Chapter 14

Spectral Integration in the Quaternionic Setting

Before we begin the study of quaternionic spectral operators, we discuss in this chapter spectral integration in the quaternionic setting. There have existed several different approaches to this topic in the literature, but these approaches required the introduction of a left multiplication on the Hilbert space (even though this multiplication was sometimes assumed to be defined only for quaternions in one complex plane and not for all $q \in \mathbb{H}$). This left multiplication was in general only partially determined by the a priori given mathematical structures; cf. also Remarks 9.3.7 and 9.4.12. It had to be extended randomly, and the necessary procedure does not generalize to the Banach space setting, in which we want to develop the theory of quaternionic spectral operators.

In this chapter we therefore develop an approach to spectral integration of intrinsic slice functions on a quaternionic right Banach space. This integration is done with respect to a spectral system instead of a spectral measure, a concept that makes specific ideas of [197]. It has a clear and intuitive interpretation in terms of the right linear structure of the space, and it is compatible with the complex theory. The prototype of a spectral system is a pair (E, J) on a Hilbert space that consists of a spectral measure E and an imaginary operator J with $E(\mathbb{H} \setminus \mathbb{R}) = -J^2$. This is exactly the structure that we used to define spectral integration on Hilbert spaces in Chapter 10. In this chapter we consider, however, spectral measures that are defined on axially symmetric subsets of $\mathbb{H}$ instead of subsets of a complex half-plane $\mathbb{C}_j^+$. Both approaches are equivalent: we can identify any axially symmetric set with its intersection with one complex half-plane $\mathbb{C}_j^+$ in order to obtain a bijective relation between these two types of sets. The two notations stress, however, two different philosophies. While the imaginary operator J was in Chapter 10 considered a multiplication by the imaginary unit j from the left, we stress in this chapter that J can also be considered a right linear multiplication

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by the entire set of imaginary units $\mathbb{S}$ form the right. This allows us to give a clear interpretation of spectral integration in terms of the right linear structure on the space.

The results in this chapter are taken from [125] and [128]. We want to point out that in this chapter and in the next one it is very important to distinguish between left and right Banach spaces. So to avoid confusion with the previous chapters we will denote the left Banach spaces by V_L, right Banach spaces by V_R and the two-sided ones by V.

14.1 Spectral Integrals of Real-Valued Slice Functions

The basic idea of spectral integration is well known: it generates a multiplication operator in a way that generalizes the multiplication by eigenvalues in the discrete case. If, for instance, $\lambda \in \sigma(A)$ of some normal operator $A : \mathbb{C}^n \to \mathbb{C}^n$, then we can define $E(\{\lambda\})$ to be the orthogonal projection of $\mathbb{C}^n$ onto the eigenspace associated with λ and we obtain $A = \sum_{\lambda\in\sigma(A)} \lambda\, E(\{\lambda\})$. Setting $E(\Delta) = \sum_{\lambda\in\Delta} E(\{\lambda\})$, one obtains a discrete measure on $\mathbb{C}$, the values of which are orthogonal projections on $\mathbb{C}^n$, and A is the integral of the identity function with respect to this measure. Changing the notation accordingly, we have

$$A = \sum_{\lambda\in\sigma(A)} \lambda\, E(\{\lambda\}) \quad \Longrightarrow \quad A = \int_{\sigma(A)} \lambda\, dE(\lambda). \tag{14.1}$$

Via functional calculus it is possible to define functions of an operator. The fundamental intuition of a functional calculus is that $f(A)$ should be defined by the action of f on the spectral values (resp. the eigenvalues) of A. For our normal operator A on $\mathbb{C}^n$ the operator $f(A)$ is the operator with the following property: if $y \in \mathbb{C}^n$ is an eigenvector of A with respect to λ, then y is an eigenvector of $f(A)$ with respect to $f(\lambda)$, just as happens, for instance, naturally for powers and polynomials of A. Using the above notation, we thus have

$$f(A) = \sum_{\lambda\in\sigma(A)} f(\lambda)\, E(\{\lambda\}) \quad \Longrightarrow \quad f(A) = \int_{\sigma(A)} f(\lambda)\, dE(\lambda). \tag{14.2}$$

In infinite-dimensional Hilbert spaces, the spectrum of a normal operator might be not discrete, so that the expressions on the left-hand side of (14.1) and (14.2) do not make sense. If E, however, is a suitable projection-valued measure, then it is possible to give the expression (14.2) a meaning by following the usual way of defining integrals via the approximation of f by simple functions. The spectral theorem then shows that for every normal operator T there exists a spectral measure such that (14.1) holds.

If we want to introduce similar concepts in the quaternionic setting, we are—even in the discrete case—faced with several unexpected phenomena.

(P1) The space of bounded linear operators on a quaternionic Banach space V_R is only a real Banach space and not a quaternionic one. Hence the expressions in (14.1) and (14.2) are defined a priori only if λ and $f(\lambda)$, respectively, are real. Otherwise, one needs to give meaning to the multiplication of the operator $E(\{\lambda\})$ by nonreal scalars.

(P2) The multiplication by a (nonreal) scalar on the right is not linear, so that $aE(\{\lambda\})$ for $a \in \mathbb{H}$ cannot be defined as $(aE(\{\lambda\}))(ya) = (E(\{\lambda\})y)a$. Moreover, the set of eigenvectors associated with a specific eigenvalue does not constitute a linear subspace of V_R: if, for instance, $Ty = ys$ with $s = s_0 + j_s s_1$ and $i \in \mathbb{S}$ with $j_s \perp i$, then $T(yi) = (Ty)i = (ys)i = (yi)\overline{s} \neq (yi)s$.

(P3) Finally, the set of eigenvalues is in general not discrete: if $s \in \mathbb{H}$ is an eigenvalue of T with $Ty = ys$ for some $y \neq 0$ and $s_j = s_0 + js_1 \in [s]$, then there exists $h \in \mathbb{H} \setminus \{0\}$ such that $s_j = h^{-1}sh$, and so

$$T(yh) = T(y)h = ysh = (yh)h^{-1}sh = (yh)s_j. \tag{14.3}$$

Thus, every $s_j \in [s]$ is also an eigenvalue of T.

As a first consequence of items (P2) and (P3), the notions of eigenvalue and eigenspace have to be adapted: linear subspaces are in the quaternionic setting not associated with individual eigenvalues s but with spheres $[s]$ of equivalent eigenvalues.

Definition 14.1.1. Let $T \in \mathcal{L}(V_R)$ and let $s \in \mathbb{H}\setminus\mathbb{R}$. We say that $[s]$ is an eigensphere of T if there exists a vector $y \in V_R \setminus \{0\}$ such that

$$(T^2 - 2s_0T + |s|^2\mathcal{I})y = \mathcal{Q}_s(T)y = 0. \tag{14.4}$$

The eigenspace of T associated with $[s]$ consists of all those vectors that satisfy (14.4).

Remark 14.1.2. For real values, things remain as we know them from the classical case: a quaternion $s \in \mathbb{R}$ is an eigenvalue of T if $Ty - ys = 0$ for some $y \neq 0$. The quaternionic right linear subspace $\ker(T - s\mathcal{I})$ is then called the eigenspace of T associated with s.

Every eigenvector y that satisfies $T(y) = ys_j$ with $s_j = s_0 + js_1$ for some $j \in \mathbb{S}$ belongs to the eigenspace associated with the eigensphere $[s]$. Note, however, that the eigenspace associated with an eigensphere $[s]$ does not consist only of eigenvectors. It contains also linear combinations of eigenvectors associated with different eigenvalues in $[s]$, as the next lemma shows.

Lemma 14.1.3. *Let $T \in \mathcal{L}(V_R)$, let $[s]$ be an eigensphere of T, and let $j \in \mathbb{S}$. A vector y belongs to the eigenspace associated with $[s]$ if and only if $y = y_1 + y_2$ such that $Ty_1 = y_1 s_j$ and $Ty_2 = y_2\overline{s_j}$, where $s_j = s_0 + js_1$.*

Proof. Observe that

$$\mathcal{Q}_s(T)y = T^2y - Ty2s_0 + y|s|^2 = T(Ty - y\overline{s_j}) - (Ty - y\overline{s_j})s_j \tag{14.5}$$

and

$$\mathcal{Q}_s(T)y = T^2y - Ty2s_0 + y|s|^2 = T(Ty - ys_j) - (Ty - ys_j)\overline{s_j}. \tag{14.6}$$

Hence $\mathcal{Q}_s(T)y = 0$ for every eigenvector associated with s_j or $\overline{s_j}$ and in turn also for every y that is the sum of two such vectors.

If, conversely, y satisfies (14.4), then we deduce from (14.5) that $Ty - y\overline{s_j}$ is a right eigenvector associated with s_j and that $Ty - ys_j$ is a right eigenvalue of T associated with $\overline{s_j}$. Since s_j and j commute, the vectors $y_1 = (Ty - y\overline{s_j})\frac{-j}{2s_1}$ and $y_2 = (Ty - ys_j)\frac{j}{2s_1}$ are right eigenvectors associated with s resp. $\overline{s_j}$, too. Hence we have obtained the desired decomposition as

$$y_1 + y_2 = (Ty - y\overline{s_j})\frac{-j}{2s_1} + (Ty - ys_j)\frac{j}{2s_1} = y(\overline{s_j} - s_j)\frac{j}{2s_1} = y. \qquad \square$$

Remark 14.1.4. If $i \in \mathbb{S}$ with $i \perp j$, then $\tilde{y}_2 := y_2(-i)$ is an eigenvector of T associated with s. Hence we can write y also as $y = y_1 + \tilde{y}_2 i$, where $y_1, \tilde{y}_2$ are both eigenvectors associated with s_j.

Since the eigenspaces of quaternionic linear operators are not associated with individual eigenvalues but instead with eigenspheres, quaternionic spectral measures must not be defined on arbitrary subsets of the quaternions. Instead, their natural domains of definition consist of axially symmetric subsets of $\mathbb{H}$, so that they associate subspaces of V_R not to sets of spectral values but to sets of spectral spheres. This is also consistent with the fact that the S-spectrum of an operator is axially symmetric.

Definition 14.1.5. We denote the σ-algebra of axially symmetric Borel sets on $\mathbb{H}$ by $\mathfrak{B}_S(\mathbb{H})$. Furthermore, we denote the set of all real-valued $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}(\mathbb{R})$-measurable functions defined on $\mathbb{H}$ by $\mathcal{M}_S(\mathbb{H}, \mathbb{R})$ and the set of all such functions that are bounded by $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$.

Remark 14.1.6. The restrictions of functions in $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ to a complex half-plane $\mathbb{C}_j^+$ are exactly the functions that were used to construct the spectral measure of a quaternionic normal operator in the previous chapters.

Definition 14.1.7. A quaternionic spectral measure on a quaternionic right Banach space V_R is a function $E : \mathfrak{B}_S(\mathbb{H}) \to \mathcal{B}(V_R)$ that satisfies

(i) $E(\Delta)$ is a continuous projection and $\|E(\Delta)\| \leq K$ for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$ with a constant $K > 0$ independent of Δ,

(ii) $E(\emptyset) = 0$ and $E(\mathbb{H}) = \mathcal{I}$,

(iii) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for every $\Delta_1, \Delta_2 \in \mathfrak{B}_S(\mathbb{H})$, and

(iv) for every sequence $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathfrak{B}_S(\mathbb{H})$ we have

$$E\left(\bigcup_{n\in\mathbb{N}}\Delta_n\right)y=\sum_{n=1}^{+\infty}E(\Delta_n)y\quad\text{for all } y\in V_R.$$

Corollary 14.1.8. *Let E be a spectral measure on V_R and let V_R^* be its dual space, the left Banach space consisting of all continuous right linear mappings from V_R to $\mathbb{H}$. For every $y\in V_R$ and $y^*\in V_R^*$, the mapping $\Delta\mapsto\langle y^*,E(\Delta)y\rangle$ is a quaternion-valued measure on $\mathfrak{B}_S(\mathbb{H})$.*

Remark 14.1.9. In the literature, authors have considered quaternionic spectral measures defined on the Borel sets $\mathfrak{B}\left(\mathbb{C}_j^+\right)$ of one of the closed complex half-planes $\mathbb{C}_j^+=\{s_0+js_1 : s_0\in\mathbb{R}, s_1\geq 0\}$, and we also did this in Chapter 10. This is equivalent to E being defined on $\mathfrak{B}_S(\mathbb{H})$. Indeed, if $\tilde{E}$ is defined on $\mathfrak{B}\left(\mathbb{C}_j^+\right)$, then setting

$$E(\Delta):=\tilde{E}\left(\Delta\cap\mathbb{C}_j^+\right)\quad\forall\Delta\in\mathfrak{B}_S(\mathbb{H})$$

yields a spectral measure in the sense of Definition 14.1.7 that is defined on $\mathfrak{B}_S(\mathbb{H})$. If, on the other hand, we start with a spectral measure E defined on $\mathfrak{B}_S(\mathbb{H})$, then setting

$$\tilde{E}(\Delta):=E([\Delta])\quad\forall\Delta\in\mathfrak{B}\left(\mathbb{C}_j^+\right)$$

yields the respective measure on $\mathfrak{B}\left(\mathbb{C}_j^+\right)$. Although both definitions are equivalent, in this chapter we prefer $\mathfrak{B}_S(\mathbb{H})$ as the domain of E because it does not suggest a dependence on the imaginary unit j.

For a function $f\in\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$, we can now define the spectral integral with respect to a spectral measure E as in the classical case. If f is a simple function, i.e., $f(s)=\sum_{k=1}^n\alpha_k\chi_{\Delta_k}(s)$ with pairwise disjoint sets $\Delta_k\in\mathfrak{B}_S(\mathbb{H})$, where χ_{Δ_k} denotes the characteristic function of Δ_k, then we set

$$\int_{\mathbb{H}}f(s)\,dE(s):=\sum_{k=1}^n\alpha_kE(\Delta_k).\tag{14.7}$$

There exists a constant $C_E>0$ that depends only on E such that

$$\left\|\int_{\mathbb{H}}f(s)\,dE(s)\right\|\leq C_E\|f\|_\infty,\tag{14.8}$$

where $\|.\|_\infty$ denotes the supremum norm. If $f\in\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ is arbitrary, then we can find a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\|f-f_n\|_\infty\to 0$ as $n\to+\infty$. In this case we can set

$$\int_{\mathbb{H}}f(s)\,dE(s):=\lim_{n\to+\infty}\int_{\mathbb{H}}f_n(s)\,dE(s),\tag{14.9}$$

where this sequence converges in the operator norm because of (14.8).

Lemma 14.1.10. *Let E be a quaternionic spectral measure on V_R. The mapping $f \mapsto \int_{\mathbb{H}} f(s)\, dE(s)$ is a continuous homomorphism from $\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ to $\mathcal{B}(V_R)$. Moreover, if T commutes with E, i.e., it satisfies $TE(\Delta) = E(\Delta)T$ for all sets $\Delta \in \mathfrak{B}_S(\mathbb{H})$, then T commutes with $\int_{\mathbb{H}} f(s)\, dE(s)$ for every $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$.*

Corollary 14.1.11. *Let E be a quaternionic spectral measure on V_R and let $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$. For every $y \in V_R$ and $y^* \in V_R^*$, we have*

$$\left\langle y^*, \left[\int_{\mathbb{H}} f\, dE\right] y \right\rangle = \int_{\mathbb{H}} f(s)\, d\langle y^*, E(s)y\rangle.$$

Proof. Let $f_n = \sum_{k=1}^{N_n} \alpha_{n,k}\chi_{\Delta_{n,k}} \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ be such that $\|f - f_n\| \to 0$ as $n \to +\infty$. Since all coefficients $\alpha_{n,k}$ are real, we have

$$\begin{aligned}\left\langle y^*, \left[\int_{\mathbb{H}} f\, dE\right] y \right\rangle &= \lim_{n\to\infty} \left\langle y^*, \left[\sum_{k=1}^{N_n} \alpha_{n,k} E(\Delta_{n,k})\right] y \right\rangle \\ &= \lim_{n\to\infty} \sum_{k=1}^{N_n} \alpha_{n,k} \left\langle y^*, E(\Delta_{n,k})y\right\rangle = \int_{\mathbb{H}} f(s)\, d\langle y^*, E(s)y\rangle. \qquad \square\end{aligned}$$

Remark 14.1.12. The above definitions are well posed and the properties given in Lemma 14.1.10 can be shown as in the classical case, so we omit their proofs. One can also deduce them directly from the classical theory: if we consider V_R a real Banach space and E a spectral measure with values in the space $\mathcal{B}_{\mathbb{R}}(V_R)$ of bounded $\mathbb{R}$-linear operators on V_R, which obviously contains $\mathcal{B}(V_R)$, then $\int_{\mathbb{H}} f(s)\, dE(s)$ defined in (14.7), resp. (14.9), is nothing but the spectral integral of f with respect to E in the classical sense. Since every α_k in (14.7) is real and since each $E(\Delta)$ is a quaternionic right linear projection, the integral of every simple function f with respect to E is a quaternionic right linear operator and hence belongs to $\mathcal{B}(V_R)$. The space $\mathcal{B}(V_R)$ is closed in $\mathcal{B}_{\mathbb{R}}(V_R)$, and hence the property of being quaternionic linear survives the approximation by simple functions such that $\int_{\mathbb{H}} f(s)\, dE(s) \in \mathcal{B}(V_R)$ for every $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ even if we consider it the integral with respect to a (real) spectral measure on the real Banach space V_R.

14.2 Imaginary Operators

The techniques introduced so far allow us to integrate real-valued functions with respect to a spectral measure. This is obviously insufficient, even for formulating the statement corresponding to (14.1) in the quaternionic setting unless $\sigma_S(T)$ is real. In order to define spectral integrals for functions that are not real-valued, we need additional information.

This fits another observation: in contrast to the complex case, even for the simple case of a normal operator on a finite-dimensional quaternionic Hilbert space, a decomposition of the space V_R into the eigenspaces of T is not sufficient to recover the entire operator T. Let $j, i \in \mathbb{S}$ with $j \neq i$ and consider, for instance, the operators T_1, T_2, and T_3 on $\mathbb{H}^2$, which are given by their matrix representations

$$T_1 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad T_2 = \begin{pmatrix} j & 0 \\ 0 & i \end{pmatrix}, \quad T_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \tag{14.10}$$

For each $\ell = 1, 2, 3$, we have $\sigma_S(T_\ell) = \mathbb{S}$ and that the only eigenspace of T_ℓ is the entire space $\mathbb{H}^2$. The spectral measure E that is associated with T_ℓ is hence given by $E(\Delta) = 0$ if $\mathbb{S} \not\subset \Delta$ and $E(\Delta) = \mathcal{I}$ if $\mathbb{S} \subset \Delta$. Obviously, the spectral measures associated with these operators agree, although these operators do not coincide.

Since the eigenspace of an operator T that is associated with some eigensphere $[s]$ contains eigenvectors associated with different eigenvalues, we need some additional information to understand how to multiply the eigensphere onto the associated eigenspace, i.e., to understand which vector in the eigenspace must be multiplied by which eigenvalue in the corresponding eigensphere $[s]$. This information will be provided by a suitable imaginary operator. Such operators generalize the properties of the anti-self-adjoint partially unitary operator J in the Teichmüller decomposition

$$T = A + JB = \frac{1}{2}(T + T^*) + \frac{1}{2}J|T - T^*| \tag{14.11}$$

(where J is an anti-self-adjoint partial isometry with $\ker J = \ker(T - T^*)$ that is determined by the polar decomposition of $\frac{!}{2}(T - T^*)$) of a normal operator on a Hilbert space to the Banach space setting.

Definition 14.2.1. An operator $J \in \mathcal{B}(V_R)$ is called *imaginary* if $-J^2$ is the projection of V_R onto $\operatorname{ran} J$ along $\ker J$. We call J *fully imaginary* if $-J^2 = \mathcal{I}$, i.e., if in addition, $\ker J = \{0\}$.

Corollary 14.2.2. *An operator $J \in \mathcal{B}(V_R)$ is an imaginary operator if and only if*

(i) $-J^2$ *is a projection and*

(ii) $\ker J = \ker J^2$.

Proof. If J is an imaginary operator, then obviously item (i) and item (ii) hold. Assume, on the other hand, that item (i) and item (ii) hold. Obviously $\operatorname{ran}(-J^2) \subset \operatorname{ran} J$. For every $x \in V_R$, we have $(-J^2)x - x \in \ker(-J^2) = \ker J$ because

$$(-J^2)\left((-J^2)x - x\right) = (-J^2)^2 x - (-J)^2 x = (-J^2)x - (-J)^2 x = 0,$$

since $(-J^2)$ is a projection. Therefore

$$0 = J(-J^2 x - x) = (-J^2)Jx - Jx,$$

and hence $y = (-J^2)y$ for every $y = Jx \in \operatorname{ran} J$. Consequently, $\operatorname{ran}(-J^2) \supset \operatorname{ran} J$, and in turn $\operatorname{ran} J = \operatorname{ran}(-J^2)$. Since $\ker J = \ker(-J^2)$, we find that $-J^2$ is the projection of V_R onto $\operatorname{ran} J$ along $\ker J$, i.e., that J is an imaginary operator. $\square$

Remark 14.2.3. The above implies that every anti-self-adjoint partially unitary operator J on a quaternionic Hilbert space $\mathcal{H}$ is an imaginary operator. Indeed, for every $y \in \ker J$, we obviously have $-J^2 y = 0$. Since the restriction of J to $\mathcal{H}_0 := \operatorname{ran} J = \ker J^{\perp}$ is unitary and J is anti-self-adjoint, we furthermore have for $y \in \mathcal{H}_0$ that $-J^2 y = J^* J y = (J|_{\mathcal{H}_0})^* (J|_{\mathcal{H}_0}) y = (J|_{\mathcal{H}_0})^{-1} (J|_{\mathcal{H}_0}) y = y$. Hence $-J^2$ is the orthogonal projection onto $\mathcal{H}_0 = \operatorname{ran} J$, and so J is an imaginary operator. In particular, every unitary anti-self-adjoint operator is fully imaginary. Cf. also Corollary 9.3.8.

Lemma 14.2.4. *If $J \in \mathcal{B}(V_R)$ is an imaginary operator, then $\sigma_S(T) \subset \{0\} \cup \{\mathbb{S}\}$.*

Proof. Since the operator $-J^2$ is a projection, its S-spectrum $\sigma_S(-J^2)$ is a subset of $\{0,1\}$. Indeed, for every projection $P \in \mathcal{B}(V)$, a simple calculation shows that the pseudo-resolvent of P at every $s \in \mathbb{H} \setminus \{0,1\}$ is given by

$$\mathcal{Q}_s(P)^{-1} = -\frac{1}{|s|^2}\left(\frac{1 - 2\mathrm{Re}(s)}{1 - 2\mathrm{Re}(s) + |s|^2} P - \mathcal{I}\right)$$

such that $s \in \rho_S(P)$. As a consequence of the spectral mapping theorem, we find that

$$-\sigma_S(J)^2 = \{-s^2 : \ s \in \sigma_S(J)\} = \sigma_S(-J^2) \subset \{0,1\}.$$

But if $-s^2 \in \{0,1\}$, then $s \in \{0\} \cup \mathbb{S}$ and hence $\sigma_S(J) \subset \{0\} \cup \mathbb{S}$. □

Remark 14.2.5. If $J = 0$, then J is an imaginary operator with $\sigma_S(T) = \{0\}$. If, on the other hand, $\ker J = \{0\}$ (i.e., if J is fully imaginary), then $\sigma_S(T) = \mathbb{S}$. In every other case we obviously have $\sigma_S(T) = \{0\} \cup \mathbb{S}$.

Our next goal is to arrive at Theorem 14.2.10, which gives a complete characterization of imaginary operators on V_R. It is the analogue of Lemma 9.3.9. Before we prove this result, however, we prove a crucial relation between the concepts of quaternionic spectral theory and the concepts of classical complex operator theory.

Every quaternionic right Banach space V_R can in a natural way be considered a complex Banach space over any of the complex planes $\mathbb{C}_j$ by restricting the multiplication by quaternionic scalars from the right to $\mathbb{C}_j$. In order to deal with the different structures on V_R, we introduce the following notation.

Definition 14.2.6. Let V_R be a quaternionic right Banach space. For $j \in \mathbb{S}$, we denote the space V_R considered as a complex Banach space over the complex field $\mathbb{C}_j$ by $V_{R,j}$. If T is a quaternionic right linear operator on V_R, then $\rho_{\mathbb{C}_j}(T)$ and $\sigma_{\mathbb{C}_j}(T)$ shall denote its *resolvent set* and *spectrum* as a $\mathbb{C}_j$-complex linear operator on $V_{R,j}$. If A is a $\mathbb{C}_j$-complex linear, but not quaternionic linear, operator on $V_{R,j}$, then we denote its *spectrum* as usual by $\sigma(A)$.

If we want to distinguish between the identity operator on V_R and the identity operator on $V_{R,j}$, we denote them by $\mathcal{I}_{V_R}$ and $\mathcal{I}_{V_{R,j}}$. We point out that the operator $\lambda \mathcal{I}_{V_{R,j}}$ for $\lambda \in \mathbb{C}_j$ acts as $\lambda \mathcal{I}_{V_{R,j}} y = y\lambda$ because the multiplication by scalars on $V_{R,j}$ is defined as the quaternionic right scalar multiplication on V_R restricted to $\mathbb{C}_j$.

Theorem 14.2.7. *Let $T \in \mathcal{L}(V_R)$ and choose $j \in \mathbb{S}$. The spectrum $\sigma_{\mathbb{C}_j}(T)$ of T considered as a closed complex linear operator on $V_{R,j}$ equals $\sigma_S(T) \cap \mathbb{C}_j$, i.e.,*

$$\sigma_{\mathbb{C}_j}(T) = \sigma_S(T) \cap \mathbb{C}_j. \tag{14.12}$$

For every λ in the resolvent set $\rho_{\mathbb{C}_j}(T)$ of T as a complex linear operator on $V_{R,j}$, the $\mathbb{C}_j$-linear resolvent of T is given by $R_\lambda(T) = \left(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T\right) \mathcal{Q}_\lambda(T)^{-1}$, i.e.,

$$R_\lambda(T)y := \mathcal{Q}_\lambda(T)^{-1} y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y. \tag{14.13}$$

For every $i \in \mathbb{S}$ with $j \perp i$, we moreover have

$$R_{\overline{\lambda}}(T)y = -[R_\lambda(T)(yi)]i. \tag{14.14}$$

Proof. Let $\lambda \in \rho_S(T) \cap \mathbb{C}_j$. The resolvent $(\lambda\mathcal{I}_{V_{R,j}} - T)^{-1}$ of T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$ is then given by (14.13). Indeed, since T and $Q_\lambda(T)^{-1}$ commute, we have for $y \in \mathcal{D}(T)$ that

$$\begin{aligned}
&(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T)\mathcal{Q}_\lambda(T)^{-1}(y\lambda - Ty)\\
&= (\overline{\lambda}\mathcal{I}_{V_{R,j}} - T)\left(\mathcal{Q}_\lambda(T)^{-1}y\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\right)\\
&= \mathcal{Q}_\lambda(T)^{-1}y\lambda\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\lambda + T^2\mathcal{Q}_\lambda(T)^{-1}y\\
&= (|\lambda|^2\mathcal{I}_{V_{R,j}} - 2\lambda_0 T + T^2)\mathcal{Q}_\lambda(T)^{-1}y = y.
\end{aligned}$$

Similarly, for $y \in V_{R,j} = V_R$, we have

$$\begin{aligned}
&(\lambda\mathcal{I}_{V_{R,j}} - T)R_\lambda(T)y\\
&= (\lambda\mathcal{I}_{V_{R,j}} - T)\left(\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\right)\\
&= \mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda}\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} + T^2\mathcal{Q}_\lambda(T)^{-1}y\\
&= (|\lambda|^2\mathcal{I}_{V_{R,j}} - 2\lambda_0 T + T^2)\mathcal{Q}_\lambda(T)^{-1}y = y.
\end{aligned}$$

Since $\mathcal{Q}_\lambda(T)^{-1}$ maps $V_{R,j}$ to $\mathcal{D}(T^2) \subset \mathcal{D}(T)$, we find that the operator $R_\lambda(T) = (\lambda\mathcal{I}_{V_{R,j}} - T)\mathcal{Q}_\lambda(T)^{-1}$ is bounded, and so λ belongs to the resolvent set$\rho_{\mathbb{C}_j}(T)$ of T considered as a $\mathbb{C}_j$-linear operator on $V_{R,j}$. Hence, $\rho_S(T) \cap \mathbb{C}_j \subset \rho_{\mathbb{C}_j}(T)$, and in turn $\sigma_{\mathbb{C}_j}(T) \subset \sigma_S(T) \cap \mathbb{C}_j$. Together with the axial symmetry of the S-spectrum, this further implies

$$\sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)} \subset (\sigma_S(T) \cap \mathbb{C}_j) \cup (\overline{\sigma_S(T) \cap \mathbb{C}_j}) = \sigma_S(T) \cap \mathbb{C}_j, \tag{14.15}$$

where $\overline{A} = \{\overline{z} : z \in A\}$.

If λ and $\overline{\lambda}$ both belong to $\rho_{\mathbb{C}_j}(T)$, then $[\lambda] \subset \rho_S(T)$ because

$$\begin{aligned}
&(\lambda\mathcal{I}_{V_{R,j}} - T)(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T)y\\
&= (y\overline{\lambda})\lambda - (Ty)\lambda - T(y\overline{\lambda}) + T^2 y\\
&= (T^2 - 2\lambda_0 T + |\lambda|^2)y
\end{aligned}$$

and hence $\mathcal{Q}_\lambda(T)^{-1} = R_\lambda(T)R_{\overline{\lambda}}(T) \in \mathcal{B}(V_R)$. Thus $\rho_S(T) \cap \mathbb{C}_j \supset \rho_{\mathbb{C}_j}(T) \cap \overline{\rho_{\mathbb{C}_j}(T)}$, and in turn

$$\sigma_S(T) \cap \mathbb{C}_j \subset \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)}. \tag{14.16}$$

The two relations (14.15) and (14.16) together yield

$$\sigma_S(T) \cap \mathbb{C}_j = \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)}. \tag{14.17}$$

What remains to show is that $\rho_{\mathbb{C}_j}(T)$ and $\sigma_{\mathbb{C}_j}(T)$ are symmetric with respect to the real axis, which then implies

$$\sigma_S(T) \cap \mathbb{C}_j = \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)} = \sigma_{\mathbb{C}_j}(T). \tag{14.18}$$

Let $\lambda \in \rho_{\mathbb{C}_j}(T)$ and choose $i \in \mathbb{S}$ with $j \perp i$. We show that $R_{\overline{\lambda}}(T)$ equals the mapping $Ay := -\left[R_\lambda(T)(yi)\right] i$. Since $\lambda i = i\overline{\lambda}$ and $i\lambda = \overline{\lambda} i$, we have for $y \in \mathcal{D}(T)$ that

$$\begin{aligned}
&A\left(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T\right) y = A\left(y\overline{\lambda} - Ty\right)\\
&= -\left[R_\lambda(T)\left((y\overline{\lambda})i - (Ty)i\right)\right] i\\
&= -\left[R_\lambda(T)((yi)\lambda - T(yi))\right] i\\
&= -\left[R_\lambda(T)(\lambda\mathcal{I}_{V_{R,j}} - T)(yi)\right] i = -yii = y.
\end{aligned}$$

Similarly, for arbitrary $y \in V_{R,j} = V_R$, we have

$$\begin{aligned}
&\left(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T\right) Ay = (Ay)\,\overline{\lambda} - T\,(Ay)\\
&= -\left[R_\lambda(T)(yi)\right] i\overline{\lambda} + T\left(\left[R_\lambda(T)(yi)\right] i\right)\\
&= -\left[R_\lambda(T)(yi)\lambda - T(R_\lambda(T)(yi))\right] i\\
&= -\left[(\lambda\mathcal{I}_{V_{R,j}} - T)R_\lambda(T)(yi)\right] i = -yii = y.
\end{aligned}$$

Hence if $\lambda \in \rho_{\mathbb{C}_j}(T)$, then $R_{\overline{\lambda}}(T) = -\left[R_\lambda(T)(yi)\right] i$ so that in particular, $\overline{\lambda} \in \rho_{\mathbb{C}_j}(T)$. Consequently, $\rho_{\mathbb{C}_j}(T)$ and in turn also $\sigma_{\mathbb{C}_j}(T)$, are symmetric with respect to the real axis, so that (14.18) holds. □

Definition 14.2.8. Let $T \in \mathcal{L}(V_R)$. We define the V_R-valued function

$$\mathcal{R}_s(T;y) = \mathcal{Q}_s(T)^{-1} y\overline{s} - T\mathcal{Q}_s(T)^{-1} y \quad \forall y \in V_R,\ s \in \rho_S(T).$$

Remark 14.2.9. By Theorem 14.2.7, the mapping $y \mapsto \mathcal{R}_s(T;y)$ coincides with the resolvent of T at s applied to y if T is considered a $\mathbb{C}_{j_s}$-linear operator on V_{R,j_s}.

Let us now turn back to characterizing imaginary operators on Banach spaces. Just as with imaginary operators on a Hilbert space, we can find three subspaces of V_R on which such an operator is simply multiplication by 0, j, or $-j$.

Theorem 14.2.10. *Let $J \in \mathcal{B}(V_R)$ be an imaginary operator. For every $j \in \mathbb{S}$, the Banach space V_R admits a direct sum decomposition as*

$$V_R = V_{J,0} \oplus V_{J,j}^{+} \oplus V_{J,j}^{-} \tag{14.19}$$

with

$$\begin{aligned} V_{J,0} &= \ker(J), \\ V_{J,j}^{+} &= \{y \in V : Jy = yj\}, \\ V_{J,j}^{-} &= \{y \in V : Jy = y(-j)\}. \end{aligned} \tag{14.20}$$

The spaces $V_{J,j}^{+}$ and $V_{J,j}^{-}$ are complex Banach spaces over $\mathbb{C}_j$ with the natural structure inherited from V_R, and for each $i \in \mathbb{S}$ with $j \perp i$, the map $y \mapsto yi$ is a $\mathbb{C}_j$-antilinear and isometric bijection between $V_{J,j}^{+}$ and $V_{J,j}^{-}$.

Conversely, let $j, i \in \mathbb{S}$ with $j \perp i$ and assume that V_R is the direct sum $V_R = V_0 \oplus V_+ \oplus V_-$ of a closed ($\mathbb{H}$-linear) subspace V_0 and two closed $\mathbb{C}_j$-linear subspaces V_+ and V_- of V_R such that $\Psi : y \mapsto yi$ is a bijection between V_+ and V_-. Let E_+ and E_- be the $\mathbb{C}_j$-linear projections onto V_+ and V_- along $V_0 \oplus V_-$, resp. $V_0 \oplus V_+$. The operator $Jy := E_+yj + E_-y(-j)$ for $y \in V_R$ is an imaginary operator on V_R.

Proof. We first assume that J is an imaginary operator and show the existence of the corresponding decomposition of V_R. Let $j \in \mathbb{S}$ and let $V_{R,j}$ denote the space V_R considered as a complex Banach over $\mathbb{C}_j$. Furthermore, let us assume that $J \neq 0$, since the statement is obviously true in this case. Then J is a bounded $\mathbb{C}_j$-linear operator on $V_{R,j}$, and by Theorem 14.2.7 and Lemma 14.2.4, the spectrum of J as an element of $\mathcal{B}(V_{R,j})$ is $\sigma_{\mathbb{C}_j}(J) = \sigma_S(J) \cap \mathbb{C}_j \subset \{0, j, -j\}$. We define now for $\tau \in \{0, j, -j\}$ the projection E_τ as the spectral projection associated with $\{\tau\}$ obtained from the Riesz–Dunford functional calculus. If we choose $0 < \varepsilon < \frac{1}{2}$, then the relation $R_z(J) = (\overline{z}\mathcal{I}_{V_{R,j}} - J)\mathcal{Q}_z(J)^{-1}$ in Theorem 14.2.7 implies

$$E_\tau y = \int_{\partial U_\varepsilon(\tau;\mathbb{C}_j)} R_z(J)y \, dz \frac{1}{2\pi i} = \int_{\partial U_\varepsilon(\tau;\mathbb{C}_j)} \mathcal{Q}_z(J)^{-1}(y\overline{z} - Jy) \, dz \frac{1}{2\pi i},$$

where $U_\varepsilon(\tau;\mathbb{C}_j)$ denotes the ball of radius ε in $\mathbb{C}_j$ that is centered at τ. (Since we assumed $\ker J \neq V$, the projections E_j and E_{-j} are not trivial. It might, however, happen that $E_0 = 0$, but this is not a problem in the following argumentation.)

We set

$$V_{J,0} = E_0 V_{R,j}, \quad V_{J,j}^{+} = E_j V_{R,j}, \quad \text{and} \quad V_{J,j}^{-} = E_{-j} V_{R,j}.$$

Obviously these are closed $\mathbb{C}_j$-linear subspaces of $V_{R,j}$, resp. V_R, and (14.19) holds.

Let us now show that the relation (14.20) holds. We first consider the subspace $V_{J,j}^{+}$. Since it is the range of the Riesz projector E_j associated with the spectral set $\{j\}$, this is a $\mathbb{C}_j$-linear subspace of $V_{R,j}$ that is invariant under J, and the restriction $J_+ := J|_{V_{J,j}^+}$ has spectrum $\sigma(J_+) = \{j\}$. Now observe that $-J_+^2 = -J^2|_{V_{J,j}^+}$ is the restriction of a projection onto an invariant subspace and hence a projection itself. Since $0 \notin \sigma(-J_+^2) = -\sigma(J_+)^2 = \{1\}$, we find that $\ker -J_+^2 = \{0\}$ and in turn $\mathcal{I}_+ := \mathcal{I}_{V_{J,j}^+} = -J_+^2$. For $y \in V_{J,j}^+$ we therefore have

$$\begin{aligned} -y &= J_+^2 y = (J_+ - j\mathcal{I}_+ + j\mathcal{I}_+)^2 y = (J_+ - j\mathcal{I}_+ + j\mathcal{I}_+)((J_+ - j\mathcal{I}_+)y + yj) \\ &= (J_+ - j\mathcal{I}_+)^2 y + (J_+ - j\mathcal{I}_+)yj + (J_+ - j\mathcal{I}_+)yj + yj^2. \end{aligned}$$

Since $j^2 = -1$, this is equivalent to

$$(J_+ - j\mathcal{I}_+)^2 y = (J_+ - j\mathcal{I}_+)y(-2j).$$

Hence $(J_+ - j\mathcal{I}_+)y$ is either 0 or an eigenvector of $J_+ - j\mathcal{I}_+$ associated with the eigenvalue $-2j$. By the spectral mapping theorem, $\sigma(J_+ - j\mathcal{I}_+) = \sigma(J_+) - j = \{0\}$. Hence $J_+ - j\mathcal{I}_+$ cannot have an eigenvector with respect to the eigenvalue $-2j$, and so $(J_+ - j\mathcal{I}_+)y = 0$. Therefore, $J_+ = \mathcal{I}_+ i$ and $Jy = J_+ y = yj$ for all $y \in V_{J,j}^+$.

With similar arguments, one shows that $Jy = y(-j)$ for every $y \in V_{J,j}^-$. Finally, $\sigma(-J_0^2) = -\sigma(J_0)^2 = \{0\}$ for $J_0 := J|_{V_{J,0}}$. Since $-J_0^2 = -J^2|_{V_{J,0}}$ is the restriction of a projection to an invariant subspace and thus a projection itself, we find that $-J_0^2$ is the zero operator, and hence $V_{J,0} = \ker(-J_0)^2 \subset \ker(J^2) = \ker J$. On the other hand, $\ker J \subset V_{J,0}$, since $V_{J,0}$ is the invariant subspace associated with the spectral value 0 of J. Thus $V_{J,0} = \ker J$, and so (14.20) is true.

Finally, if $i \in \mathbb{S}$ with $j \perp i$ and $y \in V_+$ then $(Jyi) = J(y)i = yji = (yi)(-j)$. Hence $\Psi : y \to yi$ maps $V_{J,j}^+$ to $V_{J,j}^-$. It is obviously $\mathbb{C}_j$-antilinear, isometric, and a bijection, since $y = -(yi)i$, so that the proof of the first statement is finished.

Now let $j, i \in \mathbb{S}$ with $j \perp i$ and assume that $V_R = V_0 \oplus V_+ \oplus V_-$ with subspaces V_0, V_+, and V_- as in the assumptions. We define $Jy := E_+ yj + E_- y(-j)$. Obviously, J is a continuous $\mathbb{C}_j$-linear operator on $V_{R,j}$. The mapping $\Psi : y \mapsto yi$ maps V_+ bijectively to V_-, but since $\Psi^{-1} = -\Psi$, it also maps V_- bijectively to V_+. Moreover, as an $\mathbb{H}$-linear subspace, V_0 is invariant under Ψ. For $y = y_0 + y_+ + y_- \in V_0 \oplus V_+ \oplus V_- = V_R$, we therefore obtain

$$\begin{aligned} J(yi) &= E_+(yi)j + E_-(yi)(-j) = y_- ij + y_+ i(-j) \\ &= y_-(-j)i + y_+ ji = (E_- y(-j))\, i + (E_+ yj)\, i = (Jy)i. \end{aligned}$$

If now $a \in \mathbb{H}$, then we can write $a = a_1 + a_2 i$ with $a_1, a_2 \in \mathbb{C}_j$ and find due to the $\mathbb{C}_j$-linearity of J that

$$J(ya) = J(ya_1) + J(ya_2 i) = J(y)a_1 + J(y)a_2 i = J(y)(a_1 + a_2 i) = J(y)a.$$

Hence J is quaternionic linear and therefore belongs to $\mathcal{B}(V_R)$.

Since $E_+ E_- = E_- E_+ = 0$, we furthermore observe that

$$\begin{aligned} -J^2 y &= -J(E_+ yj + E_- y(-j)) \\ &= -\left(E_+^2 yj^2 + E_+ E_- y(-j^2) + E_- E_+ y(-j^2) + E_-^2 y(-j)^2\right) = (E_+ + E_-)y. \end{aligned}$$

Hence $-J^2$ is the projection onto $V_+ \oplus V_- = \operatorname{ran}(J)$ along $\ker J = V_0$, so that J is actually an imaginary operator. □

14.3 Spectral Systems and Spectral Integrals of Intrinsic Slice Functions

As pointed out above, invariant subspaces of an operator are in the quaternionic setting not associated with spectral values but with entire spectral spheres. Hence quaternionic spectral measures associate subspaces of V_R with sets of entire spectral spheres and not with arbitrary sets of spectral values. If we want to integrate a function f that takes nonreal values with respect to a spectral measure E, then we need some additional information. We need to know how to multiply the different values that f takes on a spectral sphere onto the vectors associated with the different spectral values in this sphere. This information is given by a suitable imaginary operator. Similar to [197], we hence introduce now the notion of a spectral system.

Definition 14.3.1. A spectral system on V_R is a pair (E, J) consisting of a spectral measure and an imaginary operator J such that

(i) E and J commute, i.e., $E(\Delta)J = JE(\Delta)$ for all $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and

(ii) $E(\mathbb{H} \setminus \mathbb{R}) = -J^2$, that is, $E(\mathbb{R})$ is the projection onto $\ker J$ along $\operatorname{ran} J$, and $E(\mathbb{H} \setminus \mathbb{R})$ is the projection onto $\operatorname{ran} J$ along $\ker J$.

Definition 14.3.2. We denote by $\mathcal{SM}^\infty(\mathbb{H})$ the set of all bounded intrinsic slice functions on $\mathbb{H}$ that are measurable with respect to the usual Borel sets $\mathfrak{B}(\mathbb{H})$ on $\mathbb{H}$.

Lemma 14.3.3. *A function $f : \mathbb{H} \to \mathbb{H}$ belongs to $\mathcal{SM}^\infty(\mathbb{H})$ if and only if it is of the form $f(s) = f_0(s) + j_s f_1(s)$ with $f_0, f_1 \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ and $f_1(s) = 0$ for $s \in \mathbb{R}$.*

Proof. If $f(s) = f_0(s) + j_s f_1(s)$ with $f_0, f_1 \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ and $f_1(s) = 0$ for $s \in \mathbb{R}$, then we can set $f_0(s_0, s_1) := f_0(s_0 + js_1)$ and $f_1(s_0, s_1) = f_1(s + js_1)$ and $f_1(s_0, -s_1) := -f_1(s_0 + js_1)$ with $j \in \mathbb{S}$ arbitrary. Since $f_0(s)$ and $f_1(s)$ are $\mathfrak{B}_S(\mathbb{H})$-measurable, they are constant on each sphere $[s]$, and so this definition is independent of the chosen imaginary unit j. Since $f_1(s) = 0$ for real s, $f_1(s_0, s_1)$ is moreover well defined for $s_1 = 0$. We find that $f(s) = f_0(s) + j_s f_1(s) = f_0(s_0, s_1) + j_s f_1(s_0, s_1)$ with $f_0(s_0, s_1)$ and $f_1(s_0, s_1)$ taking real values and satisfying (2.4), so that f is actually an intrinsic slice function. Moreover, the functions $f_0(s)$ and $f_1(s)$ and the function $\varphi(s) := j_s$ if $s \notin \mathbb{R}$ and $\varphi(s) := 0$ if $s \in \mathbb{R}$ are $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable. Since $f_1(s) = 0$ if $s \in \mathbb{R}$, we have $f(s) = f_0(s) + j_s f_1(s) = f_0(s) + \varphi(s) f_1(s)$, and hence the function f is $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable too.

If, on the other hand, $f \in \mathcal{SM}^\infty(\mathbb{H})$ with $f(s) = f_0(s_0, s_1) + j_s f_1(s_0, s_1)$, then also $f_0(s) := \frac{1}{2}\left(f(s) + f(\overline{s})\right) = f_0(s_0, s_1)$ and $f_1(s) := \frac{1}{2}\varphi(s)\left(f(\overline{s}) - f(s)\right) = f_1(s_0, s_1)$ with $\varphi(s)$ as above are $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable. Moreover $f_1(s) = 0$ if

$s_1 = 0$. Since f is intrinsic, these functions take values in $\mathbb{R}$, and hence they are $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{R})$-measurable. They are, moreover, constant on each sphere $[s]$, so that the preimages $f_0^{-1}(A)$ and $f_1^{-1}(A)$ of each set $A \in \mathfrak{B}(\mathbb{R})$ are axially symmetric Borel sets in $\mathbb{H}$. Consequently, f_0 and f_1 are $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}(\mathbb{R})$-measurable. Finally, $|f|^2 = |f_0|^2 + |f_1|^2$, so that f is bounded if and only if f_0 and f_1 are bounded. □

Corollary 14.3.4. *Every function $f \in \mathcal{SM}^\infty(\mathbb{H})$ is $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}_S(\mathbb{H})$-measurable.*

Proof. Let $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Its inverse image $f^{-1}(\Delta)$ is a Borel set in $\mathbb{H}$ because f is $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable. If $s \in f^{-1}(\Delta)$, then $f(s) = f_0(s_0, s_1) + j_s f_1(s_0, s_1) \in \Delta$. The axial symmetry of Δ implies then that for every $s_j = s_0 + j s_1 \in [s]$ with $j \in \mathbb{S}$ also $f(s_j) = f_0(s_0, s_1) + j_s f_1(s_0, s_1) \in \Delta$ and hence $s_j \in f^{-1}(\Delta)$. Thus $s \in f^{-1}(\Delta)$ implies $[s] \subset f^{-1}(\Delta)$ and so $f^{-1}(\Delta) \in \mathfrak{B}_S(\mathbb{H})$. □

We observe that Lemma 14.3.3 implies that the spectral integrals of the component functions f_0 and f_1 of every $f = f_0 + j_s f_1 \in \mathcal{SM}^\infty(\mathbb{H})$ are defined by Definition 14.1.7.

Definition 14.3.5. Let (E, J) be a spectral system on V_R. For $f \in \mathcal{SM}^\infty(\mathbb{H})$ with $f(s) = f_0(s) + j_s f_1(s)$ we define the *spectral integral of f with respect to* (E, J) as

$$\int_{\mathbb{H}} f(s)\, dE_J(s) := \int_{\mathbb{H}} f_0(s)\, dE(s) + J \int_{\mathbb{H}} f_1(s)\, dE(s). \tag{14.21}$$

The estimate (14.8) generalizes to

$$\left\| \int_{\mathbb{H}} f(s)\, dE(s) \right\| \le C_E \|f_0\|_\infty + C_E \|J\| \|f_1\|_\infty \le C_{E,J} \|f\|_\infty \tag{14.22}$$

with

$$C_{E,J} := C_E(1 + \|J\|).$$

As a consequence of Lemma 14.1.10 and the fact that J and E commute, we immediately obtain the following result.

Lemma 14.3.6. *Let (E, J) be a spectral system on V_R. The mapping*

$$f \mapsto \int_{\mathbb{H}} f(s)\, dE_J(s)$$

is a continuous homomorphism from $(\mathcal{SM}^\infty(\mathbb{H}), \|.\|_\infty)$ to $\mathcal{B}(V_R)$. Moreover, if $T \in \mathcal{B}(V_R)$ commutes with E and J, then it commutes with $\int_{\mathbb{H}} f(s)\, dE_J(s)$ for every $f \in \mathcal{SM}^\infty(\mathbb{H})$.

From Corollary 14.1.11 we furthermore immediately obtain the following lemma, which is an analogue of Lemma 5.3 in [13]. See also the chapter on spectral integrals.

Corollary 14.3.7. *Let (E,J) be a spectral system on V_R and let $f = f_0 + jf_1 \in \mathcal{SM}^\infty(\mathbb{H})$. For every $y \in V_R$ and $y^* \in V_R^*$, we have*

$$\left\langle y^*, \left[\int_{\mathbb{H}} f(s)\, dE_J(s)\right] y \right\rangle = \int_{\mathbb{H}} f_0(s)\, d\langle y^*, E(s)y\rangle + \int_{\mathbb{H}} f_1(s)\, d\langle y^*, E(s)Jy\rangle .$$

Similar to the what happens for the S-functional calculus, there exists a deep relation between quaternionic and complex spectral integrals on V_R.

Lemma 14.3.8. *Let (E,J) be a spectral system on V_R, let $j \in \mathbb{S}$, let E_+ be the projection of V_R onto $V_{J,j}^+$ along $V_{J,0} \oplus V_{J,j}^-$, and let E_- be the projection of V_R onto $V_{J,j}^-$ along $V_{J,0} \oplus V_{J,j}^+$; cf. Theorem 14.2.10. For $\Delta \in \mathfrak{B}(\mathbb{C}_j)$, we set*

$$E_j(\Delta) := \begin{cases} E_+E([\Delta]) & \text{if } \Delta \subset \mathbb{C}_j^+, \\ E(\Delta) & \text{if } \Delta \subset \mathbb{R}, \\ E_-E(\Delta) & \text{if } \Delta \subset \mathbb{C}_j^-, \\ E_j(\Delta \cap \mathbb{C}_j^+) + E_j(\Delta \cap \mathbb{R}) + E_j(\Delta \cap \mathbb{C}_j^-) & \text{otherwise,} \end{cases} \tag{14.23}$$

where $\mathbb{C}_j^+$ and $\mathbb{C}_j^-$ are the open upper and lower half-plane in $\mathbb{C}_j$. Then E_j is a spectral measure on $V_{R,j}$. For every $f \in \mathcal{SM}^\infty(\mathbb{H})$, we have with $f_j := f|_{\mathbb{C}_j}$ that

$$\int_{\mathbb{H}} f(s)\, dE_J(s) = \int_{\mathbb{C}_j} f_j(z)\, dE_j(s). \tag{14.24}$$

Proof. Recall that E and J commute. For $y_+ \in V_{J,j}^+$, we thus have $JE(\Delta)y_+ = E(\Delta)Jy_+ = E(\Delta)y_+j$, so that $E(\Delta)y_+ \in V_{J,j}^+$ and in turn $E_+E(\Delta)y_+ = E(\Delta)y_+$. Similarly, we see that $E(\Delta)y_\sim \in V_{J,0} \oplus V_{J,1}^-$ for $y_\sim \in V_{J,0} \oplus V_{J,j}^-$, so that $E_+E(\Delta)y_\sim = 0$. Hence if we decompose $y \in V_R$ as $y = y_+ + y_\sim$ with $y_+ \in V_{J,j}^+$ and $y_\sim \in V_{J,0} \oplus V_{J,j}^-$ according to Theorem 14.2.10, then $E_+E(\Delta)y = E_+E(\Delta)y_+ + E_+E(\Delta)y_\sim = E(\Delta)y_+$ and $E(\Delta)E_+y = E(\Delta)y_+$, so that altogether, $E(\Delta)E_+y = E_+E(\Delta)y$. Analogous arguments show that $E_-E(\Delta) = E(\Delta)E_-$ and hence E_+, E_-, and $E(\Delta)$, $\Delta \in \mathfrak{B}_S(\mathbb{H})$, commute mutually.

Let us now show that E_j is actually a $\mathbb{C}_j$-complex linear spectral measure on $V_{R,j}$. For each $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ set $\Delta_+ := \Delta \cap \mathbb{C}_j^+$, $\Delta_- := \Delta \cap \mathbb{C}_j^-$, and $\Delta_{\mathbb{R}} := \Delta \cap \mathbb{R}$ for neatness and recall that $[\,\cdot\,]$ denotes the axially symmetric hull of a set. For every $\Delta, \sigma \in \mathfrak{B}_S(\mathbb{H})$, we have then

$$\begin{aligned} E([\Delta_+])E(\sigma_{\mathbb{R}}) = E(\Delta_{\mathbb{R}})E([\sigma_+]) = 0, \\ E([\Delta_-])E(\sigma_{\mathbb{R}}) = E(\Delta_{\mathbb{R}})E([\sigma_-]) = 0, \end{aligned} \tag{14.25}$$

because of item (iii) in Definition 14.1.7. Moreover, E_+ and E_- as well as $E([\Delta+])$, $E([\Delta_-])$, and $E(\Delta_{\mathbb{R}})$ are projections that commute mutually, as we just showed.

Since in addition, $E_+E_- = E_-E_+ = 0$, we have

$$\begin{aligned} E_j(\Delta)^2 &= (E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-]))^2 \\ &= E_+^2E([\Delta_+])^2 + E_+E([\Delta_+])E(\Delta_{\mathbb{R}}) + E_+E_-E([\Delta_+])E([\Delta_-]) \\ &\quad + E_+E(\Delta_{\mathbb{R}})E([\Delta_+]) + E(\Delta_{\mathbb{R}})^2 + E_-E(\Delta_{\mathbb{R}})E([\Delta_-]) \\ &\quad + E_-E_+E([\Delta_-])E([\Delta_+]) + E_-E([\Delta_-])E(\Delta_{\mathbb{R}}) + E_-^2E([\Delta_-])^2 \\ &= E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-]) = E_j(\Delta). \end{aligned} \tag{14.26}$$

Hence $E_j(\Delta)$ is a projection that is moreover continuous, since $\|E_j(\Delta)\| \leq K(1 + \|E_+\| + \|E_-\|)$, where $K > 0$ is the constant in Definition 14.1.7. Altogether, we find that E takes values that are uniformly bounded projections in $\mathcal{B}(V_{R,j})$.

We obviously have $E_j(\emptyset) = 0$. Since $E_+ + E_- = E(\mathbb{H} \setminus \mathbb{R})$ because of item (ii) in Definition 14.3.1, also

$$\begin{aligned} E_j(\mathbb{C}_j) &= E_+E([\mathbb{C}_j^+]) + E(\mathbb{R}) + E_-E([\mathbb{C}_j^-]) \\ &= (E_+ + E_-)E(\mathbb{H} \setminus \mathbb{R}) + E(\mathbb{R}) = E(\mathbb{H}) = \mathcal{I}. \end{aligned}$$

Using the same properties of E_+, E_-, and $E(\Delta)$ as in (14.26), we find that for $\Delta, \sigma \in \mathfrak{B}(\mathbb{C}_j)$,

$$\begin{aligned} &E_j(\Delta)E(\sigma) \\ &= \big(E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-])\big)\big(E_+E([\sigma_+]) + E(\sigma_{\mathbb{R}}) + E_-E([\sigma_-])\big) \\ &= E_+^2E([\Delta_+])E([\sigma_+]) + E_+E([\Delta_+])E(\sigma_{\mathbb{R}}) + E_+E_-E([\Delta_+])E([\sigma_-]) \\ &\quad + E_+E(\Delta_{\mathbb{R}})E([\sigma_+]) + E(\Delta_{\mathbb{R}})E(\sigma_{\mathbb{R}}) + E_-E(\Delta_{\mathbb{R}})E([\sigma_-]) \\ &\quad + E_-E_+E([\Delta_-])E([\sigma_+]) + E_-E([\Delta_-])E(\sigma_{\mathbb{R}}) + E_-^2E([\Delta_-])E([\sigma_-]) \\ &= E_+E([\Delta_+] \cap [\sigma_+]) + E(\Delta_{\mathbb{R}} \cap \sigma_{\mathbb{R}}) + E_-E([\Delta_-] \cap [\sigma_-]). \end{aligned}$$

In general it not true that $[A] \cap [B] = [A \cap B]$ for $A, B \subset \mathbb{C}_j$. (Just think, for instance, about $A = \{j\}$ and $B = \{-j\}$ with $[A] \cap [B] = \mathbb{S} \cap \mathbb{S} = \mathbb{S}$ and $[A \cap B] = [\emptyset] = \emptyset$.) For every axially symmetric set C we have, however,

$$C = \left[C \cap \mathbb{C}_i^+\right] \quad \forall i \in \mathbb{S}.$$

If A and B belong to the same complex half-plane $\mathbb{C}_i^+$, then

$$\begin{aligned} [A] \cap [B] &= \left[([A] \cap [B]) \cap \mathbb{C}_i^+\right] \\ &= \left[\left([A] \cap \mathbb{C}_i^+\right) \cap \left([B] \cap \mathbb{C}_i^+\right)\right] = [A \cap B]. \end{aligned} \tag{14.27}$$

Hence $[\Delta_+] \cap [\sigma_+] = [(\Delta \cap \sigma)_+]$ and $[\Delta_-] \cap [\sigma_-] = [(\Delta \cap \sigma)_-]$, so that altogether

$$\begin{aligned} E_j(\Delta)E_j(\sigma) &= E_+E([(\Delta \cap \sigma)_+]) + E(\Delta_{\mathbb{R}} \cap \sigma_{\mathbb{R}}) + E_-E([(\Delta \cap \sigma)_-]) \\ &= E_j(\Delta \cap \sigma). \end{aligned}$$

Finally, we find for $y \in V_{R,j} = V_R$ and every countable family $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets that

$$\begin{aligned} &E_j\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) y \\ &= E_+ E\left(\left[\bigcup_{n\in\mathbb{N}} \Delta_{n,+}\right]\right) y + E\left(\bigcup_{n\in\mathbb{N}} \Delta_{n,\mathbb{R}}\right) y + E_- E\left(\left[\bigcup_{n\in\mathbb{N}} \Delta_{n,-}\right]\right) y \\ &= E_+ E\left(\bigcup_{n\in\mathbb{N}} [\Delta_{n,+}]\right) y + E\left(\bigcup_{n\in\mathbb{N}} \Delta_{n,\mathbb{R}}\right) y + E_- E\left(\bigcup_{n\in\mathbb{N}} [\Delta_{n,-}]\right) y. \end{aligned}$$

Since the sets $\Delta_{n,+}$, $n \in \mathbb{N}$, are pairwise disjoint sets in the upper half-plane $\mathbb{C}_j^+$, their axially symmetric hulls also are disjoint because of (14.27). Similarly, the axially symmetric hulls of the sets $\Delta_{n,-}$, $n \in \mathbb{N}$ are also pairwise disjoint, so that

$$\begin{aligned} &E_j\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) y \\ &= \sum_{n\in\mathbb{N}} E_+ E_j\left([\Delta_{n,+}]\right) y + \sum_{n\in\mathbb{N}} E\left(\Delta_{n,\mathbb{R}}\right) y + \sum_{n\in\mathbb{N}} E_- E\left([\Delta_{n,-}]\right) y \\ &= \sum_{n\in\mathbb{N}} E_j(\Delta_n) y. \end{aligned}$$

Altogether, we see that E_j is actually a $\mathbb{C}_j$-linear spectral measure on $V_{\mathbb{R},j}$.

Now let us consider spectral integrals. We start with the simplest real-valued function possible: $f = \alpha\chi_\Delta$ with $\alpha \in \mathbb{R}$ and $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Since $f_j = \alpha\chi_{\Delta\cap\mathbb{C}_j}$ and $E(\Delta) = E_j(\Delta_j \cap \mathbb{C}_j)$, we have for such a function

$$\int_{\mathbb{H}} f(s)\, dE(s) = \alpha E(\Delta) = \alpha E_j(\Delta \cap \mathbb{C}_j) = \int_{\mathbb{C}_j} f_j(z)\, dE(z).$$

By linearity we find that (14.24) holds for every simple function

$$f(s) = \sum_{\ell=1}^{n} \alpha_k \chi_{\Delta(s)}$$

in $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$. Since these functions are dense in $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$, it even holds for every function in $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$. Now consider the function $\varphi(s) = j_s$ if $s \in \mathbb{H} \setminus \mathbb{R}$ and $\varphi(s) = 0$ if $s \in \mathbb{R}$. Since $\varphi_j(z) = j\chi_{\mathbb{C}_j^+} + (-j)\chi_{\mathbb{C}_j^-}$ and $E_j(\mathbb{C}_j^+) = E_+$ and $E_j^- = E_-$, the integral of φ_j with respect to E_j is

$$\begin{aligned} \int_{\mathbb{C}_j} \varphi(z)\, dE_j(z) y &= \left(j E_j(\mathbb{C}_j)\right) y + \left((-j) E_j(\mathbb{C}_j^-)\right) y \\ &= E_+ y j + E_- y(-j) = Jy \end{aligned}$$

for all $y \in V_{R,j} = V_R$. If f is now an arbitrary function in $\mathcal{SM}^\infty(\mathbb{H})$, then $f(s) = f_0(s) + \varphi(s)f_1(s)$ with $f_0, f_1 \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ and $f_1(s) = 0$ if $s \in \mathbb{R}$ by Lemma 14.3.3. By what we have shown so far and the homomorphism properties of both quaternionic and the complex spectral integrals, we thus obtain

$$\begin{aligned}
&\int_{\mathbb{H}} f(s)\, dE_J(s)\\
&= \int_{\mathbb{H}} f_0(s)\, dE(s) + J\int_{\mathbb{H}} f_1(s)\, dE(s)\\
&= \int_{\mathbb{C}_j} f_{0,j}(z)\, dE_j(z) + \left(\int_{\mathbb{C}_j} \varphi_j(z)\, dE_j(z)\right)\left(\int_{\mathbb{C}_j} f_{1,j}(z)\, dE_j(z)\right)\\
&= \int_{\mathbb{C}_j} f_{0,j}(z) + \varphi_j(z) f_{1,j}(z)\, dE_j(z) = \int_{\mathbb{C}_j} f_j(z)\, dE_j(z). \qquad \square
\end{aligned}$$

Working on a quaternionic Hilbert space, one might consider only spectral measures whose values are orthogonal projections. If J is an anti-self-adjoint partially unitary operator, as happens, for instance, in the spectral theorem for normal operators in [13], then E_j has values that are orthogonal projections.

Corollary 14.3.9. *Let $\mathcal{H}$ be a quaternionic Hilbert space, let (E,J) be a spectral system on $\mathcal{H}$, let $j \in \mathbb{S}$, and let E_j be the spectral measure defined in (14.23). If $E(\Delta)$ is for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$ an orthogonal projection on $\mathcal{H}$ and J is an anti-self-adjoint partially unitary operator, then $E_j(\Delta_j)$ is for every $\Delta_j \in \mathfrak{B}(\mathbb{C}_j)$ an orthogonal projection on $(\mathcal{H}, \langle\cdot,\cdot\rangle_j)$, where $\langle x,y\rangle_j := \{\langle x,y\rangle\}_j$ is the $\mathbb{C}_j$-part of $\langle x,y\rangle$ defined as $\{a\}_j = a_1$ if $a = a_1 + a_2 i$ with $a_1, a_2 \in \mathbb{C}_j$ and $i \in \mathbb{S}$ with $j \perp i$.*

Proof. If $x, y \in \mathcal{H}_{J,j}^+$, then

$$\langle x,y\rangle = \langle x, -J^2y\rangle = \langle Jx, Jy\rangle = \langle xj, yj\rangle = (-j)\langle x,y\rangle j,$$

so that $j\langle x,y\rangle = \langle x,y\rangle j$. Since a quaternion commutes with $j \in \mathbb{S}$ if and only if it belongs to $\mathbb{C}_j$, we have $\langle x,y\rangle \in \mathbb{C}_j$. Hence if we choose $i \in \mathbb{S}$ with $j \perp i$, then $\langle x, yi\rangle = \langle x,y\rangle i \in \mathbb{C}_j i$, so that in turn, $\langle x, yi\rangle_j = \{\langle x,y\rangle\}_j = 0$ for $x, y \in \mathcal{H}_{J,j}^+$. Since $\mathcal{H}_{J,j}^- = \{yi : y \in \mathcal{H}_{J,j}^+\}$ by Theorem 14.2.10, we obtain $\mathcal{H}_{J,j}^- \perp_j \mathcal{H}_{J,j}^+$, where $\perp_j$ denotes orthogonality in $\mathcal{H}_j$. Furthermore, we have for $x \in \mathcal{H}_0 = \ker J$ and $y \in \mathcal{H}_{J,j}^+$ that

$$\langle x,y\rangle = \langle x, Jy\rangle(-j) = \langle Jx, y\rangle j = \langle 0, y\rangle j = 0,$$

and so $\langle x,y\rangle_j = \{\langle x,y\rangle\}_j = 0$ and in turn $\mathcal{H}_{J,j}^+ \perp \mathcal{H}_0$. Similarly, we see that also $\mathcal{H}_{J,j}^- \perp_j \mathcal{H}_0$. Hence the direct sum decomposition $\mathcal{H}_j = \mathcal{H}_{J,0} \oplus \mathcal{H}_{J,j}^+ \oplus \mathcal{H}_{J,j}^-$ in (14.19) is actually a decomposition into orthogonal subspaces of $\mathcal{H}_j$. The projection E_+ of $\mathcal{H}$ onto $\mathcal{H}_{J,j}^+$ along $\mathcal{H}_{J,0} \oplus \mathcal{H}_{J,j}^-$ and the projection E_- of $\mathcal{H}$ onto $\mathcal{H}_{J,j}^-$ along $\mathcal{H}_{J,0} \oplus \mathcal{H}_{J,j}^+$ are hence orthogonal projections on $\mathcal{H}_j$.

Since the operator $E(\Delta)$ is for $\Delta \in \mathfrak{B}_S(\mathbb{H})$ an orthogonal projection on $\mathcal{H}$, it is an orthogonal projection on $\mathcal{H}_j$. A projection is orthogonal if and only if it is self-adjoint. Since E_+, E_-, and E commute mutually, we find for every $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ and $x, y \in \mathcal{H}_j = \mathcal{H}$ that

$$\begin{aligned}
&\langle x, E_j(\Delta)y\rangle_j \\
&= \langle x, E_+E([\Delta \cap \mathbb{C}_j^+])y\rangle_j + \langle x, E(\Delta \cap \mathbb{R})y\rangle_j + \langle x, E_-E([\Delta \cap \mathbb{C}_j^-])y\rangle_j \\
&= \langle E_+E([\Delta \cap \mathbb{C}_j^+])x, y\rangle_j + \langle E(\Delta \cap \mathbb{R})x, y\rangle_j + \langle E_-E([\Delta \cap \mathbb{C}_j^-])x, y\rangle_j \\
&= \langle E_j(\Delta)x, y\rangle_j.
\end{aligned}$$

Hence $E_j(\Delta)$ is an orthogonal projection on $\mathcal{H}_j$. □

We present two easy examples of spectral systems that illustrate the intuition behind the concept of a spectral system.

Example 14.3.10. We consider a compact normal operator T on a quaternionic Hilbert space $\mathcal{H}$. The spectral theorem for compact normal operators in [143] implies that the S-spectrum consists of a (possibly finite) sequence $[s_n] = s_{n,0} + \mathbb{S}s_{n,1}, n \in \Upsilon \subset \mathbb{N}$, of spectral spheres that are (apart from possibly the sphere $[0]$) isolated in $\mathbb{H}$. Moreover, it implies the existence of an orthonormal basis of eigenvectors $(b_\ell)_{\ell\in\Lambda}$ associated with eigenvalues $s_\ell = s_{\ell,0} + j_{s_\ell}s_{\ell,1}$ with $j_{s_\ell} = 0$ if $s_\ell \in \mathbb{R}$ such that

$$Ty = \sum_{\ell\in\Lambda} b_\ell s_\ell \langle b_\ell, y\rangle. \tag{14.28}$$

Each eigenvalue s_ℓ obviously belongs to one spectral sphere, namely to $[s_{n(\ell)}]$ with $s_{n(\ell),0} = s_{\ell,0}$ and $s_{n(\ell),1} = s_{\ell,1}$, and for $[s_n] \neq \{0\}$ only finitely many eigenvalues belong to the spectral sphere $[s_n]$. We can hence rewrite (14.28) as

$$Ty = \sum_{[s_n]\in\sigma_S(T)} \sum_{s_\ell\in[s_n]} b_\ell s_\ell\langle b_\ell, y\rangle = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell s_\ell \langle b_\ell, y\rangle.$$

The spectral measure E of T is then given by

$$E(\Delta)y = \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\Delta}} \sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}).$$

If $f \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$, then obviously

$$\int_{\mathbb{H}} f(s)\, dE(s)y = \sum_{n\in\Upsilon} E([s_n])yf(s_n) = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle f(s_n). \tag{14.29}$$

In particular,

$$\int_{\mathbb{H}} s_0\, dE(s)y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle s_{\ell,0}$$

and

$$\int_{\mathbb{H}} s_1\, dE(s)y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell \langle b_\ell, y\rangle s_{\ell,1}.$$

If we define

$$Jy := \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell j_{s_\ell}\langle b_\ell, y\rangle,$$

then J is an anti-self-adjoint partially unitary operator and (E, J) is a spectral system. One can check easily that E and J commute, and since $j_{s_\ell} = 0$ for $s_\ell \in \mathbb{R}$ and $j_{s_\ell} \in \mathbb{S}$ with $j_{s_\ell}^2 = -1$ otherwise, one has

$$-J^2 y = -\sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell j_{s_\ell}^2 \langle b_\ell, y\rangle = \sum_{n\in\Upsilon:[s_n]\subset\mathbb{H}\setminus\mathbb{R}}\ \sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle = E(\mathbb{H}\setminus\mathbb{R})y.$$

In particular, $\ker J = \overline{\operatorname{span}_{\mathbb{H}}\{b_\ell : s_\ell \in \mathbb{R}\}} = E(\mathbb{R})$. Note, moreover, that J is completely determined by T.

For every function $f = f_0 + jf_1 \in \mathcal{SM}^\infty(\mathbb{H})$, we have because of (14.29) and $\langle b_\ell, b_\kappa\rangle = \delta_{\ell,\kappa}$ that

$$\begin{aligned}
\int_{\mathbb{H}} f(s)\, dE_J(s)y &= \int_{\mathbb{H}} f_0(s)\, dE(s)y + J\int_{\mathbb{H}} f(s)\, dE(s)y \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle f_0(s_{n,0}, s_{n,1}) \\
&\quad + \sum_{m,n\in\Upsilon}\sum_{\substack{n(\ell)=n\\ n(\kappa)=m}} b_\ell j_{s_\ell}\langle b_\ell, b_\kappa\rangle\langle b_\kappa, y\rangle f_1(s_{m,0}, s_{m,1}) \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell f_0(s_{\ell,0}, s_{\ell,1})\langle b_\ell, y\rangle \\
&\quad + \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell j_{s_\ell} f_1(s_{\ell,0}, s_{\ell,1})\langle b_\ell, y\rangle \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\big(f_0(s_{\ell,0}, s_{\ell,1}) + j_{s_\ell} f_1(s_{\ell,0}, s_{\ell,1})\big)\langle b_\ell, y\rangle,
\end{aligned}$$

and so

$$\int_{\mathbb{H}} f(s)\, dE_J(s)y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell f(s_\ell)\langle b_\ell, y\rangle. \tag{14.30}$$

In particular,

$$\int_{\mathbb{H}} s\, dE_J(s) = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell s_\ell\langle b_\ell, y\rangle = Ty.$$

We have in particular $T = A + JB$ with $A = \int_{\mathbb{H}} s_0 \, dE(s)$ self-adjoint, $B = \int_{\mathbb{H}} s_1 \, dE(s)$ positive and J anti-self-adjoint and partially unitary as in (14.11). Moreover, E corresponds via Remark 14.1.9 to the spectral measure obtained from the spectral theorem for bounded normal operators.

We choose now $j, i \in \mathbb{S}$ with $j \perp i$, and for each $\ell \in \Lambda$ with $s_\ell \notin \mathbb{R}$ we choose $h_\ell \in \mathbb{H}$ with $|h_\ell| = 1$ such that $h_\ell^{-1} j_{s_\ell} h_\ell = j$ and in turn

$$h_\ell^{-1} s_\ell h_\ell = s_{\ell,0} + h_\ell^{-1} j_{s_\ell} h_\ell s_1 = s_{\ell,0} + j s_{\ell,1} =: s_{\ell,j}.$$

In order to simplify the notation we also set $h_\ell = 1$ and $j_{s_\ell} = 0$ if $s_\ell \in \mathbb{R}$. Then $\tilde{b}_\ell := b_\ell h_\ell, \ell \in \Lambda$ is another orthonormal basis consisting of eigenvectors of T, and since $h_\ell^{-1} = \overline{h_\ell}/|h_\ell|^2 = \overline{h_\ell}$, we have

$$\begin{aligned} Ty &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} b_\ell (h_\ell h_\ell^{-1}) s_\ell (h_\ell h_\ell^{-1}) \langle b_\ell, y \rangle \\ &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} (b_\ell h_\ell)(h_\ell^{-1} s_\ell h_\ell) \langle b_\ell h_\ell, y \rangle = \sum_{n \in \Upsilon} \sum_{n(\ell)=n} \tilde{b}_\ell s_{\ell,j} \langle \tilde{b}_\ell, y \rangle \end{aligned} \tag{14.31}$$

and similarly

$$\begin{aligned} Jy &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} b_\ell (h_\ell h_\ell^{-1}) j_\ell (h_\ell h_\ell^{-1}) \langle b_\ell, y \rangle \\ &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} (b_\ell h_\ell)(h_\ell^{-1} j_\ell h_\ell) \langle b_\ell h_\ell, y \rangle = \sum_{n \in \Upsilon} \sum_{n(\ell)=n} \tilde{b}_\ell j \langle \tilde{b}_\ell, y \rangle. \end{aligned}$$

Recall that $j\lambda = \lambda j$ for every $\lambda \in \mathbb{C}_j$ and $ji = -ij$. The splitting of $\mathcal{H}$ obtained from Theorem 14.2.10 is therefore given by

$$\mathcal{H}_{J,0} = \ker J = \overline{\operatorname{span}_{\mathbb{H}}\{\tilde{b}_\ell : \ s_\ell \in \mathbb{R}\}}, \quad \mathcal{H}_{J,j}^+ := \overline{\operatorname{span}_{\mathbb{C}_j}\{\tilde{b}_\ell : \ s_\ell \notin \mathbb{R}\}},$$

and

$$\mathcal{H}_{J,j}^- = \overline{\operatorname{span}_{\mathbb{C}_j}\{\tilde{b}_\ell i : \ s_\ell \notin \mathbb{R}\}} = \mathcal{H}_{J,j}^+ i.$$

If $\langle b_\ell, y \rangle = a_\ell = a_{\ell,1} + a_{\ell,2} i$ with $a_{\ell,1}, a_{\ell,2} \in \mathbb{C}_j$, then (14.31) implies

$$\begin{aligned} Ty &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} \tilde{b}_\ell s_{\ell,j} a_\ell \\ &= \sum_{\substack{n \in \Upsilon \\ [s_n] \subset \mathbb{R}}} \sum_{n(\ell)=n} \tilde{b}_\ell a_\ell s_\ell + \sum_{\substack{n \in \Upsilon \\ [s_n] \subset \mathbb{H} \setminus \mathbb{R}}} \sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,1} s_{\ell,j} + \sum_{\substack{n \in \Upsilon \\ [s_n] \subset \mathbb{H} \setminus \mathbb{R}}} \sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,2} i \overline{s_{\ell,j}}. \end{aligned} \tag{14.32}$$

If $f \in \mathcal{SM}^\infty(\mathbb{H})$, then the representation (14.30) of $\int_{\mathbb{H}} f(s) \, dE_J(s)$ in the basis

$\tilde{b}_\ell, \ell \in \Lambda$ implies

$$\begin{aligned}
\int f(s)\,dE_J(s)y &= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell f(s_{\ell,j})a_\ell \\
&= \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_\ell f(s_\ell) \\
&\quad + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,1} f(s_{\ell,j}) + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,2} i\overline{f(s_{\ell,j})} \\
&= \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_\ell f(s_\ell) \\
&\quad + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,1} f(s_{\ell,j}) + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,2} i f(\overline{s_{\ell,j}}),
\end{aligned} \tag{14.33}$$

since $f(s_\ell)\in\mathbb{R}$ for $s_\ell\in\mathbb{R}$ and $\overline{f(s_{\ell,j})} = f(\overline{s_{\ell,j}})$ because f is intrinsic. Note that the representations (14.32) and (14.33) show clearly that $f(T)$ is defined by letting f act on the right eigenvalues of T.

Example 14.3.11. Let us consider the space $L^2(\mathbb{R},\mathbb{H})$ of all quaternion-valued functions on $\mathbb{R}$ that are square-integrable with respect to the Lebesgue measure λ. Endowed with the pointwise multiplication $(fa)(t) = f(t)a$ for $f\in L^2(\mathbb{R},\mathbb{H})$ and $a\in\mathbb{H}$ and with the scalar product

$$\langle g, f\rangle = \int_{\mathbb{R}} \overline{g(t)}f(t)\,d\lambda(t) \quad \forall f,g\in L^2(\mathbb{R},\mathbb{H}), \tag{14.34}$$

this space is a quaternionic Hilbert space. Let us now consider a bounded measurable function $\varphi:\mathbb{R}\to\mathbb{H}$ and the multiplication operator $(M_\varphi f)(s) := \varphi(s)f(s)$. This operator is normal with $(M_\varphi)^* = M_{\overline{\varphi}}$, and its S-spectrum is the set $\overline{\varphi(\mathbb{R})}$. Indeed, writing $\varphi(t) = \varphi_0(t) + j_{\varphi(t)}\varphi_1(t)$ with $\varphi_0(t)\in\mathbb{R}$, $\varphi_1(t)>0$, and $j_{\varphi(t)}\in\mathbb{S}$ for $\varphi(t)\in\mathbb{H}\setminus\mathbb{R}$ and $j_{\varphi(t)} = 0$ for $\varphi(t)\in\mathbb{R}$, we find that

$$\begin{aligned}
\mathcal{Q}_s(M_\varphi)f(t) &= M_\varphi^2 f(t) - 2s_0 M_\varphi f(t) + |s|^2 f(t) \\
&= (\varphi^2(t) - 2s_0\varphi(t) + |s|^2)f(t) \\
&= (\varphi(t) - s_{j_{\varphi(t)}})(\varphi(t) - \overline{s_{j_{\varphi(t)}}})f(t)
\end{aligned}$$

with $s_{j_{\varphi(t)}} = s_0 + j_{\varphi(t)}s_1$, and hence

$$\mathcal{Q}_s(M_\varphi)^{-1}f(t) = (\varphi(t) - s_{j_{\varphi(t)}})^{-1}(\varphi(t) - \overline{s_{j_{\varphi(t)}}})^{-1}f(t)$$

is a bounded operator if $s\notin\overline{\varphi(\mathbb{R})}$. If we define $E(\Delta) = M_{\chi_{\varphi^{-1}(\Delta)}}$ for all $\Delta\in\mathfrak{B}_S(\mathbb{H})$, then we obtain a spectral measure on $\mathfrak{B}_S(\mathbb{H})$, namely

$$E(\Delta)f(t) = \chi_{\varphi^{-1}(\Delta)}(t)f(t).$$

If we set

$$J := M_{j_\varphi} \quad \text{i.e.,} \quad (Jf)(t) = j_{\varphi(t)} f(t),$$

then we find that (E, J) is a spectral system. Obviously J is anti-self-adjoint and partially unitary and hence an imaginary operator that commutes with E. Since $j_{\varphi(t)} = 0$ if $\varphi(t) \in \mathbb{R}$ and $j_{\varphi(t)} \in \mathbb{S}$ otherwise, we have, moreover,

$$(-J^2 f)(t) = -j_{\varphi(t)}^2 f(t) = \chi_{\varphi^{-1}(\mathbb{H}\setminus\mathbb{R})} f(t) = (E(\mathbb{H} \setminus \mathbb{R}) f)(t).$$

If $g \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$, then let $g_n(s) = \sum_{\ell=1}^{N_n} a_{n,\ell} \chi_{\Delta_{n,\ell}}(s) \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ be a sequence of simple functions that converges uniformly to g. Then

$$\int_{\mathbb{H}} g(s)\, dE(s) f(t) = \lim_{n\to\infty} \sum_{\ell=1}^{N_n} a_{n,\ell} E(\Delta_{n,\ell}) f(t) = \lim_{n\to\infty} \sum_{\ell=1}^{N_n} a_{n,\ell} \chi_{\varphi^{-1}(\Delta)}(t) f(t)$$
$$= \lim_{n\to\infty} \sum_{\ell=1}^{N_n} a_{n,\ell} \chi_{\Delta}(\varphi(t)) f(t) = \lim_{n\to\infty} (g_n \circ \varphi)(t) f(t) = (g \circ \varphi)(t) f(t).$$

Hence if $g(s) =_0 (s) + j_s f_1(s) \in \mathcal{SM}^\infty(\mathbb{H})$, then

$$\begin{aligned}
\int_{\mathbb{H}} g(s)\, dE_J(s) f(t) &= \int_{\mathbb{H}} g(s)\, dE(s) f(t) \\
&= \int_{\mathbb{H}} g_0(s)\, dE(s) f(t) + J \int_{\mathbb{H}} f_1(s)\, dE(s) f(t) \\
&= g_0(\varphi(t)) f(t) + j_{\varphi(t)} f_1(\varphi(t)) f(t) \\
&= (g_0(\varphi(t)) + j_{\varphi(t)} f_1(\varphi(t)) f(t) = (g \circ \varphi)(t) f(t),
\end{aligned}$$

and so

$$\int_{\mathbb{H}} g(s)\, dE_J(s) = M_{g\circ\varphi}.$$

Choosing $g(s) = s$, we find in particular that $T = A + JB$ with $A = \int_{\mathbb{H}} s_0\, dE(s)$ self-adjoint, $B = \int_{\mathbb{H}} s_1\, dE(s)$ positive, and J anti-self-adjoint and partially unitary as in the Teichmüller decomposition. The spectral measure E corresponds via Remark 14.1.9 to the spectral measure obtained in Theorem 11.2.1.

14.4 On the Different Approaches to Spectral Integration

The approach to spectral integration presented in this chapter specifies some ideas in [197]. We now compare this approach with the approaches in [13] and [144]. In [13], the authors consider a spectral measure E over $\mathbb{C}_j^+$ and a unitary and anti-self-adjoint operator J (i.e., a fully imaginary operator J in the terminology of this book) that commutes with E. They define a left multiplication on $\mathcal{H}$ by the imaginary unit J as $jy := Jy$ for $y \in \mathcal{H}$. (If one tries to develop the spectral theory of a normal operator T, then J is simply the extension of the imaginary operator in the Teichmüller decomposition of T to a fully imaginary operator; cf. Remark 9.3.7.) One can then define the multiplication of an operator A in $\mathcal{B}(\mathcal{H})$ by the imaginary unit j as $jA = JA$ and $Aj := AJ$, and this makes the integration of $\mathbb{C}_j$-valued functions on $f : \mathbb{C}_j^+ \to \mathbb{C}_j$ possible. The procedure

$$\int_{\mathbb{C}_j^+} f(s)\, dE(s) := \lim_{n\to+\infty} \int_{\mathbb{C}_j^+} f_n(s)\, dE(s) := \lim_{n\to+\infty} \sum_{k=1}^{N_n} \alpha_{n,k} E(\Delta_{n,k}), \qquad (14.35)$$

where $f_n := \sum_{k=1}^{N_n} \alpha_{n,k}\chi_{\Delta_{n,k}}$ with $\Delta_{n,k} \in \mathfrak{B}(\mathbb{C}_j^+)$ is a sequence of simple functions that uniformly converges to f, is in this case also well defined if the coefficients $\alpha_{n,k}$ belong to $\mathbb{C}_j$, and not only if they belong to $\mathbb{R}$.

The authors of [144] go one step further: they define a second unitary and anti-self-adjoint operator K that commutes with E and anti-commutes with J, and they define a full left multiplication on $\mathcal{H}$. They choose $i \in \mathbb{S}$ with $j \perp i$ and define $L_j := J$ and $L_i := K$ and the left multiplication

$$\mathcal{L} : \begin{cases} \mathbb{H} & \to \quad \mathcal{B}(\mathcal{H}), \\ a = a_0 + a_1 j + a_2 i + a_3 ji & \mapsto \quad L_a := a_0\mathcal{I} + a_1 j + a_2 i + a_3 ji, \end{cases}$$

so that

$$ay := L_a y = ya_0 + L_j y a_1 + L_i y a_2 + L_j L_i y a_3 \quad \forall y \in \mathcal{H}.$$

They call a pair $\mathcal{E} := (E, \mathcal{L})$ consisting of a spectral measure over $\mathbb{C}_j^+$ and a left multiplication that commutes with E an intertwining quaternionic projection-valued measure (iqPVM for short). Such iqPVMs allow one to define spectral integrals for functions $f : \mathbb{C}_j^+ \to \mathbb{H}$ with arbitrary values in $\mathbb{H}$, since the coefficients $a_{n,k}$ in (14.35) are in this case meaningful for arbitrary values $a_{n,k} \in \mathbb{H}$. The authors arrive then at the following version of the spectral theorem [144, Theorem 4.1].

Theorem 14.4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal and let $j \in \mathbb{S}$. There exists an iqPVM $\mathcal{E} = (E, \mathcal{L})$ over $\mathbb{C}_j^+$ on $\mathcal{H}$ such that*

$$T = \int_{\mathbb{C}_j^+} z\, d\mathcal{E}(z). \qquad (14.36)$$

The spectral measure E is uniquely determined by T, and the left multiplication is uniquely determined for $a \in \mathbb{C}_j$ on $\ker(T - T^*)^\perp$. Precisely, we have for every other left multiplication $\mathcal{L}'$ such that $\mathcal{E}' = (E, \mathcal{L}')$ is an iqPVM satisfying (14.36) that $L_a y = L'_a y$ for every $a \in \mathbb{C}_j$ and $y \in \ker(T - T^*)^\perp$. (Even more specifically, we have $jy = Jy$ for every $y \in \ker(T - T^*)^\perp = \operatorname{ran} J$, where J is the imaginary operator in the Teichmüller decomposition of T.)

All three approaches are consistent if things are interpreted correctly. Let us first consider a spectral measure E over $\mathbb{C}_j^+$ for some $j \in \mathbb{S}$, the values of which are orthogonal projections on a quaternionic Hilbert space $\mathcal{H}$. Furthermore, let J be a unitary anti-self-adjoint operator on $\mathcal{H}$ that commutes with E and let us interpret the application of J as multiplication by j from the left as in [13]. By Remark 14.1.9, we obtain a quaternionic spectral measure on $\mathfrak{B}_S(\mathbb{H})$ if we set $\tilde{E}(\Delta) := E\big(\Delta \cap \mathbb{C}_j^+\big)$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, and obviously we have

$$\int_{\mathbb{H}} f(s)\, d\tilde{E}(s) = \int_{\mathbb{C}_j^+} f_j(z)\, dE(z) \quad \forall f \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R}),$$

where $f_j = f|_{\mathbb{C}_j^+}$. If we set $J := \mathsf{J}\tilde{E}(\mathbb{H} \setminus \mathbb{R}) = \mathsf{J}E(\mathbb{C}_j^+ \setminus \mathbb{R})$, then J is an imaginary operator and we find that $(\tilde{E}, J)$ is a spectral system on $\mathcal{H}$. Now let $f(s) = f_0(s) + jf_1(s) \in \mathcal{SM}^\infty(\mathbb{H})$ and let again $f_j = f|_{\mathbb{C}_j^+}$, $f_{0,j} = \alpha|_{\mathbb{C}_j^+}$ and $f_{1,j} = f_1|_{\mathbb{C}_j^+}$. Since $f_1(s) = 0$ if $s \in \mathbb{R}$, we have $f_1(s) = \chi_{\mathbb{H}\setminus\mathbb{R}}(s) f_1(s)$ and in turn

$$\begin{aligned}
\int_{\mathbb{C}_j^+} f_j(z)\, dE(z) &= \int_{\mathbb{C}_j^+} f_{0,j}(z)\, dE(z) + \mathsf{J}\int_{\mathbb{C}_j^+} f_{1,j}(z)\, dE(z) \\
&= \int_{\mathbb{H}} f_0(s)\, d\tilde{E}(s) + \mathsf{J}\int_{\mathbb{H}} \chi_{\mathbb{H}\setminus\mathbb{R}}(s) f_1(s)\, d\tilde{E}(s) \\
&= \int_{\mathbb{H}} f_0(s)\, d\tilde{E}(s) + \mathsf{J}E(\mathbb{H}\setminus\mathbb{R})\int_{\mathbb{H}} f_1(s)\, d\tilde{E}(s) \\
&= \int_{\mathbb{H}} f_0(s)\, d\tilde{E}(s) + J\int_{\mathbb{H}} f_1(s)\, d\tilde{E}(s) = \int_{\mathbb{H}} f(s)\, d\tilde{E}_J(s).
\end{aligned} \tag{14.37}$$

Hence for every measurable intrinsic slice function f, the spectral integral of f with respect to the spectral system $(\tilde{E}, J)$ coincides with the spectral integral of $f|_{\mathbb{C}_j^+}$ with respect to E, where we interpret the application of J as multiplication by j from the left. Since the mapping $f \mapsto f|_{\mathbb{C}_j^+}$ is a bijection between the set of all measurable intrinsic slice functions on $\mathbb{H}$ and the set of all measurable $\mathbb{C}_j$-valued functions on $\mathbb{C}_j^+$ that map the real line into itself, both approaches are equivalent for real slice functions if we identify $\tilde{E}$ with E and f with f_j. The same identifications show that the approach in [144] is equivalent to our approach, as long as we consider only intrinsic slice functions. Indeed, if $\mathcal{E} = (E, \mathcal{L})$ is an iqPVM over $\mathbb{C}_j^+$ on $\mathcal{H}$, then $\mathsf{J}y := L_j y = jy$ is a unitary and anti-self-adjoint operator on $\mathcal{H}$. As before, we can set $\tilde{E}(\Delta) = E(\Delta \cap \mathbb{C}_j^+)$ and $J := \mathsf{J}\tilde{E}(\mathbb{H}\setminus\mathbb{R}) = L_j E(\mathbb{C}_j^+ \setminus \mathbb{R})$. We then find as in (14.37) that

$$\int_{\mathbb{C}_j^+} f_j(z)\, d\mathcal{E}(z) = \int_{\mathbb{H}} f(s)\, d\tilde{E}_J(s) \quad \forall f \in \mathcal{SM}^\infty(\mathbb{H}). \tag{14.38}$$

For intrinsic slice functions, all three approaches are hence consistent.

Let us continue our discussion of how our approach to spectral integration fits into the existing theory. We recall that every normal operator T on $\mathcal{H}$ can be decomposed as

$$T = A + JB,$$

with the self-adjoint operator $A = \frac{1}{2}(T+T^*)$, the positive operator $B = \frac{1}{2}|T-T^*|$, and the imaginary operator J with $\ker J = \ker(T-T^*)$ and $\operatorname{ran} J = \ker(T-T^*)^{\perp}$. Let $\mathcal{E} = (E, \mathcal{L})$ be an iqPVM of T obtained from Theorem 14.4.1. From [144, Theorem 3.13], we know that $\left(\int_{\mathbb{C}_j^+} \varphi(z)\, d\mathcal{E}(z)\right)^* = \int_{\mathbb{C}_j^+} \overline{\varphi(z)}\, d\mathcal{E}(z)$ and $\ker \int_{\mathbb{C}_j^+} \varphi(z)\, d\mathcal{E}(z)$ $= \operatorname{ran} E(\varphi^{-1}(0))$. Hence

$$T - T^* = \int_{\mathbb{C}_j^+} z\, d\mathcal{E}(z) - \int_{\mathbb{C}_j^+} \overline{z}\, d\mathcal{E}(z) = \int_{\mathbb{C}_j^+} 2jz_1\, d\mathcal{E}(z).$$

Since $z_1 = 0$ if and only if $z \in \mathbb{R}$, we find that $\ker J = \ker(T-T^*) = \operatorname{ran} E(\mathbb{R})$ and in turn $\operatorname{ran} J = \ker(T-T^*)^{\perp} = \operatorname{ran} E\big(\mathbb{C}_j^+ \setminus \mathbb{R}\big)$.

If we identify E with the spectral measure $\tilde{E}$ on $\mathfrak{B}_S(\mathbb{H})$ that is given by $\tilde{E}(\Delta) = E\big(\Delta \cap \mathbb{C}_j^+\big)$, then $\mathbf{J} = L_j E(\mathbb{C}_j^+ \setminus \mathbb{R})$ is an imaginary operator such that $(\tilde{E}, \mathbf{J})$ is a spectral system, as we showed above. The spectral integral of every measurable intrinsic slice function f with respect to $(\tilde{E}, \mathbf{J})$ coincides with the spectral integral of $f|_{\mathbb{C}_j^+}$ with respect to $\mathcal{E}$. Since $\operatorname{ran} E(\mathbb{C}_j^+ \setminus \mathbb{R}) = \ker(T-T^*)^{\perp} =$ $\operatorname{ran} J$ and $L_j y = \mathbf{J}y$ for all $y \in \ker(T-T^*)^{\perp}$ (this follows from the construction of $\mathcal{L}$ and in particular L_j in [144]), we moreover find that $J = \mathbf{J}$. Therefore $(\tilde{E}, J)$ is the spectral system that for integration of intrinsic slice functions is equivalent to $\mathcal{E}$. We can hence rewrite the spectral theorem in the terminology of spectral systems as follows.

Theorem 14.4.2. *Let $T = A + JB \in \mathcal{B}(\mathcal{H})$ be a normal operator. There exists a unique quaternionic spectral measure E on $\mathfrak{B}_S(\mathbb{H})$ with $E(\mathbb{H} \setminus \sigma_S(T)) = 0$, the values of which are orthogonal projections on $\mathcal{H}$, such that (E, J) is a spectral system and such that*

$$T = \int_{\mathbb{H}} s\, dE_J(s).$$

We want to point out that the spectral system (E, J) is completely determined by T—unlike the unitary anti-self-adjoint operator $\mathbf{J}$ that extends J used in [13] and unlike the iqPVM used in [144]. We also want to stress that the proof of the spectral theorem presented in Chapter 11 translates directly into the language of spectral systems: one can pass to the language of spectral systems by the identification described above without any problems.

Example 14.4.3. In order to discuss the relations described above, let us return to Example 14.3.10, in which we considered normal compact operators on a quaternionic Hilbert space given by

$$Ty = \sum_{n \in \Upsilon} \sum_{n(\ell)=n} b_\ell s_\ell \langle b_\ell, y \rangle,$$

whose spectral system (E, J) was

$$E(\Delta)y = \sum_{\substack{n\in\Upsilon \\ [s_n]\subset\Delta}} \sum_{n(\ell)=n} b_\ell \langle b_\ell, y\rangle \quad \text{and} \quad Jy = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell j_{s_\ell} \langle b_\ell, y\rangle.$$

The integral of a function $f \in \mathcal{SM}^\infty(\mathbb{H})$ with respect to (E, J) is then given by (14.30) as

$$\int_{\mathbb{H}} f(s)\, dE_J(s)y = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell f(s_\ell)\langle b_\ell, y\rangle. \tag{14.39}$$

Let $j \in \mathbb{S}$. If we set $\tilde{E}(\Delta) = E([\Delta])$ for all $\Delta \in \mathfrak{B}(\mathbb{C}_j^+)$, then $\tilde{E}$ is a quaternionic spectral measure over $\mathbb{C}_j^+$. In [13] the authors extend J to an anti-self-adjoint and unitary operator J that commutes with T and interpret applying this operator as multiplication by j from the left in order to define spectral integrals. One possibility to do this is to define $\iota(\ell) = j_{s_\ell}$ if $s_\ell \notin \mathbb{R}$ and $\iota(\ell) \in \mathbb{S}$ arbitrary if $s_\ell \in \mathbb{R}$ and to set

$$\mathsf{J}y = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell \iota(\ell)\langle b_\ell, y\rangle$$

and $iy = \mathsf{J}y$.

In [144] the authors go even one step further and extend this multiplication by scalars from the left to a full left multiplication $\mathcal{L} = (L_a)_{a\in\mathbb{H}}$ that commutes with E in order obtain an iqPVM $\mathcal{E} = (E, \mathcal{L})$. We can do this by choosing for each $\ell \in \Lambda$ an imaginary unit $\jmath(\ell) \in \mathbb{S}$ with $\jmath(\ell) \perp \iota(\ell)$ and by defining

$$\mathsf{K}y = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell \jmath(\ell)\langle b_\ell, y\rangle.$$

If we choose $i \in \mathbb{S}$ and define for $a = a_0 + a_1 j + a_2 i + a_3 ji \in \mathbb{H}$,

$$\begin{aligned} ay = L_a y &:= ya_0 + \mathsf{i}ya_1 + \mathsf{K}ya_2 + \mathsf{JK}ya_3 \\ &= \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell(a_0 + a_1\iota(\ell) + a_2\jmath(\ell) + a_3\iota(\ell)\jmath(\ell)\langle b_\ell, y\rangle, \end{aligned}$$

then $\mathcal{L} = (L_a)_{a\in\mathbb{H}}$ is obviously a left multiplication that commutes with E, and hence $\mathcal{E} = (\tilde{E}, \mathcal{L})$ is an iqPVM over $\mathbb{C}_j^+$.

Set $s_{n,j} = [s_n] \cap \mathbb{C}_j^+$. For $f_j : \mathbb{C}_j^+ \to \mathbb{H}$, the integral of f_j with respect to $\mathcal{E}$ is

$$\begin{aligned}
&\int_{\mathbb{C}_j^+} f_j(z)\, d\mathcal{E}(z) \\
&= \sum_{n\in\Upsilon} f_j(s_{n,j})\tilde{E}(\{s_{n,j}\})y = \sum_{n\in\Upsilon} f_j(s_{n,j})E([s_n])y \\
&= \sum_{n\in\Upsilon} \big(F_0(s_{n,j}) + F_1(s_{n,j})\mathsf{J} + F_2(s_{n,j})\mathsf{K} + F_3(s_{n,j})\mathsf{JK}\big) \sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\big(F_0(s_{n,j}) + F_1(s_{n,j})\iota(\ell) + F_2(s_{n,j})\jmath(\ell) + F_3(s_{n,j})\iota(\ell)\jmath(\ell)\big)\langle b_\ell, y\rangle,
\end{aligned} \tag{14.40}$$

where $F_0, \ldots, F_3$ are the real-valued component functions such that

$$f_j(z) = F_0(z) + F_1(z)j + F_2(z)i + F_3(z)ji.$$

If now f_j is the restriction of an intrinsic slice function $f(s) = f_0(s) + j_s f_1(s)$, then $F_0(s_{n(\ell),j}) = f_0(s_{\ell,j}) = f_0(s_\ell)$ and $F_1(s_{n(\ell),j}) = f_1(s_{\ell,j}) = f_1(s_\ell)$ and $F_2(z) = F_3(z) = 0$. Since moreover $F_1(s_{n(\ell),j}) = f_1(s_\ell) = 0$ if $s_\ell \in \mathbb{R}$ and $\iota(\ell) = j_{s_\ell}$ if $s_\ell \notin \mathbb{R}$, we find that (14.40) actually equals (14.39) in this case. Note, however, that for every other function f_j, the integral (14.40) depends on the random choice of the functions $\iota(\ell)$ and $\jmath(\ell)$, which are not fully determined by T.

Let us now investigate the relation between (14.40) and the right linear structure of T. Let us therefore change to the eigenbasis $\tilde{b}_\ell, \ell \in \Lambda$, with $T\tilde{b}_\ell = \tilde{b}_\ell s_{\ell,j}$ defined in Example 14.3.10. For convenience let us furthermore choose $\iota(\ell)$ and $\jmath(\ell)$ such that

$$\mathsf{J}y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell j\langle\tilde{b}_\ell, y\rangle \quad \text{and} \quad \mathsf{K}y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell i\langle\tilde{b}_\ell, y\rangle.$$

The left multiplication $\mathcal{L}$ is hence exactly the left multiplication induced by the basis $\tilde{b}_\ell, \ell \in \Lambda$, and multiplication of y by $a \in \mathbb{H}$ from the left exactly corresponds to multiplying the coordinates $\langle\tilde{b}_\ell, y\rangle$ by a from the left, i.e., $ay = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell a\langle\tilde{b}_\ell, y\rangle$. (Note, however, that unlike multiplication by scalars from the right, multiplication by scalars from the left corresponds to multiplication of the coordinates only in this basis. This relation is lost if we change the basis.)

Let us define $\langle\tilde{b}_\ell, y\rangle = a_\ell$ with $a_\ell = a_{\ell,1} + a_{\ell,2}i$ with $a_{\ell,1}, a_{\ell,2} \in \mathbb{C}_j$ and let $f_j : \mathbb{C}_j^+ \to \mathbb{H}$. If we write $f_j(z) = f_1(z) + f_2(z)i$, this time with $\mathbb{C}_j$-valued components $f_1, f_2 : \mathbb{C}_j^+ \to \mathbb{C}_j$, then (14.40) yields

$$\begin{aligned}\int_{\mathbb{C}_j^+} f_j(z)\, d\mathcal{E}(z) &= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell\big(f_1(s_{n,j}) + f_2(s_{n,j})i\big)(a_1 + a_2 i)\\ &= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell\big(a_1 f_1(s_{n,j}) + \overline{a_1} f_2(s_{n,j})i\big)\\ &\quad+ \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell\big(a_2 i\overline{f_2(s_{n,j})} - \overline{a_2} f_2(s_{n,j})\big).\end{aligned} \tag{14.41}$$

If we compare this with (14.32), then we find that $\int_{\mathbb{C}_j^+} f_j(z)\, d\mathcal{E}(z)$ corresponds to an application of f_j to the right eigenvalues of T only if $f_2 \equiv 0$ and f_1 can be extended to a function on all of $\mathbb{C}_j$ such that $f_1(\overline{s_{\ell,j}}) = \overline{f_1(s_{\ell,j})}$. This is, however, the case if and only if $f_j = f_1$ is the restriction of an intrinsic slice function to $\mathbb{C}_j^+$.

As pointed out above, spectral integrals of intrinsic slice functions defined in the sense of [13] or [144] can be considered spectral integrals with respect to a suitably chosen spectral system. The other two approaches—in particular the approach using iqPVMs in [144]—allow, however, the integration of a larger class of functions.

The authors of [144] argue in the introduction that the approach of spectral integration in [13] is complex in nature, since it allows one to integrate only $\mathbb{C}_j$-valued functions defined on $\mathbb{C}_j^+$ for some $j \in \mathbb{S}$. They argue that their approach using iqPVMs, on the other hand, is quaternionic in nature, since it allows one to integrate functions that are defined on a complex half-plane and take arbitrary values in the quaternions. It is rather the other way around. It is the approach to spectral integration using spectral systems that is quaternionic in nature, although they allow one to integrate only intrinsic slice functions, and we have three main arguments in favor of this point of view:

(i) **Spectral integration with respect to a spectral system does not require the random introduction of any undetermined structure.**

If we consider a normal operator $T = A + JB$ on a quaternionic Hilbert space, then only its spectral system J is uniquely defined. The extension of J to a unitary anti-self-adjoint operator $\mathbf{J}$ that can be interpreted as multiplication $L_j = \mathbf{J}$ by some $j \in \mathbb{S}$ from the left is not determined by T. Also, multiplication L_i by some $i \in \mathbb{S}$ with $i \perp j$ that extends L_j to the left multiplication $\mathcal{L}$ in an iqPVM $\mathcal{E} = (E, \mathcal{L})$ associated with T is not determined by T. The construction in [142] and [144] is based on the spectral theorems for quaternionic self-adjoint operators and for complex linear normal operators.

As we shall see in Chapter 15, the spectral orientation J of a spectral operator T—that is, the imaginary operator in the spectral system (E, J) associated with T—on a right Banach space can be constructed once the spectral measure E associated with T is known. Since the spectral theorems for self-adjoint operators and for complex linear operators are not available on Banach spaces, it is not clear how to extend J to a fully imaginary operator or even further to something that generalizes an iqPVM and whether this is possible at all.

(ii) **Spectral integration with respect to a spectral system has a clear interpretation in terms of the right linear structure on the space.**

The natural domain of a right linear operator is a right Banach space. If a left multiplication is defined on the Banach space, then the operator's spectral properties should be independent of this left multiplication. Integration with respect to a spectral system (E, J) has a clear and intuitive interpretation with respect to the right linear structure of the space: the spectral measure E associates (right) linear subspaces to spectral spheres, and the spectral orientation determines how to multiply the spectral values in the corresponding spectral spheres (from the right) onto the vectors in these subspaces.

The role of the left multiplication in an iqPVM in terms of the right linear structure is less clear. Indeed, we doubt that there exists a similarly clear and intuitive interpretation in view of the fact that no relation between left and right eigenvalues has been discovered up to now.

(iii) **Extending the class of integrable functions toward non-intrinsic slice functions does not seem to bring any benefit and might not even be meaningful.**

Extending the class of admissible functions for spectral integration beyond the class of measurable intrinsic slice functions seems to add little value to the theory. As pointed out above, the proof of the spectral theorem in [13] translates directly into the language of spectral systems, and hence spectral systems offer a framework that is sufficient to prove the most powerful result of spectral theory.

Even more, spectral integrals of functions that are not intrinsic slice functions cannot follow the basic intuition of spectral integration. In particular, if we define a measurable functional calculus via spectral integration, then this functional calculus only follows the fundamental intuition of a functional calculus, namely that $f(T)$ should be defined by the action of f on the spectral values of T if the underlying class of functions consists of intrinsic slice functions.