Chapter 8

The F-Functional Calculus for Unbounded Operators

Similar to the S-functional calculus, we can also extend the F-functional calculus to unbounded operators by suitably transforming the operator and the function and then applying the theory for bounded operators. Let us first specify the type of operator for which this is possible.

Let $X = X_{\mathbb{R}} \otimes \mathbb{H}$ be a quaternionic two-sided Banach space and let $T_\ell : \mathcal{D}(T_\ell) \subset X_{\mathbb{R}} \to X_{\mathbb{R}}$ be linear closed operators for $\ell = 0, \ldots, 3$ such that $T_\ell T_\kappa = T_\kappa T_\kappa$ on $\mathcal{D}(T_\ell T_\kappa) \cap \mathcal{D}(T_\kappa T_\ell)$ for $\ell, \kappa = 0, \ldots, 3$. Then

$$\mathcal{D}(T) = \bigcap_{\ell=0}^{3} \mathcal{D}(T_\ell)$$

is the domain of the quaternionic right linear operators

$$T = T_0 + \sum_{\ell=1}^{3} e_\ell T_\ell : \quad \mathcal{D}(T) \subset X \to X$$

and

$$\overline{T} = T_0 - \sum_{\ell=1}^{3} e_\ell T_\ell : \quad \mathcal{D}(T) \subset X \to X.$$

Definition 8.0.1. We denote the set of closed right linear operators with commuting components as discussed above by $\mathcal{KC}(X)$.

For operators in $\mathcal{KC}(X)$, we can characterize their *S-resolvent set* and *S-spectrum* just as in Theorem 4.5.6 as

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \quad \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) \right\}$$

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_8

with

$$\mathcal{Q}_{c,s}(T) = (s^2\mathcal{I} - 2sT_0 + T\overline{T})^{-1}.$$

Definition 8.0.2 (The F-resolvent operators for the unbounded operators)**.** Let $T \in \mathcal{KC}(X)$. For $s \in \rho_S(T)$, we define the *(left) F-resolvent operator* as

$$F_L(s,T) := -4(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-2}.$$

8.1 Relations Between F-Resolvent Operators

The following results are important since they will lead us to the definition of F-functional calculus for unbounded operators.

Proposition 8.1.1. *Let $T \in \mathcal{KC}(X)$ and assume that there exists a point $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and set $A := (T - \alpha\mathcal{I})^{-1}$ as in Theorem* 5.2.3*. For $p = (s-\alpha)^{-1}$, we have*

$$\mathcal{Q}_{c,p}(A)^{-1} = \left(A\overline{A}\right)^{-1} \mathcal{Q}_{c,s}(T)^{-1}p^{-2} = \mathcal{Q}_{c,\alpha}(T)\mathcal{Q}_{c,s}(T)^{-1}p^{-2}$$

and

$$\mathcal{Q}_{c,p}(A)^{-2} = \left(A\overline{A}\right)^{-2} \mathcal{Q}_{c,s}(T)^{-2}p^{-4} = \mathcal{Q}_{c,\alpha}(T)^2\mathcal{Q}_{c,s}(T)^{-2}p^{-4}.$$

Proof. Observe that

$$\begin{aligned}
&p^2\mathcal{I} - p\left(A + \overline{A}\right) + A\overline{A} \\
&= \left(p^2\left(A\overline{A}\right)^{-1} - p\left(A + \overline{A}\right)\left(A\overline{A}\right)^{-1} + \mathcal{I}\right)\left(A\overline{A}\right) \\
&= \left(p^2\left(A\overline{A}\right)^{-1} - p\left(A^{-1} + \overline{A}^{-1}\right) + \mathcal{I}\right)\left(A\overline{A}\right),
\end{aligned}$$

where we have used the fact that $\left(A\overline{A}\right)^{-1} = A^{-1}\overline{A}^{-1} = \overline{A}^{-1}A^{-1}$. Recalling that $A := (T - \alpha\mathcal{I})^{-1}$ and $\overline{A} := \left(\overline{T} - \alpha\mathcal{I}\right)^{-1}$, we obtain

$$\overline{A}^{-1}A^{-1} = \alpha^2\mathcal{I} - \alpha\left(T + \overline{T}\right) + T\overline{T} = \mathcal{Q}_{c,\alpha}(T)$$

and

$$A^{-1} + \overline{A}^{-1} = T + \overline{T} - 2\alpha\mathcal{I},$$

so that we obtain

$$\begin{aligned}
\mathcal{Q}_{c,p}(A)^{-1} &= \left(p^2\mathcal{I} - p\left(A + \overline{A}\right) + A\overline{A}\right)^{-1} \\
&= \left(A\overline{A}\right)^{-1}\left(p^2\left(A\overline{A}\right)^{-1} - p\left(A^{-1} + \overline{A}^{-1}\right) + \mathcal{I}\right)^{-1} \\
&= \mathcal{Q}_{c,\alpha}(T)\left(p^2\ \mathcal{Q}_{c,\alpha}(T) - p\left(T + \overline{T} - 2\alpha\mathcal{I}\right) + \mathcal{I}\right)^{-1}.
\end{aligned}$$

Observe now that $T+\overline{T}=2T_0$ and $T\overline{T}=\sum_{\ell=0}^{3}T_\ell^2$ are scalar operators and thus commute with p, so we have

$$\mathcal{Q}_{c,p}(A)^{-1}=\mathcal{Q}_{c,\alpha}(T)\left(\mathcal{Q}_{c,\alpha}(T)-p^{-1}\left(T+\overline{T}-2\alpha\mathcal{I}\right)+p^{-2}\mathcal{I}\right)^{-1}p^{-2}.$$

Finally, we also get

$$\begin{aligned}
&\mathcal{Q}_{c,\alpha}(T)-p^{-1}\left(T+\overline{T}-2\alpha\mathcal{I}\right)+p^{-2}\mathcal{I}\\
&=\alpha^2\mathcal{I}-\alpha\left(T+\overline{T}\right)+T\overline{T}-p^{-1}\left(T+\overline{T}\right)+2\alpha p^{-1}\mathcal{I}+p^{-2}\mathcal{I}\\
&=T\overline{T}-\left(p^{-1}+\alpha\right)\left(T+\overline{T}\right)+\left(p^{-2}+\alpha^2+2\alpha p^{-1}\right)\mathcal{I}\\
&=T\overline{T}-s\left(T+\overline{T}\right)+s^2\mathcal{I}=\mathcal{Q}_{c,s}(T),
\end{aligned}$$

and so

$$\mathcal{Q}_{c,p}\left(A\right)^{-1}=\mathcal{Q}_{c,\alpha}(T)\mathcal{Q}_{c,s}\left(T\right)^{-1}p^{-2}.$$

Since $\alpha\in\mathbb{R}$, we have

$$sp=s\left(s-\alpha\right)^{-1}=ps,$$

and so $\mathcal{Q}_{c,\alpha}(T)$, $\mathcal{Q}_s\left(T\right)$, and p^{-2} commute mutually. Therefore, we also obtain

$$\mathcal{Q}_{c,p}\left(A\right)^{-2}=\mathcal{Q}_{\alpha}(A)^2\mathcal{Q}_s\left(T\right)^{-2}p^{-4}. \qquad \square$$

From Proposition 8.1.1, we deduce now two important relations between the F-resolvents of T and A.

Theorem 8.1.2. *Let $T\in\mathcal{KC}(X)$, let $\alpha\in\rho_S(T)\cap\mathbb{R}\neq\emptyset$, and define $A=(T-\alpha\mathcal{I})^{-1}$. For $s\in\rho_S(T)$ with $s\notin\sigma_L(T)$ and $p=(s-\alpha)^{-1}$, we have*

$$F_L(s,T)=-\overline{A}A^2F_L(p,A)p^3. \tag{8.1}$$

Proof. We recall that $F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\mathcal{Q}_{c,p}(A)^{-2}$. Due to Proposition 8.1.1, we have

$$F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\overline{A}^{-2}A^{-2}\mathcal{Q}_{c,s}(T)^{-2}p^{-4}.$$

Since $s=p^{-1}+\alpha$ commutes with p, we have

$$F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\overline{A}^{-2}A^{-2}p^{-1}\mathcal{Q}_{c,s}(T)^{-2}p^{-3},$$

and so

$$F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\overline{A}^{-2}A^{-2}p^{-1}\left(s\mathcal{I}-\overline{T}\right)^{-1}\left(s\mathcal{I}-\overline{T}\right)\mathcal{Q}_{c,s}(T)^2p^{-3}.$$

Observe that $\overline{A}^{-2}A^{-2}=\mathcal{Q}_{\alpha,s}(T)^2$ is a scalar operator since $\alpha\in\mathbb{R}$ and hence commutes with p and so also with $\left(p\mathcal{I}-\overline{A}\right)$. Since $F_L(s,T)=-4\left(s\mathcal{I}-\overline{T}\right)\mathcal{Q}_{c,s}(T)^{-2}$, we obtain

$$F_L(p,A)=\overline{A}^{-2}A^{-2}\left(p\mathcal{I}-\overline{A}\right)p^{-1}\left(s\mathcal{I}-\overline{T}\right)^{-1}F_L(s,T)p^{-3}.$$

Replacing $s = p^{-1} + \alpha$, we get

$$
\begin{aligned}
F_L(p, A) &= \overline{A}^{-2} A^{-2} \left(p\mathcal{I} - \overline{A}\right) p^{-1} \left(p^{-1}\mathcal{I} + \alpha\mathcal{I} - \overline{T}\right)^{-1} F_L(s, T) p^{-3} \\
&= \overline{A}^{-2} A^{-2} \left(p\mathcal{I} - \overline{A}\right) p^{-1} \left(p^{-1}\mathcal{I} - \overline{A}^{-1}\right)^{-1} F_L(s, T) p^{-3}
\end{aligned}
$$

and hence

$$
F_L(p, A) = -\overline{A}^{-2} A^{-2} \left(p\mathcal{I} - \overline{A}\right) p^{-1} \left(p \left(p\mathcal{I} - \overline{A}\right)^{-1} \overline{A}\right) F_L(s, T) p^{-3}.
$$

We thus get the statement because

$$
\begin{aligned}
F_L(p, A) &= -\left(\overline{A}\right)^{-2} A^{-2} \overline{A} F_L(s, T) p^{-3} \\
&= -\overline{A}\left(\overline{A}\right)^{-2} A^{-2} F_L(s, T) p^{-3} = -\left(\overline{A}\right)^{-1} A^{-2} F_L(s, T) p^{-3}. \qquad \square
\end{aligned}
$$

Theorem 8.1.3. *Let $T \in \mathcal{KC}(X)$, let $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$, and define $A = (T - \alpha\mathcal{I})^{-1}$. For $s \in \rho_S(T)$ and $p = (s - \alpha)^{-1}$, we have*

$$
(A\overline{A})^{-1} F_L(p, A) p^4 = -4p\mathcal{Q}_{c,s}(T)^{-1} - F_L(s, T). \tag{8.2}
$$

Proof. We recall that

$$
A\overline{A} = (\alpha^2\mathcal{I} - \alpha(T + \overline{T}) + T\overline{T})^{-1} : \ \mathcal{D}(T\overline{T}) \to V
$$

and that

$$
A + \overline{A} = (T + \overline{T} - 2\alpha\mathcal{I})A\overline{A} \quad : \ \mathcal{D}(T\overline{T}) \to \mathcal{D}(T).
$$

Using the relation $s = p^{-1} + \alpha$, we get

$$
p^2\mathcal{I} - p(A + \overline{A}) + A\overline{A} = p^2\left(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}\right)(T - \alpha\mathcal{I})^{-1}(\overline{T} - \alpha\mathcal{I})^{-1}, \tag{8.3}
$$

where the right-hand side of (8.3) is the composition of the maps

$$
(T - \alpha\mathcal{I})^{-1}(\overline{T} - \alpha\mathcal{I})^{-1} : \ V \to \mathcal{D}(T\overline{T})
$$

and

$$
\left(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}\right) : \quad \mathcal{D}(T\overline{T}) \to V.
$$

We write $F_L(p, A)$ now in terms of the above positions and get

$$
\begin{aligned}
F_L(p, A) = &-4[(p\mathcal{I} - (\overline{T} - \alpha\mathcal{I})^{-1})(\overline{T} - \alpha\mathcal{I})(T - \alpha\mathcal{I})] \\
&\times (\overline{T} - \alpha\mathcal{I})(T - \alpha\mathcal{I})\left(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}\right)^{-2} p^{-8}.
\end{aligned}
$$

Due to $s = p^{-1} + \alpha$, we have

$$
\begin{aligned}
&[(p\mathcal{I} - (\overline{T} - \alpha\mathcal{I})^{-1})(\overline{T} - \alpha\mathcal{I})(T - \alpha\mathcal{I})] \\
&= [p(\alpha^2\mathcal{I} - \alpha(T + \overline{T}) + T\overline{T}) + \alpha\mathcal{I} - T] \\
&= p\mathcal{Q}_{c,s}(T) - (s\mathcal{I} - \overline{T}),
\end{aligned}
$$

from which we conclude

$$F_L(p,A) = -4(\overline{T}-\alpha\mathcal{I})(T-\alpha\mathcal{I})[p\mathcal{Q}_{c,s}(T)-(s\mathcal{I}-\overline{T})]$$
$$\times\left(s^2\mathcal{I}-s(T+\overline{T})+T\overline{T}\right)^{-2}p^{-4},$$

which gives

$$(A\overline{A})^{-1}F_L(p,A)p^4 = -4p\mathcal{Q}_{c,s}(T)^{-1}-F_L(s,T). \qquad \square$$

8.2 The F-Functional Calculus for Unbounded Operators

Let $T\in\mathcal{KC}(X)$ with $\rho_S(T)\cap\mathbb{R}\neq\emptyset$. For $\alpha\in\rho_S\cap\mathbb{R}$, we define $\Phi_\alpha:\overline{\mathbb{H}}\to\overline{\mathbb{H}}$ as

$$\Phi_\alpha(s)=(s-\alpha)^{-1},\quad \Phi_\alpha(\alpha)=\infty,\quad \Phi_\alpha(\infty)=0, \tag{8.4}$$

and set $A:=(T-\alpha\mathcal{I})^{-1}$. We recall that by Theorem 5.2.3, we have $\Phi(\overline{\sigma}_S(T))=\sigma_S(A)$ and that

$$\mathcal{SH}_L(\sigma_S(A))=\left\{f\circ\Phi_\alpha^{-1}:\quad f\in\mathcal{SH}_L(\overline{\sigma}_S(T))\right\}.$$

Definition 8.2.1 (F-functional calculus for unbounded operators)**.** Let $T\in\mathcal{KC}(X)$ with $\rho_S(T)\cap\mathbb{R}\neq\emptyset$, let $\alpha\in\rho_S(T)\cap\mathbb{R}$, and define Φ_α and A as in (8.4). For $f\in\mathcal{SH}_L(\overline{\sigma}_S(T))$ with $f(\alpha)=0$, we consider the functions

$$\phi(q):=(f\circ\Phi_\alpha^{-1})(q),$$
$$\breve{\psi}(q):=\Delta(q^2\phi(q)),$$

and define the operator $\breve{f}(T)$ for $\breve{f}=\Delta f$ as

$$\breve{f}(T):=(A\overline{A})^{-1}\breve{\psi}(A), \tag{8.5}$$

where $\breve{\psi}(A)$ is intended in the sense of Definition 7.1.11.

Remark 8.2.2. Observe that the condition $f(\alpha)=0$ is not a restriction in the above definition. Indeed, if $f(\alpha)\neq 0$, then we can consider the function $\tilde{f}(q)=f(q)-f(\alpha)$ and we find that also $\tilde{f}\in\mathcal{SH}_L(\overline{\sigma}_S(T))$ with $\breve{f}=\Delta\tilde{f}$, but now $\tilde{f}=0$. We will take this fact into account in the next result.

Theorem 8.2.3. *Let $T\in\mathcal{KC}(X)$ with $\rho_S(T)\cap\mathbb{R}\neq\emptyset$, let $\alpha\in\rho_S(T)\cap\mathbb{R}$, and define Φ_α and A as in (8.4). For $\breve{f}=\Delta f$ with $f\in\mathcal{SH}_L(\overline{\sigma}_S(T))$ with $f(\alpha)=0$, the operator $\breve{f}(T)$ defined in (8.5) satisfies*

$$\breve{f}(T)=\int_{\partial(U\cap\mathbb{C}_j)}F_L(s,T)\,ds_j\,f(s), \tag{8.6}$$

where U is any unbounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and j is any imaginary unit in $\mathbb{S}$.

In particular, $\breve{f}(T)$ is independent of α. If $f_* = f - c$ with $c \in \mathbb{H}$ such that $f_*(\beta) = 0$ with $\beta \in \rho_S(T) \cap \mathbb{R}$, we can define $\breve{f}_*(T)$ using β instead of α. Then $\breve{f} = \breve{f}_*$ and $\breve{f}(T) = \breve{f}_*(T)$.

Proof. Let $j \in \mathbb{S}$ and let U be a slice Cauchy domain as above. Furthermore, we assume that $\alpha \notin \overline{U}$. If this is not the case, we can replace U by the axially symmetric slice Cauchy domain $U \setminus \overline{B_\varepsilon(0)}$ with sufficiently small $\varepsilon > 0$ without altering the value of the integral in (8.6) by the Cauchy integral theorem.

The set $V = \Phi_\alpha(U)$ is a bounded slice Cauchy domain with $\sigma_S(T) \subset V$ and $\overline{V} \subset \mathcal{D}(f \circ \Phi_\alpha^{-1}) = \Phi(\mathcal{D}(f))$.

Using the second relation between $F_L(p, A)$ and $F_L(s, T)$, see formula (8.2), we have

$$\begin{aligned} &\int_{\partial(U\cap\mathbb{C}_j)} \left(-4p\mathcal{Q}_{c,s}(T)^{-1} - F_L(s,T)\right)\, ds_j\, f(s) \\ &= (A\overline{A})^{-1} \int_{\partial(V\cap\mathbb{C}_j)} F_L(p,A)\, dp_j\, p^2\phi(p). \end{aligned} \tag{8.7}$$

Now we work on the left-hand side:

$$\begin{aligned} &\int_{\partial(U\cap\mathbb{C}_j)} \left(-4p\mathcal{Q}_{c,s}(T)^{-1} - F_L(s,T)\right)\, ds_j\, f(s) \\ &= \int_{\partial(U\cap\mathbb{C}_j)} -4p\mathcal{Q}_{c,s}(T)^{-1} ds_j f(s) - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T) ds_j f(s) \\ &= -4\int_{\partial(U\cap\mathbb{C}_j)} (s-\alpha)^{-1}\, ds_j\, \mathcal{Q}_{c,s}(T)^{-1} f(s) \\ &\quad - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T) ds_j f(s) \\ &= -4(2\pi)\mathcal{Q}_\alpha(T)^{-1} f(\alpha) - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\, ds_j\, f(s). \end{aligned}$$

The last identity follows because $\mathcal{Q}_{c,s}(T)^{-1}\, ds_j = ds_j\, \mathcal{Q}_{c,s}(T)^{-1}$, since $T + \overline{T}$ and $T\overline{T}$ are scalar operators, so that

$$\begin{aligned} &-4\int_{\partial(U\cap\mathbb{C}_j)} (s-\alpha)^{-1}\, ds_j\, \mathcal{Q}_{c,s}(T)^{-1} f(s) \\ &= -4\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,\alpha)\, ds_j\, \mathcal{Q}_{c,s}(T)^{-1} f(s) = -4(2\pi)\mathcal{Q}_{c,\alpha}(T)^{-1} f(\alpha) \end{aligned}$$

by Cauchy's integral formula because $s \mapsto \mathcal{Q}_{c,s}(T)^{-1} f(s)$ is left slice hyperholomorphic. Indeed, for $s = u + jv$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right)\mathcal{Q}_{c,s}(T)^{-1}f(s) = \left(\frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right)\mathcal{Q}_{c,s}(T)^{-1}\right)f(s) + \mathcal{Q}_{c,s}(T)^{-1}\left(\frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right)f(s)\right) = 0$$

because $\mathcal{Q}_s(T)^{-1}$ commutes with j, since $T + \overline{T}$ and $T\overline{T}$ are scalar operators.

The identity (8.7) therefore turns into

$$\begin{aligned} &- 4(2\pi)\mathcal{Q}_\alpha(T)^{-1}f(\alpha) - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f(s) \\ &= (A\overline{A})^{-1}\int_{\partial(V\cap\mathbb{C}_j)} F_L(p,A)\,dp_j\,p^2\phi(p). \end{aligned}$$

Since by assumption $f(\alpha) = 0$, we conclude that

$$\begin{aligned} \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f(s) &= \frac{1}{2\pi}(A\overline{A})^{-1}\int_{\partial(\mathcal{V}\cap\mathbb{C}_j)} F_L(p,A)\,dp_j\,p^2\phi(p) \\ &= (A\overline{A})^{-1}\breve{\psi}(A) = \breve{f}(T). \end{aligned}$$

Finally, if $f_* = f + c$ with $f_*(\beta) = 0$ for some $\beta \in \rho_S(T)\cap\mathbb{R}$, then we find that

$$\begin{aligned} \breve{f}_*(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f_*(s) \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f(s) \\ &\quad + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,c = \breve{f}(T), \end{aligned}$$

since

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,c = 0$$

by Cauchy's integral theorem. □

8.3 Comments and Remarks

The definition of the F-functional calculus can be extended to the case of n-tuples of unbounded operators. As is well known in the case of unbounded operators, the notion of commutativity is more delicate, and one has to pay attention to the domains of the operators. The situation is simpler when just one of the operators $T_j : \mathcal{D}(T_j) \subset X \to X$, $j = 0, 1, \ldots, n$, is unbounded; see [78].

8.3.1 F-Functional Calculus for n-Tuples of Unbounded Operators

The definition of the F-functional calculus for unbounded operators is less intuitive than the S-functional calculus for unbounded operators. The reason is that the S-functional calculus is defined by a Cauchy formula, while the F-functional calculus is defined by an integral transform that maps slice monogenic functions to monogenic functions.

Definition 8.3.1 (Admissible operators)**.** Let X be a real Banach space and $X_n = X \otimes \mathbb{R}_n$. Let $T_j : \mathcal{D}(T_j) \subset X \to X$ be linear closed operators for $j = 0, 1, \dots, n$, such that $T_jT_ix = T_iT_jx$, for all $x \in \mathcal{D}(T_jT_i) \cap \mathcal{D}(T_iT_j)$ for $i, j = 0, 1, \dots, n$. Let $\mathcal{D}(T) = \bigcap_{j=0}^{n} \mathcal{D}(T_j)$ be the domain of the operator $T = T_0 + \sum_{j=1}^{n} e_jT_j \ : \mathcal{D}(T) \subset X_n \to X_n$. We say that T is an *admissible operator* if

1) $\bigcap_{j=0}^{n} \mathcal{D}(T_j)$ is dense in X_n,

2) $s\mathcal{I} - \overline{T}$ is densely defined in X_n for $s \in \mathbb{R}^{n+1}$,

3) $\mathcal{D}(T\overline{T}) \subset \mathcal{D}(T)$ is dense in X_n.

We need the following definitions:

- Let $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and let n be an odd number and let $p = (s - \alpha)^{-1}$. Set $A := (T - \alpha\mathcal{I})^{-1}$.

- Let $\alpha \in \mathbb{R}$ and define the homeomorphism $\Phi : \overline{\mathbb{R}}^{n+1} \to \overline{\mathbb{R}}^{n+1}$,

$$p = \Phi(s) = (s - \alpha)^{-1}, \quad \Phi(\infty) = 0, \quad \Phi(\alpha) = \infty. \tag{8.8}$$

Definition 8.3.2 (The F-functional calculus for n-tuples of unbounded operators)**.** Let n be an odd number and let $T : \mathcal{D}(T) \to X_n$ be an admissible operator with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{SM}_L(\overline{\sigma}_S(T))$. Let us consider the functions

$$\phi(p) := f(\Phi^{-1}(p)),$$

$$\breve{\psi}(p) := \Delta_p^{\frac{n-1}{2}} (p^{n-1}\phi(p)),$$

where Δ_p is the Laplace operator in dimension $n + 1$, and recall that

$$A := (T - \alpha\mathcal{I})^{-1}, \quad \text{for some } \alpha \in \rho_S(T) \cap \mathbb{R}.$$

With the notation above, we define

$$\widetilde{f}(T) := (A\overline{A})^{-\frac{n-1}{2}} \breve{\psi}(A) \tag{8.9}$$

for functions f such that $f(\alpha) = 0$.

The definition seems unnatural, but it is suggested by the two relations between the resolvents $F_n(p, A)$ and $F_n(s, T)$.

Theorem 8.3.3 (First relation between the F-resolvents)**.** *Let T be admissible, let $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$, let n be an odd number, and let $p = (s - \alpha)^{-1}$. Let us put $A := (T - \alpha\mathcal{I})^{-1}$ and suppose that $p \in \rho_S(A)$ and $p \neq 0$. Then we have*

$$F_n(s, T) = -(\overline{A})^{\frac{n-1}{2}} A^{\frac{n+1}{2}} F_n(p, A) p^n. \tag{8.10}$$

Theorem 8.3.4 (Second relation between $F_n(p, A)$ and $F_n(s, T)$)**.** *Let $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and let n be an odd number and let $p = (s-\alpha)^{-1}$. Recall that $A := (T-\alpha\mathcal{I})^{-1}$ for T admissible. Let $s \in \rho_S(T)$ and $p \neq 0$. Then we have*

$$(A\overline{A})^{-\frac{n-1}{2}} F_n(p, A) p^{n+1} = \gamma_n p(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n-1}{2}} - F_n(s, T), \tag{8.11}$$

where γ_n are defined in (7.29)*, i.e., $\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} \left[\left(\frac{n-1}{2}\right)!\right]^2$.*

Thanks to Theorem 8.3.3 and (8.11), we can prove that for n an odd number, if $k \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and Φ, ϕ are as above, then $\Phi(\overline{\sigma}_S(T)) = \sigma_S(A)$, and the relation $\phi(p) := f(\Phi^{-1}(p))$ determines a one-to-one correspondence between $f \in \mathcal{SM}^L(\overline{\sigma}_S(T))$ and $\phi \in \mathcal{SM}(\overline{\sigma}_S(A))$, and so the integral representation theorem of the F-functional calculus is what we expect:

Theorem 8.3.5. *Let n be an odd number and let T be admissible with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{SM}_L(\overline{\sigma}_S(T))$ and set $ds_j = -dsj$ for $j \in \mathbb{S}$. If $f(k) = 0$, then the operator $\widetilde{f}(T) := (A\overline{A})^{-\frac{n-1}{2}} \breve{\psi}(A)$, defined in* (8.9)*, does not depend on $k \in \rho_S(T) \cap \mathbb{R}$. Moreover, we have the integral formula*

$$\widetilde{f}(T) = \int_{\partial(W \cap \mathbb{C}_j)} F_n^L(s, T) ds_j f(s), \tag{8.12}$$

where W is a suitable Cauchy domain.

The reason we have defined the F-functional calculus as in (8.9) is essentially due to the relation in Theorems 8.3.3 and 8.3.4. Thanks to this relation, we can prove that $\widetilde{f}(T)$ is independent of k and admits the integral representation (8.12). A similar definition can be found for $f \in \mathcal{SM}_R(\overline{\sigma}_{\mathcal{S}}(T))$.