Chapter 12

The Spectral Theorem for Unbounded Normal Operators

In this section we will consider normal operators T that are unbounded. The strategy will be to transform T into a normal operator $Z_T \in \mathcal{B}(\mathcal{H})$ and use Theorem 11.2.1 and a change of variable argument to obtain a spectral theorem for T based on the S-spectrum. Obtaining a spectral theorem for unbounded operators in the aforementioned way has been done in the classical case, i.e., when $\mathcal{H}$ is a complex Hilbert space; see, e.g., the book of Schmüdgen [191].

12.1 Some Transformations of Operators

Given $T \in \mathcal{L}(\mathcal{H})$, we let

$$Z_T = TC_T^{1/2}, \tag{12.1}$$

where $C_T = (\mathcal{I} + T^*T)^{-1} \in \mathcal{B}(\mathcal{H})$ (the proof that C_T is bounded and positive can be carried out in a similar manner to the classical complex Hilbert case; see, e.g., Proposition 3.18(i) in [191]).

Theorem 12.1.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed operator on $\mathcal{H}$. The operator Z_T has the following properties:*

(i) $Z_T \in \mathcal{B}(\mathcal{H})$, $\|Z_T\| \leq 1$, *and*

$$C_T = (\mathcal{I} + T^*T)^{-1} = \mathcal{I} - Z_T^* Z_T. \tag{12.2}$$

(ii) $(Z_T)^* = Z_{T^*}$.

(iii) *If T is normal, then Z_T is normal.*

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Proof. The proof is based on the proof of Lemma 5.8 in [191] and is broken into three steps.

Step 1: *Prove* (i).

First note that

$$\{C_T x : \ x \in \mathcal{H}\} = \mathcal{D}(\mathcal{I} + T^*T) = \mathcal{D}(T^*T). \tag{12.3}$$

Consequently, if $x \in \mathcal{H}$, then

$$\begin{aligned} \|TC_T^{1/2}C_T^{1/2}x\|^2 &= \langle T^*TC_Tx, C_Tx\rangle \\ &\le \langle (\mathcal{I} + T^*T)C_Tx, C_Tx\rangle \\ &= \langle C_T^{-1}C_Tx, C_Tx\rangle \\ &= \langle x, C_Tx\rangle \\ &= \|C_T^{1/2}x\|^2. \end{aligned}$$

Thus if $y \in \{C_T^{1/2}x\colon x \in \mathcal{H}\}$, then

$$\|Z_Ty\| = \|TC_T^{1/2}y\| \le \|y\|. \tag{12.4}$$

Since $\ker(C_T) = \{0\}$, we have that $\ker(C_T^{1/2}) = \{0\}$, and thus $\{C_T^{1/2}x\colon x \in \mathcal{H}\}$ is a dense subset of $\mathcal{H}$. Since T is a closed operator by assumption and $C_T^{1/2} \in \mathcal{B}(\mathcal{H})$, we get that Z_T is closed as well. Thus, we have $\{C_T^{1/2}x\colon x \in \mathcal{H}\} \subseteq \mathcal{D}(T)$, $\mathcal{D}(Z_T) = \mathcal{H}$, and in view of (12.4), $\|Z_T\| \le 1$.

Next, it follows from (12.4) and $C^{1/2}T^* \subseteq Z_T^*$ that

$$\begin{aligned} (\mathcal{I} - C_T)C_T^{1/2} &= C_T^{1/2}(\mathcal{I} + T^*T)C_T - C_T^{1/2}C_T \\ &= C_T^{1/2}T^*TC_T^{1/2}C_T^{1/2} \\ &\subseteq Z_T^*Z_TC_T^{1/2}. \end{aligned}$$

Thus, $Z_T^*Z_TC_T^{1/2} = (\mathcal{I} - C_T)C_T^{1/2}$, and since $\{C_T^{1/2}x\colon x \in \mathcal{H}\}$ is a dense subset of $\mathcal{H}$, we get (12.2).

Step 2: *Prove* (ii).

Using (12.2) we get that $C_{T^*} = (\mathcal{I} + TT^*)^{-1}$. If $x \in \mathcal{D}(T^*)$, then let $y = C_{T^*}x$. Therefore,

$$x = (\mathcal{I} + TT^*)y$$

and

$$T^*x = T^*(\mathcal{I} + TT^*)y = (\mathcal{I} + T^*T)T^*y.$$

Thus, $C_{T^*}x \in \mathcal{D}(T^*)$ and hence

$$C_TT^*x = T^*y = T^*C_{T^*}x. \tag{12.5}$$

It follows easily from (12.5) and (12.2) that $p(C_{T^*})x \in \mathcal{D}(T^*)$ and

$$p(C_T)T^*x = T^*p(C_{T^*})x$$

for every real polynomial p of a real variable. By the Weierstrass approximation theorem, there exists a sequence of real polynomials $\{\phi_n\}_{n=0}^{+\infty}$ that converges uniformly to the function $t \mapsto t^{1/2}$ on $[0, 1]$. Since the continuous functional calculus is norm-preserving, we find that

$$\lim_{n\to+\infty} \|\phi_n(C_T) - C_T^{1/2}\| = \lim_{n\to+\infty} \|\phi_n(C_{T^*}) - C_{T^*}^{1/2}\| = 0.$$

Since T is a closed operator, T^* is also a closed operator. Thus, we have

$$\begin{aligned} C_T^{1/2}T^*x &= \lim_{n\to+\infty} \phi_n(C_T)T^*x = \lim_{n\to+\infty} T^*\phi_n(C_{T^*})x \\ &= T^*(C_{T^*})^{1/2}x \quad \text{for} \quad x \in \mathcal{D}(T^*). \end{aligned}$$

Since $C_T^{1/2}T^* \subseteq (TC_T^{1/2})^* = Z_{T^*}$, we get that

$$Z_{T^*}x = C_T^{1/2}T^*x = T^*(C_{T^*})^{1/2}x = (Z_T)^*x$$

for $x \in \mathcal{D}(T^*)$. Finally, since $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$, we have that $Z_{T^*}x = (Z_T)^*x$ for all $x \in \mathcal{H}$, i.e., $Z_{T^*} = (Z_T)^*$.

Step 3: *Prove* (iii).

Using (12.2) on T and T^* and the fact that $TT^* = T^*T$, we have

$$\mathcal{I} - Z_T^*Z_T = (\mathcal{I} + T^*T)^{-1} = (\mathcal{I} + TT^*)^{-1} = \mathcal{I} - Z_{T^*}^*Z_{T^*}.$$

Making use of Property (ii), we have that

$$\mathcal{I} - Z_T^*Z_T = \mathcal{I} - Z_TZ_T^*,$$

i.e., Z_T is normal. □

12.2 The Spectral Theorem for Unbounded Normal Operators

We are now ready to state and prove a spectral theorem for unbounded normal operators on a quaternionic Hilbert space.

Theorem 12.2.1. *Let T be an unbounded right linear normal operator on $\mathcal{H}$ and $j \in \mathbb{S}$. There exists a uniquely determined spectral measure E_j on $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ such that for $x \in \mathcal{D}(T)$ and $y \in \mathcal{H}$,*

$$\langle Tx, y\rangle = \int_{\Omega_j^+} \mathrm{Re}(p) d\langle E_j(p)x, y\rangle + \int_{\Omega_j^+} \mathrm{Im}(p) d\langle JE_j(p)x, y\rangle, \tag{12.6}$$

or equivalently,

$$\begin{aligned}\langle Tx, y\rangle = &\int_{\Omega_j^+} \mathrm{Re}(p)\, d\langle \Pi_0 E_j(p)x, y\rangle \\ &+ \int_{\Omega_j^+} d\langle \Pi_+^j E_j(p)x, y\rangle p \\ &+ \int_{\Omega_j^+} d\langle \Pi_-^j E_j(p)x, y\rangle \overline{p}.\end{aligned} \tag{12.7}$$

The operator J in the above equation is the imaginary operator appearing in the Teichmüller decomposition $Z_T = A + JB$ of Z_T defined in Theorem 9.3.5 *and* Π_0 *and* $\Pi_\pm^j$ *are the associated projections defined in Definition* 9.3.10. *The operator J commutes with E and satisfies $-J^2 = E(\mathbb{H} \setminus \mathbb{R})$.*

Moreover, on identifying the complex plane $\mathbb{C}_k$ with $\mathbb{C}_j$ in the natural way by the mapping φ_{kj}, we have $E_j(\varphi_{kj}(\sigma)) = E_k(\sigma), \sigma \in \mathfrak{B}(\Omega_k^+)$, for all $j, k \in \mathbb{S}$.

Proof. The proof is broken into two steps.

Step 1: *Show that a spectral measure E_j exists such that* (12.6) *holds.*

Let $\mathbb{B} = \{p \in \mathbb{H} : |p| < 1\}$, $\partial\mathbb{B} = \{p \in \mathbb{H} : |p| = 1\}$, and $\overline{\mathbb{B}} = \mathbb{B} \cup \partial\mathbb{B}$. If T is normal, then using Properties (i) and (iii) in Theorem 12.1.1, we get that $\|Z_T\| \leq 1$ and Z_T is normal, respectively. Thus, we may use Theorem 11.2.1 to obtain a uniquely determined spectral measure F on $\sigma_S(Z_T) \cap \mathbb{C}_j^+$ such that

$$f(Z_T) = \mathbb{I}(f) = \int_{\sigma_S(Z_T)\cap\mathbb{C}_j^+} f(p)\, dF(p) \tag{12.8}$$

for $f \in \mathcal{SC}_j(\sigma_S(Z_T) \cap \mathbb{C}_j^+)$. In addition, it follows from Theorem 3.1.13 that

$$\sigma_S(Z_T) \subseteq \{p \in \mathbb{H} : \ |p| \leq \|Z_T\|\}$$

and hence

$$\sigma_S(Z_T) \cap \mathbb{C}_j^+ \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_j^+.$$

If $x \in \mathcal{H}$ and $\sigma \in \mathfrak{B}(\sigma_S(Z_T) \cap \mathbb{C}_j^+)$, then in view of item (v) in Lemma 10.1.7 and (12.8), we have

$$\langle (\mathcal{I} - Z_T^* Z_T)F(\sigma)x, F(\sigma)x\rangle = \int_\sigma (1 - |p|^2) d\langle F(p)x, x\rangle. \tag{12.9}$$

Recall that $\mathcal{I} - Z_T^* Z_T = (\mathcal{I} + T^*T)^{-1}$, and so $\ker(\mathcal{I} - Z_T^* Z_T) = \{0\}$. Thus, using (12.9) with

$$\sigma = \mathbb{B} \cap \mathbb{C}_j^+,$$

we get that $\operatorname{supp} F \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_j^+$ and $F(\partial\mathbb{B} \cap \mathbb{C}_j^+) = 0$. Therefore,

$$F(\mathbb{B} \cap \mathbb{C}_j^+) = F[(\overline{\mathbb{B}} \cap \mathbb{C}_j^+) \setminus \partial(\mathbb{B} \cap \mathbb{C}_j^+)] = \mathcal{I}.$$

If $\varphi(p) = p(1-|p|^2)^{-1/2}$, then $\varphi \in \mathcal{SM}_F^{\#}(\sigma_S(Z_T) \cap \mathbb{C}_j^+)$. In view of item (iii) and (v) of Theorem 10.2.7, we have

$$\mathbb{I}(\varphi) = \mathbb{I}(f)\mathbb{I}(g),$$

where

$$f(p) = p \quad \text{and} \quad g(p) = \frac{1}{\sqrt{1-|p|^2}},$$

and $\mathcal{D}(\mathbb{I}(\varphi)) = \mathcal{D}(\mathbb{I}(g))$. Using Theorem 10.2.9, we have

$$\mathbb{I}(g) = \mathbb{I}(1/g)^{-1}.$$

Consequently, we may use item (i) in Corollary 11.2.2 to obtain

$$\mathbb{I}(g) = \{(\mathbb{I}(h))^{1/2}\}^{-1},$$

where

$$h(p) = 1 - |p|^2 \in \mathcal{SM}_j^{\infty}(\sigma_S(Z_T) \cap \mathbb{C}_j^+).$$

Putting these observations together, we obtain

$$\mathbb{I}(\varphi) = Z_T(C_T^{1/2})^{-1}. \tag{12.10}$$

Since $Z_T = TC_T^{1/2}$, we obtain $\varphi(Z_T) \subseteq T$. Using $C_T = (\mathcal{I} - Z_T^* Z_T)^{1/2}$, we get that $\mathbb{I}(\varphi) \subseteq T$. Thus, using Lemma 9.1.17, we get that

$$\mathbb{I}(\varphi) = T.$$

Let $E_j(\sigma) = F(\varphi^{-1}(\sigma))$, where

$$\varphi^{-1}(\sigma) = \{p \in \mathbb{H} : \ \varphi(p) \in \sigma\} \quad \text{for } \sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+).$$

It is readily checked that $E_j = F(\varphi^{-1})$ defines a spectral measure on $\mathbb{C}_j^+$, and thus using Lemma 10.2.11, we get (12.6). The equivalent assertion (12.7) is established in much the same way as the analogous assertion in Theorem 11.2.1.

Since the imaginary operator J in the Teichmüller decomposition of Z_T commutes with the spectral measure F, it also commutes with $E_j = F(\varphi^{-1})$. Furthermore, since φ maps $\mathbb{R}$ into itself and $\mathbb{C}_j^+ \setminus \mathbb{R}$ into itself, we obtain

$$E_j(\mathbb{C}_j^+ \setminus \mathbb{R}) = F(\varphi^{-1}(\mathbb{C}_j^+ \setminus \mathbb{R})) = F(\mathbb{C}_j^+ \setminus \mathbb{R}) = -J^2.$$

Step 2: *Show that E_j from Step 1 is unique.*

If E_j and $\widetilde{E}_j$ are spectral measures on $\sigma_S(T) \cap \mathbb{C}_j^+$ that satisfy (12.6), then $F = E_j(\varphi)$ and $\widetilde{F} = \widetilde{E}_j(\varphi)$ are both spectral measures such that for $x, y \in \mathcal{H}$,

$$\begin{aligned}\langle Z_T\, x, y\rangle &= \int_{\overline{\mathbb{B}} \cap \mathbb{C}_j^+} \mathrm{Re}(p)\, d\langle F(p)x, y\rangle + \int_{\overline{\mathbb{B}} \cap \mathbb{C}_j^+} \mathrm{Im}(p)\, d\langle JF(p)x, y\rangle \\ &= \int_{\overline{\mathbb{B}} \cap \mathbb{C}_j^+} \mathrm{Re}(p)\, d\langle \widetilde{F}(p)x, y\rangle + \int_{\overline{\mathbb{B}} \cap \mathbb{C}_j^+} \mathrm{Im}(p)\, d\langle J\widetilde{F}(p)x, y\rangle. \end{aligned} \tag{12.11}$$

Consider now a polynomial $\Phi(p) = \sum_{0\leq|\ell|\leq n} a_\ell p^{\ell_1}\overline{p}^{\ell_2}$ with real coefficients as in (9.19). In view of Lemma 10.1.7 and Remark 10.1.8, the identity (12.11) implies

$$\begin{aligned}\langle\psi(Z_T)x, y\rangle &= \int_{\sigma_S(Z_T)\cap\mathbb{C}_j^+} \psi(p)d\langle F(p)x, y\rangle \\ &= \int_{\sigma_S(Z_T)\cap\mathbb{C}_j^+} \psi(p)d\langle \widetilde{F}(p)x, y\rangle.\end{aligned}$$

Since the set of polynomials of this type is by Theorem 9.4.5 dense in $\mathcal{SC}_j(\sigma_S(Z_T)\cap\mathbb{C}_j^+)$, we have that

$$\int_{\sigma_S(Z_T)\cap\mathbb{C}_j^+} \phi(p)d\langle F(p)x, x\rangle = \int_{\sigma_S(Z_T)\cap\mathbb{C}_j^+} \phi(p)d\langle \widetilde{F}(p)x, x\rangle$$

for all $\phi \in \mathcal{SC}_j(\sigma_S(Z_T)\cap\mathbb{C}_j^+)$. Hence in view of construction of the spectral measure given in Section 11, $F = \widetilde{F}$. Therefore, $E_j = \widetilde{E}_j$. The final assertion concerning E_j and E_k is proved in a similar manner to an analogous assertion in Theorem 11.2.1. □

12.3 Some Consequences of the Spectral Theorem

We conclude this chapter with some consequences of the spectral theorem for unbounded normal operators, Moreover, in the last corollary we state the functional calculus for unbounded normal operators, which is a direct consequence of the definition and the properties of the spectral integrals, which depend of the operator J.

Corollary 12.3.1. *In the setting of Theorem 12.2.1, the following statements hold:*

(i) *If $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, then there exists a unique positive operator $W \in \mathcal{L}(\mathcal{H})$ such that $W^2 = T$.*

(ii) *$T \in \mathcal{L}(\mathcal{H})$ is self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{\mathbb{R}} t\, d\langle E(t)x, y\rangle, \quad x \in \mathcal{D}(T),\ \ y \in \mathcal{H}. \tag{12.12}$$

(iii) *$T \in \mathcal{L}(\mathcal{H})$ is anti-self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{[0,\infty)} t\, d\langle JE(t)x, y\rangle, \quad x \in \mathcal{D}(T),\ \ y \in \mathcal{H}. \tag{12.13}$$

Proof. Using Theorem 12.2.1, the proof is completed as in Corollary 11.2.2. □

Remark 12.3.2. We remind the reader that the functional calculus mentioned in Section 10 is applicable to unbounded normal operators $T \in \mathcal{L}(\mathcal{H})$. We conclude this section by stating, in the following corollary, such a functional calculus.

Corollary 12.3.3. *Let T, E_j, and J be as in Theorem 12.2.1. If $f, g \in \mathcal{SM}_E^{\#}(\Omega_j^+)$ with $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ and $\alpha, \beta \in \mathbb{R}$, then:*

(i) $\mathbb{I}(\bar{f}) = \mathbb{I}(f)^*$.

(ii) $\mathbb{I}(\alpha f + \beta g) = \overline{\alpha \mathbb{I}(f) + \beta \mathbb{I}(g)}$.

(iii) $\mathbb{I}(fg) = \overline{\mathbb{I}(f)\mathbb{I}(g)}$.

(iv) $\mathbb{I}(f)$ *is a closed normal operator on $\mathcal{H}$ and*

$$\mathbb{I}(f)^*\mathbb{I}(f) = \mathbb{I}(f\bar{f}) = \mathbb{I}(\bar{f}f).$$

(v) $\mathcal{D}(\mathbb{I}(f)\mathbb{I}(g)) = \mathcal{D}(\mathbb{I}(g)) \cap \mathcal{D}(\mathbb{I}(fg))$.

(vi) *If $x \in \mathcal{D}(\mathbb{I}(f))$ and $y \in \mathcal{D}(\mathbb{I}(g))$, then*

$$\langle \mathbb{I}(f)x, \mathbb{I}(g)y\rangle = \int_{\Omega_j^+} \mathrm{Re}(f(p)\overline{g(p)})d\langle E(p)x, y\rangle + \int_{\Omega_j^+} \mathrm{Im}(f(p)\overline{g(p)})d\langle JE(p)x, y\rangle.$$

(vii) *If $x \in \mathcal{D}(\mathbb{I}(f))$, then*

$$\|\mathbb{I}(f)x\|^2 = \int_{\Omega_j^+} |f(p)|^2 d\langle E(p)x, x\rangle.$$

Theorem 12.3.4. *Let T be as in Theorem 12.2.1 and let J be the imaginary operator in the Teichmüller decomposition of Z_T. Then there exist strongly commuting operators A and B that commute with J, where $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint and $B \in \mathcal{L}(\mathcal{H})$ is positive with* $\ker B = \ker J$ *such that*

$$T = A + JB. \tag{12.14}$$

Proof. To verify assertion (iv), let E be the spectral measure of T and define

$$A\,x = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \mathrm{Re}(p)\,dE(p)x, \quad x \in \mathcal{D}(T),$$

$$B\,x = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \mathrm{Im}(p)\,dE(p)x, \quad x \in \mathcal{D}(T).$$

If we set $E_0(\sigma) =: E(\{z \in \mathbb{C}_j^+ : \mathrm{Re}(p) \in \sigma\})$ and $E_1(\sigma) := E(\{z \in \mathbb{C}_j^+ : \mathrm{Im}(p) \in \sigma\})$ for $\sigma \in \mathfrak{B}(\mathbb{R})$, then the change of measure principle implies

$$A\,x = \int_{\mathbb{R}} t\,dE_0(t)x, \quad x \in \mathcal{D}(T),$$

$$B\,x = \int_0^{+\infty} t\,dE_1(t)x, \quad x \in \mathcal{D}(T).$$

Hence A and B are self-adjoint, and their spectral measures are E_0 and E_1. Since all projections $E(\sigma)$ with $\sigma \in \mathfrak{B}(\mathbb{C}_j^+)$ commute mutually and with J, we find that also E_0 and E_1 commute mutually and with J. Hence, A and B commute strongly, and they commute with J. Finally, we have

$$\ker B = \operatorname{ran} E_1(\{0\}) = \operatorname{ran} E(\{z \in \mathbb{C}_j^+ : \ \operatorname{Im}(z) = 0\}) = \operatorname{ran} E(\mathbb{R}) = \ker J. \qquad \square$$

Theorem 12.3.5 (Spectral mapping theorem)**.** *Let T be as in Theorem 12.2.1 and let $f \in \mathcal{SC}(\sigma_S(T))$. Then*

$$\sigma_S(f(T)) = \overline{f(\sigma_S(T))}. \tag{12.15}$$

Proof. First of all, observe that $\overline{f(\sigma_S(T))}$ is an axially symmetric set because $\sigma_S(T)$ is axially symmetric and f maps axially symmetric sets to axially symmetric sets since it is intrinsic. Let $\lambda \in \overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+$, let $\varepsilon > 0$, and choose $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon}(\tilde{\varepsilon} + 2|\operatorname{Im}(\lambda)|) < \frac{\varepsilon}{2}.$$

We can then find $z_\varepsilon \in \sigma_S(T)$ such that

$$|\lambda - f(z_\varepsilon)| < \tilde{\varepsilon},$$

and since $\lambda \in \mathbb{C}_j^+$ and f maps each complex plane $\mathbb{C}_i$ into itself, we even find that $z_\varepsilon \in \sigma_S(T) \cap \mathbb{C}_j$. (The function f, however, does not necessarily map each half-plane $\mathbb{C}_i^+$ into itself, and hence z_ε might belong to $\mathbb{C}_j^-$. In this case, $\overline{z_\varepsilon} \in \mathbb{C}_j^+$.) Then

$$\begin{aligned}\left|f(z_\varepsilon)^2 - 2\operatorname{Re}(\lambda) f(z_\varepsilon) + |\lambda|^2\right| &= |f(z_\varepsilon) - \lambda| \left|f(z_\varepsilon) - \overline{\lambda}\right| \\ &\leq |f(z_\varepsilon) - \lambda| \, |f(z_\varepsilon) - \lambda| \left|\lambda - \overline{\lambda}\right| < \tilde{\varepsilon}(\tilde{\varepsilon} + 2|\operatorname{Im}(\lambda)|) < \frac{\varepsilon}{2}.\end{aligned}$$

The map $z \mapsto \mathcal{Q}_\lambda(f(z)) := f(z)^2 - 2\operatorname{Re}(\lambda) f(z) + |\lambda|^2$ is continuous, and hence there exists $\delta > 0$ such that for $z \in \mathbb{C}_j$ with $|z - z_\varepsilon| < \delta$, we have

$$|\mathcal{Q}_\lambda(f(z)) - \mathcal{Q}_\lambda(f(z_\varepsilon))| < \frac{\varepsilon}{2}$$

and in turn

$$|\mathcal{Q}_\lambda(f(z))| \leq |\mathcal{Q}_\lambda(f(z)) - \mathcal{Q}_\lambda(f(z_\varepsilon))| + |\mathcal{Q}_\lambda(f(z_\varepsilon))| < \varepsilon.$$

Moreover,

$$|\mathcal{Q}_\lambda\left(f\left(\overline{z}\right)\right)| = \left|\mathcal{Q}_\lambda\left(\overline{f\left(z\right)}\right)\right| = \left|\overline{\mathcal{Q}_\lambda\left(f\left(z\right)\right)}\right| = |\mathcal{Q}_\lambda\left(f\left(z\right)\right)| < \varepsilon.$$

If $z_{\varepsilon,+} := [z] \cap \mathbb{C}_j^+$, that is, $z_{\varepsilon,+} = z_\varepsilon$ if $z_\varepsilon \in \mathbb{C}_j^+$ and $z_{\varepsilon,+} = \overline{z_\varepsilon}$ if $z_\varepsilon \in \mathbb{C}_j^-$, it follows that

$$\begin{aligned}U_\delta &:= \{z \in \sigma_S(T) \cap \mathbb{C}_j^+ : |z - z_{\varepsilon,+}| < \delta\} \\ &\subset \sigma_\varepsilon(\lambda) := \{z \in \sigma_S(T) \cap \mathbb{C}_j^+ : |\mathcal{Q}_z(f(z))| < \varepsilon\}.\end{aligned}$$

Since U_δ is an open set in $\sigma_S(T) \cap \mathbb{C}_j^+$, which is exactly the support of E, we find that $E(U_\delta) \neq 0$ and hence also $E(\sigma_\varepsilon) \neq 0$. We conclude from Lemma 10.2.10 that $\lambda \in \sigma_S(f(T))$, and so

$$\overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+ \subset \sigma_S(f(T)) \cap \mathbb{C}_j^+.$$

On the other hand, if $\lambda \notin \overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+$, then

$$\begin{aligned}\sigma_\varepsilon(\lambda) &= \left\{z \in \sigma_S(T) \cap \mathbb{C}_j^+ : \ \ |\mathcal{Q}_\lambda(f(z))| < \varepsilon\right\} \\ &\subset \{z \in \sigma_S(T) \cap \mathbb{C}_j : \ \ |\mathcal{Q}_\lambda(f(z))| < \varepsilon\}\end{aligned}$$

is empty for $\varepsilon > 0$ sufficiently small. Thus, Lemma 10.2.10 yields that $\lambda_0 \notin \sigma_S(f(T)) \cap \mathbb{C}_j^+$. We conclude that

$$\overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+ \supset \sigma_S(f(T)) \cap \mathbb{C}_j^+,$$

and in turn,

$$\overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+ = \sigma_S(f(T)) \cap \mathbb{C}_j^+.$$

Taking the axially symmetric hull, we arrive at (12.15). □

12.4 Comments and Remarks

Several papers have appeared in the literature that claimed to introduce a spectral theorem for normal operators on a quaternionic Hilbert space (see [107, 109, 195, 197]). However, in all of the aforementioned papers, a precise notion of spectrum is not made clear. We will now enter into a discussion concerning the papers of Teichmüller [195] and Viswanath [197].

Teichmüller's paper [195] was the first to claim a spectral theorem for normal operators; it appeared in 1936. Despite not making the notion of spectrum clear, [195] does have a number of valid and important observations (even though some details for the precise proofs may be missing) such as the decomposition $T = A + JB$ (see Theorem 9.3.5) and also the fact that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ (see (9.18)). Finally, the spectral resolution in [195] takes the form

$$N = \int_{-\infty}^{\infty} \int_0^{\infty} (\lambda' + T_0 \lambda'') dQ_{\lambda''} dP_{\lambda'}, \tag{12.16}$$

where N is a normal operator, T_0 is an "Imaginäroperator" on $\overline{\operatorname{ran} B}$, i.e., $T_0 T_0^* = \mathcal{I}_{\operatorname{ran} B}$ and $T_0^* = -T_0$ (thus T_0 is playing the role of the operator J in Theorem 12.2.1), and Q and P are projection-valued measures. This bears some resemblance to (11.15).

In 1971 the paper [197] of Viswanath also claimed to have a spectral theorem for normal operators on a quaternionic Hilbert space. It is worth noting that [195] is not cited in Viswanath's paper [197]. The approach of [197] is very different from [195] in so far as the symplectic image of a normal operator is used and the spectral theorem is allegedly deduced from the classical spectral theorem and some kind of lifting argument. Viswanath's spectral resolution takes the form

$$T = \int_{\mathbb{C}_+} \lambda dE, \tag{12.17}$$

where T is a normal operator, E is a projection-valued measure E. Viswanath claims to deduce an antecedent to the decomposition in Theorem 9.3.5 from (12.17). However, the details are not given.

Beyond the spectral theorem there is the theory of the characteristic operator function, which was initiated in [28].

On the equivalent formulations of complex and quaternionic quantum mechanics see [126]. For recent applications of the spectral theory on the S-spectrum to quantum mechanics see [170, 171] and also [168, 196]. For coherent state transforms and the Weyl equation in Clifford analysis, see [169].