Chapter 2

Slice Hyperholomorphic Functions

We will develop operator theory for quaternionic linear operators using the theory of slice hyperholomorphic functions. The most important results are the structure formula (or representation formula) and the Cauchy formulas with slice hyperholomorphic integral kernels. We will discuss the two Cauchy formulas and the associated Cauchy kernels in detail because they are the starting point for defining the S-functional calculus (in the quaternionic setting the S-functional calculus is often called the quaternionic functional calculus).

The Fueter mapping theorem is an important tool in hypercomplex analysis. It shows that the Laplace operator maps slice hyperholomorphic functions to Fueter regular functions and hence provides a method for generating Fueter regular functions. This theorem has been extended by Sce for the case of Clifford algebras with odd dimension and by Qian in the even dimension. In the literature it is often called the Fueter–Sce or Fueter–Sce–Qian theorem according to the setting. Starting from the Cauchy formula for slice hyperholomorphic functions, it is possible to give the Fueter mapping theorem an integral representation. One obtains then an integral transform that can be used to define the F-functional calculus.

We denote by $\mathbb{H}$ the algebra of quaternions. An element q of $\mathbb{H}$ is of the form

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad q_\ell \in \mathbb{R}, \ \ell = 0, 1, 2, 3,$$

where e_1, e_2 and e_3 are the generating imaginary units of $\mathbb{H}$. They satisfy the relations

$$e_1^2 = e_2^2 = e_3^2 = -1 \tag{2.1}$$

and

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2. \tag{2.2}$$

© Springer Nature Switzerland AG 2018

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_2

The real part, the imaginary part, and the modulus $|q|$ of a quaternion $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ are defined as $\mathrm{Re}(q) = q_0$, $\mathrm{Im}(q) = q_1 e_1 + q_2 e_2 + q_3 e_3$, and $|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$, respectively. The conjugate of the quaternion q is

$$\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q) = q_0 - q_1 e_1 - q_2 e_2 - q_3 e_3,$$

and it satisfies

$$|q|^2 = q\bar{q} = \bar{q}q.$$

The inverse of every nonzero element q is hence given by

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

Let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = q_1 e_1 + q_2 e_2 + q_3 e_3 : \ \ q_1^2 + q_2^2 + q_3^2 = 1\}.$$

Notice that if $j \in \mathbb{S}$, then $j^2 = -1$. For this reason the elements of $\mathbb{S}$ are also called imaginary units. The set $\mathbb{S}$ is a 2-dimensional sphere in $\mathbb{R}^4 \cong \mathbb{H}$. Given a nonreal quaternion $q = q_0 + \mathrm{Im}(q)$, we have $q = u + jv$ with $u = \mathrm{Re}(q)$, $j = \mathrm{Im}(q)/|\mathrm{Im}(q)| \in \mathbb{S}$, and $v = |\mathrm{Im}(q)|$. We can associate to q the 2-dimensional sphere

$$[q] = \{q_0 + j|\mathrm{Im}(q)| : \quad j \in \mathbb{S}\} = \{u + jv : \quad j \in \mathbb{S}\}.$$

This sphere is centered at the real point $q_0 = \mathrm{Re}(q)$ and has radius $|\mathrm{Im}(q)|$. The next lemma, which can be found in every standard textbook treating quaternions, shows that two quaternions belong to the same sphere if and only if they can be transformed into each other by multiplication by a nonzero quaternion.

Lemma 2.0.1. *Let $q \in \mathbb{H}$. A quaternion p belongs to $[q]$ if and only if there exists $h \in \mathbb{H} \setminus \{0\}$ such that $p = h^{-1} q h$.*

If $j \in \mathbb{S}$, then the set

$$\mathbb{C}_j = \{u + jv : \ \ u, v \in \mathbb{R}\}$$

is an isomorphic copy of the complex numbers. If, moreover, $i \in \mathbb{S}$ with $j \perp i$, then j, i, and $k := ji$ form a generating basis of $\mathbb{H}$, i.e., this basis also satisfies the relations (2.1) and (2.2). Hence, every quaternion $q \in \mathbb{H}$ can be written as

$$q = z_1 + z_2 i = z_1 + i\bar{z}_2$$

with unique $z_1, z_2 \in \mathbb{C}_j$, and so

$$\mathbb{H} = \mathbb{C}_j + i\mathbb{C}_j \quad \text{and} \quad \mathbb{H} = \mathbb{C}_j + \mathbb{C}_j i. \tag{2.3}$$

Moreover, we observe that

$$\mathbb{H} = \bigcup_{j \in \mathbb{S}} \mathbb{C}_j.$$

Finally, we introduce the notation $\mathbb{C}_j^+ := \{u + jv : u \in \mathbb{R}, v \geq 0\}$ for the upper half-plane in $\mathbb{C}_j$ and $\overline{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$.

2.1 Slice Hyperholomorphic Functions

The theory of slice hyperholomorphic functions is nowadays well developed. There are three possible ways to define slice hyperholomorphic functions: using the definition in [135], using the global operator of slice hyperholomorphic functions introduced in [60], or by the definition that comes from the Fueter–Sce–Qian mapping theorem. This last definition is the most appropriate for operator theory, and it is the one that we will use. In this section we therefore develop the part of the theory that it is relevant for our purposes.

Definition 2.1.1. Let $U \subseteq \mathbb{H}$.

(i) We say that U is *axially symmetric* if $[q] \subset U$ for every $q \in U$.

(ii) We say that U is a *slice domain* if $U \cap \mathbb{R} \neq \emptyset$ and if $U \cap \mathbb{C}_j$ is a domain in $\mathbb{C}_j$ for every $j \in \mathbb{S}$.

Definition 2.1.2 (Slice hyperholomorphic functions)**.** Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{H}$ is called a *left slice function* if it is of the form

$$f(q) = f_0(u,v) + j f_1(u,v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \Omega \to \mathbb{H}$ that satisfy the compatibility condition

$$f_0(u,-v) = f_0(u,v), \quad f_1(u,-v) = -f_1(u,v). \tag{2.4}$$

If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations

$$\frac{\partial}{\partial u} f_0(u,v) - \frac{\partial}{\partial v} f_1(u,v) = 0, \tag{2.5}$$

$$\frac{\partial}{\partial v} f_0(u,v) + \frac{\partial}{\partial u} f_1(u,v) = 0, \tag{2.6}$$

then f is called *left slice hyperholomorphic*. A function $f : U \to \mathbb{H}$ is called a *right slice function* if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v) j \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \Omega \to \mathbb{H}$ that satisfy (2.4). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations, then f is called *right slice hyperholomorphic*.

If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called *intrinsic*.

We denote the sets of left and right slice functions on U by $\mathcal{SF}_L(U)$ and $\mathcal{SF}_R(U)$ and the sets of left and right slice hyperholomorphic functions on U by $\mathcal{SH}_L(U)$ and $\mathcal{SH}_R(U)$, respectively. The set of intrinsic slice functions on U will be denoted by $\mathcal{FN}(U)$ and the set of slice hyperholomorphic functions on U will be denoted by $\mathcal{N}(U)$.

Remark 2.1.3. Every quaternion q can be represented as an element of a complex plane $\mathbb{C}_j$ using at least two different imaginary units $j \in \mathbb{S}$. We have $q = u + jv = u + (-j)(-v)$ and $-j$ also belongs to $\mathbb{S}$. If q is real, then we can use any imaginary unit $j \in \mathbb{S}$ to consider q an element of $\mathbb{C}_j$. The compatibility condition (2.4) ensures that the choice of this imaginary unit is irrelevant. In particular, it forces $f_1(u, v)$ to equal 0 if $v = 0$, that is if $q \in \mathbb{R}$.

Multiplication and composition with intrinsic functions preserve the slice structure and slice hyperholomorphicity. This is not true for arbitrary slice functions.

Theorem 2.1.4. *Let $U \subseteq \mathbb{H}$ be axially symmetric. The following statements hold:*

(i) *If $f \in \mathcal{NF}(U)$ and $g \in \mathcal{SF}_L(U)$, then $fg \in \mathcal{SF}_L(U)$. If $f \in \mathcal{SF}_R(U)$ and $g \in \mathcal{NF}(U)$, then $fg \in \mathcal{SF}_R(U)$.*

(ii) *If $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{N}(U)$, then $fg \in \mathcal{SH}_R(U)$.*

(iii) *If $g \in \mathcal{NF}(U)$ and $f \in \mathcal{SF}_L(g(U))$, then $f \circ g \in \mathcal{SF}_L(U)$. If $g \in \mathcal{NF}(U)$ and $f \in \mathcal{SF}_R(g(U))$, then $f \circ g \in \mathcal{SF}_R(U)$.*

(iv) *If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_L(g(U))$, then $f \circ g \in \mathcal{SH}_L(U)$. If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_R(g(U))$, then $f \circ g \in \mathcal{SH}_R(U)$.*

Proof. Let $f = f_0 + jf_1 \in \mathcal{NF}(U)$ and $g = g_0 + jg_1 \in \mathcal{SF}_L(U)$. Since f is intrinsic, the components f_0, f_1 take real values. Hence, they commute with $j \in \mathbb{S}$, and we find for $q = u + jv \in U$ that

$$\begin{aligned} f(q)g(q) &= f_0(u,v)g_0(u,v) + jf_1(u,v)g_0(u,v) \\ &\quad + f_0(u,v)jg_1(u,v) + jf_1(u,v)jg_1(u,v) \\ &= f_0(u,v)g_0(u,v) - f_1(u,v)g_1(u,v) \\ &\quad + j(f_1(u,v)g_0(u,v) + f_0(u,v)g_1(u,v)). \end{aligned}$$

The functions

$$h_0(u,v) := f_0(u,v)g_0(u,v) - f_1(u,v)g_1(u,v)$$

and

$$h_1(u,v) := f_1(u,v)g_0(u,v) + f_0(u,v)g_1(u,v)$$

satisfy the compatibility condition (2.4), as one can check easily, and hence fg

belongs to $\mathcal{SF}_L(U)$. If, moreover, f and g are slice hyperholomorphic, then

$$\begin{aligned}\frac{\partial}{\partial u}h_0(u,v) &= \left(\frac{\partial}{\partial u}f_0(u,v)\right)g_0(u,v) + f_0(u,v)\left(\frac{\partial}{\partial u}g_0(u,v)\right)\\ &\quad - \left(\frac{\partial}{\partial u}f_1(u,v)\right)g_1(u,v) - f_1(u,v)\left(\frac{\partial}{\partial u}g_1(u,v)\right)\\ &= \left(\frac{\partial}{\partial v}f_1(u,v)\right)g_0(u,v) + f_0(u,v)\left(\frac{\partial}{\partial v}g_1(u,v)\right)\\ &\quad + \left(\frac{\partial}{\partial v}f_0(u,v)\right)g_1(u,v) + f_1(u,v)\left(\frac{\partial}{\partial v}g_0(u,v)\right)\\ &= \frac{\partial}{\partial v}h_1(u,v),\end{aligned}$$

and similarly one shows that also

$$\frac{\partial}{\partial v}h_0(u,v) = -\frac{\partial}{\partial u}h_1(u,v)$$

holds. Hence $fg = h_0 + jh_1$ is left slice hyperholomorphic.

Now let $g = g_0 + jg_1 \in \mathcal{NF}(U)$ and $f = f_0 + jf_1 \in \mathcal{SF}_L(g(U))$. For $q = u + jv \in U$, we have $g(q) = g_0(u,v) + jg_1(u,v) = \tilde{u} + i\tilde{v}$ with $\tilde{u} = g_0(u,v)$, $i = j\operatorname{sgn}(g_1(u,v)) \in \mathbb{S}$ and $\tilde{v} = |g_1(u,v)|$. Thus

$$\begin{aligned}f(g(q)) &= f_0(\tilde{u},\tilde{v}) + ig_1(\tilde{u},\tilde{v})\\ &= f_0(g_0(u,v),g_1(u,v)) + jf_1(g_0(u,v),g_1(u,v)),\end{aligned}$$

because f_1 is odd in the second variable. It is immediate that the functions $h_0(u,v) = f_0(g_0(u,v),g_1(u,v))$ and $h_1(u,v) = f_1(g_0(u,v),g_1(u,v))$ satisfy the compatibility condition (2.4), and so $f \circ g \in \mathcal{SF}_L(g(U))$. If furthermore f and g are slice hyperholomorphic, then

$$\begin{aligned}\frac{\partial}{\partial u}h_0(u,v) &= \frac{\partial}{\partial g_0}f_0(g_0(u,v),g_1(u,v))\frac{\partial}{\partial u}g_0(u,v)\\ &\quad + \frac{\partial}{\partial g_1}f_0(g_0(u,v),g_1(u,v))\frac{\partial}{\partial u}g_1(u,v)\\ &= \frac{\partial}{\partial g_1}f_1(g_0(u,v),g_1(u,v))\frac{\partial}{\partial v}g_1(u,v)\\ &\quad + \frac{\partial}{\partial g_0}f_1(g_0(u,v),g_1(u,v))\frac{\partial}{\partial v}g_0(u,v)\\ &= \frac{\partial}{\partial v}h_1(u,v)\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial}{\partial u} h_1(u,v) &= \frac{\partial}{\partial g_0} f_1(g_0(u,v), g_1(u,v)) \frac{\partial}{\partial u} g_0(u,v) \\
&\quad + \frac{\partial}{\partial g_1} f_1(g_0(u,v), g_1(u,v)) \frac{\partial}{\partial u} g_1(u,v) \\
&= -\frac{\partial}{\partial g_1} f_0(g_0(u,v), g_1(u,v)) \frac{\partial}{\partial v} g_1(u,v) \\
&\quad - \frac{\partial}{\partial g_0} f_0(g_0(u,v), g_1(u,v)) \frac{\partial}{\partial v} g_0(u,v) \\
&= -\frac{\partial}{\partial v} h_0(u,v).
\end{aligned}
$$

Hence $f \circ g = h_0 + jh_1$ is left slice hyperholomorphic.

Similar arguments show that the statements for right slice functions also hold. □

Lemma 2.1.5. *Let $U \subseteq \mathbb{H}$ be axially symmetric and let f be a left (or right) slice function on U. The following statements are equivalent.*

(i) *The function f is intrinsic.*

(ii) *We have $f(U \cap \mathbb{C}_j) \subset \mathbb{C}_j$ for every $j \in \mathbb{S}$.*

(iii) *We have $f(\overline{q}) = \overline{f(q)}$ for all $q \in U$.*

Proof. Assume that $f = f_0 + jf_1$ is a left slice function. (The other case follows analogously.) The implications (i)⇒(ii) and (i)⇒(ii) are immediate. In order to show the inverse relations, we first observe that for every $q = u + jv \in U$,

$$f(q) + f(\overline{q}) = f_0(u,v) + jf_1(u,v) + f_0(u,v) - jf_1(u,v) = 2f_0(u,v)$$

and

$$f(q) - f(\overline{q}) = f_0(u,v) + jf_1(u,v) - f_0(u,v) + jf_0(u,v) = 2jf_1(u,v).$$

If (ii) holds, then $f(u+jv) \in \mathbb{C}_j$ for every $j \in \mathbb{S}$, and hence it commutes with j. Thus

$$
\begin{aligned}
jf_0(u,v) &= j(f(u+jv) + f(u-jv)) \\
&= (f(u+jv) + f(u-jv))j = 2f_0(u,v)j.
\end{aligned}
$$

Since a quaternion commutes with $j \in \mathbb{S}$ if and only if it belongs to $\mathbb{C}_j$, we have $f_0(u,v) \in \bigcap_{j\in\mathbb{S}} \mathbb{C}_j = \mathbb{R}$. For every $j \in \mathbb{S}$, we then have that

$$
\begin{aligned}
jf_0(u,v) - f_1(u,v) &= j(f(u+jv) = f(u+jv)j \\
&= f_0(u,v)j + jf_1(u,v)j = jf_0(u,v) + jf_1(u,v)j,
\end{aligned}
$$

and so $f_1(u,v) = -jf_1(u,v)j$. Thus $f_1(u,v)$ commutes with every $j \in \mathbb{S}$ and so also $f_1(u,v) \in \mathbb{R}$. Hence, f is intrinsic.

If on the other hand, (iii) holds, then for $q = u + jv \in U$ we have

$$\overline{2f_0(u,v)} = \overline{f(q) + f(\overline{q})} = f(\overline{q}) + f(q) = 2f_0(u,v)$$

and hence $f_0(u,v) \in \mathbb{R}$. We therefore also have

$$\begin{aligned} f_0(u,v) + jf_1(u,v) &= f(q) = \overline{f(\overline{q})} \\ &= \overline{f_0(u,v) - jf_1(u,v)} = f_0(u,v) + f_1(u,v)j, \end{aligned}$$

and so $jf_1(u,v) = f_1(u,v)j$. Since $j \in \mathbb{S}$ was arbitrary, we find that also $f_1(u,v) \in \mathbb{R}$ and that f is in turn intrinsic. □

If we restrict a slice hyperholomorphic function to one of the complex planes $\mathbb{C}_j$, then we obtain a function that is holomorphic in the usual sense.

Lemma 2.1.6 (The splitting lemma). *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $j, i \in \mathbb{S}$ with $i \perp j$. If $f \in \mathcal{SH}_L(U)$, then the restriction $f_j = f|_{U \cap \mathbb{C}_j}$ satisfies*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + j\frac{\partial}{\partial v} f_j(z)\right) = 0 \tag{2.7}$$

for all $z = u + jv \in U \cap \mathbb{C}_j$. Hence

$$f_j(z) = F_1(z) + F_2(z)i$$

with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$.

If $f \in \mathcal{SH}_R(U)$, then the restriction $f_j = f|_{U \cap \mathbb{C}_j}$ satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + \frac{\partial}{\partial v} f_j(z)j\right) = 0 \tag{2.8}$$

for all $z = u + jv \in U \cap \mathbb{C}_j$. Hence

$$f_j(z) = F_1(z) + iF_2(z)$$

with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$.

Proof. If $f = f_0 + jf_1$ is left slice hyperholomorphic, then

$$\begin{aligned} &\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + j\frac{\partial}{\partial v} f_j(z)\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial u} f_0(u,v) + j\frac{\partial}{\partial u} f_1(u,v) + j\frac{\partial}{\partial v} f_0(u,v) - \frac{\partial}{\partial v} f_1(u,v)\right) = 0 \end{aligned}$$

because f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5). Due to (2.3), we can write $f_j(z) = F_1(z) + F_2(z)i$ with $\mathbb{C}_j$-valued component functions F_1 and F_2. Since 1 and i are linearly independent over $\mathbb{C}_j$, the above identity applies componentwise, and hence F_1 and F_2 are holomorphic.

The right slice hyperholomorphic case can be proved similarly. □

Remark 2.1.7. The splitting lemma states that the restriction of every left slice hyperholomorphic function to a complex plane $\mathbb{C}_j$ is left holomorphic, i.e., it is a holomorphic function with values in the left vector space $\mathbb{H} = \mathbb{C}_j + \mathbb{C}_j i$ over $\mathbb{C}_j$. The restriction of a right slice hyperholomorphic function to a complex plane $\mathbb{C}_j$ is right holomorphic, i.e., it is a holomorphic function with values in the right vector space $\mathbb{H} = \mathbb{C}_j + i\mathbb{C}_j$ over $\mathbb{C}_j$.

Theorem 2.1.8 (Identity principle)**.** *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain, let $f, g : U \to \mathbb{H}$ be left (or right) slice hyperholomorphic, and set $\mathcal{Z} = \{q \in U : f(q) = g(q)\}$. If there exists $j \in \mathbb{S}$ such that $\mathcal{Z} \cap \mathbb{C}_j$ has an accumulation point in $U \cap \mathbb{C}_j$, then $f = g$.*

Proof. Assume that f and g are left slice hyperholomorphic and that $\mathcal{Z} \cap \mathbb{C}_j$ has an accumulation point in $U \cap \mathbb{C}_j$. We can furthermore assume that $g \equiv 0$. (Otherwise, we can simply replace f by $f - g$ and g by the constant zero function.) Since $U \cap \mathbb{C}_j$ is a domain in $\mathbb{C}_j$ and $f_j = f|_{U\cap\mathbb{C}_j}$ is an $\mathbb{H}$-valued (left) holomorphic function on this domain by Lemma 2.1.6, the identity theorem for holomorphic functions implies $f_j \equiv 0$. In particular, we have $f|_{U\cap\mathbb{R}} = f_j|_{U\cap\mathbb{R}} \equiv 0$.

If $i \in \mathbb{S}$ is now an arbitrary imaginary unit, then $f_i = f|_{U\cap\mathbb{C}_i}$ is again an $\mathbb{H}$-valued (left) holomorphic function on the domain $U \cap \mathbb{C}_i$ in $\mathbb{C}_i$. Since $f \equiv 0$ on $U \cap \mathbb{R} \neq \emptyset$ by the above arguments, the set of zeros of f_i has an accumulation point in $U \cap \mathbb{C}_i$. Hence, the identity theorem for holomorphic functions implies that also $f_i \equiv 0$ and in turn $f \equiv 0$ on all of U.

The right slice hyperholomorphic case follows with analogous arguments. □

The most important property of slice functions (and in particular for slice hyperholomorphic functions) is the structure formula, which is often also called representation formula.

Theorem 2.1.9 (The structure formula (or representation formula))**.** *Let $U \subseteq \mathbb{H}$ be axially symmetric and let $i \in \mathbb{S}$. A function $f : U \to \mathbb{H}$ is a left slice function on U if and only if for every $q = u + jv \in U$ we have*

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}ji\Big[f(\overline{z}) - f(z)\Big] \tag{2.9}$$

with $z = u + iv$. A function $f : U \to \mathbb{H}$ is a right slice function on U if and only if for every $q = u + jv \in U$ we have

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}\Big[f(\overline{z}) - f(z)\Big]ij \tag{2.10}$$

with $z = u + iv$.

Proof. For every left slice function f on U, we have

$$\begin{aligned} f(z) &= f(u + iv) = f_0(u, v) + if_1(u, v),\\ f(\overline{z}) &= f(u - iv) = f_0(u, v) - if_1(u, v), \end{aligned}$$

with functions f_0 and f_1 that satisfy the compatibility condition (2.4). Adding and subtracting these two equations, we get

$$f_0(u,v) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big], \quad f_1(u,v) = \frac{1}{2}i\Big[f(\overline{z}) - f(z)\Big]. \tag{2.11}$$

Since $f(q) = f_0(u,v) + jf_1(u,v)$, we obtain (2.9). If, on the other hand, f satisfies (2.9), then $f(q) = f_0(u,v) + jf_1(u,v)$ with f_0 and f_1 as in (2.11). Obviously f_0 and f_1 satisfy the compatibility condition (2.4), and hence f is a left slice function.

The statement about right slice functions can be shown with similar arguments. □

Remark 2.1.10. It is sometimes useful to rewrite (2.9) as

$$f(q) = \frac{1}{2}(1 - ij)f(z) + \frac{1}{2}(1 + ij)f(\overline{z})$$

and (2.10) as

$$f(q) = f(z)(1 - ij)\frac{1}{2} + f(\overline{z})(1 + ij)\frac{1}{2}.$$

As a consequence of the structure formula, every holomorphic function that is defined on a suitable open set in $\mathbb{C}_j$ has a slice hyperholomorphic extension.

Lemma 2.1.11. *Let $O \subset \mathbb{C}_j$ be open and symmetric with respect to the real axis. We call the set $[O] = \bigcup_{z \in O}[z]$ the axially symmetric hull of O.*

(i) *Every function $f : O \to \mathbb{H}$ has a unique extension $\mathrm{ext}_L(f)$ to a left slice function on $[O]$ and a unique extension $\mathrm{ext}_R(f)$ to a right slice function on $[O]$.*

(ii) *If $f : O \to \mathbb{H}$ is left holomorphic, i.e., it satisfies (2.7), then $\mathrm{ext}_L(f)$ is left slice hyperholomorphic.*

(iii) *If f is right holomorphic, i.e., it satisfies (2.8), then $\mathrm{ext}_R(f)$ is right slice hyperholomorphic.*

Proof. The left and right slice extensions $\mathrm{ext}_L(f)$ and $\mathrm{ext}_R(f)$ are obviously given by (2.9) resp. (2.10). Due to Theorem 2.1.9, they are also unique.

Assume that f is left holomorphic. Then $\mathrm{ext}_L(f)(q) = f_0(u,v) + if_1(u,v)$ for $q = u + jv$, with

$$f_0(u,v) = \frac{1}{2}\left[f(u - jv) + f(u + jv)\right]$$

and

$$f_1(u,v) = \frac{1}{2}j\left[f(u - jv) - f(u + jv)\right].$$

It remains to show that this actually defines a left slice hyperholomorphic function, i.e., that f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5). Because of (2.7), we have

$$\begin{aligned}\frac{\partial}{\partial u} f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial u} f(u-jv) + \frac{\partial}{\partial u} f(u+jv)\right] \\ &= \frac{1}{2}\left[j\frac{\partial}{\partial v} f(u-jv) - j\frac{\partial}{\partial v} f(u+jv)\right] = \frac{\partial}{\partial v} f_1(u,v).\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\frac{\partial}{\partial v} f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial v} f(u-jv) + \frac{\partial}{\partial v} f(u+jv)\right] \\ &= \frac{1}{2}\left[-j\frac{\partial}{\partial u} f(u-jv) + j\frac{\partial}{\partial u} f(u+jv)\right] = -\frac{\partial}{\partial u} f_1(u,v).\end{aligned}$$

Thus $\mathrm{ext}_L(f)$ is actually left slice hyperholomorphic. The right slice hyperholomorphic case can be shown with analogous arguments. □

Slice hyperholomorphic functions admit a special kind of derivative, which again yields a slice hyperholomorphic function.

Definition 2.1.12. Let $f : U \subseteq \mathbb{H} \to \mathbb{H}$ and let $q = u + jv \in U$. If q is not real, then we say that f *admits a left slice derivative in* q if

$$\partial_S f(q) := \lim_{p\to q,\, p\in\mathbb{C}_j} (p-q)^{-1}(f_j(p) - f_j(q)) \tag{2.12}$$

exists and is finite. If q is real, then we say that f *admits a left slice derivative in* q if (2.12) exists for every $j \in \mathbb{S}$.

Similarly, we say that f *admits a right slice derivative at a nonreal point* $q = u + jv \in U$ if

$$\partial_S f(q) := \lim_{p\to q,\, p\in\mathbb{C}_j} (f_j(p) - f_j(q))(p-q)^{-1} \tag{2.13}$$

exists and is finite, and we say that f *admits a right slice derivative at a real point* $q \in U$ if (2.13) exists and is finite for every $j \in \mathbb{S}$.

Remark 2.1.13. Observe that $\partial_S f(q)$ is uniquely defined and independent of the choice of $j \in \mathbb{S}$ even if q is real. If f admits a slice derivative, then f_j is $\mathbb{C}_j$-complex left resp. right differentiable, and we obtain

$$\partial_S f(q) = f_j'(q) = \frac{\partial}{\partial u} f_j(q) = \frac{\partial}{\partial u} f(q), \quad q = u + jv. \tag{2.14}$$

Proposition 2.1.14. *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $f : U \to \mathbb{H}$ be a real differentiable function.*

(i) *If $f(q) = f_0(u,v) + jf_1(u,v)$ is left (or right) slice hyperholomorphic, then it admits a left (resp. right) slice derivative and $\partial_S f$ is again left (resp. right) slice hyperholomorphic on U.*

(ii) *If f is a left (or right) slice function that admits a left (resp. right) slice derivative, then f is left (resp. right) slice hyperholomorphic.*

(iii) *If U is a slice domain, then every function that admits a left (resp. right) slice derivative is left (resp. right) slice hyperholomorphic.*

Proof. If f is a left slice hyperholomorphic function on U and $q = u + jv \in U$, then its restriction to the complex plane $\mathbb{C}_j$ can be written as $f_j(q) = F_1(q) + F_2(q)i$ for $i \in \mathbb{S}$ with $i \perp j$. By Lemma 2.1.6, the component functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$ are holomorphic, and hence

$$\begin{aligned}
&\lim_{p\to q,\, p\in\mathbb{C}_j} (p-q)^{-1}(f_j(p) - f_j(q)) \\
&= \lim_{p\to q,\, p\in\mathbb{C}_j} (p-q)^{-1}(F_1(p) + F_2(p)i - F_1(p) - F_2(q)i) \\
&= F_1'(q) + F_2'(q)i
\end{aligned}$$

exists. Therefore, f admits a left slice derivative. Moreover, this slice derivative coincides with the derivative with respect to the real part of the quaternion by (2.14), and hence

$$\partial_S f(q) = \frac{\partial}{\partial u} f(q) = \frac{\partial}{\partial u} f_0(u,v) + j \frac{\partial}{\partial u} f_1(u,v), \quad q = u + jv.$$

The functions $\frac{\partial}{\partial u} f_0(u,v)$ and $\frac{\partial}{\partial u} f_1(u,v)$ obviously satisfy the compatibility condition (2.4). Since f_0 and f_1 satisfy the Cauchy–Riemann equations, they are infinitely differentiable. Hence $\frac{\partial}{\partial u}$, and $\frac{\partial}{\partial v}$ commute with $\frac{\partial}{\partial u}$ and we obtain that also $\frac{\partial}{\partial u} f_0(u,v)$ and $\frac{\partial}{\partial u} f_1(u,v)$ satisfy the Cauchy–Riemann equations (2.5). Thus $\partial_S f$ is left slice hyperholomorphic too.

If, on the other hand, $f(q) = f_0(u,v) + j f_1(u,v)$ is a left slice function that admits a left slice derivative, we choose $j \in \mathbb{S}$. Then f_j is an $\mathbb{H}$-valued left holomorphic function on $U \cap \mathbb{C}_j$. By Lemma 2.1.11, the left slice extension $\operatorname{ext}_L(f_j)$ of f_j is therefore a left slice hyperholomorphic extension of f_j. Since f is already a left slice function, we find that $f = \operatorname{ext}_L(f_j)$, and so f is left slice hyperholomorphic.

If, finally, U is an axially symmetric slice domain and f is an arbitrary function on U that admits a left slice derivative, then we can again choose an arbitrary imaginary unit $j \in \mathbb{S}$ and find that f_j is left holomorphic. We set $\tilde{f} = \operatorname{ext}_L(f_j)$ and $g = f - \tilde{f}$. Obviously $g \equiv 0$ on $U \cap \mathbb{C}_j$. Moreover, g admits a left slice derivative, since f and $\tilde{f}$ both admit a left slice derivative. For every $i \in \mathbb{S}$, the restriction $g_i = g|_{U\cap\mathbb{C}_i}$ is a (left) holomorphic function on the domain $U \cap \mathbb{C}_i$ in $\mathbb{C}_i$. Moreover, $g|_{U\cap\mathbb{R}} \equiv 0$, and so the set of zeros of g_i has an accumulation point in $U \cap \mathbb{C}_i$. By the identity theorem for holomorphic functions, we find that $g_i \equiv 0$, and in turn $g \equiv 0$ because $i \in \mathbb{S}$ was arbitrary. Therefore, $f = \tilde{f} = \operatorname{ext}_L(f_j)$ is left slice hyperholomorphic.

The right slice hyperholomorphic case can be shown by analogous arguments. □

Important examples of slice hyperholomorphic functions are power series in the quaternionic variable: power series of the form $\sum_{n=0}^{+\infty} q^n a_n$ with $a_n \in \mathbb{H}$ are left slice hyperholomorphic, and power series of the form $\sum_{n=0}^{+\infty} a_n q^n$ are right slice hyperholomorphic. Such a power series is intrinsic if and only if the coefficients a_n are real. Conversely, every slice hyperholomorphic function can be expanded at any real point into a power series due to the splitting lemma.

Theorem 2.1.15. *Let $a \in \mathbb{R}$, let $r > 0$, and let $B_r(a) = \{q \in \mathbb{H} : |q - a| < r\}$. If $f \in \mathcal{SH}_L(B_r(a))$, then*

$$f(q) = \sum_{n=0}^{+\infty} (q-a)^n \frac{1}{n!} \partial_S^n f(a) \quad \forall q = u + jv \in B_r(a). \tag{2.15}$$

If, on the other hand, $f \in \mathcal{SH}_R(B_r(a))$, then

$$f(q) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\partial_S^n f(a)\right) (q-a)^n \quad \forall q = u + jv \in B_r(a).$$

Proof. Let $f \in \mathcal{SH}_L(B_r(a))$ and $q = u + jv \in B_r(a)$. By Lemma 2.1.6, the function $f_j = f|_{B_r(a)\cap\mathbb{C}_j}$ is left holomorphic on $B_r(a)$ and can hence be expanded into a power series. We obtain

$$f(q) = f_j(q) = \sum_{n=0}^{+\infty} (q-a)^n \frac{1}{n!} f_j^{(n)}(a).$$

But due to (2.14), we have

$$f_j^{(n)}(a) = \frac{\partial^n}{\partial^n u} f_j(a) = \frac{\partial^n}{\partial u^n} f(a) = \partial_S^n f(a).$$

The coefficients in the power series expansion are hence independent of the complex plane in which they are computed, and (2.40) holds. The right slice hyperholomorphic case follows with similar arguments. □

As pointed out above, the product of two slice hyperholomorphic functions is not slice hyperholomorphic unless the factor on the appropriate side is intrinsic. However, there exists a regularized product that preserves slice hyperholomorphicity.

Definition 2.1.16. For $f = f_0 + jf_1, g = g_0 + jg_1 \in \mathcal{SH}_L(U)$, we define their *left slice hyperholomorphic product* as

$$f *_L g = (f_0 g_0 - f_1 g_1) + j(f_0 g_1 + f_1 g_0).$$

For $f = f_0 + f_1 j, g = g_0 + g_1 j \in \mathcal{SH}_R(U)$, we define their *right slice hyperholomorphic product* as

$$f *_R g = (f_0 g_0 - f_1 g_1) + (f_0 g_1 + f_1 g_0) j.$$

Remark 2.1.17. The slice hyperholomorphic product is associative and distributive, but it is in general not commutative. If f is intrinsic, then $f *_L g$ coincides with the pointwise product fg and

$$f *_L g = fg = g *_L f. \tag{2.16}$$

Similarly, if g is intrinsic, then $f *_R g$ coincides with the pointwise product fg and

$$f *_R g = fg = g *_R f. \tag{2.17}$$

Example 2.1.18. If $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ are two left slice hyperholomorphic power series, then their slice hyperholomorphic product equals the usual product of formal power series with coefficients in a noncommutative ring:

$$\left(\sum_{n=0}^{+\infty} q^n a_n\right) *_L \left(\sum_{n=0}^{+\infty} q^n b_n\right) = (f *_L g)(q) = \sum_{n=0}^{+\infty} q^n \sum_{k=0}^{n} a_k b_{n-k}. \tag{2.18}$$

Similarly, we have for right slice hyperholomorphic power series that

$$\left(\sum_{n=0}^{+\infty} a_n q^n\right) *_R \left(\sum_{n=0}^{+\infty} b_n q^n\right) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) q^n. \tag{2.19}$$

Definition 2.1.19. We define for $f = f_0 + jf_1 \in \mathcal{SH}_L(U)$ its slice hyperholomorphic conjugate $f^c = \overline{f_0} + j\overline{f_1}$ and its symmetrization $f^s = f *_L f^c = f^c *_L f$. Similarly, we define for $f = f_0 + f_1 j \in \mathcal{SH}_R(U)$ its slice hyperholomorphic conjugate as $f^c = \overline{f_0} + \overline{f_1} j$ and its symmetrization as $f^s = f *_R f^c = f^c *_R f$.

The symmetrization of a left slice hyperholomorphic function $f = f_0 + jf_1$ is explicitly given by

$$f^s = |f_0|^2 - |f_1|^2 + j2\mathrm{Re}\left(f_0\overline{f_1}\right).$$

Hence it is an intrinsic function. It is $f^s(q) = 0$ if and only if $f(\tilde{q}) = 0$ for some $\tilde{q} \in [q]$. Furthermore, one has

$$f^c(q) = \overline{f_0(q_0, q_1)} + j_q\overline{f_1(q_0, q_1)} = \overline{f_0(q_0, q_1)} + \overline{f_1(q_0, q_1)(-j_q)} = \overline{f(\overline{q})}, \tag{2.20}$$

and an easy computation shows that

$$f *_L g(q) = f(q)g\left(f(q)^{-1}qf(q)\right) \quad \text{if } f(q) \neq 0. \tag{2.21}$$

For $f(q) \neq 0$, one has

$$\begin{aligned} f^s(q) &= f(q)f^c\left(f(q)^{-1}qf(q)\right) \\ &= f(q)\overline{f\left(\overline{f(q)^{-1}qf(q)}\right)} = f(q)\overline{f\left(f(q)^{-1}\overline{q}f(q)\right)}. \end{aligned} \tag{2.22}$$

Similar computations hold in the right slice hyperholomorphic case. Finally, if f is intrinsic, then $f^c(q) = f(q)$ and $f^s(q) = |f(q)|^2$.

As an immediate consequence of Definition 2.1.19 and the above discussion we obtain the following corollary.

Corollary 2.1.20. *The following statements are true:*

(i) *For $f \in \mathcal{SH}_L(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_L}, which satisfies $f^{-*_L} *_L f = f *_L f^{-*_L} = 1$, is given by*

$$f^{-*_L} = (f^s)^{-1} *_L f^c = (f^s)^{-1} f^c,$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(ii) *For $f \in \mathcal{SH}_R(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_R}, which satisfies $f^{-*_R} *_R f = f *_R f^{-*_R} = 1$, is given by*

$$f^{-*_R} = f^c *_R (f^s)^{-1} = f^c (f^s)^{-1},$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(iii) *If $f \in \mathcal{N}(U)$ with $f \not\equiv 0$, then $f^{-*_L} = f^{-*_R} = f^{-1}$.*

The modulus $|f^{-*_L}|$ is in a certain sense comparable to $1/|f|$. Since f^s is intrinsic, we have $|f^s(q)| = |f^s(\tilde{q})|$ for every $\tilde{q} \in [q]$. Since $f(q)qf(q)^{-1} \in [q]$ by Lemma 2.0.1, we find for $f(q) \neq 0$, because of (2.22), that

$$\begin{aligned} |f^s(q)| &= \left| f^s\left(f(q)qf(q)^{-1}\right)\right| \\ &= \left| f\left(f(q)qf(q)^{-1}\right) \overline{f\left(\overline{q}\right)}\right| = \left| f\left(f(q)qf(q)^{-1}\right)\right| \left| f\left(\overline{q}\right)\right|. \end{aligned}$$

Therefore, we have, because of (2.20), that

$$\begin{aligned} \left|f^{-*_L}(q)\right| &= \left|f^s(q)^{-1}\right| \left|f^c(q)\right| \\ &= \frac{1}{\left|f\left(f(q)qf(q)^{-1}\right)\right| \left|f\left(\overline{q}\right)\right|} \left|f\left(\overline{q}\right)\right| = \frac{1}{\left|f\left(f(q)\overline{q}f(q)^{-1}\right)\right|}, \end{aligned}$$

and so

$$\left|f^{-*_L}(q)\right| = \frac{1}{|f(\tilde{q})|} \quad \text{with } \tilde{q} = f(q)\overline{q}f(q)^{-1} \in [q]. \tag{2.23}$$

An analogous estimate holds for the slice hyperholomorphic inverse of a right slice hyperholomorphic function.

Slice hyperholomorphic functions satisfy a version of Cauchy's integral theorem and a Cauchy formula with a slice hyperholomorphic integral kernel.

Theorem 2.1.21 (Cauchy's integral theorem)**.** *Let $U \subset \mathbb{H}$ be open, let $j \in \mathbb{S}$, and let $f \in \mathcal{SH}_L(U)$ and $g \in \mathcal{SH}_R(U)$. Moreover, let $D_j \subset U \cap \mathbb{C}_j$ be an open and bounded subset of the complex plane $\mathbb{C}_j$ with $\overline{D_j} \subset U \cap \mathbb{C}_j$ such that ∂D_j is a finite union of piecewise continuously differentiable Jordan curves. Then*

$$\int_{\partial D_j} g(s)\, ds_j\, f(s) = 0,$$

where $ds_j = ds(-j)$.

Proof. If we choose $i \in \mathbb{S}$ with $i \perp j$, then we can write $f(z) = F_1(z) + F_2(z)i$ and $g(z) = G_1(z) + iG_2(z)$ for $z \in U \cap \mathbb{C}_j$ with holomorphic component functions $F_1, F_2, G_1, G_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$. By the Cauchy integral theorem for holomorphic functions, we hence obtain

$$\begin{aligned}
&\int_{\partial D_j} g(s)\, ds_j\, f(s) \\
&= \int_{\partial D_j} G_1(s)\, ds_j\, F_1(s) + \left(\int_{\partial D_j} G_1(s)\, ds_j\, F_2(s) \right) i \\
&\quad + i \int_{\partial D_j} G_2(s)\, ds_j\, F_1(s) + i \left(\int_{\partial D_j} G_1(s)\, ds_j\, F_2(s) \right) i = 0.
\end{aligned}$$ □

In order to determine the left and right slice hyperholomorphic Cauchy kernels, we start from an analogy with the classical complex case. We consider the series expansion of the complex Cauchy kernel and determine its closed form under the assumption that s and q are quaternions that do not commute.

Theorem 2.1.22. *Let $q, s \in \mathbb{H}$ with $|q| < |s|$. Then*

$$\sum_{n=0}^{+\infty} q^n s^{-n-1} = -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}) \tag{2.24}$$

and

$$\sum_{n=0}^{+\infty} s^{-n-1} q^n = -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \tag{2.25}$$

Proof. We prove only (2.24), since (2.25) follows by analogous arguments. Due to the identities $2\mathrm{Re}(s) = s + \overline{s}$ and $|s|^2 = s\overline{s}$, we have

$$\begin{aligned}
&(q^2 - 2\mathrm{Re}(s)q + |s|^2)\sum_{n=0}^{+\infty} q^n s^{-n-1} = \\
&= \sum_{n=0}^{+\infty} q^{n+2}s^{-n-1} - \sum_{n=0}^{+\infty} q^{n+1}s^{-n-1}2\mathrm{Re}(s) + \sum_{n=0}^{+\infty} q^n s^{-n-1}|s|^2 \\
&= \sum_{n=1}^{+\infty} q^{n+1}s^{-n} - \sum_{n=0}^{+\infty} q^{n+1}s^{-n} \\
&\quad - \sum_{n=0}^{+\infty} q^{n+1}s^{-n-1}\overline{s} + \sum_{n=0}^{+\infty} q^n s^{-n}\overline{s} = -q + \overline{s}.
\end{aligned}$$

Multiplication by $(q^2 - 2\mathrm{Re}(s)q - |s|^2)^{-1}$ from the left yields (2.24). □

Definition 2.1.23. We define the *left slice hyperholomorphic Cauchy kernel* as

$$S_L^{-1}(s,q) := -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}), \quad q \notin [s],$$

and the *right slice hyperholomorphic Cauchy kernel* as

$$S_R^{-1}(s,q) := -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}, \quad q \notin [s].$$

The slice hyperholomorphic Cauchy kernels $S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)$ can be written in two different ways, as the next proposition shows.

Proposition 2.1.24. *If $q, s \in \mathbb{H}$ with $q \notin [s]$, then*

$$-(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \overline{s}) = (s - \overline{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1} \tag{2.26}$$

and

$$(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(s - \overline{q}) = -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \tag{2.27}$$

Proof. Due to the identities $|q| = q\overline{q} = \overline{q}q$ and $2\mathrm{Re}(q) = \overline{q} + q$, we have

$$\begin{aligned}
&-(q - \overline{s})(s^2 - 2\mathrm{Re}(q)s + |q|^2) \\
&= -qs^2 + q(q + \overline{q})s - q^2\overline{q} + \overline{s}s^2 - \overline{s}s(q + \overline{q}) + \overline{s}q\overline{q} \\
&= q^2(s - \overline{q}) + |s|^2(s - \overline{q}) - qs^2 + q\overline{q}s - \overline{s}sq + \overline{s}q\overline{q}.
\end{aligned}$$

Since

$$\begin{aligned}
&-qs^2 + q\overline{q}s - \overline{s}sq + \overline{s}q\overline{q} = -qs^2 + |q|^2 s - |s|^2 q + \overline{s}q\overline{q} \\
&= -qs^2 + s|q|^2 - q|s|^2 + \overline{s}q\overline{q} = -qs^2 + sq\overline{q} - q\overline{s}s + \overline{s}q\overline{q} \\
&= -q(s + \overline{s})s + (s + \overline{s})q\overline{q} = -2\mathrm{Re}(s)q(s - \overline{q}),
\end{aligned}$$

we further conclude that

$$-(q-\overline{s})(s^2-2\mathrm{Re}(q)s+|q|^2)=(q^2-2\mathrm{Re}(s)q+|s|^2)(s-\overline{q}).$$

Multiplying this identity by $(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}$ on the right and by $(q^2-2\mathrm{Re}(s)q+|s|^2)^{-1}$ on the left, we obtain (2.26). Exchanging the roles of q and s and multiplying by -1 then yields (2.27). □

Proposition 2.1.24 justifies the following definition.

Definition 2.1.25. Let $q,s\in\mathbb{H}$ with $q\notin[s]$.

- We say that $S_L^{-1}(s,q)$ *is written in the form I* if

$$S_L^{-1}(s,q):=-(q^2-2\mathrm{Re}(s)q+|s|^2)^{-1}(q-\overline{s}).$$

- We say that $S_L^{-1}(s,q)$ *is written in the form II* if

$$S_L^{-1}(s,q):=(s-\overline{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}.$$

- We say that $S_R^{-1}(s,q)$ *is written in the form I* if

$$S_R^{-1}(s,q):=-(q-\overline{s})(q^2-2\mathrm{Re}(s)q+|s|^2)^{-1}.$$

- We say that $S_R^{-1}(s,q)$ *is written in the form II* if

$$S_R^{-1}(s,q):=(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}(s-\overline{q}).$$

Corollary 2.1.26. *For $q,s\in\mathbb{H}$ with $s\notin[q]$, we have*

$$S_L^{-1}(s,q)=-S_R^{-1}(q,s).$$

Lemma 2.1.27. *Let $q,s\in\mathbb{H}$ with $s\notin[q]$.*

The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,q)$ is left slice hyperholomorphic in q and right slice hyperholomorphic in s.

The right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,q)$ is left slice hyperholomorphic in s and right slice hyperholomorphic in q.

Proof. Let $q=u+jv$. We write $S_L^{-1}(s,q)$ in the form II, i.e.,

$$S_L^{-1}(s,q)=(s-\overline{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}.$$

Then $S_L^{-1}(s,q)=f_0(u,v)+jf_1(u,v)$ with

$$f_0(u,v)=(s-u)(s^2-2us+u^2+v^2)^{-1}, \tag{2.28}$$

$$f_1(u,v)=v(s^2-2us+u^2+v^2)^{-1}. \tag{2.29}$$

Obviously, f_0 and f_1 satisfy the compatibility condition (2.4). Moreover,

$$\begin{aligned}\frac{\partial}{\partial u} f_0(u,v) &= -(s^2 - 2us + u^2 + v^2)^{-1} \\ &\quad - (s-u)(s^2 - 2us + u^2 + v^2)^{-2}(-2s + 2u) \\ &= (s^2 - 2us + u^2 + v^2)^{-2}\left((s-u)^2 - v^2\right), \\ \frac{\partial}{\partial v} f_0(u,v) &= -(s-u)(s^2 - 2us + u^2 + v^2)^{-2} 2v,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial u} f_1(u,v) &= -v(s^2 - 2us + u^2 + v^2)^{-2} 2(-s + u), \\ \frac{\partial}{\partial v} f_1(u,v) &= (s^2 - 2us + u^2 + v^2)^{-1} \\ &\quad - v(s^2 - 2us + u^2 + v^2)^{-2} 2v \\ &= (s^2 - 2us + u^2 + v^2)^{-2}\left((s-u)^2 - v^2\right).\end{aligned}$$

Hence they also satisfy the Cauchy–Riemann equations (2.5), and so the mapping $q \mapsto S_L^{-1}(s,q)$ is left slice hyperholomorphic.

In order to show that $S_L^{-1}(s,q)$ is right slice hyperholomorphic in s, we write $S_L^{-1}(s,q)$ in form I, i.e.,

$$S_L^{-1}(s,q) = -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}).$$

For $s = u + jv$, we hence have $S_L^{-1}(s,q) = f_0(u,v) + f_1(u,v)j$ with

$$\begin{aligned}f_0(u,v) &= (q^2 - 2uq + u^2 + v^2)^{-1}(q - u), \\ f_1(u,v) &= (q^2 - 2uq + u^2 + v^2)^{-1} v.\end{aligned}$$

But these are exactly the functions (2.28) and (2.29) in which s is replaced by q. As we showed above, they satisfy the compatibility condition (2.4) and the Cauchy–Riemann equations (2.5), and so the mapping $s \mapsto S_L^{-1}(s,q)$ is right slice hyperholomorphic.

The properties of the right slice hyperholomorphic Cauchy kernel follow immediately, since $S_R^{-1}(s,q) = -S_L^{-1}(q,s)$ by Corollary 2.1.26. □

Lemma 2.1.28. *If s and q commute, then the left and the right slice hyperholomorphic Cauchy kernels reduce to the complex Cauchy kernel, i.e.,*

$$S_L^{-1}(s,q) = (s-q)^{-1} = S_R^{-1}(s,q) \quad \textit{if } sq = qs.$$

Proof. If q and s commute, then

$$q^2 - 2\mathrm{Re}(s)q + |s|^2 = q^2 - (s + \overline{s})q + s\overline{s} = (q - s)(q - \overline{s}).$$

Hence, we have

$$\begin{aligned} S_L^{-1}(s,q) &= -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q-\overline{s}) \\ &= -(q-s)^{-1}(q-\overline{s})^{-1}(q-\overline{s}) = (s-q)^{-1}, \end{aligned}$$

and similarly also

$$\begin{aligned} S_R^{-1}(s,q) &= -(q-\overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} \\ &= -(\overline{s}-q)(q-\overline{s})^{-1}(q-s)^{-1} = (s-q)^{-1}. \end{aligned}$$ □

Remark 2.1.29. Observe that left and right slice hyperholomorphic functions satisfy Cauchy formulas with different kernels. This is different from what happens for Fueter regular functions, where both left and right Fueter regular functions satisfy a Cauchy formula with the same integral kernel.

Definition 2.1.30 (Slice Cauchy domain)**.** An axially symmetric open set $U \subset \mathbb{H}$ is called a *slice Cauchy domain* if $U \cap \mathbb{C}_j$ is a Cauchy domain in $\mathbb{C}_j$ for every $j \in \mathbb{S}$. More precisely, U is a slice Cauchy domain if for every $j \in \mathbb{S}$ the boundary $\partial(U \cap \mathbb{C}_j)$ of $U \cap \mathbb{C}_j$ is the union a finite number of nonintersecting piecewise continuously differentiable Jordan curves in $\mathbb{C}_j$.

Remark 2.1.31. Observe that every slice Cauchy domain has only finitely many components (i.e., maximal connected subsets). Moreover, at most one of them is unbounded, and if there exists an unbounded component, then it contains a neighborhood of ∞ in $\mathbb{H}$.

Theorem 2.1.32 (The Cauchy formulas)**.** *Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$, and set $ds_j = ds(-j)$. If f is a (left) slice hyperholomorphic function on a set that contains $\overline{U}$, then*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s), \quad \text{for every } q \in U. \tag{2.30}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q), \quad \text{for every } q \in U. \tag{2.31}$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. Assume that f is left slice hyperholomorphic on a set that contains $\overline{U}$ and let $q = u + iv \in U$. Since $S_L^{-1}(s,q)$ is left slice hyperholomorphic in q, we deduce from Theorem 2.1.9 that with $p = u + jv$,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s)\\
&= \frac{1}{2}(1-ij)\left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,p)\,ds_j\,f(s)\right)\\
&\quad + \frac{1}{2}(1+ij)\left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,\overline{p})ds_j f(s)\right)\\
&= \frac{1}{2}(1-ij)\left(\frac{1}{2\pi j}\int_{\partial(U\cap\mathbb{C}_j)} (p-s)^{-1}\,ds\,f(s)\right)\\
&\quad + \frac{1}{2}(1+ij)\left(\frac{1}{2\pi j}\int_{\partial(U\cap\mathbb{C}_j)} (\overline{p}-s)^{-1}\,ds\,f(s)\right),
\end{aligned}$$

where the last identity follows from Lemma 2.1.28 because p, s, and j all belong to $\mathbb{C}_j$ and hence commute mutually. By Lemma 2.1.6, the restriction of f to $\mathbb{C}_j$ is left holomorphic. Hence it satisfies the classical Cauchy formula. Together with Theorem 2.1.9, this implies that

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s) = \frac{1}{2}(1-ij)f(p) + \frac{1}{2}(1+ij)f(\overline{p}) = f(q).$$

Since $f(q)$ is independent of U and $j \in \mathbb{S}$, the integral in (4.39) is obviously independent of U and j.

The right slice hyperholomorphic case is again shown by analogous arguments. □

Theorem 2.1.33 (Cauchy formulas on unbounded slice Cauchy domains)**.** *Let $U \subset \mathbb{H}$ be an unbounded slice Cauchy domain and let $j \in \mathbb{S}$. If $f \in \mathcal{SH}_L(\overline{U})$ and $f(\infty) := \lim_{|q|\to\infty} f(q)$ exists, then*

$$f(q) = f(\infty) + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s) \quad \textit{for every } q \in U.$$

If $f \in \mathcal{SH}_R(\overline{U})$ and $f(\infty) := \lim_{|q|\to\infty} f(q)$ exists, then

$$f(q) = f(\infty) + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,q) \quad \textit{for every } q \in U.$$

Proof. Let $f \in \mathcal{SH}_L(\overline{U})$ such that $f(\infty) := \lim_{|q|\to\infty} f(q)$ exists and let $q \in U$. For sufficiently large $r > 0$, the set $U_r := U \cap B_r(0)$ is a bounded slice Cauchy domain with $q \in U_r$ and $\mathbb{H} \setminus U_r \subset U$. By

$$\begin{aligned} f(q) &= \frac{1}{2\pi} \int_{\partial(U_r \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s) + \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s). \end{aligned}$$

Theorem 2.1.21 implies that we can vary r without changing the value of the second integral. Letting r tend to infinity, we find that it equals $f(\infty)$, and we obtain the statement. □

Finally, just like holomorphic functions, slice hyperholomorphic functions can be approximated by rational funcitons.

Definition 2.1.34. A function r is called *left rational* if it is of the form $r(q) = P(q)^{-1}Q(q)$ with polynomials $P \in \mathcal{N}(\mathbb{H})$ and $Q \in \mathcal{SH}_L(\mathbb{H})$.

A function r is called *right rational* if it is of the form $r(q) = Q(q)P(q)^{-1}$ with polynomials $P \in \mathcal{N}(\mathbb{H})$ and $Q \in \mathcal{SH}_R(\mathbb{H})$.

Finally, a function r is called *intrinsic rational* if it is of the form $r(q) = P(q)^{-1}Q(q)$ with two polynomials $P, Q \in \mathcal{N}(\mathbb{H})$.

Remark 2.1.35. The requirement that P be intrinsic is necessary because the function P^{-1} is otherwise not slice hyperholomorphic; cf. Theorem 2.1.4.

Corollary 2.1.36. *Let $f \in \mathcal{SH}_L(U)$, let $j, i \in \mathbb{S}$ with $i \perp j$, and write $f_j = F_1 + F_2 i$ with holomorphic components $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$ according to Lemma 2.1.6. Then f is left rational if and only if F_1 and F_2 are rational functions on $\mathbb{C}_j$.*

Similarly, if $f \in \mathcal{SH}_R(U)$ and we write $f_j = F_1 + iF_2$ with holomorphic components F_1 and F_2 according to Lemma 2.1.6, then f is right rational if and only if F_1, F_2 are rational functions on $\mathbb{C}_j$.

Proof. Let $f \in \mathcal{SH}_L(U)$ be left rational, i.e., $f(q) = P(q)^{-1}Q(q)$ for some intrinsic polynomial $P(q) = \sum_{n=0}^{N} q^n a_n$ with $a_n \in \mathbb{R}$ and some left slice hyperholomorphic polynomial $Q(q) = \sum_{m=0}^{M} q^m b_m$ with $b_m \in \mathbb{H}$. If we write $b_m = b_{m,1} + b_{m,2} i$ with $b_{m,1}, b_{m,2} \in \mathbb{C}_j$ and set $Q_1(q) = \sum_{m=0}^{M} q^m b_{m,1}$ and $Q_2(q) = \sum_{m=0}^{M} q^m b_{m,2}$ for $q \in U \cap \mathbb{C}_j$, we obtain $Q = Q_1 + Q_2 i$ and in turn

$$f_j(q) = P(q)^{-1}Q(q) = P(q)^{-1}Q_1(q) + P(q)^{-1}Q_2(q)i.$$

Since P has real coefficients and Q_1 and Q_2 have coefficients in $\mathbb{C}_j$, they are polynomials on $\mathbb{C}_j$, and hence $P^{-1}Q_1$ and $P^{-1}Q_2$ are rational functions on $\mathbb{C}_j$. Since furthermore, 1 and i are linearly independent over $\mathbb{C}_j$, we obtain $F_1 = P^{-1}Q_1$ and $F_2 = P^{-1}Q_2$.

In order to show the converse implication, let us assume that $F_1 = P_1^{-1}Q_1$ and $F_2 = P_2 Q_2$ are rational functions. If $P_1(q) = \sum_{n=0}^{N} q^n a_{n,1}$ with $a_{n,1}$ in $\mathbb{C}_j$, then $\overline{P_1(\overline{q})}$ is the polynomial $P_1(q) = \sum_{n=0}^{N} q^n \overline{a_{n,1}}$. The product $\tilde{P}_1(q) :=$

$P_1(q)\overline{P_1(\overline{q})}$ is again a polynomial, and since it satisfies $\tilde{P}_1(\overline{q}) = \overline{\tilde{P}_1(q)}$, it has real coefficients. Similarly, the function $\tilde{P}_2(q) := P_2(q)\overline{P_2(\overline{q})}$ is also a polynomial with real coefficients, and we have

$$F_1(q) = \tilde{P}_1(q)^{-1}\overline{P_1(\overline{q})}Q_1(q), \quad F_2(q) = \tilde{P}_2(q)^{-1}\overline{P_2(\overline{q})}Q_2(q),$$

and in turn

$$\begin{aligned} f_j(q) &= F_1(q) + F_2(q)i \\ &= \tilde{P}_1(q)^{-1}\tilde{P}_2(q)^{-1}\left(\tilde{P}_2(q)\overline{P_1(\overline{q})}Q_1(q) + \tilde{P}_1(q)\overline{P_2(\overline{q})}Q_2(q)i\right). \end{aligned}$$

The function $P(q) := \tilde{P}_1(q)\tilde{P}_2(q)$ is a polynomial with real coefficients on $\mathbb{C}_j$, the function

$$Q(q) := \tilde{P}_2(q)\overline{P_1(\overline{q})}Q_1(q) + \tilde{P}_1(q)\overline{P_2(\overline{q})}Q_2(q)i$$

is a polynomial with quaternionic coefficients on $\mathbb{C}_j$, and by construction, $f_j(q) = P(q)^{-1}Q(q)$.

Replacing the complex variable by a quaternionic variable, we can extend P to an intrinsic polynomial on $\mathbb{H}$ and Q to a left slice hyperholomorphic polynomial on $\mathbb{H}$. Due to the uniqueness of the left slice hyperholomorphic extension in Lemma 2.1.11, we then obtain

$$f = \text{ext}_L(f_j) = \text{ext}_L(P^{-1}Q) = P^{-1}Q,$$

and so f is actually left rational. The right rational case can be shown similarly. □

Theorem 2.1.37 (Runge's theorem)**.** *Let $K \subset \mathbb{H}$ be an axially symmetric compact set and let A be an axially symmetric set such that $A \cap C \neq \emptyset$ for every connected component C of $(\mathbb{H} \cup \{\infty\}) \setminus K$.*

If f is left slice hyperholomorphic on an axially symmetric open set U with $K \subset U$, then for every $\varepsilon > 0$, there exists a left rational function r whose poles lie in A such that

$$\sup\{|f(q) - r(q)| : \ q \in K\} < \varepsilon. \tag{2.32}$$

Similarly, if f is right slice hyperholomorphic on an axially symmetric open set U with $K \subset U$, then for every $\varepsilon > 0$, there exists a right rational function r whose poles lie in A such that (2.32) *holds.*

Finally, if $f \in \mathcal{N}(U)$ for some axially symmetric open set U with $K \subset U$, then for every $\varepsilon > 0$, there exists a real rational function r whose poles lie in A such that (2.32) *holds.*

Proof. Let $f \in \mathcal{SH}_L(U)$ for some axially symmetric open set U with $K \subset U$, let $j, i \in \mathbb{S}$ with $j \perp i$, and let us write $f_j = F_1 + F_2 i$ with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$ as in Lemma 2.1.6. The set $K \cap \mathbb{C}_j$ is compact in $\mathbb{C}_j$ and the set $A \cap \mathbb{C}_j$ has, due to its axial symmetry, nonempty intersection with every connected component of $(\mathbb{C}_j \cup \{\infty\}) \setminus (K \cap \mathbb{C}_j)$.

For $\varepsilon > 0$, the classical Runge's theorem for holomorphic functions implies the existence of rational functions R_1 and R_2 with poles in $A \cap \mathbb{C}_j$ such that

$$\sup\{|F_\ell(z) - R_\ell(z)| : \ z \in K \cap \mathbb{C}_j\} < \frac{\varepsilon}{4}, \quad \ell = 1, 2. \tag{2.33}$$

The left slice hyperholomorphic extension $r(q) = \operatorname{ext}_L(R_1 + R_2 i)$ is then by Lemma 3.2.10 a right rational function with poles in A, and

$$|f(z) - r(z)| \leq |F_1(z) - R_1(z)| + |F_2(z) - R_2(z)| < \frac{\varepsilon}{2}$$

for all $z \in K \cap \mathbb{C}_j$. From Theorem 2.1.9 we conclude for $q = u + kv \in K$ after setting $z = u + jv \in K \cap \mathbb{C}_j$ that

$$\begin{aligned} |f(q) - r(q)| &= \left| \frac{1}{2}(1 - kj)(f(z) + r(z)) + \frac{1}{2}(1 + kj)(f(\overline{z}) - r(\overline{z})) \right| \\ &\leq |f(z) + r(z)| + |f(\overline{z}) - r(\overline{z})) < \varepsilon. \end{aligned} \tag{2.34}$$

The right slice hyperholomorphic case can be shown by similar arguments.

What remains to show is that R can be chosen rational intrinsic if f is intrinsic. In order to do that, we first observe that in this case, $F_2 \equiv 0$, so that we can choose $R_2 \equiv 0$ in (2.33). If we set

$$\tilde{R}(z) = \frac{1}{2}\left(R_1(z) + \overline{R(\overline{z})}\right),$$

then $\tilde{R}$ is a rational function on $\mathbb{C}_j$ that satisfies $\tilde{R}(\overline{z}) = \overline{\tilde{R}(z)}$. It is hence of the form $\tilde{R}(z) = P(z)^{-1}Q(z)$ with polynomials P and Q with coefficients in $\mathbb{R}$. Its slice hyperholomorphic extension $r(q) = P(q)^{-1}Q(q)$ for $q \in \mathbb{H}$ with $P(q) \neq 0$ is then an intrinsic rational function.

As an intrinsic function, f satisfies $f(\overline{q}) = \overline{f(q)}$. Hence for $z \in K \cup \mathbb{C}_j$, we have

$$\begin{aligned} |f(z) - r(z)| &= \frac{1}{2}\left| f(z) - R_1(z) + \overline{f(\overline{z})} - \overline{R_1(\overline{z})} \right| \\ &\leq \frac{1}{2}\left(\left| f(z) - R_1(z) \right| + \left| \overline{f(\overline{z}) - R_1(\overline{z})} \right| \right) < \frac{\varepsilon}{2}. \end{aligned}$$

As in (2.34), we see then that (2.32) holds with the intrinsic rational function r. □

2.2 The Fueter Mapping Theorem in Integral Form

In order to define the F-functional calculus in Chapter 7 we recall now the Fueter mapping theorem and show its integral form. The Fueter mapping theorem in integral form was introduced in [86]. We start with recalling the definition of Fueter regularity.

Definition 2.2.1 (Cauchy–Fueter regular functions)**.** Let U be an open set in $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is *left Fueter regular* if

$$\frac{\partial}{\partial q_0} f(q) + \sum_{\ell=1}^{3} e_\ell \frac{\partial}{\partial q_\ell} f(q) = 0, \quad \text{for every } q \in U.$$

It is *right Fueter regular* if

$$\frac{\partial}{\partial q_0} f(x) + \sum_{\ell=1}^{3} \frac{\partial}{\partial q_\ell} f(q) e_\ell = 0, \quad \text{for every } q \in U.$$

It was Fueter who introduced in his paper [111] the following method for generating Fueter regular functions:

(1) We consider a holomorphic function $f(z)$ that depends on a complex variable $z = u + \iota v$ in an open set of the upper complex half-plane. (In order to distinguish it from quaternionic imaginary units, we denote the imaginary unit of the usual complex numbers by ι.) We write

$$f(z) = f_0(u, v) + \iota f_1(u, v),$$

where f_0 and f_1 are $\mathbb{R}$-valued functions that satisfy the Cauchy–Riemann system.

(2) For every quaternion q such that $u + \iota v$ belongs to the domain of f, we replace the complex imaginary unit ι in $f(z) = f_0(u, v) + \iota f_1(u, v)$ by the quaternionic imaginary unit $\frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}$, and we set $u = \mathrm{Re}(q)$ and $v = |\mathrm{Im}(q)|$. We then define

$$f(q) = f_0(q_0, |\mathrm{Im}(q)|) + \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} f_1(q_0, |\mathrm{Im}(q)|).$$

Observe that the function $f(q)$ is slice hyperholomorphic by construction.

(3) We apply the Laplace operator $\Delta = \sum_{\ell=0}^{3} \frac{\partial^2}{\partial q_\ell^2}$ to f and define $\breve{f}(q) = \Delta f(q)$.

It turns out that the function $\breve{f}(q)$ is then both left and right Fueter regular.

Observe that by construction, $f(q)$ is an intrinsic slice hyperholomorphic function on the open axially symmetric set of all quaternions $q = u + jv$ such that $u + \iota v$ belongs to the domain of f.

In modern language, the Fueter mapping theorem states that *applying the Laplace operator Δ to a slice hyperholomorphic function $f(q)$ yields the Fueter regular function*

$$\breve{f}(q) = \Delta f(q).$$

This function is left Fueter regular if f is left slice hyperholomorphic and right Fueter regular if f is right slice hyperholomorphic.

If we write f in terms of the slice hyperholomorphic Cauchy formula, we can apply Δ and commute it with the integral such that Δ is actually applied to the slice hyperholomorphic Cauchy kernel inside this integral. In this way, we obtain an integral transform with respect to the kernel $\Delta S_L^{-1}(s,p)$, resp. $\Delta S_R^{-1}(s,p)$, that maps slice hyperholomorphic functions to Fueter regular functions.

A simple formula for $\Delta S_L^{-1}(s,p)$, resp. $\Delta S_R^{-1}(s,p)$, is, however, obtained only if we write the slice hyperholomorphic Cauchy kernels in form II. As a consequence, the F-functional calculus, which is based on this integral transform, can be defined only for operators with commuting components. Otherwise, the S-resolvents cannot be written in a form that corresponds to form II of the Cauchy kernels.

Theorem 2.2.2. *Let $q, s \in \mathbb{H}$ with $q \notin [s]$ and let $\Delta = \sum_{\ell=0}^{3} \frac{\partial^2}{\partial q_\ell^2}$ be the Laplace operator in the variable q.*

(a) *Consider the left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,q)$ written in form II. Then we have*

$$\Delta S_L^{-1}(s,q) = -4(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}. \tag{2.35}$$

(b) *Consider the right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,q)$ written in form II. Then we have*

$$\Delta S_R^{-1}(s,q) = -4(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}(s-\bar{q}). \tag{2.36}$$

Proof. We show only the identity (2.35), the other one follows with similar arguments. If we write $S_L^{-1}(s,q)$ in form II, then straightforward computations yield

$$\begin{aligned}\frac{\partial^2}{\partial q_0^2} S_L^{-1}(s,q) &= 2(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}(-2s + 2q_0) \\ &\quad + 2(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}(-2s + 2q_0)^2 \\ &\quad - 2(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial q_\ell^2} S_L^{-1}(s,q) &= -4e_\ell q_\ell (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2} \\ &\quad + 8q_\ell^2 (s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3} \\ &\quad - 2(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\end{aligned}$$

for $\ell = 1, 2, 3$. Thus, we obtain

$$\begin{aligned}\Delta S_L^{-1}(s,x) &= 2(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}(-2s + 2q_0)\\ &\quad + 2(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}(-2s+2q_0)^2\\ &\quad - \sum_{\ell=1}^{3} 4e_\ell q_\ell (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + \sum_{\ell=1}^{3} 8q_\ell^2 (s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}\\ &\quad - 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}.\end{aligned}$$

Since $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}$ and $(-2s + 2q_0)$ commute, we have

$$\begin{aligned}&\Delta S_L^{-1}(s,q)\\ &= -4\left(s - q_0 + \sum_{\ell=1}^{3} 4e_\ell q_\ell\right)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 2(s-\bar{q})\left[(-2s+2q_0)^2 + \sum_{\ell=1}^{3} 4q_\ell^2\right](s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}\\ &\quad - 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &= -4(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad - 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &= -4(s-\bar{q})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-2}.\end{aligned}$$

□

Proposition 2.2.3. *Let $q \in \mathbb{H}$. The function $s \mapsto \Delta S_L^{-1}(s,q)$ is right slice hyperholomorphic on $\mathbb{H} \setminus [q]$ and the function $s \mapsto \Delta S_R^{-1}(s,q)$ is left slice hyperholomorphic on $\mathbb{H} \setminus [q]$.*

Proof. For $s = u + jv$, we have

$$\begin{aligned}\frac{\partial}{\partial u}\Delta S_L^{-1}(s,q) &= -4(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}(2s - 2\mathrm{Re}(q))\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial v}\Delta S_L^{-1}(s,q) &= -4j(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 8(s-\bar{q})(s - 2\mathrm{Re}(q)(u+jv) + |q|^2)^{-3}(2sj - 2\mathrm{Re}(q)j).\end{aligned}$$

Since j commutes with $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}$, we conclude that

$$\frac{\partial}{\partial u}\Delta S_L^{-1}(s,q) + \frac{\partial}{\partial v}\Delta S_L^{-1}(s,q)j = 0.$$

Hence $\Delta S_L^{-1}(s,q)$ is right slice hyperholomorphic in s by Proposition 2.1.14, because $\mathbb{H}\setminus[q]$ is an axially symmetric slice domain. The other case can be shown with similar arguments. □

Proposition 2.2.4. *Let $s\in\mathbb{H}$. The function $q\mapsto\Delta S_L^{-1}(s,q)$ is left Fueter regular on $\mathbb{H}\setminus[s]$ and the function $q\mapsto\Delta S_R^{-1}(s,q)$ is right Fueter regular on $\mathbb{H}\setminus[s]$.*

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial q_0}\Delta S_L^{-1}(s,q) &= 4(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\qquad -16(s-\bar q)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}(s-q_0)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial q_\ell}\Delta S_L^{-1}(s,q) &= -4e_\ell(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\qquad +16q_\ell(s-\bar q)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\partial}{\partial q_0}\Delta S_L^{-1}(s,q)+\sum_{\ell=1}^{3}e_\ell\frac{\partial}{\partial q_\ell}\Delta S_L^{-1}(s,q)\\ &= 4(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad -16(s-\bar q)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}(s-q_0)\\ &\quad +\sum_{\ell=1}^{3}4(-e_\ell^2)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad +\sum_{\ell=1}^{3}16e_\ell q_\ell(s-\bar q)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3},\end{aligned}$$

and since $(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}$ commutes with $s-q_0$, we finally obtain

$$\begin{aligned}&\frac{\partial}{\partial q_0}\Delta S_L^{-1}(s,q)+\sum_{\ell=1}^{3}e_\ell\frac{\partial}{\partial q_\ell}\Delta S_L^{-1}(s,q)\\ &= 16(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad +16\left(\left(q_0+\sum_{\ell=1}^{3}q_\ell e_\ell\right)(s-\bar q)-(s-\bar q)s\right)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}\\ &= 16(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}-16(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}=0.\end{aligned}$$

Hence $q\mapsto\Delta S_L^{-1}(s,q)$ is left Fueter regular. The right Fueter regularity of $q\mapsto\Delta S_R^{-1}(s,q)$ can be shown with analogous computations. □

Definition 2.2.5 (The Fueter kernels)**.** We define for $s \in \mathbb{H}$ with $q \notin [s]$ the F_L-*kernel* as

$$F_L(s,q) := \Delta S_L^{-1}(s,q) = -4(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2},$$

and the F_R-*kernel* as

$$F_R(s,q) := \Delta S_R^{-1}(s,q) = -4(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}(s-\bar{q}).$$

Finally, we can now prove the Fueter mapping theorem in integral form.

Theorem 2.2.6 (The Fueter mapping theorem in integral form)**.** *Let $U \subset \mathbb{H}$ be a slice Cauchy domain and choose $j \in \mathbb{S}$.*

(a) *If $f \in \mathcal{SH}_L(O)$ for some set O with $\overline{U} \subset O$, then $\breve{f}(q) = \Delta f(q)$ is left Fueter regular on U, and it admits the integral representation*

$$\breve{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} F_L(s,q)\, ds_j\, f(s) \quad \forall q \in U. \tag{2.37}$$

(b) *If $f \in \mathcal{SH}_R(O)$ for some set O with $\overline{U} \subset O$, then $\breve{f}(q) = \Delta f(q)$ is right Fueter regular on U, and it admits the integral representation*

$$\breve{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, F_R(s,q) \quad \forall q \in U. \tag{2.38}$$

The integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. The function $\breve{f}(q) = \Delta f(q)$ is Fueter regular by the Fueter mapping theorem. We can write $f(q)$ for $q \in U$ in terms of the corresponding slice hyperholomorphic Cauchy formula. If we apply the Laplacian and exchange the order of integration and differentiation, we end up with (2.37), resp. (2.38). □

2.3 Vector-Valued Slice Hyperholomorphic Functions

In this section, we generalize the notion of slice hyperholomorphicity from scalar-valued to vector-valued functions. In particular, similar to what happens for holomorphic functions, we show that the notions of weak and strong slice hyperholomorphicity are equivalent. Via the quaternionic Hahn–Banach theorem, one can prove properties of vector-valued slice hyperholomorphic functions by reducing the problems to the scalar case.

Definition 2.3.1. A *quaternionic right vector space* is an additive group $(X,+)$ that is endowed with a quaternionic right multiplication $(X,\mathbb{H}) \to X$, $(x,q) \mapsto xq$ such that for all $x,y \in X$ and all $p,q \in \mathbb{H}$,

$$x(p+q), = xp + xq \qquad (x+y)q = xq + yq, \qquad (xp)q = x(pq).$$

A *quaternionic left vector space* is an additive group $(X,+)$ that is endowed with a quaternionic left multiplication $(\mathbb{H}, X) \to X$, $(q,x) \mapsto qx$ such that for all $x, y \in X$ and all $p, q \in \mathbb{H}$,

$$(p+q)x = px + qx, \qquad q(x+y) = qx + qy, \qquad q(px) = (qp)x.$$

A *two-sided quaternionic vector space* is an additive group $(X,+)$ endowed with a quaternionic left and a quaternionic right multiplication such that X is both a left and a right vector space and such that $ax = xa$ for all $a \in \mathbb{R}$ and all $x \in X$.

Remark 2.3.2. If we start from a real vector space $X_{\mathbb{R}}$, then we can quaternionify $X_{\mathbb{R}}$ to obtain the two-sided quaternionic vector space $X = X_{\mathbb{R}} \otimes \mathbb{H}$ by setting

$$X = X_{\mathbb{R}} \otimes \mathbb{H} = \left\{ \sum_{\ell=0}^{3} x_\ell e_\ell : \ x_\ell \in X_{\mathbb{R}} \right\}$$

with the scalar multiplications

$$qx = \sum_{\ell=0}^{3} x_\ell (q e_\ell), \quad xq = \sum_{\ell=0}^{3} x_\ell (e_\ell q),$$

for $x \in X$ and $q \in \mathbb{H}$. Conversely, every two-sided quaternionic vector space X is isomorphic to the quaternionification of a real vector space, namely to $X_{\mathbb{R}} \otimes \mathbb{H}$ with the real vector space

$$X_{\mathbb{R}} = \{x \in X : \ qx = xq \quad \forall q \in \mathbb{H}\}.$$

Definition 2.3.3. A function $\|\cdot\| : X_R \to [0, +\infty)$ on a quaternionic right vector space X_R is called a *norm on* X_R, if it satisfies

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|xq\| = \|x\||q|$ for all $x \in X$ and all $q \in \mathbb{H}$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

If X_R is complete with respect to the metric induced by $\|\cdot\|$, we call X_R a *quaternionic right Banach space.*

A function $\|\cdot\| : X_L \to [0, +\infty)$ on a quaternionic left vector space X_L is called a *norm on* X_L, if it satisfies (i), (iii), and

(ii′) $\|qx\| = |q|\|x\|$ for all $x \in X$ and all $q \in \mathbb{H}$.

If X_L is complete with respect to the metric induced by $\|\cdot\|$, we call X_L a *quaternionic left Banach space.*

Finally, a two-sided quaternionic vector space X is called a quaternionic *two-sided quaternionic Banach space* if it is endowed with a norm $\|\cdot\|$ such that it is both a left and a right Banach space, that is, such that (i), (ii), (ii′) and (iii) are satisfied and such that X is complete with respect to the metric induced by $\|\cdot\|$.

Corollary 2.3.4. *A quaternionic left or right Banach space turns into a real Banach space if we restrict the left, resp. right, scalar multiplication to $\mathbb{R}$, and it turns into a complex Banach space over $\mathbb{C}_j$ with $j \in \mathbb{S}$ if we restrict the left, resp. right, scalar multiplication to $\mathbb{C}_j$.*

A two-sided quaternionic Banach space turns into a real Banach space if we restrict the scalar multiplications to $\mathbb{R}$, and it turns into a complex Banach space over $\mathbb{C}_j$ with $j \in \mathbb{S}$ if we restrict either the left or the right scalar multiplication to $\mathbb{C}_j$.

Definition 2.3.5. A function $\varphi : X_1 \to X_2$ between two quaternionic right vector spaces X_1, X_2 is called *right linear* if

$$\varphi(xq + y) = \varphi(x)q + \varphi(y) \quad \forall x, y \in X_1, q \in \mathbb{H}.$$

Similarly, a function $\varphi : X_1 \to X_2$ between two quaternionic left vector spaces X_1, X_2 is called *left linear* if

$$\varphi(qx + y) = q\varphi(x) + \varphi(y) \quad \forall x, y \in X_1, q \in \mathbb{H}.$$

A right or left linear mapping $\varphi : X_1 \to X_2$ between two quaternionic right, resp. left, Banach spaces is called *bounded* if

$$\|\varphi\| := \sup_{\|x\|_{X_1}=1} \|\varphi(x)\|_{X_2} < +\infty.$$

Definition 2.3.6. The *dual X'_R of a quaternionic right Banach space X_R* is the quaternionic left Banach space of all bounded right linear mappings from X_R to $\mathbb{H}$. The *dual X'_L of a quaternionic left Banach space X_L* is the quaternionic right Banach space of all bounded left linear mappings from X_R to $\mathbb{H}$. Finally, for a two-sided quaternionic Banach space X, we distinguish two different dual spaces: the *right dual X'_R of X* is the dual space of X as a right Banach space, and the *left dual X'_L of X* is the dual space of X as a left Banach space.

We finally recall the quaternionic Hahn–Banach theorem, which will be important in the sequel. It was first proven in [194], but a proof in English can be found in [89].

Theorem 2.3.7 (Hahn–Banach theorem)**.** *Let X_R be a quaternionic right vector space, let X_0 be a right linear subspace of X_R, and let $\rho : X_R \to [0, +\infty)$ satisfy $\rho(x + y) \leq \rho(x) + \rho(y)$ and $\rho(xq) = \rho(x)|q|$ for all $x, y \in X_R$ and all $q \in \mathbb{H}$. Moreover, let $\lambda : X_0 \to \mathbb{H}$ be a quaternionic right linear functional on X_0 such that $|\lambda(x)| \leq \rho(x)$ for all $x \in X_0$. Then there exists a right linear functional $\Lambda : X_R \to \mathbb{H}$ such that $\Lambda(x) = \lambda(x)$ for all $x \in X_0$ and such that*

$$|\Lambda(x)| \leq \rho(x) \quad \textit{for all } x \in X_R.$$

An analogous statement holds for left linear vector spaces.

Corollary 2.3.8. *The dual space of a quaternionic left or right Banach space separates points. Furthermore, both the left and the right duals of a two-sided quaternionic Banach space also separate points.*

Let us now turn our attention to slice hyperholomorphic functions with values in a quaternionic Banach space. As in the complex case, one can distinguish between strong and weak slice hyperholomorphicity.

Definition 2.3.9 (Slice hyperholomorphic vector-valued functions)**.** Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let

$$\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : \ u + \mathbb{S}v \subset U\}.$$

A function $f : U \to X_L$ with values in a quaternionic left Banach space X_L is called a *left slice function*, if is of the form

$$f(q) = f_0(u,v) + jf_1(u,v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to X_L$ that satisfy the compatibility condition (2.4). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5), then f is called *strongly left slice hyperholomorphic*.

A function $f : U \to X_R$ with values in a quaternionic right Banach space is called a *right slice function* if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v)j \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to X_R$ that satisfy the compatibility condition (2.4). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5), then f is called *strongly right slice hyperholomorphic*.

Definition 2.3.10. Let $U \subset \mathbb{H}$ be an axially symmetric open set. A function $f : U \to X_L$ with values in a quaternionic left Banach space X_L is called *weakly left slice hyperholomorphic* if Λf is left slice hyperholomorphic for every $\Lambda \in X_L'$. A function $f : U \to X_R$ with values in a quaternionic right Banach space X_R is called *weakly right slice hyperholomorphic* if Λf is right slice hyperholomorphic for every $\Lambda \in X_R'$.

Since the functionals Λ in the dual of X_L, resp. X_R, are continuous, every strongly slice hyperholomorphic function is weakly slice hyperholomorphic. As in the complex case, the converse also is true. In order to show this, we recall the following lemma. We omit the proof, since it works exactly as in the complex case (see, e.g., [179], p. 189).

Lemma 2.3.11. *Let X be a two-sided quaternionic Banach space. A sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $(\Lambda(x_n))_{n\in\mathbb{N}}$ is uniformly Cauchy for $\Lambda \in X'$, $\|\Lambda\| \leq 1$.*

Proposition 2.3.12. *Let X_L be a quaternionic left Banach space, let U be an open axially symmetric subset of $\mathbb{H}$, and let $f : U \to X_L$ be a real differentiable left slice function. Then the following statements are equivalent:*

(i) *The function f is strongly left slice hyperholomorphic.*

(ii) *The function f admits a left slice derivative, that is,*

$$\partial_S f(q) := \lim_{p\to q, p\in\mathbb{C}_j} (p-q)^{-1}(f(p)-f(q)) \tag{2.39}$$

exists for all $q = u + jv \in U$ in the topology of X_L, and it exists for every $j \in \mathbb{S}$ if q is real.

(iii) *For every $j \in \mathbb{S}$, the restriction $f_j = f|_{U\cap\mathbb{C}_j}$ of f to $U \cap \mathbb{C}_j$ satisfies*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f(q) + j\frac{\partial}{\partial v} f(q)\right) = 0, \quad \forall q = u + jv \in U \cap \mathbb{C}_j.$$

Let X_R be a quaternionic right Banach space, let U be an open axially symmetric subset of $\mathbb{H}$, and let $f : U \to X_R$ be a real differentiable right slice function. Then the following statements are equivalent:

(i) *The function f is strongly right slice hyperholomorphic.*

(ii) *The function f admits a right slice derivative, that is,*

$$\partial_S f(q) := \lim_{p\to q, p\in\mathbb{C}_j} (f(p)-f(q))(p-q)^{-1}$$

exists for all $q = u + jv \in U$ in the topology of X_R, and it exists for every $j \in \mathbb{S}$ if q is real.

(iii) *For every $j \in \mathbb{S}$, the restriction $f_j = f|_{U\cap\mathbb{C}_j}$ of f to $U \cap \mathbb{C}_j$ satisfies*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f(q) + \frac{\partial}{\partial v} f(q)j\right) = 0, \quad \forall q = u + jv \in U \cap \mathbb{C}_j.$$

Proof. Let $f : U \to X_L$ be a left slice function. The equivalence of (ii) and (iii) follows immediately from the complex theory and Corollary 2.3.4: the statement (iii) is equivalent to f_j being, for every $j \in \mathbb{S}$, a (left) holomorphic function on $\mathbb{C}_j$ with values in the complex Banach space X_L over $\mathbb{C}_j$. This is in turn equivalent to the existence of the limit

$$f_j'(q) = \lim_{p\to q, p\in\mathbb{C}_j} (p-q)^{-1}(f(p)-f(q)) = \partial_S f(q)$$

for every $q = u + jv \in U$.

Let us now show the equivalence of (i) and (iii). If (i) holds, then

$$\begin{aligned}&\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + j\frac{\partial}{\partial v} f_j(z)\right)\\ &= \frac{1}{2}\left(\frac{\partial}{\partial u} f_0(z) + j\frac{\partial}{\partial u} f_1(z) + j\frac{\partial}{\partial v} f_0(z) - \frac{\partial}{\partial v} f_1(z)\right) = 0,\end{aligned}$$

because f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5). If, on the other hand, (iii) holds, then we have because of

$$f_0(u,v) = \frac{1}{2}\left(f(u+jv) + f(u-jv)\right)$$

and

$$f_1(u,v) = \frac{1}{2}j\left(f(u-jv) - f(u+jv)\right)$$

that

$$\begin{aligned}\frac{\partial}{\partial u}f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial u}f(u-jv) + \frac{\partial}{\partial u}f(u+jv)\right]\\ &= \frac{1}{2}\left[j\frac{\partial}{\partial v}f(u-jv) - j\frac{\partial}{\partial v}f(u+jv)\right] = \frac{\partial}{\partial v}f_1(u,v).\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial v}f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial v}f(u-jv) + \frac{\partial}{\partial v}f(u+jv)\right]\\ &= \frac{1}{2}\left[-j\frac{\partial}{\partial u}f(u-jv) + j\frac{\partial}{\partial u}f(u+jv)\right] = -\frac{\partial}{\partial u}f_1(u,v).\end{aligned}$$

Hence f is actually left slice hyperholomorphic.

The right slice hyperholomorphic case can be shown with analogous arguments. □

Theorem 2.3.13. *Let $U \subset \mathbb{H}$ be an axially symmetric open set.*

(i) *Every weakly left slice hyperholomorphic function $f : U \to X_L$ with values in a quaternionic left Banach space is strongly left slice hyperholomorphic.*

(ii) *Every weakly right slice hyperholomorphic function $f : U \to X_R$ with values in a quaternionic right Banach space is strongly right slice hyperholomorphic.*

Proof. Let f be a weakly left slice hyperholomorphic function on U with values in a quaternionic left Banach space X_L. We first observe that f is a left slice function. If we choose $i \in \mathbb{S}$ and set

$$f_0(u,v) = \frac{1}{2}(f(u+iv) + f(u-iv))$$

and

$$f_1(u,v) = \frac{1}{2}i(f(u-iv) - f(u+iv))$$

for $u, v \in \mathbb{R}$ with $u + iv \in U$, then f_0 and f_1 obviously satisfy the compatibility condition (2.4). If $\Lambda \in X_L'$, then $(\Lambda \circ f)(q) := \Lambda(f(q))$ is left slice hyperholomorphic on U by our assumptions, and hence it satisfies the structure formula (2.9). If $q = u + jv \in U$, we can set $z = u + iv$ for netatness and obtain

$$\begin{aligned}
&\Lambda(f_0(u,v) + jf_1(u,v)) \\
&= \Lambda\left(\frac{1}{2}\left(f(z) + f(\overline{z})\right)\right) + ji\frac{1}{2}\left(f(\overline{z}) - f(z)\right)\Big) \\
&= \frac{1}{2}\left(\Lambda(f(z)) + \lambda(f(\overline{z}))\right) + ji\frac{1}{2}\left(\Lambda(f(\overline{z})) - \Lambda(f(z))\right) \\
&= \frac{1}{2}\left((\Lambda \circ f)(z) + (\Lambda \circ f)(\overline{z})\right) + j\frac{1}{2}\left((\Lambda \circ f)(\overline{z}) - (\Lambda \circ f)(z)\right) \\
&= (\Lambda \circ f)(q) = \Lambda(f(q)).
\end{aligned}$$

Since $\Lambda \in V_L'$ was arbitrary and V_L' separates points by Corollary 2.3.8, we find that $f(q) = f_0(u,v) + jf_1(u,v)$ and hence f is a left slice function.

The rest of the proof follows the lines of the proof in the complex case in [179, p. 189]. For every $\Lambda \in X_L'$, the function $q \mapsto \Lambda(f(q))$ is left slice hyperholomorphic on U. Its restriction to a plane $\mathbb{C}_j$ is hence left holomorphic and therefore admits a representation in terms of the Cauchy formula. If $q = u + jv \in U$ and p tends to q in $\mathbb{C}_j$, we can therefore choose $r > 0$ so small that $\overline{B_r(q)} \subset U$ and find for $p \in B_r(q) \cap \mathbb{C}_j$ that

$$\begin{aligned}
&\Lambda(f(p)) - \Lambda(f(q)) \\
&= \frac{1}{2\pi}\int_\Gamma \left((s-p)^{-1} - (s-q)^{-1}\right)\, ds_j\, \Lambda(f(s)) \\
&= \frac{1}{2\pi}\int_\Gamma (p-q)(s-p)^{-1}(s-q)^{-1}\, ds_j\, \Lambda(f(s))
\end{aligned}$$

with $\Gamma := \partial(B_r(q) \cap \mathbb{C}_j)$. Moreover, since $(\Lambda \circ f)_j'(q) = \frac{\partial}{\partial u}\Lambda(f(q))$, we also have

$$\frac{\partial}{\partial u}\Lambda(f(q)) = \frac{1}{2\pi}\int_\Gamma (s-q)^{-2} ds_j \Lambda(f(s))$$

and hence

$$\begin{aligned}
&\left|(p-q)^{-1}(\Lambda(f(p)) - \Lambda(f(q))) - \frac{\partial}{\partial u}\Lambda(f(q))\right| \\
&= \left|\frac{1}{2\pi}\int_\Gamma \left((s-p)^{-1}(s-q)^{-1} - (s-q)^2\right)\, ds_j\, \Lambda(f(s))\right|.
\end{aligned}$$

The mapping $s \mapsto \Lambda(f(s))$ is continuous on Γ. Since Γ is compact, we obtain

$$\sup_{s\in\Gamma} \|\Lambda(f(s))| < +\infty.$$

The mappings $\Lambda \mapsto \Lambda(f(s))$, $s \in \Gamma$, hence form a family of pointwise bounded linear maps from V_L' to $\mathbb{H}$. By the uniform boundedness principle, they are therefore uniformly bounded such that

$$\sup_{s\in\Gamma,\|\Lambda\|_{V_L}\leq 1} |\Lambda(f(s))| := C < +\infty.$$

Consequently, we have

$$\begin{aligned}&\left|(p-q)^{-1}(\Lambda(f(p)) - \Lambda(f(q))) - \frac{\partial}{\partial u}\Lambda(f(q))\right|\\ &\leq \frac{C}{2\pi}\int_\Gamma \left|(s-p)^{-1}(s-q)^{-1} - (s-q)^2\right| \, d|s| \longrightarrow 0\end{aligned}$$

as p approaches q in $\mathbb{C}_j$. Since the above estimate is independent of Λ, it follows that

$$\lim_{p\to q} \Lambda\left((p-q)^{-1}(f(p) - f(q))\right) = \frac{\partial}{\partial u}\Lambda(f(q)) = \frac{\partial}{\partial u}\Lambda(f(q))$$

uniformly for $\Lambda \in V_L'$ with $\|\Lambda\| \leq 1$. Thus $\Lambda\left((p-q)^{-1}(f(p) - f(q)\right)$ is in particular uniformly Cauchy as $p \to q$ for $\|\Lambda\| < 1$, and we conclude from Lemma 2.3.11 that the limit (2.39) exists, i.e., that f admits a left slice derivative at q. Since $q \in U$ was arbitrary and we already know that f is a left slice function, Proposition 2.3.12 implies that f is strongly left slice hyperholomorphic.

The right slice hyperholomorphic case can again be shown with similar arguments. □

Since weak and strong slice hyperholomorphicity are equivalent, we will refer to such functions simply as slice hyperholomorphic.

Definition 2.3.14. Let $U \subset \mathbb{H}$ be an axially symmetric open set. We denote the set of all left slice hyperholomorphic functions on U with values in a quaternionic left Banach space X_L by $\mathcal{SH}_L(U, X_L)$ and the set of all right slice hyperholomorphic function on U with values in a quaternionic right Banach space X_R by $\mathcal{SH}_R(U, X_R)$.

Corollary 2.3.15. *Let $U \subset \mathbb{H}$ be an axially symmetric open set. If X_L is a quaternionic left Banach space, then $\mathcal{SH}_L(U, X_L)$ is a quaternionic right linear space. If X_R is a quaternionic right Banach space, then $\mathcal{SH}_R(U, X_R)$ is a quaternionic left linear space.*

Since weak and strong slice hyperholomorphicity are equivalent, several results for scalar-valued slice hyperholomorphic functions can be generalized to the vector-valued case by applying functionals in the dual space in order to reduce the problems to the scalar case.

Proposition 2.3.16 (Identity principle)**.** *Let U be an axially symmetric slice domain, let f and g be two left or right slice hyperholomorphic functions on U with values in a quaternionic left, resp. right, Banach space X, and set $\mathcal{Z} := \{q \in U : f(q) = g(q)\}$. If there exists $j \in \mathbb{S}$ such that $\mathcal{Z} \cap \mathbb{C}_j$ has an accumulation point in $U \cap \mathbb{C}_j$, then $f \equiv g$ on all of U.*

Proof. The hypothesis implies $\Lambda f = \Lambda g$ on $\mathcal{Z} \cap \mathbb{C}_j$ for every element $\Lambda \in X'$. Theorem 2.1.8 thus implies that the left, resp. right, slice hyperholomorphic function $\Lambda(f-g)$ is identically zero on the entire axially symmetric slice domain U. By the Hahn–Banach theorem, we obtain $f - g = 0$ on U. □

Computations as in the scalar case show, moreover, that vector-valued slice hyperholomorphic functions also satisfy the structure formula and that they can be expanded into a Taylor series at every real point.

Proposition 2.3.17 (Structure formula (or representation formula)). *Let $U \subset \mathbb{H}$ be an axially symmetric open set, let $q = u + jv \in U$ and $z = u + iv$ for some $i \in \mathbb{S}$. If f is a left slice function on U with values in a quaternionic left Banach space X_L, then*

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2} ji\Big[f(\overline{z}) - f(z)\Big].$$

If f is a right slice function on U with values in a quaternionic right Banach space X_R, then

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}\Big[f(\overline{z}) - f(z)\Big] ij.$$

Theorem 2.3.18. *Let $a \in \mathbb{R}$, let $r > 0$, and let $B_r(a) = \{q \in \mathbb{H} : |q - a| < r\}$. If $f \in \mathcal{SH}_L(B_r(a), X_L)$ with values in a quaternionic left Banach space X_L, then*

$$f(q) = \sum_{n=0}^{+\infty} (q-a)^n \frac{1}{n!} \partial_S^n f(a) \quad \forall q = u + jv \in B_r(a). \tag{2.40}$$

If on the other hand $f \in \mathcal{SH}_R(B_r(a), X_R)$ with values in a quaternionic right Banach space X_R, then

$$f(q) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\partial_S^n f(a)\right)(q-a)^n \quad \forall q = u + jv \in B_r(a).$$

Finally, the slice hyperholomorphic Cauchy formulas hold also in the scalar case.

Theorem 2.3.19 (Vector-valued Cauchy formula). *Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$, and set $ds_j = -ds j$. If f is a left slice hyperholomorphic function with values in a quaternionic left Banach space X_L that is defined on an open axially symmetric set O with $\overline{U} \subset O$, then*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s) \quad \forall q \in U. \tag{2.41}$$

If f is a right slice hyperholomorphic function with values in a quaternionic right Banach space X_R that is defined on an open axially symmetric set O with $\overline{U} \subset O$, then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q), \quad \forall q \in U.$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. Let $f \in \mathcal{SH}_L(\overline{U}, X_L)$ and let $q \in U$. Since $\partial(U \cap \mathbb{C}_j)$ is compact and the integrand is continuous, the integral in (2.41) converges. Moreover, for every $\Lambda \in X_L'$, we have, due to the left slice hyperholomorphicity of $q \mapsto \Delta(f(q))$, that

$$\Lambda\left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s)\right) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,\Lambda(f(s)) = \Lambda(f(q)).$$

Since $\Lambda \in X_L'$ was arbitrary and X_L' separates points by the Hahn–Banach theorem, we obtain the statement. □

If one considers slice hyperholomorphic functions with values in a quaternionic Banach algebra, then the product of two slice hyperholomorphic functions is, just as in the scalar case, in general not slice hyperholomorphic. It is, however, possible to define a generalized product that preserves slice hyperholomorphicity.

Definition 2.3.20. A *two-sided quaternionic Banach algebra* is a quaternionic Banach space X that is endowed with a product $X \times X \to X$ such that:

(i) The product is associative and distributive over the sum in X.

(ii) One has $(qx)y = q(xy)$ and $x(yq) = (xy)q$ for all $x, y \in X$ and all $q \in \mathbb{H}$.

(iii) One has $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in X$.

If in addition there exists a unit with respect to the product in X, then X is called a *two-sided quaternionic Banach algebra with unit.*

Definition 2.3.21. Let $U \subset \mathbb{H}$ be an axially symmetric open set and let X be a two-sided quaternionic Banach algebra. For two functions $f, g \in \mathcal{SH}_L(U, X)$ with $f(q) = f_0 + jf_1$ and $g = g_0 + jg_1$ for $q = u + jv \in U$, we define their *left slice hyperholomorphic product* as

$$f *_L g := f_0g_0 - f_1g_1 + j\left(f_0g_1 + f_1g_0\right). \tag{2.42}$$

For two functions $f, g \in \mathcal{SH}_R(U, X)$ with $f(q) = f_0(u,v) + f_1(u,v)j$ and $g(q) = g_0(u,v) + g_1(u,v)j$ for $q = u + jv \in U$, we define their *right slice hyperholomorphic product* as

$$f *_R g := f_0g_0 - f_1g_1 + \left(f_0g_1 + f_1g_0\right)j. \tag{2.43}$$

Remark 2.3.22. It is immediate that the $*_L$-product of two left slice hyperholomorphic functions is again left slice hyperholomorphic and that the $*_R$-product of two right slice hyperholomorphic functions is again right slice hyperholomorphic. If, moreover, $U = B_r(0)$, then f, g admit power series expansions. If f and g are left slice hyperholomorphic with $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ with $a_n, b_n \in X$, then

$$(f *_L g)(q) := \sum_{n=0}^{+\infty} q^n \left(\sum_{\ell=0}^{n} a_\ell b_{n-\ell} \right).$$

Similarly, if f and g are right slice hyperholomorphic with $f(q) = \sum_{n=0}^{+\infty} a_n q^n$ and $g(q) = \sum_{n=0}^{+\infty} b_n q^n$ with $a_n, b_n \in X$, then

$$(f *_R g)(q) := \sum_{n=0}^{+\infty} \left(\sum_{\ell=0}^{n} a_\ell b_{n-\ell} \right) q^n.$$

Remark 2.3.23. The slice hyperholomorphic product can be defined in even more general settings than for functions with values in a quaternionic Banach algebra. If, for instance, $f \in \mathcal{SH}_L(U, \mathbb{H})$ and $g \in \mathcal{SH}_L(U, X_L)$ for some quaternionic left Banach space, then we can define $f *_L g \in \mathcal{SH}_L(U, X_L)$ also as in (2.42). For another example, we consider $f \in \mathcal{B}(X_1, X_2)$ and $g \in \mathcal{B}(X_2, X_3)$, where X_1, X_2, and X_3 are two-sided quaternionic Banach spaces and $\mathcal{B}(X, Y)$ denotes the set of all bounded right linear operators from X to Y. Then we can again define $f *_L g \in \mathcal{SH}_L(U, \mathcal{B}(X_1, X_3))$ by (2.42). The same can, of course, be done for right slice hyperholomorphic functions.

2.4 Comments and Remarks

The results of this chapter are spread over several papers which are quoted below. The treatment is sometimes different according to the definition of slice hyperholomorphicity that one takes. The interest in slice hyperholomorphic functions, defined in [135], arose in 2006 because of their applications to operator theory. Similar functions were, however, already used much earlier by Fueter, who considered in [110] functions of the form

$$f(q) = f_0(u + iv) + j f_1(u + iv), \quad q = u + jv,$$

where f_0, f_1 are the real-valued components of the analytic function $F(z) = f_0(z) + \iota f_1(z)$, in order to define what he called *hyperanalytic functions*. These hyperanalytic functions are nothing but intrinsic slice hyperholomorphic functions. In [111] the author generates Fueter regular functions by applying the Laplace operator to such a class of functions. The relation $\breve{f} = \Delta f$ between Fueter regular functions $\breve{f}$ and slice hyperholomorphic functions f is nowadays a modern way to state the Fueter mapping theorem. In [187], Sce extended this theorem to functions with values in a Clifford algebras of odd dimension. The extension to Clifford algebras of even dimensions needs more sophisticated arguments based on Fourier multipliers. In [175], Qian introduced the even–odd condition (2.4) in order to define entire slice hyperholomorphic functions, and he generalized the theorem of Sce. For biaxial symmetric domains, see [174].

In [135], slice hyperholomorphic functions were defined as functions that satisfy the properties shown in Lemma 2.1.6, that is, they are functions whose restrictions to complex planes $\mathbb{C}_j$ are left, resp. right, holomorphic. As we showed in Proposition 2.1.14, on axially symmetric slice domains, this definition is equivalent to Definition 2.1.2. Precisely, one can show that such functions satisfy the structure formula when they are defined on an axially symmetric slice domain. Considering only functions on axially symmetric slice domains is, however, not sufficient for developing a rich theory of quaternionic linear operators. For operator theory it is important to consider functions that are defined on axially symmetric open sets that are not necessarily slice domains, so for this reason we use Definition 2.1.2 for slice hyperholomorphicity.

There is an other approach to slice hyperholomorphic functions that refers to a global operator introduced in [60]. The global operator $G(q)$ is defined by

$$G(q) := |\underline{q}|^2 \frac{\partial}{\partial q_0} \;+\; \underline{q} \sum_{j=1}^{3} q_j \frac{\partial}{\partial q_j},$$

and if $U \subseteq \mathbb{H}$ is an open set and $f:\ U \to \mathbb{H}$ is a slice hyperholomorphic function, then

$$G(q)f(q) = 0.$$

Using as a definition of slice hyperholomorphic those functions that are in the kernel of the operators G, we have a possible definition of slice hyperholomorphic functions in several variables. Here the theory is far from being developed, because we have a system of nonconstant differential operators, and the power series expansion disappears, as the following example in [60] shows:

Example 2.4.1. Let U be an open set in $\mathbb{H} \times \mathbb{H}$ that does not intersect the real line. Then the function

$$f(q_1, q_2) = -\mathrm{Im}(q_2) + \frac{\underline{q}_2}{|\underline{q}_2|}\Big(\frac{1}{2}\mathrm{Re}(q_1)^2 - \frac{1}{2}\mathrm{Im}(q_1)^2 + \mathrm{Re}(q_2)\Big) + \frac{\underline{q}_1}{|\underline{q}_1|}\,\frac{\underline{q}_2}{|\underline{q}_2|}\mathrm{Re}(q_1)\mathrm{Im}(q_1) \tag{2.44}$$

satisfies the system

$$\begin{cases} |\underline{q}_1|^2 \frac{\partial}{\partial q_{1,0}} f(q_1, q_2) \;+ \underline{q}_1 \;\sum_{j=1}^{3} q_{1,j} \frac{\partial}{\partial q_{1,j}} f(q_1, q_2) = 0, \\ |\underline{q}_2|^2 \frac{\partial}{\partial q_{2,0}} f(q_1, q_2) \;+ \underline{q}_2 \;\sum_{j=1}^{3} q_{2,j} \frac{\partial}{\partial q_{2,j}} f(q_1, q_2) = 0. \end{cases} \tag{2.45}$$

In the paper [98] there are some results associated with the theory of slice hyperholomorphic functions in several variables, but the global operator is not used. The above example can be found also in [98].

References on function theory. The theory of slice hyperholomorphic functions is nowadays very well developed. The main monographs on this topic or containing this topic are [18, 56, 89, 96, 123, 133].

Slice hyperholomorphic functions can be defined not only over the quaternions but also over more general Clifford algebras. In the quaternionic setting, slice hyperholomorphic functions are also called slice regular, and their theory has been developed by several authors. Some of the most important contributions were published in [37–39, 58, 101, 112, 113, 130–132, 134–141, 154, 180, 181, 188–190].

Slice hyperholomorphic functions with values in a Clifford algebra are also called slice monogenic functions. The main results of their theory are contained in the papers [64, 65, 73, 90–95, 152, 198].

Several important approximation theorems for slice hyperholomorphic functions are collected in the papers [114–122] and the monograph on quaternionic approximation theory [123].

The Fueter mapping theorem provides a relation between slice hyperholomorphic functions and the classical theory of monogenic functions. Another relation is provided by the Radon transform and the dual Radon transform. Intense studies of these relations that go far beyond the results presented in Section 2.2 can be found in [61, 69, 83].

The theory of slice hyperholomorphic functions of several variables is very far from being developed, but some results can be found in the papers [3, 98, 145]. See also the paper on the Herglotz functions of several quaternionic variables [2].

Finally, the theory of slice hyperholomorphic functions has been extended to the setting of functions with values in a real alternative $*$-algebra [34, 146–149].

The Cauchy transform in the slice hyperholomorphic setting has been studied in [71].

Quaternion-valued positive definite functions on locally compact abelian groups and nuclear spaces have been considered in [17].

Slice hyperholomorphic functions are characterized by the slicewise differential equation (2.5). We, however, point out that slice hyperholomorphic functions also lie in the kernel of a global differential operator with nonconstant coefficients [60, 88, 100, 150].

References on function spaces of slice hyperholomorphic functions. Several function spaces have been extended to the slice hyperholomorphic setting. The quaternionic Hardy space $H_2(\Omega)$, where Ω is either the quaternionic unit ball $\mathbb{B}$ or the half space $\mathbb{H}^+$ of quaternions with positive real part, was introduced and studied in [12, 21, 22, 35]. We point out that the quaternionic Blaschke products were first introduced in the seminal paper [22].

The Hardy spaces $H^p(\mathbb{B})$ for arbitrary $0 < p < +\infty$ were studied in [185]. The slice hyperholomorphic Bergman spaces are studied in [59, 62, 63], the slice hyperholomorphic Fock space is considered in [31] and weighted Bergman spaces, Bloch, Besov, and Dirichlet spaces of slice hyperholomorphic functions on the unit ball $\mathbb{B}$ were introduced in [48]. Inner product spaces and Krein spaces in the quaternionic setting are studied in [26]. Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball are studied in [184]. The BMO and VMO spaces of slice hyperholomorphic functions are considered in [129]. For slice hyperholomorphic fractional Hardy spaces, see [27]. A class of quaternionic positive definite functions and their derivatives is studied in [29]. For a quaternionic analogue of the Segal–Bargmann transform, see [102].

References on slice hyperholomorphic Schur analysis. In recent years, a slice hyperholomorphic version of Schur analysis has also been developed in [1, 3, 7, 8, 12, 15, 16, 21–25, 32]. An overview of classical theory can, for example, be found in [6]. In the book [18] there is an extended introduction to the theory of Schur analysis in the slice hyperholomorphic setting. Recent results on Schur analysis, related topics and quaternionic polynomials can be found in the papers [40–46].