Chapter 11

The Spectral Theorem for Bounded Normal Operators

In this chapter we prove the spectral theorem for bounded normal operators T in $\mathcal{B}(\mathcal{H})$. Our approach has analogies with the well-known approach for complex bounded normal operators on a complex Hilbert space, see for example [163], but it has to take into account the axially symmetric structure of the S-spectrum of T and the (A, J, B)-decomposition $T = A + JB$ of the quaternionic bounded normal operators. As we will see, the spectral measures E are constructed using just the two self-adjoint operators A and B, and only later, we take into account the imaginary operator J for the spectral representation of T. We present the original proof from [13] using the Teichmüller decomposition $T = A + JB$. The following representation theorems will be used in the sequel.

Theorem 11.0.1 (Riesz representation theorem for real-valued functions)**.** *Let X be a compact Hausdorff space and let $\mathcal{C}(X, \mathbb{R})$ denote the normed space of real-valued continuous functions on X together with the supremum norm $\|\cdot\|_\infty$. Corresponding to every bounded linear functional $\psi : \mathcal{C}(X, \mathbb{R}) \to \mathbb{R}$ there exists a signed Borel measure μ on X such that*

$$\psi(f) = \int_X f(t)d\mu(t) \quad \text{for all } f \in \mathcal{C}(X, \mathbb{R}). \tag{11.1}$$

If, in addition, ψ is a positive linear functional, then μ is a positive Borel measure on X. In both cases μ is unique.

For a proof of Theorem 11.0.1 we refer to Theorem D in Section 56 of [157] for the case in which ψ is a positive bounded linear functional on X and, e.g., Chapter 21 in [182] for the more general case.

Theorem 11.0.2 (Riesz representation theorem for quaternionic Hilbert spaces)**.** *Let $\mathcal{H}$ be a quaternionic right Hilbert space with quaternionic inner product $\langle \cdot, \cdot \rangle$,*

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and let φ be a continuous right linear functional on $\mathcal{H}$. Then there exists a unique $y_\varphi \in \mathcal{H}$ such that

$$\varphi(x) = \langle x, y_\varphi \rangle, \quad \text{for all } x \in \mathcal{H}.$$

Theorem 11.0.2 can be found in [33]. We also want to mention Proposition 1.10 in [167] for a statement and proof in a more general Clifford algebra setting.

11.1 Construction of the Spectral Measure

We can now construct the spectral measures. We will use the Riesz representation theorem for continuous real-valued functions and the Riesz representation theorem for quaternionic Hilbert spaces.

In this chapter we consider a normal operator $T \in \mathcal{B}(\mathcal{H})$ and fixed imaginary unit $j \in \mathbb{S}$ and define $\Omega = \sigma_S(T)$ and

$$\Omega_j^+ := \Omega \cap \mathbb{C}_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+.$$

We recall that $\mathcal{C}(\Omega_j^+, \mathbb{R})$ denotes the space of continuous real-valued functions on Ω_j^+. By Lemma 9.4.3, every function $f_j \in \mathcal{C}(\Omega_j^+, \mathbb{R})$ is the restriction $f_j = f|_{\Omega_j^+}$ of a real-valued continuous slice function f on $\Omega = \sigma_S(T)$. We denote the set of continuous real-valued slice functions on Ω by $\mathcal{SC}(\Omega, \mathbb{R})$, and in the following, we do not distinguish between the function f_j and the function f unless that could cause confusion.

We consider for $x \in \mathcal{H}$ the mapping

$$\ell_x(g) = \langle g(T)x, x \rangle, \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}) \cong \mathcal{SC}(\Omega, \mathbb{R}),$$

where $g(T)$ is the operator obtained by the continuous function calculus introduced in Theorem 9.4.11, where $g(T)$ stands for $F_0(T)$ and $F_1(T)$. Since T is a bounded operator, its S-spectrum $\sigma_S(T)$ is a compact and nonempty set. It is readily checked that ℓ_x is a real-valued bounded linear functional on $\mathcal{C}(\Omega_j^+, \mathbb{R})$. Moreover, ℓ_x is a positive functional. Indeed, if h is a continuous nonnegative function on Ω_j^+, then we can consider the function $g(u,v) = \sqrt{h(u,v)}$ and find $g \in \mathcal{C}(\Omega_j^+, \mathbb{R})$ with $g(T) = g(T)^*$. Thus

$$\ell_x(h) = \langle h(T)x, x \rangle = \langle g(T)x, g(T)x \rangle = \|g(T)x\|^2 \geq 0.$$

Theorem 11.0.1 yields the existence of a uniquely determined positive-valued measure μ_x on the Borel sets $\mathfrak{B}(\Omega_j^+)$, so that

$$\ell_x(g) = \int_{\Omega_j^+} g(p)\, d\mu_x(p), \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}). \tag{11.2}$$

In view of (11.2), we may use the formula

$$\begin{aligned}4\langle g(T)x, y\rangle &= \langle g(T)(x+y), x+y\rangle - \langle g(T)(x-y), x-y\rangle \\ &\quad + e_1\langle g(T)(x+ye_1), x+ye_1\rangle - e_1\langle g(T)(x-ye_1), x-ye_1\rangle \\ &\quad + e_1\langle g(T)(x-ye_2), x-ye_2\rangle e_3 - e_1\langle g(T)(x+ye_2), x+ye_2\rangle e_3 \\ &\quad + \langle g(T)(x+ye_3), x+ye_3\rangle e_3 - \langle g(T)(x-ye_3), x-ye_3\rangle e_3, \quad (11.3)\end{aligned}$$

where $\{1, e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{H}$, to obtain for every $x, y \in \mathcal{H}$ a uniquely determined $\mathbb{H}$-valued measure $\mu_{x,y}$ such that

$$\langle g(T)x, y\rangle = \int_{\Omega_j^+} g(p)d\mu_{x,y}(p), \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}), \tag{11.4}$$

where

$$\begin{aligned}4\mu_{x,y} &= \mu_{x+y} - \mu_{x-y} + e_1\mu_{x+ye_1} - e_1\mu_{x-ye_1} \\ &\quad + e_1\mu_{x-ye_2}e_3 - e_1\mu_{x+ye_2}e_3 + \mu_{x+ye_3}e_3 - \mu_{x-ye_3}e_3.\end{aligned} \tag{11.5}$$

Lemma 11.1.1. *Let $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{H}$. The $\mathbb{H}$-valued measures $\mu_{x,y}$ given in (11.5) enjoy the following properties*

(i) $\mu_{x\alpha+y\beta,z} = \mu_{x,z}\alpha + \mu_{y,z}\beta$,

(ii) $\mu_{x,y\alpha+z\beta} = \bar{\alpha}\mu_{x,y} + \bar{\beta}\mu_{x,z}$,

(iii) $|\mu_{x,y}(\Omega_j^+)| \leq \|x\|\|y\|$,

(iv) $\bar{\mu}_{x,y} = \mu_{y,x}$.

Proof. Properties (i)–(iii) are easily obtained from (11.4) using the uniqueness of $\mu_{x,y}$ and the properties of $\langle\cdot,\cdot\rangle$. Property (iv) follows from properties (i) and (ii). □

It follows from properties (i) and (iii) in Lemma 11.1.1 that for every fixed $y \in \mathcal{H}$ and every fixed $\sigma \in \mathfrak{B}(\Omega_j^+)$, the mapping

$$\Phi_y(x) = \mu_{x,y}(\sigma)$$

is a continuous right linear functional on $\mathcal{H}$. Moreover, it follows from property (ii) in Lemma 11.1.1 that

$$\Phi_{y\alpha}(x) = \overline{\alpha}\,\Phi_y(x), \quad \alpha \in \mathbb{H}.$$

It follows from the Riesz representation theorem for quaternionic Hilbert spaces, see Theorem 11.0.2, that corresponding to every $x \in \mathcal{H}$, there exists a unique vector $w \in \mathcal{H}$ such that

$$\Phi_y(x) = \langle x, w\rangle, \tag{11.6}$$

i.e., $\mu_{x,y}(\sigma) = \langle x, w\rangle$. Since the left-hand side of (11.6) depends linearly on x and anti-linearly on y and the right-hand side depends linearly on x, it follows that $\Phi_y(x)$ depends linearly on x and anti-linearly on y, so

$$E(\sigma)y = w,$$

for some operator

$$E(\sigma) \in \mathcal{B}(\mathcal{H}).$$

Thus,

$$\mu_{x,y}(\sigma) = \langle x, E(\sigma)y\rangle, \quad \sigma \in \mathfrak{B}(\Omega_j^+),$$

and in view of property (iv) in Lemma 11.1.1,

$$E(\sigma) = E(\sigma)^*, \quad \sigma \in \mathfrak{B}(\Omega_j^+), \tag{11.7}$$

and hence

$$\mu_{x,y}(\sigma) = \langle E(\sigma)x, y\rangle, \quad \sigma \in \mathfrak{B}(\Omega_j^+). \tag{11.8}$$

Since μ_x is countably additive, $\mu_{x,y}$ is also countably additive. Consequently, the $\mathcal{B}(\mathcal{H})$-valued measure E is also countably additive, i.e.,

$$E\left(\bigcup_{n=0}^{+\infty} \sigma_n\right) = \sum_{n=0}^{+\infty} E(\sigma_n) \tag{11.9}$$

for every sequence of pairwise disjoint sets $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathfrak{B}(\Omega_j^+)$. The limit in (11.9) is intended with respect to the strong operator topology.

We recall that $\mathcal{SC}(\Omega)$ denotes the space of all continuous intrinsic slice functions on Ω, and we denote by

$$\mathcal{SC}_j(\Omega_j^+) := \{f_j := f|_{\Omega_j^+} : \ f \in \mathcal{SC}(\Omega)\}$$

the set of all restrictions of functions in $\mathcal{SC}(\Omega)$. Again we do not distinguish between a function f and its restriction f_j unless this could cause confusion.

Lemma 11.1.2. *Let J be the imaginary component in the $T = A+JB$ decomposition (9.17) of the normal operator $T \in \mathcal{B}(\mathcal{H})$ and let E be the spectral measure on $\mathfrak{B}(\Omega_j^+)$ with $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ defined above. The following statements hold:*

(i) *If $g \in \mathcal{C}(\Omega_j^+, \mathbb{R}) \cong \mathcal{SC}(\Omega, \mathbb{R})$, then for all $x, y \in \mathcal{H}$*

$$\langle g(T)x, y\rangle = \int_{\Omega_j^+} g(p)\, d\langle E(p)x, y\rangle. \tag{11.10}$$

(ii) *If $f = f_0 + jf_1 \in \mathcal{SC}_j(\Omega_j^+) \cong \mathcal{SC}(\Omega)$, then we have for all $x, y \in \mathcal{H}$,*

$$\langle f(T)x, y\rangle = \int_{\Omega_j^+} f_0(p)\, d\langle E(p)x, y\rangle + \int_{\Omega_j^+} f_1(p)\, d\langle JE(p)x, y\rangle. \tag{11.11}$$

(iii) *$E(\sigma)$ and J commute for all $\sigma \in \mathfrak{B}(\Omega_j^+)$ and $-J^2 = E(\Omega_j^+ \setminus \mathbb{R})$.*

Proof. Assertion (i) follows directly from (11.4) and (11.8). We will now prove assertion (11.11). In view of (11.10) and Theorems 9.4.9 and 9.4.11, we have

$$\begin{aligned}\langle f(T)x, y\rangle &= \langle \{f_0(T) + f_1(T)J\}x, y\rangle \\ &= \langle f_0(T)x, y\rangle + \langle f_1(T)Jx, y\rangle \\ &= \int_{\Omega_j^+} f_0(p)d\langle E(p)x, y\rangle + \int_{\Omega_j^+} f_1(p)d\langle E(p)Jx, y\rangle, \quad x, y \in \mathcal{H}.\end{aligned}$$

Thus, the proof of (11.11) will be complete on showing that

$$d\langle E(p)Jx, y\rangle = d\langle JE(p)x, y\rangle, \quad x, y \in \mathcal{H}.$$

To see this, let $g \in \mathcal{C}(\Omega_j^+, \mathbb{R})$ and use (11.10) and the fact that $g(T)$ and J commute to obtain

$$\int_{\Omega_j^+} g(p)d\langle E(p)Jx, y\rangle = \langle g(T)Jx, y\rangle = \langle Jg(T)x, y\rangle = \int_{\Omega_j^+} g(p)d\langle JE(p)x, y\rangle.$$

If we write $\nu = \langle E(p)Jx, y\rangle$ and $\widetilde{\nu} = \langle JE(p)x, y\rangle$ and then

$$\nu = \nu_0 \mathrm{e}_0 + \nu_1 \mathrm{e}_1 + \nu_2 \mathrm{e}_2 + \nu_3 \mathrm{e}_3$$

and

$$\widetilde{\nu} = \widetilde{\nu}_0 \mathrm{e}_0 + \widetilde{\nu}_1 \mathrm{e}_1 + \widetilde{\nu}_2 \mathrm{e}_2 + \widetilde{\nu}_3 \mathrm{e}_3,$$

where ν_ℓ and $\widetilde{\nu}_\ell$, $\ell = 0, \dots, 3$, are real signed measures and $e_\ell, = 0, \dots, 3$ is the standard basis for $\mathbb{H}$, then it follows from Theorem 11.0.1 that $\nu_\ell = \widetilde{\nu}_\ell$ for $\ell = 0, \dots, 3$. Therefore, items (iii) and (ii) hold.

Finally, we have due to (i) and due to Lemma 10.1.7(iii) that

$$BE(\mathbb{R}) = \int_{\Omega_j^+} |\mathrm{Im}(q)| \, dE(q)E(\mathbb{R}) = \int_{\Omega_j^+} |\mathrm{Im}(q)| \chi_{\mathbb{R}} \, dE(q) = 0,$$

so that

$$\operatorname{ran} E(\mathbb{R}) \subset \ker B = \ker J,$$

where B is the positive operator in the decomposition $T = A + JB$. If, on the other hand, $x \in \ker J = \ker B$, then

$$0 = \langle Bx, x\rangle = \int_{\Omega_j^+} |\mathrm{Im} q|^2 d\mu_{x,x}(q).$$

Since the measure $\mu_{x,x}(\sigma) = \langle E(\sigma)x, x\rangle$ and the function $\varphi(q) := |\mathrm{Im}(q)|^2$ are nonnegative, this implies

$$0 = \mu_{x,x}(\varphi^{-1}(\mathbb{R} \setminus \{0\}) = \mu_{x,x}(\Omega_j^+ \setminus \mathbb{R}) = \langle x, E(\Omega_j^+ \setminus \mathbb{R})x\rangle = \|E(\Omega_j^+ \setminus \mathbb{R})\|^2.$$

Hence $E(\Omega_j^+ \setminus \mathbb{R})x = 0$, and in turn, $x \in \operatorname{ran} E(\mathbb{R})$. Therefore,

$$\operatorname{ran} E(\mathbb{R}) \supset \ker B = \ker J,$$

and in turn,

$$\operatorname{ran} E(\mathbb{R}) = \ker J.$$

Since $-J^2$ is the orthogonal projection onto $(\ker J)^\perp = \operatorname{ran} J$ by Corollary 9.3.8 and $E(\Omega_j^+ \setminus \mathbb{R})$ is the orthogonal projection onto $(\operatorname{ran} E(\mathbb{R}))^\perp$, we conclude that $-J^2 = E(\Omega_j^+ \setminus \mathbb{R})$. □

The properties of the spectral measure can be checked directly as in the following result.

Theorem 11.1.3. *The $\mathcal{B}(\mathcal{H})$-valued countably additive measure E, given by (11.8), for all $\sigma, \tau \in \mathfrak{B}(\Omega_j^+)$, enjoys the following properties:*

(i) $E(\sigma) = E(\sigma)^*$.

(ii) $\|E(\sigma)\| \leq 1$.

(iii) $E(\emptyset) = 0$ *and* $E(\sigma_S(T) \cap \mathbb{C}_j^+) = \mathcal{I}$.

(iv) $E(\sigma \cap \tau) = E(\sigma)E(\tau)$.

(v) $E(\sigma)^2 = E(\sigma)$.

(vi) $E(\sigma)$ *commutes with* $f(T)$ *for all* $f \in \mathcal{SC}_j(\Omega_j^+) \cong \mathcal{SC}(\Omega)$.

(vii) $E(\sigma)$ *and* $E(\tau)$ *commute.*

Proof. The proof is broken into steps.

Step 1: *Show* (i) and (ii).

Property (i) has already been noted in (11.7). Property (ii) follows directly from property (iii) in Lemma 11.1.1. Indeed, if $x = y$ in property (iii) in Lemma 11.1.1, then

$$\mu_{x,x}(\sigma) \leq \mu_{x,x}(\Omega_j^+) \leq \|x\|^2$$

and hence

$$\langle E(\sigma)x, x\rangle \leq \|x\|^2 \quad \text{for } x \in \mathcal{H},$$

i.e., $\mathcal{I} - E(\sigma)$ is a positive operator for all $\sigma \in \mathfrak{B}(\Omega_j^+)$. Therefore, property (ii) holds.

Step 2: *Show* (iii).

Since $\mu_{x,y}(\emptyset) = 0$, we may use (11.4) to deduce $E(\emptyset) = 0$. Similarly, putting $g(p) = 1$ in (11.4) yields $g(T) = \mathcal{I}$ for all $x, y \in \mathcal{H}$ and thus

$$\langle x, y\rangle = \int_{\Omega_j^+} d\mu_{x,y} = \langle E(\Omega_j^+)x, y\rangle,$$

i.e., $E(\Omega_j^+) = \mathcal{I}$.

Step 3: *Show* (iv).

Recall that for all real-valued polynomials ϕ and ψ on Ω_j^+, we have set $\phi(T) := \phi(A,B)$ and $\psi(T) := \psi(A,B)$. Clearly we have $(\phi\psi)(T) = \phi(T)\psi(T)$, $\phi(T) = \phi(T)^*$, and $\psi(T) = \psi(T)^*$. Thus,

$$\begin{aligned}\int_{\sigma_S(T)\cap\mathbb{C}_j^+} \phi(p)\, d\mu_{\psi(T)x,x}(p) &= \langle \phi(T)\psi(T)x, x\rangle \\ &= \langle (\phi\psi)(T)x, x\rangle = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \phi(p)\psi(p) d\mu_{x,x}(p).\end{aligned} \tag{11.12}$$

Since $E(\sigma) = E(\sigma)^*$, (11.8) implies that

$$\mu_{x,x}(\sigma) \in \mathbb{R} \text{ for all } \sigma \in \mathfrak{B}(\Omega_j^+).$$

Similarly, since $\langle \psi(T)x, x\rangle$ is real, (11.8) implies that

$$\mu_{\psi(T)x,x}(\sigma) \in \mathbb{R} \text{ for all } \sigma \in \mathfrak{B}(\sigma_S(T)\cap\mathbb{C}_j^+).$$

In view of the density of real-valued polynomials in the space $\mathcal{C}(\Omega_j^+, \mathbb{R})$ and the Riesz representation theorem given in Theorem 11.0.1, (11.12) implies that

$$d\mu_{\psi(T)x,x}(p) = \psi(p) d\mu_{x,x}(p).$$

But then we may use the identity (11.5) and the fact that $\psi(p)$ is real-valued to obtain

$$d\mu_{\psi(T)x,y}(p) = \psi(p) d\mu_{x,y}(p).$$

Thus, in view of (11.8),

$$\langle E(\sigma)\psi(T)x, y\rangle = \int_\sigma \psi(p)\, d\mu_{x,y}(p) \quad \text{for } \sigma \in \mathfrak{B}(\Omega_j^+).$$

Since $E(\sigma) = E(\sigma)^*$ for $\sigma \in \mathfrak{B}(\Omega_j^+)$,

$$\begin{aligned}\int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\, d\mu_{x,E(\sigma)y} &= \langle \psi(T)x, E(\sigma)y\rangle \\ &= \langle E(\sigma)\psi(T)x, y\rangle = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\, \chi_\sigma\, d\mu_{x,y},\end{aligned}$$

where

$$\chi_\sigma(p) = \begin{cases} 1 & \text{if} \quad p \in \sigma, \\ 0 & \text{if} \quad p \notin \sigma. \end{cases}$$

Since ψ is real-valued, we also have

$$\int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\, d\mu^{(m)}_{x,E(\sigma)y} = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\,\chi_\sigma\, d\mu^{(m)}_{x,y} \quad \text{for } m = 0,\ldots,3, \tag{11.13}$$

where $\mu^{(m)}_{x,y}$ and $\mu^{(m)}_{x,E(\sigma)y}$ are real-valued signed measures given by

$$\mu_{x,y} = \sum_{m=0}^{3} \mu^{(m)}_{x,y} e_m$$

and

$$\mu^{(m)}_{x,E(\sigma)y} = \sum_{m=0}^{3} \mu^{(m)}_{x,E(\sigma)y} e_m.$$

Recall that $(e_m)_{\ell=0,\ldots,3}$ is the standard basis for $\mathbb{H}$.

In view of the density of real-valued polynomials in the space $\mathcal{C}(\Omega_j^+,\mathbb{R})$ and the Riesz representation theorem given in Theorem 11.0.1, the identity (11.13) implies that

$$d\mu^{(m)}_{x,E(\sigma)y} = \chi_\sigma\, d\mu^{(m)}_{x,y} \quad \text{for } m = 0,\ldots,3,$$

and hence

$$d\mu_{x,E(\sigma)y} = \chi_\sigma\, d\mu_{x,y}.$$

Therefore,

$$\mu_{x,E(\sigma)y}(\tau) = \int_{\Omega_j^+\cap\tau} \chi_\sigma d\mu_{x,y} = \mu_{x,y}(\sigma\cap\tau)$$

for $\sigma,\tau \in \mathfrak{B}(\Omega_j^+)$. Since

$$\mu(\sigma) = \langle E(\sigma)x, y\rangle \quad \text{for } \sigma \in \mathfrak{B}(\Omega_j^+),$$

we obtain $E(\sigma)E(\tau) = E(\sigma\cap\tau)$ for $\sigma,\tau \in \mathfrak{B}(\Omega_j^+)$.

Step 4: *Show* (v).

Property (v) can be obtained from Property (iv) when $\sigma = \tau$.

Step 5: *Show* (vi).

Let A, B, and J be as in Theorem 9.3.5. We have already observed in item (iii) of Lemma 11.1.2 that $E(\sigma)$ and J commute. One can show in a similar fashion that A and $E(\sigma)$ commute and B and $E(\sigma)$ commute. Thus, in view of the construction of $f(T)$, we have that $f(T)$ and $E(\sigma)$ commute.

Step 6: *Show* (vii).

Property (vii) follows from Property (iv) on interchanging τ and σ. □

Remark 11.1.4. The spectral measure E was constructed using only operators $g(T)$ that were generated by functions $g \in \mathcal{C}(\Omega_j^+, \mathbb{R})$, that is, by real-valued functions. By Theorem 9.4.11, for such functions, the operator $g(T)$, however, does not depend on all the information we have about T, but only on the factors A and B in the $T = A + JB$ decomposition of T. Hence E is actually a joint spectral measure of the self-adjoint operators A and B. This in particular implies that $T = A + JB$ and $T^* = A - JB$ have the same spectral measure E.

In the quaternionic setting, invariant subspaces are not associated with individual eigenvalues, but with spheres $[s]$ of equivalent eigenvalues, because the eigenvalue equation $T(x) - xs = 0$ associated with a single (nonreal) eigenvalue is not linear. The correct interpretation of the above observation is therefore that the spectral measure E associates invariant subspaces of T to sets of spectral spheres, while the imaginary operator J orients the spheres. It determines how the different spectral values in these spheres need to be multiplied onto the vectors in the associated subspaces in order to fit the operator T. A more detailed discussion of this idea will be given in Chapter 14.

11.2 The Spectral Theorem and Some Consequences

We conclude this chapter with the main result, the spectral theorem for bounded operators.

Theorem 11.2.1 (The spectral theorem for bounded normal operators)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be normal, let $J \in \mathcal{B}(\mathcal{H})$ be the imaginary operator in the Teichmüller decomposition $T = A + JB$ of Theorem 9.3.5, and fix $j \in \mathbb{S}$. Let $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ and let Π_0 and $\Pi_\pm^j$ denote the orthogonal $\mathbb{C}_j$-linear projections defined in Definition 9.3.10 corresponding to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ given in Lemma 9.3.9.*

Then there exists a unique spectral measure E_j on Ω_j^+ such that for all $x, y \in \mathcal{H}$,

$$\langle Tx, y \rangle = \int_{\Omega_j^+} \mathrm{Re}(q)\, d\langle E(q)x, y \rangle + \int_{\Omega_j^+} \mathrm{Im}(q)\, d\langle E(q)Jx, y \rangle. \tag{11.14}$$

For every function $f = f_0 + jf_1 \in \mathcal{SC}_j(\Omega_j^+)$ and $x, y \in \mathcal{H}$, we moreover have

$$\langle f(T)x, y \rangle = \int_{\Omega_j^+} f_0(p)\, d\langle E_j(p)x, y \rangle + \int_{\Omega_j^+} f_1(p)\, d\langle JE_j(p)x, y \rangle, \tag{11.15}$$

or, equivalently,

$$\begin{aligned}\langle f(T)x, y\rangle = &\int_{\Omega_j^+} d\langle \Pi_0 E_j(p)x, y\rangle\, f_0(p)\\ &+ \int_{\Omega_j^+} d\langle \Pi_+^j E_j(p)x, y\rangle\, f(p)\\ &+ \int_{\Omega_j^+} d\langle \Pi_-^j E_j(p)x, y\rangle \overline{f(p)}.\end{aligned} \tag{11.16}$$

Moreover, on identifying the complex plane $\mathbb{C}_k$ *with* $\mathbb{C}_j$ *in the natural way by the mapping* $\varphi_{kj} : u+kv \mapsto u+jv$*, we have* $E_j(\varphi_{kj}(\sigma)) = E_k(\sigma)$ *for all* $\sigma \in \mathfrak{B}(\Omega_k^+)$ *for all* $j, k \in \mathbb{S}$*.*

Proof. Formula (11.15) was established in item (ii) of Lemma 11.1.2. Formula (11.16) follows from (11.15). Indeed, if we write $y = y_0 + y_+ + y_- \in \mathcal{H}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ as in Lemma 9.3.9, then

$$\begin{aligned}\langle f(T)x, y\rangle &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)x, y\rangle + \int_{\Omega_j^+} f_1(p) d\langle JE_j(p)x, y)\rangle\\ &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)x, y\rangle + \int_{\Omega_j^+} f_1(p) d\langle E_j(p)x_+ j, y\rangle\\ &\quad + \int_{\Omega_j^+} f_1(p) d\langle E_j(p)x_-(-j), y\rangle\\ &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)x_0, y\rangle + \int_{\Omega_j^+} d\langle E_j(p)x_+, y\rangle (f_0(q) + jf_1(q))\\ &\quad + \int_{\Omega_j^+} d\langle E_j(p)x_-, y\rangle (f_0(q) - jf_1(q))\\ &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)\Pi_0 x, y\rangle + \int_{\Omega_j^+} d\langle E_j(p)\Pi_+^j x, y\rangle f(q)\\ &\quad + \int_{\Omega_j^+} d\langle E_j(p)\Pi_-^j x, y\rangle \overline{f(q)}.\end{aligned}$$

The fact that there is only one spectral measure E_j on $\sigma_S(T) \cap \mathbb{C}_j^+$ such that (11.15) holds follows directly from the uniqueness of the measure $\mu_{x,y}(\sigma) = \langle E(\sigma)x, y\rangle$ on Ω_j^+ (see (11.5)). The claimed invariance $E_j(\varphi_{jk}(\sigma)) = E_k(\sigma)$ relative to $j, k \in \mathbb{S}$ drops out easily from the aforementioned uniqueness of E_j and Theorem 9.2.3. □

Corollary 11.2.2. *In the setting of Theorem 11.2.1, the following statements hold:*

(i) *If* $T \in \mathcal{B}(\mathcal{H})$ *is a positive operator, then there exists a unique positive operator* $T^{1/2} := W \in \mathcal{B}(\mathcal{H})$ *such that* $W^2 = T$*.*

(ii) *$T \in \mathcal{B}(\mathcal{H})$ is self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{[-\|T\|,\|T\|]} t\, d\langle E_j(t)x, y\rangle, \quad x, y \in \mathcal{H}. \tag{11.17}$$

(iii) *$T \in \mathcal{B}(\mathcal{H})$ is anti-self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{[0,\|T\|]} t\, d\langle JE_j(t)x, y\rangle, \quad x, y \in \mathcal{H}. \tag{11.18}$$

(iv) *$T \in \mathcal{B}(\mathcal{H})$ is unitary if and only if*

$$\langle Tx, y\rangle = \int_{[0,\pi]} \cos(t) d\langle E_j(t)x, y\rangle + \int_{[0,\pi]} \sin(t) d\langle JE_j(t)x, y\rangle. \tag{11.19}$$

Proof. If $T \in \mathcal{B}(\mathcal{H})$ is a positive operator, then $\sigma_S(T) \subseteq [0, \|T\|]$. Thus, using Theorem 11.2.1, we have the existence of a uniquely determined spectral measure E_j such that

$$\langle Tx, y\rangle = \int_{[0,\|T\|]} t\, d\langle E_j(t)x, y\rangle. \tag{11.20}$$

Let $g(t) = t^{1/2}$ for $t \in \mathbb{R}$. Since $g \in \mathcal{C}(\sigma_S(T), \mathbb{R})$, it follows from Theorem 11.2.1 that

$$\langle Wx, y\rangle := \langle g(T)x, y\rangle = \int_{[0,\|T\|]} t^{1/2}\, d\langle E_j(t)x, y\rangle$$

satisfies $W^2 = T$. Thus, we have established the existence of a positive operator $W \in \mathcal{B}(\mathcal{H})$ such that $W^2 = T$. The proof that W is unique follows from the uniqueness of the spectral measure E_j, just as in the case that $\mathcal{H}$ is a complex Hilbert space.

The proofs of (ii)–(iv) follow readily from Theorem 11.2.1 and (9.9). □

11.3 Comments and Remarks

The spectral theorem based on the S-spectrum was proved in the following papers: the general case for bounded and unbounded normal operators was shown in [13]. A different proof for unitary operators was given in [14], and the simple case of compact normal operators was shown in [143].

Results related to the quaternionic spectral theorem can furthermore be found in [57, 74]. For quaternionic matrices, the spectral theorem based on the right spectrum was proved in [108]. The right spectrum is in the finite-dimensional case, however, equal to the S-spectrum.

The main application of the quaternionic spectral theorem is in quaternionic quantum mechanics. In the list of references there are also papers related to quaternionic quantum mechanics [107], [109], [158] in which the notion of right spectrum was used.