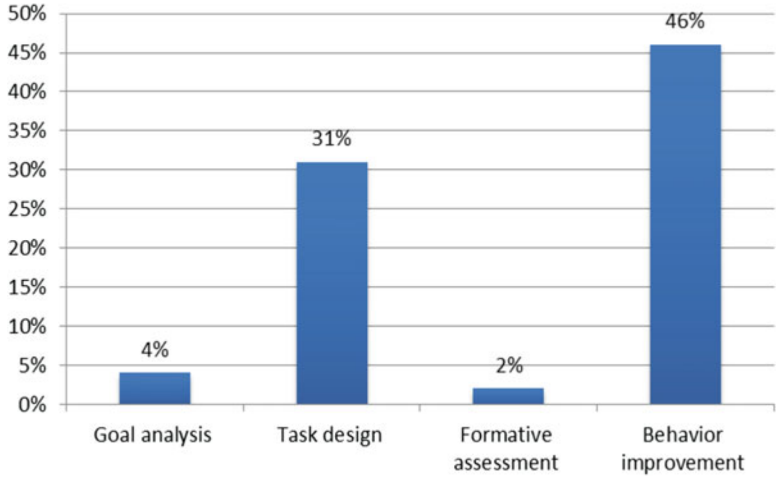


Fig. 6 Distribution of each type of practical knowledge

Figure 6 shows that *behavior improvement* occupied around half of the time (46%) devoted to practical knowledge and task design occupied around one-third of the time duration. A very small part of the duration (less than 5%) was spent on discussing *goal analysis* and *formative assessment*. This revealed that Chinese TRSs pay greater attention to discussing about task design, implementation, and improvement with less attention to goal analysis and formative assessment.

Examining transcripts of video and interviews reveals that TRSs value practical usefulness in three ways. First, facilitation is presented through the point of view of practicing teachers. Second, the design and improvement of research lessons within the lesson study community create vivid experiences and artifacts. This kind of practical learning experience supports effective teachers' learning through lesson case studies. Finally, the whole process of mentoring presents a process reflection that connects reflective incidents into a cyclic progression that refines ideas through experimental action (Ricks 2011; Schön 1983), which is different from *incident reflection*. It emphasizes the cycle of practical strategies, enactments, and effects.

5.2 *Major Features of Dynamics Between Teaching Research Specialists and Teachers During Post-lesson Debriefing*

The distribution of different types of TRSs' mentoring strategies is shown in Fig. 7. It shows that mentoring is dominated by responses to anticipated problems (50%) and general comments (34%). TRSs pay little attention to the issues that arose in observed lessons, so answers to teachers' questions and concerns occupied only 13% of mentoring activities. Dialogues and discussions about issue that arose in class occupied only 3% of the mentoring.

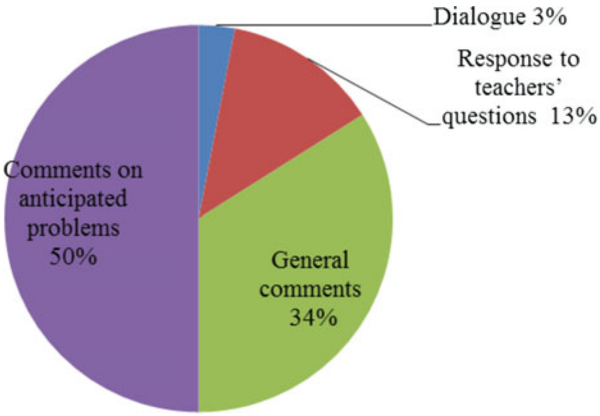


Fig. 7 The occupation of each kind of TRSs' mentoring

Thus, TRSs usually mentored teachers through general comments based on their own experiences, with less attention paid to problems that were generated in particular lessons. There were few dialogic conversations with teachers. To address this disparity, the research team suggested that TRSs should develop a dynamic and dialogic conversational norm between themselves and teachers. To facilitate this shift, TRSs are suggested to develop (1) a profound understanding of mathematical knowledge and mathematical thinking methods and a deep understanding of modern pedagogical theories and perspectives as well; (2) understandings of instructional practice, particularly in the context of current curriculum reforms; and (3) understanding what really happens in current classrooms, what new concerns or issues occur, and how students learn differently at different grades.

6 Conclusions and Discussion

6.1 Summary

Based on systematic and fine-grained analysis of more than 100 h of videotaped TRS' mentoring meetings, the research team developed a framework for analyzing and guiding mentoring activity and identified the strengths and weaknesses of TRS' mentoring strategies. The analytical framework includes two dimensions. One is mentoring content including *mathematical knowledge*, *pedagogical knowledge*, *and practical knowledge* (goal analysis, task design, informative assessment, and behavior improvement). The other is mentoring approaches including comments in general, responses to anticipated problems, responses to teachers' questions, and dialogue. The data analysis from the analytical tool indicates that the TRS spent a majority of their time on mentoring *practical knowledge*. That means that Chinese TRSs in mathematics pay greater attention to developing teachers' practical knowledge, rather than mathematical content or pedagogical knowledge in general.

Furthermore, four-core components of practical knowledge were identified: analyzing learning goals, designing instructional tasks, formative assessment of student learning, and improvement of instructional behaviors. Chinese TRS spent one-third of their mentoring time on instructional task design and around half of the time on improving instructional behaviors, but they rarely spent time on setting learning goals and formative assessment of student learning.

It may be effective for TRS to focus on practical knowledge when mentoring teachers because discussing the issues related to or emerged in lessons could benefit teacher learning (Bakkenes et al. 2010). Due to their strong background in mathematics and expertise in teaching and teaching research (Huang et al. 2012), TRSs are comfortable and confident in doing so.

However, the weaknesses of Chinese mathematics TRS' mentoring are obvious. Their discussions about task design and improvement of instructional behaviors were mainly based on their past experiences rather than evidence of student learning in class (using formative assessment to setting appropriate learning goals and adjusting instructional behaviors). This type of mentoring may lead teachers to focus on their teaching strategies and skills rather than soliciting student thinking and learning as recommended in the new curriculum (MOE 2011).

Moreover, the conversations between TRS and teachers were found to be monologs rather than dialogic in nature. The TRS paid much attention to what they knew and what they anticipated, rather than what students learned in class and what teachers concerned about their teaching. This type of conversation may inhibit the building of a dynamic and productive professional learning community of teachers and TRS.

The identified features of mentoring activities and strategies during post-lesson debriefs and the proposed framework for analyzing these activities may provide a pathway to developing a more effective and productive professional learning community of teachers and TRS and enhance lesson study in China.

6.2 Discussion

Although there have been substantial studies on the teaching development of practicing teachers over the past several decades, "the literature offers only limited empirical research-based information about the practice of didacticians" (Even 2014, p. 329). This study, grounded in data recording what TRSs are doing in a non-English-speaking county, makes a unique contribution to the field of study on practice-based teacher educators in mathematics (Adler et al. 2005). The following three points deserve to be highlighted.

First, there are a great deal of studies on pedagogical content knowledge (PCK) (Depaepe et al. 2013; Ball et al. 2008), which include knowledge of students' learning, knowledge of instructional strategies, knowledge of mathematics tasks, and knowledge of educational ends, curriculum, and context. This study explored and unpacked practical knowledge as identifying student learning goals, designing instructional task, formative assessment of student learning, and improving instructional behaviors within the context of lesson study in China. Based on the categorization of teacher knowledge in this study, practical knowledge (a combination of *knowledge in practice and knowledge of practice*) (Cochran-Smith and Lytle, 1999) is closely related to PCK, but it is built on content and pedagogical knowledge beyond a combination of them. Practical knowledge must be developed through lesson case-based situations and job-embedded practice over time. This line of exploration provides an alternative for mathematics educators to reflect on how to refine and develop PCK effectively.

Secondly, although several frameworks for analyzing and guiding post-lesson debriefing have been proposed and utilized (Lewis and Hurd 2011; Mudzimiri et al. 2014; Takahashi 2014), this study probably is the unique one that developed and validated the framework based on a great amount of videotaped debriefing meetings (more than 100 h of videos) quantitatively. Similar to other models (Lewis and Hurd 2011; Mudzimiri et al. 2014), this framework focuses on mathematical content and pedagogy, student learning, formative assessment, and analyzing and improving instruction. Yet, the framework further provides a refined analytical tool for analyzing practical knowledge including *goal analysis*, *task design, formative assessment*, and *improvement of instruction*. In addition, beyond the simple description of a combination of directive and collaborative approaches when working with teachers (Mudzimiri et al. 2014), the framework provides a continuum from monolog (comment in general) to dialogue.

Thirdly, the complexity of teacher educators' knowledge has been recognized and addressed by researchers (e.g., Jaworski 2008a; Zaslavsky 2008). According to Jaworski (2008a), there are three types of knowledge: teacher educators' knowledge (region A), which includes theories, research, and systems, and teacher's knowledge (region C) consisting of students and schools. Region B, the intersection of A and C, represents the knowledge shared by teacher educators and teachers and has elements of mathematics, didactics, and pedagogy. Although these forms of knowledge may be different according to the practices to which they relate, they provide a basis for communication between the two groups. Jaworski (2008b) further developed a model of co-learning between teachers and teacher educators in promoting classroom inquiry where "co-learning inquiry emphasizes the collaborative potential of joint activity and formation of communities of inquiry" (p.177). Although the current practice of TRS' mentoring in China tended to be one way from TRS to teachers, the framework developed in this study provides a feasible tool for monitoring mentoring practices in the formation of such a community of inquiry. It is crucial to set learning goals and discussions of instructional improvement based on student learning and formative assessment during the mentoring process. Interestingly, Huang et al. (this issue) reported an exploratory study on how a theory-driven lesson study approach could successfully shift teachers' attention to student learning and understanding.

6.3 Conclusion

This study develops a framework for characterizing teaching research specialists' mentoring during post-lesson debriefs in the context of Chinese lesson study. Based on the framework, the strengths and weaknesses of teaching research specialists' mentoring activities have been identified. Moreover, suggestions for improvement of teaching research specialists' mentoring strategies are discussed. This study makes a unique contribution to understanding of the practices of didacticians and provides an alternative way for didacticians in other cultures to reflect on effective ways of working with practicing teachers.

Acknowledgments We thank all of the teachers, teaching research specialists, and researchers who were involved in this joint effort for their dedicated time and expertise. We very much appreciate Dr. Rongjin Huang and Dr. Teresa Schmidt from Middle Tennessee State University for their help in editing and proofreading the manuscript.

References

Adler, J., Ball, D. L., Krainer, K., Lin, F. L., & Novotna, J. (2005). Mirror images of an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics, 60*, 359–381.

Bakkenes, I., Vermunt, J. D., & Wubbels, T. (2010). Teacher learning in the context of educational innovations: Learning activities and learning outcomes of experienced teachers. *Learning and Instruction, 20*, 533–548.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407.

Barlow, A. T., Burroughs, E. A., Harmon, S. E., Sutton, J. T., & Yopp, D. A. (2014). Assessing views of coaching via a video-based tool. *ZDM-The International Journal on Mathematics Education, 46*, 227–238.

Chen, G. (2006). Collaborative preparation of instruction. *Journal of Education in China, 9*, 40–41.

Cochran-Smith, M., & Lytle, S. (1999). Relationships of knowledge and practice: Teacher learning in communities. *Review of Research in Education, 24*, 249–305.

Cong, L. (2011). *Silent authority – teaching research system of Chinese basic education*. Beijing: Beijing Normal University Press.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research* (3rd ed.). Los Angeles: Sage.

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concepts have pervaded mathematics educational research. *Teaching and Teacher Education, 34*, 12–25.

Ericsson, K. A., Krampe, R., & Tesch-Romer, C. (1993). The role of deliberate practice in the acquisition of expert performance. *Psychological Review, 100*, 363–406.

Even, R. (2014). Challenges associated with the professional development of didacticians. *ZDM-The International Journal on Mathematics Education, 46*, 329–333.

Gu, F., & Gu, L. (in press). Further studies on teaching research specialists. *Global Education*.

Gu, L., & Wang, J. (2003). Teacher development in education action. *Curriculum Textbook and Pedagogy, 1*, 9–26.

Gu, L., & Zhu, L. (2012). Preliminary studies on teaching research specialists. *Global Education, 8*, 31–37.

Han, X., & Paine, L. (2010). Teaching mathematics as deliberate practice through public lessons. *The Elementary School Journal, 110*, 519–541.

Hart, L. C., Alston, A., & Murata, A. (Eds.). (2011). *Lesson study research and practice in mathematics education*. New York: Springer.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*, 372–400.

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing *keli*. *Journal of Mathematics Teacher Education, 9*, 279–298.

Huang, R., & Li, Y. (2009). Pursuing excellence in mathematics classroom instruction through exemplary lesson development in China: A case study. *ZDM Mathematics Education, 41*, 297–309.

Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*, 100–117.

Huang, R., Peng, S., Wang, L., & Li, Y. (2010). Secondary mathematics teacher professional development in China. In F. K. S. Leung & Y. Li (Eds.), *Reforms and issues in school mathematics in East Asia* (pp. 129–152). Rotterdam: Sense.

Huang, R., Li, Y., Zhang, J., & Li, X. (2011). Improving teachers' expertise in mathematics instruction through exemplary lesson development. *ZDM Mathematics Education, 43*, 805–817.

Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese lesson study. *ZDM – The International Journal on Mathematics Education, 46*, 239–251.

Huang, R., Xu, S., Su, H., Tang, B., & Strayer, J. (2012, July). Teaching researchers in China: Hybrid functions of researching, supervising and consulting. Paper presented at *12th International Conference on Mathematics Education*, July 8–15, 2012 Seoul, Korea.

Jaworski, B. (2008a). Development of the mathematics teacher educator and its relation to teaching development. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 335–361). Rotterdam: Sense.

Jaworski, B. (2008b). Building and sustaining inquiry communities in mathematics teaching development: Teachers and didacticians in collaboration. In K. Krainer & T. Wood (Eds.), *International handbook of mathematics teacher education: Participants in mathematics teacher education: Individuals, teams, communities and networks* (Vol. 3, pp. 309–330). Rotterdam: Sense.

Jaworski, B., & Huang, R. (2014). Teachers and didacticians: Key stakeholders in the processes of developing mathematics teaching. *ZDM -The International Journal on Mathematics Education, 46*, 173–188.

Lewis, C., & Hurd, J. (2011). *Lesson study step by step: How teacher learning communities improve instruction*. Portsmouth: Heinneman.

Li, Y., Huang, R., Bao, J., & Fan, Y. (2011). Facilitating mathematics teachers' professional development through ranking and promotion practices in the Chinese mainland. In N. Bednarz, D. Fiorentini, & R. Huang (Eds.), *International approaches to professional development of mathematics teachers* (pp. 72–87). Ottawa: Ottawa University Press.

Lin, C., Xin, T., & Shen, J. (1999). Observation of normal education reform based on the structure of teacher knowledge. *Teacher Education Research, 66*, 12–17.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Erlbaum.

Ministry of Education China [MOE]. (2001). *Mathematics curriculum standards (experimental version)*. Beijing: Beijing Normal University Press.

Ministry of Education China [MOE]. (2011). *Mathematics curriculum standards*. Beijing: Beijing Normal University Press.

Mudzimiri, R., Burroughs, E. A., Luebeck, J., Sutton, J., & Yopp, D. (2014). A look inside mathematics coaching: Roles, content, and dynamics. *Education Policy Analysis Archives, 22* (53), 1–32.

OECD. (2013). *PISA 2012 results: What students know and can do: Student performance in mathematics, reading and science* (Vol. 1). Paris: OECD.
Qingpu Research Institute. (2012). Teachers "action education". *Curriculum, Teaching Material and Method, 3*, 3–12.
Ricks, T. E. (2011). Process reflection during Japanese lesson study experiences by prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education, 14*(4), 251–267.
Ruth, H. (2004). Keynote speech. The 49th World Assembly World Assembly Conference. Hong Kong in July 13–17 2004.
Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4–31.
Simon, H. (1996). *The sciences of the artificial*. Cambridge MA: The MIT Press.
Stigler, J., Thompson, B., & Ji, X. (2012). This book speaks to us. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 223–231). New York: Routledge.
Sutton, J. T., Burroughs, E. A., & Yopp, D. A. (2011). Coaching knowledge: Domains and definitions. *Journal of Mathematics Education Leadership, 13*(2), 12–20.
Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 83–97.
Tang, Y. (2014). Rethinking the Chinese education: A perspective of life-oriented education. *Shanghai Research on Education, 325*, 15–19.
Tatto, M. T., Schwille, J., Senk, S., Ingvarson, L., Peck, R., & Rowley, G. (2008). *Teacher education and development study in mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics. Conceptual framework*. East Lansing: Teacher Education and Development Inter- national Study Center, College of Education, Michigan State University.
Wang, J. (2013). *Mathematics education in China: Tradition and reality*. Singapore: Galeasia Cengage Learning.
Yang, Y. (2009). How a Chinese teacher improved classroom teaching in teaching research group: A case study on Pythagoras theorem teaching in Shanghai. *ZDM— The International Journal on Mathematics Education, 41*, 279–296.
Yang, Y., & Ricks, T. E. (2013). Chinese lesson study: Developing classroom instruction through collaborations in school-based teaching research group activities. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 51–65). New York: Routledge.
Zaslavsky. (2008). Meeting the challenges of mathematics teacher education through design and use of tasks that facilitate teacher learning. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 93–114). Rotterdam: Sense.
Zhao, C. (2008). *Efficacious teaching research: The introduction of teaching research in Chinese basic education*. Shanghai: Shanghai Education Press.

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Designing and Adapting Tasks in Lesson Planning: A Critical Process of Lesson Study

Toshiakira Fujii

Contents

Abstract There is no doubt that a lesson plan is a necessary product of Lesson Study. However, the collaborative work among teachers that goes into creating that lesson plan is largely underappreciated by non-Japanese adopters of Lesson Study, possibly because the effort involved is invisible to outsiders, with our attention going to its most visible part, the live research lesson. This paper makes visible the process of lesson planning and the role and function of the lesson plan in Lesson Study, based on case studies conducted by project IMPULS at Tokyo Gakugei University in three Japanese schools. The paper identifies key features of the planning process in Lesson Study, including its focus on task design and the flow of the research lesson, and offers suggestions for educators seeking to improve Lesson Study outside Japan.

Keywords Lesson study · Lesson planning · Task design · Structured problem-solving

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This chapter is built upon an article that formally appeared in ZDM: Fujii, T. (2016). Designing and adapting tasks in lesson planning: a critical process of Lesson Study, ZDM, 48(4), 411–423.

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_33

1 Introduction

While the history of Lesson Study in Japan spans more than a century (Makinae 2010), for Japanese educators, Lesson Study is like air, part of everyday school life. This situation possibly explains why Lesson Study is regarded as being under-theorized (e.g. Elliott 2012). Educators outside Japan however, having had to learn about Lesson Study less naturally, may sometimes lose some important aspects of Lesson Study.

Lesson Study came to the attention of educators outside of Japan primarily through the publication of *The Teaching Gap* (Stigler and Hiebert 1999), which described findings from the TIMSS video study focusing on the eighth grade mathematics lessons in the USA, Germany, and Japan. Chapter "Teaching for Robust Understanding with Lesson Study" in particular, titled "Japan's Approach to the Improvement of Classroom Teaching", which is based on Yoshida's (1999) doctoral dissertation, now available in book form (Fernandez and Yoshida 2004), provoked enormous interest, not only in Lesson Study but also in the typical structure of Japanese mathematics lessons. Independently, some educators such as Lewis also noticed the significance of Japanese Lesson Study (Lewis and Tsuchida 1998).

Since then many mathematics teachers and teacher educators around the world have been involved in Lesson Study, and many books and research papers have been written on various aspects of Lesson Study (Department for Children, Schools and Families 2008; Doig and Groves, 2011; Hart et al. 2011; Lewis 2002; Lewis et al. 2009; Ono and Ferreira 2010; White and Lim 2008). However, some aspects of Lesson Study that may be taken for granted by Japanese teachers seem not to be well understood outside Japan.

This paper aims to clarify the role and function of lesson planning in the Lesson Study process, based on case studies conducted in three schools in Tokyo.

2 Background

2.1 The Lesson Study Process

Lesson Study is an approach to teacher professional development that differs sharply from the professional development practices common in other countries. Liptak (cited in Lewis 2002, p. 12) contrasted Lesson Study with traditional professional development as practised in the USA, as shown in Table 1.

Lesson Study begins with *a question*, not with *an answer* prepared by someone else. Identifying this question, which becomes the research theme for Lesson Study, is the first step in the process (see Fig. 1).

The research theme is developed through consideration of the reality of students' current state vis-à-vis educational or long-term goals for their learning and development.

Table 1 Contrasting views of professional development

Traditional professional development	Lesson Study
Begins with answer	Begins with question
Driven by outside “expert”	Driven by participants
Communication flow: trainer to teachers	Communication flow: among teachers
Hierarchical relations between trainer and teachers	Reciprocal relations among learners
Research informs practice	Practice is research

Liptak, cited in Lewis (2002, p. 12)

Fig. 1 The process of Lesson Study (Fujii 2014a, p. 113)

The second step of Lesson Study is to develop a plan to address the research theme through lessons. This means making an instructional plan for a selected unit and a detailed plan for one of the lessons in that unit in which the planning team puts forth their ideas about how to address the research theme while teaching specific academic content. That lesson is called the research lesson.

The third and fourth steps in Fig. 1, conducting the research lesson and having a detailed discussion about the lesson, occur in 1 day—a big event day for the school. Typically, it is done in a half day; one class of students stays for the research lesson while the other classes are dismissed so that every teacher can come to observe the research lesson (even the school nurse and school nutritionist usually attend). At the end of the post-lesson discussion, usually there will be final comments lasting 30 min or more by a “knowledgeable other” from outside the school, who has been invited for this purpose.

The fifth step is to reflect on the process and consolidate and carry forward the learnings from it. Teachers will usually write their reflections and publish records of Lesson Study activities in the school bulletin.

Because they are the most visible aspects of Lesson Study, some people think of the research lesson and post-lesson discussion as the most important parts of Lesson Study or even use "Lesson Study" to refer to the research lesson alone. However, these are just two of the five components of Lesson Study.

The Lesson Study cycle, with its five steps as illustrated in Fig. 1, contrasts with similar diagrams in other publications that have four steps (e.g. Lewis 2002; Lewis and Hurd 2011). These five steps, while overlapping with the four steps in the other diagrams, more accurately portray the reality of Japanese teachers' Lesson Study activity by having a closer correspondence between the titles of the steps and the activities undertaken by teachers.

Borrowing from Lewis' (2002) and Lewis and Hurd's (2011) descriptions, each step can be summarized as follows:

1. *Goal setting* Consider long-term goals for student learning and development. Identify gaps between these long-term goals and current reality. Formulate the research theme.
2. *Lesson planning* Collaboratively plan a "research lesson" designed to address the goals. Prepare a "lesson proposal"—a document that describes the research theme, content goals, connections between the current content and related content from former and later grades, rationale for the chosen approach, a detailed plan for the research lesson, anticipated student thinking, data collection, and more.
3. *Research lesson* One team member teaches the research lesson while the other members of the planning team, staff members from across the school, and, usually, an outside knowledgeable other observe and collect data.
4. *Post-lesson discussion* In a formal lesson colloquium, observers share data from the lesson to illuminate student learning, disciplinary content, lesson and unit design, and broader issues in teaching and learning.
5. *Reflection* Document the cycle to consolidate and carry forward learnings, as well as new questions for the next cycle of Lesson Study. Write a report or bulletin that includes the original research lesson proposal, student data from the research lesson, and reflections on what was learned.

There are three types of Lesson Study in Japan: school-based, district-based, and national-level Lesson Study. According to Takahashi (2006), participants' motivations or interests are different in these types of Lesson Study, but the cycle itself is basically the same. The difference is in the range, or scope, of students to be considered: school-based Lesson Study is concerned with students in the school; district-based Lesson Study is concerned with students in the district; and national-level Lesson Study is concerned with the reality of students across the country and has a research theme with a nationwide view. Lesson Study is sometimes introduced as an open lesson by a veteran teacher "jumping in" to another teacher's classroom (Takahashi 2013, p. 84). A "jumping in" lesson is just a demonstration unless the veteran teacher has a clear goal for the lesson as in Step 1, and proposes a new idea or content to be teachable, or he or she wants to demonstrate students' potential to be greater than ordinary teachers believe, so that he, or she, plans the lesson carefully as in Step 2. This kind of Lesson Study exists in Japan, and in this case the collaboration among teachers is not a critical part of Lesson Study. In any case, each step in the Lesson Study cycle is closely related to the others, with the third and fourth steps particularly related to the first and second.

In school-based Lesson Study, which is the focus of this paper, the typical Lesson Study cycle begins at the end of an academic year—i.e. in February or March in Japan—when the faculty decides upon a research theme for the next school year, which starts in April. Several research lessons are scheduled from, say, May to November. Each research lesson and its post-lesson discussion occupy only 1 day, but the teachers reflect on what they learned at the research lessons and usually write a booklet or long summary report by the end of school year.

While the importance of a lesson plan as a product of Lesson Study is certainly understood, compared to the research lesson, of which there are many public examples, the collaborative work of Japanese teachers in creating a lesson plan is generally mysterious, because it is difficult to observe. According to Lee and Takahashi (2011), "Lesson plans are central resources for these teachers in that they constantly refer to, problematize and act on them during the entire cycle of the [Lesson Study] procedure" (p. 210).

Japanese teachers spend a lot of energy and time crafting a lesson plan. Although the details vary from school to school and even from teacher to teacher, Lewis (2002, pp. 127–130) notes that a typical template for a lesson plan for a research lesson in Japan consists of the following:

1. Name of the unit
2. Unit objectives
3. Research theme
4. Current characteristics of students
5. Learning plan for the unit, which includes connections to standards and to prior and subsequent learning, the sequence of lessons in the unit and the tasks for each lesson, and explanation of unit "flow"
6. Plan for the research lesson
7. Background information and data collection forms for observers (e.g. a seating chart)

The Japanese term for the document created for a research lesson is *gakushushido-an* (学習指導案), which is usually translated as "lesson plan". In this paper we will use that common translation, although we prefer the phrase "lesson proposal", because the document is much larger and broader in scope than what is usually meant by "lesson plan". Also the word "plan" may imply a fixed script, but in Japanese Lesson Study, the teacher is expected to use his or her judgement if students respond in unanticipated ways. As Lee and Takahashi (2011) argue, researchers have taken for granted that using lesson plans, no matter how well devised, always involves judgement and interpretation, as teachers and their students face the contingencies of the lesson in the classroom. Their empirical study, in the context of Lesson Study, provided analytic descriptions of the interactive processes through which lesson plans are realized, leading to the conclusion that "classroom teachers use lesson plans as communicative resources to identify problems, specify assumptions about their teaching, and act on the evolving contingency of classroom interaction" (p. 209). However, Lee and Takahashi (2011) did not describe details of planning the lesson, including how teachers adapted or designed the task for the lesson, or how many hours they spent on planning.

In the context of Lesson Study, Lewis et al. (2009) focused on one US lesson study group, of six teachers from five different schools, that conducted a research lesson in a 2-week summer workshop. This is an experimental situation, which is different from the Japanese traditional school-based Lesson Study setting. However it is worth considering in terms of the lesson-planning activity. They documented that the group spent a total of 6 h planning the lesson: "select research lesson, do task and share solutions, anticipate student thinking, write instructional plan using template" (Lewis et al. 2009, p. 290). However they have not offered descriptions of how they designed or adapted the task for the lesson.

On the other hand, Fernandez and Yoshida (2004) described in detail the process of planning lessons in the context of Lesson Study. This ethnographic study, focused on a local elementary school in Hiroshima, vividly shows Japanese teachers' activities. However, the Lesson Study described there has the rather unique feature in that, following the research lesson being taught by a young inexperienced teacher, observed by the whole school and discussed by only the lower grade group of teachers and the principal, the lesson was revised by these teachers and then re-taught by a veteran teacher, with the whole school and an outside advisor observing the lesson and taking part in the post-lesson discussion. The notion of re-teaching is extremely problematic and sensitive. In fact, the need to revise and re-teach a lesson is one of the misconceptions identified in foreign countries implementing Japanese Lesson Study (Fujii 2014b). Whether re-teaching exists or not in the Lesson Study process affects the nature of the planning and the discussion of the lesson.

2.2 Structured Problem-Solving

The structure of Japanese mathematics lessons is often regarded as unique by non-Japanese eyes, with researchers from outside Japan having noted patterns in Japanese mathematics lessons. For example, Becker et al. (1990) identified eight components in a typical Japanese mathematics lesson, while Stigler and Hiebert (1999) identified five components and labelled these lessons as structured problem-solving. But their points of view are those of observers, while Japanese teachers usually do not think about the structure of their lessons in the same way. For instance, the first component of Stigler and Hiebert (1999), reviewing the previous lesson, is not an important activity from a Japanese teacher's point of view. Instead Japanese teachers typically consider a mathematics lesson as problem-solving in terms of the four phases shown in Table 2 (see, e.g. Shimizu 1999).

This type of lesson imposes certain demands on how to interpret the lesson plan. Phase 1, presenting the problem, means helping students understand the context of

Table 2 The four phases of a problem-solving lesson in mathematics

1. Presenting the problem for the day (5–10 min)
2. Problem solving by the students (10–20 min)
3. Comparing and discussing (*neriage*) (10–20 min)
4. Summing up by the teacher (*matome*) (5 min)

the problem or task and what it will mean to solve the task—but it specifically excludes any exposition by the teacher about how to solve the task. Instead, students are expected to work independently on the task for 10–20 min (phase2). Therefore teachers need to discuss the appropriateness of the task described in the lesson plan. The third phase, called *neriage* in Japanese, assumes that students will arrive at different solution methods and focusses on a comparison and discussion of those different solution methods. Therefore teachers need to discuss the plausibility of the anticipated student solutions listed in the lesson plan. In the fourth phase, *matome*, the teacher may say something about which strategy may be the most sophisticated and why, but it should go beyond that to include comments by the teacher concerning the mathematical and educational values of the task and lesson (Fujii et al. 1998). Therefore teachers need to discuss the reasonableness of the *matome* by the teacher as foreshadowed in the lesson plan. For a lesson to work in this way, the task should be understandable by the students with minimal teacher intervention; it should be solvable by at least some students (but not too quickly), and it should lend itself to multiple strategies.

This paper focusses on the second, planning step in the Lesson Study cycle, and aims to illuminate the nature of the collaborative work among teachers, based on three case studies where re-teaching was not part of the Lesson Study process, with particular emphasis on planning for these four phases of the research lessons.

3 Methodology

This research took place in three local public elementary schools in Tokyo, which will be referred to as schools M, S, and T. These schools were participating in the *International Math-teacher Professionalization Using Lesson Study* project (IMPULS), a recently established project funded by the Ministry of Education, Culture, Sports, Science and Technology of Japan, located at Tokyo Gakugei University, Tokyo. The purpose of this project is twofold. First, as an international centre of Lesson Study in mathematics, Tokyo Gakugei University and its network of laboratory schools help teacher professionals learn about authentic Japanese Lesson Study and thereby prepare them to create Lesson Study systems in their own countries for long-term, independent, educational improvement in mathematics teaching. Second, the project conducts research projects examining the mechanism of Japanese Lesson Study in order to maximize its impact on schools in Japan.

Although several research lessons were scheduled for each year, this study focusses on just one research lesson at each of these schools and the planning meetings for those research lessons—that is, just one Lesson Study cycle in each school.

The author observed each lesson-planning meeting and took field notes. In addition, each lesson-planning meeting was video-recorded and later transcribed; and all lesson plans and revised versions were collected and analysed with respect to their evolution.

This paper provides a descriptive analysis of the planning process undertaken by these groups of teachers in preparation for the research lessons. In a similar vein to the research carried out by Lee and Takahashi (2011), discourse-in-interaction analysis (Sacks et al. 1974) was used to examine "the methods and procedures by which participants carry out ordinary tasks of classroom teaching and collaboration among teachers" (Lee and Takahashi 2011, p. 215). The analysis began with unmotivated looking (Sacks 1992) during the observations of the planning meetings in order to identify key discussions that eventually led to consensus regarding the lesson plans.

Through this overview of the lesson-planning processes, the author came to realize that the discussions were based on the flow of the lesson. In particular, it seemed that teachers could imagine or visualize clearly what would happen at the research lesson through reading the lesson plan. Therefore it was clear that this study could focus on analysing the planning of the flow of the research lesson.

Based on the flow of Japanese problem-solving lessons, thematic content analysis (see, e.g. Braun and Clarke 2006; Fereday and Muir-Cochrane 2006) was carried out on transcripts of the lesson-planning discussions. Using the framework of the four phases of problem-solving lessons (Table 2), participants' comments were coded with appropriate keywords to track their views of the lessons. These comments were examined with respect to the role of the lesson plan and planning meetings, in order to make visible an important part of Lesson Study—namely, the planning process.

The following section is organized according to the main results obtained through the inductive process of examining the trajectory of revising lesson plans, transcribed records of planning meetings, research lesson, post-lesson discussion, and field notes.

4 Results

The results of this study are presented in three sections. First, we report on the lesson-planning meetings overall—e.g. the number of meetings and participants and the duration of meetings. Second, we examine the major component of the meetings. Finally, we identify major concerns at the meetings, such as the appropriateness of the task for the lesson, anticipated student solutions, and how to organize the comparison and discussion phase in the lesson.

4.1 The Lesson-Planning Process Overall

The dates of the research lessons held at schools M, S, and T together with the dates of the planning meetings are shown in Table 3. The planning meetings began between 4 and 6 weeks before the research lessons. Two schools, M and S, had four planning meetings, and school T had just two meetings.

It should be noted that there was no rehearsal or trial implementation of a tentative lesson plan between planning meetings. It should be noted also that this schedule fails to reveal the amount of time that the teachers may have spent thinking about their research lesson beforehand, since the grade, unit, and lesson may have been selected at the end of the previous academic year in March.

Table 4 shows the number of participants at each of the planning meetings.

In the case of school M, the regular members of planning meetings were the leader of the research steering committee, who also chaired the meeting and was the lead teacher for mathematics in the school; three Grade 3 teachers, one of whom taught the research lesson; and four Grade 4 teachers—a total of eight participants. The first planning meeting, held in the principal's office, was rather informal, since the knowledgeable other, who had given a talk at a research lesson that day, joined the meeting, together with the principal of the school. Besides these two participants, three Grade 3 teachers and two Grade 4 teachers attended. But at later meetings, in the school conference room, the only participants were the eight regular members.

At school S, which is a small school with only one class at each grade, the first meeting included five regular members: two classroom teachers for Grades 5 and 6, the music teacher, the art teacher, and the teacher for mathematics. The Grade 6 teacher was the leader of the school research steering committee and taught the research lesson. In Tokyo, in the case of mathematics only, if a school wants to divide classes into two or three groups for teaching mathematics, in order to help cater for individual differences, the school gets an extra teacher—in this case this teacher. The music teacher and the art teacher were teaching Grade 5 and 6 students; therefore the regular members were the upper year level team. At the second

Table 3 Dates of research lessons and planning meetings

	Meeting 1	Meeting 2	Meeting 3	Meeting 4	Research lesson
School M	15 May	22 May	13 June	21 June	1 July
School S	30 May	6 June	11 June	19 June	3 July
School T	28 May	4 June			26 June

Table 4 Number of participants at the planning meetings

	Meeting 1	Meeting 2	Meeting 3	Meeting 4
School M	7	8	8	8
School S	5	6	7	4
School T	8	8		

meeting, the principal joined them; at the third meeting, the knowledgeable other also joined; but the fourth meeting included only the Grade 1 teacher and the Grade 6 teacher, the music teacher, and a special needs teacher—these four constituted the school research steering committee. The venue was always a meeting room in the school.

At school T, regular members were the leader of the research steering committee, three Grade 3 teachers and three Grade 4 teachers, and the principal of the school, who attended the planning meetings—so the total number was 8. One of the Grade 4 teachers taught the research lesson. There were only two meetings, both of which were held in the principal's office.

Schools M, S, and T each organized a research steering committee. According to Takahashi and McDougal (2014), a research steering committee in Japan consists of representatives of each grade level and, in the case of the Lesson Study focusing on mathematics, the lead teacher for mathematics. In addition, representatives of special subject teams, such as music, science, and home economics, may join. The research steering committee leads the school's efforts and maintains the cohesion of ideas across the grades. Takahashi and McDougal (2014, p. 16) list roles and functions of research steering committees as follows (parenthesis added by author):

- Developing a master plan for the school research
- Scheduling and leading monthly meetings to find strategies to address the school's research theme based on the ideas of the teachers
- Publishing a monthly (not always the case) internal newsletter to record the findings from each research lesson
- Planning, editing, and publishing the school research reports, including those for the research open house
- Arranging for knowledgeable others to present lectures, teach demonstration lessons (not always the case), and give final comments at research lessons

As shown in Table 5, the duration of the planning meeting ranged from a minimum of 30 min to a maximum of 128 min.

The chairperson of the school research steering committee led most of the meetings at schools M, S, and T. As these schools were conducting Lesson Study focusing on mathematics, the lead teacher for mathematics tended to also be in charge of the school research steering committee. Besides regular members from the school, the knowledgeable other, who had given comments on a research lesson that day, attended the first meeting at school M and the third meeting at school S. Involving a knowledgeable other in this way is common; after a research lesson

Table 5 Duration of planning meetings (min)

	Meeting 1	Meeting 2	Meeting 3	Meeting 4	Total time
School M	30	128	114	81	353
School S	60	60	30	54	204
School T	78	87			165

and discussion ends, the team responsible for the next research lesson will meet with the knowledgeable other for further discussion and to get advice for their lesson.

As both of the 30-min meetings were with the knowledgeable other, these could be regarded as atypical. The average duration was 72 min, with the average duration excluding the 30-min meetings being 83 min. One reason that may account for the differences in the duration of planning meetings between schools could be that the principals of schools S and T attended and participated actively in these meetings, with teachers in both schools appearing to have great confidence in them. When teachers asked, these principals gave suggestions to help break deadlocks. As a result, the duration could become shorter. In the case of school M, some of the regular members of planning meetings were young and inexperienced. Therefore, the leader of the research steering committee, who was also the lead teacher for mathematics, sometimes needed to explain the position of the lesson in the scope and sequence of the Japanese course of study and the mathematical value of the task for use in the lesson. These factors may have had an effect on the longer duration of the meetings.

4.2 Major Components and Structure of the Planning Meetings

The first meetings held at school M and S were unusual in that the teachers discussed ideas about the research lesson in depth without a written lesson plan. At all other meetings, the discussion was based on a draft lesson plan, which had been written, either with or without the support of colleagues, by the teacher who would be teaching the lesson. Furthermore, the flow of the planning meetings followed the flow of the lesson plan. Other issues, such as the logistics of the research lesson or post-lesson discussion, were not discussed.

The format of the first draft of the lesson plan for schools M, S, and T was basically the same as Lewis' (2002) template as described earlier in this paper. In the case of school M, component 5 in Lewis' (2002) template, *Learning plan for the unit*, was missing at the beginning but was added later.

Among the seven components in Lewis' (2002) template, component 6, *Plan for the research lesson*—which we will refer to here as *Planning the flow of the research lesson* in order to distinguish it from the overall lesson plan—is the most prominent in terms of both quantity and quality. At school T, the draft lesson plan had already been prepared for the first meeting, written by the teacher who was to teach the research lesson. The items discussed at the first meeting were as follows:

1. The research theme of the school (8 min)
2. The goal of the unit; evaluation points for learning (i. interest, eagerness, and attitude; ii. mathematical way of thinking; iii. mathematical skills; and iv. knowledge and understanding); the relationship between this unit and the

research theme; other units related to this unit; students' reality; and teachers' vision of ideal students (6 min)

3. What ideal students would look like (11 min)
4. Unit and lesson plans (2 min)
5. Planning the flow of the research lesson (51 min)

These items were exactly the items written in the draft lesson plan.

In both meetings at school T, discussion relating to planning the flow of the research lesson occupied the majority of the time: 51 min (65%) of the first meeting as shown above and 87 min (78%) of the second meeting. At school S, the first meeting was held without a written lesson plan. At this stage, teachers had not yet decided exactly which unit or content to teach for the research lesson and how. From the second meeting onwards, the teachers' discussions were based on the lesson plan drafted by the teacher who was to teach the research lesson. The knowledgeable other attended the third meeting. Excluding the first and third meetings, the proportion of time spent on planning the flow of the research lesson was 74%, while when all four meetings are included, 52% of the time was spent on planning the flow of the research lesson.

At school M, the first meeting was also held without the written lesson plan. From the second meeting onwards, the discussion was based on the draft lesson plan which had been written mainly by the teacher who was to teach the research lesson, but as a team, with support from the third grade teachers. In the second, third, and fourth meetings, the proportion of time spent planning the flow of the research lesson was 74%, while if the first meeting is included, the proportion was 66%. Across the three schools, omitting meetings without the lesson plan, the average proportion of time spent on planning the flow of the research lesson was 72%, while if all meetings are included, the proportion was 63%.

Thus we have two findings: one, that the planning meetings followed the structure of the lesson plan and, two, that the discussion among teachers was particularly focussed on planning the flow of the research lesson.

The discussions specific to the flow of the research lesson during the planning meetings at the three schools could be aligned with the four phases of a problem-solving lesson (see Table 2). For example, at the second meeting at school S, a discussion on how students might grasp the given task (15 min) was related to phase 1, *Presenting the problem for the day*; discussion about likely student responses (14 min) was related to phase 2, *Problem solving by the students*; discussion about how to organize the comparison and discussion period (15 min) was obviously related to phase 3, *Comparing and discussing*; and discussion about how to conclude the lesson (5 min) was related to phase 4, *Summing up by the teacher*. Of the 49 min focussed on the flow of the research lesson, the proportions of time related to these four phases were approximately 31, 29, 31, and 10%. The other two schools showed a similar pattern.

In the next section, we will present, in more detail, what the teachers talked about regarding each phase of their lessons.

4.3 Major Concerns When Planning the Flow of the Research Lesson

Discussions by the teachers, while planning the flow of the research lesson, were classified into three key categories: appropriateness of the task, plausibility of the anticipated student solutions, and quality of the comparison and discussion (*neriage*) phase.

4.3.1 Appropriateness of the Task

Discussions about the task for the research lesson can be classified into two types. One is discussion about the task and unit from an advanced mathematical perspective, where teachers clarify the scope and sequence of relevant topics, or relationships within and expansion of the content. The second is to discuss the appropriateness of the task to the goal of the lesson, including detailed consideration of the numbers in the task, the context of the task, and so on.

When teachers talked about the position of the unit within the curriculum, they carefully referred to the National Course of Study (2008) published by the Ministry of Education, Culture, Sports, Science and Technology. According to Lewis' (2002) typical lesson plan template, this discussion is related to "connections to standards and prior and subsequent learning", which is included in the fifth component of the template, *Learning plan for the unit*, where related units in former and later grades are explained and shown by using a diagram. In fact, teachers at school M used their own diagram as they discussed why the unit was important and as they traced the students' learning path leading to the unit. In the case of school S, at the second meeting where teachers talked about sequence of units, they recalled an old version of the National Course of Study (1998) in which "speed" was placed in fifth grade. "Speed" was now in sixth grade in the National Course of Study (2008). In fact, one teacher said "At fourth grade we teach multiplication and division of decimal numbers, and in fifth grade we teach the size of per-unit quantities.[1] The closest content to speed is size of per-unit quantities.. . . We used to teach speed in fifth grade, together with the size of per-unit quantities".

Teachers also talked a lot about the task itself. The tasks in all three cases were not directly from textbooks; they were newly created or modified from tasks in the textbook. Teachers discussed why they selected the particular tasks; what roles the tasks were expected to play in the unit; and what benefits students might gain from solving the tasks: whether it helped to develop a new concept, a new way of thinking, or some important procedure.

[1]A per-unit quantity is a ratio of two quantities from different measure spaces. As a ratio, it is expressed as the amount of one measured quantity for one unit of the other measured quantity. For example, population density is typically expressed as the number of people per unit area or speed as the distance travelled per unit time.

The discussion of the curriculum was closely related to the solution of the task, because related content in the curriculum was expected to be a resource for students to solve the task. For example, in the second meeting in School S, there was the following exchange:

Teacher A: Students learned how to arrange to get the same numbers for time or distance, didn't they?

Teacher B: Yes, I suppose. However, the idea of a common multiple was learned a long time ago from the students' point of view.

Teacher C: Probably they forgot the procedure to find the common multiple.

Teacher B: When they learned division of decimal numbers, they learned the idea of per-unit. It's the same thing here. However, the idea of per-unit was not learned in the context of comparing things.

Principal: The idea of per-unit quantity was applicable for comparing crowdedness. That is a mathematical way of thinking that could be applicable for Speed.

This kind of detailed and concrete consideration of previously learned content was observed in all three schools.

Teachers also engaged in detailed discussions about the task itself, including which numbers to use and why. This aspect of Lesson Study was noted by Stigler and Hiebert (1999), who reported that teachers would talk about the "problem with which the lesson would begin, including such details as the exact wording and numbers to be used" (p. 117). However, the selection of numbers is not always from a purely mathematical point of view.

For example, in the case of School S, teachers thought about numbers both in terms of their students' reality and also from a procedural or calculation point of view. The teacher who would teach the research lesson said:

> Child A in the problem can run 40 metres in only 6 seconds. In my class there is no such fast runner. However I decided to use these numbers, because these numbers are easier for children to calculate.

Time and distance data for the first three people in the problem (A, B, C) were not changed, but data for two people (E, F) were changed from E (42 m in 6.7 s), F (28 metres in 4.9 s) to E (45 m in 6.5 s), F (50 metres in 8 s), in order to provide some faster speeds. Numbers for D, E, and F were considered hard for students to calculate, and the teachers also worried about having decimal fractions as the result of calculations. However, they decided to keep the numbers and let students use calculators if they wanted.

In the case of school M, the task was to contrast partitive and quotitive division problems obtained from one mathematical sentence. The teachers chose to use 8÷2 after also discussing 12÷3, 18÷6, 6÷2, and 10÷2 as possible candidates. They considered the numbers 8, 2, and 4 as the most easily distinguishable for students, so that students would not confuse them in using, or explaining, their ideas.

In the case of school T, the research lesson was on learning about quadrilaterals, and the task was to classify quadrilaterals. The teachers changed the plan from *asking students to draw* figures freely on dot paper to *giving students* figures already drawn by the teacher. The teacher worried that students might not construct certain figures that the teacher particularly wanted to discuss in the lesson. The teachers also discussed what would be a suitable number and what types of quadrilaterals to give. If the number of figures was too small, students would not be interested in classifying them, or they would not feel any necessity to make groups. Eventually the teachers decided on nine figures: a square, a rectangle, two parallelograms, two rhombi, an isosceles trapezium, a general trapezium, and a general quadrilateral. The team decided not to include a trapezium with a right angle. As part of their discussion, teachers simulated individual students solving the problem to get an idea of the time required. Further, they considered the quality of the problem-solving activity in terms of the appropriateness of the task and the goal of the lesson.

At all three schools, the teachers discussed the unit in reference to the curriculum, as well as discussing the main task in terms of its appropriateness within the unit, its value for clarifying mathematical ideas, and its appropriateness for accomplishing the goal of the lesson. In terms of the appropriateness of the task for the goals of the lesson, teachers considered what solutions or ideas the students would be likely to bring up. This is the topic of the next section.

4.3.2 Anticipated Student Solutions

In all three schools, teachers spent time discussing likely student responses to the main task in the research lesson. These discussions usually began by considering what was most likely from the class as a whole. They then went on to consider likely responses from students who were rather slow learners and from students who were fast learners.

In the case of school S, teachers pretended to be students in order to solve the speed task, *Who is faster?* (see Table 6), from the students' point of view. Through this activity, teachers confirmed the plausibility of the four anticipated solutions already written in the lesson plan: ① finding a common multiple of distance to compare, ② finding a common multiple of time to compare, ③ finding the amount of time per metre to compare, and ④ finding the distance per second to compare.

In the case of school T, one teacher was asked to pretend to be a student to solve the task, and the other teachers watched his activity. In the case of school M, teachers wondered whether students would be able to create two kinds of division stories or just one story. The team leader asked the other teachers if they felt uneasy partly because of their own experiences. Teachers made explicit reference to their own experiences as they tried to anticipate students' responses to the task.

Table 6 The task given: Who is faster? Let's think about the order of speed of these three children: A, B, and C

Children	Distance (m)	Time (seconds)
A	40	6
B	30	6
C	30	5

In all three schools, teachers considered how to deal with slow learners. In the case of school S, the teacher had already decided to provide hints to students who wanted them during the individual problem-solving period. The team discussed specifically what should be on these hint cards. While a hint card suggesting using common multiples was reasonable from the teachers' initial point of view, they no longer thought this might be the case when they imagined, or visualized, the lesson. They thought this strategy would eventually be rejected in favour of a better strategy: finding the distance per second. One teacher said, "Students might ask the teacher, '*Why did you not give me the best hint, if you knew*?'" The other teachers agreed that was likely to happen. So they discussed how to let students notice the per second strategy. Finally teachers thought of using 30 metres and 5 seconds as the data. "It divides beautifully". "If the teacher asks a question such as, '*Five seconds to go (30m), so if it were one second how far could you go?*, students may be able to notice the idea of *per second*'". "It will work", one teacher said, "it looks fine". Eventually the teacher decided to suggest using the "per second method" to solve the task using the data of 30 metres and 5 seconds.

In all three schools, teachers also considered how to deal with fast learners in the lesson. For instance, at school M, a teacher said, "Students who have finished solving the task, I would ask them to write mathematical sentences, possibly like $4\times2=8$ or $2\times4=8$, showing the process to get the answer".

4.3.3 The Comparison and Discussion (*Neriage*) Phase

The comparison and discussion (*neriage*) phase follows the problem-solving by the students. This phase in the structured problem-solving lesson is the most difficult for teachers to deal with. Each correct solution has equal value in terms of getting an answer. However, the ideas involved may not have equal value. The *neriage* phase is when the teacher elicits these ideas and discusses the value of each solution. The teacher at school S clearly stated, "Although each strategy is sure to get the correct answer, we should not end there … I want the students to know that getting the answer is not the final goal".

In the case of school M, teachers wanted students to compare two word problems, for partitive and quotitive division, through the use of multiplication sentences to model situations. (See the Appendix for the actual task.) The lead teacher of the research steering committee posed the question, "What should we ask to elicit a multiplication maths sentence?" For the next 17 minutes, the teachers discussed what the question should be, including its exact wording.

At school T, teachers talked about which point or theme for discussion would be best: the number of groups of quadrilaterals, where the teacher might say "this student made two groups and the other student made three groups, what made these difference? What were the thoughts behind these categorizations?" or how to characterize each group, for example, "This student made two groups. Can you see the common characteristics of the quadrilaterals in each group?" One teacher asked, "Which is the higher level of thinking?" to which another teacher responded, "Probably the number of groups is higher. This point is proposed in the lesson plan". So they decided to ask students to discuss how many groups there were and reasons behind this in the *neriage* phase.

The teams at all three schools discussed how to elevate students' mathematical thinking by comparing individual students' solutions.

5 Discussion

It is well known that Japanese teachers get together before a research lesson to discuss the lesson. What do teachers discuss? This study reveals that their discussions followed the lesson plan, which had been drafted or created before meetings, and they devoted approximately two thirds of the time to discussing the flow of the research lesson. Within that time, teachers focussed on the appropriateness of the task, anticipated student solutions, and the plan for comparing and discussing those student solutions. The teachers also referred to the Japanese National Course of Study and its guide for teachers.

5.1 The Role of the Japanese National Course of Study in Designing and Adapting the Task for the Research Lesson

At planning meetings, teachers frequently referred to the National Course of Study when they needed to confirm the role of the unit, or focus lesson, within the entire curriculum. Teachers at school S talked about the placement of speed in the previous National Course of Study. This is a more difficult conversation to have in countries lacking a clear curriculum. Lewis and Tsuchida (1998) argued that having a frugal, shared curriculum was necessary for implementing Lesson Study. With a clear curriculum sequence, teachers could identify the value of the research lesson and the unit within the curriculum: by identifying closely related content in former and later grades, teachers can understand why the research lesson is important for later learning. And identifying similar units or content in earlier grades helps teachers infer what students might do to solve the task, based on their previous learning. All three teams of teachers identified the position of the research lesson in the curriculum in order to clarify students' learning trajectory.

5.2 The Value of Discussing Anticipated Solutions

Data from the three schools revealed that teachers tried hard to anticipate student solutions in detail; and what they anticipated influenced the design of the lesson. For example, it influenced the design of the task, such as in the case of school T where the decision whether to include a trapezium with a right angle was made through considering students' anticipated solutions. Anticipating student responses also influenced how teachers decided to pose the problem. For instance, teachers at school S considered how students would react to the question of which person is faster when only times were given. Also teachers tried to predict students' difficulties and discussed how to reduce students' confusion in comparing three speeds. They eventually decided to erase the slowest person's data in order to focus on only two people.

Based on their experience, Japanese teachers know that the conditions, or characteristics, of the task influence students' thinking processes and solution methods. In the case of school T, the teachers thought that the right angle might cause students to go in a direction that was not consistent with the goal of the lesson. Anticipating student solutions at planning meetings is therefore important in Lesson Study, and this unique activity is a characteristic of task design in Lesson Study (Fujii 2015).

Teachers also think carefully about the numbers used in a task because this can strongly influence students' ways of solving the task. In the case of school S, teachers deliberately chose awkward numbers for the additional speed data, of persons D, E, and F. The teacher explained, "I want students to say that it is awkward to calculate common multiples among them". She deliberately chose numbers that would push students to calculate distance divided by time. On the other hand, the numbers for B and C were (30, 6) and (30, 5), respectively, with these chosen because the numbers "divide beautifully". The teacher clearly anticipated that some students would calculate 30÷6 and 30÷5 to get the distance per second.

Close attention to the specific numbers does not mean that teachers are sticking to a concrete level of thinking or encouraging students to think concretely. On the contrary, teachers consider the general aspect of the numbers—their quasi-variable aspects. A quasi-variable is a number deliberately used in a general way, so that it serves as a representative of many numbers, just as a variable would (Fujii and Stephens 2001, 2008; Fujii 2008, 2010). Numbers are often chosen based on their quasi-variable power or how well they can demonstrate a general truth—a general truth that is brought out during whole-class discussion.

A structured problem-solving lesson includes a *neriage*—comparison and discussion—phase for students to compare or experience their friends' methods and discuss similarities and differences between strategies as a whole class. When designing the task, there needs to be consideration of whether the task will elicit the alternative approaches needed for an effective *neriage*. Therefore teachers carefully discuss and choose appropriate numbers for the task.

Discussing students' anticipated solutions while considering the specific numbers in the task clarifies the mathematical value of the task. In their book *The Teaching Gap*, Stigler and Hiebert (1999, p. 118) have another example: teachers discuss appropriate number sentences to use in the context of teaching subtraction across 10. Subtraction across 10 can be solved by subtraction-addition (e.g. 12–9 = 10–9 + 2), subtraction-subtraction (e.g. 12–9 = 12–2–7), counting down, and counting up. In this example, the teachers believed that the subtraction-addition strategy was the most valuable for students to learn, so they examined the potential of different choices of numbers to lead to that strategy. For the same reason, almost all textbooks in Japan choose 13–9 or 12–9 to elicit the subtraction-addition strategy (Doig et al. 2011). In the case of school S, numbers were chosen to lead students to calculate distance divided by time. In the case of school T, teachers chose geometrical figures, which could lead students to classify them in terms of characteristics related to their parallel or perpendicular sides. Anticipating student solutions in Lesson Study helps clarify the mathematical value of the task and helps teachers make sure that the goal of the lesson is reached.

5.3 The Value of Designing the Neriage Phase of the Lesson

The comparison and discussion of multiple student solutions needs to be more than "show and tell" (Takahashi 2008). This *neriage* phase of a lesson should be an actualization of Vygotsky's zone of proximal development (Ohtani 2014), and the role of the teacher is critical. Teachers at the three schools, M, S, and T, discussed at length how to deepen students' ideas in the *neriage* phase. A teacher at school S said, "Although each strategy is sure to get the correct answer, we should not end there". This comment shows teachers' deliberate efforts to elevate all students' ways of thinking.

During the planning meetings, the focus of designing the *neriage* phase of the lesson was on deepening students' understanding and ways of thinking. From the point of view of mathematical value, the lesson should clarify the relative value of the different solutions, generally by contrasting these. The lesson is less likely, obviously, to do this without sufficiently rich and diverse solutions to compare. Therefore, teachers carefully examine anticipated student solutions in detail in order to make sure valuable solutions are likely to appear in the comparison and discussion phase. The value of designing the *neriage* phase of the lesson lies in its potential to elucidate or expose ways to highlight different solutions and how to compare them in order to reach the goal of the lesson.

5.4 Designing and Adapting Tasks in Lesson Planning Goes with Lesson Evaluation

As we have seen, teachers give much thought to the selection and design of the task during the planning phase of Lesson Study. The task is later evaluated during the post-lesson discussion. This is another distinguishing aspect of Lesson Study. The task is not judged based on some abstract determination of whether it is good for teaching a certain skill or concept, but based on concrete evidence from the research lesson of how the students responded to it. In the case of school S, three pairs of data points were added for students to compare, but at the post-lesson discussion, teachers

Table 7 An example of two pairs of numbers used in the context of population density

Pool	Area (m^2)	Number of people
A	200	15
B	400	45

argued about whether these additional data were useful or not. The arguments were based on how students actually responded to the task in the lesson. Similar arguments occurred at the other two schools.

In the case of school S, the arguments progressed from evaluating the task to modifying the task. In fact, the final commentator, the knowledgeable other, suggested more direct ways to manipulate numbers to identify faster speed without calculating six pairs of numbers. He gave the example shown in Table 7 of two pairs of numbers in the context of population density.

The final commentator suggested using these numbers instead of the six pairs of numbers that were used in the research lesson, as some students struggled to carry out the calculations in the time available, and then missed the educational value of the task, and the whole-class discussion. The post-lesson discussion provided a context for revising the task used at the research lesson, since points missed in planning meetings were revealed in the post-lesson discussion. This shows that the planning meetings of the Lesson Study cycle are closely related to the research lesson itself and to the post-lesson discussion.

The post-lesson discussion provided a context for revising the task used at the research lesson. However, this does not imply that re-teaching is necessarily part of Japanese Lesson Study. Based on their experience, Japanese teachers know that if students are different, then their reactions will be different. They understand that a lesson is itself an organic system; it is not like a machine. A non-organic system, such as a car, is composed of parts that may be easily replaced. However, in organic systems, like a lesson, each part supports the whole ecology. In the case of school S, important ideas missed in planning meetings were revealed in the post-lesson discussion. Teachers then regretted that their *kyozai-kenkyu* (study or research on teaching materials—see, for example, Watanabe et al. 2008) was not profound enough and broad enough to cover the idea. In other words, Japanese teachers' attitude towards research lessons and lesson plans is that their best lesson plan should be implemented at a research lesson and that a research lesson is the proving ground for teachers (c.f. Lewis and Tsuchida 1998).

6 Conclusion

It is widely understood that a lesson plan is an important product of Lesson Study, but despite much research into Lesson Study, the process of creating a lesson plan, as a collaborative effort by teachers, is largely invisible to non-Japanese adopters of

Lesson Study. This paper tries to clarify the process of lesson planning and the role and function of the lesson plan, based on case studies of Lesson Study in three Japanese schools.

In each of these case studies, we see that the planning meetings began with a lesson plan already written by the teachers and most of the time was spent discussing the flow of the research lesson. While discussing the flow of the research lesson, teachers spent time designing and adapting the task for the lesson, during which time they typically did the following: consulted the National Course of Study to clarify the position of the task in curriculum, as well as for guidelines in designing and adapting tasks; verified the mathematical value of the task by anticipating student solutions; and carefully designed the comparison and discussion (*neriage)* phase of the lesson to ensure that the goal of the lesson was reached.

In addition, teachers evaluated the task during the post-lesson discussion in light of the actual student responses in the research lesson, and they also explored how the task might be revised based on this discussion.

Some potentially interesting aspects of lesson planning were not addressed in this paper: the author did not consider the relationship between the quality of the lesson planning and the quality of the research lesson. This paper did not look at the impact of lesson planning on teachers' mathematical and pedagogical knowledge (Lee and Takahashi 2011; Lewis 2009). And the paper did not look at how the lesson-planning process exposes teachers' beliefs. The author hopes, however, that by making aspects of the planning phase of Lesson Study visible, this paper will contribute to helping educators outside Japan appreciate the full richness of Lesson Study and better understand how it can improve teaching and learning.

Acknowledgements The author would like to thank Thomas McDougal for reading and editing numerous revisions and for his invaluable comments on this paper.

Appendix: The Task Given by the Teacher

The task given by the teacher was: "let's write word problems that can be solved by $8 \div 2$. Draw a picture or diagram for the problem situation. Also, write an equation and the answer, too".

A	Division to find the group size (partitive division)
	2 people are sharing 8 strawberries. How many strawberries does each person get?
	Equation: $8 \div 2 = 4$ Answer: 4 strawberries
B	Division to find the number of groups (quotative division)
	We are going to give 2 strawberries to each person. If there are 8 strawberries, how many people will get strawberries?
	Equation: $8 \div 2 = 4$ Answer: 4 people

References

Becker, J. P., Silver, E. A., Kantowski, M. G., Travers, K. J., & Wilson, J. W. (1990). Some observations of mathematics teaching in Japanese elementary and junior high schools. *Arithmetic Teacher, 38*(2), 12–21.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101.

Department for Children, Schools and Families. (2008). *Improving practice and progression through lesson study: Handbook for head teachers, leading teachers and subject leaders.* Nottingham: DCSF Publications. Retrieved September 27, 2011, from http://teachfind.com/national-strategies/improving-practice-and-progression-through-lesson-study-handbook-headteachers-le

Doig, B., & Groves, S. (2011). Japanese Lesson Study: Teacher professional development through communities of inquiry. *Mathematics Teacher Education and Development, 13*(1), 77–93.

Doig, B., Groves, S., & Fujii, T. (2011). The critical role of task development in Lesson Study. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson Study research and practice in mathematics education* (pp. 181–199). Dordrecht: Springer.

Elliott, J. (2012). Developing a science of teaching through lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 108–125.

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods, 5*(1), 80–92.

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A case of a Japanese approach to improving instruction through school-based teacher development*. Mahwah: Lawrence Erlbaum.

Fujii, T., Kumagai, K., Shimizu, S., & Sugiyama, S. (1998). A cross-cultural study of classroom practices based on a common topic. *Tsukuba Journal of Educational Study in Mathematics, 17*, 185–194.

Fujii, T., & Stephens, M. (2001). *Fostering an understanding of algebraic generalization through numerical expressions: The role of quasi-variables*. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra (pp. 258–264). Melbourne: University of Melbourne.

Fujii, T., & Stephens, M. (2008). Using number sentences to introduce the idea of variable. In C. Green & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics-seventeenth yearbook* (pp. 127–140). Reston: National Council of Teachers of Mathematics.

Fujii, T. (2008, July 6–13). *Knowledge for teaching mathematics*. Plenary Talk at the 11th International Congress on Mathematical Education, Monterrey, Mexico.

Fujii, T. (2010). *Designing tasks in the Japanese Lesson Study: Focusing on the role of the quasi-variable*. In Proceedings of the 5th East Asia Regional conference on Mathematics Education EARCOME5, Tokyo, Japan (pp. 86–93).

Fujii, T. (2014a). *Theorizing Lesson Study in mathematics education as an emerging research area (2): Identifying components and its structure of lesson study (in Japanese)*. In Proceedings of second annual Spring conference of Japan Society of Mathematical Education (pp. 111–118).

Fujii, T. (2014b). Implementing Japanese Lesson Study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 65–83.

Fujii, T. (2015). The critical role of task design in Lesson Study. In A. Watson & M. Ohtani (Eds.), *ICMI Study 22: Task design in mathematics education* (pp. 273–286). New York: Springer.

Hart, L. C., Alston, A., & Murata, A. (2011). *Lesson study research and practice in mathematics education*. New York: Springer.

Lee, Y. A., & Takahashi, A. (2011). Lesson plans and the contingency of classroom interactions. *Human Studies, 34*(2), 209–227.

Lewis, C., & Tsuchida, I. (1998). A lesson is like a swiftly flowing river. *American Educator*, *22*(4), 12–17, 50–52.

Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: Research for Better Schools, Inc.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and north American case. *Journal of Mathematics Teacher Education, 12*, 285–304.

Lewis, C., & Hurd, J. (2011). *Lesson Study step by step: How teacher learning communities improve instruction*. Portsmouth: Heinneman.

Makinae, N. (2010). *The origin of Lesson Study in Japan*. In Y. Shimizu, Y. Sekiguchi, & K. Hino (Eds.), The *Proceedings of the 5th East Asia Regional conference on Mathematics Education: In search of excellence of Mathematics Education* (Vol. 2, pp. 140–147). Tokyo: Japan Society of Mathematics Education (JSME).

National Course of Study (in Japanese). (1998, 2008). *Ministry of Education, Culture, Sports, Science and Technology*. Retrieved April 30, 2015, from http://www.mext.go.jp/a_menu/shotou/cs/index.htm

Ohtani, M. (2014). Construction zone for the understanding of simultaneous equations: An analysis of one Japanese teacher's strategy of reflection on a task in a lesson sequence. In F. K. S. Leung, K. Park, D. Holton, & D. Clarke (Eds.), *Algebra teaching around the world* (pp. 113–128). Rotterdam: Sense Publishers.

Ono, Y., & Ferreira, J. (2010). A case study of continuing teacher professional development through lesson study in South Africa. *South African Journal of Education, 30*(1), 59–74.

Sacks, H. (1992). *Lectures on conversation*. Cambridge: Blackwell.

Sacks, H., Schegloff, E., & Jefferson, G. (1974). A simplest systematics for the organization of turn-taking for conversation. *Language, 50*(4), 693–673.

Shimizu, Y. (1999). Aspects of mathematical teacher education in Japan: Focusing on the teachers' roles. *Journal of Mathematics Teacher Education, 2*, 107–116.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Takahashi, A. (2006). Types of elementary mathematics Lesson Study in Japan: Analysis of features and characteristics (in Japanese). *Journal of Japan Society of Mathematical Education, Mathematics Education*, *88*(8), 2–14.

Takahashi, A. (2008). *Beyond show and tell: Neriage for teaching through problem-solving–Ideas from Japanese problem-solving approaches for teaching mathematics.* Paper presented at the 11th International Congress on Mathematics Education in Mexico (Section TSG 19: Research and Development in Problem Solving in Mathematics Education), Monterrey, Mexico.

Takahashi, A. (2013). *Investigation of the mechanism of Lesson Study as the core of the mathematics teacher professional development* (in Japanese). In Proceedings of first annual Spring conference of Japan Society of Mathematical Education (pp. 83–87).

Takahashi, A., & McDougal, T. (2014). *Implementing a new national curriculum: A Japanese public school's two-year Lesson-Study project*. In Using Research to Improve Instruction, Annual Perspective in Mathematics Education, NCTM (pp. 13–21).

Yoshida, M. (1999). *Lesson Study: A case study of a Japanese approach to improving instruction through school-based teacher development.* Unpublished doctoral dissertation, University of Chicago, Department of Education.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), *Inquiry into Mathematics teacher education* (Vol. 5, pp. 131–142). Raleigh: Association of Mathematics Teacher Educators (AMTE).

White, A. L., & Lim, C. S. (2008). Lesson study in Asia Pacific classrooms: Local responses to a global movement. *ZDM Mathematics Education, 40*(6), 915–925.

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A Critical Mechanism for Improving Teaching and Promoting Teacher Learning During Chinese Lesson Study: An Analysis of the Dynamics Between Enactment and Reflection

Rongjin Huang, Dovie Kimmins, and Jeremy Winters

Contents

Abstract Chinese and Japanese lesson studies are similar structurally yet different regarding the necessity of repeated teaching and knowledgeable others' input. Chinese teachers' learning through lesson study emphasizes the unity of knowing and doing and practical reasoning in deliberate practice. Utilizing a deliberate practice perspective to examine the process of lesson study, this study examined the dynamics between enactments and reflections during Chinese lesson study and its impact on improving teaching and teacher learning. Three cycles of lesson study involving three first-grade teachers occurred in the Southeastern USA, with an expert team of mathematics educators and specialists facilitating. Data included lesson plans, videotaped research lessons and debriefing sessions, and post-lesson study reflection reports. Major changes across enacted lessons and argumentations made during the debriefings contributing to the changes were identified. Data analysis

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_34

revealed that experts and teachers collaboratively made contributions to improving the lesson. Experts mainly focused on implementing learning trajectory and sequencing tasks, while teachers focused on students' learning readiness and on formative assessment. Teachers expressed enthusiasm for the lesson study process and perceived that their teaching skills and knowledge of content, pedagogy, and student thinking increased. This study uncovers the mechanism of lesson study as deliberate practice, which emphasizes the necessity of repeated teaching and immediate input of knowledgeable others, and highlights the unique impact of the dynamics between enactments and reflection on teachers' learning.

Keywords Chinese lesson study · Deliberate practice · Enactment and reflection · Knowledgeable others

1 Introduction

Research has identified key features of effective professional development (PD) (Archibald et al. 2011; Desimone 2009): (1) alignment with shared goals (school, district, and state) and assessment; (2) a focus on core content and modeling of teaching strategies for the content; (3) inclusion of opportunities for active learning of new teaching strategies; (4) provision of opportunities for collaboration among teachers; and (5) inclusion of embedded follow-up and continuous feedback. Lesson study (LS), a PD model originating in Asia, incorporating the key features of effective PD, has been adapted globally (Lewis and Lee 2017). While research has documented various effects of LS such as improving instructional practice (Lewis 2016), promoting students' learning outcomes (Huang et al. 2016; Lewis and Perry 2017), promoting teachers' learning (Huang and Shimizu 2016; Lewis and Perry 2017), and promoting the implementation of new curriculum (Takahashi and McDougal 2014), very few studies have explored theories and mechanics of why and how LS works (Elliott 2012; Huang et al. 2017; Lewis et al. 2009; Widjaja et al. 2017; Xu and Pedder 2014). Based on Clarke and Hollingsworth's (2002) *Interconnected Model for Professional Growth*, Widjaja et al. (2017) illuminated how the iterative processes of *enactment and reflection* mediate the interplay between different domains (i.e., personal domain, external domain, domain of practice, and domain of consequence) and eventually promote professional growth through LS. Using the lens of deliberate practice (Ericsson et al. 1993), Huang et al. (2017) explored how the iterative processes of enactment and reflection with immediate feedback from knowledgeable others during LS improve core instructional practice (e.g., launching of tasks, implementation of tasks, and orchestration of students' solutions) in mathematics problem solving. Although both these theories and empirical studies suggest the importance of enactment and reflection in mediating teachers' learning through LS, it is largely unclear how enactment and reflection interact for promoting teachers' learning with LS. This study aims to reveal the micro-mechanisms of the dynamics between enactment and reflection during LS.

2 Theoretical Framework

2.1 Teacher Learning Through Deliberate Practice

Cognitive scientists suggest that participation in special activities is an important factor for continued improvement and attainment of expert performance (Ericsson 2008; Ericsson et al. 1993). Ericsson et al. (1993) defined deliberate practice as special activities developed for and repeatedly pursued by individuals with feedback from experts. Engagement in deliberate practice means that one is given a task with a well-defined goal, motivated to improve, and provided with feedback and ample opportunities for repetition, resulting in gradual refinements in performance (Ericsson 2008; Ericsson et al. 1993). Deliberate practice has four characteristics (Bronkhorst et al. 2013; Ericsson et al. 1993; Gog et al. 2005). First, the practice is designed for self-improvement. Proper sequencing of challenging tasks should be set with the support from knowledgeable others (i.e., teachers, coaches, and trainers). Second, the practice is repeated to enable successive improvement and refinement. Third, the repetitive practice is followed by immediate feedback concerning different aspects that underlie the practice. Fourth, the practice requires significant effort and concentration.

To connect the principles of deliberate practice to practical implications for teacher education, Dean for Impact (2016) further illustrated five principles of deliberate practice. Four are similar to the previously mentioned four. Yet, the fifth principle suggests "Deliberate practice both produces and relies on mental models and mental representations to guide decisions. These models allow practitioners to self-monitor performance to improve their performance" (p. 12).

Researchers have explored deliberate practice in teaching and teacher education (e.g., Bronkhorst et al. 2011, 2013; Dunn and Shriner 1999; Gog et al. 2005). By analyzing daily activities of experts and experienced teachers, Dunn and Shriner (1999) found that deliberate practice could account for teaching expertise and likely results from meaningful engagement in the teaching-evaluation-revision cycle. Further, Bronkhorst et al. (2011) found that deliberate practice could be conceptualized as activities intended not only for developing teachers' expertise but also for fostering student learning and development.

2.2 Chinese Lesson Study as a Type of Deliberate Practice

Lesson study in China has existed over a century (Chen and Yang 2013; Yang 2009). While Chinese and Japanese LS are similar structurally, they have many differences (Huang et al. 2017). Among the differences, the following two features are the most salient. First, repeated teaching of the same research lesson to different groups of students is required (Huang and Bao 2006). Second, knowledgeable others (i.e., teaching research specialists and university faculty) facilitate LS throughout the entire LS process (Huang and Han 2015; Huang et al. 2014). In Chinese LS (i.e., CLS, hereafter), teachers repeat teaching of the same lesson with collaborative deliberation in planning and observation and immediate feedback from knowledgeable others to understand and improve teaching and student learning. This constitutes a deliberate practice in and of itself (Han and Paine 2010; Huang and Han 2015; Huang et al. 2017).

From a cultural perspective, Chen (2017) argues that such deliberate practice is rooted in the Chinese philosophy of unity of knowing and doing (知行合一) and epistemology of practical reasoning (实践推理). Knowledge of good teaching is not so much talked about in verbal concepts as enacted in teachers' actions in deliberate practice. Difficult and complex issues concerning teaching and learning are discussed according to real contexts rather than formal logical reasoning.

Empirically, Han and Paine (2010) found that improving teaching of mathematics as deliberate practice "gave the teachers an opportunity to refine their instruction in three core aspects: designing appropriate mathematical tasks for students, teaching the difficult mathematical idea, and using mathematically, pedagogically appropriate language" (p. 519). Recently, Huang et al. (2017) reported on a CLS conducted at the district level in the USA and found that the dynamics between enactment and reflection within CLS could help participating teachers improve their strategies for teaching problem solving and change their views about students' learning. Thus, CLS is a type of deliberate practice that enables teachers to make continued improvements in teaching and develop core instructional practices.

2.3 Teacher Argumentation and Toulmin's Model

Toulmin's model (Toulmin 1958) has been used to analyze teacher argumentation. This model consists six basic elements: the claim (C) is the position being argued for; the data (D) are the foundation or supporting evidence on which the argument is based; the warrant (W) is a general rule of inference that authorizes the step from the data to the claim; the backing (B) supports the legitimacy of the warrant; the modal qualifier (Q) represents the degree of force or strength that the data confer on a claim in virtue of the warrant; and the rebuttal (R) consists of exceptions to the applicability of the warrant. Toulmin's model has often been combined with other frameworks to address the quality and structure of argumentation.

To analyze group discussions about instructional situations, researchers (Chazan et al. 2012; Skultety et al. 2017) developed a simplified version of Toulmin's (1958) model, which consists of three components: claim, data, and justification. In the context of a post-lesson debriefing session, the *claims* are group members' remarks about modifications they wish to make to the lesson or strengths they want to remain. The claims are either explicit (i.e., group members' statements about desired modifications) or implicit (i.e., members' wondering about the appropriateness of an aspect of the lesson, without explicitly suggesting a modification). The *data* refer to the evidence that was used to prompt the claim, such as observed moments of a

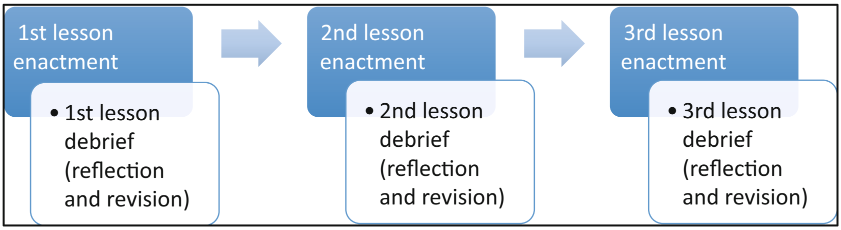

Fig. 1 Major phases of the lesson study

research lesson, recorded student work, research findings, or previous teaching experience. In the LS context, because all data are based on the observed lesson, we differentiated the data based on who makes the claims, including participating teachers, specialist, and university professors. The *justification* is the reasoning for the claim given the data, including general statements about teaching and learning as well as specific strategies, activities, and problems to connect the data to the claim. This simplified model is utilized in this study to analyze debriefing sessions to capture the LS team's pedagogical decision-making process.

2.4 A Framework of the Current Study

This study aimed to uncover the mechanisms of interplay between enactments and reflection within LS settings (see Fig. 1). On the one hand, the major phases and features of each enacted research lesson were analyzed, with the salient changes across lessons summarized. On the other hand, each post-lesson debriefing session was analyzed based on the simplified Toulmin's model with the major claims (suggested modifications), relevant justifications, and persons who provided justification. By examining the alignment between changes and claims across lessons and reflection, the implemented modifications were identified. Moreover, through examining the relationship between claims and implementation, patterns of interaction between enactments and reflection were revealed.

The purpose of this study was to reveal the micro-mechanisms of the dynamics between enactment and reflection during LS and how the interplay between them improves teaching and teacher learning. Specially, the study was designed to answer the following research questions:

1. How are the claims emerging from debriefing session enacted to strengthen the follow-up research lesson?
2. How does the follow-up enactment verify the effectiveness of claims based on the previous enactment?
3. What significant changes are made across the enactments? Who contributes to these changes?
4. What have participating teachers learned from the LS?

3 Method

3.1 Lesson Study Group

A team of mathematics teacher educators and a mathematics specialist from a midsize city in the Southeastern USA developed and implemented a hybrid model of professional development (PD) for K-6 teachers consisting of demonstration lessons, 10 full days of workshops including an intensive 5-day summer institute, and coaching or LS from Spring through Fall 2016. Twelve of 48 participants in the PD project voluntarily participated in LS. Four LS groups (three teachers in a group) were formed based on grade levels and school location. Members of LS group could be from the same grade (horizontal team) or across grades (vertical team). The four LS groups (two horizontal teams and two vertical teams) completed at least two iterations of a cycle of LS: designing a lesson, teaching/observing the lesson, reflecting, and revising the lesson.

The expert team included two professors in mathematics education from the mathematics department and one mathematics educator from the college of education at a large public university in the city (they are the authors of the chapter) and one mathematics specialist from the school system. The three university faculty are experienced in teaching pre-service teachers and conducting PD for K-12 teachers. The specialist is a veteran K-3 teacher and adjunct professor at the university. The expert team oversaw the LS activities including facilitating a lesson-planning meeting and three post-lesson debriefing meetings and commented upon lesson plans. For the purpose of this study, investigating the dynamics between enactment and reflection within the LS, we focused on one of the four groups. This LS group included three first-grade teachers from three different schools. The three schools are similar in terms of students' social-economic situation and academic achievement. The background information of the teachers and experts is shown in Table 1.

3.2 The Goals of Research Lesson

The goal of the LS was to explore how to teach first-grade students to solve contextual comparison subtraction problems. According to CCSSM 1.OA. A.1 (CCSSI 2010), first-grade students should learn to use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing by using objects, drawings, and equations. Before exploration of the comparison model, students were introduced to adding to, putting together (addition models), and taking from and taking apart (subtraction model) and the relation between addition and subtraction (taking from and missing addend). Building on these understandings, students further explored comparison subtraction (given a situation involving two quantities finding the difference, finding the bigger quantity, and finding the smaller quantity).

Table 1 Background information of the members of the LS group

Type	Name	Background
Researcher	Dr. Ross	Veteran high school mathematics teacher and mathematics and mathematics education professor. Teaching and research in pre-service teacher preparation. Extensive research experience in teacher learning through the LS approach
	Dr. Long	Veteran mathematics and mathematics education professor. Teaching and research in pre-service and in-service teacher education
	Dr. Johns	Veteran mathematics education professor. Teaching and research in pre-service and in-service K-8 teacher education
Specialist	Ms. Cook	Mathematics specialist in school district, veteran K-3 teacher, adjunct professor for university, master teacher for numerous PD projects
Teachers	Ms. Luna	Licensed K-6 teacher, with a master's degree in reading education; 6–10 years teaching
	Ms. King	Licensed K-3 teacher, with a bachelor's degree in early children education; less than 5 years teaching
	Mr. Mills	Licensed K-4 teacher, with a master's degree in education; more than 20 years teaching

Three models for interpreting subtraction are taking away, missing addend, and comparison (Sowder et al. 2014; Van de Walle et al. 2016). The first two are based on the part-part-whole model, which can be presented visually using a part-part-whole mat (Van de Walle et al. 2016). The comparison model involves the difference between two distinct sets or quantities and can be represented visually by counters or cubes. However, "it is not immediately clear to students how to associate either the addition or subtraction operation with a comparison situation" (Van de Walle et al. 2016, p. 177). The dominating part-part-whole model may negatively impact students' understanding of the comparison model of subtraction. Researchers (Van de Walle et al. 2016) have suggested that when discussing the difference between bars, asking "how many more" may help students to generate the subtraction equation. Research further suggests that using physical actions to match concrete manipulatives in a one-to-one fashion leads to visualized objects being matched in the same manner and finally to the development of number sentences using mathematical symbols (Zhou and Lin 2001). The studies reviewed suggest that teaching comparison subtraction should follow a hypothetical learning trajectory with five levels enumerated in Fig. 2. The research lesson focused on making sense of contextual comparison subtraction problems by considering this learning trajectory.

3.3 Process of Lesson Study

To (1) set the goals of the LS (developing students' understanding of contextual comparison problems and their abilities in solving them) and (2) develop an initial plan of the lesson, the LS group met. This initial meeting was facilitated by the expert team. After that, the LS group was invited to develop a detailed lesson plan

Level 1.	Physically acting out comparison subtraction (how many more or how many fewer) within a daily context
Level 2.	Using manipulatives (counters or cubes) to model comparison subtraction concretely
Level 3.	Drawing diagrams to represent the model visually
Level 4.	Creating mathematical equations
Level 5.	Varying problems regarding different unknowns (i.e., difference, bigger or smaller quantities)

Fig. 2 A learning trajectory of comparison subtraction

based on a four-column lesson plan format (including columns for phases of the lesson, anticipated students' answers, responses to students' answers, and assessment of students' understanding). Two weeks before the actual teaching, the lesson plan was sent to the expert team. Formal written feedback was sent back to the teachers one week in advance. The teachers modified their lesson plan based on the feedback, and Mr. Mills taught the research lesson. Using a classroom observation protocol, the other two teachers and experts watched the lesson, which was videotaped. Immediately after the lesson, a debriefing meeting occurred which was structured to analyze the observed lesson and make suggestions for further revision. First, the facilitator (Dr. Ross) set the norm for the debriefing meeting: the focus is on teaching, rather than teachers, and on student learning based on the evidence collected during the class. Then, the facilitator suggested the procedure of debriefing: first, the teacher who taught the research lesson introduces the learning goal of the lesson in view of the lesson plan and gives his perception of the most exciting moment of the lesson; then, other teachers share comments on positive aspects, followed by experts. Next, teachers and experts share suggestions for improving the lesson plan for the next teaching. Finally, the major points for modifying the lesson plan are summarized by the facilitator, and the teacher who will reteach the lesson makes relevant changes. The debriefing session was videotaped. Following the same process, Ms. King and Ms. Luna in their own classrooms taught the research lesson (typically, a research lesson is taught twice to different groups of students, but this LS group wanted to teach a third time). After completing the third teaching, the teachers were asked to finalize the lesson plan, and all teachers were asked to write a reflection essay based on a reflection protocol.

3.4 Data Collection

Lesson plans for each of the three enactments of the research lesson were collected. All enactments (around 1 hr. each) and post-lesson debriefing meetings (around 1 hr. each) were videotaped. Student work and exit tickets (i.e., a short assessment conducted during the last part of a lesson) were collected. After completion of the entire LS process, the participating teachers submitted a reflection essay guided by several questions. The lesson plans, videotaped enactments of the research lessons (including student work), debriefing meetings, and written reflection essays constitute the data set for this study.

3.5 Data Analysis

The data analysis includes three phases. First, the three enactments of the research lesson (videotaped) were analyzed with a focus on major phases and use of mathematical tasks. Major changes across enactments were identified, and the learning outcomes of each were evaluated based on classroom observations and/or exit tickets. The debriefing meetings were transcribed verbatim, and the transcripts were analyzed. The videos of debriefing session were checked when necessary. Based on the simplified Toulmin's model, using constant comparison (Corbin and Strauss 2008), the major claims in each debriefing session were identified and then compared across debriefing sessions, and the claim list was finalized. Finally, reflection essays were analyzed using NVivo to feature teachers' perceptions of the LS process and what they learned through the LS. Results are presented in alignment with research questions.

4 Results

4.1 Major Features of and Changes Between the Enactments

This section includes a brief description of the three enactments of the research lesson and a summary of the major features of each. Then, the major changes across the three enactments are identified.

4.1.1 Brief Description of the Three Enactments

Enactment 1

Mr. Mills began with a whole-class number talk which focused on a review of writing addition equations making use of a part-part-whole diagram, followed by an introduction to comparison problems by asking students to compare the number of girls and boys in the class which led to an activity of matching girls and boys.

Then students were given seatwork in which a comparison problem was posed: *There were 14 bears on the red truck and 8 bears in the green truck. How many more bears were on the red truck?* The problem was posed using words only, and students were provided access to a variety of manipulatives including a part-part-whole mat, linking cubes, and ten frames. The intent was to develop subtraction skills and relate addition and subtraction sentences, but a large number of students were relying on addition skills only to solve the problem, especially with the part-part-whole mats. Many needed help. When students were asked to share with the class, one shared the part-part-whole method (14 = 8 + 6), and another shared her use of ten frames to arrive at the solution (14–8 = 6). One researcher counted the number of correct answers which was 15% (3 out of 20). A second similar seatwork problem (comparing seven apples and three apples) was given with similar results.

A third comparison problem was also given as seatwork: *Mary has 3 more apples than June. Together they have 15 apples. How many apples does each girl have?* Most students were unable to develop a plan for solving the problem without extensive teacher help. The lesson ended with Mr. Mills discussing how finding a missing addend is related to subtraction and comparing numbers and by asking the students how confident they were of solving comparison problems by a show of thumbs up or thumbs down. Based on the entire lesson, there appears to be more work necessary before students will be able to solve comparison problems on their own.

Enactment 2

Ms. King began with a whole-class number talk that included a review of numerical addition and subtraction equations in reference to a picture of two different amounts of items shown. After reviewing addition, she asked which has fewer and which has more and drew a diagram matching the items in the picture to compare visually. Then, the topics of how many more and how many fewer were discussed through *counting up and counting down*, as was how these are related to addition equations with missing addend and subtraction equations.

To begin a second class discussion, a number of students were each given a picture of either an elephant or a tiger. The pictures were then physically matched to help students answer comparison questions. The whole class was asked how many more elephants than tigers and how many fewer tigers than elephants. After matching the pictures of the elephants and tigers, the students noticed that there were two elephants left over, indicating there were two more elephants than tigers and two less tigers than elephants.

This whole-class discussion of comparison was followed by individual seatwork of a similar comparison problem (bear problem from the first lesson); only in this lesson a picture showed the items being compared. Ms. King told students they could solve the problem in the same way they solved the elephants and tigers comparison problem. Students had linking cubes and ten frames available; most students used linking cubes. When sharing with the class, one student used linking cubes and counted up to solve the comparison problem, while a second student used linking cubes, ten frames, and a subtraction equation.

A second seatwork comparison problem (apple problem from first lesson) was given that described the number of items being compared in words rather than in pictures. Most students used linking cubes leading to a subtraction equation.

The lesson concluded with an exit ticket: *There were seven ducks swimming in the pond and nine ducks sitting in the grass. How many more ducks were sitting in the grass?* Students were asked to draw a picture and write a numerical equation.

Among the 16 collected exit tickets, only 25% got roughly the correct answer (9–7 = 2, 7 + 2 = 9, 9–2 = 7), while 56% added 7 and 9.

Enactment 3

Mrs. Luna began Lesson 3 with a whole-class number talk which included a review of related, previously learned skills (i.e., addition, how many altogether, and subtraction) prior to introducing the new skills (i.e., comparison, how many more and how many fewer) to be learned for the day. Students then experienced the following progression in solving comparison problems in a whole-class setting. First, the teacher asked students how many boys and how many girls were in the class. After students stated there were 9 boys and 11 girls, she asked students to figure out how many more girls than boys. The students then matched up to find how many more girls than boys and how many fewer boys than girls there were. Then the students were introduced to using linking cubes to represent comparing the number of boys to girls and were asked how to draw a picture representing this comparison. Finally, Ms. Luna asked students to find a numerical subtraction eq. (11–9 = 2) that represented the problem.

This whole-class discussion of comparison was followed by individual seatwork of a similar comparison problem that pictorially showed seven horses on one truck and four on the other. Most students immediately began to represent the number of horses on each truck with linking cubes to model the comparison problem. Some students extended this to drawing a picture that represented the matching of the horses on each of the two trucks. A couple of students voluntarily presented their work and used linking cubes to explain.

To foster student thinking more abstractly, a second seatwork comparison problem presented the number of items being compared using words only, rather than pictures (bear problem in the previous lesson, 13–8 = 5). Students used linking cubes and drew pictures to represent the number of items being compared. Two students shared their work with the class, with both writing a subtraction equation. One student shared a picture of a number line she had used, while another student demonstrated matching the 8 linking cubes with the 13 linking cubes, removing the 8 cubes that matched, leaving 5 cubes.

After summarizing key points of the comparison model, the lesson concluded with an exit ticket that asked students to solve a comparison problem involving comparing 7 and 15 by drawing a picture that represented the problem, by matching using double tape diagrams including the numbers from 1 to 20, and by writing a numerical equation. Among the 16 collected exit tickets, 10 (63%) got a correct diagram and equation (15–7 = 8 or 7 + 8 = 15), 3 added the numbers being compared (8 + 15), 1 drew a correct diagram, and the remaining 2 left it blank.

4.1.2 Changes Between the Three Research Lessons

Analyzing the major activities across the enactments, the salient changes are identified in Table 2.

Table 2 Major changes between enactments of the research lesson

Phases	From L1 to L2	From L2 to L3
Number talk (activating relevant knowledge, leading to the new topic)	Change: from focusing on properties of addition, part-part-whole, ten frames and missing addend addition to focusing on how many more or how many fewer (missing addend/counting down/taking away, using ten frame strategies)	Changes: continuing to increase the focus on the meaning of how many more or less (achieved through matching, contrasting through diagrams, modeling using linking cubes, and summarizing all these representations in a chart)
Acting out comparison (match up boys and girls)	Changes: from matching students physically and illustrating using tape diagram and missing addend addition and subtraction equation to matching animal pictures physically and modeling using linking cubes	Changes: from matching animals' pictures physically and modeling using linking cubes back to matching students physically, plus modeling with linking cubes, drawing diagrams, using number line, and writing equations
Modeling and symbolizing (using linking cubes, diagrams)	Changes: (1) adding pictures to Task 1, while leaving Task 2 without pictures	Changes: (1) changing the order of Task 1 and Task 2 so that the numbers in the first task were smaller than in second task
	(2) From using many tools to using linking cubes and ten frames and equations (removing part-part-whole)	(2) Focusing on using linking cubes and diagrams/number line and equations (removing ten frames)
Multiple-step task	Removed	Removed
Closing	Changes: adding assessment of students' learning (exit ticket) using diagrams and equations	Enrich: using diagrams and coloring number line, equations
Learning outcomes	From 15% mastering comparing 14 with 8 to 25% mastering comparing 9 with 7	From 25% mastering comparing 9 with 7 to 63% mastering comparing 15 with 7

The major features of each enactment and salient changes across enactments could be summarized as follows. In the first enactment, the emphasis was on the part-part-whole relationship during number talks, which was further enforced by using the part-part-whole mat during exploration of comparison word problems. While the boy and girl matching activity helped students to understand how many more and less concretely, when it came to finding answers to comparison problems, students were forced to think about part-part-whole addition. Moreover, the majority of students failed to solve the multiple-step and open-ended problem, which distracted students' attention from the meaning of finding the difference when given two numbers. Therefore, students actually did not understand the meaning of comparative subtraction, but were still doing part-part-whole addition. In the second teaching, due to removing the part-part-whole mat and the open-ended task, students were led to explore the meaning of comparison word problems through a number talk, animal matching activities, and various problems using linking cubes. However, when finding the difference of two given numbers, students mainly used counting on or down strategies, with a focus on missing addend addition. Although the subtraction equation come up, it was not clear why it is necessary to use subtraction. Therefore, the majority of students were still doing addition. In the third teaching, the focus on making sense of the meaning of comparison word problems (how many more or less) through a number talk and girl and boy matching activity was enforced by drawing diagrams, creating equations, and making charts to compare. Moreover, due to strategic use of linking cubes and tape diagrams with numbers 1–20, the meaning of comparison subtraction was explored from concrete (matching and linking cubes) to pictorial (tape diagram) and symbolic (subtraction equation). Students were exploring the meaning of comparison subtraction through multiple representations. In the following section, the analysis of debriefs will reveal how these changes were made.

4.2 *Major Claims, Justifications, and Resulting Enactments*

4.2.1 Major Claims and Justifications During Debriefs

During the debriefs, the teachers, mathematics specialist, and university faculty discussed strengths and weaknesses of the research lesson based on evidence of students' learning collected in the class and then provided suggestions for modifications for the enactment. The major claims made and justifications provided in debrief 1 (D1) and debrief 2 (D2) are displayed in Table 3.

There were 9 types of major claims with 15 occurrences made during the 2 debriefing meetings (e.g., Claim 1 occurred in D1 and D2 twice, while Claim 2 only occurred in D1). The positive claims include the use of pictures in the number talk and subsequent tasks (Claim 1_D1 and D2), making sense of comparison problems by an activity of matching girls and boys in the class (Claim 3_D1 and D2). The other 11 occurrences of claims were related to the cognitive demand of tasks, appropriate use of various tools, scaffolding from concrete to abstract, focusing on learning goals, and assessing student-learning outcomes. Organizing the claims in terms of the team member who made the claim and whether the claims were adopted in the follow-up enactment, the information is shown in Table 4.

Based on Table 4, we can see that overall, the teachers, the mathematics specialist, and the researchers worked collaboratively to highlight strengths and find ways to improve the lesson (Figs. 3 and 4). It is interesting to note the complementary nature of the ideas the various members brought to the discussions in the debriefing sessions. For example, one teacher suggested using linking cubes to visualize the core idea that the meaning of comparison is to find the difference between two numbers (a suggestion related to mathematical understanding), while the specialist further suggested verbalizing the difference by asking questions of how many more and how many fewer (a suggestion related to pedagogy). Further, the teachers brought the ideas of drawing circles representing objects [e.g., bears on trucks],

Table 3 Major claims and justification in research lessons 1 and 2

Category	Claim	Justification (quotes from D1 and/or D2)
Use of pictures	1. Use pictures in number talk	D1: The use of pictures will help them to talk about the concept of how many more and how many less
		D2: The number talk with pictures is really good
	2. Use pictures associated with tasks strategically	D1: In the task, we should first ask students to understand the concept using the picture and then ask students without picture
Use of tools	3. Matching up physically	D1: I think Mr. Mills did a great job in the activity of comparing boys and girls. I saw when students started comparing the boys and girls, they said matching up
		D2: Ask questions after matching the pictures of the animals, questions like how many more and how many fewer, focusing on your objective and activity of the lesson on the idea of how many more and how many fewer
	4. Use various tools appropriately	D1: Matching using linking cubes helped students to connect and solve real-word problems
		Asking students questions to put [number 14 and 8 in bear problem for Lesson 1] into part-part-whole [mat] to me needs abstract thinking. It also creates confusion, misleading to use addition
		D2: I really appreciate that you compared two numbers using linking cubes to show the difference and how to get the difference by taking the same part, namely, 7–4 = 3.
Cognitive demand of task	5. Engaging students	D1: The task [multiple-step task] is really good for engaging and challenging students. Years ago I know I never attempted some of those questions, but I saw the value of high-level tasks
	6. Appropriateness of cognitive level	D1: Drop the last task [multiple-step task], which is too challenging for the students; we should use more appropriate high-level cognitive demanding tasks
Scaffolding and transition	7. Multiple representations from concrete to abstract	D1: Students learned to solve problems of how many more or fewer from physical action to manipulative modeling; then students need to use drawing/strip diagram and finally use symbolic equation
		D2: He (a student) knew the matchup and he was able to get the answer, but moving from concrete to abstract was a challenge. Thus, those two examples [bear problem and apple problem] were really good because they can be used to scaffold

(continued)

Table 3 (continued)

Category	Claim	Justification (quotes from D1 and/or D2)
Learning goals	8. Meaning of comparison subtraction	D1: Today in this lesson, we focused on the meaning of comparisons and difference between two [given] numbers, how many more and how many fewer
		D2: When you asked questions, you should stick to the meaning of how many more and how many fewer. For example, during number talk, you should focus on the idea of how many more and how many fewer; when completing the matching up task, you should ask similar questions
Assessing student learning	9. Need of exit ticket	D1: One thing is to include exit ticket. But what questions should the exit ticket include? I am not sure, but in the exit ticket, I may ask students similar problem like the lesson
		D2: Modify the exit ticket to include food such as cokes

Table 4 Personal resources, claims, and implementation

Personal resources	Claims	Number of claims	Implemented claims
Researcher	Meaning of comparison subtraction [D1, D2]	3	3
	Appropriateness of cognitive level [D1]		
Specialist and researcher	Use pictures associated with tasks strategically [D1]	4	2
	Multiple representations from concrete to abstract [D1, D2]		
	Need of exit ticket [D2]		
Teacher and researcher	Physically matching up [D1]	2	2
	Engaging students using high cognitive tasks [D1]		
All of the three	Use pictures in number talk [D1]	5	5
	Physically matching up [D2]		
	Use various tools appropriately [D1, D2]		
	Need for exit ticket [D1]		

recording the linking cubes comparison on a tape diagram with the numbers 1–20, and writing the relationship as an equation (suggestions related to mathematics and pedagogy).

Moreover, if we examine the contributions of the experts (specialist and researchers), we see that the researchers focused on the goals of the lesson (meaning of comparison subtraction [D1, D2]) and the level of cognitive demand

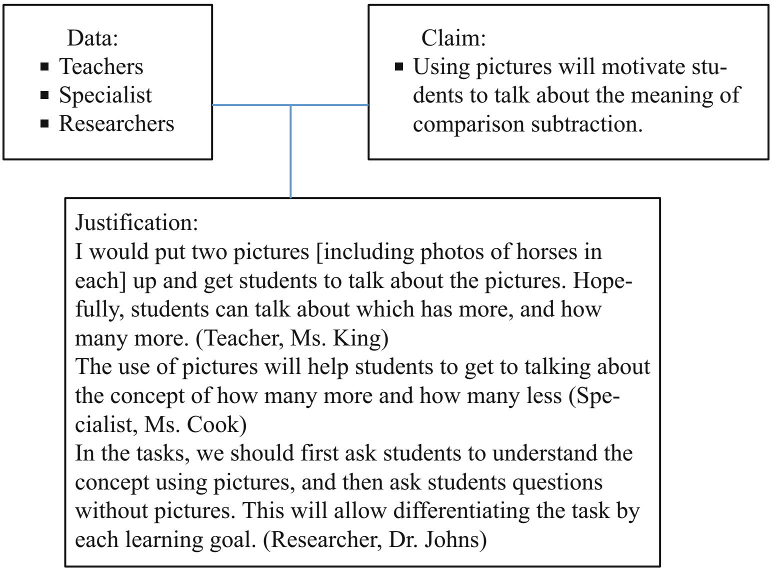

Fig. 3 Example of making a claim about using pictures in debrief 1

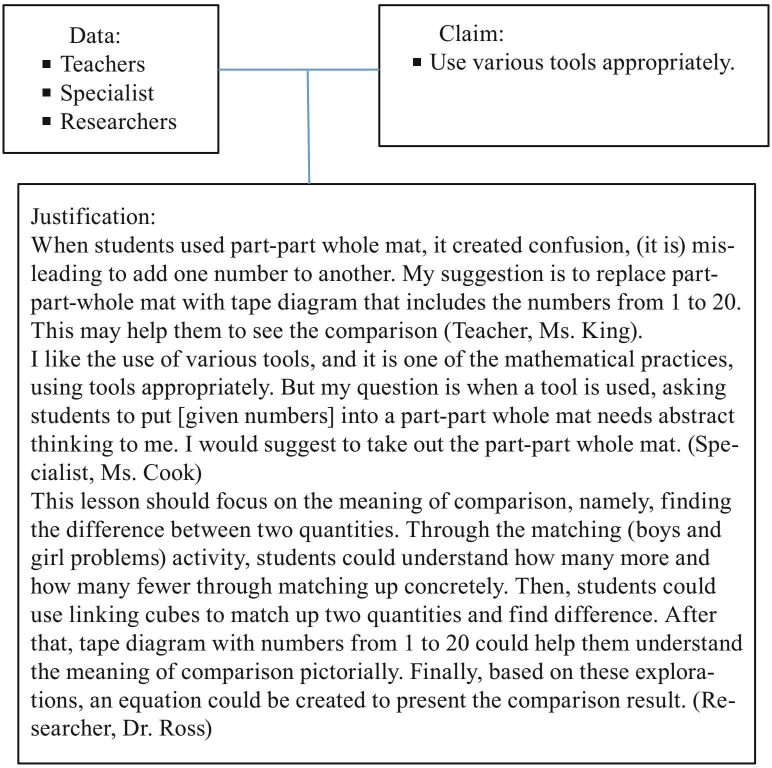

Fig. 4 Example of making claim of using tools appropriately in debrief 1

(appropriateness of cognitive level [D1]), both of which are essential components of the lesson. Regarding the cognitive demand, removing the two-step, open-ended problem created time in the lesson to focus on the learning goal of making sense of comparison problems. However, after Lesson 1 the learning goals needed further clarification, and how to achieve them needed increased thought and discussion. For example, in the first lesson, the teacher couched a comparison situation as a part-part-whole relationship (finding a missing part when given the whole) although the question involved how many more and how many less. During the first debrief, the researchers pointed out that the lesson should focus on the meaning of comparison and finding the difference between two given numbers in order to determine how many more and how many fewer. The group agreed to remove the part-part-whole mat and focus on how many more and less in the second lesson. Yet, in the second lesson, the emphasis was on making sense of how many more and how many less and finding the difference in multiple ways (including counting on or counting down to find a missing addend), with little attention on creating and explaining subtraction equations. During the second debrief, a debate occurred about the appropriateness of finding the difference by using a missing addend strategy involving counting on or down or if finding the difference should focus more exclusively on the amount that each of the numbers shared. The researchers argued that this lesson should focus on the latter based on CCSS. Then, the teachers creatively suggested ways to achieve the goals: making sense of comparison and creating subtraction equations through number talks, girl and boy matching activity, and, then, building multiple representations of comparison problems using linking cubes, drawings, and tape diagrams with numbers and number sentences simultaneously. These constructive and practical suggestions laid the foundation for the success of the third lesson with which all members of the LS group were quite satisfied.

With regard to the specialist's contribution, her rich experience in teaching and classroom observation helped her provide constructive suggestions regarding various aspects of classroom teaching (use pictures associated with tasks strategically [D1]; multiple representations from concrete to abstract [D1, D2]; need of exit ticket [D2]; use pictures in number talk [D1]; physically matching up [D2]; use various tools appropriately [D1, D2]). For example, during the first debrief, she emphasized the number talk with a picture could elicit "talk about the concept of how many more and how many less. And this will bridge from number talk to the next topic of the lesson." These ideas were emphasized in research Lessons 2 and 3.

The teachers' contributions to the debrief are fundamentally important because the researchers stated that the teachers were free to take or reject any ideas from the debriefs based on their judgment. So, teachers must understand and appreciate the ideas from external experts and take ownership of developing shared ideas, in addition to contributing ideas of their own. The following ideas were from teachers: designing an exit ticket requiring drawing objects, using a tape diagram with numbers from 1 to 20 and equation simultaneously, and creating diagrams, charts, and equations in conjunction with girl and boy matching activities and the number talk. It was the teachers' creative inputs and passion about implementing some good ideas and taking risks that made the improvement of teaching possible.

4.2.2 Improving Teaching Through the Dynamics Between Enactment and Reflections

Across the three research lessons (Table 2), there were five salient improvements: (1) a focus upon the meaning of comparison subtraction during the number talk; (2) the use of and support for the transfer between multiple representations (acting out the problem, using manipulatives, diagrams, and equations); (3) progression from simple to complex tasks; (4) appropriate use of tools (linking cubes, diagrams, number lines); and (5) use of an exit ticket to assess students' performance on contextual comparison subtraction problems. In the following discussions regarding debriefs, we illustrate how the dynamics between enactments and reflections helped to (1) shift the focus to the meaning of comparison subtraction and (2) use tools appropriately.

During the first debrief, the researcher (Ross) reminded the LS group, "In this lesson we should focus on the meanings of comparison and difference between two numbers." To achieve this goal, he suggested:

> First of all we need to develop the matching concept through physical matching activity, help students understand the meaning of how many more or less; then we need to use manipulatives (linking cubes) to model how many more or less; third, we need to draw diagrams (strip diagram or number line) to represent the relationship between two numbers. Finally, we need to use an equation to present and explain what it means by 11 is four more than seven. We need to provide relevant scaffoldings to support students to move from concrete to abstract.

Meanwhile, the specialist (Cook) realized that "asking students to put into part-part whole [mat] to me needs abstract thinking," and she recommended using ten frames. However, the teacher (Luna) pointed out "ten frames creates confusion for students when using it for putting together, it creates confusion, misleading to addition." She further suggested "using a tape diagram that has the numbers from 1 to 20." In addition, the specialist (Cook) suggested that "during the number talk, compare two numbers, for instance comparing 8 and 6, which one is more or fewer, how many more or how many fewer, we may ask this kind of question." Moreover, the open-ended, multiple-step task was believed too challenging to include. Enactments for the second teaching included the removal of the part-part-whole mat (ten frames were still used) and emphasizing the meaning of comparison subtraction (how many more/how many fewer) through a number talk, physical matching activity, and creating a subtraction equation. However, finding the difference still heavily relied on finding missing addend (counting on or counting down). Only 25% of the students could solve a contextual comparison problem using subtraction; the others still used addition.

During the second debriefing, the value of linking cubes and a number line (tape diagram with numbers from 1 to 20) was highlighted. The researchers (Ross and Long) appreciated how the teacher used linking cubes to show the difference, how to get the difference by matching and removing, and that students could correctly explain their thinking. The specialist (Cook) also confirmed the role of using linking cubes and emphasized asking questions like how many more and how many less during the number talk and comparing boys and girls. The teacher (Luna) realized the effect of drawing circles [to represent objects], drawing a number line [to record number of objects], and writing a number sentence. There was a debate about the correctness of equations of 7–4 = 3 and 7–3 = 4 or 4 + 3 = 7 to represent the solution to the question: *John has seven apples, Mary has four apples, how many more apples does John have than Mary?* Although each equation results in the same answer, the researcher (Ross) argued that only the first equation represents the relationship between the quantities presented in the problem. He explained:

> We should focus on the goal of this lesson: how many more and how many less. What does it mean? It is always to compare two quantities. We want to compare two quantities A, B, and we want students to see how to compare two quantities. We want students to understand that comparison of two different groups means to find the difference of the two quantities. We want students to develop an understanding of comparing which means to find differences of two quantities by subtraction. Students must always know that the difference [of two numbers] is equal to the bigger quantity minus the smaller quantity.

Then the specialist (Cook) suggested "The question must be asked, what is the difference between two numbers ... for instance 14–8, did the student answer this question?" She further added "in the previous lesson you showed physically by pencil matching, and then using linking cubes to find the difference by taking away [the same part], it is great job. Then you can ask [students] to write the equation ... you can ask other students if they understand this." The researcher (Ross) then recommended selecting numbers for tasks from smaller to larger purposefully.

Finally, the teams summarized the recommended changes for the next lesson. Specifically, the number talk with a picture should remain, and the activity involving physical matching to comparing the numbers of boys and girls should be reinstated. The group reiterated that the goal of focusing on the meaning of comparison subtraction leading to a subtraction equation still needed increased focus. Learning should be scaffolded to develop understanding through a number talk, physical comparing by matching, modeling using linking cubes, modeling using a number line, and finally creating an equation. In the third teaching, the teacher implemented these suggestions. Eventually, students' performance on the exit ticket increased from 25% to 63% correct.

4.3 Participant Teachers' Perception of the Lesson Study and Their Perceived Learning

4.3.1 Overall Perception of the Lesson Study

All three teachers expressed enthusiasm with regard to the LS process and product as illustrated by Ms. King, "I was proud of our work on the compare lessons and enjoyed learning and participating in watching these lessons change, evolve, and enhance the learning of the children involved in this project." They all acknowledged the benefits of the collaborative reflection process and feedback from peers and experts that provided different perspectives and drew their attention to students' learning. For example, Mr. Mills said "I can certainly see the value in sharing immediate feedback with educators as we study a particular lesson and then re-teaching the same lesson soon after, with changes and improvements to the lesson. It was creating and then actually observing the evolution of a lesson with peers, and being able to witness the changes both positive and negative that helped to improve and enhance the lesson." Ms. King further highlighted the collaborative reflective process by saying "the collaborative aspect of this study, along with the expert knowledge of the facilitators, proved to be the most advantageous elements in this process. It was refreshing to engage in a collaborative reflective revision process, and the feedback we received enabled us to greatly improve our lesson plan and our students' learning." Moreover, they also acknowledged the importance of experts' input as stated by Mr. Mills:

> The most impactful for me was when Dr. Ross mentioned using the tape diagram more as a bridge to the linking cubes manipulatives. Finally, for me, I realized how I needed to adjust my teaching and bring this idea back to my own classroom to help shift my students from the concrete to the abstract. This comment helped me reshape my lesson and improve my instruction so that my students could successfully solve comparison problems.

4.3.2 Perceived Benefits

The teachers acknowledged benefits from participating in the LS such as developing content knowledge, pedagogical knowledge, and knowledge about students' learning; enhancing specific teaching skills including utilizing mathematical discourse about mathematics practices; and professional growth, in general.

Regarding *mathematical content knowledge*, Ms. King said: "The opportunities to observe my teammates in the lesson trials not only afforded me the chance to witness effective teaching techniques, but it also allowed me to see the lesson plan come to life along with its effect on student learning. This experience enriched my understanding of the mathematical content as I was able to observe and discuss student thinking."

All three teachers acknowledged gain in *mathematics pedagogical knowledge*. Mr. Mills said: "I can see the LS becoming a valuable way to help all teachers sharpen and improve their craft and at the same time have valuable discourse on mathematical practice." More specifically, Mrs. Luna said: "Through the lesson study I learned a lot about teaching word problems in first grade. The LS gave me a whole new thinking on teaching word problems to my first graders."

Understanding student learning. Each of the three teachers realized the LS helped them better focus on and improve student learning. Ms. King said, "By and large, this [lesson study] opened my perception of their understandings and misconceptions which prompted deep reflection, discussion, and ultimately the progression of the lesson to its most productive stage." Ms. Luna further explained, "Over the course of the lesson study we were able to make the necessary changes and see our students grow their mathematical thinking of word problems. Some exciting moments for me during the lesson study were just seeing the students grow through the process."

Beyond specific knowledge growth, all recognized *their professional growth* as well. They have already utilized and will continue to implement what they learned from the LS in their classrooms and will share their experiences in the LS with their peers. For example, Mrs. Luna said, “I was able to take what I learned over the course of the lesson study and share the ideas with fellow colleagues.” Similarly, Mr. Mills said, “I am excited to share this idea with teachers in my own building as we have more time to study, share, and implement this project on a daily and weekly basis.” Furthermore, Ms. King expressed how this learning experience reshaped her professional capacity as follows:

> Nevertheless, because of this experience I feel far more capable in my abilities to design a lesson trajectory that would better serve my students’ learning. I now find myself to be more intentional in observing and reflecting upon my students’ thinking to more thoughtfully write and revise my lesson plans so that I may better support the success of their learning.

5 Discussion and Conclusion

This study in general demonstrated how Chinese lesson study, taking place in a US setting, can improve teaching that promotes students’ conceptual understanding and participating teachers’ learning as well. More specifically, the study revealed the process and effects of the dynamics between enactment and reflection facilitated by knowledgeable others. This study showed that during the *debriefing session*s, both knowledgeable others and participating teachers collaboratively worked together in critically analyzing the observed research lessons and constructively providing suggestions on improving them. However, the roles played by knowledgeable others and teachers were different. The knowledgeable others focused on the learning goals of the lesson, the learning trajectory and the selection of appropriate tasks associated with the learning trajectory, and the selection of appropriate tools. The teachers focused on specific instructional techniques for supporting students’ learning along the learning trajectory such as using a number talk, matching activity, creating a tape diagram, and designing an exit ticket. The study also showed that the ideas emerging from debriefing sessions need to be verified and refined through reenactments. As seen in the study, understanding and implementing the learning goal of this lesson of solving contextual comparison problems (finding the difference when given two quantities, rather than part-part or missing addend addition) took three iterations of enactments and debriefs. This study revealed that both the knowledgeable others’ facilitation and the interactions of enactment and debrief are crucial for maximizing effects of LS in improving teaching and promoting teachers’ learning.

5.1 Discussion

The findings about the effects of lesson study and the necessity of the involvement of knowledgeable others are well documented in other literature (e.g., Huang and Shimizu 2016; Lewis et al. 2009; Takahashi and McDougal 2016; Xu and Pedder 2015). Since CLS includes a goal-oriented design and repeated enactment of a lesson with immediate feedback from knowledgeable others, researchers have argued that it is a type of deliberate practice (Han and Paine 2010; Huang et al. 2017). We discuss two unique contributions of this study from the perspective of deliberate practice (Ericsson et al. 1993).

5.1.1 Characteristics of Knowledgeable Others and Working Modes with Teachers

According to deliberate practice, the immediate feedback from experts is critical, but it is not clear about the nature of feedback and ways of generating the feedback. Within the context of LS, this study portrayed a clear picture about the nature of feedback from both knowledgeable others and teachers. The former contributed more about overarching ideas about essential understanding of the concept (goals of learning) and progression toward achieving the learning goals (learning trajectory) and task design associated with the learning trajectory, while the latter contributed more toward providing specific techniques of designing activities (such as girl and boy matching), use of pictures and diagrams, and exit tickets. The synergy of the cogenerated ideas motivated teachers to experiment with new ideas and improve their teaching. Thus, it is crucial to build a working norm during debriefing sessions, which is honest, respectful, critical, and constructive. This can be achieved if the team focuses on teaching and student learning, rather than evaluating the teachers. Theoretically, the study contributes to understanding the role of knowledgeable others in LS, beyond the roles as final commentary providers identified by Takahashi (2014) and experts in deliberate practice in general (Ericsson et al. 1993).

5.1.2 Self-Motivation by Seeing Students' Growth and Their Own Improvements

One of the features of deliberate practice is to require participants' significant effort and concentration. In LS a group of teachers focuses on teaching the same topic repeatedly. Teachers must be motivated and have a passion about making this commitment. This study showed that with an initial interest in learning through LS, the participating teachers could enhance their interest and passion through seeing that a "small change in the lesson made a huge difference in learning outcomes" (Huang et al. 2017, p. 376). This echoes Ms. King's enjoyment from "watching these lessons change, evolve, and enhance the learning of the children involved in this project." It is the iteration process of LS that provides the opportunities for teachers to verify their ideas (generated from the debriefs) and to see the effect of student learning. This is a self-motived and rewarding process as shown in this study. Theoretically, this study enriches the understanding of deliberate practice within the context of LS.

5.2 Conclusion

This study not only makes a case that the Chinese lesson study approach could be an alternative approach for teacher professional development in other counties, but more importantly, this study sheds light upon the critical mechanism of interplay between enactment and reflections during lesson study. It paints how knowledgeable others and teachers cogenerate ideas based on their observation of the research lesson and how those ideas could be verified and refined with the goals of improving teaching that promotes student learning. Thus, it deepens an understanding of theory of teacher learning through practice (Clarke and Hollingsworth 2002). Practically, this study also provides implications for developing facilitators of lesson study: what knowledge they need to have and how they may develop their expertise (Borko et al. 2015; Carlson et al. 2017), which is a critical component of scaling up lesson study (Akiba and Wilkinson 2016).

References

Akiba, M., & Wilkinson, B. (2016). Adopting an international innovation for teacher professional development: State and district approaches to lesson study in Florida. *Journal of Teacher Education, 67*(1), 74–93.

Archibald, S., Coggshall, J. G., Croft, A., & Goe, L. (2011). *High-quality professional development for all teachers: Effectively allocating resources*. Washington, DC: National Comprehensive Center for Teacher Quality.

Borko, H., Jacobs, J., Koellner, K., & Swackhamer, L. (2015). *Mathematics professional development: Improving teaching using the problem-solving cycle and leadership preparation models*. New York: Teachers College Press.

Bronkhorst, L. H., Meijer, P. C., Koster, B., & Vermunt, J. D. (2011). Fostering learning-oriented learning and deliberate practice in teacher education. *Teaching and Teacher Education, 27*(7), 1120–1130.

Bronkhorst, L. H., Meijer, P. C., Koster, B., & Vermunt, J. D. (2013). Deliberate practice in teacher education. *European Journal of Teacher Education, 37*(1), 18–34.

Carlson, M. A., Heaton, R., & Williams, M. (2017). Translating professional development for teachers into professional development for instruction leaders. *Mathematics Teacher Educators, 6*(1), 27–39.

Chazan, D., Sela, H., & Herbst, P. (2012). Is the role of equations in the doing of word problems in school algebra changing? Initial indications from teacher study groups. *Cognition and Instruction, 30*, 1–38.

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal for Lesson and Learning Studies, 6*(4), 283–292.
Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*(3), 218–236.
Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*(8), 947–967.
Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf.
Corbin, J., & Strauss, A. (2008). *Basics of qualitative research* (3rd ed.). Los Angeles: Sage.
Dean for Impact. (2016). *Practice with purpose: The emerging science of teacher expertise*. Austin: Deans for Impact. Available at https://deansforimpact.org/wp-content/uploads/2016/12/Practice-with-Purpose_FOR-PRINT_113016.pdf.
Desimone, L. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*(3), 181–199.
Dunn, T. G., & Shriner, C. (1999). Deliberate practice in teaching: What teachers do for self-improvement. *Teaching and Teacher Education, 15*(6), 631–651.
Elliott, J. (2012). Developing a science of teaching through lesson study. *International Journal for Lesson and Learning Studies, 1*, 108–125.
Ericsson, K. A. (2008). Deliberate practice and acquisition of expert performance: A general overview. *Academic Emergency Medicine, 15*(11), 988–994.
Ericsson, K. A., Krampe, R., & Tesch-Romer, C. (1993). The role of deliberate practice in the acquisition of expert performance. *Psychological Review, 100*, 363–406.
Gog, T. V., Ericsson, K. A., Pikers, R. M. J. P., & Paas, F. (2005). Instructional design for advanced learners: Establishing connections between the theoretical frameworks of cognitive load and deliberate practice. *Educational Technology Research & Development, 53*(3), 73–81.
Han, X., & Paine, L. (2010). Teaching mathematics as deliberate practice through public lessons. *The Elementary School Journal, 110*(4), 519–541.
Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*(3), 279–298.
Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.
Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher developers, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*, 393–409.
Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese lesson study. *ZDM Mathematics Education, 46*, 239–251.
Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM Mathematics Education, 48*, 425–439.
Huang, R., Barlow, A. T., & Haupt, M. E. (2017). Improving core instructional practice in mathematics teaching through lesson study. *International Journal for Lesson and Learning Studies, 6*(4), 365–379.
Lewis, C. (2016). How does lesson study improve mathematics instruction? *ZDM Mathematics Education, 48*, 571–580.
Lewis, C., & Lee, C. (2017). The global spread of lesson study: Contextualization and adaptations. In M. Akiba & G. K. Letendre (Eds.), *International handbook of teacher quality and policy* (pp. 185–203). New York: Routledge.
Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial fractions learning. *Journal for Research in Mathematics Education, 48*(3), 261–299.
Lewis, C. C., Perry, R., & Hurd, J. (2009). Improving mathematic instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*, 285–304.

Skultety, L., Gonzálet, G., & Vargas, G. (2017). Using technology to support teachers' lesson modifications during lesson study. *Journal of Technology and Teacher Education, 25*(2), 185–213.

Sowder, J., Sowder, L., & Nickerson, S. (2014). *Reconceptualizing mathematics for elementary school teachers* (2nd ed.). New York: W.H. Freeman & Company.

Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 4–12.

Takahashi, A., & McDougal, T. (2014). Implementing a new national curriculum: A Japanese public school's two year lesson-study project. In K. Karp & A. R. McDuffie (Eds.), *Using research to improve instruction* (pp. 13–22). Reston: NCTM.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM Mathematics Education, 48*(4), 513–526.

Toulmin, S. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

Van de Walle, J., Karp, K. S., & Bay-Williams, J. M. (2016). *Elementary and middle school mathematics: Teaching developmentally (9 editions)*. Boston: Person Education.

Widjaja, W., Vale, C., Groves, S., & Doig, B. (2017). Teachers' professional growth through engagement with lesson study. *Journal of Mathematics Teacher Education, 20*, 357–383.

Xu, H., & Pedder, D. (2014). Lesson study: An international review of the research. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 29–58). London/New York: Routledge.

Xu, H., & Pedder, D. (2015). Lesson study: An international review of the research. In P. Dudley (Ed.), *Lesson study: Professional learning of our time* (pp. 29–58). Abingdon: Routledge.

Yang, Y. (2009). How a Chinese teacher improved classroom teaching in teaching research group: A case study on Pythagoras theorem teaching in Shanghai. *ZDM Mathematics Education, 41*, 279–296.

Zhou, Z., & Lin, J. (2001). Developing mathematical thinking in Chinese kindergarten children the case of addition and subtraction. *Journal of Educational Policy, Research, and Practice, 2*, 141–145.

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Race to the Top and Lesson Study Implementation in Florida: District Policy and Leadership for Teacher Professional Development

Motoko Akiba, Aki Murata, Cassie Howard, Bryan Wilkinson, and Judith Fabrega

Contents

Abstract Lesson study was introduced to school districts in Florida in the United States as part of the federal government's Race to the Top Program in 2010 to scale improvement in instruction and student learning. However, little is known about what district policy and leadership characteristics are associated with the level of lesson study implementation. Based on a mixed methods study of a statewide survey and interviews of district professional development directors, we found that district requirement of lesson study, funding provision, and future sustainability plan were significantly and positively associated with a broader implementation of lesson study within the district. The interviews revealed that the districts that implemented lesson study districtwide first internalized lesson study through communicating and funding a districtwide expectation of job-embedded, inquiry-based professional development. Following this internalization, the district leaders institutionalized it by supporting school ownership and leadership in organizing and embedding lesson

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_35

study into the school organizational structures and routines. Implications for educational leaders at local educational agencies are discussed.

Keywords Scale · System-level instructional improvement · Professional development · Lesson study · Mixed methods study

1 Introduction

With the implementation of the Common Core State Standards (CCSS) in many states since 2014 in the United States, it became urgent that instruction provided across the country reflect the goal of CCSS to ensure students' opportunity to master necessary knowledge and skills to become successful in our society. The CCSS were collectively developed by educators and experts to provide a guideline for what today's students may be expected to know and do from kindergarten to 12th grade. Their increased focus on developing students' analytical, problem-solving, and critical thinking skills along with procedural fluency is seen as a shift from previous standards that may not support conceptual understanding of the content (Common Core State Standards Initiative 2010a, b). Teacher professional development plays a critical role in supporting implementation of the ambitious instruction envisioned in the CCSS, which is still experientially unfamiliar to many teachers (Marrongelle et al. 2013).

Lesson study is one approach to teacher learning which builds on core principles of teacher ownership, collaboration, and inquiry into teaching and student learning in a chosen subject topic (Hart et al. 2011; Lewis 2002; Lewis and Hurd 2011). Lesson study was imported to the United States in the late 1990s after the publication of *Teaching Gap* (Stigler and Hiebert 1999). This comparative video study of mathematics instruction, as part of the 1995 Trends in International Mathematics and Science Study (TIMSS), identified lesson study as the driving force for Japanese teachers' student-centered problem-solving instruction that promotes conceptual understanding—a type of ambitious instruction promoted by CCSS. Since then, lesson study has been practiced by an increasing number of teacher groups, schools, and districts across the United States (Lewis et al. 2006; Perry and Lewis 2010).

Florida is the first state that promoted lesson study as a statewide model of professional development, using part of the US$700 million Race to the Top (RTTT) funding they received in 2010 (FLDOE 2010a). Race to the Top was a competitive, federal grant program in which states were awarded funding by developing innovative plans for improvement, such as adoption of rigorous standards and assessments and turning around low-achieving schools (U.S. Department of Education 2009). The current study focuses on the district implementation of lesson study across the state of Florida. School districts select, mandate, finance, and facilitate teacher professional development, serving as a driver for districtwide improvement of instruction and student learning (Coburn and Russell 2008; Elmore and Burney 1999; Firestone et al. 2005; Floden et al. 1988; Gamoran et al. 2003; Hightower et al.

2002; Knapp 2003; Little 1989, 1993; Spillane 1996; Stein and Coburn 2008; Stein and D'Amico 2002; Sykes et al. 2009). There is a need to better understand what district policy and leadership practices lead to a districtwide implementation of a promising professional development approach such as lesson study and how the districts sustain the implementation after the RTTT program ended.

To examine the characteristics of district policies and leadership practices associated with the level of lesson study implementation, we conducted a mixed methods study of a statewide survey and interviews of district professional development directors in 2014 and 2015. Conducting the survey in these 2 years allowed us to consider the possible impacts of federal funding through the RTTT program that ended in 2014 and identify factors that allowed some districts to sustain lesson study after the RTTT program. Considering the fact that many top-down reform initiatives do not sustain in many schools after the program and funding end (Datnow 2005; Fink 2000; Giles and Hargreaves 2006; McLaughlin and Mitra 2001), understanding what led some districts to sustain lesson study initiated by the Florida RTTT program will provide important leadership and policy implications.

Specifically, this study addressed the following questions:

1. What variation exists in the level of lesson study implementation across 58 Florida school districts, and how did the implementation level change after the RTTT program ended?
2. What district-level policies and leadership practices are associated with the level of lesson study implementation?
3. How did district leaders approach lesson study to achieve a districtwide implementation?

2 Background

2.1 Previous Studies on District Professional Development

Previous research on district leadership for professional development in the United States suggest that three factors could influence the implementation level of lesson study: (1) district policy on lesson study, (2) funding allocation, and (3) district professional development leadership (Coburn 2003; Firestone et al. 2005; Knapp 2003; McLaughlin and Mitra 2001; Perry and Lewis 2009, 2010).

First, given the discretion and autonomy given to districts whether and how to implement lesson study, district policy that requires lesson study would likely influence the level of lesson study implementation. District "mandate" is an important policy instrument for communicating the district priority in professional development (Knapp 2003). The district leaders may also use a policy to communicate the district vision and establish coherence in professional development (Firestone et al. 2005). Based on a comparison of three urban districts, Firestone et al. (2005) found that the district with a clear vision and emphasis on professional development implemented coherent and content-focused professional development, and the teachers in the district reported a greater influence on teaching practice. We hypothesize that the districts that require all schools to implement lesson study as a district priority would be more likely to report a higher level of lesson study implementation.

Second, districts' continued provision of funding in the forms of substitutes and stipends for teachers is a critical condition for supporting instructional improvement (Coburn 2003; McLaughlin and Mitra 2001). The importance of funding for districts to provide high-quality professional development was also identified in previous empirical studies (Akiba et al. 2015; Desimone et al. 2002; Gamoran 2003; Spillane and Thompson 1997). Considering the heavy teaching load of US teachers compared to that of teachers in other countries (Liang and Akiba 2018), provision of substitutes for planning meetings and research lessons during the regular school hours and extra payment for meeting outside the regular school hours would serve as major financial incentives for teachers to engage in lesson study (Murata 2011; Yoshida 2012). We hypothesize that the districts that provide substitute funding and teacher payment are more likely to report a higher level of lesson study implementation.

Finally, district leadership in promoting lesson study would play an important role in the implementation of lesson study districtwide (Perry and Lewis 2009, 2010). Having a designated position in charge of lesson study at the district level would ensure that ongoing support be provided to schools and teachers. Stability in professional development leadership position would also be important for providing continued support of lesson study, as previous studies have documented the detrimental effects of leadership turnover on sustaining and scaling reform initiatives (Hargreaves and Fink 2003; McLaughlin and Mitra 2001). In addition, district leaders' future plan to sustain their support of lesson study would likely send a coherent message to school leaders and teachers that lesson study is a district priority (Firestone et al. 2005) and likely draw their commitment to practice lesson study. Therefore, we hypothesize that the districts with a designated position for lesson study, stable professional development leadership, and a future sustainability plan to continuously support lesson study would be more likely to report a higher level of lesson study implementation.

In addition to the studies on district leadership for professional development, a body of research on scaling up a reform initiative provides important insights into districtwide implementation of a professional development model (Coburn 2003; Dede et al. 2005; McLaughlin and Mitra 2001). A synthesis of findings from cases of districts and schools that successfully scaled up educational innovations conducted by Dede and Honan (2005) identified four key factors to promote and support scaling up improvement at a system level: (1) coping with changes in context, leadership, and funding; (2) promoting ownership by valuing constituent input and support; (3) building human capacity; and (4) engaging in effective decision-making by interpreting data and creating and applying usable knowledge. McLaughlin and Mitra (2001) examined scaled implementations of three reforms based on promising theories of learning and instruction: Fostering Communities of Learners, Schools for Thought, and the Child Development Project. They reported the critical importance of building reform-centered knowledge and leadership capacity within the many levels of the school system so that schools and districts can adhere to core principles even if materials and practices may be adjusted to the local contexts. These studies showed that "adaptation" of an innovation is a natural part of scaling up across diverse contexts with different resources and point to the importance of ownership and capacity building so that schools and districts can effectively adapt the innovation without altering the core principles that lead to improved instruction and student learning.

2.2 *Florida Context*

Lesson study was first introduced to the Florida Department of Education (FLDOE) by the Chancellor of Public Schools who visited Japan around 2008 (Akiba 2016; Akiba and Wilkinson 2016). When the US Department of Education announced the RTTT program in 2009, FLDOE included lesson study as one of the 13 projects in its RTTT application submitted in 2010 (FLDOE 2010a). In the same year, FLDOE was awarded US$700 million and invited all 72 districts (67 regular districts, 4 university lab schools, and 1 virtual school) to submit a proposal using a state-provided template to receive part of the RTTT funding. A total of 65 districts (90% of 72 districts) submitted a proposal in late 2010 describing their 4-year plans to implement the 13 projects and budget request. All the district proposals were approved, and these 65 districts received a total of approximately US$350 million in 2011 to implement the 13 projects (FLDOE n.d.-a).

The district proposal template for "Project 1: Expand Lesson Study" prepared by FLDOE included a state policy and a compliance procedure (FLDOE 2010b). The policy states "A local education agency (LEA) with a Persistently Lowest Achieving (PLA) school will modify these schools' schedules to devote a minimum of one lesson study per month for each grade level or subject area" and specified four deliverables that LEAs with at least one PLA school are required to submit annually (FLDOE 2010b, p. 6): (a) school schedule in each PLA school that includes regularly scheduled blocks of time dedicated to lesson study for each grade level or subject area, (b) rosters of lesson study participants, (c) lesson plans used for lesson study, and (d) one improved lesson plan as a result of lesson study. The proposal template further specified the importance of sustainability by asking the districts to provide "a short description or list of factors that will contribute to the sustainability of the results of this Project (lesson study) after Race to the Top funding ends" (FLDOE 2010b, p. 7).

FLDOE specified a total of 71 schools in 25 districts as PLA schools in the 2010–2011 academic year and explained that this list would not change during the 4-year grant period for purposes of the RTTT program (FLDOE n.d.-b). Of these 25 districts with at least one PLA school, 23 districts participated in the RTTT program; thus the FLDOE requirement applied to 66 schools in these 23 districts. These schools constitute only 2% of the total of 3,450 schools in 67 regular districts across Florida; thus the state policy scope was quite limited. Despite the limited scope of the state policy, lesson study has spread across the state, and statewide surveys of district leaders reported that 668 schools across 46 districts practiced lesson study in 2014 (19% of Florida schools) and 749 schools across 39 districts (22% of Florida schools) did so in 2015 (Akiba et al. 2016).

A previous mixed methods study conducted in 2013 revealed major challenges with implementing this international innovation within different districts' organizational structures and routines of teacher professional development (Akiba and Wilkinson 2016). The authors found that many districts requested limited funding for lesson study due to the lack of awareness of time-intensive nature of lesson study. The district survey data showed that only 12 districts requested RTTT funding for lesson study and only 23 districts and 7 districts provided funding for substitutes and teacher payment, respectively—two types of funding critical for lesson study implementation (Murata 2011; Yoshida 2012). As a result, many districts promoted short-term and add-on approaches to lesson study. Yet, this study also observed a major variation across districts in using various policy instruments—mandates, inducements (funding), and capacity building to promote lesson study. Such a variation allows an examination of district policy and approaches that could lead to a districtwide practice of lesson study as a driving force for a large-scale instructional improvement.

3 Methods

3.1 Statewide District Survey

A statewide district survey named "Lesson Study District Survey" was conducted via the Qualtrics online survey tool between May and August in 2014 and 2015. These two years were selected for two purposes: (1) to examine the difference in the implementation level of lesson study within districts before and after the RTTT program ended and (2) to examine the differences in the district characteristics associated with the implementation level before and after the RTTT program ended. In both years, we followed the three stages of survey implementation: (1) identification of the district representative who is in charge of lesson study implementation in each district through web searches, emails, and phone calls; (2) administration of the online survey; and (3) follow-ups with nonrespondents through emails and phone calls. Of the total of 72 Florida districts, we decided to focus on 68 districts (67 regular districts and a virtual school district) and reached out to the district professional development directors first by emails and phone calls to identify the individuals who are in charge of lesson study implementation at the district level. Then we sent out a survey invitation email with a link to the online Lesson Study District Survey.

In both years, the survey first defined lesson study as "a continuous professional development process that involves a group of 3–6 teachers collectively engaging in four stages: (1) goal setting and study, (2) lesson planning, (3) research lesson, and (4) debriefing session." The survey included questions on seven major topics:

(1) district policy on lesson study (the number and types of schools required to practice lesson study); (2) number of schools that practiced lesson study including both required and volunteered schools; (3) funding provision for substitutes and teacher stipends; (4) district-level designated position for promoting lesson study; (5) future plan to continue the district support of lesson study in the following year; (6) other professional development programs implemented; and (7) open-ended comments about their experience with lesson study.

The survey participants received a link to a $25 online gift card of a major retailer upon completion in 2014 as an incentive. In 2015, participants received a $20 online gift card. After multiple emails and follow-up phone calls, as of August, 58 districts participated in each of the 2014 and 2015 surveys with a response rate of 85%. Of these 58 districts, 53 districts participated in both 2014 and 2015 survey (5 other districts in each year participated only in one year).

3.2 Variables

This study analyzed the implementation level of lesson study as the dependent variable and three domains of district characteristics as independent variables.

Lesson Study Implementation The professional development directors (or other district personnel in charge of lesson study) were asked in the survey, "How many schools in total practiced lesson study in your district during the 2013–2014 (2014–2015) academic year? Please include both the required schools and the schools that voluntarily practiced lesson study." Based on the number of schools reported in the survey, we computed the percentage of schools that practiced lesson study by dividing the number by the total number of regular schools in the district as the district level of lesson study implementation.

District Policy For the scope of lesson study requirement during the preceding year (2013–2014 and 2014–2015), the district professional development directors were asked to choose from three options: 1 = no school was required, 2 = only some schools (e.g., PLA or low-achieving schools) were required, and 3 = all schools in the district were required.

Funding Allocation The district professional development directors were asked whether they provided funding for (1) substitutes for teachers to participate in lesson study and (2) stipends for teachers who participate in lesson study outside the regular school hours. Their responses were coded as 1 = yes and 0 = no.

District Leadership Three variables were developed to measure the characteristics of district leadership: (1) designated position for lesson study, (2) professional development director stability, and (3) future sustainability plan. The survey respondents were asked, "Did your district have a designated facilitator or coordinator at the district level whose main responsibility was to facilitate lesson study during the 2013–14 (or 2014–2015) academic year?" and their responses were coded as 1 = yes and 0 = no. To measure the professional development director stability, we examined if there was a turnover in the position from 2013 to 2014 and from 2014 to 2015 based on the information from the district websites or email/phone communications. The districts which had the same director were coded as 1, and the districts which had two different directors from one year to the next were coded as 0. To measure future sustainability plan, the survey participants were asked, "Does your district have a plan to continue to support lesson study during the 2014–15 (or 2015–2016) academic year?" Their responses were coded as 1 = yes and 0 = no.

3.3 Interviews of Districts with a High Level of Lesson Study Implementation

To address the third research question, "How did district leaders approach lesson study to achieve a districtwide implementation?," we identified four midsized districts where the district leaders reported at least 90% of schools practiced lesson study, and we validated that their lesson study practice followed the core principles. We decided to focus on the districts that implemented lesson study districtwide in order to understand district leaders' motivation and rationales behind making various district-level decisions and how these decisions contributed to implementing lesson study districtwide and sustained the high level of implementation even after the RTTT program ended.

Of the four districts, professional development directors in three districts—Albany, Morison, and Lester (pseudonyms)—agreed to participate in the interview in Fall 2015. During the semi-structured interviews, the researchers asked a list of questions grouped into four domains: (1) influence of RTTT and district policy, (2) lesson study approach, (3) funding, and (4) sustainability and changes. The interviews lasted from 40 mins to 1 hr, which were transcribed verbatim for data analysis.

In 2015, Albany had an enrollment of 29,000 with 47% of students receiving free or reduced-price lunch (FRL) and 56% of students identifying as ethnic minority students. Morison enrolled approximately 8,000 students, 47% of whom received FRL and 52% ethnic minorities. Lester is the smallest, enrolling approximately 5,000 students, and 61% FRL students, and 29% ethnic minority students. Albany and Morison received the achievement grade of A, and Lester received B in 2015 based on the district average scores in the state assessment.

3.4 Data Analysis

To address the first question on the implementation level of lesson study, descriptive statistics and frequencies were computed. For the second research question on the relationship between district policy and leadership practice and the level of lesson study implementation, we first presented descriptive statistics on district policy and leadership practice for promoting lesson study, and then we conducted correlation, *t*-test, or ANOVA depending on the numerical nature of the variables (continuous, ordinal, or nominal). Due to the small sample size of 58 or less, we conducted only bivariate analyses.

To address the last question on how district leaders approached implementing lesson study districtwide, we created a district profile for each district summarizing the survey responses from 2014 and 2015 and the content of the RTTT district proposal and coded the interview transcripts. The coding occurred at multiple phases. In the first phase, four researchers marked the interview transcripts for broader terms that reflected our research focus. These include RTTT, policy, lesson study approach, funding, and sustainability plan. After the transcripts were coded for these broad categories, for the second phase of coding, we coded them using more specific subcategories that reflected both emergent themes and expectations based on the literature. These included the codes including district expectation, coaches' roles, lesson study templates, principal roles, teacher buy-in, securing funding from multiple sources, embedding lesson study, and lesson study as self-sustaining process. The coding at the first and second phases were discussed extensively among the four researchers to refine and finalize the subcategories. At this point, the subcategories were referenced back to the survey data for their relevance to the patterns emerging from the analysis to maintain coherence across the data analysis processes. In the final phase of the analysis, these refined codes were synthesized to identify common themes across the three cases, which were internalization, teacher professional development expectation, school ownership of lesson study process, and institutionalization. These common themes provided additional contextual explanations and examples to the findings from the surveys, to answer our research questions. Throughout each phase of coding, the researchers met and compared codes to ensure consistency of the coding process.

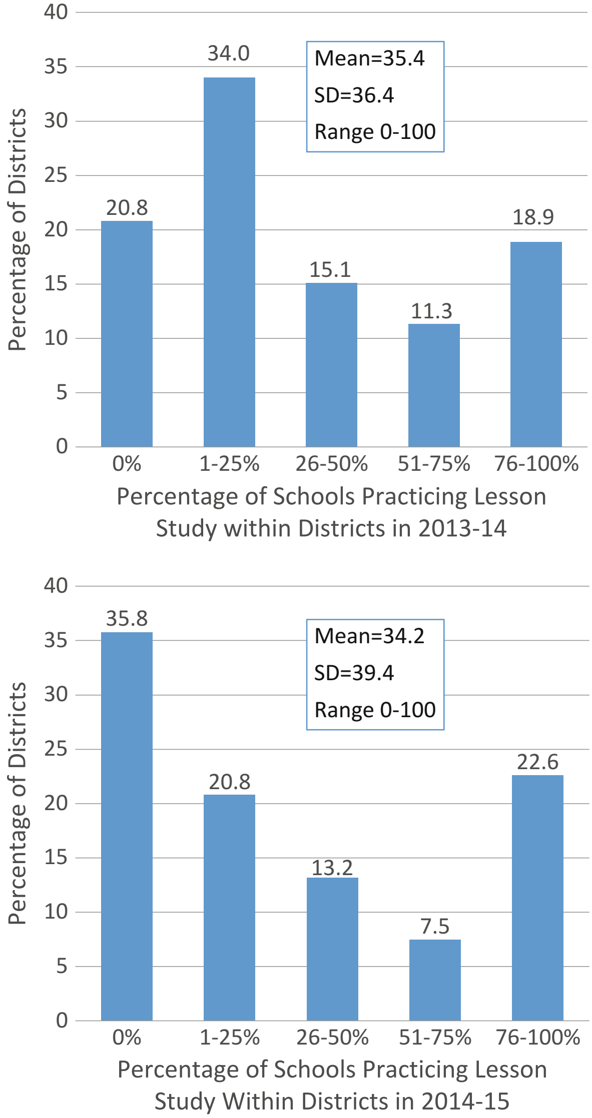
4 Results

4.1 District Level of Lesson Study Implementation

Figure 1 presents the descriptive statistics and frequencies of the level of lesson study implementation among 53 districts that participated both in 2014 and 2015 surveys in order to examine the changes in the implementation level. The original data were continuous ranging from 0 to 100%, and we present the distribution in five categories here to visually present the changes from 2014 to 2015 before and after the RTTT program ended.

We expected that the level of implementation would dramatically decline when the RTTT funding ended. However, the mean implementation level measured by the percentage of schools within districts that practiced lesson study did not change

Fig. 1 District level of lesson study implementation in 2014 and 2015



much from 2014 to 2015 (35.4% to 34.2%). The figure shows, however, that the distribution of the lesson study implementation level across 58 districts slightly changed from 2014 to 2015. As expected, the percentage of districts that stopped practicing or never practiced lesson study has increased from 20.8% to 35.8%. Yet, the percentage of districts with a high implementation level (76% to 100%) also increased from 18.9 to 22.6%. Thus, we can see a trend of bifurcation in the district level of lesson study implementation after the RTTT program ended.

4.2 District Policy and Practice in Promoting Lesson Study

Before examining the second research question on the district policy and leadership practices associated with the level of lesson study implementation, we examined the variation in the district policy and leadership practices across 58 districts in 2014 and 2015. Given the limited state requirement on lesson study, district leaders exercised their discretion in establishing policy and leadership practices for promoting lesson study in their schools. Table 1 presents the descriptive statistics of district policy and leadership practice in promoting lesson study reported in 2014 and 2015 surveys along with the district background characteristics and the districts' RTTT participation.

The descriptive statistics of the district background characteristics show that there are major variations in the size, poverty level, diversity level, and achievement level across these districts. Of the 58 districts that participated in the survey in 2014 and 2015, the district RTTT proposals were available from 52 districts. Among the 52 districts, about half specifically proposed implementation of lesson study in their 2010 district proposals to obtain RTTT funding, and 37% of these districts had at least one PLA school that was required by FLDOE to practice lesson study until 2014. The mean funding amount on lesson study the districts received in 2010 at the beginning of the RTTT program was about $70–80,000. The mean total amount of RTTT funding they received in 2010 was $4 million among the 52 districts that participated in our 2014 survey and $6 million among the 52 districts that participated in our 2015 survey.

The district policy on lesson study plays an important role in a districtwide implementation of lesson study. In 2014, 58.6% of the districts did not require any school, 24.0% required only some schools, and 17.2% required all schools to practice lesson study. In 2015, 74.1% of the districts did not require any school, 10.3% required only some schools, and 15.5% required all schools to practice lesson study (means of these three levels are presented in Table 1). Thus, there was a major overall decline in the district policy to require lesson study after the RTTT program ended in 2014 from 41.2% of districts in 2014 to only 25.8% of districts in 2015.

However, the percentage of districts that required all schools did not decline much (17.2% to 15.5%). These districts that continued to require all schools to practice lesson study seem to have figured out how to sustain lesson study without RTTT funding.

District funding allocation is a critical part of scaling up lesson study considering the limited time available during the regular school hours for teachers to engage in professional development activities. The survey data showed that in 2014, 45% of the districts provided funding for substitutes to participate in lesson study meetings and 14% of the districts provided funding for teacher payment for meeting outside the regular school hours. In 2015, the percentage of districts providing substitute funding decreased to 36%, yet the percentage providing teacher payment increased to 22%. In both years, less than half of the districts provided funding for lesson study.

Table 1 District policy and practice for promoting lesson study

		2014					2015				
		N	Mean	Min	Max	SD	N	Mean	Min	Max	SD
District background	Size (enrollment)	58	33,125	1,035	272,785	49,776	58	45,677	1,244	357,586	70,983
	Poverty	58	47.0	18.5	75.4	12.3	58	57.6	23.8	99.8	11.17
	Diversity	58	40.5	9.6	96.6	18.9	58	43.6	9.6	92.7	19.00
	Achievement	58	56.3	41.3	75.5	7.1	58	52.3	31.0	75.0	8.02
RTTT participation	LS[a] proposal	52	0.54	0	1	0.50	52	0.56	0	1	0.50
	State requirement (PLA schools)	52	0.37	0	1	0.49	52	0.37	0	1	0.49
	LS funding ($)	52	79,872	0	1,807,159	278,301	52	76,611	0	1,807,159	278,555
	Total RTTT funding ($)	52	4,443,403	5,787	37,575,662	7,127,040	52	6,331,057	5,787	73,376,735	12,235,167
District policy	LS requirement	58	1.59	1	3	0.77	58	1.41	1	3	0.75
District funding allocation	Substitutes	58	0.45	0	1	0.50	58	0.36	0	1	0.49
	Teacher payment	58	0.14	0	1	0.35	58	0.22	0	1	0.42
District leadership	Designated position for LS	58	0.53	0	1	0.50	58	0.24	0	1	0.43
	PD director stability (13–14 and 14–15)	53	0.64	0	1	0.48	53	0.74	0	1	0.45
	Future sustainability plan	58	0.59	0	1	0.50	58	0.62	0	1	0.49

Notes: [a]LS=lesson study

District leadership characteristics also play an important role in implementing lesson study especially after the RTTT funding ended. The percentage of districts with a designated position to facilitate lesson study decreased from 53% to 24% from 2014 to 2015. These positions may have been funded by the RTTT program, which were eliminated when the funding ended. Stability of the professional development director is also critical as turnover in the position who oversees lesson study could affect the continued effort to implement lesson study districtwide. From the 2012–2013 year to the 2013–2014 academic year, the same individuals served as the directors in 64% of the districts. From the 2013–2014 to 2014–2015 academic year, 74% of the districts had the same directors. Finally, the director's plan to sustain support for lesson study in the following year shows the district's long-term commitment to lesson study. In 2014, 59% of the professional development directors reported that they plan to continue their support during the 2014–2015 academic year, and in 2015, 62% of the directors reported their continued support during the 2015–2016 academic year. A slight increase in the future sustainability plan is unexpected considering the decline in the percentage of districts with a lesson study requirement, funding allocation, and a designated position from 2014 to 2015.

4.3 District Policy and Practice Associated with the Level of Lesson Study Implementation

Table 2 presents ANOVA and t-test results on the relationship between three district factors (district policy, funding allocation, and district leadership) and the level of lesson study implementation measured by the percentage of schools that practiced lesson study in each district (a continuous variable ranging from 0% to 100%). The table shows that four types of variables are significantly associated with the level of lesson study implementation. First, the districts that required all schools to practice lesson study are significantly more likely than the districts that did not require any school or required only some schools to report a higher level of lesson study implementation. As expected, district policy has a major influence on the level of lesson study implementation. Second, the districts that provided substitute funding were more likely to report a higher level of lesson study implementation in 2014, and the districts that provided teacher payment are more likely to do the same in 2015. Finally, district professional development directors' future sustainability plan to continue to support lesson study was significantly associated with the level of lesson study implementation in both years. Unexpectedly, having a designated position for lesson study and the stability of professional development directors were not significantly associated with the level of lesson study implementation reported by the district leaders.

To better understand the approaches the districts have taken to implement lesson study districtwide, the importance of these significant factors, and some possible reasons for the lack of significant relationship between the level of lesson study implementation and some district factors, we conducted interviews with three district

Table 2 Relationships between district factors and the district level of lesson study implementation

			2014		2015	
			Implementation level[a]	*F*/*t* value[b]	Implementation level[a]	*F*/*t* value[b]
District policy	Lesson study requirement	No schools	21.5	$F = 16.70$**	24.8	$F = 15.69$**
		Some schools	36.8		38.5	
		All schools	82.4		91.5	
Funding allocation	Substitutes	Yes	51.6	$t = 3.23$**	45.0	$t = 1.21$
		No	22.8		31.8	
	Teacher payment	Yes	39.8	$t = .34$	65.2	$t = 3.16$**
		No	35.1		28.3	
District leadership	Designated position for lesson study	Yes	44.3	$t = 1.98$	47.3	$t = 1.16$
		No	25.8		33.2	
	PD director stability (13–14 and 14–15)	Yes	37.1	$t = .63$	33.2	$t = -.31$
		No	30.2		37.1	
	Future sustainability plan	Yes	47.0	$t = 3.00$**	50.3	$t = 3.71$**
		No	19.7		14.0	

**p <.01

Note: [a]Implementation level was measured by the percentage of schools within districts that practiced lesson study

[b]*F* value is presented for ANOVA result on the mean difference among three or more groups, and *t* value is presented for t-test result on the mean difference between two groups

professional development directors in the districts that implemented lesson study in over 90% of the schools. In the following sections, we will present the findings from the interview analysis to contextualize the lesson study effort in these districts.

4.4 RTTT Introduction of Lesson Study, Internalization, and Institutionalization

The interviews with three professional development directors—Ms. Clark in Albany, Ms. Anderson in Morison, and Mr. Wallace in Lester (pseudonyms)—revealed the processes these districts went through in implementing lesson study. The data show that after lesson study was introduced by FLDOE, the districts internalized it through a districtwide expectation and funding and promoted institutionalization of lesson study by respecting and supporting school ownership and leadership in organizing lesson study and embedding it as part of the organizational routines.

State Influence via RTTT Participation All three professional development directors shared that the district started lesson study because of the RTTT program. A review of the district RTTT proposals showed that Albany had two PLA schools and Lester had one PLA school, and both districts submitted a lesson study implementation plan to FLDOE in their RTTT district proposals. Albany, however, did not request funding for lesson study and instead reported that they would use funding from the School Improvement Grant (SIG) (U.S. Department of Education, n.d.). Morison did not have any PLA school, so no funding was requested for lesson study in their proposal.

Despite the state initiative in introducing lesson study to these districts, these directors were not aware which PLA schools were required by FLDOE to implement lesson study nor could they recall the content of the RTTT district proposals. It could be due to the time lapse and personnel changes—it was five years ago when these proposals were submitted. However, none of these directors seem to see the FLDOE's role or RTTT program to be important beyond the initial involvement in introducing lesson study. When asked about the influence of the RTTT program, Mr. Wallace said, "Oh well, it was more of a formal process, with the documentation of lesson study." It was clear that in all of these three districts, the directors did not see RTTT to be influential in their decision or approaches to promote and implement lesson study.

Internalizing Lesson Study Through Expectation and Funding Support The survey responses of these directors showed that all schools are required to practice lesson study in their districts. The interviews revealed the decision-making processes the districts took over the years. Ms. Clark in Albany explained:

> We have an expectation of job-embedded professional development, and I think we need to keep that expectation in place. You know, this is one of the few things that we do in our districts...as a district, we expect you to, you know, to do something like lesson study. And so, I think that's a major benefit because every school does it to some extent.

Later Ms. Clark explained that lesson study is the only professional development that the district expects all teachers to be part of, and 20% of the teacher evaluation is devoted to teacher participation in lesson study through self-assessments of lesson plan, research lesson, observation and debriefing, reflection, and perceived improvement.

Ms. Anderson in Morison explained the rationale behind the expectation that all schools do one cycle of lesson study a semester:

> That's considered best practice….the way the process works here is professional learning communities are supposed to look at areas of student achievement that need to be supported. So, usually when they decided on an area that needs support, then they do their research, but usually it comes down to classroom practices need to be changed. So that's where lesson study comes in, because they located a problem; they researched that problem; they looked through the resources available to them, and then they take it to the classroom level and work on perfecting materials, perfecting the lesson, that would support the area of academic concern.

These responses show these directors' belief in the importance of job-embedded, inquiry-based professional development for improving instruction and student learning. Instead of seeing lesson study as a RTTT initiative, they believed in the benefits of lesson study and expected all schools to practice lesson study as an example of such professional development.

However, in responding to our questions about district requirement, these leaders expressed a hesitance about describing their lesson study practice as a "policy" or "requirement." Mr. Wallace in Lester explained:

> I don't know I'll use the word 'required' or 'mandatory'—those words, we try not to use those words because they create a sense that it's no longer voluntary, and we feel that it can impair involvement. So, we don't use the word 'require.' It's 'expected' that school would engage in the lesson study process.

He explained later that, as an expectation, there is no consequence for schools for not practicing lesson study. Ms. Anderson in Morison also explained that "it's more institutionalized now. It's less of a mandate, and it's more of an outgrowth of professional learning communities."

Ms. Clark further explained how schools started to see the benefit of lesson study after the initial expectation, which supported internalizing lesson study:

> At the school level they found ways to embed lesson study in their work day, and in what they do, instead of a separate entity; and when it first started it wasn't like that—they would do their professional development and then said, 'Ok. Let's start what we're doing for lesson study.' And it was very… it was an orchestrated endeavor. And slowly, as they got used to it, as job-embedded PDs became more popular, um, I think schools realize 'Oh! This is actually a useful tool to practice the things that we're learning.'

To support lesson study practice across the district, these leaders ensured that schools are given sufficient funding to meet the district expectation by drawing resources from various types of funding. Mr. Wallace in Lester explained, "We no longer have our lesson study project with Race to the Top…[so we use] Title II, and we have Focus Schools, meaning they are part of the DOE process, where the state identifies them, so there are additional funds there." Ms. Anderson in Morrison explained, "[We use] just strictly discretionary, you know—our substitute budget, we just use our substitute budget to pay for lesson study subs. Our standard allocation for substitute budget, which comes from general education funds, we used to fund lesson study."

Promoting Institutionalization of Lesson Study by Respecting School Ownership and Leadership With almost all schools practicing lesson study within the district, all three professional development directors explained how lesson study has spread since they started it in 2010. What was common in the districtwide implementation processes these leaders shared was the move from initial requirement and training to promotion of school ownership and leadership in organizing lesson study and embedding it as part of the schools' organizational routines.

> Interviewer: How often do you offer training for the schools?
> Ms. Clark: It hasn't been often, and I think it's been 2 years since last time we had it. It's pretty much being able to sustain themselves. . . you know, and they kinda embedded with their professional learning communities. So, the requirement that every school does at least one lesson study cycle—most schools do multiple cycles now. It just part of their professional learning.

Ms. Clark also explained how district-level coaches are less involved in lesson study to make the process "more authentic" and school- or teacher-driven. Likewise, she explained the changing role of the initial template that guided lesson study process by saying, "The templates aren't that useful anymore. I think what happened now is that, schools have moved beyond the templates. And so, we're really lenient about the templates at the district level; we leave it up to the schools." Her explanations show the process in which lesson study evolved from something external that needed to be guided by training, coaches, and templates to something internal that is embedded into the school organizational structures and routines through school leadership, promoting institutionalization of lesson study.

When asked about who coordinates lesson study, Mr. Wallace in Lester responded, "There would be, at the school level, you'll have your grade level teams, your PLCs, and each school has a lead team, so it's not necessarily one person assigned to oversee this initiative." Regarding the question about when lesson study groups meet, he said:

> It could be during planning time. . . it could be during specific time that's set aside for grade level teams with the Title II money where we can get subs, and they look at standards, and looking at lessons, and developing lessons. So, it depends on the schools.

To the same question, Ms. Anderson responded:

> Yes, there's no formal schedule for that. It's whenever the lesson study group meet, and when substitutes are available, so it's really covered by the way school functions, and what resources the school can pull together. Because the school principals are behind it. I mean, they work to help the lesson study groups schedule observations.

Their responses show how lesson study is now self-sustaining at school level; thus there is no need for the districts to provide specific direction or guidance to schools.

The professional development directors' responses to the question about future sustainability plan showed the continuation is assumed now that lesson study is self-sustaining and becoming institutionalized.

> Interviewer: So, is your district planning to support lesson study in the future as well?
> Ms. Anderson: Well, we have it as a district's best practice. So, I mean, there's an expectation that lesson study would continue. But schools are charged with making that happen.

Ms. Anderson also explained the process of institutionalizing lesson study by sharing findings at faculty meetings as part of the professional growth plan:

> We have lesson study groups who are presenting their findings, and the results of their lesson study to the entire faculty, because when somethings come up, they are things that can help not just their particular grade level, or professional learning community, but they have broader applications across the school. So, we know that those groups are sharing at faculty meetings, because we participate—we go to schools, and we've seen this happening a lot.

The findings from the interviews with these directors support the findings from the statewide survey. In the districts where lesson study is implemented districtwide, job-embedded inquiry-based professional development like lesson study is an expectation for all the schools. After several years of lesson study effort, the districts seem to be embracing lesson study as a core part of their professional activities and not as something required by the state. They provide sufficient funding to support schools to meet this expectation using various types of funding sources. Because the districts support and respect school ownership and leadership in organizing lesson study, factors such as designated positions for lesson study at the district level and leadership stability may not matter much for schools' practice of lesson study. Having a future sustainability plan is part of the natural response of the districts where lesson study is internalized and becoming institutionalized within and across schools.

5 Discussion and Conclusions

Lesson study was introduced to district leaders across the state by FLDOE in 2010 through the RTTT program. Five years later in 2015, district leaders across the state reported that lesson study has spread to a total of 749 schools across 39 districts—a significantly larger number of schools than the number of schools (66 schools) required by FLDOE to practice lesson study (Akiba et al. 2016). Based on a mixed methods study of a statewide survey of district professional development directors and interviews of three professional development directors in the districts where lesson study is implemented districtwide, we investigated the district policies and leadership practices associated with the level of lesson study implementation as a promising professional development initiative to promote instructional improvement across the district.

We found that district policy requiring all schools to practice lesson study, funding for substitutes and teacher payment, and districts' future sustainability plan were positively and significantly associated with the level of lesson study implementation reported by district leaders. The interviews with professional development directors of three districts where lesson study is practiced in over 90% of the schools revealed that they believed in the benefits of lesson study and all schools are expected to practice lesson study as an exemplary job-embedded inquiry-based professional development. These district leaders also secured funding from various sources to support schools to meet this expectation even after the RTTT funding ended. These district leaders internalized lesson study through the district policy and funding provision based on their belief in lesson study, instead of as a RTTT mandate. Over time, they have promoted school ownership and leadership, which promoted institutionalization of lesson study within schools. As the school leaders schedule and organize lesson study, it became embedded within their unique organizational structures and routines. Thus, district-level factors such as a district position and leadership stability seem to have become less important over time.

McLaughlin and Mitra (2001) and Firestone et al. (2005) discussed the important role of district leaders in legitimizing the reform initiative and bringing consistency and coherence to schools and teachers. All three district leaders interviewed prioritized lesson study over other initiatives through a districtwide expectation and funding. In this sense, lesson study has spread in these districts by influencing district policies and procedures and creating normative coherence across the system (Coburn 2003). At the same time, the district leaders were aware that school ownership was critical for internalizing lesson study introduced through the RTTT program and gradually withdrew district involvement through training, templates, and coaching and promoted schools' decision-making in organizing and supporting lesson study. This finding is consistent with the previous scale-up research that showed the importance of ownership and capacity building (Coburn 2003; Dede and Honan 2005; McLaughlin and Mitra 2001).

The interview data also indicate that, given the differences in school contexts regarding leadership, resources, teacher relationships, and student characteristics, perhaps one of the only feasible ways to sustain lesson study across the district is to rely on school leaders to make context-specific decisions regarding lesson study. Dede and Honan (2005) pointed out that coping with changes in context, leadership, and funding is a critical factor for scaling up improvement at a system level. This school-driven process of lesson study would likely endure changes in the district contexts including leadership turnover, changing priorities, and funding availability.

Promoting school ownership of lesson study, however, also implies that lesson study is likely to be adapted into the unique school contexts in order to be embedded in the organizational structures and routines. Such adaptation of an innovation—especially in the case of lesson study as an international innovation that emerged in different cultural and organizational contexts—has been identified as a natural part of scaling up across diverse contexts (Dede and Honan 2005; McLaughlin and Mitra 2001). In the case of lesson study, it is important that such adaptation does not involve the alteration of the core principles that lead to improvement of instruction and student learning.

The findings from this study have important implications for policy and practice of local educational authorities (LEAs) (e.g., school districts, local educational bureaus, local departments of education) for implementing reform-based professional development. First, lesson study and any other reform-based professional development may not be implemented across the LEAs and be sustained if the LEA leaders keep the ownership and focus on the "fidelity" of the professional development model. Previous research showed that many districts in Florida used a prepackaged lesson study kit filled with templates, steps, and procedures (Akiba and Wilkinson 2016). Three districts that reported a districtwide implementation of lesson study developed their own initial process to introduce lesson study to schools as a way to internalize lesson study that was introduced by the state department of education and gradually released the district involvement to promote school ownership and leadership in organizing lesson study and to embedding it into the schools' organizational structures and routines.

Second, having a LEA-wide expectation on lesson study as a practice of job-embedded inquiry-based professional development and providing necessary funding while respecting school leadership in organizing and practicing lesson study likely promote the process of internalizing and institutionalizing lesson study. This would naturally allow lesson study to be sustained over time despite changes in central and local educational contexts over time. LEAs play an important role in legitimizing a certain practice and sending a coherent message on what they value to schools and teachers. Our interviews revealed that the district leaders in Florida effectively communicated the importance of school-driven, job-embedded, and inquiry-based professional development and respected and supported school decisions on how to organize lesson study.

Finally, LEA leaders need to trust the capacity of school leaders to organize lesson study and promote instructional improvement while providing necessary supports. In the current policy climate in many countries where LEAs are held accountable for the student learning outcomes, a natural response of many LEA leaders may be to scrutinize school leadership practice by controlling school management and funding and to hold the school leaders or teachers accountable for teaching or learning outcomes. LEAs may also control professional development activities by sending instructional coaches and ensuring that teachers follow specific approaches to professional learning and instruction. Yet, as the current study indicates, such approaches will not likely draw commitment from school leaders or teachers nor build their capacity to engage in the process of improving instruction and student learning. Lesson study as a teacher-driven, collaborative, and inquiry-based professional learning process provides an opportunity for LEAs to support capacity building of school leaders and teachers by building a professional knowledge and a shared vision of effective instruction. It is up to the LEA leaders whether they see this as another reform mandate or an opportunity to develop instructional capacity and shared vision to promote systemwide improvement of instruction and student learning.

Acknowledgments This material is based on work supported by the National Science Foundation (Grant DRL-1417585). Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The authors thank Drs. Bob Floden, Miray Tekkumru Kisa, Catherine Lewis, Mary Kay Stein, and Paola Sztajn for their feedback on the manuscript. We also acknowledge the contributions of our graduate research assistants, Guillermo Farfan, Angelina Kuleshova, and Jennifer Fryer.

References

Akiba, M. (2016). Traveling teacher professional development model: Local interpretation and adaptation of lesson study in Florida. In F. Astiz & M. Akiba (Eds.), *The global and the local: New perspectives in comparative education* (pp. 77–97). Rotterdam: Sense.

Akiba, M., & Wilkinson, B. (2016). Adopting an international innovation for teacher professional development: State and district approaches to lesson study in Florida. *Journal of Teacher Education, 67*(1), 74–93.

Akiba, M., Wang, Z., & Liang, G. (2015). Organizational resources for professional development: A statewide longitudinal survey of middle school mathematics teachers. *Journal of School Leadership, 25*(March), 252–285.

Akiba, M., Wilkinson, B., Farfan, G., Howard, C., Kuleshova, A., Fryer, J., et al. (2016). *Lesson study in Florida: A longitudinal survey of district policy and practice from 2013 to 2015*. Tallahassee: Florida State University.

Coburn, C. E. (2003). Rethinking scale: Moving beyond numbers to deep and lasting change. *Educational Researcher, 32*(6), 3–12.

Coburn, C. E., & Russell, J. L. (2008). District policy and teachers' social networks. *Educational Evaluation and Policy Analysis, 30*(3), 203–235.

Common Core State Standards Initiative. (2010a). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

Common Core State Standards Initiative. (2010b). *Common core state standards for English Language Arts & Literacy in History/Social Studies, Science, and Technical Subjects*. Retrieved from http://www.corestandards.org/wp-content/uploads/ELA_Standards1.pdf

Datnow, A. (2005). The sustainability of Comprehensive Reform Models in changing district and state contexts. *Educational Administration Quarterly, 41*(1), 121–153.

Dede, C., & Honan, J. P. (2005). Scaling up success: A synthesis of themes and insights. In C. Dede, J. P. Honan, & L. C. Peters (Eds.), *Scaling up success: Lessons from technology-based educational improvement* (pp. 227–239). San Fracisco: Jossey-Bass.

Dede, C., Honan, J. P., & Peters, L. C. (2005). *Scaling up success: Lessons from technology-based educational improvement*. San Fracisco: Jossey-Bass.

Desimone, L. M., Porter, A. C., Birman, B. F., Garet, M. S., & Yoon, K. S. (2002). How do district management and implementation strategies relate to the quality of the professional development that districts provide to teachers? *Teachers College Record, 104*(7), 1265–1312.

Elmore, R. F., & Burney, D. (1999). Investing in teacher learning: Staff development and instructional improvement. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 263–291). San Francisco: Jossey Bass.

Fink, D. (2000). *Good schools/real schools: Why school reform doesn't last*. New York: Teachers College Press.

Firestone, W. A., Mangin, M. M., Martinez, M. C., & Polovsky, T. (2005). Leading coherent professional development: A comparison of three districts. *Educational Administration Quarterly, 41*(3), 413–448.

FLDOE. (2010a). *Florida's Race to the Top application for initial funding (CFDA 84.395A)*. Tallahassee: Florida Department of Education (FLDOE).

FLDOE. (2010b). *Final scope of work*. Tallahassee: Florida Department of Education (FLDOE).

FLDOE. (n.d.-a). *Race to the Top Grant Archive*, from http://www.fldoe.org/arra/RacetotheTop-archive.asp

FLDOE. (n.d.-b). *Race to the Top LEA final scope of work template definitions*, from http://www.fldoe.org/arra/RacetotheTop-archive.asp

Floden, R. E., Porter, A. C., Alford, L. E., Freeman, D. J., Susan, I., Schmidt, W. H., & Schwille, J. R. (1988). Instructional leadership at the district level: A closer look at autonomy and control. *Educational Administration Quarterly, 24*(2), 96–124.

Gamoran, A. (2003). Access to resources. In A. Gamoran, W. Anderson, P. A. Quiroz, W. G. Secada, T. Williams, & S. Ashmann (Eds.), *Transforming teaching in math and science: How schools and districts can support change* (pp. 65–86). New York: Teachers College Press.

Gamoran, A., Anderson, W., Quiroz, P. A., Secada, W. G., Williams, T., & Ashmann, S. (Eds.). (2003). *Transforming teaching in math and science: How schools and districts can support change*. New York: Teachers College Press.

Giles, C., & Hargreaves, A. (2006). The sustainability of innovative schools as learning organizations and professional learning communities during standardized reform. *Educational Administration Quarterly, 42*(1), 124–156.

Hargreaves, A., & Fink, D. (2003). Sustaining leadership. *The Phi Delta Kappan, 84*(9), 693–700.

Hart, L. C., Alston, A. S., & Murata, A. (2011). *Lesson-study research and practice in mathematics education: Learning together*. New York: Springer.

Hightower, A. M., Knapp, M. S., Marsh, J. A., & McLaughlin, M. W. (2002). *School districts and instructional renewal*. New York/London: Teachers College Press.

Knapp, M. S. (2003). Professional development as a policy pathway. *Review of Research in Education, 27*(109), 109–157.

Lewis, C. (2002). *Lesson study: A handbook for teacher-led instructional change*. Philadelphia: Research for Better Schools, Inc.

Lewis, C., & Hurd, J. (2011). *Lesson study step by step: How teacher learning communities improve instruction*. Portsmouth: Heinneman.

Lewis, C., Perry, R., Hurd, J., & O'Connell, M. P. (2006). Lesson study comes of age in North America. *Phi Delta Kappan, December, 88*, 273–281.

Liang, G., & Akiba, M. (2018). Teachers' working conditions: A cross-national analysis using the OECD TALIS and PISA data. In M. Akiba & G. K. LeTendre (Eds.), *International handbook of teacher quality and policy* (pp. 388–402). New York: Routledge/Taylor & Francis.

Little, J. W. (1989). District policy choices and teachers' professional development opportunities. *Educational Evaluation and Policy Analysis, 11*(2), 165–179.

Little, J. W. (1993). Teachers' professional development in a climate of educational reform. *Educational Evaluation and Policy Analysis, 15*(2), 129–151.

Marrongelle, K., Sztajn, P., & Smith, M. S. (2013). Scaling up professional development in an era of Common State Standards. *Journal of Teacher Education, 64*(3), 202–211.

McLaughlin, M. W., & Mitra, D. (2001). Theory-based change and change-based theory: Going deeper and going broader. *Journal of Educational Change, 2*(4), 301–323.

Murata, A. (2011). Introduction: Conceptual overview of lesson study. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 1–12). New York: Springer.

Perry, R., & Lewis, C. (2009). What is successful adaptation of lesson study in the US? *Journal of Educational Change, 10*, 365–391.

Perry, R., & Lewis, C. (2010). Building demand for research through lesson study. In C. E. Coburn & M. K. Stein (Eds.), *Research and practice in education: Building alliances, bridging the divide* (pp. 131–145). Lanham: Rowan & Littlefield Publishers, Inc.

Spillane, J. P. (1996). School districts matter: Local educational authorities and state instructional policy. *Educational Policy, 10*(1), 63–87.

Spillane, J. P., & Thompson, C. I. (1997). Reconstructing conceptions of local capacity: The local education agency's capacity for ambitious instructional reform. *Education Evaluation and Policy Analysis, 19*(2), 185–203.

Stein, M. K., & Coburn, C. (2008). Architectures for learning: A comparative analysis of two urban school districts. *American Journal of Education, 114*, 583–626.

Stein, M. K., & D'Amico, L. (2002). Inquiry at the crossroads of policy and learning: A study of a district-wide literacy initiative. *Teachers College Record, 104*(7), 1313–1344.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press, A Division of Simon & Schuster Inc.

Sykes, G., O'Day, J., & Ford, T. (2009). The district role in instructional improvement. In G. Sykes, B. Schneider, & D. N. Plank (Eds.), *Handbook of education policy research* (pp. 767–784). New York/London: Routledge.

U.S. Department of Education. (2009). *Race to the Top Program: Executive summary*. Washington, DC: Author.

U.S. Department of Education. (n.d.). *School improvement grants*. Retrieved from http://www2.ed.gov/programs/sif/index.html

Yoshida, M. (2012). Mathematics lesson study in the United States. *International Journal for Lesson and Learning Studies, 1*(2), 140–152.

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The Use of Lesson Study to Unpack Learning Trajectories and Deepen Teachers' Horizon Knowledge

Jennifer Suh, Sara Birkhead, Terrie Galanti, Rachelle Farmer, and Padmanabhan Seshaiyer

Contents

Abstract We highlight a learning trajectory-based professional development (PD) routine used during Lesson Study to deepen teachers' and coaches' understanding of the development of children's fraction knowledge through the analysis of student work. One of the key elements in the design of the coach-facilitated Lesson Study was a team of teachers from multiple grade levels who focused on a rich task that engaged all in vertical articulation of the progression of mathematical learning. Using a case study approach, we examine how the intentional introduction of the learning trajectory by the coach/mathematics teacher educator promoted the teachers' use of the learning trajectory and contributed to the expansion of teachers' horizon content knowledge, in terms of how they gained knowledge of content of students, teaching, and curriculum. In particular, we highlight the phases of the Lesson Study process, particularly the planning, the live lesson, and the debrief, which focused teachers' attention on their understanding of the learning trajectories through student work and discussion during the Lesson Study. Teachers used the curriculum standards, pacing guide, and teaching materials along with research-based learning trajectories to negotiate the appropriate conceptual corridor for

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_36

learning. Teacher's experimentation with curriculum pacing guides and learning trajectories in Lesson Study not only benefited from the jugyokenkyu, the research "kenkyu" around "jugyo" which means teaching and learning, but also deepened their understanding of how student learning progresses around fraction knowledge.

Keywords Lesson study · Learning trajectory · Mathematical knowledge of teaching · Horizon content knowledge

1 Introduction

Through Stigler and Hiebert's (1999) seminal work, *The Teaching Gap*, we, in the United States, took notice of a different form of professional development (PD) called Lesson Study that was teacher-led and focused on instructional improvement. In their recent article "Lesson Study, Improvement, and the Importing of Cultural Routines," Stigler and Hiebert (2016) discussed how global adaptations of Lesson Study have spread rapidly in the past decades and argued that Lesson Study will be developed most effectively in other countries "by fitting it into the context of related ideas that already have a place in the countries' various education systems." Stigler and Hiebert (2016) also challenged researchers to consider to use Lesson Study as a method of improvement science designed to accelerate learning by doing. It is a more user-centered and problem-centered approach to improving teaching and learning (Bryk et al. 2015). The following case study takes Stigler and Hiebert's (2016) recommendation of using Lesson Study as a networked improvement community (NIC) to build practice-based evidence. By taking on this new paradigm, we acknowledged our fellow educators as designers and researchers and as "active inquirers who are now bound together by norms and structures akin to a scientific community" (Bryk 2015, p. 469). According to Bryk (2015), randomized field trials are not designed to tell us how to make interventions work reliably for different subgroups of students and teachers working under varying contextual conditions. Improvement science, in contrast, places primacy on variability in outcomes as the central problem to address. Lesson Study has always been about focusing our attention to accumulating knowledge on improving instruction and detailing the knowledge of how to actually make it work. By working with diverse Lesson Study groups, we have the potential to learn more about how interventions and strategies work reliably over diverse contexts and populations. In fact, Lesson Study is a vehicle for improvement science that has at the core the framework of improvement science referred to the plan-do-study-act (PDSA) cycle, "a process for rapid cycles of learning from practice, coupled with three fundamental questions that drive improvement work: (1) What are we trying to accomplish? (2) How will we know that a change is an improvement? (3) What change can we make that will result in improvement?" (p. 54). Lewis (2015) described how Japanese Lesson Study centers on the fundamental improvement science questions where "Educators choose an improvement aim, agree on how they will recognize improvement, identify the changes that might produce improvement, and test these changes in lesson study cycles" (p. 57).

For our study, we focused on how the intentional use of the learning trajectory across vertical grades impacted the critical phases of Lesson Study and contributed to the improvement and expansion of teachers' horizon content knowledge of students, teaching, and curriculum.

2 Theoretical Framework

We propose examining our work using complementary perspectives of Lesson Study as a teacher-led professional development model and using the research informing how we deepen our pedagogical content knowledge while unpacking the research around hypothetical learning trajectories.

2.1 Lesson Study Deepening Mathematics Knowledge for Teaching

Lesson Study empowers teachers and coaches and provides a collaborative structure that promotes reflection and critical dialogue about pedagogical content knowledge among colleagues (Lewis 2002; Lewis et al. 2006). By focusing on a single rich mathematics task adapted for multiple grade levels within a Lesson Study team, teachers and coaches can create a rich and meaningful vertical articulation and a deepening of the understanding of the mathematical learning progression (Suh and Seshaiyer 2015).

An important part of Lesson Study is the process called *kyozaikenkyu* as "investigating what kind of materials various textbooks use to teach this topic to students, and what research suggests (if anything) about various methods for teaching the topic" (Takahashi 2006). The instructional materials may include textbooks, lesson plans, books about teaching, and possibly research articles about various methods of teaching. Improving teacher knowledge lesson through Lesson Study have been studied extensively. Takahashi, Watanabe, and Yoshida with Wang-Iverson detailed how different phases of the process deepen teachers' knowledge as it supports teachers in understanding the goals of the unit and content in the context of scope and sequence across the grades, preparing instructional materials for the learning activities, and preparing instruction that helps students achieve the targeted learning goals.

"Kyozaikenkyu refers to the careful analysis of the topic in accordance with the objective(s) of the lesson. It includes analyses of the mathematical connections both among the current and previous topics (and forthcoming ones, in some cases) and within the topic. Also included are the anticipation of students' approaches to the problem and the planning of instructional activities based on the anticipated responses" (Shimizu 1999). In Shimizu's description, we gain an insight into how the different phases of the Lesson Study provide opportunities for teachers to deepen their pedagogical content knowledge:

- Knowledge of Content and Teaching (KCT): "specific mathematical understanding and an understanding of pedagogical issues that affect student learning" (Ball et al. 2008, p. 401)
- Knowledge of Content and Students (KCS): "specific mathematical understanding and familiarity with students and their mathematical thinking" (Ball et al. 2008, p. 401)
- Knowledge of Content and Curriculum (KCC)

We propose that our intentional focus on Lesson Study using a learning trajectory-based PD provided opportunities for teachers to engage in a specific category of subject matter knowledge described as horizon content knowledge. Researchers have envisioned horizon content knowledge in a variety of ways that highlight the way in which the mathematical knowledge needed for teaching a curriculum extends beyond the knowledge specific to a single grade level. Ball et al. (2008) and Ball and Bass (2009) conceptualize horizon content knowledge (HCK) that is neither common nor specialized. It is not defined within a specific curriculum progression, but is more about having a sense of the larger mathematical environment of the discipline. Horizon knowledge is an "awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated" (Ball and Bass 2009, p. 6). They argue that knowledge of the mathematical horizon can support teachers in hearing students' mathematical insights, orienting instruction to the discipline, and making judgments about what is mathematically important. Ball et al. (2008) originally described HCK as:

> an awareness of how mathematical topics are related over the span of mathematics included in the curriculum. First-grade teachers, for example, may need to know how the mathematics they teach is related to the mathematics students will learn in third grade to be able to set the mathematical foundation for what will come later. It also includes the vision useful in seeing connections to much later mathematics ideas. (p. 403)

Hill et al. (2005) describe HCK as the "mathematical peripheral vision needed in teaching" (p. 70). This idea is also embraced by Fernández and Figueiras (2014) who state that HCK "is not only an awareness of how mathematical topics are related over the span of mathematics included in the curriculum but it also refers to the global knowledge of the evolution of the mathematical content and the relationship among its different areas needed for the teaching practice" (p. 13).

Jakobsen et al. (2012) emphasize that horizon knowledge is practice-based and describe HCK as knowledge that situates and connects to the "broader disciplinary territory" – using an analogy-like landscape. This allows for teachers to be responsive to student learning, as it "enables teachers to 'hear' students, to make judgments about the importance of particular ideas or questions, and to treat the discipline with integrity, all resources for balancing the fundamental task of connecting learners to a vast and highly developed field" (p. 4642). This mathematical knowledge "close" to the work of teaching has been shown to have positive influence on student achievement (Baumert et al. 2010; Hill et al. 2005). In discussing the issues of HCK, Jakobsen et al. (2012) argue that the advanced mathematics for teachers needs to be demonstratively related to the work of teaching in school.

2.2 Importance of Unpacking the Learning Trajectories in Mathematics

Learning trajectories (LTs) synthesize research-based evidence of student thinking to map tendencies of cognitive progress within specific mathematical domains. Confrey et al. (2009) offered the following definition of LTs.

> ...researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time. (p. 347)

Over 20 years ago, Simon (1995) introduced the term and concept of hypothetical learning trajectory (HLT) to "refer to the teacher's prediction as to the path by which learning might proceed." Steffe (2004) stated that it is "hypothetical because the actual learning trajectory is not knowable in advance. It is an expected tendency" (p. 135). It is meant to underscore the importance of having a goal and rationale for teaching decisions and the hypothetical nature of such thinking (p. 136). HLTs are also a teaching construct – something a teacher conjectures as a way to make sense of where students are and where the teacher might take them (Simon 1995). Steffe's (2004) work stated "through the construction of learning trajectories that are co-produced by children, it is possible to construct learning trajectories of children that include an account of one's own ways and means of acting and operating as a teacher."

It is important to note that in the research literature, learning trajectories are often used interchangeably with learning progressions (Lobato and Walters 2017). The National Research Council (2007) characterized learning progressions as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (p. 214). Although, learning progressions and learning trajectories are used interchangeably in research, we will use learning trajectories consistently throughout our paper to refer to this notion of "researcher-conjectured, empirically-supported description of the ordered network of constructs a student encounters through instruction." Teachers consider learning trajectories as they anticipate and plan by considering how students might respond to a task or planned activities (Smith and Stein, 2011). Often, teachers draw on their experience to anticipate student responses, and these can help determine the learning trajectories that they consider. However, these anticipated researcher-/teacher-conjectured learning trajectories are qualitatively different than district, state, or national curriculum standards used by teachers that describe the progression of a topic across a number of grade levels. Curriculum designers and textbook companies align their learning activities to standards and pacing guides; however, these are different than LTs.

More specifically, LTs have been used with teachers and researchers to better understand how students come to understand concepts (Battista 2004; Hackenberg and Tillema 2009) and to use "the learning goal, the learning activities, and the thinking and learning in which the students might engage" (Simon 1995, p. 133) to provide direction for teachers as they plan learning activities and predict the potential reasoning, misconceptions, and learning of students (Simon 1995). HLTs have also been used in professional development settings to also enhance instructional practices. Wilson et al. (2015) reported on a study using professional development where LTs bridged "guidelines for student-centered instruction with domain-specific understandings of students' thinking for teachers" (p. 227). They used two teacher professional development frameworks from the mathematics education research community: (1) *5 Practices for Orchestrating Productive Mathematics Discussions* (Smith and Stein 2011) and (2) an equipartitioning LT (Confrey 2012). Based on results from content assessments and evaluation rubrics measuring opportunities for classroom participation and questioning, Wilson and colleagues found that LTs provided a content structure within which to situate the student-centered teaching practices along with the instructional practices that supported the use of the equipartitioning LT. In addition, HLTs have been used to design curriculum (Clements and Samara 2004, 2009). Clements and Sarama's work (2004, 2009) takes on a curriculum developers' stance as they use learning trajectories to develop a specific set of sequenced instructional activities focused on early geometric reasoning in young children. Fosnot and Dolk (2002) also emphasize that teachers and curriculum designers need to have a clear and organized idea of what they want their students to know. They describe the landscape of learning as not being a linear model because "real" learning "is messy." In the landscape of learning, "knowledge of models, strategies, and big ideas, are critical to organizing a coherent hypothetical learning trajectory. The landscape of learning emphasizes the divergence from a linear progression – students may have to move "backwards" in a hypothetical learning trajectory to a previously covered idea in order to understand what is currently being taught" (p. 23).

In our study, we wanted to explore how HLT can be used during a Lesson Study and how a focus on learning trajectories can impact different phases of the process. Our driving research questions were as follows:

1. How does focusing a Lesson Study on learning trajectory across vertical grades impact the critical phases of Lesson Study – the study phases; the planning and preparation phase called "kyozaikenkyu"; the actual classroom teaching phase called "kenkyu jyugyo"; the debrief, class review phases called "jyugyo kentoukai"; and the second iteration cycle?
2. How did the intentional introduction of the learning trajectory by the coach/ mathematics teacher educator promote the teachers' use of the learning trajectory and contribute to the expansion of teachers' horizon content knowledge, in terms of how they gain knowledge of content of students, teaching, and curriculum?

We conducted Lesson Study with vertical teams of teachers and a coach in our PD and tested our conjectures about potential outcomes for vertical teams of teachers and coaches focused on the learning trajectories as they planned and implemented the research lesson and debriefed about student thinking. We wanted to study the nature of the teacher learning and how the opportunity to engage with learning

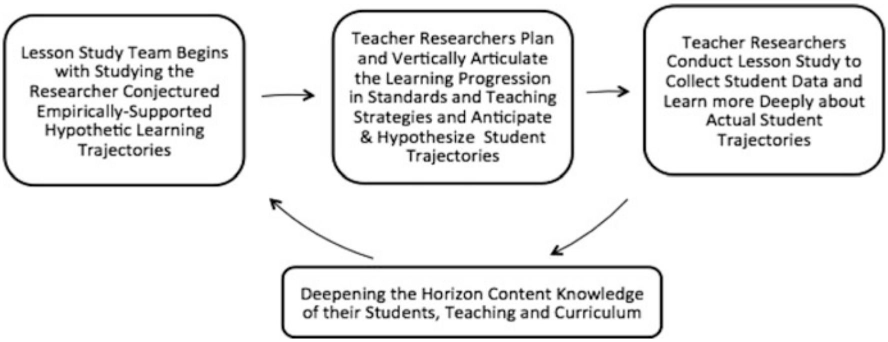


Fig. 1 Initial conjectures for our professional development design based on research on Lesson Study, HLT, and HCK

trajectories and the vertical progression of the mathematics content central to the task deepened their knowledge. The Lesson Study provided the opportunity for the teachers to think vertically about instruction, curriculum, and student thinking at multiple grade levels. By synthesizing these frameworks for horizon knowledge and learning trajectories, we formulated our initial conjectures for a PD design (see Fig. 1) which built upon the complementary aspects of both constructs.

3 The Current Study

3.1 Our Context

Fitting Lesson Study into the context of related ideas that already have a place in the countries' various education systems is critical in ensuring the success of Lesson Study (Stigler and Hiebert 2016). As such, the cultural context that matters to our study is the culture of professional learning communities in schools. In addition, our case study highlights a coach-facilitated Lesson Study structure (Suh et al. 2017) with the participation of a mathematics coach at the school site where we conducted Lesson Study. In addition to leveraging the school structures in place, we introduced our professional development routine that included experiences with rich mathematics tasks in rational number concepts, Lesson Study, the eight mathematics teaching practices from the National Council of Teachers of Mathematics (NCTM) *Principles to Actions* (2014), and unpacking the rational numbers learning trajectories (Confrey 2012).

We extended this concept of combining research-based Lesson Study and existing PD frameworks to connect content domain knowledge to instructional practice in our Math Vertical Articulation to Unpack the Learning Trajectory (VAULT) professional learning routine for teachers and coaches. The VAULT is responsive to the Consortium for Policy Research in Education (CPRE, Daro et al.

2011) recommendation that "learning trajectories need to be translated into usable tools for teachers" (p. 13). The CPRE also called for more support for research groups to collaborate with classroom teachers to validate the learning trajectories in classroom settings and demonstrate their utility for teachers and schools.

Drawing from the research base in mathematics education on learning trajectories (e.g., Clements and Sarama 2004; Confrey 2012; Simon 1995; Sztajn et al. 2012), we asked teachers and coaches to (1) create a set of related tasks across the vertical grades that map a developmental progression within a specific mathematical domain and (2) collect and analyze evidence of students' thinking and learning around a specific problem they posed during Lesson Study. Through these activities, teachers and coaches thought deeply about how their students' rational number thinking fit into a district-generated curricular learning progression.

As PD designers, we were guided by the assumption that Lesson Study could provide an opportunity for a group of teachers and coaches to work with mathematics educators to experience the coach-facilitated Lesson Study cycle (Suh et al. 2017) as a collective whole. In the diagram below, we illustrated how we intentionally designed our vertical Lesson Study to focus on the articulation of the progression of curricular standards (which were most familiar to teachers) and bring in resources for teachers and coaches to consider around the research on the learning trajectories. In other words, we wanted to bridge the use of the HLT (as a teacher-conjectured possible progression and also a researcher-documented progression of actual learners) while working with teachers as researchers in a Lesson Study context. We used the five elements to unpack an equipartitioning learning trajectory (Confrey 2012) starting with discussing the (1) underlying concepts and ideas of equipartitioning; (2) strategies, representations, and misconceptions often associated with fraction sharing problems; (3) model and definitions of what "fair share" means in an equipartitioning task; (4) complexity, patterns, and development of ideas that are in the mathematics education research literature; and (5) bridging gaps between standards. This remained the focus in our Lesson Study process (see Fig. 2).

Mathematics content seminars that were provided by university-facilitated PD across grades served as the extended phase of the "study," the first part of the Lesson Study. Vertical teams of teachers and coaches were immersed in 7 weeks of PD on rational number concepts. The seminars met every other week to delve into the rational numbers content using Lamon's (2005) book, *Teaching Fractions and Ratios for Understanding*, and the website, Learning Trajectories for the K-8 Common Core Math Standards at https://turnonccmath.net. Teachers engaged in rich mathematical problems as they considered underlying big ideas and mapped the conceptual curriculum progression for their specific mathematics domain within rational numbers. The coaches collaborated with teacher learners in school-based teams to facilitate four phases of LS (see Fig. 2). During Phase 1 of the LS, we examined the learning trajectories using the curricular standards and setting a goal for the Lesson Study, which focused on two teaching practices from NCTM's *Principles to Action*: (1) eliciting student thinking and (2) posing purposeful questions (NCTM 2014). During Phase 2 of designing the learning activity, otherwise

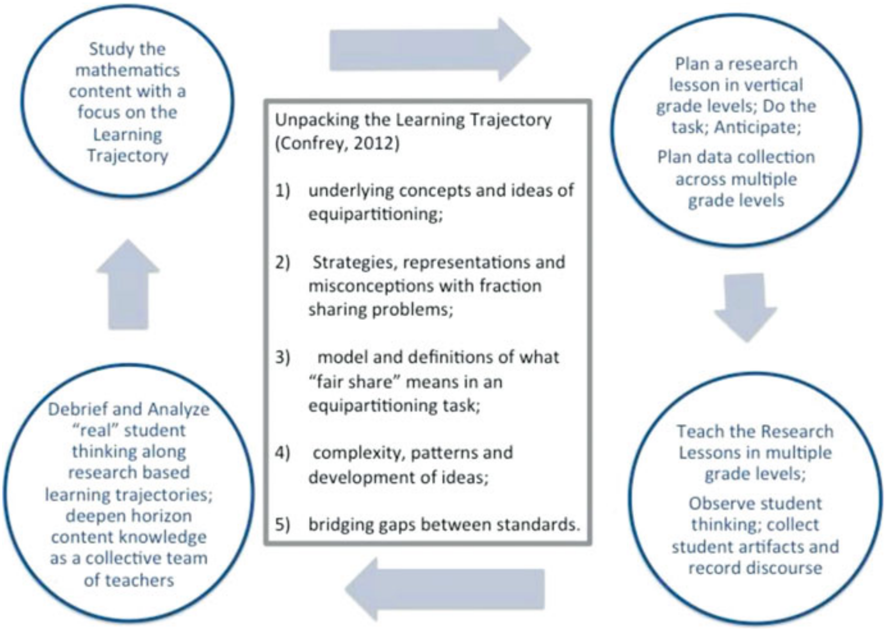


Fig. 2 Vertical Lesson Study process with learning trajectory as a focus

known as the research lesson, the teams focused on unpacking the learning progression of a chosen concept, such as equal sharing, by (a) learning about student learning trajectories related to the specific concept; (b) selecting a rich task focused on the chosen concept and anticipating possible strategies, representations, and misconceptions; (c) generating purposeful questions to ask if students were stuck or probe them to advance their thinking; and (d) connecting the mathematics goal with future learning goals along the learning progression. During Phase 3 of the LS, the host teacher taught the lesson, while the other members observed the research lesson. While the other members of the LS team observed, they collected copious notes on how students worked through the task, taking note of conversations, both that were fruitful and others that took students off-course through misunderstanding or other challenges. The observing teachers and coaches collected evidence of student thinking and knowledge along hypothetical learning trajectories relevant to their tasks. During Phase 4, debriefing the research lesson, the coaches facilitated a dialogue with the participating teachers to verify, validate, and sometimes refine the hypothetical learning trajectories based on what they observed in their research lesson. In Phase 4, modifying the research lesson, the teachers changed the lesson for the contexts of their individual classrooms (e.g., grade level, student readiness, context relevance, etc.). Coaches encouraged vertical articulation and the unpacking of the learning trajectory while probing other teachers to share how they would tailor the problem for the different grade levels. In the second iteration of the lessons, the remaining teachers taught the modified lessons in their own respective grades and added to the evidence of student thinking along the learning progression. Finally, LS teams presented their research, sharing evidence of student learning gathered from multiple grades to examine the learning trajectories across the different grade levels.

3.2 Participants

In this case study, we describe a vertical team of four teachers working with a mathematics coach and mathematics educators and the analysis of a research lesson that was taught in three different grade levels. The Lesson Study team consisted of a kindergarten teacher, Kathy, with 7 years of experience; second-grade teacher, Sera, with 6 years of experience; fourth-grade teacher, Frankie, with 2 years of experience; sixth-grade teacher, Samantha, with 2 years of teaching experience; and a mathematics coach, Clara, with 11 years of teaching and 4 years of coaching experience.

All of the participants were new to Lesson Study except for the coach. The group felt that understanding the learning trajectory in mathematics was especially important because of the nature of the transient and diverse student population they served. The coach shared that because students have moved between so many schools, assessing students' knowledge often revealed lots of conceptual gaps. It was important to the coach that teachers would not have a deficit view of the students but, instead, recognize where students were along the learning trajectory so that they can better support their learning.

3.3 Research Questions

Our research focused on the following questions:

1. How does focusing a Lesson Study on learning trajectory across vertical grades impact the critical phases of Lesson Study – the study phases; the planning and preparation phase called "kyozaikenkyu"; the actual classroom teaching phase called "kenkyu jyugyo"; the debrief, class review sessions called "jyugyo kentoukai"; and the second iteration cycle?
2. How did the intentional introduction of the learning trajectory by the coach/mathematics teacher educator promote the teachers' use of the learning trajectory and contribute to the expansion of teachers' horizon content knowledge, in terms of how they gain knowledge of content of students, teaching, and curriculum?

3.4 Data Collection and Analysis

The data sources included video clips from the research lesson debrief, student work, teacher reflections from the Lesson Study, transcription from a videotaped group presentation, researchers' observation notes written as memos, and interviews with the teachers and coach. We also asked teachers to reflect on the Lesson Study process of developing and refining a research lesson.

During the first phases of the analysis, we used the observational memos from lessons with analytic researcher memos that allowed us to not only describe our teacher-designers planning and enactment of the lessons but also summarize our thoughts about potential ways in which the Lesson Study process contributed to the teachers' ability to deepen their understanding around the learning trajectories of equipartitioning. Interviews, video clips of classroom episodes, and artifacts from lessons including lesson plans and student artifacts also helped identify the major themes related to the vertical articulation of the learning trajectory of rational numbers. We created a chronological timeline for each of the teacher-designers with linked data sources allowing us to focus on documenting the generative learning and expansion of teachers' horizon content knowledge, in terms of how they gained knowledge of content of students, teaching, and curriculum. The story line provided a comprehensive narrative for each of the participants in our case study analysis. All interviews were audio-recorded and transcribed for analysis. We systematically analyzed the data by developing initial codes and used the method of axial coding to find categories in such a way that drew emerging themes (Miles et al. 2014). To verify and compare recurring themes and categories, the research team worked individually on coding the documents before comparing preliminary codes in order to agree upon recurring themes from the reflections. Dedoose, an Internet-based data management tool (https://www.dedoose.com), was used to code and analyze the data.

4 Results

4.1 Phase 1: Study the Math

This case study is situated as part of a larger study that involved a summer content institute followed by school-based Lesson Study. During content seminars, vertical teams of teachers and coaches were immersed in rich mathematical problems as they considered underlying big ideas and mapped the conceptual learning progression for their specific mathematics domain within rational numbers. The coaches collaborated with teacher learners in school-based teams to facilitate phases of LS.

The fourth-grade teacher commented how being with vertical grade levels during the seminars help her, "I did like the fact that it was all grade levels and you could really compare [how] a high school teacher solves a problem and a kindergarten teacher solves the problem. That was, I think, one of the coolest things!" In addition to the content seminar, teachers had a chance to meet together to collaborate in school-based team to plan.

4.2 Phase 2: The Planning Phase, "Kyozaikenkyu"

The teachers and coaches engaged in what Japanese teachers would call "kyozaikenkyu" (Lewis 2014; Takahashi et al. 2005) the study of curricular resources. Kyozaikenkyu, translated as instructional materials research, is central to a Japanese educator's planning process for their everyday lessons (Watanabe et al. 2008).

During the "study and plan" phase, they pulled out the standards and focused on interpreting their curricular standards and their district pacing guide. This discussion led them to choose a task that appeared in their fourth-grade curriculum of sharing six sandwiches with eight students. But what was intriguing was that the team decided that they would not modify the problem and proposed their research question as "What would kindergarteners do with a sharing problem that would result in a non-unit fraction?" The team wanted to see how students might draw on their real experience of sharing and partitioning and see what informal knowledge they would bring to the task.

The teachers' interest in the differing access to a curricular fraction problem became a source of their underlying research question. The teachers wanted to experiment with the hypothetical learning trajectories suggested in the curriculum pacing progression. Their research question became "What would kindergarten to sixth-grade students do with a typical fourth-grade equal sharing problem that involved fractions?" (Table 1).

Teachers discussed their knowledge of the content and their students. The fourth-grade teacher shared how she had two previous lessons on fractions prior to research lesson, and the kindergarten host teacher shared that her students did not have any formal instruction on fraction but wanted to capitalize on their everyday experience of sharing fairly. "How can you make it so that they would have a fair share?"

Data from the lesson plan showed evidence that the vertical team of teachers focused on the appropriate connection to the curriculum, articulated grade-level expectations, anticipated misconceptions, and generated several questions that would help student if they encountered an impasse. In the planning phases, teachers worked together to determine the lesson objective, and although they knew that this problem was in their curriculum in the fourth grade, they wanted to see how kindergartners would attack the problem. They wrote in their lesson that they anticipated that students would "link to their understanding of 'fair share/fairness' in a real-world context. Students are eating cookies or sharing a pizza and would like to know how much they get." Meanwhile, the LS team noted that the fourth graders with their prior knowledge would "relate to previous knowledge in fractions, decimals and division from the curriculum. Ideas of fair share vs. mixed numbers. Scenarios from student's lives that could occur."

Table 1 Problem task and vertical articulated lesson goals

Kindergarten: Eight students from Ms. Thomas's class took a field trip to Washington D.C. They only packed six sandwiches for lunch. How can they share fairly?	The goal of the lesson was to explore kindergartners understanding of "fair share/fairness" in a real-world context. Students are familiar with sharing and would like to know how much they get. What would they do with leftovers? Would they know how to exhaust the whole collection of sandwiches?
Fourth grade: Eight students from Ms. Neal's class took a field trip to Washington D.C. They only packed six sandwiches for lunch. At the Washington Monument, five students shared two sandwiches At the White House, three students shared four sandwiches. Who got the most food? The least? How do you know? Explain your thinking using pictures, numbers, and/or words	The goal of the lesson was to see how fourth graders might represent, compare, and order fractions. Students will draw pictures/models to represent their divided fractions (shares). Students will use multiple representations to represent the shared sandwiches. Relate to previous knowledge in fractions, decimals, and division from the curriculum. Ideas of fair share vs mixed numbers. Scenarios from student's lives that could occur
Sixth grade: Eight students from Ms. Smith's class took a field trip to Washington D.C. They only packed six sandwiches for lunch. How can they share equally?	The goal of the lesson was to see how sixth graders solve a practical problems involving equipartitioning.
In the equipartitioning learning trajectory, students learned how to share a collection (see Bridging Standard 1.EQP.A in the equipartitioning LT) and a whole rectangle and circle (see Standards 1.G.3, 2.G.2, and 2.G.3 in the equipartitioning LT) among various numbers of people, to produce fair shares. For example, 15 coins shared among 3 people results in 5 coins per person. Likewise, one cake shared among four people results in 1/4 of the cake per person. At the end of the equipartitioning LT, students justify, for example, that three cakes shared among five people results in $^3/_5$ cake per person. They also generalize, without manipulatives, that a objects among b people results in a fair share of a/b objects per person (see Standard 5.NF.3 in the equipartitioning LT). They have also learned how to "co-split" and know that if a objects are shared among b people, then c groups (with c a whole number) can be created, with a/c objects and b/c people in each group. The fair share, however, is still a/b objects per person (see Bridging Standards 2.EQP.A and 3.EQP.A in the equipartitioning LT). https://turnonccmath.net/	

They anticipated a number of misconceptions that they would expect and noted that they would monitor for:

> Students may lack an entry point to the problem. They may not immediately relate it to fractions/division; Students may not have foundational learning in the benchmarks 0, 1/2, and 1 whole; Students may lack understanding in the meaning of the denominator (the smaller the denominator, the larger the piece); Students may not be flexible in their use of dividing manipulatives or pictures equally; Students may not have foundational learning with mixed numbers; Students may struggle writing a number sentence to show their work; Students may struggle dividing the sandwiches in equal parts; Students may not use all of the sandwiches.

They also generated several questions to ask:

> Could you draw a picture to show how the students could share the sandwiches? What tools could you use to share your thinking? How could you represent the sandwiches? How could you divide the sandwiches in a different way than what the task has asked of you? What is a number sentence that could match your thinking process? How are you sharing the sandwiches? Is this fair, unfair?

4.3 Phase 3. Teaching and Observation Phase, "Kenkyu Jyugyo": An Account of the Kindergarten Research Lesson-Partitioning Scheme

The collection of teachers' notes, photos, and video clips helped the Lesson Study team discuss some student ideas. Students quickly reached in their manipulative bag for six sandwiches, and when they realized there were not enough for eight students, they picked the ½ sized to divide the whole sandwiches. Like in Group 1 (see Fig. 3), the students place six whole sandwiches using halves, and then one of the students takes her fingers and splits them six times saying "Then we split it, split it, split it, split it, split it, split it!" Another student then notes that there are more than eight pieces and places her finger on the eighth halves. The student who originally split it counts to 12 a couple of times to be sure and then removes four halves and states that we throw it away. The group recounts to eight, and then the little girl is caught on the camera saying "We did it!" and does a victory dance!

Teachers assumed that the students quickly gravitated to cutting the six sandwiches in halves since they knew they did not have enough whole, and the next big piece in their bag was the halves. The manipulatives became the focus of our debrief

Girl says, "We have 6 sandwiches right?"	"Then we split it, split it, split it, split it, split it, split it."	Now the 6 sandwiches have been split by the student. We see 12 halves.	Then another girl counts the ½ pieces and puts her finger on the 8th piece to indicate that that would be for 8 kids.	Then the other girl who originally split it, counts all. She says, "Wait, 1,2,3,4,5,6,7,8,9,10, 11, 12" Then she said, "Wait!" and recounts to 12.	Then she removes all but 8 and says, "We throw these away." And the group counts again to be sure that there are 8 equal pieces of sandwich for the kids.

Fig. 3 Kindergarten group one came to a solution of "Split it, Split it, Split it…" Everyone got one-half and then the rest were thrown away or given to the teacher

because in some ways, teachers felt that it directed student thinking in initially splitting the sandwiches in halves. Some students experimented with the ¼ pieces when they realized they did not have 12 halves to make 6 whole sandwiches. In one of the groups, a student took the two whole leftover sandwiches and attempted to split them again into quarters and gestured cutting it in half; this would have been a ¼ of a sandwich. The kindergarteners did not have the term ¼ yet but knew that each student had ½ of a sandwich and a little more but was not able to name it. The combined amount was the ¾, but the students never reached the point of identifying the portions (see Fig. 4).

4.4 Phase 4: Debrief of the Host Lesson Debrief, Class Review Sessions Called "Jyugyo Kentoukai"

As teachers engaged in this work, one tool emerged from their district's scope and sequence material (see Fig. 5). The coach shared it with the mathematics educator, and she then brought it to the research lesson observation and debrief.

"Equal sharing problems where the answer is great than 1 are easier for young children to solve and help bridge children's understanding of whole numbers and fraction." The example given was 4 children sharing 10 candy bars (answer is more than 1) versus 4 children sharing 3 pancakes (answer less than 1 and a non-unit fraction).

Kathy shared at the lesson debrief how students were able to get to a fair share of ½ for each but was not ready to take the two whole sandwiches and split it equally

Students spends a long time drawing 8 students and as they get carried away, they actually draw 9 students and their teacher in the picture.	As the students are drawing, one boy decides to place some sandwiches on top of the drawings of students	One of the girl in the group, points with her red crayon, "Don't forget this kid!" And the group is able to place 8 halves on the drawing. There is some negotiation going on because one boy points out that you drew 2 extra people. He double counts to be sure only to leave 8 halves on the drawing.	He takes some extras and says, "These are extras!"	He continues and says, "And if they are hungry, they can cut it in the middle here" as he gestures with his hand cutting the sandwich again.	He takes another half of a sandwich and places the yellow ¼ of a sandwich to show how we can cuts some more. The lesson ends before he is able to figure out how many smaller pieces each student would get but he knows that there were extra sandwiches that could be shared.

Fig. 4 *Kindergarten* group two came to a solution of each student having half of a sandwich and if they were hungry some more

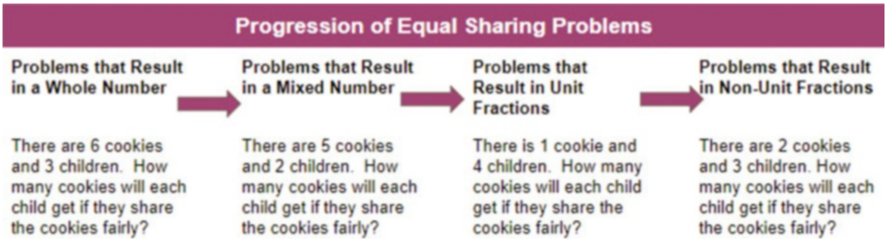


Fig. 5 District's scope and sequence

among eight students. Instead some of them removed the two others as leftovers or threw them away. This concept not only plays an important role in the fraction LT but also in the ratio and proportion and percent LT. In the website for Learning Trajectories for the K-8 Common Core Math Standards at https://turnonccmath.net, the equipartitioning learning trajectory states:

While students do not need to know these terms, the names for these may remind teachers to anticipate their development. Students justify that they have a fair-share by using the three criteria for equipartitioning:

1. Having the correct number of parts
2. Exhausting the whole, not leaving any parts unused, and
3. Having equal-sized parts (Confrey et al. n.d.)

Kathy stated in her debrief that although her students did not name the correct number of parts as ¾ she was "tickled" to see that they were able to split the pieces in halves and some were able to split to quarter pieces. She continued that although the students also did not exhaust the whole, she thought it was quite impressive and real world that some of the students said the leftover whole sandwich was for their teacher who in their mind should get a larger portion. Kathy said, "Great proportional reasoning, in my opinion!"

The coach shared how students demonstrated the notion of taking a whole and making different compositions of whole using the fractional pieces that the kindergarteners affectionately called "emoji's" referring to the quarter and a quarter and a half to make a whole or an emoji smiley face. This notion of composing and decomposing a whole using fractional parts is an essential idea in "Building on equipartitioning collections of discrete wholes leads to the idea of equipartitioning single whole circles and rectangles. Students equipartition a whole into two equal shares and name the resulting shares as, for example, "halves" or "half of"" (Confrey et al. 2009).

Another mathematical process that Kathy noticed was the use of concrete materials and acting it out. Kindergarteners gestured the cutting of the sandwich and eating it and sharing it with their friends. The LS team discussed after the research lesson that perhaps they should have had some paper sandwiches as well so students could cut the fractional parts.

4.5 The Second Iteration: An Account of the Fourth- and Sixth-Grade Research Lessons Examining the Partitioning Scheme Across the Grade Levels

The second iteration was another “live” lesson in a fourth-grade classroom. The classroom teacher modified the task slightly to present a task that would result in a mixed number with a unit fraction and a non-unit fraction. She was interested in seeing how fourth-grade students would tackle this task of *five students sharing two sandwiches* versus *three students sharing four sandwiches.*

The goal of the lesson was to see how fourth graders might represent, compare, and order fractions:

> Students will draw pictures/models to represent their divided fractions (shares). Students will use multiple representations to represent the shared sandwiches. Relate to previous knowledge in fractions, decimals, and division from the curriculum. Ideas of fair share vs mixed numbers.

Another team member was a sixth-grade teacher, and she presented the same task as the kindergarten teacher. This opportunity to present a similar task to students in grade kindergarten to sixth grade provided the team with a unique opportunity to compare and analyze how students across the grade levels tackled this task of equal partitioning (see Fig. 6). In the debrief meeting of the kindergarten lesson, teachers commented on how students were able to “split” the shares but was not able to coordinate between sharers and shares, resulting in leftovers or non-equal shares. But some of the students recognize that they did not exhaust the entire collection of sandwiches and made different attempts to subdivide even farther like taking the halves and cutting in halves again. They did not have the mathematical vocabulary for half of a half, “quarter,” but was gesturing the need to split farther. Meanwhile in the fourth- and sixth-grade student samples, students started to show what Empson and Levi (2011) call “additive coordination” where students split each share into the number of sharers and adds all the pieces as shown below ($1/5 + 1/5 = 2/5$). It was through the student samples that the sixth- grade teacher brought that showed the team what Empson and Levi (2011) called “multiplicative coordination.” Here the sixth-grade teacher shared how students anticipated needing a multiple of eight and divided the six sandwiches into four and used multiplication to show $6*4 = 24$ and then divided the 24/4 into eight and found ¾. One of the students represented their understanding of the concept of using dividing and finding the fraction that was the quotient using decimals (see Fig. 6 part h).

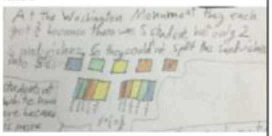
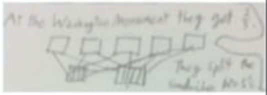
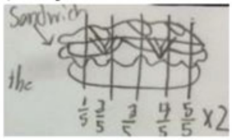
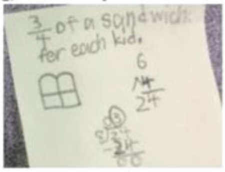
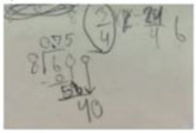
Kindergarten work samples	Fourth through sixth grade Student work samples	Sixth grade work samples
No coordination between sharers and shares (having no plan to Trial and Error to Coordinate)	Additive Coordination -one at time to grouping of items	Multiplicative coordination: Fraction as quotient and use of decimals
a) "Splitting"  b) subdividing to exhaust the collection 	c)additive coordination one at a time  d)distributing one at a time  e) 	f)multiplicative coordination  g) Fraction as quotient  h) use of decimals 

Fig. 6 Student work displaying partitioning scheme from kindergarten to sixth grade

4.6 *Deepening Teachers' Horizon Content Knowledge Around Content, Students, Teaching, and Curriculum*

In reviewing the data from the Lesson Study, we noticed that there was a parallel between the coach's use of LTs leading to an expansion of teachers' horizon content knowledge and teachers' use of LTs leading to improved understanding of their students' mathematics. As this parallel narrative emerged, we crafted the second research question, "How did the intentional introduction of the learning trajectory by the coach/mathematics teacher educator promote the teachers' use of the learning trajectory and contribute to the expansion of teachers' horizon content knowledge?"

Interviews with the teachers and the coach revealed several recurring themes. One major theme was around deepening their own knowledge through the focus on the learning trajectories with subcodes that related to the (1) vertical knowledge of content and students learned from analysis of student work, doing the math themselves, and going through the struggle and seeing the problem enacted in vertical classrooms; (2) vertical knowledge of content and teaching; and (3) vertical knowledge of content and curriculum. Other major recurring themes were about collaborating with colleagues and having the coach and the mathematics teacher educator extend their thinking around the content.

Expressing Vertical Knowledge of Content and Students The kindergarten teacher compared what she observed with her kindergarten students and her colleagues' fourth-grade students and contrasted how the older students had a better sense of coordination of the equipartitioning. She used the words, "they already knew to make a set and split them" and "already had that idea that these already needed to be in pieces and parcel them out." This comment indicates how she grasps what researchers like Steffe (2004) and Empson and Levi (2011) refer to as "coordination" scheme. She continued to share specific content knowledge around how students responded to the task, stating:

> The fourth graders were much more able to do that, but they didn't all gravitate towards the manipulatives quite as much which was interesting but what they did... When they did use the manipulatives was... They gravitated more towards the cubes, and *they already knew to make a set and split them and have each piece represent something*, whereas my younger ones like the fraction circles and they liked drawing the sandwich and cutting it into pieces... This one goes here, that one goes there... So, the *older students already had that idea that these already need to be in pieces and parcel them out. (Interview with Kathy, the kindergarten teacher)*

The fourth-grade teacher, Frankie, also commented on the difference in students understanding and strategy, though she connected it to the wide range of students' abilities to attend to in her classroom:

> I have some... kids on a second grade level all the way up to like sixth grade in math. So it was just seeing which ones had like totally grasped the concept and could just like jump here and then which ones need to still go through all the steps in order to completely master it. So that was interesting just to see where they fell. *And I guess what were the next steps for teachers to go back and be able to help them succeed. (Interview with Frankie, the fourth-grade teacher)*

She recognized the need to support students at vastly different levels of ability, but interestingly does not seem to apply value to a student's relative level. Her analysis is informational, not critical. The knowledge of the learning trajectory seems to give her a more asset-based view of her students. Instead of focusing on the gaps, she looked for the understanding that was evident in the students' work.

Much of the teachers' ability to make these statements regarding vertical nature of the fraction learning trajectory seems to be directly linked to the chance to interact deeply with the learning trajectory with the guidance of a knowledgeable other. During the debriefing, the coach facilitated a dialogue with the participating teachers to verify, validate, and sometimes dispute, the district scope and sequence based on what they observed in their research lesson. The teachers' engagement with the student work and the learning trajectories provided showed that teachers were critically thinking about the learning trajectories. The kindergarten teacher commented that despite the fact that students were not provided explicit instruction about fair share, students dove right into the task. She stated:

> So, I was impressed with their background knowledge, on how to share things fairly, when it's more than just starting with one and making it into two. Because it's a lot more than that. We started with a bunch of whole pieces and divided it into a bunch more partial pieces and you know they weren't afraid of the leftovers. The fourth graders were very uncomfortable

> with the leftovers… because they knew that or rather they'd been taught that when there's leftovers you have to split those up evenly as well. My kids saw leftovers and said we don't need this…. they had also not been introduced to the manipulative of the fraction circle before… usually we give them more time to play with it so it's not as distracting because there was one kid who took the circle and made a sandwich and was like this is my sandwich I'm not going to share it… Ya, I knew somebody would probably do something like that. But most of them used it appropriately which was nice to see. *(Interview with Kathy, the kindergarten teacher)*

Her comment shows that she recognizes a need to treat content differently based on grade level and the impact of prior knowledge at different levels. She also implies an understanding, similar to the second-grade teacher, of the way in which model use might be different at different levels. Further, during the second lesson iteration, taught in the fourth-grade classroom, teachers were surprised to see that fourth graders still struggled to name the fraction. They discussed the progression of equal sharing problems and the progression of learning from problems of sharing that resulted in whole numbers, unit fractions, and non-unit fractions. The teachers realized that this problem involving non-unit fractions was challenging for many students. They began to make generalization about how the learning trajectory mapped to the typical student in the target grades.

In a post Lesson Study interview, the kindergarten teacher described her new perspective on vertical articulation because of her work with LTs. Instead of thinking about "where are the holes" in instruction and learning, she could think about "conceptual parallels" in the work of the individual students at the different grade levels. Consistent with research characterizations of learning trajectories as not necessarily linear (Confrey and Maloney 2010), the second-grade teacher described the LTs as "guidelines and not restrictions." Her experiences during the content institute helped her to see the affordances of multiple representations as a lens on student thinking and open-ended questioning first as a learner and then as a teacher. She described her own instructional shift toward open-ended questions and multiple representations and a commitment to "let them learn."

Expressing Vertical Knowledge of Content and Teaching The second-grade teacher commented on how she expanded her skills in teaching by considering different representations while working with other colleagues teaching different grade bands.

> Being able to compare my work to my classmate's work was everything because seeing that… First of all, other people had some struggles, too, but then saying, you know, Frankie used a number line because that's her thing, and Katie drew a set model because that's her thing, and I did it with a hundreds chart because that's my thing. So really just… a region model. So it was just refreshing to be able to learn from each other, and to be able to know that I wasn't alone in it. So it gave me a lot of empathy for my kids, my students as well. *(Interview with Sera, the second-grade teacher)*

Prior to the above quotation, the teacher described that her peers were from varying grade levels, and describing the differences in individual's representations implies a difference based on the grade level they teach. This shows that the teacher recognizes the different models that are most familiar at different grade levels and implies that she assigns importance to understanding each one.

Another perspective on how the use of LTs impacted the teachers came from the coach. In discussing the affordances of planning with the LTs and seeing how well kindergarteners did with the equipartitioning task, she recounts how the fourth-grade teacher responded. She taught multiple levels of mathematics, including an advanced academic class that was learning concepts a grade level above and an inclusive special education classroom. She was the next host for the second iteration of Lesson Study and debated which class she would enact it for the group. She decided on her homeroom with a high population of inclusive special education students. The coach recounts her conversation with Frankie, "You know what, I really want to see what my homeroom kids can do." She wanted to see where her student would go with the non-unit fractions, and said, "Hey, if these kindergartners can do it, I really want to see what my special education student can do. And I also really wanna know what my other disengaged learners are able to do, when given scaffolds, when given recording sheets that scaffold thinking." From the coach's perspective, the use of LTs and the approach of looking for what students are able to understand through an asset-based approach encouraged the fourth-grade teacher to showcase her diverse learners in the second iteration of the lesson.

Expressing Vertical Knowledge of Content and Curriculum The kindergarten teacher explained how the whole focus was to see how the curriculum "spiraled up":

> What we chose to do, was we took the same basic problem and it was fraction based and we kind of spiraled it up. . . Um and we used the same basic thing for Kindergarten and then we did the same lesson in fourth grade and then we did the same one in sixth. And it was really interesting to see the understanding of fractions that the kids had so early because the problem itself was one where you... take a whole object and you portion it out and it doesn't wind up an even share. . . there's leftovers, the people sharing it are more than what you have. *(Interview with Kathy, the kindergarten teacher)*

As the other teachers modified the research lesson for their individual classrooms (e.g., grade level, student readiness, context relevance), the discussion provided evidence of vertical articulation and the unpacking of the learning trajectory. The coach probed other teachers to share how they would tailor the problem for the different grade levels. They were able to better anticipate the potential learning trajectories at different grades through the modified tasks for their classroom because they had observed student thinking in the kindergarten context and learning that focused on the learning trajectories during the content institute. Discussing with peers that taught different grade levels expanded their view of learning about how student thinking progresses.

The use of LTs not only broadened the teachers' views of students' understanding of a topic but also led them to see how exposure to curriculum over time impacts students. The second-grade teacher commented on how the county-provided learning progression of fair-sharing problems, presented with four stages (boxes), impacted their view of students' work (See Fig. 5):

> I think it was interesting to be able to look at what- because we had a very different group. We were looking at kids in the K, 2, 4, and 6. And it was interesting to see the progression and to use it as... a guideline but not as a restriction with these kids. . . . We had kindergarteners very far on the progression. We had six graders who are kind of stuck in the second little block there, they really struggled with this problem because they were so afraid of being incorrect or they were so in their own way with things. So, I think the progression gave us a nice guideline of how the students could evolve over the years but we were definitely shocked when we looked at the results of the actual student samples in comparison to the chart at how capable and how much more motivated the younger students were and how much further they were in the progression than the older students. *(Interview with Sera, the second-grade teacher)*

The teachers recognized the importance of having a coach in the LS to push them to both try new ideas and to focus on the research goals. The second-grade teacher commented:

> I think especially the days of the lesson study, we really needed Clara, our coach. We really did. We were overwhelmed. We needed that directive personality. We were throwing manipulatives into a bag and running out because we just change gears all of a sudden, we thought "oh this is going to better way and we have to try this way or we're going to regret it." And you know she was the cool level voice that day and was able to say like "Okay what's the one thing you want to do or change?" Focus on that." So, she kept us on track. *(Interview with Sera, the second-grade teacher)*

But in addition to the need for focus and direction, the presence of the coach and MTE also impacted the teachers' interaction around the content. The coach commented specifically on her intervention with the kindergarten host teacher:

> I want to go with whatever lesson you do, and Kathy, you're the kindergarten teacher, and I promise you, you can do it." and [Kathy] just made this face, she thought I was crazy, and I said, 'No and I'm telling you, I'm gonna help you.' And I kind of gave her no other option. *(Interview with Clara, the coach)*

The teacher had felt that the content was too challenging and that it was not appropriate for her students. However, through the coach's encouragement, she implemented the lesson and found success in learning about what her young mathematicians were capable of doing with a fair share problem.

5 Discussion and Conclusion

5.1 Developing a Framework for Vertical Articulation to Unpack the Learning Trajectories Through Lesson Study

This study was prompted by a Lesson Study that focused each phase of the Lesson Study on unpacking the learning progression of fractions. We have developed our PD routine using Lesson Study to focus on the Vertical Articulation to Unpack the Learning Trajectory (VAULT) to help teachers and coaches translate learning trajectories to practice (see Fig. 7).

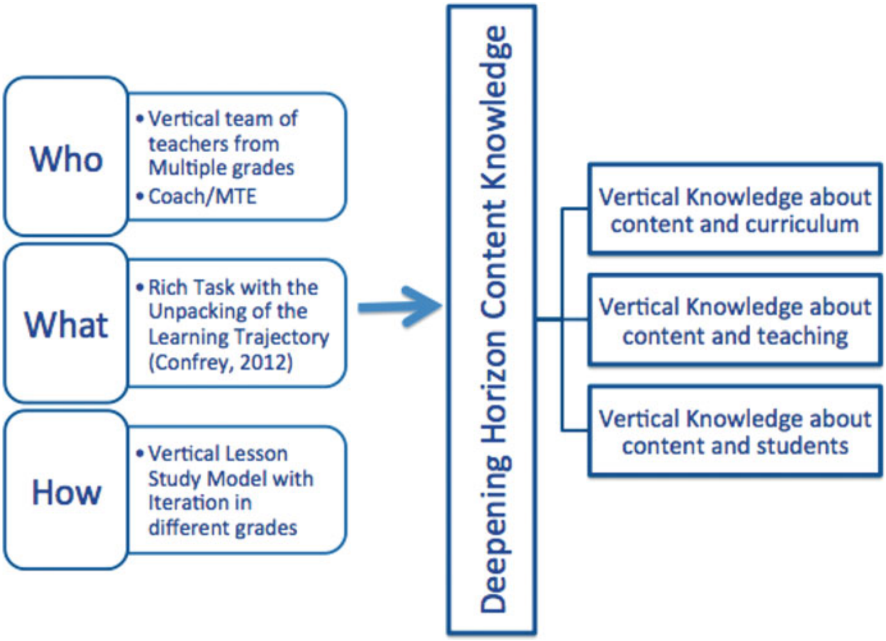


Fig. 7 Key components for the VAULT (Vertical Articulation to Unpack the Learning Trajectory) lesson model leading to deepening horizon content knowledge

The PD routine used during Lesson Study served as an effective school-based coaching tool that deepened the articulation of standards as well as helped teachers to analyze student work using learning trajectories. It is important to note the role of the coach and mathematics educators who brought the equipartitioning learning trajectory to the Lesson Study meeting. Other researchers, Huang et al. (2014) and Lewis (2016), have detailed the pivotal role of the knowledgeable others in a Lesson Study process. In our study, the coach and the mathematics educator facilitated teacher learning by providing teachers with the district-generated and research-based learning trajectories. Bridging the connection between curricular learning standards for multiple grade levels and LTs allows teachers to better attend to students who may have gaps in their learning.

Learning trajectories have the potential to foster equity, but they also have the potential limit access for students who may be classified below grade-level expectations or who think differently than the progression that has been identified or assumed (Myers et al. 2015). Although learning trajectories can offer a road map by which teachers can better assess individual student needs, they are not designed to be a generalizable measure of student achievement. Myers and colleagues asserted, "while trajectories support teachers to view student learning along a continuum, they also may allow for reifying of deficit views that justify preconceived ideas about "high" and "low" children, or ideas about students who do not follow the "typical" path as "deviants"" (p. 18). It is critical to develop teachers' understanding of how student LTs are "hypothetical" in nature and that in any given classroom, teachers will have students whose mathematical understanding may be typical or atypical to a given LT. Vertical articulation with Lesson Study gave our teachers the opportunity to observe and discuss evidence of student reasoning that was not specific to a grade-level classroom. Knowing more about the LT is one way teachers can deepen their knowledge of content, curriculum, and students to better support the sequencing of the instructional activities for each and every student that they design along the curricular standards in mathematics. This study supports Huang et al.'s (2016) call for research that connects theory and practice of LT and LS. Learning trajectory and Lesson Study have a symbiotic relation. LT helps to guide planning, teaching, and debriefing throughout the Lesson Study process as well as improving the lesson for students' understanding, proficiency, and mathematical reasoning. Lesson Study in turn can help to refine LT (Huang et al. 2016).

Our study strived to connect theories around LT and how they enhance the planning, and understanding of student thinking along with the benefits of LS to demonstrate the promising effects LT can have on improving the quality of LS and teacher and student learning. In the case study, the teachers' interaction with the LT on early equipartitioning helped them to see why students at the higher grade levels might struggle with what the teachers saw as simple problems: the students were not yet ready to deal with non-unit fractions. In this way, the LT gave teachers the understanding to support students in moving forward rather than simply casting them as failing to have the skills assessed by their grade-level standards. Coach-facilitated Lesson Study and vertical articulation became tools within VAULT by which teachers could elicit student thinking through multiple representations and purposeful questioning while considering how demonstrated understandings evolve into formal mathematical concepts over time. One implication is that horizon content knowledge requires having ways to think about ideas with the landscape in mind. It requires teachers to have a comprehensive look of the landscape like a GPS knowing where the road will take us and knowing the many pathways (hypothetical learning trajectories). Bridging the connection between curricular learning standards and learning trajectories for multiple grade levels allowed teachers to better attend to students who may have gaps in their learning. The PD routine used during Lesson Study served as an effective school-based coaching tool that deepens the articulation of standards as well as helping teachers analyze student work to map out learning trajectories.

References

Ball, D. L., & Bass, H. (2009, March). *With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures*. Paper presented at the 43rd Jahrestagung für Didaktik der Mathematik, Oldenburg, Germany.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education, 59*, 389–407.

Battista, M. T. (2004). Applying cognition-based assessment to elementary school Students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning, 6*(2), 185–204.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., et al. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180.

Bryk, A. S. (2015). Accelerating how we learn to improve. *Educational Researcher, 44*(9), 467–477.

Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. (2015). *Learning to improve: How America's schools can get better at getting better*. Cambridge, MA: Harvard University Press.

Clements, D., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*, 81–89.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Confrey, J. (2012). Articulating a learning science foundation for learning trajectories in the CCSS-M. In L. R. Van Zoerst, J. J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology Mathematics Education* (pp. 2–20). Kalamazoo: Western Michigan University.

Confrey, J., & Maloney, A. (2010). *The construction, refinement, and early validation of the equipartitioning learning trajectory*. In K. Gomez, L. Lyons, & J. Radinsky (Eds.), Proceedings of the 9th International conference of the learning sciences (Vol. 1, pp. 968–975). Chicago: International Society of the Learning Sciences.

Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009, July). *Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories*. Paper presented at 33rd Conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.

Confrey, J., Maloney, A., Nguyen, K., Lee, K.S., Panorkou, N., Corley, D., Avineri, T., Nickell, J., Neal, A., Varela, S., & Gibson, T. (n.d.). *Learning trajectories for the K-8 common core math standards*. Retrieved from https://turnonccmath.net

Daro, P., Mosher, F. A., & Corcoran, T. (2011). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction* (Consortium for policy research in education report #RR-68). Philadelphia: Consortium for policy Research in Education.

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals*. Portsmouth: Heinemann.

Fernández, S., & Figueiras, L. (2014). Horizon content knowledge: Shaping MKT for a continuous mathematical education. *REDIMAT, 3*(1), 7–29.

Fosnot, C. T., & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Portsmouth: Heinemann.

Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior, 28*(1), 1–18.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*, 371–406.

Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese Lesson Study. *ZDM Mathematics Education, 46*, 239–251.

Huang, R., Gong, Z., & Han, X. (2016). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM Mathematics Education, 48*(4), 425–439.

Jakobsen, A., Thames, M. H., Ribeiro, C. M., & Delaney, S. (2012). *Using practice to define and distinguish horizon content knowledge*. In Authors (Eds.), Pre-proceedings of 12th International Congress of Mathematics Education (pp. 4635–4644). Seoul: ICMI 12.

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah: Lawrence Erlbaum.

Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: Research for Better Schools.

Lewis, C. (2014). Lesson study with mathematical resources: A sustainable model for locally-led teacher professional learning. *Mathematics Teacher Education & Development, 16*(1), 1–20.

Lewis, C. (2015). What is improvement science? Do we need it in education? *Educational Researcher, 44*(1), 54–61.

Lewis, J. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM Mathematics Education, 48*, 527–540.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Lobato, J., & Walters, C. D. (2017). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 74–101). Reston: National Council of Teachers of Mathematics.

Miles, M. B., Huberman, A. M., & Saldana, J. (2014). *Qualitative data analysis: A methods sourcebook*. Thousand Oaks, CA: Sage Publication.

Myers, M., Sztajn, P., Wilson, P. H., & Edgington, C. (2015). From implicit to explicit: Articulating equitable learning trajectories based instruction. *Journal of Urban Mathematics Education, 8*(2), 11–22.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematics success for all*. Reston: National Council of Teachers of Mathematics.

National Research Council. (2007). In R. A. Duschl, H. A. Schweingruber, & A. W. Shouse (Eds.), *Taking science to school: Learning and teaching science in grades K-8. Committee on science learning, kindergarten through eighth grade*. Washington, DC: National Academies Press.

Shimizu, Y. (1999). Aspects of mathematics teacher education in Japan: Focusing on teachers' roles. *Journal of Mathematics Teacher Education, 2*, 107–116.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114–145.

Smith, M. S., & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions*. Reston: National Council of Teachers of Mathematics.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*(2), 129–162.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: Free Press.

Stigler, J. W., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM Mathematics Education, 48*, 581–587.

Suh, J. M., & Seshaiyer, P. (2015). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. *Journal of Mathematics Teacher Education, 18*(3), 207–229.

Suh, J. M., Birkhead, S., Baker, C., Frank, T., & Seshaiyer, P. (2017). Leveraging coach-facilitated professional development to create teacher social networks for enhancing professional practice. In M. Boston & L. West (Eds.), *Annual perspectives in mathematics education: Reflective and collaborative processes to improve mathematics teaching* (pp. 89–100). Reston: National Council of Teachers of Mathematics.

Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*(5), 147–156.

Takahashi, A. (2006, June). *Implementing lesson study in North American schools and school districts*. Paper presented at Innovation and good practices for teaching and learning Mathematics through Lesson Study at the APEC International symposium, Khon Kaen, Thailand. Retrieved from https://www.apec.org/Publications/2006/06/A-Collaborative-Study-on-Innovations-for-Teaching-and-Learning-Mathematics-in-Different-Cultures-amo

Takahashi, A., Watanabe, T., Yoshida, M., & Wang-Iverson, P. (2005). Improving content and pedagogical knowledge through kyozaikenkyu. In Building our understanding of lesson study (pp. 101–110). Philadelphia, PA: Research for Better Schools, Inc.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & M. P. Taylor (Eds.), *Inquiry into mathematics teacher education* (pp. 131–142). San Diego: Association of Mathematics Teacher Educators.

Wilson, P. H., Sztajn, P., Edgington, C., & Myers, M. (2015). Teachers' uses of a learning trajectory in student-centered instructional practices. *Journal of Teacher Education, 66*, 227–244.

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Part VI
Commentary

A Western Perspective

John Elliott

Contents

Abstract This paper constitutes an attempt to produce a critical commentary on this volume that is informed by a 'classroom action research' tradition, which originated in the work of Lawrence Stenhouse and others at the Centre for Applied Research in Education (CARE) at the University of East Anglia in England. It involved a series of projects, which engaged groups of 'teachers as researchers' in their classrooms, and stimulated the development of a research tradition that impacted across the UK and Europe and more widely in the latter part of the twentieth century. The paper begins with a summary of the main ideas embedded in this tradition of collaborative classroom action research and then goes on to discuss in their light a number of themes and issues posed by contributions in this volume. These include the respective roles of academic experts and teachers in the lesson study process, the role of teachers in constructing accounts of lesson studies and creating 'knowledge platforms', the role of teachers as researchers in relation to curriculum development, the use of learning theories to inform lesson study and the problem of globalizing lesson study methodology across cultures and systems.

Keywords Teachers as researchers · The process model of curriculum design and development · Educational aims and procedural principles · Pedagogy and the curriculum · Classrooms as laboratories · Theory informed lesson study · Lesson study and innovation transfer · Building pedagogical knowledge platforms · Innovation transfer and the globalization of lesson study

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The original version of the chapter has been revised. A correction to this chapter can be found at https://doi.org/10.1007/978-3-030-04031-4_39

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_37

1 Introduction

The ‘Western Perspective’, which informs this commentary, is the pragmatist view of ‘the teacher as a researcher’ forged by Lawrence Stenhouse during the 1960s in England through the Humanities Curriculum Project (see Stenhouse 1968, 1975) and further developed in the latter part of the twentieth century by Elliott and Adelman (1973), Ebbutt and Elliott (1985), Elliott (1976–77/2007), Somekh (2006), and other colleagues at the University of East Anglia and the Cambridge Institute of Education through a series of funded classroom action research projects and postgraduate teacher research programmes. This perspective stimulated and informed the classroom action research movement more widely in the UK and Europe (see O’Hanlon 1996, 2003, and Altrichter et al. 1993). The movement led to the creation of the *Educational Action Research Journal* in the early 1990s, aimed at supporting the development of the theory and practice of educational action research internationally (see Day et al. 2002).

The key features of Stenhouse’s conception of the ‘teacher as a researcher’, which underpinned the classroom action research movement as it evolved, can be summarized in the following terms:

1. The idea implies a *process model* (Stenhouse 1975, Ch. 7) of curriculum planning and development, in which the knowledge contents of the curriculum are selected as foci for speculative thought, rather than objective facts to be mastered. Stenhouse claimed that the structures of knowledge – procedures, concepts and criteria – that shape worthwhile curriculum content are intrinsically problematic within the subject and are therefore a focus for discussion and debate. ‘Understanding’ as a pedagogical aim for engaging learners with such content cannot be achieved, only deepened and widened, in the light of discussion between teacher and students and between the students themselves. Stenhouse argued that there must always be an element of doubt about what constitutes a valid understanding of subject matter. Both the teacher and their learners have to accept an exploration of the nature of understanding as part of their respective tasks. Educational aims for handling curriculum content, such as ‘the development of understanding’ are best viewed, according to Stenhouse, as clusters of values to be realized in the pedagogical process of engaging pupils as learners with educationally worthwhile subject matter. Such values, he claimed, could be cast as *principles of procedure* that are logically implied by an educational aim. Hence for Stenhouse, providing students with opportunities to express their personal points of view on the subject content through classroom discussion, and in the process protecting the expression of divergent and minority opinions, are important principles of procedure that are implied by the aim of ‘developing understanding’. However, such an aim also implies principles for selecting educationally worthwhile subject content; namely, that it can be linked to the lived experiences of learners in everyday life and the structures of knowledge which enable them to think about and reflect on such experiences. The research role of the teacher is concerned with identifying constraints on the realization of procedural values and principles in the teaching-learning situation and discovering action strategies for resolving them.

Stenhouse, echoing John Dewey, argued that 'there are criteria by which one can criticise and improve the process of education without reference to an end-means model that sets limits to one's efforts'. In these terms, he rejected 'technical rationality' as a model of reasoning for improving the teaching and learning process (see Stenhouse 1975, Ch. 6). The role of the 'teacher researcher' was to criticize and improve educational practice in the light of the procedural principles implicit in an educational aim. In doing so (s)he generated diagnostic and practical hypotheses for fellow teachers to test in the context of improving the teaching and learning in their own classroom. However, in doing so they depended on conversations with their students – disciplined by evidence about their experiences as learners in the classroom environment – and conversations with their peers – disciplined by evidence gathered as observers in their classroom.

According to Stenhouse (1975, Ch. 6), 'the behavioural objectives model' of curriculum planning and development distorted the nature of knowledge. In this respect, again echoing Dewey, Stenhouse embraced 'democratic rationality' as a model of reasoning for improving the teaching and learning process. Within this model of practical reasoning, 'the teacher as a researcher' reflected jointly on both the means and end of teaching while taking into account the points of view expressed by pupils and professional peers. Stenhouse was happy to depict the process of teacher action research as one of *curriculum development*, since the *process model* that shaped it synthesized 'principles for selecting content' with 'principles for engaging learners with it'. Teachers as researchers reviewed and reconceptualized worthwhile curriculum content in the process of developing strategies for realizing their pedagogical aims.

2. The idea implies that the teacher is socially situated in *professional learning communities*, which collaboratively and systematically develop shared insights into, and strategies for handling, problems and issues that arise in their classrooms about the realization of their educational aims and values in action. They do so by sharing, comparing and contrasting their case studies of teaching and learning in their classrooms. Such 'communities of practice' provide a context for 'collaborative lesson research' that is orientated towards providing platforms for the improvement of teaching and learning and curricula in schools. The aim of collaborative teacher research with support from professional researchers is to progressively construct a professional knowledge base that can be accessed by increasing numbers of teachers to support the development of their professional practice in classrooms. The scale and depth of teachers' professional development will depend on the cumulative growth of accessible 'pedagogical knowledge platforms' by teacher-researchers. This does not imply that teachers use such platforms to improve their practice without becoming teacher-researchers themselves. It simply implies that the scale and depth of teacher development are dependent on the progressive construction of publicly accessible pedagogical knowledge. Without the construction of accessible and cumulative 'pedagogical knowledge platforms' by teacher-researchers, the professional development of teachers will lack over time both *scale* and *depth*.

3. The idea implies that teaching is cast as an ‘experimental activity’ in which the publicly accessible findings of teacher research are treated as hypotheses – about how to improve the quality of students’ learning experiences and the ethical agency of teachers in realizing educational values and principles in action – to be tested and further developed in more classrooms conceived as ‘laboratories’. In this respect, we again find echoes of Dewey, with respect to his ‘laboratory model’ as opposed to an ‘apprenticeship model’ of learning to teach (see Elliott 2012). On this view, teachers learn to improve the curriculum and the teaching and learning process in their classrooms when they become active participants in the process of creating pedagogical knowledge. The research they undertake will not only benefit them as individual practitioners, but also have the potential to contribute to building ‘public platforms of professional knowledge’ that other teachers may benefit from.
4. According to Stenhouse, the growth of collaborative teacher research and the progressive construction of pedagogical knowledge platforms are dependent on ‘a common vocabulary of concepts and a syntax of theory’ as a basis for holding disciplined conversations together about the problems of teaching and learning. In other words, teacher collaborative research that is grounded in a ‘democratic model’ of reasoning needs to be informed by a pedagogical theory.

   Early in the twenty-first century, this author encountered *Learning Study* as a participatory form of classroom action research while working as an adviser on the curriculum reform process in Hong Kong. *Learning study* was developed by Lo Mun Ling and her colleagues at the Hong Kong Institute of Education in the context of the post-changeover curriculum reforms (see Lo et al. 2004; Lo 2004). It blended the Japanese tradition of lesson study as a form of collaborative teacher research with Marton and Booth’s variation theory (1997), which was developed in Sweden at Gothenburg University. A high point for this author was his appointment as Evaluator of the ‘Variation for the Improvement of Teaching and Learning’ (VITAL) Project in Hong Kong Schools (See Elliott and Yu 2008, 2013). It stimulated an interest in comparing *Japanese Lesson Study, Learning Study* and the *Teacher as Researcher* movement that stemmed from the work of Stenhouse and others in England.

2 Themes and Issues Posed for the Author by This Book

The collection of papers assembled in this book are an important resource for scholars, school teachers and policymakers who are interested in the theory and practice of lesson study, both in the context of mathematical education and more generally across the school curriculum. East Asian and North American perspectives dominate, although there are interesting accounts of lesson study in Africa, the UK, Sweden, Switzerland, Iceland and Portugal. Notably absent from this author’s point of view are accounts of mathematics lesson study in Ireland, Austria and Germany, and Spain. For this author, when viewed through the *lens* outlined above, many of the papers pose important and challenging issues about the development of the theory and practice of lesson study as it globalizes. They provide a structure for this commentary.

2.1 How the Theory and Practice of Lesson Study Shapes Up in Different Cultural Settings

There are a number of articles on the history, theory and practice of Chinese lesson study that enable the reader to explore similarities and differences between the theory and practice of Japanese and Chinese lesson research. Here details about how curriculum change in a particular historical and cultural context shapes the theory and practice of lesson study are particularly illuminating. Li's article, however, pinpoints some invariant culturally distinct features of the Chinese lesson study tradition:

> By frequently referring to some Chinese classic works, we attempt to demonstrate how Lesson Study in China has resonated strongly with a few fundamental principles in traditional Chinese education and culture regarding teaching and learning. Specifically, we identify three main themes that have underlain the forms and characteristics of Chinese Lesson Study: (1) Respecting and learning from masters and experts; (2) Teaching and learning by integrating profound theory and deliberate practice; (3) Consolidating teaching and learning into one complete process. These themes provide guidance to the roles and behaviors of all parties involved in Lesson Study activities: mentors, experts, experienced and novice teachers, and fellow teacher participants.

Within this tradition, the master teachers have expertise that they have developed over time through their active participation in a form of lesson study that has involved the joint development of theory and practice, where practice is informed by an explicit pedagogical theory, and which in turn is developed through practice. Hence, in the Chinese lesson study tradition, the development of both theory and practice has joint primacy. According to Li, the master teachers learn about different aspects of their teaching – 'educational theories, curriculum standards and instructional materials, subject matter content, student characteristics, lesson design and teaching strategies, assessment, etc.' – in the process of deliberatively developing their practice. The development of theoretical understanding is not dissociated from the development of teaching as a practice. In this way Li claims, Chinese lesson study traditionally *consolidates* the development of teachers' teaching expertise with their development as learners. There appears to be a marked similarity of perspective in this respect with Stenhouse's idea of the 'teacher as a researcher'. This might be explained by the fact that the relationship between knowledge and action in Confucian thought shares much in common with a philosophically pragmatist view of that relationship rooted in the work of Dewey and Pierce. It is this philosophical outlook that also underpins the Stenhousian perspective on the 'teacher as researcher'.

According to Li, the Chinese lesson study tradition involves not only 'sharing professional learning unidirectionally from mentors and experts to novice teachers'.

It also involves ‘mutual learning among fellow teacher participants with similar levels of experience’. He argues, consistently with Confucius, that even ‘expert and experienced teachers could potentially learn good ideas and strategies from novice teachers in Lesson Study’. Hence, in a broad sense, ‘all teachers involved in lesson study could be considered members of a learning community with equal status yet varying goals and learning needs’. On this account, Chinese lesson study does indeed appear to embody a form of ‘democratic rationality’.

In the west, it is often assumed that lesson study methodology originated in Japan. This assumption is understandable given the influence of the TIMMS and PISA tables internationally and the emergence of Japan in those tables as a high-performing nation in the field of mathematical learning, prior to the more recent emergence of high-performing Shanghai. When policymakers, the media, and academic educationalists sought reasons for Japan’s success, they discovered lesson studies as a pedagogical practice in Japanese schools, particularly in the primary sector. Interestingly, the vast majority of accounts of Japanese lesson studies depicted in this book are drawn from primary school mathematics lessons. This poses an interesting question: why is lesson study as a pedagogical practice more established in primary rather than secondary schools in Japan, particularly upper secondary schools? This does not appear to be the case with Chinese lesson study as it is portrayed in this book. Could one reason be that it is within this latter age range that teachers as individuals are increasingly held to account by high-stakes forms of summative assessment. Another might be that many primary school teachers lack sufficient subject matter knowledge to teach a subject well. By working together as a group with a subject matter specialist to study teaching materials and associated curriculum content (*kyouzai kenkyuu*) and then to develop their practice by collectively designing and observing one another teaching *research lessons*, they could compensate for individualized deficits in subject knowledge.

Lesson study in the Japanese context presupposes a ready supply of curriculum expertise (often referred to in this volume as *knowledgeable others*) to service the planning and conduct of the lesson study process. In the ‘The Origin and Development of Lesson Study in Japan’, Makinae suggests a reason for this supply, namely, the origins of Japanese lesson study in teacher training colleges, during the latter half of the nineteenth century, as a method for equipping teachers to handle the new teaching materials based on an ‘object lesson approach’ to teaching subject content. This approach was grounded, she claims, in a Pestalozzian theory of subject matter learning.

According to Makinae, a key feature of the teacher training system was the *criticism lesson*:

> In this method, the normal school (training college) students presented a lesson to the class and other students observed and discussed it – Later, the criticism lesson expanded its role from pre-service teacher training to in-service professional development. This describes how lesson study originated in Japan.

Although lesson study in Japan evolved beyond the original initial training context, it clearly maintained strong links with a source of *knowledgeable others* in the higher education institutions, which absorbed and incorporated functions in the fields of both pre- and in-service teacher education. According to Yang and Ricks (2012), Chinese mathematics teachers 'do not exhibit the quantity of formal higher education as their western and Japanese counter-parts'. Yet there is evidence to suggest, Yang and Ricks claim, that they have profounder understanding of fundamental mathematical ideas and better pedagogical content knowledge, which they are able to use more coherently for the purposes of instruction than their counterparts in the West and Japan. Yang and Ricks suggest that this may be explained by teachers' participation in the hierarchically layered – school, district and provincial – system of school-based teaching research networks. In this system, *knowledgeable others* tend to be master teachers who have developed their expertise through becoming teacher-researchers, rather than abstract academic study dissociated from deliberative action in classrooms. It is a system in which novice and in-service teachers have continuing access to feedback from *knowledgeable others* at all stages of the lesson study process. I would claim that such a system sustains the distinctive characteristics of Chinese lesson study that Li refers to in his chapter.

Makinae's paper suggests another reason for the relative containment of Japanese lesson study in the primary school sector that originated in the context of curriculum and pedagogical reforms confined to this sector, which had their roots in Pestalozzian learning theory – *the object lesson* approach. It is often remarked that Japanese lesson research is not informed by any explicit learning theory that can serve as a pedagogical resource for improving teaching and learning in classrooms. This is perhaps why it is sometimes seen to have a purely practical focus, on the development of an effective scheme of work (a lesson) as opposed to the realization of educational ideas in a form of action undertaken within the classroom. Whereas the latter might be said to place a dual emphasis on the development of theory and practice, the former appears to prioritize the 'primacy of practice', conceived in terms of technical efficacy. In Hong Kong, the rationale for synthesizing Swedish variation theory as a pedagogical tool with Japanese lesson study methodology was that the latter often appeared to focus teachers' attention on their teaching methods as opposed to the quality of their pupils' learning experiences in the classroom. Variation theory aspired to focus Hong Kong teachers' attention on the quality of students' experiences in classrooms as learners. In this respect, it gave lesson study groups a shared language for thinking about and discussing problems of teaching and learning along the lines advocated by Stenhouse (1975, 157). In this author's view, it also embodied a process model of curriculum and pedagogical design (see Elliott 2015, 152–158).

Some would argue that Japanese lesson study is not an atheoretical process. It is now tacitly rather than explicitly informed by the traditional Pestalozzian conceptions of learning and curriculum. One wonders how salient these are now as the cultural script shaping practice in primary schools. To what extent is that script now being compromised in a global context, where national educational systems are being shaped by high-stakes testing regimes that render teachers and schools accountable for their technical/instrumental effectiveness in maximizing prespecified pupil learning outcomes? Perhaps lesson study in Japan needs to incorporate a methodology that enables teachers to reflect on and analyse the cultural scripts that currently underpin the process of teaching and learning in their classrooms. I am thinking here of the seminal work on comparative cross-national lesson study coordinated by Sarkar Arani (see 2017, 10–26), which is designed to render the 'cultural script' that shapes teaching and learning in particular national settings explicit as an object of reflection for teachers.

Han and Huang in this volume write:

> Although structurally similar to Japanese Lesson Study, some unique features of Chinese LS such as emphasizing the repeatedly rehearsals of the same lesson to different groups of students, perfecting an exemplary lesson, emphasizing knowledgeable others' involvement throughout the entire of lesson study have been identified. –Yet, more empirical studies on how teachers learn from and promote students' learning through Chinese lesson study from various perspectives are needed. – Interpreting and understanding how Chinese lesson study contributes to mathematics teacher learning will help practitioners and researchers better the mechanism of Chinese lesson study and deepen understanding of the mathematical work of teaching.

This author asks: who will undertake such research and how? This leads him to the next issue this book poses.

2.2 Who Authors Accounts of Teachers' Lesson Studies?

All of the chapters in this book are authored by academic researchers and scholars located in higher education settings. Moreover, it is often very unclear how these contributors to the book are socially located in roles and relationships to the work of lesson study and the teachers who participate in it. Are they espousing the stance of a scholarly and impartial observer/spectator of the process and activities they depict? Are they evaluating lesson study as an approach to developing and improving teaching and learning? Have they facilitated and coordinated the lesson study programme for teachers, which they are now authoring an account of? Have they served a programme of lesson study as 'knowledgeable other(s)'? It is important for the reader to understand the social location of academics who write about the work of teachers. One can argue that different roles carry different 'fore-understandings' (prejudgements) about the phenomenon being described and that these ought to be rendered explicit as objects of reflection by authors who are so positioned. The authors in this volume tend to present their accounts and reports as the product of much scholarly effort, as evidenced by the long list of references at the end of chapters. The point of view presented tend stowards that of the detached and impartial academic in the role of the 'knowledgeable other'.

This author will now return to Han and Huang's plea for more empirical research into the processes involved in Chinese lesson study. His response will be to ask why the lesson studies that are to figure as objects of inquiry cannot themselves yield the empirical data Han and Huang are calling for? Cannot the teachers who participate in lesson studies produce narrative accounts of their collective and individual learning as intrinsic aspects of a lesson study text they co-author. This is not to deny the value of work co-authored by teachers with 'knowledgeable others'. However, it is a sad state of affairs when the latter marginalize the research role of teachers as active partners in constructing publicly accessible platforms of 'pedagogical knowledge'.

2.3 'Cherry-Picking' Lesson Study as It Globalizes to Western Countries

Some articles in this book provide a timely warning of the dangers of methodological distortion as lesson study globalizes to the western hemisphere. For example, the paper by Takahashi and McDougal in this volume states:

> Takahashi, the main author of this article, practiced Lesson Study as a teacher in Japan. He has nearly 20 years of experience observing activities referred to as 'Lesson Study' that looked very different from what he knows as Japanese Lesson Study. Many Lesson Study projects outside of Japan omit some of the crucial elements, which hinders their success. For example, Fujii (National Council of Teachers of Mathematics 1989) examined Lesson Study in some African countries and noted that many aspects of Japanese Lesson Study are left out. The same occurs in America; many projects omit the first crucial phase of Lesson Study, kyouzai kenkyuu ("study of teaching materials" that helps teachers gain knowledge and insight into mathematics and student thinking).

The significance of *kyouzai kenkyuu* in Japanese lesson study positioned teachers as curriculum *developers* and not simply *implementers*. Lesson study in Japan is often credited as making a significant impact on the reconstruction of textbooks and curriculum materials. In this respect, it is consistent with Stenhouse's vision of the 'teacher as researcher'. In this volume, lesson study as it internationalizes is often portrayed as having a major role in helping teachers to implement curriculum reforms shaped by new national standards. This author found little reference to lesson study having a significant role in the creation of new curricula. An interesting paper by Watanabe in this volume entitled 'Lesson Study and Textbook Revisions: What Can We Learn from the Japanese Case?' explores the evidence for crediting Japanese lesson study with making a significant impact on curricula revisions in Japan from the 1980s. He concludes:

> Although we cannot definitively conclude that Lesson Study led to specific changes in textbooks, there are some evidences that suggest its influence. We speculate that certain features of the Japanese education system may contribute to making such influences possible.

The process of lesson study that Takahashi experienced as a teacher in Japan appears to have addressed a curriculum problem that Bruner identified in the USA as far back as 1963. Bruner writes:

> The first and most obvious problem is how to construct curricula that can be taught by ordinary teachers to ordinary students and at the same time reflect clearly the basic or underlying principles of various fields of inquiry.

In western neo-liberal societies, the strong links between higher education institutions and pre-service and in-service teacher education have weakened. The former are decreasingly a major source of 'knowledgeable others'. In this respect, the thinking of Bruner (1963) on curriculum design for schools appears to have been ignored by western governments:

> Designing curricula in a way that reflects the basic structure of a field of knowledge requires the most fundamental understanding of that field. It is a task that cannot be carried out without the active participation of the ablest scholars and scientists. The experience of the past several years has shown that such scholars and scientists, working in conjunction with experienced teachers and students of child development, can prepare curricula of the sort we have been considering. Much more effort in the actual preparation of curriculum materials, in teacher training, and in supporting research will be necessary if improvements in our educational practices are to be of an order that meets the challenges of the scientific and social revolution through which we are now living.

Bruner's view is very consistent with Stenhouse's vision of the role of teachers as researchers in the curriculum development process. Curriculum experts, often located in higher education institutions, designed curricula for teachers to test and improve through research in their classrooms. Such was the vision both Bruner and Stenhouse aspired to realize as curriculum theorists and developers in partnership with teacher-researchers.

Dudley, Warwick, Vrikki, Vermunt, Mercer, Mette, van Halem and Karlsen appear to welcome the movement in the UK towards school-based CPD as a 'home' for lesson study. They portray an INSET revolution in schools that followed on from legislation accompanying the introduction of a statutory national curriculum and assessment system in the early 1990s. This legislation they argue 'transformed state schools into quasi-independent, competitive businesses regulated by a national inspectorate, Ofsted, ranked annually by examination results and inspection ratings'. In this context, teachers were subjected to statutory performance management based on lesson observation. Dudley et al. contend that this set the scene for an explosion of school initiated INSET activity 'as schools realized that if they were to gain the positive inspection grades and examination ranking needed to attract pupils and funding, they were going to need good results'. In this context, they argue that schools were encouraged to develop their own inquiry-based processes for improving teaching and learning in classrooms. It was here they claim that 'lesson study thrived' as a form of teacher research in the UK. Dudley et al. claim that by 2012, 'up to 10% of English schools had used lesson study'.

The major justification for introducing lesson study in England as a form of teacher professional development appears to have depended on demonstrating that it improves intended learning outcomes as measured by standardized tests (SATS). Although inspired by Japanese lesson study, the role of *kyouzai kenkyuu* in the latter context, this author would argue, is far from clear and in this respect echoes the concerns expressed by Takahashi in the North American context, where the neo-liberal outcomes-based educational agenda is also very evident. In a systems context shaped by this agenda, there is little space for curriculum development (*kyouzai kenkyuu*) as such. In this context, the design of the curriculum is shaped by nationally prescribed standards in the form of intended learning outcomes.

The point of introducing a form of lesson study in such a context might simply be to find the pedagogical means of implementing curricula that match the required standards. In the Camden Project depicted by Dudley et al., school-based, lesson study groups met with a group of ‘knowledgeable others’ in the field of Maths on a twice termly basis, but their experience appears to stem from assessment roles in the educational system. Could it be that their major role in the project appears to be that of helping teachers to interpret the new curriculum standards correctly as a basis for lesson planning? Securing good test results appears to have been a significant motive for doing lesson study in the project. According to Dudley et al., an experimental trial demonstrated that it ‘paid off’:

> The scores of project schools rose against the national average between 2013 and 2016 by two percentage points while the attainment of district schools not involved in the project fell against the national average by two percentage-points – a four percent difference in total. Both quantitative and qualitative data suggest that teachers had been developing their own practices in response to their LS learning experiences in ways that subsequently improved the learning of their pupils.

Dudley et al.’s account of the Camden Project’s design clearly demonstrates an intention to give participating teachers space and time to realize many of the critical features of the Japanese lesson study methodology and thereby resist pressure to simply ‘teach to the tests’. It would be good to gain access to full text lesson studies authored by Camden Project teachers, which reflectively explore the tension between the action research process they engaged in and the desire to get good results as measured by tests.

In addition to a lack of *kyouzai kenkyuu*, Takahashi noted other aspects of Japanese lesson study methodology that became diluted in the process of transference to western educational systems. These are:

- Compressed time frames for completing a lesson study cycle (e.g. from 5 weeks to a day).
- Misunderstanding the purpose of lesson study as ‘creating a perfect lesson plan’ rather than ‘gaining new knowledge’ to inform the teaching – learning process.
- Confusing demonstration lessons with research lessons. Demonstrating innovative teaching strategies to an audience of peers ‘does not necessarily help teachers to implement innovative ideas’.

Such dilutions are perhaps symptomatic of a rush to implement lesson study in short timescales, without attending to the problems of securing significant changes in the professional culture of teachers and the organizational culture of schooling. In this respect, this author would recommend reading Brosnan’s (2014) implementation study of her ‘failed attempt’ to introduce Japanese lesson study methodology in the context of a new mathematics curriculum for Irish post-primary schools. It is a rare published example of what this author depicts below as *second-order action research*. Although the paper appears to render a *pathology of innovation*, it also has a positive aspect, which Posch (2015) refers to in his response to Brosnan. ‘Luckily’, he writes, ‘after 3 years, it showed that the experience with lesson study had not been in vain but had shaken some traditional beliefs and had opened a window for change in the professional culture’.

There is clearly a need for a framework of quality criteria that will enable one to distinguish 'watered down' versions of lesson study from those which embody universal critical aspects of lesson study methodology, while acknowledging the fact that these aspects may nevertheless culturally shape up rather differently in particular educational contexts. Such quality criteria need to be couched as procedural values and principles that give form to the process of lesson study without depicting the detailed action strategies that need to be developed for overcoming system and cultural constraints in particular contexts and realizing them in practice. Such a framework will provide a basis for 'second-order action research' by those involved in coordinating and facilitating lesson study in educational systems, such as school leaders and advisors, curriculum specialists and educational researchers from the academy. In my view, the paper by Lewis, Friedkin, Emerson, Henn and Goldsmith in this volume makes an excellent start in developing such a framework. It provides a basis for planning and evaluating the lesson study process as the approach internationalizes and globalizes. The authors specify 'goals' for each phase of a lesson study cycle, which I would argue function as 'procedural principles' governing the process, by highlighting its critical aspects. They then list possible 'challenges' in the form of constraints and difficulties in realizing the appropriate goals (principles) at each phase of the process and even suggest strategies for overcoming them. This author would suggest that these are depicted as 'hypotheses' to be tested and revised through what I have termed 'second-order action research'. Lewis et al. finally pose questions to stimulate reflection on the extent to which the goals (principles) have been realized. Such questions can be used to structure summary accounts of second-order action research findings.

2.4 Using Explicit Learning Theories to Inform the Global Development of Lesson Study as a Participatory Pedagogical Science

Several papers in this volume provide accounts of how *explicit theories of learning* are used pedagogically to design and inform the lesson study process. Gunnarsson, Runesson and Håkansson describe a Swedish *learning* study on 'Identifying what is critical for learning "rate of change"'. Learning study, as noted earlier, is a form of lesson study that originated in Hong Kong and Sweden by synthesizing critical aspects of the Japanese methodology, which was considered to lack an explicit learning theory, with the theory of variation developed by Marton and Booth (1997) and colleagues at the University of Gothenburg. Although developed from phenomenographic design experiments, the theory was used as a pedagogical tool to focus the classroom research of teachers on the qualitative aspects of students' experiences as learners in their lessons. Both this author and Lo (see Elliott 2015, esp.155) have argued that such a synthesis is entirely consistent with Stenhouse's conception of the teacher as a researcher and his process model of curriculum and pedagogical design.

The reader will find other papers in this volume that depict a synthesis of variation theory and lesson study. Huang, Gong and Han's chapter on 'Implementing Mathematics Teaching That Promotes Students' Understanding Through Theory-Driven Lesson Study' shows how a Chinese lesson study, aimed at developing students' understanding of mathematical content, used variation theory as a pedagogical tool for effectively sustaining the focus of teachers on the learning process in their classrooms. Preciado Babb, Metz and Davis report interesting Canadian research into how variation theory can inform and become sustainably embedded in the concrete teaching strategies employed to enact mathematics lessons.

Variation theory is not the only theoretical perspective used to inform lesson studies reported in this volume. As lesson study internationalizes beyond Japan and China, it is inevitable that, conceptually and methodologically, it will interact with both general and subject-specific didactic theories in circulation within the adoptive country. In this volume, with respect to lesson studies informed by specific didactic theories, there are interesting papers authored by Clivaz and Ni Shuilleabhain (Switzerland) and Adler and Alshwaikh (South Africa). The volume provides examples of lesson studies that have been synthesized with a variety of theoretical perspectives. They pose the question of whether, in the context of globalization, such syntheses help to sustain lesson study as a democratic process characterized by a universal set of core values and principles of procedure.

Schoenfeld, Dosalmas, Fink, Sayavedra, Tran, Weltman, Zarkh and Zuniga-Ruiz report on a synthesis between Japanese lesson study and the theoretical framework of 'Teaching for Robust Understanding' (TRU). The latter offers a specification of the critical aspects governing an educationally worthwhile process of engaging students with curriculum content in a manner that develops their understanding. Another way of viewing this framework is that it provides teachers with a set of procedural principles implied by the aim of teaching for 'robust understanding'. Schoenfield et al. offer the following rationale for such a synthesis in the US context:

> Teachers in the US typically have little collective time to reflect on teaching practice. TRU-Lesson Study (TRU-LS) supports the growth of Teacher Learning Communities and their engagement with key ideas and practices of TRU and Lesson Study. Like Lesson Study, it profits from teachers' concerted attention to lesson design and reflection on the hypotheses reflected in the design. Like TRU-based professional development, it supports teachers to work together explicitly on key dimensions of classroom practice.

What is being asserted here are conceptual links between the methodology of Japanese lesson study and an explicit theoretical framework (TRU).

An interesting example of an attempt to synthesize a globally circulating explicit learning theory with lesson study methodology is provided by Ge Wei in this volume and entitled 'How Could Cultural-Historical Activity Theory (CHAT) Inspire Lesson Study?' Wei provides a detailed case study of a Chinese lesson study conducted by a group of teachers, which has been methodologically contextualized with neo-Vygotskian *cultural-historical activity theory*. He points out that in China, lesson study is initiated and driven by teachers. Hence, Wei's case study focuses on the teachers' accounts of their learning in the process. He claims that this data indicates that the process of methodologically contextualizing lesson study with CHAT has the impact of highlighting lesson study as a process that widens teachers' understanding of mathematical subject matter. In doing so, he argues, it captures the complexity of learning. Wei writes:

> As a dialectical theory, CHAT views human relationships as interwoven with multiple contradictions and conceptualizes learning as a dynamic and non-linear process. Expansive learning is an ideal type of this kind of learning. Expansive learning is not only a learning theory, but also a methodological instrument with which to design and promote teacher professional learning in an extensive manner.

One can see how contextualizing traditional Chinese lesson study with a globally circulating sophisticated learning process like CHAT might serve to safeguard it against the distorting impact of the global use of standardized tests to shape outcomes-based and internationally competitive educational systems. Runesson (2015) has pointed out that different explicit learning theories are underpinned by different epistemological and ontological assumptions, which when used to inform lesson studies will provide teachers with rather different foci. They will inevitably shape teachers' research to lead to rather different kinds of findings. The examples of learning theories used to inform lesson study accounts in this volume appear to be highly compatible with the Stenhousian idea of the teacher as a researcher and the process model of curriculum and pedagogical design. Yet we need second-order comparative studies of the use of such theories to inform teachers' lesson studies, especially the uses of variation theory and CHAT, to identify pedagogically significant similarities and differences.

2.5 Where Are the Knowledge Platforms?

Runesson and Gustafsson (2012) demonstrated that research findings from a learning study, informed by variation theory and conducted in Hong Kong, could be communicated to and appropriated by teachers in Sweden. They saw these findings as a contribution to developing Stenhouse's conception of teachers as *knowledge producers* capable of building 'knowledge platforms' that could be accessed and used by teachers generally. In fact, Runnesson and Gustafsson's research offers the promise of teachers participating in the construction of cross-national platforms to support curriculum and pedagogical development on a global scale.

Dudley et al. 'struck a chord' with this author when they claimed in their paper that the district level meetings of the Camden Project 'connected elements of the design cycle across groups of schools, helping to form lines of enquiry and knowledge-building in these challenging- to - teach areas of mathematics'.

The teacher lesson research groups supported by subject experts identified 'difficult to teach' mathematical content to form a cross-school agenda of themes to build lesson studies around. Dudley et al. report that:

Teaching areas in which at least a quarter of the teachers identified they had little or no confidence included:

- Place value for decimal notation of fractions.
- Written methods of division involving decimals
- Finding 100% given a percentage part
- Algebraic distinctions and terminology, providing explanations and solving simultaneous equations algebraically
- Identifying formulae, and models for algebraic equations, direct proportion and graphical representations
- Explaining the distinctions between types of numbers and illustrating particular forms of number
- Calculations involving fractions
- Dividing a quantity in a given ratio

Following the completion of a first round of lesson studies that focused on these 'difficult to teach' areas, a second district meeting enabled the LS groups to share and collectively reflect about their findings to build knowledge across cases and identify common themes to be investigated in the next round of lesson studies. Dudley et al. did not outline or provide examples of the cross-school pedagogical knowledge base that was built in the course of the Camden project and which might have been used by teachers of mathematics generally as a platform for further curriculum and pedagogical development. However, the Camden Project illustrates the value of organizational structures that will enable LS groups to draw on and contribute to the building of 'knowledge platforms' in relation to pedagogically challenging curriculum content.

This volume presents many interesting cases of lesson studies in the field of mathematics. It points to the need to develop lesson study in a range of curriculum fields as a process of pedagogical knowledge building across schools, systems and more globally. This is a long-term project for journal editors and educational publishers to address in partnership with teachers, school leaders, educational advisers, policymakers and curriculum experts and researchers in academic organizations.

References

Altrichter, H., Posch, P., & Somekh, B. (1993). *Teachers investigate their work*. London/New York: Routledge.

Brosnan, A. (2014). Introducing lesson study in introducing a new mathematics curriculum in Irish post-primary schools. *International Journal of Lesson and Learning Studies, 3*(3), 236–251.

Bruner, J. (1963). *The process of educatio*n. New York: Vintage Books (Random House), Ch.2.

Day, C., Elliott, J., Somekh, B., & Winter, R. (Eds.). (2002). *Theory and practice in action research: Some international perspectives*. Oxford: Symposium Books.

Ebbutt, D., & Elliott, J. (Eds.). (1985). *Issues in teaching for understanding*. London: School Council/Longmans.

Elliott, J. (1976–77). Developing hypotheses about classrooms from teachers' practical constructs. *Interchange, 7*(2), 2–22. Also published in Elliott, J. (2007). *Reflecting where the action is – The selected works of JOHN ELLIOTT*. London/New York: Routledge, Ch. 2.

Elliott, J. (2012). Developing a science of teaching through lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 108–125.

Elliott, J. (2015). Lesson and learning study and the idea of the teacher as a researcher. In K. Woods & S. Sithamparam (Eds.), *Realising learning* (pp. 148–167). London/New York: Routledge.

Elliott, J., & Adelman, C. (1973). Reflecting where the action is: The design of the Ford Teaching Project. *Education for Teaching, 92*, 8–20.

Elliott, J., & Yu, C. (2008). *Learning studies as an educational change strategy in Hong Kong: an independent evaluation of the 'variation for the improvement of teaching and learning' (VITAL)*. Tai Po: Centre for School Experience and Partnership, Hong Kong Institute of Education.

Elliott, J., & Yu, C. (2013). Learning studies in Hong Kong schools: A summary evaluation report on the 'Variation for the Improvement of Teaching and Learning' (VITAL) Project. *Education & Didactique, 7*(2), 147–163.

Lo, M. L. (2004). The development of the learning study approach in classroom research in Hong Kong. *Educational Research Journal, 24*(1), 165–184.

Lo, M. L., Pong, W. Y., Marton, F., Ng, F. P., & Pang, M. F. (Eds.). (2004). *For each and every one: Catering for individual differences through learning study*. Hong Kong: Hong Kong University Press.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah: Lawrence Erlbaum Associates.

O'Hanlon, C. (Ed.). (1996). *Professional development through action research in educational settings*. London: The Falmer Press.

O'Hanlon, C. (2003). *Educational inclusion as action research: An interpretative discourse*. Maidenhead: Open University Press.

Posch, P. (2015). A comment on Anne Brosnan's paper on "Introducing lesson study in introducing a new mathematics curriculum in Irish post-primary schools". *International Journal of Lesson and Learning Studies, 4*(2), 236–251.

Runesson, U. (2015). Pedagogical and learning theories and the improvement and development of learning and lesson studies. *International Journal of Lesson and Learning Studies, 4*(3), 186–193.

Runesson, U., & Gustafsson, G. (2012). Sharing and developing knowledge products from learning study. *International Journal for Lesson and Learning Studies, 1*(3), 245–260.

Sarkar Arani, M. R. (2017). Raising the quality of teaching through Kyouzai Kenkyuu: The study of teaching materials. *International Journal of Lesson and Learning Studies, 6*(1), 10–26.

Somekh, B. (2006). *Action research: A methodology for change and development*. Maidenhead: Open University Press/McGraw-Hill Education.

Stenhouse, L. (1968). The humanities curriculum project. *Journal of Curriculum Studies, 1*, 26–33.

Stenhouse, L. (1975). *An introduction to curriculum development and research*. London/New York: Heinemann.

Yang, Y., & Ricks, T. E. (2012). How crucial incidents analysis support Chinese lesson study. *International Journal for Lesson and Learning Studies, 1*(1), 41–48.

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approach with "Swedish Variation Theory" and Stenhouse's idea of "Teachers as Researchers." In 2002 and 2003, respectively, John Elliott was awarded Doctorates *Honoris Causa* by the former Hong Kong Institute of Education and the Autonomous University of Barcelona. In September 2014, he was similarly honored by the University of Jonkoping in Sweden. In 2011, he was made a fellow of the UK Academy of Social Sciences. John Elliott was the founding chief editor of the WALS journal, the *International Journal of Lesson and Learning Studies* from 2011 to 2016, during which time it was awarded recognition by the *Scopus* and *Emerging Sources Citation* (*ESCI*) indexes. He currently chairs its advisory board on behalf of WALS. His most recent publications include with Nigel Norris (Eds.) *Curriculum, Pedagogy and Educational Research: The Work of Lawrence Stenhouse*, Routledge (2012).

An Asian Perspective

Mun Ling LO

Contents

Abstract Lesson study cannot exist devoid of context; it is always a part of something bigger, for example, the school system and the culture. Focusing on the specific characteristics and features of the lesson study itself will limit our understanding within its internal horizon only. To achieve a more in-depth understanding, we must try to see how the lesson study connects with the external horizon, for example, the theoretical lens used to guide the lesson study and to analyse the research lesson, the school system in which the research lesson is enacted and the culture of which it is a part. Then, we would be able to understand why specific approaches towards conducting lesson study are preferred, why certain features vary across different countries, and unpack culturally embedded messages that have been taken for granted. It is crucial for us to develop theories because they help us to be less reliant on the expert, and more efforts should be directed towards developing theories that guide us on how to deal with the objects of learning.

Keywords Theory-informed lesson study · Culturally embedded features · Variation of lesson study · Internal horizon · External horizon

When I was invited by Rongjin to write a commentary for this book from an Asian perspective, I was happy but hesitant. It is an honour for me to be invited to participate in such an important book project, yet I do not know what the Asian perspective is! Being immersed in the Asian culture all my life, I have already taken all its features for granted. However, as one's ways of thinking are necessarily affected by the environment and one's life experiences, my perspectives will reflect the Asian culture, whether I am aware of it or not. That was why I accepted the invitation, but I will leave the readers to unpack the Asian perspectives. I believe that

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R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_38

the readers will be able to discern the 'Asian perspectives' when they contrast them with their own. Also, having started learning study in Hong Kong and having supported over 100 learning studies over almost 20 years, I cannot help but see through the eyes of a learning study researcher. So, my writing will reflect this as well.

There is a Chinese idiom story about three blind men who wished to learn about an elephant. The first blind man touched the ear of the elephant and said that an elephant was like a flat sheet. The second blind man argued that it was more like a tube as he held on to the tail of the elephant, while the third blind man insisted on his view that an elephant is shaped like a pillar because he was embracing the leg of the elephant. None of them was completely wrong, but each only learned a part but not the whole of the elephant. Lesson study is like an elephant to those of us that need to be enlightened. Although by now there are many papers on this topic, unfortunately, such publications are also under many constraints. For example, the need to focus and conform to word limits makes it impossible to paint a complete picture. This book includes chapters from researchers of different countries, with diverse experiences, under different contextual constraints, and with lesson studies at various stages of development. It presents a good collection of rich descriptions of the present situations in this field of study. The book will answer many of the questions of the patient enquirer who will read the whole book, and this book is a landmark in the development of lesson study. Still, the 'part-to-whole' relationship must be born in mind in reading the chapters, because these chapters still only represent some parts of the whole.

As I dutifully read through the chapters of this book to write the commentary, I realised that each of them has to be understood as a culturally embedded text; many of the messages and ideas need to be unpacked to be fully appreciated. One must refrain from jumping to conclusions that certain essential features are missing, because such features may be so culturally embedded that the authors have taken them for granted. Even with the same terms, they do not mean the same thing. An example is 'the object of learning' (e.g. see Gunnarsson, Runesson and Kakansson's chapter). For the researchers of learning study premised on variation theory, the 'object of learning' is not simply content. Embedded in the term 'object of learning' is a whole set of meanings. An object of learning is clarified and defined by critical features. It is dynamic because the defining critical features are only revealed when students encounter the object of learning. Its worthiness lies on how the learning of it provides a linkage to the whole of which it is a part. Also, it is not limited to content but includes capabilities, skills and attitudes (Lo 2012).

Another example is the term 'unit flow' or 'flow of the research lesson' when mentioned in lesson study (e.g. see Fujii's chapter). It does not simply mean the planned progression of the lesson and the sequencing of teaching activities as some of us might take for its meaning. This term includes the studying of curriculum and teaching materials, choosing learning tasks and examples and anticipating student learning difficulties. However, most of these would be considered by the learning study researchers under the 'object of learning', whereas the 'flow of the lesson' for them would include things like 'enacting the patterns of variation' used to bring out the critical features and carrying out the activities to bring about the patterns of variation. These are just two examples, but the readers can surely unpack many more. I raise this issue because it would be possible that researchers of different camps misunderstand each other and think that there are missing features in the other parties' approach if they do not pay particular attention to unpack the real meaning of the terms used.

The chapters also reveal how culture shapes the development of lesson study in various countries. In both China and Japan, there is an emphasis on the 'knowledgeable other' or 'expert'. In both cultures, respect for the senior is an accepted cultural norm. We can certainly see the value of an expert because their advice usually comes as a result of a fusion of many aspects and represents the sublimation of experiences and wisdom. Such insights sometimes surpass scientific analysis (Lo 2012, p. 67). In both China and Japan, since lesson study has been widespread and practised for many years, experts can be and have been nurtured. Unfortunately, in other countries, where the history of lesson study is relatively brief, the environment to nurture experts does not yet exist. Although it is possible to nurture 'experts' (see J.M. Lewis' chapter on 'Learning and Leading Lesson Study'), it takes time and opportunities. The lack of experts perhaps explains why in most countries, other than Japan and China, scientific evidences and theories are sought to support lesson study and an action research approach is usually adopted. Teaching different cycles of the research lesson is a common practice. The various teaching cycles allow evidences to be gathered to verify if the suggested amendments are indeed improvements. Explorations are underway searching for theoretical frameworks, and a whole section in this book is devoted to discussing the theoretical perspectives of lesson study. It would be interesting to take the data from one lesson study and analyse it using different explanatory frameworks and theories. The commonalities and differences would be revealed. However, I believe that there would be more commonalities than differences and that various theoretical lenses would be offering multiple ways of saying the same thing.

An aspect that shows variation across countries is the focus of the lesson study; it varies from focusing on the learning of the teacher to focusing on the learning of the students. From my reading of the chapters, lesson studies in both China and Japan have a greater focus on the learning of the teacher and similarly for lesson studies of countries that aspire towards the Japanese model. Lesson study in Japan is regarded as a platform for teachers' professional development. The research lesson is rarely retaught because its main purpose is to serve as a model lesson which provides a context for critical review and analysis (e.g. see Makinae's chapter). The experts make the final judgement about the quality of the research lesson. Similarly in China, although the research lesson may be retaught and modified, the goal is to produce the best available lesson plan. This lesson plan is then disseminated through public demonstration lessons taught to different classes of pupils. The same lesson plan, only with very slight modifications as necessitated by the dynamics of the classroom, is used. Sometimes, a teacher may even teach this demonstration lesson to different groups of unfamiliar students from other schools. This situation may seem incredible to readers of other countries, but with a national curriculum, highly motivated students, and streaming according to ability, the classes that are chosen for the demonstration lesson may be quite similar in academic achievement across the country. While the focus is on the professional development of teachers, it does not imply that the learning of students is ignored. In lesson studies of both Japan and China, meticulous attention is paid to how students learn during the research lesson, for example, by observing the reactions of target students during the lesson very carefully and making detailed notes of students' reactions. In some lesson study models, teachers draw on their knowledge of students to predict a learning trajectory and anticipate student learning difficulties (e.g. see Han and Huang's chapter and Suh, Birkhead and Galanti's chapter). However, I have not yet found any references to what students have to 'say' about their learning before or after the research lesson being used to shape the research lesson. In England, the lesson study model is modified to take into account student learning by focusing on the learning of three case students (see Dudley et al.'s chapter). These students are carefully observed and interviewed to assess their learning. The learning study model which started in Hong Kong and later further developed in Sweden and some other countries, on the other hand, focuses primarily on student learning. Special attention is paid to students' voices before, during and after the research lessons. Interviews and diagnostic tests are the norms for assessing students' ways of seeing the object of learning, and the dynamic nature of the object of learning is acknowledged. Students' difficulties with, misconceptions of, and different ways of seeing the object of learning play an important role in constituting the critical features. In learning study, the students' voices are deemed more important than the experts' voices. For example, in Hong Kong, student interviews are an essential feature of learning study. Instead of relying solely on teachers' experiences to guess the learning trajectory, pilot interviews with students from a more senior class that had already learned the topic help the learning study group to find out what difficulties students encountered in the object of learning. These students' persistent misconceptions about the object of learning become valuable resources for identifying some of the critical features. Apart from administering pretest and post-test to all students, three students considered to be of low, average, and high achievement are chosen by the class teacher for interviews just before and right after the research lesson. These interviews are attended by all members involved in the learning study, sitting at the back of the students. During the post-lesson interview, searching questions to establish how students have experienced the research lesson are asked. Questions include, for example: 'What do you think the teacher was trying to teach you in this lesson?' 'Did you learn something new in this lesson? What is it that you did not know before the lesson, but now you know or have mastered?' 'How did you learn it? What did the teacher do that helped you to learn "X"?' 'Before the lesson, this was how you described your understanding of "X".... now have you changed your mind? Can you please elaborate on it?' Also, students are asked to demonstrate what they have learned by working out some problems. In this way, the teachers get feedback directly from the students. We believe that the students are in the best position to judge whether a lesson helps them to learn or not. It is not how the teacher performed during the lesson (the enacted object of learning) but how the students experienced it (the lived object of learning) that counts. The role of the expert in making a judgement on whether the research lesson has been successful in bringing learning about is thus played down. Still, the expert has a significant role in leading the post-lesson discussion to interpret why learning has or has not taken place and how the next cycle can be improved.

If the primary purpose of lesson study is for professional development, it is natural that the wisdom of practice will guide the study. Thus, experts' opinions are sought and very much respected. If teachers' benefit from lesson study and become better teachers, then naturally, students will benefit and learn better. On the other hand, if the primary purpose of learning study is to find the best way to help students master a specific object of learning and an action research approach is adopted, gathering data for interpretation and testing hypothesis to improve learning are natural consequences. Once a better way to help students learn is found, students will learn better, and it follows that teachers will gain expertise and will teach better. So, whether the initial focus is primarily on the learning of the teachers or the learning of the students, both parties will benefit in the end.

This book includes a wide variety of contributions from different countries, with rich descriptions of how the lesson study model works for them. What I appreciate most is that many chapters give detailed accounts of the research lesson, the activities used to bring about learning, and some even include details of examples and contents used in the research lesson. It is only through such information that one can come to understand in concrete terms fully. Many useful insights are shared on both in-service teacher development and initial teacher education. I found the chapter 'Lesson Study for Pre-service teachers' by J.M. Lewis of interest. The model uses a novel idea of having the instructor teach the research lesson. In this way, the enacted lesson would not deviate too much from the intended plan, as often happens due to the insufficient mastery of technical teaching skills of a student teacher! Another exciting point reported is that against all advice about how lesson study can work best, the study succeeded. The author concludes that, ultimately, it depends on whether participants can gain positive experiences that change their initial negative attitude and have ownership at the end of the process. This conclusion prompts an important research question: 'In what ways can we ensure participants gain positive experiences?'

One obvious answer to the above question is to ensure that the process of lesson study eventually leads to better student learning. By that, it also means that a pressing research question for any lesson study group is 'How can we deal with content in ways that facilitate learning?' The importance of content has been mentioned in many chapters, and it seems to be a consensus among lesson study researchers. It is nice to see some chapters developing theories on how the content should be taught, for example, the chapter by Bahn and Winslow on the theory of didactical situations, the chapter by Schoenfeld et al. on teaching for robust understanding, the chapter by Fujii on careful selection of examples based on the variation principle and the chapter by Gunnarsson, Runesson and Hakansson on how variance and invariance can inform teacher's enactment. Nakamura's chapter points to the role of the expert in helping the student teachers see the content as a part of something bigger, the whole of which it is a part. This role was much appreciated because making explicit the relationship between this part and the whole and other parts of this whole makes for much higher-quality learning. In this way, an emerging idea of 'fusion' has been alluded to but not fully explored. For the interested readers, Lo and Chik (2016) discussed the idea of 'fusion' in some details. They differentiated the internal and external horizons of fusion. Internal horizon relates to the structure and meaning of the object of learning as experienced by the learner; it clarifies the interrelationships among an object's critical features and aspects and the part-whole relationships of these features and aspects within the object itself. Fusion in the external horizon elucidates the relationships between the object of learning and its environment as seen by the learner, and such relationships extend beyond the object's boundaries (Lo and Chik 2016: p. 296). They argued that any content (and/or an object of learning) should not be taught out of context, but both internal and external horizons of fusion should be attended to for the knowledge gained to be meaningful and worthwhile. 'Fusion' is a useful concept because we can also apply the idea to understanding the variation of lesson study across countries if we take 'lesson study' to be an 'object of learning'. To achieve a more in-depth understanding, we must try to see how the lesson study connects with its external horizon, for example, the theoretical lens used to guide the lesson study and to analyse the research lesson, the school system in which the research lesson is enacted and the culture of which it is a part. Since lesson studies aim to improve the quality of teaching and learning, which, in turn, depends on the quality of the object of learning chosen, in the future, more researches directed towards this critical aspect are needed. We can be less reliant on the expert if we can develop theoretical explanations to enlighten us on why certain acts of teaching can help students achieve a deeper understanding of the content while others do not and why some objects of learning can contribute to higher-quality learning than others. As Lo and Marton (2012) concluded in their paper, although teaching is ultimately an art, it has a scientific basis (Lo and Marton 2012).

Common to all of the chapters, we can find genuine attempts by lesson study groups to improve teaching and learning under existing constraints, to adapt lesson study to the best possible effect and to acknowledge the fact that student learning is a function of teaching. Perhaps the situation that the Japanese lesson study has never been made clear and fully unpacked (see Takahashi and McDougal's chapter) allows for explorations by different research groups, within their constraints and context, to pursue an ideal, an end product that they wish to achieve. Although the path is not clear, the goal is clear. As a result, the lesson study models thus developed are contextually viable for their local environment. The models described in this book reflect the different stages of development; some are more sophisticated, enabling very useful generalisations to be drawn (see Lewis et al.'s chapter), while some are just starting their exploration. I would consider this situation as aptly described by the saying 'every road leads to Rome'. Some roads are faster and smoother, while others may have obstacles on its way and slower, but, eventually, everyone will get there, if they will just keep going!

References

Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*. Sweden: Acta Universitatis Gothoburgensis.

Lo, M. L., & Chik, P. P. M. (2016). Two horizons of fusion. *Scandinavian Journal of Educational Research, 60*(3), 296–308.

Lo, M. L., & Marton, F. (2012). Towards a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal of Lesson and Learning Studies, 1*(1), 7–22.

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Correction to: Theory and Practice of Lesson Study in Mathematics

Rongjin Huang, Akihiko Takahashi, and João Pedro da Ponte

Correction to: R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*, Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4

This book was inadvertently published with the incorrect name for the author, Paulino Preciado-Babb in Chapters 1, 13, 17, and 37. It has been revised to Armando Paulino Preciado Babb.

The updated versions of these chapters can be found at
https://doi.org/10.1007/978-3-030-04031-4_1
https://doi.org/10.1007/978-3-030-04031-4_13
https://doi.org/10.1007/978-3-030-04031-4_17
https://doi.org/10.1007/978-3-030-04031-4_37

R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4_39

References

Abe, H. (1987). Borrowing from Japan: China's first modern educational system. In R. Hayhoe & M. Bastid (Eds.), *China's education and the industrialized world* (pp. 57–80). Armonk: M. E. Sharpe.

Adamson, B., & Walker, E. (2011). Messy collaboration: Learning from a learning study. *Teaching and Teacher Education, 27*(1), 29–36.

Adler, J. (2017). *Mathematics discourse in instruction (MDI): A discursive resource as boundary object across practices*. Paper presented at the 13th international congress on mathematical education, ICME-13, Hamburg.

Adler, J., & Ronda, E. (2015). A framework for describing mathematics discourse in instruction and interpreting differences in teaching. *African Journal of Research in Mathematics, Science and Technology Education, 19*(3), 237–254.

Adler, J., & Ronda, E. (2017a). Mathematical discourse in instruction matters. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 64–81). Abingdon: Routledge.

Adler, J., & Ronda, E. (2017b). A lesson to learn from: From research insights to teaching a lesson. In J. Adler & A. Sfard (Eds.), *Research for educational change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 133–143). Abingdon: Routledge.

Adler, J., & Ronda, E. (in preparation). *What changes in the quality of mathematics in instruction following teachers' participation in subject matter focused professional development*.

Adler, J., & Venkat, H. (2014). Teachers' mathematical discourse in instruction: Focus on examples and explanations. In H. Venkat, M. Rollnick, J. Loughran, & M. Askew (Eds.), *Exploring mathematics and science teachers' knowledge: Windows into teacher thinking* (pp. 132–146). Abingdon: Routledge.

Adler, J., Ball, D. L., Krainer, K., Lin, F. L., & Novotna, J. (2005). Mirror images of an emerging field: Researching mathematics teacher education. *Educational Studies in Mathematics, 60*, 359–381.

Akiba, M. (2016). Traveling teacher professional development model: Local interpretation and adaptation of lesson study in Florida. In F. Astiz & M. Akiba (Eds.), *The global and the local: New perspectives in comparative education* (pp. 77–97). Rotterdam: Sense.

Akiba, M., & Wilkinson, B. (2016). Adopting an international innovation for teacher professional development: State and district approaches to lesson study in Florida. *Journal of Teacher Education, 67*(1), 74–93.

Akiba, M., Ramp, L., & Wilkinson, B. (2014). *Lesson study policy and practice in Florida: Findings from a statewide district survey*. Tallahassee: Florida State University.

R. Huang et al. (eds.), *Theory and Practice of Lesson Study in Mathematics*,
Advances in Mathematics Education, https://doi.org/10.1007/978-3-030-04031-4

Akiba, M., Wang, Z., & Liang, G. (2015). Organizational resources for professional development: A statewide longitudinal survey of middle school mathematics teachers. *Journal of School Leadership, 25*(March), 252–285.

Akiba, M., Wilkinson, B., Farfan, G., Howard, C., Kuleshova, A., Fryer, J., et al. (2016). *Lesson study in Florida: A longitudinal survey of district policy and practice from 2013 to 2015*. Tallahassee: Florida State University.

Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research, 81*(2), 132–169.

Alexander, C. M., & Ambrose, R. C. (2010). Digesting student-authored story problems. *Mathematics Teaching in the Middle School, 16*(1), 27–33.

Alshwaikh, J., & Adler, J. (2017a). Researchers and teachers as learners in lesson study. In M. K. Mhlolo, S. N. Matoti, & B. Fredericks (Eds.), *SAARMSTE book of long papers* (pp. 2–14). Free State: Central University of Technology.

Alshwaikh, J., & Adler, J. (2017b). Tensions and dilemmas as source of coherence. In A. Chronaki (Ed.), *Mathematics education and life at times of crisis: Proceedings of the ninth international mathematics education and society conference (V2)* (pp. 370–381). Volos: University of Thessaly.

Alter, J., & Coggshall, J. G. (2009). Teaching as a clinical practice profession: Implications for teacher preparation and state policy. Issue Brief. *National Comprehensive Center for Teacher Quality*.

Altrichter, H., Posch, P., & Somekh, B. (1993). *Teachers investigate their work*. London/New York: Routledge.

Amador, J., & Weiland, I. (2015). What preservice teachers and knowledgeable others professionally notice during lesson study. *The Teacher Educator, 50*(2), 109–126.

An, G. (2008). The connotations and values of lesson study. *Global Education, 37*(7), 15–19 (in Chinese).

Angelini, M. L., & Álvarez, N. (2018). Spreading lesson study in pre-service teacher instruction. *International Journal for Lesson and Learning Studies, 7*(1), 23–36.

Antonini, S., Presmeg, N., Mariotti, M. A., & Zaslavsky, O. (2011). On examples in mathematical thinking and learning. *ZDM Mathematics Education, 43*(2), 191–194. https://doi.org/10.1007/s11858-011-0334-5.

Aoibhinn, N. S. (2016). Developing mathematics teachers' pedagogical content knowledge in lesson study: Case study findings. *International Journal for Lesson and Learning Studies, 5* (3), 212–226. https://doi.org/10.1108/IJLLS-11-2015-0036.

APEC Human Resources Development Working Group (n.d.). *Lesson study overview*. Retrieved March 15, 2016, from http://hrd/apec.org/index.php/Lesson_Study_Overview. (no longer available).

Archer, R. (2016). *Lesson study in initial teacher education: Students' positioning analysed through the lens of figured worlds.* Paper presented at the British Society for Research into Learning Mathematics.

Archibald, S., Coggshall, J. G., Croft, A., & Goe, L. (2011). *High-quality professional development for all teachers: Effectively allocating resources*. Washington, DC: National Comprehensive Center for Teacher Quality.

Ashlock, R. D. (1990). *Error patterns in computation*. New York: Macmillan.

Askew, M., Brown, M., Rhodes, V., Wiliam, D., & Johnson, D. (1997). *Effective teachers of numeracy: Report of a study carried out for the teacher training agency*. London: King's College, University of London.

Baba, T., & Nakai, K. (2011). *Teachers' institution and participation in a lesson Study Project in Zambia: Implication and possibilities. Paper presented at the Africa-Asia University Dialogue for Educational Development*. Report of the international experience sharing seminar (2). Actual Status and Issues of Teacher Professional Development. Retrieved from http://aadcice.hiroshima-u.ac.jp/e/publications/sosho4_2-06.pdf

Bahn, J. (2018). *Inquiry based mathematics education and lesson study: Teachers' inquiry based learning about inquiry based teaching* (Unpublished PhD dissertation), University of Copenhagen.

Bakhtin, M. M. (1981). In M. Holquist (Ed.), *The dialogic imagination: Four essays*. Austin/London: University of Texas Press.

Bakkenes, I., Vermunt, J. D., & Wubbels, T. (2010). Teacher learning in the context of educational innovation: Learning activities and learning outcomes of experienced teachers. *Learning and Instruction, 20*(6), 533–548.

Baldinger, E., Louie, N., & The Algebra Teaching Study and Mathematics Assessment Project. (2016). *TRU conversation guide: A tool for teacher learning and growth*. Berkeley/E. Lansing: Graduate School of Education, University of California, Berkeley/College of Education, Michigan State University. Retrieved from: http://ats.berkeley.edu/tools.html and/or http://map.mathshell.org/materials/pd.php

Baldry, F., & Foster, C. (in press). Lesson study partnerships in initial teacher education. In P. Wood & D. Larssen (Eds.), *Lesson study in initial teacher education. A critical perspective*. London: Emerald.

Ball, D. L. (1990a). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*, 132–144.

Ball, D. L. (1990b). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal, 90*(4), 449–466.

Ball, D. (1993). Halves, pieces and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49–84). Hillsdale: Lawrence Erlbaum Associates.

Ball, D. L. (2016). Uncovering the special mathematical work of teaching. In *Plenary lecture at the 13th International Congress on Mathematical Education (ICME)*. Germany: Hamburg. Retrieved from https://deborahloewenbergball.com/presentations-intro/#presentations

Ball, D. L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips (Ed.), *Yearbook of the national society for the study of education, constructivism in education* (pp. 193–224). Chicago: University of Chicago Press.

Ball, D. L., & Bass, H. (2009, March). *With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures*. Paper presented at the 43rd Jahrestagung für Didaktik der Mathematik, Oldenburg, Germany.

Ball, D., & Cohen, D. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice*. San Francisco: Jossey-Bass.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*, 389–407.

Bandura, A. (2001). Social cognitive theory: An agentic perspective. *Annual Review of Psychology, 52*(1), 1–26.

Barash, A., & Klein, R. (1996). Seventh grades students' algorithmic, intuitive and formal knowledge of multiplication and division of non negative rational numbers. In L. Puig & A. Gutierrez (Eds.), *Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 35–42). Valencia: University of Valencia.

Barlow, A. T., Burroughs, E. A., Harmon, S. E., Sutton, J. T., & Yopp, D. A. (2014). Assessing views of coaching via a video-based tool. *ZDM-The International Journal on Mathematics Education, 46*, 227–238.

Barnes, D., & Todd, F. (1977). *Communication and learning in small groups*. London: Routledge & Kegan Paul Ltd..

Barnett, B. G., & O'Mahony, G. R. (2006). Developing a culture of reflection: Implications for school improvement. *Reflective Practice, 7*(4), 499–523.

Bartolini Bussi, M. G., Bertolini, C., Ramploud, A., & Sun, X. (2017). Cultural transpositions of Chinese lesson study to Italy: An exploratory study on fraction in a fourth grade classroom. *International Journal for Lesson and Learning Studies, 6*(4), 380–395.

Basturk, S. (2016). Investigating the effectiveness of micro-teaching in mathematics of primary pre-service teachers. *Journal of Education and Training Studies, 4*(5), 239–249.

Batteau, V. (2017). Using lesson study in mathematics to develop primary school teachers' practices: A case study. *Quadrante, 26*, 127–157.

Battista, M. T. (2004). Applying cognition-based assessment to elementary school Students' development of understanding of area and volume measurement. *Mathematical Thinking and Learning, 6*(2), 185–204.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A., et al. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal, 47*(1), 133–180.

Becker, J. P., & Shimada, S. (1997). *The open-ended approach: A new proposal for teaching mathematics*. Reston: National Council of Teachers of Mathematics.

Becker, J. P., Silver, E. A., Kantowski, M. G., Travers, K. J., & Wilson, J. W. (1990). Some observations of mathematics teaching in Japanese elementary and junior high schools. *Arithmetic Teacher, 38*(2), 12–21.

Beijaard, D., Verloop, N., & Vermunt, J. D. (2000). Teachers' perceptions of professional identity: An exploratory study from a personal knowledge perspective. *Teaching and Teacher Education, 16*, 749–764.

Beijaard, D., Meijer, P. C., & Verloop, N. (2004). Reconsidering research on teachers' professional identity. *Teaching and Teacher Education, 20*, 107–128.

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's ways of knowing: The development of self, voice, and mind* (Vol. 15). New York: Basic Books.

Benson, B., Mudenda, V., Tindi, E., & Nakai, K. (2014). Lesson study practice of science teachers in Zambia: Its effects, enhancing and hindering factors. In Pixel (Ed.), *Conference proceedings: New perspectives in science education* (pp. 473–477). Florence: libreriauniveritaria.it.

Berg, D. H. (2008). Working memory and arithmetic calculation in children: The contributory roles of processing speed, short-term memory, and reading. *Journal of Experimental Child Psychology, 99*(4), 288–308. https://doi.org/10.1016/j.jecp.2007.12.002.

Beswick, K., Callingham, R., & Watson, J. (2012). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education, 15*(2), 131–157.

Bills, L., & Watson, A. (2008). Editorial introduction. *Educational Studies in Mathematics, 69*(2), 77–79. https://doi.org/10.1007/s10649-008-9147-z.

Bjuland, R., & Helgevold, N. (2018). Dialogic processes that enable student teachers' learning about pupil learning in mentoring conversations in a lesson study field practice. *Teaching and Teacher Education, 70*, 246–254.

Bjuland, R., & Mosvold, R. (2015). Lesson study in teacher education: Learning from a challenging case. *Teaching and Teacher Education, 52*, 83–90.

Bloch, I. (1999). L'articulation du travail mathématique du professeur et de l'élève dans l'enseignement de l'analyse en première scientifique. Détermination d'un milieu – connaissances et savoirs. *Recherches en Didactique des Mathématiques, 19*, 135–194.

Bloch, I. (2005). Quelques apports de la théorie des situations à la didactique des mathématiques dans l'enseignement secondaire et supérieur : contribution à l'étude et à l'évolution de quelques concepts issus de la théorie des situations didactiques en didactique des mathématiques. HDR, Paris 7.

Bolster, A. S. (1983). Toward a more effective model of research on teaching. *Harvard Educational Review, 53*(3), 294–308.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher, 33*(8), 3–15.

Borko, H., Eisenhart, M., Brown, C. A., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*, 194–222.

Borko, H., Jacobs, J., & Koellner, K. (2010). Contemporary approaches to teacher professional development. In E. Baker, B. McGaw, & P. Peterson (Eds.), *International encyclopaedia of education (part 7)* (3rd ed., pp. 548–555). Oxford: Elsevier Scientific Publishers.

Borko, H., Koellner, K., Jacobs, J., & Seago, N. (2011). Using video representations of teaching in practice based professional development programs. *ZDM Mathematics Education, 43*(1), 175–187.

Borko, H., Jacobs, J., Koellner, K., & Swackhamer, L. (2015). *Mathematics professional development: Improving teaching using the problem-solving cycle and leadership preparation models*. New York: Teachers College Press.

Bourdieu, P. (1990). *The logic of practice* (R. Nice, Trans.). Stanford: Stanford University Press.

Bowen, G. A. (2009). Document analysis as a qualitative research method. *Qualitative Research Journal, 9*(2), 27–40.

Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology, 3*(2), 77–101.

Britzman, D. P. (2003). *Practice makes practice: A critical study of learning to teach* (rev ed.). Albany: State University of New York Press.

Bronkhorst, L. H., Meijer, P. C., Koster, B., & Vermunt, J. D. (2011). Fostering learning-oriented learning and deliberate practice in teacher education. *Teaching and Teacher Education, 27*(7), 1120–1130.

Bronkhorst, L. H., Meijer, P. C., Koster, B., & Vermunt, J. D. (2013). Deliberate practice in teacher education. *European Journal of Teacher Education, 37*(1), 18–34.

Bronkhorst, L. H., Meijer, P. C., Koster, B., & Vermunt, J. D. (2014). Deliberate practice in teacher education. *European Journal of Teacher Education, 37*(1), 18–34.

Brosnan, A. (2014). Introducing lesson study in introducing a new mathematics curriculum in Irish post-primary schools. *International Journal of Lesson and Learning Studies, 3*(3), 236–251.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *The Journal of the Learning Sciences, 2*(2), 141–178.

Brown, S., & McIntyre, D. (1993). *Making sense of teaching*. Buckingham: Open University Press.

Bruce, C. D., & Ladky, M. S. (2011). What's going on backstage? Revealing the work of lesson study with mathematics teachers. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 243–249). New York: Springer.

Bruce, C. D., Flynn, T. C., & Bennett, S. (2016). A focus on exploratory tasks in lesson study: The Canadian 'Math for young children' project. *ZDM Mathematics Education, 48*(4), 541–554.

Bruner, J. (1963). *The process of educatio*n. New York: Vintage Books (Random House), Ch.2.

Bruner, J. (1966). *Toward a theory of instruction*. London: Harvard University Press.

Bruner, J. (1990). *Acts of meaning*. Cambridge, MA: Harvard University Press.

Bryk, A. S. (2015). Accelerating how we learn to improve. *Educational Researcher, 44*(9), 467–477.

Bryk, A. S., Sebring, P. B., Allensworth, E., Luppescu, S., & Easton, J. Q. (2010). *Organizing schools for improvement: Lessons from Chicago*. Chicago: University of Chicago Press.

Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. (2015). *Learning to improve: How America's schools can get better at getting better*. Cambridge, MA: Harvard University Press.

Bryk, A. S., Gomez, L. M., Grunow, A., & LeMahieu, P. G. (2016). *Learning to improve: How America's school can get better at getting better*. Cambridge, MA: Harvard Education Press.

Burroughs, E. A., & Luebeck, J. L. (2010). Pre-service teachers in mathematics lesson study. *The Montana Mathematics Enthusiast, 7*(2–3), 391–400.

Cai, J. (2007). What is effective mathematics teaching? A study of teachers from Australia, Mainland China, Hong Kong SAR, and the United States. *ZDM, 39*(4), 265–270.

Cai, Y. (2008). Simulation lessons, polishing lessons, and reflection lessons. *Secondary and Elementary Teacher Education, 7*, 41–42 (in Chinese).

Cai, J., & Wang, T. (2006). U.S. and Chinese teachers' conception and construction of representations: A case of teaching ratio concept. *International Journal of Science and Mathematics Education, 4*, 145–186.

Cai, J., & Wang, T. (2010). Conception of effective mathematic teaching with a cultural context: Perspectives of teachers from China and the United States. *Journal of Mathematics Teacher Education, 13*, 265–287.

Cai, J., Kaiser, G., Perry, G., & Wong, N. Y. (2009). *Effective mathematics teaching from teachers' perspectives*. Rotterdam: Sense Publishers.

Cajkler, W., & Wood, P. (2015a). Lesson study in initial teacher education. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 105–127). Oxon: Routledge.

Cajkler, W., & Wood, P. (2015b). Lesson study in initial teacher education. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (1st ed.). London: Taylor & Francis Group.

Cajkler, W., & Wood, P. (2016a). Adapting 'lesson study' to investigate classroom pedagogy in initial teacher education: What student-teachers think. *Cambridge Journal of Education, 46*(1), 1–18.

Cajkler, W., & Wood, P. (2016b). Mentors and student-teachers "lesson studying" in initial teacher education. *International Journal for Lesson and Learning Studies, 5*(2), 84–98. https://doi.org/10.1108/IJLLS-04-2015-0015.

Campbell, P. F., & Malkus, N. N. (2014). The mathematical knowledge and beliefs of elementary mathematics specialist-coaches. *ZDM Mathematics Education, 46*(2), 213–225.

Campbell, P. F., & Rowan, T. E. (1997). Teacher questions + student language + diversity =mathematical power. In M. J. Kenney (Ed.), *Multicultural and gender equity in the mathematics classroom: The gift of diversity* (pp. 60–70). Reston: National Council of Teachers of Mathematics.

Carlson, M. A., Heaton, R., & Williams, M. (2017). Translating professional development for teachers into professional development for instruction leaders. *Mathematics Teacher Educators, 6*(1), 27–39.

Carpenter, T. C., Lindquist, M. M., Brown, C. A., Kouba, V. L., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Trends and conclusions. *Arithmetic Teacher, 36*(4), 38–41.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school*. Portsmouth: Heinemann.

Cazden, C. (2001). *Classroom discourse: The language of teaching and learning*. Portsmouth: Heinemann.

Chappuis, J. (2015). *Seven strategies of assessment for learning* (2nd ed.). Hobeken: Pearson.

Chassels, C., & Melville, W. (2009). Collaborative, reflective, and iterative Japanese lesson study in an initial teacher education program: Benefits and challenges. *Canadian Journal of Education, 32*(4), 734–763.

Chazan, D., Sela, H., & Herbst, P. (2012). Is the role of equations in the doing of word problems in school algebra changing? Initial indications from teacher study groups. *Cognition and Instruction, 30*, 1–38.

Chen, Q. (1969). *History of education in modern China*. Taipei: Zhonghua Publishing House (in Chinese).

Chen, G. (2006). Collaborative preparation of instruction. *Journal of Education in China, 9*, 40–41.

Chen, X. (2017). Theorizing Chinese lesson study from a cultural perspective. *International Journal of Lesson and Learning Studies., 6*(4), 283–292.

Chen, X., & Yang, F. (2013). Chinese teachers' reconstruction of the curriculum reform through lesson study. *International Journal for Lesson and Learning Studies, 2*(3), 218–236.

Chen, C., & Zhou, Z. (Eds.). (2005). *Mathematics for 4th grade, textbooks for nine-year compulsory education*. Shanghai: Shanghai Education Press.

Cheng, E., & Lo, M. (2013). *Learning study: Its origins, operationalisation, and implications* (OECD education working papers, No. 94). Paris: OECD Publishing.

Chia, H. M. (2014). *Exploring the pedagogical flow of Chinese primary school mathematics lessons before, during and after lesson study.* Unpublished master's thesis, Universiti Sains Malaysia, Malaysia.

Chichibu, T. (2016). Impact on lesson study for initial teacher training in Japan: Focus on mentor roles and *kyouzai-kenkyuu*. *International Journal for Lesson and Learning Studies, 5*(2), 155–168.

Chiew, C. M. (2009). *Implementation of lesson study as an innovative professional development model among mathematics teachers.* Unpublished doctoral dissertation, Universiti Sains Malaysia, Malaysia.

Chiew, C. M., & Jong, C. M. (2009, July). *Perceptions of lesson study: An exploratory case study among pre-service mathematics teachers.* Paper presented at the Seminar Penyelidikan Pendidikan Institut Pendidikan Guru Malaysia Zon Utara 2009, Hotel Copthorne Orchid, Penang.

China's Ministry of Education Review Group of Curriculum Standard (CMERGCS). (2017). *A list of core competences for each subject of general high schools* [EB/OL]. Retrieved from: https://wenku.baidu.com/view/a8da25359e3143323868936b.html

Chokshi, S., & Fernadez, C. (2004, March). Challenges to importing Japanese Lesson Study: Concerns, misconceptions, and nuances. *Phi Delta Kappan, 85*, 520–525.

Clark, R. E., Kirschner, P. A., & Sweller, J. (2012). Putting students on the path to learning: The case for fully guided instruction. *American Educator, 36*(1), 6–11.

Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education, 18*, 947–967.

Clarke, D. J., Keitel, C., & Shimizu, Y. (2006). *Mathematics classrooms in twelve countries: The insider's perspective*. Rotterdam: Sense Publishers.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81–89.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach*. New York: Routledge.

Clements, D., & Sarama, J. (2013). Rethinking early mathematics: What is research-based curriculum for young children? In L. English & J. Mulligan (Eds.), *Reconceptualizing early mathematics learning: Advances in mathematics education*. Dordrecht: Springer.

Clements, D., Sarama, J., Spitler, M., Lange, A., & Wolfe, C. B. (2011). Mathematics learned by young children in an intervention based on learning trajectories: A large-scale cluster randomized trial. *Journal for Research in Mathematics Education, 42*, 127–166.

Clift, R. T., & Brady, P. (2005). In M. Cochran-Smith & K. Zeichner (Eds.), *Studying teacher education Research on methods courses and field experiences* (pp. 309–424). Mahwah: Lawrence Erlbaum.

Clivaz, S. (2014). *Des mathématiques pour enseigner? Quelle influence les connaissances mathématiques des enseignants ont-elles sur leur enseignement à l'école primaire?* La Pensée Sauvage: Grenoble.

Clivaz, S. (2015a). Des mathématiques pour enseigner? Quelques réflexions à partir d'un cas de combinaison de cadres théoriques. In L. Bacon, D. Benoit, C. Lajoie, & I. Oliveira (Eds.), *Croisements variés de concepts, d'approches et de théories: les enjeux de la création en recherche en didactique des mathématiques. Colloque du groupe de didactique des mathématiques du Québec 2014*. Montréal: UQAM.

Clivaz, S. (2015b). French Didactique des Mathématiques and Lesson Study: A profitable dialogue? *International Journal for Lesson and Learning Studies, 4*, 245–260.

Clivaz, S. (2016). Lesson study: From professional development to research in mathematics education. *Quadrante, XXV*, 97–112.

Clivaz, S. (2017). Teaching multidigit multiplication: Combining multiple frameworks to analyse a class episode. *Educational Studies in Mathematics, 96*(3), 305–325.
Cobb, P. (1995). Cultural tools and mathematical learning: A case study. *Journal for Research in Mathematics Education, 26*(4), 362–385.
Cobb, P., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.
Coburn, C. E. (2003). Rethinking scale: Moving beyond numbers to deep and lasting change. *Educational Researcher, 32*(6), 3–12.
Coburn, C. E., & Russell, J. L. (2008). District policy and teachers' social networks. *Educational Evaluation and Policy Analysis, 30*(3), 203–235.
Cochran-Smith, M., & Lytle, S. (1999). Relationships of knowledge and practice: Teacher learning community. *Review of Research in Education, 24*, 249–305.
Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resource, instruction, and research. *Education Policy Analysis Archives, 25*(2), 119–142.
Cole, M. (1985). The zone of proximal development: Where cultural and cognition create each other. In J. Wertsch (Ed.), *Culture, communication, and cognition* (pp. 146–161). New York: Cambridge University Press.
Cole, M., John-Steiner, V., Scribner, S., & Souberman, E. (Eds.). (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
Collet, V. S. (2017). Lesson study in a turnaround school: Local knowledge as a pressure-balanced valve for improved instruction. *Teachers College Record, 119*, 1–58.
Common Core State Standards Initiative. (2010a). *Common core state standards for mathematics*. http://www.corestandards.org/the-standards/mathematics
Common Core State Standards Initiative. (2010b). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/wp-content/uploads/Math_Standards1.pdf
Common Core State Standards Initiative. (2010c). *Common core state standards for English Language Arts & Literacy in History/Social Studies, Science, and Technical Subjects*. Retrieved from http://www.corestandards.org/wp-content/uploads/ELA_Standards1.pdf
Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics*. Retrieved from http://www.corestandards.org/Math/Practice
Confrey, J. (2012). Articulating a learning science foundation for learning trajectories in the CCSS-M. In L. R. Van Zoerst, J. J. Lo, & J. L. Kratky (Eds.), *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology Mathematics Education* (pp. 2–20). Kalamazoo: Western Michigan University.
Confrey, J., & Maloney, A. (2010). *The construction, refinement, and early validation of the equipartitioning learning trajectory*. In K. Gomez, L. Lyons, & J. Radinsky (Eds.), *Proceedings of the 9th International conference of the learning sciences* (Vol. 1, pp. 968–975). Chicago: International Society of the Learning Sciences.
Confrey, J., Maloney, A., Nguyen, K., Mojica, G., & Myers, M. (2009, July). *Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories*. Paper presented at 33rd Conference of the International Group for the Psychology of Mathematics Education, Thessaloniki, Greece.
Confrey, J., Maloney, A., Nguyen, K., Lee, K.S., Panorkou, N., Corley, D., Avineri, T., Nickell, J., Neal, A., Varela, S., & Gibson, T. (n.d.). *Learning trajectories for the K-8 common core math standards*. Retrieved from https://turnonccmath.net
Cong, L. (2011). *Silent authority – teaching research system of Chinese basic education*. Beijing: Beijing Normal University Press.
Corbin, J. M., & Strauss, A. L. (2008a). *Basics of qualitative research [electronic resource]: Techniques and procedures for developing grounded theory*. Los Angeles/London: SAGE.
Corbin, J., & Strauss, A. (2008b). *Basics of qualitative research* (3rd ed.). Los Angeles: Sage.
Corbin, J. M., & Strauss, A. L. (2008c). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Thousand Oaks: Sage.

Corcoran, D., & Pepperell, S. (2011). Learning to teach mathematics using lesson study. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (Vol. 50, pp. 213–230). Dordrecht: Springer.
Cordingley, P., Bell, M., Rundell, B., Evans, D., & Curtis, A. (2004). *How do collaborative and sustained CPD and sustained but not collaborative CPD affect teaching and learning?* London: EPPI-Centre, Institute of Education.
Corey, D. L., Peterson, B. E., Lewis, B. M., & Bukarau, J. (2010). Are there any places that students use their head? Principles of high-quality Japanese mathematics instruction. *Journal for Research in Mathematics Education, 41*(5), 434–478.
Corkin, D. M., Ekmekci, A., & Papakonstantinou, A. (2015). Antecedents of teachers' educational beliefs about mathematics and mathematical knowledge for teaching among in-service teachers in high poverty urban schools. *Australian Journal of Teacher Education, 40*(9), 31–62.
Corte, E. D., Greer, B., & Verschaffel, L. (1996). Mathematics teaching and learning. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 491–549). New York: Macmillan.
Cramer, K., Wyberg, T., & Leavitt, S. (2008). The role of representations in fraction addition and subtraction. *Mathematics Teaching in the Middle School, 13*(8), 490–496.
Cravens, X., & Drake, T. (2017). From Shanghai to Tennessee: Developing instructional leadership through teacher peer excellence groups. *International Journal for Lesson and Learning Studies, 6*(4), 348–364.
Cravens, X., & Wang, J. (2017). Learning from the masters: Shanghai's teacher-expertise infusion system. *International Journal for Lesson and Learning Studies, 6*(4), 306–320.
Cruickshank, D. R., & Metcalf, K. K. (1993). Improving preservice teacher assessment through on-campus laboratory experiences. *Theory Into Practice, 32*(2), 86–92.
da Ponte, J. P. (2017). Lesson studies in initial mathematics teacher education. *International Journal for Lesson and Learning Studies, 6*(2), 169–181.
Dai, Q. (2003). *Confucian thoughts and traditional mathematics in China*. Beijing: Commerce Publishing House.
Danielsson, A., & Warwick, P. (2015). Identity and discourse: Gee's discourse analysis as a way of approaching the constitution of primary science teacher identities. In L. Avraamidou & W. M. Roth (Eds.), *Studying science teacher identity: Theoretical perspectives, methodological approaches and empirical findings* (pp. 71–88). Rotterdam: Sense Publishers.
Darling-Hammond, L. (2006). *Powerful teacher education*. San Francisco: Jossey-Bass.
Darling-Hammond, L., & Richardson, N. (2009). Teacher learning: What matters. *Educational Leadership, 66*(5), 46–55.
Darling-Hammond, L., Wei, R. C., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession: A status report on teacher development in the United States and abroad*. Oxford: National Staff Development Council.
Daro, P., Mosher, F., & Corcoran, T. (2011a). *Learning trajectories in mathematics (Research Report No. 68)*. Madison: Consortium for Policy Research in Education.
Daro, P., Mosher, F. A., & Corcoran, T. (2011b). *Learning trajectories in mathematics: A foundation for standards, curriculum, assessment, and instruction (Consortium for policy research in education report #RR-68)*. Philadelphia: Consortium for policy Research in Education.
Datnow, A. (2005). The sustainability of comprehensive reform models in changing district and state contexts. *Educational Administration Quarterly, 41*(1), 121–153.
Davies, P., & Dunnill, R. (2008). Learning study as a model of collaborative practice in initial teacher education. *Journal of Education for Teaching, 34*(1), 3–16.
Davis, B., & Renert, M. (2013). Profound understanding of emergent mathematics: Broadening the construct of teachers' disciplinary knowledge. *Educational Studies in Mathematics, 82*, 245–265.

Davydov, V. V. (1999). The content and unsolved problems of activity theory. In Y. Engeström, R. Miettinent, & R.-L. Punamaki (Eds.), *Perspectives on activity theory* (pp. 39–52). New York: Cambridge University Press.

Day, C., Elliott, J., Somekh, B., & Winter, R. (Eds.). (2002). *Theory and practice in action research: Some international perspectives*. Oxford: Symposium Books.

Dean for Impact. (2016). *Practice with purpose: The emerging science of teacher expertise*. Austin: Deans for Impact. Available at https://deansforimpact.org/wp-content/uploads/2016/12/Practice-with-Purpose_FOR-PRINT_113016.pdf

Deans for Impact. (2016). *Practice with purpose: The Emerging science of teacher expertise.* September, 5, 2017*,* Retrieved from https://deansforimpact.org/resources/practice-with-purpose/

Deci, E., & Ryan, R. (1985). *Intrinsic motivation and self-determination in human behavior*. New York: Plenum.

Dede, C., & Honan, J. P. (2005). Scaling up success: A synthesis of themes and insights. In C. Dede, J. P. Honan, & L. C. Peters (Eds.), *Scaling up success: Lessons from technology-based educational improvement* (pp. 227–239). San Fracisco: Jossey-Bass.

Dede, C., Honan, J. P., & Peters, L. C. (2005). *Scaling up success: Lessons from technology-based educational improvement*. San Fracisco: Jossey-Bass.

Deng, J., & Li, C. (1995). *Resources for modern history of elementary education in Beijing*. Beijing: Beijing Education Press.

Depaepe, F., Verschaffel, L., & Kelchtermans, G. (2013). Pedagogical content knowledge: A systematic review of the way in which the concept has pervaded mathematics educational research. *Teaching and Teacher Education, 34*, 12–25.

Department for Children, Schools and Families. (2008). *Improving practice and progression through lesson study: Handbook for head teachers, leading teachers and subject leaders.* Nottingham: DCSF Publications. Retrieved September 27, 2011, from http://teachfind.com/national-strategies/improving-practice-and-progression-through-lesson-study-handbook-headteachers-le

Department for Education. (2010). *The importance of teaching: The schools white paper 2010*. London: DfE.

Department for Education. (2013). *The national curriculum for England to be taught in all local-authority-maintained schools.* Retrieved from https://www.gov.uk/government/collections/national-curriculum

Department for Education and Skills. (1997). *Excellence in schools*. London: HMSO.

Desimone, L. M. (2009). Improving impact studies of teachers' professional development: Toward better conceptualizations and measures. *Educational Researcher, 38*(3), 181–199.

Desimone, L. M., & Garet, M. S. (2015). Best practices in teachers' professional development in the United States. *Psychology, Society, and Education, 7*(3), 252–263.

Desimone, L. M., Porter, A. C., Birman, B. F., Garet, M. S., & Yoon, K. S. (2002). How do district management and implementation strategies relate to the quality of the professional development that districts provide to teachers? *Teachers College Record, 104*(7), 1265–1312.

Dewey, J. (1933). *How we think: A restatement of the relation of reflective thinking to the educative process*. Boston: D. C. Heath.

Dewey, J., & Bentley, A. (1949). *Knowing and the known. The later works, 1949–1952* (Vol. 16, pp. 1–280). Boston: Beacon Press.

DfE. (2016a). *Closing the gap test and learn: School based research testing teaching practices using experimental methods, National College of Teaching and Leadership*. https://www.gov.uk/government/publications/closing-thegap-test-and-learn

DfE. (2016b). *Eliminating unnecessary workload around planning and teaching resources*. London: Crown. Retrieved from https://www.gov.uk/government/publications/reducing-teacher-workload-planning-and-resources-group-report

Dienes, Z. (1960). *Building up mathematics*. London: Hutchinson Educational Ltd..

Ding, G., Li, M., Sun, M., Li, Y., Chen, L., Yang, F., et al. (2014). *National survey and policy analysis on the pre-service teacher education of student teachers in normal universities and colleges*. Shanghai: East China University Press (in Chinese).

Doig, B., & Groves, S. (2011). Japanese lesson study: Teacher professional development through communities of inquiry. *Mathematics Teacher Education and Development, 13*(1), 77–93.

Doig, B., Groves, S., & Fujii, T. (2011). The critical role of task development in Lesson study. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 181–199). Dordrecht: Springer.

Dong, L. W. (2015). Developing sense of quantity in mathematics experiment. *China Mathematics Education, 11*, 2–6 (in Chinese).

Dorier, J.-L. (2012). La démarche d'investigation en classe de mathématiques : quel renouveau pour le questionnement didactique ? In B. Calmettes (Ed.), *Démarches d'investigation. Références, représentations, pratiques et formation*. Paris: L'Harmattan.

DosAlmas, A., & Lewis, C. (2017). Bowling with walnuts: What we can learn from Kyouzai Kenkyuu (study of teaching materials). *International Journal for Lesson and Learning Studies, 6*(1), 27–31.

Dossey, J. A., Mullis, I. V. S., Lindquist, M. M., & Chambers, D. L. (1988). *The mathematics report card: Are we measuring up? Trends and achievement based on the 1986 national assessment*. Princeton: Educational Testing Service.

Dudley, P. (2005). *Getting started with research lesson study*. Nottingham: National College for School Leadership.

Dudley, P. (2008). *Improving practice and progression through lesson study: A handbook for headteachers, leading teachers and subject leaders*. London: DCSF.

Dudley, P. (2011). Lesson study development in England: From school networks to national policy. *International Journal for Lesson and Learning Studies, 1*(1), 85–100.

Dudley, P. (2013). Teacher learning in lesson study: What interaction-level discourse analysis revealed about how teachers utilised imagination, tacit knowledge of teaching and fresh evidence of students learning, to develop practice knowledge and so enhance their students' learning. *Teaching and Teacher Education, 34*, 107–121.

Dudley, P. (2014). *Lesson study: Professional learning for our time* (pp. 29–58). London/New York: Routledge.

Dudley, P. (2014a). How lesson study works and why it creates excellent learning and teaching. In P. Dudley (Ed.), *Lesson study* (pp. 1–28). London: Routledge.

Dudley, P. (2014b). *Lesson study: Professional Learning for our time' a resume*. Retrieved from http://lessonstudy.co.uk/2014/09/lesson-study-professional-learning-for-our-time

Dudley, P. (2014c). Lesson study handbook (revised).

Dudley, P. (Ed.). (2015a). *Lesson study: Professional learning for our time*. Oxon: Routledge.

Dudley, P. (2015b). How lesson study works and why it creates excellent learning and teaching. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 1–28). Oxon: Routledge.

Dunn, T. G., & Shriner, C. (1999). Deliberate practice in teaching: What teachers do for self-improvement. *Teaching and Teacher Education, 15*(6), 631–651.

Earl, L., Fullan, M., Leithwood, K., Watson, N., Jantzi, D., Levin, B., & Torrance, N. (2003). *Watching and learning 3: Final report of the external evaluation of England's national literacy and numeracy strategies*. Toronto: Ontario Institute for Studies in Education.

Ebbutt, D., & Elliott, J. (Eds.). (1985). *Issues in teaching for understanding*. London: School Council/Longmans.

Editorial Team. (2016). *Mathematics (Grade4-Vol.2)*. Beijing: Beijing Normal University Press.

Edwards, C. (1993). *The hundred languages of children: The Reggio Emilia approach to early childhood education*. Norwood: Ablex Publishing Corporation.

Edwards, A. (2012). The role of common knowledge in achieving collaboration across practices. *Learning, Culture and Social Interaction, 1*(1), 22–32.

Edwards, D., & Mercer, N. (1987). *Common knowledge: The development of understanding*. London: Methuen.

Elipane, L. (2011). Incorporating lesson study in pre-service mathematics teachers education. *Proceedings of the 35th Conference of the International Group of the Psychology of Mathematics Education, 1*, 305–312.

Elipane, L. E. (2012). *Integrating the essential elements of lesson study in pre-service mathematics teacher education*. IND Skriftserie: Copenhagen University.

Elliot, J. (2012). Developing a science of teaching through lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 108–125. https://doi.org/10.1108/20468251211224163.

Elliott, J. (1976–77). Developing hypotheses about classrooms from teachers' practical constructs. *Interchange*, *7*(2), 2–22. Also published in Elliott, J. (2007). *Reflecting where the action is – The selected works of JOHN ELLIOTT*. London/New York: Routledge, Ch. 2.

Elliott, J. (1991). *Action research for educational change*. Buckingham: Open University Press.

Elliott, J. (2015). Lesson and learning study and the idea of the teacher as a researcher. In K. Woods & S. Sithamparam (Eds.), *Realising learning* (pp. 148–167). London/New York: Routledge.

Elliott, J., & Adelman, C. (1973). Reflecting where the action is: The design of the Ford teaching project. *Education for Teaching, 92*, 8–20.

Elliott, J., & Yu, C. (2008). *Learning studies as an educational change strategy in Hong Kong: An independent evaluation of the 'variation for the improvement of teaching and learning' (VITAL)*. Tai Po: Centre for School Experience and Partnership, Hong Kong Institute of Education.

Elliott, J., & Yu, C. (2013). Learning studies in Hong Kong schools: A summary evaluation. Report on the 'variation for the improvement of teaching and learning' (VITAL) project. *Education & Didactique, 7*(2), 147–163.

Ellis, A. B., Ozgur, Z., Vinsonhaler, R., Dogan, M. F., Carolan, T., Lockwood, E., Lynch, A., Sabouri, P., Knuth, E., & Zaslavsky, O. (in press). Student thinking with examples: The criteria-affordances-purposes-strategies framework. *The Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2017.06.003

Elmore, R. F., & Burney, D. (1999). Investing in teacher learning: Staff development and instructional improvement. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 263–291). San Francisco: Jossey Bass.

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals*. Portsmouth: Heinemann.

Engeström, Y. (1987). *Learning by expanding: An activity-theoretical approach to developmental research*. Orienta-Konsultit Oy: Helsinki.

Engeström, Y. (1996). Developmental work research as educational research. *Nordisk Pedagogik: Journal of Nordic Educational Research, 16*(5), 131–143.

Engeström, Y. (2001). Expansive learning at work: Toward an activity theoretical reconceptualization. *Journal of Education and Work, 14*(1), 133–156.

Engeström, Y. (2005). *Developmental work research: Expanding activity theory in practice*. Berlin: Lehmanns Media.

Engeström, Y. (2007). Enriching the theory of expansive learning: Lessons from journeys toward coconfiguration. *Mind, Culture, and Activity, 14*(1–2), 23–39.

Engeström, Y. (2015). *Learning by expanding: An activity-theoretical approach to developmental research* (2nd ed.). Cambridge: Cambridge University Press.

Engeström, Y. (2016). *Studies in expansive learning: Learning what is not yet there*. Cambridge: Cambridge University Press.

Engeström, Y., & Sannino, A. (2010). A. Studies of expansive learning: Foundations, findings and future challenges. *Educational Research Review, 5*(1), 1–24.

Engle, R. W., Cantor, J., & Carullo, J. J. (1992). Individual differences in working memory and comprehension: A test of four hypotheses. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 18*, 972–992.

Engstrom, Y. (2003). Activity theory and individual and social transformation. In Y. Engstrom, R. Miettinen, & R. L. Punamaki (Eds.), *Perspectives on activity theory*. Cambridge: Cambridge University Press.

Enokizono, K. (1982). Shousuu no kakezan (multiplication of decimal numbers). *Atarashii Sansuu Kenkyuu, 134*, 53–55.

Eraut, M. (1994). *Developing professional knowledge and competence*. London: Falmer Press.

Ericsson, K. A. (2003). Deliberate practice and the acquisition and maintenance of expert performance in medicine and related domains. *Academic Medicine, 79*(10), 70–81.

Ericsson, K. A. (2005). Recent advances in expertise research: A commentary on the contributions to the special issue. *Applied Cognitive Psychology, 19*(2), 233–241.

Ericsson, K. A. (2008). Deliberate practice and acquisition of expert performance: A general overview. *Academic Emergency Medicine, 15*(11), 988–994.

Ericsson, K. A., & Pool, R. (2016). *Peak*. Boston: Houghton Mifflin Harcourt.

Ericsson, K. A., Krampe, R., & Tesch-Romer, C. (1993). The role of deliberate practice in the acquisition of expert performance. *Psychological Review, 100*(3), 363–406.

Esser, J. K. (1998). Alive and well after 25 years: A review of groupthink research. *Organisational Behaviour and Human Decision Processes, 73*(2–3), 116–141.

Even, R. (2014). Challenges associated with the professional development of didacticians. *ZDM-The International Journal on Mathematics Education, 46*, 329–333.

Fan, Q. (2013). *Survey on pedagogical content knowledge among pre-service mathematics teachers from normal universities and colleges* (Unpublished master thesis). Xian: Shaanxi Normal University (in Chinese).

Fang, Y. (2017). School-based teaching research and lesson-case study in mediating the second-cycle curriculum reform in Shanghai. *International Journal for Lesson and Learning Studies, 6* (4), 293–305.

Fang, Y., Lee, C. K. E., & Haron, S. T. S. (2008). Lesson study in mathematics: Three cases for Singapore. In K. Y. Wong, P. Y. Lee, B. Kaur, P. Y. Foong, & S. F. Ng (Eds.), *Mathematics education: The Singapore journey* (pp. 104–129). Singapore: World Scientific.

Fang, Y. P., Lee, K. E. C., & Haron, S. T. (2009). Lesson study in mathematics: Three cases from Singapore. In K. Y. Wong, P. Y. Lee, B. Kaur, P. Y. Foong, & S. F. Ng (Eds.), *Mathematics education – the Singapore journey* (pp. 104–129). Singapore: World Scientific.

Fang, Y. P., Lee, K. E. C., & Yang, Y. (2011). Developing video cases from research lessons as curriculum and pedagogical support for teacher learning – A case of long division. *International Journal of Lesson and Learning Studies, 1*(1), 65–84.

Farrell, T. S. C. (2002). Lesson planning. In J. C. Richards & W. A. Renandya (Eds.), *Methodology in language teaching: An anthology of current practice* (pp. 30–39). New York: Cambridge University Press.

Fei, L. F. (2014). Sense of quantity as an important content of teaching quantitative unit: A case of knowing millimeter. *Elementary Teaching Design, 8*, 4–5 (in Chinese).

Feiman Nemser, S. (1983). Learning to teach. In L. Shulman & G. Sykes (Eds.), *Handbook of teaching and policy* (pp. 150–170). New York: Longman.

Femandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. London: Routledge.

Fendel, D. M. (1987). *Understanding the structure of elementary school mathematics*. Newton: Allyn and Bacon.

Fendel, D., Resek, D., Alper, L., & Fraser, S. (1997). *Interactive mathematics program: Integrated high school mathematics: Year 1*. Berkeley: Key Curriculum Press.

Fereday, J., & Muir-Cochrane, E. (2006). Demonstrating rigor using thematic analysis: A hybrid approach of inductive and deductive coding and theme development. *International Journal of Qualitative Methods, 5*(1), 80–92.

Ferguson, P. (1985, September–October). Teacher education in Japan: An historical and comparative perspective. *Journal of Teacher Education*, 211–224.

Fernandez, C. (2002a). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 16*(1), 49–65.

Fernandez, C. (2002b). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 53*(5), 393–405.

Fernandez, C. (2002c). Learning from Japanese approaches to professional development. *Journal of Teacher Education, 53*, 393–405.

Fernandez, C. (2005). Lesson study: A means for elementary teachers to develop the knowledge of mathematics needed for reform-minded teaching? *Mathematical Thinking and Learning, 7*(4), 265–289.

Fernández, M. L. (2005). *Exploring "lesson study" in teacher preparation.* Paper presented at the Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education.

Fernández, S., & Figueiras, L. (2014). Horizon content knowledge: Shaping MKT for a continuous mathematical education. *REDIMAT, 3*(1), 7–29.

Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Mahwah/New York: Lawrence Erlbaum Associates, Inc./Routledge.

Fernandez, C., & Yoshida, M. (2012). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. New York: Routledge.

Fernandez, M. L., & Zilliox, J. (2011). Investigating approaches to lesson study in prospective mathematics teacher education. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 85–102). Dordrecht: Springer.

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A U.S.–Japan lesson study collaborative reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*(2), 171–185.

Fink, D. (2000). *Good schools/real schools: Why school reform doesn't last*. New York: Teachers College Press.

Firestone, W. A., Mangin, M. M., Martinez, M. C., & Polovsky, T. (2005). Leading coherent professional development: A comparison of three districts. *Educational Administration Quarterly, 41*(3), 413–448.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3–17. https://doi.org/10.2307/748969.

FLDOE. (2010a). *Florida's Race to the Top application for initial funding (CFDA 84.395A)*. Tallahassee: Florida Department of Education (FLDOE).

FLDOE. (2010b). *Final scope of work*. Florida Department of Education (FLDOE): Tallahassee.

FLDOE. (n.d.-a). *Race to the top grant archive*, from http://www.fldoe.org/arra/RacetotheTop-archive.asp

FLDOE. (n.d.-b). *Race to the Top LEA final scope of work template definitions*, from http://www.fldoe.org/arra/RacetotheTop-archive.asp

Floden, R. E., Porter, A. C., Alford, L. E., Freeman, D. J., Susan, I., Schmidt, W. H., & Schwille, J. R. (1988). Instructional leadership at the district level: A closer look at autonomy and control. *Educational Administration Quarterly, 24*(2), 96–124.

Fong, H., Ramakrishnan, C., & Choo, M. (2003). *My pals are here!* Singapore: Federal Publications.

Forsythe, S., & Baldry, F. (2017). *Developing effective Learning Circles: An account of how student teachers participated in a shared dialogue of a team teaching experience*. Paper presented at the European Conference on Educational Research 2017, Copenhagen.

Fosnot, C. T., & Dolk, M. (2002). *Young mathematicians at work: Constructing fractions, decimals, and percents*. Portsmouth: Heinemann.

Foster, C., Wake, G., & Swan, M. (2014). Mathematical knowledge for teaching problem solving: Lessons from lesson study. In S. Oesterle, C. Nicol, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PMENA 36* (Vol. 3, pp. 97–104). Vancouver: PME.

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (2002). Chapter 16: Advancing children's mathematical thinking. In J. Sowder & B. Schappelle (Eds.), *Lessons learned from research* (pp. 135–142). Reston: The National Council of Teachers of Mathematics.

Franke, M., Kazemi, E., Shih, J., Biagetti, S., & Battey, D. (2005). Changing teachers' professional work in mathematics: One school's journey. In T. A. Romberg, T. P. Carpenter, & F. Dremock (Eds.), *Understanding mathematics and science matters* (pp. 209–229). Mahwah: Erlbaum.

Frobisher, L., Monaghan, J., Orton, A., Orton, J., Roper, T., & Threlfall, J. (1999). *Learning to teach number*. Cheltenham: Stanley Thornes.

Fujii, T. (2008, July 6–13). *Knowledge for teaching mathematics*. Plenary Talk at the 11th International Congress on Mathematical Education, Monterrey, Mexico.

Fujii, T. (2010). *Designing tasks in the Japanese Lesson Study: Focusing on the role of the quasi-variable*. In Proceedings of the 5th East Asia Regional conference on Mathematics Education EARCOME5, Tokyo, Japan (pp. 86–93).

Fujii, T. (2014a). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 2–18.

Fujii, H. (2014b). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 65–83.

Fujii, T. (2014c). Theorizing Lesson Study in mathematics education as emerging research area: Identifying components and its structure of Lesson Study. In Japan Society of Science Education (JSSE) (Ed.), *Proceedings of second annual spring conference of Japan Society of Mathematical Education* (Vol. X, pp. 111–118). (in Japanese).

Fujii, T. (2014d). Implementing Japanese lesson study in foreign countries: Misconceptions revealed. *Mathematics Teacher Education and Development, 16*(1), 4–21.

Fujii, T. (2014e). *Theorizing Lesson Study in mathematics education as an emerging research area (2): Identifying components and its structure of lesson study (in Japanese)*. In Proceedings of second annual Spring conference of Japan Society of Mathematical Education (pp. 111–118).

Fujii, T. (2015a). *Mathematics 4B, Tokyo Shoseki* (in Japanese).

Fujii, T. (2015b). The critical role of task design in lesson study. In A. Watson & M. Ohtani (Eds.), *ICMI Study 22: Task design in mathematics education* (pp. 273–286). New York: Springer.

Fujii, T. (2016). Designing and adapting tasks in lesson planning: A critical process of lesson study. *ZDM Mathematics Education, 48*(4), 411–423.

Fujii, T. (2017). Unifying lesson study with teaching mathematics through problem solving. In International math-teacher professionalization using lesson study. In *Essential mathematics for the next generation* (pp. 85–103). Tokyo: Tokyo Gakugei University Press.

Fujii, T. (2018). Lesson study and teaching mathematics trough problem solving: The two wheels of a cart. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Da Ponte, A. Ni Shuilleabhain, A. Takahashi, & T. Fujii (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Fujii, T., & Iitaka, S. (2012). *Mathematics international (grade 1-grade 9)*. Retrieved from www.globaledresources.com

Fujii, T., & Stephens, M. (2001). *Fostering an understanding of algebraic generalization through numerical expressions: The role of quasi-variables*. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra* (pp. 258–264). Melbourne: University of Melbourne.

Fujii, T., & Stephens, M. (2008). Using number sentences to introduce the idea of variable. In C. Green & R. Rubenstein (Eds.), *Algebra and algebraic thinking in school mathematics-seventeenth yearbook* (pp. 127–140). Reston: National Council of Teachers of Mathematics.

Fujii, T., Kumagai, K., Shimizu, S., & Sugiyama, S. (1998). A cross-cultural study of classroom practices based on a common topic. *Tsukuba Journal of Educational Study in Mathematics, 17*, 185–194.

Gamoran, A. (2003). Access to resources. In A. Gamoran, W. Anderson, P. A. Quiroz, W. G. Secada, T. Williams, & S. Ashmann (Eds.), *Transforming teaching in math and science: How schools and districts can support change* (pp. 65–86). New York: Teachers College Press.

Gamoran, A., Anderson, W., Quiroz, P. A., Secada, W. G., Williams, T., & Ashmann, S. (Eds.). (2003). *Transforming teaching in math and science: How schools and districts can support change*. New York: Teachers College Press.

Ganesh, B., & Matteson, S. M. (2010). The benefits of reteaching lessons in preservice methods classes. *Action in Teacher Education, 32*(4), 52–60.

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What makes professional development effective? Results from a national sample of teachers. *American Educational Research Journal, 38*(4), 915–945.

Gersten, R., Taylor, M. J., Keys, T. D., Rolfhus, E., & Newman-Gonchar, R. (2014). *Summary of research on the effectiveness of math professional development approaches. (REL 2014–010).* Washington, DC: U.S. Department of Education, Institute of Education Sciences, National Center for Education Evaluation and Regional Assistance, Regional Educational Laboratory Southeast.

Giles, C., & Hargreaves, A. (2006). The sustainability of innovative schools as learning organizations and professional learning communities during standardized reform. *Educational Administration Quarterly, 42*(1), 124–156.

Glaser, B. J. (1965). The constant comparative method of qualitative analysis. *Social Problems, 12*(4), 436–445.

Glaser, B. G., & Strauss, A. L. (2017). *Discovery of grounded theory: Strategies for qualitative research*. New York: Routledge.

Gog, T. V., Ericsson, K. A., Pikers, R. M. J. P., & Paas, F. (2005). Instructional design for advanced learners: Establishing connections between the theoretical frameworks of cognitive load and deliberate practice. *Educational Technology Research & Development, 53*(3), 73–81.

Goh, R., & Fang, Y. (2017). Improving English language teaching through lesson study: Case study of teacher learning in a Singapore primary school grade level team. *International Journal for Lesson and Learning Studies, 6*(2), 135–150.

Goh, S. C., Tan, K. A., & Lim, C. S. (2007). *Engaging in lesson study: Our experience*. In C. S. Lim et al. (Eds.), *Proceedings of the 4th East Asia Regional Conference on Mathematics Education [EARCOME4]* (pp. 574–579). Penang: Universiti Sains Malaysia.

Goldenberg, P., & Mason, J. (2008). Spreading light on and with example spaces. *Educational Studies in Mathematics, 69*(2), 183–194.

Goldsmith, L. T., Doerr, H. M., & Lewis, C. C. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education, 17*(1), 5–36.

Graeber, A. O., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 20*(1), 95–102.

Graff, N. (2011). "An effective and agonizing way to learn": Backwards design and new teachers' preparation for planning curriculum. *Teacher Education Quarterly, 38*(3), 151–168. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23479622.

Grevholm, B. (2006). Matematikdidaktikens möjligheter i en forskningsbaserat lärarutbildning. In S. Ongstad (Ed.), *Fag og didaktik i lærerutdanning. Kunnskap i grenseland* (pp. 183–206). Oslo: Universitetsforlaget.

Griffiths, J. (2016). Bridging the school placement gap with peer micro-teaching lesson study. *International Journal for Lesson and Learning Studies, 5*(3), 227–238.

Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184–205.

Grossman, P., & Thompson, C. (2004). *Curriculum materials: Scaffolds for new teacher learning*. Stanford, CA: Center for the Study of Teaching and Policy. Retrieved from http://depts.washington.edu/ctpmail/Reports.html

Grossman, P. L., Wineburg, S. S., & Woolworth, S. (2001). Toward a theory of teacher community. *Teachers College Record, 103*(6), 942–1012.

Grossman, P., Compton, C., Shahan, E., Ronfeldt, M., Igra, D., & Shaing, J. (2007). Preparing practitioners to respond to resistance: A cross-professional view. *Teachers and Teaching: Theory and Practice, 13*(2), 109–123.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009a). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*(9), 2055–2100.

Grossman, P., Hammerness, K., & McDonald, M. (2009b). Redefining teaching, re-imagining teacher education. *Teachers and Teaching: Theory and Practice, 15*(2), 273–289.

Groves, S., Doig, B., Vale, C., & Widjaja, W. (2016). Critical factors in the adaptation and implementation of Japanese lesson study in the Australian context. *ZDM: The International Journal on Mathematics Education, 48*(4), 501–512.

Gu, M. (2003). Traditions and evolutions of teacher education. *Teacher Education Research, 15*(3), 1–6 (in Chinese).

Gu, M. (2014). *Cultural foundations of Chinese education*. Leiden: Koninklijke Brill NV.

Gu, F., & Gu, L. (2016). Characterizing mathematics teaching research specialists' mentoring in the context of Chinese lesson study. *ZDM Mathematics Education, 48*(4), 441–454. https://doi.org/10.1007/s11858-016-0756-1.

Gu, F., & Gu, L. (in press). Further studies on teaching research specialists. *Global Education*.

Gu, L. Y., & Wang, J. (2003a). Teachers' growth in educational action: A study on the model of teacher education based on curriculum. *Global Education, 185*(1), 44–49 (in Chinese).

Gu, L., & Wang, J. (2003b). Teacher development in education action. *Curriculum Textbook and Pedagogy, 1*, 9–26.

Gu, L., & Wang, J. (2003c). Teachers' growths in educational actions: A study on the model of teacher education based on curriculum (Part 1). *Curriculum, Textbook, Pedagogy, 1*, 9–15 (in Chinese).

Gu, L., & Wang, J. (2003d). Teachers' growths in educational actions: A study on the model of teacher education based on curriculum (Part 2). *Curriculum, Textbook, Pedagogy, 2*, 14–19 (in Chinese).

Gu, L., & Yang, Y. (2003). School-based research on teachers' professional growths. *Research on Educational Development, 6*, 1–7 (in Chinese).

Gu, L., & Zhu, L. (2012). Preliminary studies on teaching research specialists. *Global Education, 8*, 31–37.

Gu, L., Huang, R., & Marton, F. (2004a). Teaching with variation: An effective way of mathematics teaching in China. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–348). Singapore: World Scientific.

Gu, L., Huang, R., & Marton, F. (2004b). Teaching with variation: A Chinese way of promoting effective mathematics learning. In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 309–347). Hackensack: World Scientific.

Gu, L., Huang, R., & Marton, F. (2004c). In L. Fan, N. Y. Wong, J. Cai, & S. Li (Eds.),. How Chinese learn mathematics: Perspectives from insiders *Teaching with variation: An effective way of mathematics teaching in China* (pp. 309–348). Singapore: World Scientific.

Gu, F., Huang, R., & Gu, L. (2017). Theory and development of teaching through variation in mathematics in China. In R. Huang & Y. Li (Eds.), *Teaching and learning through variations* (pp. 13–42). Rotterdam: Sense.

Gunnarsdóttir, G. H., & Pálsdóttir, G. (2010). Mathematics teacher education at Iceland University of Education. In B. Sriraman, C. Bergsten, S. Goodchild, G. Pálsdóttir, B. D. Søndergaard, & L. Haapasalo (Eds.), *The first sourcebook on Nordic research in mathematics education* (pp. 467–477). Charlotte: Information Age Publishing.

Gunnarsdóttir, G. H., & Pálsdóttir, G. (2011a). *Lesson study in teacher education: A tool to establish a learning community*. In M. Pytlak, E. Swoboda, & T. Rowland (Eds.), *Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 2660–2669). Rzeszów: University of Rzeszów.

Gunnarsdóttir, G. H., & Pálsdóttir, G. (2011b). *Lesson study in teacher education: A tool to establish a learning community.* Paper presented at the Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education.
Gunnarsdóttir, G. H., & Pálsdóttir, G. (2013). *New teachers´ ideas on professional development*. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *CERME8 Proceedings of the Seventh Congress of the European Society for Research in Mathematics Education* (pp. 3085–3094). Ankara: Middle East Technical University.
Gunnarsdóttir, G. H., & Pálsdóttir G. (2016). Lesson study in mathematics teacher education. In T. E. Rangnes & H. Alrö (Eds.), *Matematiklæring for framtida. Festskrift til Marit Johnsen-Höines* (bls. 61–85). Bergen: Caspar Forlag AS.
Guskey, T. R. (1986). Staff development and the process of teacher change. *Educational Researcher, 15*(5), 5–12.
Guskey, T. R., & Yoon, K. S. (2009). What works in professional development. *Phi Delta Kappan, 90*(7), 495–500.
Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *The Journal of Mathematical Behavior, 28*(1), 1–18.
Hadfield, M., Jopling, M., & Emira, M. (2011). *Evaluation of the national strategies' primary leading teachers programme*. London: Department for Education.
Hairon, S., & Dimmock, C. (2012). Singapore schools and professional learning communities: Teacher professional development and school leadership in an Asian hierarchical system. *Educational Review, 64*(4), 405–424.
Hammerness, K., Darling-Hammond, L., & Bransford, J. (2005). How teachers learn and develop. In L. Darling-Hammond & J. Bransford (Eds.), *Preparing teachers for a changing world: What teachers should learn and be able to do* (pp. 358–389). San Francisco: Jossey Bass.
Han, X., & Paine, L. (2010). Teaching mathematics as deliberate practice through public lessons. *The Elementary School Journal, 110*(4), 519–541.
Han, X., Gong, Z., & Huang, R. (2017a). Teaching mathematical concepts through variation and learning progression: A case study of division of fractions. In R. Huang & Y. Li (Eds.), *Teaching and learning through variations* (pp. 267–293). Rotterdam: Sense.
Han, X., Gong, Z., & Huang, R. (2017b). Teaching and learning mathematics through variation in lesson study. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 267–293). Boston: Sense Publishers.
Hargreaves, D. (1999). The knowledge creating school. *British Journal of Educational Studies, 47* (2), 122–144.
Hargreaves, A., & Fink, D. (2003). Sustaining leadership. *The Phi Delta Kappan, 84*(9), 693–700.
Hargreaves, A., & Fink, D. (2009). What works in professional development. *Phi Delta Kappan, 90* (7), 495–500.
Hart, P. (1994). *Government: A study of small groups and policy failure*. Baltimore: The Johns Hopkins University Press.
Hart, L. C., Alston, A., & Murata, A. (Eds.). (2011). *Lesson study research and practice in mathematics education: Learning together*. Dordrecht/New York: Springer.
Hashimoto, Y. (1987). *Classroom practice of problem solving in Japanese elementary school*. In J. P. Becker & T. Miwa (Eds.), *Proceedings of the U.S.-Japan seminar of mathematical problem solving* (pp. 94–101). Carbondale: Board of Trustees of Southern Illinois University.
Hatton, N., & Smith, D. (1995). Reflection in teacher education: Towards definition and implementation. *Teaching and Teacher Education, 11*(1), 33–49.
He, X. B. (2013). The components, and characteristics and development of core capacity of teaching research specialists. *Journal for Introduction to Education, 10*(1), 38–40 (in Chinese).
Hersant, M., & Perrin-Glorian, M. (2005). Characterization of an ordinary teaching practice with the help of the theory of didactical situations. *Educational Studies in Mathematics, 59*, 113–151.

Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An "experiment" model for teaching and teacher preparations in mathematics. *Journal of Mathematics Teacher Education, 6*(3), 201–222.

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B., Garnier, H., Smith, M., et al. (2005). Mathematics teaching in the United States today (and tomorrow): Results from the TIMSS 1999 Video Study. *Educational Evaluation and Policy Analysis, 27*(2), 111–132.

Hightower, A. M., Knapp, M. S., Marsh, J. A., & McLaughlin, M. W. (2002). *School districts and instructional renewal*. New York/London: Teachers College Press.

Hill, H. C. (2010). The nature and predictors of elementary teachers' mathematical knowledge for teaching. *Journal for Research in Mathematics Education, 41*, 513–545.

Hill, H. C. (2011). The nature and effects of middle school mathematics teacher learning experiences. *Teachers College Record, 113*(1), 205–234.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*, 371–406.

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education, 39*, 372–400.

Hino, K. (2007). Toward the problem-centered classroom: Trends in mathematical problem solving in Japan. *ZDM, The International Journal of Mathematics Education, 39*, 503–514.

Hironaka, H., & Sugiyama, Y. (Eds.) (2006). *Mathematics 4A for elementary school.* Translated into English by Yoshida et al. Tokyo Shoseki, Co., Ltd.

Hodgen, J. (2011). Knowledge and identity: A situated theory of mathematics knowlege in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching*. London: Springer.

Hogan, D., & Gopinathan, S. (2008). Knowledge management, sustainable innovation, and pre-service teacher education in Singapore. *Teachers and Teaching, 14*(4), 369–384.

Hogan, D., Chan, M., Rahim, R., Towndrow, P., & Kwek, D. (2012). Understanding classroom talk in secondary 3 mathematics classes in Singapore. In B. Kaur (Ed.), *Connections, reasoning and communication: New directions in mathematics education*. Singapore: Springer.

Hord, S. M., & Sommers, W. A. (Eds.). (2008). *Leading professional learning communities: Voices from research and practice*. Thousand Oaks: Corwin Press.

Horn, I. S. (2005). Learning on the job: A situated account of teacher learning in high school mathematics departments. *Cognition and Instruction, 23*(2), 207–236.

Horn, I. S., & Kane, B. D. (2015). Opportunities for professional learning in mathematics teacher workgroup conversations: Relationships to instructional expertise. *Journal of the Learning Sciences, 24*(3), 373–418, 1–46.

Horn, I. S., & Little, J. W. (2010). Attending to problems of practice: Routines and resources for professional learning in teachers' workplace interactions. *American Educational Research Journal, 47*(1), 181–217.

Horn, I. S., Garner, B., Kane, B. D., & Brasel, J. (2017). A taxonomy of instructional learning opportunities in teachers' workgroup conversations. *Journal of Teacher Education, 68*(1), 41–54.

Hou, D. (2016). An empirical study of improving students' teaching skill under the background of colleges and universities transformation. *Journal of Northwest Normal University (Social Sciences), 53*(6), 109–114.

Hsieh, F., Lu, S., Hsieh, C., Tang, S., & Wang, T. (2018). The conception of mathematics teachers' literacy for teaching from a historical perspective. In Y. Li & R. Huang (Eds.), *How Chinese acquire and improve mathematics knowledge for teaching* (pp. 37–56). Leiden: Koninklijke Brill NV.

Hu, S. (1932). *Guidelines for teaching practice in New China*. Beijing: Zhonghua Publishing House (in Chinese).

Hu, G. (2005). Professional development of secondary EFL teachers: Lesson from China. *Teachers College Record, 107*(4), 654–705.

Huang, Y. (2016). The missing and improvement of pre-service teachers' practical knowledge. *Teacher Education Research, 28*(5), 85–90 (in Chinese).

Huang, J. H. (2017a). *Shanghai elementary mathematics textbook, grade 4(1)*. Shanghai: Shanghai Education Publisher (in Chinese).

Huang, J. H. (2017b). *Teaching reference materials, grade 4(1)*. Shanghai: Shanghai Education Publisher (in Chinese).

Huang, R., & Bao, J. (2006). Towards a model for teacher's professional development in China: Introducing keli. *Journal of Mathematics Teacher Education, 9*, 279–298. Singapore: World Scientific.

Huang, R., & Cai, J. (2011). Pedagogical representations to teach linear relations in Chinese and U.S. classrooms. *Parallel or hierarchical? The Journal of Mathematical Behavior, 30*(2), 149–165.

Huang, R., & Han, X. (2015). Developing mathematics teachers' competence through parallel lesson study. *International Journal for Lesson and Learning Studies, 4*(2), 100–117.

Huang, R., & Li, Y. (2009). Pursuing excellence in mathematics classroom instruction through exemplary lesson development in China: A case study. *ZDM Mathematics Education, 41*, 297–309.

Huang, R., & Li, Y. (Eds.). (2017). *Teaching and learning mathematics through variation: Confucian heritage meets western theories*. Boston: Sense Publishers.

Huang, R., & Shimizu, Y. (2016). Improving teaching, developing teachers and teacher educators, and linking theory and practice through lesson study in mathematics: An international perspective. *ZDM Mathematics Education, 48*(4), 393–409, 439–587. https://doi.org/10.1007/s11858-016-0795-7

Huang, R., Peng, S., Wang, L., & Li, Y. (2010). Secondary mathematics teacher professional development in China. In F. K. S. Leung & Y. Li (Eds.), *Reforms and issues in school mathematics in East Asia* (pp. 129–152). Rotterdam: Sense Publishers.

Huang, R., Li, Y., Zhang, J., & Li, X. (2011). Improving teachers' expertise in mathematics instruction through exemplary lesson development. *ZDM Mathematics Education, 43*, 805–817.

Huang, R., Xu, S., Su, H., Tang, B., & Strayer, J. (2012, July). Teaching researchers in China: Hybrid functions of researching, supervising and consulting. Paper presented at *12th International Conference on Mathematics Education*, July 8–15, 2012 Seoul, Korea.

Huang, X., Sun, L., & Wu, S. (2013). To enhance pre-service teachers' professional accomplishment in education of science. *Teacher Education Research, 25*(5), 56–61.

Huang, R., Su, H., & Xu, S. (2014). Developing teachers' and teaching researchers' professional competence in mathematics through Chinese lesson study. *ZDM Mathematics Education, 46*(4), 239–251.

Huang, R., Gong, Z., & Han, X. (2016a). Implementing mathematics teaching that promotes students' understanding through theory-driven lesson study. *ZDM Mathematics Education, 48*, 425–439.

Huang, R., Ye, L., & Prince, K. (2016b). Professional development system and practices of mathematics teachers in Mainland China. In B. Kaur & K. O. Nam (Eds.), *Professional development of mathematics teachers: An Asian perspective* (pp. 17–32). New York: Springer.

Huang, R., Fang, Y., & Chen, X. (Eds.). (2017a). Theory and practice of Chinese lesson study and its adaption in other countries [Special Issue]. *International Journal for Lesson and Learning Studies, 6*(4).

Huang, R., Fang, Y., & Chen, X. (2017b). Chinese lesson study: A deliberate practice, a research methodology, and an improvement science. *International Journal for Lesson and Learning Studies, 6*(4), 270–282. https://doi.org/10.1108/IJLLS-08-2017-0037.

Huang, R., Haupt, M., & Barlow, A. (2017c). Developing high-leverage practices as deliberate practice through lesson study. *International Journal for Lesson and Learning Studies, 6*(4), 365–379.

Huang, R., Barlow, A. T., & Haupt, M. E. (2017d). Improving core instructional practice in mathematics teaching through lesson study. *International Journal for Lesson and Learning Studies, 6*(4), 365–379.

Huang, R., Zhang, J., Mok, I., Zhou, Y., Wu, Z., & Zhao, W. (2017e). The competence of teaching research specialists and their development in China. *International Journal for Lesson and Learning Studies, Special issue, 6*(4), 321–335.

Hucker, J. (1994). *Creating paths to mathematical literacy: A.S.B./A.P.P.A.travelling fellowship 1994 report*. Auckland: Author.

Hunting, R. (1984). Understanding equivalent fractions. *Journal of Science and Mathematics Educations in S E Asia, 7*(1), 26–33.

Hutchins, E. (1996). Learning to navigate. In S. Chaiklen & J. Lave (Eds.), *Understanding practice: Perspectives on activity and context* (pp. 35–63). Cambridge, UK: Cambridge University Press.

Iksan, Z. H., & Rahim, M. B. (2017). Reflection on teaching and learning of mathematics through lesson study and video critique. *Advances in Social Sciences Research Journal, 4*(1), 50–63.

Il'enkov, E. V. (1977). *Dialectical logic: Essays in its history and theory*. Moscow: Progress.

Inagaki, T. (1995). *Meiji kyouju rironshi kenkyu* [A historical research on teaching theory in meiji-era]. Tokyo: Hyuuron-Sya. [in Japanese].

Inprasitha, M., Isoda, M., Wang-Iverson, P., & Yeap, B. H. (2015). *Lesson study: Challenges in mathematics education*. Singapore: World Scientific.

Isoda, M. (2015). The science of lesson study in the problem solving approach. In M. Imprashita, M. Isoda, P. Wang-Iverson, & B. Har Yeap (Eds.), *Lesson study: Challenges in mathematics education* (pp. 81–108). Singapore: World Scientific.

Isoda, M., Stephens, M., Ohara, Y., & Miyakawa, T. (Eds.). (2007). *Japanese lesson study in mathematics: Its impact, diversity and potential for educational improvement*. Singapore: World Scientific Publishing.

Jacob, R., Hill, H., & Corey, D. (2017). The impact of a professional development program on teachers' mathematical knowledge for teaching, instruction, and student achievement. *Journal of Research on Educational Effectiveness, 10*(2), 379–407.

Jacobs, C., Martin, S. N., & Otieno, T. C. (2008). A science lesson plan analysis instrument for formative and summative program evaluation of a teacher education program. *Science Education, 92*, 1096–1126.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education, 41*, 169–202.

Jakobsen, A., Thames, M. H., Ribeiro, C. M., & Delaney, S. (2012). *Using practice to define and distinguish horizon content knowledge*. In Authors (Eds.), Pre-proceedings of 12th International Congress of Mathematics Education (pp. 4635–4644). Seoul: ICMI 12.

James, M., McCormick, R., Black, P., Carmichael, P., Drummond, M. J., Fox, A., MacBeath, J., Marshall, B., Pedder, D., Proctor, R., Swaffield, S., Swann, J., & Wiliam, D. (2007). *Improving learning how to learn: Classrooms, schools and networks, TLRP improving learning series*. London: Routledge.

Janis, I. (1982). *Groupthink: Psychological studies of policy decisions and fiascos* (2nd ed.). New York: Houghton Mifflin.

Jansen, A., & Spitzer, S. M. (2009). Prospective middle school mathematics teachers' reflective thinking skills: Descriptions of their students' thinking and interpretations of their teaching. *Journal of Mathematics Teacher Education, 12*(2), 133–151.

Japan Society of Mathematical Education. (2001). Sansu jugyou no houhou ni kansuru chousa no kekka (Results of the survey on mathematics teaching approaches). *Arithmetic Education, 83* (2), 31–42.

Jaworski, B. (2007). Introducing LCM – learning communities in mathematics. In B. Jaworski, A. B. Fuglestad, R. Bjuland, T. Breiteig, S. Goodchild, & B. Grevholm (Eds.), *Læringsfællesskap i matematikk: Learning communities in mathematics* (pp. 13–25). Bergen: Caspar Forlag AS.

Jaworski, B. (2008a). Development of the mathematics teacher educator and its relation to teaching development. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 335–361). Rotterdam: Sense.

Jaworski, B. (2008b). Building and sustaining inquiry communities in mathematics teaching development: Teachers and didacticians in collaboration. In K. Krainer & T. Wood (Eds.), *International handbook of mathematics teacher education: Participants in mathematics teacher education: Individuals, teams, communities and networks* (Vol. 3, pp. 309–330). Rotterdam: Sense.

Jaworski, B., & Huang, R. (2014). Teachers and didacticians: Key stakeholders in the processes of developing mathematics teaching. *ZDM -The International Journal on Mathematics Education, 46*, 173–188.

Jay, J. K., & Johnson, K. L. (2002). Capturing complexity: A typology of reflective practice for teacher education. *Teaching and Teacher Education, 18*, 73–85.

Ji, M. (2016). How to realize inheritance and innovation in Shanghai teaching and research. *People Education, 20*, 20–23.

Jiang, X. (2008). The evolution of the "Backbone Teacher" training system since 1949 and its implications. *Contemporary Educational Science, 14*, 10–12 (in Chinese).

Jiang, Y. G., & Liu, G. B. (2017). How to achieve the teaching goals effectively: A case study of elementary mathematics teaching. *Education and Teaching Research, 31*(6), 85–90 (in Chinese).

Jigyel, K., & Afamasaga-Fuata'I, K. (2007). Students' conceptions of models of fractions and equivalence. *Australian Mathematics Teacher, 63*(4), 17–25.

Jita, L. C., Maree, J. G., & Ndlalane, T. C. (2008). Lesson study (Jyugyo Kenkyu) from Japan to South Africa: A science and mathematics intervention program for secondary school teachers. In B. Atweh et al. (Eds.), *Internationalisation and globalisation in mathematics and science education* (pp. 465–486). Dordrecht: Springer.

Ju, X., Tong, F., & Zhang, S. (1994). *A compilation of materials for the history of modern education in China*. Shanghai: Shanghai Education Press (in Chinese).

JUMP Math. (2018). *JUMP Math: Multiplying potential*. https://jumpmath.org/jump/en/

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *The Journal of Mathematical Behavior, 14*, 365–378.

Karagöz-Akar, G. (2016). Prospective secondary mathematics teachers' perspectives and mathematical knowledge for teaching. *EURASIA Journal of Mathematics, Science & Technology Education, 12*(1), 3–24.

Karp, A. (2010). Analyzing and attempting to overcome prospective teachers' difficulties during problem-solving instruction. *Journal of Teacher Education, 13*, 121–139.

Kauffman, D., Johnson, S. M., Kardos, S. M., Liu, E., & Peske, H. G. (2002). Lost at sea: New teachers' experiences with curriculum and assessment. *Teachers College Record, 104*(2), 273–230.

Kazemi, E., & Franke, M. L. (2004). Teacher learning in mathematics: Using student work to promote collective inquiry. *Journal of Mathematics Teacher Education, 7*(3), 203–235.

Kazemi, E., & Hubbard, A. (2008). New directions for the design of professional development: Attending to the coevolution of teachers' participation across contexts. *Journal of Teacher Education, 59*(5), 428–441.

Kervin, K. (2007). Exploring the use of slow motion animation (Slowmation) as a teaching strategy to develop year 4 students' understandings of equivalent Fractions. *Contemporary Issues in Technology and Teacher Education, 7*(2), 100–106.

Kieran, C., Krainer, K., & Shaughnessy, J. M. (2013). Linking research to practice: Teachers as key stakeholders in mathematics education research. In M. A. Clements, A. J. Bishop, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Third international handbook of mathematics education* (pp. 361–392). New York: Springer.

Kikuchi, H. (1982). Koment: Bunsuu no warizan (comment: Division of fractions). *Atarashii Sansuu Kenkyuu, 137*, 57.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Kilpatrick, J., Martin, W. G., & Schifter, D. (Eds.). (2003). *A research companion to principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

Kilpatrick, J., Mesa, V., & Sloane, F. (2006, November). *US algebra teaching and learning viewed internationally*. In *In: Paper presented at the 2nd IEA international research conference*. Brookings: Institution, Washington, DC.

Knapp, M. S. (2003). Professional development as a policy pathway. *Review of Research in Education, 27*(109), 109–157.

Koh, K. H. (2011). Improving teachers' assessment literacy through professional development. *Teaching Education, 22*(3), 255–276.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM Mathematics Education, 40*, 873–892.

Krystal, B. (2018). Developing teachers' mathematical-task knowledge and practice through lesson study. *International Journal for Lesson and Learning Studies, 7*(2), 136–149.

Kullberg, A. (2012). Can findings from learning studies be shared by others? *International Journal for Lesson and Learning Studies, 1*(3), 232–244.

Kullberg, A., Runesson, U., & Mårtensson, P. (2014). Different possibilities to learn from the same task. In C. Nicol, S. Oesterle, P. Liljedahl, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 8(4), pp. 139–150). Vancouver: PME.

Kullberg, A., Runesson, K., & Marton, F. (2017). What is made possible to learn when using the variation theory of learning in teaching mathematics? *ZDM Mathematics Education, 49*, 559–569. https://doi.org/10.1007/s11858-017-0858-4.

Kuno, H. (2015). Evolving the curriculum through lesson study. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 104–117). London: Routledge.

Kunter, M., Klusmann, U., Baumert, J., Richter, D., Voss, T., & Hachfeld, A. (2013). Professional competence of teachers: Effects on instructional quality and student development. *Journal of Educational Psychology, 105*(3), 805–820.

Lakatos, I. (1970). Falsification and the methodology of scientific research programmes. In I. Lakatos & A. Musgrave (Eds.), *Criticism and the growth of knowledge* (pp. 91–195). Cambridge: Cambridge University Press.

Lamb, P. (2015). Peer-learning between pre-service teachers: embracing Lesson Study. *International Journal for Lesson and Learning Studies, 4*(4), 343–361.

Lamon, S. J. (2005). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. Mahwah: Lawrence Erlbaum.

Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review, 55*(2), 178–194.

Lampert, M., & Clark, C. M. (1990). Expert knowledge and expert thinking in teaching: A response to Floden and Klinzig. *Educational Researcher, 19*(5), 21–21.

Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning in and for practice. *Elementary School Journal, 109*(5), 491–509.

Lampert, M., Franke, M. L., Kazemi, E., Ghousseini, H., Turrou, A. C., Beasley, H., et al. (2013). Keeping it complex: Using rehearsals to support novice teacher learning of ambitious teaching. *Journal of Teacher Education, 64*(3), 226–243.

Langley, G. J., Moen, R. D., Nolan, K. M., Nolan, T. W., Norman, C. L., & Provost, L. P. (2009). *The improvement guide*. San Francisco: Jossey-Bass.

Lannin, J. K., Webb, M., Chval, K., Arbaugh, F., Hicks, S., Taylor, C., & Bruton, R. (2013). The development of beginning mathematics teacher pedagogical content knowledge. *Journal of Mathematics Teacher Education, 16*(6), 403–426.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (1998). *Variables and patterns: Introducing algebra–teacher's guide*. Boston: Dale Seymour Publications.
Larkin, D. (2013). 10 things to know about mentoring student teachers. *The Phi Delta Kappan, 94* (7), 38–43. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23611699.
Larssen, D. L. S., Cajkler, W., Mosvold, R., Bjuland, R., Helgevold, N., Fauskanger, J., Wood, P., Baldry, F., Jakobsen, A., Bugge, H. E., Næsheim-Bjørkvik, G., & Norton, J. (2018). A literature review of lesson study in initial teacher education. Perspectives about learning and observations. *International Journal for Lesson and Learning Studies, 7*(1), 8–22. https://doi.org/10.1108/IJLLS-06-2017-0030.
Lasut, M. (2013). Effect of implementation lesson study to improve students' learning achievement in Calculus I of mathematics department. *Journal of Education and Practice, 4*(20), 182–188.
Latham, N. I., & Vogt, W. P. (2007). Do professional development schools reduce teacher attrition? Evidence from a longitudinal study of 1000 graduates. *Journal of Teacher Education, 58*(2), 153–167. https://doi.org/10.1177/0022487106297840.
Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. Cambridge/London: University of Cambridge Press.
Leavy, A., & Hourigan, M. (2016). Using lesson study to support knowledge development in initial teacher education: Insights from early number classrooms. *Teaching and Teacher Education, 57*, 161–175.
Lee, H.-J. (2005). Understanding and assessing preservice teachers' reflective thinking. *Teaching and Teacher Education, 21*(6), 699–715.
Lee, C. K. E., & Lo, M. L. (2013). The role of lesson study in facilitating curriculum reform. *International Journal for Lesson and Learning Studies, 2*, 200–206.
Lee, Y. A., & Takahashi, A. (2011). Lesson plans and the contingency of classroom interactions. *Human Studies, 34*(2), 209–227.
Legge, J. (1869). *The Chinese classics: Translated into English with preliminary essays and explanatory notes by James Legge. Vol. 1. The life and teachings of Confucius.* Second Edn. London: N. Trübner. Retrieved on June 4, 2018 at http://oll.libertyfund.org/titles/2270
Leon, J., Medina-Garrido, E., & Núñez, J. L. (2017). Teaching quality in math class: The development of a scale and the analysis of its relationship with engagement and achievement. *Frontiers in Psychology, 8*, 1–14.
Leong, Y., Ho, W., & Cheng, L. (2015). Concrete-pictorial-abstract: Surveying its origins and charting its future. *The Mathematics Educator, 16*(1), 1–19.
Leontiev, A. N. (1978). The problem of activity and psychology. In A. N. Leont'ev (Ed.), *Activity, consciousness, and personality* (pp. 45–74). Englewood Cliffs: Prentice Hall.
Lesson Study Alliance. (2012). *LessonNote*. Chicago.
Lewis, C. (1998, Winter). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 22*, 12–17 & 50–51.
Lewis, C. (2000a). *Lesson study: The core of Japanese professional development*. Paper presented at the Annual Meeting of the American Educational Research Association, New Orleans, LA.
Lewis, C. (2000b). Lesson Study: The core of Japanese professional development. In *AERA annual meeting*, April 2000.
Lewis, C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia: RBS Research for Better Schools, Inc..
Lewis, C. (2009). What is the nature of knowledge development in lesson study? *Educational Action Research, 17*, 95–110.
Lewis, C. (2014). Lesson study with mathematical resources: A sustainable model for locally-led teacher professional learning. *Mathematics Teacher Education and Development, 16*(1), 1–20.
Lewis, C. C. (2015). What is improvement sciences? Do we need it in education? *Educational Researcher, 44*(1), 54–61.
Lewis, C. (2016a). How does lesson study improve mathematics instruction? *ZDM Mathematics Education, 48*, 571–580.

Lewis, C. (2016b). *What learning occurs at each stage of the lesson study process*. Presentation given at the World association of lesson studies annual conference, 3rd Sept 2016, University of Exeter.

Lewis, J. M. (2016c). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM Mathematics Education, 48*(4), 527–540, 1–14. https://doi.org/10.1007/s11858-015-0753-9

Lewis, C., & Hurd, J. (2011). *Lesson study step by step: How teacher learning communities improve instruction*. Portsmouth: Heinemann.

Lewis, C., & Lee, C. (2017). The global spread of lesson study: Contextualization and adaptations. In M. Akiba & G. K. Letendre (Eds.), *International handbook of teacher quality and policy* (pp. 185–203). New York: Routledge.

Lewis, C., & Perry, R. (2006). Professional development through lesson study: Progress and challenges in the U.S. *Tsukuba Journal of Educational Study in Mathematics, 25*, 89–106.

Lewis, C., & Perry, R. (2014). Lesson study with mathematical resources: A sustainable model for locally-led teacher professional learning. *Mathematics Teacher Education and Development, 16* (1), 22–42.

Lewis, C., & Perry, R. (2015). A randomized trial of lesson study with mathematical resource kits: Analysis of impact on teachers' beliefs and learning community. In E. J. Cai & Middleton (Eds.), *Design, results, and implications of large-scale studies in mathematics education* (pp. 133–155). New York: Springer.

Lewis, C., & Perry, R. (2017). Lesson study to scale up research-based knowledge: A randomized, controlled trial of fractions learning. *Journal for Research in Mathematics Education, 48*(3), 261–299.

Lewis, C., & Takahashi, A. (2013). Facilitating curriculum reforms through lesson study. *International Journal for Lesson and Learning Studies, 2*, 207–217.

Lewis, C., & Tsuchida, I. (1997a). Planned educational change in Japan: The case of elementary science instruction. *Journal of Education Policy, 12*, 313–331.

Lewis, C., & Tsuchida, I. (1997b). The case elementary science instruction. *Journal of Education Policy, 12*(5), 313–331.

Lewis, C., & Tsuchida, I. (1998a). A lesson is like a swiftly owing river. *American Educator, 22*(4), 12–17.

Lewis, C., & Tsuchida, I. (1998b). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 12(Winter), 12–17*, 50–52.

Lewis, C., & Tsuchida, I. (1998c, Winter). A river is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 12–17*, 50–52.

Lewis, C., & Tsuchida, I. (1998d, Winter). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 22*, 14–17. & 50–52.

Lewis, C., & Tsuchida, I. (1998e). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 22*(4), 12–17. 50–52.

Lewis, C., & Tsuchida, I. (1998f, Winter). A lesson is like a swiftly flowing river: Research lessons and the improvement of Japanese education. *American Educator, 14–17*, 50–52.

Lewis, C., & Tsuchida, I. (1998g, Winter). A lesson is like a swiftly flowing river: How research lessons improve Japanese education. *American Educator, 22*, 12–17 & 50–52.

Lewis, C., Perry, R., & Hurd, J. (2004). A deeper look at lesson study. *Educational Leadership, 61* (5), 18–23.

Lewis, C., Perry, R., & Murata, A. (2006a). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14.

Lewis, C., Perry, R., Hurd, J., & O'Connell, M. P. (2006b, December). Lesson study comes of age in North America. *Phi Delta Kappan, 88*(04), 273–281.

Lewis, C., Perry, R., & Hurd, J. (2009). Improving mathematics instruction through lesson study: A theoretical model and North American case. *Journal of Mathematics Teacher Education, 12*(4), 285–304.

Lewis, C., Akita, K., & Sato, M. (2010). Lesson study as a human science. In W. R. Penuel & K. O'Connor (Eds.), *Learning research as a human science* (National Society for the study of education yearbook) (Vol. 109, pp. 222–237). New York: Teachers College, Columbia University.

Lewis, C., Perry, R., & Friedkin, S. (2011). Using Japanese curriculum materials to support lesson study outside Japan: Toward coherent curriculum. *Educational studies in Japan: International yearbook: ESJ, 6*(Classrooms and Schools in Japan), 5–19.

Lewis, C., Perry, R., Friedkin, S., & Roth, J. (2012). Improving teaching does improve teachers: Evidence from lesson study. *Journal of Teacher Education, 63*, 368–375.

Lewis, J. M., Fischman, D., Riggs, I., & Wasserman, K. (2013). Teachers learning in lesson study. *The Mathematics Enthusiast, 3*(10), 583–619.

Li, J. (1998). The development of a Chinese style normal education in modern China. *Teacher Education Research, 56*, 46–54 (in Chinese).

Li, S. (1999). Does practice make perfect? *For the Learning of Mathematics, 19*(3), 33–35.

Li, Y. (2008). What do students need to learn about division of fractions? *Mathematics Teaching in the Middle School, 13*, 546–552.

Li, S. (2014). Teaching research in secondary and elementary schools in the past 60 years: Reflections and prospects. *Contemporary Educational Science, 17*, 17–21 (in Chinese).

Li, S., & Dai, Q. (2009). Chinese traditional culture and mathematics education. In J. Wang (Ed.), *Mathematics education in China: Traditions and reality* (pp. 1–30). Nanjing: Jiangsu Education Publishing House. (in Chinese).

Li, Y., & Huang, R. (2009). Examining and understanding prospective mathematics teacher preparation in China from an international perspective. *Journal of Zhejiang Education Institutes, 1*, 37–44 (in Chinese).

Li, Y., & Huang, R. (2018). *How Chinese acquire and improve mathematics knowledge for teaching*. Boston: Sense Publishers.

Li, Y., & Li, J. (2009). Mathematics classroom instruction excellence through the platform of teaching contests. *ZDM Mathematics Education, 41*(3), 263–277.

Li, S., Huang, R., & Shin, Y. (2008). Discipline knowledge preparation for prospective secondary mathematics teachers: An East Asian perspective. In P. Sullivan & T. Wood (Eds.), *Knowledge and beliefs in mathematics teaching and teaching development* (pp. 63–86). Rotterdam: Sense Publishers.

Li, Y., Chen, X., & Khum, G. (2009). Mathematics teachers' practices and thinking in lesson plan development: A case of teaching fraction division. *ZDM Mathematics Education, 41*, 717–731.

Li, Y., Huang, R., Bao, J., & Fan, Y. (2011). Facilitating mathematics teachers' professional development through ranking and promotion practices in the Chinese mainland. In N. Bednarz, D. Fiorentini, & R. Huang (Eds.), *International approaches to professional development of mathematics teachers* (pp. 72–87). Ottawa: Ottawa University Press.

Li, X., Li, S., & Zhang, D. (2015). Cultural roots, traditions, and characteristics of contemporary mathematics education in China. In B. Sriraman et al. (Eds.), *The first sourcebook on Asian research in mathematics education* (pp. 67–88). Charlotte: Information Age Publishing.

Liang, G., & Akiba, M. (2018). Teachers' working conditions: A cross-national analysis using the OECD TALIS and PISA data. In M. Akiba & G. K. LeTendre (Eds.), *International handbook of teacher quality and policy* (pp. 388–402). New York: Routledge/Taylor & Francis.

Lieberman, A., & Miller, L. (2008). *Teacher in professional communities*. New York: Teachers College Press.

Lim, C. S. (2006). *Promoting peer collaboration among pre-service mathematics teachers through lesson study process*. In Y. Suan et al. (Eds.), *Proceedings of XII IOSTE symposium: Science and technology in the service of mankind* (pp. 590–593). Penang: School of Educational Studies, Universiti Sains Malaysia.

Lim, C. S., White, A. L., & Chiew, C. M. (2005). *Promoting mathematics teacher collaboration through lesson study: What can we learn from two countries' experience?* Paper presented at the

8th International Conference of The Mathematics Education into the 21st Century Project, Johor Bahru.

Lin, C., Xin, T., & Shen, J. (1999). Observation of normal education reform based on the structure of teacher knowledge. *Teacher Education Research, 66*, 12–17.

Linell, P. (2009). *Rethinking language, mind, and world dialogically: Interactional and contextual theories of human sense-making*. Charlotte: Information Age Publishing.

Linn, M. C., & Eylon, B.-S. (2011). *Science learning and instruction: Taking advantage of technology to promote knowledge integration*. New York: Routledge.

Linn, M., Eylon, B., & Davis, E. (2004). Internet environments for science education. In M. Linn, E. Davis, & P. Bell (Eds.), *The knowledge integration perspective on learning* (pp. 29–46). Mahwah: Lawrence Erlbaum Associates.

Linn, M. C., Davis, E. A., & Bell, P. (2013). *Internet environments for science education*. New York: Routledge.

Little, J. W. (1989). District policy choices and teachers' professional development opportunities. *Educational Evaluation and Policy Analysis, 11*(2), 165–179.

Little, J. W. (1993). Teachers' professional development in a climate of educational reform. *Educational Evaluation and Policy Analysis, 15*(2), 129–151.

Little, J. W. (1999). Organizing schools for teacher learning. In L. Darling-Hammond & G. Sykes (Eds.), *Teaching as the learning profession: Handbook of policy and practice* (pp. 233–262). San Francisco: Jossey-Bass.

Little, J. W. (2002). Locating learning in teachers' communities of practice: Opening up problems of analysis in records of everyday work. *Teaching and Teacher Education, 18*(8), 917–946.

Little, J. W. (2003). Inside teacher community: Representations of classroom practice. *Teachers College Record, 105*(6), 913–945.

Littleton, K., & Mercer, N. (2013). *Interthinking: Putting talk to work*. Abingdon: Routledge.

Liu, J., & Li, Y. (2010). Mathematics curriculum reform in the Chinese mainland: Changes and challenges. In F. K. S. Leung & Y. Li (Eds.), *Reforms and issues in school mathematics in East Asia*. Rotterdam: Sense Publisher.

Livy, S. L., Vale, C., & Herbert, S. (2016). Developing primary pre-service teachers' mathematical content knowledge during practicum teaching. *Australian Journal of Teacher Education, 41*(2), 152–173.

Lo, M. L. (2004). The development of the learning study approach in classroom research in Hong Kong. *Educational Research Journal, 24*(1), 165–184.

Lo, M. L. (2012). *Variation theory and the improvement of teaching and learning*. Sweden: Acta Universitatis Gothoburgensis.

Lo, M. L. (2016). You can only see what you have chosen to see: Overcoming the limitations inherent in our theoretical lenses. *International Journal for Lesson and Learning Studies, 5*(3), 170–179.

Lo, M. L., & Chik, P. P. M. (2016). Two horizons of fusion. *Scandinavian Journal of Educational Research, 60*(3), 296–308.

Lo, M. L., & Marton, F. (2012). Towards a science of the art of teaching: Using variation theory as a guiding principle of pedagogical design. *International Journal of Lesson and Learning Studies, 1*(1), 7–22.

Lo, M. L., Pong, W. Y., Marton, F., Ng, F. P., & Pang, M. F. (Eds.). (2004). *For each and every one: Catering for individual differences through learning study*. Hong Kong: Hong Kong University Press.

Lo, M. L., Pong, W. Y., & Chik, P. (Eds.). (2005). *For each and every one. Catering for individual differences through learning study*. Hong Kong: Hong Kong University Press.

Lobato, J., & Walters, C. D. (2017). A taxonomy of approaches to learning trajectories and progressions. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 74–101). Reston: National Council of Teachers of Mathematics.

Lortie, D. (1975a). *School teacher: A sociological study*. Chicago: University of Chicago press.

Lortie, D. (1975b). *Schoolteacher: A sociological analysis*. Chicago: University of Chicago Press.

Loucks-Horsley, S., Stiles, K. E., Mundry, S., Hewson, P. W., & Love, N. (2010). *Designing professional development for teachers of science and mathematics* (3rd ed.). Thousand Oaks: Corwin.

Loughran, J., Mulhall, P., & Berry, A. (2008). Exploring pedagogical content knowledge in science teacher education. *International Journal of Science Education, 30*(10), 1301–1320.

Ma, L. (1999a). *Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States*. Mahwah: Lawrence Erlbaum.

Ma, L. (1999b). *Knowing and teaching elementary mathematics*. Hillsdale: Erlbaum.

Mak, W. F. (2016). *Effect of lesson study incorporating phase-based instruction on Form one students' achievement and learning motivation in Geometry.* Unpublished master's thesis, Universiti Sains Malaysia, Malaysia.

Makinae, N. (2010a). *The origin of lesson study in Japan*. Proceedings of EARCOME5, Japan Society of Mathematics Education.

Makinae, N. (2010b). The origin of lesson study in Japan. In Y. Shimizu, Y. Sekiguchi, & K. Hino (Eds.), *The Proceedings of the 5th East Asia Regional conference on mathematics education: In search of excellence of mathematics education* (Vol. 2, pp. 140–147). Tokyo: Japan Society of Mathematics Education (JSME).

Maloney, A. P., Confrey, J., & Nguyen, K. H. (2014). *Learning over time: Learning trajectories in mathematics*. Charlotte: Informational Age Publishing.

Margolinas, C. (2002). Situations, milieux, connaissances: Analyse de l'activité du professeur. In J.-L. Dorier, M. Artaud, M. Artigue, R. Berthelot, & R. Floris (Eds.), *Actes de la 11e école d'été de didactique des mathématiques*. La Pensée Sauvage: Grenoble.

Margolinas, C. (2004). Modeling the teacher's situation in the classroom. In H. Fujita, Y. Hashimoto, B. Hodgson, P. Lee, S. Lerman, & T. Sawada (Eds.), *Proceedings of the ninth international congress on mathematical education*. Dordrecht: Springer Netherlands.

Margolinas, C., Coulange, L., & Bessot, A. (2005). What can the teacher learn in the classroom? *Educational Studies in Mathematics, 59*, 205–234.

Marrongelle, K., Sztajn, P., & Smith, M. S. (2013). Scaling up professional development in an era of common state standards. *Journal of Teacher Education, 64*(3), 202–211.

Martin, D., & Clerc-Georgy, A. (2015). Use of theoretical concepts in lesson study: An example from teacher training. *International Journal for Lesson and Learning Studies, 4*(3), 261–273.

Martin, L., & Towers, J. (2016). Folding back and growing mathematical understanding: A longitudinal study of learning. *International Journal for Lesson and Learning Studies, 5*(4), 281–294.

Marton, F. (2015). *Necessary conditions of learning*. London/New York: Routledge.

Marton, F., & Booth, S. (1997). *Learning and awareness*. Mahwah: Lawrence Erlbaum Associates.

Marton, F., & Häggström. (2017). Teaching through variation: A European perspective. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 389–406). Boston: Sense Publishers.

Marton, F., & Pang, M. F. (2003). Beyond "lesson study": Comparing two ways of facilitating the grasp of economic concepts. *Instructional Science, 31*(3), 175–194.

Marton, F., & Pang, M. F. (2006). On some necessary conditions of learning. *The Journal of the Learning Sciences, 15*(2), 193–220.

Marton, F., & Runesson, U. (2015). The idea and practice of learning study. In K. Wood & S. Sithampram (Eds.), *Realising learning. Teachers' professional development through lesson study and learning study* (pp. 103–121). New York: Routledge.

Marton, F., & Säljö, R. (1984). Approaches to learning. In F. Marton, D. Hounsell, & N. Entwistle (Eds.), *The experience of learning* (pp. 39–58). Edinburgh: Scottish Academic Press.

Marton, F., & Tsui, A. B. M. (Eds.). (2004a). *Classroom discourse and the space of learning*. London: Lawrence Erlbaum.

Marton, F., & Tsui, A. B. M. (2004b). *Classroom discourse and the space of learning*. Mahwah: Lawrence Erlbaum.

Marton, F., Tsui, A. B., Chik, P. P., Ko, P. Y., & Lo, M. L. (2004a). *Classroom discourse and the space of learning*. Mahwah: Routledge.

Marton, F., & Tsui, A. B. M. (with Chik, P. P. M., Ko, P. Y., Lo, M. L., Mok, I. A. C., Ng, F. P., Pang, M.F., et al.) (Eds.). (2004b). *Classroom discourse and the space of learning.* Mahwah: Lawrence Erlbaum.

Måseide, P. (2003). Medical talk and moral order: Social interaction and collaborative clinical work. *Text, 2*(3), 369–403.

Mason, J. (2001a). Tunja sequences as examples of employing students' powers to generalize. *The Mathematics Teacher, 94*(3), 164–168.

Mason, J. (2001b). *Researching your own practice: The discipline of noticing*. London: Routledge.

Mathematics Teaching Research Group at Jiading Secondary School. (1953, March–April). A report on improving the mathematics teaching at Jiading Secondary School. *Bulletin of Mathematics*, 37–42 (in Chinese).

Mathis. (2018). *Secondary preservice mathematics teachers' curricular reasoning: What influences their decisions.* Unpublished master's thesis. Brigham Young University, Provo, Utah.

Matsuzawa Elementary School. (2011). *School research report by the Matsuzawa elementary school.* Available from http://www.impuls-tgu.org/cms/uploads/File/resource/MatsuzawaLeafletDec12011.pdf

Matthews, M. E., Hlas, C. S., & Finken, T. M. (2009). Using lesson study and four-column lesson planning with pre-service teachers. *Mathematics Teacher, 102*(7), 504–508.

McDougal, T., & Takahashi, A. (2014). *Teaching mathematics through problem solving*. Retrieved from http://www.nais.org/MagazinesNewsletters/ITMagazine/Pages/Teaching-Mathematics-Through-ProblemSolving.aspx

McKinsey. (2007). *How the world's best-performing school systems come out on top*. London: McKinsey & company.

McLaughlin, M. W., & Mitra, D. (2001). Theory-based change and change-based theory: Going deeper and going broader. *Journal of Educational Change, 2*(4), 301–323.

McLaughlin, M. W., & Talbert, J. E. (2001). *Professional communities and the work of high school teaching*. Chicago: University of Chicago Press.

McLaughlin, M., & Talbert, J. (2006). *Building school-based teacher learning communities: Professional strategies to improve student achievement*. New York: Teachers College Press.

McMahon, M. T., & Hines, E. (2008). Lesson study with preservice teachers. *Mathematics Teacher, 102*(3), 186–191.

Mecoli, S. (2013). The influence of the pedagogical content knowledge theoretical framework on research on preservice teacher education. *The Journal of Education, 193*(3), 21–27. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/24636918.

Meng, C. C., & Sam, L. C. (2013). Developing pre-service teachers' technological pedagogical content knowledge for teaching mathematics with the Geometer's sketchpad through lesson study. *Journal of Education and Learning, 2*(1), 1–8. https://doi.org/10.5539/jel.v2n1p1.

Mercer, N. (1995). *The guided construction of knowledge: Talk amongst teachers and learners*. Clevedon: Multilingual Matters.

Mercer, N. (2000). *Words and minds: How we use language to think together*. London: Routledge.

Mercer, N. (2004). Sociocultural discourse analysis: Analysing classroom talk as a social mode of thinking. *Journal of Applied linguistics and Professional Practice, 1*(2), 137–168.

Mercer, N. (2014). The social brain, language, and goal-directed collective thinking: A social conception of cognition and its implications for understanding how we think, teach, and learn. *Educational Psychologist, 48*(3), 148–168.

Mercer, N., & Littleton, K. (2007). *Dialogue and the development of children's thinking: A sociocultural approach*. London: Routledge.

Mercer, N., Littleton, K., & Wegerif, R. (2004). Methods for studying the processes of interaction and collaborative activity in computer-based educational activities. *Technology, Pedagogy and Education, 13*(2), 193–209.

Metz, M., Preciado-Babb, P., Sabbaghan, S., Davis, B., Pinchbeck, G., & Aljarrah, A. (2016). Transcending traditional/ reform dichotomies in mathematics education. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 1252–1258). Tucson: The University of Arizona.

Metz, M., Preciado-Babb, P., Sabbaghan, S., Davis, B., & Ashebir, A. (2017). Using variation to critique and adapt mathematical tasks. In P. Preciado Babb, L. Yeworiew, & S. Sabbaghan (Eds.), *Selected proceedings of the IDEAS conference: Leading educational change* (pp. 169–178). Calgary: Werklund School of Education, University of Calgary.

Meyer, R. D., & Wilkerson, T. L. (2011). Lesson study: The impact on teachers' knowledge for teaching mathematics. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 15–26). Dordrecht: Springer.

Mighton, J., Sabourin, S., & Klebanov, A. (2009a). *JUMP Math 1.1: Assessment & practice*. Toronto: JUMP Math.

Mighton, J., Sabourin, S., & Klebanov, A. (2009b). *JUMP Math 5.1: Assessment & practice*. Toronto: JUMP Math.

Mighton, J., Sabourin, S., & Klebanov, A. (2010a). *JUMP Math: Teacher's guide: Workbook 2*. Toronto: JUMP Math.

Mighton, J., Sabourin, S., & Klebanov, A. (2010b). *JUMP Math: Teacher's guide: Workbook 5*. Toronto: JUMP Math.

Miheso-O'Connor Khakasa, M., & Berger, M. (2016). Status of teachers' proficiency in mathematical knowledge for teaching at secondary school level in Kenya. *International Journal of Science and Mathematics Education, 14*, 419–435.

Miles, M. B., Huberman, A. M., & Saldana, J. (2014). *Qualitative data analysis: A methods sourcebook*. Thousand Oaks, CA: Sage Publication.

Miller, J. B., & Stiver, I. (2015). *The healing connection: How women form relationships in therapy and in life*. Boston: Beacon Press.

Mills College Lesson Study Group. (2005). *How many seats? Excerpts of a lesson study cycle [DVD]*. Oakland: Mills College Lesson Study Group.

Ministry of Education. (2007). *Primary mathematics syllabus*. Singapore: Author.

Ministry of Education (MoE). (2001). *Mathematics curriculum standards for compulsory education (trial)*. Beijing: Beijing Normal University Publisher (in Chinese).

Ministry of Education (MoE). (2011). *Mathematics curriculum standards for compulsory education*. Beijing: Beijing Normal University Publisher (in Chinese).

Ministry of Education China [MOE]. (2001). *Mathematics curriculum standards (experimental version)*. Beijing: Beijing Normal University Press.

Ministry of Education China [MOE]. (2011). *Mathematics curriculum standards*. Beijing: Beijing Normal University Press.

Ministry of Education of the People's Republic of China. (1986). *Tentative regulations on secondary school teacher professional titles*.

Ministry of Education, P. R. China. (2011). *Mathematics curriculum standards for compulsory education (grades 1–9)*. Beijing: Beijing Normal University Press.

Miwa, T. (1992). *The teaching mathematical problem solving in Japan and the US*. Tokyo: Toyokanshuppan (in Japanese).

Miyakawa, T., & Winsløw, C. (2009a). Didactical designs for students' proportional reasoning. *Educational Studies in Mathematics, 72*(2), 199–218.

Miyakawa, T., & Winsløw, C. (2009b). Didactical designs for students' proportional reasoning: An "open approach" lesson and a "fundamental situation". *Educational Studies in Mathematics, 72*, 199–218.

Miyakawa, T., & Winsløw, C. (2018). Paradidactical infrastructure for sharing and documenting mathematics teacher knowledge: A case study of "practice research" in Japan. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-017-9394-y.

Miyazaki, K, Kumazawa, A., Odaka, T., & Okamoto, K. (1969). Teaching methods in Modernization of mathematics education, *Kindaishinsho*. (in Japanese).

Mogensen, A. (2011). *Point-driven mathematics teaching: Studying and intervening in Danish classrooms* (PhD dissertation), Roskilde University. Retrieved from http://milne.ruc.dk/imfufatekster/pdf/484web.pdf. 5 Feb 2018.

Mon, C. C., Dali, M. H., & Sam, L. C. (2016). Implementation of lesson study as an innovative professional development model among Malaysian school teachers. *Malaysian Journal of Learning and Instruction, 13*(1), 83–111.

Monbusyo. (1873). *Sho-gaku sanjutsu-syo* [Elementary arithmetic textbook]. Tokyo: Shihangakkou. [in Japanese].

Morris, A. K., & Hiebert, J. (2011). Creating shared instructional products: An alternative approach to improving teaching. *Educational Researcher, 40*(1), 5–14.

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122–147.

Moss, J., Messina, R., Morley, E., & Tepylo, D. (2012). Building the sustaining professional collaborations: Using Japanese Lesson Study to improve the teaching and learning of mathematics. In J. M. Bay-Williams & W. R. Speer (Eds.), *Professional collaborations in mathematics teaching and learning: Seeking success for all* (pp. 297–309). Reston: National Council of Teachers of Mathematics.

Mostofo, J. (2014). The impact of using lesson study with pre-service mathematics teachers. *Journal of Instructional Research*, 355–363.

Mosvold, R., & Bjuland, R. (2011). An activity theory view on learning studies. *International Journal of Early Childhood, 43*, 261–275.

Mourshed, M., Chijioke, C., & Barber, M. (2010). *How the world's most improved school systems keep getting better*. London: McKinsey and Company.

Mudzimiri, R., Burroughs, E. A., Luebeck, J., Sutton, J., & Yopp, D. (2014). A look inside mathematics coaching: Roles, content, and dynamics. *Education Policy Analysis Archives, 22* (53), 1–32.

Munby, H., Russell, T., & Martin, A. K. (2001). Teachers'knowledge and how it develops. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 877–905). Washington, D.C.: American Educational Research Association.

Munter, C. (2014). Developing visions of high-quality mathematics instruction. *Journal for Research in Mathematics Education, 45*(5), 584–635.

Murata, A. (2011a). Conceptual overview of lesson study: Introduction. In L. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 1–12). New York: Springer.

Murata, A. (2011b). Introduction: Conceptual overview of lesson study. In C. L. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 1–12). New York: Springer.

Murata, A., & Pothen, B. (2011a). Lesson study in preservice elementary mathematics courses: Connecting emerging practice and understanding. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 103–116). Dordrecht: Springer.

Murata, A., & Pothen, B. E. (2011b). Lesson study in preservice elementary mathematics methods courses: Connecting emerging practice and understanding. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 103–116). New York: Springer.

Murata, A. & Takahashi, A. (2002a). Vehicle to connect theory, research and practice: How teacher thinking changes in district-level lesson study in Japan. In Editors (Eds.), *Proceedings of the twenty-fourth annual meeting of North American chapter of the international group of the Psychology of Mathematics Education* (pp. 1879–1887). Athens: USA.

Murata, A., & Takahashi, A. (2002b). Vehicle to connect theory, research, and practice: How teacher thinking changes in district-level lesson study in Japan. In D. L. Haury (Ed.), *Twenty-fourth annual meeting of North American chapter of the international* group *of the Psychology of Mathematics Education, Columbus, OH, 2002b* (Vol. 1–4, pp. 1879–1888): ERIC/CSMEE Publications.

Murata, A., Bofferding, L., Pothen, B., Taylor, M., & Wischnia, S. (2012). Making connections among student learning, content, and teaching: Teacher talk paths in elementary mathematics lesson study. *Journal for Research in Mathematics Education, 43*, 616–650.

Murphy, R., Weinhardt, F., Wyness, G. and Rolfe, H. (2017). *Lesson study: Evaluation report and executive summary*. Education Endowment Foundation. Available at: https://educationendowmentfoundation.org.uk/public/files/Projects/Evaluation_Reports/Lesson_Study.pdf. Accessed 20 Feb 2017.

Myers, J. (2012a). Lesson study as a means for facilitating preservice teacher reflectivity. *International Journal for the Scholarship of Teaching and Learning, 6*(1), Article 15. https://doi.org/10.20429/ijsotl.2012.060115

Myers, J. (2012b). The effects of lesson study on classroom observations and perceptions of lesson effectiveness. *The Journal of Effective Teaching, 12*(3), 94–104.

Myers, M., Sztajn, P., Wilson, P. H., & Edgington, C. (2015). From implicit to explicit: Articulating equitable learning trajectories based instruction. *Journal of Urban Mathematics Education, 8* (2), 11–22.

Nagasaki, E., & Senuma, H. (1986). *Suugaku kyouiku ni okeru mondai kaiketu ni tuiteno kenkyuu no doukou (1) (Research trends on problem solving in mathematics education)*. Tokyo: National Institute for Educational Research.

Nakajima, K. (1982). Koment: Shousuu no kakezan (comment: Multiplication of decimal numbers). *Atarashii Sansuu Kenkyuu, 134*, 56.

Nakano, K. (1939). *Sanjutsu no jugyou kenkyu (lesson study in elementary school mathematics)*. Tokyo: Kenbunkan.

Nakatome, T. (Ed.) (1984). *Kounai kensyu wo tsukuru: Nihonn no kounai kensyu keiei no sougouteki kenkyu* [Developing teacher training in school: A comprehensive study of management of teacher training in Japanese school]. Tokyo: Eideru Kenkyusyo. [in Japanese].

Nassaji, H., & Wells, G. (2000). What's the use of 'triadic dialogue'?: An investigation of teacher-student interaction. *Applied Linguistics, 21*, 376–406.

National Council of Teachers of Mathematics. (1980). *An agenda for action*. Reston: NCTM.

National Council of Teachers of Mathematics. (1989). *Curriculm and evaluation standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions: Ensuring mathematics success for all*. Reston: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics [NCTM]. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston: Author.

National Course of Study (in Japanese). (1998, 2008). *Ministry of Education, Culture, Sports, Science and Technology*. Retrieved April 30, 2015, from http://www.mext.go.jp/a_menu/shotou/cs/index.htm

National Education Policy Research Institute, J. K. K. S. K. (2011). *Report of survey research on improvement of teacher quality* [Kyouin no Shitsu no Koujou ni Kansuru Chosa Kenkyuu]. Retrieved from Tokyo.

National Governors Association Center for Best Practices, Council of Chief State School Officers. (2010). *Common core state standards: Standards for mathematical process*. Washington, DC: Author.

National People's Congress. (1986). *Compulsory education law of the People's Republic of China.* Retrieved on June 3, 2018 from http://www.lawinfochina.com/display.aspx?lib=law&id=1166&CGid

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

National Research Council. (2007). In R. A. Duschl, H. A. Schweingruber, & A. W. Shouse (Eds.), *Taking science to school: Learning and teaching science in grades K-8. Committee on science learning, kindergarten through eighth grade*. Washington, DC: National Academies Press.

National Research Council. (2010). *The teacher development continuum in the United States and China: Summary of a workshop*. Washington, DC: The National Academies Press.

NCTM. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s*. Reston: National Council of Teachers of Mathematics.

NCTM. (2000). *Principles and standards for school mathematics*. Reston: National Council of Teachers of Mathematics.

NEA Foundation for the Improvement of Education. (2003). *Using data about classroom practice and student work to improve professional development for educators*. Washington, DC: Author.

Nelson. (2018). *Canadian tests of basic skills (CTBS).* Nelson. www.assess.nelson.com/

Ni, Y. (2001). Semantic domains of rational numbers and the acquisition of fraction equivalence. *Contemporary Educational Psychology, 26*, 400–417.

Ni Shuilleabhain, A. (2015). *Developing mathematics teachers' pedagogical content knowledge through lesson study: A multiple case study at a time of curriculum change.* Doctor of Philosophy Ph.D, Trinity College Dublin.

Ni Shuilleabhain, A. (2016). Developing mathematics teachers' pedagogical content knowledge in lesson study: Case study findings. *International Journal for Lesson and Learning Studies, 5*, 212–226.

Ni Shuilleabhain, A., & Clivaz, S. (2017). Analyzing teacher learning in lesson study: Mathematical knowledge for teaching and levels of teacher activity. *Quadrante, 26*, 99–125.

Ni Shuilleabhain, A., & Seery, A. (2017). Enacting curriculum reform through lesson study: A case study of mathematics teacher learning. *Professional Development in Education, 44*, 1–15.

Nilsson, P. (2008). Teaching for understanding: The complex nature of pedagogical content knowledge in pre-service education. *International Journal of Science Education, 30*(10), 1281–1299.

Niss, M. (2003). The Danish "KOM" project and possible consequenses for teacher education. In R. Strässer, G. Brandell, B. Grevholm, & O. Helenius (Eds.), *Educating for the future: Proceedings of an international symposium on mathematics teacher education*. The Royal Swedish Academy of Sciences: Stockholm.

Nohda, N. (1995). Teaching and evaluating using 'Open-ended problems in classroom. *Zentralblatt für Didaktik der Mathematik, 27*(2), 57–60.

Nohda, N. (2000). *Teaching by open-approach method in Japanese mathematics classroom*. Proceeding of the 24th conference of the international group for the psychology of mathematics education (Vol. 1, pp. 39–54). Hiroshima: Hiroshima University.

Nolan, E. C., Dixon, J. K., Roy, G. J., & Andreasen, J. B. (2016). *Making sense of mathematics for teaching-grades 3–5*. Bloomington: Solution Tree Press.

Norman, P. (2011). Planning for what kind of teaching? Supporting cooperating teachers as teachers of planning. *Teacher Education Quarterly, 38*(3), 49–68. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23479617.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings: Learning cultures and computers*. Dordrecht: Kluwer.

Noyes, A. (2013). The effective mathematics department: Adding value and increasing participation? *School Effectiveness and School Improvement, 24*(1), 1–17.

Ntow, F., & Adler, J. (2017). An exploration into teachers' take up of professional development teaching resources. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 313–320). Singapore: PME.

Nurfaidah, S., Lengkanawati, N. S., & Sukyadi, D. (2017). Levels of reflection in EFL pre-service teachers' teaching journal. *Indonesian Journal of Applied Linguistics, 7*(1), 80–92.

O'Hanlon, C. (Ed.). (1996). *Professional development through action research in educational settings*. London: The Falmer Press.

O'Hanlon, C. (2003). *Educational inclusion as action research: An interpretative discourse*. Maidenhead: Open University Press.

OECD. (2013). *PISA 2012 results: What students know and can do: Student performance in mathematics, reading and science* (Vol. 1). Paris: OECD.

OECD. (2017). *Empowering and enabling teachers to improve equity and outcomes for all*. Paris: OECD Publishing.

Ohtani, M. (2014). Construction zone for the understanding of simultaneous equations: An analysis of one Japanese teacher's strategy of reflection on a task in a lesson sequence. In F. K. S. Leung, K. Park, D. Holton, & D. Clarke (Eds.), *Algebra teaching around the world* (pp. 113–128). Rotterdam: Sense Publishers.

Ólafsson, R.F. (2014). *TALIS 2013: Starfsaðstæður, viðhorf og kennsluhættir kennara og skólastjóra á Íslandi í alþjóðlegum samanburði.* [OECD Teaching and Learning International Survey]. Reykjavík: Námsmatsstofnun. Available at https://mms.is/sites/mms.is/files/talis_skyrsla_2014.pdf

Olteanu, L. (2017). Distributive law as object of learning through direct and inverse tasks. *International Journal for Lesson and Learning Studies, 6*(1), 56–65. https://doi.org/10.1108/IJLLS-05-2016-0014.

Ong, E. G. (2010). *Changes in mathematics teachers' questioning techniques through the lesson study process.* Unpublished doctoral dissertation, Universiti Sains Malaysia, Malaysia.

Ono, Y., & Ferreira, J. (2010). A case study of continuing teacher professional development through lesson study in South Africa. *South African Journal of Education, 30*(1), 59–74.

Opfer, V. D., & Pedder, D. (2010). Benefits, status and effectiveness of continuous professional development for teachers in England. *The Curriculum Journal, 21*(4), 413–431.

Opfer, V. D., & Pedder, D. (2011). Conceptualizing teacher professional learning. *Review of Educational Research, 81*(3), 376–407.

Organisation for Economic Co-operation and Development. (2014). *Education at a glance 2014: OECD Indicators*. Paris: OECD Publishing. https://doi.org/10.1787/eag-2014-en.

Ott, J. M., Snook, D. L., & Gibson, D. L. (1991). Understanding partitive division of fractions. *The Arithmetic Teacher, 39*, 7–11.

Paavola, S., & Hakkarainen, K. (2005). The knowledge creation metaphor–An emergent epistemological approach to learning. *Science & Education, 14*(6), 535–557.

Paavola, S., Lipponen, L., & Hakkarainen, K. (2004). Models of innovative knowledge communities and three metaphors of learning. *Review of Educational Research, 74*, 557–576.

Paine, L. W. (1990). The teacher as virtuoso: A Chinese model for teaching. *Teachers College Record, 92*(1), 49–81.

Paine, L. W., & Fang, Y. P. (2006). Reform as hybrid model of teaching and teacher development in China. *International Journal for Education Research, 45*(4–5), 279–289.

Paine, L., & Ma, L. (1993). Teachers working together: A dialogue on organizational and cultural perspectives of Chinese teachers. *International Journal of Educational Research, 19*(8), 675–697.

Paine, L., Fang, Y., & Wilson, S. (2003). Entering a culture of teaching: Teacher induction in Shanghai. In E. D. Britton, L. Paine, & S. Raizen (Eds.), *Comprehensive teacher induction: Systems for early career learning* (pp. 20–83). Dordrecht: Kluwer.

Pang, Y. (2011). *A study of prospective teachers' mathematical knowledge for teaching and how to develop it* (Unpublished master thesis). Shanghai: East China Normal University (in Chinese).

Pang, J. (2016). Improving mathematics instruction and supporting teacher learning in Korea through lesson study using five practices. *ZDM: The International Journal on Mathematics Education, 48*(4), 471–483. https://doi.org/10.1007/s11858-016-0768-x.

Pang, M. F., & Ki, W. W. (2016). Revisiting the idea of "critical aspects". *Scandinavian Journal of Educational Research, 60*(3), 323–336.

Pang, M. F., & Marton, F. (2005). Learning theory as teaching resource: Another example of radical enhancement of students' understanding of economic aspects of the world around them. *Instructional Science, 33*(2), 159–191.

Pang, M. F., & Marton, F. (2013). Interaction between the learners' initial grasp of the object of learning and the learning resource afforded. *Instructional Science, 41*(6), 1065–1082.
Pang, M. F., & Marton, F. (2017). Chinese lesson study, learning study and keys to learning. *International Journal for Lesson and Learning Studies, 6*(4), 336–347.
Pang, M. F., Marton, F., Bao, J., & Ki, W. W. (2016). Teaching to add three-digit numbers in Hong Kong and Shanghai: Illustration of differences in the systematic use of variation and invariance. *ZDM Mathematics Education, 48*(4), 455–470.
Pang, M. F., Bao, J., & Ki, W. W. (2017). 'Bianshi' and the variation theory of learning. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 43–67). Boston: Sense Publishers.
Parks, A. N. (2008). Messy learning: Preservice teachers' lesson-study conversations about mathematics and students. *Teaching and Teacher Education, 24*, 1200–1216.
Parks, A. N. (2009). Collaborating about what? An instructor's look at preservice lesson study. *Teacher Education Quarterly, 36*(4), 81–97.
Parsons, M., & Stephenson, M. (2005). Developing reflective practice in student teachers: Collaboration and critical partnerships. *Teachers and Teaching: Theory and Practice, 11*(1), 95–116. https://doi.org/10.1080/1354060042000337110.
Patton, M. Q. (2002). *Qualitative research & evaluation methods* (3rd ed.). Thousand Oaks: Sage.
Payne, J. N. (1976). Review of research on fractions. In R. Lesh (Ed.), *Number and measurement* (pp. 145–188). Athens: University of Georgia.
Pernilla, M., & Henrik, H. (2018). Challenging teachers' ideas about what students need to learn: Teachers' collaborative work in subject didactic groups. *International Journal for Lesson and Learning Studies, 7*(2), 98–110.
Perrin-Glorian, M. -J. (2008). *From producing optimal teaching to analysing usual classroom situations. Development of a fundamental concept in the theory of didactical situations: the notion of milieu*. Retrieved from https://www.unige.ch/math/EnsMath/Rome2008/WG5/Papers/PERRIN.pdf. 5 Feb 2018.
Perry, R. R., & Lewis, C. C. (2008). What is successful adaptation of lesson study in the US? *Journal of Educational Change, 10*, 365–391.
Perry, R. R., & Lewis, C. C. (2009). What is successful adaptation of lesson study in the US? *Journal of Educational Change, 10*(4), 365–391. https://doi.org/10.1007/s10833-008-9069-7.
Perry, R., & Lewis, C. (2010). Building demand for research through lesson study. In C. E. Coburn & M. K. Stein (Eds.), *Research and practice in education: Building alliances, bridging the divide* (pp. 131–145). Lanham: Rowan & Littlefield Publishers, Inc..
Perry, R. R., & Lewis, C. C. (2011). *Improving the mathematical content base of lesson study summary of results*. Oakland: Mills College Lesson Study Group. Retrieved from http://www.lessonresearch.net/IESAbstract10.pdf
Pestalozzi, J. (1801). *Wie gertrud ihre kinder lehrt* [trans: Nagao T. and Fukuda H. in 1976, Tokyo: Meiji-tosyo]. [in Japanese].
Peterson, B. E. (2005). Student teaching in Japan: The lesson. *Journal of Mathematics Teacher Education, 8*(1), 61–74.
Peterson, B. E. (2010). Mathematics student teaching in Japan: Where's the management. In G. Anthony & B. Grevhold (Eds.), *Teachers of mathematics: Recruitment and retention, professional development and identity* (pp. 135–144). Kristiansand: Writings from Swedish Society for Research in Mathematics Education No. 8.
Petit, M., Laird, R. E., & Marsden, E. L. (2010). *A focus on fractions: Bringing research to the classroom*. New York: Taylor & Francis.
Pickering, A. (1995). *The mangle of practice*. Chicago: University of Chicago.
Pillay, V., & Adler, J. (2015). Evaluation as key to describing the enacted object of learning. *International Journal for Lesson and Learning Studies, 4*(3), 224–244.
Plummer, J. S., & Peterson, B. E. (2009). A preservice secondary teacher's moves to protect her view of herself as a mathematics expert. *School Science and Mathematics, 109*(5), 247–257.

Porter, A. C., Garet, M. S., Desimone, L. M., & Birman, B. F. (2003). Providing effective professional development: Lessons from the Eisenhower program. *Science Educator, 12*(1), 23–40.

Posch, P. (2015). A comment on Anne Brosnan's paper on "Introducing lesson study in introducing a new mathematics curriculum in Irish post-primary schools". *International Journal of Lesson and Learning Studies, 4*(2), 236–251.

Posthuma, B. (2012). Mathematics teachers' reflective practice within the context of adapted lesson study. *Pythagoras*, *33*(3). https://doi.org/10.4102/pythagoras.v33i3.140

Potari, D. (2011). Response to part II: Emerging issues from lesson study approaches in prospective mathematics teacher education. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education* (pp. 127–132). Dordrecht: Springer.

Pournara, C., Hodgen, J., Adler, J., & Pillay, V. (2015). Can improving teachers' knowledge of mathematics lead to gains in learners' attainment in mathematics? *South African Journal of Education, 35*(3), 10. https://doi.org/10.15700/saje.v35n3a1083.

Powell, A., Francisco, J., & Maher, C. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behaviour, 22*(4), 405–435.

Pratt, S. S., Lupton, T. M., & Richardson, K. (2015). Division quilts: A measurement model of division. *Teaching Children Mathematics, 22*(2), 102–109.

Preciado-Babb, A. P., & Liljedahl, P. (2012). Three cases of teachers' collaborative design: Perspectives from those involved. *Canadian Journal of Science, Mathematics, and Technology Education, 12*(1), 22–35.

Preciado-Babb, A. P., Metz, M., Sabbaghan, S., & Davis, B. (2016a). Fine-grained, continuous assessment for the diverse classroom: A key factor to increase performance in mathematics. In *Proceedings of the American Education Research Association Annual Meeting 2016* (pp. 1–20). http://www.aera.net/Publications/Online-Paper-Repository/AERA-Online-Paper-Repository/Owner/973698

Preciado-Babb, A. P., Aljarrah, A., Sabbaghan, S., Metz, M., Pinchbeck, G., & Davis, B. (2016b). Teachers' perceived difficulties for creating mathematical extensions at the border of students' discernments. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American chapter of the International Group for the Psychology of Mathematics Education* (pp. 514–517). Tucson: The University of Arizona.

Preciado-Babb, A. P., Metz, M., Sabbaghan, S., & Davis, B. (2017). The role of continuous assessment and effective teacher response in engaging all students. In R. Hunter, M. Civil, N. P. Herbel-Eisenmann, & D. Wagner (Eds.), *Mathematical discourse that breaks barriers and creates space for marginalized learners* (pp. 101–120). Rotterdam: Sense Publishers.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM Mathematics Education, 40*, 165–178.

Puchner, L. D., & Taylor, A. R. (2006). Lesson study, collaboration and teacher efficacy: Stories from two school-based math lesson study groups. *Teaching and Teacher Education, 22*(7), 922–934.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1), 4–15.

Qingpu Research Institute. (2012). Teachers "action education". *Curriculum, Teaching Material and Method, 3*, 3–12.

Quaresma, M., Winsløw, C., Clivaz, S., Ponte, J. P., Ni Shuilleabhain, A., & Takahashi, A. (Eds.). (2018a). *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Quaresma, M., Winsløw, C., Clivaz, S., da Ponte, J., Ní Shúilleabháin, A., & Takahashi, A. (Eds.). (2018b). *Mathematics lesson study around the world: Theoretical and methodological issues*. New York: Springer.

Quaresma, M., Winsløw, C., Clivaz, S., da Ponte, J. P., Ní Shúilleabháin, A., & Takahashi, A. (Eds.). (2018c). *Mathematics lesson study around the world: Theoretical and methodological issues (ICMI-13 monographs)*. Dordrecht: Springer.

Rasmussen, K. (2016). Lesson study in prospective mathematics teacher education: Didactic and paradidactic technology in the post-lesson reflection. *Journal of Mathematics Teacher Education, 19*(4), 301–324.

Reagan, T. (2000). *Non-Western educational traditions: Alternative approaches to educational thought and practice*. Mahwah: Lawrence Erlbaum Associates, Publishers.

Reys, B. J., Reys, R. E., & Chavez, O. (2004). Why mathematics textbooks matter. *Educational Leadership, 61*(5), 61–66.

Richards, J. C. (1998). *Beyond training: Perspectives on language teacher education*. New York: Cambridge University Press.

Ricks, T. E. (2003). *An investigation of reflective processes during lesson study by mathematics preservice teachers.* Unpublished master's thesis. Brigham Young University, Provo, Utah.

Ricks, T. E. (2011). Process reflection during Japanese lesson study experiences by prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education, 14*, 251–267.

Robinson, N., & Leikin, R. (2012). One teacher, two lessons: The lesson study process. *International Journal of Science and Mathematics Education, 10*(1), 139–161.

Robinson, V., Hohepa, M., & Lloyd, C. (2009). *School leadership and student outcomes: Identifying what works and why best evidence synthesis*. Auckland: New Zealand Ministry of Education.

Roditi, E. (2011). *Recherches sur les pratiques enseignantes en mathématiques: apports d'une intégration de diverses approches et perspectives*. Paris V: Université René Descartes.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). *Developing mathematics teaching: The genesis of the knowledge quartet*. Gent: Rijksuniversiteit Gent.

Rowland, T., Ruthven, K., & Hodgen, J. (2011). Knowing and identity: A situated theory of mathematics knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching*. Dordrecht: Springer Netherlands.

Ruan, C. (2012). An investigation of teacher education in the late Qing Dynasty around the promulgation of the Presented School Regulations. *Guangdong Social Sciences, 4*, 133–139.

Rui, J. F. (2015). The practical strategy of accumulating the experience on sense of quantity in mathematical experiments. *Basic Education Research, 5*, 49–51 (in Chinese).

Runesson, U. (2005). Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *The Cambridge Journal of Education, 35*(1), 69–87.

Runesson, U. (2006). What is it possible to learn? On variation as a necessary condition for learning. *Scandinavian Journal of Educational Research, 50*(4), 397–410. https://doi.org/10.1080/00313830600823753.

Runesson, U. (2015). Pedagogical and learning theories and the improvement and development of learning and lesson studies. *International Journal of Lesson and Learning Studies, 4*(3), 186–193.

Runesson, U. (2016). Pedagogical learning theories in lesson and learning studies – Revisited. *International Journal for Lesson and Learning Studies, 5*(4), 295–299.

Runesson, U., & Gustafsson, G. (2012). Sharing and developing knowledge products from Learning Study. *International Journal for Lesson and Learning Studies, 1*(3), 245–260.

Ruth, H. (2004). Keynote speech. The 49th World Assembly World Assembly Conference. Hong Kong in July 13–17 2004.

Ryan, R. M., & Deci, E. L. (2000). Self-determination theory and the facilitation of intrinsic motivation, social development, and well-being. *American Psychologist, 55*(1), 68.

Sacks, H. (1992). *Lectures on conversation*. Cambridge: Blackwell.

Sacks, H., Schegloff, E., & Jefferson, G. (1974). A simplest systematics for the organization of turn-taking for conversation. *Language, 50*(4), 693–673.

Sahin-Taskin, C. (2017). Exploring pre-service teachers' perceptions of lesson planning in primary education. *Journal of Education and Practice, 8*(12), 57–63.

Sakamoto Elementary School. (1983). *Sansuuka mondai kaiketu nouryoku no ikusei (fostering students' problem solving capacity in elementary school mathematics)*. Tokyo: Meiji Tosho.

Sandholtz, J. (2011). Preservice teachers' conceptions of effective and ineffective teaching practices. *Teacher Education Quarterly, 38*(3), 27–47. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/23479616.

Sarkar Arani, M. R. (2017). Raising the quality of teaching through Kyouzai Kenkyuu – The study of teaching materials. *International Journal for Lesson and Learning Studies, 6*(1), 10–26.

Sato, M. (2008). Philosophy on the restoration of schools: The vision, principles and activity system of the learning community. *Journal of All India Association for Educational Research, 20* (3&4), 14–26.

Sawada, D. (1999). Mathematics as problem solving: A Japanese way. *Teaching Children Mathematics, 6*(1), 54–58.

Schipper, T., Goei, S. L., de Vries, S., & van Veen, K. (2017). Professional growth in adaptive teaching competence as a result of Lesson Study. *Teaching and Teacher Education, 68*, 289–303.

Schleppenbach, M., Flevares, L. M., Sims, L. M., & Perry, M. (2007). Teachers' responses to student mistakes in Chinese and U.S. mathematics classrooms. *Elementary School Journal, 108*, 131–147.

Schmidt, M. (2010). Learning from teaching experience: Dewey's theory and preservice teachers' learning. *Journal of Research in Music Education*, *58*(2), 131–146. Retrieved from http://www.jstor.org.ezproxy.library.yorku.ca/stable/40666239

Schoenfeld, A. H. (2002). A highly interactive discourse structure. *Social Constructivist Teaching, 9*, 131–169.

Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications*. New York: Routledge.

Schoenfeld, A. H. (2013). Classroom observations in theory and practice. *ZDM, The International Journal on Mathematics Education, 45*, 607–621. https://doi.org/10.1007/s11858-012-0483-1.

Schoenfeld, A. H. (2014, November). What makes for powerful classrooms, and how can we support teachers in creating them? *Educational Researcher, 43*(8), 404–412. https://doi.org/10.3102/0013189X1455.

Schoenfeld, A. H. (2015). Thoughts on scale. *ZDM, The International Journal on Mathematics Education, 47*, 161–169. https://doi.org/10.1007/s11858-014-0662-3.

Schoenfeld, A. H. (2017). Teaching for robust understanding of essential mathematics. In T. McDougal (Ed.), *Essential mathematics for the next generation: What and how students should learn* (pp. 104–129). Tokyo: Tokyo Gagukei University.

Schoenfeld, A. H., & The Teaching for Robust Understanding Project. (2016). *The Teaching for Robust Understanding (TRU) observation guide: A tool for teachers, coaches, administrators, and professional learning communities*. Berkeley: Graduate School of Education, University of California, Berkeley. Retrieved from: http://TRU.berkeley.edu or http://map.mathshell.org/ or http://ats.berkeley.edu/

Schoenfeld, A. H., Floden, R. E., & The Algebra Teaching Study and Mathematics Assessment Project. (2014). *The TRU Math Scoring Rubric*. Berkeley/E. Lansing: Graduate School of Education, University of California, Berkeley and College of Education, Michigan State University. Retrieved from http://ats.berkeley.edu/tools.html

Schoenfeld, A. H., Floden, R. B., & The Algebra Teaching Study and Mathematics Assessment Project. (2018). *On classroom observations*. Manuscript submitted for publication.

Schoenfeld, A. H., Baldinger, E., Disston, J., Donovan, S., Dosalmas, A., Driskill, M., Fink, H., Foster, D., Haumersen, R., Lewis, C., Louie, N., Mertens, A., Murray, E., Narasimhan, L., Ortega, C., Reed, M., Ruiz, S., Sayavedra, A., Sola, T., Tran, K., Weltman, A., Wilson, D., & Zarkh, A. (in press). Learning with and from TRU: Teacher educators and the teaching for robust understanding framework. In K. Beswick (Ed.), *Handbook of mathematics teacher education, volume 4: The mathematics teacher educator as a developing professional*. Rotterdam: Sense Publishers.

Schön, D. (1983a). *The reflective practitioner: How professionals think in action*. London: Temple Smith.
Schön, D. A. (1983b). *The reflective practitioner: How professionals think in action*. New York: Basic Books.
Scribner, S. (1997). A sociocultural approach to the study of mind. In E. Toback, R. J. Flamagne, M. B. Parlee, L. M. W. Martin, & A. S. Kapelman (Eds.), *Mind and social practice: Selected writings of Sylvia Scribner* (pp. 266–280). New York: Cambridge University Press.
Scribner, S. (1999). Knowledge at work. In R. McCormick & C. Paechter (Eds.), *Learning and knowledge* (pp. 103–111). London: Paul Chapman.
Secondary Mathematical Instruction Council (SMIC), China Education Association. (2012). *Guidance for secondary school teaching research activity at city and county levels*. September 4, 2012. Retrievable at: http://www.wendangku.net/doc/7547a01dff00bed5b9f31da6.html
Seleznyov, S. (2018). Lesson study: An exploration of its translation beyond Japan. *International Journal for Lesson and Learning Studies, 7*(3), 217–229.
Selling, S. K., Garcia, N., & Ball, D. L. (2016). What does it take to develop assessments of mathematical knowledge for teaching?: Unpacking the mathematical work of teaching. *The Mathematics Enthusiast, 13*, 35–51.
Senior, R. (2006). *The experience of language teaching*. New York: Cambridge University Press.
Setagaya Elementary School Attached to Tokyo Gakugei University. (1986). *Kenkyuu Kiyou* (vol. 26). Tokyo.
Setagaya Elementary School Attached to Tokyo Gakugei University. (1987). *Kenkyuu Kiyou* (vol. 26). Tokyo.
Setagaya Elementary School Attached to Tokyo Gakugei University. (1988). *Kenkyuu Kiyou* (vol. 26). Tokyo.
Setagaya Elementary School Attached to Tokyo Gakugei University (1989). *Kenkyuu Kiyou* (vol. 26). Tokyo.
Sfard, A. (1998). On two metaphors for learning and the dangers of choosing just one. *Educational Researcher, 27*, 4–13.
Shanghai Education Committee [SHEC]. (2014). *Shanghai elementary and secondary school mathematics curriculum*. Shanghai: Shanghai Education Publisher (in Chinese).
Sheldon, E. (1871). *A manual of elementary instruction for the use of public and private schools and normal classes: Containing a graduated course of object lessons for training the senses and developing the faculties of children* (6th ed.). New York: Charles Scribner and Co..
Shen, X. J., & Tan, N. J. (2014). The cultivation of sense of quantity is inseparable from real experience: The lesson record and comments on knowing millimeter. *Hunan Education, 6*, 52–57 (in Chinese).
Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting video clips to promote mathematics teachers' discussion of student thinking. *Journal of Teacher Education, 60*(3), 213–230.
Shi, M. M., & Tan, N. J. (2014). *Improvement of sense of quantity: Only slow work can make a fine work: Lesson record and comment on grams and kilograms* (Vol. 9, pp. 50–55). Hunan Education (in Chinese).
Shifter, D., Bastable, V., & Russell, S. J. (2010). *Developing mathematical ideas*. Reston: National Council of Teachers of Mathematics.
Shimizu, Y. (1999). Aspects of mathematics teacher education in Japan: Focusing on the teachers' roles. *Journal of Mathematics Teacher Education, 2*(1), 107–116.
Shimizu, Y. (2002). Sharing a new approach to teaching mathematics with the teachers from outside the school: The role of lesson study at "fuzoku" schools. In *US-Japan cross cultural seminar on the professionalization of teachers through lesson study, Park City, Utah*, July 2002.
Shön, D. A. (1983). *The reflective practitioner*. New York: Basic Books.
Shotou-kyouiku Kenkyukai. (1904). *Kyouiku Kenkyu* [Journal of Educational Research] vol. 1. [in Japanese].

Shu, X. (1981). *Resources for modern history of education in China*. Beijing: People's Education Press. (in Chinese).

Shuilleabhain, A. N. (2016). Developing mathematics teachers' pedagogical content knowledge in lesson study: Case study findings. *International Journal for Lesson and Learning Studies, 5*(3), 212–226. https://doi.org/10.1108/IJLLS-11-2015-0036.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4–31.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Revue, 57*, 1–22.

Siegler, R., Carpenter, T., Fennell, F., Geary, D., Lewis, J., Okamoto, Y., Thompson, L., & Wray, J. (2010). *Developing effective fractions instruction for kindergarten through 8th grade: A practice guide (NCEE#2010–4039).* Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education. Retrieved from https://ies.ed.gov/ncee/wwc/pdf/practice_guides/fractions_pg_093010.pdf

Sierpinska, A. (2007). *I need the teacher to tell me if I am right or wrong*. Proceedings of the 31st conference of the international group for the psychology of mathematics education, Seoul, South Korea, July 8–13, 2007 (Vol. 1 pp. 45–64).

Silvia, E. M. (1983). A look at division with fractions. *The Arithmetic Teacher, 30*, 38–41.

Sim, L., & Walsh, D. (2009). Lesson Study with preservice teachers: Lessons from lessons. *Teaching and Teacher Education, 25*, 724–733.

Simon, M. A. (1995a). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24*, 233–254.

Simon, M. (1995b). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*, 114–145.

Simon, H. (1996). *The sciences of the artificial*. Cambridge, MA: The MIT Press.

Sinclair, N., Watson, A., Zazkis, R., & Mason, J. (2011). The structuring of personal example spaces. *The Journal of Mathematical Behavior, 30*(4), 291–303. https://doi.org/10.1016/j.jmathb.2011.04.001.

Skultety, L., Gonzálet, G., & Vargas, G. (2017). Using technology to support teachers' lesson modifications during lesson study. *Journal of Technology and Teacher Education, 25*(2), 185–213.

Smith, M. S., & Stein, M. K. (2011). *Five practices for orchestrating productive mathematics discussions*. Reston: National Council of Teachers of Mathematics.

Somekh, B. (2006). *Action research: A methodology for change and development*. Maidenhead: Open University Press/McGraw-Hill Education.

Son, J. W., & Crespo, S. (2009). Prospective teachers' reasoning and response to a student's nontraditional strategy when dividing fractions. *Journal of Mathematics Teacher Education, 12*, 235–261.

Sorenson, P. D., Newton, L. R., & Harrison, C. (2006). *The professional development of teachers through interaction with digital video*. Paper presented at the BERA annual conference 2006, Sept 2006. Location.

Souma, K. (1983). Problem-solving approach: Mathematics education and problem-solving. *Journal of Japan Society of Mathematical Education, 65*(9), 2–11 in Japanese.

Sowder, J., Sowder, L., & Nickerson, S. (2010). *Reconceptualizing mathematics for elementary school teachers*. New York: W. H. Freeman & Company.

Sowder, J., Sowder, L., & Nickerson, S. (2014). *Reconceptualizing mathematics for elementary school teachers* (2nd ed.). New York: W.H. Freeman & Company.

Speer, N. M., King, K. D., & Howell, H. (2015). Definitions of mathematical knowledge for teaching: Using these constructs in research on secondary and college mathematics teachers. *Journal of Mathematics Teacher Education, 18*, 105–122.

Spillane, J. P. (1996). School districts matter: Local educational authorities and state instructional policy. *Educational Policy, 10*(1), 63–87.

Spillane, J. P., & Thompson, C. I. (1997). Reconstructing conceptions of local capacity: The local education agency's capacity for ambitious instructional reform. *Education Evaluation and Policy Analysis, 19*(2), 185–203.

Stafford-Plummer, J. (2002). *An analysis of the influence of lesson study on preservice secondary mathematics teachers′ view of self-as mathematics expert*. Unpublished master's thesis. Brigham Young University, Provo, Utah. http://scholarsarchive.byu.edu/etd/62

Star, S. L., & Griesemer, J. R. (1989). Institutional ecology, translations, and boundary objects: Amateurs and professionals in Berkeley's Museum of Vertebrate Zoology, 1907–1939. *Social Studies of Science, 19*(3), 387–420.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: using video to improve preservice mathematics teachers' ability to notice. *Journal of Teacher Education, 11*, 107–125.

Steffe, L. P. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning, 6*(2), 129–162.

Stein, M. K., & Coburn, C. (2008). Architectures for learning: A comparative analysis of two urban school districts. *American Journal of Education, 114*, 583–626.

Stein, M. K., & D'Amico, L. (2002). Inquiry at the crossroads of policy and learning: A study of a district-wide literacy initiative. *Teachers College Record, 104*(7), 1313–1344.

Stein, M. K., & Lane, S. (1996). Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. *Educational Research and Evaluation, 2*, 50–80.

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10*, 313–340.

Steinbring, H. (1998). Elements of epistemological knowledge for mathematics teachers. *Journal of Mathematics Teacher Education, 1*(2), 157–189.

Steinbring, H. (2005). *The construction of new mathematical knowledge in classroom interaction: An epistemological perspective*. New York: Springer.

Steinbring, H. (2006). What makes a sign a mathematical sign?–An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics, 61*(1–2), 133–162.

Stenhouse, L. (1968). The humanities curriculum project. *Journal of Curriculum Studies, 1*, 26–33.

Stenhouse, L. (1975). *An introduction to curriculum development and research*. London/New York: Heinemann.

Stenhouse, L. (1981). What counts as research? *British Journal of Educational Studies, 29*(2), 103–114.

Stenhouse, L. (1985). *Research as a basis for teaching: Readings from the work of Lawrence Stenhouse*. London/Portsmouth: Heinemann Educational Books.

Stetsenko, A. (2005). Activity as object-related: Resolving the dichotomy of individual and collective planes of activity. *Mind, Culture, and Activity, 12*(1), 70–88.

Stiegler, J. W., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM – The International Journal of Mathematics Education, 48*(4), 581–587.

Stigler, J., & Hiebert, J. (1997). Understanding and improving classroom mathematics instruction: An overview of the TIMSS video study. *Phi Delta Kappan, 79*(1), 14–21.

Stigler, J., & Hiebert, J. (1999a). *The teaching gap*. New York: Free Press.

Stigler, J. W., & Hiebert, J. (1999b). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press, A Division of Simon & Schuster Inc..

Stigler, J., & Hiebert, J. (2009). Closing the teaching gap. *Phi Delta Kappan, 91*(3), 32–37.

Stigler, J., & Hiebert, J. (2016). Lesson study, improvement, and the importing of cultural routines. *ZDM Mathematics Education, 48*(4), 581–587.

Stigler, J., & Perry, M. (1988). Cross cultural studies of mathematics teaching and learning recent findings and new direction. In D. A. Grouws, T. J. Cooney, & D. Jones (Eds.), *Perspectives on research on effective mathematics teaching* (pp. 194–223). Reston: The National Council of Teacher of Mathematics.

Stigler, J., Thompson, B., & Ji, X. (2012). This book speaks to us. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 223–231). New York: Routledge.
Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht: Kluwer.
Suh, J., & Seshaiyer, P. (2014). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. *Journal of Mathematics Teacher Education, 18*, 1–23.
Suh, J., & Seshaiyer, P. (2015). Examining teachers' understanding of the mathematical learning progression through vertical articulation during lesson study. *Journal of Mathematics Teacher Education, 18*, 207–229.
Suh, J. M., Birkhead, S., Baker, C., Frank, T., & Seshaiyer, P. (2017). Leveraging coach-facilitated professional development to create teacher social networks for enhancing professional practice. In M. Boston & L. West (Eds.), *Annual perspectives in mathematics education: Reflective and collaborative processes to improve mathematics teaching* (pp. 89–100). Reston: National Council of Teachers of Mathematics.
Sun, C. R., Dong, L. W., & Zhu, G. F. (2016). Measurement framework construction and strategy analysis of mathematics experiment conception in middle school. *Curriculum, Teaching Materials, and Methods, 36*(7), 90–95 (in Chinese).
Suo, Q. (1953, March–April). The experiences of the mathematics teaching research group at Anyang no.1 secondary school in organizing demonstration teaching. *Bulletin of Mathematics*, 37–42 (in Chinese).
Suratno, T. (2012). Lesson study in Indonesia: An Indonesia University of education experience. *International Journal for Lesson and Learning Studies, 1*(3), 196–215.
Suratno, T., & Iskandar, S. (2010). Teacher reflection in Indonesia: Lessons learnt from a lesson study program. *US-China Education Review, 7*(12), 39–48.
Sutton, J. T., Burroughs, E. A., & Yopp, D. A. (2011). Coaching knowledge: Domains and definitions. *Journal of Mathematics Education Leadership, 13*(2), 12–20.
Sykes, G., O'Day, J., & Ford, T. (2009). The district role in instructional improvement. In G. Sykes, B. Schneider, & D. N. Plank (Eds.), *Handbook of education policy research* (pp. 767–784). New York/London: Routledge.
Sztajn, P., Confrey, J., Wilson, P. H., & Edgington, C. (2012). Learning trajectory based instruction: Toward a theory of teaching. *Educational Researcher, 41*, 147–156.
Takahashi, A. (2000). Current trends and issues in lesson study in Japan and the United States. *Journal of Japan Society of Mathematical Education, 82*(12), 15–21.
Takahashi, A. (2006a). *Characteristics of Japanese mathematics lessons*. Retrieved from http://www.criced.tsukuba.ac.jp/math/sympo_2006/takahashi.pdf
Takahashi, A. (2006b). Types of elementary mathematics Lesson Study in Japan: Analysis of features and characteristics (in Japanese). *Journal of Japan Society of Mathematical Education, Mathematics Education, 88*(8), 2–14.
Takahashi, A. (2006c, June). *Implementing lesson study in North American schools and school districts*. Paper presented at Innovation and good practices for teaching and learning Mathematics through Lesson Study at the APEC International symposium, Khon Kaen, Thailand. Retrieved from https://www.apec.org/Publications/2006/06/A-Collaborative-Study-on-Innovations-for-Teaching-and-Learning-Mathematics-in-Different-Cultures-amo
Takahashi, A. (2008). *Beyond show and tell: Neriage for teaching through problem-solving–Ideas from Japanese problem-solving approaches for teaching mathematics.* Paper presented at the 11th International Congress on Mathematics Education in Mexico (Section TSG 19: Research and Development in Problem Solving in Mathematics Education), Monterrey, Mexico.
Takahashi, A. (2013). *Investigation of the mechanism of Lesson Study as the core of the mathematics teacher professional development* (in Japanese). In Proceedings of first annual Spring conference of Japan Society of Mathematical Education (pp. 83–87).

Takahashi, A. (2014a). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 4–21.

Takahashi, A. (2014b). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), *1–17*.

Takahashi, A. (2014c). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*(1), 83–97.

Takahashi, A. (2014e). Supporting the effective implementation of a new mathematics curriculum: A case study of school-based lesson study at a Japanese public elementary school. In I. Y. Li & G. Lappan (Eds.), *Mathematics curriculum in school education* (pp. 417–441). New York: Springer.

Takahashi, A. (2015). Personal communication, July 1, 2015.

Takahashi, A., & McDougal, T. (2014). Implementing a new national curriculum: A Japanese public school's two-year lesson-study project. In A. R. McDuffie & K. S. Karp (Eds.), *Annual perspectives in mathematics education (APME): Using research to improve instruction* (pp. 13–21). Reston: National Council of Teachers of Mathematics.

Takahashi, A., & McDougal, T. (2016). Collaborative lesson research: Maximizing the impact of lesson study. *ZDM: The International Journal on Mathematics Education, 48*(4), 513–526.

Takahashi, A., & Mcdougal, T. (2018). Collaborative lesson research (CLR). In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Da Ponte, A. Ni Shuilleabhain, A. Takahashi, & T. Fujii (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Takahashi, A., & Yoshida, M. (2004a). Ideas for establishing lesson-study communities. *Teaching Children Mathematics, 10*(9), 436–443.

Takahashi, A., & Yoshida, M. (2004b). How can we start lesson study?: Ideas for establishing lesson study communities. *Teaching Children Mathematics, 10*(9), 436–443.

Takahashi, A., Watanabe, T., Yoshida, M., & Wang-Iverson, P. (2005a). Improving content and pedagogical knowledge through kyozaikenkyu. In P. Wang-Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 77–84). Philadelphia: Research for Better Schools.

Takahashi, A., Watanabe, T., Yoshida, M., & Wand-Iverson, P. (2005b). Improving content and pedagogical knowledge through kyozaikenkyu. In P. Wang-Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 101–110). Philadelphia: Research for Better Schools.

Takahashi, A., Watanabe, T., Yoshida, M., & Wang-Iverson, P. (2005c). Improving content and pedagogical knowledge through kyozaikenkyu. In *Building our understanding of lesson study* (pp. 101–110). Philadelphia, PA: Research for Better Schools, Inc..

Takayanagi, M. (1982). Gakushuuji no hyouka: Bunsuu no warizan (assessment during a lesson: Division of fractions). *Atarashii Sansuu Kenkyuu, 134*, 54–56.

Tan, D. L., & Zhu, J. M. (2016). Teaching evaluation for middle school mathematics experiment. *Curriculum, Teaching Materials, and Methods, 36*(8), 108–113 (in Chinese).

Tang, Y. (2014). Rethinking the Chinese education: A perspective of life-oriented education. *Shanghai Research on Education, 325*, 15–19.

Tashevska, S. (2008). *Some lesson planning problems for new teachers of English*. CELTA Syllabus and Assessment Guidelines. Retrieved from http://www.cambridge.efl.org_teaching

Tatto, M. T., Schwille, J., Senk, S., Ingvarson, L., Peck, R., & Rowley, G. (2008). *Teacher education and development study in mathematics (TEDS-M): Policy, practice, and readiness to teach primary and secondary mathematics. Conceptual framework*. East Lansing: Teacher Education and Development Inter- national Study Center, College of Education, Michigan State University.

Tepylo, D. H., & Moss, J. (2011a). Examining change in teacher mathematical knowledge through lesson study. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education: Learning together* (pp. 59–77). New York: Springer.

Tepylo, D. H., & Moss, J. (2011b). Examining change in teacher mathematical knowledge through lesson study. In L. C. Hart, A. S. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education*. Dordrecht: Springer Netherlands.

Thompson, P. (2013). Learner-centered education and "cultural translation". *International Journal of Educational Development, 33*, 48–58.

Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education, 31*, 5–25.

Tirosh, D., Fishbein, E., Graeber, A., & Wilson, J.W. (1998). *Prospective elementary teachers' conceptions of rational numbers*. Retrieved from http://jwilson.coe.uga.edu/texts.folder/tirosh/pros.el.tchrs.html

Tokyo Shoseki (n.d.) *Atarashii sansu* (New elementary school mathematics). Tokyo: Tokyo Shoseki Publishing.

Toulmin, S. (1958). *The uses of argument*. Cambridge: Cambridge University Press.

Travers, K. J., & Westbury, I. (1989). *The IEA study of mathematics I: Analysis of mathematics curricula*. New York: Pergamon Press.

Truxaw, M. P. (2005). *Orchestrating whole group discourse to mediate mathematical meaning*. DigitalCommons@UConn.

Tsubota, K. (2004). *Sansu jugyo kenkyu saikou (re-thinking lesson study in elementary school mathematics)*. Tokyo: Toyokan Shuppan.

Tsui, A. B. M., & Law, D. Y. K. (2007). Learning as boundary-crossing in school–university partnership. *Teaching and Teacher Education, 23*(8), 1289–1301.

Tsukamoto, A. (1869). *Hisan kunmou* [Enlightenment on written calculation]. Tokyo: Numazu Gakkou. [in Japanese].

Tyutou-kyouiku Kenkyukai. (1908). *Tyutou Kyouiku* [Journal of Secondary Education] vol. 1. [in Japanese].

U.S. Department of Education. (2009). *Race to the top program: Executive summary*. Washington, DC: Author.

U.S. Department of Education. (n.d.). *School improvement grants*. Retrieved from http://www2.ed.gov/programs/sif/index.html

Unal, H., & Jakubowski, E. (2007). Middle and secondary preservice mathematics teachers' comparative analysis of TIMSS videotape lesson study. *Turkish Online Journal of Educational Technology – TOJET, 6*(3), 61–69.

Valli, L. (1997). Listening to other voices: A description of teacher reflection in the United States. *Peabody Journal of Education, 72*(1), 67–88.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2004). *Elementary and middle school mathematics*. Boston: Allyn and Bacon.

Van de Walle, J. A., Karp, K. S., & Bay-Williams, J. M. (2009). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). Boston: Allyn & Bacon.

Van de Walle, J., Karp, K. S., & Bay-Williams, J. M. (2016). *Elementary and middle school mathematics: Teaching developmentally (9 editions)*. Boston: Person Education.

Van den Heuvel-Panhuizen, M. (2000). *Mathematics education in the Netherlands: A guided tour.* Freudenthal Institute Cd-rom for ICME9. Utrecht: Utrecht University.

Van den Heuvel-Panhuizen, M., & Treffers, A. (2009). Mathe-didactical reflections on youngchildren's understanding and application of subtraction-related principles. *Mathematical Thinking and Learning, 11*(1–2), 102–112.

van Gog, T., Ericsson, K. A., Rikers, R. M. J. P., & Paas, F. (2005). Instructional design for advanced learners: Establishing connections between the theoretical frameworks of cognitive load and deliberate practice. *Educational Technology Research and Development, 53*(3), 73–81.

Van Manen, M. (1977). Linking ways of knowing with ways of being practical. *Curriculum Inquiry, 6*(3), 205–228.

Vdovina, E., & Gaibisso, L. C. (2013). Developing critical thinking in the English language classroom: A lesson plan. *ELTA Journal, 1*(1), 54–68.

Venkat, H., & Adler, J. (2012). Coherence and connections in teachers' mathematical discourses in instruction. *Pythagoras, 33*(3), 25–32. https://doi.org/10.4102/pythagoras.v33i3.188.

Verhoef, N., Tall, D., Coenders, F., & van Smaalen, D. (2014). The complexities of a lesson study in a Dutch situation: Mathematics teacher learning. *International Journal of Science and Mathematics Education, 12*(4), 859–881.

Vermunt, J. D. (2013, 22 May). *Teacher learning and student learning: are they related?* Inaugural lecture given at the University of Cambridge, Faculty of Education.

Vermunt, J. D., & Endedijk, M. D. (2011). Patterns in teacher learning in different phases of the professional career. *Learning and Individual Differences, 21*(3), 294–302.

Vermunt, J. D., & Vermetten, Y. J. (2004). Patterns in student learning: Relationships between learning strategies, conceptions of learning, and learning orientations. *Educational Psychology Review, 16*(4), 359–384.

Vermunt, J. D., Vrikki, M., Mercer, N., & Warwick, P. (2015). *UK teachers' perceptions of Lesson Study and its effects on teacher learning: a survey study*. Paper presented at the 16th conference of the European association of research on learning and instruction, Limassol, Cyprus.

Vermunt, J. D., Vrikki, M., Warwick, P., & Mercer, N. (2017). Connecting teacher identity formation to patterns in teacher learning. In D. J. Clandinin & J. Husu (Eds.), *The SAGE handbook of research on teacher education* (pp. 143–159). London: SAGE.

Vikström, A., Billström, A., Fazeli, P., Holm, M., Jonsson, K., Karlsson, G., & Rydström, P. (2013). Teachers' solutions: A learning study about solution chemistry in Grade 8. *International Journal for Lesson and Learning Studies, 2*(1), 26–40.

Virkkunen, J., & Newnham, D. S. (2013). *The change laboratory: A tool for collaborative development of work and education*. Rotterdam: Sense.

Vrikki, M., Warwick, P., Vermunt, J. D., Mercer, N., & Van Halem, N. (2017). Teacher learning in the context of lesson study: A video-based analysis of teacher discussions. *Teaching and Teacher Education, 61*, 211–224.

Vygotsky, L. (1962). *Thought and language*. Cambridge, MA: MIT.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

Wakabayashi, T., & Shirai, T. (eds.) (1883). *Kaisei kyojujutsu* [Revision of teaching methods]. Tokyo: Fukyu-Sya. [in Japanese].

Wake, G., Swan, M., & Foster, C. (2016). Professional learning through the collaborative design of problem solving lessons. *Journal of Mathematics Teacher Education, 19*, 243–260.

Wang, B. (1997). Teacher preparation and teacher education in China: Commemorating the 100th anniversary of teacher education in China. *Teacher Education Research, 54*, 3–9. (in Chinese).

Wang, Q. (2008). *Instructional skills of mathematics*. Shanghai: East China Normal University Press (in Chinese).

Wang, J. (2013a). *Mathematics education in China: Tradition and reality*. Singapore: Galeasia Cengage Learning.

Wang, M. (2013b). A preliminary examination of teachers' teaching research during the Republic of China era. *Journal of East China Normal University (Educational Sciences), 31*(1), 89–95 (in Chinese).

Wang, R., & Gao, J. (2012a). Lesson study: Domestic experiences and multiple forms (Part 1). *Research on Educational Development, 8*, 31–36 (in Chinese).

Wang, R., & Gao, J. (2012b). Lesson study: Domestic experiences and multiple forms (Part 2). *Research on Educational Development, 10*, 44–49 (in Chinese).

Wang-Iverson, P., & Yoshida, M. (2005). *Building our understanding of lesson study*. Philadelphia: Research for Better Schools.

Ward, J. R., & McCotter, S. S. (2004). Reflection as a visible outcome for preservice teachers. *Teaching and Teacher Education, 20*(3), 243–257.

Warfield, V. (2014). *Invitation to didactique*. New York: Springer.

Warwick, P., Vrikki, M., Vermunt, J. D., Mercer, N., & Van Halem, N. (2016a). Connecting observations of student and teacher learning: An examination of dialogic processes in lesson study discussions in mathematics. *ZDM Mathematics Education, 44*, 1–15.

Warwick, P., Vrikki, M., Vermunt, J. D., Mercer, N., & van Halem, N. (2016b). Connecting observations of student and teacher learning: An examination of dialogic processes in lesson study discussions in mathematics. *ZDM – The International Journal of Mathematics Education, 48*(4), 555–569.

Warwick, P., Vrikki, M., Færøyvik Karlsen, A-M., Dudley, P., & Vermunt, J. D. (2018). '*Instead of numbers, I would make it decimals': The role of pupil voice in teacher learning in Lesson Study*. Under review.

Watanabe, T. (2002a). Learning from Japanese lesson study. *Educational Leadership, 59*(6), 36–39.

Watanabe, T. (2002b). The role of outside experts in lesson study. In C. Lewis (Ed.), *Lesson study: A handbook of teacher-led instuctional imporvement*. Philadelphia: Research for Better Schools.

Watanabe, T. (2014). Transformation of Japanese elementary mathematics textbooks: 1958–2012. In Y. Li, E. A. Silver, & S. Li (Eds.), *Transforming mathematics instruction: Multiple approaches and practices* (pp. 199–216). Cham: Springer.

Watanabe, T., & Wang-Iverson, P. (2005). The role of knowledgeable others. In P. Wang-Iverson & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 85–91). Philadelphia: Research for Better Schools.

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh & P. M. Taylor (Eds.), *Inquiry into mathematics teacher education* (*Association of Mathematics Teacher Educators (AMTE) Monograph Series*) (Vol. 5, pp. 131–142). San Diego: AMTE.

Watson, A. (2017). Pedagogy of variations: Synthesis of various notions of variation pedagogy. In R. Huang & Y. Li (Eds.), *Teaching and learning mathematics through variation: Confucian heritage meets western theories* (pp. 85–103). Boston: Sense Publishers.

Watson, A., & Mason, J. (2006). Seeing an exercise as a single mathematical object: Using variation to structure sense making. *Mathematical Thinking and Learning, 8*(2), 91–111.

Webb, M. M. (2006). *What does it mean to preservice mathematics teachers to anticipate student responses?* Unpublished master's thesis. Brigham Young University, Provo, Utah. http://scholarsarchive.byu.edu/etd/1122

Webster-Wright, A. (2009). Reframing professional development through understanding authentic professional learning. *Review of Educational Research, 79*(2), 702–739.

Wegerif, R. (2007). *Dialogic education and technology: Expanding the space for learning*. New York: Springer.

Wei, G. (2017a). Review of learning by expanding: An activity-theoretical approach to developmental research. *Frontiers of Education in China, 12*(1), 130–132.

Wei, G (2017b). *Dynamics of teacher practical knowledge: From cultural-historical activity theoretical perspective* (Unpublished Doctoral Dissertation), Peking University.

Wei, R. C., Darling-Hammond, L., Andree, A., Richardson, N., & Orphanos, S. (2009). *Professional learning in the learning profession.* A status report on teacher professional development in the United States and abroad. Technical report. Dallas: National Staff Development Council.

Wells, G. (1999). *Dialogic inquiry: Toward a sociocultural practice and theory of education*. New York: Cambridge University Press.

Wells, G. (2002). The role of dialogue in activity theory. *Mind, Culture, and Activity, 9*(1), 43–66.

Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. New York: Cambridge University Press.

Wertsch, J. V. (1985). *Vygotsky and the social formation of mind*. Cambridge, MA: Harvard University Press.

West, B. T. (2009). Analyzing longitudinal data with the linear mixed models procedure in SPSS. *Evaluation & the Health Professions, 32*(3), 207–228.

Westenskow, A., & Moyer-Packenham, P. (2016). Using an iceberg intervention model to understand equivalent fraction learning when students with mathematical learning difficulties using different manipulatives. *International Journal for Technology in Mathematics Education, 23*(2), 45–62.

White, A. L., & Lim, C. S. (2008). Lesson study in Asia Pacific classrooms: Local responses to a global movement. *ZDM – The International Journal on Mathematics Education, 40*(6), 915–925.

White, P., Wilson, S., & Mitchelmore, M. (2011). Teaching for abstraction: Collaborative teacher learning. In J. Dindyal, L. P. Cheng, & S. F. Ng (Eds.), *Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 761–768). Singapore: MERGA.

Widjaja, W., Vale, C., Groves, S., & Doig, B. (2017). Teachers' professional growth through engagement with lesson study. *Journal of Mathematics Teacher Education, 20*(4), 357–383.

Wiliam, D. (2011). *Embedded formative assessment*. Bloomington: Solution Tree Press.

Wiliam, D., & Leahy, S. (2015). *Embedding formative assessment: Practical techniques for K – 12 classrooms*. West Palm Beach: Learning Sciences International.

Wilkie, K. J. (2016). Learning to teach upper primary school algebra: Changes to teachers' mathematical knowledge for teaching functional thinking. *Mathematics Education Research Journal, 28*(2), 245–275.

Wilson, S. M., & Berne, J. (1999). Teacher learning and the acquisition of professional knowledge: An examination of research on contemporary professional development. In A. Iran-Nejad & P. D. Pearson (Eds.), *Review of research in education* (Vol. 24, pp. 173–209). Washington, DC: AERA.

Wilson, P. H., Sztajn, P., Edgington, C., & Myers, M. (2015). Teachers' uses of a learning trajectory in student-centered instructional practices. *Journal of Teacher Education, 66*, 227–244.

Winsløw, C., Bahn, J., & Rasmussen, K. (2018a). Theorizing lesson study: Two related frameworks and two Danish case studies. In M. Quaresma, C. Winsløw, S. Clivaz, J. da Ponte, A. N. Shúilleabháin, & A. Takahashi (Eds.), *Mathematics lesson study around the world. ICME-13 monographs*. Cham: Springer.

Winsløw, C., Bahn, J., & Rasmussen, K. (2018b). Theorizing lesson study: Two related frameworks and two Danish case-studies. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. Da Ponte, A. Ni Shuilleabhain, A. Takahashi, & T. Fujii (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues*. Cham: Springer.

Wong, M., & Evans, D. (2007). Students' conceptual understanding of equivalent fractions. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice (Proceedings of the 30th annual conference of the Mathematics Education Group of Australasia)* (pp. 824–833). Adelaide: MERGA.

Wood, K. (2017). Is there really any difference between lesson and learning study? Both focus on neriage. *International Journal for Lesson and Learning Studies, 6*(2), 118–123. https://doi.org/10.1108/IJLLS-02-2017-0008.

Wood, K. (2018a). The many faces of lesson study and learning study. *International Journal for Lesson and Learning Studies, 7*(1), 2–7. https://doi.org/10.1108/IJLLS-10-2017-0047.

Wood, K. (2018b). *What and how do teachers learn by taking part in lesson study?* Presentation to European Association for Research on Learning and Instruction (EARLI) Special Interest Group 9: Phenomenography and Variation Theory, Birmingham University, UK, 16–18 September.

Woodbury, S., & Gess-Newsome, J. (2002). Overcoming the paradox of change without difference: A model of change in the arena of fundamental school reform. *Educational Policy, 16*, 763–782.

Wu, X., & Zheng, Y. (2012). Exploring students' mathematical competences based on the new curriculum. *China Teaching Journal, 4*, 52–55 (in Chinese).

Wylie, C., & Wiliam, D. (2007). *Analyzing diagnostic items: What makes a student response interpretable?* Paper presented at Annual Meeting of the National Council on Measurement in Education (NCME), April 2006, Chicago. http://www.dylanwiliam.org/Dylan_Wiliams_website/Papers.html

Xu, H., & Pedder, D. (2014). Lesson Study. An international review of research. In P. Dudley (Ed.), *Lesson study: Professional learning for our time* (pp. 29–58). London: Routledge.
Xu, H., & Pedder, D. (2015a). Lesson study: An international review of the research. In P. Dudley (Ed.), *Lesson study: Professional learning for our time*. London/New York: Routledge.
Xu, H., & Pedder, D. (2015b). Lesson study: An international review of the research. In P. Dudley (Ed.), *Lesson study: Professional learning of our time* (pp. 29–58). Abingdon: Routledge.
Yamagata-Lynch, L. (2010). *Activity systems analysis methods*. New York: Springer.
Yan, G. (2003). An analysis of teacher education in modern Chinese compulsory education. *Teacher Education Research, 15*(1), 61–66 (in Chinese).
Yan, L. (2014). *Research on pre-service mathematics teachers' ability to analyze and deal with students' errors* (Unpublished master thesis). Xian: Shanxi Normal university (in Chinese).
Yang, Y. (2008). How should teachers conduct lesson study. *Research on Educational Development, 8*, 72–82 (in Chinese).
Yang, Y. (2009). How a Chinese teacher improved classroom teaching in teaching research group: A case study on Pythagoras theorem teaching in Shanghai. *ZDM— The International Journal on Mathematics Education, 41*(3), 279–296.
Yang, Y., & Ricks, T. E. (2011). How crucial incidents analysis support Chinese lesson study. *International Journal for Lesson and Learning Studies, 1*(1), 41–48.
Yang, Y., & Ricks, T. E. (2012). How crucial incidents analysis support Chinese lesson study. *International Journal for Lesson and Learning Studies, 1*(1), 41–48.
Yang, Y., & Ricks, T. E. (2013). Chinese Lesson Study: Developing classroom instruction through collaborations in school-based teaching research group activities. In Y. Li & R. Huang (Eds.), *How Chinese teach mathematics and improve teaching* (pp. 51–65). New York: Routledge.
Yang, Y., Li, J., Gao, H., & Xu, Q. (2012). Teacher education and the professional development of mathematics teachers. In J. P. Wang (Ed.), *Mathematics education in China: Tradition and reality* (pp. 205–238). Singapore: Cengage Learning Asia.
Yildiz, A., & Baltaci, S. (2017). Reflections from the lesson study for the development of techno-pedagogical competencies in teaching fractal geometry. *European Journal of Educational Research, 6*(1), 41–50.
Yin, R. K. (2009). *Case study research: Design and methods*. Thousand Oaks: Sage.
Yin, R. K. (2017). *Case study research and applications: Design and methods*. Thousand Oaks: Sage.
Ylonen, A., Dudley, P., & Lang, J. (2015). *London schools excellence fund new curriculum higher order mathematics lesson study programme final report*. London: Greater London Authority.
Yokosuka, K. (1990). *Jugyo kenkyu yougo jiten (dictionary of lesson study terms)*. Tokyo: Kyouiku Shuppan.
Yoshida, M. (1999a). *Lesson study: An ethnographic investigation of school-based teacher development in Japan*. Doctoral dissertation, University of Chicago.
Yoshida, M. (1999b). *Lesson Study: A case study of a Japanese approach to improving instruction through school-based teacher development.* Unpublished doctoral dissertation, University of Chicago, Department of Education.
Yoshida, M. (2002). *Lesson study: An introduction*. Madison: Global Education Resources.
Yoshida, M. (2012). Mathematics lesson study in the United States: Current status and ideas for conducting high quality and effective lesson study. *International Journal for Lesson and Learning Studies, 1*(2), 140–152.
Yoshida, M., & Jackson, W. C. (2011). Response to part V: Ideas for developing mathematical pedagogical content knowledge through Lesson Study. In L. C. Hart, A. Alston, & A. Murata (Eds.), *Lesson study research and practice in mathematics education*. New York: Springer.
Youchu, H. (2016). A qualitative study on the development of pre-service teachers' mathematical knowledge for teaching in a history-based course. *EURASIA Journal of Mathematics, Science & Technology Education, 12*(9), 2599–2616.
Yu, Z. Q. (2017). What is more important than the goal of mathematics classroom teaching in elementary school? *People's Education, 27*, 55–58 (in Chinese).

Yu, P., & Dong, L. W. (2016). Analysis of nature of mathematics experiment in middle school. *Curriculum, Teaching Materials, and Methods, 36*(8), 89–95 (in Chinese).

Yuan, Z., & Li, X. (2015). "Same content different lesson constructs" activities and their impact on prospective mathematics teachers' professional development – A case study of Nadine. In L. Fan, N. Wong, J. Cai, & S. Li (Eds.), *How Chinese teach mathematics – Perspectives from insiders* (pp. 565–588). Singapore: World Scientific.

Zaslavsky. (2008). Meeting the challenges of mathematics teacher education through design and use of tasks that facilitate teacher learning. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teacher education: The mathematics teacher educator as a developing professional* (Vol. 4, pp. 93–114). Rotterdam: Sense.

Zaslavsky, O. (in press). There is more to examples than meets the eye: Thinking with and through mathematical examples in different settings. *The Journal of Mathematical Behavior*. https://doi.org/10.1016/j.jmathb.2017.10.001.

Zeichner, K. (2010). Rethinking the connections between campus courses and field experiences in college and university-based teacher education. *Journal of Teacher Education, 89*(11), 89–99.

Zhao, C. (2008). *Efficacious teaching research: The introduction of teaching research in Chinese basic education*. Shanghai: Shanghai Education Press.

Zhao, Y. (2014). *Who's afraid of the big bad dragon? Why China has the best (and worst) education system in the world*. New York: Jossey-Bass.

Zhao, W. K., & Zhang, J. Y. (2016). Teaching design of mathematics experiment in middle school. *Curriculum, Teaching Materials, and Methods, 36*(8), 102–107 (in Chinese).

Zhou, G. R. (2014). *Survey and analysis of Grade 6 students' sense of quantity*. Unpublished master thesis. Chongqing: Southwest University (in Chinese).

Zhou, Z., & Lin, J. (2001). Developing mathematical thinking in Chinese kindergarten children the case of addition and subtraction. *Journal of Educational Policy, Research, and Practice, 2*, 141–145.

Zhu, W. (1992). Confucius and traditional Chinese education: An assessment. In R. Hayhoe (Ed.), *Education and modernization: The Chinese experience* (pp. 3–22). New York: Pergamon Press.

Zodik, I., & Zaslavsky, O. (2008). Characteristics of teachers' choice of examples in and for the mathematics classroom. *Educational Studies in Mathematics, 69*(2), 165–182. https://doi.org/10.1007/s10649-008-9140-6.

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