Chapter 7

The F-Functional Calculus for Bounded Operators

The Fueter mapping theorem in integral form introduced in [86], see Chapter 2.2, provides an integral transform that turns slice hyperholomorphic functions into Fueter regular ones. By formally replacing the scalar variable in this integral transform by an operator T, we obtain a functional calculus for Fueter regular functions that is based on the theory of slice hyperholomorphic functions. The F-functional calculus was introduced and studied in the following papers [54, 78, 81, 86].

7.1 The F-Resolvent Operators and the F-Functional Calculus

We begin our discussion with the feasibility of this functional calculus.

Definition 7.1.1. For $m \in \mathbb{N}$ and $q \in \mathbb{H}$ we consider the *Fueter regular polynomials*

$$\mathcal{P}_m(q) := \Delta q^m. \tag{7.1}$$

Lemma 7.1.2. *We have* $\mathcal{P}_0 \equiv \mathcal{P}_1 \equiv 0$ *and* $\mathcal{P}_2 \equiv -4$*. Furthermore, for even* $m \geq 2$*, we have*

$$\mathcal{P}_m(q) = m(m-1)q^{m-2} + 2\mathrm{Re}\left(\sum_{\ell=1}^{3}\sum_{\kappa=1}^{m-1}\sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}\right), \tag{7.2}$$

and for odd $m \geq 2$ *we have*

$$\mathcal{P}_m(q) = m(m-1)q^{m-2} + 2\mathrm{Im}\left(\sum_{\ell=1}^{3}\sum_{\kappa=1}^{m-1}\sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}\right). \tag{7.3}$$

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
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Proof. The identities $\mathcal{P}_0 \equiv \mathcal{P}_1 \equiv 0$ and $\mathcal{P}_2 = -4$ follow by straightforward computations. Thus assume that $m \geq 2$.

For $q = q_0 + \underline{q} = q_0 + \sum_{\ell=1}^{3} q_\ell e_\ell \in \mathbb{H}$, we have

$$q^m = \sum_{k=0}^{m} \binom{m}{k} q_0^k \underline{q}^{m-k},$$

and so

$$\begin{aligned}\frac{\partial^2}{\partial q_0^2} q^m &= \sum_{k=2}^{m} \binom{m}{k} k(k-1) q_0^{k-2} \underline{q}^{m-k} = \sum_{k=2}^{m} \frac{m!}{(m-k)!(k-2)!} q_0^{k-2} \underline{q}^{m-k} \\ &= m(m-1) \sum_{k=0}^{m-2} \frac{(m-2)!}{(m-k)!(k-2)!} q_0^k \underline{q}^{m-2-k} = m(m-1) q^{m-2}.\end{aligned}$$

Furthermore, observe that for $1 \leq \ell \leq 3$ we have

$$\frac{\partial}{\partial q_\ell} \underline{q}^r = \sum_{\kappa=0}^{r-1} \underline{q}^\kappa e_\ell \underline{q}^{r-1-\kappa}. \tag{7.4}$$

For $r = 1$, we have $\frac{\partial}{\partial q_\ell} \underline{q} = e_\ell$, and so (7.4) holds. If, on the other hand, (7.4) holds for $r-1$, then

$$\begin{aligned}\frac{\partial}{\partial q_\ell} \underline{q}^r &= \left(\frac{\partial}{\partial q_\ell} q \right) q^{r-1} + q \left(\frac{\partial}{\partial q_\ell} q^{r-1} \right) \\ &= e_\ell q^{r-1} + \sum_{\kappa=0}^{r-2} \underline{q}^{k+1} e_\ell \underline{q}^{r-2-\kappa} = \sum_{\kappa=0}^{r-1} \underline{q}^\kappa e_\ell \underline{q}^{r-1-\kappa}.\end{aligned}$$

Applying this identity twice, we obtain

$$\begin{aligned}\frac{\partial}{\partial q_\ell} \underline{q}^m &= \sum_{\kappa=1}^{m-1} \left(\frac{\partial}{\partial q_\ell} \underline{q}^\kappa \right) e_\ell \underline{q}^{m-1-\kappa} + \sum_{\kappa=0}^{m-2} \underline{q}^\kappa e_\ell \left(\frac{\partial}{\partial q_\ell} \underline{q}^{m-1-\kappa} \right) \\ &= \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} + \sum_{\kappa=0}^{m-2} \sum_{s=0}^{m-2-\kappa} \underline{q}^\kappa e_\ell \underline{q}^s e_\ell \underline{q}^{m-2-\kappa-s} \\ &= \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} + \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^{m-1-\kappa} e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^s \\ &= \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} + (-1)^m \overline{\sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}},\end{aligned}$$

where the last identity follows from

$$\begin{aligned}\overline{\underline{q}^{m-1-\kappa} e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^s} &= \overline{\underline{q}^s}\, \overline{e_\ell}\, \overline{\underline{q}^{\kappa-1-s}}\, \overline{e_\ell}\, \overline{\underline{q}^{m-1-\kappa}} \\ &= (-1)^m \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}\end{aligned}$$

because $\overline{\underline{q}} = -\underline{q}$, since $\underline{q}$ is purely imaginary. Therefore, we obtain

$$\begin{aligned}\Delta q^m = \frac{\partial^2}{\partial q_0^2} q^m + \sum_{\ell=1}^{3} \frac{\partial}{\partial q_\ell^2} q^m &= m(m-1)q^{m-2} \\ &+ \sum_{\ell=1}^{3}\sum_{\kappa=1}^{m-1}\sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} \\ &+ (-1)^m \overline{\sum_{\kappa=1}^{m-1}\sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}},\end{aligned}$$

which yields (7.2) resp. (7.3). □

Definition 7.1.3 (Fueter kernel series)**.** Let $s, q \in \mathbb{H}$. We define the *left Fueter kernel series* as

$$\sum_{m\geq 2} \mathcal{P}_m(q) s^{-1-m},$$

and the *right Fueter kernel series* as

$$\sum_{m\geq 2} s^{-1-m} \mathcal{P}_m(q).$$

Proposition 7.1.4. *For $s, q \in \mathbb{H}$ with $|q| < |s|$, the left and right Fueter kernel series converge.*

Proof. Because of (7.2) and (7.3), we have for $m \geq 2$ that

$$\begin{aligned}|\mathcal{P}_m(q)| &\leq m(m-1)|q|^{m-2} + 2\sum_{\ell=1}^{3}\sum_{k=1}^{m-1}\sum_{s=0}^{\kappa-1} |q|^{m-2} \\ &= m(m-1)|q|^{m-2} + 3m(m-1)|q|^{m-2} = 4m(m-1)|q|^{m-2}.\end{aligned}$$

If $|q| < |s|$, we therefore have for the left Fueter kernel series

$$\sum_{m\geq 2} |\mathcal{P}_m(q)s^{-1-m}| \leq 4\sum_{m\geq 2} m(m-1)|q|^{m-2}|s^{-1-m}| < +\infty,$$

and the convergence of the right Fueter kernel series is shown similarly. □

The Fueter kernel series are the Taylor series expansions of the Fueter kernels $F_L(s,q)$ and $F_R(s,q)$ introduced in Definition 2.2.5. They are their slice hyperholomorphic Taylor expansions in the variable s at infinity and the Fueter regular Taylor expansions in the variable q at 0; cf. the Comments and Remarks in Section 7.6.

Lemma 7.1.5. *For $|q| < |s|$, we have*

$$F_L(s,q) = \sum_{n=0}^{+\infty} \mathcal{P}_m(q)s^{-1-m} \quad \text{and} \quad F_R(s,q) = \sum_{n=0}^{+\infty} s^{-1-m}\mathcal{P}_m(q).$$

Proof. Due to the Taylor series expansion $S_L^{-1}(s,q) = \sum_{n=0}^{+\infty} q^n s^{-1-n}$ of the left Cauchy kernel in Theorem 2.1.22, we have

$$F_L(s,q) = \Delta S_L^{-1}(s,q) = \sum_{n=0}^{+\infty} \Delta q^n s^{-1-n} = \sum_{n=2}^{+\infty} \mathcal{P}_n(q)s^{-1-n},$$

where we are allowed to exchange the Laplacian with the sum because of the uniform convergence shown in Proposition 7.1.4.

The series of the right Fueter kernel follows similarly from the Taylor series expansion of the right Cauchy kernel. □

Because of the above considerations, we can define the Fueter kernel operator series by formally replacing q in the Fueter kernel series by the operator T with commuting components.

Definition 7.1.6 (Fueter kernel operator series)**.** Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$ with $\|T\| < |s|$, we define the *left Fueter kernel operator series* as

$$\sum_{m\geq 2} \mathcal{P}_m(T)s^{-1-m}$$

and the *right Fueter kernel operator series* as

$$\sum_{m\geq 2} s^{-1-m}\mathcal{P}_m(T).$$

Proposition 7.1.7. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$ with $\|T\| < |s|$, we have*

$$\sum_{m\geq 2} \mathcal{P}_m(T)s^{-1-m} = -4(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-2} \tag{7.5}$$

and

$$\sum_{m\geq 2} s^{-1-m}\mathcal{P}_m(T) = -4\mathcal{Q}_{c,s}(T)^{-2}(s\mathcal{I} - \overline{T}) \tag{7.6}$$

with $\mathcal{Q}_{c,s}(T) = s^2\mathcal{I} - 2sT_0 + T\overline{T}$, where $\overline{T} = T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell$.

Proof. Using Theorem 2.1.22 and Theorem 2.2.2 we get

$$\begin{aligned}\sum_{m\geq 2} \mathcal{P}_m(q)s^{-1-m} &= \Delta \sum_{m=0}^{+\infty} q^m s^{-1-m} \\ &= \Delta S_L^{-1}(s,q) = -4(s-\overline{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}.\end{aligned}$$

The fact that the components of T commute allows us to substitute T for q; thus we get the statement. □

Remark 7.1.8. We point out an important fact related to the Fueter mapping theorem in integral form. As we could observe in the proof of Theorem 2.2.2, the computation

$$\begin{aligned} &-\Delta(q^2-2q\mathrm{Re}(s)+|s|^2)^{-1}(q-\overline{s}) \\ &=-4(s-\overline{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2} \end{aligned}$$

can be carried out in a natural way only if we write $S_L^{-1}(s,q)$ in form II. The function $S_L^{-1}(s,q)$ can be written in two different ways because the components of q commute. Unfortunately, form II involves the term $|q|^2=\overline{q}q=q\overline{q}$, and this identity requires that the components of x commute. This has implications on the functional calculus when one tries to replace q by an operator T. In this case we have to require that the components of T commute. When we write $S_L^{-1}(s,q)$ in form I, then we can replace q by an operator T whose components do not necessarily commute, because only actual powers q and not powers of its components appear. But in this case the explicit computation of $\Delta S_L^{-1}(s,q)$ does not yield a simple closed from.

Recall that the S-resolvent set of $T\in\mathcal{BC}(X)$ can, by Theorem 4.5.6, be characterized as

$$\rho_S(T)=\left\{s\in\mathbb{H}:\ \ \mathcal{Q}_{c,s}(T)^{-1}\in\mathcal{B}(X)\right\},$$

where

$$\mathcal{Q}_{c,s}(T)=s^2\mathcal{I}-2sT_0+T\overline{T}.$$

Definition 7.1.9 (F-resolvent operators)**.** Let $T\in\mathcal{BC}(X)$. For $s\in\rho_S(T)$, we define the *left F-resolvent operator* as

$$F_L(s,T):=-4\left(s\mathcal{I}-\overline{T}\right)\mathcal{Q}_{c,s}(T)^{-2}, \tag{7.7}$$

and the *right F-resolvent operator* as

$$F_R(s,T):=-4\mathcal{Q}_{c,s}(T)^{-2}\left(s\mathcal{I}-\overline{T}\right). \tag{7.8}$$

Lemma 7.1.10. *Let $T\in\mathcal{BC}(X)$.*

(i) *The left F-resolvent operator $F_L(s,T)$ is a $\mathcal{B}(X)$-valued right slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

(ii) *The right F-resolvent operator $F_R(s,T)$ is a $\mathcal{B}(X)$-valued left slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

Proof. The statement follows by computations that are similar to those in Lemma 3.1.11. □

If f is a left or right slice hyperholomorphic function, then the function $\breve{f}=\Delta f$ is a left, resp. right, Fueter regular function by the Fueter mapping theorem. We showed in Theorem 2.2.6 that $\breve{f}$ can be represented as the integral transform of f involving the left, resp. right, Fueter kernel. If we replace in this integral representation the Fueter kernel by the F-resolvent operator, we obtain the F-functional calculus.

Definition 7.1.11 (The F-functional calculus for bounded operators)**.** Let $T \in \mathcal{BC}(X)$ and set $ds_j = ds(-j)$ for $j \in \mathbb{S}$. For every function $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\, ds_j\, f(s), \tag{7.9}$$

where U is an arbitrary bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and $j \in \mathbb{S}$ is an arbitrary imaginary unit.

For every function $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, F_R(s,T) \tag{7.10}$$

with U and j as above.

Theorem 7.1.12. *The F-functional calculus is well defined, that is, the integrals in (7.9) and (7.10) depend neither on the imaginary unit $j \in \mathbb{S}$ nor on the slice Cauchy domain U.*

Proof. We discuss only the case $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, since the other one follows by analogous arguments.

Since $F_L(s,T)$ is a right slice hyperholomorphic function in s and f is left slice hyperholomorphic, the independence from U follows from the Cauchy integral theorem, cf. also the proof of Theorem 3.2.6.

In order to show the independence from the imaginary unit, we choose $j, i \in \mathbb{S}$ with $j \neq i$ and two bounded slice Cauchy domains U_p, U_s with $\sigma_S(T) \subset U_q$, $\overline{U_q} \subset U_s$, and $\overline{U_s} \subset \mathcal{D}(f)$. Then every $s \in \partial(U_s \cap \mathbb{C}_j)$ belongs to the unbounded slice Cauchy domain $\mathbb{H} \setminus U_q$. Since we have $\lim_{q\to+\infty} F_L(q,T) = 0$, the slice hyperholomorphic Cauchy formula implies

$$\begin{aligned} F_L(s,T) &= \frac{1}{2\pi} \int_{\partial(\mathbb{H}\setminus U_q\cap\mathbb{C}_i)} F_L(q,T)\, dq_i\, S_R^{-1}(q,s) \\ &= \frac{1}{2\pi} \int_{\partial(U_q\cap\mathbb{C}_i)} F_L(q,T)\, dq_i\, S_L^{-1}(s,q), \end{aligned}$$

where the last identity holds because $\partial(\mathbb{H}\setminus U_q\cap\mathbb{C}_i) = -\partial(U_q\cap\mathbb{C}_i)$ and $S_R^{-1}(q,s) = -S_L^{-1}(s,q)$. Thus

$$\begin{aligned} \breve{f}(T) &= \frac{1}{2\pi} \int_{\partial(U_s\cap\mathbb{C}_j)} F_L(s,T)\, ds_j f(s) \\ &= \frac{1}{2\pi} \int_{\partial(U_s\cap\mathbb{C}_j)} \left(\frac{1}{2\pi} \int_{\partial(U_q\cap\mathbb{C}_i)} F_L(q,T)\, dq_i\, S_L^{-1}(s,q) \right) ds_j f(s). \end{aligned}$$

Since the integrand is continuous and the path of integration is bounded, Fubini's theorem allows us to exchange the order of integration, and we obtain

$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_i)} F_L(q,T)\,dq_i\left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j f(s)\right)\\ &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_i)} F_L(q,T)\,dq_i\,f(q). \qquad \square\end{aligned}$$

Remark 7.1.13. In the above theorem we have shown that the F-functional calculus is well defined, in the sense that the integrals in (7.9) and (7.10) depend neither on the imaginary unit $j\in\mathbb{S}$ nor on the slice Cauchy domain U. However, if $f\in\mathcal{SH}_L(U)$, it might happen that $\breve{f}=\Delta f=\Delta g=\breve{g}$ for some $g\in\mathcal{SH}_L(U)$ with $f\neq g$, and we did not show that then $\breve{f}(T)=\breve{g}(T)$. The function $f-g$ is in this case a left slice hyperholomorphic function in $\ker\Delta$. If U is connected, we hence have $f(s)-g(s)=s\alpha+\beta$ with $\alpha,\beta\in\mathbb{H}$ and so

$$\begin{aligned}\breve{f}(T)-\breve{g}(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,(f(s)-g(s))\\ &= \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,(s\alpha-\beta),\end{aligned}$$

where we used Cauchy's integral theorem and the slice hyperholomorphicity of $F_L(s,T)$ in s in order to change the domain of integration to $B_r(0)$ with $\|T\|<r$. From the power series expansion $F_L(s,T)=\sum_{m\geq 2}\mathcal{P}_m(T)s^{-1-m}$ in (7.5), we conclude now that

$$\begin{aligned}\breve{f}(T)-\breve{g}(T) &= \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)}\sum_{m\geq 2} P_m(T)s^{-1-m}ds_j(s\alpha+\beta)\\ &= \sum_{m\geq 2} P_m(T)\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)} s^{-1-m}ds_j(s\alpha+\beta)=0\end{aligned}$$

by Cauchy's integral theorem since the integrand tends to 0 at infinity. If, however, U is not connected, then $f(s)-g(s)=\sum_{\ell=1}^n\chi_{U_\ell}(s)(s\alpha_\ell-\beta_\ell)$, where $U_\ell,\ell=1,\dots,n$ are the connected components of U and χ_{U_ℓ} denotes the characteristic function of U_ℓ. Hence, we have

$$\breve{f}(T)-\breve{g}(T)=\sum_{\ell=1}^n\frac{1}{2\pi}\int_{\partial(U_\ell\cap\mathbb{C}_j)} F_L(s,T)\,ds_j(s\alpha_\ell-\beta_\ell),$$

and we cannot use the same arguments as above in order to show that the terms in the sum vanish, because $F_L(s,T)$ is not slice hyperholomorphic on $\mathbb{H}\setminus U_\ell$ since this set contains part of the S-spectrum of T. In this case, the terms vanish because of Lemma 7.4.1. The proof of this lemma makes, however, use of the monogenic functional calculus by A. McIntosh. This functional calculus makes the assumptions that $T = T_1e_1 + T_2e_2 + T_3e_3$, that is, $T_0 = 0$, with commuting components T_ℓ that have real spectrum. Only if this condition is satisfied we have $\breve{f}(T) = \breve{g}(T)$ also if U is not connected. If this condition is not satisfied, it is in general not true and it is easy to construct counter-examples even using matrices in $\mathbb{H}^{2\times 2}$.

We conclude this section with some algebraic properties of the F-functional calculus.

Proposition 7.1.14. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum.*

(i) *If $\breve{f} = \Delta f$ and $\breve{g} = \Delta g$ with $f, g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then*
$$(\breve{f}a + \breve{g})(T) = \breve{f}(T)a + \breve{g}(T).$$

(ii) *If $\breve{f} = \Delta f$ and $\breve{g} = \Delta g$ with $f, g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then*
$$(a\breve{f} + \breve{g})(T) = a\breve{f}(T) + \breve{g}(T).$$

Proof. The above identities follow immediately from the linearity of the integrals in (7.9), resp. (7.10). □

Proposition 7.1.15. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum.*

(i) *Let $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$ and assume that $f(q) = \sum_{\ell=0}^{+\infty} q^\ell a_\ell$ with $a_\ell \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*
$$\breve{f}(T) = \sum_{\ell=2}^{+\infty} \mathcal{P}_\ell(T) a_\ell.$$

(ii) *Let $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_R(\sigma_S(T))$ and assume that $f(q) = \sum_{\ell=0}^{+\infty} a_\ell q^\ell$ with $a_\ell \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*
$$\breve{f}(T) = \sum_{\ell=2}^{+\infty} a_\ell \mathcal{P}_\ell(T).$$

Proof. We prove (i), but (ii) is shown similarly. We choose an imaginary unit $j \in \mathbb{S}$ and a radius $0 < R < r$ such that $\sigma_S(T) \subset B_R(0)$. Then the series expansion of f converges uniformly on $\partial(B_R(0) \cap \mathbb{C}_j)$, and so
$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_j)} F_L(s, T)\, ds_j \sum_{\ell=0}^{+\infty} s^\ell a_\ell \\ &= \frac{1}{2\pi} \sum_{\ell=0}^{+\infty} \int_{\partial(B_R(0)\cap\mathbb{C}_j)} F_L(s, T)\, ds_j\, s^\ell a_\ell.\end{aligned}$$

Replacing $F_L(s,T)$ by its series expansion, we further obtain

$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi}\sum_{\ell=0}^{+\infty}\int_{\partial(B_R(0)\cap\mathbb{C}_j)}\sum_{k=2}^{+\infty}\mathcal{P}_k(T)s^{-1-k}\,ds_j\ s^\ell a_\ell\\ &= \frac{1}{2\pi}\sum_{\ell=0}^{+\infty}\sum_{k=2}^{+\infty}\mathcal{P}_k(T)\int_{\partial(B_R(0)\cap\mathbb{C}_j)}s^{-1-k}\,ds_j\ s^\ell a_\ell = \sum_{\ell\geq 0}\mathcal{P}_\ell(T)a_\ell,\end{aligned}$$

because the integral $\int_{\partial(B_R(0)\cap\mathbb{C}_j)} s^{-1-k}\,ds_j\ s^\ell$ equals 2π if $\ell = k$, and 0 otherwise. □

Theorem 7.1.16. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1,2,3$, have real spectrum. Let $\breve{f} = \Delta f$ and $\breve{f}_m = \Delta f_m, m \in \mathbb{N}$, with $f, f_m \in \mathcal{SH}_L(\sigma_S(T))$ and assume that f_m tends uniformly to f on an axially symmetric open set O that contains $\sigma_S(T)$. Then $\breve{f}_m$ tends uniformly to $\breve{f}$ on $\sigma_S(T)$ and $\breve{f}_m(T) \to \breve{f}(T)$ in $\mathcal{B}(X)$.*

Proof. Let U be a slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset O$ and choose $j \in \mathbb{S}$. Then

$$\breve{f}_m(q) - \breve{f}(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}(f_m(s) - f(s))\,ds_j\,F_L(s,q).$$

Since $\operatorname{dist}(\sigma_S(T), \partial(U\cap\mathbb{C}_j)) > 0$, we have

$$C := \sup_{\substack{s\in\partial(U\cap\mathbb{C}_j)\\ q\in\sigma_S(T)}} |F_L(s,q)| < +\infty,$$

and so

$$\left|\breve{f}_m(q) - \breve{f}(q)\right| \leq \frac{C}{2\pi}|\partial(U\cap\mathbb{C}_j)| \sup_{s\in\partial(U\cap\mathbb{C}_j)}|f_m(s) - f(s)|,$$

and hence $\breve{f}_m \to \breve{f}$ uniformly on $\sigma_S(T)$. Similarly, we have

$$\begin{aligned}&\left\|\breve{f}_m(T) - \breve{f}(T)\right\| = \left\|\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}(f_m(s) - f(s))\,ds_j\,F_L(s,T)\right\|\\ &\leq \frac{|\partial(U\cap\mathbb{C}_j)|}{2\pi}\sup_{s\in\partial(U\cap\mathbb{C}_j)}\|F_L(s,T)\|\sup_{s\in\partial(U\cap\mathbb{C}_j)}|f_m(s) - f(s)| \overset{m\to+\infty}{\longrightarrow} 0.\end{aligned}$$ □

7.2 Bounded Perturbations of the F-Resolvent

We point out that the inverses of the F-resolvents

$$\begin{aligned}F_L(s,T)^{-1} &= -\frac{1}{4}\mathcal{Q}_{c,s}(T)S_L(s,T) = -\frac{1}{4}\mathcal{Q}_{c,s}(T)^2(s\mathcal{I} - \overline{T})^{-1},\\ F_R(s,T)^{-1} &= -\frac{1}{4}S_R(s,T)\mathcal{Q}_{c,s}(T) = -\frac{1}{4}(s\mathcal{I} - \overline{T})^{-1}\mathcal{Q}_{c,s}(T)^2,\end{aligned}$$

exist for every $s \notin \sigma_L(\overline{T})$.

Lemma 7.2.1. *Let $T, Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, $Z = Z_1e_1 + Z_2e_2 + Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1, 2, 3$, have real spectrum. Assume that $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Then there exists a positive constant $C_{Z,T}(s)$ depending on s and also on the operators T and Z such that*

$$\|F_L(s,T)^{-1} - F_L(s,Z)^{-1}\| \leq C_{Z,T}(s)(|s| + \vartheta)^{-2}\|T - Z\|, \tag{7.11}$$

$$\|F_R(s,T)^{-1} - F_R(s,Z)^{-1}\| \leq C_{Z,T}(s)(|s| + \vartheta)^{-2}\|T - Z\|, \tag{7.12}$$

where $\vartheta := \max\{\|T\|, \|Z\|\}$.

Proof. Since we have for $s \in \rho_S(T)$ that

$$F_L(s,T) := -4S_L^{-1}(s,T)\mathcal{Q}_{c,s}(T)^{-1}, \tag{7.13}$$

the inverse $F_L(s,T)^{-1}$ exists for $s \notin \sigma_L(\overline{T})$, and it is given by

$$F_L(s,T)^{-1} = -\frac{1}{4}\mathcal{Q}_{c,s}(T)S_L(s,T), \tag{7.14}$$

while the inverse of the operator $F_L(s,Z)$ exists for $s \notin \sigma_L(\overline{Z})$, and it is given by

$$F_L(s,Z)^{-1} = -\frac{1}{4}\mathcal{Q}_{c,s}(Z)S_L(s,Z). \tag{7.15}$$

We have

$$\begin{aligned}
&-4\left(F_L(s,T)^{-1} - F_L(s,Z)^{-1}\right)\\
&= \mathcal{Q}_{c,s}(T)S_L(s,T) - \mathcal{Q}_{c,s}(Z)S_L(s,Z)\\
&= \mathcal{Q}_{c,s}(T)S_L(s,T) - \mathcal{Q}_{c,s}(T)S_L(s,Z)\\
&\quad + \mathcal{Q}_{c,s}(T)S_L(s,Z) - \mathcal{Q}_{c,s}(Z)S_L(s,Z)\\
&= \mathcal{Q}_{c,s}(T)\ [S_L(s,T) - S_L(s,Z)]\\
&\quad + [-s(T+\overline{T}) + T\overline{T} + s(Z+\overline{Z}) - Z\overline{Z})]\ S_L(s,Z)\\
&= \mathcal{Q}_{c,s}(T)\ [S_L(s,T) - S_L(s,Z)]\\
&\quad + [s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(\overline{T} - \overline{Z})]\ S_L(s,Z),
\end{aligned}$$

and taking the norm, we get

$$\begin{aligned}
&\left\|F_L(s,T)^{-1} - F_L(s,Z)^{-1}\right\|\\
&\leq (|s|^2 + 2|s|\,\|T\| + \|T\overline{T}\|)\|S_L(s,T) - S_L(s,Z)\|\\
&\quad + \left[2|s|\,\|Z - T\| + \|T - Z\|(\|\overline{T}\| + \|Z\|)\right]\ \|S_L(s,Z)\|\\
&\leq (|s| + \vartheta)^2\|S_L(s,T) - S_L(s,Z)\|\\
&\quad + \left[2(|s| + \vartheta)\|Z - T\|)\right]\ \|S_L(s,Z)\|.
\end{aligned}$$

Now observe that

$$
\begin{aligned}
&(|s|+\vartheta)^{-1}\|S_L(s,Z)\| \\
&\leq (|s|+\vartheta)^{-1}\big[|s|+\|(s\mathcal{I}-\overline{Z})\|\,\|Z\|\,\|(s\mathcal{I}-\overline{Z})^{-1}\|\big] := M_Z(s),
\end{aligned}
\tag{7.16}
$$

where $M_Z(s)$ is a continuous function, since $s \notin \sigma_L(\overline{Z})$. Using Lemma 4.6.3, we get

$$
\big\|F_L(s,T)^{-1} - F_L(s,Z)^{-1}\big\| \leq \frac{1}{4}[K_Z(s)+2M(s)](|s|+\vartheta)^2\|Z-T\|, \tag{7.17}
$$

and $K_{T,Z}(s)$ is defined in (4.25). We can argue similarly for $F_R(s,T)^{-1}$. □

Lemma 7.2.2. *Let $T, Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, $Z = Z_1e_1 + Z_2e_2 + Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1,2,3$, have real spectrum. Assume that $s \in \rho_S(T)$, let $s \notin \sigma_L\left(\overline{T}\right) \cup \sigma_L\left(\overline{Z}\right)$, and suppose that*

$$
\|T-Z\| < \frac{1}{C_{Z,T}(s)}(|s|+\vartheta)^{-2}\|F_L(s,T)\|^{-1},
$$

where $C_{Z,T}(s)$ is defined in Lemma 7.2.1. Then $s \in \rho_S(Z)$ and

$$
F_L(s,Z) - F_L(s,T) = F_L(s,T)\sum_{m=1}^{+\infty}\left[\left(F_L(s,T)^{-1} - F_L(s,Z)^{-1}\right)F_L^{-1}(s,T)\right]^m.
$$

An analogous statement holds for $F_R^{-1}(s,T)$.

Proof. By Lemma 3.1.12 and formula (3.2) with

$$
A := (F_L(s,T))^{-1}, \quad B := (F_L(s,Z))^{-1}, \quad A^{-1} = F_L(s,T), \tag{7.18}
$$

we have for $B^{-1} = F_L(s,Z)$ that

$$
F_L(s,Z) = F_L(s,T)\sum_{m=0}^{+\infty}\left[(F_L(s,T))^{-1} - (F_L(s,Z))^{-1}F_L(s,T)\right]^m. \tag{7.19}
$$

Using the hypothesis, we find that the series converges, since

$$
\begin{aligned}
&\|(F_L(s,T) - F_L(s,Z))F_L^{-1}(s,T)\| \\
&\leq \|(F_L(s,T) - F_L(s,Z))\|\,\|F_L^{-1}(s,T)\| \\
&\leq C_{Z,T}(s)(|s|+\vartheta)^2\|Z-T\|\,\|F_L^{-1}(s,T)\| < 1.
\end{aligned}
$$

□

Theorem 7.2.3. *Let $T, Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, $Z = Z_1e_1 + Z_2e_2 + Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1,2,3$, have real spectrum. Assume that $s \in \rho_S(T)$ and $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Let $\varepsilon > 0$ and let* *us consider the* ε*-neighborhood* $B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$ *of* $\sigma_S(T)\cup\sigma_L(\overline{T})$*. Then there exists* $\delta>0$ *such that, for* $\|T-Z\|<\delta$*, we have*

$$\sigma_S(Z)\subseteq B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$$

and

$$\|F_L(s,Z)-F_L(s,T)\|<\varepsilon,\quad \textit{for } s\notin B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T})).$$

An analogous statement holds for the right F-resolvent.

Proof. Let $T,Z\in\mathcal{BC}(X)$ and let $\varepsilon>0$. Thanks to Lemma 3.1.12 there exists $\eta>0$ such that if

$$\|T-Z\|<\eta,$$

then $\sigma_L(\overline{Z})\subset B_\varepsilon(\sigma_L(\overline{T}))$. So we can always choose η such that $\sigma_L(\overline{Z})\subset B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$. Consider the function $C_{Z,T}(s)$ defined in Lemma 7.2.1. The constant

$$C_\varepsilon:=\sup_{s\notin B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))}C_{Z,T}(s)$$

is finite because $s\notin B(\sigma_S(T)\cup\sigma_L(\overline{T},\varepsilon)$ and

$$\lim_{s\to\infty}\|(s\mathcal{I}-\overline{Z})^{-1}\|=\lim_{s\to\infty}\|(s\mathcal{I}-\overline{T})^{-1}\|=0.$$

Observe that since $s\in\rho_S(T)$, the map $s\mapsto\|F_L(s,T)\|$ is continuous, and

$$\lim_{s\to\infty}\|F_L(s,T)\|=0,$$

and so for s in the complement set of $B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T})$ we have that there exists a positive constant M_ε such that

$$\|F_L(s,T)\|\le M_\varepsilon.$$

From Lemma 7.2.2, we find that if $\delta_1>0$ is such that

$$\|Z-T\|<\frac{1}{C_\varepsilon M_\varepsilon}:=\delta_3,$$

then $s\in\rho_S(Z)$ and

$$\begin{aligned}\|F_L^{-1}(s,Z)&-F_L^{-1}(s,T)\|\\ &\le\frac{\|F_L^{-1}(s,T)\|^2\ \|F_L(s,T)-F_L(s,Z)\|}{1-\|F_L^{-1}(s,T)\|\ \|F_L(s,T)-\mathcal{F}(s,Z)\|}\\ &\le\frac{M_\varepsilon^2C_{n,\varepsilon}\|Z-T\|}{1-M_\varepsilon C_{n,\varepsilon}\|Z-T\|}<\varepsilon\end{aligned}$$

if we take

$$\|Z-T\|<\delta_4:=\frac{\varepsilon}{C_{n,\varepsilon}(M_\varepsilon^2+\varepsilon M_\varepsilon)}.$$

To get the statement it suffices to set $\delta=\min\{\eta,\delta_3,\delta_4\}$. □

Theorem 7.2.4. *Let $T, Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1+T_2e_2+T_3e_3$, $Z = Z_1e_1+Z_2e_2+Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1, 2, 3$, have real spectrum. Let $f \in \mathcal{SH}_L(\sigma_S(T))$ (or $f \in \mathcal{SH}_R(\sigma_S(T))$) and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for $\|Z - T\| < \delta$, we have $f \in \mathcal{SH}_L(\sigma_S(Z))$ (or $f \in \mathcal{SH}_R(\sigma_S(Z))$) and*

$$\|\breve{f}(Z) - \breve{f}(T)\| < \varepsilon.$$

Proof. We recall that

$$\breve{f}(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s, T)\; ds_j\; f(s),$$

where $U \subset \mathbb{H}$ is a bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and where $j \in \mathbb{S}$. Let furthermore $B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$ be contained in U. By Lemma 7.2.3 there exists $\delta_1 > 0$ such that $\sigma_S(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently $f \in \mathcal{SH}_L(\sigma_S(Z))$ if $\|Z - T\| < \delta_1$. By Lemma 7.2.3, $F_L(s, T)$ is uniformly close to $F_L(s, Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_j)$ for $j \in \mathbb{S}$ if $\|Z - T\|$ is small enough. So for some positive $\delta \le \delta_1$, we get

$$\|\breve{f}(T) - \breve{f}(Z)\| \le \frac{1}{2\pi}\|\int_{\partial(U\cap\mathbb{C}_j)} [F_L(s, T) - F_L(s, Z)]\; ds_j\; f(s)\| < \varepsilon.$$

We can argue similarly if $f \in \mathcal{SH}_R(U)$. □

7.3 The F-Resolvent Equations

The F-resolvents satisfy a relation that can be considered a generalized resolvent equation. In particular, they allow one to show that the F-functional calculus is capable of generating projections onto subspaces that are invariant under the operator.

Theorem 7.3.1 (Left and right F-resolvent equations)**.** *Let $T \in \mathcal{BC}(X)$ and let $s \in \rho_S(T)$. The F-resolvent operators satisfy the equations*

$$F_L(s, T)s - TF_L(s, T) = -4\mathcal{Q}_{c,s}(T)^{-1} \tag{7.20}$$

and

$$sF_R(s, T) - F_R(s, T)T = -4\mathcal{Q}_{c,s}(T)^{-1}. \tag{7.21}$$

Proof. We prove relation (7.20), since (7.21) follows with similar computations. We have

$$F_L(s, T)s = -4(s\mathcal{I} - \overline{T})s\mathcal{Q}_{c,s}(T)^{-2}$$

and

$$TF_L(s, T) = -4(Ts - T\overline{T})\mathcal{Q}_{c,s}(T)^{-2}.$$

Taking the difference, we obtain

$$\begin{aligned} F_L(s, T)s - TF_L(s, T) &= -4(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})\mathcal{Q}_{c,s}(T)^{-2} \\ &= -4\mathcal{Q}_{c,s}(T)^{-1}. \end{aligned}$$

□

Lemma 7.3.2. *Let $T \in \mathcal{BC}(X)$. For $q, s \in \rho_S(T)$, with $s \notin [q]$ and with the position $\mathcal{Q}_s(q) = q^2 - 2\mathrm{Re}(s)q + |s|^2$, the following equation holds:*

$$F_R(s,T)S_L^{-1}(q,T) + S_R^{-1}(s,T)F_L(q,T) - 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1}$$
$$= \big[\,(F_R(s,T) - F_L(q,T))\,q - \bar{s}\,(F_R(s,T) - F_L(q,T))\,\big]\,\mathcal{Q}_s(q)^{-1}. \quad (7.22)$$

Proof. We consider the S-resolvent equation (3.7) and write the S-resolvent operators in the form (4.20) and (4.21) for operators with commuting components. If we multiply it on the left by $-4\mathcal{Q}_{c,s}(T)^{-1}$, we get

$$F_R(s,T)S_L^{-1}(q,T) = \Big[\,\big(F_R(s,T) + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q$$
$$- \bar{s}\,\big(F_R(s,T) + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,\Big]\,\mathcal{Q}_s(q)^{-1}.$$

If we multiply the S-resolvent equation on the right by $-4\mathcal{Q}_{c,q}(T)^{-1}$, we get

$$S_R^{-1}(s,T)F_L(q,T) = \Big[\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} - F_L(q,T)\big)\,q$$
$$- \bar{s}\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} - F_L(q,T)\big)\,\Big]\,\mathcal{Q}_s(q)^{-1}.$$

Adding these two equations yields

$$\begin{aligned}
&F_R(s,T)S_L^{-1}(q,T) + S_R^{-1}(s,T)F_L(q,T) \\
&= \left[(F_R(s,T) - F_L(q,T))\,q - \bar{s}\,(F_R(s,T) - F_L(q,T))\right]\mathcal{Q}_s(q)^{-1} \\
&\quad + \Big[\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q \\
&\qquad - \bar{s}\Big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\Big)\Big]\,\mathcal{Q}_s(q)^{-1}.
\end{aligned}$$

The proof is concluded if we verify that

$$\begin{aligned}
&\Big[\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q \\
&\quad - \bar{s}\Big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\Big)\Big]\,\mathcal{Q}_s(q)^{-1} \\
&= 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1}.
\end{aligned}$$

This follows from

$$\begin{aligned}
&\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q \\
&\quad - \bar{s}\Big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\Big) \\
&= -4\Big[\Big(\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}(q\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1}\Big)q \\
&\quad - \bar{s}\Big(\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}(q\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= -4\Big[\mathcal{Q}_{c,s}(T)^{-1}(s-q)\mathcal{Q}_{c,q}(T)^{-1}q - \bar{s}\mathcal{Q}_{c,s}(T)^{-1}(s-q)\mathcal{Q}_{c,q}(T)^{-1}\Big]\\
&= -4\Big[\mathcal{Q}_{c,s}(T)^{-1}(sq-q^2)\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}(\bar{s}s-\bar{s}q)\mathcal{Q}_{c,q}(T)^{-1}\Big]\\
&= -4\Big[\mathcal{Q}_{c,s}(T)^{-1}(sq-q^2-\bar{s}s+\bar{s}q)\mathcal{Q}_{c,q}(T)^{-1}\Big]\\
&= 4\mathcal{Q}_{c,s}(T)^{-1}(q^2-2\mathrm{Re}(s)q+|s|^2)\mathcal{Q}_{c,q}(T)^{-1} = 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1}\mathcal{Q}_s(q). \;\Box
\end{aligned}
$$

Theorem 7.3.3 (The F-resolvent equation)**.** *Let $T \in \mathcal{BC}(X)$. For all quaternions $q, s \in \rho_S(T)$ with $s \notin [q]$, the following equation holds:*

$$
\begin{aligned}
&F_R(s,T)S_L^{-1}(q,T) + S_R^{-1}(s,T)F_L(q,T)\\
&\quad - \frac{1}{4}\Big(sF_R(s,T)F_L(q,T)q - sF_R(s,T)TF_L(q,T)\\
&\quad - F_R(s,T)TF_L(q,T)q + F_R(s,T)T^2F_L(q,T)\Big)\\
&= \big[\,(F_R(s,T)-F_L(q,T))\,q - \bar{s}\,(F_R(s,T)-F_L(q,T))\,\big]\mathcal{Q}_s(q)^{-1}.
\end{aligned}
\tag{7.23}
$$

Proof. The identities (7.20) and (7.21) yield

$$
\begin{aligned}
-4^2\mathcal{Q}_s(T)^{-1}\mathcal{Q}_q(T)^{-1} &= (sF_R(s,T)-F_R(s,T)T)(F_L(q,T)q-TF_L(q,T))\\
&= sF_R(s,T)F_L(q,T)q - sF_R(s,T)TF_L(q,T)\\
&\quad - F_R(s,T)TF_L(q,T)q + F_R(s,T)T^2F_L(q,T).
\end{aligned}
$$

Applying this identity in (7.22), we obtain (7.23). □

7.4 The Riesz Projectors for the F-Functional Calculus

In the sequel we will need the following lemma, which is based on the monogenic functional calculus; see the book [159] for more details (or some of the papers [160, 161, 166], where the calculus was introduced).

Lemma 7.4.1. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1,2,3$, have real spectrum. Let G be a bounded slice Cauchy domain such that $(\partial G)\cap\sigma_S(T) = \emptyset$. For every $j \in \mathbb{S}$, we then have*

$$
\int_{\partial(G\cap\mathbb{C}_j)} s\,ds_j\,F_R(s,T) = 0 \quad \text{and} \quad \int_{\partial(G\cap\mathbb{C}_j)} F_L(q,T)\,dq_j\,q = 0.
$$

Proof. Since $\mathcal{P}_1(q) = \Delta q = 0$, we have

$$
\int_{\partial(G\cap\mathbb{C}_j)} s\,ds_j\,F_R(s,p) = \mathcal{P}_1(p) = 0
$$

and

$$
\int_{\partial(G\cap\mathbb{C}_j)} F_L(q,p)\,dq_j\,q = \mathcal{P}_1(q) = 0
$$

for all $p \notin \partial G$ and $j \in \mathbb{S}$. We observe that at this point we need the Cauchy–Fueter functional calculus, described in the next section, to represent $F_L(p,T)$. We consider only the case of $F_L(p,T)$; the other case can be shown in a similar way. We recall that $F_L(p,q)$ is left Fueter regular in q on $\mathbb{H}\setminus[p]$ for every p, so we can use Definition 7.5.6 and write

$$F_L(p,T) = \int_{\partial\Omega} \mathcal{G}(\omega,T) D\omega F_L(p,\omega),$$

where the open set Ω contains the left spectrum of T, $\mathcal{G}(\omega,T)$ is the Cauchy–Fueter resolvent operator. Using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\partial(G\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \\
&= \int_{\partial(G\cap\mathbb{C}_j)} \int_{\partial\Omega} \Big(\mathcal{G}(\omega,T) D\omega F_L(q,\omega)\Big)\, dq_j\, q \\
&= \int_{\partial\Omega} \mathcal{G}(\omega,T) D\omega \Big(\int_{\partial(G\cap\mathbb{C}_j)} F_L(p,\omega)\, dp_j\, q \Big) = 0,
\end{aligned}$$

which concludes the proof. □

Theorem 7.4.2. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1,2,3$, have real spectrum. Let $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with*

$$\operatorname{dist}(\sigma_1,\sigma_2) > 0.$$

Let $G_1, G_2 \subset \mathbb{H}$ be two bounded slice Cauchy domains such that $\sigma_1 \subset G_1$ and $\overline{G_1} \subset G_2$ and such that $\operatorname{dist}(G_2,\sigma_2) > 0$. *Then the operator*

$$\begin{aligned}
\breve{P} &:= -\frac{1}{4(2\pi)} \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j q^2 \\
&= -\frac{1}{4(2\pi)} \int_{\partial(G_2\cap\mathbb{C}_j)} s^2 ds_j F_R(s,T)
\end{aligned}$$

is a projection that commutes with T, i.e., we have

$$\breve{P}^2 = \breve{P} \quad \text{and} \quad T\breve{P} = \breve{P}T.$$

Proof. If we multiply the F-resolvent equation (7.23) by s on the left and by q on the right, we get

$$\begin{aligned}
&sF_R(s,T)S_L^{-1}(q,T)q + sS_R^{-1}(s,T)F_L(q,T)q \\
&\quad - \frac{1}{4}\Big(s^2F_R(s,T)F_L(q,T)q^2 - s^2F_R(s,T)TF_L(q,T)q \\
&\quad - sF_R(s,T)TF_L(q,T)q^2 + sF_R(s,T)T^2F_L(q,T)q\Big) \\
&= s\left[(F_R(s,T) - F_L(q,T))\, q - \bar{s}\, (F_R(s,T) - F_L(q,T))\right] \mathcal{Q}_s(q)^{-1} q.
\end{aligned}$$

If we multiply this equation by ds_j on the left, integrate it over $\partial(G_2 \cap \mathbb{C}_j)$ with respect to ds_j, and then multiply it by dq_j on the right and integrate over $\partial(G_1 \cap \mathbb{C}_j)$ with respect to dq_j, we obtain

$$\begin{aligned}
&\int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, F_R(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} S_L^{-1}(q,T)\, dq_j\, q \\
&\quad + \int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, S_R^{-1}(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \\
&\quad - \frac{1}{4}\bigg(\int_{\partial(G_2\cap\mathbb{C}_j)} s^2\, ds_j\, F_R(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q^2 \\
&\quad - \int_{\partial(G_2\cap\mathbb{C}_j)} s^2\, ds_j\, F_R(s,T)T \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \\
&\quad - \int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, F_R(s,T)T \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q^2 \\
&\quad + \int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, F_R(s,T)T^2 \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \bigg) \\
&= \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\Big[(F_R(s,T) - F_L(q,T))\, q \\
&\qquad\qquad - \bar{s}\, (F_R(s,T) - F_L(q,T)) \Big] \mathcal{Q}_s(q)^{-1}\, dq_j\, q.
\end{aligned}$$

By Lemma 7.4.1, this simplifies to

$$\begin{aligned}
&- \frac{1}{4} \int_{\partial(G_2\cap\mathbb{C}_j)} s^2\, ds_j\, F_R(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q^2 \\
&= \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\Big[(F_R(s,T) - F_L(q,T))\, q \\
&\quad - \bar{s}\, (F_R(s,T) - F_L(q,T)) \Big] \mathcal{Q}_s(q)^{-1}\, dq_j q,
\end{aligned}$$

which equals

$$\begin{aligned}
4(2\pi)^2 \breve{P}^2 &= \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\big[F_R(s,T)q - \bar{s}F_R(s,T)\big] \mathcal{Q}_s(q)^{-1}\, dq_j\, q \\
&\quad - \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\big[F_L(q,T)q - \bar{s}F_L(q,T)\big] \mathcal{Q}_s(q)^{-1}\, dq_j\, q.
\end{aligned}$$

Since $\overline{G_1} \subset G_2$, for every $s \in \partial(G_2 \cap \mathbb{C}_j)$ the functions

$$q \mapsto q\mathcal{Q}_s(q)^{-1} = q(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$$

and

$$q \mapsto \mathcal{Q}_s(q)^{-1} = (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$$

are intrinsic slice hyperholomorphic on $\overline{G_1}$. By the Cauchy integral theorem, we therefore have

$$\int_{\partial(G_1\cap\mathbb{C}_j)} q\mathcal{Q}_s(q)^{-1}\,dq_j\,q = 0 \quad \text{and} \quad \int_{\partial(G_1\cap\mathbb{C}_j)} \mathcal{Q}_s(q)^{-1}\,dq_j\,q = 0,$$

and it follows that

$$\int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\big[F_R(s,T)q - \bar{s}F_R(s,T)\big]\,\mathcal{Q}_s(q)^{-1}\,dq_j\,q = 0.$$

Thus, we obtain

$$\begin{aligned}\breve{P}^2 = -\frac{1}{4(2\pi)^2}\int_{\partial(G_2\cap\mathbb{C}_j)} s\,ds_j \\ \times \int_{\partial(G_1\cap\mathbb{C}_j)} \big[\,(F_L(q,T)q - \bar{s}F_L(q,T))\,\big]\,\mathcal{Q}_s(q)^{-1}\,dq_j q,\end{aligned}$$

and by exchanging the order of integration and applying Lemma 4.1.2, we finally obtain

$$\breve{P}^2 = -\frac{1}{4(2\pi)}\int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\,dq_j\,q^2 = \breve{P}.$$

We furthermore deduce from (7.21) that

$$\begin{aligned}\breve{P}T &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^2\,ds_j\,F_R(s,T)T \\ &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^3\,ds_j\,F_R(s,T) - \frac{16}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^2\,ds_j\,\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

Since $s^3\chi_{G_1}(s)$ is intrinsic slice hyperholomorphic, this equals

$$\begin{aligned}\breve{P} &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^2\,ds_j\,F_R(s,T)T \\ &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,s^3 - \frac{16}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} \mathcal{Q}_s(T)^{-2}\,ds_j\,s^2 \\ &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} TF_L(s,T)\,ds_j\,s^2 = T\breve{P},\end{aligned}$$

where we applied (7.20) in the third identity. □

7.5 The Cauchy–Fueter Functional Calculus

We recall the Cauchy formula for Cauchy–Fueter regular functions (or Fueter regular functions), and we use it to define the Cauchy–Fueter functional calculus.

We will not give all the details but just the main definitions. The function $\mathcal{G}(q)$ defined by

$$\mathcal{G}(q) = \frac{q^{-1}}{|q|^2} = \frac{\bar{q}}{|q|^4} \tag{7.24}$$

is called the *Cauchy–Fueter kernel*, and it is left and right Fueter regular on $\mathbb{H}\setminus\{0\}$.

Theorem 7.5.1 (Cauchy–Fueter formula)**.** *Let f be a left Fueter regular function on an open set that contains $\overline{U}$. If U is a four-dimensional compact, oriented manifold with smooth boundary ∂U, then*

$$f(q) = \frac{1}{2\pi^2} \int_{\partial U} \mathcal{G}(p-q)\, Dp\, f(p), \quad q \in U, \tag{7.25}$$

the differential form Dp is given by $Dp = \eta(p)dS(p)$, where $\eta(p)$ is the outer unit normal to ∂U at the point p, and $dS(p)$ is the scalar element of surface area on ∂U. If f is a right Fueter regular function on U, then

$$f(q) = \frac{1}{2\pi^2} \int_{\partial U} f(p)\, Dp\, \mathcal{G}(p-q), \quad q \in U. \tag{7.26}$$

Fueter regular functions do not admit power series expansions, but there exist series expansions in terms of suitable homogeneous functions. For every triple $\nu = (n_1, n_2, n_3)$ with $|\nu| := n_1 + n_2 + n_3 = n$, we define

$$\partial_\nu = \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} \quad \text{and} \quad \mathcal{G}_\nu(q) = \partial_\nu \mathcal{G}(q),$$

where $\mathcal{G}(q)$ is the Cauchy–Fueter kernel. Furthermore, we define the set $\Gamma(\nu)$ as the set of all n-tuples $(\lambda_1, \ldots, \lambda_n)$ with exactly n_1 entries that equal 1, exactly n_2 entries that equal 2, and exactly n_3 entries that equal 3. In other words, if we set $\lambda_1, \ldots, \lambda_{n_1} = 1$ and $\lambda_{n_1+1}, \ldots, \lambda_{n_1+n_2} = 2$ and $\lambda_{n_1+n_2+1}, \ldots \lambda_n = 3$, then

$$\Gamma(\nu) = \{(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) : \quad \sigma \in \text{perm}(n)\},$$

where perm(n) denotes the group of permutations of n elements. Furthermore, let us denote by σ_n the set of all triples $\nu = (n_1, n_2, n_3)$ with $|v| = n_1 + n_2 + n_3 = n$. For every $n > 0$, the set σ_n contains $\frac{1}{2}(n+1)(n+2)$ triples. If $n = 0$, we set $\nu = \emptyset$ and $P_\nu \equiv 1$. For every $\nu \in \sigma_n$ and for $q = q_0 + \sum_{\ell=1}^{3} q_\ell e_\ell$, we define

$$P_\nu(q) = \frac{1}{n!} \sum_{(\lambda_1, \ldots, \lambda_n) \in \Gamma(n)} (q_0 e_{\lambda_1} - q_{\lambda_1}) \cdots (q_0 i_{\lambda_n} - q_{\lambda_n}).$$

The polynomials $P_\nu(q)$ play the role of the powers z^n in the Taylor expansion of a function $\sum_{n=0}^{+\infty} a_n z^n$ holomorphic at the origin.

Let $\mathcal{U}_n$ be the quaternionic right vector space of functions $f : \mathbb{H} \to \mathbb{H}$ that are left Fueter regular and homogeneous of degree $n \geq 0$ over $\mathbb{R}$, i.e., such that $f(\alpha q) = \alpha^n f(q)$ for every $\alpha \in \mathbb{R}$. We have the following result.

Theorem 7.5.2. *The polynomials P_ν, $\nu \in \sigma_n$, are left Fueter regular and form a basis for $\mathcal{U}_n$. Moreover, if $f \in \mathcal{U}_n$, then*

$$f(q) = \sum_{\nu\in\sigma_n} (-1)^n P_\nu(q)\partial_\nu f(0). \tag{7.27}$$

If f is right Fueter regular, then the polynomials P_ν are right Fueter regular and

$$f(q) = \sum_{\nu\in\sigma_n} (-1)^n \partial_\nu f(0) P_\nu(q).$$

The introduction of the polynomials P_ν and the derivatives $\mathcal{G}_\nu$ allows one to prove two results that generalize the Taylor and the Laurent expansion series.

Theorem 7.5.3. *Let $f : U \subseteq \mathbb{H} \to \mathbb{H}$ be left Fueter regular, $p \in U$. Then there exists a ball $|q-p| < \delta$ with radius $\delta < \operatorname{dist}(p, \partial U)$ in which f can be represented by a uniformly convergent series of the form*

$$f(q) = \sum_{n=0}^{+\infty} \sum_{\nu\in\sigma_n} P_\nu(q-p)a_\nu,$$

where

$$a_\nu = (-1)^n \partial_\nu f(p) = \frac{1}{2\pi^2}\int_{|q-p|=\delta} \mathcal{G}_\nu(q-p)\, , Dq\, f(q).$$

If $f : U \to \mathbb{H}$ is right Fueter regular, then

$$f(q) = \sum_{n=0}^{+\infty} \sum_{\nu\in\sigma_n} a_\nu P_\nu(q-p),$$

where

$$a_\nu = (-1)^n \partial_\nu f(p) = \frac{1}{2\pi^2}\int_{|q-p|=\delta} f(q)\, Dq\, \mathcal{G}_\nu(q-p).$$

Let T be a quaternionic bounded linear operator with commuting components on a two-sided quaternionic Banach space X. Recall that such a set is denoted by $\mathcal{BC}(X)$. In this case, we consider the function $\mathcal{G}(q,p) := G(p-q)$ written in series expansion, and we replace p by T. We get

$$\mathcal{G}(q,T) = \sum_{n\geq 0}\sum_{\nu\in\sigma_n} P_\nu(T)\mathcal{G}_\nu(q) = \sum_{n\geq 0}\sum_{\nu\in\sigma_n} \mathcal{G}_\nu(q)P_\nu(T). \tag{7.28}$$

The expansions hold for $\|T\| < |q|$ and define a bounded operator. It is natural to give the following definition:

Definition 7.5.4. The maximal open set $\rho(T)$ in $\mathbb{H}$ on which the series (7.28) converges in the operator norm to a bounded operator is called the *resolvent set of T*. The *spectral set $\sigma(T)$ of T* is defined as the complement set in $\mathbb{H}$ of the resolvent set.

Definition 7.5.5. A function f is said to be *locally right Cauchy–Fueter regular* on the spectral set $\sigma(T)$ of an operator $T \in \mathcal{BC}(X)$ if there exists an open set $U \subset \mathbb{H}$ containing $\sigma(T)$ whose boundary ∂U is a rectifiable 3-cell and such that f is regular in every connected component of U. We denote by $\mathcal{CF}_L(\sigma(T))$ the set of locally left Cauchy–Fueter regular functions on $\sigma(T)$. We denote by $\mathcal{CF}_R(\sigma(T))$ the set of locally right Cauchy–Fueter regular functions on $\sigma(T)$.

Definition 7.5.6 (The Cauchy–Fueter functional calculus)**.** Let $f \in \mathcal{CF}_L(\sigma(T))$ and $T \in \mathcal{BC}(V)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum. We define

$$f(T) := \frac{1}{2\pi^2} \int_{\partial U} \mathcal{G}(q,T) Dq f(q).$$

Let $f \in \mathcal{CF}_R(\sigma(T))$ and $T \in \mathcal{BC}(V)$. We define

$$f(T) := \frac{1}{2\pi^2} \int_{\partial U} f(q) Dq \mathcal{G}(q,T),$$

where U is an open set in $\mathbb{H}$ containing $\sigma(T)$ as in Definition 7.5.5.

The definition is well posed, since the integrals that define the Cauchy–Fueter functional calculus do not depend on the open set U. This is a consequence of the Cauchy–Fueter regularity of the operator-valued function $\mathcal{G}(q,T)$. We point out that the series expansion of the Cauchy-Fueter resolvent operator in (7.28) has a closed form if T has commuting components, namely

$$\mathcal{G}(q,T) = (q\mathcal{I} - T)^{-2}(\overline{q\mathcal{I} - T})^{-1}.$$

This operator is then associated with the left spectrum of T. A closed form of the sum $\mathcal{G}(q,T)$ in the general case, without the assumption that the components of T commute, would naturally lead to a notion of spectrum of the operator T for the case of Fueter regularity. But if we want to replace operators with noncommuting components, then it is not clear what is the closed formula for the Cauchy–Fueter resolvent. Observe that for the slice hyperholomorphic case, a closed form of the series $\sum_{n\geq 0} T^n s^{-1-n}$ can be found. It is

$$\sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I})$$

for $\|T\| < |s|$, and this identity does not depend on the commutativity of the components of T. This is one of its great advantages.

7.6 Comments and Remarks

Comments on the references. The F-functional calculus has been developed in the papers [20, 54, 78, 81, 86]. It is based on the Fueter mapping theorem in integral form, and it is a monogenic functional calculus in the spirit of McIntosh and collaborators, see [159–161, 164, 166], but it is associated with slice hyperholomorphicity. The W-functional calculus is also a monogenic functional calculus, and it was introduced in the paper [70].

7.6.1 The F-Functional Calculus for n-Tuples of Operators

The F-functional calculus can be extended to the case of n-tuples of commuting operators. Because of the structure of the Fueter–Sce mapping theorem in integral form, the F-functional calculus depends on the dimension of the Clifford algebra. The Fueter–Sce–Qian mapping theorem, one should say, was proved by Michele Sce [187] for n odd and by Tao Qian [176] for the case in which n is even. Later on, Fueter's theorem was generalized to the case in which a slice hyperholomorphic function f is multiplied by a monogenic homogeneous polynomial of degree k, see [162] [172] [173], and to the case in which the function f is defined on an open set U not necessarily chosen in the upper complex plane; see [175–177]. We need to recall the definition of monogenic functions.

Definition 7.6.1 (Monogenic functions)**.** Let U be an open set in $\mathbb{R}^{n+1}$. A real differentiable function $f : U \to \mathbb{R}_n$ is *left monogenic* if

$$\frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} f(x) = 0.$$

It is *right monogenic* if

$$\frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x) e_i = 0.$$

We recall the theorem of Sce to produce monogenic functions from complex-valued functions (the case of odd dimension of $\mathbb{R}_n$):

We consider a holomorphic function $f(z)$ that depends on a complex variable $z = u + \iota v$ in an open set of the upper complex half-plane. We write

$$f(z) = f_0(u, v) + \iota f_1(u, v),$$

where f_0 and f_1 are $\mathbb{R}$-valued functions that satisfy the Cauchy–Riemann system. For every paravector x such that $u + \iota v$ belongs to the domain of f, we replace the complex imaginary unit ι in $f(z) = f_0(u, v) + \iota f_1(u, v)$ by the Clifford imaginary unit $j = \underline{x}/|\underline{x}|$ and we set $u = x_0$ and $v = |\underline{x}|$. We then define

$$f(x) = f_0(x_0, |\underline{x}|) + j f_1(x_0, |\underline{x}|).$$

This function in slice hyperholomorphic with values in the Clifford algebra $\mathbb{R}_n$ (or slice monogenic). Then we apply the $(n-1)/2$th power of the Laplace operator $\Delta^{(n-1)/2}$ in dimension $n+1$ to f. The function

$$\breve{f}(x_0, |\underline{x}|) := \Delta^{(n-1)/2}(f_0(x_0, |\underline{x}|) + j f_1(x_0, |\underline{x}|))$$

is then left monogenic, i.e., it is in the kernel of the Dirac operator. If we replace $f_0(x_0, |\underline{x}|) + jf_1(x_0, |\underline{x}|)$ by $f_0(x_0, |\underline{x}|) + f_1(x_0, |\underline{x}|)j$ in the above procedure, we obtain a right monogenic function.

Proposition 7.6.2. *Let n be an odd number and let x, $s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let $S_L^{-1}(s, x)$ and $S_R^{-1}(s, x)$ be the slice hyperholomorphic Cauchy kernels in form II. Then:*

- *The function $\Delta^{\frac{n-1}{2}} S_L^{-1}(s, x)$ is a left monogenic function in the variable x and right slice hyperholomorphic in s.*
- *The function $\Delta^{\frac{n-1}{2}} S_R^{-1}(s, x)$ is a right monogenic function in the variable x and left slice hyperholomorphic in s.*

Based on the explicit computations of functions

$$(s, x) \mapsto \Delta^{\frac{n-1}{2}} S_L^{-1}(s, x) \quad \text{and} \quad (s, x) \mapsto \Delta^{\frac{n-1}{2}} S_R^{-1}(s, x),$$

for $s \notin [x]$, we define the F_n-kernels.

Definition 7.6.3 (The F_n-kernels)**.** Let n be an odd number and let $x, s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the F_n^L*-kernel* as

$$F_n^L(s, x) := \Delta^{\frac{n-1}{2}} S_L^{-1}(s, x) = \gamma_n (s - \bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and the F_n^R*-kernel* as

$$F_n^R(s, x) := \Delta^{\frac{n-1}{2}} S_R^{-1}(s, x) = \gamma_n (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}} (s - \bar{x}),$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} \Big[\Big(\frac{n-1}{2}\Big)!\Big]^2. \tag{7.29}$$

Theorem 7.6.4 (The Fueter–Sce mapping theorem in integral form)**.** *Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain and choose $j \in \mathbb{S}$. Let n be an odd number.*

(a) *If $f \in \mathcal{SM}_L(O)$ for some set O with $\overline{U} \subset O$, then the left monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} F_n^L(s, x) ds_j f(s). \tag{7.30}$$

(b) *If $f \in \mathcal{SM}_R(O)$ for some set O with $\overline{U} \subset O$, then the right monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s) ds_j F_n^R(s, x). \tag{7.31}$$

The integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

We refer to the section on comments and remarks at the end of chapter on the properties of the S-functional calculus for bounded operators for the definition of the Clifford algebra $\mathbb{R}_n$ and for the functional setting on paravector operators $T = T_0 + T_1e_1 + \cdots + T_ne_n$. In the sequel, we will consider bounded paravector operators T, with commuting components $T_\ell \in \mathcal{B}(X)$ for $\ell = 0, 1, \ldots, n$. Such subsets of $\mathcal{B}(X_n)$ will be denoted by $\mathcal{BC}^{0,1}(X_n)$. The F-functional calculus is based on the commutative version of the S-spectrum (often called F-spectrum in the literature). So we define the F-resolvent operators.

Definition 7.6.5 (F-resolvent operators)**.** Let n be an odd number and let $T \in \mathcal{BC}^{0,1}(X_n)$. For $s \in \rho_S(T)$ we define the *left F-resolvent operator* by

$$F_n^L(s,T) := \gamma_n(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-\frac{n+1}{2}}, \tag{7.32}$$

and the *right F-resolvent operator* by

$$F_n^R(s,T) := \gamma_n \mathcal{Q}_{c,s}(T)^{-\frac{n+1}{2}}(s\mathcal{I} - \overline{T}), \tag{7.33}$$

where the constants γ_n are given by (7.29).

Let $T \in \mathcal{BC}^{0,1}(X_n)$. We denote by $\mathcal{SM}_L(\sigma_S(T))$, $\mathcal{SM}_R(\sigma_S(T))$ the set of all left (or right) slice hyperholomorphic functions f with $\sigma_S(T) \subset \mathcal{D}(f)$.

Definition 7.6.6 (The F-functional calculus for bounded operators)**.** Let n be an odd number, let $T \in \mathcal{BC}^{0,1}(X_n)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1, \ldots, n$, have real spectrum. Set $ds_j = ds/j$. For every function $f \in \mathcal{SM}_L(\sigma_S(T))$, we define

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_n^L(s,T)\, ds_j\, f(s). \tag{7.34}$$

For every $f \in \mathcal{SM}_R(\sigma_S(T))$, we define

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, F_n^R(s,T), \tag{7.35}$$

where $j \in \mathbb{S}$ and U is a slice Cauchy domain U.

The definition of the F-functional calculus is well posed, since the integrals in (7.9) and (7.35) depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

7.6.2 The Inverse Fueter–Sce Mapping Theorem

In recent years, new problems related to the inversion of the Fueter–Sce mapping theorem have been solved. For the sake of simplicity here we mention the inversion problem of axially monogenic functions. The results can be found in [83], while more general cases are treated in the papers [84, 85, 87, 103].

Definition 7.6.7 (Axially monogenic function)**.** Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$, and let $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in U$, for $\underline{\omega} \in \mathbb{S}$. Assume that $\breve{f} : U \to \mathbb{R}_n$ is a monogenic function, i.e., it is in the kernel of the Dirac operator. We say that $\breve{f}$ is an axially monogenic function if there exist two functions $A = A(x_0, r)$ and $B = B(x_0, r)$, independent of $\underline{\omega} \in \mathbb{S}^{n-1}$ and with values in $\mathbb{R}_n$, such that

$$\breve{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r).$$

We denote by $\mathcal{AM}(U)$ the set of left axially monogenic functions on the open set U.

The problem is as follows: suppose that $\breve{f}$ is an axially monogenic function and f is a slice monogenic function such that $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. Determine a slice monogenic function f in terms of the components $A(x_0, r)$ and $B(x_0, r)$ of the axially monogenic function $\breve{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r)$. It is important to give the definition of a Fueter–Sce primitive.

Definition 7.6.8 (Fueter–Sce primitive)**.** Let n be an odd number and let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain. Suppose that $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is a left slice monogenic function. We say that f is a *Fueter–Sce primitive* of $\breve{f} \in \mathcal{M}(U)$ if $\Delta^{\frac{n-1}{2}} f(x) = \breve{f}(x)$ on U.

The definition of a Fueter–Sce primitive of $\breve{f}$ is well posed, since slice monogenic functions are infinitely differentiable. The monogenic Cauchy kernel $\mathcal{G}(x)$ is defined for $x \in \mathbb{R}^{n+1} \setminus \{0\}$ as

$$\mathcal{G}(x) = \frac{1}{A_{n+1}} \frac{\overline{x}}{|x|^{n+1}}, \tag{7.36}$$

where $A_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$ is the area of the unit sphere in $\mathbb{R}^{n+1}$. As we will see, $\mathcal{G}(x)$ plays a crucial role in the inversion formula of monogenic functions.

Definition 7.6.9 (The kernels $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$)**.** Let $\mathcal{G}(x)$ be the monogenic Cauchy kernel defined in (7.36) with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, and for $\underline{y} = r\underline{\omega} \in \mathbb{R}^n$ we assume $r = 1$ and $\underline{\omega} \in \mathbb{S}^{n-1}$. We define the kernels

$$\mathcal{N}_n^+(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x - \underline{\omega})\, dS(\underline{\omega}), \quad \mathcal{N}_n^-(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x - \underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}), \tag{7.37}$$

where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

Theorem 7.6.10 (The structure of the Fueter–Sce primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$)**.** *Let n be an odd number and denote by $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ the Fueter–Sce primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$, respectively. Consider the functions*

$$\mathcal{W}_n^+(x_0) := \frac{\mathcal{C}_n}{\mathcal{K}_n}\, D^{-(n-1)} \frac{x_0}{(x_0^2 + 1)^{(n+1)/2}},$$

$$\mathcal{W}_n^-(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n}\, D^{-(n-1)} \frac{1}{(x_0^2 + 1)^{(n+1)/2}},$$

where $\frac{\mathcal{C}_n}{\mathcal{K}_n}$ *is a given constant and the symbol* $D^{-(n-1)}$ *stands for* $(n-1)$ *integrations with respect to* x_0*. Then, by replacing* x_0 *by* x *in* $\mathcal{W}_n^+(x_0)$ *and in* $\mathcal{W}_n^-(x_0)$*, we get* $\mathcal{W}_n^+(x)$ *and* $\mathcal{W}_n^-(x)$*, respectively. Moreover, the functions* $\mathcal{W}_n^+(x)$ *and* $\mathcal{W}_n^-(x)$ *are extendable to slice monogenic functions defined for all* $x \in \{x_0 + r\underline{\omega} : (x_0, r) \neq (0,1)\}$*.*

The Fueter–Sce primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ can be explicitly computed. For example, when $n = 3$ they are given by

$$\mathcal{W}_3^+(x) = \frac{1}{2\pi} \arctan x, \quad \mathcal{W}_3^-(x) = -\frac{1}{2\pi}\, x \arctan x.$$

Theorem 7.6.11 (The inverse Fueter–Sce mapping theorem)**.** *Let* $\breve{f}(x) = A(x_0, \rho) + \underline{\omega} B(x_0, \rho)$ *be an axially monogenic function defined on an axially symmetric domain* $U \subseteq \mathbb{R}^{n+1}$*. Let* Γ *be the boundary of an open bounded subset* $\mathcal{V}$ *of the half-plane* $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ *and let*

$$V = \{x = u + \underline{\omega} v,\ (u, v) \in \mathcal{V},\ \underline{\omega} \in \mathbb{S}^{n-1}\} \subset U.$$

Moreover, suppose that Γ *is a regular curve whose parametric equations* $y_0 = y_0(s)$*,* $\rho = \rho(s)$ *are expressed in terms of the arc length* $s \in [0, L]$*,* $L > 0$*. Then the function*

$$\begin{aligned} f(x) = &\int_\Gamma \mathcal{W}_n^-\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0\, A(y_0, \rho) - d\rho\, B(y_0, \rho)) \\ &- \int_\Gamma \mathcal{W}_n^+\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0 B(y_0, \rho) - d\rho A(y_0, \rho)). \end{aligned} \tag{7.38}$$

is a Fueter–Sce primitive of $\breve{f}(x)$ *on* V*, where* $\mathcal{W}_n^+$ *and* $\mathcal{W}_n^-$ *are as in Theorem 7.6.10.*

This theorem has several generalizations, and this topic is still under investigation.