# Chapter 1

# Introduction

Since the discovery of the $S$-spectrum in 2006 the theory of slice hyperholomorphic functions has become the underlining function theory on which new functional calculi for quaternionic operators and for $n$-tuples of operators have been developed. These calculi are the $S$-functional calculus and the $F$-functional calculus, and both are based on the notion of $S$-spectrum. In 2014 it was proved that the quaternionic spectral theorem is also naturally based on the $S$-spectrum. These facts restore the analogy with the classical case in which the holomorphic functional calculus and the spectral theorem are based on the same notion of spectrum.

*So to replace complex spectral theory with quaternionic spectral theory we have to replace the classical spectrum with the $S$-spectrum.* The quaternionic spectral theory contains as a particular case the complex spectral theory.

## 1.1 What is Quaternionic Spectral Theory?

To orient the reader in this new spectral theory we summarize some of the fundamental concepts and facts such as the notion of slice hyperholomorphic functions, the problem of the spectrum of a quaternionic linear operator, the $S$-functional calculus, the $F$-functional calculus, the spectral theorem on the $S$-spectrum, and spectral operators.

*Slice hyperholomorphicity* is the crucial notion of hyperholomorphicity for the quaternionic spectral theory based on the $S$-spectrum. We denote by $\mathbb{H}$ the algebra of quaternions; the imaginary units in $\mathbb{H}$ are denoted by $e_1$, $e_2$, and $e_3$, respectively; and an element in $\mathbb{H}$ is of the form $q = q_0 + e_1q_1 + e_2q_2 + e_3q_3$, for $q_\ell \in \mathbb{R}$, $\ell = 0, 1, 2, 3$. The real part, the imaginary part, and the modulus of a quaternion are defined as $\mathrm{Re}(q) = q_0$, $\underline{q} = \mathrm{Im}(q) = e_1q_1 + e_2q_2 + e_3q_3$, $|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$. The conjugate of the quaternion $q$ is defined by $\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q) = q_0 - e_1q_1 - e_2q_2 - e_3q_3$ and it satisfies $|q|^2 = q\bar{q} = \bar{q}q$. Let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e., $\mathbb{S} = \{q = e_1q_1 + e_2q_2 + e_3q_3 \text{ such that } \sum_{\ell=1}^{3} q_\ell^2 = 1\}$.

 
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Every quaternion $q$ belongs to a suitable complex plane: if we set

$$j_q := \begin{cases} \frac{\underline{q}}{|\underline{q}|} & \text{if } \underline{q} \neq 0, \\ \text{any } j \in \mathbb{S} & \text{if } \underline{q} = 0, \end{cases}$$

then $q = u + j_q v$ with $u = \mathrm{Re}(q)$ and $v = |\mathrm{Im}(q)|$. For every $q = u + j_q v \in \mathbb{H}$ we define the set $[q] := \{u + jv \mid j \in \mathbb{S}\}$.

*Slice continuous functions.* Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{H}$ is called a left slice function if it is of the form

$$f(q) = f_0(u,v) + j f_1(u,v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{H}$ that satisfy the compatibility condition

$$f_0(u,-v) = f_0(u,v), \quad f_1(u,-v) = -f_1(u,v). \tag{1.1}$$

*Slice hyperholomorphic function.* If in addition the components $f_0$ and $f_1$ of the slice function $f$ satisfy the Cauchy–Riemann equations

$$\frac{\partial}{\partial u} f_0(u,v) - \frac{\partial}{\partial v} f_1(u,v) = 0, \tag{1.2}$$

$$\frac{\partial}{\partial v} f_0(u,v) + \frac{\partial}{\partial u} f_1(u,v) = 0, \tag{1.3}$$

then $f$ is called left slice hyperholomorphic. An analogous notion is given also for right slice continuous and right slice hyperholomorphic functions. If $f$ is a left (or right) slice function such that $f_0$ and $f_1$ are real-valued, then $f$ is called intrinsic.

*The problem of the spectrum.* Several attempts have been made by several authors in the past decades to prove the quaternionic spectral theorem: see, e.g., [195, 197]. However, the notion of spectrum was unclear. It is easy to explain the difficulties in giving an appropriate definition of spectrum of a quaternionic linear operator if one tries to adapt the classical notion of spectrum. Consider, for example, a right linear bounded quaternionic operator $T : X \to X$ acting on a two-sided quaternionic Banach space $X$, that is,

$$T(w_1\alpha + w_2\beta) = T(w_1)\alpha + T(w_2)\beta,$$

for all $\alpha, \beta \in \mathbb{H}$ and $w_1,\ w_2 \in X$. The symbol $\mathcal{B}(X)$ denotes the Banach space of all bounded right linear operators endowed with the natural norm.

The left spectrum $\sigma_L(T)$ of $T$ is related to the resolvent operator $(s\mathcal{I} - T)^{-1}$, that is,

$$\sigma_L(T) = \{s \in \mathbb{H} : \ s\mathcal{I} - T \text{ is not invertible in } \mathcal{B}(X)\},$$

where the notation $s\mathcal{I}$ in $\mathcal{B}(X)$ means that $(s\mathcal{I})(v) = sv$.

The right spectrum $\sigma_R(T)$ of $T$ is associated with the right eigenvalue problem, i.e., the search for quaternions $s$ such that there exists a nonzero vector $v \in X$ satisfying

$$T(v) = vs.$$

Now observe that the operator $\mathcal{I}s - T$ associated with the right eigenvalue problem is not linear, so it is not clear what resolvent operator is to be considered in this case. There is just one case in which the quaternionic spectral theorem is proved by specifying the spectrum, and it is the case of quaternionic normal matrices; see [108] and [199]. In this case the right spectrum $\sigma_R(T)$ has been used, but this is the case in which we have just the eigenvalues. The left spectrum $\sigma_L(T)$, which is associated with a linear resolvent operator, is not useful because it is not clear what notion of hyperholomorphicity is associated to the map $s \to (s\mathcal{I} - T)^{-1}$. Moreover, in quaternionic quantum mechanics the right spectrum $\sigma_R(T)$ is the most useful notion of spectrum for studying the bounded states of a systems (where there are just the eigenvalues).

*The S-functional calculus.* The notion of $S$-spectrum for quaternionic linear operators turned out to be the correct notion of spectrum, and it was discovered from the Cauchy formulas of slice hyperholomorphic functions with slice hyperholomorphic kernels. Moreover, the right spectrum $\sigma_R$ of a matrix is equal to the $S$-spectrum. More generally, the right eigenvalues $\sigma_R$ are equal to the $S$-eigenvalues. We limit the discussion to the case of quaternionic operators, but the following definition of $S$-spectrum can be adapted to the case of $n$-tuples of noncommuting operators. If $T$ is a linear bounded quaternionic operator then the $S$-spectrum is defined as

$$\sigma_S(T) = \{s \in \mathbb{H} : \ T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} \text{ is not invertible}\},$$

while the $S$-resolvent set is $\rho_S(T) := \mathbb{H} \setminus \sigma_S(T)$. Due to the noncommutativity of the quaternions, there are two resolvent operators associated with a quaternionic linear operator $T$: the left $S$-resolvent operator is defined as

$$S_L^{-1}(s,T) := -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \quad s \in \rho_S(T), \tag{1.4}$$

and the right $S$-resolvent operator is

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}, \quad s \in \rho_S(T). \tag{1.5}$$

The $S$-resolvent equation involves both the $S$-resolvent operators:

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T) = {} & [[S_R^{-1}(s,T) - S_L^{-1}(p,T)]p \\ & - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(p,T)]](p^2 - 2s_0p + |s|^2)^{-1}, \end{aligned}$$

for $s$, $p \in \rho_S(T)$. The $S$-functional calculus, or quaternionic functional calculus, is based on the Cauchy formula for slice hyperholomorphic functions. We denote by

$\mathcal{SH}_L(\sigma_S(T))$ the set of left slice hyperholomorphic functions $f: U \to \mathbb{H}$, where $U$ is a suitable open set that contains the $S$-spectrum of $T$; in the case of bounded operators, the $S$-spectrum is a bounded and nonempty set in $\mathbb{H}$. Analogously, we define $\mathcal{SH}_R(\sigma_S(T))$ for right slice hyperholomorphic functions. The formulations of the quaternionic functional calculus are defined as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\ ds_j\ f(s), \quad f \in \mathcal{SH}_L(\sigma_S(T)), \tag{1.6}$$

and

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\ ds_j\ S_R^{-1}(s,T), \quad f \in \mathcal{SH}_R(\sigma_S(T)), \tag{1.7}$$

where $ds_j = -dsj$, for $j \in \mathbb{S}$. The functional calculus is well defined, since the integrals depend neither on the open set $U$ nor on $j \in \mathbb{S}$. It is important to note that in the definition of the quaternionic functional calculus it is not required that the linear operator $T$ be written in components $T = T_0 + e_1T_1 + e_2T_2 + e_3T_3$ where $T_\ell$, for $\ell = 0, 1, 2, 3$, are bounded linear operators on a real Banach space. Moreover, in the case that $T$ is represented as $T = T_0+e_1T_1+e_2T_2+e_3T_3$, it is not even required that the operators $T_\ell$, for $\ell = 0, 1, 2, 3$, commute among themselves.

*The commutative version of the $S$-spectrum.* We will denote by $\mathcal{BC}(X)$ the subclass of $\mathcal{B}(X)$ that consists of those quaternionic operators $T$ that can be written as $T = T_0 + e_1T_1 + e_2T_2 + e_3T_3$, where the operators $T_\ell$, $\ell = 0, 1, 2, 3$, commute among themselves, and we set $\overline{T} = T_0 - e_1T_1 - e_2T_2 - e_3T_3$. In this case, the $S$-spectrum has an alternative definition that takes into account the commutativity of $T_\ell$, for $\ell = 0, 1, 2, 3$. In the literature the commutative definition of the $S$-spectrum is often called the $F$-spectrum because it is used for the $F$-functional calculus. Let $T \in \mathcal{BC}(X)$. We define the commutative version of the $S$-spectrum (or $F$-spectrum $\sigma_F(T)$) of $T$ as

$$\sigma_S(T) = \{s \in \mathbb{H} : \ s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} \text{ is not invertible}\}.$$

The S-resolvent set $\rho_S(T)$ is defined as $\rho_S(T) = \mathbb{H} \setminus \sigma_S(T)$.

*The $F$-functional calculus.* A deep result in hypercomplex analysis is the Fueter-Sce mapping theorem, which in modern language says that *if we apply the Laplace operator to a slice hyperholomorphic function $f : U \subseteq \mathbb{H} \to \mathbb{H}$, we obtain a Cauchy–Fueter regular function $\breve{f} : U \subseteq \mathbb{H} \to \mathbb{H}$, that is,*

$$\breve{f}(q) = \Delta f(q), \quad q \in U.$$

Applying the Laplace operator to the Cauchy kernels of slice hyperholomorphic functions, we obtain two new kernels that allow us to write the Fueter–Sce mapping theorem in integral form. Using such an integral transform, we define a functional calculus that starting from slice hyperholomorphic functions, defines

Cauchy-Fueter regular functions of a linear operator $\breve{f}(T)$ for $T \in \mathcal{BC}(X)$. Precisely, for $T \in \mathcal{BC}(X)$, we define the left $F$-resolvent operator as

$$F_L(s,T) := -4(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-2}, \quad s \in \rho_S(T),$$

and the right $F$-resolvent operator as

$$F_R(s,T) := -4(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-2}(s\mathcal{I} - \overline{T}), \quad s \in \rho_S(T).$$

So the formulations of the quaternionic $F$-functional calculus for bounded operators are defined as follows:

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\, ds_j\, f(s), \quad f \in \mathcal{SH}_L(\sigma_S(T)), \tag{1.8}$$

and

$$\breve{f}(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, F_R(s,T), \quad f \in \mathcal{SH}_R(\sigma_S(T)), \tag{1.9}$$

where $ds_j = -ds j$, for $j \in \mathbb{S}$, and the integrals depends neither on the open set $U$ nor on $j \in \mathbb{S}$. The $F$-resolvent equation in this case is more complicated, and it also involves the two $S$-resolvent operators in their commutative version. We point out that both the $S$-functional calculus and the $F$-functional calculus can be extended to the case of unbounded operators; moreover, the $S$-functional calculus can be used to define the quaternionic $H^\infty$-functional calculus.

*The spectral theorem based on the $S$-spectrum.* If $T \in \mathcal{B}(\mathcal{H})$ is a bounded normal quaternionic linear operator on a quaternionic Hilbert space $\mathcal{H}$, then there exist three quaternionic linear operators $A$, $J$, $B$ such that $T = A + JB$, where $A$ is self-adjoint and $B$ is positive, $J$ is an anti-self-adjoint partial isometry (called an imaginary operator). Moreover, $A$, $B$, and $J$ mutually commute.

There exists a unique spectral measure $E_j$ on $\Omega_j^+ := \sigma_S(T) \cap \mathbb{C}_j^+$ such that for every slice continuous intrinsic function $f = f_0 + f_1 j$,

$$\langle f(T)x, y\rangle = \int_{\Omega_j^+} f_0(q)\, d\langle E_j(q)x, y\rangle + \int_{\Omega_j^+} f_1(q)\, d\langle JE_j(q)x, y\rangle, \quad x, y \in \mathcal{H}. \tag{1.10}$$

In this book we treat also the spectral theorem for unbounded quaternionic normal operators on a Hilbert space, and we define a functional calculus for a much larger class of functions with respect to the class of continuous functions. This functional calculus is deduced by the theory of spectral integrals depending on an imaginary operator $J$.

*Spectral operators in Banach spaces.* We develop furthermore a concise theory of spectral integration in quaternionic Banach spaces based on the notion of spectral systems and use this tool to study quaternionic spectral operators. Analogously to the classical theory of Dunford, such operators have a canonical decomposition into a scalar and a radical part. The first one can be represented as the spectral integral with respect to the spectral system of the operator, and the second one is quasi-nilpotent. We also study the transformation of this decomposition under the $S$-functional calculus.

## 1.2 Some Historical Remarks on the $S$-Spectrum

It has been known since the 1930s, with the paper of G. Birkhoff and J. von Neumann on the logic of quantum mechanics, that quantum mechanics can be formulated over the real, the complex, and the quaternionic numbers. Since then, several papers and books have treated this topic. However, it is interesting, and somewhat surprising, that for a long time, an appropriate notion of quaternionic spectrum was not present in the literature.

Thus we believe that it is interesting to explain the facts, presented several times in some talks of the authors, that led to discovery of the $S$-spectrum for quaternionic linear operators, of the $S$-functional calculus, and of some of the difficulties that, in our opinion, prevented these objects from being found earlier.

The quaternionic spectral theory based on the notion of $S$-spectrum not only is relevant for researchers in quaternionic analysis but is applicable, as a particular case, to vector operators, such as the gradient operator or its variations, and has applications, for example, to fractional diffusion processes.

### 1.2.1 The Discovery of the $S$-Spectrum

The $S$-spectrum was discovered by F. Colombo and I. Sabadini in 2006. They had been working for several years on the spectral theory for quaternionic linear operators, starting with the aforementioned paper of G. Birkhoff and J. von Neumann; see [36]. They soon realized that the notion of spectrum of a quaternionic linear operator was poorly understood and as a consequence, the quaternionic spectral theory could not be developed.

The only thing that was clear was that the existing notions of left spectrum and right spectrum of a quaternionic linear operator were insufficient to construct the quaternionic spectral theory. The main reason is that left spectrum and right spectrum mimic the definition of eigenvalues in the complex case, but they do not shed light on the true nature of the quaternionic spectrum.

Thus they started to investigate what could be the quaternionic version of the Riesz–Dunford functional calculus, of the evolution operator, and of the spectral theorem. After more then 10 years of exhausting research and 70 years after [36], in 2006 they understood that the $S$-spectrum was the correct notion of spectrum for quaternionic spectral theory.

A crucial fact in classical operator theory is that the holomorphic functional calculus (called Riesz–Dunford functional calculus) and the spectral theorem are based on the same notion of spectrum. In the quaternionic setting things were different indeed: for right linear operators with commuting components, the functional calculus based on the Cauchy–Fueter formula is based on the left spectrum. However, in quaternionic quantum mechanics physicists used the notion of right spectrum, which consists just of eigenvalues.

After several attempts in various directions they started to believe that since the physicists were unable to deduce from physical arguments the quaternionic spectrum, they could use hypercomplex analysis to find it. More precisely, they thought that from the Cauchy formula of a "suitable new notion of quaternionic hyperholomorphicity" one could read the precise notion of quaternionic spectrum, which, in analogy with the complex case, could have worked for both the new hyperholomorphic functional calculus and the quaternionic spectral theorem.

In 1998 in the paper [82], F. Colombo and I. Sabadini investigated the quaternionic functional calculus based on the Cauchy formula for Fueter regular functions, and it was clear that the Fueter spectrum was incompatible with the quaternionic spectral theorem. Moreover, it was realized that this calculus was to some extent the quaternionic version of the monogenic functional calculus already introduced and studied by A. McIntosh and his collaborators; see [160, 161] and the book [159]. It was then clear that a different notion of hyperholomorphicity was needed. After so many years of intensive and unfruitful research, in 2006 D.C. Struppa showed them the new definition of slice regularity, and later they also discussed with G. Gentili this new notion of regularity introduced in the paper [135] (which is an announcement of the paper [136]). This notion requires that all the restrictions of a quaternionic-valued function to every complex plane are holomorphic maps. Thus the usual Cauchy formula for holomorphic functions holds on each complex plane $\mathbb{C}_j$ and the Cauchy kernel has the series expansion

$$\sum_{n=0}^{\infty} q^n s^{-1-n} = \frac{1}{s-q}, \quad \text{for } q,\ s \in \mathbb{C}_j,\ j \in \mathbb{S},\ |q| < |s|;$$

the above expansion obviously holds just for those quaternions $q$ and $s$ that belong to $\in \mathbb{C}_j$ and such that $|q| < |s|$.

At this point, the crucial idea was to replace $q$ in the series $\sum_{n=0}^{\infty} q^n s^{-1-n}$ by a quaternionic linear operator $T$, and to look for a closed formula for this non-commutative power series. To obtain a closed formula for the formal power series

$$\sum_{n=0}^{\infty} T^n s^{-1-n}$$

was not an easy task. F. Colombo and I. Sabadini proceeded as follows: first they found a closed form for the series expansion $\sum_{n=0}^{\infty} q^n s^{-1-n}$, that is,

$$\sum_{n=0}^{\infty} q^n s^{-1-n} = -(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \overline{s}), \quad \text{for } |q| < |s|, \tag{1.11}$$

where formula (1.11) holds for all quaternions $q$, $s \in \mathbb{H}$ such that $|q| < |s|$, and then they observed that the right-hand side of (1.11) does not depend on the commutativity of the components of the quaternion $q$, because it contains just the powers of $q$. This second crucial fact led to the natural definition of the so-called $S$-spectrum

$$\sigma_S(T) = \{s \in \mathbb{H} : \ T^2 - 2T\mathrm{Re}(s) + |s|^2\mathcal{I} \text{ is not invertible}\},$$

where $\mathcal{I}$ is the identity operator, and of the $S$-resolvent operator

$$S_L^{-1}(s,T) := -(T^2 - 2T\text{Re}(s) + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}). \tag{1.12}$$

It is also interesting to note that in 2006, when (1.12) was introduced, the Cauchy formula with slice hyperholomorphic kernel was not known, and so the sum of the series (1.11) was obtained with direct computations, using the Niven algorithm. This procedure is explained in Note 4.18.3 of the book [89]. The fact that the notion of slice hyperholomorphicity works very well for quaternionic operator theory was one of our motivations for its development.

The existence of the $S$-spectrum, with further considerations, appeared in 2007 in the paper [66] with G. Gentili and D.C. Struppa. This paper is an announcement of the results of the paper [68] (unfortunately, as may happen, published only in 2010), containing a version of the $S$-functional calculus defined just for slice hyperholomorphic functions that admit a power series expansion at the origin. This calculus is the starting point for the general definition of the $S$-functional calculus based on the Cauchy formula with slice hyperholomorphic kernels, which was completely described in [79]. The paper [79] together with the formulations of the $S$-functional calculus, see [80], and the $S$-resolvent equation, see [10], constitute the heart of the $S$-functional calculus. Finally, it is worthwhile to mention that the case of unbounded operators was treated in [97], [67] and with a direct approach in [124]. The study of the quaternionic evolution operator is in the paper [76], while the $H^\infty$-functional calculus is in [30] and [52]. Finally, the main results about the spectral theorem based on the $S$-spectrum are proved in [13, 14].

The authors would like to thank G. Gentili, I. Sabadini, and D.C. Struppa for their comments on this note about the discovery of the $S$-spectrum.

### 1.2.2 Why Did It Take So Long to Understand the $S$-Spectrum?

After over 20 years of research, it is now clear that this new spectral theory based on the $S$-spectrum is very natural because it generalizes complex spectral theory and because the $S$-functional calculus (which is the slice hyperholomorphic functional calculus) and the quaternionic spectral theorem are based on the same notion of spectrum. There are several reasons why it took so long to discover the $S$-spectrum. We recall two of them.

Complex analysis and Cauchy–Fueter analysis are based on functions in the kernel of a constant-coefficient differential operator. This fact was misleading in the search for a new definition of hyperholomorphicity because one is tempted to look for a constant-coefficient quaternionic differential operator, not necessarily of first order, in order to find a "new notion of hyperholomorphicity" from which we could read the quaternionic spectrum. In [60] it was shown that slice hyperholomorphic functions are functions in the kernel of a first-order quaternionic differential operator *with nonconstant coefficients*. This fact was unexpected.

Another interesting fact is that in the paper of Fueter [111] there was a partial solution to the problem. In fact, in [111] Fueter gives a procedure to construct Fueter regular functions starting from holomorphic functions. His procedure consists of two steps: from holomorphic functions he constructs what he calls *hyperanalytic functions*; then he applies the Laplace operator to such hyperanalytic functions and thereby obtains Fueter regular functions. Fueter's hyperanalytic functions are what nowadays are called intrinsic slice hyperholomorphic functions but for some reason these functions have never been systematically studied. The Cauchy formula for hyperanalytic functions, which is the Cauchy formula for the $S$-functional calculus and from which one could read the $S$-spectrum, has never been investigated to the best of our knowledge.

## 1.3 The Fueter–Sce–Qian theorem and spectral theories

In this section we want to put the spectral theory on the $S$-spectrum into the perspective of the spectral theories that arise from the Fueter–Sce–Qian mapping theorem. In classical complex operator theory, the Cauchy formula of holomorphic functions is a fundamental tool for defining functions of operators. Moreover, the Cauchy–Riemann operator factorizes the Laplace operator, so holomorphic functions play also a crucial role in harmonic analysis and in boundary value problems. In higher dimensions, for quaternion-valued functions or, more generally, for Clifford-valued functions, there appear two different notions of hyperholomorphicity. The first one is called slice hyperholomorphicity and the second one is known under different names, depending to the dimension of the algebra and the range of the functions: Cauchy–Fueter regularity for quaternion-valued and monogenicity for Clifford-algebra-valued functions. The Fueter–Sce–Qian mapping theorem reveals a fundamental relation between the different notions of hyperholomorphicity. It will be explained in detail later on (see, for example, the section on the Fueter mapping in integral form), but it can be illustrated by the following diagram:

$$Hol(\Omega) \xrightarrow{F_1} \mathcal{N}(U) \xrightarrow{F_2} \mathcal{AM}(U).$$

Applying the mapping $F_1$, we can use any function in the set $Hol(\Omega)$ of holomorphic functions on a suitable open set $\Omega$ in $\mathbb{C}$ to generate a function in the set $\mathcal{N}(U)$ of all intrinsic slice hyperholomorphic functions on a certain open subset $U$ of $\mathbb{H}$. Applying a second transformation $F_2$, we can transform any intrinsic slice hyperholomorphic function into an axially Fueter-regular resp. an axially monogenic function.

When considering quaternion-valued functions, the mapping $F_2$ that transforms an intrinsic slice hyperholomorphic function into a Fueter regular one is the application of the Laplace operator, i.e., $F_2 = \Delta$. When we work with Clifford-algebra-valued functions, then $F_2 = \Delta^{(n-1)/2}$, where $n$ is the number of generating units of the Clifford algebra. The Fueter–Sce–Qian mapping theorem can be adapted to the more general case in which $\mathcal{N}(U)$ is replaced by slice hyperholomorphic functions and the axially regular (or axially monogenic) functions $\mathcal{AM}(U)$ are replaced by monogenic functions. The generalization of holomorphicity to quaternion- or Clifford-algebra-valued functions produces two different notions of hyperholomorphicity that are useful for different purposes. Precisely, we have that:

- The Cauchy formula for slice hyperholomorphic functions leads to the definition of the $S$-spectrum and the $S$-functional calculus for quaternionic linear operators. Moreover, the spectral theorem for quaternionic linear operators is based on the $S$-spectrum. The aim of this book and of the monograph [56], is to give a systematic treatment of this theory and of its applications.

- The Cauchy formula associated with Cauchy–Fueter regularity resp. monogenicity leads to the notion of monogenic spectrum and produces the Cauchy–Fueter functional calculus for quaternion-valued functions and the monogenic functional calculus for Clifford-algebra-valued functions. This theory has applications in harmonic analysis in higher dimension and in boundary value problems. For an overview on the monogenic functional calculus and its applications see [159] and for applications to boundary values problems see [155] and the references contained in those books.

We want to stress that these two approaches start from two totally different perspectives: while the first one develops the spectral theory of a single quaternionic- resp. Clifford-linear operator, the latter develops a joint spectral theory for $n$-tuples of real-linear operators. However, the $F$-functional calculus provides a relation between these two approaches and shows that they are consistent under reasonable assumptions. In this book we treat the quaternionic spectral theory on the $S$-spectrum, so very often we will refer to it as quaternionic spectral theory because no confusion arises with respect to the monogenic spectral theory.