Chapter 4

Properties of the S-Functional Calculus for Bounded Operators

In this chapter we will show that most of the properties that hold for the Riesz-Dunford functional calculus can be extended to the S-functional calculus. The proofs of the quaternionic results require several additional efforts with respect to the classical case.

4.1 Algebraic Properties and Riesz Projectors

An immediate consequence of Definition 3.2.5 is that the S-functional calculus for left slice hyperholomorphic functions is quaternionic right linear and that the S-functional calculus for right slice hyperholomorphic functions is quaternionic left linear.

Lemma 4.1.1. *Let $T \in \mathcal{B}(X)$.*

(i) *If $f, g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \textit{and} \quad (fa)(T) = f(T)a.$$

(ii) *If $f, g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \textit{and} \quad (af)(T) = af(T).$$

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Proof. If $f, g \in \mathcal{SH}_L(U)$ and $a \in \mathbb{H}$, then we have

$$\begin{aligned}(f+g)(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,(f(s)+g(s))\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\\ &\quad+\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,g(s) = f(T)+g(T)\end{aligned}$$

and

$$\begin{aligned}(fa)(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)a\\ &= \left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\right)a = f(T)a.\end{aligned}$$

The right slice hyperholomorphic case follows by similar computations. □

Since the product of two slice hyperholomorphic functions is not necessarily slice hyperholomorphic, we cannot expect to obtain a product rule for arbitrary slice hyperholomorphic functions. However, if $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$, then $fg \in \mathcal{SH}_L(\sigma_S(T))$, and if $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T)$, then $fg \in \mathcal{SH}_R(\sigma_S(T))$. In order to show that the S-functional calculus is at least in these cases compatible with the multiplication of functions, we need the following lemma.

Lemma 4.1.2. *Let $B \in \mathcal{B}(X)$. For all $q, s \in \mathbb{H}$ with $q \notin [s]$, we have*

$$(\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} = (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q}). \quad (4.1)$$

If, moreover, f is an intrinsic slice hyperholomorphic function and U is a bounded slice Cauchy domain with $\overline{U} \subset \mathcal{D}(f)$, then

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,(\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} = Bf(q)$$

for every $q \in U$ and $j \in \mathbb{S}$.

Proof. Since $s\overline{s} = |s|^2$ and $s+\overline{s} = 2\mathrm{Re}(s)$ are real, they commute with the operator B. Hence, for all $q, s \in \mathbb{H}$ with $q \notin [s]$, we have that

$$\begin{aligned}&(s^2 - 2\mathrm{Re}(q)s + |q|^2)(\overline{s}B - Bq)\\ &= s|s|^2B - 2\mathrm{Re}(q)|s|^2B + |q|^2\overline{s}B - s^2Bq + 2\mathrm{Re}(q)sBq - |q|^2Bq\\ &= sB|s|^2 - B|s|^2(q+\overline{q}) + \overline{s}B|q|^2 - s^2Bq + sB(q+\overline{q})q - B|q|^2q\\ &= (sB - B\overline{q})|s|^2 - s(s+\overline{s})Bq + (s+\overline{s})B\overline{q}q + (sB - B\overline{q})q^2\\ &= (sB - B\overline{q})|s|^2 - (sB - B\overline{q})2\mathrm{Re}(s)q + (sB - B\overline{q})q^2\\ &= (sB - B\overline{q})(q^2 - 2\mathrm{Re}(s)q + |s|^2).\end{aligned}$$

Multiplication by $(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$ from the right and multiplication by $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}$ from the left yields (4.1).

Let now f be an intrinsic slice hyperholomorphic function, let $U \subset \mathcal{D}(f)$ be a bounded slice Cauchy domain, let $q = u + iv \in U$, and let $j \in \mathbb{S}$. An application of (4.1) gives

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} ds_j\, f(s)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q}),\end{aligned}$$

where ds_j and $f(s)$ commute because $f(s) \in \mathbb{C}_j$ for $s \in \mathbb{C}_j$, since f is intrinsic. Now observe that $f(s)$ is intrinsic slice hyperholomorphic on $\mathcal{D}(f)$, that $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}$ is intrinsic slice hyperholomorphic in s on $\mathbb{H} \setminus [q]$, and that $sB - B\overline{q}$ is left slice hyperholomorphic in s on all of $\mathbb{H}$. Hence their product $F(s) := f(s)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q})$ is left slice hyperholomorphic on $\mathcal{D}(f) \setminus [q]$. By Proposition 2.3.12, the restriction F_j of this function to the complex plane $\mathbb{C}_j$ is therefore a left holomorphic function with values in the complex left Banach space X over $\mathbb{C}_j$.

Assume now that $q \notin \mathbb{R}$. Then F_j has two poles in $U \cap \mathbb{C}_j$, namely $q_j = u + jv$ and $\overline{q_j}$. From the residue theorem we therefore deduce that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}\\ &= \mathrm{Res}(F_j, q_j) + \mathrm{Res}(F_j, \overline{q_j}).\end{aligned}$$

Since s and q_j belong to the same complex plane, they commute, so that we have

$$(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1} = (s - q_j)^{-1}(s - \overline{q_j})^{-1},$$

and in turn

$$\begin{aligned}\mathrm{Res}(F_j, q_j) &= \lim_{s\to q_j, s\in\mathbb{C}_j} (s - q_j)F_j(s)\\ &= f(q_j)(q_j - \overline{q_j})^{-1}(q_jB - B\overline{q}) = f(q_j)(2vj)^{-1}(vjB + Bvi)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}(F_j, \overline{q_j}) &= \lim_{s\to \overline{q_j}, s\in\mathbb{C}_j} (s - \overline{q_j})F_j(s)\\ &= f(\overline{q_j})(\overline{q_j} - q_j)^{-1}(\overline{q_j}B - B\overline{q}) = f(\overline{q_j})(-2jv)^{-1}(-vjB + Bvi).\end{aligned}$$

Thus we have

$$\begin{aligned}\mathrm{Res}(F_j, q_j) + \mathrm{Res}(F_j, \overline{q_j}) &= f(q_j)\frac{1}{2}B - f(q_j)\frac{1}{2}jBi + f(\overline{q_j})\frac{1}{2}B + f(\overline{q_j})\frac{1}{2}jBi\\ &= \frac{1}{2}(f(q_j) + f(\overline{q_j}))B + \frac{1}{2}(-f(q_j) + f(\overline{q_j}))jBi.\end{aligned}$$

Since $f(q_j) = f_0(u,v) + f_1(u,v)j$ with $f_0(u,v), f_1(u,v) \in \mathbb{R}$, we finally obtain

$$\begin{aligned}\operatorname{Res}(F_j, q_j) + \operatorname{Res}(F_j, \overline{q_j}) &= f_0(u,v)B + (-f_1(u,v)j)jBi \\ &= B(f_0(u,v) + f_1(u,v)i) = Bf(q)\end{aligned}$$

and hence

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} = Bf(q).$$

If, on the other hand, $q \in \mathbb{R}$, then also $f(q) \in \mathbb{R}$. Since $q = \overline{q}$ commutes in this case with B, we moreover have

$$F(s) = (s-q)^{-1} f(s)B,$$

and so

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} \\ &= \operatorname{Res}(F_j, q) = \lim_{s\to q,\, s\in\mathbb{C}_j} (s-q)F(s)B = f(q)B = Bf(q).\end{aligned}$$ □

Theorem 4.1.3 (Product rule). *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ or let $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. Then*

$$(fg)(T) = f(T)g(T).$$

Proof. Let $f \in \mathcal{N}(\sigma_S(T))$, let $g \in \mathcal{SH}_L(\sigma_S(T))$, and let U_q and U_s be bounded slice Cauchy domains that contain $\sigma_S(T)$ such that $\overline{U_q} \subset U_s$ and $\overline{U_s} \subset \mathcal{D}(f)\cap\mathcal{D}(g)$. The subscripts q and s refer to the respective variables of integration in the following computation. We choose $j \in \mathbb{S}$ and we set $\Gamma_s := \partial(U_s \cap \mathbb{C}_j)$ and $\Gamma_q := \partial(U_q \cap \mathbb{C}_j)$ for neatness. By Theorem 3.2.11, we can write $f(T)$ using both the left and right S-resolvent operators, and so

$$\begin{aligned}f(T)g(T) &= \frac{1}{2\pi}\int_{\Gamma_s} f(s)\, ds_j\, S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\, dq_j\, g(q) \\ &= \frac{1}{2\pi}\int_{\Gamma_s} f(s)\, ds_j \left[\frac{1}{2\pi}\int_{\Gamma_q} S_R^{-1}(s,T)S_L^{-1}(q,T)\, dq_j\, g(q)\right].\end{aligned}$$

For simplicity we set $\mathcal{Q}_s(q)^{-1} := (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$. If we apply (3.7) in the above integral, we obtain

$$
\begin{aligned}
f(T)g(T) = &\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_R^{-1}(s,T)q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\\
&-\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_L^{-1}(q,T)q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\\
&-\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} \overline{s}S_R^{-1}(s,T)\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\\
&+\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} \overline{s}S_L^{-1}(q,T)\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q).
\end{aligned}
$$

We observe that

$$
\begin{aligned}
&\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_R^{-1}(s,T)q\mathcal{Q}_s(q)^{-1}dq_j\,g(q)\\
&=\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j\,S_R^{-1}(s,T)\left[\int_{\Gamma_q} q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\right]=0
\end{aligned}
$$

and

$$
\begin{aligned}
&-\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \left[\int_{\Gamma_q} \overline{s}\,S_R^{-1}(s,T)\mathcal{Q}_s(q)^{-1}dq_j\,g(q)\right]\\
&=-\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j\,\overline{s}\,S_R^{-1}(s,T)\left[\int_{\Gamma_q} \mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\right]=0
\end{aligned}
$$

by Cauchy's integral theorem, because the functions $\mathcal{Q}_s(q)^{-1}$ and $q\mathcal{Q}_s(q)^{-1}$ are for every $s \in \Gamma_s$ right slice hyperholomorphic on an open set that contains $\overline{U_q}$, since we chose $\overline{U_q} \subset U_s$. Therefore, we have

$$
\begin{aligned}
f(T)g(T) = &-\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_L^{-1}(q,T)q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\\
&+\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} \overline{s}S_L^{-1}(q,T)\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\\
=&\frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_q} f(s)\,ds_j\left[\overline{s}S_L^{-1}(q,T)-S_L^{-1}(q,T)q\right]\mathcal{Q}_s(q)^{-1}dq_j\,g(q).
\end{aligned}
$$

The integrand in the last integral is continuous and hence bounded on $\Gamma_s \times \Gamma_q$. We can thus apply Fubini's theorem and change the order of integration, so that

$$
f(T)g(T) = \frac{1}{(2\pi)^2}\int_{\Gamma_q}\left[\int_{\Gamma_s} f(s)\,ds_j\,[\overline{s}S_L^{-1}(q,T)-S_L^{-1}(q,T)q]\mathcal{Q}_s(q)^{-1}\right]dq_j\,g(q).
$$

Applying Lemma 4.1.2 with $B = S_L^{-1}(q,T)$, we obtain

$$f(T)g(T) = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j\,f(q)g(q) = (fg)(T).$$

The product rule for the S-functional calculus for right slice hyperholomorphic functions can be shown with analogous computations using the second version (3.8) of the S-resolvent equation. □

Corollary 4.1.4. *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. If $f^{-1} \in \mathcal{N}(\sigma_S(T))$, then $f(T)$ is invertible and $f(T)^{-1} = f^{-1}(T)$.*

Proof. From Theorem 4.1.3, we deduce that

$$\mathcal{I} = 1(T) = \left(ff^{-1}\right)(T) = f(T)f^{-1}(T)$$

if we consider f and f^{-1} left slice hyperholomorphic functions and that

$$\mathcal{I} = 1(T) = \left(f^{-1}f\right)(T) = f^{-1}(T)f(T)$$

if we consider them right slice hyperholomorphic functions. Hence $f(T)$ is invertible with $f(T)^{-1} = f(T)^{-1}$. □

Finally, the S-functional calculus has the capability to define the quaternionic Riesz projectors and allows one in turn to identify invariant subspaces of T that are associated with sets of spectral values.

Theorem 4.1.5 (Riesz's projectors)**.** *Let $T \in \mathcal{B}(X)$ and assume that $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with*

$$\operatorname{dist}(\sigma_1,\sigma_2) > 0.$$

We choose an open axially symmetric set O with $\sigma_1 \subset O$ and $\overline{O} \cap \sigma_2 = \emptyset$ and define $\chi_{\sigma_1}(s) = 1$ for $s \in O$ and $\chi_{\sigma_2}(s) = 0$ for $s \notin O$. Then $\chi_{\sigma_1} \in \mathcal{N}(\sigma_S(T))$, and

$$P_{\sigma_1} := \chi_{\sigma_1}(T) = \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j$$

is a continuous projection that commutes with T. Hence $P_{\sigma_1}X$ is a right linear subspace of X that is invariant under T.

Proof. The function χ_{σ_1} obviously belongs to $\mathcal{N}(\sigma_S(T))$, and by Theorem 4.1.3, we have

$$P_{\sigma_1}^2 = \chi_{\sigma_1}(T)\chi_{\sigma_1}(T) = (\chi_{\sigma_1}\chi_{\sigma_1})(T) = \chi_{\sigma_1}(T) = P_{\sigma_1}.$$

Hence P_{σ_1} is a projection in $\mathcal{B}(X)$. Since it is right linear, its range $P_{\sigma_1}X$ is a closed right linear subspace of X. Moreover, we have

$$TP_{\sigma_1} = s(T)\chi_{\sigma_1}(T) = (s\chi_{\sigma_1})(T) = (\chi_{\sigma_1}s)(T) = \chi_{\sigma_1}(T)s(T) = P_{\sigma_1}T.$$

For every $x \in P_{\sigma_1}X$, we thus obtain

$$Tx = TP_{\sigma_1}x = P_{\sigma_1}Tx \quad \text{for all } x \in P_{\sigma_1}X,$$

and hence $P_{\sigma_1}X$ is invariant under T.

We can show these properties explicitly, which we shall do now so that the reader can see the analogy with the Riesz projectors of the F-functional calculus in Theorem 7.4.2. Let us choose two bounded Cauchy slice domains U_q and U_s such that $\sigma \subset U_q$ and $\overline{U_q} \subset U_s$ and $\overline{U_s} \subset O$. We choose $j \in \mathbb{S}$ and we set $\Gamma_s := \partial(U_s \cap \mathbb{C}_j)$ and $\Gamma_q := \partial(U_q \cap \mathbb{C}_j)$ for neatness. By Theorem 3.2.11, we then have

$$P_{\sigma_1} = \frac{1}{2\pi} \int_{\Gamma_s} ds_j\, S_R^{-1}(s,T) = \frac{1}{2\pi} \int_{\Gamma_q} S_L^{-1}(q,T)\, dq_j,$$

and so

$$\begin{aligned} P_{\sigma_1}^2 &= \frac{1}{2\pi} \int_{\Gamma_s} ds_j\, S_R^{-1}(s,T) \frac{1}{2\pi} \int_{\Gamma_q} S_L^{-1}(q,T)\, dq_j \\ &= \frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j \left[\int_{\Gamma_q} S_R^{-1}(s,T) S_L^{-1}(q,T)\, dq_j \right]. \end{aligned}$$

For simplicity we set $\mathcal{Q}_s(q)^{-1} := (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$. If we apply (3.7) in the above integral, we obtain

$$\begin{aligned} f(T)g(T) = &\frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j \int_{\Gamma_q} S_R^{-1}(s,T) q \mathcal{Q}_s(q)^{-1}\, dq_j \\ &- \frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j \int_{\Gamma_q} S_L^{-1}(q,T) q \mathcal{Q}_s(q)^{-1}\, dq_j \\ &- \frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j \int_{\Gamma_q} \overline{s} S_R^{-1}(s,T) \mathcal{Q}_s(q)^{-1}\, dq_j \\ &+ \frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j \int_{\Gamma_q} \overline{s} S_L^{-1}(q,T) \mathcal{Q}_s(q)^{-1}\, dq_j. \end{aligned}$$

We observe that

$$\begin{aligned} &\frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j \int_{\Gamma_q} S_R^{-1}(s,T) q \mathcal{Q}_s(q)^{-1} dq_j \\ &= \frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j\, S_R^{-1}(s,T) \left[\int_{\Gamma_q} q \mathcal{Q}_s(q)^{-1}\, dq_j \right] = 0 \end{aligned}$$

and

$$\begin{aligned} &- \frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j \left[\int_{\Gamma_q} \overline{s}\, S_R^{-1}(s,T) \mathcal{Q}_s(q)^{-1} dq_j \right] \\ &= - \frac{1}{(2\pi)^2} \int_{\Gamma_s} ds_j\, \overline{s}\, S_R^{-1}(s,T) \left[\int_{\Gamma_q} \mathcal{Q}_s(q)^{-1}\, dq_j \right] = 0 \end{aligned}$$

by Cauchy's integral theorem, because the functions $\mathcal{Q}_s(q)^{-1}$ and $q\mathcal{Q}_s(q)^{-1}$ are for every $s \in \Gamma_s$ right slice hyperholomorphic on an open set that contains $\overline{U_q}$; since we chose $\overline{U_q} \subset U_s$. Therefore, we have

$$\begin{aligned} P_{\sigma_1}^2 &= -\frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} S_L^{-1}(q,T)q\mathcal{Q}_s(q)^{-1}\,dq_j \\ &\quad + \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} \overline{s}S_L^{-1}(q,T)\mathcal{Q}_s(q)^{-1}\,dq_j \\ &= \frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_q} ds_j \left[\overline{s}S_L^{-1}(q,T) - S_L^{-1}(q,T)q\right]\mathcal{Q}_s(q)^{-1}dq_j. \end{aligned}$$

The integrand in the last integral is continuous and hence bounded on $\Gamma_s \times \Gamma_q$. We can thus apply Fubini's theorem and change the order of integration so that

$$P_{\sigma_1}^2 = \frac{1}{(2\pi)^2}\int_{\Gamma_q}\left[\int_{\Gamma_s} ds_j\,[\overline{s}S_L^{-1}(q,T) - S_L^{-1}(q,T)q]\mathcal{Q}_s(q)^{-1}\right] dq_j.$$

Applying Lemma 4.1.2 with $B = S_L^{-1}(q,T)$ and $f(q) = 1$, we obtain

$$P_{\sigma_1}^2 = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j = P_{\sigma_1}.$$

We furthermore have, because of (3.4), that

$$\begin{aligned} TP_{\sigma_1} &= \frac{1}{2\pi}\int_{\Gamma_q} TS_L^{-1}(q,T)\,dq_j \\ &= \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j q - \frac{1}{2\pi}\int_{\Gamma_q}\mathcal{I}\,dq_j = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j q \end{aligned}$$

by Cauchy's integral theorem and similarly

$$\begin{aligned} P_{\sigma_1}T &= \frac{1}{2\pi}\int_{\Gamma_s} ds_j\,S_R^{-1}(s,T)T \\ &= \frac{1}{2\pi}\int_{\Gamma_s} sds_j\,S_R^{-1}(s,T) - \frac{1}{2\pi}\int_{\Gamma_s} ds_j\mathcal{I} = \frac{1}{2\pi}\int_{\Gamma_s} sds_j\,S_R^{-1}(s,T). \end{aligned}$$

By Theorem 3.2.11, we thus have

$$TP_{\sigma_1} = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j\,q = \frac{1}{2\pi}\int_{\Gamma_s} s\,ds_j\,S_R^{-1}(s,T) = P_{\sigma_1}T. \qquad \square$$

4.2 The Spectral Mapping Theorem and the Composition Rule

Similar to the product rule, the spectral mapping theorem does not hold for arbitrary slice hyperholomorphic functions. This is not surprising: it is clear that it can hold only for slice hyperholomorphic functions that preserve the fundamental geometry of the S-spectrum, namely its axial symmetry. Again, the class of intrinsic slice hyperholomorphic functions stands out here, since it is this class of functions that maps axially symmetric sets to axially symmetric sets.

Theorem 4.2.1 (The spectral mapping theorem)**.** *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. Then*

$$\sigma_S(f(T)) = f(\sigma_S(T)) = \{f(s) : \ s \in \sigma_S(T)\}.$$

Proof. Let U be a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and let $s = u + jv \in \sigma_S(T)$. For $q \in U \setminus [s]$, we define

$$\tilde{g}(q) = (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(f(q)^2 - 2\mathrm{Re}(f(s))f(q) + |f(s)|^2).$$

Since f is intrinsic slice hyperholomorphic, the function

$$q \mapsto f(q)^2 - 2\mathrm{Re}(f(s))f(q) + |f(s)|^2$$

is intrinsic slice hyperholomorphic too. If we multiply it by the intrinsic rational function $(q^2-2\mathrm{Re}(s)q+|s|^2)^{-1}$, we obtain again an intrinsic slice hyperholomorphic function, and hence $\tilde{g}$ belongs to $\mathcal{N}(U) \setminus [s]$.

We can extend $\tilde{g}$ to a function $g \in \mathcal{N}(U)$. Indeed, if $s \notin \mathbb{R}$ and $i \in \mathbb{S}$, then the function $\tilde{g}_i = \tilde{g}|_{\mathbb{C}_i}$ has the singularities $s_i = u + iv$ and $\overline{s_i} = u - iv$ in $U \cap \mathbb{C}_i$. However, we have

$$\begin{aligned}
&\lim_{z\to s_i, z\in\mathbb{C}_i} \tilde{g}_i(z)\\
&= \lim_{z\to s_i, z\in\mathbb{C}_i} (z^2 - 2\mathrm{Re}(s_i)z + |s_i|^2)^{-1}(f(z)^2 - 2\mathrm{Re}(f(s_i))f(z) + |f(s_i)|^2)\\
&= \lim_{z\to s_i, z\in\mathbb{C}_i} (z - s_i)^{-1}(f(z) - f(s_i))(z - \overline{s_i})^{-1}\left(f(z) - \overline{f(s_i)}\right)\\
&= f_i'(s_i)(s_i - \overline{s_i})^{-1}\left(f(s_i) - \overline{f(s_i)}\right) = f_i'(s_i)\frac{f_1(u,v)}{v}
\end{aligned}$$

because s_i, z, $f(s_i)$, and $f(z)$ belong to the same complex plane, since f is intrinsic, and hence they mutually commute. Since $f(\overline{s_i}) = \overline{f(s_i)}$ because f is intrinsic, we also have

$$\begin{aligned}
&\lim_{z\to \overline{s_i}, z\in\mathbb{C}_i} \tilde{g}_i(z)\\
&= \lim_{z\to \overline{s_i}, z\in\mathbb{C}_i} (z^2 - 2\mathrm{Re}(s_i)z + |s_i|^2)^{-1}(f(z)^2 - 2\mathrm{Re}(f(s_i))f(z) + |f(s_i)|^2)\\
&= \lim_{z\to \overline{s_i}, z\in\mathbb{C}_i} (z - \overline{s_i})^{-1}\left(f(z) - \overline{f(s_i)}\right)(z - s_i)^{-1}(f(z) - f(s_i))\\
&= f_i'(\overline{s_i})(\overline{s_i} - s_i)^{-1}\left(f(\overline{s_i}) - \overline{f(\overline{s_i})}\right) = f_i'(\overline{s_i})\frac{f_1(u,v)}{v}.
\end{aligned}$$

Thus s_i and $\overline{s_i}$ are removable singularities of $\tilde{g}_i$, and since $\overline{s_i} = s_{-i}$, the function

$$g(q) = \begin{cases} \tilde{g}(q) & \text{if } q \in U \setminus [s], \\ \partial_S f(q) \frac{f_1(u,v)}{v} & \text{if } q = u + iv \in [s], \end{cases}$$

is well defined. Obviously, it is an intrinsic slice function, and its restriction g_i to any complex plane $\mathbb{C}_i$ is holomorphic. By Lemma 2.1.6, the function g is intrinsic slice hyperholomorphic.

If, on the other hand, $s \in \mathbb{R}$, then the point s is for every $i \in \mathbb{S}$ the only singularity of the function $\tilde{g}_i$. Since $\overline{f(s)} = f(\overline{s}) = f(s)$, we have $f(s) \in \mathbb{R}$ and hence $\mathrm{Re}(s) = s$ and $\mathrm{Re}(f(s)) = f(s)$ such that

$$\begin{aligned} \lim_{z \to s, z \in \mathbb{C}_i} \tilde{g}_i(z) &= \lim_{z \to s, z \in \mathbb{C}_i} (z^2 - 2sz + s^2)^{-1}(f(z)^2 - 2f(s)f(z) + f(s)^2) \\ &= \lim_{z \to \infty, z \in \mathbb{C}_i} (z - s)^{-2} (f(z) - f(s))^2 = (f_i'(s))^2. \end{aligned}$$

Therefore, the singularity s of $\tilde{g}_i$ is removable for every $i \in \mathbb{S}$, and since $(f_i'(s))^2 = (\partial_S f(s))^2$ does not depend on the imaginary unit i, the function

$$g(q) = \begin{cases} \tilde{g}(q) & \text{if } q \in U \setminus [s], \\ (\partial_S f(s))^2 & \text{if } q = s, \end{cases}$$

is well defined. Obviously, g is an intrinsic slice function and $g_i = g|_{U \cap \mathbb{C}_i}$ is holomorphic on $U \cap \mathbb{C}_i$ for every $i \in \mathbb{S}$. By Lemma 2.1.6, the function g is also in this case intrinsic slice hyperholomorphic.

The product rule implies

$$f(T)^2 - 2\mathrm{Re}(f(s))f(T) + |f(s)|^2\mathcal{I} = (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})g(T).$$

If the operator $f(T)^2 - 2\mathrm{Re}(f(s))f(T) + |f(s)|^2\mathcal{I}$ were invertible, then

$$g(T)(f(T)^2 - 2\mathrm{Re}(f(s))f(T) + |f(s)|^2\mathcal{I})^{-1}$$

would therefore be the inverse of $T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}$. Since we assumed $s \in \sigma_S(T)$, this is impossible, and hence $f(s) \in \sigma_S(f(T))$. Thus

$$f(\sigma_S(T)) \subset \sigma_S(f(T)).$$

If, on the other hand, $s \notin f(\sigma_S(T))$, then we can consider the function

$$h(q) := (f^2(q) - 2\mathrm{Re}(s)f(q) + |s|^2)^{-1},$$

which is an intrinsic slice hyperholomorphic function. Its poles are the spheres $[q] \subset U$ such that $f([q]) = [f(q)] = [s]$. Since we assumed $s \notin f(\sigma_S(T))$, it does not have any poles on $\sigma_S(T)$. Thus it belongs to $\mathcal{N}(\sigma_S(T))$, and Corollary 4.1.4 implies

$$h(T) = \left(f(T)^2 - 2\mathrm{Re}(s)f(T) + |s|^2\right)^{-1} \in \mathcal{B}(X).$$

We find that $s \in \rho_S(T)$ and in turn also

$$\sigma_S(f(T)) \subset f(\sigma_S(T)). \qquad \square$$

The spectral mapping theorem allows us to generalize the Gelfand formula for the spectral radius to quaternionic linear operators.

Definition 4.2.2. Let $T \in \mathcal{B}(X)$. Then the *S-spectral radius* of T is defined to be the nonnegative real number

$$r_S(T) := \sup\{|s| : \ s \in \sigma_S(T)\}.$$

Theorem 4.2.3. *For $T \in \mathcal{B}(X)$, we have*

$$r_S(T) = \lim_{n\to+\infty} \|T^n\|^{\frac{1}{n}}.$$

Proof. The mapping $q \mapsto q^{-1}$ is intrinsic slice hyperholomorphic, and hence $q \mapsto S_L^{-1}(q^{-1}, T)$ is slice hyperholomorphic on the set

$$U := \{q \in \mathbb{H} : \ q^{-1} \in \rho_S(T)\}.$$

Since $\mathbb{H} \setminus B_{r_S(T)}(0) \subset \rho_S(T)$, the set U contains the ball $B_{1/r_S(T)}(0)$. By Theorem 2.1.15, the function $S_L^{-1}(q^{-1}, T)$ admits a power series expansion at 0 that converges on $B_{1/r_S(T)}(0)$. Because of Theorem 3.1.6, it is given by

$$S_L^{-1}\left(q^{-1}, T\right) = \sum_{n=0}^{+\infty} T^n q^{n+1}, \quad |q| < \frac{1}{r_S(T)}.$$

For $s \in \mathbb{H}$ with $|s| > r_S(T)$, we thus have $\left\|T^n s^{-n-1}\right\| \to 0$ as $n \to +\infty$ because the above series converges. In particular, we have

$$C(s) = \sup_{n\in\mathbb{N}} \|T^n s^{-n-1}\| < +\infty.$$

Therefore,

$$\begin{aligned}\limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} \frac{1}{|s|} &= \limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} |s|^{-\frac{n+1}{n}} \\ &= \limsup_{n\to+\infty} \|T^n s^{-n-1}\|^{\frac{1}{n}} \leq \limsup_{n\to+\infty} C(s)^{\frac{1}{n}} = 1,\end{aligned}$$

and hence

$$\limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} \leq |s|.$$

Since s was arbitrary with $|s| > r_S(T)$, we obtain

$$\limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} \leq r_S(T).$$

Moreover, Theorem 4.2.1 implies

$$\sigma_S(T^n) = \sigma_S(T)^n$$

and we conclude from Theorem 3.1.13 that

$$\begin{aligned} r_S(T)^n &= \sup\{|s|^n : \ s \in \sigma_S(T)\} \\ &= \sup\{|s| : \ s \in \sigma_S(T^n)\} = r_S(T^n) \leq \|T^n\| \end{aligned}$$

for every $n \in \mathbb{N}$. Therefore, we get

$$r_S(T) \leq \liminf_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq r_S(T) \tag{4.2}$$

and in turn $r_S(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$, where (4.2) also implies the existence of the limit. □

Finally, the spectral mapping theorem also allows us to generalize the composition rule.

Theorem 4.2.4 (Composition rule)**.** *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. If $g \in \mathcal{SH}_L(\sigma_S(f(T))$, then $g \circ f \in \mathcal{SH}_L(\sigma_S(T))$, and if $g \in \mathcal{SH}_R(f(\sigma_S(T)))$, then $g \circ f \in \mathcal{SH}_R(\sigma_S(T))$. In both cases,*

$$g(f(T)) = (g \circ f)(T).$$

Proof. If $g \in \mathcal{SH}_L(f(\sigma_S(T)))$, then $\mathcal{D}(g)$ is open and axially symmetric. Since f is continuous and intrinsic, the inverse image of every open axially symmetric set under f is again open and axially symmetric. The set $f^{-1}(\mathcal{D}(g))$ is therefore an axially symmetric open set, and it contains $\sigma_S(T)$, since $f(\sigma_S(T)) = \sigma_S(f(T)) \subset \mathcal{D}(g)$ because of Theorem 4.2.1. By Theorem 2.1.4, the composition $g \circ f$ is a left slice hyperholomorphic function with domain $f^{-1}(\mathcal{D}(g))$, and so it belongs to $\mathcal{SH}_L(\sigma_S(T))$.

Let U be a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and let W be another bounded slice Cauchy domain such that $\sigma_S(T) \subset \overline{f(U)} \subset W$ and $\overline{W} \subset \mathcal{D}(g)$. (Such slice Cauchy domains exist because of Remark 3.2.4.) The mapping $s \mapsto S_L^{-1}(q, f(s))$ is left slice hyperholomorphic on

$$\{s \in \mathcal{D}(f) : \ f(s) \notin [q]\} = \{s \in \mathcal{D}(f) : \ q \notin [f(s)]\}$$

by Theorem 2.1.4. If $q \notin \sigma_S(f(T)) = f(\sigma_S(T))$, then $s \mapsto S_L^{-1}(q, f(s))$ therefore belongs to $\mathcal{SH}_L(\sigma_S(T))$. Since the S-functional calculus is compatible with algebraic operations, we have

$$\begin{aligned}S_L^{-1}(q,f(T)) &= -\mathcal{Q}_q(f(T))^{-1}(f(T)-\overline{q}\mathcal{I})\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,\left[-\mathcal{Q}_q(f(s))^{-1}(f(s)-\overline{q})\right]\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,S_L^{-1}(q,f(s))\end{aligned}$$

with $\mathcal{Q}_s(f(s))^{-1} = (f(s)^2 - 2\mathrm{Re}(q)f(s)+|q|^2)^{-1}$ and an arbitrary imaginary unit $j\in\mathbb{S}$. Therefore,

$$\begin{aligned}g(f(T)) &= \frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(q,f(T))\,dq_j\,g(q)\\ &= \frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)}\left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,S_L^{-1}(q,f(s))\right]dq_j\,g(q).\end{aligned}$$

Since the integrand in the last integral is continuous and hence bounded on the compact set $\partial(W\cap\mathbb{C}_j)\times\partial(U\cap\mathbb{C}_j)$, we can apply Fubini's theorem to change the order of integration and obtain

$$\begin{aligned}g(f(T)) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\left[\frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(p,f(s))\,dp_j\,g(p)\right]\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,g(f(s))\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,(g\circ f)(s) = (g\circ f)(T).\end{aligned}$$

□

4.3 Convergence in the S-Resolvent Sense

The following definition and the next result show that the notion of convergence in the resolvent sense is meaningful also in the quaternionic setting. This notion is important for unbounded operators.

Definition 4.3.1 (Converges in the norm S-resolvent sense)**.** Let T_m, $m\in\mathbb{N}$, and T belong to $\mathcal{B}(X)$ and suppose that $\rho_S(T)=\rho_S(T_m)$ for all $m\in\mathbb{N}$. We say that T_m *converges to* T *in the norm left* S*-resolvent sense* if $S_L^{-1}(s,T_m)\to S_L^{-1}(s,T)$ in $\mathcal{B}(X)$ as $m\to+\infty$ for all $s\in\rho_S(T)$ and that T_m *converges to* T *in the norm right* S*-resolvent sense* if $S_R^{-1}(s,T_m)\to S_R^{-1}(s,T)$ in $\mathcal{B}(X)$ as $m\to+\infty$ for all $s\in\rho_S(T)$.

Theorem 4.3.2. *Let* $T_m\in\mathcal{B}(X)$*,* $m\in\mathbb{N}$ *be uniformly bounded,* $T\in\mathcal{B}(X)$*, and suppose that* $\rho_S(T)=\rho_S(T_m)$ *for all* $m\in\mathbb{N}$*. The following statements are then equivalent:*

(i) *T_m converges to T in $\mathcal{B}(X)$.*

(ii) *T_m converges to T in the norm left S-resolvent sense.*

(iii) *T_m converges to T in the norm right S-resolvent sense.*

In each of these cases, the convergence $S_L^{-1}(s,T_m) \to S_L^{-1}(s,T)$ or $S_R^{-1}(s,T_m) \to S_R^{-1}(s,T)$ is uniform for s on compact subsets of $\rho_S(T)$.

Proof. Assume first that (i) holds. Then

$$\begin{aligned} S_L^{-1}(s,T) - S_L^{-1}(s,T_m) = &-\mathcal{Q}_s(T)^{-1}\left(T - \overline{s}\mathcal{I} - T_m + \overline{s}\mathcal{I}\right) \\ &- \left(\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right)\left(T_m - \overline{s}\mathcal{I}\right) \end{aligned}$$

and hence

$$\begin{aligned} &\|S_L^{-1}(s,T) - S_L^{-1}(s,T_m)\| \\ &\qquad \le \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T - T_m\| + \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\|. \end{aligned} \tag{4.3}$$

We observe that

$$\begin{aligned} &\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1} \\ &\qquad = \mathcal{Q}_s(T_m)^{-1}\left(T_m^2 - T^2 - 2s_0(T - T_m)\right)\mathcal{Q}_s(T)^{-1} \\ &\qquad = \mathcal{Q}_s(T_m)^{-1}\left(T_m(T_m - T) + (T_m - T)T + 2s_0(T - T_m)\right)\mathcal{Q}_s(T)^{-1}. \end{aligned} \tag{4.4}$$

Hence if we can show that there exists a positive constant C_s such that $\|Q_s(T_m)\| \le C_s$ for all $m \in \mathbb{N}$, then we will obtain $\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\| \to 0$ and in turn, due to (4.3), that $S_L^{-1}(s,T) \to S_L^{-1}(s,T)$ in $\mathcal{B}(X)$. We point out that

$$\begin{aligned} \mathcal{Q}_s(T_m) &= \mathcal{Q}_s(T)\left(\mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T_m)\right) \\ &= \mathcal{Q}_s(T)\left(\mathcal{I} - \mathcal{Q}_s(T)^{-1}(T^2 - T_m^2 - 2s_0(T_m - T))\right). \end{aligned} \tag{4.5}$$

For $A \in \mathcal{B}(X)$ with $\|A\| < 1$, the operator $(\mathcal{I} - A)^{-1} = \sum_{n=0}^{+\infty} A^n \in \mathcal{B}(X)$ exists and satisfies $\left\|(\mathcal{I} - A)^{-1}\right\| \le (1 - \|A\|)^{-1}$. Since $T_m \to T$, we find that

$$A_m := \mathcal{Q}_s(T)^{-1}(T^2 - T_m^2 - 2s_0(T_m - T)) \to 0$$

in $\mathcal{B}(X)$ as $m \to +\infty$ and hence $\|A_m\| \le 1/2$ for sufficiently large m. For such m, the operator $\mathcal{I} - A_m$ is invertible with $\left\|(\mathcal{I} - A_m)^{-1}\right\| \le 2$, and because of (4.5), we obtain

$$\mathcal{Q}_s(T_m)^{-1} = (\mathcal{I} - A_m)^{-1}\mathcal{Q}_s(T)^{-1} \tag{4.6}$$

and in turn

$$\left\|Q_s(T_m)^{-1}\right\| \le \left\|(\mathcal{I} - A_m)^{-1}\right\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \le 2\left\|\mathcal{Q}_s(T)^{-1}\right\|. \tag{4.7}$$

Therefore,

$$C_s := \sup_{m \in \mathbb{N}} \left\|\mathcal{Q}_s(T_m)^{-1}\right\| < +\infty,$$

and we conclude that (ii) holds.

The convergence $S_L^{-1}(s,T_m) \to S_L^{-1}(s,T)$ is even uniform in s on every compact set K, since because of (4.3), we have

$$\begin{aligned}
&\sup_{s\in K} \|S_L^{-1}(s,T) - S_L^{-1}(s,T_m)\| \\
&\leq \sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T - T_m\| + \sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\|.
\end{aligned}$$

Since $\mathcal{Q}_s(T)^{-1}$ is continuous on $\rho_S(T)$, we have $\sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1}\right\| < +\infty$, and so the first summand converges to 0 uniformly in s as $m \to +\infty$. For the second summand, we have because of (4.4) that

$$\begin{aligned}
&\sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\leq \sup_{s\in K} \left\|\mathcal{Q}_s(T_m)^{-1}\right\| \|T_m\|\|T_m - T\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\quad + \sup_{s\in K} \left\|\mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - T\|\|T\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\quad + \sup_{s\in K} |2s_0| \left\|\mathcal{Q}_s(T_m)^{-1}\right\| \|T - T_m\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\|.
\end{aligned}$$

Because of (4.7), we have $\left\|Q_s(T_m)^{-1}\right\| \leq 2\left\|\mathcal{Q}_s(T)^{-1}\right\|$ for m sufficiently large, and so

$$\begin{aligned}
&\sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\leq \sup_{s\in K} 2\left\|\mathcal{Q}_s(T)^{-1}\right\|^2 (\|T\| + \|T_m\| + 2|s_0|\|)\, (\|T_m\| + |\overline{s}|)\|T - T_m\|\| \\
&\leq C\|T - T_m\|
\end{aligned}$$

because $\mathcal{Q}_s(T)^{-1}$ and s_0 and $\overline{s}$ depend continuously on s and are hence bounded on the compact set K.

Conversely, we suppose now that (ii) holds and we show that $\|T - T_m\| \to 0$. Since T and T_m are uniformly bounded, there exists $\alpha \in \rho_S(T) \cap \bigcup_{m\in\mathbb{N}} \rho_S(T_m)$. We then have $S_L^{-1}(\alpha,T) = (\alpha\mathcal{I} - T)^{-1}$ and $S_L^{-1}(\alpha,T_m) = (\alpha\mathcal{I} - T_m)^{-1}$, and so

$$\begin{aligned}
&\|T - T_m\| = \|\alpha - T_m - (\alpha - T)\| \\
&\leq \|\alpha - T_m\| \left\|(\alpha\mathcal{I} - T)^{-1} - (\alpha\mathcal{I} - T_m)^{-1}\right\| \|\alpha - T\| \\
&= \|\alpha\mathcal{I} - T_m\|\|\alpha\mathcal{I} - T\| \left\|S_L^{-1}(\alpha,T) - S_L^{-1}(\alpha,T_m)\right\| \to 0
\end{aligned}$$

because $\|T_m\|$ is uniformly bounded.

The equivalence of (i) and (iii) is shown with similar arguments. □

Remark 4.3.3. Since by the above theorem convergence in the norm left S-resolvent sense and convergence in the norm right S-resolvent sense are equivalent, we will not distinguish between them in the following and just say that T_m converges to T in the norm S-resolvent sense.

Theorem 4.3.4. *Let T_m, $m \in \mathbb{N}$, and T belong to $\mathcal{B}(X)$ with $\rho_S(T) = \rho_S(T_m)$ for all $m \in \mathbb{N}$ and suppose that T_m converges to T in the norm S-resolvent sense. If $f \in \mathcal{SH}_L(\sigma_S(T))$ or $f \in \mathcal{SH}_R(\sigma_S(T))$, then*

$$\|f(T) - f(T_m)\| \to 0 \quad \text{as} \quad m \to +\infty.$$

Proof. If $f \in \mathcal{SH}_L(\sigma_S(T))$, then

$$f(T_m) - f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} \left(S_L^{-1}(s, T_m) - S_L^{-1}(s, T)\right) ds_j \, f(s)$$

with $j \in \mathbb{S}$ and a suitable bounded slice Cauchy domain U. Since $f(s)$ is continuous, it is bounded on the compact set $\partial(U \cap \mathbb{C}_j)$, and hence there exists a positive constant $C > 0$ such that

$$\|f(T_m) - f(T)\| \leq C \max_{s \in \partial(U \cap \mathbb{C}_j)} \left\|S_L^{-1}(s, T_m) - S_L^{-1}(s, T)\right\| \to 0,$$

since $\left\|S_L^{-1}(s, T_m) - S_L^{-1}(s, T)\right\| \to 0$ converges uniformly to 0 on the compact set $\partial(U \cap \mathbb{C}_j)$ by Theorem 4.3.2. Similarly, we prove the statement for $f \in \mathcal{SH}_R(\sigma_S(T))$. □

4.4 The Taylor Formula for the S-Functional Calculus

Consider a bounded operator T and let N be a small perturbation operator that furthermore commutes with T. Then $f(T + N)$ can be represented as a power series in N that formally corresponds to a Taylor series expansion in the operator. In this section we show that the Taylor formula can be extended to quaternionic operators, but before we can state the main theorem, several preliminary results are needed. This result is the quaternionic analogue of Theorem VII.10 in [104], and it was proved in [55] in the more general setting of paravector operators on a two-sided Clifford module.

Before we are able to show the Taylor expansion in the operator, we need to determine the slice derivatives of the S-resolvents. We start by finding explicit formulas for the functions

$$S_L^n(s, q) := (s - q)^{*_L n} \quad \text{and} \quad S_R^n(s, q) := (s - q)^{*_R n}.$$

Lemma 4.4.1. *Let $s \in \mathbb{H}$. For $n \geq 0$, we have*

$$S_L^n(s, q) = \sum_{k=0}^{n} \binom{n}{k} (-q)^k s^{n-k} \quad \text{and} \quad S_R^n(s, q) = \sum_{k=0}^{n} \binom{n}{k} s^{n-k} (-q)^k. \tag{4.8}$$

With $\mathcal{Q}_s(q) = q^2 - 2s_0 q + |s|^2$, we moreover have

$$S_L^{-n}(s, q) = \mathcal{Q}_s(q)^{-n} (\overline{s} - q)^{*_L n} \quad \text{and} \quad S_R^{-n}(s, q) = (\overline{s} - q)^{*_R n} \mathcal{Q}_s(q)^{-n}. \tag{4.9}$$

Furthermore, for $m, n \geq 0$, we have

$$S_L^{-n}(s,q) *_L S_L^{-m}(\overline{s},q) = \mathcal{Q}_s(q)^{-(n+m)} \left[(\overline{s}-q)^{*_L n} *_L (s-q)^{*_L m}\right]$$

and

$$S_R^{-n}(s,q) *_R S_R^{-m}(\overline{s},q) = \left[(\overline{s}-q)^{*_R n} *_R (s-q)^{*_R m}\right] \mathcal{Q}_s(q)^{-(n+m)}.$$

Proof. For $n = 0$, we have $(s-q)^{*_L 0} \equiv 1$, and hence (4.8) is obviously true. Assume that it holds for $n-1$. Then (2.18) implies

$$\begin{aligned} S_L^n(s,q) &= (s-q)^{*_L n} = (s-q)^{*_L (n-1)} *_L (s-q) \\ &= (s-q)^{*_L (n-1)} *_L s + (s-q)^{*_L (n-1)} *_L (-q) \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} (-q)^k s^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} (-q)^{k+1} s^{n-1-k} \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} (-q)^k s^{n-k} + \sum_{k=1}^{n} \binom{n-1}{k-1} (-q)^k s^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-q)^k s^{n-k}, \end{aligned}$$

and (4.8) follows by induction.

We also prove the identity (4.9) by induction. It is obviously true for $n = 0$. Assume that it holds for $n-1$ and observe that $\mathcal{Q}_s(q)^{-1} \in \mathcal{N}(\mathbb{H} \setminus [s])$. Then by (2.16) and Corollary 2.1.20 we have $S_L^{-1}(s,q) = (s-q)^{-*_L}$, so this implies

$$\begin{aligned} S_L^{-n}(s,q) &= (s-q)^{-*_L(n-1)} *_L (s-q)^{-*_L} \\ &= \left[\mathcal{Q}_s(q)^{-(n-1)}(\overline{s}-q)^{*_L(n-1)}\right] *_L \left[\mathcal{Q}_s(q)^{-1}(\overline{s}-q)\right] \\ &= \mathcal{Q}_s(q)^{-(n-1)} *_L (\overline{s}-q)^{*_L(n-1)} *_L \mathcal{Q}_s(q)^{-1} *_L (\overline{s}-q) \\ &= \mathcal{Q}_s(q)^{-(n-1)} *_L \mathcal{Q}_s(q)^{-1} *_L (\overline{s}-q)^{*_L(n-1)} *_L (\overline{s}-q) \\ &= \mathcal{Q}_s(q)^{-n}(\overline{s}-q)^{*_L n}. \end{aligned}$$

Finally (2.16) also implies for $m, n \geq 0$ that

$$\begin{aligned} S_L^{-n}(s,q) *_L S_L^{-m}(\overline{s},q) &= \left[\mathcal{Q}_s(q)^{-n}(\overline{s}-q)^{*_L n}\right] *_L \left[\mathcal{Q}_s(q)^{-m}(s-q)^{*_L m}\right] \\ &= \mathcal{Q}_s(q)^{-n} *_L (\overline{s}-q)^{*_L n} *_L \mathcal{Q}_s(q)^{-m} *_L (s-q)^{*_L m} \\ &= \mathcal{Q}_s(q)^{-n} *_L \mathcal{Q}_s(q)^{-m} *_L (\overline{s}-q)^{*_L n} *_L (s-q)^{*_L m} \\ &= \mathcal{Q}_s(q)^{-(n+m)} \left[(\overline{s}-q)^{*_L n} *_L (s-q)^{*_L m}\right]. \end{aligned}$$

The right slice hyperholomorphic case can be shown by similar computations. □

Corollary 4.4.2. *Let $s = s_0 + j_s s_1 \in \mathbb{H}$ and $n, m \in \mathbb{N}_0$. If $q \in \mathbb{C}_{j_s}$, then*

$$(s-q)^{*_L m} *_L (\overline{s}-q)^{*_L n} = (s-q)^m (\overline{s}-q)^n \tag{4.10}$$

and

$$S_L^{-m}(s,q) *_L S_L^{-n}(\overline{s},q) = (s-q)^{-m}(\overline{s}-q)^{-n}. \tag{4.11}$$

Moreover, for every $n \in \mathbb{N}_0$, the function

$$P(q) := \sum_{k=0}^{n} (\overline{s}-q)^{*_L(k+1)} *_L (s-q)^{*_L(n-k+1)} \tag{4.12}$$

is a polynomial with real coefficients. Analogous statements hold for right slice hyperholomorphic powers $S_R^m(s,q)$ of $s-q$.

Proof. If $q \in \mathbb{C}_{j_s}$, then s, $\overline{s}$, and q commute. Hence it follows from (4.8) and the binomial theorem that $(s-q)^{*_L m} = (s-q)^m$ and $(\overline{s}-q)^{*_L n} = (\overline{s}-q)^n$. From (2.16), we deduce (4.10). Since q and s commute, we also find that

$$\mathcal{Q}_s(q)^{-1} = (q-s)^{-1}(q-\overline{q})^{-1},$$

and so (4.9) implies

$$S_L^{-m}(s,q) = (s-q)^{-m}(\overline{s}-q)^{-m}(\overline{s}-q)^{m} = (s-q)^{-m}.$$

An analogous computation shows that $S_L^{-n}(\overline{s},q) = (\overline{s}-q)^{-n}$.

For arbitrary left slice hyperholomorphic functions f and g, it is because of (2.21) immediate that $(f *_L g)(q) = f(q)g(q)$ at a point q if $f(q) \in \mathbb{C}_{j_q}$. Since $(s-q)^{-m}$ belongs to $\mathbb{C}_{j_q}$ if $q \in \mathbb{C}_{j_s}$, we furthermore find that

$$S_L^{-m}(s,q) *_L S_L^{-n}(\overline{s},q) = (s-q)^{-m} *_L (\overline{s}-q)^{-n} = (s-q)^{-m}(\overline{s}-q)^{-n}.$$

Finally, we consider $P(q)$. The restriction P_{j_s} of this function to the plane $\mathbb{C}_{j_s}$ is the complex polynomial $P_{j_s}(z) = \sum_{k=0}^{n}(\overline{s}-z)^{k+1}(s-z)^{n-k+1}$. From the relation

$$P_{j_s}(\overline{q}) = \sum_{k=0}^{n}(\overline{s}-\overline{q})^{k+1}(s-\overline{q})^{n-k+1} = \overline{\sum_{k=0}^{n}(s-q)^{k+1}(\overline{s}-q)^{n-k+1}} = \overline{P_{j_s}(q)},$$

we deduce that its coefficients are real. Consequently, $P = \mathrm{ext}_L(P_j)$ is a polynomial with real coefficients on $\mathbb{H}$, where ext_L means the extension with the representation formula. We can show the analogous statement for right slice hyperholomorphic powers $S_R^m(s,q)$ of $s-q$ with similar arguments. □

We need now to formally replace the scalar variable q in the functions introduced above by the operator T in a way that is consistent with the S-functional calculus. Recall, however, that the product rule $(fg)(T) = f(T)g(T)$ holds only if $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ or if $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. This is due to the fact that for $f, g \in \mathcal{SH}_L(\sigma_S(T))$ or for $f, g \in \mathcal{SH}_R(\sigma_S(T))$, the product fg does not in general belong to $\mathcal{SH}_L(\sigma_S(T))$ resp. $\mathcal{SH}_R(\sigma_S(T))$.

If, on the other hand, one considers the left slice hyperholomorphic product $f *_L g$ of two left slice hyperholomorphic functions (or equivalently, the right slice hyperholomorphic product of two right slice hyperholomorphic functions), then it is not clear to which operation between operators it corresponds. Some considerations actually suggest that such an operation does not exist.

However, for power series of an operator variable, we can use the formulas (2.18) and (2.19) to define their $*_L$-product resp. $*_R$-product.

Definition 4.4.3. Let $T \in \mathcal{B}(X)$. For $F = \sum_{n=0}^{+\infty} T^n a_n$ and $G = \sum_{n=0}^{+\infty} T^n b_n$ with $a_\ell, b_\ell \in \mathbb{H}$ for $\ell \in \mathbb{N}$, we define

$$F *_L G := \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

For $\tilde{F} = \sum_{n=0}^{+\infty} a_n T^n$ and $\tilde{G} = \sum_{n=0}^{+\infty} b_n T^n$, we define

$$\tilde{F} *_R \tilde{G} := \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) T^n.$$

Remark 4.4.4. For $F = \sum_{n=0}^{+\infty} T^n a_n$ and $G = \sum_{n=0}^{+\infty} T^n b_n$ note that $F *_L G = FG$ if $a_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. In this case, the coefficients a_n commute with the operator T, and hence

$$F *_L G = \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} T^k a_k T^{n-k} b_{n-k} = FG.$$

Similarly, $\tilde{F} *_R \tilde{G} = \tilde{F}\tilde{G}$ if $b_n \in \mathbb{R}$ for every $n \in \mathbb{N}$.

Corollary 4.4.5. *Let $T \in \mathcal{B}(X)$ and let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ be two left slice hyperholomorphic power series that converge on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$f(T) *_L g(T) = (f *_L g)(T).$$

Similarly, for two right slice hyperholomorphic power series $\tilde{f}(q) = \sum_{n=0}^{+\infty} a_n q^n$ and $\tilde{g}(q) = \sum_{n=0}^{+\infty} b_n q^n$ that converge on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$, we have

$$\tilde{f}(T) *_R \tilde{g}(T) = (\tilde{f} *_R \tilde{g})(T).$$

Proof. By the properties of the S-functional calculus, we have $f(T) = \sum_{n=0}^{+\infty} T^n a_n$ and $g(T) = \sum_{n=0}^{+\infty} T^n b_n$. Hence

$$\begin{aligned} f(T) *_L g(T) &= \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j \sum_{n=0}^{+\infty} s^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f *_L g(s) = (f *_L g)(T). \end{aligned}$$

An analogous computation shows the right slice hyperholomorphic case. □

Observe that $S_L^n(s,T)$ and $S_R^n(s,T)$ and slice hyperholomorphic products of such expressions are well defined because of Definition 4.4.3. In analogy with (4.4.1), we furthermore give the following definition.

Definition 4.4.6. Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. For $n, m \geq 0$, we define

$$S_L^{-n}(s,T) := \mathcal{Q}_s(T)^{-n}(\overline{s}\mathcal{I} - T)^{*_L n}$$

and

$$S_L^{-n}(s,T) *_L S_L^{-m}(\overline{s},T) := \mathcal{Q}_s(T)^{-(n+m)}\left[(\overline{s}\mathcal{I} - T)^{*_L n} *_L (s\mathcal{I} - T)^{*_L m}\right].$$

Similarly, we define

$$S_R^{-n}(s,T) := (\overline{s}\mathcal{I} - T)^{*_R n}\mathcal{Q}_s(T)^{-n}$$

and

$$S_R^{-n}(s,T) *_R S_R^{-m}(\overline{s},T) := \left[(\overline{s}\mathcal{I} - T)^{*_R n} *_R (s\mathcal{I} - T)^{*_R m}\right]\mathcal{Q}_s(T)^{-(n+m)}.$$

Remark 4.4.7. Since the function $Q_s(q)^{-n}$ is intrinsic, the above definitions, due to the product rule, are compatible with the S-functional calculus, that is,

$$\left[S_L^{-n}(s,\cdot)\right](T) = S_L^{-n}(s,T) \quad \text{and} \quad \left[S_R^{-n}(s,\cdot)\right](T) = S_R^{-n}(s,T)$$

as well as

$$\left[S_L^{-n}(s,\cdot) *_L S_L^{-m}(\overline{s},\cdot)\right](T) = S_L^{-n}(s,T) *_L S_L^{-m}(\overline{s},T)$$

and

$$\left[S_R^{-n}(s,\cdot) *_L S_R^{-m}(\overline{s},\cdot)\right](T) = S_R^{-n}(s,T) *_L S_R^{-m}(\overline{s},T).$$

Proposition 4.4.8. *Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. Then*

$$\partial_S^m S_L^{-1}(s,T) = (-1)^m m!\, S_L^{-(m+1)}(s,T) \tag{4.13}$$

and

$$\partial_S^m S_R^{-1}(s,T) = (-1)^m m!\, S_R^{-(m+1)}(s,T), \tag{4.14}$$

for every $m \geq 0$.

Proof. Recall that the slice derivative, see Definition 2.1.12, coincides with the partial derivative with respect to the real part s_0 of s. We show only (4.43), since (4.44) follows by analogous computations.

We prove the statement by induction. For $m = 0$, the identity (4.43) is obvious. We assume that $\partial_S^{m-1}S_L^{-1}(s,T) = (-1)^{m-1}(m-1)!\, S_L^{-m}(s,T)$ and we compute $\partial_S^m S_L^{-1}(s,T)$. We represent $S_L^{-m}(s,T)$ using the S-functional calculus. If we choose the path of integration in the complex plane $\mathbb{C}_{j_s}$, then we find because of (4.11) that

$$\begin{aligned}\partial_S S_L^{-m}(s,T) &= \partial_S \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{j_s})} S_L^{-1}(p,T)\,dp_j\,S_L^{-m}(s,p)\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{j_s})} S_L^{-1}(p,T)\,dp_j\,\frac{\partial}{\partial s_0}(s-p)^{-m}\\ &= -m\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{j_s})} S_L^{-1}(p,T)\,dp_j\,(s-p)^{-(m+1)} = -m\,S_L^{-(m+1)}(s,T),\end{aligned}$$

and in turn,

$$\begin{aligned}\partial_S^m S_L^{-1}(s,T) &= \partial_S\left(\partial_S^{m-1}S_L^{-1}(s,T)\right)\\ &= (-1)^{m-1}(m-1)!\partial_S S_L^{-m}(s,T) = (-1)^m m!\,S_L^{-(m+1)}(s,T). \qquad \square\end{aligned}$$

Remark 4.4.9. We point out that Proposition 4.4.8 also holds for unbounded closed operators. In this case, we have to modify the definition of $S_L^{-m}(s,T)$ by commuting every occurrence of T with $\mathcal{Q}_s(T)^{-m}$ just as we did in the definition of the left S-resolvent operator. Otherwise $S_L^{-m}(s,T)$ is defined only on $\mathcal{D}(T^m)$ and not on the entire space V.

Let us now turn our attention to the Taylor series expansion of $f(T+N)$ in the operator variable. In order for such an expansion to hold, it is essential that adding a somewhat small operator N not to perturb the S-spectrum of T a lot. The following result clarifies how one has to measure the distance between a point $s\in\rho_S(T)$ and the S-spectrum of T.

Lemma 4.4.10. *Let $A\subset\mathbb{H}$ be axially symmetric and let $s=s_0+js_1\in\mathbb{H}$. Then*

$$\operatorname{dist}(s,A)=\operatorname{dist}(s,A\cap\mathbb{C}_j)=\operatorname{dist}\left(s,A\cap\mathbb{C}_j^{\geq}\right),$$

where $\operatorname{dist}(s,A):=\inf\{|s-q|:q\in A\}$ *and* $\mathbb{C}_j^{\geq}=\{q_0+jq_1:q_0\in\mathbb{R},q_1\geq 0\}$.

Proof. For $q=q_0+j_qq_1\in A$, define $q_j=q_0+jq_1$. We choose $i\in\mathbb{S}$ with $j\perp i$ and set $k=ji$. Then $q=q_0+\tilde{q}_1j+\tilde{q}_2i+\tilde{q}_3k$ with $\tilde{q}_1^2+\tilde{q}_2^2+\tilde{q}_3^2=|\underline{q}|^2=q_1^2$, and in turn

$$\begin{aligned}|s-q_j|^2 &= (s_0-q_0)^2+(s_1-q_1)^2\\ &= (s_0-q_0)^2+s_1^2-2s_1q_1+q_1^2\\ &= (s_0-q_0)^2+s_1^2-2s_1\sqrt{\tilde{q}_1^2+\tilde{q}_2^2+\tilde{q}_3^2}+\tilde{q}_1^2+\tilde{q}_2^2+\tilde{q}_2^3\\ &\leq (s_0-q_0)^2+s_1^2-2s_1\tilde{q}_1+\tilde{q}_1^2+\tilde{q}_2^2+\tilde{q}_3{}^2\\ &= (s_0-q_0)^2+(s_1-\tilde{q}_1)^2+\tilde{q}_2^2=|s-q|^2.\end{aligned}$$

Since A is axially symmetric, we have $\{q_j:q\in A\}=A\cap\mathbb{C}_j^{\geq}$. Consequently,

$$\inf_{q\in A}|s-q|\leq\inf_{q\in A\cap\mathbb{C}_j^{\geq}}|s-q|\leq\inf_{q\in A}|s-q_j|\leq\inf_{q\in A}|s-q|,$$

and in turn,

$$\operatorname{dist}(s,A)=\inf_{q\in A}|s-q|=\inf_{q\in A\cap\mathbb{C}_j^{\geq}}|s-q|=\operatorname{dist}(s,A\cap\mathbb{C}_j^{\geq}).\qquad\square$$

Proposition 4.4.11. *Let $T\in\mathcal{B}(X)$ and let $C\subset\mathbb{H}$ with $\operatorname{dist}(C,\sigma_S(T))>\varepsilon$ for some $\varepsilon>0$. Then there exists a positive constant K_T such that*

$$\left\|S_L^{-m}(s,T)*_L S_L^{-n}(\overline{s},T)\right\|\leq\frac{K_T}{\varepsilon^{m+n}}\tag{4.15}$$

and

$$\left\|S_R^{-m}(s,T)*_L S_R^{-n}(\overline{s},T)\right\|\leq\frac{K_T}{\varepsilon^{m+n}},\tag{4.16}$$

for every $s\in C$ and $m,n\geq 0$.

Proof. Let U be a bounded slice Cauchy domain with $\sigma_S(T)\subset U$ with $\operatorname{dist}(C,\overline{U})>\varepsilon$. We choose $s=s_0+js_1\in C$. By Corollary 4.4.2, we have

$$S_L^{-m}(s,q)*_L S_L^{-n}(\overline{s},q)=(s-q)^{-m}(\overline{s}-q)^{-n}$$

for every $x\in\mathbb{C}_j$. Lemma 4.4.10 implies $\operatorname{dist}(s,\overline{U}\cap\mathbb{C}_j)=\operatorname{dist}(s,\overline{U})>\varepsilon$. Since $\overline{U}\cap\mathbb{C}_j$ is symmetric with respect to the real axis, we also have $\operatorname{dist}(\overline{s},\overline{U}\cap\mathbb{C}_j)>\varepsilon$, and we deduce

$$\begin{aligned}
&\left\|S_L^{-m}(s,T)*_L S_L^{-n}(\overline{s},T)\right\|\\
&=\left\|\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}S_L^{-1}(p,T)\,dp_j\,S_L^{-m}(s,p)*_L S_L^{-n}(\overline{s},p)\right\|\\
&=\left\|\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}S_L^{-1}(p,T)\,dp_j\,(s-p)^{-m}(\overline{s}-p)^{-n}\right\|\\
&\leq\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}\left\|S_L^{-1}(p,T)\right\|\,d|p|\,\left|(s-p)^{-m}(\overline{s}-p)^{-n}\right|\\
&\leq\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}\left\|S_L^{-1}(p,T)\right\|\,d|p|\,\frac{1}{\varepsilon^{m+n}}.
\end{aligned}$$

Hence if we set

$$K_T:=\sup_{i\in\mathbb{S}}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_i)}\left\|S_L^{-1}(p,T)\right\|\,d|p|,$$

which depends neither on the point $s\in C$ nor on the numbers $n,m\geq 0$, then

$$\left\|S_L^{-m}(s,T)*_L S_L^{-n}(\overline{s},T)\right\|\leq\frac{K_T}{\varepsilon^{m+n}}.\qquad\square$$

Theorem 4.4.12. *Let $T \in \mathcal{B}(X)$ and let $N \in \mathcal{B}(X)$ be such that T and N commute and such that $\sigma_S(N)$ is contained in the open ball $B_\varepsilon(0)$. If $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$, then $s \in \rho_S(T+N)$ and*

$$\mathcal{Q}_s(T)^{-1} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n,$$

where the series converges in the operator norm.

Proof. We first show the convergence of the series

$$\Sigma(s,T,N) := \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n.$$

Since $\sigma_S(N)$ is compact, there exists $\theta \in (0,1)$ such that $\sigma_S(N) \subset B_{\theta\varepsilon}(0) \subset B_\varepsilon(0)$. Applying the S-functional calculus, we obtain

$$\begin{aligned} \|N^m\| &= \left\| \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} S_L^{-1}(s,N)\, ds_j\, s^m \right\| \\ &\leq \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} \left\| S_L^{-1}(s,N) \right\| \, d|s|\, |s|^m \\ &= \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} \left\| S_L^{-1}(s,N) \right\| \, d|s|\, (\theta\varepsilon)^m \end{aligned}$$

for every $m \geq 0$. Hence

$$\|N^m\| \leq K_N (\theta\varepsilon)^m \tag{4.17}$$

with

$$K_N := \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} \left\| S_L^{-1}(s,N) \right\| \, d|s|.$$

From Proposition 4.4.11, we deduce

$$\begin{aligned} &\sum_{n=0}^{+\infty} \left\| \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n \right\| \\ &\leq \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left\| S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right\| \|N^n\| \\ &\leq \sum_{n=0}^{+\infty} (n+1) \frac{K_T}{\varepsilon^{n+2}} K_N (\theta\varepsilon)^n \leq \frac{K_T K_N}{\varepsilon^2} \sum_{n=0}^{+\infty} (n+1)\theta^n. \end{aligned}$$

By the root test, this last series converges because $0 < \theta < 1$. The comparison test yields the convergence of the original series $\Sigma(s,T,N)$ in the operator norm.

From Definition 4.4.6 and the fact that T and N commute, we deduce

$$\begin{aligned}\mathcal{Q}_s(T+N) &= T^2 + 2TN + N^2 - 2s_0T - 2s_0N + |s|^2\mathcal{I}\\ &= \mathcal{Q}_s(T) + (2T - 2s_0)N + |s|^2\mathcal{I}.\end{aligned}$$

If we define

$$\Lambda_T(k,n,s) := (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)}$$

for neatness, we therefore have

$$\begin{aligned}&\Sigma(s,T,N)\mathcal{Q}_s(T+N)\\ &= \left(\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T)\right)N^n\right)\mathcal{Q}_s(T+N)\\ &= \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^n\mathcal{Q}_s(T)\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^{n+1}(2T - 2s_0\mathcal{I})\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^{n+2}.\end{aligned}$$

Applying Corollary 4.4.2 and the S-functional calculus, we see that each of the coefficients $\sum_{k=0}^{n}\Lambda_T(k,n,s) = \sum_{k=0}^{n}(\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)}$ is a polynomial in T with real coefficients and hence commutes with the operator $\mathcal{Q}_s(T)$. Remark 4.4.4 implies

$$\begin{aligned}&\Sigma(s,T,N)\mathcal{Q}_s(T+N)\\ &= \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+1)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^n\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s) *_L (2T - 2s_0\mathcal{I})\right)N^{n+1}\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^{n+2}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) \right) N^n \\
&\quad + \sum_{n=1}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n-1} \Lambda_T(k,n-1,s) *_L (2T - 2s_0\mathcal{I}) \right) N^n \\
&\quad + \sum_{n=2}^{+\infty} \mathcal{Q}_s(T)^{-n} \sum_{k=0}^{n-2} \Lambda_T(k,n-2,s) N^n .
\end{aligned}
$$

The identity

$$
\begin{aligned}
&\mathcal{Q}_s(T)^{-n} \left(\sum_{k=0}^{n-2} \Lambda_T(k,n-2,s) \right) \\
&= \mathcal{Q}_s(T)^{-n} \left(\sum_{k=0}^{n-2} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k-1)} \right) \\
&= \mathcal{Q}_s(T)^{-n} \left(\sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L k} *_L (s\mathcal{I} - T)^{*_L(n-k)} \right) \\
&= \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} \right) \\
&= \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=1}^{n-1} \Lambda_T(k,n,s) \right),
\end{aligned}
$$

finally yields

$$
\begin{aligned}
&\Sigma(s,T,N)\mathcal{Q}_s(T+N) \\
&= \mathcal{Q}_s(T)^{-1} \Lambda_T(0,0,s) N^0 \\
&\quad + \mathcal{Q}_s(T)^{-2} \left(\sum_{k=0}^{1} \Lambda_T(k,1,s) + \Lambda(0,0,s) *_L (2T - 2s_0\mathcal{I}) \right) N \\
&\quad + \sum_{n=2}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) \right. \\
&\quad \left. + \sum_{k=0}^{n-1} \Lambda_T(k,n-1,s) *_L (2T - 2s_0\mathcal{I}) + \sum_{k=1}^{n-1} \Lambda_T(k,n,s) \right) N^n .
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
\mathcal{Q}_s(T)^{-1} \Lambda_T(0,0,s) N^0 &= \mathcal{Q}_s(T)^{-1} \left((\overline{s}\mathcal{I} - T) *_L (s\mathcal{I} - T) \right) \\
&= \mathcal{Q}_s(T)^{-1} \mathcal{Q}_s(T) = \mathcal{I}.
\end{aligned}
$$

Because of $2T - 2s_0\mathcal{I} = -(s\mathcal{I} - T) - (\overline{s}\mathcal{I} - T)$, we have

$$\begin{aligned}
&\sum_{k=0}^{1} \Lambda_T(k,1,s) + \Lambda(0,0,s) *_L (2T - 2s_0\mathcal{I}) \\
&= (\overline{s}\mathcal{I} - T) *_L (s\mathcal{I} - T)^{*_L 2} + (\overline{s}\mathcal{I} - T)^{*_L 2} *_L (s\mathcal{I} - T) \\
&\quad - (\overline{s}\mathcal{I} - T)^{*_L 2} *_L (s\mathcal{I} - T) - (\overline{s}\mathcal{I} - T) *_L (s\mathcal{I} - T)^{*_L 2} = 0.
\end{aligned}$$

Finally, we also find again because of $2T - 2s_0\mathcal{I} = -(s\mathcal{I} - T) - (\overline{s}\mathcal{I} - T)$ that

$$\begin{aligned}
\sum_{k=0}^{n} \Lambda_T(k,n,s) + \sum_{k=0}^{n-1} \Lambda_T(k,n-1,s) *_L (2T - 2s_0\mathcal{I}) + \sum_{k=1}^{n-1} \Lambda_T(k,n,s) \\
= \sum_{k=0}^{n} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} \\
- \sum_{k=0}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+2)} *_L (s\mathcal{I} - T)^{*_L(n-k)} \\
- \sum_{k=0}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} \\
+ \sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} = 0,
\end{aligned}$$

where the last identity follows after an index shift k to $k+1$ in the second sum. Altogether, we obtain

$$\Sigma(s,T,N)\mathcal{Q}_s(T+N) = \mathcal{I}.$$

From Corollary 4.4.2 and the S-functional calculus, we already concluded that each of the coefficients $\sum_{k=0}^{n} \Lambda_T(k,n,s)$ in $\Sigma(s,T,N)$ is a polynomial in T with real coefficients and thus commutes with both T and N. Hence it also commutes with $\mathcal{Q}_s(T+N)$, and so also

$$\mathcal{Q}_s(T+N)\Sigma(s,T,N) = \Sigma(s,T,N)\mathcal{Q}_s(T+N) = \mathcal{I}.$$

Hence $\mathcal{Q}_s(T+N)$ is invertible, which implies $s \in \rho_S(T+N)$. □

Theorem 4.4.13. *Let $T, N \in \mathcal{B}(X)$ be such that $\sigma_S(N) \subset B_\varepsilon(0)$ and such that T and N commute. For every $s \in \rho_S(T)$ with* $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$, *the identities*

$$S_L^{-1}(s, T+N) = \sum_{n=0}^{+\infty} N^n \, S_L^{-(n+1)}(s,T)$$

and

$$S_R^{-1}(s, T+N) = \sum_{n=0}^{+\infty} S_R^{-(n+1)}(s,T) \, N^n$$

hold, where the series converge uniformly on every set C with $\operatorname{dist}(C, \sigma_S(T)) > \varepsilon$.

Proof. In (4.17), we showed the existence of two constants $K_N \geq 0$ and $\theta \in (0,1)$ such that $\|N\|^m \leq K_N(\theta\varepsilon)^m$ for every $m \in \mathbb{N}_0$. Moreover, for every $C \subset \mathbb{H}$ with $\operatorname{dist}(C, \sigma_S(T)) > \varepsilon$, Proposition 4.4.11 implies the existence of a constant K_T such that $\|S_L^{-m}(s,T)\| \leq K_T/\varepsilon^m$ for every $s \in C$ and $m \in \mathbb{N}_0$. Therefore, the estimate

$$\sum_{n=n_0}^{\infty} \left\| N^n \, S_L^{-(n+1)}(s,T) \right\| \leq \sum_{n=n_0}^{+\infty} \|N^n\| \left\| S_L^{-(n+1)}(s,T) \right\|$$

$$\leq \sum_{n=n_0}^{+\infty} K_N(\theta\varepsilon)^n \frac{K_T}{\varepsilon^{n+1}} = \frac{K_T K_N}{\varepsilon} \sum_{n=n_0}^{+\infty} \theta^n \overset{n_0 \to \infty}{\longrightarrow} 0$$

holds for every $s \in C$ and implies the uniform convergence of the series on C.

Let $s \in \rho_S(T)$ with $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$. We have

$$\begin{aligned}
&\left((T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}\right) \sum_{n=0}^{+\infty} N^n \, S_L^{-(n+1)}(s,T) \\
&= \left(T^2 - 2s_0T + |s|^2\mathcal{I}\right) \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + (2T - 2s_0\mathcal{I})\, N \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + N^2 \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&= \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-n} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + \sum_{n=0}^{+\infty} N^{n+1}\, (2T - 2s_0\mathcal{I})\, (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + \sum_{n=0}^{+\infty} N^{n+2} (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)}.
\end{aligned}$$

Shifting the indices yields

$$
\begin{aligned}
&\big((T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}\big)\sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T)\\
&= \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-n}(\overline{s}\mathcal{I} - T)^{*_L(n+1)}\\
&\quad + \sum_{n=1}^{+\infty} N^n\,(2T - 2s_0\mathcal{I})\,(T^2 - 2s_0T + |s|^2\mathcal{I})^{-n}(\overline{s}\mathcal{I} - T)^{*_L n}\\
&\quad + \sum_{n=2}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n-1)}(\overline{s}\mathcal{I} - T)^{*_L(n-1)}\\
&= \overline{s}\mathcal{I} - T + N(T^2 - s_0T + |s|^2\mathcal{I})^{-1}(\overline{s}\mathcal{I} - T)^{*_L 2}\\
&\quad + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(2T - 2s_0\mathcal{I})(\overline{s}\mathcal{I} - T) +\\
&\quad + \sum_{n=2}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-n}\Big[(\overline{s}\mathcal{I} - T)^{*_L(n+1)}\\
&\qquad\quad + (2T - 2s_0\mathcal{I})\,(\overline{s}\mathcal{I} - T)^{*_L n}\\
&\qquad\quad + (T^2 - 2s_0T + |s|^2\mathcal{I})(\overline{s}\mathcal{I} - T)^{*_L(n-1)}\Big].
\end{aligned}
$$

The last series equals 0 because Remark 4.4.4 and the identity

$$
(T^2 - 2s_0T + |s|^2\mathcal{I}) = (s\mathcal{I} - T) *_L (\overline{s}\mathcal{I} - T)
$$

imply

$$
\begin{aligned}
&(\overline{s}\mathcal{I} - T)^{*_L(n+1)} + (2T - 2s_0\mathcal{I})\,(\overline{s}\mathcal{I} - T)^{*_L n}\\
&\quad + (T^2 - 2s_0T + |s|^2\mathcal{I})(\overline{s}\mathcal{I} - T)^{*_L(n-1)}\\
&= (\overline{s}\mathcal{I} - T)^{*_L(n+1)} + (2T - 2s_0\mathcal{I}) *_L (\overline{s}\mathcal{I} - T)^{*_L n} + (s\mathcal{I} - T) *_L (\overline{s}\mathcal{I} - T)^{*_L n}\\
&= (\overline{s}\mathcal{I} - T + 2T - 2s_0\mathcal{I} + s\mathcal{I} - T) *_L (\overline{s}\mathcal{I} - T)^{*_L(n-1)} = 0.
\end{aligned}
$$

Hence, we finally obtain

$$
\begin{aligned}
&\big((T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}\big)\sum_{n=0}^{+\infty} N^n S_L^{-(n+1)}(s,T)\\
&= \overline{s}\mathcal{I} - T + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(\overline{s}^2\mathcal{I} - 2T\overline{s} + T^2)\\
&\quad + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(2T\overline{s} - 2s_0\overline{s}\mathcal{I} - 2T^2 + 2s_0T)\\
&= \overline{s}\mathcal{I} - T + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(-T^2 + 2s_0T - |s|^2\mathcal{I}) = \overline{s}\mathcal{I} - T - N.
\end{aligned}
$$

Since $\mathcal{Q}_s(T+N) = (T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}$ is invertible by Theorem 4.4.12, this is equivalent to

$$\sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T) = \mathcal{Q}_s(T+N)^{-1}(\overline{s}\mathcal{I} - T - N) = S_L^{-1}(s, T+N).$$

The identity for the right S-resolvent can be shown with analogous computations. □

Theorem 4.4.14 (The Taylor formulas)**.** *Let $T, N \in \mathcal{B}(X)$ with $\sigma_S(N) \subset B_\varepsilon(0)$ such that T and N commute and set*

$$C_\varepsilon(\sigma_S(T)) := \{s \in \mathbb{H} :\ \ \mathrm{dist}(s, \sigma_S(T)) \le \varepsilon\}.$$

If $f \in \mathcal{SH}_L(C_\varepsilon(\sigma_S(T)))$, then $f \in \mathcal{SH}_L(\sigma_S(T+N))$ and

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!}\left(\partial_S^n f\right)(T).$$

Similarly, if $f \in \mathcal{SH}_R(C_\varepsilon(\sigma_S(T)))$, then $f \in \mathcal{SH}_R(\sigma_S(T+N))$ and

$$f(T+N) = \sum_{n=0}^{+\infty} \frac{1}{n!}\left(\partial_S^n f\right)(T) N^n.$$

Proof. We prove just the first Taylor formula; the second one is obtained with similar computations. By Theorem 4.4.12, we have $\sigma_S(T+N) \subset C_\varepsilon(\sigma_S(T))$, and so the function f belongs to $\mathcal{SH}_L(\sigma_S(T+N))$. If U is a bounded slice Cauchy domain with $C_\varepsilon(\sigma_S(T)) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$, then we find due to Theorem 4.4.13 that

$$\begin{aligned} f(T+N) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s, T+N)\, ds_j\, f(s) \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} \sum_{n=0}^{+\infty} N^n S_L^{-(n+1)}(s,T)\, ds_j\, f(s) \\ &= \sum_{n=0}^{+\infty} N^n \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-(n+1)}(s,T)\, ds_j\, f(s). \end{aligned}$$

By Proposition 4.4.8, we have

$$S_L^{-(n+1)}(s,T) = (-1)^n \frac{1}{n!}\partial_S^n S_L^{-1}(s,T),$$

and so

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{(-1)^n}{n!}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} \partial_S^n S_L^{-1}(s,T)\, ds_j\, f(s).$$

After integrating the nth term in the sum n times by parts, we finally obtain

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, (\partial_S^n f)(s)$$
$$= \sum_{n=0}^{+\infty} N^n \frac{1}{n!} (\partial_S^n f)(T). \qquad \square$$

4.5 Bounded Operators with Commuting Components

If the components of T commute, then the S-spectrum can be characterized by a different operator, which is often easier to handle in the applications. The S-resolvent operators can in this case be expressed in a form that corresponds to replacing the scalar variable q in the slice hyperholomorphic Cauchy kernels by the operator T when they are written in form II; see Chapter 2.

We saw in Remark 2.3.2 that every two-sided quaternionic vector space X is essentially of the form $X = X_{\mathbb{R}} \otimes \mathbb{H}$, where $X_{\mathbb{R}}$ is the real vector space consisting of the vectors that commute with all quaternions. If $x = \sum_{\ell=0}^{3} x_\ell e_\ell$ with $x_\ell \in X_{\mathbb{R}}$, where we set $e_0 = 1$ for neatness, then we can write any operator $T \in \mathcal{B}(X)$ as $T = \sum_{\ell=0}^{3} T_\ell e_\ell$ with components $T_\ell \in \mathcal{B}(X_{\mathbb{R}})$, where this operator acts as

$$Tx = \left(\sum_{\ell=0}^{3} T_\ell e_\ell\right)\left(\sum_{\kappa=0}^{3} x_\kappa e_\kappa\right) = \sum_{\ell,\kappa=0}^{3} T_\ell(x_\kappa) e_\ell e_\kappa.$$

We obtain $\mathcal{B}(X) = \mathcal{B}(X_{\mathbb{R}}) \otimes \mathbb{H}$, and hence we call any operator in $\mathcal{B}(X_{\mathbb{R}})$ a scalar operator on X.

Definition 4.5.1. We define $\mathcal{BC}(X)$ to be the space of all operators $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{B}(X)$ with components $T_\ell \in \mathcal{B}(X_{\mathbb{R}}), \ell = 0, \dots, 3,$ that mutually commute.

Definition 4.5.2. For $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$, we set

$$\overline{T} := T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell.$$

The following statement shows that for an operator $T \in \mathcal{BC}(X)$ the analogues of the scalar identities $s + \overline{s} = 2\mathrm{Re}(s)$ and $s\overline{s} = \overline{s}s = |s|^2$ hold. This motivates the idea that we can write the S-resolvent for such operators also by formally replacing q by T in the slice hyperholomorphic Cauchy kernels when they are written in form II.

Lemma 4.5.3. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. Then $2T_0 = T + \overline{T}$ and $T\overline{T} = \overline{T}T = \sum_{\ell=0}^{3} T_\ell^2$.*

Proof. We obviously have

$$T + \overline{T} = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell + T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell = 2T_0.$$

Since the components T_ℓ mutually commute and $e_\ell e_\kappa = -e_\kappa e_\ell$ for $1 \leq \ell, \kappa \leq 3$ with $\ell \neq \kappa$, we also have

$$\begin{aligned} T\overline{T} &= \left(T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell\right)\left(T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\ &= T_0^2 - \sum_{\ell=1}^{3} T_0 T_\ell e_\ell + \sum_{\ell=1}^{3} T_\ell T_0 e_\ell - \sum_{\ell,\kappa=1}^{3} T_\ell T e_\ell e_\kappa \\ &= T_0^2 - \sum_{\ell=1}^{3} T_\ell^2 e_\ell^2 + \sum_{\substack{\ell=1,2,3 \\ \ell<\kappa}} (T_\ell T_\kappa - T_\kappa T_\ell) e_\ell e_\kappa = \sum_{\ell=0}^{3} T_\ell^2. \end{aligned}$$

□

Lemma 4.5.4. *If $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$, then the following statements are equivalent:*

(i) *The operator T is invertible.*

(ii) *The operator $\overline{T}$ is invertible.*

(iii) *The operator $T\overline{T}$ is invertible.*

In this case we have

$$\overline{T}^{-1} = \overline{T^{-1}} \quad \textit{and} \quad T^{-1} = (T\overline{T})^{-1}\overline{T}. \tag{4.18}$$

Proof. If $T\overline{T}$ is invertible, then $(T\overline{T})^{-1} = \left(\sum_{\ell=0}^{3} T_\ell^2\right)^{-1}$ commutes with T and $\overline{T}$, and hence

$$\left(T\overline{T}\right)^{-1} \overline{T}T = \left(\overline{T}T\right)^{-1} \overline{T}T = \mathcal{I}$$

and

$$T\left(T\overline{T}\right)^{-1} \overline{T} = \left(T\overline{T}\right)^{-1} T\overline{T} = \mathcal{I}.$$

Thus (iii) implies (i), and the second identity in (4.18) holds.

If, on the other hand, T is invertible and $T^{-1} = B_0 + \sum_{\kappa=1}^{3} B_\kappa e_\kappa \in \mathcal{B}(X)$, then

$$\begin{aligned} \mathcal{I} = T^{-1}T &= \left(B_0 + \sum_{\kappa=1}^{3} B_\kappa e_\kappa\right)\left(T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\ &= B_0 T_0 - \sum_{\ell=1}^{3} B_\ell T_\ell + (B_2 T_3 - B_3 T_2) e_1 \\ &\quad + (B_3 T_1 - B_1 T_3) e_2 + (B_1 T_2 - B_2 T_1) e_3. \end{aligned}$$

We conclude that

$$\mathcal{I} = B_0T_0 - \sum_{\ell=1}^{3} B_\ell T_\ell$$

and

$$B_\ell T_\kappa - B_\kappa T_\ell = 0$$

for $1 \leq \ell < \kappa \leq 3$. Therefore,

$$\begin{aligned}\overline{B}\,\overline{T} &= \left(B_0 - \sum_{\ell=1}^{3} B_\ell e_\ell\right)\left(T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell\right)\\ &= B_0T_0 - \sum_{\ell=1}^{3} B_\ell T_\ell + (B_2T_3 - B_3T_2)e_1\\ &\quad + (B_3T_1 - B_1T_3)e_2 + (B_1T_2 - B_2T_1)e_3 = \mathcal{I},\end{aligned}$$

and similarly we see that also $\overline{T}\,\overline{B} = \mathcal{I}$. Hence (i) implies (ii) and $\overline{T}^{-1} = \overline{T^{-1}}$. Since $\overline{\overline{T}} = T$, we can exchange the roles of T and $\overline{T}$ and find that (ii) implies (i). Finally, we see that in this case, $(T\overline{T})^{-1} = \overline{T}^{-1}T^{-1} \in \mathcal{B}(X)$, and we find that (i) and (ii) also imply (iii). □

Definition 4.5.5. Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$, we define the operator

$$\mathcal{Q}_{c,s}(T) := s^2\mathcal{I} - 2sT_0 + T\overline{T}.$$

Theorem 4.5.6. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. Then $\mathcal{Q}_{c,s}(T)$ is invertible if and only if $\mathcal{Q}_s(T)^{-1}$ is invertible, and so*

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \ \ \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) \right\}. \tag{4.19}$$

Moreover, for $s \in \rho_S(T)$, we have

$$S_L^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1} \tag{4.20}$$

and

$$S_R^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}). \tag{4.21}$$

Proof. We observe that for $s \in \mathbb{H}$, we have $\overline{\mathcal{Q}_s(T)} = \mathcal{Q}_s(\overline{T})$ and $\overline{\mathcal{Q}_{c,s}(T)} = \mathcal{Q}_{c,\overline{s}}(T)$, and so

$$\begin{aligned}\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)} &= (s^2\mathcal{I} - 2sT_0 + T\overline{T})(\overline{s}^2\mathcal{I} - 2\overline{s}T_0 + T\overline{T})\\ &= |s|^4\mathcal{I} - 2s|s|^2T_0 + s^2T\overline{T} - 2|s|^2T_0\overline{s} + 4|s|^2T_0^2 - 2sT_0T\overline{T}\\ &\quad + \overline{s}^2T\overline{T} - 2\overline{s}T_0T\overline{T} + (T\overline{T})^2\\ &= |s|^4\mathcal{I} - 2s_0|s|^2T - 2s_0|s|^2\overline{T} + 2\mathrm{Re}(s^2)T\overline{T}\\ &\quad + 4|s|^2T_0^2 - 2s_0T^2\overline{T} - 2s_0T\overline{T}^2 + T^2\overline{T}^2,\end{aligned}$$

where we used in the last identity that $2s_0 = s + \overline{s}$, that $|s|^2 = s\overline{s}$, and that $2T_0 = T + \overline{T}$. Since, for $s = s_0 + j_s s_1$, we have

$$2\mathrm{Re}(s^2)T\overline{T} = 2s_0^2 T\overline{T} - 2s_1^2 T\overline{T}$$

and

$$4|s|^2 T_0^2 = |s|^2(T+\overline{T})^2 = |s|^2 T^2 + 2s_0^2 T\overline{T} + s_1^2 T\overline{T} + |s|^2\overline{T}^2,$$

we further find that

$$\begin{aligned}\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)} &= |s|^2(|s|^2\mathcal{I} - 2s_0T + T^2)\\ &\quad - 2s_0\overline{T}(|s|^2\mathcal{I} - 2s_0T + T^2)\\ &\quad + \overline{T}^2(|s|^2\mathcal{I} - 2s_0T + T^2) = \mathcal{Q}_s(T)\overline{\mathcal{Q}_s(T)}.\end{aligned}$$

From Lemma 4.5.4, we conclude that the invertibility of $\mathcal{Q}_{c,s}(T)$ is equivalent to the invertibility of $\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)} = \mathcal{Q}_s(T)\overline{\mathcal{Q}_s(T)}$, which is in turn equivalent to the invertibility of $\mathcal{Q}_s(T)$, and hence (4.19) holds.

Because of Lemma 4.5.3, we furthermore have

$$\begin{aligned}(\overline{s}\mathcal{I} - T)\mathcal{Q}_{c,s}(T) &= (\overline{s}\mathcal{I} - T)\left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)\\ &= |s|^2 s\mathcal{I} - Ts^2 - 2|s|^2T_0 + 2TT_0s + \overline{s}T\overline{T} - T^2\overline{T}\\ &= |s|^2 s\mathcal{I} - Ts^2 - |s|^2T - |s|^2\overline{T} + T^2s + T\overline{T}s + \overline{s}T\overline{T} - T^2\overline{T}\\ &= |s|^2\left(s\mathcal{I} - \overline{T}\right) - 2s_0T\left(s\mathcal{I} - \overline{T}\right) + T^2\left(s\mathcal{I} - \overline{T}\right)\\ &= \left(T^2 - 2s_0T + |s|^2\mathcal{I}\right)\left(s\mathcal{I} - \overline{T}\right) = \mathcal{Q}_s(T)(s\mathcal{I} - \overline{T}),\end{aligned} \tag{4.22}$$

and so

$$S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}(\overline{s}\mathcal{I} - T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1}.$$

Similar computations show that also the identity (4.21) holds. □

Definition 4.5.7 (SC-resolvent operators)**.** Let $T \in \mathcal{BC}(X)$. For $s \in \rho_S(T)$, we define the *left* and *right SC-resolvent operator* of T as

$$S_{c,L}^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1}$$

and

$$S_{c,R}^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}).$$

Corollary 4.5.8. *Let $T \in \mathcal{BC}(X)$. For $f \in \mathcal{SH}_L(\sigma_S(T))$, we have*

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_{c,L}^{-1}(s,T)\,ds_j\,f(s),$$

and for $f \in \mathcal{SH}_R(\sigma_S(T))$ we have

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_{c,R}^{-1}(s,T)$$

for any imaginary unit $j \in \mathbb{S}$ and any bounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$.

Remark 4.5.9. The S-functional calculus for operators with commuting components defined by the above integrals that involve the SC-resolvents is often also referred to as the SC-functional calculus. Similarly, the S-spectrum is sometimes called the F-spectrum when it is characterized by the operator $\mathcal{Q}_{c,s}(T)^{-1}$, in order to stress that one is using the simpler characterization that holds only for operators with commuting components.

4.6 Perturbations of the SC-Resolvent Operators

In order to study bounded perturbations of the F-resolvent operators (see Chapter 7), we study in this section a preliminary result about the perturbations of the S-resolvent operators $S_{c,L}^{-1}(s,T)$ and $S_{c,R}^{-1}(s,T)$. This will be used in the sequel. We recall that the left spectrum $\sigma_L(T)$ and the left resolvent sets $\rho_L(T)$ were defined in Definition 3.3.1. The following corollary of Lemma 3.1.12 will be used in the sequel.

Corollary 4.6.1. *Let $T \in \mathcal{BC}(X)$. If $s \in \rho_S(T) \cap \rho_L(\overline{T})$, then*

$$\left(S_{c,L}^{-1}(s,T)\right)^{-1} = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right) T \left(s\mathcal{I} - \overline{T}\right)^{-1},$$

$$\left(S_{c,R}^{-1}(s,T)\right)^{-1} = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right)^{-1} T \left(s\mathcal{I} - \overline{T}\right).$$

Proof. By Theorem 4.5.6, we have

$$S_{c,L}^{-1}(s,T) = \left(s\mathcal{I} - \overline{T}\right) \mathcal{Q}_{c,s}(T)^{-1} = \left(s\mathcal{I} - \overline{T}\right) \left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)^{-1}.$$

Since $\mathcal{Q}_{c,s}(T) = s\left(s\mathcal{I} - \overline{T}\right) - (s\mathcal{I} - T)T$, we thus obtain

$$\begin{aligned}\left(S_{c,L}^{-1}(s,T)\right)^{-1} &= \left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)\left(s\mathcal{I} - \overline{T}\right)^{-1} \\ &= s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right) T \left(s\mathcal{I} - \overline{T}\right)^{-1}.\end{aligned}$$

Similar computations show the identity for the right S-resolvent. □

Definition 4.6.2. Let $T \in \mathcal{BC}(X)$. For $s \in \rho_L(\overline{T})$, we define

$$S_{c,L}(s,T) = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right) T \left(s\mathcal{I} - \overline{T}\right)^{-1},$$

$$S_{c,R}(s,T) = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right)^{-1} T \left(s\mathcal{I} - \overline{T}\right)^{-1}.$$

Lemma 4.6.3. *Let T, $Z \in \mathcal{BC}(X)$. If $s \notin \sigma_L\left(\overline{T}\right) \cup \sigma_L\left(\overline{Z}\right)$, then*

$$\|S_{c,L}(s,T) - S_{c,L}(s,Z)\| \leq K_{T,Z}(s)\|T - Z\|, \tag{4.23}$$

$$\|S_{c,R}(s,T) - S_{c,R}(s,Z)\| \leq K_{T,Z}(s)\|T - Z\|, \tag{4.24}$$

with

$$K_{T,Z}(s) := \left\|(s\mathcal{I} - \overline{Z})^{-1}\right\| \left(\|Z\| + \left\|s\mathcal{I} - \overline{T}\right\| \left[1 + \|T\| \left\|(s\mathcal{I} - \overline{T})^{-1}\right\|\right]\right). \tag{4.25}$$

Proof. We consider the chain of equalities

$$\begin{aligned}
&S_{c,L}(s,T) - S_{c,L}(s,Z)\\
&= (s\mathcal{I} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})T(s\mathcal{I} - \overline{T})^{-1}\\
&= (s\mathcal{I} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})Z(s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + (s\mathcal{I} - \overline{T})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})T(s\mathcal{I} - \overline{T})^{-1}\\
&= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})\left[Z(s\mathcal{I} - \overline{Z})^{-1} - T(s\mathcal{I} - \overline{T})^{-1}\right]\\
&= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})\Big[(Z - T)(s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + T\left((s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})^{-1}\right)\Big]\\
&= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})\Big[(Z - T)(s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + T\left(s\mathcal{I} - \overline{Z}\right)^{-1}\left(\overline{Z} - \overline{T}\right)\left(s\mathcal{I} - \overline{T}\right)^{-1}\Big].
\end{aligned}$$

Taking the norm and observing that $\|T - Z\| = \left\|\overline{T} - \overline{Z}\right\|$, we have

$$\begin{aligned}
\|S_{c,L}(s,T) - S_{c,L}(s,Z)\| &\leq \|T - Z\| \Big(\|Z\| \, \|(s\mathcal{I} - \overline{Z})^{-1}\|\\
&\quad + \|s\mathcal{I} - \overline{T}\|\Big[\|(s\mathcal{I} - \overline{Z})^{-1}\| + \|T\| \, \|(s\mathcal{I} - \overline{Z})^{-1}\|\|(s\mathcal{I} - \overline{T})^{-1}\|\Big]\Big),
\end{aligned}$$

and so (4.23) holds. The second estimate is shown with similar arguments. □

Lemma 4.6.4. *Let T, $Z \in \mathcal{BC}(X)$, let $s \in \rho_S(T)$ with $s \notin \sigma_L\left(\overline{T}\right) \cup \sigma_L\left(\overline{Z}\right)$, and suppose that*

$$\|T - Z\| < \frac{1}{K_{Z,T}(s)} \|S_{c,L}^{-1}(s,T)\|^{-1},$$

with $K_{Z,T}(s)$ as in Lemma 4.6.3. Then $s \in \rho_S(Z)$ and

$$\begin{aligned}
&S_{c,L}^{-1}(s,Z) - S_{c,L}^{-1}(s,T)\\
&= S_{c,L}^{-1}(s,T) \sum_{m=1}^{+\infty} \left[(S_{c,L}(s,T) - S_L(s,Z))S_{c,L}^{-1}(s,T)\right]^m.
\end{aligned} \tag{4.26}$$

Similarly, if

$$\|T - Z\| < \frac{1}{K_{Z,T}(s)} \left\|S_R^{-1}(s,T)\right\|^{-1},$$

then $s \in \rho_S(Z)$ *and*

$$\begin{aligned} &S_{c,R}^{-1}(s,Z) - S_{c,R}^{-1}(s,T) \\ &= S_{c,R}^{-1}(s,T) \sum_{m=1}^{+\infty} \left[(S_{c,R}(s,T) - S_{c,R}(s,Z))S_{c,R}^{-1}(s,T)\right]^m. \end{aligned} \tag{4.27}$$

Proof. If we apply Lemma 3.1.12 with $A = S_{c,L}(s,T)$ and $B = S_{c,L}(s,Z)$, then we obtain

$$S_{c,L}^{-1}(s,Z) = S_{c,L}^{-1}(s,T) \sum_{m=0}^{+\infty} \left[(S_{c,L}(s,T) - S_{c,L}(s,Z))S_{c,L}^{-1}(s,T)\right]^m. \tag{4.28}$$

This series converges, since

$$\left\|(S_{c,L}(s,T) - S_{c,L}(s,Z))S_{c,L}^{-1}(s,T)\right\| \leq K_{Z,T}(s) \left\|T - Z\|\|S_{c,L}^{-1}(s,T)\right\| < 1,$$

and we obtain $s \in \rho_S(Z)$ as

$$\mathcal{Q}_{c,s}(T)^{-1} = (s\mathcal{I} - \overline{T})^{-1} S_L^1(s,T).$$

We can show the statement for the right S-resolvent with similar arguments. □

Definition 4.6.5. Let $O \subset \mathbb{H}$. We denote by $B_\varepsilon(O)$ for $\varepsilon > 0$ the ε-neighborhood of O defined as

$$B_\varepsilon(O) := \{q \in \mathbb{H} : \inf_{s \in O} |s - q| < \varepsilon\}.$$

Theorem 4.6.6. *Let* $T, Z \in \mathcal{BC}(X)$, *let* $s \in \rho_S(T)$, *and assume also that* $s \notin \sigma_L\left(\overline{T}\right) \cup \sigma_L\left(\overline{Z}\right)$. *For every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $\|T - Z\| < \delta$, *we have*

$$\sigma_S(Z) \subseteq B_\varepsilon\left(\sigma_S(T)\right),$$

and for $s \notin B_\varepsilon\left(\sigma_S(T) \cup \sigma_L\left(\overline{T}\right)\right)$,

$$\begin{aligned} \|S_{c,L}^{-1}(s,Z) - S_{c,L}^{-1}(s,T)\| &< \varepsilon, \\ \|S_{c,R}^{-1}(s,Z) - S_{c,R}^{-1}(s,T)\| &< \varepsilon. \end{aligned}$$

Proof. Let $T, Z \in \mathcal{BC}(X)$ and let $\varepsilon > 0$. Thanks to Lemma 3.1.12 there exists $\eta > 0$ such that if

$$\|T - Z\| < \eta,$$

then $\sigma_L(\overline{Z}) \subset B_\varepsilon\left(\sigma_L\left(\overline{T}\right)\right)$, where $B_\varepsilon\left(\sigma_L\left(\overline{T}\right)\right)$ is the ε-neighborhood of $\sigma_L(\overline{T})$. We can hence always choose η such that

$$\sigma_L(\overline{Z}) \subset B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T})).$$

Consider the function $K_{T,Z}(s)$ defined in Lemma 4.6.3 and observe that the constant K_ε defined by

$$K_\varepsilon = \sup_{s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)} K_{T,Z}(s)$$

is finite, since $s \notin B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T}))$, since due to the above observation $\sigma_L(\overline{Z}) \subset B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T}))$ and since

$$\lim_{s\to\infty} \|(s\mathcal{I} - \overline{Z})^{-1}\| = \lim_{s\to\infty} \|(s\mathcal{I} - \overline{T})^{-1}\| = 0.$$

Observe that since $s \in \rho_{\mathcal{S}}(T)$, the map $s \mapsto \|S_{c,L}^{-1}(s,T)\|$ is continuous and that

$$\lim_{s\to\infty} \|S_{c,L}^{-1}(s,T)\| = 0.$$

For s in the complement of $B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T}))$ we have thus that there exists a positive constant N_ε such that

$$\|S_{c,L}^{-1}(s,T)\| \leq N_\varepsilon.$$

If $\delta_1 > 0$ is such that $\|Z - T\| < \frac{1}{K_\varepsilon N_\varepsilon} := \delta_1$, then we can conclude from Lemma 4.6.4 that $s \in \rho_S(Z)$ and that

$$\begin{aligned}
&\|S_{c,L}^{-1}(s,Z) - S_{c,L}^{-1}(s,T)\| \\
&\leq \frac{\|S_{c,L}^{-1}(s,T)\|^2 \; \|S_{c,L}(s,T) - S_{c,L}(s,Z)\|}{1 - \|S_{c,L}^{-1}(s,T)\| \; \|S_{c,L}(s,T) - S_{c,L}(s,Z)\|} \\
&\leq \frac{N_\varepsilon^2 K_\varepsilon \|Z - T\|}{1 - N_\varepsilon K_\varepsilon \|Z - T\|} < \varepsilon
\end{aligned}$$

if

$$\|Z - T\| < \delta_2 := \frac{\varepsilon}{K_\varepsilon(N_\varepsilon^2 + \varepsilon N_\varepsilon)}.$$

To get the statement, it suffices to set $\delta = \min\{\eta, \delta_1, \delta_2\}$.

For the right S-resolvent, we can argue similarly. □

Theorem 4.6.7. *Let $T, Z \in \mathcal{BC}(X)$, let $f \in \mathcal{SH}_L(\sigma_S(T))$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for $\|Z - T\| < \delta$, we have $f \in \mathcal{SH}_L(\sigma_S(Z))$ and*

$$\|f(Z) - f(T)\| < \varepsilon.$$

Proof. We recall that the operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S^{-1}_{c,L}(s,T)\, ds_j\, f(s),$$

where $U \subset \mathbb{H}$ is any bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and where $j \in \mathbb{S}$. Suppose furthermore that U contains an ε-neighborhood of $\sigma_S(T) \cup \sigma_L(\overline{T})$.

By Lemma 4.6.6 there exists $\delta_1 > 0$ such that $\sigma_S(Z) \subset U$ if we have $\|Z-T\| < \delta_1$. Consequently, $f \in \mathcal{SH}_L(\sigma_S(Z))$ for $\|Z - T\| < \delta_1$. Due to Lemma 4.6.6, $S^{-1}_{c,L}(s,T)$ is uniformly close to $S^{-1}_{c,L}(s,Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_j)$ for $j \in \mathbb{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get

$$\|f(T) - f(Z)\| = \frac{1}{2\pi} \| \int_{\partial(U\cap\mathbb{C}_j)} \left[S^{-1}_{c,L}(s,T) - S^{-1}_{c,L}(s,Z)\right]\, ds_j\, f(s)\| < \varepsilon. \quad \square$$

4.7 Some Examples

We end this chapter with some examples in which we compute the S-spectrum of different operators. In particular, we illustrate how the characterization of the S-spectrum of operators with commuting components in Theorem 4.5.6 simplifies its computation.

Example 4.7.1. Let us consider a, b, α, $\beta \in \mathbb{R}$ and the two matrices

$$T_1 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \quad T_2 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}.$$

It is easy to verify that $T_1T_2 = T_2T_1$. We can thus consider the operator

$$T = T_1e_1 + T_2e_2 = \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix},$$

with commuting components on $\mathbb{H}^2$. We have

$$\overline{T} = -\begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix},$$

so that $T + \overline{T} = 0$ and

$$T\overline{T} = \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}. \tag{4.29}$$

The S-spectrum is associated with the equation $\mathcal{Q}_{c,s}(T)x = 0$, that is,

$$\left(s^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}\right) x = 0 \quad \text{for } x \neq 0. \tag{4.30}$$

Observe that the matrix $T\overline{T}$ in (4.29) has only real entries. If $s = u + jv$, we can consider the matrix $T\overline{T}$ therefore a $\mathbb{C}_j$-complex matrix, and we find that s satisfies (4.30) if and only if $-s^2$ is an eigenvalue of $T\overline{T}$. Standard computations show that the only eigenvalue of $T\overline{T}$ is $a^2 + \alpha^2$ and we conclude that

$$\sigma_S(T) = \left\{ j\sqrt{a^2+\alpha^2} : \ j \in \mathbb{S} \right\}.$$

Example 4.7.2. We illustrate in this example how the computation of the S-spectrum of an operator with commuting components is simplified by the characterization given in Theorem 4.5.6. We consider the two commuting matrices

$$T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

and the associated quaternionic operator

$$T = e_1 T_1 + e_2 T_2 = \begin{bmatrix} e_2 & e_1 + e_2 \\ 0 & e_1 + 2e_2 \end{bmatrix}.$$

Since we have

$$\overline{T} = \begin{bmatrix} -e_2 & -e_1 - e_2 \\ 0 & -e_1 - 2e_2 \end{bmatrix},$$

it is immediate that $T + \overline{T} = 0$ and that

$$T\overline{T} = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix}.$$

In order to compute the S-spectrum using Theorem 4.5.6, we have to solve the equation $\mathcal{Q}_{c,s}(T)^{-1}x = 0$. For $x = (y,z)^T$, this turns into

$$\begin{bmatrix} s^2+1 & 4 \\ 0 & s^2+5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0, \quad \text{for} \quad \begin{bmatrix} y \\ z \end{bmatrix} \neq 0.$$

This gives the two equations

$$\begin{aligned} (s^2+1)y + 4z &= 0, \\ (s^2+5)z &= 0. \end{aligned} \tag{4.31}$$

If $s = u + jv$, then we can choose $i \in \mathbb{S}$ with $i \perp j$ and write $y = y_1 + y_2 i$ and $z_1 + z_2 i$ with $y_\ell, z_\ell \in \mathbb{C}_j$. Since 1 and i are linearly independent over $\mathbb{C}_j$ and the system (4.31) contains only coefficients in $\mathbb{C}_j$, it is equivalent to

$$\begin{aligned} (s^2+1)y_\ell + 4z_\ell &= 0, \\ (s^2+5)z_\ell &= 0, \end{aligned} \qquad \ell = 1, 2.$$

We are hence left with a $\mathbb{C}_j$-complex linear system of equations that can be solved easily. Its solutions are j and $\sqrt{5}j$, and thus

$$\sigma_S(T) = \left\{ j, \ \sqrt{5}j : \quad j \in \mathbb{S} \right\}.$$

The same result can be obtained by solving the equation

$$(T^2 - 2s_0T + |s|^2\mathcal{I})x = 0,$$

that is,

$$\begin{bmatrix} -1 - 2s_0e_2 + |s|^2 & -4 - 2s_0(e_1 + e_2) \\ 0 & -5 - 2s_0(e_1 + 2e_2) + |s|^2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0.$$

This corresponds to the two equations

$$\begin{aligned} (-1 - 2s_0e_2 + |s|^2)y - (4 + 2s_0(e_1 + e_2))z &= 0, \\ (-5 - 2s_0(e_1 + 2e_2) + |s|^2)z &= 0. \end{aligned}$$

Observe, however, that the coefficients of this system do not belong to one single complex plane, so that it cannot be reduced to a complex linear system of two equations. If we suppose that $\mathrm{Re}(s) = 0$, we find that either $s = j$, or $s = \sqrt{5}j$ with $j \in \mathbb{S}$. If $s_0 \neq 0$, then very long calculations show that there are no solutions; thus the S-spectrum coincides in both cases.

Example 4.7.3. We compute the equations for determining the S-spectrum of a bounded operator T with commuting components on a Banach space X. We use both the commutative and the noncommutative approaches and we see that the computations are again simpler in the first case.

Let $T = e_1T_1 + e_2T_2 \in \mathcal{B}(X)$, where T_1, T_2 are commuting bounded operators on $X_{\mathbb{R}}$. We determine the S-eigenvalue equation. We have

$$\overline{T} = -e_1T_1 - e_2T_2,$$

so

$$T + \overline{T} = 0,$$

and since $T_1T_2 = T_2T_1$, we also have

$$T\overline{T} = T_1^2 + T_2^2.$$

The point S-spectrum $\sigma_S(T)$ consists of quaternions s such that $\mathcal{Q}_{c,s}(T)$ has a bounded inverse. Hence we need to solve the equation

$$(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})x = y$$

for every $y \in X$, which simplifies in our case to

$$(s^2\mathcal{I} + T_1^2 + T_2^2)x = y. \tag{4.32}$$

If $s = u + jv$, the operator $T\overline{T} = T_1^2 + T_2^2$ can be considered an operator on the $\mathbb{C}_j$-complex Banach space $X_{\mathbb{R}} \otimes \mathbb{C}_j := X_{\mathbb{R}} + jX_{\mathbb{R}}$, and (4.32) is then exactly an eigenvalue equation of this operator. We can choose $i \in \mathbb{S}$ with $i \perp j$ and write

$x = x_1 + x_2 i$ and $y = y_1 + y_2 i$ with $x_\ell, y_\ell \in X_{\mathbb{R}} \otimes \mathbb{C}_j$. Since 1 and i are linearly independent over $\mathbb{C}_j$, we find that (4.32) is equivalent to

$$(s^2\mathcal{I} + T_1^2 + T_2^2)x_\ell = y_\ell, \quad \ell = 1, 2. \tag{4.33}$$

Hence s belongs to $\sigma_S(T)$ if and only if $-s^2$ belongs to the classical spectrum $\sigma(T\overline{T})$ of $T\overline{T}$. Because of the axial symmetry of the S-spectrum, $\sigma_S(T)$ is then given by

$$\sigma_S(T) = \left\{u + iv : \ u + jv \in \sigma\left(T\overline{T}\right), \ i \in \mathbb{S}\right\}.$$

In case one considers the noncommutative definition of the S-spectrum, we have $T^2 = -T_1^2 - T_2^2$, so that the equation

$$(T^2 - 2s_0 T + |s|^2\mathcal{I})x = y$$

becomes

$$\left(-T_1^2 - T_2^2 - 2s_0(e_1T_1 + e_2T_2) + |s|^2\mathcal{I}\right)x = y.$$

Observe that this is again a system that is more complicated than the eigenvalue equation of a complex linear operator. If we write $x = x_0 + \sum_{\ell=1}^3 x_\ell e_\ell$ and $y = x_0 + \sum_{\ell=1}^3 y_\ell e_\ell$ and set

$$A := |s|^2\mathcal{I} - T_1^2 - T_2^2,$$

we can rewrite the above equation in terms of its real components and obtain

$$\begin{aligned} &Ax_0 + 2\mathrm{Re}(s)T_1x_1 + \mathrm{Re}(s)T_2x_2 \\ &\quad + e_1(-2\mathrm{Re}(s)T_1x_0 + Ax_1 - 2\mathrm{Re}(s)T_2x_3) \\ &\quad + e_2(-2\mathrm{Re}(s)T_2x_0 + Ax_2 + 2\mathrm{Re}(s)T_1x_3) \\ &\quad + e_1e_2(Ax_3 - 2\mathrm{Re}(s)T_1x_2 + 2\mathrm{Re}(s)T_2x_1) = y_0 + \sum_{\ell=1}^3 y_\ell e_\ell. \end{aligned}$$

Thus the S-spectrum of T is given by the system of equations

$$\begin{cases} (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_0 + 2\mathrm{Re}(s)T_1x_1 + \mathrm{Re}(s)T_2x_2 = y_0, \\ -2\mathrm{Re}(s)T_1x_0 + (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_1 - 2\mathrm{Re}(s)T_2x_3 = y_1, \\ -2\mathrm{Re}(s)T_2x_0 + (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_2 + 2\mathrm{Re}(s)T_1x_3 = y_2, \\ (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_3 - 2\mathrm{Re}(s)T_1x_2 + 2\mathrm{Re}(s)T_2x_1 = y_3. \end{cases} \tag{4.34}$$

This system is much more complicated than the eigenvalue equation in (4.32), but it gives the same solution.

Example 4.7.4 (Fractional powers)**.** The slice hyperholomorphic logarithm on $\mathbb{H}$ is defined as

$$\log s := \ln|s| + j\arg(s) \quad \text{for } s = u + jv \in \mathbb{H} \setminus (-\infty, 0],$$

where $\arg(s) = \arccos(\mathrm{Re}(s)/|s|)$ is the unique angle $\varphi \in [0, \pi]$ such that $s = |s|e^{j\varphi}$. Observe that for $s = \mathrm{Re}(s) \in [0, +\infty)$, we have

$$\arccos(\mathrm{Re}(s)/|s|) = \arccos(1) = 0,$$

and so $\log s = \ln s$. Therefore, $\log s$ is well defined on the positive real axis and does not depend on the choice of the imaginary unit j. One has

$$e^{\log s} = s \quad \text{for } s \in \mathbb{H}$$

and

$$\log e^s = s \quad \text{for } s \in \mathbb{H} \text{ with } |\underline{s}| < \pi.$$

The quaternionic logarithm is both left and right slice hyperholomorphic (and actually even intrinsic) on $\mathbb{H} \setminus (-\infty, 0]$, and for every $j \in \mathbb{S}$, its restriction to the complex plane $\mathbb{C}_j$ coincides with the principal branch of the complex logarithm on $\mathbb{C}_j$. We define the fractional powers of exponent $\alpha \in \mathbb{R}$ of a quaternion s as

$$s^\alpha := e^{\alpha \log s} = e^{\alpha(\ln|s| + j\arccos(u/|s|))}, \quad s = u + jv \in \mathbb{H} \setminus (-\infty, 0].$$

This function is obviously also left and right slice hyperholomorphic on the set $\mathbb{H} \setminus (-\infty, 0]$. So we can define the fractional powers of bounded operators and in particular of matrices by the S-functional calculus. We can define fractional powers of a bounded vector operator $T = e_1T_1 + e_2T_2 + e_3T_3$ using the S-functional calculus,

$$T^\alpha = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} s^\alpha \, ds_j \, S_R^{-1}(s, T) \tag{4.35}$$

if $\sigma_S(T) \subset U$ is contained in the domain of s^α. Since $s \mapsto s^\alpha$ is an intrinsic slice hyperholomorphic function, we also have

$$T^\alpha = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s, T) \, ds_j \, s^\alpha.$$

These formulas were introduced in [50], and the theory of fractional powers of quaternionic operators was further developed in the papers [51, 52]. These operators are a natural tool to define fractional Fourier laws, and they have applications in fractional diffusion and fractional evolution problems.

4.8 Comments and Remarks

Comments on the references. The complete list of the papers in which the S-functional calculus for bounded operators has been developed is [10, 55, 66, 68, 79, 80, 127]. In the case we consider intrinsic functions, the S-functional calculus can be defined for a one-sided Banach space, as has been shown in [125]. In the paper [125], the author has also developed the theory of spectral operators in Banach spaces; see also [128].

The S-functional calculus can be defined also for n-tuples of noncommuting operators using slice hyperholomorphic functions with values in a Clifford algebra (also called slice monogenic functions); see [75, 97]. The commutative version of the S-functional calculus, that is, the S-functional calculus for operators with commuting components, is studied in [77].

The S-functional calculus was the starting point for the development of various quaternionic functional calculi. We mention the Philips functional calculus for generators of strongly continuous groups, which is based on the quaternionic version of the Laplace–Stieltjes transform; see [11]. Groups and semigroups of quaternionic linear operators have been considered in [19, 76, 153].

In the paper [30], the authors introduce the H^∞-functional calculus based on the S-spectrum. This is the quaternionic analogue of the calculus introduced by McIntosh [165]. In [30] is also considered the H^∞-functional calculus for n-tuples of noncommuting operators.

A more general version of the H^∞-functional calculus, the study of the fractional powers of quaternionic linear operators, is treated in [51, 52]. Here the authors also show how the fractional powers of quaternionic linear operators define new fractional diffusion and evolution processes. For a more direct approach to fractional powers of quaternionic operators that include the Kato formula, see the paper [50].

4.8.1 The S-Functional Calculus for n-Tuples of Operators

The notion of S-spectrum and also the definition of the S-functional calculus can be extended to n-tuples of not necessarily commuting operators. For this setting we need slice hyperholomorphic functions with values in a Clifford algebra (slice monogenic functions). Slice monogenicity is similar to the quaternionic setting; see the book [89]. We explain here the basic concepts. Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \dots, e_n$ satisfying the relations $e_\ell e_m + e_m e_\ell = 0$, $\ell \neq m$, $e_\ell^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$, where $A = \{\ell_1 \cdots \ell_r\} \in \mathcal{P}\{1, 2, \dots, n\}$, $\ell_1 < \cdots < \ell_r$ is a multi-index, and $e_A = e_{\ell_1} e_{\ell_2} \cdots e_{\ell_r}$, $e_\emptyset = 1$. An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{\ell=1}^n x_\ell e_\ell \in \mathbb{R}_n$, called a *paravector*, and the real part x_0 of x will also be denoted by $\operatorname{Re}(x)$. The *norm* of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \cdots + x_n^2$. The *conjugate* of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{\ell=1}^n x_\ell e_\ell$. We denote by $\mathbb{S}$ the sphere

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \cdots + e_n x_n : \ x_1^2 + \cdots + x_n^2 = 1\};$$

for $j \in \mathbb{S}$ we obviously have $j^2 = -1$. Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, let us set $j_x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$, the set

$$[x] := \{y \in \mathbb{R}^{n+1} : \ y = x_0 + j|\underline{x}|, \ j \in \mathbb{S}\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R}+j\mathbb{R}$ passing through 1 and $j \in \mathbb{S}$ will be denoted by $\mathbb{C}_j$, and an element belonging to $\mathbb{C}_j$ will be indicated by $u + jv$, for u, $v \in \mathbb{R}$. With an abuse of notation we will write $x \in \mathbb{R}^{n+1}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f:\ U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ can be interpreted as a function of the paravector x. With the above notations, the definition of the slice hyperholomorphic functions $f:\ U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is analogous to the notion of slice hyperholomorphic functions for quaternionic-valued functions. We adapt the definition of slice hyperholomorphicity to the Clifford-algebra-valued case; in this case functions are often called *slice monogenic*. The definition of an *axially symmetric* set is as in the quaternionic setting, i.e., we say that $U \subseteq \mathbb{R}^{n+1}$ is axially symmetric if $[x] \subset U$ for all $x \in U$.

Definition 4.8.1 (Slice hyperholomorphic functions with values in $\mathbb{R}_n$ (or slice monogenic functions))**.** Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{R}_n$ is called a *left slice function* if it is of the form

$$f(q) = f_0(u, v) + jf_1(u, v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ that satisfy the compatibility conditions

$$f_0(u, -v) = f_0(u, v), \quad f_1(u, -v) = -f_1(u, v). \tag{4.36}$$

If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations

$$\frac{\partial}{\partial u} f_0(u, v) - \frac{\partial}{\partial v} f_1(u, v) = 0, \tag{4.37}$$

$$\frac{\partial}{\partial v} f_0(u, v) + \frac{\partial}{\partial u} f_1(u, v) = 0, \tag{4.38}$$

then f is called *left slice hyperholomorphic* (or *left slice monogenic*). A function $f : U \to \mathbb{R}_n$ is called a *right slice function* if it is of the form

$$f(q) = f_0(u, v) + f_1(u, v)j \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ that satisfy (4.36). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equation, then f is called *right slice hyperholomorphic* (or *right slice monogenic*). If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called *intrinsic*. We denote the sets of left and right slice hyperholomorphic functions on U by $\mathcal{SM}_L(U)$ and $\mathcal{SM}_R(U)$, respectively.

Also for slice monogenic functions we have a Cauchy formula that is analogous to the quaternionic case. Let $x, s \in \mathbb{R}^{n+1}$ with $x \notin [s]$ be paravectors. The Cauchy kernels in form I and in form II are the same as in the quaternionic case when the quaternions are replaced by the paravectors. For example, for the form I we have

$$S_L^{-1}(s, x) := -(x^2 - 2\text{Re}(s)x + |s|^2)^{-1}(x - \overline{s})$$

and

$$S_R^{-1}(s, x) := -(x - \bar{s})(x^2 - 2\text{Re}(s)x + |s|^2)^{-1}.$$

Theorem 4.8.2 (The Cauchy formulas for slice monogenic functions)**.** *Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$, and set $ds_j = ds(-j)$. If f is a (left) slice monogenic function on a set that contains $\overline{U}$, then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,x)\, ds_j\, f(s), \quad \text{for every } x \in U. \tag{4.39}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,x), \quad \text{for every } x \in U. \tag{4.40}$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

To define the S-functional calculus for n-tuples of operators, we consider a Banach space X over $\mathbb{R}$ with norm $\|\cdot\|$. It is possible to endow X with an operation of multiplication by elements of $\mathbb{R}_n$ that gives a two-sided module over $\mathbb{R}_n$. A two-sided module V over $\mathbb{R}_n$ is called a Banach module over $\mathbb{R}_n$ if there exists a constant $C \geq 1$ such that $\|va\| \leq C\|v\||a|$ and $\|av\| \leq C|a|\|v\|$ for all $v \in V$ and $a \in \mathbb{R}_n$. By X_n we denote $X \otimes \mathbb{R}_n$ over $\mathbb{R}_n$; X_n turns out to be a two-sided Banach module.

An element in X_n is of type $\sum_A v_A \otimes e_A$ (where $A = \ell_1 \cdots \ell_r$, $i_\ell \in \{1, 2, \dots, n\}$, $\ell_1 < \cdots < \ell_r$ is a multi-index). Multiplication of an element $v \in X_n$ by a scalar $a \in \mathbb{R}_n$ is defined by $va = \sum_A v_A \otimes (e_A a)$ and $av = \sum_A v_A \otimes (a e_A)$. For simplicity, we will write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. Finally, we define $\|v\|^2_{X_n} = \sum_A \|v_A\|^2_X$.

We denote by $\mathcal{B}(X)$ the space of bounded $\mathbb{R}$-homomorphisms of the Banach space X to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(X)}$. Given $T_A \in \mathcal{B}(X)$, we can introduce the operator $T = \sum_A T_A e_A$ and its action on $v = \sum v_B e_B \in X_n$ as $T(v) = \sum_{A,B} T_A(v_B) e_A e_B$. The operator T is a right-module homomorphism that is a bounded linear map on X_n.

In the sequel, we will consider operators of the form (called *paravector operators*)

$$T = T_0 + \sum_{\ell=1}^{n} e_\ell T_\ell,$$

where $T_\ell \in \mathcal{B}(X)$ for $\ell = 0, 1, \dots, n$. The subset of such operators in $\mathcal{B}(X_n)$ will be denoted by $\mathcal{B}^{0,1}(X_n)$. We define $\|T\|_{\mathcal{B}^{0,1}(X_n)} = \sum_\ell \|T_\ell\|_{\mathcal{B}(X)}$. Note that, in the sequel, we will omit the subscript $\mathcal{B}^{0,1}(X_n)$ in the norm of an operator. Note also that $\|TS\| \leq \|T\|\|S\|$. The Cauchy kernel operator series are the power series expansions of the S-resolvent operators.

Theorem 4.8.3. *Let $T \in \mathcal{B}^{0,1}(X_n)$ and let $s \in \mathbb{H}$. Then for $\|T\| < |s|$, we have*

$$\sum_{m\geq 0} T^m s^{-1-m} = -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{4.41}$$

$$\sum_{m\geq 0} s^{-1-m} T^m = -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}. \tag{4.42}$$

We observe that the sums of the above series are independent of the fact that the components of the paravector operator T commute. Moreover, the operators on the right-hand sides of (4.41) and (4.42) are defined on a subset of $\mathbb{R}^{n+1}$ that is larger than $\{s \in \mathbb{R}^{n+1} \ : \ \|T\| < |s|\}$. So we define the S-spectrum, the S-resolvent set, and the S-resolvent operators for the paravector operator $T \in \mathcal{B}^{0,1}(V_n)$.

Definition 4.8.4 (The S-spectrum and the S-resolvent set)**.** Let $T \in \mathcal{B}^{0,1}(X_n)$. We define the S-*spectrum* $\sigma_S(T)$ of T as

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} : \ T^2 - 2\,Re(s)T + |s|^2\mathcal{I} \text{ is not invertible}\}.$$

The S-*resolvent set* $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

Definition 4.8.5 (The S-resolvent operators)**.** Let $T \in \mathcal{B}^{0,1}(X_n)$ and $s \in \rho_S(T)$. We define the *left* S*-resolvent operator* as

$$S_L^{-1}(s,T) := -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{4.43}$$

and the *right* S*-resolvent operator* as

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}. \tag{4.44}$$

Definition 4.8.6 (The S-functional calculus for n-tuples of operators)**.** Let X_n be a two-sided Banach module and $T \in \mathcal{B}^{0,1}(X_n)$. Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain that contains $\sigma_S(T)$ and set $ds_j = -dsj$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\ ds_j\ f(s), \quad \textit{for } f \in \mathcal{SM}_L(\sigma_S(T)), \tag{4.45}$$

and

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\ ds_j\ S_R^{-1}(s,T), \quad \textit{for } f \in \mathcal{SM}_R(\sigma_S(T)), \tag{4.46}$$

where $\mathcal{SM}_L(\sigma_S(T))$ (resp. $\mathcal{SM}_R(\sigma_S(T))$) are left (resp. right) slice hyperholomorphic Clifford-algebra-valued functions defined on a suitable open set that contains the S-spectrum of the paravector operator T.

Most of the results that hold for the quaternionic S-functional calculus extend to the S-functional calculus for n-tuples of operators.

4.8.2 The *W*-Functional Calculus for Quaternionic Operators

Using the notion of slice hyperholomorphic functions it is possible to define a transform that maps slice hyperholomorphic functions into Fueter regular functions of plane wave type. This transform is different from the Fueter mapping theorem in integral form. With such an integral transform we can define the W-functional calculus. This calculus was introduced in [70] for monogenic functions. Here we reformulate it for the quaternionic setting. Using the Cauchy formula for slice hyperholomorphic functions it is possible to define an integral transform that associates to a slice hyperholomorphic function a Fueter regular function. Inspired by [192], we introduce an integral transform that associates to a slice hyperholomorphic function a Fueter regular function of plane wave type. The following result is immediate; see [192], Section 1.1.

Proposition 4.8.7. *Suppose that the differentiable functions $(g_1, -g_2)$ satisfy the Cauchy–Riemann system in an open set of the complex plane identified with the set D of the pairs $(u,p) \in \mathbb{R}^2$:*

$$\partial_u g_1(u,p) = -\partial_p g_2(u,p), \quad \partial_p g_1(u,p) = \partial_u g_2(u,p). \tag{4.47}$$

Let

$$U_D = \{x \in \mathbb{H} : \ x = u + \underline{\omega} p, \ (u,p) \in D, \ \underline{\omega} \in \mathbb{S}\}$$

and define the function $\tilde{G} : U_D \subseteq \mathbb{H} \to \mathbb{H}$ by

$$\tilde{G}(x) := g_1(u,p) - \underline{\omega}\, g_2(u,p). \tag{4.48}$$

Then $\tilde{G}(x)$ is slice hyperholomorphic in U_D.

When necessary, we will identify $\mathbb{H}$ with $\mathbb{R}^2 \times \mathbb{S}$ by setting $x \mapsto (x_0, p, \underline{\omega})$, and instead of $\tilde{G}(x)$ we will write $\tilde{G}(x_0, p, \underline{\omega})$ (keeping the symbol $\tilde{G}$ for the function). Starting from the slice hyperholomorphic function $\tilde{G}(u, p, \underline{\omega})$ in (4.48) we can construct a Fueter regular function of plane wave type by the substitution

$$u = \langle \underline{x}, \underline{\omega} \rangle, \quad p = x_0.$$

Suppose that the functions $(g_1, -g_2)$ satisfy the Cauchy–Riemann system and let us define the function

$$G(x_0, \langle \underline{x}, \underline{\omega} \rangle, \underline{\omega}) := g_1(\langle \underline{x}, \underline{\omega} \rangle, x_0) + \underline{\omega}\, g_2(\langle \underline{x}, \underline{\omega} \rangle, x_0), \quad \text{for } \underline{\omega} \in \mathbb{S}. \tag{4.49}$$

We recall a simple result stated in [192]:

Proposition 4.8.8. *The function G defined in (4.49) is left Fueter regular in the variable $x = x_0 + \underline{x}$.*

Definition 4.8.9. A function of the form (4.49) is called a Fueter plane wave function.

Definition 4.8.10 (The W-kernels)**.** Let $S_L^{-1}(s,x)$, $S_R^{-1}(s,x)$ be the Cauchy kernels of left and right slice hyperholomorphic functions, respectively, and let $\underline{\omega} \in \mathbb{S}$. For $\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega} \notin [s]$ we define

$$\begin{aligned} W_{\underline{\omega}}^L(s,x) &:= S_L^{-1}(s, \langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega}) \\ &= -[(\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega})^2 - 2s_0(\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega}) + |s|^2]^{-1}(\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega} - \overline{s}) \end{aligned}$$

and

$$W^R_{\underline{\omega}}(s,x) := S^{-1}_R(s, \langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega})$$
$$= -(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega} - \bar{s})[(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega})^2 - 2\mathrm{Re}(s)(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}) + |s|^2]^{-1},$$

where $\underline{\omega} \in \mathbb{S}$ is considered a parameter.

Observe that $W^L_{\underline{\omega}}$ and $W^R_{\underline{\omega}}$ are obtained by the change of variable $x \to \langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}$ in the Cauchy kernels of slice hyperholomorphic functions and $\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}$ is still a paravector.

The following theorem is a direct consequence of the Cauchy formula of slice hyperholomorphic functions.

Theorem 4.8.11. *Let $\underline{\omega} \in \mathbb{S}$ be a parameter and let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. We furthermore assume that $\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega} \in U$. If f is a left slice hyperholomorphic function on a set that contains $\overline{U}$, the integral*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} W^L_{\underline{\omega}}(s,x)\,ds_j\,f(s), \quad \textit{for every } q \in U, \tag{4.50}$$

depends neither on U nor on the imaginary unit $j \in \mathbb{S}$. If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, the integral

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,W^R_{\underline{\omega}}(s,x), \quad \textit{for every } q \in U, \tag{4.51}$$

depends neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Thanks to Theorem 4.8.11 we can define the W-transform, which maps slice hyperholomorphic functions into Fueter regular functions.

Definition 4.8.12 (The W-transforms). Let $\underline{\omega} \in \mathbb{S}$ be a parameter and let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. Assume that $\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega} \in U$. If f is a left slice hyperholomorphic function on a set that contains $\overline{U}$, then we define the *left W^L-transform* as

$$\breve{f}_{\underline{\omega}}(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} W^L_{\underline{\omega}}(s,x)\,ds_j\,f(s), \quad \text{for every } q \in U. \tag{4.52}$$

If f is a right slice hyperholomorphic function then we define the *right W^R-transform* as

$$\breve{f}_{\underline{\omega}}(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,W^R_{\underline{\omega}}(s,x), \quad \text{for every } q \in U. \tag{4.53}$$

We observe that the W-transform defines a transformation between slice hyperholomorphic functions and Fueter regular functions that depends on a parameter on the unit sphere $\mathbb{S}$. This transform can be extended to the more general case of Clifford-algebra-valued functions.

- For every $\underline{\omega} \in \mathbb{S}$ the function $W^L_{\underline{\omega}}(s,x)$ is right slice hyperholomorphic in s and left Fueter regular in x for every x, s such that $(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}) \notin [s]$. Moreover, the W^L-transform maps left slice hyperholomorphic functions f into left Fueter regular plane wave functions $\breve{f}_{\underline{\omega}}$.
- For every $\underline{\omega} \in \mathbb{S}$ the function $W^R_{\underline{\omega}}(s,x)$ is left slice hyperholomorphic in s and right Fueter regular in x for every x, s such that $(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}) \notin [s]$. Moreover, the W^R-transform maps right slice hyperholomorphic functions f into right Fueter regular plane wave functions $\breve{f}_{\underline{\omega}}$.

Theorem 4.8.13. *Let $T = T_0 + e_1T_1 + e_2T_2 + e_3T_3 \in \mathcal{B}(X)$. Assume that $\underline{\omega} \in \mathbb{S}$ and define the operator*

$$A_{\underline{\omega}} := \sum_{j=1}^{3} T_j\omega_j - T_0\underline{\omega}.$$

Then $A_{\underline{\omega}}$ belongs to $\mathcal{B}(X)$, and the operator $A^2_{\underline{\omega}} - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I}$ is invertible for $s \in \mathbb{H}$ with $\|T\| < |s|$ for all $\underline{\omega} \in \mathbb{S}$. Moreover, for $s \in \mathbb{H}$ with $\|T\| < |s|$ and for all $\underline{\omega} \in \mathbb{S}$, we have

$$\sum_{m\geq 0} A^m_{\underline{\omega}} s^{-1-m} = -(A^2_{\underline{\omega}} - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}(A_{\underline{\omega}} - \bar{s}\mathcal{I}), \tag{4.54}$$

$$\sum_{m\geq 0} s^{-1-m} A^m_{\underline{\omega}} = -(A_{\underline{\omega}} - \bar{s}\mathcal{I})(A^2_{\underline{\omega}} - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}. \tag{4.55}$$

The above theorem motivates the notion of W-spectrum.

Definition 4.8.14 (The W-spectrum and the W-resolvent set)**.** Let $T \in \mathcal{B}(X)$ and let $\underline{\omega} \in \mathbb{S}$. We define the operators

$$A_{\underline{\omega}} = \sum_{j=1}^{3} T_j\omega_j - T_0\underline{\omega} \quad \text{and} \quad Q_{\underline{\omega}}(T,s) := A^2_{\underline{\omega}} - 2s_0A_{\underline{\omega}} + |s|^2\mathcal{I}.$$

We define the *W-spectrum* $\sigma_W(T)$ of T as:

$$\sigma_W(T,\underline{\omega}) = \left\{ s \in \mathbb{R}^{n+1} : \; Q_{\underline{\omega}}(T,s) \text{ is not invertible in } \mathcal{B}(X) \right\}.$$

The *W-resolvent set* $\rho_W(T)$ is defined by

$$\rho_W(T,\underline{\omega}) = \mathbb{H} \setminus \sigma_W(T,\underline{\omega}).$$

The theorem on the structure of the W-spectrum holds also in this case. Let $T \in \mathcal{B}(X)$, $\underline{\omega} \in \mathbb{S}$, and let $p = p_0 + p_1 j \in [p_0 + p_1 j] \subset \mathbb{H} \setminus \mathbb{R}$, such that $p \in \sigma_W(T,\underline{\omega})$. Then all the elements of the 2-sphere $[p_0 + p_1 j]$ belong to $\sigma_W(T,\underline{\omega})$. Thus the W-spectrum consists of real points and/or 2-spheres. In the case of bounded operators, the W-spectrum, for all $\underline{\omega} \in \mathbb{S}$, is a compact nonempty set.

Definition 4.8.15 (The W-resolvent operators)**.** Let $T \in \mathcal{B}(X)$, let $\underline{\omega} \in \mathbb{S}$, and let $A_{\underline{\omega}} := \sum_{j=1}^{3} T_j \omega_j - T_0 \underline{\omega}$. For $s \in \rho_W(T)$ we define the *left W-resolvent operator* by

$$W_{\underline{\omega}}^L(s,T) = -(A_{\underline{\omega}}^2 - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}(A_{\underline{\omega}} - \overline{s}\mathcal{I}), \tag{4.56}$$

and the *right W-resolvent operator* by

$$W_{\underline{\omega}}^R(s,T) = -(A_{\underline{\omega}} - \overline{s}\mathcal{I})(A_{\underline{\omega}}^2 - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}. \tag{4.57}$$

Definition 4.8.16 (The W-functional calculus for bounded operators)**.** Let $T \in \mathcal{B}(V)$ and let $\underline{\omega} \in \mathbb{S}$. Let j be an arbitrary imaginary unit and U an arbitrary slice Cauchy domain U as in Remark 3.2.4. For every function $f \in \mathcal{SH}_L(\sigma_W(T,\underline{\omega}))$, we define

$$\breve{f}_{\underline{\omega}}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} W_{\underline{\omega}}^L(s,T)\, ds_j\, f(s). \tag{4.58}$$

For every $f \in \mathcal{SH}_R(\sigma_W(T,\underline{\omega}))$, we define

$$\breve{f}_{\underline{\omega}}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, W_{\underline{\omega}}^R(s,T), \tag{4.59}$$

with the obvious meaning of the symbols $\mathcal{SH}_L(\sigma_W(T,\underline{\omega}))$ and $\mathcal{SH}_R(\sigma_W(T,\underline{\omega}))$.

The definition of the W-functional calculus is well posed, since the integrals in (4.58) and (4.59) depend neither on the open set U nor on the imaginary unit $j \in \mathbb{S}$.

The W-functional calculus is a functional calculus that is based on slice hyperholomorphic functions, but it produces operators $\breve{f}_{\underline{\omega}}(T)$ for Fueter regular functions $\breve{f}_{\underline{\omega}}(s)$. The W-functional calculus and the F-functional calculus are Fueter functional calculi. In the case of Clifford-algebra-valued functions these two calculi become monogenic functional calculi in the spirit of the monogenic functional calculus introduced and studied by A. McIntosh and his collaborators in a series of papers [160, 161, 166], and in the book [159].