Chapter 5
Design for Reasoning with Uncertainty

Hana Manor Braham and Dani Ben-Zvi

Abstract The uncertainty involved in drawing conclusions based on a single sample is at the heart of informal statistical inference. Given only the sample evidence, there is always uncertainty regarding the true state of the situation. An "Integrated Modelling Approach" (IMA) was developed and implemented to help students understand the relationship between sample and population in an authentic context. This chapter focuses on the design of one activity in the IMA learning trajectory that aspires to assist students to reason with the uncertainty involved in drawing conclusions from a single sample to a population. It describes design principles and insights arising from the implementation of the activity with two students (age 12, grade 6). Implications for research and practice are also discussed.

Keywords Informal statistical inference · Model and modeling
Sample and population · Statistics education · Uncertainty

5.1 Introduction

Data are everywhere and drawing inferences from data is part of daily life. Every student must therefore have a sense of the potential in drawing reliable statistical inferences from samples, appreciate the purpose of such activity, and deal with the complexities of an uncertain world. However, studies indicate that students can hold contradictory views regarding the relationships between samples and their population (Pfannkuch 2008) and respond in a deterministic way while reasoning about data (Ben-Zvi et al. 2012).

This study is part of the *Connections* Project (2005–2020)—a longitudinal design-based research (Cobb et al. 2003) that studies children's statistical reasoning in an

H. M. Braham (✉) · D. Ben-Zvi
Faculty of Education, The University of Haifa, Haifa, Israel
e-mail: hana.manor@gmail.com

D. Ben-Zvi
e-mail: dbenzvi@univ.haifa.ac.il

G. Burrill and D. Ben-Zvi (eds.), *Topics and Trends in Current Statistics Education Research*, ICME-13 Monographs, https://doi.org/10.1007/978-3-030-03472-6_5

inquiry-based and technology-enhanced statistics learning environment for grades 4–9 (Ben-Zvi et al. 2007). The purpose of this chapter is to present the *Integrated Modelling Approach* (IMA) —a pedagogic design approach that intends to help students understand the relationships between samples and populations. More specifically, it describes the design principles of a sixth-grade activity in the IMA learning trajectory and how they contributed to the progression of students' reasoning with uncertainty.

This chapter begins by describing the instructional design principles of the Connections statistical reasoning learning environment that forms the basis of the IMA learning trajectory. We then address the challenge of facilitating young students' reasoning with uncertainty while they are involved in making informal statistical inferences (ISI) . We describe how the IMA shaped the design of the experimental learning trajectory, provide a detailed description of one activity and illustrate its impact by describing the progression in reasoning with uncertainty of a pair of sixth grade students. This example shows how the students invented methods to face the uncertainty involved in making informal inferences from a sample to a population. Finally, we discuss the challenges in designing activities that foster students' abilities to envision a process of repeated samples (Shaughnessy 2007; Thompson et al. 2007).

We argue that even relatively young students are able to make sense of complex ideas that form the basis of ISIs, such as, uncertainty and the relationship between data and chance. Furthermore, fostering students' exploration of two types of uncertainties (contextual and statistical uncertainty) and the connections between them may facilitate students' understanding of the relationship between a process of repeated samples and a single sample in the inference process.

5.2 Scientific Background

We start this section by describing the design principles of our approach followed by the core statistical ideas of this study—uncertainty in informal statistical inference. Based on these foundations we present the *Integrated Modelling Approach* for supporting the reasoning with sample-population relationships.

5.2.1 Design Principles

Current theories of learning suggest that under certain conditions students who are engaged in carefully designed learning environments may become motivated to construct knowledge from the learning process (Ben-Zvi et al. 2018; Greeno and Engeström 2014). Statistics educators and researchers have recommended the implementation and use of certain statistical learning environments to support the development of students' statistical reasoning. Garfield and Ben-Zvi (2009) pointed out several principles of an effective statistical reasoning learning environment (SRLE) to develop students' statistical reasoning. For our study, we adopted four of those principles: Focus on key statistical ideas, use real and motivating data, use inquiry-based activities to develop students' statistical reasoning, and integrate the use of appropriate technological tools.

Focusing on key statistical ideas (such as, distribution, center, variability, uncertainty, and sampling) can stimulate students to encounter them in different contexts and create various representations that illustrate their interrelationships (Garfield and Ben-Zvi 2008). Making connections between existing context knowledge and the results of data analysis can help students develop understanding of key statistical ideas (Wild and Pfannkuch 1999). Using real and motivating data (Edelson and Reiser 2006) through exploratory data analysis (EDA) activities (Pfannkuch and Wild 2004) can help students formulate research questions and conjectures about their explored phenomenon, examine evidence from data in relation to their contextual conjectures, and become critical thinkers in making inferences. Collecting real and authentic data can make the investigation more relevant for students (Herrington and Oliver 2000). Using dynamic visual displays as analytical tools with appropriate technological tools (Garfield et al. 2000) can involve students in the organization, description, interpretation, representation, analysis and creation of inferences of data situations (Ben-Zvi and Arcavi 2001; Ben-Zvi 2006).

5.2.2 Uncertainty in Informal Statistical Inference

We first discuss the nature of reasoning with uncertainty in the context of making informal statistical inferences and then consider the challenge of facilitating students' reasoning with uncertainty.

5.2.2.1 Reasoning with Uncertainty

"Statistical inference moves beyond the data in hand to draw conclusions about some wider universe, taking into account that variation is everywhere and that conclusions are uncertain" (Moore 2007, p. xxviii). Given only sample evidence, the statistician is always unsure of any assertion he makes about the true state of the situation. The theory of statistical inference provides ways to assess this uncertainty and calculate the probability of error.

Students, even at a relatively young age, should have a sense of the power and purpose in drawing reliable statistical inferences from samples. Given that statistical inference is challenging for most students (Garfield and Ben-Zvi 2008), Informal Statistical Inference (ISI) and Informal Inferential Reasoning (IIR) became a recent focus of research (Pratt and Ainley 2008; Makar et al. 2011). ISI is a data-based generalization that includes consideration of uncertainty and does not involve formal procedures (Makar and Rubin 2009, 2018). IIR is the reasoning process that leads to the formulation of ISIs that includes "the cognitive activities involved in informally drawing conclusions or making predictions about 'some wider universe' from patterns, representations, statistical measures, and statistical models of random samples, while attending to the strength and limitations of the sampling and the drawn inferences" (Ben-Zvi et al. 2007, p. 2).

Uncertainty is at the heart of formal and informal statistical inference. To understand the uncertainty involved in taking a sample, one needs to envision a process of repeated sampling and its relation to the individual sample (Arnold et al. 2018; Saldanha and McAllister 2014). However, research suggests that students tend to focus on individual samples and their statistical summaries instead of the distribution of sample statistics (Saldanha and Thompson 2002).

5.2.2.2 Facilitating Students' Reasoning with Uncertainty

Given the importance of IIR, a significant question is how to facilitate young students' reasoning with uncertainty during sampling and making ISIs. The literature contains examples of two types of settings that have been frequently used to study IIR: (1) scientific inquiry learning environments in which students create surveys and are engaged in real world data inquiries to learn about a wider phenomenon (e.g., Ben-Zvi 2006; Lehrer and Romberg 1996; Makar et al. 2011; Makar and Rubin 2009; Pfannkuch 2006); (2) probability learning environments in which students are engaged in manipulating chance devices such as spinners to learn how probability is used by statisticians in problem solving (e.g., Pratt 2000).

The first setting has considerable potential for students to improve their use of data as evidence to draw conclusions. When students study topics close to their world in an authentic and relevant activity, they can gain important insights into how statistical tools can be used to argue, investigate, and communicate foundational statistical ideas. These settings can also sensitize students to the uncertainty involved in drawing conclusions from samples and the limitations of what can be inferred about the population. However, these settings may lack probabilistic considerations, which contribute to understanding the uncertainty involved in making inferences from samples to populations.

The second setting can encourage and support reasoning with uncertainty. When students manipulate chance devices they can easily build probability models of the expected distribution and observe simulation data generated by the model. They can then compare simulation data with empirical data to draw conclusions. This comparison strategy introduces students to the logic of statistical inference and the role of chance variation. Probability settings, however, may lack aspects of an authentic data exploration and exclude the relevance of the situation.

We suggest that integrating these two settings in making ISIs is important to further support students' reasoning with uncertainty during sampling. Therefore, we developed an *Integrated Modelling Approach* (IMA) aimed to help students understand the relationships between sample and population. Before presenting the IMA, we first present our conceptual framework for reasoning with sampling.

5.3 An Integrated Modelling Approach for Supporting Sample Population Relationships

5.3.1 Suggested Conceptual Framework for Reasoning with Sampling

We developed an initial conceptual framework (Fig. 5.1) that represents reasoning with sampling during ISIs. According to this framework, reasoning with sampling is an integration of two types of reasoning: (1) reasoning within samples to infer to a population; (2) reasoning between repeated samples.

The first type of reasoning, reasoning *within* sample, is the reasoning involved when exploring real sample data. This includes, for example, looking for signal and noise in data, as well as searching for patterns, trends, and relationships between attributes to learn about real world phenomenon in the population. The second type of reasoning, reasoning *between* samples, is the reasoning involved while drawing repeated collections of samples from the population or from a model of the population. This includes, for example, exploration of sampling variability and examination of the role of sample size on sampling variability. According to this framework, reasoning with sampling creates connections and integration between these two types of reasoning, for example, the relationship between the sampling variability and the likelihood of a single sample statistic.

Our study design was motivated by the hypothesis that integrating between the two types of reasoning with sampling may stimulate students to face both contextual and statistical uncertainty. Contextual uncertainty is the situation in which people are unsure about their context knowledge. The contextual uncertainty stems from a conflict between context knowledge and the sample data at hand. Such a conflict may

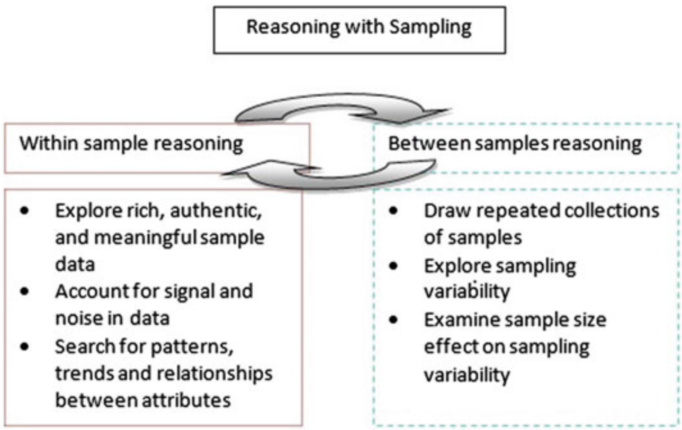

Fig. 5.1 A framework of reasoning with sampling

affect confidence about context knowledge and the ability to infer from a sample at hand. When students infer from a sample to a population their contextual uncertainty may be expressed by probabilistic language (Makar and Rubin 2009) using phrases like: “might be”, “it seems” or “sort of” or by suggesting a subjective confidence level. The subjective confidence level is how certain one feels about inferences in a numeric level (e.g., a number from 1 to 10 or a percent from 0 to 100), which is not calculated but based on subjective estimation. Statistical uncertainty is a situation in which people are unsure about sampling issues such as the behavior of random samples. The statistical uncertainty can be examined and even quantified. For example, the behavior of random samples can be examined by observing sampling variability, and confidence level can be quantified by calculating the probability of getting a statistic as extreme as or more extreme than a specific result, given a specified null hypothesis.

5.3.2 The Integrated Modelling Approach (IMA)

Based on these ideas, an *Integrated Modelling Approach* (IMA) was developed by us to guide the design and analysis of a learning trajectory aimed at supporting students’ IIR. It is comprised of *data* and *model* worlds to help students learn about the relationship between sample and population. The data world is designed to foster reasoning *within sample,* and the model world is designed to foster reasoning *between samples* (Fig. 5.1).

In the *data world*, students collect a real sample by a random sampling process to study a particular phenomenon in the population. In this world, students choose a research theme, pose questions, select attributes, collect and analyze data, make informal inferences about a population, and express their level of confidence in the data. However, they may not account for probabilistic considerations, such as the chance variability that stems from the random sampling process.

In the *model world*, students build a model (a probability distribution) for an explored (hypothetical) population and generate random samples from this model. They study the model and the random process that produces the outcome from this model. The details vary from sample to sample due to randomness, but the variability is controlled. Given a certain distribution of the population, the likelihood of certain results can be estimated.

In the IMA learning trajectory, students iteratively create connections between the two worlds by working on the same problem context in both worlds and by using TinkerPlots (Konold and Miller 2011). TinkerPlots is dynamic interactive statistics software developed to support young students’ statistical reasoning through investigation of data and statistical concepts. The dynamic nature of this software encourages learners to explore data in different repeated representations while testing various hypotheses. TinkerPlots includes a “Sampler” , that allows learners to design and run probability simulations to explore relationships between data and chance, by means of one technological tool (Konold and Kazak 2008). For a detailed description of the IMA approach see Manor Braham and Ben-Zvi (2015).

5.4 Method

5.4.1 The Research Question

This chapter focuses on the design of one activity (The Hidden Model of Social Networks—HMSN) in the IMA learning trajectory that serves as a scaffold for bringing the two worlds closer to the students and fosters students' reasoning with uncertainty. The focus is on the question: *How can reasoning with uncertainty be promoted in a way that is meaningful for young students while they are making ISIs?* More specifically we ask about the role the HMSN Activity played in the development of reasoning with uncertainty in students.

5.4.2 Methodology

To address this question, we carried out an illustrative case study of two sixth grade students. We explored their reasoning with uncertainty while making ISIs under the design principles of the activity. Data collection included student responses, gestures (captured using Camtasia), and artifacts (e.g., data representations drawn by them) , as well as researcher's observations. All students' verbalizations were carefully transcribed. Interpretive micro-analysis (e.g., Meira 1998), a microgenetic method (Chinn and Sherin 2014), was used to analyze the data. It is a systematic, qualitative, and detailed analysis of the transcripts, which takes into account verbal, gestural, and symbolic actions within the situations in which they occurred. The validation of the data analysis was performed by a small group of statistics education researchers (including the co-authors). The researchers discussed, presented, advanced, or rejected hypotheses, interpretations, and inferences about the reasoning and articulations of the students. The goal of such an analysis was to explore articulations of uncertainty by the students. Initial interpretations grounded in data were reviewed by the researchers and triangulated by a group of expert and novice peers. During these triangulation meetings, hypotheses that were posed by the researchers were advanced or rejected, until a consensus was reached. In order to achieve the necessary "trustworthiness" (Lincoln and Guba 1985), triangulation was achieved only after multiple sources of data validated a specific result (Schoenfeld 2007).

5.4.3 The Participants

This study involved a pair of boys (grade 6, aged 12), Shon and Yam, in a private school in northern Israel. The students were selected due to their superior communication skills that provide a window into their statistical reasoning. They participated in a *Connections* unit in fifth grade when they collected and investigated data about their peers using TinkerPlots. Following the growing samples heuristic (Ben-Zvi et al. 2012), they were gradually introduced to samples of increasing size to support their reasoning about ISI and sampling.

5.5 The Hidden Model of Social Networks (Hmsn)

To put the HMSN activity in context, we provide a general description of the entire learning trajectory as well as the rationale and place of the HMSN activity in the learning trajectory.

5.5.1 The Entire IMA Learning Trajectory

The learning trajectory[1] encompassed eight activities that initially introduced the two worlds separately. In the data world, the students planned a statistical investigation where they chose a research theme, posed research and survey questions, formulated a conjecture, and decided about the sampling method and sample size (Activity 1). Shon and Yam decided to study the use of technological tools among fourth to ninth grade students in their school. Both Shon and Yam played a lot of computer games, and their research choice arose from their desire to convince the school headmaster to authorize playing computer games at school. Shon and Yam suggested that there were some types of computer games, which they called “wise games,” that can develop thinking and therefore may potentially have a positive influence on students. They decided to explore the relationship between two attributes: whether a student is attentive and whether a favorite type of computer game is “wise.” However, they suggested a biased sampling method of taking two students from each class in grades 4–9 by asking the teachers to choose one attentive child and one non-attentive child. Therefore, we added an activity (Activity 2) to explore the meaning of biased sampling versus random sampling. We also used this activity to expose students to the idea of sampling distribution. The students refined their sampling method, reformulated their conjectures, and implemented a survey in their school (Activity 3). They explored their real data using TinkerPlots (Activity 4). In the model world, they used the Sampler in TinkerPlots to build a hypothetical model for the population distribu-

[1] The actual IMA learning trajectory can be viewed at http://connections.edtech.haifa.ac.il/Research/theimalt.

tion based on their conjecture. They drew samples from this model, compared them to the model and their real sample data, and explored sampling distributions (Activity 5). To encourage them to examine the connections between the worlds, they were asked "what if" questions on hypothetical real data results while exploring generated random samples.

Since students found it difficult to connect between generated random samples and the real sample, they were given a sixth activity (HMSN Activity 6, which is the focus of the current study). They were asked to study a hidden TinkerPlots model, built by other students, and explore random samples drawn from this model. They then returned to their own investigation and once again explored different sample sizes drawn from their model to compare between them and decide about the minimal sample size needed to draw conclusions about the population. According to their chosen sample size, the students collected more data (Activity 7). Finally, they simultaneously explored data and models in the two worlds by examining the real larger sample data in relation to their conclusions in the model world. They used their estimation of the likelihood to get a specific result given a sample size and a certain distribution of the population, in their conclusions about the population from the real larger sample data (Activity 8).

5.5.2 *Rationale of the HMSN Activity*

The shift from the data world (Activity 4) to the model world (Activity 5) was challenging for the students. The motivation for the students to move to the model world was that in the model world they would be able to explore two issues: (a) the relationship between random sample and population; (b) the minimal sample size that provides for reliable inferences about the population of interest. While the students explored random model-generated samples they became confused between model-generated samples and real samples. It was challenging for the students to understand what they can gain from exploring the random model-generated samples and how it can help them in investigating real samples. Therefore, we designed a scaffolding activity, the HMSN Activity, to provide a practical purpose for students to study the behavior of many model-generated samples and connect between the repeated sampling and the inferences that are based on a single sample.

5.5.3 *The HMSN Activity*

A hidden Sampler is a TinkerPlots software option that locks the Sampler to keep students from changing any of its settings and to prevent them from revealing the contents of hidden population devices. In the Hidden Model of Social Networks (HMSN) Activity, students are asked to use the Sampler in TinkerPlots to draw many random samples from a hidden Sampler (Fig. 5.2) to make ISIs. The hidden

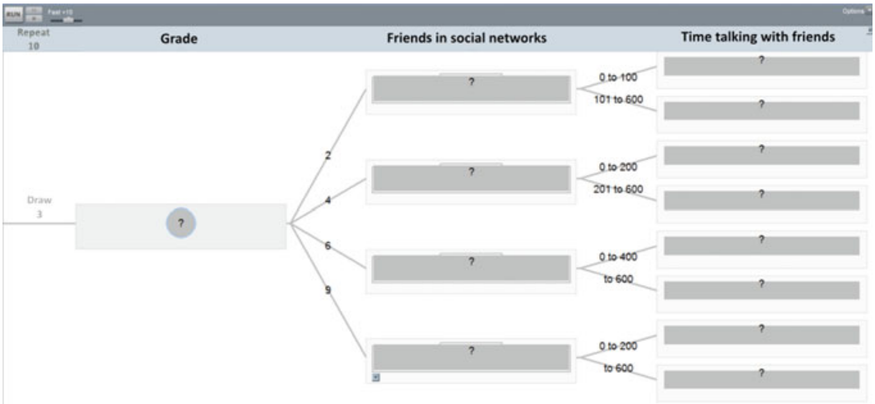

Fig. 5.2 The TinkerPlots Sampler hidden model in the HMSN Activity

Sampler contained three interconnected attributes concerning teenager use of social networks: grade, number of friends in social networks (#FSN), and average time spent on social networks (minutes per day). The students were asked to draw ISIs based on growing sample sizes, beginning their exploration with a small sample size of 10. Each time they wanted to increase the sample size, they had to explain the rationale.

The rationale for including a hidden population was to make the sample-population relationship resemble these relationships in real situations in which the population is unknown. We decided to restrict students to drawing relatively small samples so they would notice the large sampling variability and explore ways to reduce it. We thought this small sample restriction would seem reasonable to the students, since they were aware of the necessity to understand the technical statistical issue of making inferences based on small samples (Ainley et al. 2015). Those students were aware from previous activities of the fact that in real situations one could not collect all data but needed to make inferences on populations from samples. Unlike real life, in the HMSN Activity, the students were able to draw many random samples of a chosen size, and gradually increase the sample size to discover the minimal sample size that can be used for reliable inferences. We hypothesized that following engagement in the HMSN activity, it would be easier for students to enter the model world and the required probabilistic reasoning in the fifth activity.

5.6 Key Features of the HMSN Activity

The main goal of the activity was to develop reasoning with uncertainty of students engaged in sampling during the process of making ISIs. We wanted to motivate and support students in the development of ways to describe, control, and quantify the uncertainty involved in making ISIs from a single random sample. Guided by the IMA, this activity aimed to support a smooth transition between two types of reasoning: reasoning within samples that occurs in the data world and reasoning between repeated samples that occurs in the model world. In this section we explain the activity design principles and describe the concepts and situations we hypothesize may play a central role in students' reasoning with uncertainty.

5.6.1 Cognitive Conflict Between Data and Context

Most of the theoretical models developed to explain conceptual change (e.g., Strike and Posner 1985; Chi et al. 1994) emphasize the role of cognitive conflict as essential component for conceptual change. Cognitive conflict is generated by dissatisfaction with existing concepts and ideas (Posner et al. 1982). It occurs when a learner cannot use his existing knowledge to solve a problem or explain a phenomenon and is therefore motivated to learn new concepts and ideas (Lee and Kwon 2001). In the learning of statistics, conflicts that take place between former knowledge and current understanding of data analysis can give rise to uncertainty about the explored phenomenon. This can foster and result in new statistical understandings to reduce uncertainty, for example by looking for more data or considering other intervening variables.

Our supposition was that creating conflicts between sample data and context knowledge may motivate students to move from within-sample reasoning (in the data world) to between-samples reasoning (in the model world). To create conflicts we reasoned that an exploration of real and meaningful data was essential. We wanted to ensure that students will easily recognize data that contradicts their experience and be motivated to explore and explain the contradiction.

Shon and Yam were enthusiastic computer users and therefore deeply interested in the theme of this Activity. The research theme they chose in Activities 1-4 was the use and benefit of technological tools among fourth to ninth grade students. The data was also real for the students since the hidden sampler was built by two other students in their class. Those other students built the model while keeping in mind real data that they collected in their school. Therefore, we expected that Shon and Yam would be interested to explore the hidden model data and be equipped with knowledge about its context.

We assumed the students would struggle with sample data that did not make sense in relation to their context knowledge. We hoped that in order to find solutions to those conflicts and handle the uncertainty in data, students would use the TinkerPlots option of generating more samples or consider increasing the sample size. To increase the incidence of conflicts between data and context, the students were asked to begin their explorations with a small sample (size 10). We hoped the students would notice the "noise" (Konold and Kazak 2008) in the data and be motivated to handle the uncertainty by repeated sampling and increased sample size.

5.6.2 Growing Samples

Growing samples is an instructional approach mentioned by Konold and Pollatsek (2002), developed by Bakker (2004, 2007) and elaborated by Ben-Zvi (2006). According to the growing samples method students explore small data sets to infer about a wider set of data. They are gradually given more data and asked what can be inferred regarding the bigger sample or the entire population. Therefore, students learn about the limitations of what can be inferred by the teacher's "what-if" questions. This instructional approach was found fruitful in supporting students' reasoning with key statistical concepts such as distribution, variability, and sampling (Ben-Zvi et al. 2012).

In the HSMN Activity, students were exposed to increased sample sizes while expressing considerable uncertainty in small samples due to the limitations of what can be inferred about the hidden Sampler from these small samples. The rationale of using the growing samples heuristic was that it focused the students' attention on inferences (Bakker 2004) and motivated them to develop key statistical ideas and concepts that underlie between-samples reasoning, such as the role of sample size in the confidence level or the connection between sample size and sampling variability.

5.7 Learning Progression of Students

We identified three main thematic and chronological stages in Shon and Yam's expressions of uncertainty: *examine*, *control*, and *quantify uncertainty*. During these stages students gradually refined their way of thinking about uncertainty while learning to integrate the data and model worlds. Due to conflicts they identified, within certain sample results, between data and context knowledge, the students invented these stages (examine, control or quantify) to deal with uncertainty. In this section we describe each one of those stages detailing the conflict and the methods invented by the students to tackle their challenges.

5.7.1 Stage I: Examine Uncertainty

5.7.1.1 The First Conflict

Before Shon and Yam drew the first sample (size 10) from the hidden sampler, their initial conjecture was that older students would have more friends in social networks. Observing the sample data, the students were puzzled since the data was in contradiction to their hunch and prior knowledge regarding friends in social networks. For example, they noticed that a fourth grade student had the highest #FSN and that a ninth grade student had no #FSN. Shon commented: "something doesn't make

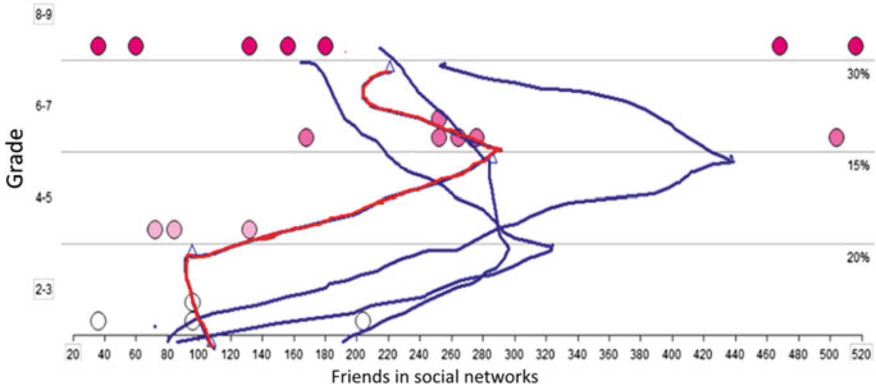

Fig. 5.3 The students' comparison of the MTLs of #FSN over four samples of size 20. A dot represents a case, a blue triangle represents the mean number of friends in a grade, a line connects the four means in each sample, and the red line is the MTL of the fourth sample

sense." Grappling with this *contextual uncertainty*, the boys added means of #FSN for every grade to find a signal in the data. In their search for a pattern in the data, they used TinkerPlots' drawing tool to connect the means to one another with a "*mean trend line*" (MTL).

5.7.1.2 Students Invent the First Method to Compare Between Samples

To motivate Shon and Yam to consider repeated sampling, we asked them: "What will happen if you drew another sample? Could it help you somehow?" Yam said, in response: "We can take one more [sample]," and Shon excitedly added: "Yes, yes, let's do it [draw from the Sampler] many times." The students began drawing additional random samples from the Sampler hidden model. To examine the uncertainty caused by the sample variability, Shon and Yam invented a "*Capture MTLs*" method. They plotted the MTL for each sample they drew (Fig. 5.3), compared their position, and noticed the large variability between them. Struggling with this *statistical uncertainty*, Yam reflected: "It [the MTL] is very different each time." The students consequently asked to increase the sample size from 10 to 20, and Shon stated that, "a sample size of ten is too small."

5.7.1.3 Reasoning with Uncertainty for Stage I: Examine Uncertainty

During this stage, the students grappled with two types of uncertainty: the contextual uncertainty that stems from a conflict between the data and their prior knowledge as well as statistical uncertainty that stemmed from the large sampling variability they observed and their inability to control random samples. In order to deal with the contextual uncertainty, they added the MTL to find the signal in the data. In order to deal with the statistical uncertainty, they asked to increase the sample size.

5.7.2 Stage II: Control Uncertainty

5.7.2.1 Second Conflict

The boys used the same method, "*Capture MTLs*", to examine the variability between the MTLs of #FSN for samples of size 20. After drawing three random samples from the Sampler, they noticed the smaller variability between the MTLs in comparison to samples of size 10. During these explorations of samples (size 20), they referred to the similarities and differences in location, shape and "peak" (maximal mean) between the MTLs. They noticed a trend in the #FSN (The number of friends increased from grade 2 to 7 and decreased from grade 7 to 9). They explained that the reason for the decrease from grade 7 to 9 is that ninth graders have usually more homework and exams and therefore have less time to communicate with friends on social networks. However, they still expressed their statistical uncertainty and wanted to further increase the sample size.

A fourth MTL surprised them (the red line in Fig. 5.3) and destabilized their relative confidence regarding the MTL's trend (e.g., unlike the previous samples, the number of friends decreased from grade 2 to 5 in the fourth sample). Yam said, "It [this fourth sample] is very bad." Instead of drawing more samples, they asked to increase the sample size once again. At this point, the researcher tried to motivate the students to draw more samples by asking: "Do you feel more confident in your conclusions about certain grades?" In response, the students decided to draw many samples of size 20 and examine the variability between the means of #FSN within the grades.

5.7.2.2 Students Invent a Second Method to Compare Between Samples

The students developed a new graphical method, "*Capture Means*", to capture the variability between the means in order to examine whether they could control the uncertainty in the repeated sampling process. According to their "*Capture Means*" method, when the mean result of a particular grade could be captured inside a drawn circle, they concluded that the variability within that grade was small. They drew several samples of size 20, and Yam noticed that in grade 6, "It [the mean variability] is relatively stable because it [the mean] is usually in the area of the circle. That's why I say that they [the means over several samples, Fig. 5.4] are relatively stable." However, the students noticed that the mean results from the three other grades could not be captured inside a drawn circle. Therefore, they expressed higher statistical uncertainty regarding the sample size and the resulting conclusions. Due

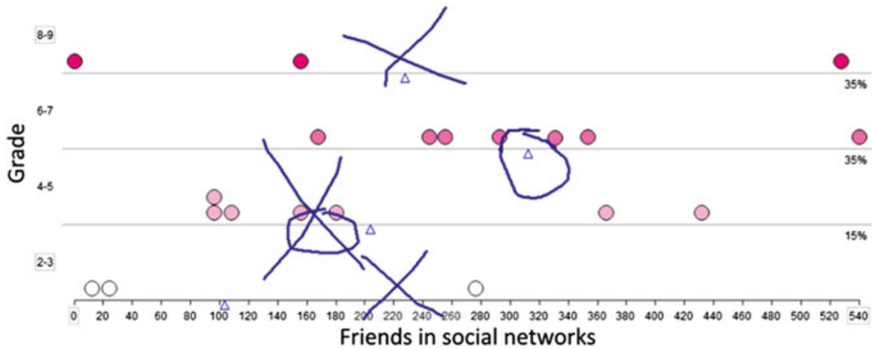

Fig. 5.4 “Stable” and “constantly varying” mean signals over several samples of size 20

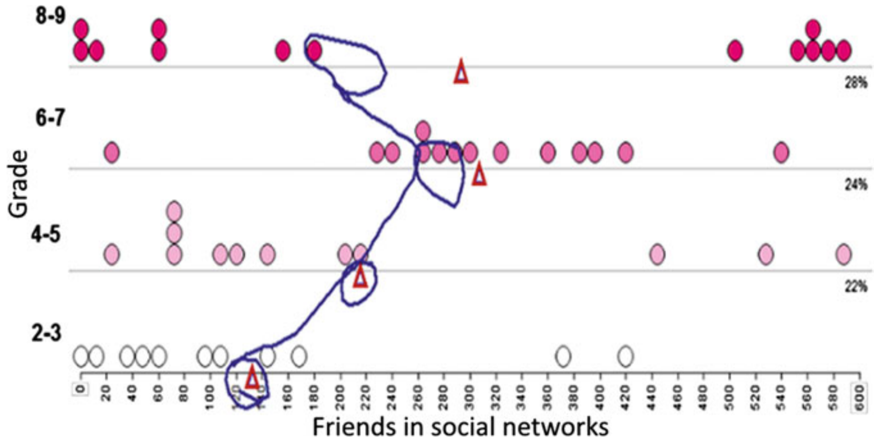

Fig. 5.5 The hypothetical MTL of #FSN over several samples size 50

to the outcome of only one circle (i.e., stable mean) over many samples, they asked to increase the sample size to 50.

Shon and Yam applied their “*Capture Means*” method on larger samples of size 50, drew a circle for each grade capturing the means of that grade over many samples (Fig. 5.5). They noticed that, “grade 9 [means] stay in this area [the top blue circle in Fig. 5.5]. It [grade 4 means] really jump around this spot [Yam drew a circle around grade 4 means].” Encouraged by these results, the boys expressed a higher confidence level and were satisfied with the sample size. Shon said: “In my opinion, [sample of] 50 will be enough.” Their confidence about the MTL’s stability increased, and they connected the four circles (Fig. 5.5) saying they were “absolutely certain.”

5.7.2.3 Reasoning with Uncertainty for Stage II: Control Uncertainty

In the beginning of the second stage, comparing samples (size 20) and observing smaller variability between MTLs, decreased the students' statistical uncertainty. A surprising sample showing an MTL that was incongruous with their former knowledge increased their contextual uncertainty causing them to feel unsure about their context knowledge. To handle the contextual uncertainty, they decided to invent the "*Capture Means*" method and examine whether they could control the means of #FSN in certain grades. Examining several samples of size 20 with the "*Capture Means*" increased their statistical uncertainty regarding conclusions that could be drawn from random samples of size 20. To deal with the statistical uncertainty, they increased the sample size to 50 and with the "*Capture Means*" method decreased their statistical uncertainty. To examine their conclusions in relation to their former knowledge, they drew a new MTL. The similarity between the MTL and their hypothesis decreased their contextual uncertainty.

5.7.3 Stage III: Quantify Uncertainty

5.7.3.1 Third Conflict

During the next meeting, the students' confidence encouraged them to refine their hypothetical MTL for samples of size 50. They drew a few random samples but were surprised that several of them showed a significantly different trend than the hypothetical trend. They therefore decided to differentiate between two main trends: "type 0" trend (the MTL of #FSN is increasing between grade 2 to 7 and is decreasing between grade 7 to 9) and "type 1" trend (the MTL of #FSN is decreasing between grade 2 to 5, increasing between grade 5 to 7 and increasing between grade 7 to 9). They complained: "We can't draw an inference because it is different all the time." They tried to deal with the growing uncertainty about the trend by drawing bigger random samples of size 65 and noticed that there were more samples of "type 0" than "type 1" trend.

5.7.3.2 Students Invent a Third Method to Compare Between Samples

To *quantify* their uncertainty about the trend, the boys invent a third method to compare between samples. They calculated the difference between the numbers of samples within each trend, and called this difference a "breakpoint." For example, if the first and second samples showed "type 0" trend and the third sample showed "type 1" trend, they said that the breakpoint is one (2-1). They decided that when this breakpoint equals a certain number, determined in advance, it would point at the more likely trend. Setting the breakpoint to three, the boys strengthened their previous assumption and chose "type 0" trend over "type 1" estimating their subjective confidence level to be 80%. They explained their high confidence level in "type 0" trend by referring to the difference between the number of samples with "type 0" trend to those with "type 1" trend. They even found a way to increase their confidence level in "type 0" trend by setting the breakpoint to five.

Yam: Because we had three times more [cases of "type 0" than "type 1"]. There are still times it's like this ["type 1"], but most of the time it's like this ["type 0"].

Shon: We will wait until it [the breakpoint] will be more than five. Here again… we'll wait until it will arrive at five… If there's one more time [a sample with "type 0" trend], then I believe 90% [that the trend is of "type 0"].

At this point they generalized the meaning of the breakpoint to be an estimate of their confidence level; a bigger breakpoint results in a higher confidence level.

5.7.3.3 Reasoning with Uncertainty for Stage III: Quantify Uncertainty

In the third stage the students felt unsure about their context knowledge because they observed some MTLs that were incongruous with their former knowledge, a fact that increased their contextual uncertainty. Furthermore, they felt unsure about the ability to infer from random samples of size 50, a fact that increased their statistical uncertainty. To deal with the uncertainties, the students increased the sample size to 65 and quantified the sampling variability by calculating the difference between the number of samples in each trend. However, to express their level of confidence in their inference, they didn't make calculations but used a subjective confidence level of 80%.

5.8 Discussion

The main question of this chapter is: How can reasoning with uncertainty be promoted in a way that is meaningful for young students while they are making ISIs? In this chapter we presented the IMA and the design principles of one activity in the IMA learning trajectory. Our analysis illustrated how the students' reasoning with uncertainty was refined during their engagement in this activity. In the following section we discuss the research conditions and its limitations followed by the pedagogical and theoretical implications of our analysis regarding the relationship between: a) one sample and repeated samples, and b) data and chance. We also discuss our main design challenges to highlight the characteristics of the activity that cultivated the progress of the students learning.

5.8.1 Limitations

The purpose of describing the design of a learning trajectory and an analysis such as the one presented in this chapter is that researchers and teachers could learn from it and adjust the activities to their circumstances. Therefore, it is important to provide the conditions and limitations of this research. This chapter is based on our analysis and experience with only a small number of students who had superior communication and reasoning skills. Therefore, our findings are only a proof of principle. More research is needed to determine how this activity in particular and the IMA in general can be performed with less intervention in a classroom setting. We are currently conducting another study with sixth grade students in a classroom setting to test the idiosyncrasy and the generality of the case presented in this chapter.

Researchers and teachers will also need to consider that our students were involved in EDA activities during the previous year. Throughout that year we exposed them to ideas of sample size and inferences that can be drawn from a sample. We think that in the IMA learning trajectory, students' experience with an exploratory approach to data is essential for entering the model world and dealing with the complex idea of uncertainty (Pfannkuch et al. 2012). Reasoning with uncertainty in the context of informal statistical inference is an ongoing discourse aimed to convince others regarding inferences that can be made and the level of confidence in making those inferences. The fact that our students were used to an environment of open and critical discourse from the previous year prepared them to deepen their reasoning with uncertainty and inferences this year.

5.8.2 Implications

5.8.2.1 Pedagogical Implications

Our case study demonstrates how reasoning with uncertainty developed through students' iterations between the data and the model worlds. In our analyzed data, the students' expressions of contextual and statistical uncertainties shaped their movement between the worlds. The *contextual uncertainty*, which occurred in the data world, stemmed from the conflict between the boys' context knowledge and the data in relation to a specific sample. For example, when the boys explored a sample size of 10, Shon doubted that a fourth grade student had the biggest #FSN and thought that "it is strange." Such a conflict increased the boys' uncertainty about the ability to infer from a sample. The *statistical uncertainty,* which occurred in the model world, stemmed from sampling variability. Disconcerted by small sample sizes and restricted by the activity design, the boys invented graphical methods to examine the variability between means and MTLs over many samples. These situations increased the boys' uncertainty about the ability to infer from a single sample of a certain size.

In order to understand the uncertainty involved in taking a sample, one needs to envision a process of repeated sampling and its relation to the individual sample (Saldanha and McAllister 2014). The relationship between the individual sample and repeated samples may emerge during the construction of the relationship between contextual and statistical uncertainties. Furthermore, articulations of statistical uncertainty may emerge from the need to face and explain the contextual uncertainty and evolve with repeated sampling. For example, the need to elucidate conflicts between data and prior knowledge and the ability of the tool (TinkerPlots) to draw repeated samples assisted students in examining whether the conflicts happened due to chance and impelled them to face statistical uncertainty.

Distinguishing between the two types of uncertainties may be important from a pedagogical point of view. As we depict in this study, facing a contextual uncertainty may motivate students to examine statistical uncertainty by drawing repeated samples and observing sampling variability. Therefore, we argue that designing activities that promote conflicts between data and context knowledge and encouraging students to consider repeated sampling may be fruitful in understanding the relationship between sampling variability and confidence in a single sample.

5.8.2.2 Theoretical Implications

The findings of this study are consistent with the argument that students must be able to integrate between data and chance in order to understand informal statistical inference (e.g., Konold and Kazak 2008; Pfannkuch et al. 2018). This is due to the fact that making ISIs involves connecting probability-based notions of uncertainty and inferences that are drawn from data (Makar and Rubin 2009, 2018). Although researchers agree that EDA is an appropriate method for exploring statistics, a criticism of the EDA pedagogical approach towards informal statistical inference is its data-centric perspective (Prodromou and Pratt 2006) that does not foster students' appreciation of the power of their inferences as does the model-based perspective (Horvath and Lehrer 1998; Pfannkuch et al. 2018). This study responds to the challenge of reconnecting data and chance bi-directionally with an *Integrated Modelling Approach* that adds elements of a model-based perspective to the EDA approach. We suggest that engaging students with iterations between the data and model worlds in the IMA, as presented in the HMSN Activity Section, may help them integrate ideas of data and chance.

Figure 5.6 summarizes the students' iterations between the data and model worlds and between data and chance. Pronounced conflicts between data and context knowledge that were expressed by contextual uncertainty in the data world (the left column in Fig. 5.6) played an important role in the boys' motivation to examine chance, as well as invent and refine their methods of examining, controlling, and quantifying the statistical uncertainty in the model world (the right column in Fig. 5.6). During the first stage, exploring sample data that contradicted their previous knowledge in the data world played an important role in the boys' motivation to move to the model world, draw repeated samples, and invent the "*Capture MTLs*" method to examine

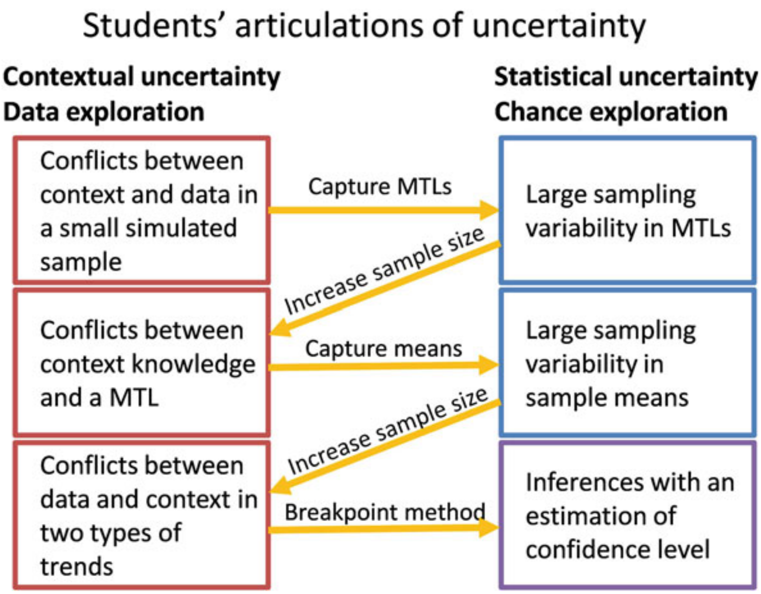

Fig. 5.6 The students' reasoning with uncertainty through iterations between the worlds

the statistical uncertainty in sampling variability. By examining sampling variability, they actually explored whether the conflicts they observed between context and data were due to chance. The large sampling variability compelled them to increase sample size. During the second stage, a surprising sample showing MTL that made no sense in the data world forced the boys to invent the "*Capture Means*" method to control the statistical uncertainty. The quantification of the statistical uncertainty in the third stage resulted from their contextual uncertainty regarding the hypothetical MTL.

Shuttling between the worlds, the students were able to make meaningful connections between inferences they can draw about a phenomenon from samples of a certain size and the idea of repeated samples and sampling variability. Our case study depicts that, in carefully designed activities that cultivate the idea of repeated sampling (Shaughnessy 2007; Thompson et al. 2007), even relatively young students can be exposed to and make some sense of complex ideas behind ISI such as the relationship between sample size, sampling variability, and confidence level in a sample of a certain size.

5.8.2.3 Design Implications

One challenge in cultivating reasoning with uncertainty in the context of ISI is how to motivate students to deal with statistical uncertainty. In other words, how to create situations in which students see utility (Ainley and Pratt 2010) in drawing many samples to examine uncertainty. Utility of an idea is an understanding what it is useful for and what power it offers in addressing problems with respect to a ‘project’ in which the student is currently engaged. Moving to the model world and envisioning a process of repeated sampling and its relation to the individual sample (Saldanha and McAllister 2014) is not easy or natural for the students (Saldanha and Thompson 2002). One reason for that may be that the idea of repeated sampling is too theoretical for students, and they usually don’t find utility in the action (Ainley et al. 2006). However, in the HMSN Activity, students found utility in drawing many samples and used it to increase their informal confidence level in the inference that could be made from a single sample. Informal confidence level is an estimation of how certain one feels about informal inferences. It is uttered by a numeric level that is not calculated but based on a relative number of repeating samples that indicate a particular result.

We hypothesize that the combination of engaging students with real and meaningful data that motivates them to deal with contextual uncertainty and the possibility in TinkerPlots of drawing many random samples of different sizes, assisted students in comparing between samples.

Although the HMSN Activity included an artificial task and took the students away from their focus project, they were aware of the statistical idea of sampling and the need to examine the power it had on estimating the level of confidence in samples in their ongoing project. We think so since the students were engaged before the HMSN Activity in inquiry-based activities based on real sample data in both the data and the model worlds. During those activities they were dealing with the questions: “Can one trust random samples?” and “What is a sufficient sample size on which one can make reliable inferences on the population?” So in this context, the artificial task in the HMSN Activity was connected to their ongoing project. Furthermore, after the HMSN Activity, the students returned to work on their real data and used what they learned about sampling and uncertainty to find the minimal sample size on which they could make reliable inferences about the explored population. We believe that in such a learning environment, if students find utility in drawing many samples as a way to face uncertainty, there is a greater chance that they will understand this concept and also utilize it in other contexts.

A second challenge was in motivating students to invent methods to compare between samples. We didn’t want to prescribe a comparison solution prior to their experience. Furthermore, we thought that by inventing methods to explore and compare between samples, students would have the opportunity to struggle with the rationale of examining many samples and their relationship to a single sample. We suggest that since these students were used to an exploratory approach to data (EDA) , it seemed natural for them to look for and invent different methods to compare between samples. In the previous year and in the first four activities of the IMA learning trajectory, the students looked for different methods to analyze data in order to convince their peers about their inferences. The current activity with its use of TinkerPlots enabled students not only to find methods to analyze sample data but also to draw many samples and find innovative ways to compare between them.

Inventing methods to compare between samples has conceptual consequences. First, observing many samples and deciding how to compare between them can support the concept of aggregate view (Konold et al. 2015; Aridor and Ben-Zvi 2018). For example, while students invented and examined their *MTL method*, they realized the importance of the location and "peaks" of the line, in addition to its shape. Second, while students are engaged in inventing methods to compare between samples, they can learn different ways to notice and describe sampling variability and its relation to sample size.

However, inventing methods to compare between samples invited also meta-conceptual questions such as: How can we compare between samples? What does it mean to compare between samples? What is a good method for comparison? What information is missing in our method? For example, when the boys began to compare the MTLs they realized that there was a similarity between the shapes of the MTLs, but there were differences in the MTLs location and "peaks". Therefore, they looked for other ways to compare between samples and invented the "*Capture Means*" method that helped them focus on the variability of the #FSN means locations within the grades, over many samples.

Although on a small scale, this study sheds light on new ways to combine data and chance, in order to support students' informal inferential reasoning. Helping students make connections between data and chance using the IMA pedagogy will inevitably bring with it new challenges regarding learning to make ISIs and smoothing the transitions between the data and model worlds. However, these difficulties can be embraced as essential steps in the development of the reasoning of students who are engaged in a modern society in which drawing inferences from data becomes part of everyday life.

References

Ainley, J., Gould, R., & Pratt, D. (2015). Learning to reason from samples: Commentary from the perspectives of task design and the emergence of "big data". *Educational Studies in Mathematics, 88*(3), 405–412.

Ainley, J., & Pratt, D. (2010). It's not what you know, it's recognising the power of what you know: Assessing understanding of utility. In C. Reading (Ed.), *Proceedings of the 8th International Conference on the Teaching of Statistics* (ICOTS). Ljubljana, Slovenia: International Statistical Institute and International Association for Statistical Education. Retrieved from http://www.Stat.Auckland.Ac.Nz/~IASE/Publications.

Ainley, J., Pratt, D., & Hansen, A. (2006). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal, 32*(1), 23–38.

Aridor, K., & Ben-Zvi, D. (2018). Statistical modeling to promote students' aggregate reasoning with sample and sampling. *ZDM—International Journal on Mathematics Education*. https://doi.org/10.1007/s11858-018-0994-5.

Arnold, P., Confrey, J., Jones, R. S., Lee, H. S., & Pfannkuch, M. (2018). Statistics learning trajectories. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 295–326). Cham: Springer.

Bakker, A. (2004). *Design research in statistics education: On symbolizing and computer tools*. Utrecht, The Netherlands: CD-β Press, Center for Science and Mathematics Education.

Bakker, A. (2007). Diagrammatic reasoning and hypostatic abstraction in statistics education. *Semiotica, 2007*(164), 9–29.
Ben-Zvi, D. (2006). Scaffolding students' informal inference and argumentation. In A. Rossman & B. Chance (Eds.), *Proceedings of the Seventh International Conference on Teaching Statistics (CD-ROM)*, Salvador, Bahia, Brazil, July 2006. Voorburg, The Netherlands: International Statistical Institute and International Association for Statistical Education.
Ben-Zvi, D., & Arcavi, A. (2001). Junior high school students' construction of global views of data and data representations. *Educational Studies in Mathematics, 45*, 35–65.
Ben-Zvi, D., Aridor, K., Makar, K., & Bakker, A. (2012). Students' emergent articulations of uncertainty while making informal statistical inferences. *ZDM—The International Journal on Mathematics Education, 44*(7), 913–925.
Ben-Zvi, D., Gil, E., & Apel, N. (2007). What is hidden beyond the data? Helping young students to reason and argue about some wider universe. In D. Pratt & J. Ainley (Eds.), Reasoning about informal inferential statistical reasoning: A collection of current research studies. *Proceedings of the Fifth International Research Forum on Statistical Reasoning, Thinking, and Literacy (SRTL5)*, University of Warwick, UK, August 2007.
Ben-Zvi, D., Gravemeijer, K., & Ainley, J. (2018). Design of statistics learning environments. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 473–502). Cham: Springer.
Chi, M. T. H., Slotta, J. D., & de Leeuw, N. (1994). From things to processes: A theory of conceptual change for learning science concepts. *Learning and Instruction, 4*, 27–44.
Chinn, C. A., & Sherin, B. L. (2014). Microgenetic methods. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 171–190). New York: Cambridge University Press.
Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(1), 9–13.
Edelson, D. C., & Reiser, B. J. (2006). Making authentic practices accessible to learners: Design challenges and strategies. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences*. New York: Cambridge University Press.
Garfield, J., & Ben-Zvi, D. (2008). *Developing students' statistical reasoning: Connecting research and teaching practice*. New York: Springer.
Garfield, J., & Ben-Zvi, D. (2009). Helping students develop statistical reasoning: Implementing a statistical reasoning learning environment. *Teaching Statistics, 31*(3), 72–77.
Garfield, J., Chance, B., & Snell, J. L. (2000). Technology in college statistics courses. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 357–370). Dordrecht, The Netherlands: Kluwer Academic Publishers.
Greeno, J. G., & Engeström, Y. (2014). Learning in activity. In R. K. Sawyer (Ed.), *The Cambridge handbook of the learning sciences* (pp. 128–150). New York: Cambridge University Press.
Herrington, J., & Oliver, R. (2000). An instructional design framework for authentic learning environments. *Educational Technology Research and Development, 48*(3), 23–48.
Horvath, J. K., & Lehrer, R. (1998). A model-based perspective on the development of children's understanding of chance and uncertainty. In S. P. LaJoie (Ed.), *Reflections on statistics: Agendas for learning, teaching and assessment in K-12* (pp. 121–148). Mahwah, NJ: Lawrence Erlbaum Associates.
Konold, C., Higgins, T., Russell, S. J., & Khalil, K. (2015). Data seen through different lenses. *Educational Studies in Mathematics, 88*(3), 305–325.
Konold, C., & Kazak, S. (2008). Reconnecting data and chance. *Technology Innovations in Statistics Education, 2*(1). http://repositories.cdlib.org/uclastat/cts/tise/vol2/iss1/art1/.
Konold, C., & Miller, C. (2011). *TinkerPlots™ 2.0*. Amherst, MA: University of Massachusetts.
Konold, C., & Pollatsek, A. (2002). Data analysis as the search for signals in noisy processes. *Journal for Research in Mathematics Education, 33*(4), 259–289.
Lee, G., & Kwon, J. (2001). *What do we know about students' cognitive conflict in science classroom: A theoretical model of cognitive conflict process*. Paper presented at Annual Meeting of the Association for the Education of Teachers in Science, Costa Mesa, CA, January 18–21, 2001.

Lehrer, R., & Romberg, T. (1996). Exploring children's data modeling. *Cognition and Instruction, 14*(1), 69–108.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Beverly Hills, CA: Sage.

Makar, K., Bakker, A., & Ben-Zvi, D. (2011). The reasoning behind informal statistical inference. *Mathematical Thinking and Learning, 13*(1–2), 152–173.

Makar, K., & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal, 8*(1), 82–105.

Makar, K., & Rubin, A. (2018). Learning about statistical inference. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International handbook of research in statistics education* (pp. 261–294). Cham: Springer.

Manor Braham, H., & Ben-Zvi, D. (2015). Students' articulations of uncertainty in informally exploring sampling distributions. In A. Zieffler & E. Fry (Eds.), *Reasoning about uncertainty: Learning and teaching informal inferential reasoning* (pp. 57–94). Minneapolis, Minnesota: Catalyst Press.

Meira, L. (1998). Making sense of instructional devices: The emergence of transparency in mathematical activity. *Journal for Research in Mathematics Education, 29*(2), 121–142.

Moore, D. (2007). *The basic practice of statistics* (4th ed.). New York: W. H. Freeman.

Pfannkuch, M. (2006). Informal inferential reasoning. In A. Rossman & B. Chance (Eds.), *Proceedings of the 7th International Conference on Teaching Statistics (ICOTS) [CD-ROM]*, Salvador, Bahia, Brazil.

Pfannkuch, M. (2008, July). Building sampling concepts for statistical inference: A case study. In *The 11th International Congress on Mathematics Education*. Monterrey, Mexico.

Pfannkuch, M., Ben-Zvi, D., & Budgett, S. (2018). Innovations in statistical modeling to connect data, chance and context. *ZDM—International Journal on Mathematics Education*. https://doi.org/10.1007/s11858-018-0989-2.

Pfannkuch, M., & Wild, C. (2004). Towards an understanding of statistical thinking. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 17–46). Netherlands: Springer.

Pfannkuch, M., Wild, C., & Parsonage, R. (2012). A conceptual pathway to confidence intervals. *ZDM—The International Journal on Mathematics Education*, *44*(7), 899–911.

Posner, G. J., Strike, K. A., Hewson, P. W., & Gertzog, W. A. (1982). Accommodation of a scientific conception: Toward a theory of conceptual change. *Science Education, 66*(2), 211–227.

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in Mathematics Education, 31*(5), 602–625.

Pratt, D., & Ainley, J. (2008). Introducing the special issue on informal inferential reasoning. *Statistics Education Research Journal, 7*(2), 3–4.

Prodromou, T., & Pratt, D. (2006). The role of causality in the co-ordination of two perspectives on distribution within a virtual simulation. *Statistics Education Research Journal, 5*(2), 69–88.

Saldanha, L., & McAllister, M. (2014). Using re-sampling and sampling variability in an applied context as a basis for making statistical inference with confidence. In K. Makar, B. de Sousa, & R. Gould (Eds.), *Sustainability in statistics education. Proceedings of the Ninth International Conference on Teaching Statistics (ICOTS-9)*.Voorburg, The Netherlands: International Statistical Institute and International Association for Statistical Education.

Saldanha, L., & Thompson, P. (2002). Conceptions of sample and their relationship to statistical inference. *Educational Studies in Mathematics, 51*(3), 257–270.

Schoenfeld, A. H. (2007). Method. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 69–107). Charlotte, NC: Information Age Publishing.

Shaughnessy, M. (2007). Research on statistics learning and reasoning. In F. Lester (Ed.), *Second handbook of research on the teaching and learning of mathematics* (Vol. 2, pp. 957–1009). Charlotte, NC: Information Age Publishing.

Strike, K. A., & Posner, G. J. (1985). A conceptual change view of learning and understanding. In L. West & L. Pines (Eds.), *Cognitive structure and conceptual change* (pp. 259–266). Orlando, FL: Academic Press.

Thompson, P. W., Liu, Y., & Saldanha, L. A. (2007). Intricacies of statistical inference and teachers' understandings of them. In M. Lovett & P. Shah (Eds.), *Thinking with data* (pp. 207–231). New York: Lawrence Erlbaum Associates.
Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry (with discussion). *International Statistical Review, 67,* 223–265.