Chapter 6

The H^∞-Functional Calculus

The H^∞-functional calculus is an extension of the Riesz–Dunford functional calculus for bounded operators to unbounded sectorial operators, and it was introduced by A. McIntosh in [165]; see also [5]. This calculus is connected with pseudo-differential operators, with Kato's square root problem, and with the study of evolution equations and, in particular, the characterization of maximal regularity and with the fractional powers of differential operators. For an overview and more problems associated with this functional calculus for the classical case, see the book [156] and the references therein.

In this chapter we consider the quaternionic version of the H^∞-functional calculus introduced in [30], where with suitable conditions on the operators T we can study the quaternionic analogue of the results in [165]. A more general treatment of the H^∞-functional calculus for quaternionic operators has been done in [51, 52], where also the fractional powers of quaternionic linear operators are considered and new fractional diffusion and evolution processes are defined. We will mention such applications at the end of this chapter, see also [128].

6.1 The Rational Functional Calculus

The H^∞-functional calculus is defined using a version of the S-functional calculus for sectorial operators and on the rational functional calculus for intrinsic rational slice hyperholomorphic functions.

Definition 6.1.1 (Intrinsic rational slice hyperholomorphic function)**.** Let P and Q be intrinsic polynomials. An *intrinsic rational slice hyperholomorphic function* is defined as

$$R(p) := P(p)Q(p)^{-1}.$$

Observe that since $P(p)$ and $Q(p)^{-1}$ are intrinsic slice hyperholomorphic functions, the $\star$-product of $P(p)$ and $Q(p)^{-1}$ is equal to $P(p)Q(p)^{-1}$, and it is an intrinsic slice hyperholomorphic function.

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_6

Definition 6.1.2 (Rational functional calculus)**.** Assume that the rational function $R(p) = P(p)Q(p)^{-1}$ has no poles on the S-spectrum of T. Let T be a closed densely defined operator. We define the *rational functional calculus* as

$$R(T) := P(T)Q(T)^{-1}.$$

The operator $R(T)$ is closed and densely defined, and its domain is $\mathcal{D}(T^m)$, where

$$m := \max\{0, \deg P - \deg Q\}.$$

An important example of an intrinsic rational function, useful in the sequel, is

$$\psi(s) = \left(\frac{s}{1+s^2}\right)^k, \quad k \in \mathbb{N}.$$

We recall that slice hyperholomorphic rational functions have poles that are real points and/or spheres. This is compatible with the structure of the S-spectrum of T, which consists of real points and/or spheres. With ψ as above, we have

$$\psi(T) = \Big(T(\mathcal{I} + T^2)^{-1}\Big)^k, \quad k \in \mathbb{N}.$$

We summarize in the following the properties of the rational functional calculus. The proofs are similar to the classical results, and for this reason we omit them.

Proposition 6.1.3. *Let T be a linear quaternionic operator that is single-valued on a quaternionic Banach space X. Let P and Q be intrinsic quaternionic polynomials of order n and m, respectively. Then*

(i) *If $P \not\equiv 0$ then $P(T)Q(T) = (PQ)(T)$.*

(ii) *If $P(T)$ is injective and $Q \not\equiv 0$, then*

$$\mathcal{D}(P(T)^{-1}) \cap \mathcal{D}(Q(T)) \subset \mathcal{D}(P(T)^{-1}Q(T)) \cap \mathcal{D}(Q(T)P(T)^{-1})$$

and

$$P(T)^{-1}Q(T)v = Q(T)P(T)^{-1}v, \quad \forall v \in \mathcal{D}(Q(T)) \cap \mathcal{D}(P(T)^{-1}).$$

(iii) *Suppose that T is a closed linear operator with $\rho_S(T) \neq \emptyset$. Then $P(T)$ is closed and $P(\sigma_S(T)) = \sigma_S(P(T))$.*

For rational functions we have the following result, whose proof is similar to the classical case.

Proposition 6.1.4. *Let T be a linear quaternionic operator that is single-valued on a quaternionic Banach space X with $\rho_S(T) \neq \emptyset$. Let $0 \not\equiv R = PQ^{-1}$ and $R_1 = P_1 Q_1^{-1}$ be intrinsic rational functions. Then we have:*

(i) *$R(T)$ is a closed operator.*

(ii) $R(\overline{\sigma_S}(T)) \subset \overline{\sigma_S}(R(T))$, *where* $\overline{\sigma_S}(T) = \sigma_S(T) \cup \{\infty\}$ *denotes the extended S-spectrum of T.*

(iii) $R(T)R_1(T) \subset (RR_1)(T)$ *and equality holds if*

$$(\deg(P) - \deg(Q))(\deg(P_1) - \deg(Q_1)) \geq 0.$$

(iv) $R(T) + R_1(T) \subset (R + R_1)(T)$ *and equality holds if*

$$\deg(PQ_1 + P_1Q) = \max\{\deg(PQ_1), \deg(P_1Q)\}.$$

6.2 The S-Functional Calculus for Operators of Type ω

We show that at least for a suitable subclass of closed densely defined operators, we can extend the formulas of the S-functional calculus for bounded operators. In order to do this, we recall that the definitions the S-resolvent operators are given in the previous chapter for unbounded operators.

Definition 6.2.1 (Argument function)**.** Let $s \in \mathbb{H} \setminus \{0\}$. We define $\arg(s)$ as the unique number $\theta \in [0, \pi]$ such that $s = |s|e^{\theta j_s}$.

Observe that $\theta = \arg(s)$ does not depend on the choice of j_s if $s \in \mathbb{R} \setminus \{0\}$, since $p = |p|e^{0j}$ for every $j \in \mathbb{S}$ if $p > 0$ and $p = |p|e^{\pi j}$ for every $j \in \mathbb{S}$ if $p < 0$. Let $\vartheta \in [0, \pi]$. We define the sets

$$\begin{aligned} \mathcal{S}_\vartheta &= \{s \in \mathbb{H} : \ |\arg(p)| \leq \vartheta \text{ or } s = 0\}, \\ \mathcal{S}^0_\vartheta &= \{s \in \mathbb{H} : \ |\arg(p)| < \vartheta\}. \end{aligned} \tag{6.1}$$

Definition 6.2.2 (Operator of type ω)**.** Let $\omega \in [0, \pi)$. We say that the linear operator $T : D(T) \subset X \to X$ is of type ω if

(i) T is closed and densely defined,

(ii) $\sigma_S(T) \subset \mathcal{S}_\vartheta \cup \{\infty\}$,

(iii) for every $\vartheta \in (\omega, \pi]$ there exists a positive constant C_ϑ such that

$$\|S_L^{-1}(s, T)\| \leq \frac{C_\vartheta}{|s|}, \quad \|S_R^{-1}(s, T)\| \leq \frac{C_\vartheta}{|s|} \quad \text{for all nonzero } s \in \mathcal{S}^0_\vartheta.$$

We now introduce the following subsets of the set of slice hyperholomorphic functions, which consist of bounded slice hyperholomorphic functions.

Definition 6.2.3. Let $\mu \in (0, \pi]$. We set

$$\mathcal{SH}_L^\infty(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_L(\mathcal{S}^0_\mu) \quad \text{such that } \|f\|_\infty := \sup_{s \in \mathcal{S}^0_\mu} |f(s)| < \infty\},$$

$$\mathcal{SH}_R^\infty(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_R(\mathcal{S}^0_\mu) \quad \text{such that } \|f\|_\infty := \sup_{s \in \mathcal{S}^0_\mu} |f(s)| < \infty\},$$

$$\mathcal{N}^\infty(\mathcal{S}^0_\mu) = \{f \in \mathcal{N}(\mathcal{S}^0_\mu) \quad \text{such that } \|f\|_\infty := \sup_{s \in \mathcal{S}^0_\mu} |f(s)| < \infty\}.$$

In order to define bounded functions of operators of type ω, we need to introduce suitable subclasses of bounded slice hyperholomorphic functions:

Definition 6.2.4. With the notation introduced in Definition 6.2.3, we define

$$\Psi_L(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}^\infty_L(\mathcal{S}^0_\mu) : \ \exists\, \alpha > 0,\ c > 0 : |f(s)| \le \frac{c|s|^\alpha}{1+|s|^{2\alpha}} \text{ for all } s \in \mathcal{S}^0_\mu\},$$

$$\Psi_R(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}^\infty_R(\mathcal{S}^0_\mu) : \ \exists\, \alpha > 0,\ c > 0 : |f(s)| \le \frac{c|s|^\alpha}{1+|s|^{2\alpha}} \text{ for all } s \in \mathcal{S}^0_\mu\},$$

$$\Psi(\mathcal{S}^0_\mu) = \{f \in \mathcal{N}^\infty(\mathcal{S}^0_\mu) : \ \exists\, \alpha > 0,\ c > 0 : |f(s)| \le \frac{c|s|^\alpha}{1+|s|^{2\alpha}} \text{ for all } s \in \mathcal{S}^0_\mu\}.$$

The following theorem is a crucial step for the definition of the S-functional calculus for operators of type ω, because it shows that the following integrals depend neither on the path that we choose nor on the complex plane $\mathbb{C}_j$, $j \in \mathbb{S}$.

Theorem 6.2.5. *Let T be an operator of type ω. Let $j \in \mathbb{S}$, and let $\mathcal{S}^0_\mu$ be as in (6.1). Choose a piecewise smooth path Γ in $\mathcal{S}^0_\mu \cap \mathbb{C}_j$ that goes from $\infty e^{j\theta}$ to $\infty e^{-j\theta}$, where $\omega < \theta < \mu$. Then the integrals*

$$\frac{1}{2\pi}\int_\Gamma S_L^{-1}(s,T)\, ds_j\, \psi(s), \quad \textit{for all } \psi \in \Psi_L(\mathcal{S}^0_\mu), \tag{6.2}$$

$$\frac{1}{2\pi}\int_\Gamma \psi(s)\, ds_j\, S_R^{-1}(s,T), \quad \textit{for all } \psi \in \Psi_R(\mathcal{S}^0_\mu), \tag{6.3}$$

depend neither on Γ nor on $j \in \mathbb{S}$, and they define bounded operators.

Proof. We reason on the integral (6.2), since (6.3) can be treated in a similar way.

The growth estimates on ψ and on the resolvent operator imply that the integral (6.2) exists and defines a bounded right-linear operator.

The independence of the choice of θ and of the choice of the path Γ in the complex plane $\mathbb{C}_j$ follows from Cauchy's integral theorem.

In order to show that the integral (6.2) is independent of the choice of the imaginary unit $j \in \mathbb{S}$, we take an arbitrary $i \in \mathbb{S}$ with $j \neq i$.

Let $B(0, r)$ be the ball centered at the origin with radius r; let $a_0 > 0$ and $\theta_0 \in (0, \pi)$, $n \in \mathbb{N}$. We define the sector $\Sigma(\theta_0, a_0)$ as

$$\Sigma(\theta_0, a_0) := \{s \in \mathbb{H} : \ \arg(s - a_n) \ge \theta_n\}.$$

Let $\theta_0 < \theta_s < \theta_p < \pi$ and set $U_s := \Sigma(\theta_s, 0) \cup B(0, a_0/2)$ and $U_p := \Sigma(\theta_p, 0) \cup B(0, a_0/3)$, where the indices s and p denote the variables of integration over the boundary of the respective set. Suppose that U_p and U_s are Cauchy domains and $\partial(U_s \cap \mathbb{C}_j)$ and $\partial(U_p \cap \mathbb{C}_i)$ are paths that are contained in the sector. Observe that

$\psi(s)$ is right slice hyperholomorphic on $\overline{U_p}$, and hence by Theorem 2.1, we have

$$\psi(T) = \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_j)} \psi(s)\, ds_j\, S_R^{-1}(s,T) \tag{6.4}$$

$$= \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_j)} \left(\int_{\partial(U_p\cap\mathbb{C}_i)} \psi(p)\, dp_i\, S_R^{-1}(p,s)\right) ds_j\, S_R^{-1}(s,T) \tag{6.5}$$

$$= \frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_j)} \psi(p)\, dp_i \left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_j)} S_R^{-1}(p,s)\, ds_j\, S_R^{-1}(s,T)\right) \tag{6.6}$$

$$= \frac{1}{2\pi}\int_{\partial(U_p\cap\mathbb{C}_i)} \psi(p)\, dp_i\, S_R^{-1}(p,T). \tag{6.7}$$

To exchange the order of integration we apply Fubini's theorem. The last equation follows as an application of the S-functional calculus for unbounded operators, introduced in the previous chapter, since $S_R^{-1}(p,\infty) = \lim_{s\to\infty} S_R^{-1}(p,s) = 0$. So we get the statement. □

Thanks to the above theorem the following definitions are well posed.

Definition 6.2.6 (The S-functional calculus for operators of type ω)**.** Let T be an operator of type ω. Let $j \in \mathbb{S}$, and let $\mathcal{S}_\mu^0$ be the sector defined above. Choose a piecewise smooth path Γ in $\mathcal{S}_\mu^0\cap\mathbb{C}_j$ that goes from $\infty e^{j\theta}$ to $\infty e^{-j\theta}$, for $\omega < \theta < \mu$. Then

$$\psi(T) := \frac{1}{2\pi}\int_\Gamma S_L^{-1}(s,T)\, ds_j\, \psi(s), \quad \text{for all } \psi \in \Psi_L(\mathcal{S}_\mu^0), \tag{6.8}$$

$$\psi(T) := \frac{1}{2\pi}\int_\Gamma \psi(s)\, ds_j\, S_R^{-1}(s,T), \quad \text{for all } \psi \in \Psi_R(\mathcal{S}_\mu^0). \tag{6.9}$$

From the definition of the functional calculus the linearity properties follow immediately. In fact, if T is an operator of type ω, then $\psi(T)$, defined in (6.8) and (6.9), satisfy

$$(\psi a + \varphi b)(T) = \psi(T)a + \varphi(T)b, \quad \text{for all } \psi,\varphi \in \Psi_L(\mathcal{S}_\mu^0),$$
$$(a\psi + b\varphi)(T) = a\psi(T) + b\varphi(T), \quad \text{for all } \psi,\varphi \in \Psi_R(\mathcal{S}_\mu^0).$$

For functions ψ that belong to $\Psi(\mathcal{S}_\mu^0)$ both representations can be used. Moreover,

$$\psi(T) := \frac{1}{2\pi}\int_\Gamma \psi(s)\, ds_i\, S_R^{-1}(s,T)$$
$$= \frac{1}{2\pi}\int_\Gamma S_L^{-1}(s,T)\, ds_i\, \psi(s), \quad \text{for all } \psi \in \Psi(\mathcal{S}_\mu^0).$$

Using the S-resolvent equation with similar computations as in the case of bounded operators, adapted to this case, we can prove the product rule:

Theorem 6.2.7. *Let T be an operator of type ω. Then*

$$(\psi\varphi)(T) = \psi(T)\varphi(T), \quad \text{for all } \psi \in \Psi(\mathcal{S}^0_\mu),\ \varphi \in \Psi_L(\mathcal{S}^0_\mu),$$
$$(\psi\varphi)(T) = \psi(T)\varphi(T), \quad \text{for all } \psi \in \Psi_R(\mathcal{S}^0_\mu),\ \varphi \in \Psi(\mathcal{S}^0_\mu).$$

6.3 The H^∞-Functional Calculus

To define the H^∞ functional calculus we suppose that T is an operator of type ω, and moreover, we assume that it is one-to-one and with dense range. Here we will consider slice hyperholomorphic functions defined on the open sector $\mathcal{S}^0_\mu$, for $0 \leq \omega < \mu \leq \pi$, which can grow at infinity as $|s|^k$ and at the origin as $|s|^{-k}$ for $k \in \mathbb{N}$. This enlarges the class of functions to which the functional calculus can be applied. Precisely we make the following definition.

Definition 6.3.1 (Operators of type Ω)**.** Let ω be a real number such that $0 \leq \omega \leq \pi$. We denote by Ω the set of linear operators T acting on a two-sided quaternionic Banach space such that:

(i) T is a linear operator of type ω;

(ii) T is one-to-one and with dense range.

Then we define the following function spaces according to the set of operators defined above:

Definition 6.3.2. Let ω and μ be real numbers such that $0 \leq \omega < \mu \leq \pi$. We set

$$\mathcal{F}_L(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_L(\mathcal{S}^0_\mu) : \ |f(s)| \leq C(|s|^k + |s|^{-k}) \text{ for some } k > 0 \text{ and } C > 0\},$$
$$\mathcal{F}_R(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_R(\mathcal{S}^0_\mu) : \ |f(s)| \leq C(|s|^k + |s|^{-k}) \text{ for some } k > 0 \text{ and } C > 0\},$$
$$\mathcal{F}(\mathcal{S}^0_\mu) = \{f \in \mathcal{N}(\mathcal{S}^0_\mu) : \quad |f(s)| \leq C(|s|^k + |s|^{-k}) \text{ for some } k > 0 \text{ and } C > 0\}.$$

To extend the functional calculus we consider a quaternionic two-sided Banach space X, the operators in the class Ω, and

- the noncommutative algebra $\mathcal{F}_L(\mathcal{S}^0_\mu)$ (resp. $\mathcal{F}_R(\mathcal{S}^0_\mu)$);
- the S-functional calculus Φ for operators of type ω:

$$\Phi : \Psi_L(\mathcal{S}^0_\mu) \text{ (resp. } \Psi_R(\mathcal{S}^0_\mu)) \to \mathcal{B}(X), \quad \Phi : \phi \to \phi(T);$$

- the commutative subalgebra of $\mathcal{F}_L(\mathcal{S}^0_\mu)$ consisting of intrinsic rational functions;

Furthermore, the functions in $\mathcal{F}_L(\mathcal{S}^0_\mu)$ have at most polynomial growth. So taking an intrinsic rational functions ψ, the operator $\psi(T)$ can be defined by the rational functional calculus.

We assume also that $\psi(T)$ is injective.

Definition 6.3.3 (H^∞-functional calculus)**.** Let X be a two-sided quaternionic Banach space and let $T \in \Omega$. For $k \in \mathbb{N}$ consider the function

$$\psi(s) := \left(\frac{s}{1+s^2}\right)^{k+1}.$$

For $f \in \mathcal{F}_L(\mathcal{S}^0_\mu)$ and T right linear, we define the *extended functional calculus* as

$$f(T) := (\psi(T))^{-1}(\psi f)(T). \tag{6.10}$$

For $f \in \mathcal{F}_R(\mathcal{S}^0_\mu)$ and T left linear, we define the *extended functional calculus* as

$$f(T) := (f\psi)(T)(\psi(T))^{-1}. \tag{6.11}$$

We say that ψ *regularizes* f.

In the previous definition the operator $(\psi f)(T)$ (resp. $(f\psi)(T)$) is defined using the S-functional calculus Φ for operators of type ω, and $\psi(T)$ is defined by the rational functional calculus.

Theorem 6.3.4. *The definition of the functional calculus in* (6.10) *and in* (6.11) *does not depend on the choice of the intrinsic rational slice hyperholomorphic function* ψ.

Proof. Let us prove (6.10). Suppose that ψ and ψ' are two different regularizers and set

$$A := (\psi(T))^{-1}(\psi f)(T) \quad \text{and} \quad B := (\psi'(T))^{-1}(\psi' f)(T).$$

Observe that since the functions ψ and ψ' commute, because there are intrinsic rational functions, one has

$$\psi(T)\psi'(T) = (\psi\psi')(T) = (\psi'\psi)(T) = \psi'(T)\psi(T),$$

so we get

$$(\psi'(T))^{-1}(\psi(T))^{-1} = (\psi(T))^{-1}(\psi'(T))^{-1}.$$

It is now easy to see that

$$\begin{aligned} A = (\psi(T))^{-1}(\psi f)(T) &= (\psi(T))^{-1}(\psi'(T))^{-1}(\psi'(T))(\psi f)(T) = \\ &= (\psi'(T))^{-1}(\psi(T))^{-1}(\psi\psi' f)(T) \\ &= (\psi'(T))^{-1}(\psi(T))^{-1}\psi(T)(\psi' f)(T) \\ &= (\psi'(T))^{-1}(\psi' f)(T) = B, \end{aligned}$$

where we used the fact that from the product rule, see Proposition 6.1.4, we have that the inverse of $\psi(T)$ is $(1/\psi)(T)$. The proof of (6.11) follows in a similar way. □

We now state an important result for functions in $\mathcal{F}_L(\mathcal{S}^0_\mu)$ (the same result with obvious changes holds for functions in $\mathcal{F}_R(\mathcal{S}^0_\mu)$).

Theorem 6.3.5. *Let $f \in \mathcal{F}(\mathcal{S}_\mu^0)$ and $g \in \mathcal{F}_L(\mathcal{S}_\mu^0)$. Then we have*

$$\begin{aligned} f(T) + g(T) &\subset (f+g)(T), \\ f(T)g(T) &\subset (fg)(T), \end{aligned}$$

and $\mathcal{D}(f(T)g(T)) = \mathcal{D}((fg)(T)) \bigcap \mathcal{D}(g(T))$.

Proof. Let us take ψ_1 and ψ_2 that regularize f and g, respectively. Observe that the function $\psi := \psi_1\psi_2$ regularizes f, g, $f+g$, and fg because ψ_1, ψ_2, and f commute among themselves. Observe that

$$\begin{aligned} f(T) + g(T) &= (\psi(T))^{-1}(\psi f)(T) + (\psi(T))^{-1}(\psi g)(T) \\ &\subset (\psi(T))^{-1}[(\psi f)(T) + (\psi g)(T)] \\ &= (\psi(T))^{-1}[\psi(f+g)](T) = (f+g)(T). \end{aligned}$$

We can consider now the product rule

$$\begin{aligned} f(T)g(T) &= (\psi_1(T))^{-1}(\psi_1 f)(T)\ (\psi_2(T))^{-1}(\psi_2 g)(T) \\ &\subset (\psi_1(T))^{-1}(\psi_2(T))^{-1}[(\psi_1 f)(T)(\psi_2 g)(T)] \\ &= (\psi_2(T)\psi_1(T))^{-1}[\psi_1(T)\psi_2(T)(fg)](T) \\ &= (\psi(T))^{-1}(\psi fg)(T) = (fg)(T), \end{aligned}$$

where we have used $\psi := \psi_1\psi_2$. Regarding the domains, it is as in the complex case. □

6.4 Boundedness of the H^∞-Functional Calculus

The following convergence theorem is stated for functions in $\mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$, but it holds also for functions in $\mathcal{SH}_R^\infty(\mathcal{S}_\mu^0)$ and is the quaternionic analogue of the theorem in Section 5 in [165]. The proof follows the proof of the convergence theorem in [165, p. 216]; we just point out that the convergence theorem is based on the principle of uniform boundedness that holds also for quaternionic operators.

Theorem 6.4.1 (A Convergence theorem)**.** *Suppose that $0 \le \omega < \mu \le \pi$ and that T is a linear operator of type ω such that it is one-to-one and with dense range. Let f_α be a net in $\mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$ and let $f \in \mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$ and assume that:*

(i) *there exists a positive constant M such that $\|f_\alpha(T)\| \le M$;*

(ii) *for every $0 < \delta < \lambda < \infty$,*

$$\sup\{|f_\alpha(s) - f(s)| \text{ such that } s \in \mathcal{S}_\mu^0 \text{ and } \delta \le |s| \le \lambda\} \to 0.$$

Then $f(T) \in \mathcal{B}(V)$ and $f_\alpha(T)u \to f(T)u$ for all $u \in V$, and moreover, $\|f(T)\| \le M$.

In the following we discuss the boundedness of the H^∞ functional calculus. The crucial tool to show the boundedness of the H^∞ functional calculus is the so-called quadratic estimates; see [165].

Definition 6.4.2 (Quadratic estimate)**.** Let T be a right linear operator of type ω on a quaternionic Hilbert space $\mathcal{H}$ and let $\psi \in \Psi(\mathcal{S}_\mu^0)$, where $0 \leq \omega < \mu \leq \pi$. We say that T *satisfies a quadratic estimate with respect to* ψ if there exists a positive constant β such that

$$\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t} \leq \beta^2 \|u\|^2, \quad \text{for all } u \in \mathcal{H},$$

where we write $\|u\|$ for $\|u\|_{\mathcal{H}}$.

Let us introduce the notation

$$\Psi^+(\mathcal{S}_\mu^0) = \{\psi \in \Psi(\mathcal{S}_\mu^0) : \quad \psi(t) > 0 \text{ for all } t \in (0,\infty)\}$$

and

$$\psi_t(s) = \psi(ts), \quad t \in (0,\infty).$$

Theorem 6.4.3. *Let $0 \leq \omega < \mu \leq \pi$ and assume that T is a right linear operator in Ω. Suppose that T and its adjoint T^* satisfy the quadratic estimates with respect to the functions ψ and $\widetilde{\psi} \in \Psi^+(\mathcal{S}_\mu^0)$. Suppose that f belongs to $\mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$. Then the operator $f(T)$ is bounded, and there exists a positive constant C such that*

$$\|f(T)\| \leq C\|f\|_\infty \quad \textit{for all } f \in \mathcal{SH}_L^\infty(\mathcal{S}_\mu^0).$$

Proof. We follow the proof of Theorem on p. 221 in [165], and we point out the differences. We observe that we choose the functions ψ, $\widetilde{\psi}$, and η in the space of intrinsic functions $\Psi^+(\mathcal{S}_\mu^0)$ because the pointwise product

$$\varphi(s) := \psi(s)\widetilde{\psi}(s)\eta(s)$$

has to be slice hyperholomorphic, and moreover, η has to be such that

$$\int_0^\infty \varphi(t)\,\frac{dt}{t} = 1.$$

For $f \in \mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$ let us define

$$f_{\varepsilon,R}(s) = \int_\varepsilon^R (\varphi_t f)(s)\frac{dt}{t}.$$

Using the quadratic estimates it follows that there exists a positive constant C such that

$$\|f_{\varepsilon,R}(T)\| \leq C\|f\|_\infty.$$

The convergence theorem (Theorem 6.4.1) gives the formula

$$f(T)u = \lim_{\varepsilon\to 0}\lim_{R\to\infty} f_{\varepsilon,R}(T)u \quad \text{for all } u \in \mathcal{H},$$

where $(\eta_t f)(T)$ is defined by the S-functional calculus

$$(\eta_t f)(T) = \frac{1}{2\pi}\int_\Gamma S_L^{-1}(s,T)\, ds_i\, \eta_t(s) f(s), \quad \text{for all } f \in \Psi_L(\mathcal{S}_\mu^0),$$

since $\eta_t f \in \Psi_L(\mathcal{S}_\mu^0)$ because η_t is intrinsic. Precisely, the quadratic estimates and some computations show that there exists a positive constant C_β such that

$$|\langle f_{\varepsilon,R}(T)u, v\rangle| \leq C_\beta \sup_{t\in(0,\infty)} \|(\eta_t f)(T)\|\|u\|\|v\|.$$

Since

$$\begin{aligned}\|(\eta_t f)(T)\| &\leq \frac{1}{2\pi}\|f\|_\infty \sup_{i\in\mathbb{S}}\int_\Gamma \|S_L^{-1}(s,T)\||ds_i|\,|\eta_t(s)| \\ &\leq \frac{1}{2\pi}\sup_{i\in\mathbb{S}}\|f\|_\infty \int_\Gamma \frac{C_\eta}{|s|}\,|ds_i|\,\frac{c|s|^\alpha}{1+|s|^{2\alpha}} \\ &\leq C_T(\mu,\eta)\|f\|_\infty,\end{aligned}$$

from the above estimates we get the statement. □

6.5 Comments and Remarks

To study fractional diffusion and fractional evolution problems we need a more involved and refined version of the H^∞-functional calculus in the quaternionic setting, which is beyond the aim of this book. For more details see the papers [50–52], where the fractional powers of quaternionic operators and applications are treated. In the paper [53], the authors introduced the so-called S-spectrum approach to fractional diffusion processes, which allows one to study very general fractional diffusion problems. This strategy is largely explained in the monograph [56]. The new approach to fractional diffusion problems will be explained without too many technical details in the following subsection.

6.5.1 Comments on Fractional Diffusion Processes

We denote by u the temperature on and by $\mathbf{q}$ the heat flow, and we set the thermal diffusivity equal to 1. The heat equation is then deduced from the two laws

$$\mathbf{q} = -\nabla u \quad \text{(Fourier's law)}, \tag{6.12}$$

$$\partial_t u + \operatorname{div}\mathbf{q} = 0 \quad \text{(conservation of energy)}, \tag{6.13}$$

where u and $\mathbf{q}$ are defined on $\mathbb{R}^3$, and Fourier's law is substituted into the equation for conservation of energy, that is,

$$\partial_t u - \Delta u = 0.$$

The fractional heat equation is an alternative model that takes into account non-local interactions, and it is obtained by replacing the negative Laplacian in the heat equation by its fractional power, so that

$$\partial_t u + (-\Delta)^\alpha u = 0, \quad \alpha \in (0,1), \tag{6.14}$$

where the fractional Laplacian is given by

$$(-\Delta)^\alpha u(x) = c(n,\alpha) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2\alpha}} dy,$$

and the integral is defined in the sense of the principal value, $c(n,\alpha)$ is a known constant, and $u : \mathbb{R}^n \to \mathbb{R}$ must belong to a suitable function space.

The approach with the fractional powers of quaternionic operators defined via the H^∞-functional calculus is different, very general, and in the case $\mathbf{q} = -\nabla u$ it reduces to the fractional Laplace operator.

Precisely, we identify

$$\mathbb{R}^3 \cong \{s \in \mathbb{H} : \ \mathrm{Re}(s) = 0\},$$

and we consider the gradient ∇ the quaternionic Nabla operator

$$\nabla = e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_3 \partial_{x_3}.$$

Instead of replacing the negative Laplacian in the heat equation by $(-\Delta)^\alpha$, we want to replace the gradient in (6.12) by its fractional power ∇^α, and then we replace it in the law of conservation of energy. We proceed as follows:

- Since s^α is not defined on $(-\infty, 0)$, and on $L^2(\mathbb{R}^3, \mathbb{H})$ it is $\sigma_S(\nabla) = \mathbb{R}$, we consider the projections of the fractional powers of ∇^α, indicated by $f_\alpha(\nabla)$, to the subspace associated with the subset $[0 + \infty)$ of the S-spectrum of ∇, on which the function s^α is well defined.
- Then we take just the vector part $\mathrm{Vect}(f_\alpha(\nabla)) = e_1 T_1 + e_2 T_2 + e_3 T_3$ of the quaternionic operator $f_\alpha(\nabla) = T_0 + e_1 T_1 + e_2 T_2 + e_3 T_3$ so that we can apply the divergence operator.

We point out that the above procedure applied to the gradient operator gives the classical result. Indeed, the definition of ∇^α only on the subspace associated to $[0, \infty)$ is given by

$$f_\alpha(\nabla) v = \frac{1}{2\pi} \int_{-j\mathbb{R}} S_L^{-1}(s, \nabla)\, ds_j\, s^\alpha \nabla v,$$

for $v : \mathbb{R}^3 \to \mathbb{H}$ in $\mathcal{D}(\nabla)$. This corresponds to the Balakrishnan formula, which is a consequence of the quaternionic H^∞-functional calculus, in which only positive spectral values are taken into account. With this definition and the surprising expression for the left S-resolvent operator

$$S_L^{-1}(-jt, \nabla) = (-jt + \nabla)\underbrace{(-t^2 + \Delta)^{-1}}_{=R_{-t^2}(-\Delta)},$$

the operator $f_\alpha(\nabla)$, with some computations, becomes

$$f_\alpha(\nabla)v = \underbrace{\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2 v}_{\mathrm{Scal} f_\alpha(\nabla)v} + \underbrace{\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v}_{=\mathrm{Vec} f_\alpha(\nabla)v}.$$

We define the scalar part of the operator $f_\alpha(\nabla)$ applied to v as

$$\mathrm{Scal} f_\alpha(\nabla)v := \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2 v,$$

and the vector part as

$$\mathrm{Vec} f_\alpha(\nabla)v := \frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v.$$

Now we observe hat

$$\mathrm{div}\mathrm{Vec} f_\alpha(\nabla)v = -\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}+1}v.$$

This proves that in the case of the gradient, we get the same result, which is the fractional Laplacian. The fractional heat equation for $\alpha \in (1/2, 1)$,

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0,$$

can hence be written as

$$\partial_t u(t,x) - 2\mathrm{div}\,(\mathrm{Vec} f_\beta(\nabla)u) = 0, \quad \beta = 2\alpha - 1.$$

We point out that the operator $f_\alpha(\nabla)$ can be applied to vector-valued functions v. For an application to the heat equation it is applied to the scalar-valued function u that represents the temperature. The quaternionic fractional powers approach is very general, and it is applicable to a large class of operators such as

$$\widetilde{\nabla} = e_1\, a(x)\partial_{x_1} + e_2\, b(x)\partial_{x_2} + e_3\, c(x)\partial_{x_3},$$

where a, b, c are suitable real-valued functions that depend on the space variables $x = (x_1, x_2, x_3)$ and possibly also on time. For every suitable vector operator T, we define a new fractional evolution equation as

$$\partial_t u(t,x) - 2\mathrm{div}\,(\mathrm{Vec} f_\beta(T)u) = 0.$$

For example, a new fractional evolution equation can be deduced when we consider the following Fourier's law:

$$T = e_1 x_1 \partial_{x_1} + e_2 x_2 \partial_{x_2} + e_3 x_3 \partial_{x_3}.$$

Working in the space $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu)$ with

$$\mathbb{R}^3_+ = \{e_1 x_1 + e_2 x_2 + e_3 x_3 : \ \ x_\ell > 0\}$$

and $d\mu = (x_1 x_2 x_3)^{-1} dx$, we get the operator

$$\begin{aligned} &\mathrm{Vec} f_\beta(T) v(\xi) \\ &\quad = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} -|y|^{2\alpha} e^{e_1 \sum_{k=1}^3 \xi_k y_k} e^{-e_1 x \cdot y} \begin{pmatrix} e^{x_1} v_{\xi_1}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_2} v_{\xi_2}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_3} v_{\xi_3}(e^{x_1}, e^{x_2}, e^{x_3}) \end{pmatrix} dx\, dy. \end{aligned}$$

We point out that the fractional powers of the operator $\mathbf{q}(x, \partial_x)$ are very useful for inhomogeneous materials, and this approach has several advantages: It modifies the Fourier law but keeps the law of conservation of energy, and it is applicable to a large class of operators that includes the gradient but also operators with variable coefficients such as the operator $\mathbf{q}(x, \partial_x)$. Moreover, $\mathbf{q}$ can also depend on time.

The fact that we keep the evolution equation in divergence form allows an immediate definition of the weak solution of the fractional evolution problem.

To represent the fractional powers of an operator T we have to write an explicit expression for the inverse of the operator $T^2 - 2s_0 T + |s|^2 \mathcal{I}$, and this can be done on bounded or unbounded domains.