Chapter 15

Bounded Quaternionic Spectral Operators

We turn our attention now to the study of quaternionic linear spectral operators, in which we generalize the complex linear theory in [106]. The results presented in this chapter can be found in [125] and in [128].

15.1 The Spectral Decomposition of a Spectral Operator

A complex spectral operator is a bounded operator A that has a spectral resolution, i.e., there exists a spectral measure E defined on the Borel sets $\mathfrak{B}(\mathbb{C})$ on $\mathbb{C}$ such that $\sigma_S(A|_\Delta) \subset \overline{\Delta}$ with $A_\Delta = A|_{\operatorname{ran} E(\Delta)}$ for all $\Delta \in \mathfrak{B}(\mathbb{C})$. Chapter 14 showed that spectral systems take over the role of spectral measures in the quaternionic setting. If E is a spectral measure that reduces an operator $T \in \mathcal{B}(V_R)$, then there will in general exist infinitely many imaginary operators J such that (E, J) is a spectral system. We thus have to find a criterion for identifying the one among them that fits the operator T and that can hence serve as its spectral orientation. A first and quite obvious requirement is that T and J commute. This is, however, not sufficient. Indeed, if J and T commute, then also $-J$ and T commute. More generally, every operator that is of the form $\tilde{J} := -E(\Delta)J + E(\mathbb{H} \setminus \Delta)J$ with $\Delta \in \mathfrak{B}_S(\mathbb{H})$ is an imaginary operator such that $(E, \tilde{J})$ is a spectral system that commutes with T.

We develop a second criterion by analogy with the finite-dimensional case. Let $T \in \mathcal{B}(\mathbb{H}^n)$ be the operator on $\mathbb{H}^n$ that is given by the diagonal matrix $T = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and let us assume $\lambda_\ell \notin \mathbb{R}$ for $\ell = 1, \ldots, n$. We intuitively identify the operator $J = \operatorname{diag}(j_{\lambda_1}, \ldots, j_{\lambda_n})$ as the spectral orientation for T, cf. also Example 14.3.10. Obviously J commutes with T. Moreover, if $s_0 \in \mathbb{R}$ and

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$s_1 > 0$ are arbitrary, then the operator $(s_0\mathcal{I} - s_1 J) - T$ is invertible. Indeed, one has

$$(s_0\mathcal{I} - s_1 J) - T = \operatorname{diag}(\overline{s_{j_{\lambda_1}}} - \lambda_1, \ldots, \overline{s_{j_{\lambda_n}}} - \lambda_n),$$

where $s_{j_{\lambda_\ell}} = s_0 + j_{\lambda_\ell} s_1$. Since $\overline{s_{j_{\lambda_\ell}}} - \lambda_\ell = (s_0 - \lambda_{\ell,0}) + j_{\lambda_\ell}(-s_1 - \lambda_{\ell,1})$ and both $s_1 > 0$ and $\lambda_{\ell,1} > 0$ for all $\ell = 1, \ldots, n$, each of the diagonal elements has an inverse, and so

$$((s_0\mathcal{I} - s_1 J) - T)^{-1} = \operatorname{diag}\left((\overline{s_{j_{\lambda_1}}} - \lambda_1)^{-1}, \ldots, (\overline{s_{j_{\lambda_n}}} - \lambda_n)^{-1}\right).$$

This invertibility is the criterion that uniquely identifies J.

Definition 15.1.1. An operator $T \in \mathcal{B}(V_R)$ is called a *spectral operator* if there exists a spectral decomposition for T, i.e., a spectral system (E, J) on V_R such that the following three conditions hold:

(i) The spectral system (E, J) commutes with T, i.e., $E(\Delta)T = TE(\Delta)$ for all $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and $TJ = JT$.

(ii) If we set $T_\Delta := T|_{V_\Delta}$ with $V_\Delta = E(\Delta)V_R$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, then

$$\sigma_S(T_\Delta) \subset \overline{\Delta} \quad \text{for all } \Delta \in \mathfrak{B}_S(\mathbb{H}).$$

(iii) For all $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, the operator $((s_0\mathcal{I} - s_1 J) - T)|_{V_1}$ has a bounded inverse on $V_1 := E(\mathbb{H} \setminus \mathbb{R})V_R = \operatorname{ran} J$.

The spectral measure E is called a *spectral resolution* for T, and the imaginary operator J is called a *spectral orientation* of T.

A first easy result, which we shall use frequently, is that the restriction of a spectral operator to an invariant subspace $E(\Delta)V_R$ is again a spectral operator.

Lemma 15.1.2. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator on V_R and let (E, J) be a spectral decomposition for T. For every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, the operator $T_\Delta = T|_{V_\Delta}$ with $V_\Delta = \operatorname{ran} E(\Delta)$ is a spectral operator on V_Δ. A spectral decomposition for T_Δ is (E_Δ, J_Δ) with $E_\Delta(\sigma) = E(\sigma)|_{V_\Delta}$ and $J_\Delta = J|_{V_\Delta}$.*

Proof. Since $E(\Delta)$ commutes with $E(\sigma)$ for $\sigma \in \mathfrak{B}_S(\mathbb{H})$ and J, the restrictions $E_\Delta(\sigma) = E(\sigma)|_{V_\Delta}$ and $J_\Delta = J|_{V_\Delta}$ are right linear operators on V_Δ. It is immediate that E_Δ is a spectral measure on V_Δ. Moreover,

$$\ker J_\Delta = \ker J \cap V_\Delta = \operatorname{ran} E(\mathbb{R}) \cap V_\Delta = \operatorname{ran} E_\Delta(\mathbb{R})$$

and

$$\operatorname{ran} J_\Delta = \operatorname{ran} J \cap V_\Delta = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) \cap V_\Delta = \operatorname{ran} E_\Delta(\mathbb{H} \setminus \mathbb{R}).$$

Since

$$-J_\Delta^2 = -J^2|_{V_\Delta} = E(\mathbb{H} \setminus \mathbb{R})|_{V_\Delta} = E_\Delta(\mathbb{H} \setminus \mathbb{R}),$$

the operator $-J_\Delta^2$ is the projection of V_Δ onto $\operatorname{ran} J_\Delta$ along $\ker J_\Delta$. Hence J_Δ is an imaginary operator on V_Δ. Moreover, (E_Δ, J_Δ) is a spectral system. Since

$$E_\Delta(\sigma)T_\Delta E(\Delta) = E(\sigma)TE(\Delta) = TE(\sigma)E(\Delta) = T_\Delta E_\Delta(\sigma)E(\Delta),$$

and similarly

$$J_\Delta T_\Delta E(\Delta) = JTE(\Delta) = TJE(\Delta) = T_\Delta J_\Delta E(\Delta),$$

this spectral system commutes with T_Δ.

If $\sigma \in \mathfrak{B}_S(\mathbb{H})$ and we set $V_{\Delta,\sigma} = \operatorname{ran} E_\Delta(\sigma)$, then

$$V_{\Delta,\sigma} = \operatorname{ran} E(\sigma)|_{V_\Delta} = \operatorname{ran} E(\sigma)E(\Delta) = \operatorname{ran} E(\sigma \cap \Delta) = V_{\Delta\cap\sigma}.$$

Thus $T_\Delta|_{V_{\Delta,\sigma}} = T|_{V_{\sigma\cap\Delta}}$ and so $\sigma_S(T_{\Delta,\sigma}) = \sigma_S(T_{\Delta\cap\sigma}) \subset \Delta \cup \sigma \subset \sigma$. Hence E_Δ is a spectral resolution for T_Δ. Finally, for $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, the operator $s_0\mathcal{I} - s_1 J - T$ leaves the subspace $V_{\Delta,1} := \operatorname{ran} E_\Delta(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} E(\Delta \cap (\mathbb{H} \setminus \mathbb{R}))$ invariant because it commutes with E. Hence the restriction of $(s_0\mathcal{I} - s_1 J - T)|_{V_1}^{-1}$ to $V_{\Delta,1} \subset V_1 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ is a bounded linear operator on $V_{\Delta,1}$. It obviously is the inverse of $(s_0\mathcal{I} - s_1 J_\Delta - T_\Delta)|_{V_{\Delta,1}}$. Therefore (E_Δ, J_Δ) is actually a spectral decomposition for T_Δ, which hence is in turn a spectral operator. □

The remainder of this section considers the questions of uniqueness and existence of the spectral decomposition (E, J) of T. We recall the V_R-valued right slice hyperholomorphic function $\mathcal{R}_s(T; y) := \mathcal{Q}_s(T)^{-1} y\bar{s} - T\mathcal{Q}_s(T)^{-1} y$ on $\rho_S(T)$ for $T \in \mathcal{L}(V_R)$ and $y \in V_R$, which was defined in Definition 14.2.8. If T is bounded, then $\mathcal{Q}_s(T)^{-1}$ and T commute, and we have

$$\mathcal{R}_s(T; y) := \mathcal{Q}_s(T)^{-1}(y\bar{s} - Ty).$$

Definition 15.1.3. Let $T \in \mathcal{B}(V_R)$ and let $y \in V_R$. A V_R-valued right slice hyperholomorphic function f defined on an axially symmetric open set $\mathcal{D}(f) \subset \mathbb{H}$ with $\rho_S(T) \subset \mathcal{D}(f)$ is called a *slice hyperholomorphic extension* of $\mathcal{R}_s(T; y)$ if

$$(T^2 - 2s_0 T + |s|^2\mathcal{I})f(s) = y\bar{s} - Ty \quad \forall s \in \mathcal{D}(f). \tag{15.1}$$

Obviously such an extension satisfies

$$f(s) = \mathcal{R}_s(T; y) \quad \text{for } s \in \rho_S(T).$$

Definition 15.1.4. Let $T \in \mathcal{B}(V_R)$ and let $y \in V_R$. The function $\mathcal{R}_s(T; y)$ is said to have the *single-valued extension property* if every two slice hyperholomorphic extensions f and g of $\mathcal{R}_s(T; y)$ satisfy $f(s) = g(s)$ for $s \in \mathcal{D}(f) \cap \mathcal{D}(g)$. In this case,

$$\rho_S(y) := \bigcup \{\mathcal{D}(f) : \ f \text{ is a slice hyperholomorphic extension of } \mathcal{R}_s(T; y)\}$$

is called the *S-resolvent set* of y, and $\sigma_S(y) = \mathbb{H} \setminus \rho_S(y)$ is called the *S-spectrum* of y.

Since it is the union of axially symmetric sets, $\rho_S(y)$ is axially symmetric. Moreover, there exists a unique maximal extension of $\mathcal{R}_s(T;y)$ to $\rho_S(y)$. We shall denote this extension by $y(s)$.

We shall see soon that the single-valued extension property holds for $\mathcal{R}_s(T;y)$ for every $y \in V_R$ if T is a spectral operator. This is, however, not true for an arbitrary operator $T \in \mathcal{B}(V_R)$. A counterexample can be constructed analogously to [106, p. 1932].

Lemma 15.1.5. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution for T. Let $s \in \mathbb{H}$ and let $\Delta \subset \mathbb{H}$ be a closed axially symmetric set such that $s \notin \Delta$. If $y \in V_R$ satisfies $(T^2 - 2s_0T + |s|^2\mathcal{I})y = 0$, then*

$$E(\Delta)y = 0 \quad \textit{and} \quad E([s])y = y.$$

Proof. Assume that $y \in V_R$ satisfies $(T^2 - 2s_0T + |s|^2\mathcal{I})y = 0$ and let T_Δ be the restriction of T to the subspace $V_\Delta = E(\Delta)V$. Since $s \notin \Delta$, we have $s \in \rho_S(T_\Delta)$, and so $\mathcal{Q}_s(T_\Delta)$ is invertible. Since $\mathcal{Q}_s(T_\Delta)^{-1} = \mathcal{Q}_s(T)^{-1}|_{V_\Delta}$, we have

$$\mathcal{Q}_s(T_\Delta)^{-1}(T^2 - 2s_0T + |s|^2\mathcal{I})E(\Delta) = E(\Delta),$$

from which we deduce

$$\begin{aligned} E(\Delta)y &= \mathcal{Q}_s(T_\Delta)^{-1}(T^2 - 2s_0T + |s|^2\mathcal{I})E(\Delta)y \\ &= \mathcal{Q}_s(T_\Delta)^{-1}E(\Delta)(T^2 - 2s_0T + |s|^2\mathcal{I})y = 0. \end{aligned}$$

Now define for $n \in \mathbb{N}$ the closed axially symmetric set

$$\Delta_n = \left\{ p \in \mathbb{H} : \ \operatorname{dist}(p,[s]) \geq \frac{1}{n} \right\}.$$

By the above, we have $E(\Delta_n)y = 0$ and in turn

$$(\mathcal{I} - E([s]))y = \lim_{n\to\infty} E(\Delta_n)y = 0,$$

so that $y = E([s])y$. □

Lemma 15.1.6. *If $T \in \mathcal{B}(V_R)$ is a spectral operator, then for every $y \in V_R$, the function $\mathcal{R}_s(T;y)$ has the single-valued extension property.*

Proof. Let $y \in V_R$ and let f and g be two slice hyperholomorphic extensions of $\mathcal{R}_s(T;y)$. We set $h(s) = f(s) - g(s)$ for $s \in \mathcal{D}(h) = \mathcal{D}(f) \cap \mathcal{D}(g)$.

If $s \in \mathcal{D}(h)$, then there exists an axially symmetric neighborhood $U \subset \mathcal{D}(h)$ of s, and for every $p \in U$ we have

$$\begin{aligned} (T^2 - 2p_0T + |p|^2\mathcal{I})h(p) &= (T^2 - 2p_0T + |p|^2\mathcal{I})f(p) - (T^2 - 2p_0T + |p|^2)g(p) \\ &= (y\overline{p} - Ty) - (y\overline{p} - Ty) = 0. \end{aligned}$$

If E is a spectral resolution of T, then we can conclude from the above and Lemma 15.1.5 that $E([p])h(p) = h(p)$ for $p \in U$. We consider now a sequence $s_n \in U$ with $s_n \neq s$ for $n \in U$ such that $s_n \to s$ as $n \to \infty$ and obtain

$$0 = E([s])E([s_n])h(s_n) = E([s])h(s_n) \to E([s])h(s) = h(s).$$

Hence $f(s) = g(s)$, and $\mathcal{R}_s(T, y)$ has the single-valued extension property. □

Corollary 15.1.7. *If $T \in \mathcal{B}(V_R)$ is a spectral operator, then for every $y \in V_R$, the function $\mathcal{R}_s(T; y)$ has a unique maximal slice hyperholomorphic extension to $\rho_S(y)$. We denote this maximal slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$ by $y(\cdot)$.*

Corollary 15.1.8. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let $y \in V_R$. Then $\sigma_S(y) = \emptyset$ if and only if $y = 0$.*

Proof. If $y = 0$, then $y(s) = 0$ is the maximal slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$. It is defined on all of $\mathbb{H}$, and hence $\sigma_S(y) = \emptyset$.

Now assume that $\sigma_S(y) = \emptyset$ for some $y \in V_R$ such that the maximal slice hyperholomorphic extension $y(\cdot)$ of $\mathcal{R}_s(T; y)$ is defined on all of $\mathbb{H}$. For every $w^* \in V_R^*$, the function $s \to \langle w^*, y(s)\rangle$ is an entire right slice hyperholomorphic function. From the fact that $\mathcal{R}_s(T; y)$ equals the resolvent of T as a bounded operator on V_{R,j_s}, we deduce $\lim_{s\to\infty} \mathcal{R}_s(T; y) = 0$ and then

$$\lim_{s\to\infty} \langle w^*, y(s)\rangle = \lim_{s\to\infty} \langle w^*, \mathcal{R}_s(T; y)\rangle = 0.$$

Liouville's theorem for slice hyperholomorphic functions therefore implies that $\langle w^*, y(s)\rangle = 0$ for all $s \in \mathbb{H}$. Since w^* was arbitrary, we obtain $y(s) = 0$ for all $s \in \mathbb{H}$.

Finally, we can choose $s \in \rho_S(T)$ such that the operator $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2\mathcal{I}$ is invertible, and we find because of (15.1) that

$$\begin{aligned} 0 = y(s)s - Ty(s) &= \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)y(s)s - T\mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)y(s) \\ &= \mathcal{Q}_s(T)^{-1}\left(\mathcal{Q}_s(T)y(s)s - T\mathcal{Q}_s(T)y(s)\right) \\ &= \mathcal{Q}_s(T)^{-1}((y\overline{s} - Ty)s - T(y\overline{s} - Ty)) \\ &= \mathcal{Q}_s(T)^{-1}\left(T^2y - Ty2s_0 + y|s|^2\right) = \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)y = y. \end{aligned}$$ □

Theorem 15.1.9. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution for T. If $\Delta \in \mathfrak{B}_S(\mathbb{H})$ is closed, then*

$$E(\Delta)V_R = \{y \in V_R : \ \sigma_S(y) \subset \Delta\}.$$

Proof. Let $V_\Delta = E(\Delta)V_R$ and let T_Δ be the restriction of T to V_Δ. Since Δ is closed, Definition 15.1.1 implies $\sigma_S(T_\Delta) \subset \Delta$. Moreover $\mathcal{Q}_s(T_\Delta) = \mathcal{Q}_s(T)|_{V_\Delta}$ for $s \in \mathbb{H}$. If $y \in V_\Delta$, then

$$\mathcal{Q}_s(T)\mathcal{R}_s(T; y) = \mathcal{Q}_s(T_\Delta)\mathcal{Q}_s(T_\Delta)^{-1}(y\overline{s} - T_\Delta y) = y\overline{s} - Ty$$

for $s \in \rho_S(T_\Delta)$, and hence $\mathcal{R}_s(T_\Delta; y)$ is a slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$ to $\rho_S(T_\Delta) \supset \mathbb{H} \setminus \Delta$. Thus $\sigma_S(y) \subset \Delta$. Since $y \in V_R$ was arbitrary, we obtain $E(\Delta)V_R \subset \{y \in V_R : \sigma_S(y) \subset \Delta\}$.

In order to show the converse relation, we assume that $\sigma_S(y) \subset \Delta$. We consider a closed subset $\sigma \in \mathfrak{B}_S(\mathbb{H})$ of the complement of Δ and set $T_\sigma = T|_{V_\sigma}$ with $V_\sigma = E(\sigma)V_R$. As above, $\mathcal{R}_s(T_\sigma; E(\sigma)y)$ is then a slice hyperholomorphic extension of $\mathcal{R}_s(T; E(\sigma)y)$ to $\mathbb{H} \setminus \sigma$. If, on the other hand, $y(s)$ is the unique maximal slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$, then

$$\begin{aligned}\mathcal{Q}_s(T)E(\sigma)y(s) &= E(\sigma)\mathcal{Q}_s(T)y(s)\\ &= E(\sigma)(y\overline{s} - Ty) = (E(\sigma)y)\overline{s} - T(E(\sigma)y)\end{aligned}$$

for $s \in \mathbb{H}\setminus\Delta$, and so $E(\sigma)y(s)$ is a slice hyperholomorphic extension of $\mathcal{R}_s(T; E(\sigma)y)$ to $\mathbb{H} \setminus \Delta$. Combining these two extensions, we find that $\mathcal{R}_s(T; E(\sigma)y)$ has a slice hyperholomorphic extension to all of $\mathbb{H}$. Hence $\sigma_S(E(\sigma)y) = \emptyset$, so that $E(\Delta)y = 0$ by Corollary 15.1.8.

Let us now choose an increasing sequence of closed subsets $\sigma_n \in \mathfrak{B}_S(\mathbb{H})$ of $\mathbb{H} \setminus \Delta$ such that $\bigcup_{n\in\mathbb{N}} \sigma_n = \mathbb{H} \setminus \Delta$. By the above arguments, $E(\sigma_n)y = 0$ for every $n \in \mathbb{N}$. Hence

$$E(\mathbb{H} \setminus \Delta)y = \lim_{n\to\infty} E(\Delta_n)y = 0,$$

so that in turn $E(\Delta)y = y$. We thus obtain $E(\Delta)V_R \supset \{y \in V_R : \sigma_S(y) \subset \Delta\}$. □

The following corollaries are immediate consequences of Theorem 15.1.9.

Corollary 15.1.10. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution of T. Then $E(\sigma_S(T)) = \mathcal{I}$.*

Corollary 15.1.11. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let $\Delta \in \mathfrak{B}_S(\mathbb{H})$ be closed. The set of all $y \in V_R$ with $\sigma_S(y) \subset \Delta$ is a closed right subspace of V_R.*

Lemma 15.1.12. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator. If $A \in \mathcal{B}(V_R)$ commutes with T, then A commutes with every spectral resolution E for T. Moreover, $\sigma_S(Ay) \subset \sigma_S(y)$ for all $y \in V_R$.*

Proof. For $y \in V_R$ we have

$$\begin{aligned}(T^2 - 2s_0T + |s|^2\mathcal{I})Ay(s) &= A(T^2 - 2s_0T + |s|^2\mathcal{I})y(s)\\ &= A(y\overline{s} - Ty) = (Ay)\overline{s} - T(Ay).\end{aligned}$$

The function $Ay(s)$ is therefore a slice hyperholomorphic extension of $\mathcal{R}_s(T; Ay)$ to $\rho_S(y)$, and so $\sigma_S(Ay) \subset \sigma_S(y)$. From Theorem 15.1.9 we deduce that

$$AE(\Delta)V \subset E(\Delta)V$$

for every closed axially symmetric subset Δ of $\mathbb{H}$.

If σ and Δ are two disjoint closed axially symmetric sets, we therefore have

$$E(\Delta)AE(\Delta) = AE(\Delta) \quad \text{and} \quad E(\Delta)AE(\sigma) = E(\Delta)E(\sigma)AE(\sigma) = 0.$$

If we choose again an increasing sequence of closed sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$ with $\mathbb{H}\setminus\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$, we therefore have

$$E(\Delta)AE(\mathbb{H}\setminus\Delta)y = \lim_{n\to\infty} E(\Delta)AE(\Delta_n)y = 0 \quad \forall y \in V_R$$

and hence

$$E(\Delta)A = E(\Delta)A[E(\Delta) + E(\mathbb{H}\setminus\Delta)] = E(\Delta)AE(\Delta) = AE(\Delta). \tag{15.2}$$

Since Δ was an arbitrary closed set in $\mathfrak{B}_S(\mathbb{H})$ and since the sigma-algebra $\mathfrak{B}_S(\mathbb{H})$ is generated by sets of this type, we finally conclude that (15.2) holds for every set $\sigma \in \mathfrak{B}_S(\mathbb{H})$. □

Lemma 15.1.13. *The spectral resolution E of a spectral operator $T \in \mathcal{B}(V_R)$ is uniquely determined.*

Proof. Let E and $\tilde{E}$ be two spectral resolutions of T. For every closed set $\Delta \in \mathfrak{B}_S(\mathbb{H})$, Theorem 15.1.9 implies

$$\tilde{E}(\Delta)E(\Delta) = E(\Delta) \quad \text{and} \quad E(\Delta)\tilde{E}(\Delta) = \tilde{E}(\Delta),$$

and we deduce from Lemma 15.1.12 that $E(\Delta) = \tilde{E}(\Delta)$. Since the sigma algebra $\mathfrak{B}_S(\mathbb{H})$ is generated by the closed sets in $\mathfrak{B}_S(\mathbb{H})$, we obtain $E = \tilde{E}$, and hence the spectral resolution of T is uniquely determined. □

Before we consider the uniqueness of the spectral orientation, we observe that for certain operators, the existence of a spectral resolution already implies the existence of a spectral orientation and is hence sufficient for them to be a spectral operator.

Proposition 15.1.14. *Let $T \in \mathcal{B}(V_R)$ and assume that there exists a spectral resolution E for T. If $\sigma_S(T)\cap\mathbb{R} = \emptyset$, then there exists an imaginary operator $J \in \mathcal{B}(V_R)$ that is a spectral orientation for T such that T is a spectral operator with spectral resolution (E, J). Moreover, this spectral orientation is unique.*

Proof. Since $\sigma_S(T)$ is closed with $\sigma_S(T) \cap \mathbb{R} = \emptyset$, we have $\operatorname{dist}(\sigma_S(T), \mathbb{R}) > 0$. We choose $j \in \mathbb{S}$ and consider T a complex linear operator on $V_{R,j}$. Because of Theorem 14.2.7, the spectrum of T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$ is $\sigma_{\mathbb{C}_j}(T) = \sigma_S(T) \cap \mathbb{C}_j$. Since $\operatorname{dist}(\sigma_S(T), \mathbb{R}) > 0$, the sets

$$\sigma_+ = \sigma_{\mathbb{C}_j}(T) \cap \mathbb{C}_j^+ \quad \text{and} \quad \sigma_- = \sigma_{\mathbb{C}_j}(T) \cap \mathbb{C}_j^-$$

are open and closed subsets of $\sigma_{\mathbb{C}_j}(T)$ such that $\sigma_+ \cup \sigma_- = \sigma_{\mathbb{C}_j}(T)$. Via the Riesz–Dunford functional calculus we can hence associate spectral projections E_+ and

E_- onto closed invariant $\mathbb{C}_j$-linear subspaces of $V_{R,j}$ to σ_+ and σ_-. The resolvent of T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$ at $z \in \rho_{\mathbb{C}_j}(T)$ is $R_z(T)y := \mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)$, and hence these projections are given by

$$\begin{aligned} E_+ y &:= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\, dz \frac{1}{2\pi j}, \\ E_- y &:= \int_{\Gamma_-} \mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\, dz \frac{1}{2\pi j}, \end{aligned} \tag{15.3}$$

where Γ_+ is a positively oriented Jordan curve that surrounds σ_+ in $\mathbb{C}_j^+$ and Γ_- is a positively oriented Jordan curve that surrounds σ_- in $\mathbb{C}_j^-$. We set

$$Jy := E_- y(-j) + E_+ yj.$$

From Theorem 14.2.10 we deduce that J is an imaginary operator on V_R if $\Psi : y \mapsto yi$ is a bijection between $V_+ := E_+ V_R$ and $V_- := E_- V_R$ for $i \in \mathbb{S}$ with $i \perp j$. This is indeed the case: due to the symmetry of $\sigma_{\mathbb{C}_j}(T) = \sigma_S(T) \cap \mathbb{C}_j$ with respect to the real axis, we obtain $\sigma_+ = \overline{\sigma_-}$, so that we can choose $\Gamma_-(t) = \overline{\Gamma_+(1-t)}$ for $t \in [0,1]$ in (15.3). Because of the relation (14.14) established in Theorem 14.2.7, the resolvent $R_z(T)$ of T as an operator on $V_{R,j}$ satisfies $R_{\overline{z}}(T)y = -\left[R_z(T)(yi)\right] i$, and so

$$\begin{aligned} E_- y &= \int_{\Gamma_-} R_z(T) y \, dz \frac{1}{2\pi j} = - \int_{\Gamma_+} R_{\overline{z}}(T) y \, d\overline{z} \frac{1}{2\pi j} \\ &= \int_{\Gamma_+} \left[R_z(T)(yi)\right] i \, d\overline{z} \frac{1}{2\pi j} = \int_{\Gamma_+} \left[R_z(T)(yi)\right] \, dz \frac{1}{2\pi j} (-i) = \left[E_+(yi)\right](-i). \end{aligned}$$

Hence we have

$$(E_- y)\, i = E_+(yi) \quad \forall y \in V_R. \tag{15.4}$$

If $y \in V_-$, then $yi = (E_- y)i = E_+(yi)$, and so $yi \in V_+$. Replacing y by yi in (15.4), we find that also $(E_- yi)\, i = -E_+(y)$ and in turn $E_-(yi) = E_+(y)i$. For $y \in V_+$ we thus find that $yi = E_+(y)i = E_-(yi)$, and so $yi \in V_-$. Hence Ψ maps V_+ to V_- and V_- to V_+, and since $\Psi^{-1} = -\Psi$, it is even bijective. We conclude that J is actually an imaginary operator.

Let us now show that (i) in Definition 15.1.1 holds. For every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, the operator $\mathcal{Q}_z(T)^{-1}$ commutes with $E(\Delta)$. Hence

$$\begin{aligned} E(\Delta) E_+ y &= \int_{\Gamma_+} E(\Delta) \mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\, dz \frac{1}{2\pi j} \\ &= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(E(\Delta) y\overline{z} - TE(\Delta)y)\, dz \frac{1}{2\pi j} = E_+ E(\Delta) y \end{aligned} \tag{15.5}$$

for every $y \in V_{R,j} = V_R$, and so $E_+ E(\Delta) = E(\Delta) E_+$. Similarly, one can show that also $E(\Delta) E_- = E_- E(\Delta)$. By construction, the operator J hence commutes with T and with $E(\Delta)$ for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, since

$$TJy = TE_-y(-j) + TE_+yj = E_-Ty(-j) + E_+Tyj = JTy$$

and

$$\begin{aligned} E(\Delta)Jy &= E(\Delta)E_-y(-j) + E(\Delta)E_+yj \\ &= E_-E(\Delta)y(-j) + E_+E(\Delta)yj = JE(\Delta)y. \end{aligned}$$

Moreover, since $\sigma_S(T) \cap \mathbb{R} = \emptyset$, Corollary 15.1.10 implies $\operatorname{ran} E(\mathbb{R}) = \{0\} = \ker J$ and $\operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = V_R = \operatorname{ran} J$. Hence (E, J) is actually a spectral system that moreover commutes with T.

Let us now show condition (iii) of Definition 15.1.1. If $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, then set $s_j := s_0 + js_1$. Since $E_+ + E_- = \mathcal{I}$, we then have

$$\begin{aligned} &((s_0\mathcal{I} - s_1J) - T)y \\ &= (E_+ + E_-)ys_0 - (E_+y)js_1 - (E_-y)(-j)s_1 - T(E_+ + E_-)y \\ &= (E_+y)(s_0 - s_1j) - T(E_+y) + (E_-y))(s_0 + s_1j) - T(E_-y) \\ &= (E_+y)\overline{s_j} - T(E_+y) + (E_-y)s_j - T(E_-y) \\ &= (\overline{s_j}\mathcal{I}_{V_{R,j}} - T)E_+y + (s_j\mathcal{I}_{V_{R,j}} - T)E_-y. \end{aligned}$$

Since E_+ and E_- are the Riesz projectors associated to σ_+ and σ_-, the spectrum $\sigma(T_+)$ of $T_+ := T|_{V_+}$ is $\sigma_+ \subset \mathbb{C}_j^+$ and the spectrum $\sigma(T_-)$ of $T_- := T|_{V_-}$ is $\sigma_- \subset \mathbb{C}_j^-$. Since s_j has positive imaginary part, we find that $\overline{s_j} \in \mathbb{C}_j^- \subset \rho(T_+)$ and $s_j \in \mathbb{C}_j^+ \subset \rho(T_-)$, so that $R_{\overline{s}_j}(T_+) := \left(\overline{s_j}\mathcal{I}_{V_+} - T_+\right)^{-1} \in \mathcal{B}(V_+)$ and $R_{s_j}(T)^{-1} := \left(s_j\mathcal{I}_{V_-} - T_-\right)^{-1} \in \mathcal{B}(V_-)$ exist. Since $E_+|_{V_+} = \mathcal{I}_{V_+}$ and $E_-|_{V_+} = 0$, they satisfy the relations

$$E_+R_{\overline{s_j}}(T_+)E_+ = R_{\overline{s_j}}(T_+)E_+ \quad \text{and} \quad E_-R_{\overline{s_j}}(T_+)E_+ = 0 \tag{15.6}$$

and similarly also

$$E_-R_{s_j}(T_-)E_- = R_{s_j}(T_-)E_- \quad \text{and} \quad E_+R_{s_j}(T_-)E_- = 0. \tag{15.7}$$

Setting $R(s_0, s_1) := R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-$, we obtain a bounded $\mathbb{C}_j$-linear operator that is defined on the entire space $V_{R,j} = V_R$. Because E_+ and E_- commute with T and satisfy $E_+E_- = E_-E_+ = 0$ and because (15.6) and (15.7) hold, we obtain for every $y \in V_R$,

$$\begin{aligned} &R(s_0, s_1)((s_0\mathcal{I} - s_1J) - T)y \\ &= \left[R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-\right]\left[(\overline{s_j}\mathcal{I}_{V_{R,j}} - T)E_+y + (s_j\mathcal{I}_{V_{R,j}} - T)E_-y\right] \\ &= R_{\overline{s_j}}(T_+)(\overline{s_j}\mathcal{I}_{V_{R,j}} - T_+)E_+y + R_{s_j}(T_-)E_-(s_j\mathcal{I}_{V_{R,j}} - T_-)E_-y \\ &= E_+y + E_-y = y \end{aligned}$$

and

$$\begin{aligned}
&((s_0\mathcal{I} - s_1 J) - T)R(s_0, s_1)y \\
&= \left[(\overline{s_j}\mathcal{I}_{V_{R,j}} - T)E_+ + (s_j\mathcal{I}_{V_{R,j}} - T)E_-\right]\left[R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-\right]y \\
&= (\overline{s_j}\mathcal{I}_{V_+} - T_+)R_{\overline{s_j}}(T_+)E_+y + (s_j\mathcal{I}_{V_-} - T_-)R_{s_j}(T_-)E_-y \\
&= E_+y + E_-y = y.
\end{aligned}$$

Hence $R(s_0, s_1) \in \mathcal{B}(V_{R,j})$ is the $\mathbb{C}_j$-linear bounded inverse of $(s_0\mathcal{I} - s_1 J) - T$. Since $(s_0\mathcal{I} - s_1 J) - T$ is quaternionic right linear, its inverse is quaternionic right linear too, so that even $((s_0\mathcal{I} - s_1 J) - T)^{-1} \in \mathcal{B}(V_R)$. Therefore, J is actually a spectral orientation for T, and T is in turn a spectral operator with spectral decomposition (E, J).

Finally, we show the uniqueness of the spectral orientation J. Assume that $\widetilde{J}$ is an arbitrary spectral orientation for T. We show that $\widetilde{V_+} := V^+_{\widetilde{J},j}$ equals $V_+ = V^+_{J,j}$. Theorem 14.2.10 implies then $J = \widetilde{J}$ because $\ker J = \ker \widetilde{J} = \operatorname{ran} E(\mathbb{R}) = \{0\}$ and $V^-_{J,j} = V_+ i = \widetilde{V_+} i = V^-_{\widetilde{J},j}$.

Since $\widetilde{J}$ commutes with T, we have $\widetilde{J}E_+ = E_+\widetilde{J}$, since

$$\begin{aligned}
\widetilde{J}E_+y &= \int_{\Gamma_+} \widetilde{J}\mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\,dz\frac{1}{2\pi j} \\
&= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(\widetilde{J}y\overline{z} - T\widetilde{J}y)\,dz\frac{1}{2\pi j} = E_+\widetilde{J}y.
\end{aligned} \tag{15.8}$$

The projection E_+ therefore leaves $\widetilde{V_+}$ invariant because

$$\widetilde{J}(E_+y) = E_+\big(\widetilde{J}y\big) = (E_+y)j \in \widetilde{V_+}$$

for every $y \in \widetilde{V_+}$. Hence $E_+|_{\widetilde{V_+}}$ is a projection on $\widetilde{V_+}$.

We show now that $\ker E_+|_{\widetilde{V_+}} = \{0\}$, so that $E_+|_{\widetilde{V_+}} = \mathcal{I}_{V_+}$ and hence $\widetilde{V_+} \subset \operatorname{ran} E_+ = V_+$. We do this by constructing a slice hyperholomorphic extension of $\mathcal{R}_s(T;y)$ that is defined on all of $\mathbb{H}$ and applying Corollary 15.1.8 tor any $y \in \ker E_+|_{\widetilde{V_+}}$.

Let $y \in \ker E_+|_{\widetilde{V_+}}$. Since $\ker E_+|_{\widetilde{V_+}} \subset \ker E_+ = \operatorname{ran} E_- = V_-$, we obtain $y \in V_-$. For $z = z_0 + z_1 j \in \mathbb{C}_j$, we define the function

$$f_j(z;y) := \begin{cases} R_z(T_-)y, & z_1 \geq 0, \\ \Big(z_0\mathcal{I} + z_1\widetilde{J} - T)\Big)^{-1} y, & z_1 < 0. \end{cases}$$

This function is (right) holomorphic on $\mathbb{C}_j$. On $\mathbb{C}_j^+$ this is obvious because the resolvent of T_- is a holomorphic function. For $z_1 < 0$, we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial}{\partial z_0} f_j(z;y) + \frac{\partial}{\partial z_1} f_j(z;y) j\right) \\
&= \frac{1}{2}\left(-\left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2} y - \left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2} \widetilde{J}yj\right) \\
&= \frac{1}{2}\left(-\left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2} y - \left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2} yj^2\right) = 0,
\end{aligned}$$

since $\widetilde{J}y = yj$ because $y \in \widetilde{V_+} = V^+_{\widetilde{J},j}$. The slice extension $f(s;y)$ of $f_j(s;y)$ obtained from Lemma 2.1.11 is a slice hyperholomorphic extension of $\mathcal{R}_s(T;y)$ to all of $\mathbb{H}$ in the sense of Definition 15.1.3. Indeed, since

$$\mathcal{Q}_z(T)|_{V_-} = \mathcal{Q}_z(T_-) = (\mathcal{I}_{V_-}\overline{z} - T_-)(\mathcal{I}_{V_-}z - T_-),$$

we find for $s \in \mathbb{C}_j^+$ that

$$\begin{aligned}
&\mathcal{Q}_s(T)f(s;y) = \mathcal{Q}_s(T_-)f_j(s;y) \\
&= (\overline{s}\mathcal{I}_{V_-} - T_-)(s\mathcal{I}_{V_-} - T_-)R_s(T_-)y \\
&= (\overline{s}\mathcal{I}_{V_-} - T_-)y = ys - T_-y = ys - Ty.
\end{aligned}$$

On the other hand, the facts that T and $\widetilde{J}$ commute and that $-\widetilde{J}^2 = \mathcal{I}$ because $\widetilde{J}$ is an imaginary operator with $\operatorname{ran}\widetilde{J} = V_R$ imply

$$\begin{aligned}
&\left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right) \\
&= s_0^2\mathcal{I} - s_0s_1\widetilde{J} - s_0T + s_0s_1\widetilde{J} - s_1^2\widetilde{J}^2 - s_1\widetilde{J}T - s_0T + s_1T\widetilde{J} + T^2 \\
&= |s|^2\mathcal{I} - 2s_0T + T^2 = \mathcal{Q}_s(T).
\end{aligned}$$

For $s = s_1 + (-j)s_1 \in \mathbb{C}_j^-$, we find thus because of $y \in \widetilde{V_+} = V^+_{\widetilde{J},j}$ that

$$\begin{aligned}
\mathcal{Q}_s(T)f(s;y) &= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right) f_j(s;y) \\
&= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)^{-1} y \\
&= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right) y = ys_0 + yjs_1 - T = y\overline{s} - Ty.
\end{aligned}$$

Finally, for $s \notin \mathbb{C}_j$, the representation formula yields

$$\begin{aligned}
\mathcal{Q}_s(T)f(s;y) &= \mathcal{Q}_s(T)f_j(s_j;y)(1 - jj_s)\frac{1}{2} + \mathcal{Q}_s(T)f_j\left(\overline{s_j};y\right)(1 + jj_s)\frac{1}{2} \\
&= \left(y\overline{s_j} - Ty\right)(1 - jj_s)\frac{1}{2} + (ys_j - Ty)(1 + jj_s)\frac{1}{2} \\
&= y\left(\overline{s_j}(1 - jj_s) + s(1 + jj_s)\right)\frac{1}{2} - Ty\left((1 - jj_s) + (1 + jj_s)\right)\frac{1}{2} \\
&= y(s_j + \overline{s_j} + (s_j - \overline{s_j})jj_s)\frac{1}{2} - Ty = y(s_0 - s_1j_s) - Ty = y\overline{s} - Ty.
\end{aligned}$$

From Corollary 15.1.8, we hence deduce that $y = 0$, and so $\ker E_+|_{\widetilde{V_+}} = \{0\}$. Since $E_+|_{\widetilde{V_+}}$ is a projection on $\widetilde{V_+}$, we have $\widetilde{V_+} = \ker E_+|_{\widetilde{V_+}} \oplus \operatorname{ran} E_+|_{\widetilde{V_+}} = \{0\} \oplus \operatorname{ran} E_+|_{\widetilde{V_+}}$. We conclude that $\widetilde{V_+} = \operatorname{ran} E_+|_{\widetilde{V_+}} \subset \operatorname{ran} E_+ = V_+$. We therefore have

$$V_R = \widetilde{V_+} \oplus \widetilde{V_+} i \subset V_+ \oplus V_+ i = V_R.$$

This implies $V_+ = \widetilde{V_+}$ and in turn $J = \widetilde{J}$. □

Corollary 15.1.15. *Let $T \in \mathcal{B}(V_R)$ and assume that there exists a spectral resolution for T as in Proposition 15.1.14. If $\sigma_S(T) = \Delta_1 \cup \Delta_2$ with closed sets $\Delta_1, \Delta_2 \in \mathfrak{B}_S(\mathbb{H})$ such that $\Delta_1 \subset \mathbb{R}$ and $\Delta_2 \cap \mathbb{R} = \emptyset$, then there exists a unique imaginary operator $J \in \mathcal{B}(V_R)$ that is a spectral orientation for T such that T is a spectral operator with spectral decomposition (E, J).*

Proof. Let $T_2 = T_2|_{V_2}$, where $V_2 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} E(\Delta_2)$. Then the spectral measure $E_2(\Delta) := E(\Delta)|_{V_2}$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$ is by Lemma 15.1.2 a spectral resolution for T_2. Since $\sigma_S(T_2) \subset \Delta_2$ and $\Delta_2 \cap \mathbb{R} = \emptyset$, Proposition 15.1.14 implies the existence of a unique spectral orientation J_2 for T_2.

The fact that (E_2, J_2) is a spectral system implies $\operatorname{ran} J_2 = \operatorname{ran} E_2(\mathbb{H}\setminus\mathbb{R})V_2 = V_2$ because $E_2(\mathbb{H} \setminus \mathbb{R}) = E(\mathbb{H} \setminus \mathbb{R})|_{V_2} = \mathcal{I}_{V_2}$. If we set $J = J_2 E(\mathbb{H} \setminus \mathbb{R})$, we find that $\ker J = \operatorname{ran} E(\mathbb{R})$ and $\operatorname{ran} J = V_2 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. We also have

$$\begin{aligned} E(\Delta)J &= E(\Delta \cap \mathbb{R}) J_2 E(\mathbb{H} \setminus \mathbb{R}) + E(\Delta \setminus \mathbb{R}) J_2 E(\mathbb{H} \setminus \mathbb{R}) \\ &= E_2(\Delta \setminus \mathbb{R}) J_2 E(\mathbb{H} \setminus \mathbb{R}) = J_2 E_2(\Delta \setminus \mathbb{R}) E(\mathbb{H} \setminus \mathbb{R}) \\ &= J_2 E(\Delta \setminus \mathbb{R}) E(\mathbb{H} \setminus \mathbb{R}) = J_2 E(\mathbb{H} \setminus \mathbb{R}) E(\Delta \setminus \mathbb{R}) = J E(\Delta), \end{aligned}$$

where the last identity used that $E(\mathbb{H} \setminus \mathbb{R}) E(\Delta \cap \mathbb{R}) = 0$. Moreover, we have

$$-J^2 = -J_2 E(\mathbb{H} \setminus \mathbb{R}) J_2 E(\mathbb{H} \setminus \mathbb{R}) = -J_2^2 E(\mathbb{H} \setminus \mathbb{R}) = E(\mathbb{H} \setminus \mathbb{R}),$$

so that $-J^2$ is a projection onto $\operatorname{ran} J = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ along $\ker J = \operatorname{ran} E(\mathbb{R})$. Hence, J is an imaginary operator and (E, J) is a spectral system on V_R. Finally, for every $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we have

$$\left((s_0 \mathcal{I} - s_1 J - T)|_{V_2}\right)^{-1} = (s_0 \mathcal{I}_{V_2} - s_1 J_2 - T_2)^{-1} \in \mathcal{B}(V_2),$$

and hence (E, J) is actually a spectral decomposition for T.

In order to show the uniqueness of J we consider an arbitrary spectral orientation $\widetilde{J}$ for T. Then

$$\ker \widetilde{J} = E(\mathbb{R}) V_R = \ker J \quad \text{and} \quad \operatorname{ran} \widetilde{J} = E(\mathbb{H} \setminus \mathbb{R}) V_R = \operatorname{ran} J. \tag{15.9}$$

By Lemma 15.1.2, the operator $\widetilde{J}|_{V_2}$ is a spectral orientation for T_2. The spectral orientation of T_2 is, however, unique by Proposition 15.1.14, and hence $\widetilde{J}|_{V_2} = J_2 = J|_{V_2}$. We conclude that $\widetilde{J} = J$. □

Finally, we can now show the uniqueness of the spectral orientation of an arbitrary spectral operator.

Theorem 15.1.16. *The spectral decomposition* (E, J) *of a spectral operator* $T \in \mathcal{B}(V_R)$ *is uniquely determined.*

Proof. The uniqueness of the spectral resolution E has already been shown in Lemma 15.1.13. Let J and $\widetilde{J}$ be two spectral orientations for T. Since (15.9) holds also in this case, we can reduce the problem to showing that $J|_{V_1} = \widetilde{J}|_{V_1}$ with $V_1 := \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. The operator $T_1 := T|_{V_1}$ is a spectral operator on V_1. By Lemma 15.1.2, (E_1, J_1) and $(E_1, \widetilde{J}_1)$ with $E_1(\Delta) = E(\Delta)|_{V_1}$ and $J_1 = J|_{V_1}$ and $\widetilde{J}_1 := \widetilde{J}|_{V_1}$ are spectral decompositions of T_1. Since $E_0(\mathbb{R}) = 0$, it is hence sufficient to show the uniqueness of the spectral orientation of a spectral operator under the assumption $E(\mathbb{R}) = 0$.

Therefore, let T be a spectral operator with spectral decomposition (E, J) such that $E(\mathbb{R}) = 0$. If $\operatorname{dist}(\sigma_S(T), \mathbb{R}) > 0$, then we already know that the statement holds. We have shown this in Proposition 15.1.14. Otherwise, we choose a sequence of pairwise disjoint sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$ with $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ that cover $\sigma_S(T) \setminus \mathbb{R}$. We can choose, for instance,

$$\Delta_n := \left\{ s \in \mathbb{H} : \ -\|T\| \leq s_0 \leq \|T\|, \ \frac{\|T\|}{n+1} < s_1 \leq \frac{\|T\|}{n} \right\}.$$

By Corollary 15.1.10 and since $E(\mathbb{R}) = 0$, we have

$$E(\sigma_S(T) \setminus \mathbb{R}) = E(\sigma_S(T) \setminus \mathbb{R}) + E(\sigma_S(T) \cap \mathbb{R}) = E(\sigma_S(T)) = \mathcal{I}.$$

We therefore obtain $\sum_{n=0}^{+\infty} E(\Delta_n) y = E\left(\bigcup_{n \in \mathbb{N}} \Delta_n\right) y = y$ for every $y \in V_R$ because we have $\sigma_S(T) \setminus \mathbb{R} \subset \bigcup_{n \in \mathbb{N}} \Delta_n$.

Since $E(\Delta_n)$ and J commute, the operator J leaves $V_{\Delta_n} := \operatorname{ran} E(\Delta_n)$ invariant. Hence $J_{\Delta_n} = J|_{V_{\Delta_n}}$ is a bounded operator on V_{Δ_n}, and we have

$$Jy = J \sum_{n=0}^{+\infty} E(\Delta_n) y = \sum_{n=1}^{+\infty} J E(\Delta_n) y = \sum_{n=1}^{+\infty} J_{\Delta_n} E(\Delta_n) y.$$

Similarly, we see that also $\widetilde{J_{\Delta_n}} := \widetilde{J}|_{V_{\Delta_n}}$ is a bounded operator on V_{Δ_n} and that $\widetilde{J} y = \sum_{n=1}^{+\infty} \widetilde{J}_{\Delta_n} E(\Delta_n) y$.

Now observe that T_{Δ_n} is a spectral operator. Its spectral resolution is given by $E_n(\Delta) := E(\Delta)|_{V_{\Delta_n}}$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, as one can check easily. Its spectral orientation is given by J_{Δ_n}: for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, we have

$$E_n(\Delta) J_{\Delta_n} E(\Delta_n) = E(\Delta) J E(\Delta_n) = J E(\Delta) E(\Delta_n) = J_{\Delta_n} E_n(\Delta) E(\Delta_n)$$

and hence $E_n(\Delta) J_{\Delta_n} = J_{\Delta_n} E(\Delta_n)$ on V_{Δ_n}. Since $\ker J_{\Delta_n} = \{0\} = E_n(\mathbb{R})$ and $\operatorname{ran} J_{\Delta_n} = V_{\Delta_n} = E_n(\mathbb{H} \setminus \mathbb{R})$, the pair (E, J_{Δ_n}) is actually a spectral system. Furthermore, the operators T_{Δ_n} and J_{Δ_n} commute, since

$$T_{\Delta_n} J_{\Delta_n} E(\Delta_n) = T J E(\Delta_n) = J T E(\Delta_n) = J_{\Delta_n} T_{\Delta_n} E(\Delta_n).$$

Finally, for all $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we obtain

$$(s_0\mathcal{I}_{V_{\Delta_n}} - s_1 J_{\Delta_n} - T_{\Delta_n})^{-1} = (s_0\mathcal{I} - s_1 J - T)^{-1}|_{V_{\Delta_n}},$$

so that (E_n, J_{Δ_n}) is actually a spectral decomposition for T_{Δ_n}. However, the same arguments show that also $\left(E_n, \widetilde{J}_{\Delta_n}\right)$ is a spectral decomposition for T_{Δ_n}. Since, however, $\sigma_S(T_{\Delta_n}) \subset \Delta_n$ and $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$, Proposition 15.1.14 implies that the spectral orientation of T_{Δ_n} is unique such that $J_{\Delta_n} = \widetilde{J}_{\Delta_n}$. We thus obtain

$$Jy = \sum_{n=1}^{+\infty} J_{\Delta_n} E(\Delta_n) y = \sum_{n=1}^{+\infty} \widetilde{J}_{\Delta_n} E(\Delta_n) y = \widetilde{J} y. \qquad \square$$

Remark 15.1.17. In Proposition 15.1.14 and Corollary 15.1.15 we showed that under certain assumptions the existence of a spectral resolution E for T already implies the existence of a spectral orientation and is hence a sufficient condition for T to be a spectral operator. One may wonder whether this is true in general. An intuitive approach for showing this follows the idea of the proof of Theorem 15.1.16. We can cover $\sigma_S(T)\setminus\mathbb{R}$ by pairwise disjoint sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$ with $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ for each $n \in \mathbb{N}$. On each of the subspaces $V_n := \operatorname{ran} E(\Delta_n)$, the operator T induces the operator $T_n := T|_{V_n}$ with $\sigma_S(T_n) \subset \overline{\Delta_n}$. Since $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$, we can then define $\Delta_{n,+} := \Delta_n \cap \mathbb{C}_j^+$ and $\Delta_{n,-} := \Delta_n \cap \mathbb{C}_j^-$ for an arbitrary imaginary unit $j \in \mathbb{S}$ and consider the Riesz projectors $E_{n,+} := \chi_{\Delta_{n,+}}(T_n)$ and $E_{n,-} := \chi_{\Delta_{n,-}}(T_n)$ of T_n on $V_{n,j}$ associated with $\Delta_{n,+}$ and $\Delta_{n,-}$. Just as we did in the proof of Proposition 15.1.14, we can then construct a spectral orientation for T_n by setting $J_n y = E_{n,+} y j + E_{n,-} y(-j)$ for $y \in V_n$. The spectral orientation of J must then be

$$Jy = \sum_{n=1}^{+\infty} J_n E(\Delta_n) y = \sum_{n=1}^{+\infty} E_{n,+} E(\Delta_n) y j + E_{n,-} E(\Delta_n) y(-j). \tag{15.10}$$

If T is a spectral operator, then $E_{n,+} = E_+|_{V_n}$ and $E_{n,-} = E_-|_{V_n}$, where E_+ and E_- are as usual the projections of V_R onto $V_{J,j}^+$ and $V_{J,j}^-$ along $V_0 \oplus V_{J,j}^-$ resp. $V_0 \oplus V_{J,j}^+$. Hence the Riesz projectors $E_{n,+}$ and $E_{n,-}$ are uniformly bounded in $n \in \mathbb{N}$, and the above series converges. The spectral orientation of T can therefore be constructed as described above if T is a spectral operator.

This procedure, however, fails if the Riesz projectors $E_{n,+}$ and $E_{n,-}$ are not uniformly bounded, because the convergence of the above series is in this case not guaranteed. The next example presents an operator for which the above series does actually not converge for this reason although the operator has a quaternionic spectral resolution. Hence the existence of a spectral resolution does not in general imply the existence of a spectral orientation.

Example 15.1.18. Let $\ell^2(\mathbb{H})$ be the space of all square-summable sequences with quaternionic entries and choose $j, i \in \mathbb{S}$ with $j \perp i$. We define an operator T on

$\ell^2(\mathbb{H})$ by the following rule: if $(b_n)_{n\in\mathbb{N}} = T(a_n)_{n\in\mathbb{N}}$, then

$$\begin{pmatrix} b_{2m-1} \\ b_{2m} \end{pmatrix} = \frac{1}{m^2}\begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix}\begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix}. \tag{15.11}$$

For neatness, let us denote the matrix in the above equation by J_m and let us set $T_m := \frac{1}{m^2}J_m$, that is,

$$J_m := \begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix} \quad \text{and} \quad T_m := \frac{1}{m^2}\begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix}.$$

Since all matrix norms are equivalent, there exists a constant $C > 0$ such that

$$\|M\| \leq C \max_{\ell,\kappa\in\{1,2\}} |m_{\ell,\kappa}| \quad \forall M = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} \in \mathbb{H}^{2\times 2}, \tag{15.12}$$

such that $\|J_m\| \leq 2Cm$. We thus find for (15.11) that

$$\|(b_{2m-1}, b_{2m})^T\|_2 \leq \frac{2C}{m}\|(a_{2m-1}, a_{2m})^T\|_2 \leq 2C\|(a_{2m-1}, a_{2m})^T\|_2,$$

and in turn

$$\begin{aligned} \|T(a_n)_{n\in\mathbb{N}}\|^2_{\ell^2(\mathbb{H})} &= \sum_{m=1}^{+\infty} |b_{2m-1}|^2 + |b_{2m}|^2 \\ &\leq \sum_{m=1}^{+\infty} 4C^2\left(|a_{2m-1}|^2 + |a_{2m}|^2\right) = 4C^2\,\|(a_n)_{n\in\mathbb{N}}\|^2_{\ell^2(\mathbb{H})}. \end{aligned} \tag{15.13}$$

Hence T is a bounded right-linear operator on $\ell^2(\mathbb{H})$.

We show now that the S-spectrum of T is the set $\Lambda = \{0\} \cup \bigcup_{n\in\mathbb{N}} \frac{1}{n^2}\mathbb{S}$. For $s \in \mathbb{H}$, the operator $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2$ is given by the following relation: if $(c_n)_{n\in\mathbb{N}} = \mathcal{Q}_s(T)(a_n)_{n\in\mathbb{N}}$, then

$$\begin{pmatrix} c_{2m-1} \\ c_{2m} \end{pmatrix} = \begin{pmatrix} -\frac{1}{m^2} - 2j\frac{s_0}{m^2} + |s|^2 & -4j\frac{s_0}{m} \\ 0 & -\frac{1}{m^2} - 2j\frac{s_0}{m^2} + |s|^2 \end{pmatrix}\begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix}. \tag{15.14}$$

The inverse of the above matrix is

$$\begin{aligned} \mathcal{Q}_s(T_m)^{-1} &= \begin{pmatrix} \frac{m^4}{|s|^2m^4 - 2is_0m^2 - 1} & \frac{4im^7s_0}{|s|^4m^8 + 2\left(s_0^2 - s_1^2\right)m^4 + 1} \\ 0 & \frac{m^4}{|s|^2m^4 + 2is_0m^2 - 1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\left(s_j - \frac{j}{m^2}\right)\left(\overline{s_j} - \frac{j}{m^2}\right)} & \frac{4is_0}{m\left(s_j + \frac{j}{m^2}\right)\left(s_j - \frac{j}{m^2}\right)\left(\overline{s_j} + \frac{j}{m^2}\right)\left(\overline{s_j} - \frac{j}{m^2}\right)} \\ 0 & \frac{1}{\left(s_j + \frac{j}{m^2}\right)\left(\overline{s_j} + \frac{j}{m^2}\right)} \end{pmatrix} \end{aligned}$$

with $s_j = s_0 + js_1$. Hence $\mathcal{Q}_s(T_m)^{-1}$ exists for $s_j \neq \frac{1}{m^2}j$. We have

$$\left|s_j - \frac{j}{m^2}\right|\left|\overline{s_j} - \frac{j}{m^2}\right| = \left|s_j + \frac{j}{m^2}\right|\left|\overline{s_j} + \frac{j}{m^2}\right| \geq 2\left|s_j - \frac{j}{m^2}\right| = 2\mathrm{dist}\left(s, \left[\frac{j}{m}\right]\right),$$

and so

$$\left\|\mathcal{Q}_s(T_m)^{-1}\right\| \leq C \max\left\{\frac{1}{2\mathrm{dist}\left(s, \left[\frac{j}{m^2}\right]\right)}, \frac{|s_0|}{m\left(\mathrm{dist}\left(s, \left[\frac{j}{m^2}\right]\right)\right)^2}\right\}, \tag{15.15}$$

where C is the constant in (15.12). If $s \notin \Lambda$, then $0 < \mathrm{dist}(s, \Lambda) \leq \mathrm{dist}\left(s, \left[\frac{j}{m^2}\right]\right)$ and hence the matrices $\mathcal{Q}_s(T_m)^{-1}$ are for $m \in \mathbb{N}$ uniformly bounded by

$$\left\|\mathcal{Q}_s(T_m)^{-1}\right\| \leq C \max\left\{\frac{1}{2\mathrm{dist}\,(s, \Lambda)}, \frac{|s_0|}{\mathrm{dist}\,(s, \Lambda)^2}\right\}.$$

The operator $\mathcal{Q}_s(T)^{-1}$ is then given by the relation

$$\begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix} = \mathcal{Q}_s(T_m)^{-1} \begin{pmatrix} c_{2m-1} \\ c_{2m} \end{pmatrix}, \tag{15.16}$$

for $(a_n)_{n\in\mathbb{N}} = Q_s(T)^{-1}(c_n)_{n\in\mathbb{N}}$. A computation similar to the one in (15.13) shows that this operator is bounded on $\ell^2(\mathbb{H})$. Thus $s \in \rho_S(T)$ if $s \notin \Lambda$ and in turn $\sigma_S(T) \subset \Lambda$.

For every $m \in \mathbb{N}$, we set $s_m = \frac{1}{m^2} j$. The sphere $[s_m] = \frac{1}{m^2}\mathbb{S}$ is an eigensphere of T and the associated eigenspace V_m is the right-linear span of $\mathbf{e}_{2m-1}$ and $\mathbf{e}_{2m}$, where $\mathbf{e}_n = (\delta_{n,\ell})_{\ell\in\mathbb{N}}$, as one can see easily from (15.14). A straightforward computation, moreover, shows that the vectors $y_{2m-1} := \mathbf{e}_{2m-1}$ and $y_{2m} := -\mathbf{e}_{2m-1} i + \frac{1}{m}\mathbf{e}_{2m} i$ are eigenvectors of T with respect to the eigenvalue s_m. Hence $[s_m] \subset \sigma_S(T)$. Since $\sigma_S(T)$ is closed, we finally obtain $\Lambda = \overline{\bigcup_{m\in\mathbb{N}} [s_m]} \subset \sigma_S(T)$ and in turn $\sigma_S(T) = \Lambda$.

Let E_m for $m \in \mathbb{N}$ be the orthogonal projection of $\ell^2(\mathbb{H})$ onto the subspace $V_m := \mathrm{span}_{\mathbb{H}}\{\mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}$, that is, $E_m(a_n)_{n\in\mathbb{N}} = \mathbf{e}_{2m-1}a_{2m-1} + \mathbf{e}_{2m}a_{2m}$. We define for every set $\Delta \in \mathfrak{B}_S(\mathbb{H})$ the operator

$$E(\Delta) = \sum_{m\in I_\Delta} E_m \quad \text{with} \quad I_\Delta := \left\{m \in \mathbb{N} : \frac{1}{m^2}\mathbb{S} \subset \Delta\right\}.$$

It is immediate that E is a spectral measure on $\ell^2(\mathbb{H})$, that $\|E(\Delta)\| \leq 1$ for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and that $E(\Delta)$ commutes with T for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Moreover, if $s \notin \overline{\Delta}$, then the pseudo-resolvent $\mathcal{Q}_s(T_\Delta)^{-1}$ of $T_\Delta = T|_{V_\Delta}$ with $V_\Delta = \mathrm{ran}\, E(\Delta)$ is given by

$$\mathcal{Q}_s(T_\Delta)^{-1} = \left(\sum_{m\in I_\Delta} \mathcal{Q}_s(T_m)^{-1} E_m\right)\Bigg|_{\mathrm{ran}\, E(\Delta)}.$$

Since $0 < \mathrm{dist}\left(s, \bigcup_{m\in I_\Delta}\left[\frac{j}{m^2}\right]\right) = \inf_{m\in I_\Delta} \mathrm{dist}\left(s, \left[\frac{j}{m^2}\right]\right)$, the operators $\mathcal{Q}_s(T_m)^{-1}$ are uniformly bounded for $m \in I_\Delta$. Computations similar to (15.13) show that $\mathcal{Q}_s(T_\Delta)^{-1}$ is a bounded operator on V_Δ. Hence $s \in \rho_S(T_\Delta)$ and in turn $\sigma_S(T_\Delta) \subset \overline{\Delta}$. Altogether we obtain that E is a spectral resolution for T.

In order to construct a spectral orientation for T, we first observe that J_m is a spectral orientation for T_m. For $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we have

$$s_0 \mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m = \begin{pmatrix} s_0 - \left(s_1 + \frac{1}{m^2}\right) j & -\left(s_1 + \frac{1}{m^2}\right) 2mj \\ 0 & s_0 + \left(s_1 + \frac{1}{m^2}\right) j \end{pmatrix},$$

the inverse of which is given by the matrix

$$\left(s_0 \mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m\right)^{-1} = \begin{pmatrix} \frac{1}{s_0 - \left(s_1 + \frac{1}{m^2}\right) j} & \frac{2jm\left(\frac{1}{m^2} + s_1\right)}{s_0^2 + \left(\frac{1}{m^2} + s_1\right)^2} \\ 0 & \frac{1}{s_0 + \left(\frac{1}{m^2} + s_1\right) j} \end{pmatrix}.$$

Since $s_1 > 0$, each entry has nonzero denominator, and hence we have that the operator $(s_0 \mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m)^{-1}$ belongs to $\mathcal{B}(\mathbb{H}^2)$.

If $J \in \mathcal{B}(\ell^2(\mathbb{H}))$ is a spectral orientation for T, then the restriction $J|_{V_m}$ of J to $V_m = \operatorname{span}_{\mathbb{H}}\{\mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}$ is also a spectral orientation for T_m. The uniqueness of the spectral orientation implies $J|_{V_m} = J_m$ and hence

$$J = \sum_{m=1}^{+\infty} J|_{V_m} E\left(\frac{1}{m^2}\mathbb{S}\right) = \sum_{m=1}^{+\infty} J_m E_m.$$

This series does not, however, converge, because the operators J_{V_m} are not uniformly bounded. Hence, it does not define a bounded operator on $\ell^2(\mathbb{H})$. Indeed, the sequence $a_{2m-1} = 0$, $a_{2m} = m^{-\frac{3}{2}}$, for instance, belongs to $\ell^2(\mathbb{H})$, but

$$\begin{aligned} \left\| \sum_{m=1}^{+\infty} J_m E_m (a_n)_{n\in\mathbb{N}} \right\|_{\ell^2(\mathbb{H})}^2 &= \sum_{m=1}^{+\infty} \left\| \begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix} \begin{pmatrix} 0 \\ m^{-\frac{3}{2}} \end{pmatrix} \right\|_2^2 \\ &= 2 \sum_{m=1}^{+\infty} 4\frac{1}{m} + \frac{1}{m^3} = +\infty. \end{aligned}$$

Hence there cannot exist a spectral orientation for T, and in turn T is not a spectral operator on $\ell^2(\mathbb{H})$.

We conclude this example with a remark on its geometric intuition. Let us identify $\mathbb{H}^2 \cong \mathbb{C}_j^4$, which is for every $i \in \mathbb{S}$ with $i \perp j$ spanned by the basis vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_4 = \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

The vectors $y_{m,1} = b_1$ and $y_{m,2} = -b_2 + \frac{1}{m} b_4$ are eigenvectors of J_m with respect to j, and the vectors $y_1 i = b_2$ and $y_{m,2} = b_1 - \frac{1}{m} b_3$ are eigenvectors of J_m with respect to $-j$. We thus obtain $V_{J_m,j}^+ = \operatorname{span}_{\mathbb{C}_j}\{b_1, -b_2 + \frac{1}{m} b_4\}$ and $V_{J_m,j}^- = V_{J_m,j}^+ i$. However, as m tends to infinity, the vector y_2 tends to $y_1 i$ and $y_2 i$ tends to y_1. Hence intuitively, in the limit $V_{J_m,j}^- = V_{J_m,j}^+ i = V_{J_m,j}^+$, and consequently the projections of $\mathbb{H}^2 = \mathbb{C}_j^4$ onto $V_{J_m,j}^+$ along $V_{J_m,j}^-$ become unbounded.

Finally, the notion of quaternionic spectral operator is backward compatible with the complex theory on $V_{R,j}$.

Theorem 15.1.19. *An operator $T \in \mathcal{B}(V_R)$ is a quaternionic spectral operator if and only if it is a spectral operator on $V_{R,j}$ for some (and hence every) $j \in \mathbb{S}$. (See [106] for the complex theory.) If furthermore (E, J) is the quaternionic spectral decomposition of T and E_j is the spectral resolution of T as a complex $\mathbb{C}_j$-linear operator on $V_{R,j}$, then*

$$\begin{aligned} E(\Delta) &= E_j(\Delta \cap \mathbb{C}_j) \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}), \\ Jy &= E_j(\mathbb{C}_j^{+,\circ}\backslash)yj + E_j(\mathbb{C}_j^{-,\circ})y(-j) \quad \forall y \in V_R \end{aligned} \tag{15.17}$$

with

$$\mathbb{C}_j^{\pm,\circ} := \mathbb{C}_j^{\pm} \setminus \mathbb{R} = \{z_0 + jz_1 : \ z_0 \in \mathbb{R}, z_1 > 0\}.$$

Conversely, E_j is the spectral measure on V_R determined by (E, J) that was constructed in Lemma 14.3.8.

Proof. Let us first assume that $T \in \mathcal{B}(V_R)$ is a quaternionic spectral operator with spectral decomposition (E, J) in the sense of Definition 15.1.1 and let $j \in \mathbb{S}$. Let E_+ be the projection of $\operatorname{ran} J = V_{J,j}^+ \oplus V_{J,j}^-$ onto $V_{J,j}^+$ along $V_{J,j}^-$ and let E_- be the projection of $\operatorname{ran} J$ onto $V_{J,j}^-$ along $V_{J,j}^+$; cf. Theorem 14.2.10. Since T and $E(\Delta)$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$ commute with J, they leave the spaces $V_{J,j}^+$ and $V_{J,j}^-$ invariant, and hence they commute with E_+ and E_-. By Lemma 14.3.8, the set function E_j on $\mathbb{C}_j$ defined in (14.23), which is given by

$$E_j(\Delta) = E_+ E\left(\left[\Delta \cap \mathbb{C}_j^{+,\circ}\right]\right) + E(\Delta \cap \mathbb{R}) + E_- E\left(\left[\Delta \cap \mathbb{C}_j^{-,\circ}\right]\right), \tag{15.18}$$

for $\Delta \in \mathfrak{B}(\mathbb{C}_j)$, is a spectral measure on $V_{R,j}$. Since the spectral measure E and the projections E_+ and E_- commute with T, the spectral measure E_j commutes with T too.

If $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ is a subset of $\mathbb{C}_j^{+,\circ}$, then $Jy = yj$ for $y \in V_{j,\Delta} := \operatorname{ran} E_j(\Delta)$, since $\operatorname{ran} E_j(\Delta) = \operatorname{ran}(E_+ E([\Delta])) \subset V_{J,j}^+$. For $z = z_0 + jz_1 \in \mathbb{C}_j$ and $y \in V_{j,\Delta}$, we thus have

$$\begin{aligned} (z\mathcal{I}_{V_{j,\Delta}} - T)y &= yz_0 + yjz_1 - Ty \\ &= yz_0 + Jyz_1 - Ty = (z_0\mathcal{I}_{V_{j,\Delta}} + z_1 J - T)y. \end{aligned}$$

If $z \in \mathbb{C}_j^{-,\circ}$, then the inverse of $(z_0\mathcal{I}_{V_{R,j}} + z_1 J - T)|_{\operatorname{ran} J}$ exists because J is the spectral orientation of T. We thus have $R_z(T_\Delta) = (z_0\mathcal{I}_{V_{R,j}} + z_1 J - T)^{-1}|_{V_{j,\Delta}}$, and so $\mathbb{C}_j^{-,\circ} \subset \rho(T_\Delta)$. If, on the other hand, $z \in \mathbb{C}_j^+ \setminus \overline{\Delta}$, then $z \in \rho_S(T_{[\Delta]})$, where $T_{[\Delta]} = T|_{V_{[\Delta]}}$ with $V_{[\Delta]} = \operatorname{ran} E([\Delta])$. Hence $\mathcal{Q}_z(T_{[\Delta]})$ has a bounded inverse on $V_{[\Delta]}$. By the construction of E_j we have $V_{j,\Delta} = E_+ V_{[\Delta]}$, and since $T_{[\Delta]}$ and E_+ commute, $\mathcal{Q}_z(T_{[\Delta]})^{-1}$ leaves $V_{j,\Delta}$ invariant, so that $\mathcal{Q}_z(T_{[\Delta]})^{-1}\big|_{V_{j,\Delta}}$ defines a bounded $\mathbb{C}_j$-linear operator on $V_{j,\Delta}$. Because of Theorem 14.2.7, the resolvent of T_Δ at z is therefore given by

$$R_z(T)y = \mathcal{Q}_s(T_{[\Delta]})^{-1}(y\overline{z} - T_\Delta y) \quad \forall y \in V_{j,\Delta}.$$

Altogether, we conclude that $\rho(T_\Delta) \supset \mathbb{C}_j^{-,\circ} \cup \left(\mathbb{C}_j^{+} \setminus \overline{\Delta}\right) = \mathbb{C}_j \setminus \overline{\Delta}$ and in turn $\sigma(T_\Delta) \subset \overline{\Delta}$. Similarly, we see that $\sigma(T_\Delta) \subset \overline{\Delta}$ if $\Delta \subset \mathbb{C}_j^{-,\circ}$. If, on the other hand, $\Delta \subset \mathbb{R}$, then $E_j(\Delta) = E(\Delta)$, so that T_Δ is a quaternionic linear operator with $\sigma_S(T_\Delta) \subset \overline{\Delta}$. By Theorem 14.2.7, we have $\sigma(T_\Delta) = \sigma_{\mathbb{C}_j}(T_\Delta) = \sigma_S(T) \subset \overline{\Delta}$. Finally, if $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ is arbitrary and $z \notin \overline{\Delta}$, we can set $\Delta_+ := \Delta \cap \mathbb{C}_j^{+,\circ}$, $\Delta_- := \Delta \cap \mathbb{C}_j^{-,\circ}$, and $\Delta_{\mathbb{R}} := \Delta \cap \mathbb{R}$. Then z belongs to the resolvent sets of each of the operators T_{Δ_+}, T_{Δ_-}, and $T_{\Delta_{\mathbb{R}}}$, and we obtain

$$R_z(T) = R_z(T_{\Delta_+})E_j(\Delta_+) + R_z(T_{\Delta_{\mathbb{R}}})E(\Delta_{\mathbb{R}}) + R_z(T_{\Delta_-})E_j(\Delta_-).$$

We thus have $\sigma(T_\Delta) \subset \overline{\Delta}$. Hence T is a spectral operator on $V_{R,j}$, and E_j is its ($\mathbb{C}_j$-complex) spectral resolution on $V_{R,j}$.

Now assume that T is a bounded quaternionic linear operator on V_R and that for some $j \in \mathbb{S}$ there exists a $\mathbb{C}_j$-linear spectral resolution E_j for T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$. Following Definition 6 of [104, Chapter XV.2], an analytic extension of $R_z(T)y$ with $y \in V_{R,j} = V_R$ is a holomorphic function f defined on a set $\mathcal{D}(f)$ such that $(z\mathcal{I}_{V_{R,j}} - T)f(z) = y$ for $z \in \mathcal{D}(f)$. The resolvent $\rho(y)$ is the domain of the unique maximal analytic extension of $R_z(T)y$, and the spectrum $\sigma(y)$ is the complement of $\rho(y)$ in $\mathbb{C}_j$. (We defined the quaternionic counterparts of these concepts in Definition 15.1.3 and Definition 15.1.4.) Analogously to Theorem 15.1.9, we have

$$E_j(\Delta)V_{R,j} = \{y \in V_{R,j} = V_R : \ \sigma(y) \subset \Delta\}, \quad \forall \Delta \in \mathfrak{B}(\mathbb{C}_j). \tag{15.19}$$

Let $y \in V_{R,j}$, let $i \in \mathbb{S}$ with $j \perp i$, and let f be the unique maximal analytic extension of $R_z(T)y$ defined on $\rho(y)$. The mapping $z \mapsto f(\overline{z})\, i$ is then holomorphic on $\overline{\rho(y)}$: for every $z \in \overline{\rho(y)}$, we have $\overline{z} \in \rho(y)$ and in turn

$$\lim_{h\to 0} \left(f\left(\overline{z+h}\right) i - f(\overline{z})\, i\right) h^{-1} = \lim_{h\to 0} \left(f(\overline{z} + \overline{h}) - f(\overline{z})\right) \overline{h}^{-1} i = f'(\overline{z})\, i.$$

Since T is quaternionic linear, we moreover have for $z \in \overline{\rho(y)}$ that

$$\left(z\mathcal{I}_{V_{R,j}} - T\right)(f(\overline{z})\, i) = f(\overline{z})\, iz - T\,(f(\overline{z})\, i) = \left(f(\overline{z})\, \overline{z} - T\,(f(\overline{z}))\,\right) i = yi.$$

Hence $z \mapsto f(\overline{z})\, i$ is an analytic extension of $R_z(T)(yi)$ that is defined on $\overline{\rho(y)}$. Consequently $\rho(yi) \supset \overline{\rho(y)}$, and in turn $\sigma(yi) \subset \overline{\sigma(y)}$. If $\tilde{f}$ is the maximal analytic extension of $R_z(T)(yi)$, then similar arguments show that $z \mapsto \tilde{f}(\overline{z})\,(-i)$ is an analytic extension of $R_z(T)y$. Since this function is defined on $\overline{\rho(yi)}$, we obtain

$\rho(y) \supset \overline{\rho(yi)}$ and in turn $\sigma(y) \subset \overline{\sigma(yi)}$. Altogether, we obtain $\sigma(y) = \overline{\sigma(yi)}$ and $\tilde{f}(z) = f(\overline{z})\, i$. From (15.19) we deduce

$$\begin{aligned} \operatorname{ran} E_j\left(\overline{\Delta}\right) &= \left\{y \in V_{R,j} = V_R : \ \sigma(y) \subset \overline{\Delta}\right\} \\ &= \{yi \in V_{R,j} = V_R : \ \sigma(y) \subset \Delta\} = (\operatorname{ran} E_j(\Delta))\, i. \end{aligned} \tag{15.20}$$

In order to construct the quaternionic spectral resolution of T, we define now

$$E(\Delta) := E_j(\Delta \cap \mathbb{C}_j), \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}).$$

Obviously this operator is a bounded $\mathbb{C}_j$-linear projection on $V_R = V_{R,j}$. We show now that it is also quaternionic linear. Due to the axial symmetry of Δ, the identity (15.20) implies

$$\begin{aligned} (\operatorname{ran} E(\Delta))i = (\operatorname{ran} E_j(\Delta \cap \mathbb{C}_j))\, i &= \operatorname{ran} E_j\left(\overline{\Delta \cap \mathbb{C}_j}\right) \\ &= \operatorname{ran} E_j\left(\Delta \cap \mathbb{C}_j\right) = \operatorname{ran} E(\Delta). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} (\ker E(\Delta))i = (\ker E_j(\Delta \cap \mathbb{C}_j))i &= (\operatorname{ran} E_j(\mathbb{C}_j \setminus \Delta))i \\ &= \operatorname{ran} E_j\left(\overline{\mathbb{C}_j \setminus \Delta}\right) = \operatorname{ran} E_j\left(\mathbb{C}_j \setminus \Delta\right) \\ &= \ker E_j(\Delta \cap \mathbb{C}_j) = \ker E(\Delta). \end{aligned}$$

If we write $y \in V_R$ as $y = y_0 + y_1$ with $y_0 \in \ker E(\Delta)$ and $y_1 \in \operatorname{ran} E(\Delta)$, we thus have

$$E(\Delta)(yi) = E(\Delta)(y_0 i) + E(\Delta)(y_1 i) = y_1 i = (E(\Delta)y)i.$$

Writing $a \in \mathbb{H}$ as $a = a_1 + ia_2$ with $a_1, a_2 \in \mathbb{C}_j$, we find due to the $\mathbb{C}_j$-linearity of $E(\Delta)$ that even

$$E(\Delta)(ya) = (E(\Delta)y)a_1 + (E(\Delta)yi)a_2 = (E(\Delta)y)a_1 + (E(\Delta)y)ia_2 = (E(\Delta)y)a.$$

Hence the set function $\Delta \mapsto E(\Delta)$ defined in (15.18) takes values that are bounded quaternionic linear projections on V_R. It is immediate that it moreover satisfies items (i) to (iv) in Definition 14.1.7 because E_j is a spectral measure on $V_{R,j}$ and hence has the respective properties. Consequently, E is a quaternionic spectral measure. Since E_j commutes with T, also E commutes with T. From Theorem 14.2.7 and the fact that $\sigma(T|_{\operatorname{ran} E_j(\Delta_j)}) \subset \overline{\Delta_j}$ for $\Delta_j \in \mathfrak{B}(\mathbb{C}_j)$, we deduce for $T_\Delta = T|_{\operatorname{ran} E(\Delta)} = T|_{\operatorname{ran} E_j(\Delta \cap \mathbb{C}_j)}$ that

$$\sigma_S(T_\Delta) = \left[\sigma_{\mathbb{C}_j}(T_\Delta)\right] \subset \left[\overline{\Delta \cap \mathbb{C}_j}\right] = \overline{[\Delta \cap \mathbb{C}_j]} = \overline{\Delta}.$$

Therefore E is a spectral resolution for T.

Let us now set $V_0 = \operatorname{ran} E_j(\mathbb{R})$ as well as $V_+ := \operatorname{ran} E_j\left(\mathbb{C}_j^{+,\circ}\right)$ and $V_- := \operatorname{ran} E_j\left(\mathbb{C}_j^{-,\circ}\right)$. Then $V_{R,j} = V_0 \oplus V_+ \oplus V_-$ is a decomposition of V_R into closed

$\mathbb{C}_j$-linear subspaces. The space $V_0 = \operatorname{ran} E_j(\mathbb{R}) = \operatorname{ran} E(\mathbb{R})$ is even a quaternionic right linear subspace of V_R because $E(\mathbb{R})$ is a quaternionic right linear operator. Moreover, (15.20) shows that $y \mapsto yi$ is a bijection from V_+ to V_-. By Theorem 14.2.10, the operator

$$Jy = E_j\left(\mathbb{C}_j^{+,\circ}\right) yj + E_j\left(\mathbb{C}_j^{-,\circ}\right) y(-j)$$

is an imaginary operator on V_R. Since E_j commutes with T and $E(\Delta)$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, also J commutes with T and $E(\Delta)$. Moreover, $\ker J = V_0 = \operatorname{ran} E(\mathbb{R})$ and $\operatorname{ran} J = \operatorname{ran} E_j(\mathbb{C}_j^{+,\circ}) \oplus \operatorname{ran} E_j(\mathbb{C}_j^{-,\circ}) = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$, and hence (E, J) is a spectral system that commutes with T. Finally, we have $\sigma(T_+) \subset \mathbb{C}_j^+$ for $T_+ = T|_{V_+} = T|_{\operatorname{ran} E_j(\mathbb{C}_j^{+,\circ})}$, and hence the resolvent of $R_z(T_+)$ exists for every $z \in \mathbb{C}_j^{-,\circ}$. Similarly, the resolvent $R_z(T_-)$ with $T_- = T|_{V_-} = T|_{\operatorname{ran} E_j(\mathbb{C}_j^{-,\circ})}$ exists for every $z \in \mathbb{C}_j^{+,\circ}$. For $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$ we can hence set $s_j = s_0 + js_1$ and define by

$$R(s_0, s_1) := \left(R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-\right)\Big|_{V_+ \oplus V_-}$$

with $E_+ = E_j(\mathbb{C}_j^{+,\circ})$ and $E_- = E_j(\mathbb{C}_j^{-,\circ})$ a bounded operator on $V_+ \oplus V_- = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. Since T leaves V_+ and V_- invariant, we then have for $y = y_+ + y_- \in V_+ \oplus V_-$ that

$$\begin{aligned}
&R(s_0, s_1)(s_0\mathcal{I} - s_1 J - T)y \\
&= R(s_0, s_1)\left(y_+ s_0 - Jy_+ s_1 - Ty_+ + y_- s_0 - Jy_- s_1 - Ty_-\right) \\
&= R(s_0, s_1)\left(y_+ \overline{s_j} - Ty_+\right) + R(s_0, s_1)\left(y_- s_j - Ty_-\right) \\
&= R_{\overline{s_j}}(T_+)\left(y_+ \overline{s_j} - T_+ y_+\right) + R_{s_j}(T_-)\left(y_- s_j - T_- y_-\right) = y_+ + y_- = y.
\end{aligned}$$

Similarly we find that

$$\begin{aligned}
&(s_0\mathcal{I} - s_1 J - T)R(s_0, s_1)y \\
&= (s_0\mathcal{I} - s_1 J - T)R_{\overline{s_j}}(T_+)y_+ + (s_0\mathcal{I} - s_1 J - T)R_{s_j}(T_-)y_- \\
&= R_{\overline{s_j}}(T_+)y_+ s_0 - J(R_{\overline{s_j}}(T_+)y_+)s_1 - TR_{\overline{s_j}}(T_+)y_+ \\
&\quad + R_{s_j}(T_-)y_- s_0 - J(R_{s_j}(T_-)y_-)s_1 - TR_{s_j}(T_-)y_- \\
&= R_{\overline{s_j}}(T_+)y_+(s_0 - js_1) - R_{\overline{s_j}}(T_+)T_+ y_+ \\
&\quad + R_{s_j}(T_-)y_-(s_0 + js_1) - R_{s_j}(T_-)T_- y_- \\
&= R_{\overline{s_j}}(T_+)\left(y_+ \overline{s} - T_+ y_+\right) + R_{s_j}(T_-)\left(y_- s - T_- y_-\right) = y_+ + y_- = y.
\end{aligned}$$

Hence $R(s_0, s_1)$ is the bounded inverse of $(s_0\mathcal{I} - s_1 J - T)|_{\operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})}$, and so J is actually a spectral orientation for T. Consequently, T is a quaternionic spectral operator, and the relation (15.17) holds.

□

Remark 15.1.20. We want to stress that Theorem 15.1.19 showed a one-to-one relation between quaternionic spectral operators on V_R and $\mathbb{C}_j$-complex spectral operators on $V_{R,j}$ that are furthermore compatible with the quaternionic scalar multiplication. It did not show a one-to-one relation between quaternionic spectral operators on V_R and $\mathbb{C}_j$-complex spectral operators on $V_{R,j}$. There exist $\mathbb{C}_j$-complex spectral operators on $V_{R,j}$ that are not quaternionic linear and hence cannot be quaternionic spectral operators.

15.2 Canonical Reduction and Intrinsic S-Functional Calculus for Quaternionic Spectral Operators

As in the complex case, every bounded quaternionic spectral operator T can be decomposed into the sum $T = S + N$ of a scalar operator S and a quasi-nilpotent operator N. The intrinsic S-functional calculus for a spectral operator can then be expressed as a Taylor series similar to the one that involves functions of S obtained via spectral integration and powers of N. Analogously to the complex case in [106], the operator $f(T)$ is therefore already determined by the values of f on $\sigma_S(T)$ and not only by its values on a neighborhood of $\sigma_S(T)$.

Definition 15.2.1. An operator $S \in \mathcal{B}(V_R)$ is said to be *of scalar type* if it is a spectral operator and satisfies the identity

$$S = \int s \, dE_J(s), \tag{15.21}$$

where (E, J) is the spectral decomposition of S.

Remark 15.2.2. If we start from a spectral system (E, J) and S is the operator defined by (15.21), then S is an operator of scalar type and (E, J) is its spectral decomposition. This can easily be checked by direct calculations or indirectly via the following argument: by Lemma 14.3.8, we can choose $j \in \mathbb{S}$ and obtain

$$S = \int_{\mathbb{H}} s \, dE_J(s) = \int_{\mathbb{C}_j} z \, dE_j(z),$$

where E_j is the spectral measure constructed in (14.23). From the complex theory in [106], we deduce that S is a spectral operator on $V_{R,j}$ with spectral decomposition E_j that is furthermore quaternionic linear. By Theorem 15.1.19, this is equivalent to S being a quaternionic spectral operator on V_R with spectral decomposition (E, J).

Lemma 15.2.3. *Let S be an operator of scalar type with spectral decomposition (E, J). An operator $A \in \mathcal{B}(V_R)$ commutes with S if and only if it commutes with the spectral system (E, J).*

Proof. If $A \in \mathcal{B}(V_R)$ commutes with (E, J), then it commutes with $S = \int_{\mathbb{H}} s \, dE_J(s)$ because of Lemma 14.3.6. If, on the other hand, A commutes with S, then it also commutes with E by Lemma 15.1.12. By Lemma 14.1.10, it commutes in turn with the operator $f(T) = \int_{\mathbb{H}} f(s)\, dE(s)$ for every $f \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$. If we define

$$S_0 := \int_{\mathbb{H}} \mathrm{Re}(s)\, dE(s) \quad \text{and} \quad S_1 := \int_{\mathbb{H}} \underline{s}\, dE_J(S) = J \int_{\mathbb{H}} |\underline{s}|\, dE(s),$$

where $\underline{s} = j_s s_1$ denotes the imaginary part of a quaternion s, then $AS = SA$ and $AS_0 = S_0 A$ and in turn

$$AS_1 = A(S - S_0) = AS - AS_0 = SA - S_0 A = (S - S_0)A = S_1 A.$$

We can now choose pairwise disjoint sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$, $n \in \mathbb{N}$, such that $\sigma_S(T) \setminus \mathbb{R} = \bigcup_{n\in\mathbb{N}} \Delta_n$ and such that $\mathrm{dist}(\Delta_n, \mathbb{R}) > 0$ for every $n \in \mathbb{N}$. Then $s \mapsto |\underline{s}|^{-1}\chi_{\Delta_n}(s)$ belongs to $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ for every $n \in \mathbb{N}$, and in turn

$$\begin{aligned}
AJE(\Delta_n) &= AJ \left(\int_{\mathbb{H}} |\underline{s}||\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) \\
&= AJ \left(\int_{\mathbb{H}} |\underline{s}|\, dE(s) \right) \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) \\
&= AS_1 \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) \\
&= S_1 \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) A \\
&= J \left(\int_{\mathbb{H}} |\underline{s}|\, dE(s) \right) \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) A \\
&= J \left(\int_{\mathbb{H}} |\underline{s}||\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) A = JE(\Delta_n) A.
\end{aligned}$$

Since $\sigma_S(S) \setminus \mathbb{R} \subset \bigcup_{n\in\mathbb{N}} \Delta_n$, we have $\sum_{n=0}^{+\infty} E(\Delta_n) y = E(\sigma_S(T) \setminus \mathbb{R}) y = E(\mathbb{H} \setminus \mathbb{R}) y$ for all $y \in V_R$ by Corollary 15.1.10. Since $J = JE(\mathbb{H} \setminus \mathbb{R})$, we hence obtain

$$\begin{aligned}
AJy = AJE(\mathbb{H} \setminus \mathbb{R}) y &= \sum_{n=1}^{+\infty} AJE(\Delta_n) y \\
&= \sum_{n=1}^{+\infty} JE(\Delta_n) Ay = JE(\mathbb{H} \setminus \mathbb{R}) Ay = JAy,
\end{aligned}$$

which finishes the proof. □

Definition 15.2.4. An operator $N \in \mathcal{B}(V_R)$ is called *quasi-nilpotent* if

$$\lim_{n\to\infty} \|N^n\|^{\frac{1}{n}} = 0. \tag{15.22}$$

The following corollaries are immediate consequences of Gelfand's formula

$$r(T) = \lim_{n\to+\infty} \|T^n\|^{\frac{1}{n}},$$

for the spectral radius $r(T) = \max_{s\in\sigma_S(T)} |s|$ of T.

Corollary 15.2.5. *An operator $N \in \mathcal{B}(V_R)$ is quasi-nilpotent if and only if $\sigma_S(T) = \{0\}$.*

Corollary 15.2.6. *Let $S, N \in \mathcal{B}(V_R)$ be commuting operators and let N be quasi-nilpotent. Then $\sigma_S(S+N) = \sigma_S(S)$.*

We are now ready to show the main result of this section: the canonical reduction of a spectral operator, the quaternionic analogue of Theorem 5 in [106, Chapter XV.4.3].

Theorem 15.2.7. *An operator $T \in \mathcal{B}(V_R)$ is a spectral operator if and only if it is the sum $T = S + N$ of a bounded operator S of scalar type and a quasi-nilpotent operator N that commutes with S. Furthermore, this decomposition is unique, and T and S have the same S-spectrum and the same spectral decomposition (E, J).*

Proof. Let us first show that every operator $T \in \mathcal{B}(V_R)$ that is the sum $T = S+N$ of an operator S of scalar type and a quasi-nilpotent operator N commuting with S is a spectral operator. If (E, J) is the spectral decomposition of S, then Lemma 15.2.3 implies $E(\Delta)N = NE(\Delta)$ for all $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and $JN = NJ$. Since $T = S + N$, we find that also T commutes with (E, J).

Let now $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Then $T_\Delta = S_\Delta + N_\Delta$, where as usual the subscript Δ denotes the restriction of an operator to $V_\Delta = E(\Delta)V_R$. Since N_Δ inherits the property of being quasi-nilpotent from N and commutes with S_Δ, we deduce from Corollary 15.2.6,that

$$\sigma_S(T_\Delta) = \sigma_S(S_\Delta + N_\Delta) = \sigma_S(S_\Delta) \subset \overline{\Delta}.$$

Thus (E, J) satisfies items (i) and (ii) of Definition 15.1.1. It remains to show that also item (iii) holds true. Therefore, let $V_0 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ and set $T_0 = T|_{V_0}$, $S_0 = S|_{V_0}$, $N_0 = N|_{V_0}$, and $J_0 = J|_{V_0}$ and choose $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$. Since (E, J) is the spectral resolution of S, the operator $s_0\mathcal{I}_{V_0} - s_1J_0 - S_0$ has a bounded inverse $R(s_0, s_1) = (s_0\mathcal{I}_{V_0} - s_1J_0 - S_0)^{-1} \in \mathcal{B}(V_0)$. The operator N_0 is quasi-nilpotent because N is quasi-nilpotent, and hence it satisfies (15.22). The root test thus shows the convergence of the series $\sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n+1}$ in $\mathcal{B}(V_0)$.

Since T_0, N_0, S_0, and J_0 commute mutually, we have

$$\begin{aligned}
&(s_0\mathcal{I}_{V_0} - s_1 J_0 - T_0)\sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1}\\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1}(s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0 - N_0)\\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1}(s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0) - \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1} N_0\\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n} - \sum_{n=0}^{+\infty} N_0^{n+1} R(s_0,s_1)^{n+1} = \mathcal{I}_{V_0}.
\end{aligned}$$

We find that $s_0\mathcal{I}_0 - s_1 J_0 - T_0$ has a bounded inverse for $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, so that J is a spectral orientation for T. Hence, T is a spectral operator and T and S have the same spectral decomposition (E, J).

Since the spectral decomposition of T is uniquely determined, $S = \int_{\mathbb{H}} s\, dE_J(s)$ and in turn also $N = T - S$ are uniquely determined. Moreover, Corollary 15.2.6 implies that $\sigma_S(T) = \sigma_S(S)$.

Now assume that T is a spectral operator and let (E, J) be its spectral decomposition. We set

$$S := \int_{\mathbb{H}} s\, dE_J(s) \quad \text{and} \quad N := T - S.$$

By Remark 15.2.2, the operator S is of scalar type, and its spectral decomposition is (E, J). Since T commutes with (E, J), it commutes with S by Lemma 15.2.3. Consequently, $N = T - S$ also commutes with S and with T. What remains to show is that N is quasi-nilpotent. In view of Corollary 15.2.5, it is sufficient to show that $\sigma_S(N)$ is for every $\varepsilon > 0$ contained in the open ball $B_\varepsilon(0)$ of radius ε centered at 0 .

For arbitrary $\varepsilon > 0$, we choose $\alpha > 0$ such that $0 < (1 + C_{E,J})\alpha < \varepsilon$, where $C_{E,J} > 0$ is the constant in (14.22). We decompose $\sigma_S(T)$ into the union of disjoint axially symmetric Borel sets $\Delta_1, \ldots, \Delta_n \in \mathfrak{B}_S(\mathbb{H})$ such that for each $\ell \in \{1, \ldots, n\}$, the set Δ_ℓ is contained in a closed axially symmetric set whose intersection with every complex half-plane is a half-disk of diameter α. More precisely, we assume that there exist points $s_1, \ldots, s_n \in \mathbb{H}$ such that for all $\ell = 1, \ldots, n$,

$$\Delta_\ell \subset B_\alpha^+([s_\ell]) = \{p \in \mathbb{H} : \ \operatorname{dist}(p, [s_\ell]) \le \alpha \text{ and } p_1 \ge s_{\ell,1}\}.$$

Observe that we have either $s_\ell \in \mathbb{R}$ or $B_\alpha^+([s_\ell]) \cap \mathbb{R} = \emptyset$.

We set $V_{\Delta_\ell} = E(\Delta_\ell)V_R$. Since T and S commute with $E(\Delta_\ell)$, also $N = T - S$ does, and so $NV_{\Delta_\ell} \subset V_{\Delta_\ell}$. Hence $N_{\Delta_\ell} = N|_{V_{\Delta_\ell}} \in \mathcal{B}(V_{\Delta_\ell})$. If s belongs to $\rho_S(N_{\Delta_\ell})$

for all $\ell \in \{1, \dots, n\}$, we can set

$$\mathcal{Q}(s)^{-1} := \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell),$$

where

$$\mathcal{Q}_s(N_{\Delta_\ell})^{-1} = \left(N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}\right)^{-1} \in \mathcal{B}(V_{\Delta_\ell})$$

is the pseudo-resolvent of N_{Δ_ℓ} as s. The operator $\mathcal{Q}(s)^{-1}$ commutes with $E(\Delta_\ell)$ for every $\ell \in \{1, \dots, n\}$, so that

$$\begin{aligned}
&(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) \mathcal{Q}(s)^{-1} \\
&= \sum_{\ell=1}^{n} (N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}) \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell) = \sum_{\ell=1}^{n} E(\Delta_\ell) = \mathcal{I}_{V_R}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{Q}(s)^{-1}(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) \\
&= \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell)(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) \\
&= \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1}(N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}) E(\Delta_\ell) \\
&= \sum_{\ell=1}^{n} E(\Delta_\ell) = \mathcal{I}_{V_R}.
\end{aligned}$$

Therefore, we find $s \in \rho_S(N)$ such that $\bigcap_{\ell=1}^{n} \rho_S(N_{\Delta_\ell}) \subset \rho_S(N)$ and in turn $\sigma_S(N) \subset \bigcup_{\ell=1}^{n} \sigma_S(N_{\Delta_\ell})$. It is hence sufficient to show that $\sigma_S(N_{\Delta_\ell}) \subset B_\varepsilon(0)$ for all $\ell = 1, \dots, n$.

We distinguish two cases: if $s_\ell \in \mathbb{R}$, then we write

$$N_{\Delta_\ell} = (T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}).$$

Since $s_\ell \in \mathbb{R}$, we have for $p \in \mathbb{H}$ that

$$\begin{aligned}
&\mathcal{Q}_p(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) \\
&= (T_{\Delta_\ell}^2 - 2s_\ell T_{\Delta_\ell} + s_\ell^2 \mathcal{I}_{V_{\Delta_\ell}} - 2p_0(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (p_0^2 + p_1^2)\mathcal{I}_{V_{\Delta_\ell}} \\
&= T_{\Delta_\ell}^2 - 2(p_0 - s_\ell) T_{\Delta_\ell} + \left((p_0 - s_\ell)^2 + p_1^2\right) \mathcal{I}_{V_{\Delta_\ell}} = \mathcal{Q}_{p - s_\ell}(T_{\Delta_\ell})
\end{aligned}$$

and thus

$$\begin{aligned}
\sigma_S(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_\ell}) &= \{p - s_\ell \in \mathbb{H} : p \in \sigma_S(T_{\Delta_\ell})\} \\
&\subset \{p - s_\ell \in \mathbb{H} : p \in B_\alpha^+(s_\ell)\} = B_\alpha(0).
\end{aligned} \tag{15.23}$$

Moreover, the function $f(s) = (s_\ell - s)\chi_{\Delta_\ell}(s)$ is an intrinsic slice function because $s_\ell \in \mathbb{R}$. Since it is bounded, its integral with respect to (E, J) is defined and

$$s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell} = \left(\int_{\mathbb{H}} (s_\ell - s)\chi_{\Delta_\ell}(s)\, dE_J(s) \right)\Big|_{V_{\Delta_\ell}}.$$

We thus have

$$\|s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}\| \leq C_{E,J} \|(s_\ell - s)\chi_{\Delta_\ell}(s)\|_\infty \leq C_{E,J}\alpha \tag{15.24}$$

because $\Delta_\ell \subset B_\alpha([s_\ell])^+ = \overline{B_\alpha(s_\ell)}$. Since the operator $T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}$ and the operator $s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}$ commute, we conclude from Theorem 4.4.12 together with (15.23) and (15.24) that

$$\begin{aligned}
&\sigma_S(T_{\Delta\ell}) = \sigma_S\left((T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}) \right) \\
&\subset \left\{ s \in \mathbb{H} : \operatorname{dist}\left(s, \sigma_S\big(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}\big) \right) \leq C_{E,J}\alpha \right\} \subset B_{\alpha(1+C_{E,J})}(0) \subset B_\varepsilon(0).
\end{aligned}$$

If $s_\ell \notin \mathbb{R}$, then let us write

$$N_{\Delta_\ell} = (T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1} J_{\Delta_\ell}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1} J_{\Delta_\ell} - S_{\Delta_\ell}) \tag{15.25}$$

with $J_{\Delta_\ell} = J|_{V_{\Delta_\ell}}$. Since $E(\Delta_\ell)$ and J commute, J_{Δ_ℓ} is an imaginary operator on V_{Δ_ℓ} and it moreover commutes with T_{Δ_ℓ}. Since $-J_{\Delta_\ell}^2 = -J^2|_{V_\Delta} = E(\mathbb{H}\setminus\mathbb{R})|_{V_{\Delta_\ell}} = \mathcal{I}_{V_{\Delta_\ell}}$ because $\Delta_\ell \subset \mathbb{H} \setminus \mathbb{R}$, we find for $s = s_0 + j_s s_1 \in \mathbb{H}$ with $s_1 \geq 0$ that

$$\begin{aligned}
&\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big)\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big) \\
&= s_0^2 - s_1^2 J_{\Delta_\ell}^2 - 2s_0 T_{\Delta_\ell} + T_{\Delta_\ell}^2 = \mathcal{Q}_s(T_{\Delta_\ell}).
\end{aligned} \tag{15.26}$$

Because of condition (iii) in Definition 15.1.1, the operator $(s_0\mathcal{I} - s_1 J - T)|_{\operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})}$ is invertible if $s_1 > 0$. Since this operator commutes with $E(\Delta_\ell)$, the restriction of its inverse to V_{Δ_ℓ} is the inverse of $(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell})$ in $\mathcal{B}(V_{\Delta_\ell})$. Hence if $s_1 > 0$, then $(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell})$, and we conclude from (15.26) that

$$\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big)^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \quad \Longleftrightarrow \quad \mathcal{Q}_s(T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell}). \tag{15.27}$$

If, on the other hand, $s_1 = 0$, then both factors on the left-hand side of (15.26) agree, and so (15.27) holds also in this case. Hence $s \in \rho_S(T_{\Delta_\ell})$ if and only if the operator $(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T)$ has an inverse in $\mathcal{B}(V_{\Delta_\ell})$. Since

$$\sigma_S(T_{\Delta_\ell}) \subset \overline{\Delta_\ell} \subset B_\alpha^+([s_\ell]) \subset \{s = s_0 + j_s s_1 \in \mathbb{H} : \ \ s_1 \geq s_{\ell,1}\},$$

the operator $s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}$ is in particular invertible for every quaternion $s \in \mathbb{H}$ with $0 \leq s_1 < s_{\ell,1}$. Since J_{Δ_ℓ} is a spectral orientation for T_{Δ_ℓ}, this operator is also invertible if $s_1 < 0$, and hence we even obtain

$$(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \quad \forall s_0, s_1 \in \mathbb{R} : \ \ s_1 < s_{\ell,1}. \tag{15.28}$$

We can use these observations to deduce a spectral mapping property: a straightforward computation using the facts that T_{Δ_ℓ} and J_{Δ_ℓ} commute and that $J_{\Delta_\ell}^2 = -\mathcal{I}_{V_{\Delta_\ell}}$ shows that

$$\begin{aligned} &\mathcal{Q}_s(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}) \\ &= \Big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_1 + s_{\ell,1})J_{\Delta_\ell} - T_{\Delta_\ell}\Big) \\ &\quad \cdot \Big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_{\ell,1} - s_1)J_{\Delta_\ell} - T_{\Delta_\ell}\Big). \end{aligned} \tag{15.29}$$

If $s_1 > 0$, then the second factor is invertible because of (15.28). Hence we have $s \in \rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})$ if and only if the first factor in (15.29) is also invertible, i.e., if and only if

$$\big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_1 + s_{\ell,1})J_{\Delta_\ell} - T_{\Delta_\ell}\big)^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \tag{15.30}$$

exists. If, on the other hand, $s_1 = 0$, then both factors in (15.29) agree. Hence also in this case, s belongs to $\rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})$ if and only if the operator in (15.30) exists. By (15.27), the existence of (15.30) is, however, equivalent to

$$s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})\mathbb{S} \subset \rho_S(T_\Delta),$$

so that

$$\rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}) = \{s \in \mathbb{H} : \ \ s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})j_s \in \rho_S(T_{\Delta_\ell})\}$$

and in turn

$$\begin{aligned} &\sigma_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}) \\ &= \{s \in \mathbb{H} : \ \ s_0 + s_{\ell,1} + (s_1 + s_{\ell,1})j_s \in \sigma_S(T_{\Delta_\ell})\} \\ &\subset \{s \in \mathbb{H} : \ \ s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})j_s \in B_\alpha^+(s_\ell)\} = B_\alpha(0). \end{aligned}$$

For the second operator in (15.25), we have again

$$s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell} = \left(\int_{\mathbb{H}} (s_{\ell,0} + i_s s_{\ell,1} - s)\chi_{\Delta_\ell}(s)\, dE_J(s)\right)\Big|_{V_{\Delta_\ell}},$$

and so

$$\|s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell}\| \le C_{E,J}\|(s_{\ell,0} + i_s s_{\ell,1} - s)\chi_{\Delta_\ell}(s)\|_\infty \le C_{E,J}\alpha.$$

Since the operators $T_{\Delta_\ell} - s_\ell\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}$ and $s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell}$ commute, we conclude as before from Theorem 4.4.12 that $\sigma_S(T_{\Delta_\ell}) \subset B_{\alpha(1+C_{E,J})}(0) = B_\varepsilon(0)$.

Altogether, we obtain that N is quasi-nilpotent, which concludes the proof. □

Remark 15.2.8. Twice we applied Theorem 4.4.12 in the above proof, even though we are working on a right Banach space and the theory in Chapter 4 was developed on a two-sided Banach space. Using Theorem 14.2.7, one can, however, define the S-functional calculus also on right-sided Banach spaces, so that this result is actually applicable. For details, we refer to [125].

Definition 15.2.9. Let $T \in \mathcal{B}(V_R)$ be a spectral operator and decompose $T = S+N$ as in Theorem 15.2.7. The scalar operator S is called the *scalar part of* T, and the quasi-nilpotent operator N is called the *radical part of* T.

Remark 15.2.10. Let $T \in \mathcal{B}(V_R)$ be a spectral operator. The canonical decomposition of T into its scalar part and its radical part obviously coincides for every $j \in \mathbb{S}$ with the canonical decomposition of T as a $\mathbb{C}_j$-linear spectral operator on V_j.

The remainder of this section discusses the S-functional calculus for spectral operators. Similar to the complex case, one can express $f(T)$ for every intrinsic function f as a formal Taylor series in the radical part N of T. The Taylor coefficients are spectral integrals of f with respect to the spectral decomposition of T. Hence these coefficients, and in turn also $f(T)$, depend only on the values of f on the S-spectrum $\sigma_S(T)$ of T and not on the values of f on an entire neighborhood of $\sigma_S(T)$. The operator $f(T)$ is again a spectral operator, and its spectral decomposition can easily be constructed from the spectral decomposition of T.

In the following we consider an operator that is again defined on a two-sided Banach space V.

Proposition 15.2.11. *Let $S \in \mathcal{B}(V)$ be an operator of scalar type on a two-sided quaternionic Banach space V. If $f \in \mathcal{N}(\sigma_S(S))$, then*

$$f(S) = \int_{\mathbb{H}} f(s)\, dE_J(s), \tag{15.31}$$

where $f(S)$ is intended in the sense of the S-functional calculus.

Proof. Since $1(T) = \mathcal{I} = \int_{\mathbb{H}} 1\, dE_J(s)$ and $s(S) = S = \int_{\mathbb{H}} s\, dE_J(s)$, the product rule and the $\mathbb{R}$-linearity of both the S-functional calculus and the spectral integration imply that (15.31) holds for every intrinsic polynomial. It in turn also holds for every intrinsic rational function in $\mathcal{N}(\sigma_S(S))$, i.e., for every function r of the form $r(s) = p(s)q(s)^{-1}$ with intrinsic polynomials p and q such that $q(s) \neq 0$ for every $s \in \sigma_S(S)$.

Let now $f \in \mathcal{N}(\sigma_S(S))$ be arbitrary and let U be a bounded axially symmetric open set such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$. Runge's theorem for slice hyperholomorphic functions implies the existence of a sequence of intrinsic rational functions $r_n \in \mathcal{N}(\overline{U})$ such that $r_n \to f$ uniformly on $\overline{U}$. Because of Lemma 14.3.6, we thus have

$$\int_{\mathbb{H}} f(s)\, dE_J(s) \ = \lim_{n\to+\infty} \int_{\mathbb{H}} r_n(s)\, dE_J(s) = \lim_{n\to+\infty} r_n(S) = f(S). \qquad \square$$

Theorem 15.2.12. *Let $T \in \mathcal{B}(V)$ be a spectral operator on a two-sided quaternionic Banach space V with spectral decomposition (E, J) and let $T = S + N$ be the decomposition of T into scalar and radical parts. If $f \in \mathcal{N}(\sigma_S(T))$, then*

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \int_{\mathbb{H}} (\partial_S^n f)(s)\, dE_J(s), \tag{15.32}$$

where $f(T)$ is intended in the sense of the S-functional calculus and the series converges in the operator norm.

Proof. Since $T = S + N$ with $SN = NS$ and $\sigma_S(N) = \{0\}$, it follows from Theorem 4.4.14 that

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} (\partial_S^n f)(S).$$

What remains to show is that

$$(\partial_S^n f)(S) = \int_{\mathbb{H}} (\partial_S^n f)(s)\, dE_J(s), \tag{15.33}$$

but this follows immediately from Proposition 15.2.11. □

The operator $f(T)$ is again a spectral operator, and its radical part can be easily obtained from the above series expansion.

Definition 15.2.13. A spectral operator $T \in \mathcal{B}(V)$ on a two-sided quaternionic Banach space V is said to be of *type* $m \in \mathbb{N}$ if its radical part satisfies $N^{m+1} = 0$.

Lemma 15.2.14. *A spectral operator $T \in \mathcal{B}(V)$ on a two-sided quaternionic Banach space V with spectral resolution (E, J) and radical part N is of type m if and only if*

$$f(T) = \sum_{n=0}^{m} N^n \frac{1}{n!} \int_{\mathbb{H}} (\partial_S^n f)(s)\, dE_J(s) \quad \forall f \in \mathcal{N}(\sigma_S(T)). \tag{15.34}$$

In particular, T is a scalar operator if and only if it is of type 0.

Proof. If T is of type m, then the above formula follows immediately from Theorem 15.2.12 and $N^{m+1} = 0$. If, on the other hand, (15.34) holds, then we choose $f(s) = \frac{1}{m!} s^m$ in (15.32) and (15.34) and subtract these two expressions. We obtain

$$0 = N^{m+1} \int_{\mathbb{H}} dE_J(s) = N^{m+1}.$$ □

Theorem 15.2.15. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator with spectral decomposition (E, J). If $f \in \mathcal{N}(\sigma_S(T))$, then $f(T)$ is a spectral operator, and the spectral decomposition $(\tilde{E}, \tilde{J})$ of $f(T)$ is given by*

$$\tilde{E}(\Delta) = E\big(f^{-1}(\Delta)\big) \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}) \quad \text{and} \quad \tilde{J} = \int_{\mathbb{H}} j_{f(s)}\, dE_J(s),$$

where $j_{f(s)} = 0$ *if* $f(s) \in \mathbb{R}$ *and* $j_{f(s)} = \underline{f(s)}/|\underline{f(s)}|$ *if* $f(s) \in \mathbb{H} \setminus \mathbb{R}$. *For every* $g \in \mathcal{SM}^{\infty}(\mathbb{H})$ *we have*

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}_{\tilde{J}}(s) = \int_{\mathbb{H}} (g \circ f)(s)\, dE_J(s), \tag{15.35}$$

and if S *is the scalar part of* T, *then* $f(S)$ *is the scalar part of* $f(T)$.

Proof. We first show that $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$. By Corollary 14.3.4, the function f is $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}_S(\mathbb{H})$-measurable, so that $\tilde{E}$ is a well-defined spectral measure on $\mathfrak{B}_S(\mathbb{H})$.

The operator $\tilde{J}$ obviously commutes with E. Moreover, writing $f(s) = f_0(s) + j_s f_1(s)$ as in Lemma 14.3.3, we have $j_{f(s)} = j_s \operatorname{sgn}(f_1(s))$. If we set $\Delta_+ = \{s \in \mathbb{H} : f_1(s) > 0\}$, $\Delta_- = \{s \in \mathbb{H} : f_1(s) < 0\}$, and $\Delta_0 = \{s \in \mathbb{H} : f_1(s) = 0\}$, we therefore have

$$\tilde{J} = JE(\Delta_+) - JE(\Delta_-).$$

Since $f_1(s) = 0$ for every $s \in \mathbb{R}$, we have $\mathbb{R} \subset \Delta_0$ and hence $V_+ = \operatorname{ran} E(\Delta_+) \subset \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} J$ and similarly also $V_- = \operatorname{ran} E(\Delta_-) \subset \operatorname{ran} J$. Since J and E commute, V_+ and V_- are invariant subspaces of J contained in $\operatorname{ran} J$, so that J_+ and J_- define bounded surjective operators on V_+, resp. V_-. Moreover, $\ker J = \operatorname{ran} E(\mathbb{R})$, and hence $\ker J|_{V_+} = V_+ \cap \ker J = \{0\}$ and $\ker J|_{V_-} = V_- \cap \ker J = \{0\}$, so that $\ker \tilde{J} = \operatorname{ran} E(\Delta_0)$ and $\operatorname{ran} \tilde{J} = \operatorname{ran} E(\Delta_+) \oplus \operatorname{ran} E(\Delta_-) = \operatorname{ran} E(\Delta_+ \cup \Delta_-)$.

Now observe that $f(s) \in \mathbb{R}$ if and only if $f_1(s) = 0$. Hence $f^{-1}(\mathbb{R}) = \Delta_0$ and $f^{-1}(\mathbb{H} \setminus \mathbb{R}) = \Delta_+ \cup \Delta_-$, and we obtain

$$\operatorname{ran} \tilde{J} = \operatorname{ran} E(\Delta_+ \cup \Delta_-) = \operatorname{ran} E\left(f^{-1}(\mathbb{H} \setminus \mathbb{R})\right) = \operatorname{ran} \tilde{E}(\mathbb{H} \setminus \mathbb{R})$$

and

$$\ker \tilde{J} = \operatorname{ran} E(\Delta_0) = \operatorname{ran} E\left(f^{-1}(\mathbb{R})\right) = \operatorname{ran} \tilde{E}(\mathbb{R}).$$

Moreover, since $E(\Delta_+)E(\Delta_-) = E(\Delta_-)E(\Delta_+) = 0$ and $-J^2 = E(\mathbb{H} \setminus \mathbb{R})$, we have

$$\begin{aligned} -\tilde{J}^2 &= -J^2 E(\Delta_+)^2 - (-J^2) E(\Delta_-)^2 \\ &= E(\mathbb{H} \setminus \mathbb{R}) E(\Delta_+) + E(\mathbb{H} \setminus \mathbb{R}) E(\Delta_-) \\ &= E(\Delta_+ \cup \Delta_-) = \tilde{E}(\mathbb{H} \setminus \mathbb{R}), \end{aligned}$$

where we used that $\Delta_+ \subset \mathbb{H} \setminus \mathbb{R}$ and $\Delta_- \subset \mathbb{H} \setminus \mathbb{R}$ as $\mathbb{R} \subset \Delta_0$. Hence $-\tilde{J}^2$ is the projection onto $\operatorname{ran} \tilde{J}$ along $\ker \tilde{J}$, and so $\tilde{J}$ is actually an imaginary operator, and $(\tilde{E}, \tilde{J})$ in turn is a spectral system.

Let $g = \sum_{\ell=0}^{n} a_\ell \chi_{\Delta_\ell} \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ be a simple function. Then $(g \circ f)(s) = \sum_{\ell=0}^{n} a_\ell \chi_{f^{-1}(\Delta_\ell)}(s)$ is also a simple function in $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ and

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}(s) = \sum_{\ell=0}^{n} a_\ell \tilde{E}(\Delta_\ell) = \sum_{\ell=0}^{n} a_\ell E\left(f^{-1}(\Delta_\ell)\right) = \int_{\mathbb{H}} (g \circ f)(s)\, dE(s).$$

Due to the density of simple functions in $(\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R}), \|.\|_\infty)$, we hence obtain

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}(s) = \int_{\mathbb{H}} (g\circ f)(s)\, dE(s), \quad \forall g \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R}).$$

If $g \in \mathcal{SM}^\infty(\mathbb{H})$, then we deduce from Lemma 14.3.3 that $g(s) = \gamma(s) + j_s\delta(s)$ with $\gamma, \delta \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ and $j_s = \underline{s}/|\underline{s}|$ if $s \notin \mathbb{R}$ and $j_s = \delta(s) = 0$ if $s \in \mathbb{R}$. We then have $(g\circ f)(s) = \gamma(f(s)) + j_{f(s)}\delta(f(s))$, and we obtain

$$\begin{aligned}\int_{\mathbb{H}} g(s)\, d\tilde{E}_{\tilde{J}}(s) &= \int_{\mathbb{H}} \gamma(s)\, d\tilde{E}(s) + \tilde{J}\int_{\mathbb{H}} \delta(s)\, d\tilde{E}(s)\\ &= \int_{\mathbb{H}} (\gamma\circ f)(s)\, dE(s) + \tilde{J}\int_{\mathbb{H}} (\delta\circ f)(s)\, dE(s)\\ &= \int_{\mathbb{H}} (\gamma\circ f)(s)\, dE_J(s) + \int_{\mathbb{H}} j_{f(s)}\, dE_J(s)\int_{\mathbb{H}} (\delta\circ f)(s)\, dE(s)\\ &= \int_{\mathbb{H}} (\gamma\circ f)(s) + j_{f(s)}(\delta\circ f)(s)\, dE_J(s) = \int_{\mathbb{H}} (g\circ f)(s)\, dE_J(s),\end{aligned}$$

and hence (15.35) holds. Choosing in particular $g(s) = s$, we deduce from Proposition 15.2.11 that

$$f(S) = \int_{\mathbb{H}} f(s)\, dE_J(s) = \int_{\mathbb{H}} s\, d\tilde{E}_{\tilde{J}}(s).$$

By Remark 15.2.2, $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$.

Theorem 15.2.12 implies $f(T) = f(S) + \Theta$ with

$$\Theta := \sum_{n=1}^{+\infty} N^n \frac{1}{n!}(\partial_S^n f)(S).$$

If we can show that Θ is a quasi-nilpotent operator, then the statement of the theorem follows from Theorem 15.2.7. We first observe that each term in the sum is a quasi-nilpotent operator because N^n and $(\partial_S^n f)(S)$ commute due to Lemmas 14.3.6 and 15.2.3, so that

$$0 \le \lim_{k\to\infty}\left\|\left(N^n\frac{1}{n!}(\partial_S^n f)(S)\right)^k\right\|^{\frac{1}{k}} \le \left\|\frac{1}{n!}(\partial_S^n f)(S)\right\|\left(\lim_{k\to\infty}\left\|N^{nk}\right\|^{\frac{1}{nk}}\right)^n = 0.$$

Corollary 15.2.5 thus implies $\sigma_S\left(N^n\frac{1}{n!}(\partial_S^n f(S))\right) = \{0\}$.

By induction we conclude from Taylor's formula and Corollary 15.2.5 that for each $m \in \mathbb{N}$m, the finite sum $\Theta_1(m) := \sum_{n=1}^{m} N^n\frac{1}{n!}(\partial_S^n f)(S)$ is quasi-nilpotent and satisfies $\sigma_S(\Theta(m)) = \{0\}$.

Since the series Θ converges in the operator norm, for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that $\Theta_2(m_\varepsilon) := \sum_{n=m_\varepsilon+1}^{+\infty} N^n\frac{1}{n!}(\partial_S^n f)(S)$ satisfies $\|\Theta_2(m_\varepsilon)\| < \varepsilon$.

Hence $\sigma_S(\Theta_2(m_\varepsilon)) \subset B_\varepsilon(0)$, and since $\Theta = \Theta_1(m_\varepsilon) + \Theta_2(m_\varepsilon)$ and $\Theta_1(m_\varepsilon)$ and $\Theta_2(m_\varepsilon)$ commute, we conclude from Theorem 4.4.12 that $\sigma_S(\Theta) \subset B_\varepsilon(0)$. Since $\varepsilon > 0$ was arbitrary, we obtain $\sigma_S(\Theta) = \{0\}$. By Corollary 15.2.5, Θ is quasi-nilpotent.

We have shown that $f(T) = f(S) + \Theta$, that $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$, and that Θ is quasi-nilpotent. From Theorem 15.2.7 we therefore deduce that $f(T)$ is a spectral operator with spectral decomposition $(\tilde{E}, \tilde{J})$, that $f(S)$ is its scalar part, and that Θ is its radical part. This concludes the proof. □

Corollary 15.2.16. *Let $T \in \mathcal{B}(\mathcal{H})$ be a spectral operator and let $f \in \mathcal{N}(\sigma_S(T))$. If T is of type $m \in \mathbb{N}$, then $f(T)$ is of type m too.*

Proof. If $T = S + N$ is the decomposition of T into its scalar and radical parts and T is of type m such that $N^{m+1} = 0$, then the radical part Θ of $f(T)$ is given, due to Lemma 15.2.14 and Theorem 15.2.15, by

$$\Theta = f(T) - f(S) = \sum_{n=1}^{+\infty} N^n \frac{1}{n!}(\partial_S^n f)(S) = \sum_{n=1}^{m} N^n \frac{1}{n!}(\partial_S^n f)(S).$$

Obviously also $\Theta^{m+1} = 0$. □