Chapter 13

Spectral Theorem for Unitary Operators

The spectral theorem for unitary operators is a particular case of the spectral theorem for bounded normal operators proved in Chapter 11. However, as in the complex case, the spectral theorem for unitary operators can be deduced from the quaternionic version of Herglotz's theorem proved in [16]. The spectral theorem for unitary operators based on Herglotz's theorem was proved in [14].

13.1 Herglotz's Theorem in the Quaternionic Setting

We recall some classical results and also their quaternionic analogues, which will be useful in proving a spectral theorem for quaternionic unitary operators. We need to recall some classical results in order to prove the quaternionic version of Herglotz's theorem.

Theorem 13.1.1 (Herglotz's theorem)**.** *The function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{C}^{s\times s}$ is positive definite if and only if there exists a unique $\mathbb{C}^{s\times s}$-valued measure μ on $[0, 2\pi]$ such that*

$$r(n) = \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}. \tag{13.1}$$

Theorem 13.1.2. *Let μ and ν be $\mathbb{C}^{s\times s}$-valued measures on $[0, 2\pi]$. If*

$$\int_0^{2\pi} e^{int} d\mu(t) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z},$$

then $\mu = \nu$.

In the above theorems we used the imaginary unit i for the complex plane. Given $P \in \mathbb{H}^{s\times s}$, there exist unique $P_1, P_2 \in \mathbb{C}^{s\times s}$ such that $P = P_1 + P_2 j$. Recall

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the bijective homomorphism $\chi : \mathbb{H}^{s\times s} \to \mathbb{C}^{2s\times 2s}$ given by

$$\chi P = \begin{pmatrix} P_1 & P_2 \\ -\overline{P_2} & \overline{P_1} \end{pmatrix}, \quad \text{where } P = P_1 + P_2 j. \tag{13.2}$$

Definition 13.1.3. Given an $\mathbb{H}$-valued measure ν, we may always write $\nu = \nu_1 + \nu_2 j$, where ν_1 and ν_2 are uniquely determined $\mathbb{C}$-valued measures. We call a measure $d\nu$ on $[0, 2\pi]$ *q-positive* if the $\mathbb{C}^{2\times 2}$-valued measure

$$\mu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3 \end{pmatrix}, \quad \text{where } \nu_3(t) = \nu_1(2\pi - t),\ t \in [0, 2\pi], \tag{13.3}$$

is positive and in addition,

$$\nu_2(t) = -\nu_2(2\pi - t), \quad t \in [0, 2\pi].$$

Remark 13.1.4. If ν is q-positive, then $\nu = \nu_1 + \nu_2 j$, where ν_1 is a uniquely determined positive measure and ν_2 is a uniquely determined $\mathbb{C}$-valued measure.

Remark 13.1.5. If $r = (r(n))_{n\in\mathbb{Z}}$ is an $\mathbb{H}$-valued sequence on $\mathbb{Z}$ such that

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t),$$

where $d\nu$ is a q-positive measure, then r is Hermitian, i.e., $\overline{r(-n)} = r(n)$.

The following result is a particular case of [16, Theorem 5.5] ($\mathbb{H}^{s\times s}$-valued positive sequences for $s > 1$ were also considered in [16]).

Theorem 13.1.6 (Herglotz's theorem for the quaternions)**.** *The function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{H}$ is positive definite if and only if there exists a unique q-positive measure ν on $[0, 2\pi]$ such that*

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}. \tag{13.4}$$

Proof. We give the proof for the general case. Let $(r(n))_{n\in\mathbb{Z}}$ be a positive definite sequence and write $r(n) = r_1(n) + r_2(n)j$, where $r_1(n), r_2(n) \in \mathbb{C}^{s\times s}$, $n \in \mathbb{Z}$. Put $R(n) = \chi r(n)$, $n \in \mathbb{Z}$. It is easily seen that $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence if and only if $(r(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{H}^{s\times s}$-valued sequence. Thus by Theorem 13.1.1, there exists a unique positive $\mathbb{C}^{2s\times 2s}$-valued measure μ on $[0, 2\pi]$ such that

$$R(n) = \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}. \tag{13.5}$$

Write

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12}^* & \mu_{22} \end{pmatrix} : \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix} \to \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix}.$$

It follows from

$$R(n) = \begin{pmatrix} r_1(n) & r_2(n) \\ -\overline{r_2(n)} & \overline{r_1(n)} \end{pmatrix}, \quad n \in \mathbb{Z},$$

and (13.5) that

$$r_1(n) = \int_0^{2\pi} e^{int} d\mu_{11}(t) = \int_0^{2\pi} e^{-int} d\bar{\mu}_{22}(t), \quad n \in \mathbb{Z},$$

and hence

$$\int_0^{2\pi} e^{int} d\mu_{11}(t) = \int_0^{2\pi} e^{int} d\bar{\mu}_{22}(2\pi - t), \quad n \in \mathbb{Z}.$$

Thus, Theorem 13.1.2 yields that $d\mu_{11}(t) = d\bar{\mu}_{22}(2\pi - t)$ for $t \in [0, 2\pi)$. Similarly,

$$r_2(n) = \int_0^{2\pi} e^{int} d\mu_{12}(t) = -\int_0^{2\pi} e^{-int} d\mu_{12}(t)^T, \quad n \in \mathbb{Z},$$

and hence

$$\int_0^{2\pi} e^{int} d\mu_{12}(t) = \int_0^{2\pi} e^{int} (-d\mu_{12}(2\pi - t)^T), \quad n \in \mathbb{Z}.$$

Thus, Theorem 13.1.2 yields that $d\mu_{12}(t) = -d\mu_{12}(2\pi - t)^T$ for $t \in [0, 2\pi)$.

It is easy to show that

$$\begin{pmatrix} I_s & -jI_s \end{pmatrix} R(n) \begin{pmatrix} I_s \\ jI_s \end{pmatrix} = 2r(n),$$

and hence (13.5) yields

$$\begin{aligned} 2r(n) &= \int_0^{2\pi} \begin{pmatrix} e^{int} & -je^{int} \end{pmatrix} \begin{pmatrix} d\mu_{11}(t) + d\mu_{12}(t)j \\ d\mu_{12}(t)^* + d\mu_{22}(t)j \end{pmatrix} \\ &= \int_0^{2\pi} e^{int} d\mu_{11}(t) + \int_0^{2\pi} e^{int} d\mu_{12}(t)j - \int_0^{2\pi} e^{-int} d\mu_{12}(t)^T j \\ &\qquad + \int_0^{2\pi} e^{-int} d\bar{\mu}_{22}(t) \\ &= \int_0^{2\pi} e^{int} d\mu_{11}(t) + \int_0^{2\pi} e^{int} d\mu_{12}(t)j - \int_0^{2\pi} e^{int} d\mu_{12}(2\pi - t)^T j \\ &\qquad + \int_0^{2\pi} e^{int} d\bar{\mu}_{22}(2\pi - t) \\ &= 2\int_0^{2\pi} e^{int} d\mu_{11}(t) + 2\int_0^{2\pi} e^{int} d\mu_{12}(t)j, \quad n \in \mathbb{Z}, \end{aligned}$$

where the last line follows from $d\mu_{11}(t) = d\bar{\mu}_{22}(2\pi - t)$ and $d\mu_{12}(t) = -d\mu_{12}(2\pi - t)^T$. If we put $\nu = \mu_{11} + \mu_{12}j$, then ν is a q-positive measure that satisfies (13.4).

Conversely, suppose $\nu = \nu_1 + \nu_2 j$ is a q-positive measure on $[0, 2\pi]$ and put

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}.$$

Since ν is q-positive,

$$\mu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3 \end{pmatrix}, \quad \text{where } d\nu_3(t) = d\bar{\nu}_1(2\pi - t),\ t \in [0, 2\pi),$$

is a positive $\mathbb{C}^{2s\times 2s}$-valued measure on $[0, 2\pi]$ and

$$d\nu_2(t) = -d\nu_2(2\pi - t)^T, \quad t \in [0, 2\pi).$$

Since μ is a positive $\mathbb{C}^{2s\times 2s}$-valued measure, $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence, where

$$R(n) := \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}.$$

Moreover, $R(n)$ can be written in the form

$$R(n) = \begin{pmatrix} r_1(n) & r_2(n) \\ -\overline{r_2(n)} & \overline{r_1(n)} \end{pmatrix}, \quad n \in \mathbb{Z},$$

where

$$r_1(n) = \int_0^{2\pi} e^{int} d\nu_1(t), \quad n \in \mathbb{Z};$$
$$r_2(n) = \int_0^{2\pi} e^{int} d\nu_2(t), \quad n \in \mathbb{Z}.$$

Thus, $R(n) = \chi r(n)$, where

$$r(n) = r_1(n) + r_2(n)j = \int_0^{2\pi} e^{int} d\nu(t).$$

Since $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence, we get that $(r(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{H}^{s\times s}$-valued sequence.

Finally, suppose that the q-positive measure ν were not unique, i.e., that there existed $\tilde{\nu}$ such that $\tilde{\nu} \neq \nu$ and

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t) = \int_0^{2\pi} e^{int} d\tilde{\nu}(t), \quad n \in \mathbb{Z}.$$

Write $\nu = \nu_1 + \nu_2 j$ and $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2 j$ as in Remark 13.1.4. If we consider $R(n) = \chi r(n), n \in \mathbb{Z}$, then it follows from Theorem 13.1.1 that $\nu_1 = \tilde{\nu}_1$ and $\nu_2 = \tilde{\nu}_2$ and hence that $\nu = \tilde{\nu}$, a contradiction. □

Remark 13.1.7. For every $i \in \mathbb{S}$, there exists $j \in \mathbb{S}$ such that $ij = -ji$. Thus, $\mathbb{H} = \mathbb{C}_i \oplus \mathbb{C}_i j$, and we may rewrite (13.4) as

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}, \tag{13.6}$$

where $\nu = \nu_1 + \nu_2 j$ is a q-positive measure (in the sense that

$$\mu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3 \end{pmatrix}$$

is positive). Here $\nu_3(t) = \nu_1(2\pi - t)$.

For our purpose the scalar case will be important.

13.2 Preliminaries for the Spectral Resolution

We start with a preliminary result.

Lemma 13.2.1. *Let U be a unitary operator on $\mathcal{H}$ and let $r_x(n) = \langle U^n x, x\rangle$ for $x \in \mathcal{H}$. Then $r_x = (r_x(n))_{n\in\mathbb{Z}}$ is an $\mathbb{H}$-valued positive definite sequence.*

Proof. If $\{p_0, \ldots, p_N\} \subset \mathbb{H}$, then

$$\begin{aligned}
\sum_{m,n=0}^{N} \bar{p}_m r_x(n-m) p_n &= \sum_{m,n=0}^{N} \bar{p}_m \langle U^{n-m} x, x\rangle p_n \\
&= \sum_{m,n=0}^{N} \langle U^{n-m} x p_n, x p_m\rangle \\
&= \sum_{m,n=0}^{N} \langle U^{n} x p_n, U^m x p_m\rangle \\
&= \langle \sum_{n=0}^{N} U^n x p_n, \sum_{m=0}^{N} U^m x p_m\rangle \\
&= \left\| \sum_{n=0}^{N} U^n x p_n \right\|^2 \geq 0.
\end{aligned}$$

Thus, r_x is a positive definite $\mathbb{H}$-valued sequence. □

Let r_x be as in Lemma 13.2.1. It follows from Theorem 13.1.6 that there exists a unique q-positive measure $d\nu_x$ such that

$$r_x(n) = \langle U^n x, x\rangle = \int_0^{2\pi} e^{int} d\nu_x(t), \quad n \in \mathbb{Z}. \tag{13.7}$$

One can check that

$$\begin{aligned}4\langle U^n x, y\rangle &= \langle U^n(x+y), x+y\rangle - \langle U^n(x-y), x-y\rangle \qquad (13.8)\\ &\quad + i\langle U^n(x+yi), x+yi\rangle\\ &\quad - i\langle U^n(x-yi), x-yi\rangle + i\langle U^n(x-yj), x-yj\rangle k \qquad (13.9)\\ &\quad - i\langle U^n(x+yj), x+yj\rangle k\\ &\quad + \langle U^n(x+yk), x+yk\rangle k - \langle U^n(x-yk), x-yk\rangle k, \qquad (13.10)\end{aligned}$$

and hence letting

$$\begin{aligned}4\nu_{x,y} &:= \nu_{x+y} - \nu_{x-y} + i\nu_{x+yi} - i\nu_{x-yi} + i\nu_{x-yj}k - i\nu_{x+yj}k\\ &\quad + \nu_{x+yk}k - \nu_{x-yk}k, \qquad (13.11)\end{aligned}$$

then

$$\langle U^n x, y\rangle = \int_0^{2\pi} e^{int} d\nu_{x,y}(t), \quad x, y \in \mathcal{H} \text{ and } n \in \mathbb{Z}. \qquad (13.12)$$

Theorem 13.2.2. *The $\mathbb{H}$-valued measures $\nu_{x,y}$ defined on $\mathbf{B}([0,2\pi])$ enjoy the following properties:*

(i) $\nu_{x\alpha+y\beta,z} = \nu_{x,z}\alpha + \nu_{y,z}\beta, \quad \alpha, \beta \in \mathbb{H}$;

(ii) $\nu_{x,y\alpha+z\beta} = \bar{\alpha}\nu_{x,y} + \bar{\beta}\nu_{x,z}, \quad \alpha, \beta \in \mathbb{C}_i$;

(iii) $\nu_{x,y}([0,2\pi]) \leq \|x\|\|y\|$;

where $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{H}$.

Proof. Formula (13.12) yields

$$\begin{aligned}\int_0^{2\pi} e^{int} d\nu_{x\alpha+y\beta,z}(t) &= \langle U^n x, z\rangle\alpha + \langle U^n y, z\rangle\beta\\ &= \int_0^{2\pi} e^{int}(d\nu_{x,z}(t)\alpha + d\nu_{y,z}(t)\beta), \quad n \in \mathbb{Z}.\end{aligned}$$

The uniqueness of the q-positive measure proved in Theorem 13.1.6 allows us to conclude that

$$\nu_{x\alpha+y\beta,z}(t) = \nu_{x,z}(t)\alpha + \nu_{y,z}(t)\beta,$$

and hence we have proved (i). Property (ii) is proved in a similar fashion, observing that $\bar{\alpha}$, $\bar{\beta}$ commute with e^{int}.

If $n = 0$ in (13.12), then

$$\langle x, y\rangle = \int_0^{2\pi} d\nu_{x,y}(t) = \nu_{x,y}([0,2\pi]),$$

and thus we can use an analogue of the Cauchy–Schwarz inequality (see Lemma 5.6 in [33]) to obtain

$$\nu_{x,y}([0,2\pi]) \leq \|x\|\|y\|,$$

and hence we have proved (iii). □

Remark 13.2.3. In contrast to the classical complex Hilbert space setting, $\nu_{x,y}$ need not equal $\bar{\nu}_{y,x}$.

It follows from statements (i), (ii), and (iii) in Theorem 13.2.2 that $\phi(x) = \nu_{x,y}(\sigma)$, where $y \in \mathcal{H}$ and $\sigma \in \mathbf{B}([0, 2\pi])$ are fixed, is a continuous right linear functional. Moreover, an analogue of the Riesz representation theorem (see Theorem 6.1 in [33] or Theorem 7.6 in [47]) gives that corresponding to every $x \in \mathcal{H}$, there exists a uniquely determined vector $w \in \mathcal{H}$ such that

$$\phi(x) = \langle x, w \rangle,$$

i.e., $\nu_{x,y}(\sigma) = \langle x, w \rangle$. Use (i) and (ii) in Theorem 13.2.2 to deduce that $w = E(\sigma)^* y$. The uniqueness of E follows readily from the construction. Thus, we have

$$\nu_{x,y}(\sigma) = \langle E(\sigma)x, y \rangle, \quad x, y \in \mathcal{H} \text{ and } \sigma \in \mathbf{B}([0, 2\pi]), \tag{13.13}$$

whence

$$\langle U^n x, y \rangle = \int_0^{2\pi} e^{int} \langle dE(t)x, y \rangle. \tag{13.14}$$

To prove the main properties of the operator E we need a uniqueness result on quaternionic measures that is a corollary of the following:

Theorem 13.2.4. *Let μ and ν be $\mathbb{C}$-valued measures on $[0, 2\pi]$. If*

$$r(n) = \int_0^{2\pi} e^{int} d\mu(t) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}, \tag{13.15}$$

then $\mu = \nu$.

Proof. See, e.g., Theorem 1.9.5 in [186]. □

Theorem 13.2.5. *Let μ and ν be $\mathbb{H}$-valued measures on $[0, 2\pi]$. If*

$$r(n) = \int_0^{2\pi} e^{int} d\mu(t) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}, \tag{13.16}$$

then $\mu = \nu$.

Proof. Write $r(n) = r_1(n) + r_2(n)j$, $\mu = \mu_1 + \mu_2 j$, and $\nu = \nu_1 + \nu_2 j$, where $r_1(n), r_2(n) \in \mathbb{C}$ and $\mu_1, \mu_2, \nu_1, \nu_2$ are $\mathbb{C}$-valued measures on $[0, 2\pi]$. It follows from (13.16) that

$$r_1(n) = \int_0^{2\pi} e^{int} d\mu_1(t) = \int_0^{2\pi} e^{int} d\nu_1(t), \quad n \in \mathbb{Z},$$

and

$$r_2(n) = \int_0^{2\pi} e^{int} d\mu_2(t) = \int_0^{2\pi} e^{int} d\nu_2(t), \quad n \in \mathbb{Z}.$$

Use Theorem 13.2.4 to conclude that $\mu_1 = \nu_1$, $\mu_2 = \nu_2$ and hence that $\mu = \nu$. □

Theorem 13.2.6. *The operator E given in (13.13) enjoys the following properties:*

(i) $\|E(\sigma)\| \le 1$;

(ii) $E(\emptyset) = 0$ *and* $E([0, 2\pi]) = \mathcal{I}$;

(iii) *If* $\sigma \cap \tau = \emptyset$, *then* $E(\sigma \cup \tau) = E(\sigma) + E(\tau)$;

(iv) $E(\sigma \cap \tau) = E(\sigma)E(\tau)$;

(v) $E(\sigma)^2 = E(\sigma)$;

(vi) $E(\sigma)$ *commutes with* U *for all* $\sigma \in \mathbf{B}([0, 2\pi])$.

Proof. Use (13.13) with $y = E(\sigma)x$ and (iii) in Theorem (13.2.2) to obtain

$$\|E(\sigma)x\|^2 \le \|x\|\|E(\sigma)x\|,$$

whence we have shown (i). The first part of property (ii) follows directly from the fact that $\nu_{x,y}(\emptyset) = 0$. The last part follows from (13.14) when $n = 0$. Statement (iii) follows easily from the additivity of the measure $\nu_{x,y}$.

We will now prove property (iv). It follows from (13.14) that

$$\begin{aligned}\langle U^{n+m}x, y\rangle &= \int_0^{2\pi} e^{int}e^{imt}\langle dE(t)x, y\rangle \\ &= \langle U^n(U^m x), y\rangle \\ &= \int_0^{2\pi} e^{int} d\langle E(t)U^m x, y\rangle.\end{aligned}$$

Using the uniqueness in Theorem 13.2.5 we obtain

$$e^{imt} d\langle E(t)x, y\rangle = \langle dE(t)U^m x, y\rangle,$$

and hence denoting by $\mathbf{1}_\sigma$ the characteristic function of the set σ, we have

$$\int_0^{2\pi} \mathbf{1}_\sigma(t)e^{imt}\langle dE(t)x, y\rangle = \langle E(\sigma)U^m x, y\rangle.$$

But

$$\int_0^{2\pi} \mathbf{1}_\sigma(t)e^{imt}\langle dE(t)x, y\rangle = \langle U^k x, E(\sigma)^* y\rangle = \int_0^{2\pi} e^{imt} d\langle E(t)x, E(\sigma)^* y\rangle.$$

Using the uniqueness in Theorem 13.2.5 once more, we get

$$\mathbf{1}_\sigma(t) d\langle E(t)x, y\rangle = \langle dE(t)x, E(\sigma)^* y\rangle$$

and hence

$$\int_0^{2\pi} \mathbf{1}_\tau(t)\mathbf{1}_\sigma(t)\langle dE(t)x, y\rangle = \langle E(t)x, E(\sigma)^* y\rangle$$

and thus
$$\langle E(\sigma \cap \tau)x, y\rangle = \langle E(\sigma)E(\tau)x, y\rangle.$$
Property (v) is obtained from (iv) by letting $\sigma = \tau$.

Finally, since U is unitary, one can check that

$$\langle U(x \pm U^*y), x \pm U^*y\rangle = \langle U(Ux \pm y), Ux \pm y\rangle,$$

and hence from (13.12) and the uniqueness in Theorem 13.2.5 we obtain $\nu_{x\pm U^*y} = \nu_{Ux\pm y}$. Similarly,

$$\nu_{x\pm U^*yi} = \nu_{Ux\pm yi},$$
$$\nu_{x\pm U^*yj} = \nu_{Ux\pm yj},$$

and
$$\nu_{x\pm U^*yk} = \nu_{Ux\pm yk}.$$
It follows from (13.11) that
$$\nu_{x,U^*y} = \nu_{Ux,y}.$$
Now use (13.13) to obtain

$$\langle E(\sigma)x, U^*y\rangle = \langle E(\sigma)Ux, y\rangle,$$

i.e.,
$$\langle UE(\sigma)x, y\rangle = \langle E(\sigma)Ux, y\rangle, \quad x, y \in \mathcal{H}. \qquad \square$$

Given any quaternionic Hilbert space $\mathcal{H}$, there exists a subspace $\mathcal{M} \subset \mathcal{H}$ on $\mathbb{C}$ such that for every $x \in \mathcal{H}$ we have

$$x = x_1 + x_2 j, \quad x_1, x_2 \in \mathcal{M}.$$

Theorem 13.2.7. *Let U be a unitary operator on a quaternionic Hilbert space $\mathcal{H}$ and let E be the corresponding operator given by* (13.13)*. E is self-adjoint if and only if $U : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is as above.*

Proof. If $E = E^*$, then it follows from (13.13) that $\nu_{x,y} = \bar{\nu}_{y,x}$ for all $x, y \in \mathcal{H}$. In particular, we get $\nu_{x,x} = \bar{\nu}_{x,x}$, i.e.,

$$\nu_x = \bar{\nu}_x, \quad x \in \mathcal{H}. \tag{13.17}$$

Since ν_x is a q-positive measure, we may write $\nu_x = \alpha_x + \beta_x j$, where α_x is a positive Borel measure on $[0, 2\pi]$ and β_x is a complex Borel measure on $[0, 2\pi]$. It follows from (13.17) that
$$\beta_x = -\beta_x,$$
i.e., $\beta_x = 0$. Thus, we may make use of the spectral theorem for unitary operators on a complex Hilbert space (see, e.g., Section 31.7 in [163]) to deduce that $U : \mathcal{M} \to \mathcal{M}$. Conversely, if $U : \mathcal{M} \to \mathcal{M}$, then the spectral theorem for unitary operators on a complex Hilbert space yields that $E = E^*$. $\square$

If $U : \mathbb{H}^n \to \mathbb{H}^n$ is unitary, then (13.14) and Theorem 13.2.6 assert that

$$U = \sum_{a=1}^{n} e^{i\theta_a} P_a, \tag{13.18}$$

where $\theta_1, \dots, \theta_n \in [0, 2\pi]$ and $P_1, \dots, P_n$ are oblique projections (i.e., $(P_a)^2 = P_a$ but $(P_a)^*$ need not equal P_a). Corollary 6.2 in Zhang [199] asserts, in particular, the existence of $V : \mathbb{H}^n \to \mathbb{H}^n$ that is unitary and $\theta_1, \dots, \theta_n \in [0, 2\pi]$ such that

$$U = V^* \text{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})V. \tag{13.19}$$

In the following remark we will explain how (13.18) and (13.19) are consistent.

Remark 13.2.8. Let $U : \mathbb{H}^n \to \mathbb{H}^n$ be unitary. Let V and $\theta_1, \dots, \theta_n$ be as above. If we let $e_a = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{H}^n$, where the 1 is the ath position, then we can rewrite (13.19) as

$$U = \sum_{a=1}^{n} V^* e^{i\theta_a} e_a e_a^* V.$$

Note that $V^* e^{i\theta_a} e_a e_a^* V = e^{i\theta_a} V^* e_a e_a^* V$ if and only if $V : \mathbb{C}^n \to \mathbb{C}^n$. In this case $U : \mathbb{C}^n \to \mathbb{C}^n$ and

$$U = \sum_{a=1}^{n} e^{i\theta_a} P_a,$$

where P_a denotes the orthogonal projection given by $V^* e^{i\theta_a} e_a e_a^* V$.

Remark 13.2.9. Observe that in the proof of the spectral theorem for U^n we have taken the imaginary units i, j, k for the quaternions and we have determined spectral measures $\langle dE(t)x, y\rangle$ that are supported on the unit circle in $\mathbb{C}_i$. If one uses other orthogonal units i', j', and $k' \in \mathbb{S}$ to represent quaternions, then the spectral measures are supported on the unit circle in $\mathbb{C}_{i'}$.

Observe that (13.14) provides a vehicle to define a functional calculus for unitary operators on a quaternionic Hilbert space. For a continuous $\mathbb{H}$-valued function f on the unit circle, which will be approximated by the polynomials $\sum_k e^{ikt} a_k$. We will consider a subclass of continuous quaternionic-valued functions defined as follows, see [142]: It is important to note that every polynomial of the form $P(u + jv) = \sum_{k=0}^{n} (u + jv)^n a_n$, $a_n \in \mathbb{H}$ is a slice continuous function in the whole of $\mathbb{H}$. A trigonometric polynomial of the form $P(e^{jt}) = \sum_{m=-n}^{n} e^{jmt} a_m$ is a slice continuous function on $\partial\mathbb{B}$, where $\mathbb{B}$ denotes the unit ball of quaternions.

Let us now denote by $\mathcal{PS}(\sigma_S(T))$ the set of slice continuous functions $f(u + iv) = \alpha(u, v) + i\beta(u, v)$, where α, β are polynomials in the variables u, v.

In the sequel we will work in the complex plane $\mathbb{C}_i$ and we denote by $\mathbb{T}_i$ the boundary of $\mathbb{B} \cap \mathbb{C}_i$. Any other choice of an imaginary unit in the unit sphere $\mathbb{S}$ will provide an analogous result.

Remark 13.2.10. For every $i \in \mathbb{S}$, there exists $j \in \mathbb{S}$ such that $ij = -ji$. Bearing in mind Remark 13.1.7, we can construct $\nu^{(j)}_{x,y}$ such that (13.12) can also be written as

$$\langle U^n x, y\rangle = \int_0^{2\pi} e^{int} d\nu^{(j)}_{x,y}(t), \quad x, y \in \mathcal{H} \text{ and } n \in \mathbb{Z}. \tag{13.20}$$

Consequently, (13.14) can be written as

$$\langle U^n x, y\rangle = \int_0^{2\pi} e^{int} \langle E_j(t)x, y\rangle, \tag{13.21}$$

where E_j is given by

$$\nu^{(j)}_{x,y}(\sigma) = \langle E_j(\sigma)x, y\rangle, \quad x, y \in \mathcal{H} \text{ and } \sigma \in \mathrm{B}(\mathbb{T}_i).$$

Moreover, the E_j satisfy properties (i)–(v) listed in Theorem 13.2.6.

13.3 Further Properties of Quaternionic Riesz Projectors

An axially symmetric set $\sigma \subseteq \sigma_S(T)$ that is both open and closed in $\sigma_S(T)$ in its relative topology, is called an S-spectral set. Denote by Ω_σ an axially symmetric domain that contains the spectral set σ but not any other points of the S-spectrum. We recall the Riesz projectors

$$\mathcal{P}(\sigma) = \frac{1}{2\pi} \int_{\partial(\Omega_\sigma \cap \mathbb{C}_j)} S^{-1}_L(s, T) ds_j$$

and the fact that $\mathcal{P}(\sigma)$ can be given using the right S-resolvent operator $S^{-1}_R(s, T)$, that is,

$$\mathcal{P}(\sigma) = \frac{1}{2\pi} \int_{\partial(\Omega_\sigma \cap \mathbb{C}_j)} ds_j S^{-1}_R(s, T).$$

We have the following properties.

Theorem 13.3.1. *Let T be a quaternionic linear operator. Then the family of operators $\mathcal{P}(\sigma)$ has the following properties:*

(i) $(\mathcal{P}(\sigma))^2 = \mathcal{P}(\sigma)$;

(ii) $T\mathcal{P}(\sigma) = \mathcal{P}(\sigma)T$;

(iii) $\mathcal{P}(\sigma_S(T)) = \mathcal{I}$;

(iv) $\mathcal{P}(\emptyset) = 0$;

(v) $\mathcal{P}(\sigma \cup \delta) = \mathcal{P}(\sigma) + \mathcal{P}(\delta); \quad \sigma \cap \delta = \emptyset$;

(vi) $\mathcal{P}(\sigma \cap \delta) = \mathcal{P}(\sigma)\mathcal{P}(\delta)$.

Proof. Properties (i) and (ii) are proved in Theorem 4.1.5. Property (iii) follows from the quaternionic functional calculus, since

$$T^m = \frac{1}{2\pi} \int_{\partial(\Omega\cap\mathbb{C}_j)} S_L^{-1}(s,T) ds_j \, s^m, \quad m \in \mathbb{N}_0,$$

for $\sigma_S(T) \subset \Omega$, which for $m = 0$ gives

$$\mathcal{I} = \frac{1}{2\pi} \int_{\partial(\Omega\cap\mathbb{C}_j)} S_L^{-1}(s,T) ds_j.$$

Property (iv) is a consequence of the functional calculus as well.

Property (v) follows from

$$\begin{aligned} \mathcal{P}(\sigma \cup \delta) &= \frac{1}{2\pi} \int_{\partial(\Omega_{\sigma\cup\delta}\cap\mathbb{C}_j)} S_L^{-1}(s,T) ds_j \\ &= \frac{1}{2\pi} \int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} S_L^{-1}(s,T) ds_j + \frac{1}{2\pi} \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} S_L^{-1}(s,T) ds_j \\ &= \mathcal{P}(\sigma) + \mathcal{P}(\delta). \end{aligned}$$

To prove (vi), assume that $\sigma \cap \delta \neq \emptyset$, and for simplicity set

$$\mathcal{Q}_s(p)^{-1} := (p^2 - 2\mathrm{Re}(s)p + |s|^2)^{-1}, \quad p \notin [s],$$

and consider

$$\begin{aligned} \mathcal{P}(\sigma)\mathcal{P}(\delta) &= \frac{1}{(2\pi)^2} \int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j S_R^{-1}(s,T) \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} S_L^{-1}(p,T) dp_j \\ &= \frac{1}{(2\pi)^2} \int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [S_R^{-1}(s,T) - S_L^{-1}(p,T)] p \mathcal{Q}_s(p)^{-1} dp_j \\ &\quad - \frac{1}{(2\pi)^2} \int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(p,T)] \mathcal{Q}_s(p)^{-1} dp_j, \end{aligned}$$

where we have used the S-resolvent equation (see Theorem 3.1.15). We rewrite the above relation as

$$\begin{aligned} \mathcal{P}(\sigma)\mathcal{P}(\delta) &= -\frac{1}{(2\pi)^2} \int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s} S_R^{-1}(s,T) - S_R^{-1}(s,T) p] \mathcal{Q}_s(p)^{-1} dp_j \\ &\quad + \frac{1}{(2\pi)^2} \int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s} S_L^{-1}(p,T) - S_L^{-1}(p,T) p] \mathcal{Q}_s(p)^{-1} dp_j \\ &:= \mathcal{J}_1 + \mathcal{J}_2. \end{aligned}$$

Moreover, we have

$$\begin{aligned}
\mathcal{J}_1 &= -\frac{1}{(2\pi)^2}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s}S_R^{-1}(s,T) - S_R^{-1}(s,T)p]\mathcal{Q}_s(p)^{-1}dp_j \\
&= \frac{1}{2\pi}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j S_R^{-1}(s,T), \quad \text{for } s\in\Omega_\delta\cap\mathbb{C}_j \\
&= \frac{1}{2\pi}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j, \quad \text{for } s\in\Omega_\delta\cap\mathbb{C}_j,
\end{aligned}$$

while $\mathcal{J}_1 = 0$ when $s\notin\Omega_\delta\cap\mathbb{C}_j$, since

$$\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s}S_R^{-1}(s,T) - S_R^{-1}(s,T)p]\mathcal{Q}_s(p)^{-1}dp_j = 0.$$

Similarly, one can show that

$$\mathcal{J}_2 = \frac{1}{2\pi}\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} S_L^{-1}(p,T)dp_j, \quad \text{for } p\in\Omega_\sigma\cap\mathbb{C}_j,$$

while $\mathcal{J}_2 = 0$ when $p\notin\Omega_\sigma\cap\mathbb{C}_j$. The integrals $\mathcal{J}_1$, $\mathcal{J}_2$ are either both zero or both nonzero, so with a change of variable we get

$$\mathcal{J}_1+\mathcal{J}_2 = \frac{1}{2\pi}\int_{\partial(\Omega_{\sigma\cap\delta}\cap\mathbb{C}_j)} S_L^{-1}(r,T)dr_j = \mathcal{P}(\sigma\cap\delta). \qquad \square$$

We recall that if U is a unitary operator on $\mathcal{H}$, then the S-spectrum of U belongs to the unit sphere of the quaternions; see Theorem 9.2.7. We denote the Borel sets in $[0,2\pi]$ by $\mathbf{B}([0,2\pi])$.

Lemma 13.3.2. *Let $x,y\in\mathcal{H}$ and let $\mathcal{P}(\sigma)$ be the projector associated with the unitary operator U. We define*

$$m_{x,y}(\sigma) := \langle\mathcal{P}(\sigma)x,y\rangle, \quad x,\, y\in\mathcal{H},\ \sigma\in\mathbf{B}([0,2\pi]).$$

Then the $\mathbb{H}$-valued measures $m_{x,y}$ defined on $\mathbf{B}([0,2\pi])$ enjoy the following properties:

(i) $m_{x\alpha+y\beta,z} = m_{x,z}\alpha + m_{y,z}\beta$;

(ii) $m_{x,y\alpha+z\beta} = \overline{\alpha}m_{x,y} + \overline{\beta}m_{x,z}$;

(iii) $m_{x,y}([0,2\pi]) \le \|x\|\|y\|$;

where $x,y,z\in\mathcal{H}$ and $\alpha,\beta\in\mathbb{H}$.

Proof. Properties (i) and (ii) follow from the properties of the quaternionic scalar product, while (iii) follows from Property (iii) in Theorem 13.3.1 and the Cauchy–Schwarz inequality. $\square$

13.4 The Spectral Resolution

We are now in a position to prove the spectral theorem for quaternionic unitary operators.

Theorem 13.4.1 (The spectral theorem for quaternionic unitary operators)**.** *Let U be a unitary operator on a right linear quaternionic Hilbert space $\mathcal{H}$. Let $i, j \in \mathbb{S}$, i orthogonal to j. Then there exists a unique spectral measure E_j defined on the Borel sets of $\mathbb{T}_i$ such that for every slice continuous function $f \in \mathcal{S}(\sigma_S(U))$, we have*

$$f(U) = \int_0^{2\pi} f(e^{it}) dE_j(t).$$

Proof. Let us consider a polynomial $P(t) = \sum_{m=-n}^{n} e^{imt} a_m$ defined on $\mathbb{T}_i$. Then using (13.21), we have

$$\langle U^m x, y\rangle = \int_0^{2\pi} e^{imt} \langle dE_j(t)x, y\rangle, \quad x, y, \in \mathcal{H}.$$

By linearity, we can define

$$\langle P(U)x, y\rangle = \int_0^{2\pi} P(e^{it}) \langle dE_j(t)x, y\rangle, \quad x, y, \in \mathcal{H}.$$

The map $\Psi : \mathcal{PS}(\sigma_S(U)) \to \mathcal{H}$ defined by $\psi_U(P) = P(U)$ is $\mathbb{R}$-linear. By fixing a basis for $\mathbb{H}$, e.g., the basis $1, i', j', k'$, each slice continuous function f can be decomposed using intrinsic functions, i.e., $f = f_0 + f_1 i' + f_2 j' + f_3 k'$ with $f_\ell \in \mathcal{S}_{\mathbb{R}}(\sigma_S(U))$, $\ell = 0, \dots, 3$. For these functions the spectral mapping theorem holds; thus $f_\ell(\sigma_S(U)) = \sigma_S(f_\ell(U))$, and so $\|f_\ell(U)\| = \|f_\ell\|_\infty$. The map ψ is continuous, and so there exists $C > 0$, which does not depend on ℓ, such that

$$\|P(U)\|_{\mathcal{H}} \leq C \max_{t \in \sigma_S(U)} |P(t)|.$$

A slice continuous function $f \in \mathcal{S}(\sigma_S(U))$ is defined on an axially symmetric subset $K \subseteq \mathbb{T}$, and thus it can be written as a function $f(e^{jt}) = \alpha(\cos t, \sin t) + j\beta(\cos t, \sin t)$. By fixing a basis of $\mathbb{H}$, e.g., $1, i', j', k'$, f can be decomposed into four slice continuous intrinsic functions $f_\ell(\cos t, \sin t) = \alpha_\ell(\cos t, \sin t) + j\beta_\ell(\cos t, \sin t)$, $\ell = 0, \dots, 3$, such that $f = f_0 + f_1 i' + f_2 j' + f_3 k'$.

By the Weierstrass approximation theorem for trigonometric polynomials, see, e.g., Theorem 8.15 in [183], each function f_ℓ can be approximated by a sequence of polynomials

$$\tilde{R}_{\ell n} = \tilde{a}_{\ell n}(\cos t, \sin t) + j\tilde{b}_{\ell n}(\cos t, \sin t),$$

$\ell = 0, \dots, 3$, which tend uniformly to f_ℓ. These polynomials do not necessarily belong to the class of the continuous slice functions, since $\tilde{a}_{\ell n}, \tilde{b}_{\ell n}$ do not satisfy, in general, the even and odd conditions of slice continuous functions. However, by setting

$$a_{\ell n}(u,v) = \frac{1}{2}(\tilde{a}_{\ell n}(u,v) + \tilde{a}_{\ell n}(u,-v)),$$
$$b_{\ell n}(u,v) = \frac{1}{2}(\tilde{b}_{\ell n}(u,-v) - \tilde{b}_{\ell n}(u,v)),$$

we obtain that the sequence of polynomials $a_{\ell n} + j' b_{\ell n}$ still converges to f_ℓ, $\ell = 0, \dots, 3$. By putting $R_{\ell n} = a_{\ell n}(\cos t, \sin t) + j b_{\ell n}(\cos t, \sin t)$, $\ell = 0, \dots, 3$, and $R_n = R_{0n} + R_{1n} i' + R_{2n} j' + R_{3n} k'$ we have a sequence of slice continuous polynomials $R_n(e^{jt})$ converging to $f(e^{jt})$ uniformly on $\mathbb{R}$.

By the previous discussion, $\{R_n(U)\}$ is a Cauchy sequence in the space of bounded linear operators, since

$$\|R_n(U) - R_m(U)\| \le C \max_{t \in \sigma_S(U)} |R_n(t) - R_m(t)|;$$

so $R_n(U)$ has a limit, which we denote by $f(U)$. □

Remark 13.4.2. Fix $j \in \mathbb{S}$. It is worth pointing out that $f(u + jv) = (u + jv)^{-1}$ is an intrinsic function on $\mathbb{C}_j \cap \partial\mathbb{B}$, where $\partial\mathbb{B} = \{q \in \mathbb{H} : |q| = 1\}$, since

$$f(u + jv) = \frac{u}{u^2 + v^2} + \left(\frac{-v}{u^2 + v^2}\right) j.$$

Thus, using Theorem 13.4.1, we may write

$$U^{-1} = \int_0^{2\pi} e^{-it} dE_j(t) \tag{13.1}$$

and

$$U = \int_0^{2\pi} e^{it} dE_j(t). \tag{13.2}$$

We are now ready to prove the following fundamental result, which shows the relation between the spectral measures and the S-spectrum.

Theorem 13.4.3. *Fix $i, j \in \mathbb{S}$, with i orthogonal to j. Let U be a unitary operator on a right linear quaternionic Hilbert space $\mathcal{H}$ and let $E(t) = E_j(t)$ be its spectral measure. Assume that $\sigma_S^0(U) \cap \mathbb{C}_i$ is contained in the arc of the unit circle in $\mathbb{C}_i$ with endpoints t_0 and t_1. Then*

$$\mathcal{P}(\sigma_S^0(U)) = E(t_1) - E(t_0).$$

Proof. The spectral theorem implies that the operator $S_R^{-1}(s, U)$ can be written as

$$S_R^{-1}(s, U) = \int_0^{2\pi} S_R^{-1}(e^{it}, s) dE(t).$$

The Riesz projector is given by

$$\mathcal{P}(\sigma_S^0(U)) = \frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i S_R^{-1}(s,U),$$

where Ω_0 is an open set containing $\sigma_S^0(U)$ such that $\partial(\Omega_0\cap\mathbb{C}_i)$ is a smooth closed curve in $\mathbb{C}_i$. Write

$$\mathcal{P}(\sigma_S^0(U)) = \frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i\Big(\int_0^{2\pi} S_R^{-1}(e^{it},s)dE(t)\Big)$$

and use Fubini's theorem to obtain

$$\mathcal{P}(\sigma_S^0(U)) = \int_0^{2\pi}\Big(\frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i S_R^{-1}(e^{it},s)\Big)dE(t).$$

It follows from the Cauchy formula that

$$\frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i S_R^{-1}(e^{it},s) = \mathbf{1}_{[t_0,t_1]},$$

where $\mathbf{1}_{[t_0,t_1]}$ is the characteristic function of the set $[t_0,t_1]$, and thus the statement follows, since

$$\mathcal{P}(\sigma_S^0(U)) = \int_0^{2\pi}\mathbf{1}_{[t_0,t_1]}dE(t) = E(t_1)-E(t_2). \qquad \square$$

We will close by establishing a connection between the spectral resolutions for a unitary operator presented in Theorem 11.2.1 and Theorem 13.4.1. Let $U\in\mathcal{B}(\mathcal{H})$ be unitary. Since $U\in\mathcal{B}(\mathcal{H})$ is normal, we may write

$$U = A + JB,$$

where A, J, and B are as in Theorem 9.3.5. Thus, Theorem 11.2.1 asserts the existence of a spectral measure E (in the usual sense) on $\Omega := [0,\pi]\cap\sigma_S(U)$ such that if $n\in\mathbb{Z}$, then

$$\langle U^n x,y\rangle = \int_\Omega \cos(n\theta)d\langle E(\theta)x,y\rangle + \int_\Omega \sin(n\theta)d\langle JE(\theta)x,y\rangle, \quad x,y\in\mathcal{H}. \tag{13.3}$$

On the other hand, Theorem 13.4.1 asserts the existence of a $\mathcal{B}(\mathcal{H})$-valued measure F that satisfies most of the properties of a spectral measure (see Theorem 13.2.6) such that if $n\in\mathbb{Z}$, then

$$\langle U^n x,y\rangle = \int_0^{2\pi} e^{in\theta}d\langle F(\theta)x,y\rangle, \quad x,y\in\mathcal{H}. \tag{13.4}$$

Consequently, if we let $d\nu_x(\theta) := d\langle E(\theta)x, x\rangle$ and $d\mu_x(\theta) := d\langle F(\theta)x, x\rangle$, then $d\nu_x$ is a positive measure and $d\mu_x := d\mu_x^{(0)} + d\mu_x^{(1)} j$ is a q-positive measure (and hence $d\mu_x^{(0)}$ is a positive measure). Now (13.3) implies that

$$\frac{1}{2}\langle (U^n + U^{*n})x, x\rangle = \int_0^{\pi} \cos(n\theta) d\nu_x(\theta),$$

while (13.4) implies that

$$\frac{1}{2}\langle (U^n + U^{*n})x, x\rangle = \int_0^{2\pi} \cos(n\theta) d\mu_x^{(0)}(\theta).$$

Since $d\mu_x^{(0)}$ and $d\nu_x$ are positive measures, the uniqueness assertion in Theorem 13.1.1 forces $d\mu_x^{(0)} = d\nu_x$ and hence $d\langle E(\theta)x, x\rangle = \mathrm{Re}\langle F(\theta)x, x\rangle$.

13.5 Comments and Remarks

Theorem 13.1.6 is taken from [16], and it helped give rise to a spectral theorem for unitary operators based on the S-spectrum in [14]. In addition, Theorem 13.1.6 can be used to generate a quaternionic analogue of the Herglotz representation on a slice (see Theorem 8.1 in [16]). More precisely, if $f : \mathbb{B} \to \mathbb{H}$ is slice hyperholomorphic with $\mathrm{Re}(f(p)) \geq 0$ for all $p \in \mathbb{B} := \{p \in \mathbb{H} : |p| < 1\}$ and $i, j \in \mathbb{S}$ with i and j orthogonal, then there exists a $\mathbb{C}_j$-valued measure $d\mu_j(t) = d\mu_1(t) + d\mu_2(t)j$ of finite total variation with μ_1 positive and μ_2 signed such that the restriction $f_j(z) = f|_{\mathbb{C}_j} = F(z) + G(z)j$ admits the representation

$$f_j(z) = i[\mathrm{Im}F(0) + \mathrm{Im}G(0)j] + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu_j(t). \tag{13.5}$$

A half-space analogue of (13.5) was treated in [9] (albeit with stronger conditions on f and the corresponding measure).