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Preface

Classical operator theory in Banach and Hilbert spaces has been stimulated by several problems in mathematics and physics. Moreover, the theory of holomorphic functions plays a crucial role in operator theory and in particular in the definition of functions of operators. A great impulse was given to the development of operator theory at the beginning of the last century when quantum mechanics was formulated; in particular, the spectral theorem for unbounded normal operators on a Hilbert space was one of the most important achievements. In 1936, Birkhoff and von Neumann showed that quantum mechanics can be formulated on real, complex, and quaternionic numbers. So a natural problem was to understand what notion of spectrum one should use in quaternionic operator theory. This problem was solved only in 2006 with the discovery of the S-spectrum for quaternionic linear operators, and since then the quaternionic spectral theory has grown rapidly. The aim of this book is to give a systematic foundation of quaternionic spectral theory based on the S-spectrum and to present the theory of slice hyperholomorphic functions, which will be used in the treatment of quaternionic operator theory.

This book treats four main topics: the S-functional calculus, the F-functional calculus, the quaternionic spectral theorem, and the theory of quaternionic spectral operators. The S-functional calculus is the natural extension to the quaternionic setting of the Riesz–Dunford functional calculus, and it can be used to define the quaternionic H^∞-functional calculus for quaternionic or vector sectorial operators. The H^∞-functional calculus has important applications in fractional diffusion processes because it allows one to define fractional powers of vector operators such as the gradient or a generalization of the gradient operator with nonconstant coefficients. The F-functional calculus is based on an integral transform, called the Fueter-Sce mapping theorem in integral form, and it defines Fueter-regular functions of quaternionic operators. This calculus is based on slice hyperholomorphic functions and on the so-called F-resolvent operators that allow us to define, via an integral formula, functions of a quaternionic operator. We treat the spectral theorem for quaternionic normal operators based on the S-spectrum, which was proved in 2014 and published in 2016. The quaternionic spectral theorem for unbounded anti-selfadjoint operators is a very important tool for formulating quaternionic quantum mechanics. We conclude the book with the theory of spectral operators in Banach spaces that has been developed in the last two years.

To orient the reader who is not familiar with quaternionic or vectors analysis we have summarized some of the fundamental concepts of this theory in the first chapter of this book, with some historical comments on the discovery of the S-spectrum.

Since the theory of slice hyperholomorphic functions is a crucial tool in quaternionic operator theory, we dedicate the second chapter to the formulation of the function theory, and we prove the most important results that are used in the book.

Chapters 3–6 are devoted to the S-functional calculus, in Chapters 7 and 8 we develop the F-functional calculus, in Chapters 9–13 we treat the quaternionic spectral theorem, and finally the theory of spectral operators in Banach spaces is developed in Chapters 14 and 15. At the end of the chapters there are comments and remarks about the extension of the theory of slice hyperholomorphic functions with values in a Clifford algebra (slice monogenic functions) and its applications to the S-functional calculus for n-tuples of not necessarily commuting operators and to the F-functional calculus for n-tuples of commuting operators. We will also make comments on the links between the theory of slice hyperholomorphic functions and the classical theory of Cauchy–Fueter regular functions (or the Dirac monogenic function theory). The natural continuation of this book is the monograph [56], in which we further develop theoretical aspects of quaternionic operator theory and give applications to fractional diffusion processes. At the end of this book one can find a table of contents of the monograph [56].

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Contents

Chapter 1

Introduction

Since the discovery of the S-spectrum in 2006 the theory of slice hyperholomorphic functions has become the underlining function theory on which new functional calculi for quaternionic operators and for n-tuples of operators have been developed. These calculi are the S-functional calculus and the F-functional calculus, and both are based on the notion of S-spectrum. In 2014 it was proved that the quaternionic spectral theorem is also naturally based on the S-spectrum. These facts restore the analogy with the classical case in which the holomorphic functional calculus and the spectral theorem are based on the same notion of spectrum.

So to replace complex spectral theory with quaternionic spectral theory we have to replace the classical spectrum with the S-spectrum. The quaternionic spectral theory contains as a particular case the complex spectral theory.

1.1 What is Quaternionic Spectral Theory?

To orient the reader in this new spectral theory we summarize some of the fundamental concepts and facts such as the notion of slice hyperholomorphic functions, the problem of the spectrum of a quaternionic linear operator, the S-functional calculus, the F-functional calculus, the spectral theorem on the S-spectrum, and spectral operators.

Slice hyperholomorphicity is the crucial notion of hyperholomorphicity for the quaternionic spectral theory based on the S-spectrum. We denote by $\mathbb{H}$ the algebra of quaternions; the imaginary units in $\mathbb{H}$ are denoted by e_1, e_2, and e_3, respectively; and an element in $\mathbb{H}$ is of the form $q = q_0 + e_1q_1 + e_2q_2 + e_3q_3$, for $q_\ell \in \mathbb{R}$, $\ell = 0, 1, 2, 3$. The real part, the imaginary part, and the modulus of a quaternion are defined as $\text{Re}(q) = q_0$, $\underline{q} = \text{Im}(q) = e_1q_1+e_2q_2+e_3q_3$, $|q|^2 = q_0^2+q_1^2+q_2^2+q_3^2$. The conjugate of the quaternion q is defined by $\bar{q} = \text{Re}(q) - \text{Im}(q) = q_0 - e_1q_1 - e_2q_2 - e_3q_3$ and it satisfies $|q|^2 = q\bar{q} = \bar{q}q$. Let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e., $\mathbb{S} = \{q = e_1q_1 + e_2q_2 + e_3q_3 \text{ such that } \sum_{\ell=1}^{3} q_\ell^2 = 1\}$.

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Every quaternion q belongs to a suitable complex plane: if we set

$$j_q := \begin{cases} \frac{\underline{q}}{|\underline{q}|} & \text{if } \underline{q} \neq 0, \\ \text{any } j \in \mathbb{S} & \text{if } \underline{q} = 0, \end{cases}$$

then $q = u + j_q v$ with $u = \mathrm{Re}(q)$ and $v = |\mathrm{Im}(q)|$. For every $q = u + j_q v \in \mathbb{H}$ we define the set $[q] := \{u + jv \mid j \in \mathbb{S}\}$.

Slice continuous functions. Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} = \{(u, v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{H}$ is called a left slice function if it is of the form

$$f(q) = f_0(u, v) + j f_1(u, v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{H}$ that satisfy the compatibility condition

$$f_0(u, -v) = f_0(u, v), \quad f_1(u, -v) = -f_1(u, v). \tag{1.1}$$

Slice hyperholomorphic function. If in addition the components f_0 and f_1 of the slice function f satisfy the Cauchy–Riemann equations

$$\frac{\partial}{\partial u} f_0(u, v) - \frac{\partial}{\partial v} f_1(u, v) = 0, \tag{1.2}$$

$$\frac{\partial}{\partial v} f_0(u, v) + \frac{\partial}{\partial u} f_1(u, v) = 0, \tag{1.3}$$

then f is called left slice hyperholomorphic. An analogous notion is given also for right slice continuous and right slice hyperholomorphic functions. If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called intrinsic.

The problem of the spectrum. Several attempts have been made by several authors in the past decades to prove the quaternionic spectral theorem: see, e.g., [195, 197]. However, the notion of spectrum was unclear. It is easy to explain the difficulties in giving an appropriate definition of spectrum of a quaternionic linear operator if one tries to adapt the classical notion of spectrum. Consider, for example, a right linear bounded quaternionic operator $T : X \to X$ acting on a two-sided quaternionic Banach space X, that is,

$$T(w_1\alpha + w_2\beta) = T(w_1)\alpha + T(w_2)\beta,$$

for all $\alpha, \beta \in \mathbb{H}$ and $w_1,\ w_2 \in X$. The symbol $\mathcal{B}(X)$ denotes the Banach space of all bounded right linear operators endowed with the natural norm.

The left spectrum $\sigma_L(T)$ of T is related to the resolvent operator $(s\mathcal{I} - T)^{-1}$, that is,

$$\sigma_L(T) = \{s \in \mathbb{H} : \ s\mathcal{I} - T \text{ is not invertible in } \mathcal{B}(X)\},$$

where the notation $s\mathcal{I}$ in $\mathcal{B}(X)$ means that $(s\mathcal{I})(v) = sv$.

The right spectrum $\sigma_R(T)$ of T is associated with the right eigenvalue problem, i.e., the search for quaternions s such that there exists a nonzero vector $v \in X$ satisfying

$$T(v) = vs.$$

Now observe that the operator $\mathcal{I}s - T$ associated with the right eigenvalue problem is not linear, so it is not clear what resolvent operator is to be considered in this case. There is just one case in which the quaternionic spectral theorem is proved by specifying the spectrum, and it is the case of quaternionic normal matrices; see [108] and [199]. In this case the right spectrum $\sigma_R(T)$ has been used, but this is the case in which we have just the eigenvalues. The left spectrum $\sigma_L(T)$, which is associated with a linear resolvent operator, is not useful because it is not clear what notion of hyperholomorphicity is associated to the map $s \to (s\mathcal{I} - T)^{-1}$. Moreover, in quaternionic quantum mechanics the right spectrum $\sigma_R(T)$ is the most useful notion of spectrum for studying the bounded states of a systems (where there are just the eigenvalues).

The S-functional calculus. The notion of S-spectrum for quaternionic linear operators turned out to be the correct notion of spectrum, and it was discovered from the Cauchy formulas of slice hyperholomorphic functions with slice hyperholomorphic kernels. Moreover, the right spectrum σ_R of a matrix is equal to the S-spectrum. More generally, the right eigenvalues σ_R are equal to the S-eigenvalues. We limit the discussion to the case of quaternionic operators, but the following definition of S-spectrum can be adapted to the case of n-tuples of noncommuting operators. If T is a linear bounded quaternionic operator then the S-spectrum is defined as

$$\sigma_S(T) = \{s \in \mathbb{H} : \ T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} \text{ is not invertible}\},$$

while the S-resolvent set is $\rho_S(T) := \mathbb{H} \setminus \sigma_S(T)$. Due to the noncommutativity of the quaternions, there are two resolvent operators associated with a quaternionic linear operator T: the left S-resolvent operator is defined as

$$S_L^{-1}(s,T) := -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \quad s \in \rho_S(T), \tag{1.4}$$

and the right S-resolvent operator is

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}, \quad s \in \rho_S(T). \tag{1.5}$$

The S-resolvent equation involves both the S-resolvent operators:

$$\begin{aligned} S_R^{-1}(s,T)S_L^{-1}(p,T) = {} & [[S_R^{-1}(s,T) - S_L^{-1}(p,T)]p \\ & - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(p,T)]](p^2 - 2s_0p + |s|^2)^{-1}, \end{aligned}$$

for s, $p \in \rho_S(T)$. The S-functional calculus, or quaternionic functional calculus, is based on the Cauchy formula for slice hyperholomorphic functions. We denote by

$\mathcal{SH}_L(\sigma_S(T))$ the set of left slice hyperholomorphic functions $f : U \to \mathbb{H}$, where U is a suitable open set that contains the S-spectrum of T; in the case of bounded operators, the S-spectrum is a bounded and nonempty set in $\mathbb{H}$. Analogously, we define $\mathcal{SH}_R(\sigma_S(T))$ for right slice hyperholomorphic functions. The formulations of the quaternionic functional calculus are defined as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\ ds_j\ f(s), \quad f \in \mathcal{SH}_L(\sigma_S(T)), \tag{1.6}$$

and

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\ ds_j\ S_R^{-1}(s,T), \quad f \in \mathcal{SH}_R(\sigma_S(T)), \tag{1.7}$$

where $ds_j = -dsj$, for $j \in \mathbb{S}$. The functional calculus is well defined, since the integrals depend neither on the open set U nor on $j \in \mathbb{S}$. It is important to note that in the definition of the quaternionic functional calculus it is not required that the linear operator T be written in components $T = T_0 + e_1T_1 + e_2T_2 + e_3T_3$ where T_ℓ, for $\ell = 0, 1, 2, 3$, are bounded linear operators on a real Banach space. Moreover, in the case that T is represented as $T = T_0+e_1T_1+e_2T_2+e_3T_3$, it is not even required that the operators T_ℓ, for $\ell = 0, 1, 2, 3$, commute among themselves.

The commutative version of the S-spectrum. We will denote by $\mathcal{BC}(X)$ the subclass of $\mathcal{B}(X)$ that consists of those quaternionic operators T that can be written as $T = T_0 + e_1T_1 + e_2T_2 + e_3T_3$, where the operators T_ℓ, $\ell = 0, 1, 2, 3$, commute among themselves, and we set $\overline{T} = T_0 - e_1T_1 - e_2T_2 - e_3T_3$. In this case, the S-spectrum has an alternative definition that takes into account the commutativity of T_ℓ, for $\ell = 0, 1, 2, 3$. In the literature the commutative definition of the S-spectrum is often called the F-spectrum because it is used for the F-functional calculus. Let $T \in \mathcal{BC}(X)$. We define the commutative version of the S-spectrum (or F-spectrum $\sigma_F(T)$) of T as

$$\sigma_S(T) = \{s \in \mathbb{H} :\ \ s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} \text{ is not invertible}\}.$$

The S-resolvent set $\rho_S(T)$ is defined as $\rho_S(T) = \mathbb{H} \setminus \sigma_S(T)$.

The F-functional calculus. A deep result in hypercomplex analysis is the Fueter-Sce mapping theorem, which in modern language says that *if we apply the Laplace operator to a slice hyperholomorphic function $f : U \subseteq \mathbb{H} \to \mathbb{H}$, we obtain a Cauchy–Fueter regular function $\breve{f} : U \subseteq \mathbb{H} \to \mathbb{H}$, that is,*

$$\breve{f}(q) = \Delta f(q), \quad q \in U.$$

Applying the Laplace operator to the Cauchy kernels of slice hyperholomorphic functions, we obtain two new kernels that allow us to write the Fueter–Sce mapping theorem in integral form. Using such an integral transform, we define a functional calculus that starting from slice hyperholomorphic functions, defines

Cauchy-Fueter regular functions of a linear operator $\breve{f}(T)$ for $T \in \mathcal{BC}(X)$. Precisely, for $T \in \mathcal{BC}(X)$, we define the left F-resolvent operator as

$$F_L(s,T) := -4(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-2}, \quad s \in \rho_S(T),$$

and the right F-resolvent operator as

$$F_R(s,T) := -4(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-2}(s\mathcal{I} - \overline{T}), \quad s \in \rho_S(T).$$

So the formulations of the quaternionic F-functional calculus for bounded operators are defined as follows:

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\, ds_j\, f(s), \quad f \in \mathcal{SH}_L(\sigma_S(T)), \tag{1.8}$$

and

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, F_R(s,T), \quad f \in \mathcal{SH}_R(\sigma_S(T)), \tag{1.9}$$

where $ds_j = -ds j$, for $j \in \mathbb{S}$, and the integrals depends neither on the open set U nor on $j \in \mathbb{S}$. The F-resolvent equation in this case is more complicated, and it also involves the two S-resolvent operators in their commutative version. We point out that both the S-functional calculus and the F-functional calculus can be extended to the case of unbounded operators; moreover, the S-functional calculus can be used to define the quaternionic H^∞-functional calculus.

The spectral theorem based on the S-spectrum. If $T \in \mathcal{B}(\mathcal{H})$ is a bounded normal quaternionic linear operator on a quaternionic Hilbert space $\mathcal{H}$, then there exist three quaternionic linear operators A, J, B such that $T = A + JB$, where A is self-adjoint and B is positive, J is an anti-self-adjoint partial isometry (called an imaginary operator). Moreover, A, B, and J mutually commute.

There exists a unique spectral measure E_j on $\Omega_j^+ := \sigma_S(T) \cap \mathbb{C}_j^+$ such that for every slice continuous intrinsic function $f = f_0 + f_1 j$,

$$\langle f(T)x, y\rangle = \int_{\Omega_j^+} f_0(q)\, d\langle E_j(q)x, y\rangle + \int_{\Omega_j^+} f_1(q)\, d\langle JE_j(q)x, y\rangle, \quad x, y \in \mathcal{H}. \tag{1.10}$$

In this book we treat also the spectral theorem for unbounded quaternionic normal operators on a Hilbert space, and we define a functional calculus for a much larger class of functions with respect to the class of continuous functions. This functional calculus is deduced by the theory of spectral integrals depending on an imaginary operator J.

Spectral operators in Banach spaces. We develop furthermore a concise theory of spectral integration in quaternionic Banach spaces based on the notion of spectral systems and use this tool to study quaternionic spectral operators. Analogously to the classical theory of Dunford, such operators have a canonical decomposition into a scalar and a radical part. The first one can be represented as the spectral integral with respect to the spectral system of the operator, and the second one is quasi-nilpotent. We also study the transformation of this decomposition under the S-functional calculus.

1.2 Some Historical Remarks on the S-Spectrum

It has been known since the 1930s, with the paper of G. Birkhoff and J. von Neumann on the logic of quantum mechanics, that quantum mechanics can be formulated over the real, the complex, and the quaternionic numbers. Since then, several papers and books have treated this topic. However, it is interesting, and somewhat surprising, that for a long time, an appropriate notion of quaternionic spectrum was not present in the literature.

Thus we believe that it is interesting to explain the facts, presented several times in some talks of the authors, that led to discovery of the S-spectrum for quaternionic linear operators, of the S-functional calculus, and of some of the difficulties that, in our opinion, prevented these objects from being found earlier.

The quaternionic spectral theory based on the notion of S-spectrum not only is relevant for researchers in quaternionic analysis but is applicable, as a particular case, to vector operators, such as the gradient operator or its variations, and has applications, for example, to fractional diffusion processes.

1.2.1 The Discovery of the S-Spectrum

The S-spectrum was discovered by F. Colombo and I. Sabadini in 2006. They had been working for several years on the spectral theory for quaternionic linear operators, starting with the aforementioned paper of G. Birkhoff and J. von Neumann; see [36]. They soon realized that the notion of spectrum of a quaternionic linear operator was poorly understood and as a consequence, the quaternionic spectral theory could not be developed.

The only thing that was clear was that the existing notions of left spectrum and right spectrum of a quaternionic linear operator were insufficient to construct the quaternionic spectral theory. The main reason is that left spectrum and right spectrum mimic the definition of eigenvalues in the complex case, but they do not shed light on the true nature of the quaternionic spectrum.

Thus they started to investigate what could be the quaternionic version of the Riesz–Dunford functional calculus, of the evolution operator, and of the spectral theorem. After more then 10 years of exhausting research and 70 years after [36], in 2006 they understood that the S-spectrum was the correct notion of spectrum for quaternionic spectral theory.

A crucial fact in classical operator theory is that the holomorphic functional calculus (called Riesz–Dunford functional calculus) and the spectral theorem are based on the same notion of spectrum. In the quaternionic setting things were different indeed: for right linear operators with commuting components, the functional calculus based on the Cauchy–Fueter formula is based on the left spectrum. However, in quaternionic quantum mechanics physicists used the notion of right spectrum, which consists just of eigenvalues.

After several attempts in various directions they started to believe that since the physicists were unable to deduce from physical arguments the quaternionic spectrum, they could use hypercomplex analysis to find it. More precisely, they thought that from the Cauchy formula of a "suitable new notion of quaternionic hyperholomorphicity" one could read the precise notion of quaternionic spectrum, which, in analogy with the complex case, could have worked for both the new hyperholomorphic functional calculus and the quaternionic spectral theorem.

In 1998 in the paper [82], F. Colombo and I. Sabadini investigated the quaternionic functional calculus based on the Cauchy formula for Fueter regular functions, and it was clear that the Fueter spectrum was incompatible with the quaternionic spectral theorem. Moreover, it was realized that this calculus was to some extent the quaternionic version of the monogenic functional calculus already introduced and studied by A. McIntosh and his collaborators; see [160, 161] and the book [159]. It was then clear that a different notion of hyperholomorphicity was needed. After so many years of intensive and unfruitful research, in 2006 D.C. Struppa showed them the new definition of slice regularity, and later they also discussed with G. Gentili this new notion of regularity introduced in the paper [135] (which is an announcement of the paper [136]). This notion requires that all the restrictions of a quaternionic-valued function to every complex plane are holomorphic maps. Thus the usual Cauchy formula for holomorphic functions holds on each complex plane $\mathbb{C}_j$ and the Cauchy kernel has the series expansion

$$\sum_{n=0}^{\infty} q^n s^{-1-n} = \frac{1}{s-q}, \quad \text{for } q,\, s \in \mathbb{C}_j,\ j \in \mathbb{S},\ |q| < |s|;$$

the above expansion obviously holds just for those quaternions q and s that belong to $\in \mathbb{C}_j$ and such that $|q| < |s|$.

At this point, the crucial idea was to replace q in the series $\sum_{n=0}^{\infty} q^n s^{-1-n}$ by a quaternionic linear operator T, and to look for a closed formula for this non-commutative power series. To obtain a closed formula for the formal power series

$$\sum_{n=0}^{\infty} T^n s^{-1-n}$$

was not an easy task. F. Colombo and I. Sabadini proceeded as follows: first they found a closed form for the series expansion $\sum_{n=0}^{\infty} q^n s^{-1-n}$, that is,

$$\sum_{n=0}^{\infty} q^n s^{-1-n} = -(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \overline{s}), \quad \text{for } |q| < |s|, \tag{1.11}$$

where formula (1.11) holds for all quaternions q, $s \in \mathbb{H}$ such that $|q| < |s|$, and then they observed that the right-hand side of (1.11) does not depend on the commutativity of the components of the quaternion q, because it contains just the powers of q. This second crucial fact led to the natural definition of the so-called S-spectrum

$$\sigma_S(T) = \{s \in \mathbb{H} : \ T^2 - 2T\mathrm{Re}(s) + |s|^2\mathcal{I} \text{ is not invertible}\},$$

where $\mathcal{I}$ is the identity operator, and of the S-resolvent operator

$$S_L^{-1}(s,T) := -(T^2 - 2T\mathrm{Re}(s) + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}). \tag{1.12}$$

It is also interesting to note that in 2006, when (1.12) was introduced, the Cauchy formula with slice hyperholomorphic kernel was not known, and so the sum of the series (1.11) was obtained with direct computations, using the Niven algorithm. This procedure is explained in Note 4.18.3 of the book [89]. The fact that the notion of slice hyperholomorphicity works very well for quaternionic operator theory was one of our motivations for its development.

The existence of the S-spectrum, with further considerations, appeared in 2007 in the paper [66] with G. Gentili and D.C. Struppa. This paper is an announcement of the results of the paper [68] (unfortunately, as may happen, published only in 2010), containing a version of the S-functional calculus defined just for slice hyperholomorphic functions that admit a power series expansion at the origin. This calculus is the starting point for the general definition of the S-functional calculus based on the Cauchy formula with slice hyperholomorphic kernels, which was completely described in [79]. The paper [79] together with the formulations of the S-functional calculus, see [80], and the S-resolvent equation, see [10], constitute the heart of the S-functional calculus. Finally, it is worthwhile to mention that the case of unbounded operators was treated in [97], [67] and with a direct approach in [124]. The study of the quaternionic evolution operator is in the paper [76], while the H^∞-functional calculus is in [30] and [52]. Finally, the main results about the spectral theorem based on the S-spectrum are proved in [13, 14].

The authors would like to thank G. Gentili, I. Sabadini, and D.C. Struppa for their comments on this note about the discovery of the S-spectrum.

1.2.2 Why Did It Take So Long to Understand the S-Spectrum?

After over 20 years of research, it is now clear that this new spectral theory based on the S-spectrum is very natural because it generalizes complex spectral theory and because the S-functional calculus (which is the slice hyperholomorphic functional calculus) and the quaternionic spectral theorem are based on the same notion of spectrum. There are several reasons why it took so long to discover the S-spectrum. We recall two of them.

Complex analysis and Cauchy–Fueter analysis are based on functions in the kernel of a constant-coefficient differential operator. This fact was misleading in the search for a new definition of hyperholomorphicity because one is tempted to look for a constant-coefficient quaternionic differential operator, not necessarily of first order, in order to find a "new notion of hyperholomorphicity" from which we could read the quaternionic spectrum. In [60] it was shown that slice hyperholomorphic functions are functions in the kernel of a first-order quaternionic differential operator *with nonconstant coefficients*. This fact was unexpected.

Another interesting fact is that in the paper of Fueter [111] there was a partial solution to the problem. In fact, in [111] Fueter gives a procedure to construct Fueter regular functions starting from holomorphic functions. His procedure consists of two steps: from holomorphic functions he constructs what he calls *hyperanalytic functions*; then he applies the Laplace operator to such hyperanalytic functions and thereby obtains Fueter regular functions. Fueter's hyperanalytic functions are what nowadays are called intrinsic slice hyperholomorphic functions but for some reason these functions have never been systematically studied. The Cauchy formula for hyperanalytic functions, which is the Cauchy formula for the S-functional calculus and from which one could read the S-spectrum, has never been investigated to the best of our knowledge.

1.3 The Fueter–Sce–Qian theorem and spectral theories

In this section we want to put the spectral theory on the S-spectrum into the perspective of the spectral theories that arise from the Fueter–Sce–Qian mapping theorem. In classical complex operator theory, the Cauchy formula of holomorphic functions is a fundamental tool for defining functions of operators. Moreover, the Cauchy–Riemann operator factorizes the Laplace operator, so holomorphic functions play also a crucial role in harmonic analysis and in boundary value problems. In higher dimensions, for quaternion-valued functions or, more generally, for Clifford-valued functions, there appear two different notions of hyperholomorphicity. The first one is called slice hyperholomorphicity and the second one is known under different names, depending to the dimension of the algebra and the range of the functions: Cauchy–Fueter regularity for quaternion-valued and monogenicity for Clifford-algebra-valued functions. The Fueter–Sce–Qian mapping theorem reveals a fundamental relation between the different notions of hyperholomorphicity. It will be explained in detail later on (see, for example, the section on the Fueter mapping in integral form), but it can be illustrated by the following diagram:

$$Hol(\Omega) \xrightarrow{F_1} \mathcal{N}(U) \xrightarrow{F_2} \mathcal{AM}(U).$$

Applying the mapping F_1, we can use any function in the set $Hol(\Omega)$ of holomorphic functions on a suitable open set Ω in $\mathbb{C}$ to generate a function in the set $\mathcal{N}(U)$ of all intrinsic slice hyperholomorphic functions on a certain open subset U of $\mathbb{H}$. Applying a second transformation F_2, we can transform any intrinsic slice hyperholomorphic function into an axially Fueter-regular resp. an axially monogenic function.

When considering quaternion-valued functions, the mapping F_2 that transforms an intrinsic slice hyperholomorphic function into a Fueter regular one is the application of the Laplace operator, i.e., $F_2 = \Delta$. When we work with Clifford-algebra-valued functions, then $F_2 = \Delta^{(n-1)/2}$, where n is the number of generating units of the Clifford algebra. The Fueter–Sce–Qian mapping theorem can be adapted to the more general case in which $\mathcal{N}(U)$ is replaced by slice hyperholomorphic functions and the axially regular (or axially monogenic) functions $\mathcal{AM}(U)$ are replaced by monogenic functions. The generalization of holomorphicity to quaternion- or Clifford-algebra-valued functions produces two different notions of hyperholomorphicity that are useful for different purposes. Precisely, we have that:

- The Cauchy formula for slice hyperholomorphic functions leads to the definition of the S-spectrum and the S-functional calculus for quaternionic linear operators. Moreover, the spectral theorem for quaternionic linear operators is based on the S-spectrum. The aim of this book and of the monograph [56], is to give a systematic treatment of this theory and of its applications.

- The Cauchy formula associated with Cauchy–Fueter regularity resp. monogenicity leads to the notion of monogenic spectrum and produces the Cauchy–Fueter functional calculus for quaternion-valued functions and the monogenic functional calculus for Clifford-algebra-valued functions. This theory has applications in harmonic analysis in higher dimension and in boundary value problems. For an overview on the monogenic functional calculus and its applications see [159] and for applications to boundary values problems see [155] and the references contained in those books.

We want to stress that these two approaches start from two totally different perspectives: while the first one develops the spectral theory of a single quaternionic- resp. Clifford-linear operator, the latter develops a joint spectral theory for n-tuples of real-linear operators. However, the F-functional calculus provides a relation between these two approaches and shows that they are consistent under reasonable assumptions. In this book we treat the quaternionic spectral theory on the S-spectrum, so very often we will refer to it as quaternionic spectral theory because no confusion arises with respect to the monogenic spectral theory.

Chapter 2

Slice Hyperholomorphic Functions

We will develop operator theory for quaternionic linear operators using the theory of slice hyperholomorphic functions. The most important results are the structure formula (or representation formula) and the Cauchy formulas with slice hyperholomorphic integral kernels. We will discuss the two Cauchy formulas and the associated Cauchy kernels in detail because they are the starting point for defining the S-functional calculus (in the quaternionic setting the S-functional calculus is often called the quaternionic functional calculus).

The Fueter mapping theorem is an important tool in hypercomplex analysis. It shows that the Laplace operator maps slice hyperholomorphic functions to Fueter regular functions and hence provides a method for generating Fueter regular functions. This theorem has been extended by Sce for the case of Clifford algebras with odd dimension and by Qian in the even dimension. In the literature it is often called the Fueter–Sce or Fueter–Sce–Qian theorem according to the setting. Starting from the Cauchy formula for slice hyperholomorphic functions, it is possible to give the Fueter mapping theorem an integral representation. One obtains then an integral transform that can be used to define the F-functional calculus.

We denote by $\mathbb{H}$ the algebra of quaternions. An element q of $\mathbb{H}$ is of the form

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3, \quad q_\ell \in \mathbb{R}, \ \ \ell = 0, 1, 2, 3,$$

where e_1, e_2 and e_3 are the generating imaginary units of $\mathbb{H}$. They satisfy the relations

$$e_1^2 = e_2^2 = e_3^2 = -1 \tag{2.1}$$

and

$$e_1 e_2 = -e_2 e_1 = e_3, \quad e_2 e_3 = -e_3 e_2 = e_1, \quad e_3 e_1 = -e_1 e_3 = e_2. \tag{2.2}$$

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_2

The real part, the imaginary part, and the modulus $|q|$ of a quaternion $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ are defined as $\mathrm{Re}(q) = q_0$, $\mathrm{Im}(q) = q_1e_1 + q_2e_2 + q_3e_3$, and $|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$, respectively. The conjugate of the quaternion q is

$$\bar{q} = \mathrm{Re}(q) - \mathrm{Im}(q) = q_0 - q_1e_1 - q_2e_2 - q_3e_3,$$

and it satisfies

$$|q|^2 = q\bar{q} = \bar{q}q.$$

The inverse of every nonzero element q is hence given by

$$q^{-1} = \frac{\bar{q}}{|q|^2}.$$

Let us denote by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = q_1e_1 + q_2e_2 + q_3e_3 : \ q_1^2 + q_2^2 + q_3^2 = 1\}.$$

Notice that if $j \in \mathbb{S}$, then $j^2 = -1$. For this reason the elements of $\mathbb{S}$ are also called imaginary units. The set $\mathbb{S}$ is a 2-dimensional sphere in $\mathbb{R}^4 \cong \mathbb{H}$. Given a nonreal quaternion $q = q_0 + \mathrm{Im}(q)$, we have $q = u + jv$ with $u = \mathrm{Re}(q)$, $j = \mathrm{Im}(q)/|\mathrm{Im}(q)| \in \mathbb{S}$, and $v = |\mathrm{Im}(q)|$. We can associate to q the 2-dimensional sphere

$$[q] = \{q_0 + j|\mathrm{Im}(q)| : \quad j \in \mathbb{S}\} = \{u + jv : \quad j \in \mathbb{S}\}.$$

This sphere is centered at the real point $q_0 = \mathrm{Re}(q)$ and has radius $|\mathrm{Im}(q)|$. The next lemma, which can be found in every standard textbook treating quaternions, shows that two quaternions belong to the same sphere if and only if they can be transformed into each other by multiplication by a nonzero quaternion.

Lemma 2.0.1. *Let $q \in \mathbb{H}$. A quaternion p belongs to $[q]$ if and only if there exists $h \in \mathbb{H} \setminus \{0\}$ such that $p = h^{-1}qh$.*

If $j \in \mathbb{S}$, then the set

$$\mathbb{C}_j = \{u + jv : \ u, v \in \mathbb{R}\}$$

is an isomorphic copy of the complex numbers. If, moreover, $i \in \mathbb{S}$ with $j \perp i$, then j, i, and $k := ji$ form a generating basis of $\mathbb{H}$, i.e., this basis also satisfies the relations (2.1) and (2.2). Hence, every quaternion $q \in \mathbb{H}$ can be written as

$$q = z_1 + z_2 i = z_1 + i\bar{z}_2$$

with unique $z_1, z_2 \in \mathbb{C}_j$, and so

$$\mathbb{H} = \mathbb{C}_j + i\mathbb{C}_j \quad \text{and} \quad \mathbb{H} = \mathbb{C}_j + \mathbb{C}_j i. \tag{2.3}$$

Moreover, we observe that

$$\mathbb{H} = \bigcup_{j \in \mathbb{S}} \mathbb{C}_j.$$

Finally, we introduce the notation $\mathbb{C}_j^+ := \{u + jv : u \in \mathbb{R}, v \geq 0\}$ for the upper half-plane in $\mathbb{C}_j$ and $\overline{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$.

2.1 Slice Hyperholomorphic Functions

The theory of slice hyperholomorphic functions is nowadays well developed. There are three possible ways to define slice hyperholomorphic functions: using the definition in [135], using the global operator of slice hyperholomorphic functions introduced in [60], or by the definition that comes from the Fueter–Sce–Qian mapping theorem. This last definition is the most appropriate for operator theory, and it is the one that we will use. In this section we therefore develop the part of the theory that it is relevant for our purposes.

Definition 2.1.1. Let $U \subseteq \mathbb{H}$.

(i) We say that U is *axially symmetric* if $[q] \subset U$ for every $q \in U$.

(ii) We say that U is a *slice domain* if $U \cap \mathbb{R} \neq \emptyset$ and if $U \cap \mathbb{C}_j$ is a domain in $\mathbb{C}_j$ for every $j \in \mathbb{S}$.

Definition 2.1.2 (Slice hyperholomorphic functions)**.** Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f : U \to \mathbb{H}$ is called a *left slice function* if it is of the form

$$f(q) = f_0(u,v) + jf_1(u,v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \Omega \to \mathbb{H}$ that satisfy the compatibility condition

$$f_0(u,-v) = f_0(u,v), \quad f_1(u,-v) = -f_1(u,v). \tag{2.4}$$

If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations

$$\frac{\partial}{\partial u} f_0(u,v) - \frac{\partial}{\partial v} f_1(u,v) = 0, \tag{2.5}$$

$$\frac{\partial}{\partial v} f_0(u,v) + \frac{\partial}{\partial u} f_1(u,v) = 0, \tag{2.6}$$

then f is called *left slice hyperholomorphic*. A function $f : U \to \mathbb{H}$ is called a *right slice function* if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v)j \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \Omega \to \mathbb{H}$ that satisfy (2.4). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations, then f is called *right slice hyperholomorphic*.

If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called *intrinsic*.

We denote the sets of left and right slice functions on U by $\mathcal{SF}_L(U)$ and $\mathcal{SF}_R(U)$ and the sets of left and right slice hyperholomorphic functions on U by $\mathcal{SH}_L(U)$ and $\mathcal{SH}_R(U)$, respectively. The set of intrinsic slice functions on U will be denoted by $\mathcal{FN}(U)$ and the set of slice hyperholomorphic functions on U will be denoted by $\mathcal{N}(U)$.

Remark 2.1.3. Every quaternion q can be represented as an element of a complex plane $\mathbb{C}_j$ using at least two different imaginary units $j \in \mathbb{S}$. We have $q = u + jv = u + (-j)(-v)$ and $-j$ also belongs to $\mathbb{S}$. If q is real, then we can use any imaginary unit $j \in \mathbb{S}$ to consider q an element of $\mathbb{C}_j$. The compatibility condition (2.4) ensures that the choice of this imaginary unit is irrelevant. In particular, it forces $f_1(u, v)$ to equal 0 if $v = 0$, that is if $q \in \mathbb{R}$.

Multiplication and composition with intrinsic functions preserve the slice structure and slice hyperholomorphicity. This is not true for arbitrary slice functions.

Theorem 2.1.4. *Let $U \subseteq \mathbb{H}$ be axially symmetric. The following statements hold:*

(i) *If $f \in \mathcal{NF}(U)$ and $g \in \mathcal{SF}_L(U)$, then $fg \in \mathcal{SF}_L(U)$. If $f \in \mathcal{SF}_R(U)$ and $g \in \mathcal{NF}(U)$, then $fg \in \mathcal{SF}_R(U)$.*

(ii) *If $f \in \mathcal{N}(U)$ and $g \in \mathcal{SH}_L(U)$, then $fg \in \mathcal{SH}_L(U)$. If $f \in \mathcal{SH}_R(U)$ and $g \in \mathcal{N}(U)$, then $fg \in \mathcal{SH}_R(U)$.*

(iii) *If $g \in \mathcal{NF}(U)$ and $f \in \mathcal{SF}_L(g(U))$, then $f \circ g \in \mathcal{SF}_L(U)$. If $g \in \mathcal{NF}(U)$ and $f \in \mathcal{SF}_R(g(U))$, then $f \circ g \in \mathcal{SF}_R(U)$.*

(iv) *If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_L(g(U))$, then $f \circ g \in \mathcal{SH}_L(U)$. If $g \in \mathcal{N}(U)$ and $f \in \mathcal{SH}_R(g(U))$, then $f \circ g \in \mathcal{SH}_R(U)$.*

Proof. Let $f = f_0 + jf_1 \in \mathcal{NF}(U)$ and $g = g_0 + jg_1 \in \mathcal{SF}_L(U)$. Since f is intrinsic, the components f_0, f_1 take real values. Hence, they commute with $j \in \mathbb{S}$, and we find for $q = u + jv \in U$ that

$$\begin{aligned} f(q)g(q) &= f_0(u,v)g_0(u,v) + jf_1(u,v)g_0(u,v) \\ &\quad + f_0(u,v)jg_1(u,v) + jf_1(u,v)jg_1(u,v) \\ &= f_0(u,v)g_0(u,v) - f_1(u,v)g_1(u,v) \\ &\quad + j(f_1(u,v)g_0(u,v) + f_0(u,v)g_1(u,v)). \end{aligned}$$

The functions

$$h_0(u,v) := f_0(u,v)g_0(u,v) - f_1(u,v)g_1(u,v)$$

and

$$h_1(u,v) := f_1(u,v)g_0(u,v) + f_0(u,v)g_1(u,v)$$

satisfy the compatibility condition (2.4), as one can check easily, and hence fg

belongs to $\mathcal{SF}_L(U)$. If, moreover, f and g are slice hyperholomorphic, then

$$\begin{aligned}\frac{\partial}{\partial u}h_0(u,v) &= \left(\frac{\partial}{\partial u}f_0(u,v)\right)g_0(u,v) + f_0(u,v)\left(\frac{\partial}{\partial u}g_0(u,v)\right)\\ &\quad - \left(\frac{\partial}{\partial u}f_1(u,v)\right)g_1(u,v) - f_1(u,v)\left(\frac{\partial}{\partial u}g_1(u,v)\right)\\ &= \left(\frac{\partial}{\partial v}f_1(u,v)\right)g_0(u,v) + f_0(u,v)\left(\frac{\partial}{\partial v}g_1(u,v)\right)\\ &\quad + \left(\frac{\partial}{\partial v}f_0(u,v)\right)g_1(u,v) + f_1(u,v)\left(\frac{\partial}{\partial v}g_0(u,v)\right)\\ &= \frac{\partial}{\partial v}h_1(u,v),\end{aligned}$$

and similarly one shows that also

$$\frac{\partial}{\partial v}h_0(u,v) = -\frac{\partial}{\partial u}h_1(u,v)$$

holds. Hence $fg = h_0 + jh_1$ is left slice hyperholomorphic.

Now let $g = g_0 + jg_1 \in \mathcal{NF}(U)$ and $f = f_0 + jf_1 \in \mathcal{SF}_L(g(U))$. For $q = u + jv \in U$, we have $g(q) = g_0(u,v) + jg_1(u,v) = \tilde{u} + i\tilde{v}$ with $\tilde{u} = g_0(u,v)$, $i = j\operatorname{sgn}(g_1(u,v)) \in \mathbb{S}$ and $\tilde{v} = |g_1(u,v)|$. Thus

$$\begin{aligned}f(g(q)) &= f_0(\tilde{u},\tilde{v}) + ig_1(\tilde{u},\tilde{v})\\ &= f_0(g_0(u,v),g_1(u,v)) + jf_1(g_0(u,v),g_1(u,v)),\end{aligned}$$

because f_1 is odd in the second variable. It is immediate that the functions $h_0(u,v) = f_0(g_0(u,v),g_1(u,v))$ and $h_1(u,v) = f_1(g_0(u,v),g_1(u,v))$ satisfy the compatibility condition (2.4), and so $f \circ g \in \mathcal{SF}_L(g(U))$. If furthermore f and g are slice hyperholomorphic, then

$$\begin{aligned}\frac{\partial}{\partial u}h_0(u,v) &= \frac{\partial}{\partial g_0}f_0(g_0(u,v),g_1(u,v))\frac{\partial}{\partial u}g_0(u,v)\\ &\quad + \frac{\partial}{\partial g_1}f_0(g_0(u,v),g_1(u,v))\frac{\partial}{\partial u}g_1(u,v)\\ &= \frac{\partial}{\partial g_1}f_1(g_0(u,v),g_1(u,v))\frac{\partial}{\partial v}g_1(u,v)\\ &\quad + \frac{\partial}{\partial g_0}f_1(g_0(u,v),g_1(u,v))\frac{\partial}{\partial v}g_0(u,v)\\ &= \frac{\partial}{\partial v}h_1(u,v)\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial u}h_1(u,v) &= \frac{\partial}{\partial g_0}f_1(g_0(u,v),g_1(u,v))\frac{\partial}{\partial u}g_0(u,v)\\
&\quad + \frac{\partial}{\partial g_1}f_1(g_0(u,v),g_1(u,v))\frac{\partial}{\partial u}g_1(u,v)\\
&= -\frac{\partial}{\partial g_1}f_0(g_0(u,v),g_1(u,v))\frac{\partial}{\partial v}g_1(u,v)\\
&\quad - \frac{\partial}{\partial g_0}f_0(g_0(u,v),g_1(u,v))\frac{\partial}{\partial v}g_0(u,v)\\
&= -\frac{\partial}{\partial v}h_0(u,v).
\end{aligned}$$

Hence $f\circ g = h_0 + jh_1$ is left slice hyperholomorphic.

Similar arguments show that the statements for right slice functions also hold. □

Lemma 2.1.5. *Let $U\subseteq\mathbb{H}$ be axially symmetric and let f be a left (or right) slice function on U. The following statements are equivalent.*

(i) *The function f is intrinsic.*

(ii) *We have $f(U\cap\mathbb{C}_j)\subset\mathbb{C}_j$ for every $j\in\mathbb{S}$.*

(iii) *We have $f(\overline{q}) = \overline{f(q)}$ for all $q\in U$.*

Proof. Assume that $f = f_0 + jf_1$ is a left slice function. (The other case follows analogously.) The implications (i)⇒(ii) and (i)⇒(ii) are immediate. In order to show the inverse relations, we first observe that for every $q = u + jv\in U$,

$$f(q) + f(\overline{q}) = f_0(u,v) + jf_1(u,v) + f_0(u,v) - jf_1(u,v) = 2f_0(u,v)$$

and

$$f(q) - f(\overline{q}) = f_0(u,v) + jf_1(u,v) - f_0(u,v) + jf_0(u,v) = 2jf_1(u,v).$$

If (ii) holds, then $f(u+jv)\in\mathbb{C}_j$ for every $j\in\mathbb{S}$, and hence it commutes with j. Thus

$$\begin{aligned}
jf_0(u,v) &= j(f(u+jv) + f(u-jv))\\
&= (f(u+jv) + f(u-jv))j = 2f_0(u,v)j.
\end{aligned}$$

Since a quaternion commutes with $j\in\mathbb{S}$ if and only if it belongs to $\mathbb{C}_j$, we have $f_0(u,v)\in\bigcap_{j\in\mathbb{S}}\mathbb{C}_j = \mathbb{R}$. For every $j\in\mathbb{S}$, we then have that

$$\begin{aligned}
jf_0(u,v) - f_1(u,v) &= j(f(u+jv) = f(u+jv)j\\
&= f_0(u,v)j + jf_1(u,v)j = jf_0(u,v) + jf_1(u,v)j,
\end{aligned}$$

and so $f_1(u,v) = -jf_1(u,v)j$. Thus $f_1(u,v)$ commutes with every $j \in \mathbb{S}$ and so also $f_1(u,v) \in \mathbb{R}$. Hence, f is intrinsic.

If on the other hand, (iii) holds, then for $q = u + jv \in U$ we have

$$\overline{2f_0(u,v)} = \overline{f(q) + f(\overline{q})} = f(\overline{q}) + f(q) = 2f_0(u,v)$$

and hence $f_0(u,v) \in \mathbb{R}$. We therefore also have

$$\begin{aligned} f_0(u,v) + jf_1(u,v) &= f(q) = \overline{f(\overline{q})} \\ &= \overline{f_0(u,v) - jf_1(u,v)} = f_0(u,v) + f_1(u,v)j, \end{aligned}$$

and so $jf_1(u,v) = f_1(u,v)j$. Since $j \in \mathbb{S}$ was arbitrary, we find that also $f_1(u,v) \in \mathbb{R}$ and that f is in turn intrinsic. □

If we restrict a slice hyperholomorphic function to one of the complex planes $\mathbb{C}_j$, then we obtain a function that is holomorphic in the usual sense.

Lemma 2.1.6 (The splitting lemma). *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $j, i \in \mathbb{S}$ with $i \perp j$. If $f \in \mathcal{SH}_L(U)$, then the restriction $f_j = f|_{U\cap\mathbb{C}_j}$ satisfies*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + j\frac{\partial}{\partial v} f_j(z)\right) = 0 \tag{2.7}$$

for all $z = u + jv \in U \cap \mathbb{C}_j$. Hence

$$f_j(z) = F_1(z) + F_2(z)i$$

with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$.

If $f \in \mathcal{SH}_R(U)$, then the restriction $f_j = f|_{U\cap\mathbb{C}_j}$ satisfies

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + \frac{\partial}{\partial v} f_j(z)j\right) = 0 \tag{2.8}$$

for all $z = u + jv \in U \cap \mathbb{C}_j$. Hence

$$f_j(z) = F_1(z) + iF_2(z)$$

with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$.

Proof. If $f = f_0 + jf_1$ is left slice hyperholomorphic, then

$$\begin{aligned} &\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + j\frac{\partial}{\partial v} f_j(z)\right) \\ &= \frac{1}{2}\left(\frac{\partial}{\partial u} f_0(u,v) + j\frac{\partial}{\partial u} f_1(u,v) + j\frac{\partial}{\partial v} f_0(u,v) - \frac{\partial}{\partial v} f_1(u,v)\right) = 0 \end{aligned}$$

because f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5). Due to (2.3), we can write $f_j(z) = F_1(z) + F_2(z)i$ with $\mathbb{C}_j$-valued component functions F_1 and F_2. Since 1 and i are linearly independent over $\mathbb{C}_j$, the above identity applies componentwise, and hence F_1 and F_2 are holomorphic.

The right slice hyperholomorphic case can be proved similarly. □

Remark 2.1.7. The splitting lemma states that the restriction of every left slice hyperholomorphic function to a complex plane $\mathbb{C}_j$ is left holomorphic, i.e., it is a holomorphic function with values in the left vector space $\mathbb{H} = \mathbb{C}_j + \mathbb{C}_j i$ over $\mathbb{C}_j$. The restriction of a right slice hyperholomorphic function to a complex plane $\mathbb{C}_j$ is right holomorphic, i.e., it is a holomorphic function with values in the right vector space $\mathbb{H} = \mathbb{C}_j + i\mathbb{C}_j$ over $\mathbb{C}_j$.

Theorem 2.1.8 (Identity principle)**.** *Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain, let $f, g : U \to \mathbb{H}$ be left (or right) slice hyperholomorphic, and set $\mathcal{Z} = \{q \in U : f(q) = g(q)\}$. If there exists $j \in \mathbb{S}$ such that $\mathcal{Z} \cap \mathbb{C}_j$ has an accumulation point in $U \cap \mathbb{C}_j$, then $f = g$.*

Proof. Assume that f and g are left slice hyperholomorphic and that $\mathcal{Z}\cap\mathbb{C}_j$ has an accumulation point in $U\cap\mathbb{C}_j$. We can furthermore assume that $g \equiv 0$. (Otherwise, we can simply replace f by $f-g$ and g by the constant zero function.) Since $U\cap\mathbb{C}_j$ is a domain in $\mathbb{C}_j$ and $f_j = f|_{U\cap\mathbb{C}_j}$ is an $\mathbb{H}$-valued (left) holomorphic function on this domain by Lemma 2.1.6, the identity theorem for holomorphic functions implies $f_j \equiv 0$. In particular, we have $f|_{U\cap\mathbb{R}} = f_j|_{U\cap\mathbb{R}} \equiv 0$.

If $i \in \mathbb{S}$ is now an arbitrary imaginary unit, then $f_i = f|_{U\cap\mathbb{C}_i}$ is again an $\mathbb{H}$-valued (left) holomorphic function on the domain $U \cap \mathbb{C}_i$ in $\mathbb{C}_i$. Since $f \equiv 0$ on $U \cap \mathbb{R} \neq \emptyset$ by the above arguments, the set of zeros of f_i has an accumulation point in $U \cap \mathbb{C}_i$. Hence, the identity theorem for holomorphic functions implies that also $f_i \equiv 0$ and in turn $f \equiv 0$ on all of U.

The right slice hyperholomorphic case follows with analogous arguments. □

The most important property of slice functions (and in particular for slice hyperholomorphic functions) is the structure formula, which is often also called representation formula.

Theorem 2.1.9 (The structure formula (or representation formula))**.** *Let $U \subseteq \mathbb{H}$ be axially symmetric and let $i \in \mathbb{S}$. A function $f : U \to \mathbb{H}$ is a left slice function on U if and only if for every $q = u + jv \in U$ we have*

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}ji\Big[f(\overline{z}) - f(z)\Big] \tag{2.9}$$

with $z = u + iv$. A function $f : U \to \mathbb{H}$ is a right slice function on U if and only if for every $q = u + jv \in U$ we have

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}\Big[f(\overline{z}) - f(z)\Big]ij \tag{2.10}$$

with $z = u + iv$.

Proof. For every left slice function f on U, we have

$$\begin{aligned} f(z) &= f(u + iv) = f_0(u, v) + if_1(u, v),\\ f(\overline{z}) &= f(u - iv) = f_0(u, v) - if_1(u, v), \end{aligned}$$

with functions f_0 and f_1 that satisfy the compatibility condition (2.4). Adding and subtracting these two equations, we get

$$f_0(u,v) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big], \quad f_1(u,v) = \frac{1}{2}i\Big[f(\overline{z}) - f(z)\Big]. \tag{2.11}$$

Since $f(q) = f_0(u,v) + jf_1(u,v)$, we obtain (2.9). If, on the other hand, f satisfies (2.9), then $f(q) = f_0(u,v)+jf_1(u,v)$ with f_0 and f_1 as in (2.11). Obviously f_0 and f_1 satisfy the compatibility condition (2.4), and hence f is a left slice function.

The statement about right slice functions can be shown with similar arguments. □

Remark 2.1.10. It is sometimes useful to rewrite (2.9) as

$$f(q) = \frac{1}{2}(1 - ij)f(z) + \frac{1}{2}(1 + ij)f(\overline{z})$$

and (2.10) as

$$f(q) = f(z)(1 - ij)\frac{1}{2} + f(\overline{z})(1 + ij)\frac{1}{2}.$$

As a consequence of the structure formula, every holomorphic function that is defined on a suitable open set in $\mathbb{C}_j$ has a slice hyperholomorphic extension.

Lemma 2.1.11. *Let $O \subset \mathbb{C}_j$ be open and symmetric with respect to the real axis. We call the set $[O] = \bigcup_{z\in O}[z]$ the axially symmetric hull of O.*

(i) *Every function $f : O \to \mathbb{H}$ has a unique extension $\operatorname{ext}_L(f)$ to a left slice function on $[O]$ and a unique extension $\operatorname{ext}_R(f)$ to a right slice function on $[O]$.*

(ii) *If $f : O \to \mathbb{H}$ is left holomorphic, i.e., it satisfies (2.7), then $\operatorname{ext}_L(f)$ is left slice hyperholomorphic.*

(iii) *If f is right holomorphic, i.e., it satisfies (2.8), then $\operatorname{ext}_R(f)$ is right slice hyperholomorphic.*

Proof. The left and right slice extensions $\operatorname{ext}_L(f)$ and $\operatorname{ext}_R(f)$ are obviously given by (2.9) resp. (2.10). Due to Theorem 2.1.9, they are also unique.

Assume that f is left holomorphic. Then $\operatorname{ext}_L(f)(q) = f_0(u,v) + if_1(u,v)$ for $q = u + jv$, with

$$f_0(u,v) = \frac{1}{2}\left[f(u - jv) + f(u + jv)\right]$$

and

$$f_1(u,v) = \frac{1}{2}j\left[f(u - jv) - f(u + jv)\right].$$

It remains to show that this actually defines a left slice hyperholomorphic function, i.e., that f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5). Because of (2.7), we have

$$\begin{aligned}\frac{\partial}{\partial u} f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial u} f(u-jv) + \frac{\partial}{\partial u} f(u+jv)\right] \\ &= \frac{1}{2}\left[j\frac{\partial}{\partial v} f(u-jv) - j\frac{\partial}{\partial v} f(u+jv)\right] = \frac{\partial}{\partial v} f_1(u,v).\end{aligned}$$

Similarly, we also have

$$\begin{aligned}\frac{\partial}{\partial v} f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial v} f(u-jv) + \frac{\partial}{\partial v} f(u+jv)\right] \\ &= \frac{1}{2}\left[-j\frac{\partial}{\partial u} f(u-jv) + j\frac{\partial}{\partial u} f(u+jv)\right] = -\frac{\partial}{\partial u} f_1(u,v).\end{aligned}$$

Thus $\mathrm{ext}_L(f)$ is actually left slice hyperholomorphic. The right slice hyperholomorphic case can be shown with analogous arguments. □

Slice hyperholomorphic functions admit a special kind of derivative, which again yields a slice hyperholomorphic function.

Definition 2.1.12. Let $f : U \subseteq \mathbb{H} \to \mathbb{H}$ and let $q = u + jv \in U$. If q is not real, then we say that f *admits a left slice derivative in* q if

$$\partial_S f(q) := \lim_{p\to q,\, p\in\mathbb{C}_j} (p-q)^{-1}(f_j(p) - f_j(q)) \tag{2.12}$$

exists and is finite. If q is real, then we say that f *admits a left slice derivative in* q if (2.12) exists for every $j \in \mathbb{S}$.

Similarly, we say that f *admits a right slice derivative at a nonreal point* $q = u + jv \in U$ if

$$\partial_S f(q) := \lim_{p\to q,\, p\in\mathbb{C}_j} (f_j(p) - f_j(q))(p-q)^{-1} \tag{2.13}$$

exists and is finite, and we say that f *admits a right slice derivative at a real point* $q \in U$ if (2.13) exists and is finite for every $j \in \mathbb{S}$.

Remark 2.1.13. Observe that $\partial_S f(q)$ is uniquely defined and independent of the choice of $j \in \mathbb{S}$ even if q is real. If f admits a slice derivative, then f_j is $\mathbb{C}_j$-complex left resp. right differentiable, and we obtain

$$\partial_S f(q) = f_j'(q) = \frac{\partial}{\partial u} f_j(q) = \frac{\partial}{\partial u} f(q), \quad q = u + jv. \tag{2.14}$$

Proposition 2.1.14. *Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let $f : U \to \mathbb{H}$ be a real differentiable function.*

(i) *If $f(q) = f_0(u,v) + jf_1(u,v)$ is left (or right) slice hyperholomorphic, then it admits a left (resp. right) slice derivative and $\partial_S f$ is again left (resp. right) slice hyperholomorphic on U.*

(ii) *If f is a left (or right) slice function that admits a left (resp. right) slice derivative, then f is left (resp. right) slice hyperholomorphic.*

(iii) *If U is a slice domain, then every function that admits a left (resp. right) slice derivative is left (resp. right) slice hyperholomorphic.*

Proof. If f is a left slice hyperholomorphic function on U and $q = u+jv \in U$, then its restriction to the complex plane $\mathbb{C}_j$ can be written as $f_j(q) = F_1(q)+F_2(q)i$ for $i \in \mathbb{S}$ with $i \perp j$. By Lemma 2.1.6, the component functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$ are holomorphic, and hence

$$\begin{aligned}&\lim_{p\to q,\, p\in\mathbb{C}_j} (p-q)^{-1}(f_j(p) - f_j(q))\\ &= \lim_{p\to q,\, p\in\mathbb{C}_j} (p-q)^{-1}(F_1(p) + F_2(p)i - F_1(p) - F_2(q)i)\\ &= F_1'(q) + F_2'(q)i\end{aligned}$$

exists. Therefore, f admits a left slice derivative. Moreover, this slice derivative coincides with the derivative with respect to the real part of the quaternion by (2.14), and hence

$$\partial_S f(q) = \frac{\partial}{\partial u} f(q) = \frac{\partial}{\partial u} f_0(u,v) + j\frac{\partial}{\partial u} f_1(u,v), \quad q = u + jv.$$

The functions $\frac{\partial}{\partial u} f_0(u,v)$ and $\frac{\partial}{\partial u} f_1(u,v)$ obviously satisfy the compatibility condition (2.4). Since f_0 and f_1 satisfy the Cauchy–Riemann equations, they are infinitely differentiable. Hence $\frac{\partial}{\partial u}$, and $\frac{\partial}{\partial v}$ commute with $\frac{\partial}{\partial u}$ and we obtain that also $\frac{\partial}{\partial u} f_0(u,v)$ and $\frac{\partial}{\partial u} f_1(u,v)$ satisfy the Cauchy–Riemann equations (2.5). Thus $\partial_S f$ is left slice hyperholomorphic too.

If, on the other hand, $f(q) = f_0(u,v) + jf_1(u,v)$ is a left slice function that admits a left slice derivative, we choose $j \in \mathbb{S}$. Then f_j is an $\mathbb{H}$-valued left holomorphic function on $U \cap \mathbb{C}_j$. By Lemma 2.1.11, the left slice extension $\mathrm{ext}_L(f_j)$ of f_j is therefore a left slice hyperholomorphic extension of f_j. Since f is already a left slice function, we find that $f = \mathrm{ext}_L(f_j)$, and so f is left slice hyperholomorphic.

If, finally, U is an axially symmetric slice domain and f is an arbitrary function on U that admits a left slice derivative, then we can again choose an arbitrary imaginary unit $j \in \mathbb{S}$ and find that f_j is left holomorphic. We set $\tilde{f} = \mathrm{ext}_L(f_j)$ and $g = f - \tilde{f}$. Obviously $g \equiv 0$ on $U \cap \mathbb{C}_j$. Moreover, g admits a left slice derivative, since f and $\tilde{f}$ both admit a left slice derivative. For every $i \in \mathbb{S}$, the restriction $g_i = g|_{U\cap\mathbb{C}_i}$ is a (left) holomorphic function on the domain $U \cap \mathbb{C}_i$ in $\mathbb{C}_i$. Moreover, $g|_{U\cap\mathbb{R}} \equiv 0$, and so the set of zeros of g_i has an accumulation point in $U \cap \mathbb{C}_i$. By the identity theorem for holomorphic functions, we find that $g_i \equiv 0$, and in turn $g \equiv 0$ because $i \in \mathbb{S}$ was arbitrary. Therefore, $f = \tilde{f} = \mathrm{ext}_L(f_j)$ is left slice hyperholomorphic.

The right slice hyperholomorphic case can be shown by analogous arguments. □

Important examples of slice hyperholomorphic functions are power series in the quaternionic variable: power series of the form $\sum_{n=0}^{+\infty} q^n a_n$ with $a_n \in \mathbb{H}$ are left slice hyperholomorphic, and power series of the form $\sum_{n=0}^{+\infty} a_n q^n$ are right slice hyperholomorphic. Such a power series is intrinsic if and only if the coefficients a_n are real. Conversely, every slice hyperholomorphic function can be expanded at any real point into a power series due to the splitting lemma.

Theorem 2.1.15. *Let $a \in \mathbb{R}$, let $r > 0$, and let $B_r(a) = \{q \in \mathbb{H} : |q - a| < r\}$. If $f \in \mathcal{SH}_L(B_r(a))$, then*

$$f(q) = \sum_{n=0}^{+\infty} (q-a)^n \frac{1}{n!} \partial_S^n f(a) \quad \forall q = u + jv \in B_r(a). \tag{2.15}$$

If, on the other hand, $f \in \mathcal{SH}_R(B_r(a))$, then

$$f(q) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\partial_S^n f(a)\right) (q-a)^n \quad \forall q = u + jv \in B_r(a).$$

Proof. Let $f \in \mathcal{SH}_L(B_r(a))$ and $q = u+jv \in B_r(a)$. By Lemma 2.1.6, the function $f_j = f|_{B_r(a)\cap\mathbb{C}_j}$ is left holomorphic on $B_r(a)$ and can hence be expanded into a power series. We obtain

$$f(q) = f_j(q) = \sum_{n=0}^{+\infty} (q-a)^n \frac{1}{n!} f_j^{(n)}(a).$$

But due to (2.14), we have

$$f_j^{(n)}(a) = \frac{\partial^n}{\partial^n u} f_j(a) = \frac{\partial^n}{\partial u^n} f(a) = \partial_S^n f(a).$$

The coefficients in the power series expansion are hence independent of the complex plane in which they are computed, and (2.40) holds. The right slice hyperholomorphic case follows with similar arguments. □

As pointed out above, the product of two slice hyperholomorphic functions is not slice hyperholomorphic unless the factor on the appropriate side is intrinsic. However, there exists a regularized product that preserves slice hyperholomorphicity.

Definition 2.1.16. For $f = f_0 + jf_1, g = g_0 + jg_1 \in \mathcal{SH}_L(U)$, we define their *left slice hyperholomorphic product* as

$$f *_L g = (f_0 g_0 - f_1 g_1) + j(f_0 g_1 + f_1 g_0).$$

For $f = f_0 + f_1 j, g = g_0 + g_1 j \in \mathcal{SH}_R(U)$, we define their *right slice hyperholomorphic product* as

$$f *_R g = (f_0 g_0 - f_1 g_1) + (f_0 g_1 + f_1 g_0)j.$$

Remark 2.1.17. The slice hyperholomorphic product is associative and distributive, but it is in general not commutative. If f is intrinsic, then $f *_L g$ coincides with the pointwise product fg and

$$f *_L g = fg = g *_L f. \tag{2.16}$$

Similarly, if g is intrinsic, then $f *_R g$ coincides with the pointwise product fg and

$$f *_R g = fg = g *_R f. \tag{2.17}$$

Example 2.1.18. If $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ are two left slice hyperholomorphic power series, then their slice hyperholomorphic product equals the usual product of formal power series with coefficients in a noncommutative ring:

$$\left(\sum_{n=0}^{+\infty} q^n a_n\right) *_L \left(\sum_{n=0}^{+\infty} q^n b_n\right) = (f *_L g)(q) = \sum_{n=0}^{+\infty} q^n \sum_{k=0}^{n} a_k b_{n-k}. \tag{2.18}$$

Similarly, we have for right slice hyperholomorphic power series that

$$\left(\sum_{n=0}^{+\infty} a_n q^n\right) *_R \left(\sum_{n=0}^{+\infty} b_n q^n\right) = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) q^n. \tag{2.19}$$

Definition 2.1.19. We define for $f = f_0 + jf_1 \in \mathcal{SH}_L(U)$ its slice hyperholomorphic conjugate $f^c = \overline{f_0} + j\overline{f_1}$ and its symmetrization $f^s = f *_L f^c = f^c *_L f$. Similarly, we define for $f = f_0 + f_1 j \in \mathcal{SH}_R(U)$ its slice hyperholomorphic conjugate as $f^c = \overline{f_0} + \overline{f_1} j$ and its symmetrization as $f^s = f *_R f^c = f^c *_R f$.

The symmetrization of a left slice hyperholomorphic function $f = f_0 + jf_1$ is explicitly given by

$$f^s = |f_0|^2 - |f_1|^2 + j2\mathrm{Re}\left(f_0\overline{f_1}\right).$$

Hence it is an intrinsic function. It is $f^s(q) = 0$ if and only if $f(\tilde{q}) = 0$ for some $\tilde{q} \in [q]$. Furthermore, one has

$$f^c(q) = \overline{f_0(q_0, q_1)} + j_q\overline{f_1(q_0, q_1)} = \overline{f_0(q_0, q_1)} + \overline{f_1(q_0, q_1)(-j_q)} = \overline{f(\overline{q})}, \tag{2.20}$$

and an easy computation shows that

$$f *_L g(q) = f(q)g\left(f(q)^{-1}qf(q)\right) \quad \text{if } f(q) \neq 0. \tag{2.21}$$

For $f(q) \neq 0$, one has

$$\begin{aligned} f^s(q) &= f(q)f^c\left(f(q)^{-1}qf(q)\right) \\ &= f(q)\overline{f\left(\overline{f(q)^{-1}qf(q)}\right)} = f(q)\overline{f\left(f(q)^{-1}\overline{q}f(q)\right)}. \end{aligned} \tag{2.22}$$

Similar computations hold in the right slice hyperholomorphic case. Finally, if f is intrinsic, then $f^c(q) = f(q)$ and $f^s(q) = |f(q)|^2$.

As an immediate consequence of Definition 2.1.19 and the above discussion we obtain the following corollary.

Corollary 2.1.20. *The following statements are true:*

(i) *For $f \in \mathcal{SH}_L(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_L}, which satisfies $f^{-*_L} *_L f = f *_L f^{-*_L} = 1$, is given by*

$$f^{-*_L} = (f^s)^{-1} *_L f^c = (f^s)^{-1} f^c,$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(ii) *For $f \in \mathcal{SH}_R(U)$ with $f \not\equiv 0$, its slice hyperholomorphic inverse f^{-*_R}, which satisfies $f^{-*_R} *_R f = f *_R f^{-*_R} = 1$, is given by*

$$f^{-*_R} = f^c *_R (f^s)^{-1} = f^c (f^s)^{-1},$$

and it is defined on $U \setminus [\mathcal{Z}_f]$, where $\mathcal{Z}_f = \{s \in U : f(s) = 0\}$.

(iii) *If $f \in \mathcal{N}(U)$ with $f \not\equiv 0$, then $f^{-*_L} = f^{-*_R} = f^{-1}$.*

The modulus $|f^{-*_L}|$ is in a certain sense comparable to $1/|f|$. Since f^s is intrinsic, we have $|f^s(q)| = |f^s(\tilde{q})|$ for every $\tilde{q} \in [q]$. Since $f(q)qf(q)^{-1} \in [q]$ by Lemma 2.0.1, we find for $f(q) \neq 0$, because of (2.22), that

$$\begin{aligned} |f^s(q)| &= \left|f^s\left(f(q)qf(q)^{-1}\right)\right| \\ &= \left|f\left(f(q)qf(q)^{-1}\right)\overline{f(\overline{q})}\right| = \left|f\left(f(q)qf(q)^{-1}\right)\right| \left|f(\overline{q})\right|. \end{aligned}$$

Therefore, we have, because of (2.20), that

$$\begin{aligned} \left|f^{-*_L}(q)\right| &= \left|f^s(q)^{-1}\right| \left|f^c(q)\right| \\ &= \frac{1}{\left|f\left(f(q)qf(q)^{-1}\right)\right| \left|f(\overline{q})\right|} \left|f(\overline{q})\right| = \frac{1}{\left|f\left(f(q)\overline{q}f(q)^{-1}\right)\right|}, \end{aligned}$$

and so

$$\left|f^{-*_L}(q)\right| = \frac{1}{|f(\tilde{q})|} \quad \text{with } \tilde{q} = f(q)\overline{q}f(q)^{-1} \in [q]. \tag{2.23}$$

An analogous estimate holds for the slice hyperholomorphic inverse of a right slice hyperholomorphic function.

Slice hyperholomorphic functions satisfy a version of Cauchy's integral theorem and a Cauchy formula with a slice hyperholomorphic integral kernel.

Theorem 2.1.21 (Cauchy's integral theorem). *Let $U \subset \mathbb{H}$ be open, let $j \in \mathbb{S}$, and let $f \in \mathcal{SH}_L(U)$ and $g \in \mathcal{SH}_R(U)$. Moreover, let $D_j \subset U \cap \mathbb{C}_j$ be an open and bounded subset of the complex plane $\mathbb{C}_j$ with $\overline{D_j} \subset U \cap \mathbb{C}_j$ such that ∂D_j is a finite union of piecewise continuously differentiable Jordan curves. Then*

$$\int_{\partial D_j} g(s)\, ds_j\, f(s) = 0,$$

where $ds_j = ds(-j)$.

Proof. If we choose $i \in \mathbb{S}$ with $i \perp j$, then we can write $f(z) = F_1(z) + F_2(z)i$ and $g(z) = G_1(z) + iG_2(z)$ for $z \in U \cap \mathbb{C}_j$ with holomorphic component functions $F_1, F_2, G_1, G_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$. By the Cauchy integral theorem for holomorphic functions, we hence obtain

$$\begin{aligned}
&\int_{\partial D_j} g(s)\, ds_j\, f(s)\\
&= \int_{\partial D_j} G_1(s)\, ds_j\, F_1(s) + \left(\int_{\partial D_j} G_1(s)\, ds_j\, F_2(s)\right) i\\
&\quad + i\int_{\partial D_j} G_2(s)\, ds_j\, F_1(s) + i\left(\int_{\partial D_j} G_1(s)\, ds_j\, F_2(s)\right) i = 0.
\end{aligned}$$

□

In order to determine the left and right slice hyperholomorphic Cauchy kernels, we start from an analogy with the classical complex case. We consider the series expansion of the complex Cauchy kernel and determine its closed form under the assumption that s and q are quaternions that do not commute.

Theorem 2.1.22. *Let $q, s \in \mathbb{H}$ with $|q| < |s|$. Then*

$$\sum_{n=0}^{+\infty} q^n s^{-n-1} = -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}) \tag{2.24}$$

and

$$\sum_{n=0}^{+\infty} s^{-n-1} q^n = -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \tag{2.25}$$

Proof. We prove only (2.24), since (2.25) follows by analogous arguments. Due to the identities $2\mathrm{Re}(s) = s + \overline{s}$ and $|s|^2 = s\overline{s}$, we have

$$\begin{aligned}
&(q^2 - 2\mathrm{Re}(s)q + |s|^2) \sum_{n=0}^{+\infty} q^n s^{-n-1} = \\
&= \sum_{n=0}^{+\infty} q^{n+2} s^{-n-1} - \sum_{n=0}^{+\infty} q^{n+1} s^{-n-1} 2\mathrm{Re}(s) + \sum_{n=0}^{+\infty} q^n s^{-n-1} |s|^2 \\
&= \sum_{n=1}^{+\infty} q^{n+1} s^{-n} - \sum_{n=0}^{+\infty} q^{n+1} s^{-n} \\
&\quad - \sum_{n=0}^{+\infty} q^{n+1} s^{-n-1}\overline{s} + \sum_{n=0}^{+\infty} q^n s^{-n}\overline{s} = -q + \overline{s}.
\end{aligned}$$

Multiplication by $(q^2 - 2\mathrm{Re}(s)q - |s|^2)^{-1}$ from the left yields (2.24). □

Definition 2.1.23. We define the *left slice hyperholomorphic Cauchy kernel* as

$$S_L^{-1}(s, q) := -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}), \quad q \notin [s],$$

and the *right slice hyperholomorphic Cauchy kernel* as

$$S_R^{-1}(s, q) := -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}, \quad q \notin [s].$$

The slice hyperholomorphic Cauchy kernels $S_L^{-1}(s, q)$ and $S_R^{-1}(s, q)$ can be written in two different ways, as the next proposition shows.

Proposition 2.1.24. *If $q, s \in \mathbb{H}$ with $q \notin [s]$, then*

$$-(q^2 - 2q\mathrm{Re}(s) + |s|^2)^{-1}(q - \overline{s}) = (s - \bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1} \tag{2.26}$$

and

$$(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(s - \bar{q}) = -(q - \overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \tag{2.27}$$

Proof. Due to the identities $|q| = q\overline{q} = \overline{q}q$ and $2\mathrm{Re}(q) = \overline{q} + q$, we have

$$\begin{aligned}
&- (q - \overline{s})(s^2 - 2\mathrm{Re}(q)s + |q|^2) \\
&= -qs^2 + q(q + \overline{q})s - q^2\overline{q} + \overline{s}s^2 - \overline{s}s(q + \overline{q}) + \overline{s}q\overline{q} \\
&= q^2(s - \overline{q}) + |s|^2(s - \overline{q}) - qs^2 + q\overline{q}s - \overline{s}sq + \overline{s}q\overline{q}.
\end{aligned}$$

Since

$$\begin{aligned}
&- qs^2 + q\overline{q}s - \overline{s}sq + \overline{s}q\overline{q} = -qs^2 + |q|^2 s - |s|^2 q + \overline{s}q\overline{q} \\
&= -qs^2 + s|q|^2 - q|s|^2 + \overline{s}q\overline{q} = -qs^2 + sq\overline{q} - q\overline{s}s + \overline{s}q\overline{q} \\
&= -q(s + \overline{s})s + (s + \overline{s})q\overline{q} = -2\mathrm{Re}(s)q(s - \overline{q}),
\end{aligned}$$

we further conclude that

$$-(q-\overline{s})(s^2-2\mathrm{Re}(q)s+|q|^2)=(q^2-2\mathrm{Re}(s)q+|s|^2)(s-\overline{q}).$$

Multiplying this identity by $(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}$ on the right and by $(q^2-2\mathrm{Re}(s)q+|s|^2)^{-1}$ on the left, we obtain (2.26). Exchanging the roles of q and s and multiplying by -1 then yields (2.27). □

Proposition 2.1.24 justifies the following definition.

Definition 2.1.25. Let $q,s\in\mathbb{H}$ with $q\notin[s]$.

- We say that $S_L^{-1}(s,q)$ *is written in the form I* if

$$S_L^{-1}(s,q):=-(q^2-2\mathrm{Re}(s)q+|s|^2)^{-1}(q-\overline{s}).$$

- We say that $S_L^{-1}(s,q)$ *is written in the form II* if

$$S_L^{-1}(s,q):=(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}.$$

- We say that $S_R^{-1}(s,q)$ *is written in the form I* if

$$S_R^{-1}(s,q):=-(q-\overline{s})(q^2-2\mathrm{Re}(s)q+|s|^2)^{-1}.$$

- We say that $S_R^{-1}(s,q)$ *is written in the form II* if

$$S_R^{-1}(s,q):=(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}(s-\bar{q}).$$

Corollary 2.1.26. *For $q,s\in\mathbb{H}$ with $s\notin[q]$, we have*

$$S_L^{-1}(s,q)=-S_R^{-1}(q,s).$$

Lemma 2.1.27. *Let $q,s\in\mathbb{H}$ with $s\notin[q]$.*

The left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,q)$ is left slice hyperholomorphic in q and right slice hyperholomorphic in s.

The right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,q)$ is left slice hyperholomorphic in s and right slice hyperholomorphic in q.

Proof. Let $q=u+jv$. We write $S_L^{-1}(s,q)$ in the form II, i.e.,

$$S_L^{-1}(s,q)=(s-\overline{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}.$$

Then $S_L^{-1}(s,q)=f_0(u,v)+jf_1(u,v)$ with

$$f_0(u,v)=(s-u)(s^2-2us+u^2+v^2)^{-1}, \tag{2.28}$$

$$f_1(u,v)=v(s^2-2us+u^2+v^2)^{-1}. \tag{2.29}$$

Obviously, f_0 and f_1 satisfy the compatibility condition (2.4). Moreover,

$$\begin{aligned}\frac{\partial}{\partial u} f_0(u,v) &= -(s^2-2us+u^2+v^2)^{-1}\\ &\quad - (s-u)(s^2-2us+u^2+v^2)^{-2}(-2s+2u)\\ &= (s^2-2us+u^2+v^2)^{-2}\left((s-u)^2-v^2\right),\\ \frac{\partial}{\partial v} f_0(u,v) &= -(s-u)(s^2-2us+u^2+v^2)^{-2}2v,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial u} f_1(u,v) &= -v(s^2-2us+u^2+v^2)^{-2}2(-s+u),\\ \frac{\partial}{\partial v} f_1(u,v) &= (s^2-2us+u^2+v^2)^{-1}\\ &\quad - v(s^2-2us+u^2+v^2)^{-2}2v\\ &= (s^2-2us+u^2+v^2)^{-2}\left((s-u)^2-v^2\right).\end{aligned}$$

Hence they also satisfy the Cauchy–Riemann equations (2.5), and so the mapping $q \mapsto S_L^{-1}(s,q)$ is left slice hyperholomorphic.

In order to show that $S_L^{-1}(s,q)$ is right slice hyperholomorphic in s, we write $S_L^{-1}(s,q)$ in form I, i.e.,

$$S_L^{-1}(s,q) = -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q-\overline{s}).$$

For $s = u + jv$, we hence have $S_L^{-1}(s,q) = f_0(u,v) + f_1(u,v)j$ with

$$\begin{aligned}f_0(u,v) &= (q^2-2uq+u^2+v^2)^{-1}(q-u),\\ f_1(u,v) &= (q^2-2uq+u^2+v^2)^{-1}v.\end{aligned}$$

But these are exactly the functions (2.28) and (2.29) in which s is replaced by q. As we showed above, they satisfy the compatibility condition (2.4) and the Cauchy–Riemann equations (2.5), and so the mapping $s \mapsto S_L^{-1}(s,q)$ is right slice hyperholomorphic.

The properties of the right slice hyperholomorphic Cauchy kernel follow immediately, since $S_R^{-1}(s,q) = -S_L^{-1}(q,s)$ by Corollary 2.1.26. □

Lemma 2.1.28. *If s and q commute, then the left and the right slice hyperholomorphic Cauchy kernels reduce to the complex Cauchy kernel, i.e.,*

$$S_L^{-1}(s,q) = (s-q)^{-1} = S_R^{-1}(s,q) \quad \textit{if } sq = qs.$$

Proof. If q and s commute, then

$$q^2 - 2\mathrm{Re}(s)q + |s|^2 = q^2 - (s+\overline{s})q + s\overline{s} = (q-s)(q-\overline{s}).$$

Hence, we have

$$\begin{aligned} S_L^{-1}(s,q) &= -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(q - \overline{s}) \\ &= -(q-s)^{-1}(q-\overline{s})^{-1}(q-\overline{s}) = (s-q)^{-1}, \end{aligned}$$

and similarly also

$$\begin{aligned} S_R^{-1}(s,q) &= -(q-\overline{s})(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} \\ &= -(\overline{s}-q)(q-\overline{s})^{-1}(q-s)^{-1} = (s-q)^{-1}. \end{aligned}$$ □

Remark 2.1.29. Observe that left and right slice hyperholomorphic functions satisfy Cauchy formulas with different kernels. This is different from what happens for Fueter regular functions, where both left and right Fueter regular functions satisfy a Cauchy formula with the same integral kernel.

Definition 2.1.30 (Slice Cauchy domain)**.** An axially symmetric open set $U \subset \mathbb{H}$ is called a *slice Cauchy domain* if $U \cap \mathbb{C}_j$ is a Cauchy domain in $\mathbb{C}_j$ for every $j \in \mathbb{S}$. More precisely, U is a slice Cauchy domain if for every $j \in \mathbb{S}$ the boundary $\partial(U \cap \mathbb{C}_j)$ of $U \cap \mathbb{C}_j$ is the union a finite number of nonintersecting piecewise continuously differentiable Jordan curves in $\mathbb{C}_j$.

Remark 2.1.31. Observe that every slice Cauchy domain has only finitely many components (i.e., maximal connected subsets). Moreover, at most one of them is unbounded, and if there exists an unbounded component, then it contains a neighborhood of ∞ in $\mathbb{H}$.

Theorem 2.1.32 (The Cauchy formulas)**.** *Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$, and set $ds_j = ds(-j)$. If f is a (left) slice hyperholomorphic function on a set that contains $\overline{U}$, then*

$$f(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s), \quad \textit{for every } q \in U. \tag{2.30}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q), \quad \textit{for every } q \in U. \tag{2.31}$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. Assume that f is left slice hyperholomorphic on a set that contains $\overline{U}$ and let $q = u + iv \in U$. Since $S_L^{-1}(s,q)$ is left slice hyperholomorphic in q, we deduce from Theorem 2.1.9 that with $p = u + jv$,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s)\\
&= \frac{1}{2}(1-ij)\left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,p)\,ds_j\,f(s)\right)\\
&\quad + \frac{1}{2}(1+ij)\left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,\overline{p})ds_j f(s)\right)\\
&= \frac{1}{2}(1-ij)\left(\frac{1}{2\pi j}\int_{\partial(U\cap\mathbb{C}_j)} (p-s)^{-1}\,ds\,f(s)\right)\\
&\quad + \frac{1}{2}(1+ij)\left(\frac{1}{2\pi j}\int_{\partial(U\cap\mathbb{C}_j)} (\overline{p}-s)^{-1}\,ds\,f(s)\right),
\end{aligned}$$

where the last identity follows from Lemma 2.1.28 because p, s, and j all belong to $\mathbb{C}_j$ and hence commute mutually. By Lemma 2.1.6, the restriction of f to $\mathbb{C}_j$ is left holomorphic. Hence it satisfies the classical Cauchy formula. Together with Theorem 2.1.9, this implies that

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s) = \frac{1}{2}(1-ij)f(p) + \frac{1}{2}(1+ij)f(\overline{p}) = f(q).$$

Since $f(q)$ is independent of U and $j \in \mathbb{S}$, the integral in (4.39) is obviously independent of U and j.

The right slice hyperholomorphic case is again shown by analogous arguments. □

Theorem 2.1.33 (Cauchy formulas on unbounded slice Cauchy domains)**.** *Let $U \subset \mathbb{H}$ be an unbounded slice Cauchy domain and let $j \in \mathbb{S}$. If $f \in \mathcal{SH}_L(\overline{U})$ and $f(\infty) := \lim_{|q|\to\infty} f(q)$ exists, then*

$$f(q) = f(\infty) + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s) \quad \textit{for every } q \in U.$$

If $f \in \mathcal{SH}_R(\overline{U})$ and $f(\infty) := \lim_{|q|\to\infty} f(q)$ exists, then

$$f(q) = f(\infty) + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,q) \quad \textit{for every } q \in U.$$

Proof. Let $f \in \mathcal{SH}_L(\overline{U})$ such that $f(\infty) := \lim_{|q|\to\infty} f(q)$ exists and let $q \in U$. For sufficiently large $r > 0$, the set $U_r := U \cap B_r(0)$ is a bounded slice Cauchy domain with $q \in U_r$ and $\mathbb{H} \setminus U_r \subset U$. By

$$f(q) = \frac{1}{2\pi} \int_{\partial(U_r \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s)$$
$$= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s) + \frac{1}{2\pi} \int_{\partial(B_r(0) \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s).$$

Theorem 2.1.21 implies that we can vary r without changing the value of the second integral. Letting r tend to infinity, we find that it equals $f(\infty)$, and we obtain the statement. □

Finally, just like holomorphic functions, slice hyperholomorphic functions can be approximated by rational funcitons.

Definition 2.1.34. A function r is called *left rational* if it is of the form $r(q) = P(q)^{-1}Q(q)$ with polynomials $P \in \mathcal{N}(\mathbb{H})$ and $Q \in \mathcal{SH}_L(\mathbb{H})$.

A function r is called *right rational* if it is of the form $r(q) = Q(q)P(q)^{-1}$ with polynomials $P \in \mathcal{N}(\mathbb{H})$ and $Q \in \mathcal{SH}_R(\mathbb{H})$.

Finally, a function r is called *intrinsic rational* if it is of the form $r(q) = P(q)^{-1}Q(q)$ with two polynomials $P, Q \in \mathcal{N}(\mathbb{H})$.

Remark 2.1.35. The requirement that P be intrinsic is necessary because the function P^{-1} is otherwise not slice hyperholomorphic; cf. Theorem 2.1.4.

Corollary 2.1.36. *Let $f \in \mathcal{SH}_L(U)$, let $j, i \in \mathbb{S}$ with $i \perp j$, and write $f_j = F_1 + F_2 i$ with holomorphic components $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$ according to Lemma 2.1.6. Then f is left rational if and only if F_1 and F_2 are rational functions on $\mathbb{C}_j$.*

Similarly, if $f \in \mathcal{SH}_R(U)$ and we write $f_j = F_1 + iF_2$ with holomorphic components F_1 and F_2 according to Lemma 2.1.6, then f is right rational if and only if F_1, F_2 are rational functions on $\mathbb{C}_j$.

Proof. Let $f \in \mathcal{SH}_L(U)$ be left rational, i.e., $f(q) = P(q)^{-1}Q(q)$ for some intrinsic polynomial $P(q) = \sum_{n=0}^{N} q^n a_n$ with $a_n \in \mathbb{R}$ and some left slice hyperholomorphic polynomial $Q(q) = \sum_{m=0}^{M} q^m b_m$ with $b_m \in \mathbb{H}$. If we write $b_m = b_{m,1} + b_{m,2} i$ with $b_{m,1}, b_{m,2} \in \mathbb{C}_j$ and set $Q_1(q) = \sum_{m=0}^{M} q^m b_{m,1}$ and $Q_2(q) = \sum_{m=0}^{M} q^m b_{m,2}$ for $q \in U \cap \mathbb{C}_j$, we obtain $Q = Q_1 + Q_2 i$ and in turn

$$f_j(q) = P(q)^{-1}Q(q) = P(q)^{-1}Q_1(q) + P(q)^{-1}Q_2(q)i.$$

Since P has real coefficients and Q_1 and Q_2 have coefficients in $\mathbb{C}_j$, they are polynomials on $\mathbb{C}_j$, and hence $P^{-1}Q_1$ and $P^{-1}Q_2$ are rational functions on $\mathbb{C}_j$. Since furthermore, 1 and i are linearly independent over $\mathbb{C}_j$, we obtain $F_1 = P^{-1}Q_1$ and $F_2 = P^{-1}Q_2$.

In order to show the converse implication, let us assume that $F_1 = P_1^{-1}Q_1$ and $F_2 = P_2 Q_2$ are rational functions. If $P_1(q) = \sum_{n=0}^{N} q^n a_{n,1}$ with $a_{n,1}$ in $\mathbb{C}_j$, then $\overline{P_1(\overline{q})}$ is the polynomial $P_1(q) = \sum_{n=0}^{N} q^n \overline{a_{n,1}}$. The product $\tilde{P}_1(q) :=$

$P_1(q)\overline{P_1(\overline{q})}$ is again a polynomial, and since it satisfies $\tilde{P}_1(\overline{q}) = \overline{\tilde{P}_1(q)}$, it has real coefficients. Similarly, the function $\tilde{P}_2(q) := P_2(q)\overline{P_2(\overline{q})}$ is also a polynomial with real coefficients, and we have

$$F_1(q) = \tilde{P}_1(q)^{-1}\overline{P_1(\overline{q})}Q_1(q), \quad F_2(q) = \tilde{P}_2(q)^{-1}\overline{P_2(\overline{q})}Q_2(q),$$

and in turn

$$\begin{aligned} f_j(q) &= F_1(q) + F_2(q)i \\ &= \tilde{P}_1(q)^{-1}\tilde{P}_2(q)^{-1}\left(\tilde{P}_2(q)\overline{P_1(\overline{q})}Q_1(q) + \tilde{P}_1(q)\overline{P_2(\overline{q})}Q_2(q)i\right). \end{aligned}$$

The function $P(q) := \tilde{P}_1(q)\tilde{P}_2(q)$ is a polynomial with real coefficients on $\mathbb{C}_j$, the function

$$Q(q) := \tilde{P}_2(q)\overline{P_1(\overline{q})}Q_1(q) + \tilde{P}_1(q)\overline{P_2(\overline{q})}Q_2(q)i$$

is a polynomial with quaternionic coefficients on $\mathbb{C}_j$, and by construction, $f_j(q) = P(q)^{-1}Q(q)$.

Replacing the complex variable by a quaternionic variable, we can extend P to an intrinsic polynomial on $\mathbb{H}$ and Q to a left slice hyperholomorphic polynomial on $\mathbb{H}$. Due to the uniqueness of the left slice hyperholomorphic extension in Lemma 2.1.11, we then obtain

$$f = \operatorname{ext}_L(f_j) = \operatorname{ext}_L(P^{-1}Q) = P^{-1}Q,$$

and so f is actually left rational. The right rational case can be shown similarly. □

Theorem 2.1.37 (Runge's theorem)**.** *Let $K \subset \mathbb{H}$ be an axially symmetric compact set and let A be an axially symmetric set such that $A \cap C \neq \emptyset$ for every connected component C of $(\mathbb{H} \cup \{\infty\}) \setminus K$.*

If f is left slice hyperholomorphic on an axially symmetric open set U with $K \subset U$, then for every $\varepsilon > 0$, there exists a left rational function r whose poles lie in A such that

$$\sup\{|f(q) - r(q)| : \quad q \in K\} < \varepsilon. \tag{2.32}$$

Similarly, if f is right slice hyperholomorphic on an axially symmetric open set U with $K \subset U$, then for every $\varepsilon > 0$, there exists a right rational function r whose poles lie in A such that (2.32) *holds.*

Finally, if $f \in \mathcal{N}(U)$ for some axially symmetric open set U with $K \subset U$, then for every $\varepsilon > 0$, there exists a real rational function r whose poles lie in A such that (2.32) *holds.*

Proof. Let $f \in \mathcal{SH}_L(U)$ for some axially symmetric open set U with $K \subset U$, let $j, i \in \mathbb{S}$ with $j \perp i$, and let us write $f_j = F_1 + F_2 i$ with holomorphic functions $F_1, F_2 : U \cap \mathbb{C}_j \to \mathbb{C}_j$ as in Lemma 2.1.6. The set $K \cap \mathbb{C}_j$ is compact in $\mathbb{C}_j$ and the set $A \cap \mathbb{C}_j$ has, due to its axial symmetry, nonempty intersection with every connected component of $(\mathbb{C}_j \cup \{\infty\}) \setminus (K \cap \mathbb{C}_j)$.

For $\varepsilon > 0$, the classical Runge's theorem for holomorphic functions implies the existence of rational functions R_1 and R_2 with poles in $A \cap \mathbb{C}_j$ such that

$$\sup\{|F_\ell(z) - R_\ell(z)| : \quad z \in K \cap \mathbb{C}_j\} < \frac{\varepsilon}{4}, \quad \ell = 1, 2. \tag{2.33}$$

The left slice hyperholomorphic extension $r(q) = \operatorname{ext}_L(R_1 + R_2 i)$ is then by Lemma 3.2.10 a right rational function with poles in A, and

$$|f(z) - r(z)| \leq |F_1(z) - R_1(z)| + |F_2(z) - R_2(z)| < \frac{\varepsilon}{2}$$

for all $z \in K \cap \mathbb{C}_j$. From Theorem 2.1.9 we conclude for $q = u + kv \in K$ after setting $z = u + jv \in K \cap \mathbb{C}_j$ that

$$\begin{aligned} |f(q) - r(q)| &= \left|\frac{1}{2}(1 - kj)(f(z) + r(z)) + \frac{1}{2}(1 + kj)(f(\overline{z}) - r(\overline{z}))\right| \\ &\leq |f(z) + r(z)| + |f(\overline{z}) - r(\overline{z})) < \varepsilon. \end{aligned} \tag{2.34}$$

The right slice hyperholomorphic case can be shown by similar arguments.

What remains to show is that R can be chosen rational intrinsic if f is intrinsic. In order to do that, we first observe that in this case, $F_2 \equiv 0$, so that we can choose $R_2 \equiv 0$ in (2.33). If we set

$$\tilde{R}(z) = \frac{1}{2}\left(R_1(z) + \overline{R(\overline{z})}\right),$$

then $\tilde{R}$ is a rational function on $\mathbb{C}_j$ that satisfies $\tilde{R}(\overline{z}) = \overline{\tilde{R}(z)}$. It is hence of the form $\tilde{R}(z) = P(z)^{-1}Q(z)$ with polynomials P and Q with coefficients in $\mathbb{R}$. Its slice hyperholomorphic extension $r(q) = P(q)^{-1}Q(q)$ for $q \in \mathbb{H}$ with $P(q) \neq 0$ is then an intrinsic rational function.

As an intrinsic function, f satisfies $f(\overline{q}) = \overline{f(q)}$. Hence for $z \in K \cup \mathbb{C}_j$, we have

$$\begin{aligned} |f(z) - r(z)| &= \frac{1}{2}\left|f(z) - R_1(z) + \overline{f(\overline{z})} - \overline{R_1(\overline{z})}\right| \\ &\leq \frac{1}{2}\left(\left|f(z) - R_1(z)\right| + \left|\overline{f(\overline{z}) - R_1(\overline{z})}\right|\right) < \frac{\varepsilon}{2}. \end{aligned}$$

As in (2.34), we see then that (2.32) holds with the intrinsic rational function r. □

2.2 The Fueter Mapping Theorem in Integral Form

In order to define the F-functional calculus in Chapter 7 we recall now the Fueter mapping theorem and show its integral form. The Fueter mapping theorem in integral form was introduced in [86]. We start with recalling the definition of Fueter regularity.

Definition 2.2.1 (Cauchy–Fueter regular functions)**.** Let U be an open set in $\mathbb{H}$. A real differentiable function $f : U \to \mathbb{H}$ is *left Fueter regular* if

$$\frac{\partial}{\partial q_0} f(q) + \sum_{\ell=1}^{3} e_\ell \frac{\partial}{\partial q_\ell} f(q) = 0, \quad \text{for every } q \in U.$$

It is *right Fueter regular* if

$$\frac{\partial}{\partial q_0} f(x) + \sum_{\ell=1}^{3} \frac{\partial}{\partial q_\ell} f(q) e_\ell = 0, \quad \text{for every } q \in U.$$

It was Fueter who introduced in his paper [111] the following method for generating Fueter regular functions:

(1) We consider a holomorphic function $f(z)$ that depends on a complex variable $z = u + \iota v$ in an open set of the upper complex half-plane. (In order to distinguish it from quaternionic imaginary units, we denote the imaginary unit of the usual complex numbers by ι.) We write

$$f(z) = f_0(u, v) + \iota f_1(u, v),$$

where f_0 and f_1 are $\mathbb{R}$-valued functions that satisfy the Cauchy–Riemann system.

(2) For every quaternion q such that $u + \iota v$ belongs to the domain of f, we replace the complex imaginary unit ι in $f(z) = f_0(u, v) + \iota f_1(u, v)$ by the quaternionic imaginary unit $\frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|}$, and we set $u = \mathrm{Re}(q)$ and $v = |\mathrm{Im}(q)|$. We then define

$$f(q) = f_0(q_0, |\mathrm{Im}(q)|) + \frac{\mathrm{Im}(q)}{|\mathrm{Im}(q)|} f_1(q_0, |\mathrm{Im}(q)|).$$

Observe that the function $f(q)$ is slice hyperholomorphic by construction.

(3) We apply the Laplace operator $\Delta = \sum_{\ell=0}^{3} \frac{\partial^2}{\partial q_\ell^2}$ to f and define $\breve{f}(q) = \Delta f(q)$.

It turns out that the function $\breve{f}(q)$ is then both left and right Fueter regular.

Observe that by construction, $f(q)$ is an intrinsic slice hyperholomorphic function on the open axially symmetric set of all quaternions $q = u + jv$ such that $u + \iota v$ belongs to the domain of f.

In modern language, the Fueter mapping theorem states that *applying the Laplace operator* Δ *to a slice hyperholomorphic function* $f(q)$ *yields the Fueter regular function*

$$\breve{f}(q) = \Delta f(q).$$

This function is left Fueter regular if f is left slice hyperholomorphic and right Fueter regular if f is right slice hyperholomorphic.

If we write f in terms of the slice hyperholomorphic Cauchy formula, we can apply Δ and commute it with the integral such that Δ is actually applied to the slice hyperholomorphic Cauchy kernel inside this integral. In this way, we obtain an integral transform with respect to the kernel $\Delta S_L^{-1}(s,p)$, resp. $\Delta S_R^{-1}(s,p)$, that maps slice hyperholomorphic functions to Fueter regular functions.

A simple formula for $\Delta S_L^{-1}(s,p)$, resp. $\Delta S_R^{-1}(s,p)$, is, however, obtained only if we write the slice hyperholomorphic Cauchy kernels in form II. As a consequence, the F-functional calculus, which is based on this integral transform, can be defined only for operators with commuting components. Otherwise, the S-resolvents cannot be written in a form that corresponds to form II of the Cauchy kernels.

Theorem 2.2.2. *Let $q,s\in\mathbb{H}$ with $q\notin[s]$ and let $\Delta=\sum_{\ell=0}^{3}\frac{\partial^2}{\partial q_\ell^2}$ be the Laplace operator in the variable q.*

(a) *Consider the left slice hyperholomorphic Cauchy kernel $S_L^{-1}(s,q)$ written in form II. Then we have*

$$\Delta S_L^{-1}(s,q)=-4(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}. \tag{2.35}$$

(b) *Consider the right slice hyperholomorphic Cauchy kernel $S_R^{-1}(s,q)$ written in form II. Then we have*

$$\Delta S_R^{-1}(s,q)=-4(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}(s-\bar{q}). \tag{2.36}$$

Proof. We show only the identity (2.35), the other one follows with similar arguments. If we write $S_L^{-1}(s,q)$ in form II, then straightforward computations yield

$$\begin{aligned}\frac{\partial^2}{\partial q_0^2}S_L^{-1}(s,q)&=2(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}(-2s+2q_0)\\&\quad+2(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}(-2s+2q_0)^2\\&\quad-2(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial q_\ell^2}S_L^{-1}(s,q)&=-4e_\ell q_\ell(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\&\quad+8q_\ell^2(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}\\&\quad-2(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\end{aligned}$$

for $\ell = 1, 2, 3$. Thus, we obtain

$$\begin{aligned}\Delta S_L^{-1}(s,x) &= 2(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}(-2s + 2q_0)\\ &\quad + 2(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}(-2s+2q_0)^2\\ &\quad - \sum_{\ell=1}^{3} 4e_\ell q_\ell (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + \sum_{\ell=1}^{3} 8q_\ell^2 (s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}\\ &\quad - 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}.\end{aligned}$$

Since $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}$ and $(-2s + 2q_0)$ commute, we have

$$\begin{aligned}&\Delta S_L^{-1}(s,q)\\ &= -4\left(s - q_0 + \sum_{\ell=1}^{3} 4e_\ell q_\ell\right)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 2(s-\bar{q})\left[(-2s+2q_0)^2 + \sum_{\ell=1}^{3} 4q_\ell^2\right](s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}\\ &\quad - 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &= -4(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad - 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &= -4(s-\bar{q})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-2}.\end{aligned}$$

□

Proposition 2.2.3. *Let $q \in \mathbb{H}$. The function $s \mapsto \Delta S_L^{-1}(s,q)$ is right slice hyperholomorphic on $\mathbb{H} \setminus [q]$ and the function $s \mapsto \Delta S_R^{-1}(s,q)$ is left slice hyperholomorphic on $\mathbb{H} \setminus [q]$.*

Proof. For $s = u + jv$, we have

$$\begin{aligned}\frac{\partial}{\partial u}\Delta S_L^{-1}(s,q) &= -4(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 8(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-3}(2s - 2\mathrm{Re}(q))\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial v}\Delta S_L^{-1}(s,q) &= -4j(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}\\ &\quad + 8(s-\bar{q})(s - 2\mathrm{Re}(q)(u+jv) + |q|^2)^{-3}(2sj - 2\mathrm{Re}(q)j).\end{aligned}$$

Since j commutes with $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}$, we conclude that

$$\frac{\partial}{\partial u}\Delta S_L^{-1}(s,q) + \frac{\partial}{\partial v}\Delta S_L^{-1}(s,q)j = 0.$$

Hence $\Delta S_L^{-1}(s,q)$ is right slice hyperholomorphic in s by Proposition 2.1.14, because $\mathbb{H}\setminus[q]$ is an axially symmetric slice domain. The other case can be shown with similar arguments. □

Proposition 2.2.4. *Let $s\in\mathbb{H}$. The function $q\mapsto\Delta S_L^{-1}(s,q)$ is left Fueter regular on $\mathbb{H}\setminus[s]$ and the function $q\mapsto\Delta S_R^{-1}(s,q)$ is right Fueter regular on $\mathbb{H}\setminus[s]$.*

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial q_0}\Delta S_L^{-1}(s,q) &= 4(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad -16(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}(s-q_0)\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial q_\ell}\Delta S_L^{-1}(s,q) &= -4e_\ell(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad +16q_\ell(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{\partial}{\partial q_0}\Delta S_L^{-1}(s,q)+\sum_{\ell=1}^{3}e_\ell\frac{\partial}{\partial q_\ell}\Delta S_L^{-1}(s,q)\\ &=4(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad -16(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}(s-q_0)\\ &\quad +\sum_{\ell=1}^{3}4(-e_\ell^2)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad +\sum_{\ell=1}^{3}16e_\ell q_\ell(s-\bar{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3},\end{aligned}$$

and since $(s^2-2\mathrm{Re}(q)s+|q|^2)^{-1}$ commutes with $s-q_0$, we finally obtain

$$\begin{aligned}&\frac{\partial}{\partial q_0}\Delta S_L^{-1}(s,q)+\sum_{\ell=1}^{3}e_\ell\frac{\partial}{\partial q_\ell}\Delta S_L^{-1}(s,q)\\ &=16(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\\ &\quad +16\left(\left(q_0+\sum_{\ell=1}^{3}q_\ell e_\ell\right)(s-\bar{q})-(s-\bar{q})s\right)(s^2-2\mathrm{Re}(q)s+|q|^2)^{-3}\\ &=16(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}-16(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}=0.\end{aligned}$$

Hence $q\mapsto\Delta S_L^{-1}(s,q)$ is left Fueter regular. The right Fueter regularity of $q\mapsto\Delta S_R^{-1}(s,q)$ can be shown with analogous computations. □

Definition 2.2.5 (The Fueter kernels)**.** We define for $s \in \mathbb{H}$ with $q \notin [s]$ the F_L-*kernel* as

$$F_L(s,q) := \Delta S_L^{-1}(s,q) = -4(s-\bar{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2},$$

and the F_R-*kernel* as

$$F_R(s,q) := \Delta S_R^{-1}(s,q) = -4(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}(s-\bar{q}).$$

Finally, we can now prove the Fueter mapping theorem in integral form.

Theorem 2.2.6 (The Fueter mapping theorem in integral form)**.** *Let $U \subset \mathbb{H}$ be a slice Cauchy domain and choose $j \in \mathbb{S}$.*

(a) *If $f \in \mathcal{SH}_L(O)$ for some set O with $\overline{U} \subset O$, then $\breve{f}(q) = \Delta f(q)$ is left Fueter regular on U, and it admits the integral representation*

$$\breve{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} F_L(s,q)\, ds_j\, f(s) \quad \forall q \in U. \tag{2.37}$$

(b) *If $f \in \mathcal{SH}_R(O)$ for some set O with $\overline{U} \subset O$, then $\breve{f}(q) = \Delta f(q)$ is right Fueter regular on U, and it admits the integral representation*

$$\breve{f}(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, F_R(s,q) \quad \forall q \in U. \tag{2.38}$$

The integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. The function $\breve{f}(q) = \Delta f(q)$ is Fueter regular by the Fueter mapping theorem. We can write $f(q)$ for $q \in U$ in terms of the corresponding slice hyperholomorphic Cauchy formula. If we apply the Laplacian and exchange the order of integration and differentiation, we end up with (2.37), resp. (2.38). □

2.3 Vector-Valued Slice Hyperholomorphic Functions

In this section, we generalize the notion of slice hyperholomorphicity from scalar-valued to vector-valued functions. In particular, similar to what happens for holomorphic functions, we show that the notions of weak and strong slice hyperholomorphicity are equivalent. Via the quaternionic Hahn–Banach theorem, one can prove properties of vector-valued slice hyperholomorphic functions by reducing the problems to the scalar case.

Definition 2.3.1. A *quaternionic right vector space* is an additive group $(X,+)$ that is endowed with a quaternionic right multiplication $(X,\mathbb{H}) \to X$, $(x,q) \mapsto xq$ such that for all $x,y \in X$ and all $p,q \in \mathbb{H}$,

$$x(p+q), = xp + xq \qquad (x+y)q = xq + yq, \qquad (xp)q = x(pq).$$

A *quaternionic left vector space* is an additive group $(X,+)$ that is endowed with a quaternionic left multiplication $(\mathbb{H},X)\to X$, $(q,x)\mapsto qx$ such that for all $x,y\in X$ and all $p,q\in\mathbb{H}$,

$$(p+q)x = px+qx, \qquad q(x+y) = qx+qy, \qquad q(px) = (qp)x.$$

A *two-sided quaternionic vector space* is an additive group $(X,+)$ endowed with a quaternionic left and a quaternionic right multiplication such that X is both a left and a right vector space and such that $ax = xa$ for all $a\in\mathbb{R}$ and all $x\in X$.

Remark 2.3.2. If we start from a real vector space $X_{\mathbb{R}}$, then we can quaternionify $X_{\mathbb{R}}$ to obtain the two-sided quaternionic vector space $X = X_{\mathbb{R}}\otimes\mathbb{H}$ by setting

$$X = X_{\mathbb{R}}\otimes\mathbb{H} = \left\{\sum_{\ell=0}^{3} x_\ell e_\ell : \ \ x_\ell\in X_{\mathbb{R}}\right\}$$

with the scalar multiplications

$$qx = \sum_{\ell=0}^{3} x_\ell(qe_\ell), \quad xq = \sum_{\ell=0}^{3} x_\ell(e_\ell q),$$

for $x\in X$ and $q\in\mathbb{H}$. Conversely, every two-sided quaternionic vector space X is isomorphic to the quaternionification of a real vector space, namely to $X_{\mathbb{R}}\otimes\mathbb{H}$ with the real vector space

$$X_{\mathbb{R}} = \{x\in X: \ \ qx = xq \quad \forall q\in\mathbb{H}\}.$$

Definition 2.3.3. A function $\|\cdot\| : X_R\to[0,+\infty)$ on a quaternionic right vector space X_R is called a *norm on* X_R, if it satisfies

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|xq\| = \|x\||q|$ for all $x\in X$ and all $q\in\mathbb{H}$,

(iii) $\|x+y\|\leq\|x\|+\|y\|$ for all $x,y\in X$.

If X_R is complete with respect to the metric induced by $\|\cdot\|$, we call X_R a *quaternionic right Banach space.*

A function $\|\cdot\| : X_L\to[0,+\infty)$ on a quaternionic left vector space X_L is called a *norm on* X_L, if it satisfies (i), (iii), and

(ii′) $\|qx\| = |q|\|x\|$ for all $x\in X$ and all $q\in\mathbb{H}$.

If X_L is complete with respect to the metric induced by $\|\cdot\|$, we call X_L a *quaternionic left Banach space.*

Finally, a two-sided quaternionic vector space X is called a quaternionic *two-sided quaternionic Banach space* if it is endowed with a norm $\|\cdot\|$ such that it is both a left and a right Banach space, that is, such that (i), (ii), (ii′) and (iii) are satisfied and such that X is complete with respect to the metric induced by $\|\cdot\|$.

Corollary 2.3.4. *A quaternionic left or right Banach space turns into a real Banach space if we restrict the left, resp. right, scalar multiplication to $\mathbb{R}$, and it turns into a complex Banach space over $\mathbb{C}_j$ with $j \in \mathbb{S}$ if we restrict the left, resp. right, scalar multiplication to $\mathbb{C}_j$.*

A two-sided quaternionic Banach space turns into a real Banach space if we restrict the scalar multiplications to $\mathbb{R}$, and it turns into a complex Banach space over $\mathbb{C}_j$ with $j \in \mathbb{S}$ if we restrict either the left or the right scalar multiplication to $\mathbb{C}_j$.

Definition 2.3.5. A function $\varphi : X_1 \to X_2$ between two quaternionic right vector spaces X_1, X_2 is called *right linear* if

$$\varphi(xq + y) = \varphi(x)q + \varphi(y) \quad \forall x, y \in X_1, q \in \mathbb{H}.$$

Similarly, a function $\varphi : X_1 \to X_2$ between two quaternionic left vector spaces X_1, X_2 is called *left linear* if

$$\varphi(qx + y) = q\varphi(x) + \varphi(y) \quad \forall x, y \in X_1, q \in \mathbb{H}.$$

A right or left linear mapping $\varphi : X_1 \to X_2$ between two quaternionic right, resp. left, Banach spaces is called *bounded* if

$$\|\varphi\| := \sup_{\|x\|_{X_1}=1} \|\varphi(x)\|_{X_2} < +\infty.$$

Definition 2.3.6. The *dual X'_R of a quaternionic right Banach space X_R* is the quaternionic left Banach space of all bounded right linear mappings from X_R to $\mathbb{H}$. The *dual X'_L of a quaternionic left Banach space X_L* is the quaternionic right Banach space of all bounded left linear mappings from X_R to $\mathbb{H}$. Finally, for a two-sided quaternionic Banach space X, we distinguish two different dual spaces: the *right dual X'_R of X* is the dual space of X as a right Banach space, and the *left dual X'_L of X* is the dual space of X as a left Banach space.

We finally recall the quaternionic Hahn–Banach theorem, which will be important in the sequel. It was first proven in [194], but a proof in English can be found in [89].

Theorem 2.3.7 (Hahn–Banach theorem)**.** *Let X_R be a quaternionic right vector space, let X_0 be a right linear subspace of X_R, and let $\rho : X_R \to [0, +\infty)$ satisfy $\rho(x + y) \leq \rho(x) + \rho(y)$ and $\rho(xq) = \rho(x)|q|$ for all $x, y \in X_R$ and all $q \in \mathbb{H}$. Moreover, let $\lambda : X_0 \to \mathbb{H}$ be a quaternionic right linear functional on X_0 such that $|\lambda(x)| \leq \rho(x)$ for all $x \in X_0$. Then there exists a right linear functional $\Lambda : X_R \to \mathbb{H}$ such that $\Lambda(x) = \lambda(x)$ for all $x \in X_0$ and such that*

$$|\Lambda(x)| \leq \rho(x) \quad \textit{for all } x \in X_R.$$

An analogous statement holds for left linear vector spaces.

Corollary 2.3.8. *The dual space of a quaternionic left or right Banach space separates points. Furthermore, both the left and the right duals of a two-sided quaternionic Banach space also separate points.*

Let us now turn our attention to slice hyperholomorphic functions with values in a quaternionic Banach space. As in the complex case, one can distinguish between strong and weak slice hyperholomorphicity.

Definition 2.3.9 (Slice hyperholomorphic vector-valued functions)**.** Let $U \subseteq \mathbb{H}$ be an axially symmetric open set and let

$$\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : \ u + \mathbb{S}v \subset U\}.$$

A function $f : U \to X_L$ with values in a quaternionic left Banach space X_L is called a *left slice function*, if is of the form

$$f(q) = f_0(u,v) + jf_1(u,v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to X_L$ that satisfy the compatibility condition (2.4). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5), then f is called *strongly left slice hyperholomorphic*.

A function $f : U \to X_R$ with values in a quaternionic right Banach space is called a *right slice function* if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v)j \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to X_R$ that satisfy the compatibility condition (2.4). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5), then f is called *strongly right slice hyperholomorphic*.

Definition 2.3.10. Let $U \subset \mathbb{H}$ be an axially symmetric open set. A function $f : U \to X_L$ with values in a quaternionic left Banach space X_L is called *weakly left slice hyperholomorphic* if Λf is left slice hyperholomorphic for every $\Lambda \in X_L'$. A function $f : U \to X_R$ with values in a quaternionic right Banach space X_R is called *weakly right slice hyperholomorphic* if Λf is right slice hyperholomorphic for every $\Lambda \in X_R'$.

Since the functionals Λ in the dual of X_L, resp. X_R, are continuous, every strongly slice hyperholomorphic function is weakly slice hyperholomorphic. As in the complex case, the converse also is true. In order to show this, we recall the following lemma. We omit the proof, since it works exactly as in the complex case (see, e.g., [179], p. 189).

Lemma 2.3.11. *Let X be a two-sided quaternionic Banach space. A sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy if and only if $(\Lambda(x_n))_{n\in\mathbb{N}}$ is uniformly Cauchy for $\Lambda \in X'$, $\|\Lambda\| \leq 1$.*

Proposition 2.3.12. *Let X_L be a quaternionic left Banach space, let U be an open axially symmetric subset of $\mathbb{H}$, and let $f : U \to X_L$ be a real differentiable left slice function. Then the following statements are equivalent:*

(i) *The function f is strongly left slice hyperholomorphic.*

(ii) *The function f admits a left slice derivative, that is,*

$$\partial_S f(q) := \lim_{p \to q, p \in \mathbb{C}_j} (p-q)^{-1}(f(p) - f(q)) \tag{2.39}$$

exists for all $q = u + jv \in U$ in the topology of X_L, and it exists for every $j \in \mathbb{S}$ if q is real.

(iii) *For every $j \in \mathbb{S}$, the restriction $f_j = f|_{U \cap \mathbb{C}_j}$ of f to $U \cap \mathbb{C}_j$ satisfies*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f(q) + j \frac{\partial}{\partial v} f(q)\right) = 0, \quad \forall q = u + jv \in U \cap \mathbb{C}_j.$$

Let X_R be a quaternionic right Banach space, let U be an open axially symmetric subset of $\mathbb{H}$, and let $f : U \to X_R$ be a real differentiable right slice function. Then the following statements are equivalent:

(i) *The function f is strongly right slice hyperholomorphic.*

(ii) *The function f admits a right slice derivative, that is,*

$$\partial_S f(q) := \lim_{p \to q, p \in \mathbb{C}_j} (f(p) - f(q))(p-q)^{-1}$$

exists for all $q = u + jv \in U$ in the topology of X_R, and it exists for every $j \in \mathbb{S}$ if q is real.

(iii) *For every $j \in \mathbb{S}$, the restriction $f_j = f|_{U \cap \mathbb{C}_j}$ of f to $U \cap \mathbb{C}_j$ satisfies*

$$\frac{1}{2}\left(\frac{\partial}{\partial u} f(q) + \frac{\partial}{\partial v} f(q) j\right) = 0, \quad \forall q = u + jv \in U \cap \mathbb{C}_j.$$

Proof. Let $f : U \to X_L$ be a left slice function. The equivalence of (ii) and (iii) follows immediately from the complex theory and Corollary 2.3.4: the statement (iii) is equivalent to f_j being, for every $j \in \mathbb{S}$, a (left) holomorphic function on $\mathbb{C}_j$ with values in the complex Banach space X_L over $\mathbb{C}_j$. This is in turn equivalent to the existence of the limit

$$f_j'(q) = \lim_{p \to q, p \in \mathbb{C}_j} (p-q)^{-1}(f(p) - f(q)) = \partial_S f(q)$$

for every $q = u + jv \in U$.

Let us now show the equivalence of (i) and (iii). If (i) holds, then

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial}{\partial u} f_j(z) + j \frac{\partial}{\partial v} f_j(z)\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial u} f_0(z) + j \frac{\partial}{\partial u} f_1(z) + j \frac{\partial}{\partial v} f_0(z) - \frac{\partial}{\partial v} f_1(z)\right) = 0,
\end{aligned}$$

because f_0 and f_1 satisfy the Cauchy–Riemann equations (2.5). If, on the other hand, (iii) holds, then we have because of

$$f_0(u,v) = \frac{1}{2}\left(f(u+jv) + f(u-jv)\right)$$

and

$$f_1(u,v) = \frac{1}{2}j\left(f(u-jv) - f(u+jv)\right)$$

that

$$\begin{aligned}\frac{\partial}{\partial u}f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial u}f(u-jv) + \frac{\partial}{\partial u}f(u+jv)\right]\\ &= \frac{1}{2}\left[j\frac{\partial}{\partial v}f(u-jv) - j\frac{\partial}{\partial v}f(u+jv)\right] = \frac{\partial}{\partial v}f_1(u,v).\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial v}f_0(u,v) &= \frac{1}{2}\left[\frac{\partial}{\partial v}f(u-jv) + \frac{\partial}{\partial v}f(u+jv)\right]\\ &= \frac{1}{2}\left[-j\frac{\partial}{\partial u}f(u-jv) + j\frac{\partial}{\partial u}f(u+jv)\right] = -\frac{\partial}{\partial u}f_1(u,v).\end{aligned}$$

Hence f is actually left slice hyperholomorphic.

The right slice hyperholomorphic case can be shown with analogous arguments. □

Theorem 2.3.13. *Let $U \subset \mathbb{H}$ be an axially symmetric open set.*

(i) *Every weakly left slice hyperholomorphic function $f : U \to X_L$ with values in a quaternionic left Banach space is strongly left slice hyperholomorphic.*

(ii) *Every weakly right slice hyperholomorphic function $f : U \to X_R$ with values in a quaternionic right Banach space is strongly right slice hyperholomorphic.*

Proof. Let f be a weakly left slice hyperholomorphic function on U with values in a quaternionic left Banach space X_L. We first observe that f is a left slice function. If we choose $i \in \mathbb{S}$ and set

$$f_0(u,v) = \frac{1}{2}(f(u+iv) + f(u-iv))$$

and

$$f_1(u,v) = \frac{1}{2}i(f(u-iv) - f(u+iv))$$

for $u, v \in \mathbb{R}$ with $u + iv \in U$, then f_0 and f_1 obviously satisfy the compatibility condition (2.4). If $\Lambda \in X_L'$, then $(\Lambda \circ f)(q) := \Lambda(f(q))$ is left slice hyperholomorphic on U by our assumptions, and hence it satisfies the structure formula (2.9). If $q = u + jv \in U$, we can set $z = u + iv$ for netatness and obtain

$$\begin{aligned}
&\Lambda(f_0(u,v) + jf_1(u,v))\\
&= \Lambda\left(\frac{1}{2}\left(f(z) + f(\overline{z})\right)) + ji\frac{1}{2}\left(f(\overline{z}) - f(z)\right)\right)\\
&= \frac{1}{2}\left(\Lambda(f(z)) + \lambda(f(\overline{z}))\right) + ji\frac{1}{2}\left(\Lambda(f(\overline{z})) - \Lambda(f(z))\right)\\
&= \frac{1}{2}\left((\Lambda\circ f)(z) + (\Lambda\circ f)(\overline{z})\right) + j\frac{1}{2}\left((\Lambda\circ f)(\overline{z}) - (\Lambda\circ f)(z)\right)\\
&= (\Lambda\circ f)(q) = \Lambda(f(q)).
\end{aligned}$$

Since $\Lambda \in V_L'$ was arbitrary and V_L' separates points by Corollary 2.3.8, we find that $f(q) = f_0(u,v) + jf_1(u,v)$ and hence f is a left slice function.

The rest of the proof follows the lines of the proof in the complex case in [179, p. 189]. For every $\Lambda \in X_L'$, the function $q \mapsto \Lambda(f(q))$ is left slice hyperholomorphic on U. Its restriction to a plane $\mathbb{C}_j$ is hence left holomorphic and therefore admits a representation in terms of the Cauchy formula. If $q = u + jv \in U$ and p tends to q in $\mathbb{C}_j$, we can therefore choose $r > 0$ so small that $\overline{B_r(q)} \subset U$ and find for $p \in B_r(q) \cap \mathbb{C}_j$ that

$$\begin{aligned}
&\Lambda(f(p)) - \Lambda(f(q))\\
&= \frac{1}{2\pi}\int_\Gamma \left((s-p)^{-1} - (s-q)^{-1}\right)\, ds_j\, \Lambda(f(s))\\
&= \frac{1}{2\pi}\int_\Gamma (p-q)(s-p)^{-1}(s-q)^{-1}\, ds_j\, \Lambda(f(s))
\end{aligned}$$

with $\Gamma := \partial(B_r(q) \cap \mathbb{C}_j)$. Moreover, since $(\Lambda\circ f)_j'(q) = \frac{\partial}{\partial u}\Lambda(f(q))$, we also have

$$\frac{\partial}{\partial u}\Lambda(f(q)) = \frac{1}{2\pi}\int_\Gamma (s-q)^{-2} ds_j \Lambda(f(s))$$

and hence

$$\begin{aligned}
&\left|(p-q)^{-1}(\Lambda(f(p)) - \Lambda(f(q))) - \frac{\partial}{\partial u}\Lambda(f(q))\right|\\
&= \left|\frac{1}{2\pi}\int_\Gamma \left((s-p)^{-1}(s-q)^{-1} - (s-q)^{2}\right)\, ds_j\, \Lambda(f(s))\right|.
\end{aligned}$$

The mapping $s \mapsto \Lambda(f(s))$ is continuous on Γ. Since Γ is compact, we obtain

$$\sup_{s\in\Gamma} \|\Lambda(f(s))| < +\infty.$$

The mappings $\Lambda \mapsto \Lambda(f(s))$, $s \in \Gamma$, hence form a family of pointwise bounded linear maps from V_L' to $\mathbb{H}$. By the uniform boundedness principle, they are therefore uniformly bounded such that

$$\sup_{s\in\Gamma, \|\Lambda\|_{V_L}\leq 1} |\Lambda(f(s))| := C < +\infty.$$

Consequently, we have

$$\begin{aligned}&\left|(p-q)^{-1}(\Lambda(f(p)) - \Lambda(f(q))) - \frac{\partial}{\partial u}\Lambda(f(q))\right| \\ &\leq \frac{C}{2\pi}\int_\Gamma \left|(s-p)^{-1}(s-q)^{-1} - (s-q)^2\right| \, d|s| \longrightarrow 0\end{aligned}$$

as p approaches q in $\mathbb{C}_j$. Since the above estimate is independent of Λ, it follows that

$$\lim_{p\to q} \Lambda\left((p-q)^{-1}(f(p) - f(q))\right) = \frac{\partial}{\partial u}\Lambda(f(q)) = \frac{\partial}{\partial u}\Lambda(f(q))$$

uniformly for $\Lambda \in V_L'$ with $\|\Lambda\| \leq 1$. Thus $\Lambda\left((p-q)^{-1}(f(p) - f(q)\right)$ is in particular uniformly Cauchy as $p \to q$ for $\|\Lambda\| < 1$, and we conclude from Lemma 2.3.11 that the limit (2.39) exists, i.e., that f admits a left slice derivative at q. Since $q \in U$ was arbitrary and we already know that f is a left slice function, Proposition 2.3.12 implies that f is strongly left slice hyperholomorphic.

The right slice hyperholomorphic case can again be shown with similar arguments. □

Since weak and strong slice hyperholomorphicity are equivalent, we will refer to such functions simply as slice hyperholomorphic.

Definition 2.3.14. Let $U \subset \mathbb{H}$ be an axially symmetric open set. We denote the set of all left slice hyperholomorphic functions on U with values in a quaternionic left Banach space X_L by $\mathcal{SH}_L(U, X_L)$ and the set of all right slice hyperholomorphic function on U with values in a quaternionic right Banach space X_R by $\mathcal{SH}_R(U, X_R)$.

Corollary 2.3.15. *Let $U \subset \mathbb{H}$ be an axially symmetric open set. If X_L is a quaternionic left Banach space, then $\mathcal{SH}_L(U, X_L)$ is a quaternionic right linear space. If X_R is a quaternionic right Banach space, then $\mathcal{SH}_R(U, X_R)$ is a quaternionic left linear space.*

Since weak and strong slice hyperholomorphicity are equivalent, several results for scalar-valued slice hyperholomorphic functions can be generalized to the vector-valued case by applying functionals in the dual space in order to reduce the problems to the scalar case.

Proposition 2.3.16 (Identity principle)**.** *Let U be an axially symmetric slice domain, let f and g be two left or right slice hyperholomorphic functions on U with values in a quaternionic left, resp. right, Banach space X, and set $\mathcal{Z} := \{q \in U : f(q) = g(q)\}$. If there exists $j \in \mathbb{S}$ such that $\mathcal{Z} \cap \mathbb{C}_j$ has an accumulation point in $U \cap \mathbb{C}_j$, then $f \equiv g$ on all of U.*

Proof. The hypothesis implies $\Lambda f = \Lambda g$ on $\mathcal{Z} \cap \mathbb{C}_j$ for every element $\Lambda \in X'$. Theorem 2.1.8 thus implies that the left, resp. right, slice hyperholomorphic function $\Lambda(f-g)$ is identically zero on the entire axially symmetric slice domain U. By the Hahn–Banach theorem, we obtain $f - g = 0$ on U. □

Computations as in the scalar case show, moreover, that vector-valued slice hyperholomorphic functions also satisfy the structure formula and that they can be expanded into a Taylor series at every real point.

Proposition 2.3.17 (Structure formula (or representation formula)). *Let $U \subset \mathbb{H}$ be an axially symmetric open set, let $q = u + jv \in U$ and $z = u + iv$ for some $i \in \mathbb{S}$. If f is a left slice function on U with values in a quaternionic left Banach space X_L, then*

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}ji\Big[f(\overline{z}) - f(z)\Big].$$

If f is a right slice function on U with values in a quaternionic right Banach space X_R, then

$$f(q) = \frac{1}{2}\Big[f(\overline{z}) + f(z)\Big] + \frac{1}{2}\Big[f(\overline{z}) - f(z)\Big]ij.$$

Theorem 2.3.18. *Let $a \in \mathbb{R}$, let $r > 0$, and let $B_r(a) = \{q \in \mathbb{H} : |q - a| < r\}$. If $f \in \mathcal{SH}_L(B_r(a), X_L)$ with values in a quaternionic left Banach space X_L, then*

$$f(q) = \sum_{n=0}^{+\infty} (q-a)^n \frac{1}{n!} \partial_S^n f(a) \quad \forall q = u + jv \in B_r(a). \tag{2.40}$$

If on the other hand $f \in \mathcal{SH}_R(B_r(a), X_R)$ with values in a quaternionic right Banach space X_R, then

$$f(q) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\partial_S^n f(a)\right)(q-a)^n \quad \forall q = u + jv \in B_r(a).$$

Finally, the slice hyperholomorphic Cauchy formulas hold also in the scalar case.

Theorem 2.3.19 (Vector-valued Cauchy formula). *Let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$, and set $ds_j = -ds j$. If f is a left slice hyperholomorphic function with values in a quaternionic left Banach space X_L that is defined on an open axially symmetric set O with $\overline{U} \subset O$, then*

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,q)\, ds_j\, f(s) \quad \forall q \in U. \tag{2.41}$$

If f is a right slice hyperholomorphic function with values in a quaternionic right Banach space X_R that is defined on an open axially symmetric set O with $\overline{U} \subset O$, then

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,q), \quad \forall q \in U.$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Proof. Let $f \in \mathcal{SH}_L(\overline{U}, X_L)$ and let $q \in U$. Since $\partial(U \cap \mathbb{C}_j)$ is compact and the integrand is continuous, the integral in (2.41) converges. Moreover, for every $\Lambda \in X_L'$, we have, due to the left slice hyperholomorphicity of $q \mapsto \Delta(f(q))$, that

$$\Lambda\left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,f(s)\right)$$
$$= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j\,\Lambda(f(s)) = \Lambda(f(q)).$$

Since $\Lambda \in X_L'$ was arbitrary and X_L' separates points by the Hahn–Banach theorem, we obtain the statement. □

If one considers slice hyperholomorphic functions with values in a quaternionic Banach algebra, then the product of two slice hyperholomorphic functions is, just as in the scalar case, in general not slice hyperholomorphic. It is, however, possible to define a generalized product that preserves slice hyperholomorphicity.

Definition 2.3.20. A *two-sided quaternionic Banach algebra* is a quaternionic Banach space X that is endowed with a product $X \times X \to X$ such that:

(i) The product is associative and distributive over the sum in X.

(ii) One has $(qx)y = q(xy)$ and $x(yq) = (xy)q$ for all $x, y \in X$ and all $q \in \mathbb{H}$.

(iii) One has $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in X$.

If in addition there exists a unit with respect to the product in X, then X is called a *two-sided quaternionic Banach algebra with unit.*

Definition 2.3.21. Let $U \subset \mathbb{H}$ be an axially symmetric open set and let X be a two-sided quaternionic Banach algebra. For two functions $f, g \in \mathcal{SH}_L(U, X)$ with $f(q) = f_0 + jf_1$ and $g = g_0 + jg_1$ for $q = u + jv \in U$, we define their *left slice hyperholomorphic product* as

$$f *_L g := f_0 g_0 - f_1 g_1 + j\left(f_0 g_1 + f_1 g_0\right). \tag{2.42}$$

For two functions $f, g \in \mathcal{SH}_R(U, X)$ with $f(q) = f_0(u, v) + f_1(u, v)j$ and $g(q) = g_0(u, v) + g_1(u, v)j$ for $q = u + jv \in U$, we define their *right slice hyperholomorphic product* as

$$f *_R g := f_0 g_0 - f_1 g_1 + \left(f_0 g_1 + f_1 g_0\right) j. \tag{2.43}$$

Remark 2.3.22. It is immediate that the $*_L$-product of two left slice hyperholomorphic functions is again left slice hyperholomorphic and that the $*_R$-product of two right slice hyperholomorphic functions is again right slice hyperholomorphic. If, moreover, $U = B_r(0)$, then f, g admit power series expansions. If f and g are left slice hyperholomorphic with $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ with $a_n, b_n \in X$, then

$$(f *_L g)(q) := \sum_{n=0}^{+\infty} q^n \left(\sum_{\ell=0}^{n} a_\ell b_{n-\ell} \right).$$

Similarly, if f and g are right slice hyperholomorphic with $f(q) = \sum_{n=0}^{+\infty} a_n q^n$ and $g(q) = \sum_{n=0}^{+\infty} b_n q^n$ with $a_n, b_n \in X$, then

$$(f *_R g)(q) := \sum_{n=0}^{+\infty} \left(\sum_{\ell=0}^{n} a_\ell b_{n-\ell} \right) q^n.$$

Remark 2.3.23. The slice hyperholomorphic product can be defined in even more general settings than for functions with values in a quaternionic Banach algebra. If, for instance, $f \in \mathcal{SH}_L(U, \mathbb{H})$ and $g \in \mathcal{SH}_L(U, X_L)$ for some quaternionic left Banach space, then we can define $f *_L g \in \mathcal{SH}_L(U, X_L)$ also as in (2.42). For another example, we consider $f \in \mathcal{B}(X_1, X_2)$ and $g \in \mathcal{B}(X_2, X_3)$, where X_1, X_2, and X_3 are two-sided quaternionic Banach spaces and $\mathcal{B}(X, Y)$ denotes the set of all bounded right linear operators from X to Y. Then we can again define $f *_L g \in \mathcal{SH}_L(U, \mathcal{B}(X_1, X_3))$ by (2.42). The same can, of course, be done for right slice hyperholomorphic functions.

2.4 Comments and Remarks

The results of this chapter are spread over several papers which are quoted below. The treatment is sometimes different according to the definition of slice hyperholomorphicity that one takes. The interest in slice hyperholomorphic functions, defined in [135], arose in 2006 because of their applications to operator theory. Similar functions were, however, already used much earlier by Fueter, who considered in [110] functions of the form

$$f(q) = f_0(u + iv) + j f_1(u + iv), \quad q = u + jv,$$

where f_0, f_1 are the real-valued components of the analytic function $F(z) = f_0(z) + \imath f_1(z)$, in order to define what he called *hyperanalytic functions*. These hyperanalytic functions are nothing but intrinsic slice hyperholomorphic functions. In [111] the author generates Fueter regular functions by applying the Laplace operator to such a class of functions. The relation $\breve{f} = \Delta f$ between Fueter regular functions $\breve{f}$ and slice hyperholomorphic functions f is nowadays a modern way to state the Fueter mapping theorem. In [187], Sce extended this theorem to functions with values in a Clifford algebras of odd dimension. The extension to Clifford algebras of even dimensions needs more sophisticated arguments based on Fourier multipliers. In [175], Qian introduced the even–odd condition (2.4) in order to define entire slice hyperholomorphic functions, and he generalized the theorem of Sce. For biaxial symmetric domains, see [174].

In [135], slice hyperholomorphic functions were defined as functions that satisfy the properties shown in Lemma 2.1.6, that is, they are functions whose restrictions to complex planes $\mathbb{C}_j$ are left, resp. right, holomorphic. As we showed in Proposition 2.1.14, on axially symmetric slice domains, this definition is equivalent to Definition 2.1.2. Precisely, one can show that such functions satisfy the structure formula when they are defined on an axially symmetric slice domain. Considering only functions on axially symmetric slice domains is, however, not sufficient for developing a rich theory of quaternionic linear operators. For operator theory it is important to consider functions that are defined on axially symmetric open sets that are not necessarily slice domains, so for this reason we use Definition 2.1.2 for slice hyperholomorphicity.

There is an other approach to slice hyperholomorphic functions that refers to a global operator introduced in [60]. The global operator $G(q)$ is defined by

$$G(q) := |\underline{q}|^2 \frac{\partial}{\partial q_0} \ + \ \underline{q} \sum_{j=1}^{3} q_j \frac{\partial}{\partial q_j},$$

and if $U \subseteq \mathbb{H}$ is an open set and $f: \ U \to \mathbb{H}$ is a slice hyperholomorphic function, then

$$G(q)f(q) = 0.$$

Using as a definition of slice hyperholomorphic those functions that are in the kernel of the operators G, we have a possible definition of slice hyperholomorphic functions in several variables. Here the theory is far from being developed, because we have a system of nonconstant differential operators, and the power series expansion disappears, as the following example in [60] shows:

Example 2.4.1. Let U be an open set in $\mathbb{H} \times \mathbb{H}$ that does not intersect the real line. Then the function

$$f(q_1, q_2) = -\mathrm{Im}(q_2) + \frac{\underline{q}_2}{|\underline{q}_2|}\Big(\frac{1}{2}\mathrm{Re}(q_1)^2 - \frac{1}{2}\mathrm{Im}(q_1)^2 + \mathrm{Re}(q_2)\Big) + \frac{\underline{q}_1}{|\underline{q}_1|}\,\frac{\underline{q}_2}{|\underline{q}_2|}\mathrm{Re}(q_1)\mathrm{Im}(q_1) \tag{2.44}$$

satisfies the system

$$\begin{cases} |\underline{q}_1|^2 \frac{\partial}{\partial q_{1,0}} f(q_1, q_2) \ + \underline{q}_1 \ \sum_{j=1}^{3} q_{1,j} \frac{\partial}{\partial q_{1,j}} f(q_1, q_2) = 0, \\ |\underline{q}_2|^2 \frac{\partial}{\partial q_{2,0}} f(q_1, q_2) \ + \underline{q}_2 \ \sum_{j=1}^{3} q_{2,j} \frac{\partial}{\partial q_{2,j}} f(q_1, q_2) = 0. \end{cases} \tag{2.45}$$

In the paper [98] there are some results associated with the theory of slice hyperholomorphic functions in several variables, but the global operator is not used. The above example can be found also in [98].

References on function theory. The theory of slice hyperholomorphic functions is nowadays very well developed. The main monographs on this topic or containing this topic are [18, 56, 89, 96, 123, 133].

Slice hyperholomorphic functions can be defined not only over the quaternions but also over more general Clifford algebras. In the quaternionic setting, slice hyperholomorphic functions are also called slice regular, and their theory has been developed by several authors. Some of the most important contributions were published in [37–39, 58, 101, 112, 113, 130–132, 134–141, 154, 180, 181, 188–190].

Slice hyperholomorphic functions with values in a Clifford algebra are also called slice monogenic functions. The main results of their theory are contained in the papers [64, 65, 73, 90–95, 152, 198].

Several important approximation theorems for slice hyperholomorphic functions are collected in the papers [114–122] and the monograph on quaternionic approximation theory [123].

The Fueter mapping theorem provides a relation between slice hyperholomorphic functions and the classical theory of monogenic functions. Another relation is provided by the Radon transform and the dual Radon transform. Intense studies of these relations that go far beyond the results presented in Section 2.2 can be found in [61, 69, 83].

The theory of slice hyperholomorphic functions of several variables is very far from being developed, but some results can be found in the papers [3, 98, 145]. See also the paper on the Herglotz functions of several quaternionic variables [2].

Finally, the theory of slice hyperholomorphic functions has been extended to the setting of functions with values in a real alternative $*$-algebra [34, 146–149].

The Cauchy transform in the slice hyperholomorphic setting has been studied in [71].

Quaternion-valued positive definite functions on locally compact abelian groups and nuclear spaces have been considered in [17].

Slice hyperholomorphic functions are characterized by the slicewise differential equation (2.5). We, however, point out that slice hyperholomorphic functions also lie in the kernel of a global differential operator with nonconstant coefficients [60, 88, 100, 150].

References on function spaces of slice hyperholomorphic functions. Several function spaces have been extended to the slice hyperholomorphic setting. The quaternionic Hardy space $H_2(\Omega)$, where Ω is either the quaternionic unit ball $\mathbb{B}$ or the half space $\mathbb{H}^+$ of quaternions with positive real part, was introduced and studied in [12, 21, 22, 35]. We point out that the quaternionic Blaschke products were first introduced in the seminal paper [22].

The Hardy spaces $H^p(\mathbb{B})$ for arbitrary $0 < p < +\infty$ were studied in [185]. The slice hyperholomorphic Bergman spaces are studied in [59, 62, 63], the slice hyperholomorphic Fock space is considered in [31] and weighted Bergman spaces, Bloch, Besov, and Dirichlet spaces of slice hyperholomorphic functions on the unit ball $\mathbb{B}$ were introduced in [48]. Inner product spaces and Krein spaces in the quaternionic setting are studied in [26]. Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball are studied in [184]. The BMO and VMO spaces of slice hyperholomorphic functions are considered in [129]. For slice hyperholomorphic fractional Hardy spaces, see [27]. A class of quaternionic positive definite functions and their derivatives is studied in [29]. For a quaternionic analogue of the Segal–Bargmann transform, see [102].

References on slice hyperholomorphic Schur analysis. In recent years, a slice hyperholomorphic version of Schur analysis has also been developed in [1, 3, 7, 8, 12, 15, 16, 21–25, 32]. An overview of classical theory can, for example, be found in [6]. In the book [18] there is an extended introduction to the theory of Schur analysis in the slice hyperholomorphic setting. Recent results on Schur analysis, related topics and quaternionic polynomials can be found in the papers [40–46].

Chapter 3

The S-Spectrum and the S-Functional Calculus

The fundamental difficulty in developing a mathematically rigorous theory of quaternionic linear operators was the identification of suitable notions of quaternionic spectrum and quaternionic resolvent operator. In this chapter we study the properties of the S-spectrum and the S-resolvent operators, and we introduce the quaternionic S-functional calculus.

3.1 The S-Spectrum and the S-Resolvent Operators

We begin with some remarks on the algebraic structure of the space of bounded linear operators on a quaternionic Banach space.

Definition 3.1.1. We denote the set of all bounded right linear operators on a quaternionic right Banach space X_R endowed with the natural norm by $\mathcal{B}(X_R)$.

Remark 3.1.2. One can also consider left linear operators instead of right linear operators. The theory we develop in the following also applies in this case with obvious modifications.

The set $\mathcal{B}(X_R)$ is a real Banach space with the operations

$$(T + U)(x) = T(x) + U(x), \quad (Ta)(x) = T(xa),$$

for $x \in X$ and $a \in \mathbb{R}$. However, defining $(Tq)(x) = T(xq)$ does not yield a right linear operator if $q \in \mathbb{H} \setminus \mathbb{R}$, since

$$(Tq)(xp) = T(xpq) \neq T(xqp) = T(xq)p = ((Tq)x)p$$

if p and q do not belong to the same complex plane $\mathbb{C}_j$, for $j \in \mathbb{S}$. The space $\mathcal{B}(X_R)$ of all bounded right linear operators on a quaternionic right Banach space X_R is therefore not a quaternionic linear space.

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_3

For this reason, in the following we work in a two-sided quaternionic Banach space X. In this case, we can define

$$(Tq)x = T(qx) \quad \text{and} \quad (qT)(x) = q(T(x))$$

for $q \in \mathbb{H}$ and $T \in \mathcal{B}(X)$. Then $\mathcal{B}(X)$ is a two-sided quaternionic Banach space, too. Together with the multiplication $(TU)(x) = T(U(x))$ it is also a two-sided quaternionic Banach algebra.

Our first goal is to identify an appropriate notion of spectrum for quaternionic linear operators and then to generalize the Riesz–Dunford functional calculus to this setting. The spectrum of a complex linear operator A is the complement of its resolvent set

$$\rho(A) = \left\{ z \in \mathbb{C} : \ (z\mathcal{I} - A)^{-1} \text{ is bounded} \right\}.$$

The resolvent operator $R_z(A) := (z\mathcal{I} - A)^{-1}$ of A, which determines the resolvent set and the spectrum of A, is the inverse of the operator associated with the eigenvalue equation of A for $z \in \mathbb{C}$. However, as pointed out in the introduction, if T is a quaternionic right linear operator and q is a nonreal quaternion, then the right eigenvalue equation $T(x) - xq = 0$ is not right linear. Hence its associated operator cannot be used to determine a notion of spectrum of T.

In order to determine the correct notion of spectrum for a quaternionic linear operator, we follow a different path. The complex Cauchy kernel is the function $1/(z-\xi)$. We observe that formally replacing the scalar variable ξ by the operator A yields exactly the resolvent $R_z(A)$ of A at z. This analogy is the fundamental principle on which the Riesz–Dunford functional calculus is built. In order to determine the proper notion of a quaternionic resolvent and in turn a quaternionic spectrum that allows us to generalize the Riesz–Dunford functional calculus, we formally replace the quaternionic variable q in the slice hyperholomorphic Cauchy kernels by the operator T. So we consider the series expansions (2.24) and (2.25), that is, from the series

$$\sum_{n=0}^{+\infty} q^n s^{-n-1} \quad \text{and} \quad \sum_{n=0}^{+\infty} s^{-n-1} q^n$$

and we give the following definition.

Definition 3.1.3. Let $T \in \mathcal{B}(X)$ and $s \in \mathbb{H}$. We call the series

$$\sum_{n=0}^{+\infty} T^n s^{-n-1} \quad \text{and} \quad \sum_{n=0}^{+\infty} s^{-n-1} T^n$$

the *left* and *right Cauchy kernel operator series*, respectively.

Lemma 3.1.4. *Let $T \in \mathcal{B}(X)$. For $\|T\| < |s|$, the left and the right Cauchy kernel operator series converge in the operator norm.*

Proof. We have

$$\sum_{n=0}^{+\infty} \left\|T^n s^{-n-1}\right\| \le |s|^{-1} \sum_{n=0}^{+\infty} \left(\|T\||s|^{-1}\right)^n.$$

Thus, the left Cauchy kernel operator series converges if $\|T\| < |s|$. The same argument shows the convergence of the right Cauchy kernel operator series. □

Our goal is now to determine the closed form of the Cauchy kernel operator series. We start by showing the closed form of a second important series.

Theorem 3.1.5. *Let $T \in \mathcal{B}(X)$ and let $s \in \mathbb{H}$ with $\|T\| < |s|$. Then*

$$\left(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}\right)^{-1} = \sum_{n=0}^{+\infty} T^n \sum_{k=0}^{n} \overline{s}^{-k-1} s^{-n+k-1}, \tag{3.1}$$

where this series converges in the operator norm.

Proof. Let us denote the coefficients in (3.1) for neatness by

$$a_n = \sum_{k=0}^{n} \overline{s}^{-k-1} s^{-n+k-1}.$$

We have

$$|a_n| \le \sum_{k=0}^{n} |\overline{s}|^{k-1} |s|^{-n+k-1} = (n+1)|s|^{-n-2},$$

and so

$$\sum_{n=0}^{+\infty} \|T^n a_n\| \le \sum_{n=0}^{+\infty} \|T\|^n |s|^{-n-2}(n+1).$$

Since $\|T\| < |s|$, the ratio test implies the convergence of this series, since

$$\lim_{n\to\infty} \frac{\|T\|^{n+1}|s|^{-n-3}(n+2)}{\|T\|^n|s|^{-n-2}(n+1)} = \lim_{n\to\infty} \frac{(n+2)\|T\|}{(n+1)|s|} = \frac{\|T\|}{|s|} < 1,$$

and hence the series (3.1) converges in the operator norm. Moreover, we have

$$\begin{aligned}
&(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}) \sum_{n=0}^{+\infty} T^n a_n \\
&= \sum_{n=2}^{+\infty} T^n a_{n-2} - \sum_{n=1}^{+\infty} T^n a_{n-1} 2\mathrm{Re}(s) + \sum_{n=0}^{+\infty} T^n a_n |s|^2 \\
&= \sum_{n=2}^{+\infty} T^n (a_{n-2} - a_{n-1} 2\mathrm{Re}(s) + a_n |s|^2) \\
&\quad + T(a_0 2\mathrm{Re}(s) + a_1|s|^2) + \mathcal{I} a_0 |s|^2.
\end{aligned}$$

For $n \geq 2$, we have, because of $2\mathrm{Re}(s) = s + \overline{s}$ and $|s|^2 = s\overline{s} = \overline{s}s$, that

$$\begin{aligned}
&a_{n-2} - a_{n-1}2\mathrm{Re}(s) + a_n|s|^2\\
&= \sum_{k=0}^{n-2} \overline{s}^{-k-1}s^{-n+1+k} - \sum_{k=0}^{n-1} \overline{s}^{-k-1}2\mathrm{Re}(s)s^{-n+k} + \sum_{k=0}^{n} \overline{s}^{-k-1}|s|^2 s^{-n+k-1}\\
&= \sum_{k=1}^{n-1} \overline{s}^{-k}s^{-n+k} - \sum_{k=0}^{n-1} \overline{s}^{-k}s^{-n+k} - \sum_{k=0}^{n-1} \overline{s}^{-k-1}s^{-n+k+1} + \sum_{k=0}^{n} \overline{s}^{-k}s^{-n+k}\\
&= -s^{-n} + s^{-n} = 0.
\end{aligned}$$

Similarly, we also have, because of $s^{-1} = |s|^{-2}\overline{s}$ and $\overline{s}^{-1} = |s|^{-2}s$, that

$$\begin{aligned}
a_0 2\mathrm{Re}(s) - a_1|s|^2 &= |s|^{-2}(s+\overline{s}) - \left(\overline{s}^{-1}s^{-2} + \overline{s}^{-2}s^{-1}\right)|s|^2\\
&= \overline{s}^{-1} + s^{-1} - |s|^{-2}\left(s^{-1} + \overline{s}^{-1}\right)|s|^2 = 0,
\end{aligned}$$

and so altogether,

$$(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}) \sum_{n=0}^{+\infty} T^n a_n = \mathcal{I}a_0|s|^2 = \mathcal{I}.$$

Since the coefficients a_n satisfy $\overline{a_n} = a_n$, they are real, and hence they commute with T. Therefore, also

$$\sum_{n=0}^{+\infty} T^n a_n (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}) = (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}) \sum_{n=0}^{+\infty} T^n a_n = \mathcal{I},$$

and hence (3.1) holds. □

Theorem 3.1.6. *Let $T \in \mathcal{B}(X)$ and let $s \in \mathbb{H}$ with $\|T\| < |s|$.*

(i) *The left Cauchy kernel series equals*

$$\sum_{n=0}^{+\infty} T^n s^{-n-1} = -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}).$$

(ii) *The right Cauchy kernel series equals*

$$\sum_{n=0}^{+\infty} s^{-n-1}T^n = -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}.$$

Proof. We show just (i), since the other case can be shown with similar arguments. We prove the identity

$$\overline{s}\mathcal{I} - T = (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}) \sum_{n=0}^{+\infty} T^n s^{-1-n}.$$

Since $T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}$ is invertible by Theorem 3.1.5, this is equivalent to (i). The quaterions $2\mathrm{Re}(s) = s + \overline{s}$ and $|s|^2 = s\,\overline{s} = \overline{s}\,s$ are real, and hence they commute with the operator T, so we get

$$\begin{aligned}
&(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}) \sum_{n=0}^{+\infty} T^n s^{-n-1} \\
&= \sum_{n=0}^{+\infty} T^{n+2} s^{-n-1} - \sum_{n=0}^{+\infty} T^{n+1} s^{-n-1}(s+\overline{s}) + \sum_{n=0}^{+\infty} T^n s^{-n-1} s\overline{s} \\
&= \sum_{n=1}^{+\infty} T^{n+1} s^{-n} - \sum_{n=0}^{+\infty} T^{n+1} s^{-n} - \sum_{n=0}^{+\infty} T^{n+1} s^{-n-1}\overline{s} + \sum_{n=0}^{+\infty} T^n s^{-n}\overline{s} \\
&= \overline{s}\mathcal{I} - T.
\end{aligned}$$

□

The previous result motivates the following definition.

Definition 3.1.7. Let $T \in \mathcal{B}(X)$. For $s \in \mathbb{H}$, we set

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}.$$

We define the *S-resolvent set* $\rho_S(T)$ of T as

$$\rho_S(T) := \{s \in \mathbb{H} : \ \mathcal{Q}_s(T) \text{ is invertible in } \mathcal{B}(X)\},$$

and we define the *S-spectrum* $\sigma_S(T)$ of T as

$$\sigma_S(T) := \mathbb{H} \setminus \rho_S(T).$$

For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1} \in \mathcal{B}(X)$ is called the *pseudo-resolvent operator* of T at s.

As the following result shows, the S-spectrum has a structure that is compatible with the structure of slice hyperholomorphic functions and with the symmetry of the set of right eigenvalues of T. Moreover, it generalizes the set of right eigenvalues just as the classical spectrum generalizes the set of eigenvalues of a complex linear operator.

Proposition 3.1.8. *Let $T \in \mathcal{B}(X)$. The sets $\rho_S(T)$ and $\sigma_S(T)$ are axially symmetric.*

Proof. If $s = u + jv \in \mathbb{H}$ and $\tilde{s} = u + iv \in [s]$, then

$$\mathcal{Q}_{\tilde{s}}(T) = T^2 - 2uT + (u^2 + v^2)^2\mathcal{I} = \mathcal{Q}_s(T).$$

Hence $\mathcal{Q}_{\tilde{s}}(T)$ is invertible if and only if $\mathcal{Q}_s(T)$ is invertible, and so $s \in \rho_S(T)$ if and only if $\tilde{s} \in \rho_S(T)$. Therefore, $\rho_S(T)$ and $\sigma_S(T)$ are axially symmetric. □

Proposition 3.1.9. *Let $T \in \mathcal{B}(X)$. Then* $\ker \mathcal{Q}_s(T) \neq \{0\}$ *if and only if s is a right eigenvalue of T. In particular, every right eigenvalue belongs to $\sigma_S(T)$.*

Proof. If s is a right eigenvalue of T, then there exists $x \neq 0$ such that $Tx = xs$. Since $2\mathrm{Re}(s) = s + \overline{s}$, we have

$$\mathcal{Q}_s(T)x = T^2x - Tx2\mathrm{Re}(s) + x|s|^2 = xs^2 - xs(s+\overline{s}) + xs\overline{s} = 0,$$

and so $x \in \ker \mathcal{Q}_s(T) \neq \{0\}$. In particular, this implies that $\mathcal{Q}_s(T)$ is not invertible, and so $s \in \sigma_S(T)$.

Assume on the other hand that $x \in \ker \mathcal{Q}_s(T)$ for some $x \neq 0$. If $Tx = xs$, then s is a right eigenvalue. Otherwise, $\tilde{x} := Tx - xs \neq 0$ and

$$T\tilde{x} - \tilde{x}\overline{s} = T^2x - Txs - Tx\overline{s} + xs\overline{s} = \mathcal{Q}_s(T)x = 0.$$

Hence $\overline{s}$ is a right eigenvalue of T, and since the set of right eigenvalues is axially symmetric, we find that also s is a right eigenvalue. More precisely, if $s = u + jv \in \mathbb{C}_j$, then we can choose $i \in \mathbb{S}$ with $i \perp j$ and obtain

$$T(\tilde{x}i) = (T\tilde{x})i = (x\overline{s})i = (xi)s. \qquad \square$$

On the S-resolvent set we can now define the slice hyperholomorphic resolvents. Since we distinguish between left and right slice hyperholomorphicity, two different resolvent operators are associated with an operator T in the quaternionic setting.

Definition 3.1.10. Let $T \in \mathcal{B}(X)$. For $s \in \rho_S(T)$, we define the *left S-resolvent operator* as

$$S_L^{-1}(s,T) = -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I}),$$

and the *right S-resolvent operator* as

$$S_R^{-1}(s,T) = -(T - \overline{s}\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

Lemma 3.1.11. *Let $T \in \mathcal{B}(X)$.*

(i) *The left S-resolvent $S_L^{-1}(s,T)$ is a $\mathcal{B}(X)$-valued right slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

(ii) *The right S-resolvent $S_R^{-1}(s,T)$ is a $\mathcal{B}(X)$-valued left slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

Proof. We prove only (i), since (ii) is shown similarly. We have

$$S_L^{-1}(s,T) = f_0(u,v) + f_1(u,v)j$$

for every $s = u + jv \in \rho_S(T)$ with the $\mathcal{B}(X)$-valued functions

$$f_0(u,v) = -(T^2 - 2uT + (u^2+v^2)\mathcal{I})^{-1}(T - u\mathcal{I}),$$
$$f_1(u,v) = -(T^2 - 2uT + (u^2+v^2)\mathcal{I})^{-1}v.$$

Since f_0 and f_1 obviously satisfy the compatibility condition (2.4), the function $S_L^{-1}(s,T)$ is a $\mathcal{B}(X)$-valued right slice function on $\rho_S(T)$.

We verify that the restriction of $S_L^{-1}(s,T)$ to any complex plane belongs to the kernel of the Cauchy–Riemann operator on this plane if it is applied from the right. This is, by Proposition 2.3.12, equivalent to the right slice hyperholomorphicity of $S_L^{-1}(s,T)$. For $s = u + jv \in \rho_S(T)$, we have

$$\frac{\partial}{\partial u}\mathcal{Q}_s(T) = -2T + 2u\mathcal{I}, \quad \frac{\partial}{\partial v}\mathcal{Q}_s(T) = 2v\mathcal{I}.$$

Hence $\mathcal{Q}_s(T)$, $\frac{\partial}{\partial u}\mathcal{Q}_s(T)$ and $\frac{\partial}{\partial v}\mathcal{Q}_s(T)$ commute, and standard computations give

$$\frac{\partial}{\partial \eta}\mathcal{Q}_s(T)^{-1} = -\mathcal{Q}_s(T)^{-2}\frac{\partial}{\partial \eta}\mathcal{Q}_s(T)$$

for $\eta = u, v$. Therefore,

$$\frac{\partial}{\partial u}S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-2}(-2T + 2u\mathcal{I})(T - \overline{s}\mathcal{I}) + \mathcal{Q}_s(T)^{-1}$$

and

$$\frac{\partial}{\partial v}S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-2}2v(T - \overline{s}\mathcal{I}) - \mathcal{Q}_s(T)^{-1}j.$$

So finally we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial}{\partial u}S_L^{-1}(s,T) + \frac{\partial}{\partial v}S_L^{-1}(s,T)j\right)\\
&= \frac{1}{2}\Big(\mathcal{Q}_s(T)^{-2}(-2T + 2u\mathcal{I})(T - \overline{s}\mathcal{I}) + \mathcal{Q}_s(T)^{-1}\\
&\qquad + \mathcal{Q}_s(T)^{-2}2v(T - \overline{s}\mathcal{I})j - \mathcal{Q}_s(T)^{-1}j^2\Big)\\
&= \mathcal{Q}_s(T)^{-2}\left(-T(T - (u + vj)\mathcal{I}) + (T - (u + vj)\mathcal{I})\overline{s}\right) + \mathcal{Q}_s(T)^{-1}\\
&= -\mathcal{Q}_s(T)^{-1} + \mathcal{Q}_s(T)^{-1} = 0.
\end{aligned}$$

□

The S-spectrum has properties that are analogous to the properties of the usual spectrum of a complex linear operator. We first need the following result on invertibility of operators; see, for instance, Theorem 10.12 in [183].

Lemma 3.1.12. *The set* $\mathrm{Inv}(\mathcal{B}(X))$ *of invertible elements in* $\mathcal{B}(X)$ *is an open set in the uniform operator topology on* $\mathcal{B}(X)$*. If* $\mathrm{Inv}(\mathcal{B}(X))$ *contains an element* A*, then it contains the ball*

$$U(A) = \left\{B \in \mathcal{B}(X) : \ \|A - B\| < \left\|A^{-1}\right\|^{-1}\right\}.$$

If $B \in U(A)$*, then the inverse is given by the series*

$$B^{-1} = A^{-1}\sum_{m=0}^{+\infty}[(A - B)A^{-1}]^m. \tag{3.2}$$

Furthermore, the map $A \mapsto A^{-1}$ *is a homeomorphism from* $\mathrm{Inv}(\mathcal{B}(X))$ *onto* $\mathrm{Inv}(\mathcal{B}(X))$ *in the uniform operator topology.*

Proof. If $\|A - B\| < \left\|A^{-1}\right\|^{-1}$, then the series (3.2) converges and

$$\begin{aligned} BA^{-1}\sum_{m=0}^{+\infty}\left[(A-B)A^{-1}\right]^m &= \left(-(A-B)A^{-1}+\mathcal{I}\right)\sum_{m=0}^{+\infty}\left[(A-B)A^{-1}\right] \\ &= -\sum_{m=0}^{+\infty}\left[(A-B)A^{-1}\right]^{m+1}+\sum_{m=0}^{+\infty}\left[(A-B)A^{-1}\right]^m = \mathcal{I}. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} \left(A^{-1}\sum_{m=0}^{+\infty}\left[(A-B)A^{-1}\right]^m\right)B &= \left(\sum_{m=0}^{+\infty}\left[A^{-1}(A-B)\right]^m A^{-1}\right)B \\ &= \left(\sum_{m=0}^{+\infty}\left[A^{-1}(A-B)\right]^m\right)\left(-A^{-1}(A-B)+\mathcal{I}\right] = \mathcal{I}. \end{aligned}$$

Now observe that the series (3.2) converges uniformly on every ball $B_\varepsilon(A)$ with $0 < \varepsilon < \left\|A^{-1}\right\|^{-1}$. Hence if $A_n \to A \in \mathrm{Inv}(\mathcal{B}(X))$ with respect to $\|\cdot\|$, then for sufficiently large n we have

$$\lim_{n\to+\infty} A_n^{-1} = A^{-1}\sum_{m=0}^{+\infty}\lim_{n\to+\infty}\left[(A-A_n)A^{-1}\right]^m = A^{-1}.$$

The mapping $A \mapsto A^{-1}$ is therefore continuous on $\mathrm{Inv}(\mathcal{B}(X))$. Since it is self-inverse, it is even a homeomorphism. □

Theorem 3.1.13 (Compactness of the S-spectrum)**.** *Let* $T \in \mathcal{B}(X)$*. The* S*-spectrum* $\sigma_S(T)$ *of* T *is a nonempty compact set contained in the closed ball* $\overline{B_{\|T\|}(0)}$*.*

Proof. For $\|T\| < r$, the series $S_L^{-1}(s,T) = \sum_{n=0}^{+\infty} T^n s^{-n-1}$ converges uniformly on $\partial B_r(0)$. For $j \in \mathbb{S}$, we therefore have

$$\int_{\partial(B_r(0)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j = \sum_{n=0}^{+\infty} T^n \int_{\partial(B_r(0)\cap\mathbb{C}_j)} s^{-n-1}\,ds_j = 2\pi\,\mathcal{I}, \tag{3.3}$$

because $\int_{\partial(B_r(0)\cap\mathbb{C}_j)} s^{-n-1} ds_j$ equals 2π if $n = 0$ and 0 otherwise. If $\overline{B_r(0)}$ were a subset of $\rho_S(T)$, then $S_L^{-1}(s,T)$ would be right slice hyperholomorphic on $\overline{B_r(0)}$ by Lemma 3.1.11. Cauchy's integral theorem would then imply that the integral in (3.3) vanishes. Since this is obviously not the case, we conclude that $\overline{B_r(0)} \not\subset \rho_S(T)$ and in turn $\emptyset \neq \sigma_S(T) \cap \overline{B_r(0)}$. In particular, $\sigma_S(T)$ is not empty.

We can consider $\mathcal{B}(X)$ a real Banach algebra if we restrict the scalar multiplication to $\mathbb{R}$. The set $\mathrm{Inv}(\mathcal{B}(X))$ of invertible elements of this real Banach algebra is open thanks to Lemma 3.1.12. Since $\tau : s \mapsto \mathcal{Q}_s(T)$ is a continuous function with values in $\mathcal{B}(X)$, we find that $\rho_S(T) = \tau^{-1}(\mathrm{Inv}(\mathcal{B}(X)))$ is open in $\mathbb{H}$ and that $\sigma_S(T)$ in turn is closed.

Finally, Lemma 3.1.5 implies $|s| \leq \|T\|$ for every $s \in \sigma_S(T)$. Thus $\sigma_S(T)$ is a closed subset of the compact set $\overline{B_{\|T\|}(0)}$ and therefore compact itself. □

In the quaternionic setting the S-resolvent equation contains both S-resolvent operators. We need a preliminary result.

Theorem 3.1.14. *Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. The left S-resolvent operator satisfies the* left S-resolvent equation

$$S_L^{-1}(s,T)s - TS_L^{-1}(s,T) = \mathcal{I}, \tag{3.4}$$

and the right S-resolvent operator satisfies the right S-resolvent equation

$$sS_R^{-1}(s,T) - S_R^{-1}(s,T)T = \mathcal{I}. \tag{3.5}$$

Proof. Since $2\mathrm{Re}(s)$ and $|s|^2$ are real, they commute with the operator T. Therefore,

$$T\mathcal{Q}_s(T) = \mathcal{Q}_s(T)T,$$

and in turn

$$\mathcal{Q}_s(T)^{-1}T = T\mathcal{Q}_s(T)^{-1}.$$

Thus

$$\begin{aligned}
&S_L^{-1}(s,T)s - TS_L^{-1}(s,T)\\
&= -\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I})s + T\mathcal{Q}_s(T)^{-1}(T - \overline{s}\mathcal{I})\\
&= \mathcal{Q}_s(T)^{-1}\left(-(T - \overline{s}\mathcal{I})s + T(T - \overline{s}\mathcal{I})\right)\\
&= \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T) = \mathcal{I}.
\end{aligned}$$

The right S-resolvent equation follows by similar computations. □

The left and right S-resolvent equations cannot be considered generalizations of the classical resolvent equation

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda)R_\lambda(A)R_\mu(A), \quad \text{for } \lambda, \mu \in \rho(A), \tag{3.6}$$

where $R_\lambda(A) = (\lambda\mathcal{I} - A)^{-1}$ is the resolvent operator of A at $\lambda \in \rho(A)$. This equation provides the possibility to split the product of two resolvent operators into a sum of the factors. This is not the case for the left and the right S-resolvent equations.

The proper generalization of (3.6), which preserves this philosophy, is the S-resolvent equation that we show in the following theorem. It is remarkable that this equation involves both the left and right S-resolvent operators and that no generalization of (3.6) that includes just one of them has ever been found.

Theorem 3.1.15 (The S-resolvent equation)**.** *Let $T \in \mathcal{B}(X)$ and let $s, q \in \rho_S(T)$ with $q \notin [s]$. Then the equation*

$$S_R^{-1}(s,T)S_L^{-1}(p,T) = \big[\big(S_R^{-1}(s,T) - S_L^{-1}(q,T)\big)\, q \\ -\overline{s}\left(S_R^{-1}(s,T) - S_L^{-1}(q,T)\right)\big]\,(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} \quad (3.7)$$

holds. Equivalently, it can also be written as

$$S_R^{-1}(s,T)S_L^{-1}(q,T) = (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1} \\ \cdot \big[\big(S_L^{-1}(q,T) - S_R^{-1}(s,T)\big)\,\overline{q} - s\left(S_L^{-1}(q,T) - S_R^{-1}(s,T)\right)\big]. \quad (3.8)$$

Proof. We show that

$$S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2) \\ = (S_R^{-1}(s,T) - S_L^{-1}(q,T))q - \overline{s}(S_R^{-1}(s,T) - S_L^{-1}(q,T)), \quad (3.9)$$

which is equivalent to (3.7). The left S-resolvent equation (3.4) implies

$$S_L^{-1}(q,T)q = TS_L^{-1}(q,T) + \mathcal{I}.$$

Applying this identity twice, we obtain

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2) \\ &= S_R^{-1}(s,T)[TS_L^{-1}(q,T) + \mathcal{I}]q - 2s_0S_R^{-1}(s,T)[TS_L^{-1}(q,T) + \mathcal{I}] \\ &\quad + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T) \\ &= S_R^{-1}(s,T)T[TS_L^{-1}(q,T) + \mathcal{I}] + S_R^{-1}(s,T)q \\ &\quad - 2s_0S_R^{-1}(s,T)[TS_L^{-1}(q,T) + \mathcal{I}] + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T) \\ &= [S_R^{-1}(s,T)T]TS_L^{-1}(q,T) + S_R^{-1}(s,T)T + S_R^{-1}(s,T)q \\ &\quad - 2s_0[[S_R^{-1}(s,T)T]S_L^{-1}(q,T) + S_R^{-1}(s,T)] \\ &\quad + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T). \end{aligned}$$

Now the right S-resolvent equation (3.5) implies

$$S_R^{-1}(s,T)T = sS_R^{-1}(s,T) - \mathcal{I}.$$

Applying this identity twice, we obtain

$$\begin{aligned} &S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2) \\ &= [[sS_R^{-1}(s,T) - \mathcal{I}]T]S_L^{-1}(q,T) + sS_R^{-1}(s,T) - \mathcal{I}] + S_R^{-1}(s,T)q \\ &\quad - 2s_0[[sS_R^{-1}(s,T) - \mathcal{I}]S_L^{-1}(q,T) + S_R^{-1}(s,T)] \\ &+ |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T) \\ &= [s[sS_R^{-1}(s,T) - \mathcal{I}] - T]S_L^{-1}(q,T) + sS_R^{-1}(s,T) - \mathcal{I}] + S_R^{-1}(s,T)q \\ &\quad - 2s_0[[sS_R^{-1}(s,T)S_L^{-1}(q,T) - S_L^{-1}(q,T)] + S_R^{-1}(s,T)] \\ &\quad + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T). \end{aligned}$$

Collecting like terms, we end up with

$$\begin{aligned}&S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2-2s_0q+|s|^2)\\&=(s^2-2s_0s+|s|^2)S_R^{-1}(s,T)S_L^{-1}(q,T)\\&\quad+[S_R^{-1}(s,T)-S_L^{-1}(q,T)]q-\overline{s}[S_R^{-1}(s,T)-S_L^{-1}(q,T)],\end{aligned}$$

and since $s^2-2s_0s+|s|^2=0$, we obtain (3.9). With similar computations we can show that also (3.8) holds. □

3.2 Definition of the S-Functional Calculus

We can now define the S-functional calculus for a bounded quaternionic linear operator T on a two-sided quaternionic Banach space X. The S-functional calculus is the quaternionic version of the Riesz–Dunford functional calculus for complex linear operators. We consider a function f that is slice hyperholomorphic on $\sigma_S(T)$, and we use the slice hyperholomorphic Cauchy formula. In order to define $f(T)$ we formally replace the scalar variable q by the operator T, in Cauchy kernels $S_L^{-1}(s,q)$ and $S_R^{-1}(s,q)$, and we replace in the Cauchy formulas the S-resolvent operators $S_L^{-1}(s,T)$ and $S_R^{-1}(s,T)$. It is nontrivial to motivate the fact that we can replace q by T, as we will see. The main references in which the formulations and the properties of S-functional calculus for quaternionic operators have been studied are [10, 79, 80].

Before we define the S-functional calculus, we show that the procedure described above is actually meaningful. In particular, it must be consistent with functions of T that we can define explicitly, that is, with polynomials in T.

Lemma 3.2.1. *Let $T\in\mathcal{B}(X)$, let $m\in\mathbb{N}\cup\{0\}$, and let $U\subset\mathbb{H}$ be a bounded slice Cauchy domain with $\sigma_S(T)\subset U$. For every imaginary unit $j\in\mathbb{S}$, we have*

$$T^m=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}S_L^{-1}(s,T)\,ds_j\,s^m$$

and also

$$T^m=\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}s^m\,ds_j\,S_R^{-1}(s,T).$$

Proof. Let us first consider the case that U is a ball $B_r(0)$ with $\|T\|<r$. Then $S_L^{-1}(s,T)=\sum_{n=0}^{+\infty}T^ns^{-n-1}$ for every $s\in\partial B_r(0)$ by Theorem 3.1.6, and this series converges uniformly on $\partial B_r(0)$. Thus

$$\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)}S_L^{-1}(s,T)\,ds_j\,s^m=\frac{1}{2\pi}\sum_{n=0}^{+\infty}T^n\int_{\partial(B_r(0)\cap\mathbb{C}_j)}s^{-1-n+m}ds_j=T^m,$$

because

$$\int_{\partial(B_r(0)\cap\mathbb{C}_j)} s^{-1-n+m}\,ds_j = \begin{cases} 0 & \text{if } n \neq m,\\ 2\pi & \text{if } n = m.\end{cases}$$

Now let U be an arbitrary bounded slice Cauchy domain that contains $\sigma_S(T)$. Then there exists a radius r such that $\overline{U} \subset B_r(0)$. The left S-resolvent $S_L^{-1}(s,T)$ is then right slice hyperholomorphic and the monomial s^m is left slice hyperholomorphic on the bounded slice Cauchy domain $B_r(0) \setminus U$. We conclude from Cauchy's integral theorem that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,s^m - \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,s^m\\ &= \frac{1}{2\pi}\int_{\partial((B_r(0)\setminus U)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,s^m = 0,\end{aligned}$$

and so

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,s^m = \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,s^m = T^m.$$

The second identity, which involves the right S-resolvent $S_R^{-1}(s,T)$, follows by similar arguments from the corresponding series expansion of the right S-resolvent operator. □

Theorem 3.2.2. *Let $T \in \mathcal{B}(X)$, let U be a bounded slice Cauchy domain that contains $\sigma_S(T)$, and let $j \in \mathbb{S}$. For every left slice hyperholomorphic polynomial $P(q) = \sum_{\ell=0}^n q^\ell a_\ell$ with $a_\ell \in \mathbb{H}$, we set $P(T) = \sum_{\ell=0}^n T^\ell a_\ell$. Then*

$$P(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,P(s). \tag{3.10}$$

Similarly, we set $P(T) = \sum_{\ell=0}^n a_\ell T^\ell$ for every right slice hyperholomorphic polynomial $P(q) = \sum_{\ell=0}^n a_\ell q^\ell$ with $a_\ell \in \mathbb{H}$. Then

$$P(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} P(s)\,ds_j\,S_R^{-1}(s,T). \tag{3.11}$$

In particular, the operators in (3.10) and (3.11) coincide for every intrinsic polynomial $P(q) = \sum_{\ell=0}^n q^\ell a_\ell$ with real coefficients $a_\ell \in \mathbb{R}$.

Proof. For $P(q) = \sum_{\ell=0}^n q^\ell a_\ell$, Lemma 3.2.1 implies

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,P(s) &= \sum_{\ell=0}^n \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,s^\ell\right] a_\ell\\ &= \sum_{\ell=0}^n T^\ell a_\ell = P(T).\end{aligned}$$

The case of a right slice hyperholomorphic polynomial follows with analogous computations. Finally, if $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ is an intrinsic polynomial with real coefficients, then the coefficients a_ℓ commute with the operator T and hence

$$\begin{aligned}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,P(s) &= \sum_{\ell=0}^{n} T^\ell a_\ell \\ &= \sum_{\ell=0}^{n} a_\ell T^\ell = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} P(s)\,ds_j\,S_R^{-1}(s,T).\end{aligned}$$ □

The S-functional calculus applies to functions that are slice hyperholomorphic on the S-spectrum of T. We introduce the following notation for this class of functions.

Definition 3.2.3. Let $T \in \mathcal{B}(X)$. We denote by $\mathcal{SH}_L(\sigma_S(T))$, $\mathcal{SH}_R(\sigma_S(T))$, and $\mathcal{N}(\sigma_S(T))$ the sets of all left, right, and intrinsic slice hyperholomorphic functions f with $\sigma_S(T) \subset \mathcal{D}(f)$, where $\mathcal{D}(f)$ is the domain of the function f.

Remark 3.2.4. The set $\mathcal{D}(f)$ is an axially symmetric open set that contains the compact axially symmetric set $\sigma_S(T)$. If we choose $j \in \mathbb{S}$, then $\mathcal{D}(f) \cap \mathbb{C}_j$ is an open set in $\mathbb{C}_j$ that contains the compact set $\sigma_S(T) \cap \mathbb{C}_j$ and hence there exists a bounded Cauchy domain U_j in $\mathbb{C}_j$ such that $\sigma_S(T) \cap \mathbb{C}_j \subset U_j$ and $\overline{U_j} \subset \mathcal{D}(f) \cap \mathbb{C}_j$. Since $\sigma_S(T) \cap \mathbb{C}_j$ and $\mathcal{D}(f) \cap \mathbb{C}_j$ are symmetric with respect to the real line, we can also choose U_j symmetric with respect to the real line. Taking the axially symmetric hull, we obtain a bounded slice Cauchy domain $U := [U_j]$ with

$$\sigma_S(T) \subset U \quad \text{and} \quad \overline{U} \subset \mathcal{D}(f).$$

Definition 3.2.5 (S-functional calculus)**.** Let $T \in \mathcal{B}(X)$. For every function $f \in \mathcal{SH}_L(\sigma_S(T))$, we define

$$f(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s), \tag{3.12}$$

where j is an arbitrary imaginary unit in $\mathbb{S}$ and U is an arbitrary slice Cauchy domain U as in Remark 3.2.4. For every $f \in \mathcal{SH}_R(\sigma_S(T))$, we define

$$f(T) := \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T), \tag{3.13}$$

where j is again an arbitrary imaginary unit in $\mathbb{S}$ and U is an arbitrary slice Cauchy domain as in Remark 3.2.4.

Theorem 3.2.2 shows that the S-functional calculus is meaningful, because it is consistent with polynomials in T. As the next crucial result shows, it is, moreover, well defined.

Theorem 3.2.6. *Let $T \in \mathcal{B}(X)$. For every $f \in \mathcal{SH}_L(\sigma_S(T))$, the integral in (3.12) that defines the operator $f(T)$ is independent of the choice of the slice Cauchy domain U and the imaginary unit $j \in \mathbb{S}$. Similarly, for every $f \in \mathcal{SH}_R(\sigma_S(T))$, the integral in (3.13) that defines the operator $f(T)$ is also independent of the choice of U and $j \in \mathbb{S}$.*

Proof. Let $f \in \mathcal{SH}_L(\sigma_S(T))$. We first show that the integral (3.12) does not depend on the slice Cauchy domain U. Let U' be another bounded slice Cauchy domain with $\sigma_S(T) \subset U'$ and $\overline{U'} \subset \mathcal{D}(f)$, and let us assume for the moment that $\overline{U'} \subset U$. Then $O = U \setminus U'$ is again a bounded slice Cauchy domain, and we have $\overline{O} \subset \rho_S(T)$ and $\overline{O} \subset \mathcal{D}(f)$. Hence the function f is left slice hyperholomorphic and the left S-resolvent is right slice hyperholomorphic on $\overline{O}$. Cauchy's integral theorem therefore implies

$$\begin{aligned} 0 =& \frac{1}{2\pi} \int_{\partial(O\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) \\ =& \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) \\ & - \frac{1}{2\pi} \int_{\partial(U'\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s). \end{aligned}$$

If $\overline{U'} \not\subset U$, then $O := U \cap U'$ is an axially symmetric open set that contains $\sigma_S(T)$. As in Remark 3.2.4, we can hence find a third slice Cauchy domain U'' with $\sigma_S(T) \subset U''$ and $\overline{U''} \subset O = U \cap U'$. The above arguments show that the integrals over the boundaries of all three sets agree.

In order to show the independence of the imaginary unit, we choose two units $i, j \in \mathbb{S}$ and two slice Cauchy domains $U_q, U_s \subset \mathcal{D}(f)$ with $\sigma_S(T) \subset U_q$ and $\overline{U_q} \subset U_s$. (The subscripts q and s are chosen in order to indicate the respective variable of integration in the following computation.) The set $U_q^c := \mathbb{H} \setminus U_q$ is then an unbounded axially symmetric slice Cauchy domain with $\overline{U_q^c} \subset \rho_S(T)$. The left S-resolvent is right slice hyperholomorphic on $\rho_S(T)$ and also at infinity because

$$\lim_{s\to\infty} S_L^{-1}(s,T) = \lim_{s\to\infty} \sum_{n=0}^{+\infty} T^n s^{-n-1} = 0.$$

The right slice hyperholomorphic Cauchy formula implies therefore

$$S_L^{-1}(s,T) = \frac{1}{2\pi} \int_{\partial(U_q^c\cap\mathbb{C}_i)} S_L^{-1}(q,T)\, dq_i\, S_R^{-1}(q,s)$$

for every $s \in U^c$. Since $\partial(U_q^c \cap \mathbb{C}_j) = -\partial(U_q \cap \mathbb{C}_j)$ and $S_R^{-1}(q,s) = -S_L^{-1}(s,q)$ by

Corollary 2.1.26, we therefore obtain

$$\begin{aligned} f(T) &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s) \\ &= \frac{1}{(2\pi)^2}\int_{\partial(U_s\cap\mathbb{C}_j)} \left(\int_{\partial(U_q^c\cap\mathbb{C}_i)} S_L^{-1}(q,T)\,dq_i\,S_R^{-1}(q,s)\right) ds_j\,f(s) \\ &= \frac{1}{(2\pi)^2}\int_{\partial(U_q\cap\mathbb{C}_i)} S_L^{-1}(q,T)\,dq_i \left(\int_{\partial(U_s\cap\mathbb{C}_j)} S_L^{-1}(q,s)\,ds_j\,f(s)\right) \\ &= \frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_i)} S_L^{-1}(q,T)\,dq_i f(q), \end{aligned}$$

where the last identity follows again from the slice hyperholomorphic Cauchy formula because we chose $\overline{U_q} \subset U_s$. □

Theorem 3.2.6 shows that the S-functional calculus is well defined for every left or right slice hyperholomorphic function, and Theorem 3.2.1 shows that it is consistent with polynomials. Moreover, if $f \in \mathcal{N}(\sigma_S(T))$, then (3.12) and (3.13) give the same operator. We will show this by uniform approximation of f with intrinsic rational functions. Hence we first need to show that the S-functional calculus is consistent with the limits of uniformly convergent sequences of slice hyperholomorphic functions and that both versions of the S-functional calculus are consistent for intrinsic rational functions.

Theorem 3.2.7. *Let $T\in\mathcal{B}(X)$. Let $f_n, f\in\mathcal{SH}_L(\sigma_S(T))$, or let $f_n, f\in\mathcal{SH}_R(\sigma_S(T))$ for $n \in \mathbb{N}$. If there exists a bounded slice Cauchy domain U with $\sigma_S(T) \subset U$ such that $f_n \to f$ uniformly on $\overline{U}$, then $f_n(T)$ converges to $f(T)$ in $\mathcal{B}(X)$.*

Proof. Since $f_n \to f$ uniformly on $\overline{U}$, we can exchange limit and integration and obtain

$$\begin{aligned} \lim_{n\to\infty} f_n(T) &= \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f_n(s) \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s) = f(T). \end{aligned}$$ □

Lemma 3.2.8 (Product rule)**.** *Let $T \in \mathcal{B}(X)$ and let P be a slice hyperholomorphic intrinsic polynomial. If $g \in \mathcal{SH}_L(\sigma_S(T))$, then $Pg \in \mathcal{SH}_L(\sigma_S(T))$ and*

$$(Pg)(T) = P(T)g(T). \tag{3.14}$$

Similarly, if $f \in \mathcal{SH}_R(\sigma_S(T))$, then $fP \in \mathcal{SH}_R(\sigma_S(T))$ and

$$(fP)(T) = f(T)P(T). \tag{3.15}$$

Proof. We consider only the case $g \in \mathcal{SH}_L(\sigma_S(T))$ because the right slice hyperholomorphic case follows as usual by similar arguments. Since P is intrinsic, the function Pg belongs to $\mathcal{SH}_L(\sigma_S(T))$ by Theorem 2.1.4. By Theorem 3.2.2, we can represent the operator $P(T)$ by the slice hyperholomorphic Cauchy integral (3.11) that involves right S-resolvent operator. The identity (3.14) follows now the same computations as in the proof of the general product rule in Theorem 4.1.3. Since these computations are a quite long, we prefer not to replicate them here and refer instead to the proof of Theorem 4.1.3. □

Remark 3.2.9. The reader might wonder why we refer to the proof of the general product rule instead of showing it right away here. The reason is that at this stage, we are not yet able to do this: since the S-resolvent equations involve both the left and right S-resolvent operators, it is essential to know that (3.12) and (3.13) are consistent for every intrinsic function f, i.e., that they give the same operator $f(T)$. So far, we know only that this holds for intrinsic polynomials; cf. Theorem 3.2.2. The special case of the product rule in Lemma 3.2.8 is, however, essential for the proof of the compatibility of the S-functional calculus with intrinsic rational functions and in turn for the proof of the compatibility of (3.12) and (3.13) for arbitrary intrinsic functions. Hence it cannot be postponed. The overall strategy consists, therefore, in proving the following statements, each of which builds upon the previous one.

(1) The S-functional calculi for left and right slice hyperholomorphic functions coincide for intrinsic polynomials; cf. Theorem 3.2.2.

(2) The product rule holds if an intrinsic polynomial is involved; cf. Lemma 3.2.8.

(3) The S-functional calculi for left and right slice hyperholomorphic functions are consistent for intrinsic rational functions.

(4) The S-functional calculi for left and right slice hyperholomorphic functions are consistent for arbitrary intrinsic slice hyperholomorphic functions.

Only at this stage can we prove the general product rule. The computations for the special case in (2) and for the general product rule are, however, identical, so that we prefer to show them just once.

Lemma 3.2.10. *Let $T \in \mathcal{B}(X)$. If P is an intrinsic polynomial such that $P^{-1} \in \mathcal{N}(\sigma_S(T))$, then $P^{-1}(T) = P(T)^{-1}$. Moreover, if $r(q) = P(q)^{-1}Q(q)$ is an intrinsic rational function and $P^{-1} \in \mathcal{N}(\sigma_S(T))$, then* (3.12) *and* (3.13) *give the same operator $r(T) = P(T)^{-1}Q(T)$.*

Proof. Let $0 \neq P \in \mathcal{N}(\mathbb{H})$ be a polynomial with real coefficients such that $P^{-1} \in \mathcal{N}(\sigma_S(T))$. Then Lemma 3.2.8 implies

$$\mathcal{I} = 1(T) = (PP^{-1})(T) = P(T)P^{-1}(T)$$

if we consider P and P^{-1} left slice hyperholomorphic functions and

$$\mathcal{I} = 1(T) = (P^{-1}P)(T) = P^{-1}(T)P(T)$$

if we consider them right slice hyperholomorphic functions. Hence $P(T)$ is invertible and

$$P(T)^{-1} = P^{-1}(T)$$

for both versions of the S-functional calculus.

For an intrinsic rational function $r(q) = P(q)^{-1}Q(q)$, we obtain again from Lemma 3.2.8 that $r(T) = P(T)^{-1}Q(T)$ if we consider it a right slice hyperholomorphic function and $r(T) = Q(T)P(T)^{-1}$ if we consider it a left slice hyperholomorphic function. Since $P(T)$ and $Q(T)$ are polynomials in T with real coefficients, they commute, and we find that in both cases, $r(T) = P(T)^{-1}Q(T)$. □

Theorem 3.2.11. *Let $T \in \mathcal{B}(X)$. If $f \in \mathcal{N}(\sigma_S(T))$, then both versions of S-functional calculus give the same operator $f(T)$. Precisely, we have*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T).$$

Proof. Let $f \in \mathcal{N}(\sigma_S(T))$ and let U be a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$. Then $\overline{U}$ is compact and therefore Theorem 2.1.37 implies the existence of a sequence r_n of intrinsic rational functions such that $f = \lim_{n\to\infty} r_n$ uniformly on $\overline{U}$. From Theorem 3.2.7 and Lemma 3.2.10, we conclude that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\\
&= \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,r_n(s)\\
&= \lim_{n\to\infty}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} r_n(s)\,ds_j\,S_R^{-1}(s,T)\\
&= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,S_R^{-1}(s,T).
\end{aligned}$$
□

3.3 Comments and Remarks

In the following subsections we want to point out two facts. First we present the series expansion of the resolvent operator associated with the left spectrum; then we show how the S-resolvent equation has been obtained.

3.3.1 The Left spectrum $\sigma_L(T)$ and the Left Resolvent Operator

Definition 3.3.1. Let T be a right linear operator in $\mathcal{B}(X)$. We define the *left resolvent set* of T as

$$\rho_L(T) := \left\{ s \in \mathbb{H} : \ (s\mathcal{I} - T)^{-1} \in \mathcal{B}(X) \right\}$$

and the *left spectrum* of T as

$$\sigma_L(T) := \mathbb{H} \setminus \rho_L(T).$$

Moreover, we define

$$R_T(s) := (s\mathcal{I} - T)^{-1}$$

for all $s \in \rho_L(T)$.

Theorem 3.3.2. *Let $T \in \mathcal{B}(V)$ and let $s \in \mathbb{H}$ be such that $\|T\| < |s|$.*

(i) *Then the operator $\sum_{n=0}^{\infty}(s^{-1}T)^n s^{-1}\mathcal{I}$ is the right and left algebraic inverse of $s\mathcal{I} - T$, and the series converges in the operator norm.*

(ii) *The left spectrum $\sigma_L(T)$ is contained in the ball $\{s \in \mathbb{H} : |s| \leq \|T\|\}$.*

Proof. We prove point (i). It follows from the computations that

$$\begin{aligned}
&(s\mathcal{I} - T)\sum_{n=0}^{\infty}(s^{-1}T)^n s^{-1}\mathcal{I} \\
&= s\mathcal{I}\sum_{n=0}^{\infty}(s^{-1}T)^n s^{-1}\mathcal{I} - T\sum_{n\geq 0}(s^{-1}T)^n s^{-1}\mathcal{I} \\
&= s\mathcal{I}s^{-1}\mathcal{I} + Ts^{-1}\mathcal{I} + T(s^{-1}T)s^{-1}\mathcal{I} + \cdots \\
&\quad - Ts^{-1}\mathcal{I} - T(s^{-1}T)s^{-1}\mathcal{I} - T(s^{-1}T)^2 s^{-1}\mathcal{I} - \cdots = \mathcal{I}.
\end{aligned}$$

Similarly, we can prove that

$$\sum_{n=0}^{\infty}(s^{-1}T)^n s^{-1}\mathcal{I}(s\mathcal{I} - T) = \mathcal{I}.$$

Finally, we observe that for $\|T\| < |s|$, the following series converges:

$$\begin{aligned}
\|\sum_{n=0}^{\infty}(s^{-1}T)^n s^{-1}\mathcal{I}\| &\leq \sum_{n=0}^{\infty}\|(s^{-1}T)^n s^{-1}\mathcal{I}\| \\
&\leq \sum_{n=0}^{\infty}\|(s^{-1}T)\|^n |s^{-1}| \\
&\leq \sum_{n=0}^{\infty}\|T\|^n |s^{-1}|^{n+1}.
\end{aligned}$$

We prove point (ii). Since the series $\sum_{n=0}^{\infty}(s^{-1}T)^n s^{-1}\mathcal{I}$ converges if and only if $|s^{-1}|\|T\| < 1$, we get the statement. □

We observe that since

$$R_T(s) := (s\mathcal{I} - T)^{-1} = \sum_{n=0}^{\infty}(s^{-1}T)^n s^{-1}\mathcal{I}, \quad \text{for } |s| < \|T\|,$$

it is difficult to say whether there exists some notion of hyperholomorphicity associated with this resolvent operator.

3.3.2 Power Series Expansions and the S-Resolvent Equation

Finding the S-resolvent equation was a quite difficult task. A hint on its structure came from computations with the power series expansions of the S-resolvent operators, which we show in the following lemmas.

Lemma 3.3.3. *Let $B \in \mathcal{B}(X)$ and let $s, q \in \mathbb{H}$. For $|q| < |s|$, we have*

$$\sum_{m=0}^{+\infty} q^m B s^{-1-m} = -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(qB - B\overline{s}) \tag{3.16}$$

and

$$\sum_{m=0}^{+\infty} s^{-1-m} B q^m = -(Bq - \overline{s}B)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \tag{3.17}$$

Moreover, (3.16) can be rewritten as

$$\sum_{m=0}^{+\infty} q^m B s^{-1-m} = (\overline{q}B - Bs)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}, \tag{3.18}$$

and (3.17) can be written as

$$\sum_{m=0}^{+\infty} s^{-1-m} B q^m = (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q}). \tag{3.19}$$

Proof. We have

$$\begin{aligned}
&(q^2 - 2\mathrm{Re}(s)q + |s|^2) \sum_{m=0}^{+\infty} q^m B s^{-1-m} \\
&= \sum_{m=0}^{+\infty} q^{m+2} B s^{-1-m} - \sum_{m=0}^{+\infty} q^{m+1} B s^{-1-m} 2\mathrm{Re}(s) + \sum_{m=0}^{+\infty} q^m B s^{-1-m} |s|^2 \\
&= \sum_{m=2}^{+\infty} q^m B s^{-1-m} (s^2 - 2\mathrm{Re}(s)s + |s|^2) \\
&\quad - qBs^{-1}(s + \overline{s}) + Bs^{-1}|s|^2 + qBs^{-2}|s|^2 = -qB + B\overline{s},
\end{aligned}$$

because for every $s \in \mathbb{H}$,

$$s^2 - 2\mathrm{Re}(s)s + |s|^2 = 0.$$

Multiplication by $(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$ from the left yields (3.16). The identity

(3.18), on the other hand, follows from

$$\begin{aligned}
&\sum_{m=0}^{+\infty} q^m B s^{-1-m}(s^2 - 2\mathrm{Re}(q)s + |q|^2)\\
&= \sum_{m=0}^{+\infty} q^m B s^{1-m} - \sum_{m=0}^{+\infty} 2\mathrm{Re}(q) q^m B s^{-m} + \sum_{m=0}^{+\infty} |q|^2 q^m B s^{-1-m}\\
&= \sum_{m=0}^{+\infty} (q^2 - 2\mathrm{Re}(q)q + |q|)^2 q^m B s^{-1-m} + Bs + qB - (q+\overline{q})B = Bs - \overline{q}B
\end{aligned}$$

and multiplication of this equality by $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}$ from the right. With similar computations one can verify (3.17) and (3.19). □

Corollary 3.3.4. *Let $B \in \mathcal{B}(X)$. For $s, q \in \mathbb{H}$ with $|q| < |s|$, we have*

$$\begin{aligned}
\sum_{j=0}^{m} q^j B s^{-1-j} =& -(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(qB - B\overline{s})\\
&+ q^{m+1}(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(qB - B\overline{s})s^{-1-m}
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
\sum_{j=0}^{m} s^{-1-j} B q^j =& -(Bq - \overline{s}B)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}\\
&+ s^{-1-m}(Bq - \overline{s}B)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} q^{m+1}.
\end{aligned} \tag{3.21}$$

Moreover, (3.20) can also be written as

$$\begin{aligned}
\sum_{j=0}^{m} q^j B s^{-1-j} =& (\overline{q}B - Bs)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}\\
&+ q^{m+1}(\overline{q}B - Bs)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}s^{-1-m},
\end{aligned} \tag{3.22}$$

and (3.21) can also written as

$$\begin{aligned}
\sum_{j=0}^{m} s^{-1-j} B q^j =& (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q})\\
&- s^{-1-m}(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q})q^{m+1}.
\end{aligned} \tag{3.23}$$

Proof. We have

$$\begin{aligned}
\sum_{j=0}^{m} q^j A s^{-1-j} &= \sum_{j=0}^{+\infty} q^j A s^{-1-j} - \sum_{j=m+1}^{+\infty} q^j A s^{-1-j}\\
&= \sum_{j=0}^{+\infty} q^j A s^{-1-j} - q^{m+1}\left(\sum_{j=0}^{+\infty} q^j A s^{-1-j}\right) s^{-1-m},
\end{aligned}$$

and applying (3.16) for the series yields (3.20). Similarly, the identities (3.21), (3.22), and (3.23) follow from (3.17), (3.18), and (3.19), respectively. □

We prove now the S-resolvent equation when the S-resolvent operators admit the power series expansions

$$S_L^{-1}(q,T) = \sum_{m=0}^{+\infty} T^m q^{-1-m} \quad \text{and} \quad S_R^{-1}(s,T) = \sum_{m=0}^{+\infty} s^{-1-m} T^m,$$

which is in particular the case for $\|T\| < |q| < |s|$. Then we have

$$S_R^{-1}(s,T) S_L^{-1}(q,T) = \left(\sum_{j=0}^{+\infty} s^{-1-j} T^j \right) \left(\sum_{j=0}^{+\infty} T^j q^{-1-j} \right), \tag{3.24}$$

and setting

$$\Lambda_m(s,q;T) := \sum_{j=0}^{m} s^{-1-j} (T^m q^{-1-m}) q^j,$$

we can write (3.24) as

$$S_R^{-1}(s,T) S_L^{-1}(q,T) = \sum_{m=0}^{+\infty} \Lambda_m(s,q;T).$$

Since q and $(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$ commute, we deduce from (3.21) with $B = T^m q^{-1-m}$ that

$$\begin{aligned}
\Lambda_m(s,q;T) &= -((T^m q^{-1-m})q - \overline{s}(T^m q^{-1-m}))(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} \\
&\quad + s^{-1-m}((T^m q^{-1-m})q - \overline{s}(T^m q^{-1-m}))(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} q^{m+1} \\
&= -[(T^m q^{-1-m})q - \overline{s}(T^m q^{-1-m}) \\
&\quad + (s^{-1-m}T^m)q - \overline{s}(s^{-1-m}T^m)](q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&S_R^{-1}(s,T) S_L^{-1}(q,T) = \sum_{m=0}^{+\infty} \Lambda_m(s,q;T) = \\
&= -\Bigg[\left(\sum_{m=0}^{+\infty} T^m q^{-1-m} \right) q - \overline{s} \sum_{m=0}^{+\infty} T^m q^{-1-m} \\
&\quad + \left(\sum_{m=0}^{+\infty} s^{-1-m} T^m \right) q - \overline{s} \sum_{m=0}^{+\infty} s^{-1-m} T^m \Bigg] (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}
\end{aligned}$$

and (3.7) follows. To prove that the resolvent equation can be written in the second form (3.8), we observe that $\Lambda_m(s,q;T)$ can also be written using (3.19) as

$$\Lambda_m(s,q;T) = (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(s(T^m q^{-1-m}) - (T^m q^{-1-m})\overline{q})$$
$$- s^{-1-m}(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(s(T^m q^{-1-m}) - (T^m q^{-1-m})\overline{q})q^{m+1}.$$

Substituting this in the sum $\sum_{m=0}^{+\infty}\Lambda_m(s,q;T)$, we get the second version of the resolvent equation.

Chapter 4

Properties of the S-Functional Calculus for Bounded Operators

In this chapter we will show that most of the properties that hold for the Riesz-Dunford functional calculus can be extended to the S-functional calculus. The proofs of the quaternionic results require several additional efforts with respect to the classical case.

4.1 Algebraic Properties and Riesz Projectors

An immediate consequence of Definition 3.2.5 is that the S-functional calculus for left slice hyperholomorphic functions is quaternionic right linear and that the S-functional calculus for right slice hyperholomorphic functions is quaternionic left linear.

Lemma 4.1.1. *Let $T \in \mathcal{B}(X)$.*

(i) *If $f, g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \textit{and} \quad (fa)(T) = f(T)a.$$

(ii) *If $f, g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(f+g)(T) = f(T) + g(T) \quad \textit{and} \quad (af)(T) = af(T).$$

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_4

Proof. If $f, g \in \mathcal{SH}_L(U)$ and $a \in \mathbb{H}$, then we have

$$\begin{aligned}(f+g)(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,(f(s)+g(s))\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\\ &\quad + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,g(s) = f(T)+g(T)\end{aligned}$$

and

$$\begin{aligned}(fa)(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)a\\ &= \left(\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\right)a = f(T)a.\end{aligned}$$

The right slice hyperholomorphic case follows by similar computations. □

Since the product of two slice hyperholomorphic functions is not necessarily slice hyperholomorphic, we cannot expect to obtain a product rule for arbitrary slice hyperholomorphic functions. However, if $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$, then $fg \in \mathcal{SH}_L(\sigma_S(T))$, and if $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T)$, then $fg \in \mathcal{SH}_R(\sigma_S(T))$. In order to show that the S-functional calculus is at least in these cases compatible with the multiplication of functions, we need the following lemma.

Lemma 4.1.2. *Let $B \in \mathcal{B}(X)$. For all $q, s \in \mathbb{H}$ with $q \notin [s]$, we have*

$$(\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} = (s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q}). \tag{4.1}$$

If, moreover, f is an intrinsic slice hyperholomorphic function and U is a bounded slice Cauchy domain with $\overline{U} \subset \mathcal{D}(f)$, then

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\,(\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1} = Bf(q)$$

for every $q \in U$ and $j \in \mathbb{S}$.

Proof. Since $s\overline{s} = |s|^2$ and $s+\overline{s} = 2\mathrm{Re}(s)$ are real, they commute with the operator B. Hence, for all $q, s \in \mathbb{H}$ with $q \notin [s]$, we have that

$$\begin{aligned}&(s^2 - 2\mathrm{Re}(q)s + |q|^2)(\overline{s}B - Bq)\\ &= s|s|^2B - 2\mathrm{Re}(q)|s|^2B + |q|^2\overline{s}B - s^2Bq + 2\mathrm{Re}(q)sBq - |q|^2Bq\\ &= sB|s|^2 - B|s|^2(q+\overline{q}) + \overline{s}B|q|^2 - s^2Bq + sB(q+\overline{q})q - B|q|^2q\\ &= (sB - B\overline{q})|s|^2 - s(s+\overline{s})Bq + (s+\overline{s})B\overline{q}q + (sB - B\overline{q})q^2\\ &= (sB - B\overline{q})|s|^2 - (sB - B\overline{q})2\mathrm{Re}(s)q + (sB - B\overline{q})q^2\\ &= (sB - B\overline{q})(q^2 - 2\mathrm{Re}(s)q + |s|^2).\end{aligned}$$

Multiplication by $(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$ from the right and multiplication by $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}$ from the left yields (4.1).

Let now f be an intrinsic slice hyperholomorphic function, let $U \subset \mathcal{D}(f)$ be a bounded slice Cauchy domain, let $q = u + iv \in U$, and let $j \in \mathbb{S}$. An application of (4.1) gives

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} ds_j\, f(s)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q}),\end{aligned}$$

where ds_j and $f(s)$ commute because $f(s) \in \mathbb{C}_j$ for $s \in \mathbb{C}_j$, since f is intrinsic. Now observe that $f(s)$ is intrinsic slice hyperholomorphic on $\mathcal{D}(f)$, that $(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}$ is intrinsic slice hyperholomorphic in s on $\mathbb{H} \setminus [q]$, and that $sB - B\overline{q}$ is left slice hyperholomorphic in s on all of $\mathbb{H}$. Hence their product $F(s) := f(s)(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1}(sB - B\overline{q})$ is left slice hyperholomorphic on $\mathcal{D}(f) \setminus [q]$. By Proposition 2.3.12, the restriction F_j of this function to the complex plane $\mathbb{C}_j$ is therefore a left holomorphic function with values in the complex left Banach space X over $\mathbb{C}_j$.

Assume now that $q \notin \mathbb{R}$. Then F_j has two poles in $U \cap \mathbb{C}_j$, namely $q_j = u + jv$ and $\overline{q_j}$. From the residue theorem we therefore deduce that

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}\\ &= \mathrm{Res}(F_j, q_j) + \mathrm{Res}(F_j, \overline{q_j}).\end{aligned}$$

Since s and q_j belong to the same complex plane, they commute, so that we have

$$(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-1} = (s - q_j)^{-1}(s - \overline{q_j})^{-1},$$

and in turn

$$\begin{aligned}\mathrm{Res}(F_j, q_j) &= \lim_{s\to q_j, s\in\mathbb{C}_j} (s - q_j)F_j(s)\\ &= f(q_j)(q_j - \overline{q_j})^{-1}(q_jB - B\overline{q}) = f(q_j)(2vj)^{-1}(vjB + Bvi)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}(F_j, \overline{q_j}) &= \lim_{s\to \overline{q_j}, s\in\mathbb{C}_j} (s - \overline{q_j})F_j(s)\\ &= f(\overline{q_j})(\overline{q_j} - q_j)^{-1}(\overline{q_j}B - B\overline{q}) = f(\overline{q_j})(-2jv)^{-1}(-vjB + Bvi).\end{aligned}$$

Thus we have

$$\begin{aligned}\mathrm{Res}(F_j, q_j) + \mathrm{Res}(F_j, \overline{q_j}) &= f(q_j)\frac{1}{2}B - f(q_j)\frac{1}{2}jBi + f(\overline{q_j})\frac{1}{2}B + f(\overline{q_j})\frac{1}{2}jBi\\ &= \frac{1}{2}(f(q_j) + f(\overline{q_j}))B + \frac{1}{2}(-f(q_j) + f(\overline{q_j}))jBi.\end{aligned}$$

Since $f(q_j) = f_0(u,v) + f_1(u,v)j$ with $f_0(u,v), f_1(u,v) \in \mathbb{R}$, we finally obtain

$$\begin{aligned}\operatorname{Res}(F_j, q_j) + \operatorname{Res}(F_j, \overline{q_j}) &= f_0(u,v)B + (-f_1(u,v)j)jBi \\ &= B(f_0(u,v) + f_1(u,v)i) = Bf(q)\end{aligned}$$

and hence

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\operatorname{Re}(s)q + |s|^2)^{-1} = Bf(q).$$

If, on the other hand, $q \in \mathbb{R}$, then also $f(q) \in \mathbb{R}$. Since $q = \overline{q}$ commutes in this case with B, we moreover have

$$F(s) = (s-q)^{-1} f(s) B,$$

and so

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, (\overline{s}B - Bq)(q^2 - 2\operatorname{Re}(s)q + |s|^2)^{-1} \\ &= \operatorname{Res}(F_j, q) = \lim_{s\to q, s\in\mathbb{C}_j} (s-q)F(s)B = f(q)B = Bf(q).\end{aligned}$$ □

Theorem 4.1.3 (Product rule)**.** *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ or let $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. Then*

$$(fg)(T) = f(T)g(T).$$

Proof. Let $f \in \mathcal{N}(\sigma_S(T))$, let $g \in \mathcal{SH}_L(\sigma_S(T))$, and let U_q and U_s be bounded slice Cauchy domains that contain $\sigma_S(T)$ such that $\overline{U_q} \subset U_s$ and $\overline{U_s} \subset \mathcal{D}(f)\cap\mathcal{D}(g)$. The subscripts q and s refer to the respective variables of integration in the following computation. We choose $j \in \mathbb{S}$ and we set $\Gamma_s := \partial(U_s \cap \mathbb{C}_j)$ and $\Gamma_q := \partial(U_q \cap \mathbb{C}_j)$ for neatness. By Theorem 3.2.11, we can write $f(T)$ using both the left and right S-resolvent operators, and so

$$\begin{aligned}f(T)g(T) &= \frac{1}{2\pi}\int_{\Gamma_s} f(s)\, ds_j\, S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\, dq_j\, g(q) \\ &= \frac{1}{2\pi}\int_{\Gamma_s} f(s)\, ds_j \left[\frac{1}{2\pi}\int_{\Gamma_q} S_R^{-1}(s,T)S_L^{-1}(q,T)\, dq_j\, g(q)\right].\end{aligned}$$

For simplicity we set $\mathcal{Q}_s(q)^{-1} := (q^2 - 2\operatorname{Re}(s)q + |s|^2)^{-1}$. If we apply (3.7) in the above integral, we obtain

$$\begin{aligned} f(T)g(T) = & \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_R^{-1}(s,T)q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q) \\ & - \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_L^{-1}(q,T)q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q) \\ & - \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} \overline{s}S_R^{-1}(s,T)\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q) \\ & + \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} \overline{s}S_L^{-1}(q,T)\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q). \end{aligned}$$

We observe that

$$\begin{aligned} & \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_R^{-1}(s,T)q\mathcal{Q}_s(q)^{-1}dq_j\,g(q) \\ & = \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j\,S_R^{-1}(s,T)\left[\int_{\Gamma_q} q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\right] = 0 \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j\left[\int_{\Gamma_q} \overline{s}\,S_R^{-1}(s,T)\mathcal{Q}_s(q)^{-1}dq_j\,g(q)\right] \\ & = -\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j\,\overline{s}\,S_R^{-1}(s,T)\left[\int_{\Gamma_q} \mathcal{Q}_s(q)^{-1}\,dq_j\,g(q)\right] = 0 \end{aligned}$$

by Cauchy's integral theorem, because the functions $\mathcal{Q}_s(q)^{-1}$ and $q\mathcal{Q}_s(q)^{-1}$ are for every $s \in \Gamma_s$ right slice hyperholomorphic on an open set that contains $\overline{U_q}$, since we chose $\overline{U_q} \subset U_s$. Therefore, we have

$$\begin{aligned} f(T)g(T) = & -\frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} S_L^{-1}(q,T)q\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q) \\ & + \frac{1}{(2\pi)^2}\int_{\Gamma_s} f(s)\,ds_j \int_{\Gamma_q} \overline{s}S_L^{-1}(q,T)\mathcal{Q}_s(q)^{-1}\,dq_j\,g(q) \\ = & \frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_q} f(s)\,ds_j\left[\overline{s}S_L^{-1}(q,T) - S_L^{-1}(q,T)q\right]\mathcal{Q}_s(q)^{-1}dq_j\,g(q). \end{aligned}$$

The integrand in the last integral is continuous and hence bounded on $\Gamma_s \times \Gamma_q$. We can thus apply Fubini's theorem and change the order of integration, so that

$$f(T)g(T) = \frac{1}{(2\pi)^2}\int_{\Gamma_q}\left[\int_{\Gamma_s} f(s)\,ds_j\,[\overline{s}S_L^{-1}(q,T) - S_L^{-1}(q,T)q]\mathcal{Q}_s(q)^{-1}\right]dq_j\,g(q).$$

Applying Lemma 4.1.2 with $B = S_L^{-1}(q,T)$, we obtain

$$f(T)g(T) = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j\,f(q)g(q) = (fg)(T).$$

The product rule for the S-functional calculus for right slice hyperholomorphic functions can be shown with analogous computations using the second version (3.8) of the S-resolvent equation. □

Corollary 4.1.4. *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. If $f^{-1} \in \mathcal{N}(\sigma_S(T))$, then $f(T)$ is invertible and $f(T)^{-1} = f^{-1}(T)$.*

Proof. From Theorem 4.1.3, we deduce that

$$\mathcal{I} = 1(T) = \left(ff^{-1}\right)(T) = f(T)f^{-1}(T)$$

if we consider f and f^{-1} left slice hyperholomorphic functions and that

$$\mathcal{I} = 1(T) = \left(f^{-1}f\right)(T) = f^{-1}(T)f(T)$$

if we consider them right slice hyperholomorphic functions. Hence $f(T)$ is invertible with $f(T)^{-1} = f(T)^{-1}$. □

Finally, the S-functional calculus has the capability to define the quaternionic Riesz projectors and allows one in turn to identify invariant subspaces of T that are associated with sets of spectral values.

Theorem 4.1.5 (Riesz's projectors)**.** *Let $T \in \mathcal{B}(X)$ and assume that $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with*

$$\operatorname{dist}(\sigma_1, \sigma_2) > 0.$$

We choose an open axially symmetric set O with $\sigma_1 \subset O$ and $\overline{O} \cap \sigma_2 = \emptyset$ and define $\chi_{\sigma_1}(s) = 1$ for $s \in O$ and $\chi_{\sigma_2}(s) = 0$ for $s \notin O$. Then $\chi_{\sigma_1} \in \mathcal{N}(\sigma_S(T))$, and

$$P_{\sigma_1} := \chi_{\sigma_1}(T) = \frac{1}{2\pi}\int_{\partial(O\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j$$

is a continuous projection that commutes with T. Hence $P_{\sigma_1}X$ is a right linear subspace of X that is invariant under T.

Proof. The function χ_{σ_1} obviously belongs to $\mathcal{N}(\sigma_S(T))$, and by Theorem 4.1.3, we have

$$P_{\sigma_1}^2 = \chi_{\sigma_1}(T)\chi_{\sigma_1}(T) = (\chi_{\sigma_1}\chi_{\sigma_1})(T) = \chi_{\sigma_1}(T) = P_{\sigma_1}.$$

Hence P_{σ_1} is a projection in $\mathcal{B}(X)$. Since it is right linear, its range $P_{\sigma_1}X$ is a closed right linear subspace of X. Moreover, we have

$$TP_{\sigma_1} = s(T)\chi_{\sigma_1}(T) = (s\chi_{\sigma_1})(T) = (\chi_{\sigma_1}s)(T) = \chi_{\sigma_1}(T)s(T) = P_{\sigma_1}T.$$

For every $x \in P_{\sigma_1}X$, we thus obtain

$$Tx = TP_{\sigma_1}x = P_{\sigma_1}Tx \quad \text{for all } x \in P_{\sigma_1}X,$$

and hence $P_{\sigma_1}X$ is invariant under T.

We can show these properties explicitly, which we shall do now so that the reader can see the analogy with the Riesz projectors of the F-functional calculus in Theorem 7.4.2. Let us choose two bounded Cauchy slice domains U_q and U_s such that $\sigma \subset U_q$ and $\overline{U_q} \subset U_s$ and $\overline{U_s} \subset O$. We choose $j \in \mathbb{S}$ and we set $\Gamma_s := \partial(U_s \cap \mathbb{C}_j)$ and $\Gamma_q := \partial(U_q \cap \mathbb{C}_j)$ for neatness. By Theorem 3.2.11, we then have

$$P_{\sigma_1} = \frac{1}{2\pi}\int_{\Gamma_s} ds_j\, S_R^{-1}(s,T) = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\, dq_j,$$

and so

$$\begin{aligned} P_{\sigma_1}^2 &= \frac{1}{2\pi}\int_{\Gamma_s} ds_j\, S_R^{-1}(s,T)\frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\, dq_j \\ &= \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \left[\int_{\Gamma_q} S_R^{-1}(s,T)S_L^{-1}(q,T)\, dq_j\right]. \end{aligned}$$

For simplicity we set $\mathcal{Q}_s(q)^{-1} := (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$. If we apply (3.7) in the above integral, we obtain

$$\begin{aligned} f(T)g(T) = {} & \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} S_R^{-1}(s,T)q\mathcal{Q}_s(q)^{-1}\, dq_j \\ & - \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} S_L^{-1}(q,T)q\mathcal{Q}_s(q)^{-1}\, dq_j \\ & - \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} \overline{s}S_R^{-1}(s,T)\mathcal{Q}_s(q)^{-1}\, dq_j \\ & + \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} \overline{s}S_L^{-1}(q,T)\mathcal{Q}_s(q)^{-1}\, dq_j. \end{aligned}$$

We observe that

$$\begin{aligned} & \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} S_R^{-1}(s,T)q\mathcal{Q}_s(q)^{-1} dq_j \\ & = \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j\, S_R^{-1}(s,T)\left[\int_{\Gamma_q} q\mathcal{Q}_s(q)^{-1}\, dq_j\right] = 0 \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \left[\int_{\Gamma_q} \overline{s}\, S_R^{-1}(s,T)\mathcal{Q}_s(q)^{-1} dq_j\right] \\ & = -\frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j\, \overline{s}\, S_R^{-1}(s,T)\left[\int_{\Gamma_q} \mathcal{Q}_s(q)^{-1}\, dq_j\right] = 0 \end{aligned}$$

by Cauchy's integral theorem, because the functions $\mathcal{Q}_s(q)^{-1}$ and $q\mathcal{Q}_s(q)^{-1}$ are for every $s \in \Gamma_s$ right slice hyperholomorphic on an open set that contains $\overline{U_q}$; since we chose $\overline{U_q} \subset U_s$. Therefore, we have

$$\begin{aligned} P_{\sigma_1}^2 &= -\frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} S_L^{-1}(q,T)q\mathcal{Q}_s(q)^{-1}\,dq_j \\ &\quad + \frac{1}{(2\pi)^2}\int_{\Gamma_s} ds_j \int_{\Gamma_q} \overline{s}S_L^{-1}(q,T)\mathcal{Q}_s(q)^{-1}\,dq_j \\ &= \frac{1}{(2\pi)^2}\int_{\Gamma_s}\int_{\Gamma_q} ds_j\left[\overline{s}S_L^{-1}(q,T) - S_L^{-1}(q,T)q\right]\mathcal{Q}_s(q)^{-1}dq_j. \end{aligned}$$

The integrand in the last integral is continuous and hence bounded on $\Gamma_s \times \Gamma_q$. We can thus apply Fubini's theorem and change the order of integration so that

$$P_{\sigma_1}^2 = \frac{1}{(2\pi)^2}\int_{\Gamma_q}\left[\int_{\Gamma_s} ds_j\,[\overline{s}S_L^{-1}(q,T) - S_L^{-1}(q,T)q]\mathcal{Q}_s(q)^{-1}\right] dq_j.$$

Applying Lemma 4.1.2 with $B = S_L^{-1}(q,T)$ and $f(q) = 1$, we obtain

$$P_{\sigma_1}^2 = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j = P_{\sigma_1}.$$

We furthermore have, because of (3.4), that

$$\begin{aligned} TP_{\sigma_1} &= \frac{1}{2\pi}\int_{\Gamma_q} TS_L^{-1}(q,T)\,dq_j \\ &= \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j q - \frac{1}{2\pi}\int_{\Gamma_q}\mathcal{I}\,dq_j = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j q \end{aligned}$$

by Cauchy's integral theorem and similarly

$$\begin{aligned} P_{\sigma_1}T &= \frac{1}{2\pi}\int_{\Gamma_s} ds_j\,S_R^{-1}(s,T)T \\ &= \frac{1}{2\pi}\int_{\Gamma_s} sds_j\,S_R^{-1}(s,T) - \frac{1}{2\pi}\int_{\Gamma_s} ds_j\mathcal{I} = \frac{1}{2\pi}\int_{\Gamma_s} sds_j\,S_R^{-1}(s,T). \end{aligned}$$

By Theorem 3.2.11, we thus have

$$TP_{\sigma_1} = \frac{1}{2\pi}\int_{\Gamma_q} S_L^{-1}(q,T)\,dq_j\,q = \frac{1}{2\pi}\int_{\Gamma_s} s\,ds_j\,S_R^{-1}(s,T) = P_{\sigma_1}T. \qquad \square$$

4.2 The Spectral Mapping Theorem and the Composition Rule

Similar to the product rule, the spectral mapping theorem does not hold for arbitrary slice hyperholomorphic functions. This is not surprising: it is clear that it can hold only for slice hyperholomorphic functions that preserve the fundamental geometry of the S-spectrum, namely its axial symmetry. Again, the class of intrinsic slice hyperholomorphic functions stands out here, since it is this class of functions that maps axially symmetric sets to axially symmetric sets.

Theorem 4.2.1 (The spectral mapping theorem). *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. Then*

$$\sigma_S(f(T)) = f(\sigma_S(T)) = \{f(s) : \ s \in \sigma_S(T)\}.$$

Proof. Let U be a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and let $s = u + jv \in \sigma_S(T)$. For $q \in U \setminus [s]$, we define

$$\tilde{g}(q) = (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}(f(q)^2 - 2\mathrm{Re}(f(s))f(q) + |f(s)|^2).$$

Since f is intrinsic slice hyperholomorphic, the function

$$q \mapsto f(q)^2 - 2\mathrm{Re}(f(s))f(q) + |f(s)|^2$$

is intrinsic slice hyperholomorphic too. If we multiply it by the intrinsic rational function $(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$, we obtain again an intrinsic slice hyperholomorphic function, and hence $\tilde{g}$ belongs to $\mathcal{N}(U) \setminus [s]$.

We can extend $\tilde{g}$ to a function $g \in \mathcal{N}(U)$. Indeed, if $s \notin \mathbb{R}$ and $i \in \mathbb{S}$, then the function $\tilde{g}_i = \tilde{g}|_{\mathbb{C}_i}$ has the singularities $s_i = u + iv$ and $\overline{s_i} = u - iv$ in $U \cap \mathbb{C}_i$. However, we have

$$\begin{aligned}
&\lim_{z \to s_i, z \in \mathbb{C}_i} \tilde{g}_i(z) \\
&= \lim_{z \to s_i, z \in \mathbb{C}_i} (z^2 - 2\mathrm{Re}(s_i)z + |s_i|^2)^{-1}(f(z)^2 - 2\mathrm{Re}(f(s_i))f(z) + |f(s_i)|^2) \\
&= \lim_{z \to s_i, z \in \mathbb{C}_i} (z - s_i)^{-1}(f(z) - f(s_i))(z - \overline{s_i})^{-1}\left(f(z) - \overline{f(s_i)}\right) \\
&= f_i'(s_i)(s_i - \overline{s_i})^{-1}\left(f(s_i) - \overline{f(s_i)}\right) = f_i'(s_i)\frac{f_1(u,v)}{v}
\end{aligned}$$

because s_i, z, $f(s_i)$, and $f(z)$ belong to the same complex plane, since f is intrinsic, and hence they mutually commute. Since $f(\overline{s_i}) = \overline{f(s_i)}$ because f is intrinsic, we also have

$$\begin{aligned}
&\lim_{z \to \overline{s_i}, z \in \mathbb{C}_i} \tilde{g}_i(z) \\
&= \lim_{z \to \overline{s_i}, z \in \mathbb{C}_i} (z^2 - 2\mathrm{Re}(s_i)z + |s_i|^2)^{-1}(f(z)^2 - 2\mathrm{Re}(f(s_i))f(z) + |f(s_i)|^2) \\
&= \lim_{z \to \overline{s_i}, z \in \mathbb{C}_i} (z - \overline{s_i})^{-1}\left(f(z) - \overline{f(s_i)}\right)(z - s_i)^{-1}(f(z) - f(s_i)) \\
&= f_i'(\overline{s_i})(\overline{s_i} - s_i)^{-1}\left(f(\overline{s_i}) - \overline{f(\overline{s_i})}\right) = f_i'(\overline{s_i})\frac{f_1(u,v)}{v}.
\end{aligned}$$

Thus s_i and $\overline{s_i}$ are removable singularities of $\tilde{g}_i$, and since $\overline{s_i} = s_{-i}$, the function

$$g(q) = \begin{cases} \tilde{g}(q) & \text{if } q \in U \setminus [s], \\ \partial_S f(q) \frac{f_1(u,v)}{v} & \text{if } q = u + iv \in [s], \end{cases}$$

is well defined. Obviously, it is an intrinsic slice function, and its restriction g_i to any complex plane $\mathbb{C}_i$ is holomorphic. By Lemma 2.1.6, the function g is intrinsic slice hyperholomorphic.

If, on the other hand, $s \in \mathbb{R}$, then the point s is for every $i \in \mathbb{S}$ the only singularity of the function $\tilde{g}_i$. Since $\overline{f(s)} = f(\overline{s}) = f(s)$, we have $f(s) \in \mathbb{R}$ and hence $\mathrm{Re}(s) = s$ and $\mathrm{Re}(f(s)) = f(s)$ such that

$$\begin{aligned} \lim_{z \to s, z \in \mathbb{C}_i} \tilde{g}_i(z) &= \lim_{z \to s, z \in \mathbb{C}_i} (z^2 - 2sz + s^2)^{-1}(f(z)^2 - 2f(s)f(z) + f(s)^2) \\ &= \lim_{z \to \infty, z \in \mathbb{C}_i} (z - s)^{-2} \left(f(z) - f(s)\right)^2 = (f_i'(s))^2. \end{aligned}$$

Therefore, the singularity s of $\tilde{g}_i$ is removable for every $i \in \mathbb{S}$, and since $\left(f_i'(s)\right)^2 = \left(\partial_S f(s)\right)^2$ does not depend on the imaginary unit i, the function

$$g(q) = \begin{cases} \tilde{g}(q) & \text{if } q \in U \setminus [s], \\ \left(\partial_S f(s)\right)^2 & \text{if } q = s, \end{cases}$$

is well defined. Obviously, g is an intrinsic slice function and $g_i = g|_{U \cap \mathbb{C}_i}$ is holomorphic on $U \cap \mathbb{C}_i$ for every $i \in \mathbb{S}$. By Lemma 2.1.6, the function g is also in this case intrinsic slice hyperholomorphic.

The product rule implies

$$f(T)^2 - 2\mathrm{Re}(f(s))f(T) + |f(s)|^2\mathcal{I} = (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})g(T).$$

If the operator $f(T)^2 - 2\mathrm{Re}(f(s))f(T) + |f(s)|^2\mathcal{I}$ were invertible, then

$$g(T)(f(T)^2 - 2\mathrm{Re}(f(s))f(T) + |f(s)|^2\mathcal{I})^{-1}$$

would therefore be the inverse of $T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}$. Since we assumed $s \in \sigma_S(T)$, this is impossible, and hence $f(s) \in \sigma_S(f(T))$. Thus

$$f(\sigma_S(T)) \subset \sigma_S(f(T)).$$

If, on the other hand, $s \notin f(\sigma_S(T))$, then we can consider the function

$$h(q) := (f^2(q) - 2\mathrm{Re}(s)f(q) + |s|^2)^{-1},$$

which is an intrinsic slice hyperholomorphic function. Its poles are the spheres $[q] \subset U$ such that $f([q]) = [f(q)] = [s]$. Since we assumed $s \notin f(\sigma_S(T))$, it does not have any poles on $\sigma_S(T)$. Thus it belongs to $\mathcal{N}(\sigma_S(T))$, and Corollary 4.1.4 implies

$$h(T) = \left(f(T)^2 - 2\mathrm{Re}(s)f(T) + |s|^2\right)^{-1} \in \mathcal{B}(X).$$

We find that $s \in \rho_S(T)$ and in turn also

$$\sigma_S(f(T)) \subset f(\sigma_S(T)). \qquad \square$$

The spectral mapping theorem allows us to generalize the Gelfand formula for the spectral radius to quaternionic linear operators.

Definition 4.2.2. Let $T \in \mathcal{B}(X)$. Then the *S-spectral radius* of T is defined to be the nonnegative real number

$$r_S(T) := \sup\{|s| : \ s \in \sigma_S(T)\}.$$

Theorem 4.2.3. *For $T \in \mathcal{B}(X)$, we have*

$$r_S(T) = \lim_{n\to+\infty} \|T^n\|^{\frac{1}{n}}.$$

Proof. The mapping $q \mapsto q^{-1}$ is intrinsic slice hyperholomorphic, and hence $q \mapsto S_L^{-1}(q^{-1}, T)$ is slice hyperholomorphic on the set

$$U := \{q \in \mathbb{H} : \ q^{-1} \in \rho_S(T)\}.$$

Since $\mathbb{H} \setminus B_{r_S(T)}(0) \subset \rho_S(T)$, the set U contains the ball $B_{1/r_S(T)}(0)$. By Theorem 2.1.15, the function $S_L^{-1}(q^{-1}, T)$ admits a power series expansion at 0 that converges on $B_{1/r_S(T)}(0)$. Because of Theorem 3.1.6, it is given by

$$S_L^{-1}\left(q^{-1}, T\right) = \sum_{n=0}^{+\infty} T^n q^{n+1}, \quad |q| < \frac{1}{r_S(T)}.$$

For $s \in \mathbb{H}$ with $|s| > r_S(T)$, we thus have $\left\|T^n s^{-n-1}\right\| \to 0$ as $n \to +\infty$ because the above series converges. In particular, we have

$$C(s) = \sup_{n\in\mathbb{N}} \|T^n s^{-n-1}\| < +\infty.$$

Therefore,

$$\begin{aligned}\limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} \frac{1}{|s|} &= \limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} |s|^{-\frac{n+1}{n}} \\ &= \limsup_{n\to+\infty} \|T^n s^{-n-1}\|^{\frac{1}{n}} \leq \limsup_{n\to+\infty} C(s)^{\frac{1}{n}} = 1,\end{aligned}$$

and hence

$$\limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} \leq |s|.$$

Since s was arbitrary with $|s| > r_S(T)$, we obtain

$$\limsup_{n\to+\infty} \|T^n\|^{\frac{1}{n}} \leq r_S(T).$$

Moreover, Theorem 4.2.1 implies

$$\sigma_S(T^n) = \sigma_S(T)^n$$

and we conclude from Theorem 3.1.13 that

$$\begin{aligned} r_S(T)^n &= \sup\{|s|^n : \ s \in \sigma_S(T)\} \\ &= \sup\{|s| : \ s \in \sigma_S(T^n)\} = r_S(T^n) \leq \|T^n\| \end{aligned}$$

for every $n \in \mathbb{N}$. Therefore, we get

$$r_S(T) \leq \liminf_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}} \leq r_S(T) \tag{4.2}$$

and in turn $r_S(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$, where (4.2) also implies the existence of the limit. □

Finally, the spectral mapping theorem also allows us to generalize the composition rule.

Theorem 4.2.4 (Composition rule)**.** *Let $T \in \mathcal{B}(X)$ and let $f \in \mathcal{N}(\sigma_S(T))$. If $g \in \mathcal{SH}_L(\sigma_S(f(T))$, then $g \circ f \in \mathcal{SH}_L(\sigma_S(T))$, and if $g \in \mathcal{SH}_R(f(\sigma_S(T)))$, then $g \circ f \in \mathcal{SH}_R(\sigma_S(T))$. In both cases,*

$$g(f(T)) = (g \circ f)(T).$$

Proof. If $g \in \mathcal{SH}_L(f(\sigma_S(T)))$, then $\mathcal{D}(g)$ is open and axially symmetric. Since f is continuous and intrinsic, the inverse image of every open axially symmetric set under f is again open and axially symmetric. The set $f^{-1}(\mathcal{D}(g))$ is therefore an axially symmetric open set, and it contains $\sigma_S(T)$, since $f(\sigma_S(T)) = \sigma_S(f(T)) \subset \mathcal{D}(g)$ because of Theorem 4.2.1. By Theorem 2.1.4, the composition $g \circ f$ is a left slice hyperholomorphic function with domain $f^{-1}(\mathcal{D}(g))$, and so it belongs to $\mathcal{SH}_L(\sigma_S(T))$.

Let U be a bounded slice Cauchy domain such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and let W be another bounded slice Cauchy domain such that $\sigma_S(T) \subset \overline{f(U)} \subset W$ and $\overline{W} \subset \mathcal{D}(g)$. (Such slice Cauchy domains exist because of Remark 3.2.4.) The mapping $s \mapsto S_L^{-1}(q, f(s))$ is left slice hyperholomorphic on

$$\{s \in \mathcal{D}(f) : \ f(s) \notin [q]\} = \{s \in \mathcal{D}(f) : \ q \notin [f(s)]\}$$

by Theorem 2.1.4. If $q \notin \sigma_S(f(T)) = f(\sigma_S(T))$, then $s \mapsto S_L^{-1}(q, f(s))$ therefore belongs to $\mathcal{SH}_L(\sigma_S(T))$. Since the S-functional calculus is compatible with algebraic operations, we have

$$\begin{aligned} S_L^{-1}(q, f(T)) &= -\mathcal{Q}_q(f(T))^{-1}(f(T) - \overline{q}\mathcal{I}) \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\ \left[-\mathcal{Q}_q(f(s))^{-1}(f(s) - \overline{q})\right] \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, S_L^{-1}(q, f(s)) \end{aligned}$$

with $\mathcal{Q}_s(f(s))^{-1} = (f(s)^2 - 2\mathrm{Re}(q)f(s) + |q|^2)^{-1}$ and an arbitrary imaginary unit $j \in \mathbb{S}$. Therefore,

$$\begin{aligned} g(f(T)) &= \frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(q, f(T))\, dq_j\, g(q) \\ &= \frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)} \left[\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, S_L^{-1}(q, f(s))\right] dq_j\, g(q). \end{aligned}$$

Since the integrand in the last integral is continuous and hence bounded on the compact set $\partial(W\cap\mathbb{C}_j)\times\partial(U\cap\mathbb{C}_j)$, we can apply Fubini's theorem to change the order of integration and obtain

$$\begin{aligned} g(f(T)) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\ \left[\frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(p, f(s))\, dp_j\, g(p)\right] \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, g(f(s)) \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, (g\circ f)(s) = (g\circ f)(T). \end{aligned}$$

□

4.3 Convergence in the S-Resolvent Sense

The following definition and the next result show that the notion of convergence in the resolvent sense is meaningful also in the quaternionic setting. This notion is important for unbounded operators.

Definition 4.3.1 (Converges in the norm S-resolvent sense)**.** Let T_m, $m \in \mathbb{N}$, and T belong to $\mathcal{B}(X)$ and suppose that $\rho_S(T) = \rho_S(T_m)$ for all $m \in \mathbb{N}$. We say that T_m *converges to* T *in the norm left* S*-resolvent sense* if $S_L^{-1}(s, T_m) \to S_L^{-1}(s,T)$ in $\mathcal{B}(X)$ as $m \to +\infty$ for all $s \in \rho_S(T)$ and that T_m *converges to* T *in the norm right* S*-resolvent sense* if $S_R^{-1}(s, T_m) \to S_R^{-1}(s,T)$ in $\mathcal{B}(X)$ as $m \to +\infty$ for all $s \in \rho_S(T)$.

Theorem 4.3.2. *Let* $T_m \in \mathcal{B}(X)$, $m \in \mathbb{N}$ *be uniformly bounded,* $T \in \mathcal{B}(X)$, *and suppose that* $\rho_S(T) = \rho_S(T_m)$ *for all* $m \in \mathbb{N}$. *The following statements are then equivalent:*

(i) *T_m converges to T in $\mathcal{B}(X)$.*

(ii) *T_m converges to T in the norm left S-resolvent sense.*

(iii) *T_m converges to T in the norm right S-resolvent sense.*

In each of these cases, the convergence $S_L^{-1}(s,T_m) \to S_L^{-1}(s,T)$ or $S_R^{-1}(s,T_m) \to S_R^{-1}(s,T)$ is uniform for s on compact subsets of $\rho_S(T)$.

Proof. Assume first that (i) holds. Then

$$\begin{aligned} S_L^{-1}(s,T) - S_L^{-1}(s,T_m) = & -\mathcal{Q}_s(T)^{-1}\left(T - \overline{s}\mathcal{I} - T_m + \overline{s}\mathcal{I}\right) \\ & - \left(\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right)\left(T_m - \overline{s}\mathcal{I}\right) \end{aligned}$$

and hence

$$\begin{aligned} &\|S_L^{-1}(s,T) - S_L^{-1}(s,T_m)\| \\ &\qquad \le \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T - T_m\| + \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\|. \end{aligned} \tag{4.3}$$

We observe that

$$\begin{aligned} &\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1} \\ &\qquad = \mathcal{Q}_s(T_m)^{-1}\left(T_m^2 - T^2 - 2s_0(T - T_m)\right)\mathcal{Q}_s(T)^{-1} \\ &\qquad = \mathcal{Q}_s(T_m)^{-1}\left(T_m(T_m - T) + (T_m - T)T + 2s_0(T - T_m)\right)\mathcal{Q}_s(T)^{-1}. \end{aligned} \tag{4.4}$$

Hence if we can show that there exists a positive constant C_s such that $\|Q_s(T_m)\| \le C_s$ for all $m \in \mathbb{N}$, then we will obtain $\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\| \to 0$ and in turn, due to (4.3), that $S_L^{-1}(s,T) \to S_L^{-1}(s,T)$ in $\mathcal{B}(X)$. We point out that

$$\begin{aligned} \mathcal{Q}_s(T_m) &= \mathcal{Q}_s(T)\left(\mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T_m)\right) \\ &= \mathcal{Q}_s(T)\left(\mathcal{I} - \mathcal{Q}_s(T)^{-1}(T^2 - T_m^2 - 2s_0(T_m - T))\right). \end{aligned} \tag{4.5}$$

For $A \in \mathcal{B}(X)$ with $\|A\| < 1$, the operator $(\mathcal{I} - A)^{-1} = \sum_{n=0}^{+\infty} A^n \in \mathcal{B}(X)$ exists and satisfies $\left\|(\mathcal{I} - A)^{-1}\right\| \le (1 - \|A\|)^{-1}$. Since $T_m \to T$, we find that

$$A_m := \mathcal{Q}_s(T)^{-1}(T^2 - T_m^2 - 2s_0(T_m - T)) \to 0$$

in $\mathcal{B}(X)$ as $m \to +\infty$ and hence $\|A_m\| \le 1/2$ for sufficiently large m. For such m, the operator $\mathcal{I} - A_m$ is invertible with $\left\|(\mathcal{I} - A_m)^{-1}\right\| \le 2$, and because of (4.5), we obtain

$$\mathcal{Q}_s(T_m)^{-1} = (\mathcal{I} - A_m)^{-1}\mathcal{Q}_s(T)^{-1} \tag{4.6}$$

and in turn

$$\left\|Q_s(T_m)^{-1}\right\| \le \left\|(\mathcal{I} - A_m)^{-1}\right\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \le 2\left\|\mathcal{Q}_s(T)^{-1}\right\|. \tag{4.7}$$

Therefore,

$$C_s := \sup_{m \in \mathbb{N}} \left\|\mathcal{Q}_s(T_m)^{-1}\right\| < +\infty,$$

and we conclude that (ii) holds.

The convergence $S_L^{-1}(s,T_m) \to S_L^{-1}(s,T)$ is even uniform in s on every compact set K, since because of (4.3), we have

$$\begin{aligned}
&\sup_{s\in K} \|S_L^{-1}(s,T) - S_L^{-1}(s,T_m)\| \\
&\le \sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T - T_m\| + \sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\|.
\end{aligned}$$

Since $\mathcal{Q}_s(T)^{-1}$ is continuous on $\rho_S(T)$, we have $\sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1}\right\| < +\infty$, and so the first summand converges to 0 uniformly in s as $m \to +\infty$. For the second summand, we have because of (4.4) that

$$\begin{aligned}
&\sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\le \sup_{s\in K} \left\|\mathcal{Q}_s(T_m)^{-1}\right\| \|T_m\|\|T_m - T\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\quad + \sup_{s\in K} \left\|\mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - T\|\|T\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\quad + \sup_{s\in K} |2s_0| \left\|\mathcal{Q}_s(T_m)^{-1}\right\| \|T - T_m\| \left\|\mathcal{Q}_s(T)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\|.
\end{aligned}$$

Because of (4.7), we have $\left\|Q_s(T_m)^{-1}\right\| \le 2\left\|\mathcal{Q}_s(T)^{-1}\right\|$ for m sufficiently large, and so

$$\begin{aligned}
&\sup_{s\in K} \left\|\mathcal{Q}_s(T)^{-1} - \mathcal{Q}_s(T_m)^{-1}\right\| \|T_m - \overline{s}\mathcal{I}\| \\
&\le \sup_{s\in K} 2\left\|\mathcal{Q}_s(T)^{-1}\right\|^2 (\|T\| + \|T_m\| + 2|s_0|\|)\,(\|T_m\| + |\overline{s}|)\|T - T_m\|\| \\
&\le C\|T - T_m\|
\end{aligned}$$

because $\mathcal{Q}_s(T)^{-1}$ and s_0 and $\overline{s}$ depend continuously on s and are hence bounded on the compact set K.

Conversely, we suppose now that (ii) holds and we show that $\|T - T_m\| \to 0$. Since T and T_m are uniformly bounded, there exists $\alpha \in \rho_S(T) \cap \bigcup_{m\in\mathbb{N}} \rho_S(T_m)$. We then have $S_L^{-1}(\alpha,T) = (\alpha\mathcal{I} - T)^{-1}$ and $S_L^{-1}(\alpha,T_m) = (\alpha\mathcal{I} - T_m)^{-1}$, and so

$$\begin{aligned}
\|T - T_m\| &= \|\alpha - T_m - (\alpha - T)\| \\
&\le \|\alpha - T_m\| \left\|(\alpha\mathcal{I} - T)^{-1} - (\alpha\mathcal{I} - T_m)^{-1}\right\| \|\alpha - T\| \\
&= \|\alpha\mathcal{I} - T_m\|\|\alpha\mathcal{I} - T\| \left\|S_L^{-1}(\alpha,T) - S_L^{-1}(\alpha,T_m)\right\| \to 0
\end{aligned}$$

because $\|T_m\|$ is uniformly bounded.

The equivalence of (i) and (iii) is shown with similar arguments. □

Remark 4.3.3. Since by the above theorem convergence in the norm left S-resolvent sense and convergence in the norm right S-resolvent sense are equivalent, we will not distinguish between them in the following and just say that T_m converges to T in the norm S-resolvent sense.

Theorem 4.3.4. *Let T_m, $m \in \mathbb{N}$, and T belong to $\mathcal{B}(X)$ with $\rho_S(T) = \rho_S(T_m)$ for all $m \in \mathbb{N}$ and suppose that T_m converges to T in the norm S-resolvent sense. If $f \in \mathcal{SH}_L(\sigma_S(T))$ or $f \in \mathcal{SH}_R(\sigma_S(T))$, then*

$$\|f(T) - f(T_m)\| \to 0 \quad \text{as} \quad m \to +\infty.$$

Proof. If $f \in \mathcal{SH}_L(\sigma_S(T))$, then

$$f(T_m) - f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} \left(S_L^{-1}(s,T_m) - S_L^{-1}(s,T)\right) \, ds_j \, f(s)$$

with $j \in \mathbb{S}$ and a suitable bounded slice Cauchy domain U. Since $f(s)$ is continuous, it is bounded on the compact set $\partial(U \cap \mathbb{C}_j)$, and hence there exists a positive constant $C > 0$ such that

$$\|f(T_m) - f(T)\| \leq C \max_{s\in\partial(U\cap\mathbb{C}_j)} \left\|S_L^{-1}(s,T_m) - S_L^{-1}(s,T)\right\| \to 0,$$

since $\left\|S_L^{-1}(s,T_m) - S_L^{-1}(s,T)\right\| \to 0$ converges uniformly to 0 on the compact set $\partial(U \cap \mathbb{C}_j)$ by Theorem 4.3.2. Similarly, we prove the statement for $f \in \mathcal{SH}_R(\sigma_S(T))$. □

4.4 The Taylor Formula for the S-Functional Calculus

Consider a bounded operator T and let N be a small perturbation operator that furthermore commutes with T. Then $f(T + N)$ can be represented as a power series in N that formally corresponds to a Taylor series expansion in the operator. In this section we show that the Taylor formula can be extended to quaternionic operators, but before we can state the main theorem, several preliminary results are needed. This result is the quaternionic analogue of Theorem VII.10 in [104], and it was proved in [55] in the more general setting of paravector operators on a two-sided Clifford module.

Before we are able to show the Taylor expansion in the operator, we need to determine the slice derivatives of the S-resolvents. We start by finding explicit formulas for the functions

$$S_L^n(s,q) := (s-q)^{*_L n} \quad \text{and} \quad S_R^n(s,q) := (s-q)^{*_R n}.$$

Lemma 4.4.1. *Let $s \in \mathbb{H}$. For $n \geq 0$, we have*

$$S_L^n(s,q) = \sum_{k=0}^{n} \binom{n}{k} (-q)^k s^{n-k} \quad \text{and} \quad S_R^n(s,q) = \sum_{k=0}^{n} \binom{n}{k} s^{n-k} (-q)^k. \tag{4.8}$$

With $\mathcal{Q}_s(q) = q^2 - 2s_0 q + |s|^2$, we moreover have

$$S_L^{-n}(s,q) = \mathcal{Q}_s(q)^{-n} (\overline{s} - q)^{*_L n} \quad \text{and} \quad S_R^{-n}(s,q) = (\overline{s} - q)^{*_R n} \mathcal{Q}_s(q)^{-n}. \tag{4.9}$$

Furthermore, for $m, n \geq 0$, we have

$$S_L^{-n}(s,q) *_L S_L^{-m}(\overline{s},q) = \mathcal{Q}_s(q)^{-(n+m)} \left[(\overline{s}-q)^{*_L n} *_L (s-q)^{*_L m}\right]$$

and

$$S_R^{-n}(s,q) *_R S_R^{-m}(\overline{s},q) = \left[(\overline{s}-q)^{*_R n} *_R (s-q)^{*_R m}\right] \mathcal{Q}_s(q)^{-(n+m)}.$$

Proof. For $n = 0$, we have $(s-q)^{*_L 0} \equiv 1$, and hence (4.8) is obviously true. Assume that it holds for $n-1$. Then (2.18) implies

$$\begin{aligned}
S_L^n(s,q) &= (s-q)^{*_L n} = (s-q)^{*_L (n-1)} *_L (s-q) \\
&= (s-q)^{*_L (n-1)} *_L s + (s-q)^{*_L (n-1)} *_L (-q) \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (-q)^k s^{n-k} + \sum_{k=0}^{n-1} \binom{n-1}{k} (-q)^{k+1} s^{n-1-k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (-q)^k s^{n-k} + \sum_{k=1}^{n} \binom{n-1}{k-1} (-q)^k s^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-q)^k s^{n-k},
\end{aligned}$$

and (4.8) follows by induction.

We also prove the identity (4.9) by induction. It is obviously true for $n = 0$. Assume that it holds for $n-1$ and observe that $\mathcal{Q}_s(q)^{-1} \in \mathcal{N}(\mathbb{H} \setminus [s])$. Then by (2.16) and Corollary 2.1.20 we have $S_L^{-1}(s,q) = (s-q)^{-*_L}$, so this implies

$$\begin{aligned}
S_L^{-n}(s,q) &= (s-q)^{-*_L (n-1)} *_L (s-q)^{-*_L} \\
&= \left[\mathcal{Q}_s(q)^{-(n-1)} (\overline{s}-q)^{*_L (n-1)}\right] *_L \left[\mathcal{Q}_s(q)^{-1} (\overline{s}-q)\right] \\
&= \mathcal{Q}_s(q)^{-(n-1)} *_L (\overline{s}-q)^{*_L (n-1)} *_L \mathcal{Q}_s(q)^{-1} *_L (\overline{s}-q) \\
&= \mathcal{Q}_s(q)^{-(n-1)} *_L \mathcal{Q}_s(q)^{-1} *_L (\overline{s}-q)^{*_L (n-1)} *_L (\overline{s}-q) \\
&= \mathcal{Q}_s(q)^{-n} (\overline{s}-q)^{*_L n}.
\end{aligned}$$

Finally (2.16) also implies for $m, n \geq 0$ that

$$\begin{aligned}
S_L^{-n}(s,q) *_L S_L^{-m}(\overline{s},q) &= \left[\mathcal{Q}_s(q)^{-n} (\overline{s}-q)^{*_L n}\right] *_L \left[\mathcal{Q}_s(q)^{-m} (s-q)^{*_L m}\right] \\
&= \mathcal{Q}_s(q)^{-n} *_L (\overline{s}-q)^{*_L n} *_L \mathcal{Q}_s(q)^{-m} *_L (s-q)^{*_L m} \\
&= \mathcal{Q}_s(q)^{-n} *_L \mathcal{Q}_s(q)^{-m} *_L (\overline{s}-q)^{*_L n} *_L (s-q)^{*_L m} \\
&= \mathcal{Q}_s(q)^{-(n+m)} \left[(\overline{s}-q)^{*_L n} *_L (s-q)^{*_L m}\right].
\end{aligned}$$

The right slice hyperholomorphic case can be shown by similar computations. □

Corollary 4.4.2. *Let $s = s_0 + j_s s_1 \in \mathbb{H}$ and $n, m \in \mathbb{N}_0$. If $q \in \mathbb{C}_{j_s}$, then*

$$(s-q)^{*_L m} *_L (\overline{s}-q)^{*_L n} = (s-q)^m (\overline{s}-q)^n \tag{4.10}$$

and

$$S_L^{-m}(s,q) *_L S_L^{-n}(\overline{s},q) = (s-q)^{-m}(\overline{s}-q)^{-n}. \tag{4.11}$$

Moreover, for every $n \in \mathbb{N}_0$, *the function*

$$P(q) := \sum_{k=0}^{n} (\overline{s}-q)^{*_L(k+1)} *_L (s-q)^{*_L(n-k+1)} \tag{4.12}$$

is a polynomial with real coefficients. Analogous statements hold for right slice hyperholomorphic powers $S_R^m(s,q)$ *of* $s-q$.

Proof. If $q \in \mathbb{C}_{j_s}$, then s, $\overline{s}$, and q commute. Hence it follows from (4.8) and the binomial theorem that $(s-q)^{*_L m} = (s-q)^m$ and $(\overline{s}-q)^{*_L n} = (\overline{s}-q)^n$. From (2.16), we deduce (4.10). Since q and s commute, we also find that

$$\mathcal{Q}_s(q)^{-1} = (q-s)^{-1}(q-\overline{q})^{-1},$$

and so (4.9) implies

$$S_L^{-m}(s,q) = (s-q)^{-m}(\overline{s}-q)^{-m}(\overline{s}-q)^m = (s-q)^{-m}.$$

An analogous computation shows that $S_L^{-n}(\overline{s},q) = (\overline{s}-q)^{-n}$.

For arbitrary left slice hyperholomorphic functions f and g, it is because of (2.21) immediate that $(f *_L g)(q) = f(q)g(q)$ at a point q if $f(q) \in \mathbb{C}_{j_q}$. Since $(s-q)^{-m}$ belongs to $\mathbb{C}_{j_q}$ if $q \in \mathbb{C}_{j_s}$, we furthermore find that

$$S_L^{-m}(s,q) *_L S_L^{-n}(\overline{s},q) = (s-q)^{-m} *_L (\overline{s}-q)^{-n} = (s-q)^{-m}(\overline{s}-q)^{-n}.$$

Finally, we consider $P(q)$. The restriction P_{j_s} of this function to the plane $\mathbb{C}_{j_s}$ is the complex polynomial $P_{j_s}(z) = \sum_{k=0}^{n}(\overline{s}-z)^{k+1}(s-z)^{n-k+1}$. From the relation

$$P_{j_s}(\overline{q}) = \sum_{k=0}^{n}(\overline{s}-\overline{q})^{k+1}(s-\overline{q})^{n-k+1} = \overline{\sum_{k=0}^{n}(s-q)^{k+1}(\overline{s}-q)^{n-k+1}} = \overline{P_{j_s}(q)},$$

we deduce that its coefficients are real. Consequently, $P = \operatorname{ext}_L(P_j)$ is a polynomial with real coefficients on $\mathbb{H}$, where ext_L means the extension with the representation formula. We can show the analogous statement for right slice hyperholomorphic powers $S_R^m(s,q)$ of $s-q$ with similar arguments. □

We need now to formally replace the scalar variable q in the functions introduced above by the operator T in a way that is consistent with the S-functional calculus. Recall, however, that the product rule $(fg)(T) = f(T)g(T)$ holds only if $f \in \mathcal{N}(\sigma_S(T))$ and $g \in \mathcal{SH}_L(\sigma_S(T))$ or if $f \in \mathcal{SH}_R(\sigma_S(T))$ and $g \in \mathcal{N}(\sigma_S(T))$. This is due to the fact that for $f,g \in \mathcal{SH}_L(\sigma_S(T))$ or for $f,g \in \mathcal{SH}_R(\sigma_S(T))$, the product fg does not in general belong to $\mathcal{SH}_L(\sigma_S(T))$ resp. $\mathcal{SH}_R(\sigma_S(T))$.

If, on the other hand, one considers the left slice hyperholomorphic product $f *_L g$ of two left slice hyperholomorphic functions (or equivalently, the right slice hyperholomorphic product of two right slice hyperholomorphic functions), then it is not clear to which operation between operators it corresponds. Some considerations actually suggest that such an operation does not exist.

However, for power series of an operator variable, we can use the formulas (2.18) and (2.19) to define their $*_L$-product resp. $*_R$-product.

Definition 4.4.3. Let $T \in \mathcal{B}(X)$. For $F = \sum_{n=0}^{+\infty} T^n a_n$ and $G = \sum_{n=0}^{+\infty} T^n b_n$ with $a_\ell, b_\ell \in \mathbb{H}$ for $\ell \in \mathbb{N}$, we define

$$F *_L G := \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right).$$

For $\tilde{F} = \sum_{n=0}^{+\infty} a_n T^n$ and $\tilde{G} = \sum_{n=0}^{+\infty} b_n T^n$, we define

$$\tilde{F} *_R \tilde{G} := \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) T^n.$$

Remark 4.4.4. For $F = \sum_{n=0}^{+\infty} T^n a_n$ and $G = \sum_{n=0}^{+\infty} T^n b_n$ note that $F *_L G = FG$ if $a_n \in \mathbb{R}$ for every $n \in \mathbb{N}$. In this case, the coefficients a_n commute with the operator T, and hence

$$F *_L G = \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) = \sum_{n=0}^{+\infty} \sum_{k=0}^{n} T^k a_k T^{n-k} b_{n-k} = FG.$$

Similarly, $\tilde{F} *_R \tilde{G} = \tilde{F}\tilde{G}$ if $b_n \in \mathbb{R}$ for every $n \in \mathbb{N}$.

Corollary 4.4.5. *Let $T \in \mathcal{B}(X)$ and let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ be two left slice hyperholomorphic power series that converge on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$f(T) *_L g(T) = (f *_L g)(T).$$

Similarly, for two right slice hyperholomorphic power series $\tilde{f}(q) = \sum_{n=0}^{+\infty} a_n q^n$ and $\tilde{g}(q) = \sum_{n=0}^{+\infty} b_n q^n$ that converge on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$, we have

$$\tilde{f}(T) *_R \tilde{g}(T) = (\tilde{f} *_R \tilde{g})(T).$$

Proof. By the properties of the S-functional calculus, we have $f(T) = \sum_{n=0}^{+\infty} T^n a_n$ and $g(T) = \sum_{n=0}^{+\infty} T^n b_n$. Hence

$$\begin{aligned} f(T) *_L g(T) &= \sum_{n=0}^{+\infty} T^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j \sum_{n=0}^{+\infty} s^n \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \\ &= \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f *_L g(s) = (f *_L g)(T). \end{aligned}$$

An analogous computation shows the right slice hyperholomorphic case. □

Observe that $S^n_L(s,T)$ and $S^n_R(s,T)$ and slice hyperholomorphic products of such expressions are well defined because of Definition 4.4.3. In analogy with (4.4.1), we furthermore give the following definition.

Definition 4.4.6. Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. For $n, m \geq 0$, we define

$$S_L^{-n}(s,T) := \mathcal{Q}_s(T)^{-n}(\overline{s}\mathcal{I} - T)^{*_L n}$$

and

$$S_L^{-n}(s,T) *_L S_L^{-m}(\overline{s},T) := \mathcal{Q}_s(T)^{-(n+m)}\left[(\overline{s}\mathcal{I} - T)^{*_L n} *_L (s\mathcal{I} - T)^{*_L m}\right].$$

Similarly, we define

$$S_R^{-n}(s,T) := (\overline{s}\mathcal{I} - T)^{*_R n}\mathcal{Q}_s(T)^{-n}$$

and

$$S_R^{-n}(s,T) *_R S_R^{-m}(\overline{s},T) := \left[(\overline{s}\mathcal{I} - T)^{*_R n} *_R (s\mathcal{I} - T)^{*_R m}\right]\mathcal{Q}_s(T)^{-(n+m)}.$$

Remark 4.4.7. Since the function $Q_s(q)^{-n}$ is intrinsic, the above definitions, due to the product rule, are compatible with the S-functional calculus, that is,

$$\left[S_L^{-n}(s,\cdot)\right](T) = S_L^{-n}(s,T) \quad \text{and} \quad \left[S_R^{-n}(s,\cdot)\right](T) = S_R^{-n}(s,T)$$

as well as

$$\left[S_L^{-n}(s,\cdot) *_L S_L^{-m}(\overline{s},\cdot)\right](T) = S_L^{-n}(s,T) *_L S_L^{-m}(\overline{s},T)$$

and

$$\left[S_R^{-n}(s,\cdot) *_L S_R^{-m}(\overline{s},\cdot)\right](T) = S_R^{-n}(s,T) *_L S_R^{-m}(\overline{s},T).$$

Proposition 4.4.8. *Let $T \in \mathcal{B}(X)$ and let $s \in \rho_S(T)$. Then*

$$\partial_S^m S_L^{-1}(s,T) = (-1)^m m!\, S_L^{-(m+1)}(s,T) \tag{4.13}$$

and

$$\partial_S^m S_R^{-1}(s,T) = (-1)^m m!\, S_R^{-(m+1)}(s,T), \tag{4.14}$$

for every $m \geq 0$.

Proof. Recall that the slice derivative, see Definition 2.1.12, coincides with the partial derivative with respect to the real part s_0 of s. We show only (4.43), since (4.44) follows by analogous computations.

We prove the statement by induction. For $m = 0$, the identity (4.43) is obvious. We assume that $\partial_S^{m-1} S_L^{-1}(s,T) = (-1)^{m-1}(m-1)!\, S_L^{-m}(s,T)$ and we compute $\partial_S^m S_L^{-1}(s,T)$. We represent $S_L^{-m}(s,T)$ using the S-functional calculus. If we choose the path of integration in the complex plane $\mathbb{C}_{j_s}$, then we find because of (4.11) that

$$\begin{aligned}\partial_S S_L^{-m}(s,T) &= \partial_S \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{j_s})} S_L^{-1}(p,T)\,dp_j\,S_L^{-m}(s,p)\\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{j_s})} S_L^{-1}(p,T)\,dp_j\,\frac{\partial}{\partial s_0}(s-p)^{-m}\\ &= -m\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_{j_s})} S_L^{-1}(p,T)\,dp_j\,(s-p)^{-(m+1)} = -m\,S_L^{-(m+1)}(s,T),\end{aligned}$$

and in turn,

$$\begin{aligned}\partial_S^m S_L^{-1}(s,T) &= \partial_S\left(\partial_S^{m-1}S_L^{-1}(s,T)\right)\\ &= (-1)^{m-1}(m-1)!\partial_S S_L^{-m}(s,T) = (-1)^m m!\,S_L^{-(m+1)}(s,T). \qquad \square\end{aligned}$$

Remark 4.4.9. We point out that Proposition 4.4.8 also holds for unbounded closed operators. In this case, we have to modify the definition of $S_L^{-m}(s,T)$ by commuting every occurrence of T with $\mathcal{Q}_s(T)^{-m}$ just as we did in the definition of the left S-resolvent operator. Otherwise $S_L^{-m}(s,T)$ is defined only on $\mathcal{D}(T^m)$ and not on the entire space V.

Let us now turn our attention to the Taylor series expansion of $f(T+N)$ in the operator variable. In order for such an expansion to hold, it is essential that adding a somewhat small operator N not to perturb the S-spectrum of T a lot. The following result clarifies how one has to measure the distance between a point $s\in\rho_S(T)$ and the S-spectrum of T.

Lemma 4.4.10. *Let $A\subset\mathbb{H}$ be axially symmetric and let $s=s_0+js_1\in\mathbb{H}$. Then*

$$\operatorname{dist}(s,A) = \operatorname{dist}(s,A\cap\mathbb{C}_j) = \operatorname{dist}\left(s, A\cap\mathbb{C}_j^{\geq}\right),$$

where $\operatorname{dist}(s,A) := \inf\{|s-q| : q\in A\}$ *and* $\mathbb{C}_j^{\geq} = \{q_0+jq_1 : q_0\in\mathbb{R}, q_1\geq 0\}$.

Proof. For $q=q_0+j_q q_1\in A$, define $q_j=q_0+jq_1$. We choose $i\in\mathbb{S}$ with $j\perp i$ and set $k=ji$. Then $q=q_0+\tilde q_1 j+\tilde q_2 i+\tilde q_3 k$ with $\tilde q_1^2+\tilde q_2^2+\tilde q_3^2=|\underline{q}|^2=q_1^2$, and in turn

$$\begin{aligned}|s-q_j|^2 &= (s_0-q_0)^2+(s_1-q_1)^2\\ &= (s_0-q_0)^2+s_1^2-2s_1q_1+q_1^2\\ &= (s_0-q_0)^2+s_1^2-2s_1\sqrt{\tilde q_1^2+\tilde q_2^2+\tilde q_3^2}+\tilde q_1^2+\tilde q_2^2+\tilde q_2^3\\ &\leq (s_0-q_0)^2+s_1^2-2s_1\tilde q_1+\tilde q_1^2+\tilde q_2^2+\tilde q_3{}^2\\ &= (s_0-q_0)^2+(s_1-\tilde q_1)^2+\tilde q_2^2 = |s-q|^2.\end{aligned}$$

Since A is axially symmetric, we have $\{q_j : q\in A\} = A\cap\mathbb{C}_j^{\geq}$. Consequently,

$$\inf_{q\in A}|s-q| \leq \inf_{q\in A\cap\mathbb{C}_j^{\geq}}|s-q| \leq \inf_{q\in A}|s-q_j| \leq \inf_{q\in A}|s-q|,$$

and in turn,

$$\operatorname{dist}(s,A) = \inf_{q\in A}|s-q| = \inf_{q\in A\cap\mathbb{C}_j^{\geq}}|s-q| = \operatorname{dist}(s,A\cap\mathbb{C}_j^{\geq}). \qquad \square$$

Proposition 4.4.11. *Let $T\in\mathcal{B}(X)$ and let $C\subset\mathbb{H}$ with $\operatorname{dist}(C,\sigma_S(T))>\varepsilon$ for some $\varepsilon>0$. Then there exists a positive constant K_T such that*

$$\left\|S_L^{-m}(s,T)*_L S_L^{-n}(\overline{s},T)\right\| \leq \frac{K_T}{\varepsilon^{m+n}} \tag{4.15}$$

and

$$\left\|S_R^{-m}(s,T)*_L S_R^{-n}(\overline{s},T)\right\| \leq \frac{K_T}{\varepsilon^{m+n}}, \tag{4.16}$$

for every $s\in C$ and $m,n\geq 0$.

Proof. Let U be a bounded slice Cauchy domain with $\sigma_S(T)\subset U$ with $\operatorname{dist}(C,\overline{U})>\varepsilon$. We choose $s=s_0+js_1\in C$. By Corollary 4.4.2, we have

$$S_L^{-m}(s,q)*_L S_L^{-n}(\overline{s},q) = (s-q)^{-m}(\overline{s}-q)^{-n}$$

for every $x\in\mathbb{C}_j$. Lemma 4.4.10 implies $\operatorname{dist}(s,\overline{U}\cap\mathbb{C}_j)=\operatorname{dist}(s,\overline{U})>\varepsilon$. Since $\overline{U}\cap\mathbb{C}_j$ is symmetric with respect to the real axis, we also have $\operatorname{dist}(\overline{s},\overline{U}\cap\mathbb{C}_j)>\varepsilon$, and we deduce

$$\begin{aligned}
&\left\|S_L^{-m}(s,T)*_L S_L^{-n}(\overline{s},T)\right\| \\
&= \left\|\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(p,T)\,dp_j\,S_L^{-m}(s,p)*_L S_L^{-n}(\overline{s},p)\right\| \\
&= \left\|\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(p,T)\,dp_j\,(s-p)^{-m}(\overline{s}-p)^{-n}\right\| \\
&\leq \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} \left\|S_L^{-1}(p,T)\right\|\,d|p|\,\left|(s-p)^{-m}(\overline{s}-p)^{-n}\right| \\
&\leq \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} \left\|S_L^{-1}(p,T)\right\|\,d|p|\,\frac{1}{\varepsilon^{m+n}}.
\end{aligned}$$

Hence if we set

$$K_T := \sup_{i\in\mathbb{S}}\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_i)}\left\|S_L^{-1}(p,T)\right\|\,d|p|,$$

which depends neither on the point $s\in C$ nor on the numbers $n,m\geq 0$, then

$$\left\|S_L^{-m}(s,T)*_L S_L^{-n}(\overline{s},T)\right\| \leq \frac{K_T}{\varepsilon^{m+n}}. \qquad \square$$

Theorem 4.4.12. *Let $T \in \mathcal{B}(X)$ and let $N \in \mathcal{B}(X)$ be such that T and N commute and such that $\sigma_S(N)$ is contained in the open ball $B_\varepsilon(0)$. If $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$, then $s \in \rho_S(T+N)$ and*

$$\mathcal{Q}_s(T)^{-1} = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n,$$

where the series converges in the operator norm.

Proof. We first show the convergence of the series

$$\Sigma(s,T,N) := \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n.$$

Since $\sigma_S(N)$ is compact, there exists $\theta \in (0,1)$ such that $\sigma_S(N) \subset B_{\theta\varepsilon}(0) \subset B_\varepsilon(0)$. Applying the S-functional calculus, we obtain

$$\begin{aligned}
\|N^m\| &= \left\| \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} S_L^{-1}(s,N)\, ds_j\, s^m \right\| \\
&\leq \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} \left\| S_L^{-1}(s,N) \right\|\, d|s|\, |s|^m \\
&= \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} \left\| S_L^{-1}(s,N) \right\|\, d|s|\, (\theta\varepsilon)^m
\end{aligned}$$

for every $m \geq 0$. Hence

$$\|N^m\| \leq K_N (\theta\varepsilon)^m \tag{4.17}$$

with

$$K_N := \frac{1}{2\pi} \int_{\partial(B_{\theta\varepsilon}(0)\cap\mathbb{C}_j)} \left\| S_L^{-1}(s,N) \right\|\, d|s|.$$

From Proposition 4.4.11, we deduce

$$\begin{aligned}
&\sum_{n=0}^{+\infty} \left\| \left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right) N^n \right\| \\
&\leq \sum_{n=0}^{+\infty} \sum_{k=0}^{n} \left\| S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T) \right\| \|N^n\| \\
&\leq \sum_{n=0}^{+\infty} (n+1) \frac{K_T}{\varepsilon^{n+2}} K_N (\theta\varepsilon)^n \leq \frac{K_T K_N}{\varepsilon^2} \sum_{n=0}^{+\infty} (n+1)\theta^n.
\end{aligned}$$

By the root test, this last series converges because $0 < \theta < 1$. The comparison test yields the convergence of the original series $\Sigma(s,T,N)$ in the operator norm.

From Definition 4.4.6 and the fact that T and N commute, we deduce

$$\begin{aligned}\mathcal{Q}_s(T+N) &= T^2 + 2TN + N^2 - 2s_0T - 2s_0N + |s|^2\mathcal{I}\\ &= \mathcal{Q}_s(T) + (2T - 2s_0)N + |s|^2\mathcal{I}.\end{aligned}$$

If we define

$$\Lambda_T(k,n,s) := (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)}$$

for neatness, we therefore have

$$\begin{aligned}&\Sigma(s,T,N)\mathcal{Q}_s(T+N)\\ &= \left(\sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} S_L^{-(k+1)}(s,T) *_L S_L^{-(n-k+1)}(\overline{s},T)\right)N^n\right)\mathcal{Q}_s(T+N)\\ &= \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^n\mathcal{Q}_s(T)\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^{n+1}(2T - 2s_0\mathcal{I})\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^{n+2}.\end{aligned}$$

Applying Corollary 4.4.2 and the S-functional calculus, we see that each of the coefficients $\sum_{k=0}^{n}\Lambda_T(k,n,s) = \sum_{k=0}^{n}(\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)}$ is a polynomial in T with real coefficients and hence commutes with the operator $\mathcal{Q}_s(T)$. Remark 4.4.4 implies

$$\begin{aligned}&\Sigma(s,T,N)\mathcal{Q}_s(T+N)\\ &= \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+1)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^n\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s) *_L (2T - 2s_0\mathcal{I})\right)N^{n+1}\\ &\quad + \sum_{n=0}^{+\infty}\mathcal{Q}_s(T)^{-(n+2)}\left(\sum_{k=0}^{n}\Lambda_T(k,n,s)\right)N^{n+2}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) \right) N^n \\
&\quad + \sum_{n=1}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n-1} \Lambda_T(k,n-1,s) *_L (2T - 2s_0\mathcal{I}) \right) N^n \\
&\quad + \sum_{n=2}^{+\infty} \mathcal{Q}_s(T)^{-n} \sum_{k=0}^{n-2} \Lambda_T(k,n-2,s) N^n.
\end{aligned}
$$

The identity

$$
\begin{aligned}
&\mathcal{Q}_s(T)^{-n} \left(\sum_{k=0}^{n-2} \Lambda_T(k,n-2,s) \right) \\
&= \mathcal{Q}_s(T)^{-n} \left(\sum_{k=0}^{n-2} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k-1)} \right) \\
&= \mathcal{Q}_s(T)^{-n} \left(\sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L k} *_L (s\mathcal{I} - T)^{*_L(n-k)} \right) \\
&= \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} \right) \\
&= \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=1}^{n-1} \Lambda_T(k,n,s) \right),
\end{aligned}
$$

finally yields

$$
\begin{aligned}
&\Sigma(s,T,N)\mathcal{Q}_s(T+N) \\
&= \mathcal{Q}_s(T)^{-1} \Lambda_T(0,0,s) N^0 \\
&\quad + \mathcal{Q}_s(T)^{-2} \left(\sum_{k=0}^{1} \Lambda_T(k,1,s) + \Lambda(0,0,s) *_L (2T - 2s_0\mathcal{I}) \right) N \\
&\quad + \sum_{n=2}^{+\infty} \mathcal{Q}_s(T)^{-(n+1)} \left(\sum_{k=0}^{n} \Lambda_T(k,n,s) \right. \\
&\quad \left. + \sum_{k=0}^{n-1} \Lambda_T(k,n-1,s) *_L (2T - 2s_0\mathcal{I}) + \sum_{k=1}^{n-1} \Lambda_T(k,n,s) \right) N^n.
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
\mathcal{Q}_s(T)^{-1} \Lambda_T(0,0,s) N^0 &= \mathcal{Q}_s(T)^{-1} \left((\overline{s}\mathcal{I} - T) *_L (s\mathcal{I} - T) \right) \\
&= \mathcal{Q}_s(T)^{-1} \mathcal{Q}_s(T) = \mathcal{I}.
\end{aligned}
$$

Because of $2T - 2s_0\mathcal{I} = -(s\mathcal{I} - T) - (\overline{s}\mathcal{I} - T)$, we have

$$\begin{aligned}
&\sum_{k=0}^{1} \Lambda_T(k,1,s) + \Lambda(0,0,s) *_L (2T - 2s_0\mathcal{I}) \\
&= (\overline{s}\mathcal{I} - T) *_L (s\mathcal{I} - T)^{*_L 2} + (\overline{s}\mathcal{I} - T)^{*_L 2} *_L (s\mathcal{I} - T) \\
&\quad - (\overline{s}\mathcal{I} - T)^{*_L 2} *_L (s\mathcal{I} - T) - (\overline{s}\mathcal{I} - T) *_L (s\mathcal{I} - T)^{*_L 2} = 0.
\end{aligned}$$

Finally, we also find again because of $2T - 2s_0\mathcal{I} = -(s\mathcal{I} - T) - (\overline{s}\mathcal{I} - T)$ that

$$\begin{aligned}
\sum_{k=0}^{n} \Lambda_T(k,n,s) &+ \sum_{k=0}^{n-1} \Lambda_T(k,n-1,s) *_L (2T - 2s_0\mathcal{I}) + \sum_{k=1}^{n-1} \Lambda_T(k,n,s) \\
&= \sum_{k=0}^{n} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} \\
&\quad - \sum_{k=0}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+2)} *_L (s\mathcal{I} - T)^{*_L(n-k)} \\
&\quad - \sum_{k=0}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} \\
&\quad + \sum_{k=1}^{n-1} (\overline{s}\mathcal{I} - T)^{*_L(k+1)} *_L (s\mathcal{I} - T)^{*_L(n-k+1)} = 0,
\end{aligned}$$

where the last identity follows after an index shift k to $k+1$ in the second sum. Altogether, we obtain

$$\Sigma(s,T,N)\mathcal{Q}_s(T+N) = \mathcal{I}.$$

From Corollary 4.4.2 and the S-functional calculus, we already concluded that each of the coefficients $\sum_{k=0}^{n} \Lambda_T(k,n,s)$ in $\Sigma(s,T,N)$ is a polynomial in T with real coefficients and thus commutes with both T and N. Hence it also commutes with $\mathcal{Q}_s(T+N)$, and so also

$$\mathcal{Q}_s(T+N)\Sigma(s,T,N) = \Sigma(s,T,N)\mathcal{Q}_s(T+N) = \mathcal{I}.$$

Hence $\mathcal{Q}_s(T+N)$ is invertible, which implies $s \in \rho_S(T+N)$. □

Theorem 4.4.13. *Let $T, N \in \mathcal{B}(X)$ be such that $\sigma_S(N) \subset B_\varepsilon(0)$ and such that T and N commute. For every $s \in \rho_S(T)$ with* $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$, *the identities*

$$S_L^{-1}(s, T+N) = \sum_{n=0}^{+\infty} N^n \, S_L^{-(n+1)}(s,T)$$

and

$$S_R^{-1}(s, T+N) = \sum_{n=0}^{+\infty} S_R^{-(n+1)}(s,T) \, N^n$$

hold, where the series converge uniformly on every set C with $\operatorname{dist}(C, \sigma_S(T)) > \varepsilon$.

Proof. In (4.17), we showed the existence of two constants $K_N \geq 0$ and $\theta \in (0,1)$ such that $\|N\|^m \leq K_N(\theta\varepsilon)^m$ for every $m \in \mathbb{N}_0$. Moreover, for every $C \subset \mathbb{H}$ with $\operatorname{dist}(C, \sigma_S(T)) > \varepsilon$, Proposition 4.4.11 implies the existence of a constant K_T such that $\|S_L^{-m}(s,T)\| \leq K_T/\varepsilon^m$ for every $s \in C$ and $m \in \mathbb{N}_0$. Therefore, the estimate

$$\sum_{n=n_0}^{\infty} \left\| N^n\, S_L^{-(n+1)}(s,T) \right\| \leq \sum_{n=n_0}^{+\infty} \|N^n\| \left\| S_L^{-(n+1)}(s,T) \right\| \leq \sum_{n=n_0}^{+\infty} K_N(\theta\varepsilon)^n \frac{K_T}{\varepsilon^{n+1}} = \frac{K_T K_N}{\varepsilon} \sum_{n=n_0}^{+\infty} \theta^n \overset{n_0\to\infty}{\longrightarrow} 0$$

holds for every $s \in C$ and implies the uniform convergence of the series on C.

Let $s \in \rho_S(T)$ with $\operatorname{dist}(s, \sigma_S(T)) > \varepsilon$. We have

$$\begin{aligned}
&\big((T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}\big) \sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T) \\
&= \big(T^2 - 2s_0T + |s|^2\mathcal{I}\big) \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + (2T - 2s_0\mathcal{I})\, N \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + N^2 \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&= \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-n} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + \sum_{n=0}^{+\infty} N^{n+1}\, (2T - 2s_0\mathcal{I})\, (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)} \\
&\quad + \sum_{n=0}^{+\infty} N^{n+2} (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n+1)} (\overline{s}\mathcal{I} - T)^{*_L(n+1)}.
\end{aligned}$$

Shifting the indices yields

$$\begin{aligned}
&\big((T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}\big)\sum_{n=0}^{+\infty} N^n\, S_L^{-(n+1)}(s,T)\\
&= \sum_{n=0}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-n} (\overline{s}\mathcal{I} - T)^{*_L(n+1)}\\
&\quad + \sum_{n=1}^{+\infty} N^n\,(2T - 2s_0\mathcal{I})\,(T^2 - 2s_0T + |s|^2\mathcal{I})^{-n} (\overline{s}\mathcal{I} - T)^{*_L n}\\
&\quad + \sum_{n=2}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-(n-1)} (\overline{s}\mathcal{I} - T)^{*_L(n-1)}\\
&= \overline{s}\mathcal{I} - T + N(T^2 - s_0T + |s|^2\mathcal{I})^{-1}(\overline{s}\mathcal{I} - T)^{*_L 2}\\
&\quad + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(2T - 2s_0\mathcal{I})(\overline{s}\mathcal{I} - T) +\\
&\quad + \sum_{n=2}^{+\infty} N^n (T^2 - 2s_0T + |s|^2\mathcal{I})^{-n}\Big[(\overline{s}\mathcal{I} - T)^{*_L(n+1)}\\
&\qquad\qquad + (2T - 2s_0\mathcal{I})\,(\overline{s}\mathcal{I} - T)^{*_L n}\\
&\qquad\qquad + (T^2 - 2s_0T + |s|^2\mathcal{I})(\overline{s}\mathcal{I} - T)^{*_L(n-1)}\Big].
\end{aligned}$$

The last series equals 0 because Remark 4.4.4 and the identity

$$(T^2 - 2s_0T + |s|^2\mathcal{I}) = (s\mathcal{I} - T) *_L (\overline{s}\mathcal{I} - T)$$

imply

$$\begin{aligned}
&(\overline{s}\mathcal{I} - T)^{*_L(n+1)} + (2T - 2s_0\mathcal{I})\,(\overline{s}\mathcal{I} - T)^{*_L n}\\
&\quad + (T^2 - 2s_0T + |s|^2\mathcal{I})(\overline{s}\mathcal{I} - T)^{*_L(n-1)}\\
&= (\overline{s}\mathcal{I} - T)^{*_L(n+1)} + (2T - 2s_0\mathcal{I}) *_L (\overline{s}\mathcal{I} - T)^{*_L n} + (s\mathcal{I} - T) *_L (\overline{s}\mathcal{I} - T)^{*_L n}\\
&= (\overline{s}\mathcal{I} - T + 2T - 2s_0\mathcal{I} + s\mathcal{I} - T) *_L (\overline{s}\mathcal{I} - T)^{*_L(n-1)} = 0.
\end{aligned}$$

Hence, we finally obtain

$$\begin{aligned}
&\big((T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}\big)\sum_{n=0}^{+\infty} N^n S_L^{-(n+1)}(s,T)\\
&= \overline{s}\mathcal{I} - T + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(\overline{s}^2\mathcal{I} - 2T\overline{s} + T^2)\\
&\quad + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(2T\overline{s} - 2s_0\overline{s}\mathcal{I} - 2T^2 + 2s_0T)\\
&= \overline{s}\mathcal{I} - T + N(T^2 - 2s_0T + |s|^2\mathcal{I})^{-1}(-T^2 + 2s_0T - |s|^2\mathcal{I}) = \overline{s}\mathcal{I} - T - N.
\end{aligned}$$

Since $\mathcal{Q}_s(T+N) = (T+N)^2 - 2s_0(T+N) + |s|^2\mathcal{I}$ is invertible by Theorem 4.4.12, this is equivalent to

$$\sum_{n=0}^{+\infty} N^n \, S_L^{-(n+1)}(s,T) = \mathcal{Q}_s(T+N)^{-1}(\overline{s}\mathcal{I} - T - N) = S_L^{-1}(s, T+N).$$

The identity for the right S-resolvent can be shown with analogous computations. □

Theorem 4.4.14 (The Taylor formulas)**.** *Let $T, N \in \mathcal{B}(X)$ with $\sigma_S(N) \subset B_\varepsilon(0)$ such that T and N commute and set*

$$C_\varepsilon(\sigma_S(T)) := \{s \in \mathbb{H} : \ \operatorname{dist}(s, \sigma_S(T)) \le \varepsilon\}.$$

If $f \in \mathcal{SH}_L(C_\varepsilon(\sigma_S(T)))$, then $f \in \mathcal{SH}_L(\sigma_S(T+N))$ and

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \left(\partial_S^n f\right)(T).$$

Similarly, if $f \in \mathcal{SH}_R(C_\varepsilon(\sigma_S(T)))$, then $f \in \mathcal{SH}_R(\sigma_S(T+N))$ and

$$f(T+N) = \sum_{n=0}^{+\infty} \frac{1}{n!} \left(\partial_S^n f\right)(T) N^n.$$

Proof. We prove just the first Taylor formula; the second one is obtained with similar computations. By Theorem 4.4.12, we have $\sigma_S(T+N) \subset C_\varepsilon(\sigma_S(T))$, and so the function f belongs to $\mathcal{SH}_L(\sigma_S(T+N))$. If U is a bounded slice Cauchy domain with $C_\varepsilon(\sigma_S(T)) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$, then we find due to Theorem 4.4.13 that

$$\begin{aligned} f(T+N) &= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s, T+N)\, ds_j\, f(s) \\ &= \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} \sum_{n=0}^{+\infty} N^n S_L^{-(n+1)}(s,T)\, ds_j\, f(s) \\ &= \sum_{n=0}^{+\infty} N^n \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-(n+1)}(s,T)\, ds_j\, f(s). \end{aligned}$$

By Proposition 4.4.8, we have

$$S_L^{-(n+1)}(s,T) = (-1)^n \frac{1}{n!} \partial_S^n S_L^{-1}(s,T),$$

and so

$$f(T+N) = \sum_{n=0}^{+\infty} N^n \frac{(-1)^n}{n!} \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} \partial_S^n S_L^{-1}(s,T)\, ds_j\, f(s).$$

After integrating the nth term in the sum n times by parts, we finally obtain

$$\begin{aligned} f(T+N) &= \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, (\partial_S^n f)(s) \\ &= \sum_{n=0}^{+\infty} N^n \frac{1}{n!} (\partial_S^n f)(T). \end{aligned}$$ □

4.5 Bounded Operators with Commuting Components

If the components of T commute, then the S-spectrum can be characterized by a different operator, which is often easier to handle in the applications. The S-resolvent operators can in this case be expressed in a form that corresponds to replacing the scalar variable q in the slice hyperholomorphic Cauchy kernels by the operator T when they are written in form II; see Chapter 2.

We saw in Remark 2.3.2 that every two-sided quaternionic vector space X is essentially of the form $X = X_{\mathbb{R}} \otimes \mathbb{H}$, where $X_{\mathbb{R}}$ is the real vector space consisting of the vectors that commute with all quaternions. If $x = \sum_{\ell=0}^{3} x_\ell e_\ell$ with $x_\ell \in X_{\mathbb{R}}$, where we set $e_0 = 1$ for neatness, then we can write any operator $T \in \mathcal{B}(X)$ as $T = \sum_{\ell=0}^{3} T_\ell e_\ell$ with components $T_\ell \in \mathcal{B}(X_{\mathbb{R}})$, where this operator acts as

$$Tx = \left(\sum_{\ell=0}^{3} T_\ell e_\ell\right)\left(\sum_{\kappa=0}^{3} x_\kappa e_\kappa\right) = \sum_{\ell,\kappa=0}^{3} T_\ell(x_\kappa) e_\ell e_\kappa.$$

We obtain $\mathcal{B}(X) = \mathcal{B}(X_{\mathbb{R}}) \otimes \mathbb{H}$, and hence we call any operator in $\mathcal{B}(X_{\mathbb{R}})$ a scalar operator on X.

Definition 4.5.1. We define $\mathcal{BC}(X)$ to be the space of all operators $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{B}(X)$ with components $T_\ell \in \mathcal{B}(X_{\mathbb{R}}), \ell = 0,\dots,3$, that mutually commute.

Definition 4.5.2. For $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$, we set

$$\overline{T} := T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell.$$

The following statement shows that for an operator $T \in \mathcal{BC}(X)$ the analogues of the scalar identities $s + \overline{s} = 2\mathrm{Re}(s)$ and $s\overline{s} = \overline{s}s = |s|^2$ hold. This motivates the idea that we can write the S-resolvent for such operators also by formally replacing q by T in the slice hyperholomorphic Cauchy kernels when they are written in form II.

Lemma 4.5.3. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. Then $2T_0 = T + \overline{T}$ and $T\overline{T} = \overline{T}T = \sum_{\ell=0}^{3} T_\ell^2$.*

Proof. We obviously have

$$T + \overline{T} = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell + T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell = 2T_0.$$

Since the components T_ℓ mutually commute and $e_\ell e_\kappa = -e_\kappa e_\ell$ for $1 \leq \ell, \kappa \leq 3$ with $\ell \neq \kappa$, we also have

$$\begin{aligned} T\overline{T} &= \left(T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell\right)\left(T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\ &= T_0^2 - \sum_{\ell=1}^{3} T_0 T_\ell e_\ell + \sum_{\ell=1}^{3} T_\ell T_0 e_\ell - \sum_{\ell,\kappa=1}^{3} T_\ell T e_\ell e_\kappa \\ &= T_0^2 - \sum_{\ell=1}^{3} T_\ell^2 e_\ell^2 + \sum_{\substack{\ell=1,2,3 \\ \ell<\kappa}} (T_\ell T_\kappa - T_\kappa T_\ell) e_\ell e_\kappa = \sum_{\ell=0}^{3} T_\ell^2. \end{aligned}$$

□

Lemma 4.5.4. *If $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$, then the following statements are equivalent:*

(i) *The operator T is invertible.*

(ii) *The operator $\overline{T}$ is invertible.*

(iii) *The operator $T\overline{T}$ is invertible.*

In this case we have

$$\overline{T}^{-1} = \overline{T^{-1}} \quad \textit{and} \quad T^{-1} = (T\overline{T})^{-1}\overline{T}. \tag{4.18}$$

Proof. If $T\overline{T}$ is invertible, then $(T\overline{T})^{-1} = \left(\sum_{\ell=0}^{3} T_\ell^2\right)^{-1}$ commutes with T and $\overline{T}$, and hence

$$\left(T\overline{T}\right)^{-1}\overline{T}T = \left(\overline{T}T\right)^{-1}\overline{T}T = \mathcal{I}$$

and

$$T\left(T\overline{T}\right)^{-1}\overline{T} = \left(T\overline{T}\right)^{-1}T\overline{T} = \mathcal{I}.$$

Thus (iii) implies (i), and the second identity in (4.18) holds.

If, on the other hand, T is invertible and $T^{-1} = B_0 + \sum_{\kappa=1}^{3} B_\kappa e_\kappa \in \mathcal{B}(X)$, then

$$\begin{aligned} \mathcal{I} = T^{-1}T &= \left(B_0 + \sum_{\kappa=1}^{3} B_\kappa e_\kappa\right)\left(T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell\right) \\ &= B_0 T_0 - \sum_{\ell=1}^{3} B_\ell T_\ell + (B_2 T_3 - B_3 T_2) e_1 \\ &\quad + (B_3 T_1 - B_1 T_3) e_2 + (B_1 T_2 - B_2 T_1) e_3. \end{aligned}$$

We conclude that

$$\mathcal{I} = B_0T_0 - \sum_{\ell=1}^{3} B_\ell T_\ell$$

and

$$B_\ell T_\kappa - B_\kappa T_\ell = 0$$

for $1 \leq \ell < \kappa \leq 3$. Therefore,

$$\begin{aligned}\overline{B}\,\overline{T} &= \left(B_0 - \sum_{\ell=1}^{3} B_\ell e_\ell\right)\left(T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell\right)\\ &= B_0T_0 - \sum_{\ell=1}^{3} B_\ell T_\ell + (B_2T_3 - B_3T_2)e_1\\ &\quad + (B_3T_1 - B_1T_3)e_2 + (B_1T_2 - B_2T_1)e_3 = \mathcal{I},\end{aligned}$$

and similarly we see that also $\overline{T}\,\overline{B} = \mathcal{I}$. Hence (i) implies (ii) and $\overline{T}^{-1} = \overline{T^{-1}}$. Since $\overline{\overline{T}} = T$, we can exchange the roles of T and $\overline{T}$ and find that (ii) implies (i). Finally, we see that in this case, $(T\overline{T})^{-1} = \overline{T}^{-1}T^{-1} \in \mathcal{B}(X)$, and we find that (i) and (ii) also imply (iii). □

Definition 4.5.5. Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$, we define the operator

$$\mathcal{Q}_{c,s}(T) := s^2\mathcal{I} - 2sT_0 + T\overline{T}.$$

Theorem 4.5.6. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. Then $\mathcal{Q}_{c,s}(T)$ is invertible if and only if $\mathcal{Q}_s(T)^{-1}$ is invertible, and so*

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \ \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) \right\}. \tag{4.19}$$

Moreover, for $s \in \rho_S(T)$, we have

$$S_L^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1} \tag{4.20}$$

and

$$S_R^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}). \tag{4.21}$$

Proof. We observe that for $s \in \mathbb{H}$, we have $\overline{\mathcal{Q}_s(T)} = \mathcal{Q}_s(\overline{T})$ and $\overline{\mathcal{Q}_{c,s}(T)} = \mathcal{Q}_{c,\overline{s}}(T)$, and so

$$\begin{aligned}\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)} &= (s^2\mathcal{I} - 2sT_0 + T\overline{T})(\overline{s}^2\mathcal{I} - 2\overline{s}T_0 + T\overline{T})\\ &= |s|^4\mathcal{I} - 2s|s|^2T_0 + s^2T\overline{T} - 2|s|^2T_0\overline{s} + 4|s|^2T_0^2 - 2sT_0T\overline{T}\\ &\quad + \overline{s}^2T\overline{T} - 2\overline{s}T_0T\overline{T} + (T\overline{T})^2\\ &= |s|^4\mathcal{I} - 2s_0|s|^2T - 2s_0|s|^2\overline{T} + 2\mathrm{Re}(s^2)T\overline{T}\\ &\quad + 4|s|^2T_0^2 - 2s_0T^2\overline{T} - 2s_0T\overline{T}^2 + T^2\overline{T}^2,\end{aligned}$$

where we used in the last identity that $2s_0 = s + \overline{s}$, that $|s|^2 = s\overline{s}$, and that $2T_0 = T + \overline{T}$. Since, for $s = s_0 + j_s s_1$, we have

$$2\mathrm{Re}(s^2)T\overline{T} = 2s_0^2 T\overline{T} - 2s_1^2 T\overline{T}$$

and

$$4|s|^2T_0^2 = |s|^2(T+\overline{T})^2 = |s|^2T^2 + 2s_0^2T\overline{T} + s_1^2T\overline{T} + |s|^2\overline{T}^2,$$

we further find that

$$\begin{aligned}\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)} &= |s|^2(|s|^2\mathcal{I} - 2s_0T + T^2)\\ &\quad - 2s_0\overline{T}(|s|^2\mathcal{I} - 2s_0T + T^2)\\ &\quad + \overline{T}^2(|s|^2\mathcal{I} - 2s_0T + T^2) = \mathcal{Q}_s(T)\overline{\mathcal{Q}_s(T)}.\end{aligned}$$

From Lemma 4.5.4, we conclude that the invertibility of $\mathcal{Q}_{c,s}(T)$ is equivalent to the invertibility of $\mathcal{Q}_{c,s}(T)\overline{\mathcal{Q}_{c,s}(T)} = \mathcal{Q}_s(T)\overline{\mathcal{Q}_s(T)}$, which is in turn equivalent to the invertibility of $\mathcal{Q}_s(T)$, and hence (4.19) holds.

Because of Lemma 4.5.3, we furthermore have

$$\begin{aligned}(\overline{s}\mathcal{I} - T)\mathcal{Q}_{c,s}(T) &= (\overline{s}\mathcal{I} - T)\left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)\\ &= |s|^2 s\mathcal{I} - Ts^2 - 2|s|^2T_0 + 2TT_0s + \overline{s}T\overline{T} - T^2\overline{T}\\ &= |s|^2 s\mathcal{I} - Ts^2 - |s|^2T - |s|^2\overline{T} + T^2s + T\overline{T}s + \overline{s}T\overline{T} - T^2\overline{T}\\ &= |s|^2\left(s\mathcal{I} - \overline{T}\right) - 2s_0T\left(s\mathcal{I} - \overline{T}\right) + T^2\left(s\mathcal{I} - \overline{T}\right)\\ &= \left(T^2 - 2s_0T + |s|^2\mathcal{I}\right)\left(s\mathcal{I} - \overline{T}\right) = \mathcal{Q}_s(T)(s\mathcal{I} - \overline{T}),\end{aligned} \tag{4.22}$$

and so

$$S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}(\overline{s}\mathcal{I} - T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1}.$$

Similar computations show that also the identity (4.21) holds. □

Definition 4.5.7 (SC-resolvent operators)**.** Let $T \in \mathcal{BC}(X)$. For $s \in \rho_S(T)$, we define the *left* and *right SC-resolvent operator* of T as

$$S_{c,L}^{-1}(s,T) = (s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-1}$$

and

$$S_{c,R}^{-1}(s,T) = \mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T}).$$

Corollary 4.5.8. *Let $T \in \mathcal{BC}(X)$. For $f \in \mathcal{SH}_L(\sigma_S(T))$, we have*

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_{c,L}^{-1}(s,T)\,ds_j\,f(s),$$

and for $f \in \mathcal{SH}_R(\sigma_S(T))$ we have

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S^{-1}_{c,R}(s,T)$$

for any imaginary unit $j \in \mathbb{S}$ and any bounded slice Cauchy domain U with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$.

Remark 4.5.9. The S-functional calculus for operators with commuting components defined by the above integrals that involve the SC-resolvents is often also referred to as the SC-functional calculus. Similarly, the S-spectrum is sometimes called the F-spectrum when it is characterized by the operator $\mathcal{Q}_{c,s}(T)^{-1}$, in order to stress that one is using the simpler characterization that holds only for operators with commuting components.

4.6 Perturbations of the SC-Resolvent Operators

In order to study bounded perturbations of the F-resolvent operators (see Chapter 7), we study in this section a preliminary result about the perturbations of the S-resolvent operators $S^{-1}_{c,L}(s,T)$ and $S^{-1}_{c,R}(s,T)$. This will be used in the sequel. We recall that the left spectrum $\sigma_L(T)$ and the left resolvent sets $\rho_L(T)$ were defined in Definition 3.3.1. The following corollary of Lemma 3.1.12 will be used in the sequel.

Corollary 4.6.1. *Let $T \in \mathcal{BC}(X)$. If $s \in \rho_S(T) \cap \rho_L(\overline{T})$, then*

$$\left(S^{-1}_{c,L}(s,T)\right)^{-1} = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right) T \left(s\mathcal{I} - \overline{T}\right)^{-1},$$

$$\left(S^{-1}_{c,R}(s,T)\right)^{-1} = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right)^{-1} T \left(s\mathcal{I} - \overline{T}\right).$$

Proof. By Theorem 4.5.6, we have

$$S^{-1}_{c,L}(s,T) = \left(s\mathcal{I} - \overline{T}\right) \mathcal{Q}_{c,s}(T)^{-1} = \left(s\mathcal{I} - \overline{T}\right) \left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)^{-1}.$$

Since $\mathcal{Q}_{c,s}(T) = s\left(s\mathcal{I} - \overline{T}\right) - (s\mathcal{I} - T)T$, we thus obtain

$$\begin{aligned}\left(S^{-1}_{c,L}(s,T)\right)^{-1} &= \left(s^2\mathcal{I} - 2sT_0 + T\overline{T}\right)\left(s\mathcal{I} - \overline{T}\right)^{-1} \\ &= s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right) T \left(s\mathcal{I} - \overline{T}\right)^{-1}.\end{aligned}$$

Similar computations show the identity for the right S-resolvent. □

Definition 4.6.2. Let $T \in \mathcal{BC}(X)$. For $s \in \rho_L(\overline{T})$, we define

$$S_{c,L}(s,T) = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right) T \left(s\mathcal{I} - \overline{T}\right)^{-1},$$

$$S_{c,R}(s,T) = s\mathcal{I} - \left(s\mathcal{I} - \overline{T}\right)^{-1} T \left(s\mathcal{I} - \overline{T}\right)^{-1}.$$

Lemma 4.6.3. *Let T, $Z \in \mathcal{BC}(X)$. If $s \notin \sigma_L\left(\overline{T}\right) \cup \sigma_L\left(\overline{Z}\right)$, then*

$$\|S_{c,L}(s,T) - S_{c,L}(s,Z)\| \leq K_{T,Z}(s)\|T - Z\|, \tag{4.23}$$
$$\|S_{c,R}(s,T) - S_{c,R}(s,Z)\| \leq K_{T,Z}(s)\|T - Z\|, \tag{4.24}$$

with

$$K_{T,Z}(s) := \left\|(s\mathcal{I} - \overline{Z})^{-1}\right\| \left(\|Z\| + \left\|s\mathcal{I} - \overline{T}\right\| \left[1 + \|T\| \left\|(s\mathcal{I} - \overline{T})^{-1}\right\|\right]\right). \tag{4.25}$$

Proof. We consider the chain of equalities

$$\begin{aligned}
&S_{c,L}(s,T) - S_{c,L}(s,Z)\\
&= (s\mathcal{I} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})T(s\mathcal{I} - \overline{T})^{-1}\\
&= (s\mathcal{I} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})Z(s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + (s\mathcal{I} - \overline{T})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})T(s\mathcal{I} - \overline{T})^{-1}\\
&= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})\left[Z(s\mathcal{I} - \overline{Z})^{-1} - T(s\mathcal{I} - \overline{T})^{-1}\right]\\
&= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})\Big[(Z - T)(s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + T\left((s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})^{-1}\right)\Big]\\
&= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})\Big[(Z - T)(s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + T\left(s\mathcal{I} - \overline{Z}\right)^{-1}\left(\overline{Z} - \overline{T}\right)\left(s\mathcal{I} - \overline{T}\right)^{-1}\Big].
\end{aligned}$$

Taking the norm and observing that $\|T - Z\| = \left\|\overline{T} - \overline{Z}\right\|$, we have

$$\begin{aligned}
\|S_{c,L}(s,T) - S_{c,L}(s,Z)\| \leq \|T - Z\| \Big(&\|Z\|\,\|(s\mathcal{I} - \overline{Z})^{-1}\|\\
&+ \|s\mathcal{I} - \overline{T}\|\Big[\|(s\mathcal{I} - \overline{Z})^{-1}\| + \|T\|\,\|(s\mathcal{I} - \overline{Z})^{-1}\|\|(s\mathcal{I} - \overline{T})^{-1}\|\Big]\Big),
\end{aligned}$$

and so (4.23) holds. The second estimate is shown with similar arguments. □

Lemma 4.6.4. *Let T, $Z \in \mathcal{BC}(X)$, let $s \in \rho_S(T)$ with $s \notin \sigma_L\left(\overline{T}\right) \cup \sigma_L\left(\overline{Z}\right)$, and suppose that*

$$\|T - Z\| < \frac{1}{K_{Z,T}(s)}\|S_{c,L}^{-1}(s,T)\|^{-1},$$

with $K_{Z,T}(s)$ as in Lemma 4.6.3. Then $s \in \rho_S(Z)$ and

$$\begin{aligned}
&S_{c,L}^{-1}(s,Z) - S_{c,L}^{-1}(s,T)\\
&= S_{c,L}^{-1}(s,T)\sum_{m=1}^{+\infty}\left[(S_{c,L}(s,T) - S_L(s,Z))S_{c,L}^{-1}(s,T)\right]^m.
\end{aligned} \tag{4.26}$$

Similarly, if

$$\|T - Z\| < \frac{1}{K_{Z,T}(s)} \left\| S_R^{-1}(s,T) \right\|^{-1},$$

then $s \in \rho_S(Z)$ *and*

$$\begin{aligned} &S_{c,R}^{-1}(s,Z) - S_{c,R}^{-1}(s,T) \\ &= S_{c,R}^{-1}(s,T) \sum_{m=1}^{+\infty} \left[(S_{c,R}(s,T) - S_{c,R}(s,Z)) S_{c,R}^{-1}(s,T) \right]^m. \end{aligned} \tag{4.27}$$

Proof. If we apply Lemma 3.1.12 with $A = S_{c,L}(s,T)$ and $B = S_{c,L}(s,Z)$, then we obtain

$$S_{c,L}^{-1}(s,Z) = S_{c,L}^{-1}(s,T) \sum_{m=0}^{+\infty} \left[(S_{c,L}(s,T) - S_{c,L}(s,Z)) S_{c,L}^{-1}(s,T) \right]^m. \tag{4.28}$$

This series converges, since

$$\left\| (S_{c,L}(s,T) - S_{c,L}(s,Z)) S_{c,L}^{-1}(s,T) \right\| \leq K_{Z,T}(s) \left\| T - Z \| \| S_{c,L}^{-1}(s,T) \right\| < 1,$$

and we obtain $s \in \rho_S(Z)$ as

$$\mathcal{Q}_{c,s}(T)^{-1} = (s\mathcal{I} - \overline{T})^{-1} S_L^1(s,T).$$

We can show the statement for the right S-resolvent with similar arguments. □

Definition 4.6.5. Let $O \subset \mathbb{H}$. We denote by $B_\varepsilon(O)$ for $\varepsilon > 0$ the ε-neighborhood of O defined as

$$B_\varepsilon(O) := \{q \in \mathbb{H} : \inf_{s \in O} |s - q| < \varepsilon\}.$$

Theorem 4.6.6. *Let* $T, Z \in \mathcal{BC}(X)$*, let* $s \in \rho_S(T)$*, and assume also that* $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$*. For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $\|T - Z\| < \delta$*, we have*

$$\sigma_S(Z) \subseteq B_\varepsilon(\sigma_S(T)),$$

and for $s \notin B_\varepsilon\left(\sigma_S(T) \cup \sigma_L(\overline{T})\right)$*,*

$$\begin{aligned} \|S_{c,L}^{-1}(s,Z) - S_{c,L}^{-1}(s,T)\| &< \varepsilon, \\ \|S_{c,R}^{-1}(s,Z) - S_{c,R}^{-1}(s,T)\| &< \varepsilon. \end{aligned}$$

Proof. Let $T, Z \in \mathcal{BC}(X)$ and let $\varepsilon > 0$. Thanks to Lemma 3.1.12 there exists $\eta > 0$ such that if

$$\|T - Z\| < \eta,$$

then $\sigma_L(\overline{Z}) \subset B_\varepsilon\left(\sigma_L\left(\overline{T}\right)\right)$, where $B_\varepsilon\left(\sigma_L\left(\overline{T}\right)\right)$ is the ε-neighborhood of $\sigma_L(\overline{T})$. We can hence always choose η such that

$$\sigma_L(\overline{Z}) \subset B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T})).$$

Consider the function $K_{T,Z}(s)$ defined in Lemma 4.6.3 and observe that the constant K_ε defined by

$$K_\varepsilon = \sup_{s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)} K_{T,Z}(s)$$

is finite, since $s \notin B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T}))$, since due to the above observation $\sigma_L(\overline{Z}) \subset B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T}))$ and since

$$\lim_{s\to\infty} \|(s\mathcal{I} - \overline{Z})^{-1}\| = \lim_{s\to\infty} \|(s\mathcal{I} - \overline{T})^{-1}\| = 0.$$

Observe that since $s \in \rho_{\mathcal{S}}(T)$, the map $s \mapsto \|S_{c,L}^{-1}(s,T)\|$ is continuous and that

$$\lim_{s\to\infty} \|S_{c,L}^{-1}(s,T)\| = 0.$$

For s in the complement of $B_\varepsilon(\sigma_S(T) \cup \sigma_L(\overline{T}))$ we have thus that there exists a positive constant N_ε such that

$$\|S_{c,L}^{-1}(s,T)\| \leq N_\varepsilon.$$

If $\delta_1 > 0$ is such that $\|Z - T\| < \frac{1}{K_\varepsilon N_\varepsilon} := \delta_1$, then we can conclude from Lemma 4.6.4 that $s \in \rho_S(Z)$ and that

$$\begin{aligned}
&\|S_{c,L}^{-1}(s,Z) - S_{c,L}^{-1}(s,T)\| \\
&\leq \frac{\|S_{c,L}^{-1}(s,T)\|^2 \; \|S_{c,L}(s,T) - S_{c,L}(s,Z)\|}{1 - \|S_{c,L}^{-1}(s,T)\| \; \|S_{c,L}(s,T) - S_{c,L}(s,Z)\|} \\
&\leq \frac{N_\varepsilon^2 K_\varepsilon \|Z - T\|}{1 - N_\varepsilon K_\varepsilon \|Z - T\|} < \varepsilon
\end{aligned}$$

if

$$\|Z - T\| < \delta_2 := \frac{\varepsilon}{K_\varepsilon(N_\varepsilon^2 + \varepsilon N_\varepsilon)}.$$

To get the statement, it suffices to set $\delta = \min\{\eta, \delta_1, \delta_2\}$.

For the right S-resolvent, we can argue similarly. □

Theorem 4.6.7. *Let $T, Z \in \mathcal{BC}(X)$, let $f \in \mathcal{SH}_L(\sigma_S(T))$, and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for $\|Z - T\| < \delta$, we have $f \in \mathcal{SH}_L(\sigma_S(Z))$ and*

$$\|f(Z) - f(T)\| < \varepsilon.$$

Proof. We recall that the operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S^{-1}_{c,L}(s,T)\ ds_j\ f(s),$$

where $U \subset \mathbb{H}$ is any bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and where $j \in \mathbb{S}$. Suppose furthermore that U contains an ε-neighborhood of $\sigma_S(T) \cup \sigma_L(\overline{T})$.

By Lemma 4.6.6 there exists $\delta_1 > 0$ such that $\sigma_S(Z) \subset U$ if we have $\|Z-T\| < \delta_1$. Consequently, $f \in \mathcal{SH}_L(\sigma_S(Z))$ for $\|Z - T\| < \delta_1$. Due to Lemma 4.6.6, $S^{-1}_{c,L}(s,T)$ is uniformly close to $S^{-1}_{c,L}(s,Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_j)$ for $j \in \mathbb{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get

$$\|f(T) - f(Z)\| = \frac{1}{2\pi} \| \int_{\partial(U\cap\mathbb{C}_j)} \left[S^{-1}_{c,L}(s,T) - S^{-1}_{c,L}(s,Z)\right]\ ds_j\ f(s)\| < \varepsilon. \quad \square$$

4.7 Some Examples

We end this chapter with some examples in which we compute the S-spectrum of different operators. In particular, we illustrate how the characterization of the S-spectrum of operators with commuting components in Theorem 4.5.6 simplifies its computation.

Example 4.7.1. Let us consider a, b, α, $\beta \in \mathbb{R}$ and the two matrices

$$T_1 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \quad T_2 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}.$$

It is easy to verify that $T_1T_2 = T_2T_1$. We can thus consider the operator

$$T = T_1e_1 + T_2e_2 = \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix},$$

with commuting components on $\mathbb{H}^2$. We have

$$\overline{T} = - \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix},$$

so that $T + \overline{T} = 0$ and

$$T\overline{T} = \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}. \tag{4.29}$$

The S-spectrum is associated with the equation $\mathcal{Q}_{c,s}(T)x = 0$, that is,

$$\left(s^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}\right) x = 0 \quad \text{for } x \neq 0. \tag{4.30}$$

Observe that the matrix $T\overline{T}$ in (4.29) has only real entries. If $s = u + jv$, we can consider the matrix $T\overline{T}$ therefore a $\mathbb{C}_j$-complex matrix, and we find that s satisfies (4.30) if and only if $-s^2$ is an eigenvalue of $T\overline{T}$. Standard computations show that the only eigenvalue of $T\overline{T}$ is $a^2 + \alpha^2$ and we conclude that

$$\sigma_S(T) = \left\{ j\sqrt{a^2+\alpha^2} : \ j \in \mathbb{S} \right\}.$$

Example 4.7.2. We illustrate in this example how the computation of the S-spectrum of an operator with commuting components is simplified by the characterization given in Theorem 4.5.6. We consider the two commuting matrices

$$T_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

and the associated quaternionic operator

$$T = e_1T_1 + e_2T_2 = \begin{bmatrix} e_2 & e_1+e_2 \\ 0 & e_1+2e_2 \end{bmatrix}.$$

Since we have

$$\overline{T} = \begin{bmatrix} -e_2 & -e_1-e_2 \\ 0 & -e_1-2e_2 \end{bmatrix},$$

it is immediate that $T + \overline{T} = 0$ and that

$$T\overline{T} = \begin{bmatrix} 1 & 4 \\ 0 & 5 \end{bmatrix}.$$

In order to compute the S-spectrum using Theorem 4.5.6, we have to solve the equation $\mathcal{Q}_{c,s}(T)^{-1}x = 0$. For $x = (y,z)^T$, this turns into

$$\begin{bmatrix} s^2+1 & 4 \\ 0 & s^2+5 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0, \quad \text{for } \begin{bmatrix} y \\ z \end{bmatrix} \neq 0.$$

This gives the two equations

$$\begin{aligned} (s^2+1)y + 4z &= 0, \\ (s^2+5)z &= 0. \end{aligned} \tag{4.31}$$

If $s = u + jv$, then we can choose $i \in \mathbb{S}$ with $i \perp j$ and write $y = y_1 + y_2 i$ and $z_1 + z_2 i$ with $y_\ell, z_\ell \in \mathbb{C}_j$. Since 1 and i are linearly independent over $\mathbb{C}_j$ and the system (4.31) contains only coefficients in $\mathbb{C}_j$, it is equivalent to

$$\begin{aligned} (s^2+1)y_\ell + 4z_\ell &= 0, \\ (s^2+5)z_\ell &= 0, \end{aligned} \qquad \ell = 1,2.$$

We are hence left with a $\mathbb{C}_j$-complex linear system of equations that can be solved easily. Its solutions are j and $\sqrt{5}j$, and thus

$$\sigma_S(T) = \left\{ j, \ \sqrt{5}j : \quad j \in \mathbb{S} \right\}.$$

The same result can be obtained by solving the equation

$$(T^2 - 2s_0 T + |s|^2\mathcal{I})x = 0,$$

that is,

$$\begin{bmatrix} -1 - 2s_0e_2 + |s|^2 & -4 - 2s_0(e_1 + e_2) \\ 0 & -5 - 2s_0(e_1 + 2e_2) + |s|^2 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0.$$

This corresponds to the two equations

$$\begin{aligned} (-1 - 2s_0e_2 + |s|^2)y - (4 + 2s_0(e_1 + e_2))z &= 0, \\ (-5 - 2s_0(e_1 + 2e_2) + |s|^2)z &= 0. \end{aligned}$$

Observe, however, that the coefficients of this system do not belong to one single complex plane, so that it cannot be reduced to a complex linear system of two equations. If we suppose that $\mathrm{Re}(s) = 0$, we find that either $s = j$, or $s = \sqrt{5}j$ with $j \in \mathbb{S}$. If $s_0 \neq 0$, then very long calculations show that there are no solutions; thus the S-spectrum coincides in both cases.

Example 4.7.3. We compute the equations for determining the S-spectrum of a bounded operator T with commuting components on a Banach space X. We use both the commutative and the noncommutative approaches and we see that the computations are again simpler in the first case.

Let $T = e_1T_1 + e_2T_2 \in \mathcal{B}(X)$, where T_1, T_2 are commuting bounded operators on $X_{\mathbb{R}}$. We determine the S-eigenvalue equation. We have

$$\overline{T} = -e_1T_1 - e_2T_2,$$

so

$$T + \overline{T} = 0,$$

and since $T_1T_2 = T_2T_1$, we also have

$$T\overline{T} = T_1^2 + T_2^2.$$

The point S-spectrum $\sigma_S(T)$ consists of quaternions s such that $\mathcal{Q}_{c,s}(T)$ has a bounded inverse. Hence we need to solve the equation

$$(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})x = y$$

for every $y \in X$, which simplifies in our case to

$$(s^2\mathcal{I} + T_1^2 + T_2^2)x = y. \tag{4.32}$$

If $s = u + jv$, the operator $T\overline{T} = T_1^2 + T_2^2$ can be considered an operator on the $\mathbb{C}_j$-complex Banach space $X_{\mathbb{R}} \otimes \mathbb{C}_j := X_{\mathbb{R}} + jX_{\mathbb{R}}$, and (4.32) is then exactly an eigenvalue equation of this operator. We can choose $i \in \mathbb{S}$ with $i \perp j$ and write

$x = x_1 + x_2 i$ and $y = y_1 + y_2 i$ with $x_\ell, y_\ell \in X_{\mathbb{R}} \otimes \mathbb{C}_j$. Since 1 and i are linearly independent over $\mathbb{C}_j$, we find that (4.32) is equivalent to

$$(s^2\mathcal{I} + T_1^2 + T_2^2)x_\ell = y_\ell, \quad \ell = 1, 2. \tag{4.33}$$

Hence s belongs to $\sigma_S(T)$ if and only if $-s^2$ belongs to the classical spectrum $\sigma(T\overline{T})$ of $T\overline{T}$. Because of the axial symmetry of the S-spectrum, $\sigma_S(T)$ is then given by

$$\sigma_S(T) = \left\{u + iv : \ u + jv \in \sigma\left(T\overline{T}\right),\ i \in \mathbb{S}\right\}.$$

In case one considers the noncommutative definition of the S-spectrum, we have $T^2 = -T_1^2 - T_2^2$, so that the equation

$$(T^2 - 2s_0 T + |s|^2\mathcal{I})x = y$$

becomes

$$\left(-T_1^2 - T_2^2 - 2s_0(e_1T_1 + e_2T_2) + |s|^2\mathcal{I}\right)x = y.$$

Observe that this is again a system that is more complicated than the eigenvalue equation of a complex linear operator. If we write $x = x_0 + \sum_{\ell=1}^3 x_\ell e_\ell$ and $y = x_0 + \sum_{\ell=1}^3 y_\ell e_\ell$ and set

$$A := |s|^2\mathcal{I} - T_1^2 - T_2^2,$$

we can rewrite the above equation in terms of its real components and obtain

$$\begin{aligned} &Ax_0 + 2\mathrm{Re}(s)T_1x_1 + \mathrm{Re}(s)T_2x_2 \\ &\quad + e_1(-2\mathrm{Re}(s)T_1x_0 + Ax_1 - 2\mathrm{Re}(s)T_2x_3) \\ &\quad + e_2(-2\mathrm{Re}(s)T_2x_0 + Ax_2 + 2\mathrm{Re}(s)T_1x_3) \\ &\quad + e_1e_2(Ax_3 - 2\mathrm{Re}(s)T_1x_2 + 2\mathrm{Re}(s)T_2x_1) = y_0 + \sum_{\ell=1}^3 y_\ell e_\ell. \end{aligned}$$

Thus the S-spectrum of T is given by the system of equations

$$\begin{cases} (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_0 + 2\mathrm{Re}(s)T_1x_1 + \mathrm{Re}(s)T_2x_2 = y_0, \\ -2\mathrm{Re}(s)T_1x_0 + (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_1 - 2\mathrm{Re}(s)T_2x_3 = y_1, \\ -2\mathrm{Re}(s)T_2x_0 + (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_2 + 2\mathrm{Re}(s)T_1x_3 = y_2, \\ (|s|^2\mathcal{I} - T_1^2 - T_2^2)x_3 - 2\mathrm{Re}(s)T_1x_2 + 2\mathrm{Re}(s)T_2x_1 = y_3. \end{cases} \tag{4.34}$$

This system is much more complicated than the eigenvalue equation in (4.32), but it gives the same solution.

Example 4.7.4 (Fractional powers)**.** The slice hyperholomorphic logarithm on $\mathbb{H}$ is defined as

$$\log s := \ln|s| + j\arg(s) \quad \text{for } s = u + jv \in \mathbb{H} \setminus (-\infty, 0],$$

where $\arg(s) = \arccos(\mathrm{Re}(s)/|s|)$ is the unique angle $\varphi \in [0, \pi]$ such that $s = |s|e^{j\varphi}$. Observe that for $s = \mathrm{Re}(s) \in [0, +\infty)$, we have

$$\arccos(\mathrm{Re}(s)/|s|) = \arccos(1) = 0,$$

and so $\log s = \ln s$. Therefore, $\log s$ is well defined on the positive real axis and does not depend on the choice of the imaginary unit j. One has

$$e^{\log s} = s \quad \text{for } s \in \mathbb{H}$$

and

$$\log e^s = s \quad \text{for } s \in \mathbb{H} \text{ with } |\underline{s}| < \pi.$$

The quaternionic logarithm is both left and right slice hyperholomorphic (and actually even intrinsic) on $\mathbb{H} \setminus (-\infty, 0]$, and for every $j \in \mathbb{S}$, its restriction to the complex plane $\mathbb{C}_j$ coincides with the principal branch of the complex logarithm on $\mathbb{C}_j$. We define the fractional powers of exponent $\alpha \in \mathbb{R}$ of a quaternion s as

$$s^\alpha := e^{\alpha \log s} = e^{\alpha(\ln|s| + j\arccos(u/|s|))}, \quad s = u + jv \in \mathbb{H} \setminus (-\infty, 0].$$

This function is obviously also left and right slice hyperholomorphic on the set $\mathbb{H} \setminus (-\infty, 0]$. So we can define the fractional powers of bounded operators and in particular of matrices by the S-functional calculus. We can define fractional powers of a bounded vector operator $T = e_1T_1 + e_2T_2 + e_3T_3$ using the S-functional calculus,

$$T^\alpha = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} s^\alpha \, ds_j \, S_R^{-1}(s, T) \tag{4.35}$$

if $\sigma_S(T) \subset U$ is contained in the domain of s^α. Since $s \mapsto s^\alpha$ is an intrinsic slice hyperholomorphic function, we also have

$$T^\alpha = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s, T) \, ds_j \, s^\alpha.$$

These formulas were introduced in [50], and the theory of fractional powers of quaternionic operators was further developed in the papers [51, 52]. These operators are a natural tool to define fractional Fourier laws, and they have applications in fractional diffusion and fractional evolution problems.

4.8 Comments and Remarks

Comments on the references. The complete list of the papers in which the S-functional calculus for bounded operators has been developed is [10, 55, 66, 68, 79, 80, 127]. In the case we consider intrinsic functions, the S-functional calculus can be defined for a one-sided Banach space, as has been shown in [125]. In the paper [125], the author has also developed the theory of spectral operators in Banach spaces; see also [128].

The S-functional calculus can be defined also for n-tuples of noncommuting operators using slice hyperholomorphic functions with values in a Clifford algebra (also called slice monogenic functions); see [75, 97]. The commutative version of the S-functional calculus, that is, the S-functional calculus for operators with commuting components, is studied in [77].

The S-functional calculus was the starting point for the development of various quaternionic functional calculi. We mention the Philips functional calculus for generators of strongly continuous groups, which is based on the quaternionic version of the Laplace–Stieltjes transform; see [11]. Groups and semigroups of quaternionic linear operators have been considered in [19, 76, 153].

In the paper [30], the authors introduce the H^∞-functional calculus based on the S-spectrum. This is the quaternionic analogue of the calculus introduced by McIntosh [165]. In [30] is also considered the H^∞-functional calculus for n-tuples of noncommuting operators.

A more general version of the H^∞-functional calculus, the study of the fractional powers of quaternionic linear operators, is treated in [51, 52]. Here the authors also show how the fractional powers of quaternionic linear operators define new fractional diffusion and evolution processes. For a more direct approach to fractional powers of quaternionic operators that include the Kato formula, see the paper [50].

4.8.1 The S-Functional Calculus for n-Tuples of Operators

The notion of S-spectrum and also the definition of the S-functional calculus can be extended to n-tuples of not necessarily commuting operators. For this setting we need slice hyperholomorphic functions with values in a Clifford algebra (slice monogenic functions). Slice monogenicity is similar to the quaternionic setting; see the book [89]. We explain here the basic concepts. Let $\mathbb{R}_n$ be the real Clifford algebra over n imaginary units $e_1, \dots, e_n$ satisfying the relations $e_\ell e_m + e_m e_\ell = 0$, $\ell \neq m$, $e_\ell^2 = -1$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$, where $A = \{\ell_1 \cdots \ell_r\} \in \mathcal{P}\{1, 2, \dots, n\}$, $\ell_1 < \cdots < \ell_r$ is a multi-index, and $e_A = e_{\ell_1} e_{\ell_2} \cdots e_{\ell_r}$, $e_\emptyset = 1$. An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{\ell=1}^n x_\ell e_\ell \in \mathbb{R}_n$, called a *paravector*, and the real part x_0 of x will also be denoted by $\mathrm{Re}(x)$. The *norm* of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \cdots + x_n^2$. The *conjugate* of x is defined by $\bar{x} = x_0 - \underline{x} = x_0 - \sum_{\ell=1}^n x_\ell e_\ell$. We denote by $\mathbb{S}$ the sphere

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \cdots + e_n x_n : \ x_1^2 + \cdots + x_n^2 = 1\};$$

for $j \in \mathbb{S}$ we obviously have $j^2 = -1$. Given an element $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, let us set $j_x = \underline{x}/|\underline{x}|$ if $\underline{x} \neq 0$, and given an element $x \in \mathbb{R}^{n+1}$, the set

$$[x] := \{y \in \mathbb{R}^{n+1} : \ y = x_0 + j|\underline{x}|, \ j \in \mathbb{S}\}$$

is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R}+j\mathbb{R}$ passing through 1 and $j \in \mathbb{S}$ will be denoted by $\mathbb{C}_j$, and an element belonging to $\mathbb{C}_j$ will be indicated by $u+jv$, for u, $v \in \mathbb{R}$. With an abuse of notation we will write $x \in \mathbb{R}^{n+1}$. Thus, if $U \subseteq \mathbb{R}^{n+1}$ is an open set, a function $f: \ U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ can be interpreted as a function of the paravector x. With the above notations, the definition of the slice hyperholomorphic functions $f: \ U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is analogous to the notion of slice hyperholomorphic functions for quaternionic-valued functions. We adapt the definition of slice hyperholomorphicity to the Clifford-algebra-valued case; in this case functions are often called *slice monogenic*. The definition of an *axially symmetric* set is as in the quaternionic setting, i.e., we say that $U \subseteq \mathbb{R}^{n+1}$ is axially symmetric if $[x] \subset U$ for all $x \in U$.

Definition 4.8.1 (Slice hyperholomorphic functions with values in $\mathbb{R}_n$ (or slice monogenic functions))**.** Let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric open set and let $\mathcal{U} = \{(u,v) \in \mathbb{R}^2 : u + \mathbb{S}v \subset U\}$. A function $f: U \to \mathbb{R}_n$ is called a *left slice function* if it is of the form

$$f(q) = f_0(u,v) + jf_1(u,v) \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ that satisfy the compatibility conditions

$$f_0(u,-v) = f_0(u,v), \quad f_1(u,-v) = -f_1(u,v). \tag{4.36}$$

If in addition f_0 and f_1 satisfy the Cauchy–Riemann equations

$$\frac{\partial}{\partial u} f_0(u,v) - \frac{\partial}{\partial v} f_1(u,v) = 0, \tag{4.37}$$

$$\frac{\partial}{\partial v} f_0(u,v) + \frac{\partial}{\partial u} f_1(u,v) = 0, \tag{4.38}$$

then f is called *left slice hyperholomorphic* (or *left slice monogenic*). A function $f: U \to \mathbb{R}_n$ is called a *right slice function* if it is of the form

$$f(q) = f_0(u,v) + f_1(u,v)j \quad \text{for } q = u + jv \in U$$

with two functions $f_0, f_1 : \mathcal{U} \to \mathbb{R}_n$ that satisfy (4.36). If in addition f_0 and f_1 satisfy the Cauchy–Riemann equation, then f is called *right slice hyperholomorphic* (or *right slice monogenic*). If f is a left (or right) slice function such that f_0 and f_1 are real-valued, then f is called *intrinsic*. We denote the sets of left and right slice hyperholomorphic functions on U by $\mathcal{SM}_L(U)$ and $\mathcal{SM}_R(U)$, respectively.

Also for slice monogenic functions we have a Cauchy formula that is analogous to the quaternionic case. Let $x, s \in \mathbb{R}^{n+1}$ with $x \notin [s]$ be paravectors. The Cauchy kernels in form I and in form II are the same as in the quaternionic case when the quaternions are replaced by the paravectors. For example, for the form I we have

$$S_L^{-1}(s,x) := -(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}(x - \overline{s})$$

and

$$S_R^{-1}(s,x) := -(x - \bar{s})(x^2 - 2\mathrm{Re}(s)x + |s|^2)^{-1}.$$

Theorem 4.8.2 (The Cauchy formulas for slice monogenic functions)**.** *Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$, and set $ds_j = ds(-j)$. If f is a (left) slice monogenic function on a set that contains $\overline{U}$, then*

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,x)\, ds_j\, f(s), \quad \textit{for every } x \in U. \tag{4.39}$$

If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, then

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,x), \quad \textit{for every } x \in U. \tag{4.40}$$

These integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

To define the S-functional calculus for n-tuples of operators, we consider a Banach space X over $\mathbb{R}$ with norm $\|\cdot\|$. It is possible to endow X with an operation of multiplication by elements of $\mathbb{R}_n$ that gives a two-sided module over $\mathbb{R}_n$. A two-sided module V over $\mathbb{R}_n$ is called a Banach module over $\mathbb{R}_n$ if there exists a constant $C \geq 1$ such that $\|va\| \leq C\|v\||a|$ and $\|av\| \leq C|a|\|v\|$ for all $v \in V$ and $a \in \mathbb{R}_n$. By X_n we denote $X \otimes \mathbb{R}_n$ over $\mathbb{R}_n$; X_n turns out to be a two-sided Banach module.

An element in X_n is of type $\sum_A v_A \otimes e_A$ (where $A = \ell_1 \cdots \ell_r$, $i_\ell \in \{1, 2, \ldots, n\}$, $\ell_1 < \cdots < \ell_r$ is a multi-index). Multiplication of an element $v \in X_n$ by a scalar $a \in \mathbb{R}_n$ is defined by $va = \sum_A v_A \otimes (e_A a)$ and $av = \sum_A v_A \otimes (a e_A)$. For simplicity, we will write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. Finally, we define $\|v\|^2_{X_n} = \sum_A \|v_A\|^2_X$.

We denote by $\mathcal{B}(X)$ the space of bounded $\mathbb{R}$-homomorphisms of the Banach space X to itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(X)}$. Given $T_A \in \mathcal{B}(X)$, we can introduce the operator $T = \sum_A T_A e_A$ and its action on $v = \sum v_B e_B \in X_n$ as $T(v) = \sum_{A,B} T_A(v_B) e_A e_B$. The operator T is a right-module homomorphism that is a bounded linear map on X_n.

In the sequel, we will consider operators of the form (called *paravector operators*)

$$T = T_0 + \sum_{\ell=1}^{n} e_\ell T_\ell,$$

where $T_\ell \in \mathcal{B}(X)$ for $\ell = 0, 1, \ldots, n$. The subset of such operators in $\mathcal{B}(X_n)$ will be denoted by $\mathcal{B}^{0,1}(X_n)$. We define $\|T\|_{\mathcal{B}^{0,1}(X_n)} = \sum_\ell \|T_\ell\|_{\mathcal{B}(X)}$. Note that, in the sequel, we will omit the subscript $\mathcal{B}^{0,1}(X_n)$ in the norm of an operator. Note also that $\|TS\| \leq \|T\|\|S\|$. The Cauchy kernel operator series are the power series expansions of the S-resolvent operators.

Theorem 4.8.3. *Let $T \in \mathcal{B}^{0,1}(X_n)$ and let $s \in \mathbb{H}$. Then for $\|T\| < |s|$, we have*

$$\sum_{m\geq 0} T^m s^{-1-m} = -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \bar{s}\mathcal{I}), \tag{4.41}$$

$$\sum_{m\geq 0} s^{-1-m} T^m = -(T - \bar{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}. \tag{4.42}$$

We observe that the sums of the above series are independent of the fact that the components of the paravector operator T commute. Moreover, the operators on the right-hand sides of (4.41) and (4.42) are defined on a subset of $\mathbb{R}^{n+1}$ that is larger than $\{s \in \mathbb{R}^{n+1} \ : \ \|T\| < |s|\}$. So we define the S-spectrum, the S-resolvent set, and the S-resolvent operators for the paravector operator $T \in \mathcal{B}^{0,1}(V_n)$.

Definition 4.8.4 (The S-spectrum and the S-resolvent set)**.** Let $T \in \mathcal{B}^{0,1}(X_n)$. We define the S-*spectrum* $\sigma_S(T)$ of T as

$$\sigma_S(T) = \{s \in \mathbb{R}^{n+1} : \ T^2 - 2\,Re(s)T + |s|^2\mathcal{I} \text{ is not invertible}\}.$$

The S-*resolvent set* $\rho_S(T)$ is defined by

$$\rho_S(T) = \mathbb{R}^{n+1} \setminus \sigma_S(T).$$

Definition 4.8.5 (The S-resolvent operators)**.** Let $T \in \mathcal{B}^{0,1}(X_n)$ and $s \in \rho_S(T)$. We define the *left S-resolvent operator* as

$$S_L^{-1}(s,T) := -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I}), \tag{4.43}$$

and the *right S-resolvent operator* as

$$S_R^{-1}(s,T) := -(T - \overline{s}\mathcal{I})(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}. \tag{4.44}$$

Definition 4.8.6 (The S-functional calculus for n-tuples of operators)**.** Let X_n be a two-sided Banach module and $T \in \mathcal{B}^{0,1}(X_n)$. Let $U \subset \mathbb{R}^{n+1}$ be a bounded slice Cauchy domain that contains $\sigma_S(T)$ and set $ds_j = -ds j$. We define

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\; ds_j\; f(s), \quad \textit{for } f \in \mathcal{SM}_L(\sigma_S(T)), \tag{4.45}$$

and

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\; ds_j\; S_R^{-1}(s,T), \quad \textit{for } f \in \mathcal{SM}_R(\sigma_S(T)), \tag{4.46}$$

where $\mathcal{SM}_L(\sigma_S(T))$ (resp. $\mathcal{SM}_R(\sigma_S(T))$) are left (resp. right) slice hyperholomorphic Clifford-algebra-valued functions defined on a suitable open set that contains the S-spectrum of the paravector operator T.

Most of the results that hold for the quaternionic S-functional calculus extend to the S-functional calculus for n-tuples of operators.

4.8.2 The W-Functional Calculus for Quaternionic Operators

Using the notion of slice hyperholomorphic functions it is possible to define a transform that maps slice hyperholomorphic functions into Fueter regular functions of plane wave type. This transform is different from the Fueter mapping theorem in integral form. With such an integral transform we can define the W-functional calculus. This calculus was introduced in [70] for monogenic functions. Here we reformulate it for the quaternionic setting. Using the Cauchy formula for slice hyperholomorphic functions it is possible to define an integral transform that associates to a slice hyperholomorphic function a Fueter regular function. Inspired by [192], we introduce an integral transform that associates to a slice hyperholomorphic function a Fueter regular function of plane wave type. The following result is immediate; see [192], Section 1.1.

Proposition 4.8.7. *Suppose that the differentiable functions $(g_1, -g_2)$ satisfy the Cauchy–Riemann system in an open set of the complex plane identified with the set D of the pairs $(u, p) \in \mathbb{R}^2$:*

$$\partial_u g_1(u,p) = -\partial_p g_2(u,p), \quad \partial_p g_1(u,p) = \partial_u g_2(u,p). \tag{4.47}$$

Let

$$U_D = \{x \in \mathbb{H} : \ \ x = u + \underline{\omega} p, \ (u,p) \in D, \ \underline{\omega} \in \mathbb{S}\}$$

and define the function $\tilde{G} : U_D \subseteq \mathbb{H} \to \mathbb{H}$ by

$$\tilde{G}(x) := g_1(u,p) - \underline{\omega}\, g_2(u,p). \tag{4.48}$$

Then $\tilde{G}(x)$ is slice hyperholomorphic in U_D.

When necessary, we will identify $\mathbb{H}$ with $\mathbb{R}^2 \times \mathbb{S}$ by setting $x \mapsto (x_0, p, \underline{\omega})$, and instead of $\tilde{G}(x)$ we will write $\tilde{G}(x_0, p, \underline{\omega})$ (keeping the symbol $\tilde{G}$ for the function). Starting from the slice hyperholomorphic function $\tilde{G}(u, p, \underline{\omega})$ in (4.48) we can construct a Fueter regular function of plane wave type by the substitution

$$u = \langle \underline{x}, \underline{\omega} \rangle, \quad p = x_0.$$

Suppose that the functions $(g_1, -g_2)$ satisfy the Cauchy–Riemann system and let us define the function

$$G(x_0, \langle \underline{x}, \underline{\omega} \rangle, \underline{\omega}) := g_1(\langle \underline{x}, \underline{\omega} \rangle, x_0) + \underline{\omega}\, g_2(\langle \underline{x}, \underline{\omega} \rangle, x_0), \quad \text{for } \underline{\omega} \in \mathbb{S}. \tag{4.49}$$

We recall a simple result stated in [192]:

Proposition 4.8.8. *The function G defined in (4.49) is left Fueter regular in the variable $x = x_0 + \underline{x}$.*

Definition 4.8.9. A function of the form (4.49) is called a Fueter plane wave function.

Definition 4.8.10 (The W-kernels)**.** Let $S_L^{-1}(s,x)$, $S_R^{-1}(s,x)$ be the Cauchy kernels of left and right slice hyperholomorphic functions, respectively, and let $\underline{\omega} \in \mathbb{S}$. For $\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega} \notin [s]$ we define

$$\begin{aligned} W_{\underline{\omega}}^L(s,x) &:= S_L^{-1}(s, \langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega}) \\ &= -[(\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega})^2 - 2s_0(\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega}) + |s|^2]^{-1}(\langle \underline{x}, \underline{\omega} \rangle - x_0 \underline{\omega} - \overline{s}) \end{aligned}$$

and

$$\begin{aligned}W^R_{\underline{\omega}}(s,x) &:= S^{-1}_R(s, \langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}) \\ &= -(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega} - \bar{s})[(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega})^2 - 2\mathrm{Re}(s)(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}) + |s|^2]^{-1},\end{aligned}$$

where $\underline{\omega} \in \mathbb{S}$ is considered a parameter.

Observe that $W^L_{\underline{\omega}}$ and $W^R_{\underline{\omega}}$ are obtained by the change of variable $x \to \langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}$ in the Cauchy kernels of slice hyperholomorphic functions and $\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}$ is still a paravector.

The following theorem is a direct consequence of the Cauchy formula of slice hyperholomorphic functions.

Theorem 4.8.11. *Let $\underline{\omega} \in \mathbb{S}$ be a parameter and let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. We furthermore assume that $\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega} \in U$. If f is a left slice hyperholomorphic function on a set that contains $\overline{U}$, the integral*

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} W^L_{\underline{\omega}}(s,x)\,ds_j\, f(s), \quad \textit{for every } q \in U, \tag{4.50}$$

depends neither on U nor on the imaginary unit $j \in \mathbb{S}$. If f is a right slice hyperholomorphic function on a set that contains $\overline{U}$, the integral

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\, W^R_{\underline{\omega}}(s,x), \quad \textit{for every } q \in U, \tag{4.51}$$

depends neither on U nor on the imaginary unit $j \in \mathbb{S}$.

Thanks to Theorem 4.8.11 we can define the W-transform, which maps slice hyperholomorphic functions into Fueter regular functions.

Definition 4.8.12 (The W-transforms)**.** Let $\underline{\omega} \in \mathbb{S}$ be a parameter and let $U \subset \mathbb{H}$ be a bounded slice Cauchy domain, let $j \in \mathbb{S}$ and set $ds_j = ds(-j)$. Assume that $\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega} \in U$. If f is a left slice hyperholomorphic function on a set that contains $\overline{U}$, then we define the *left W^L-transform* as

$$\breve{f}_{\underline{\omega}}(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} W^L_{\underline{\omega}}(s,x)\,ds_j\, f(s), \quad \text{for every } q \in U. \tag{4.52}$$

If f is a right slice hyperholomorphic function then we define the *right W^R-transform* as

$$\breve{f}_{\underline{\omega}}(x) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} f(s)\,ds_j\, W^R_{\underline{\omega}}(s,x), \quad \text{for every } q \in U. \tag{4.53}$$

We observe that the W-transform defines a transformation between slice hyperholomorphic functions and Fueter regular functions that depends on a parameter on the unit sphere $\mathbb{S}$. This transform can be extended to the more general case of Clifford-algebra-valued functions.

- For every $\underline{\omega} \in \mathbb{S}$ the function $W^L_{\underline{\omega}}(s,x)$ is right slice hyperholomorphic in s and left Fueter regular in x for every x, s such that $(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}) \notin [s]$. Moreover, the W^L-transform maps left slice hyperholomorphic functions f into left Fueter regular plane wave functions $\breve{f}_{\underline{\omega}}$.
- For every $\underline{\omega} \in \mathbb{S}$ the function $W^R_{\underline{\omega}}(s,x)$ is left slice hyperholomorphic in s and right Fueter regular in x for every x, s such that $(\langle \underline{x}, \underline{\omega}\rangle - x_0\underline{\omega}) \notin [s]$. Moreover, the W^R-transform maps right slice hyperholomorphic functions f into right Fueter regular plane wave functions $\breve{f}_{\underline{\omega}}$.

Theorem 4.8.13. *Let $T = T_0 + e_1T_1 + e_2T_2 + e_3T_3 \in \mathcal{B}(X)$. Assume that $\underline{\omega} \in \mathbb{S}$ and define the operator*

$$A_{\underline{\omega}} := \sum_{j=1}^{3} T_j\omega_j - T_0\underline{\omega}.$$

Then $A_{\underline{\omega}}$ belongs to $\mathcal{B}(X)$, and the operator $A^2_{\underline{\omega}} - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I}$ is invertible for $s \in \mathbb{H}$ with $\|T\| < |s|$ for all $\underline{\omega} \in \mathbb{S}$. Moreover, for $s \in \mathbb{H}$ with $\|T\| < |s|$ and for all $\underline{\omega} \in \mathbb{S}$, we have

$$\sum_{m\geq 0} A^m_{\underline{\omega}} s^{-1-m} = -(A^2_{\underline{\omega}} - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}(A_{\underline{\omega}} - \overline{s}\mathcal{I}), \tag{4.54}$$

$$\sum_{m\geq 0} s^{-1-m} A^m_{\underline{\omega}} = -(A_{\underline{\omega}} - \overline{s}\mathcal{I})(A^2_{\underline{\omega}} - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}. \tag{4.55}$$

The above theorem motivates the notion of W-spectrum.

Definition 4.8.14 (The W-spectrum and the W-resolvent set)**.** Let $T \in \mathcal{B}(X)$ and let $\underline{\omega} \in \mathbb{S}$. We define the operators

$$A_{\underline{\omega}} = \sum_{j=1}^{3} T_j\omega_j - T_0\underline{\omega} \quad \text{and} \quad Q_{\underline{\omega}}(T,s) := A^2_{\underline{\omega}} - 2s_0A_{\underline{\omega}} + |s|^2\mathcal{I}.$$

We define the *W-spectrum* $\sigma_W(T)$ of T as:

$$\sigma_W(T,\underline{\omega}) = \left\{ s \in \mathbb{R}^{n+1} : \ \ Q_{\underline{\omega}}(T,s) \text{ is not invertible in } \mathcal{B}(X) \right\}.$$

The *W-resolvent set* $\rho_W(T)$ is defined by

$$\rho_W(T,\underline{\omega}) = \mathbb{H} \setminus \sigma_W(T,\underline{\omega}).$$

The theorem on the structure of the W-spectrum holds also in this case. Let $T \in \mathcal{B}(X)$, $\underline{\omega} \in \mathbb{S}$, and let $p = p_0 + p_1j \in [p_0 + p_1j] \subset \mathbb{H} \setminus \mathbb{R}$, such that $p \in \sigma_W(T,\underline{\omega})$. Then all the elements of the 2-sphere $[p_0 + p_1j]$ belong to $\sigma_W(T,\underline{\omega})$. Thus the W-spectrum consists of real points and/or 2-spheres. In the case of bounded operators, the W-spectrum, for all $\underline{\omega} \in \mathbb{S}$, is a compact nonempty set.

Definition 4.8.15 (The W-resolvent operators)**.** Let $T \in \mathcal{B}(X)$, let $\underline{\omega} \in \mathbb{S}$, and let $A_{\underline{\omega}} := \sum_{j=1}^{3} T_j \omega_j - T_0 \underline{\omega}$. For $s \in \rho_W(T)$ we define the *left W-resolvent operator* by

$$W_{\underline{\omega}}^L(s,T) = -(A_{\underline{\omega}}^2 - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}(A_{\underline{\omega}} - \overline{s}\mathcal{I}), \tag{4.56}$$

and the *right W-resolvent operator* by

$$W_{\underline{\omega}}^R(s,T) = -(A_{\underline{\omega}} - \overline{s}\mathcal{I})(A_{\underline{\omega}}^2 - 2\mathrm{Re}(s)A_{\underline{\omega}} + |s|^2\mathcal{I})^{-1}. \tag{4.57}$$

Definition 4.8.16 (The W-functional calculus for bounded operators)**.** Let $T \in \mathcal{B}(V)$ and let $\underline{\omega} \in \mathbb{S}$. Let j be an arbitrary imaginary unit and U an arbitrary slice Cauchy domain U as in Remark 3.2.4. For every function $f \in \mathcal{SH}_L(\sigma_W(T,\underline{\omega}))$, we define

$$\breve{f}_{\underline{\omega}}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} W_{\underline{\omega}}^L(s,T)\, ds_j\, f(s). \tag{4.58}$$

For every $f \in \mathcal{SH}_R(\sigma_W(T,\underline{\omega}))$, we define

$$\breve{f}_{\underline{\omega}}(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, W_{\underline{\omega}}^R(s,T), \tag{4.59}$$

with the obvious meaning of the symbols $\mathcal{SH}_L(\sigma_W(T,\underline{\omega}))$ and $\mathcal{SH}_R(\sigma_W(T,\underline{\omega}))$.

The definition of the W-functional calculus is well posed, since the integrals in (4.58) and (4.59) depend neither on the open set U nor on the imaginary unit $j \in \mathbb{S}$.

The W-functional calculus is a functional calculus that is based on slice hyperholomorphic functions, but it produces operators $\breve{f}_{\underline{\omega}}(T)$ for Fueter regular functions $\breve{f}_{\underline{\omega}}(s)$. The W-functional calculus and the F-functional calculus are Fueter functional calculi. In the case of Clifford-algebra-valued functions these two calculi become monogenic functional calculi in the spirit of the monogenic functional calculus introduced and studied by A. McIntosh and his collaborators in a series of papers [160, 161, 166], and in the book [159].

Chapter 5

The S-Functional Calculus for Unbounded Operators

The S-functional calculus can also be defined for unbounded operators. We consider a two-sided quaternionic Banach space X and we introduce a notation for the set of unbounded operators that we consider in this chapter. These results are taken from the papers [67, 97], where we reduce the case of unbounded operators, with suitable transformations, to the case of bounded operators. The direct approach has been studied in the more recent paper [124], while the S-resolvent equation is in [50].

Definition 5.0.1. Let X be a two-sided Banach space. A right linear operator $T : \mathcal{D}(T) \subset X \to X$ is called *closed* if its graph is closed in $X \oplus X$. We denote the set of closed right linear operators $T : \mathcal{D}(T) \subset X \to X$ by $\mathcal{K}(X)$.

When we deal with operators of this type we have to pay attention to the domains on which they are defined. We illustrate this with the following example.

Example 5.0.2. We define powers of $T \in \mathcal{K}(X)$ as usual by $T^0 = \mathcal{I}$ with $\mathcal{D}(T^0) = X$ and $T^{n+1}x := T(T^n x)$ for $x \in \mathcal{D}(T^{n+1}) = \{x \in \mathcal{D}(T^n) : T^n x \in \mathcal{D}(T)\}$. Moreover, we define for every intrinsic polynomial $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ with $a_\ell \in \mathbb{R}$, the operator $P(T)x := \sum_{\ell=0}^{n} a_\ell T^\ell x$ with $\mathcal{D}(P(T)) = \mathcal{D}(T^n)$.

The operator $P(T) : \mathcal{D}(T^n) \subset X \to X$ is then a closed operator. This follows immediately from the corresponding result for $\mathbb{R}$-linear operators, because every quaternionic linear operator is also $\mathbb{R}$-linear, and the topology on X does not depend on whether we consider X a vector space over $\mathbb{H}$ or over $\mathbb{R}$. The situation is, however, fundamentally different if we consider polynomials with quaternionic coefficients.

The operator T is right linear and hence related to the right multiplication, but it does not have any relation to the left multiplication on the space X. If $x, y \in \mathcal{D}(T)$ and $a \in \mathbb{H}$, then $T(xa + y) = T(x)a + T(y)$ due to the right linearity

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of T, so that $xa + y \in \mathcal{D}(T)$ and $\mathcal{D}(T)$ is in turn a right linear subspace of X. However, since in general $T(ax) \neq aT(x)$, it is not clear that $ax \in \mathcal{D}(T)$ for any $a \in \mathbb{H}$ and any $x \in \mathcal{D}(T)$, so that $\mathcal{D}(T)$ is not a left linear and in particular not a two-sided subspace of X. The same holds obviously also true for the domains $\mathcal{D}(T^n)$ of powers T^n of T, in general, $\mathcal{D}(T^n)$ is a right linear, but not a left linear, subspace of V.

We can now define, for every right slice hyperholomorphic polynomial $P(q) = \sum_{\ell=0}^{n} a_\ell q^\ell$ with $a_\ell \in \mathbb{H}$, the operator $P(T)x := \sum_{\ell=0}^{n} a_\ell T^\ell x$, and we find that the domain of this operator is again $\mathcal{D}(P(T)) = \mathcal{D}(T^n)$. However, for a left slice hyperholomorphic polynomial $P(q) = \sum_{\ell=0}^{n} q^\ell a_\ell$ with $a_\ell \in \mathbb{H}$, setting $P(T)x := \sum_{\ell=0}^{n} T^\ell a_\ell x$ might not be possible in a straightforward manner. Since the domain $\mathcal{D}(T^n)$ is in general not a left linear subspace of X, we do not necessarily have $a_\ell x \in \mathcal{D}(T^n)$ for $x \in \mathcal{D}(T^n)$. In this case, the expression $P(T)x = \sum_{\ell=0}^{n} T^\ell a_\ell x$ is meaningless, so that the domain of the operator $P(T) = \sum_{\ell=0}^{n} T^\ell a_\ell$ is in general not the entire subspace $\mathcal{D}(T^n)$.

5.1 The S-Spectrum and the S-Resolvent Operators

As for bounded operators, we define for $T \in \mathcal{K}(X)$ and $s \in \mathbb{H}$ the operator

$$\mathcal{Q}_s(T) := T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I}$$

that maps $\mathcal{D}(T^2) \subset X$ to X.

Definition 5.1.1. Let $T \in \mathcal{L}(X)$. We define the S-*resolvent set* of T as

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \ \ \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(X) \right\}$$

and the S-*spectrum* $\sigma_S(T)$ of T as

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$

The following result in particular implies that $\mathcal{Q}_s(T)$ is closed for every $s \in \mathbb{H}$, whenever $\rho_S(T) \cap \mathbb{R} \neq \emptyset$.

Theorem 5.1.2. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. For every intrinsic polynomial $P \in \mathcal{N}(\mathbb{H})$, the operator $P(T)$ is closed.*

Proof. We choose any $\alpha \in \rho_S(T) \cap \mathbb{R}$ and we consider the homomorphism $\Phi_\alpha : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ defined by

$$\mu = \Phi_\alpha(q) = (q - \alpha)^{-1}, \quad \Phi_\alpha(\infty) = 0, \quad \Phi_\alpha(\alpha) = \infty.$$

Since $P \in \mathcal{N}(\mathbb{H})$ and α is a real number, P can be written as

$$P(q) = \sum_{\ell=0}^{n} b_\ell (q-\alpha)^{n-\ell} = (q-\alpha)^n \sum_{\ell=0}^{n} b_\ell (q-\alpha)^{-\ell},$$

where the b_ℓ are real numbers. The homomorphism Φ_α maps P to $\mu^{-n}R(\mu)$, that is, $P(q) = \mu^{-n}R(\mu)$ with $\mu = \Phi_\alpha(q)$, where $R(\mu) = \sum_{\ell=0}^{n} b_\ell \mu^\ell$.

We define now $A := (T - \alpha\mathcal{I})^{-1}$, which formally corresponds to $\Phi_\alpha(T)$. The operator A is a one-to-one map from X onto $\mathcal{D}(T)$, and hence the operator A^ℓ maps $\mathcal{D}(T^n)$ onto $\mathcal{D}(T^{n+\ell}) \subset \mathcal{D}(T^n)$ for every $\ell \in \mathbb{N}$. For $x \in \mathcal{D}(T^n)$, we thus have $R(A)x \in \mathcal{D}(T^n)$. It is, moreover, easy to see that

$$P(T)x = (T - \alpha\mathcal{I})^n R(A)x = R(A)(T - \alpha\mathcal{I})^n x, \tag{5.1}$$

for all $x \in \mathcal{D}(T^n)$.

Consider now a sequence $(x_k)_{k\in\mathbb{N}}$ in $\mathcal{D}(T^n)$ such that we have $x_k \to x$ and $P(T)x_k \to y$ in X. In order to show that $P(T)$ is closed, we must show that $x \in \mathcal{D}(T^n)$ and that $P(T)x = y$. If we set $z_k := R(A)x_k$, then we have

$$\lim_{k\to+\infty} (T - \alpha\mathcal{I})^n z_k = \lim_{k\to+\infty} (T - \alpha\mathcal{I})^n R(A)x_k = \lim_{k\to+\infty} P(T)x_k = y$$

due to (5.1). Since $R(A)$ is bounded because A is bounded, the limit

$$\lim_{k\to+\infty} z_k = \lim_{k\to+\infty} R(A)x_k = R(A)x := z \in X$$

also exists. The operator $(T - \alpha\mathcal{I})^n$ is, however, closed because it has a bounded inverse, namely A^n, and so we conclude that $z = R(A)x$ belongs to $\mathcal{D}((T-\alpha\mathcal{I})^n) = \mathcal{D}(T^n)$ and that

$$y = (T - \alpha\mathcal{I})^n z = (T - \alpha\mathcal{I})^n R(A)x. \tag{5.2}$$

What remains to show is that $x \in \mathcal{D}(P(T)) = \mathcal{D}(T^n)$, because in this case, (5.1) and (5.2) imply $y = P(T)x$ and so the closedness of $P(T)$.

If we write $R(A)$ explicitly, we obtain

$$R(A)x = b_0 x + \sum_{\ell=1}^{n} b_\ell A^\ell x \tag{5.3}$$

with $b_0 = a_n \neq 0$. We already know that $R(A)x$ belongs to $\mathcal{D}(T)$. Moreover, also $\sum_{\ell=1}^{n} b_\ell A^\ell x = A \sum_{\ell=1}^{n} b_\ell A^{\ell-1} x$ belongs to $\mathcal{D}(T)$, since A maps X onto $\mathcal{D}(T)$. We conclude from (5.3) that also $x \in \mathcal{D}(T)$ because it is a linear combination of vectors in $\mathcal{D}(T)$. Even more, if we assume that $x \in \mathcal{D}(T^k)$ with $1 \leq k < n$, then

$$(T - \alpha\mathcal{I})^k R(A)x = b_0(T - \alpha\mathcal{I})^k + \sum_{\ell=1}^{k} b_\ell (T - \alpha\mathcal{I})^{k-\ell} + \sum_{\ell=k+1}^{n} b_\ell A^{\ell-k} x.$$

As before, we see that $(T - \alpha\mathcal{I})^k$ is a linear combination of vectors in $\mathcal{D}(T)$, and hence $(T - \alpha\mathcal{I})^k x \in \mathcal{D}(T)$ and so $x \in \mathcal{D}(T^{k+1})$. By induction, we find that $x \in \mathcal{D}(T^n) = \mathcal{D}(P(T))$. □

For closed operators, the definition of the S-resolvent operators needs a little modification. If we define the left S-resolvent operator as in the case of bounded operators, we obtain

$$S_L^{-1}(s,T)x := -\mathcal{Q}_s(T)^{-1}(T-\overline{s}\mathcal{I})x, \tag{5.4}$$

which is defined only for $x \in \mathcal{D}(T)$ and not on all of X. However, for $x \in \mathcal{D}(T)$, we have $\mathcal{Q}_s(T)^{-1}Tx = T\mathcal{Q}_s(T)^{-1}x$, and so we can commute T and $\mathcal{Q}_s(T)^{-1}$ in order to obtain an operator that is defined on all of X.

Definition 5.1.3 (The S-resolvent operators of a closed operator)**.** Let $T \in \mathcal{K}(X)$. For $s \in \rho_S(T)$, we define the *left S-resolvent operator of T* at s as

$$S_L^{-1}(s,T)x := \mathcal{Q}_s(T)^{-1}\overline{s}x - T\mathcal{Q}_s(T)^{-1}x, \quad \text{for all } x \in X, \tag{5.5}$$

and the *right S-resolvent operator of T* at s as

$$S_R^{-1}(s,T)x := -(T-\mathcal{I}\overline{s})\mathcal{Q}_s(T)^{-1}x, \quad \text{for all } x \in X. \tag{5.6}$$

Remark 5.1.4. For $s \in \rho_S(T)$, the operator $\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T^2)$. Hence $T\mathcal{Q}_s(T)^{-1}$ is a bounded operator and $S_L^{-1}(S,T)$, and so $S_R^{-1}(s,T)$ are bounded, too.

A second difference between the left and right S-resolvent operators is that the right S-resolvent equation holds only on $\mathcal{D}(T)$.

Theorem 5.1.5 (The S-resolvent equations)**.** *Let $T \in \mathcal{K}(X)$. For $s \in \rho_S(T)$, the left S-resolvent operator satisfies the identity*

$$S_L^{-1}(s,T)sx - TS_L^{-1}(s,T)x = x, \quad \textit{for all } x \in X. \tag{5.7}$$

Moreover, the right S-resolvent operator satisfies the identity

$$sS_R^{-1}(s,T)x - S_R^{-1}(s,T)Tx = x, \quad \textit{for all } \ x \in \mathcal{D}(T). \tag{5.8}$$

Proof. We have for $x \in \mathcal{D}(T)$ that

$$\begin{aligned}
&sS_R^{-1}(s,T)x - S_R^{-1}(s,T)Tx \\
&= -s(T-\mathcal{I}\overline{s})\mathcal{Q}_s(T)^{-1}x + (T-\mathcal{I}\overline{s})\mathcal{Q}_s(T)^{-1}Tx \\
&= (-sT+|s|^2\mathcal{I})\mathcal{Q}_s(T)^{-1}x + (T^2-\overline{s}T)\mathcal{Q}_s(T)^{-1}x \\
&= (T^2-2\mathrm{Re}(s)T+|s|^2\mathcal{I})\mathcal{Q}_s(T)^{-1}x = x.
\end{aligned}$$

Similar computations establish (5.7). □

Remark 5.1.6. We can extend (5.8) to an equation that holds on the entire space X, similarly to how we could extend (5.4) to a bounded operator on the entire space X. This equation is

$$sS_R^{-1}(s,T)x + (T^2-\overline{s}T)\mathcal{Q}_s(T)^{-1}x = x, \quad \text{for all } x \in X.$$

Theorem 5.1.7 (S-resolvent equation). *Let $T \in \mathcal{K}(X)$. If $s, q \in \rho_S(T)$ with $s \notin [q]$, then*

$$S_R^{-1}(s,T)S_L^{-1}(q,T) = \big[[S_R^{-1}(s,T) - S_L^{-1}(q,T)]q - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(q,T)]\big](q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}. \quad (5.9)$$

Proof. As in the case of bounded operators, the S-resolvent equation is deduced from the left and right S-resolvent equations. However, we have to pay attention to being consistent with the domains of definition of every operator that appears in the proof.

We show that for every $x \in X$, one has

$$S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2)x = [S_R^{-1}(s,T) - S_L^{-1}(q,T)]qx - \overline{s}[S_R^{-1}(s,T) - S_L^{-1}(q,T)]x. \quad (5.10)$$

We then obtain (5.9) by replacing x by $(q^2 - 2s_0q + |s|^2)^{-1}x$. For $w \in X$, the left S-resolvent equation (5.7) implies

$$S_R^{-1}(s,T)S_L^{-1}(q,T)qw = S_R^{-1}(s,T)TS_L^{-1}(q,T)w + S_R^{-1}(s,T)w.$$

The pseudo-resolvent $\mathcal{Q}_s(T)^{-1}$ maps X onto $\mathcal{D}(T^2)$. Therefore, the left S-resolvent operator $S_L^{-1}(s,T) = \mathcal{Q}_s(T)^{-1}\overline{s} - T\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T)$, and so $S_L^{-1}(q,T)w \in \mathcal{D}(T)$. The right S-resolvent equation (5.8) yields

$$S_R^{-1}(s,T)S_L^{-1}(q,T)qw = sS_R^{-1}(s,T)S_L^{-1}(q,T)w - S_L^{-1}(q,T)w + S_R^{-1}(s,T)w. \quad (5.11)$$

If we apply this identity with $w = qx$, we get

$$\begin{aligned}
&S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2)x \\
&= S_R^{-1}(s,T)S_L^{-1}(q,T)q^2x - 2s_0S_R^{-1}(s,T)S_L^{-1}(q,T)qx \\
&\quad + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T)x \\
&= sS_R^{-1}(s,T)S_L^{-1}(q,T)qx - S_L^{-1}(q,T)qx + S_R^{-1}(s,T)qx \\
&\quad - 2s_0S_R^{-1}(s,T)S_L^{-1}(q,T)qx + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T)x.
\end{aligned}$$

Applying identity (5.11) again with $w = x$ gives

$$\begin{aligned}
&S_R^{-1}(s,T)S_L^{-1}(q,T)(q^2 - 2s_0q + |s|^2)x \\
&= s^2S_R^{-1}(s,T)S_L^{-1}(q,T)x - sS_L^{-1}(q,T)x + sS_R^{-1}(s,T)x \\
&\quad - S_L^{-1}(q,T)qx + S_R^{-1}(s,T)qx \\
&\quad - 2s_0sS_R^{-1}(s,T)S_L^{-1}(q,T)x + 2s_0S_L^{-1}(q,T)x - 2s_0S_R^{-1}(s,T)x \\
&\quad + |s|^2S_R^{-1}(s,T)S_L^{-1}(q,T)x \\
&= (s^2 - 2s_0s + |s|^2)S_R^{-1}(s,T)S_L^{-1}(q,T)x \\
&\quad - (2s_0 - s)[S_R^{-1}(s,T)x - S_L^{-1}(q,T)x] \\
&\quad + [S_R^{-1}(s,T) - S_L^{-1}(q,T)]qx.
\end{aligned}$$

The identity $2s_0 = s + \overline{s}$ implies $s^2 - 2s_0 s + |s|^2 = 0$ and $2s_0 - s = \overline{s}$, and hence we obtain the desired equation (5.10). □

5.2 Definition of the S-Functional Calculus

Before we define the S-functional calculus for closed operators, we have to define a notion of spectrum that takes the possible unboundedness of T into account.

Definition 5.2.1. Let $T \in \mathcal{K}(X)$. We define the *extended S-spectrum* of T as

$$\overline{\sigma}_S(T) := \begin{cases} \sigma_S(T) & \text{if } T \text{ is bounded.} \\ \sigma_S(T) \bigcup \{\infty\} & \text{if } T \text{ is unbounded.} \end{cases}$$

Remark 5.2.2. We recall that a function is said to be left slice hyperholomorphic at infinity if $\mathbb{H} \setminus B_r(0) \subset \mathcal{D}(f)$ for some $r > 0$ and the limit $f(\infty) := \lim_{q\to\infty} f(q)$ exists.

Hence if $T \in \mathcal{K}(X)$ is unbounded, then $f \in \mathcal{SH}_L(\overline{\sigma}_S(T))$ if and only if f is left slice hyperholomorphic with $\sigma_S(T) \subset \mathcal{D}(f)$ and if furthermore, $\mathbb{H} \setminus B_r(0) \subset \mathcal{D}(f)$ for some $r > 0$ and $f(\infty) = \lim_{q\to\infty} f(q)$ exists. The characterization of functions in $\mathcal{SH}_R(\overline{\sigma}_S(T))$ and $\mathcal{N}(\overline{\sigma}_S(T))$ is in this case, of course, similar.

Theorem 5.2.3. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. For $\alpha \in \rho_S(T) \cap \mathbb{R}$, we define the function $\Phi_\alpha : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ given by*

$$\Phi_\alpha(s) = (s - \alpha)^{-1}, \quad \Phi_\alpha(\alpha) = \infty, \quad \Phi_\alpha(\infty) = 0, \tag{5.12}$$

and set $A := (T - \alpha\mathcal{I})^{-1} = -S_L^{-1}(\alpha, T)$, which formally corresponds to $\Phi_\alpha(T)$. Then

$$\sigma_S(A) = \overline{\sigma}_S(A) = \Phi_\alpha(\overline{\sigma}_S(T)). \tag{5.13}$$

For $s \in \rho_S(T)$ and $q = \Phi_\alpha(s)$, we moreover have

$$S_L^{-1}(s, T) = q\mathcal{I} - S_L^{-1}(q, A)q^2 \tag{5.14}$$

and

$$S_R^{-1}(s, T) = q\mathcal{I} - q^2 S_R^{-1}(q, A). \tag{5.15}$$

Proof. Let s, $q \in \mathbb{H}$ and $\alpha \in \mathbb{R}$ be such that $q = (s - \alpha)^{-1}$. Then the identities

$$\operatorname{Re}(s)|q|^2 = \alpha|q|^2 + \operatorname{Re}(q), \tag{5.16}$$

$$|q|^2|s|^2 = \alpha^2|q|^2 + 2\operatorname{Re}(q)\alpha + 1, \tag{5.17}$$

$$-q^{-2} = (2\,(\alpha - \operatorname{Re}(s))\,\overline{q} + 1)\,|q|^{-2}, \tag{5.18}$$

$$-\overline{s}q^{-2} = \left(\left(\alpha^2 - |s|^2\right)\overline{q} + \alpha\right)|q|^{-2}, \tag{5.19}$$

can be verified by direct calculations.

We choose $\alpha \in \rho_S(T) \cap \mathbb{R}$ and set $A := (T - \alpha\mathcal{I})^{-1}$. Assume now that $q \in \rho_S(A) \setminus \{0\}$, that is,

$$\mathcal{Q}_q(A)^{-1} = (A^2 - 2\mathrm{Re}(q)A + |q|^2\mathcal{I})^{-1} \in \mathcal{B}(X).$$

Then

$$\begin{aligned}\mathcal{Q}_q(A)^{-1} &= \left[(T-\alpha\mathcal{I})^{-2} - 2\mathrm{Re}(q)(T-\alpha\mathcal{I})^{-1} + |q|^2\mathcal{I}\right]^{-1}\\ &= \left[\left[\mathcal{I} - 2\mathrm{Re}(q)(T-\alpha\mathcal{I}) + |q|^2(T-\alpha\mathcal{I})^2\right](T-\alpha\mathcal{I})^{-2}\right]^{-1}.\end{aligned}$$

For $x \in \mathcal{D}(T^2)$, we have, because of (5.16) and (5.17), that

$$\begin{aligned}&\left[\mathcal{I} - 2\mathrm{Re}(q)(T-\alpha\mathcal{I}) + |q|^2(T-\alpha\mathcal{I})^2\right]x\\ &= |q|^2T^2x - 2(\alpha|q|^2 + \mathrm{Re}(q))Tx + \left(|q|^2\alpha^2 + 2\mathrm{Re}(q)\alpha\mathcal{I} + 1\right)x\\ &= |q|^2T^2x - 2|q|^2\mathrm{Re}(s)Tx + |q|^2|s|^2x\\ &= |q|^2(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})x = |q|^2\mathcal{Q}_s(T)x.\end{aligned}$$

Since $(T-\alpha\mathcal{I})^{-2}$ maps V to $\mathcal{D}(T^2)$, we obtain

$$\mathcal{Q}_q(A)^{-1} = \left[|q|^2\mathcal{Q}_s(T)(T-\alpha\mathcal{I})^{-2}\right]^{-1} = |q|^{-2}(T-\alpha\mathcal{I})^2\mathcal{Q}_s(T)^{-1}, \tag{5.20}$$

and applying $A = (T-\alpha\mathcal{I})^{-2}$ from the right, we conclude that

$$\mathcal{Q}_s(T)^{-1} = |q|^2A^2\mathcal{Q}_s(A)^{-1} \in \mathcal{B}(X).$$

Hence $s \in \rho_S(T)$. If, on the other hand, $s \in \rho_S(T)$, then we can perform the above computations in the inverse order to see that also in this case (5.20) holds. Since $\mathcal{Q}_s(T)^{-1}$ maps X to $\mathcal{D}(T^2) = \mathcal{D}\left((T-\alpha\mathcal{I})^2\right)$ and $(T-\alpha\mathcal{I})^2$ is closed, we find that $\mathcal{Q}_q(A)^{-1}$ is bounded and hence $q \in \rho_S(A)$. We even have $q \in \rho_S(A) \setminus \{0\}$, since $s \in \mathbb{H}$, and so $q \neq 0 = \Phi_\alpha(\infty)$.

Altogether, we have $\Phi_\alpha(\rho_S(T)) = \rho_S(A) \setminus \{0\}$ and in turn

$$\Phi_\alpha(\sigma_S(T) \cup \{\infty\}) = \sigma_S(A) \cup \{0\}.$$

Finally, $0 \in \sigma_S(T)$ if and only if $A^{-1} = T - \alpha\mathcal{I}$, and hence also T is unbounded, which is equivalent to $\infty \in \overline{\sigma}_S(T)$. Therefore (5.13) holds.

In order to prove (5.14), we recall (5.20) and obtain

$$S_L^{-1}(q,A) = |q|^{-2}(T-\alpha\mathcal{I})^2\mathcal{Q}_s(T)^{-1}\overline{q} - |q|^{-2}(T-\alpha\mathcal{I})\mathcal{Q}_s(T)^{-1}.$$

Now observe that

$$\begin{aligned}(T-\alpha\mathcal{I})^2\mathcal{Q}_s(T)^{-1} &= (T^2 - 2\alpha T + \alpha^2\mathcal{I})\mathcal{Q}_s(T)^{-1}\\ &= (T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})\mathcal{Q}_s(T)^{-1}\\ &\quad + \left(-2(\alpha - \mathrm{Re}(s))T + (\alpha^2 - |s|^2)\mathcal{I}\right)\mathcal{Q}_s(T)^{-1}\\ &= \mathcal{I} + \left(-2(\alpha - \mathrm{Re}(s))T + (\alpha^2 - |s|^2)\mathcal{I}\right)\mathcal{Q}_s(T)^{-1},\end{aligned} \tag{5.21}$$

and so

$$\begin{aligned}S_L^{-1}(q,A) &= |q|^{-2}\mathcal{I}\overline{q} + |q|^{-2}\left(-2(\alpha - s_0)T + (\alpha^2 - |s|^2)\mathcal{I}\right)\mathcal{Q}_s(T)^{-1}\overline{q}\\ &\quad - |q|^{-2}(T-\alpha)Q_s(T)^{-1}\\ &= \mathcal{I}q^{-1} - T\mathcal{Q}_s(T)^{-1}(2(\alpha - \mathrm{Re}(s))\overline{q} + 1)|q|^{-2}\\ &\quad + \mathcal{Q}_s(T)^{-1}((\alpha^2 - |s|^2)\overline{q} + \alpha)|q|^{-2}.\end{aligned}$$

From (5.18) and (5.19), we finally conclude that

$$S_L^{-1}(q,A) = \mathcal{I}q^{-1} - \mathcal{Q}_s(T)^{-1}\overline{s}q^{-2} + T\mathcal{Q}_s(T)^{-1}q^{-2}$$

and thus

$$S_L^{-1}(q,A) = \mathcal{I}q^{-1} - S_L^{-1}(s,T)q^{-2}. \tag{5.22}$$

Since $s \in \rho_S(T)$, we have $q = \Phi(s) \neq 0$, and so (5.22) is equivalent to (5.14).

It remains to prove the relation (5.15). Using (5.20), we have for $q = \Phi_\alpha(s)$ with $s \in \rho_S(T)$ that

$$\begin{aligned}S_R^{-1}(q,A) &= (\overline{q}\mathcal{I} - A)\mathcal{Q}_q(A)^{-1}\\ &= (\overline{q}\mathcal{I} - A)|q|^{-2}(T - \alpha\mathcal{I})^2\mathcal{Q}_s(T)^{-1}\\ &= \overline{q}|q|^{-2}(T-\alpha\mathcal{I})^2\mathcal{Q}_s(T)^{-1} - |q|^{-2}(T-\alpha\mathcal{I})\mathcal{Q}_s(T)^{-1}.\end{aligned}$$

Applying the identity (5.21), we find, similar to the case of the left S-resolvent, that

$$\begin{aligned}S_R^{-1}(q,A) &= \overline{q}|q|^{-2}\mathcal{I} - |q|^{-2}\left(2\overline{q}(\alpha - \mathrm{Re}(s)) + 1\right)T\mathcal{Q}_s(T)^{-1}\\ &\quad + |q|^{-2}\left(-\overline{q}(\alpha^2 - |s|^2) + \alpha\right)\mathcal{Q}_s(T)^{-1}.\end{aligned}$$

Applying again (5.18) and (5.19), we obtain

$$S_R^{-1}(q,A) = q^{-1}\mathcal{I} + q^{-2}T\mathcal{Q}_s(T)^{-1} - q^{-2}\overline{s}\mathcal{Q}_s(T)^{-1}$$

and in turn

$$S_R^{-1}(q,A) = q^{-1}\mathcal{I} - q^{-2}S_R^{-1}(s,T),$$

which is equivalent to (5.15), since $q = \Phi(s) \neq 0$. □

Corollary 5.2.4. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. For $\alpha \in \rho_S(T) \cap \mathbb{R}$, let Φ_α and A be as in Theorem 5.2.3. The mapping $f \mapsto f \circ \Phi_\alpha^{-1}$ determines one-to-one correspondences between $\mathcal{SH}_L(\overline{\sigma}_S(T))$ and $\mathcal{SH}_L(\sigma_S(A))$, between $\mathcal{SH}_R(\overline{\sigma}_S(T))$ and $\mathcal{SH}_R(\sigma_S(A))$, and between $\mathcal{N}(\overline{\sigma}_S(T))$ and $\mathcal{N}(\sigma_S(A))$. Precisely, we have*

$$\begin{aligned}\mathcal{SH}_L(\sigma_S(A)) &= \left\{f \circ \Phi_\alpha^{-1} : f \in \mathcal{SH}_L(\overline{\sigma}_S(T))\right\},\\ \mathcal{SH}_R(\sigma_S(A)) &= \left\{f \circ \Phi_\alpha^{-1} : f \in \mathcal{SH}_R(\overline{\sigma}_S(T))\right\},\\ \mathcal{N}(\sigma_S(A)) &= \left\{f \circ \Phi_\alpha^{-1} : f \in \mathcal{N}(\overline{\sigma}_S(T))\right\}.\end{aligned}$$

Proof. The above relations are immediate consequences of (5.13) in Theorem 5.2.3. □

Definition 5.2.5 (The S-functional calculus for closed operators)**.** Let $T \in \mathcal{K}(X)$ with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. We choose an arbitrary $\alpha \in \rho_S(T) \cap \mathbb{R}$ and we define, as in Theorem 5.2.3, the function $\Phi_\alpha : \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ by

$$\Phi_\alpha(s) = (s-\alpha)^{-1}, \quad \Phi_\alpha(\alpha) = \infty, \quad \Phi_\alpha(\infty) = 0,$$

and the operator $A := (T - \alpha\mathcal{I})^{-1} = -S_L^{-1}(\alpha, T) \in \mathcal{B}(X)$, which formally corresponds to $\Phi_\alpha(T)$. For every function $f \in \mathcal{SH}_L(\overline{\sigma}_S(T))$ and every function $f \in \mathcal{SH}_R(\overline{\sigma}_S(T))$, we define

$$f(T) := f \circ \Phi_\alpha^{-1}(A), \tag{5.23}$$

where $f \circ \Phi_\alpha^{-1}(A)$ is intended in the sense of the S-functional calculus for bounded quaternionic linear operators in Definition 3.2.5.

Remark 5.2.6. By Theorem 4.2.1 and Theorem 4.2.4, the above approach is consistent with the S-functional calculus for bounded operators.

The S-functional calculus for closed operators admits also for unbounded operators an integral representation that corresponds to the integrals in (3.12) and (3.13) for bounded operators.

Theorem 5.2.7. *Let $T \in \mathcal{K}(X)$ be unbounded and let $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. Assume that $f \in \mathcal{SH}_L(\overline{\sigma}_S(T))$ and $f(T)$ is the operator defined in Definition 5.2.5. Then*

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} S_L^{-1}(s,T)\, ds_j\, f(s) \tag{5.24}$$

for every unbounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and every imaginary unit $j \in \mathbb{S}$. Similarly, if $f \in \mathcal{SH}_R(\overline{\sigma}_S(T))$ and $f(T)$ is the operator defined in Definition 5.2.5, then

$$f(T) = f(\infty)\mathcal{I} + \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, S_R^{-1}(s,T), \tag{5.25}$$

for every unbounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and every imaginary unit $j \in \mathbb{S}$.

Proof. Let $f \in \mathcal{SH}_L(\overline{\sigma}_S(T))$, let $\alpha \in \rho_S(T) \cap \mathbb{R}$, and set $A = (T - \alpha\mathcal{I})^{-1}$ and $f(T) := f \circ \Phi_\alpha^{-1}(A)$ as in Definiton 5.2.5. Furthermore, let $j \in \mathbb{S}$ and let U be an unbounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$. We can assume that $\alpha \notin U$. Otherwise, Cauchy's integral theorem allows us to replace U by $U' = U \setminus B_\varepsilon(\alpha)$ with sufficiently small $\varepsilon > 0$ without changing the value of the integral in (5.24).

The set $V := \Phi_\alpha(U)$ is a bounded slice Cauchy domain that contains $\sigma_S(A)$ by Theorem 5.2.3. Thus, after the change of variables $q = \Phi_\alpha(s)$ in the integral in (5.24), we find due to the relation (5.14) that

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,T)\,ds_j\,f(s)\\
&= -\frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_j)} \left(q\mathcal{I} - S_L^{-1}(q,A)q^2\right) q^{-2}\,dq_j\,\left(f\circ\Phi_\alpha^{-1}\right)(q)\\
&= -\frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_j)} q^{-1}\,dq_j\,\left(f\circ\Phi_\alpha^{-1}\right)(q)\\
&\quad + \frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_j)} S_L^{-1}(q,A)\,dq_j\,\left(f\circ\Phi_\alpha^{-1}\right)(q)\\
&= -\left(f\circ\Phi_\alpha^{-1}\right)(0)\mathcal{I} + \left(f\circ\Phi_\alpha^{-1}\right)(A),
\end{aligned}$$

where the last identity follows from Cauchy's integral formula because

$$\begin{aligned}
&-\frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_j)} q^{-1}\,dq_j\,\left(f\circ\Phi_\alpha^{-1}\right)(q)\\
&= -\frac{1}{2\pi}\int_{\partial(V\cap\mathbb{C}_j)} S_L^{-1}(q,0)\,dq_j\,\left(f\circ\Phi_\alpha^{-1}\right)(q) = \left(f\circ\Phi_\alpha^{-1}\right)(0),
\end{aligned}$$

since $0 \in \sigma_S(A) \subset V$. Since $f\circ\Phi_\alpha^{-1}(A) = f(T)$ and $f\circ\Phi_\alpha^{-1}(0) = f(\infty)$, we obtain

$$\frac{1}{2\pi}\int_{\partial(W\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j f(s) = -\mathcal{I}f(\infty) + f(T),$$

which is exactly (5.24). The right slice hyperholomorphic case can be shown by similar computations using the identity (5.15). □

Corollary 5.2.8. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T)\cap\mathbb{R} \neq \emptyset$. For every function $f \in \mathcal{SH}_L(\overline{\sigma}_S(T))$ and every function $f \in \mathcal{SH}_R(\overline{\sigma}_S(T))$, the operator $f(T)$ defined in (5.23) does not depend on the choice of $\alpha \in \rho_S(T)\cap\mathbb{R}$.*

Proof. The fact that the operator $f(T)$ defined in (5.23) is independent of $\alpha \in \rho_S(T)\cap\mathbb{R}$ follows from the validity of formulas (5.24) and (5.25), since these integrals are independent of α. □

We conclude this chapter with the algebraic properties of the S-functional calculus. These are immediate consequences of the respective properties of the S-functional calculus for bounded operators.

Theorem 5.2.9. *Let $T \in \mathcal{K}(X)$ with $\rho_S(T)\cap\mathbb{R} \neq \emptyset$.*

(i) *If $f, g \in \mathcal{SH}_L(\overline{\sigma}_S(T))$ and $a \in \mathbb{H}$, then*

$$(fa+g)(T) = f(T)a + g(T).$$

Similarly, if $f, g \in \mathcal{SH}_R(\overline{\sigma}_S(T))$ *and* $a \in \mathbb{H}$, *then*

$$(af + g)(T) = af(T) + g(T).$$

(ii) *If* $f \in \mathcal{N}(\overline{\sigma}_S(T))$ *and* $g \in \mathcal{SH}_L(\overline{\sigma}_S(T))$ *or* $f \in \mathcal{SH}_R(\overline{\sigma}_S(T))$ *and* $g \in \mathcal{N}(\overline{\sigma}_S(T))$, *then*

$$(fg)(T) = f(T)g(T).$$

Proof. Let $\alpha \in \rho_S(T) \cap \mathbb{R}$, set $A := (T - \alpha\mathcal{I})^{-1}$, and define Φ_α as in (5.12). If $f, g \in \mathcal{SH}_L(\overline{\sigma}_S(T))$ and $a \in \mathbb{H}$, then we conclude from Lemma 4.1.1 that

$$\begin{aligned}(fa + g)(T) &= (fa + g) \circ \Phi_\alpha^{-1}(A) = \left((f \circ \Phi_\alpha^{-1})a + g \circ \Phi_\alpha^{-1}\right)(A) \\ &= f \circ \Phi_\alpha^{-1}(A)a + g \circ \Phi_\alpha^{-1}(A) = f(T)a + g(T).\end{aligned}$$

Similarly, if $f \in \mathcal{N}(\overline{\sigma}_S(T))$ and $g \in \mathcal{SH}_L(\overline{\sigma}_S(T))$, then $f \circ \Phi_\alpha^{-1} \in \mathcal{N}(\sigma_S(A))$ by Corollary 5.2.4, and we conclude from Theorem 4.1.3 that

$$\begin{aligned}(fg)(T) &= (fg) \circ \Phi_\alpha^{-1}(A) = \left(\left(f \circ \Phi_\alpha^{-1}\right)\left(g \circ \Phi_\alpha^{-1}\right)\right)(A) \\ &= \left(f \circ \Phi_\alpha^{-1}\right)(A)\left(g \circ \Phi_\alpha^{-1}\right)(A) = f(T)g(T).\end{aligned}$$

The statements for right slice hyperholomorphic functions follow by analogous arguments. □

Theorem 5.2.10 (Spectral mapping and product rule)**.** *Let* $T \in \mathcal{K}(X)$ *with* $\rho_S(T) \cap \mathbb{R} \neq \emptyset$. *If* $f \in \mathcal{N}(\overline{\sigma}_S(T))$, *then*

$$\sigma_S(f(T)) = f(\overline{\sigma}_S(T)),$$

and for every $g \in \mathcal{SH}(f(\sigma_S(T)))$ *and* $g \in \mathcal{SH}_R(f(\sigma_S(T)))$, *we have*

$$(g \circ f)(T) = g(f(T)).$$

Proof. Let $\alpha \in \rho_S(T) \cap \mathbb{R}$, set $A := (T - \alpha\mathcal{I})^{-1}$, and define Φ_α as in (5.12). Since $f \in \mathcal{N}(\overline{\sigma}_S(T))$, Corollary 5.2.4 implies $f \circ \Phi_\alpha^{-1} \in \mathcal{N}(\overline{\sigma}_S(A))$. From (5.13) and Theorem 4.2.1, we thus conclude that

$$\begin{aligned}f(\overline{\sigma}_S(T)) &= f \circ \Phi_\alpha^{-1} \circ \Phi_\alpha(\overline{\sigma}_S(T)) \\ &= f \circ \Phi_\alpha^{-1}(\sigma_S(A)) = \sigma_S(f \circ \Phi_\alpha(A)) = \sigma_S(f(T)).\end{aligned}$$

Since $\sigma_S(f(T)) = \sigma_S(f \circ \Phi_\alpha^{-1}(A))$, we moreover have for $g \in \mathcal{SH}_L(f(\sigma_S(T)))$ or $g \in \mathcal{SH}_R(f(\sigma_S(T))$ by Theorem 4.2.4 that

$$g \circ f(T) = g \circ f \circ \Phi_\alpha^{-1}(A) = g(f \circ \Phi_\alpha^{-1}(A)) = g(f(T)). \qquad \square$$

5.3 Comments and Remarks

The gradient operator defined on most common function spaces is a closed operator. The S-spectrum is associated with the spectrum of the operator $s^2\mathcal{I} + \Delta$.

Example 5.3.1 (The gradient operator)**.** We consider the operator

$$T = \partial_{x_1} e_1 + \partial_{x_2} e_2 + \partial_{x_3} e_3$$

on a suitable Banach space X, and we determine the operator associated with the S-spectrum of T. We have

$$\overline{T} = -\partial_{x_1} e_1 - \partial_{x_2} e_2 - \partial_{x_3} e_3.$$

Thus $T + \overline{T} = 0$, and since $\partial_{x_\ell}\partial_{x_\kappa} = \partial_{x_\kappa}\partial_{x_\ell}$ for all $\kappa, \ell = 1, 2, 3$, we have

$$T\overline{T} = \Delta,$$

where Δ is the Laplace operator. The S-spectrum is associated with the invertibility of the operator

$$s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} = s^2\mathcal{I} + \Delta \tag{5.26}$$

in the Banach space $X = X_{\mathbb{R}} \otimes \mathbb{H}$. If $s = u + jv$, then we can consider the scalar operator Δ an operator on $X_{\mathbb{R}} \otimes \mathbb{C}_j$. The operator in (5.26) is then invertible if and only if $s^2\mathcal{I} + \Delta$ is invertible on $X_{\mathbb{R}} \otimes \mathbb{C}_j$, that is, if and only if $-s^2$ belongs to the classical resolvent set $\rho(\Delta)$ of Δ. Because of the axial symmetry of the S-spectrum, we obtain

$$\sigma_S(T) = \left\{ s = u + iv : \quad -(u + jv)^2 \in \sigma(\Delta),\ i \in \mathbb{S} \right\}.$$

Chapter 6

The H^∞-Functional Calculus

The H^∞-functional calculus is an extension of the Riesz–Dunford functional calculus for bounded operators to unbounded sectorial operators, and it was introduced by A. McIntosh in [165]; see also [5]. This calculus is connected with pseudo-differential operators, with Kato's square root problem, and with the study of evolution equations and, in particular, the characterization of maximal regularity and with the fractional powers of differential operators. For an overview and more problems associated with this functional calculus for the classical case, see the book [156] and the references therein.

In this chapter we consider the quaternionic version of the H^∞-functional calculus introduced in [30], where with suitable conditions on the operators T we can study the quaternionic analogue of the results in [165]. A more general treatment of the H^∞-functional calculus for quaternionic operators has been done in [51, 52], where also the fractional powers of quaternionic linear operators are considered and new fractional diffusion and evolution processes are defined. We will mention such applications at the end of this chapter, see also [128].

6.1 The Rational Functional Calculus

The H^∞-functional calculus is defined using a version of the S-functional calculus for sectorial operators and on the rational functional calculus for intrinsic rational slice hyperholomorphic functions.

Definition 6.1.1 (Intrinsic rational slice hyperholomorphic function)**.** Let P and Q be intrinsic polynomials. An *intrinsic rational slice hyperholomorphic function* is defined as

$$R(p) := P(p)Q(p)^{-1}.$$

Observe that since $P(p)$ and $Q(p)^{-1}$ are intrinsic slice hyperholomorphic functions, the $\star$-product of $P(p)$ and $Q(p)^{-1}$ is equal to $P(p)Q(p)^{-1}$, and it is an intrinsic slice hyperholomorphic function.

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_6

Definition 6.1.2 (Rational functional calculus)**.** Assume that the rational function $R(p) = P(p)Q(p)^{-1}$ has no poles on the S-spectrum of T. Let T be a closed densely defined operator. We define the *rational functional calculus* as

$$R(T) := P(T)Q(T)^{-1}.$$

The operator $R(T)$ is closed and densely defined, and its domain is $\mathcal{D}(T^m)$, where

$$m := \max\{0, \deg P - \deg Q\}.$$

An important example of an intrinsic rational function, useful in the sequel, is

$$\psi(s) = \left(\frac{s}{1+s^2}\right)^k, \quad k \in \mathbb{N}.$$

We recall that slice hyperholomorphic rational functions have poles that are real points and/or spheres. This is compatible with the structure of the S-spectrum of T, which consists of real points and/or spheres. With ψ as above, we have

$$\psi(T) = \Big(T(\mathcal{I} + T^2)^{-1}\Big)^k, \quad k \in \mathbb{N}.$$

We summarize in the following the properties of the rational functional calculus. The proofs are similar to the classical results, and for this reason we omit them.

Proposition 6.1.3. *Let T be a linear quaternionic operator that is single-valued on a quaternionic Banach space X. Let P and Q be intrinsic quaternionic polynomials of order n and m, respectively. Then*

(i) *If $P \not\equiv 0$ then $P(T)Q(T) = (PQ)(T)$.*

(ii) *If $P(T)$ is injective and $Q \not\equiv 0$, then*

$$\mathcal{D}(P(T)^{-1}) \cap \mathcal{D}(Q(T)) \subset \mathcal{D}(P(T)^{-1}Q(T)) \cap \mathcal{D}(Q(T)P(T)^{-1})$$

and

$$P(T)^{-1}Q(T)v = Q(T)P(T)^{-1}v, \quad \forall v \in \mathcal{D}(Q(T)) \cap \mathcal{D}(P(T)^{-1}).$$

(iii) *Suppose that T is a closed linear operator with $\rho_S(T) \neq \emptyset$. Then $P(T)$ is closed and $P(\sigma_S(T)) = \sigma_S(P(T))$.*

For rational functions we have the following result, whose proof is similar to the classical case.

Proposition 6.1.4. *Let T be a linear quaternionic operator that is single-valued on a quaternionic Banach space X with $\rho_S(T) \neq \emptyset$. Let $0 \not\equiv R = PQ^{-1}$ and $R_1 = P_1Q_1^{-1}$ be intrinsic rational functions. Then we have:*

(i) *$R(T)$ is a closed operator.*

(ii) $R(\overline{\sigma_S}(T)) \subset \overline{\sigma_S}(R(T))$, *where* $\overline{\sigma_S}(T) = \sigma_S(T) \cup \{\infty\}$ *denotes the extended* S*-spectrum of* T.

(iii) $R(T)R_1(T) \subset (RR_1)(T)$ *and equality holds if*

$$(\deg(P) - \deg(Q))(\deg(P_1) - \deg(Q_1)) \geq 0.$$

(iv) $R(T) + R_1(T) \subset (R + R_1)(T)$ *and equality holds if*

$$\deg(PQ_1 + P_1Q) = \max\{\deg(PQ_1), \deg(P_1Q)\}.$$

6.2 The S-Functional Calculus for Operators of Type ω

We show that at least for a suitable subclass of closed densely defined operators, we can extend the formulas of the S-functional calculus for bounded operators. In order to do this, we recall that the definitions the S-resolvent operators are given in the previous chapter for unbounded operators.

Definition 6.2.1 (Argument function)**.** Let $s \in \mathbb{H} \setminus \{0\}$. We define $\arg(s)$ as the unique number $\theta \in [0, \pi]$ such that $s = |s|e^{\theta j_s}$.

Observe that $\theta = \arg(s)$ does not depend on the choice of j_s if $s \in \mathbb{R} \setminus \{0\}$, since $p = |p|e^{0j}$ for every $j \in \mathbb{S}$ if $p > 0$ and $p = |p|e^{\pi j}$ for every $j \in \mathbb{S}$ if $p < 0$. Let $\vartheta \in [0, \pi]$. We define the sets

$$\begin{aligned} \mathcal{S}_\vartheta &= \{s \in \mathbb{H} : \ |\arg(p)| \leq \vartheta \text{ or } s = 0\}, \\ \mathcal{S}^0_\vartheta &= \{s \in \mathbb{H} : \ |\arg(p)| < \vartheta\}. \end{aligned} \tag{6.1}$$

Definition 6.2.2 (Operator of type ω)**.** Let $\omega \in [0, \pi)$. We say that the linear operator $T : D(T) \subset X \to X$ is of type ω if

(i) T is closed and densely defined,

(ii) $\sigma_S(T) \subset \mathcal{S}_\vartheta \cup \{\infty\}$,

(iii) for every $\vartheta \in (\omega, \pi]$ there exists a positive constant C_ϑ such that

$$\|S_L^{-1}(s,T)\| \leq \frac{C_\vartheta}{|s|}, \quad \|S_R^{-1}(s,T)\| \leq \frac{C_\vartheta}{|s|} \quad \text{for all nonzero } s \in \mathcal{S}^0_\vartheta.$$

We now introduce the following subsets of the set of slice hyperholomorphic functions, which consist of bounded slice hyperholomorphic functions.

Definition 6.2.3. Let $\mu \in (0, \pi]$. We set

$$\mathcal{SH}_L^\infty(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_L(\mathcal{S}^0_\mu) \quad \text{such that } \|f\|_\infty := \sup_{s \in \mathcal{S}^0_\mu} |f(s)| < \infty\},$$

$$\mathcal{SH}_R^\infty(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_R(\mathcal{S}^0_\mu) \quad \text{such that } \|f\|_\infty := \sup_{s \in \mathcal{S}^0_\mu} |f(s)| < \infty\},$$

$$\mathcal{N}^\infty(\mathcal{S}^0_\mu) = \{f \in \mathcal{N}(\mathcal{S}^0_\mu) \quad \text{such that } \|f\|_\infty := \sup_{s \in \mathcal{S}^0_\mu} |f(s)| < \infty\}.$$

In order to define bounded functions of operators of type ω, we need to introduce suitable subclasses of bounded slice hyperholomorphic functions:

Definition 6.2.4. With the notation introduced in Definition 6.2.3, we define

$$\Psi_L(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}^\infty_L(\mathcal{S}^0_\mu): \ \exists\, \alpha > 0,\ c > 0 : |f(s)| \leq \frac{c|s|^\alpha}{1+|s|^{2\alpha}} \text{ for all } s \in \mathcal{S}^0_\mu\},$$

$$\Psi_R(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}^\infty_R(\mathcal{S}^0_\mu): \ \exists\, \alpha > 0,\ c > 0 : |f(s)| \leq \frac{c|s|^\alpha}{1+|s|^{2\alpha}} \text{ for all } s \in \mathcal{S}^0_\mu\},$$

$$\Psi(\mathcal{S}^0_\mu) = \{f \in \mathcal{N}^\infty(\mathcal{S}^0_\mu): \ \exists\, \alpha > 0,\ c > 0 : |f(s)| \leq \frac{c|s|^\alpha}{1+|s|^{2\alpha}} \text{ for all } s \in \mathcal{S}^0_\mu\}.$$

The following theorem is a crucial step for the definition of the S-functional calculus for operators of type ω, because it shows that the following integrals depend neither on the path that we choose nor on the complex plane $\mathbb{C}_j$, $j \in \mathbb{S}$.

Theorem 6.2.5. *Let T be an operator of type ω. Let $j \in \mathbb{S}$, and let $\mathcal{S}^0_\mu$ be as in (6.1). Choose a piecewise smooth path Γ in $\mathcal{S}^0_\mu \cap \mathbb{C}_j$ that goes from $\infty e^{j\theta}$ to $\infty e^{-j\theta}$, where $\omega < \theta < \mu$. Then the integrals*

$$\frac{1}{2\pi}\int_\Gamma S_L^{-1}(s,T)\, ds_j\, \psi(s), \quad \textit{for all } \psi \in \Psi_L(\mathcal{S}^0_\mu), \tag{6.2}$$

$$\frac{1}{2\pi}\int_\Gamma \psi(s)\, ds_j\, S_R^{-1}(s,T), \quad \textit{for all } \psi \in \Psi_R(\mathcal{S}^0_\mu), \tag{6.3}$$

depend neither on Γ nor on $j \in \mathbb{S}$, and they define bounded operators.

Proof. We reason on the integral (6.2), since (6.3) can be treated in a similar way.

The growth estimates on ψ and on the resolvent operator imply that the integral (6.2) exists and defines a bounded right-linear operator.

The independence of the choice of θ and of the choice of the path Γ in the complex plane $\mathbb{C}_j$ follows from Cauchy's integral theorem.

In order to show that the integral (6.2) is independent of the choice of the imaginary unit $j \in \mathbb{S}$, we take an arbitrary $i \in \mathbb{S}$ with $j \neq i$.

Let $B(0,r)$ be the ball centered at the origin with radius r; let $a_0 > 0$ and $\theta_0 \in (0,\pi)$, $n \in \mathbb{N}$. We define the sector $\Sigma(\theta_0, a_0)$ as

$$\Sigma(\theta_0, a_0) := \{s \in \mathbb{H}: \ \arg(s - a_n) \geq \theta_n\}.$$

Let $\theta_0 < \theta_s < \theta_p < \pi$ and set $U_s := \Sigma(\theta_s, 0) \cup B(0, a_0/2)$ and $U_p := \Sigma(\theta_p, 0) \cup B(0, a_0/3)$, where the indices s and p denote the variables of integration over the boundary of the respective set. Suppose that U_p and U_s are Cauchy domains and $\partial(U_s \cap \mathbb{C}_j)$ and $\partial(U_p \cap \mathbb{C}_i)$ are paths that are contained in the sector. Observe that

$\psi(s)$ is right slice hyperholomorphic on $\overline{U_p}$, and hence by Theorem 2.1, we have

$$\psi(T) = \frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} \psi(s)\, ds_j\, S_R^{-1}(s,T) \tag{6.4}$$

$$= \frac{1}{(2\pi)^2} \int_{\partial(U_s \cap \mathbb{C}_j)} \left(\int_{\partial(U_p \cap \mathbb{C}_i)} \psi(p)\, dp_i\, S_R^{-1}(p,s) \right) ds_j\, S_R^{-1}(s,T) \tag{6.5}$$

$$= \frac{1}{2\pi} \int_{\partial(U_p \cap \mathbb{C}_j)} \psi(p)\, dp_i \left(\frac{1}{2\pi} \int_{\partial(U_s \cap \mathbb{C}_j)} S_R^{-1}(p,s)\, ds_j\, S_R^{-1}(s,T) \right) \tag{6.6}$$

$$= \frac{1}{2\pi} \int_{\partial(U_p \cap \mathbb{C}_i)} \psi(p)\, dp_i\, S_R^{-1}(p,T). \tag{6.7}$$

To exchange the order of integration we apply Fubini's theorem. The last equation follows as an application of the S-functional calculus for unbounded operators, introduced in the previous chapter, since $S_R^{-1}(p,\infty) = \lim_{s\to\infty} S_R^{-1}(p,s) = 0$. So we get the statement. □

Thanks to the above theorem the following definitions are well posed.

Definition 6.2.6 (The S-functional calculus for operators of type ω)**.** Let T be an operator of type ω. Let $j \in \mathbb{S}$, and let $\mathcal{S}^0_\mu$ be the sector defined above. Choose a piecewise smooth path Γ in $\mathcal{S}^0_\mu \cap \mathbb{C}_j$ that goes from $\infty e^{j\theta}$ to $\infty e^{-j\theta}$, for $\omega < \theta < \mu$. Then

$$\psi(T) := \frac{1}{2\pi} \int_\Gamma S_L^{-1}(s,T)\, ds_j\, \psi(s), \quad \text{for all } \psi \in \Psi_L(\mathcal{S}^0_\mu), \tag{6.8}$$

$$\psi(T) := \frac{1}{2\pi} \int_\Gamma \psi(s)\, ds_j\, S_R^{-1}(s,T), \quad \text{for all } \psi \in \Psi_R(\mathcal{S}^0_\mu). \tag{6.9}$$

From the definition of the functional calculus the linearity properties follow immediately. In fact, if T is an operator of type ω, then $\psi(T)$, defined in (6.8) and (6.9), satisfy

$$(\psi a + \varphi b)(T) = \psi(T)a + \varphi(T)b, \quad \text{for all } \psi,\varphi \in \Psi_L(\mathcal{S}^0_\mu),$$
$$(a\psi + b\varphi)(T) = a\psi(T) + b\varphi(T), \quad \text{for all } \psi,\varphi \in \Psi_R(\mathcal{S}^0_\mu).$$

For functions ψ that belong to $\Psi(\mathcal{S}^0_\mu)$ both representations can be used. Moreover,

$$\psi(T) := \frac{1}{2\pi} \int_\Gamma \psi(s)\, ds_i\, S_R^{-1}(s,T)$$
$$= \frac{1}{2\pi} \int_\Gamma S_L^{-1}(s,T)\, ds_i\, \psi(s), \quad \text{for all } \psi \in \Psi(\mathcal{S}^0_\mu).$$

Using the S-resolvent equation with similar computations as in the case of bounded operators, adapted to this case, we can prove the product rule:

Theorem 6.2.7. *Let T be an operator of type ω. Then*

$$(\psi\varphi)(T) = \psi(T)\varphi(T), \quad \text{for all } \psi \in \Psi(\mathcal{S}^0_\mu),\ \varphi \in \Psi_L(\mathcal{S}^0_\mu),$$
$$(\psi\varphi)(T) = \psi(T)\varphi(T), \quad \text{for all } \psi \in \Psi_R(\mathcal{S}^0_\mu),\ \varphi \in \Psi(\mathcal{S}^0_\mu).$$

6.3 The H^∞-Functional Calculus

To define the H^∞ functional calculus we suppose that T is an operator of type ω, and moreover, we assume that it is one-to-one and with dense range. Here we will consider slice hyperholomorphic functions defined on the open sector $\mathcal{S}^0_\mu$, for $0 \leq \omega < \mu \leq \pi$, which can grow at infinity as $|s|^k$ and at the origin as $|s|^{-k}$ for $k \in \mathbb{N}$. This enlarges the class of functions to which the functional calculus can be applied. Precisely we make the following definition.

Definition 6.3.1 (Operators of type Ω)**.** Let ω be a real number such that $0 \leq \omega \leq \pi$. We denote by Ω the set of linear operators T acting on a two-sided quaternionic Banach space such that:

(i) T is a linear operator of type ω;

(ii) T is one-to-one and with dense range.

Then we define the following function spaces according to the set of operators defined above:

Definition 6.3.2. Let ω and μ be real numbers such that $0 \leq \omega < \mu \leq \pi$. We set

$$\mathcal{F}_L(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_L(\mathcal{S}^0_\mu) : \ |f(s)| \leq C(|s|^k + |s|^{-k}) \text{ for some } k > 0 \text{ and } C > 0\},$$
$$\mathcal{F}_R(\mathcal{S}^0_\mu) = \{f \in \mathcal{SH}_R(\mathcal{S}^0_\mu) : \ |f(s)| \leq C(|s|^k + |s|^{-k}) \text{ for some } k > 0 \text{ and } C > 0\},$$
$$\mathcal{F}(\mathcal{S}^0_\mu) = \{f \in \mathcal{N}(\mathcal{S}^0_\mu) : \quad |f(s)| \leq C(|s|^k + |s|^{-k}) \text{ for some } k > 0 \text{ and } C > 0\}.$$

To extend the functional calculus we consider a quaternionic two-sided Banach space X, the operators in the class Ω, and

- the noncommutative algebra $\mathcal{F}_L(\mathcal{S}^0_\mu)$ (resp. $\mathcal{F}_R(\mathcal{S}^0_\mu)$);
- the S-functional calculus Φ for operators of type ω:

$$\Phi : \Psi_L(\mathcal{S}^0_\mu) \text{ (resp. } \Psi_R(\mathcal{S}^0_\mu)) \to \mathcal{B}(X), \quad \Phi : \phi \to \phi(T);$$

- the commutative subalgebra of $\mathcal{F}_L(\mathcal{S}^0_\mu)$ consisting of intrinsic rational functions;

Furthermore, the functions in $\mathcal{F}_L(\mathcal{S}^0_\mu)$ have at most polynomial growth. So taking an intrinsic rational functions ψ, the operator $\psi(T)$ can be defined by the rational functional calculus.

We assume also that $\psi(T)$ is injective.

Definition 6.3.3 (H^∞-functional calculus)**.** Let X be a two-sided quaternionic Banach space and let $T \in \Omega$. For $k \in \mathbb{N}$ consider the function

$$\psi(s) := \Big(\frac{s}{1+s^2}\Big)^{k+1}.$$

For $f \in \mathcal{F}_L(\mathcal{S}^0_\mu)$ and T right linear, we define the *extended functional calculus* as

$$f(T) := (\psi(T))^{-1}(\psi f)(T). \tag{6.10}$$

For $f \in \mathcal{F}_R(\mathcal{S}^0_\mu)$ and T left linear, we define the *extended functional calculus* as

$$f(T) := (f\psi)(T)(\psi(T))^{-1}. \tag{6.11}$$

We say that ψ *regularizes* f.

In the previous definition the operator $(\psi f)(T)$ (resp. $(f\psi)(T)$) is defined using the S-functional calculus Φ for operators of type ω, and $\psi(T)$ is defined by the rational functional calculus.

Theorem 6.3.4. *The definition of the functional calculus in* (6.10) *and in* (6.11) *does not depend on the choice of the intrinsic rational slice hyperholomorphic function* ψ.

Proof. Let us prove (6.10). Suppose that ψ and ψ' are two different regularizers and set

$$A := (\psi(T))^{-1}(\psi f)(T) \quad \text{and} \quad B := (\psi'(T))^{-1}(\psi' f)(T).$$

Observe that since the functions ψ and ψ' commute, because there are intrinsic rational functions, one has

$$\psi(T)\psi'(T) = (\psi\psi')(T) = (\psi'\psi)(T) = \psi'(T)\psi(T),$$

so we get

$$(\psi'(T))^{-1}(\psi(T))^{-1} = (\psi(T))^{-1}(\psi'(T))^{-1}.$$

It is now easy to see that

$$\begin{aligned} A = (\psi(T))^{-1}(\psi f)(T) &= (\psi(T))^{-1}(\psi'(T))^{-1}(\psi'(T))(\psi f)(T) = \\ &= (\psi'(T))^{-1}(\psi(T))^{-1}(\psi\psi' f)(T) \\ &= (\psi'(T))^{-1}(\psi(T))^{-1}\psi(T)(\psi' f)(T) \\ &= (\psi'(T))^{-1}(\psi' f)(T) = B, \end{aligned}$$

where we used the fact that from the product rule, see Proposition 6.1.4, we have that the inverse of $\psi(T)$ is $(1/\psi)(T)$. The proof of (6.11) follows in a similar way. □

We now state an important result for functions in $\mathcal{F}_L(\mathcal{S}^0_\mu)$ (the same result with obvious changes holds for functions in $\mathcal{F}_R(\mathcal{S}^0_\mu)$).

Theorem 6.3.5. *Let $f \in \mathcal{F}(\mathcal{S}_\mu^0)$ and $g \in \mathcal{F}_L(\mathcal{S}_\mu^0)$. Then we have*

$$f(T) + g(T) \subset (f+g)(T),$$
$$f(T)g(T) \subset (fg)(T),$$

and $\mathcal{D}(f(T)g(T)) = \mathcal{D}((fg)(T)) \bigcap \mathcal{D}(g(T))$.

Proof. Let us take ψ_1 and ψ_2 that regularize f and g, respectively. Observe that the function $\psi := \psi_1\psi_2$ regularizes f, g, $f+g$, and fg because ψ_1, ψ_2, and f commute among themselves. Observe that

$$\begin{aligned} f(T) + g(T) &= (\psi(T))^{-1}(\psi f)(T) + (\psi(T))^{-1}(\psi g)(T) \\ &\subset (\psi(T))^{-1}[(\psi f)(T) + (\psi g)(T)] \\ &= (\psi(T))^{-1}[\psi(f+g)](T) = (f+g)(T). \end{aligned}$$

We can consider now the product rule

$$\begin{aligned} f(T)g(T) &= (\psi_1(T))^{-1}(\psi_1 f)(T)\ (\psi_2(T))^{-1}(\psi_2 g)(T) \\ &\subset (\psi_1(T))^{-1}(\psi_2(T))^{-1}[(\psi_1 f)(T)(\psi_2 g)(T)] \\ &= (\psi_2(T)\psi_1(T))^{-1}[\psi_1(T)\psi_2(T)(fg)](T) \\ &= (\psi(T))^{-1}(\psi fg)(T) = (fg)(T), \end{aligned}$$

where we have used $\psi := \psi_1\psi_2$. Regarding the domains, it is as in the complex case. □

6.4 Boundedness of the H^∞-Functional Calculus

The following convergence theorem is stated for functions in $\mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$, but it holds also for functions in $\mathcal{SH}_R^\infty(\mathcal{S}_\mu^0)$ and is the quaternionic analogue of the theorem in Section 5 in [165]. The proof follows the proof of the convergence theorem in [165, p. 216]; we just point out that the convergence theorem is based on the principle of uniform boundedness that holds also for quaternionic operators.

Theorem 6.4.1 (A Convergence theorem)**.** *Suppose that $0 \leq \omega < \mu \leq \pi$ and that T is a linear operator of type ω such that it is one-to-one and with dense range. Let f_α be a net in $\mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$ and let $f \in \mathcal{SH}_L^\infty(\mathcal{S}_\mu^0)$ and assume that:*

(i) *there exists a positive constant M such that $\|f_\alpha(T)\| \leq M$;*

(ii) *for every $0 < \delta < \lambda < \infty$,*

$$\sup\{|f_\alpha(s) - f(s)| \text{ such that } s \in \mathcal{S}_\mu^0 \text{ and } \delta \leq |s| \leq \lambda\} \to 0.$$

Then $f(T) \in \mathcal{B}(V)$ and $f_\alpha(T)u \to f(T)u$ for all $u \in V$, and moreover, $\|f(T)\| \leq M$.

In the following we discuss the boundedness of the H^∞ functional calculus. The crucial tool to show the boundedness of the H^∞ functional calculus is the so-called quadratic estimates; see [165].

Definition 6.4.2 (Quadratic estimate)**.** Let T be a right linear operator of type ω on a quaternionic Hilbert space $\mathcal{H}$ and let $\psi \in \Psi(\mathcal{S}^0_\mu)$, where $0 \leq \omega < \mu \leq \pi$. We say that T *satisfies a quadratic estimate with respect to* ψ if there exists a positive constant β such that

$$\int_0^\infty \|\psi(tT)u\|^2 \frac{dt}{t} \leq \beta^2 \|u\|^2, \quad \text{for all } u \in \mathcal{H},$$

where we write $\|u\|$ for $\|u\|_{\mathcal{H}}$.

Let us introduce the notation

$$\Psi^+(\mathcal{S}^0_\mu) = \{\psi \in \Psi(\mathcal{S}^0_\mu) : \quad \psi(t) > 0 \text{ for all } t \in (0,\infty)\}$$

and

$$\psi_t(s) = \psi(ts), \quad t \in (0,\infty).$$

Theorem 6.4.3. *Let $0 \leq \omega < \mu \leq \pi$ and assume that T is a right linear operator in Ω. Suppose that T and its adjoint T^* satisfy the quadratic estimates with respect to the functions ψ and $\widetilde{\psi} \in \Psi^+(\mathcal{S}^0_\mu)$. Suppose that f belongs to $\mathcal{SH}^\infty_L(\mathcal{S}^0_\mu)$. Then the operator $f(T)$ is bounded, and there exists a positive constant C such that*

$$\|f(T)\| \leq C\|f\|_\infty \quad \textit{for all } f \in \mathcal{SH}^\infty_L(\mathcal{S}^0_\mu).$$

Proof. We follow the proof of Theorem on p. 221 in [165], and we point out the differences. We observe that we choose the functions ψ, $\widetilde{\psi}$, and η in the space of intrinsic functions $\Psi^+(\mathcal{S}^0_\mu)$ because the pointwise product

$$\varphi(s) := \psi(s)\widetilde{\psi}(s)\eta(s)$$

has to be slice hyperholomorphic, and moreover, η has to be such that

$$\int_0^\infty \varphi(t)\, \frac{dt}{t} = 1.$$

For $f \in \mathcal{SH}^\infty_L(\mathcal{S}^0_\mu)$ let us define

$$f_{\varepsilon,R}(s) = \int_\varepsilon^R (\varphi_t f)(s) \frac{dt}{t}.$$

Using the quadratic estimates it follows that there exists a positive constant C such that

$$\|f_{\varepsilon,R}(T)\| \leq C\|f\|_\infty.$$

The convergence theorem (Theorem 6.4.1) gives the formula

$$f(T)u = \lim_{\varepsilon\to 0}\lim_{R\to\infty} f_{\varepsilon,R}(T)u \quad \text{for all } u\in\mathcal{H},$$

where $(\eta_t f)(T)$ is defined by the S-functional calculus

$$(\eta_t f)(T) = \frac{1}{2\pi}\int_\Gamma S_L^{-1}(s,T)\,ds_i\,\eta_t(s)f(s), \quad \text{for all } f\in\Psi_L(\mathcal{S}_\mu^0),$$

since $\eta_t f\in\Psi_L(\mathcal{S}_\mu^0)$ because η_t is intrinsic. Precisely, the quadratic estimates and some computations show that there exists a positive constant C_β such that

$$|\langle f_{\varepsilon,R}(T)u, v\rangle| \le C_\beta \sup_{t\in(0,\infty)} \|(\eta_t f)(T)\|\|u\|\|v\|.$$

Since

$$\begin{aligned}\|(\eta_t f)(T)\| &\le \frac{1}{2\pi}\|f\|_\infty \sup_{i\in\mathbb{S}}\int_\Gamma \|S_L^{-1}(s,T)\||ds_i|\,|\eta_t(s)|\\ &\le \frac{1}{2\pi}\sup_{i\in\mathbb{S}}\|f\|_\infty\int_\Gamma \frac{C_\eta}{|s|}\,|ds_i|\,\frac{c|s|^\alpha}{1+|s|^{2\alpha}}\\ &\le C_T(\mu,\eta)\|f\|_\infty,\end{aligned}$$

from the above estimates we get the statement. □

6.5 Comments and Remarks

To study fractional diffusion and fractional evolution problems we need a more involved and refined version of the H^∞-functional calculus in the quaternionic setting, which is beyond the aim of this book. For more details see the papers [50–52], where the fractional powers of quaternionic operators and applications are treated. In the paper [53], the authors introduced the so-called S-spectrum approach to fractional diffusion processes, which allows one to study very general fractional diffusion problems. This strategy is largely explained in the monograph [56]. The new approach to fractional diffusion problems will be explained without too many technical details in the following subsection.

6.5.1 Comments on Fractional Diffusion Processes

We denote by u the temperature on and by $\mathbf{q}$ the heat flow, and we set the thermal diffusivity equal to 1. The heat equation is then deduced from the two laws

$$\mathbf{q} = -\nabla u \quad \text{(Fourier's law)}, \tag{6.12}$$

$$\partial_t u + \operatorname{div}\mathbf{q} = 0 \quad \text{(conservation of energy)}, \tag{6.13}$$

where u and $\mathbf{q}$ are defined on $\mathbb{R}^3$, and Fourier's law is substituted into the equation for conservation of energy, that is,

$$\partial_t u - \Delta u = 0.$$

The fractional heat equation is an alternative model that takes into account non-local interactions, and it is obtained by replacing the negative Laplacian in the heat equation by its fractional power, so that

$$\partial_t u + (-\Delta)^\alpha u = 0, \quad \alpha \in (0, 1), \tag{6.14}$$

where the fractional Laplacian is given by

$$(-\Delta)^\alpha u(x) = c(n, \alpha) P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2\alpha}} dy,$$

and the integral is defined in the sense of the principal value, $c(n, \alpha)$ is a known constant, and $u : \mathbb{R}^n \to \mathbb{R}$ must belong to a suitable function space.

The approach with the fractional powers of quaternionic operators defined via the H^∞-functional calculus is different, very general, and in the case $\mathbf{q} = -\nabla u$ it reduces to the fractional Laplace operator.

Precisely, we identify

$$\mathbb{R}^3 \cong \{s \in \mathbb{H} : \ \mathrm{Re}(s) = 0\},$$

and we consider the gradient ∇ the quaternionic Nabla operator

$$\nabla = e_1 \partial_{x_1} + e_2 \partial_{x_2} + e_3 \partial_{x_3}.$$

Instead of replacing the negative Laplacian in the heat equation by $(-\Delta)^\alpha$, we want to replace the gradient in (6.12) by its fractional power ∇^α, and then we replace it in the law of conservation of energy. We proceed as follows:

- Since s^α is not defined on $(-\infty, 0)$, and on $L^2(\mathbb{R}^3, \mathbb{H})$ it is $\sigma_S(\nabla) = \mathbb{R}$, we consider the projections of the fractional powers of ∇^α, indicated by $f_\alpha(\nabla)$, to the subspace associated with the subset $[0 + \infty)$ of the S-spectrum of ∇, on which the function s^α is well defined.
- Then we take just the vector part $\mathrm{Vect}(f_\alpha(\nabla)) = e_1 T_1 + e_2 T_2 + e_3 T_3$ of the quaternionic operator $f_\alpha(\nabla) = T_0 + e_1 T_1 + e_2 T_2 + e_3 T_3$ so that we can apply the divergence operator.

We point out that the above procedure applied to the gradient operator gives the classical result. Indeed, the definition of ∇^α only on the subspace associated to $[0, \infty)$ is given by

$$f_\alpha(\nabla) v = \frac{1}{2\pi} \int_{-j\mathbb{R}} S_L^{-1}(s, \nabla) \, ds_j \, s^\alpha \nabla v,$$

for $v : \mathbb{R}^3 \to \mathbb{H}$ in $\mathcal{D}(\nabla)$. This corresponds to the Balakrishnan formula, which is a consequence of the quaternionic H^∞-functional calculus, in which only positive spectral values are taken into account. With this definition and the surprising expression for the left S-resolvent operator

$$S_L^{-1}(-jt, \nabla) = (-jt + \nabla)\underbrace{(-t^2 + \Delta)^{-1}}_{=R_{-t^2}(-\Delta)},$$

the operator $f_\alpha(\nabla)$, with some computations, becomes

$$f_\alpha(\nabla)v = \underbrace{\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2 v}_{\mathrm{Scal} f_\alpha(\nabla)v} + \underbrace{\frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v}_{=\mathrm{Vec} f_\alpha(\nabla)v}.$$

We define the scalar part of the operator $f_\alpha(\nabla)$ applied to v as

$$\mathrm{Scal} f_\alpha(\nabla)v := \frac{1}{2}(-\Delta)^{\frac{\alpha}{2}-1}\nabla^2 v,$$

and the vector part as

$$\mathrm{Vec} f_\alpha(\nabla)v := \frac{1}{2}(-\Delta)^{\frac{\alpha-1}{2}}\nabla v.$$

Now we observe hat

$$\mathrm{div}\mathrm{Vec} f_\alpha(\nabla)v = -\frac{1}{2}(-\Delta)^{\frac{\alpha}{2}+1} v.$$

This proves that in the case of the gradient, we get the same result, which is the fractional Laplacian. The fractional heat equation for $\alpha \in (1/2, 1)$,

$$\partial_t u(t,x) + (-\Delta)^\alpha u(t,x) = 0,$$

can hence be written as

$$\partial_t u(t,x) - 2\mathrm{div}\,(\mathrm{Vec} f_\beta(\nabla)u) = 0, \quad \beta = 2\alpha - 1.$$

We point out that the operator $f_\alpha(\nabla)$ can be applied to vector-valued functions v. For an application to the heat equation it is applied to the scalar-valued function u that represents the temperature. The quaternionic fractional powers approach is very general, and it is applicable to a large class of operators such as

$$\widetilde{\nabla} = e_1\, a(x)\partial_{x_1} + e_2\, b(x)\partial_{x_2} + e_3\, c(x)\partial_{x_3},$$

where a, b, c are suitable real-valued functions that depend on the space variables $x = (x_1, x_2, x_3)$ and possibly also on time. For every suitable vector operator T, we define a new fractional evolution equation as

$$\partial_t u(t,x) - 2\mathrm{div}\,(\mathrm{Vec} f_\beta(T)u) = 0.$$

For example, a new fractional evolution equation can be deduced when we consider the following Fourier's law:

$$T = e_1 x_1 \partial_{x_1} + e_2 x_2 \partial_{x_2} + e_3 x_3 \partial_{x_3}.$$

Working in the space $L^2(\mathbb{R}^3_+, \mathbb{H}, d\mu)$ with

$$\mathbb{R}^3_+ = \{e_1 x_1 + e_2 x_2 + e_3 x_3 : \ \ x_\ell > 0\}$$

and $d\mu = (x_1 x_2 x_3)^{-1} dx$, we get the operator

$$\begin{aligned} &\mathrm{Vec} f_\beta(T) v(\xi) \\ &\quad = \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} -|y|^{2\alpha} e^{e_1 \sum_{k=1}^3 \xi_k y_k} e^{-e_1 x \cdot y} \begin{pmatrix} e^{x_1} v_{\xi_1}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_2} v_{\xi_2}(e^{x_1}, e^{x_2}, e^{x_3}) \\ e^{x_3} v_{\xi_3}(e^{x_1}, e^{x_2}, e^{x_3}) \end{pmatrix} dx\, dy. \end{aligned}$$

We point out that the fractional powers of the operator $\mathbf{q}(x, \partial_x)$ are very useful for inhomogeneous materials, and this approach has several advantages: It modifies the Fourier law but keeps the law of conservation of energy, and it is applicable to a large class of operators that includes the gradient but also operators with variable coefficients such as the operator $\mathbf{q}(x, \partial_x)$. Moreover, $\mathbf{q}$ can also depend on time.

The fact that we keep the evolution equation in divergence form allows an immediate definition of the weak solution of the fractional evolution problem.

To represent the fractional powers of an operator T we have to write an explicit expression for the inverse of the operator $T^2 - 2s_0 T + |s|^2 \mathcal{I}$, and this can be done on bounded or unbounded domains.

Chapter 7

The F-Functional Calculus for Bounded Operators

The Fueter mapping theorem in integral form introduced in [86], see Chapter 2.2, provides an integral transform that turns slice hyperholomorphic functions into Fueter regular ones. By formally replacing the scalar variable in this integral transform by an operator T, we obtain a functional calculus for Fueter regular functions that is based on the theory of slice hyperholomorphic functions. The F-functional calculus was introduced and studied in the following papers [54, 78, 81, 86].

7.1 The F-Resolvent Operators and the F-Functional Calculus

We begin our discussion with the feasibility of this functional calculus.

Definition 7.1.1. For $m \in \mathbb{N}$ and $q \in \mathbb{H}$ we consider the *Fueter regular polynomials*

$$\mathcal{P}_m(q) := \Delta q^m. \tag{7.1}$$

Lemma 7.1.2. *We have $\mathcal{P}_0 \equiv \mathcal{P}_1 \equiv 0$ and $\mathcal{P}_2 \equiv -4$. Furthermore, for even $m \geq 2$, we have*

$$\mathcal{P}_m(q) = m(m-1)q^{m-2} + 2\mathrm{Re}\left(\sum_{\ell=1}^{3}\sum_{\kappa=1}^{m-1}\sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}\right), \tag{7.2}$$

and for odd $m \geq 2$ we have

$$\mathcal{P}_m(q) = m(m-1)q^{m-2} + 2\mathrm{Im}\left(\sum_{\ell=1}^{3}\sum_{\kappa=1}^{m-1}\sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}\right). \tag{7.3}$$

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_7

Proof. The identities $\mathcal{P}_0 \equiv \mathcal{P}_1 \equiv 0$ and $\mathcal{P}_2 = -4$ follow by straightforward computations. Thus assume that $m \geq 2$.

For $q = q_0 + \underline{q} = q_0 + \sum_{\ell=1}^{3} q_\ell e_\ell \in \mathbb{H}$, we have

$$q^m = \sum_{k=0}^{m} \binom{m}{k} q_0^k \underline{q}^{m-k},$$

and so

$$\frac{\partial^2}{\partial q_0^2} q^m = \sum_{k=2}^{m} \binom{m}{k} k(k-1) q_0^{k-2} \underline{q}^{m-k} = \sum_{k=2}^{m} \frac{m!}{(m-k)!(k-2)!} q_0^{k-2} \underline{q}^{m-k}$$
$$= m(m-1) \sum_{k=0}^{m-2} \frac{(m-2)!}{(m-k)!(k-2)!} q_0^k \underline{q}^{m-2-k} = m(m-1) q^{m-2}.$$

Furthermore, observe that for $1 \leq \ell \leq 3$ we have

$$\frac{\partial}{\partial q_\ell} \underline{q}^r = \sum_{\kappa=0}^{r-1} \underline{q}^\kappa e_\ell \underline{q}^{r-1-\kappa}. \tag{7.4}$$

For $r = 1$, we have $\frac{\partial}{\partial q_\ell} \underline{q} = e_\ell$, and so (7.4) holds. If, on the other hand, (7.4) holds for $r-1$, then

$$\frac{\partial}{\partial q_\ell} \underline{q}^r = \left(\frac{\partial}{\partial q_\ell} \underline{q} \right) \underline{q}^{r-1} + \underline{q} \left(\frac{\partial}{\partial q_\ell} \underline{q}^{r-1} \right)$$
$$= e_\ell \underline{q}^{r-1} + \sum_{\kappa=0}^{r-2} \underline{q}^{k+1} e_\ell \underline{q}^{r-2-\kappa} = \sum_{\kappa=0}^{r-1} \underline{q}^\kappa e_\ell \underline{q}^{r-1-\kappa}.$$

Applying this identity twice, we obtain

$$\frac{\partial}{\partial q_\ell} \underline{q}^m = \sum_{\kappa=1}^{m-1} \left(\frac{\partial}{\partial q_\ell} \underline{q}^\kappa \right) e_\ell \underline{q}^{m-1-\kappa} + \sum_{\kappa=0}^{m-2} \underline{q}^\kappa e_\ell \left(\frac{\partial}{\partial q_\ell} \underline{q}^{m-1-\kappa} \right)$$
$$= \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} + \sum_{\kappa=0}^{m-2} \sum_{s=0}^{m-2-\kappa} \underline{q}^\kappa e_\ell \underline{q}^s e_\ell \underline{q}^{m-2-\kappa-s}$$
$$= \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} + \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^{m-1-\kappa} e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^s$$
$$= \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} + (-1)^m \overline{\sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}},$$

where the last identity follows from

$$\overline{\underline{q}^{m-1-\kappa} e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^s} = \overline{\underline{q}^s}\, \overline{e_\ell}\, \overline{\underline{q}^{\kappa-1-s}}\, \overline{e_\ell}\, \overline{\underline{q}^{m-1-\kappa}}$$
$$= (-1)^m \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}$$

because $\overline{\underline{q}} = -\underline{q}$, since $\underline{q}$ is purely imaginary. Therefore, we obtain

$$\Delta q^m = \frac{\partial^2}{\partial q_0^2} q^m + \sum_{\ell=1}^{3} \frac{\partial}{\partial q_\ell^2} q^m = m(m-1)q^{m-2} + \sum_{\ell=1}^{3} \sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa} + (-1)^m \overline{\sum_{\kappa=1}^{m-1} \sum_{s=0}^{\kappa-1} \underline{q}^s e_\ell \underline{q}^{\kappa-1-s} e_\ell \underline{q}^{m-1-\kappa}},$$

which yields (7.2) resp. (7.3). □

Definition 7.1.3 (Fueter kernel series)**.** Let $s, q \in \mathbb{H}$. We define the *left Fueter kernel series* as

$$\sum_{m\geq 2} \mathcal{P}_m(q) s^{-1-m},$$

and the *right Fueter kernel series* as

$$\sum_{m\geq 2} s^{-1-m} \mathcal{P}_m(q).$$

Proposition 7.1.4. *For $s, q \in \mathbb{H}$ with $|q| < |s|$, the left and right Fueter kernel series converge.*

Proof. Because of (7.2) and (7.3), we have for $m \geq 2$ that

$$\begin{aligned} |\mathcal{P}_m(q)| &\leq m(m-1)|q|^{m-2} + 2\sum_{\ell=1}^{3} \sum_{k=1}^{m-1} \sum_{s=0}^{\kappa-1} |q|^{m-2} \\ &= m(m-1)|q|^{m-2} + 3m(m-1)|q|^{m-2} = 4m(m-1)|q|^{m-2}. \end{aligned}$$

If $|q| < |s|$, we therefore have for the left Fueter kernel series

$$\sum_{m\geq 2} |\mathcal{P}_m(q) s^{-1-m}| \leq 4 \sum_{m\geq 2} m(m-1)|q|^{m-2}|s^{-1-m}| < +\infty,$$

and the convergence of the right Fueter kernel series is shown similarly. □

The Fueter kernel series are the Taylor series expansions of the Fueter kernels $F_L(s, q)$ and $F_R(s, q)$ introduced in Definition 2.2.5. They are their slice hyperholomorphic Taylor expansions in the variable s at infinity and the Fueter regular Taylor expansions in the variable q at 0; cf. the Comments and Remarks in Section 7.6.

Lemma 7.1.5. *For $|q| < |s|$, we have*

$$F_L(s,q) = \sum_{n=0}^{+\infty} \mathcal{P}_m(q) s^{-1-m} \quad \textit{and} \quad F_R(s,q) = \sum_{n=0}^{+\infty} s^{-1-m}\mathcal{P}_m(q).$$

Proof. Due to the Taylor series expansion $S_L^{-1}(s,q) = \sum_{n=0}^{+\infty} q^n s^{-1-n}$ of the left Cauchy kernel in Theorem 2.1.22, we have

$$F_L(s,q) = \Delta S_L^{-1}(s,q) = \sum_{n=0}^{+\infty} \Delta q^n s^{-1-n} = \sum_{n=2}^{+\infty} \mathcal{P}_n(q) s^{-1-n},$$

where we are allowed to exchange the Laplacian with the sum because of the uniform convergence shown in Proposition 7.1.4.

The series of the right Fueter kernel follows similarly from the Taylor series expansion of the right Cauchy kernel. □

Because of the above considerations, we can define the Fueter kernel operator series by formally replacing q in the Fueter kernel series by the operator T with commuting components.

Definition 7.1.6 (Fueter kernel operator series)**.** Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$ with $\|T\| < |s|$, we define the *left Fueter kernel operator series* as

$$\sum_{m\geq 2} \mathcal{P}_m(T) s^{-1-m}$$

and the *right Fueter kernel operator series* as

$$\sum_{m\geq 2} s^{-1-m}\mathcal{P}_m(T).$$

Proposition 7.1.7. *Let $T = T_0 + \sum_{\ell=1}^{3} T_\ell e_\ell \in \mathcal{BC}(X)$. For $s \in \mathbb{H}$ with $\|T\| < |s|$, we have*

$$\sum_{m\geq 2} \mathcal{P}_m(T) s^{-1-m} = -4(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-2} \tag{7.5}$$

and

$$\sum_{m\geq 2} s^{-1-m}\mathcal{P}_m(T) = -4\mathcal{Q}_{c,s}(T)^{-2}(s\mathcal{I} - \overline{T}) \tag{7.6}$$

with $\mathcal{Q}_{c,s}(T) = s^2\mathcal{I} - 2sT_0 + T\overline{T}$, where $\overline{T} = T_0 - \sum_{\ell=1}^{3} T_\ell e_\ell$.

Proof. Using Theorem 2.1.22 and Theorem 2.2.2 we get

$$\begin{aligned}\sum_{m\geq 2} \mathcal{P}_m(q) s^{-1-m} &= \Delta \sum_{m=0}^{+\infty} q^m s^{-1-m} \\ &= \Delta S_L^{-1}(s,q) = -4(s-\overline{q})(s^2 - 2\mathrm{Re}(q)s + |q|^2)^{-2}.\end{aligned}$$

The fact that the components of T commute allows us to substitute T for q; thus we get the statement. □

Remark 7.1.8. We point out an important fact related to the Fueter mapping theorem in integral form. As we could observe in the proof of Theorem 2.2.2, the computation

$$\begin{aligned}&-\Delta(q^2-2q\mathrm{Re}(s)+|s|^2)^{-1}(q-\overline{s})\\&=-4(s-\overline{q})(s^2-2\mathrm{Re}(q)s+|q|^2)^{-2}\end{aligned}$$

can be carried out in a natural way only if we write $S_L^{-1}(s,q)$ in form II. The function $S_L^{-1}(s,q)$ can be written in two different ways because the components of q commute. Unfortunately, form II involves the term $|q|^2=\overline{q}q=q\overline{q}$, and this identity requires that the components of x commute. This has implications on the functional calculus when one tries to replace q by an operator T. In this case we have to require that the components of T commute. When we write $S_L^{-1}(s,q)$ in form I, then we can replace q by an operator T whose components do not necessarily commute, because only actual powers q and not powers of its components appear. But in this case the explicit computation of $\Delta S_L^{-1}(s,q)$ does not yield a simple closed from.

Recall that the S-resolvent set of $T\in\mathcal{BC}(X)$ can, by Theorem 4.5.6, be characterized as

$$\rho_S(T)=\left\{s\in\mathbb{H}:\ \ \mathcal{Q}_{c,s}(T)^{-1}\in\mathcal{B}(X)\right\},$$

where

$$\mathcal{Q}_{c,s}(T)=s^2\mathcal{I}-2sT_0+T\overline{T}.$$

Definition 7.1.9 (F-resolvent operators)**.** Let $T\in\mathcal{BC}(X)$. For $s\in\rho_S(T)$, we define the *left F-resolvent operator* as

$$F_L(s,T):=-4\left(s\mathcal{I}-\overline{T}\right)\mathcal{Q}_{c,s}(T)^{-2}, \tag{7.7}$$

and the *right F-resolvent operator* as

$$F_R(s,T):=-4\mathcal{Q}_{c,s}(T)^{-2}\left(s\mathcal{I}-\overline{T}\right). \tag{7.8}$$

Lemma 7.1.10. *Let $T\in\mathcal{BC}(X)$.*

(i) *The left F-resolvent operator $F_L(s,T)$ is a $\mathcal{B}(X)$-valued right slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

(ii) *The right F-resolvent operator $F_R(s,T)$ is a $\mathcal{B}(X)$-valued left slice hyperholomorphic function of the variable s on $\rho_S(T)$.*

Proof. The statement follows by computations that are similar to those in Lemma 3.1.11. □

If f is a left or right slice hyperholomorphic function, then the function $\breve{f}=\Delta f$ is a left, resp. right, Fueter regular function by the Fueter mapping theorem. We showed in Theorem 2.2.6 that $\breve{f}$ can be represented as the integral transform of f involving the left, resp. right, Fueter kernel. If we replace in this integral representation the Fueter kernel by the F-resolvent operator, we obtain the F-functional calculus.

Definition 7.1.11 (The F-functional calculus for bounded operators)**.** Let $T \in \mathcal{BC}(X)$ and set $ds_j = ds(-j)$ for $j \in \mathbb{S}$. For every function $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, we set

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\, ds_j\, f(s), \tag{7.9}$$

where U is an arbitrary bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and $j \in \mathbb{S}$ is an arbitrary imaginary unit.

For every function $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_R(\sigma_S(T))$, we set

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_j)} f(s)\, ds_j\, F_R(s,T) \tag{7.10}$$

with U and j as above.

Theorem 7.1.12. *The F-functional calculus is well defined, that is, the integrals in (7.9) and (7.10) depend neither on the imaginary unit $j \in \mathbb{S}$ nor on the slice Cauchy domain U.*

Proof. We discuss only the case $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$, since the other one follows by analogous arguments.

Since $F_L(s,T)$ is a right slice hyperholomorphic function in s and f is left slice hyperholomorphic, the independence from U follows from the Cauchy integral theorem, cf. also the proof of Theorem 3.2.6.

In order to show the independence from the imaginary unit, we choose $j, i \in \mathbb{S}$ with $j \neq i$ and two bounded slice Cauchy domains U_p, U_s with $\sigma_S(T) \subset U_q$, $\overline{U_q} \subset U_s$, and $\overline{U_s} \subset \mathcal{D}(f)$. Then every $s \in \partial(U_s \cap \mathbb{C}_j)$ belongs to the unbounded slice Cauchy domain $\mathbb{H} \setminus U_q$. Since we have $\lim_{q\to+\infty} F_L(q,T) = 0$, the slice hyperholomorphic Cauchy formula implies

$$\begin{aligned} F_L(s,T) &= \frac{1}{2\pi} \int_{\partial(\mathbb{H}\setminus U_q\cap\mathbb{C}_i)} F_L(q,T)\, dq_i\, S_R^{-1}(q,s) \\ &= \frac{1}{2\pi} \int_{\partial(U_q\cap\mathbb{C}_i)} F_L(q,T)\, dq_i\, S_L^{-1}(s,q), \end{aligned}$$

where the last identity holds because $\partial(\mathbb{H}\setminus U_q\cap\mathbb{C}_i) = -\partial(U_q\cap\mathbb{C}_i)$ and $S_R^{-1}(q,s) = -S_L^{-1}(s,q)$. Thus

$$\begin{aligned} \breve{f}(T) &= \frac{1}{2\pi} \int_{\partial(U_s\cap\mathbb{C}_j)} F_L(s,T)\, ds_j f(s) \\ &= \frac{1}{2\pi} \int_{\partial(U_s\cap\mathbb{C}_j)} \left(\frac{1}{2\pi} \int_{\partial(U_q\cap\mathbb{C}_i)} F_L(q,T)\, dq_i\, S_L^{-1}(s,q) \right) ds_j f(s). \end{aligned}$$

Since the integrand is continuous and the path of integration is bounded, Fubini's theorem allows us to exchange the order of integration, and we obtain

$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi}\int_{\partial(U_q\cap\mathbb{C}_i)} F_L(q,T)\,dq_i\left(\frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_j)} S_L^{-1}(s,q)\,ds_j f(s)\right)\\ &= \frac{1}{2\pi}\int_{\partial(U_s\cap\mathbb{C}_i)} F_L(q,T)\,dq_i\,f(q). \qquad \square\end{aligned}$$

Remark 7.1.13. In the above theorem we have shown that the F-functional calculus is well defined, in the sense that the integrals in (7.9) and (7.10) depend neither on the imaginary unit $j\in\mathbb{S}$ nor on the slice Cauchy domain U. However, if $f\in\mathcal{SH}_L(U)$, it might happen that $\breve{f}=\Delta f=\Delta g=\breve{g}$ for some $g\in\mathcal{SH}_L(U)$ with $f\neq g$, and we did not show that then $\breve{f}(T)=\breve{g}(T)$. The function $f-g$ is in this case a left slice hyperholomorphic function in $\ker\Delta$. If U is connected, we hence have $f(s)-g(s)=s\alpha+\beta$ with $\alpha,\beta\in\mathbb{H}$ and so

$$\begin{aligned}\breve{f}(T)-\breve{g}(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,(f(s)-g(s))\\ &= \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,(s\alpha-\beta),\end{aligned}$$

where we used Cauchy's integral theorem and the slice hyperholomorphicity of $F_L(s,T)$ in s in order to change the domain of integration to $B_r(0)$ with $\|T\|<r$. From the power series expansion $F_L(s,T)=\sum_{m\geq 2}\mathcal{P}_m(T)s^{-1-m}$ in (7.5), we conclude now that

$$\begin{aligned}\breve{f}(T)-\breve{g}(T) &= \frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)}\sum_{m\geq 2}P_m(T)s^{-1-m}ds_j(s\alpha+\beta)\\ &= \sum_{m\geq 2}P_m(T)\frac{1}{2\pi}\int_{\partial(B_r(0)\cap\mathbb{C}_j)} s^{-1-m}ds_j(s\alpha+\beta)=0\end{aligned}$$

by Cauchy's integral theorem since the integrand tends to 0 at infinity. If, however, U is not connected, then $f(s)-g(s)=\sum_{\ell=1}^{n}\chi_{U_\ell}(s)(s\alpha_\ell-\beta_\ell)$, where $U_\ell,\ell=1,\dots,n$ are the connected components of U and χ_{U_ℓ} denotes the characteristic function of U_ℓ. Hence, we have

$$\breve{f}(T)-\breve{g}(T)=\sum_{\ell=1}^{n}\frac{1}{2\pi}\int_{\partial(U_\ell\cap\mathbb{C}_j)} F_L(s,T)\,ds_j(s\alpha_\ell-\beta_\ell),$$

and we cannot use the same arguments as above in order to show that the terms in the sum vanish, because $F_L(s,T)$ is not slice hyperholomorphic on $\mathbb{H}\setminus U_\ell$ since this set contains part of the S-spectrum of T. In this case, the terms vanish because of Lemma 7.4.1. The proof of this lemma makes, however, use of the monogenic functional calculus by A. McIntosh. This functional calculus makes the assumptions that $T = T_1e_1 + T_2e_2 + T_3e_3$, that is, $T_0 = 0$, with commuting components T_ℓ that have real spectrum. Only if this condition is satisfied we have $\breve{f}(T) = \breve{g}(T)$ also if U is not connected. If this condition is not satisfied, it is in general not true and it is easy to construct counter-examples even using matrices in $\mathbb{H}^{2\times 2}$.

We conclude this section with some algebraic properties of the F-functional calculus.

Proposition 7.1.14. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum.*

(i) *If $\breve{f} = \Delta f$ and $\breve{g} = \Delta g$ with $f, g \in \mathcal{SH}_L(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(\breve{f}a + \breve{g})(T) = \breve{f}(T)a + \breve{g}(T).$$

(ii) *If $\breve{f} = \Delta f$ and $\breve{g} = \Delta g$ with $f, g \in \mathcal{SH}_R(\sigma_S(T))$ and $a \in \mathbb{H}$, then*

$$(a\breve{f} + \breve{g})(T) = a\breve{f}(T) + \breve{g}(T).$$

Proof. The above identities follow immediately from the linearity of the integrals in (7.9), resp. (7.10). □

Proposition 7.1.15. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum.*

(i) *Let $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_L(\sigma_S(T))$ and assume that $f(q) = \sum_{\ell=0}^{+\infty} q^\ell a_\ell$ with $a_\ell \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$\breve{f}(T) = \sum_{\ell=2}^{+\infty} \mathcal{P}_\ell(T) a_\ell.$$

(ii) *Let $\breve{f} = \Delta f$ with $f \in \mathcal{SH}_R(\sigma_S(T))$ and assume that $f(q) = \sum_{\ell=0}^{+\infty} a_\ell q^\ell$ with $a_\ell \in \mathbb{H}$, where this series converges on a ball $B_r(0)$ with $\sigma_S(T) \subset B_r(0)$. Then*

$$\breve{f}(T) = \sum_{\ell=2}^{+\infty} a_\ell \mathcal{P}_\ell(T).$$

Proof. We prove (i), but (ii) is shown similarly. We choose an imaginary unit $j \in \mathbb{S}$ and a radius $0 < R < r$ such that $\sigma_S(T) \subset B_R(0)$. Then the series expansion of f converges uniformly on $\partial(B_R(0) \cap \mathbb{C}_j)$, and so

$$\begin{aligned} \breve{f}(T) &= \frac{1}{2\pi} \int_{\partial(B_R(0)\cap\mathbb{C}_j)} F_L(s,T)\, ds_j \, \sum_{\ell=0}^{+\infty} s^\ell a_\ell \\ &= \frac{1}{2\pi} \sum_{\ell=0}^{+\infty} \int_{\partial(B_R(0)\cap\mathbb{C}_j)} F_L(s,T)\, ds_j \, s^\ell a_\ell. \end{aligned}$$

Replacing $F_L(s,T)$ by its series expansion, we further obtain

$$\begin{aligned}\breve{f}(T) &= \frac{1}{2\pi}\sum_{\ell=0}^{+\infty}\int_{\partial(B_R(0)\cap\mathbb{C}_j)}\sum_{k=2}^{+\infty}\mathcal{P}_k(T)s^{-1-k}\,ds_j\,s^\ell a_\ell\\ &= \frac{1}{2\pi}\sum_{\ell=0}^{+\infty}\sum_{k=2}^{+\infty}\mathcal{P}_k(T)\int_{\partial(B_R(0)\cap\mathbb{C}_j)}s^{-1-k}\,ds_j\,s^\ell a_\ell = \sum_{\ell\geq 0}\mathcal{P}_\ell(T)a_\ell,\end{aligned}$$

because the integral $\int_{\partial(B_R(0)\cap\mathbb{C}_j)}s^{-1-k}\,ds_j\,s^\ell$ equals 2π if $\ell = k$, and 0 otherwise. □

Theorem 7.1.16. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1,2,3$, have real spectrum. Let $\breve{f} = \Delta f$ and $\breve{f}_m = \Delta f_m, m \in \mathbb{N}$, with $f, f_m \in \mathcal{SH}_L(\sigma_S(T))$ and assume that f_m tends uniformly to f on an axially symmetric open set O that contains $\sigma_S(T)$. Then $\breve{f}_m$ tends uniformly to $\breve{f}$ on $\sigma_S(T)$ and $\breve{f}_m(T) \to \breve{f}(T)$ in $\mathcal{B}(X)$.*

Proof. Let U be a slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset O$ and choose $j \in \mathbb{S}$. Then

$$\breve{f}_m(q) - \breve{f}(q) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}(f_m(s) - f(s))\,ds_j\,F_L(s,q).$$

Since $\operatorname{dist}(\sigma_S(T), \partial(U\cap\mathbb{C}_j)) > 0$, we have

$$C := \sup_{\substack{s\in\partial(U\cap\mathbb{C}_j)\\ q\in\sigma_S(T)}} |F_L(s,q)| < +\infty,$$

and so

$$\left|\breve{f}_m(q) - \breve{f}(q)\right| \leq \frac{C}{2\pi}|\partial(U\cap\mathbb{C}_j)| \sup_{s\in\partial(U\cap\mathbb{C}_j)}|f_m(s) - f(s)|,$$

and hence $\breve{f}_m \to \breve{f}$ uniformly on $\sigma_S(T)$. Similarly, we have

$$\begin{aligned}&\left\|\breve{f}_m(T) - \breve{f}(T)\right\| = \left\|\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)}(f_m(s) - f(s))\,ds_j\,F_L(s,T)\right\|\\ &\leq \frac{|\partial(U\cap\mathbb{C}_j)|}{2\pi}\sup_{s\in\partial(U\cap\mathbb{C}_j)}\|F_L(s,T)\|\sup_{s\in\partial(U\cap\mathbb{C}_j)}|f_m(s) - f(s)| \overset{m\to+\infty}{\longrightarrow} 0.\end{aligned}$$

□

7.2 Bounded Perturbations of the *F*-Resolvent

We point out that the inverses of the F-resolvents

$$\begin{aligned}F_L(s,T)^{-1} &= -\frac{1}{4}\mathcal{Q}_{c,s}(T)S_L(s,T) = -\frac{1}{4}\mathcal{Q}_{c,s}(T)^2(s\mathcal{I} - \overline{T})^{-1},\\ F_R(s,T)^{-1} &= -\frac{1}{4}S_R(s,T)\mathcal{Q}_{c,s}(T) = -\frac{1}{4}(s\mathcal{I} - \overline{T})^{-1}\mathcal{Q}_{c,s}(T)^2,\end{aligned}$$

exist for every $s \notin \sigma_L(\overline{T})$.

Lemma 7.2.1. *Let $T, Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, $Z = Z_1e_1 + Z_2e_2 + Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1, 2, 3$, have real spectrum. Assume that $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Then there exists a positive constant $C_{Z,T}(s)$ depending on s and also on the operators T and Z such that*

$$\|F_L(s,T)^{-1} - F_L(s,Z)^{-1}\| \le C_{Z,T}(s)(|s| + \vartheta)^{-2}\|T - Z\|, \tag{7.11}$$

$$\|F_R(s,T)^{-1} - F_R(s,Z)^{-1}\| \le C_{Z,T}(s)(|s| + \vartheta)^{-2}\|T - Z\|, \tag{7.12}$$

where $\vartheta := \max\{\|T\|, \|Z\|\}$.

Proof. Since we have for $s \in \rho_S(T)$ that

$$F_L(s,T) := -4S_L^{-1}(s,T)\mathcal{Q}_{c,s}(T)^{-1}, \tag{7.13}$$

the inverse $F_L(s,T)^{-1}$ exists for $s \notin \sigma_L(\overline{T})$, and it is given by

$$F_L(s,T)^{-1} = -\frac{1}{4}\mathcal{Q}_{c,s}(T)S_L(s,T), \tag{7.14}$$

while the inverse of the operator $F_L(s,Z)$ exists for $s \notin \sigma_L(\overline{Z})$, and it is given by

$$F_L(s,Z)^{-1} = -\frac{1}{4}\mathcal{Q}_{c,s}(Z)S_L(s,Z). \tag{7.15}$$

We have

$$\begin{aligned}
&-4\left(F_L(s,T)^{-1} - F_L(s,Z)^{-1}\right)\\
&= \mathcal{Q}_{c,s}(T)S_L(s,T) - \mathcal{Q}_{c,s}(Z)S_L(s,Z)\\
&= \mathcal{Q}_{c,s}(T)S_L(s,T) - \mathcal{Q}_{c,s}(T)S_L(s,Z)\\
&\quad + \mathcal{Q}_{c,s}(T)S_L(s,Z) - \mathcal{Q}_{c,s}(Z)S_L(s,Z)\\
&= \mathcal{Q}_{c,s}(T)\ [S_L(s,T) - S_L(s,Z)]\\
&\quad + [-s(T + \overline{T}) + T\overline{T} + s(Z + \overline{Z}) - Z\overline{Z})]\ S_L(s,Z)\\
&= \mathcal{Q}_{c,s}(T)\ [S_L(s,T) - S_L(s,Z)]\\
&\quad + [s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(\overline{T} - \overline{Z})]\ S_L(s,Z),
\end{aligned}$$

and taking the norm, we get

$$\begin{aligned}
&\left\|F_L(s,T)^{-1} - F_L(s,Z)^{-1}\right\|\\
&\le (|s|^2 + 2|s|\,\|T\| + \|T\overline{T}\|)\|S_L(s,T) - S_L(s,Z)\|\\
&\quad + \left[2|s|\,\|Z - T\| + \|T - Z\|(\|\overline{T}\| + \|Z\|)\right]\ \|S_L(s,Z)\|\\
&\le (|s| + \vartheta)^2\|S_L(s,T) - S_L(s,Z)\|\\
&\quad + \left[2(|s| + \vartheta)\|Z - T\|)\right]\ \|S_L(s,Z)\|.
\end{aligned}$$

Now observe that

$$(|s|+\vartheta)^{-1}\|S_L(s,Z)\| \\ \leq (|s|+\vartheta)^{-1}\big[|s|+\|(s\mathcal{I}-\overline{Z})\|\,\|Z\|\,\|(s\mathcal{I}-\overline{Z})^{-1}\|\big] := M_Z(s), \tag{7.16}$$

where $M_Z(s)$ is a continuous function, since $s \notin \sigma_L(\overline{Z})$. Using Lemma 4.6.3, we get

$$\big\|F_L(s,T)^{-1}-F_L(s,Z)^{-1}\big\| \leq \frac{1}{4}[K_Z(s)+2M(s)](|s|+\vartheta)^2\|Z-T\|, \tag{7.17}$$

and $K_{T,Z}(s)$ is defined in (4.25). We can argue similarly for $F_R(s,T)^{-1}$. □

Lemma 7.2.2. *Let $T,Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1+T_2e_2+T_3e_3$, $Z = Z_1e_1+Z_2e_2+Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1,2,3$, have real spectrum. Assume that $s \in \rho_S(T)$, let $s \notin \sigma_L\left(\overline{T}\right) \cup \sigma_L\left(\overline{Z}\right)$, and suppose that*

$$\|T-Z\| < \frac{1}{C_{Z,T}(s)}(|s|+\vartheta)^{-2}\|F_L(s,T)\|^{-1},$$

where $C_{Z,T}(s)$ is defined in Lemma 7.2.1. Then $s \in \rho_S(Z)$ and

$$F_L(s,Z)-F_L(s,T) = F_L(s,T)\sum_{m=1}^{+\infty}\left[\left(F_L(s,T)^{-1}-F_L(s,Z)^{-1}\right)F_L^{-1}(s,T)\right]^m.$$

An analogous statement holds for $F_R^{-1}(s,T)$.

Proof. By Lemma 3.1.12 and formula (3.2) with

$$A := (F_L(s,T))^{-1}, \quad B := (F_L(s,Z))^{-1}, \quad A^{-1} = F_L(s,T), \tag{7.18}$$

we have for $B^{-1} = F_L(s,Z)$ that

$$F_L(s,Z) = F_L(s,T)\sum_{m=0}^{+\infty}\left[(F_L(s,T))^{-1}-(F_L(s,Z))^{-1}F_L(s,T)\right]^m. \tag{7.19}$$

Using the hypothesis, we find that the series converges, since

$$\|(F_L(s,T)-F_L(s,Z))F_L^{-1}(s,T)\| \\ \leq \|(F_L(s,T)-F_L(s,Z))\|\,\|F_L^{-1}(s,T)\| \\ \leq C_{Z,T}(s)(|s|+\vartheta)^2\|Z-T\|\,\|F_L^{-1}(s,T)\| < 1.$$ □

Theorem 7.2.3. *Let $T,Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1+T_2e_2+T_3e_3$, $Z = Z_1e_1+Z_2e_2+Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1,2,3$, have real spectrum. Assume that $s \in \rho_S(T)$ and $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Let $\varepsilon > 0$ and let* *us consider the* ε*-neighborhood* $B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$ *of* $\sigma_S(T)\cup\sigma_L(\overline{T})$*. Then there exists* $\delta>0$ *such that, for* $\|T-Z\|<\delta$*, we have*

$$\sigma_S(Z)\subseteq B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$$

and

$$\|F_L(s,Z)-F_L(s,T)\|<\varepsilon,\quad \textit{for } s\notin B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T})).$$

An analogous statement holds for the right F*-resolvent.*

Proof. Let $T,Z\in\mathcal{BC}(X)$ and let $\varepsilon>0$. Thanks to Lemma 3.1.12 there exists $\eta>0$ such that if

$$\|T-Z\|<\eta,$$

then $\sigma_L(\overline{Z})\subset B_\varepsilon(\sigma_L(\overline{T}))$. So we can always choose η such that $\sigma_L(\overline{Z})\subset B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$. Consider the function $C_{Z,T}(s)$ defined in Lemma 7.2.1. The constant

$$C_\varepsilon:=\sup_{s\notin B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))}C_{Z,T}(s)$$

is finite because $s\notin B(\sigma_S(T)\cup\sigma_L(\overline{T},\varepsilon)$ and

$$\lim_{s\to\infty}\|(s\mathcal{I}-\overline{Z})^{-1}\|=\lim_{s\to\infty}\|(s\mathcal{I}-\overline{T})^{-1}\|=0.$$

Observe that since $s\in\rho_S(T)$, the map $s\mapsto\|F_L(s,T)\|$ is continuous, and

$$\lim_{s\to\infty}\|F_L(s,T)\|=0,$$

and so for s in the complement set of $B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T})$ we have that there exists a positive constant M_ε such that

$$\|F_L(s,T)\|\leq M_\varepsilon.$$

From Lemma 7.2.2, we find that if $\delta_1>0$ is such that

$$\|Z-T\|<\frac{1}{C_\varepsilon M_\varepsilon}:=\delta_3,$$

then $s\in\rho_S(Z)$ and

$$\begin{aligned}\|F_L^{-1}(s,Z)-F_L^{-1}(s,T)\|&\\ &\leq\frac{\|F_L^{-1}(s,T)\|^2\ \|F_L(s,T)-F_L(s,Z)\|}{1-\|F_L^{-1}(s,T)\|\ \|F_L(s,T)-\mathcal{F}(s,Z)\|}\\ &\leq\frac{M_\varepsilon^2C_{n,\varepsilon}\|Z-T\|}{1-M_\varepsilon C_{n,\varepsilon}\|Z-T\|}<\varepsilon\end{aligned}$$

if we take

$$\|Z-T\|<\delta_4:=\frac{\varepsilon}{C_{n,\varepsilon}(M_\varepsilon^2+\varepsilon M_\varepsilon)}.$$

To get the statement it suffices to set $\delta=\min\{\eta,\delta_3,\delta_4\}$. □

Theorem 7.2.4. *Let $T, Z \in \mathcal{BC}(X)$ be such that $T = T_1e_1+T_2e_2+T_3e_3$, $Z = Z_1e_1+Z_2e_2+Z_3e_3$, and assume that the operators T_ℓ, Z_ℓ, $\ell = 1, 2, 3$, have real spectrum. Let $f \in \mathcal{SH}_L(\sigma_S(T))$ (or $f \in \mathcal{SH}_R(\sigma_S(T))$) and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that for $\|Z - T\| < \delta$, we have $f \in \mathcal{SH}_L(\sigma_S(Z))$ (or $f \in \mathcal{SH}_R(\sigma_S(Z))$) and*

$$\|\breve{f}(Z) - \breve{f}(T)\| < \varepsilon.$$

Proof. We recall that

$$\breve{f}(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\, ds_j\, f(s),$$

where $U \subset \mathbb{H}$ is a bounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and where $j \in \mathbb{S}$. Let furthermore $B_\varepsilon(\sigma_S(T)\cup\sigma_L(\overline{T}))$ be contained in U. By Lemma 7.2.3 there exists $\delta_1 > 0$ such that $\sigma_S(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently $f \in \mathcal{SH}_L(\sigma_S(Z))$ if $\|Z-T\| < \delta_1$. By Lemma 7.2.3, $F_L(s,T)$ is uniformly close to $F_L(s,Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_j)$ for $j \in \mathbb{S}$ if $\|Z - T\|$ is small enough. So for some positive $\delta \le \delta_1$, we get

$$\|\breve{f}(T) - \breve{f}(Z)\| \le \frac{1}{2\pi}\Big\|\int_{\partial(U\cap\mathbb{C}_j)} [F_L(s,T) - F_L(s,Z)]\, ds_j\, f(s)\Big\| < \varepsilon.$$

We can argue similarly if $f \in \mathcal{SH}_R(U)$. □

7.3 The F-Resolvent Equations

The F-resolvents satisfy a relation that can be considered a generalized resolvent equation. In particular, they allow one to show that the F-functional calculus is capable of generating projections onto subspaces that are invariant under the operator.

Theorem 7.3.1 (Left and right F-resolvent equations)**.** *Let $T \in \mathcal{BC}(X)$ and let $s \in \rho_S(T)$. The F-resolvent operators satisfy the equations*

$$F_L(s,T)s - TF_L(s,T) = -4\mathcal{Q}_{c,s}(T)^{-1} \tag{7.20}$$

and

$$sF_R(s,T) - F_R(s,T)T = -4\mathcal{Q}_{c,s}(T)^{-1}. \tag{7.21}$$

Proof. We prove relation (7.20), since (7.21) follows with similar computations. We have

$$F_L(s,T)s = -4(s\mathcal{I} - \overline{T})s\mathcal{Q}_{c,s}(T)^{-2}$$

and

$$TF_L(s,T) = -4(Ts - T\overline{T})\mathcal{Q}_{c,s}(T)^{-2}.$$

Taking the difference, we obtain

$$\begin{aligned} F_L(s,T)s - TF_L(s,T) &= -4(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})\mathcal{Q}_{c,s}(T)^{-2} \\ &= -4\mathcal{Q}_{c,s}(T)^{-1}. \end{aligned}$$

□

Lemma 7.3.2. *Let $T \in \mathcal{BC}(X)$. For $q, s \in \rho_S(T)$, with $s \notin [q]$ and with the position $\mathcal{Q}_s(q) = q^2 - 2\mathrm{Re}(s)q + |s|^2$, the following equation holds:*

$$F_R(s,T)S_L^{-1}(q,T) + S_R^{-1}(s,T)F_L(q,T) - 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1} = \big[\,(F_R(s,T) - F_L(q,T))\,q - \bar{s}\,(F_R(s,T) - F_L(q,T))\,\big]\mathcal{Q}_s(q)^{-1}. \tag{7.22}$$

Proof. We consider the S-resolvent equation (3.7) and write the S-resolvent operators in the form (4.20) and (4.21) for operators with commuting components. If we multiply it on the left by $-4\mathcal{Q}_{c,s}(T)^{-1}$, we get

$$F_R(s,T)S_L^{-1}(q,T) = \Big[\,\big(F_R(s,T) + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q - \bar{s}\,\big(F_R(s,T) + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\Big]\mathcal{Q}_s(q)^{-1}.$$

If we multiply the S-resolvent equation on the right by $-4\mathcal{Q}_{c,q}(T)^{-1}$, we get

$$S_R^{-1}(s,T)F_L(q,T) = \Big[\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} - F_L(q,T)\big)\,q - \bar{s}\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} - F_L(q,T)\big)\Big]\mathcal{Q}_s(q)^{-1}.$$

Adding these two equations yields

$$\begin{aligned}
&F_R(s,T)S_L^{-1}(q,T) + S_R^{-1}(s,T)F_L(q,T)\\
&= \left[(F_R(s,T) - F_L(q,T))\,q - \bar{s}\,(F_R(s,T) - F_L(q,T))\right]\mathcal{Q}_s(q)^{-1}\\
&\quad + \Big[\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q\\
&\qquad - \bar{s}\Big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\Big)\Big]\mathcal{Q}_s(q)^{-1}.
\end{aligned}$$

The proof is concluded if we verify that

$$\begin{aligned}
&\Big[\,\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q\\
&\quad - \bar{s}\Big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\Big)\Big]\mathcal{Q}_s(q)^{-1}\\
&= 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1}.
\end{aligned}$$

This follows from

$$\begin{aligned}
&\big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\big)\,q\\
&\quad - \bar{s}\Big(S_R^{-1}(s,T)(-4)\mathcal{Q}_{c,q}(T)^{-1} + 4\mathcal{Q}_{c,s}(T)^{-1}S_L^{-1}(q,T)\Big)\\
&= -4\Big[\Big(\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}(q\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1}\Big)q\\
&\quad - \bar{s}\Big(\mathcal{Q}_{c,s}(T)^{-1}(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}(q\mathcal{I} - \overline{T})\mathcal{Q}_{c,q}(T)^{-1}\Big)\Big]
\end{aligned}$$

$$
\begin{aligned}
&= -4\Big[\mathcal{Q}_{c,s}(T)^{-1}(s-q)\mathcal{Q}_{c,q}(T)^{-1}q - \bar{s}\mathcal{Q}_{c,s}(T)^{-1}(s-q)\mathcal{Q}_{c,q}(T)^{-1}\Big]\\
&= -4\Big[\mathcal{Q}_{c,s}(T)^{-1}(sq-q^2)\mathcal{Q}_{c,q}(T)^{-1} - \mathcal{Q}_{c,s}(T)^{-1}(\bar{s}s-\bar{s}q)\mathcal{Q}_{c,q}(T)^{-1}\Big]\\
&= -4\Big[\mathcal{Q}_{c,s}(T)^{-1}(sq-q^2-\bar{s}s+\bar{s}q)\mathcal{Q}_{c,q}(T)^{-1}\Big]\\
&= 4\mathcal{Q}_{c,s}(T)^{-1}(q^2-2\mathrm{Re}(s)q+|s|^2)\mathcal{Q}_{c,q}(T)^{-1} = 4\mathcal{Q}_{c,s}(T)^{-1}\mathcal{Q}_{c,q}(T)^{-1}\mathcal{Q}_s(q). \square
\end{aligned}
$$

Theorem 7.3.3 (The F-resolvent equation)**.** *Let $T \in \mathcal{BC}(X)$. For all quaternions $q, s \in \rho_S(T)$ with $s \notin [q]$, the following equation holds:*

$$
\begin{aligned}
&F_R(s,T)S_L^{-1}(q,T) + S_R^{-1}(s,T)F_L(q,T)\\
&\quad - \frac{1}{4}\Big(sF_R(s,T)F_L(q,T)q - sF_R(s,T)TF_L(q,T)\\
&\quad - F_R(s,T)TF_L(q,T)q + F_R(s,T)T^2F_L(q,T)\Big)\\
&= \big[\,(F_R(s,T)-F_L(q,T))\,q - \bar{s}\,(F_R(s,T)-F_L(q,T))\,\big]\mathcal{Q}_s(q)^{-1}.
\end{aligned}
\tag{7.23}
$$

Proof. The identities (7.20) and (7.21) yield

$$
\begin{aligned}
-4^2\mathcal{Q}_s(T)^{-1}\mathcal{Q}_q(T)^{-1} &= (sF_R(s,T)-F_R(s,T)T)(F_L(q,T)q - TF_L(q,T))\\
&= sF_R(s,T)F_L(q,T)q - sF_R(s,T)TF_L(q,T)\\
&\quad - F_R(s,T)TF_L(q,T)q + F_R(s,T)T^2F_L(q,T).
\end{aligned}
$$

Applying this identity in (7.22), we obtain (7.23). □

7.4 The Riesz Projectors for the *F*-Functional Calculus

In the sequel we will need the following lemma, which is based on the monogenic functional calculus; see the book [159] for more details (or some of the papers [160, 161, 166], where the calculus was introduced).

Lemma 7.4.1. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1,2,3$, have real spectrum. Let G be a bounded slice Cauchy domain such that $(\partial G) \cap \sigma_S(T) = \emptyset$. For every $j \in \mathbb{S}$, we then have*

$$
\int_{\partial(G\cap\mathbb{C}_j)} s\,ds_j\,F_R(s,T) = 0 \quad \textit{and} \quad \int_{\partial(G\cap\mathbb{C}_j)} F_L(q,T)\,dq_j\,q = 0.
$$

Proof. Since $\mathcal{P}_1(q) = \Delta q = 0$, we have

$$
\int_{\partial(G\cap\mathbb{C}_j)} s\,ds_j\,F_R(s,p) = \mathcal{P}_1(p) = 0
$$

and

$$
\int_{\partial(G\cap\mathbb{C}_j)} F_L(q,p)\,dq_j\,q = \mathcal{P}_1(q) = 0
$$

for all $p \notin \partial G$ and $j \in \mathbb{S}$. We observe that at this point we need the Cauchy–Fueter functional calculus, described in the next section, to represent $F_L(p,T)$. We consider only the case of $F_L(p,T)$; the other case can be shown in a similar way. We recall that $F_L(p,q)$ is left Fueter regular in q on $\mathbb{H}\setminus[p]$ for every p, so we can use Definition 7.5.6 and write

$$F_L(p,T) = \int_{\partial\Omega} \mathcal{G}(\omega,T) D\omega F_L(p,\omega),$$

where the open set Ω contains the left spectrum of T, $\mathcal{G}(\omega,T)$ is the Cauchy–Fueter resolvent operator. Using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\partial(G\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \\
&= \int_{\partial(G\cap\mathbb{C}_j)} \int_{\partial\Omega} \Big(\mathcal{G}(\omega,T) D\omega F_L(q,\omega)\Big)\, dq_j\, q \\
&= \int_{\partial\Omega} \mathcal{G}(\omega,T) D\omega \Big(\int_{\partial(G\cap\mathbb{C}_j)} F_L(p,\omega)\, dp_j\, q\Big) = 0,
\end{aligned}$$

which concludes the proof. □

Theorem 7.4.2. *Let $T \in \mathcal{BC}(X)$ be such that $T = T_1e_1 + T_2e_2 + T_3e_3$, and assume that the operators T_ℓ, $\ell = 1,2,3$, have real spectrum. Let $\sigma_S(T) = \sigma_1 \cup \sigma_2$ with*

$$\operatorname{dist}(\sigma_1,\sigma_2) > 0.$$

Let $G_1, G_2 \subset \mathbb{H}$ be two bounded slice Cauchy domains such that $\sigma_1 \subset G_1$ and $\overline{G_1} \subset G_2$ and such that $\operatorname{dist}(G_2,\sigma_2) > 0$. *Then the operator*

$$\begin{aligned}
\breve{P} &:= -\frac{1}{4(2\pi)} \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j q^2 \\
&= -\frac{1}{4(2\pi)} \int_{\partial(G_2\cap\mathbb{C}_j)} s^2 ds_j F_R(s,T)
\end{aligned}$$

is a projection that commutes with T, i.e., we have

$$\breve{P}^2 = \breve{P} \quad \text{and} \quad T\breve{P} = \breve{P}T.$$

Proof. If we multiply the F-resolvent equation (7.23) by s on the left and by q on the right, we get

$$\begin{aligned}
&sF_R(s,T)S_L^{-1}(q,T)q + sS_R^{-1}(s,T)F_L(q,T)q \\
&\quad - \frac{1}{4}\Big(s^2F_R(s,T)F_L(q,T)q^2 - s^2F_R(s,T)TF_L(q,T)q \\
&\quad - sF_R(s,T)TF_L(q,T)q^2 + sF_R(s,T)T^2F_L(q,T)q\Big) \\
&= s\left[(F_R(s,T) - F_L(q,T))\, q - \bar{s}\,(F_R(s,T) - F_L(q,T))\right] \mathcal{Q}_s(q)^{-1}q.
\end{aligned}$$

If we multiply this equation by ds_j on the left, integrate it over $\partial(G_2 \cap \mathbb{C}_j)$ with respect to ds_j, and then multiply it by dq_j on the right and integrate over $\partial(G_1 \cap \mathbb{C}_j)$ with respect to dq_j, we obtain

$$\begin{aligned}
&\int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, F_R(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} S_L^{-1}(q,T)\, dq_j\, q \\
&\quad + \int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, S_R^{-1}(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \\
&\quad - \frac{1}{4}\Bigg(\int_{\partial(G_2\cap\mathbb{C}_j)} s^2\, ds_j\, F_R(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q^2 \\
&\quad - \int_{\partial(G_2\cap\mathbb{C}_j)} s^2\, ds_j\, F_R(s,T)T \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \\
&\quad - \int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, F_R(s,T)T \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q^2 \\
&\quad + \int_{\partial(G_2\cap\mathbb{C}_j)} s\, ds_j\, F_R(s,T)T^2 \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q \Bigg) \\
&= \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\Big[(F_R(s,T) - F_L(q,T))\, q \\
&\qquad\qquad - \bar{s}\,(F_R(s,T) - F_L(q,T)) \Big] \mathcal{Q}_s(q)^{-1}\, dq_j\, q.
\end{aligned}$$

By Lemma 7.4.1, this simplifies to

$$\begin{aligned}
&- \frac{1}{4} \int_{\partial(G_2\cap\mathbb{C}_j)} s^2\, ds_j\, F_R(s,T) \int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\, dq_j\, q^2 \\
&= \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\Big[(F_R(s,T) - F_L(q,T))\, q \\
&\quad - \bar{s}\,(F_R(s,T) - F_L(q,T)) \Big] \mathcal{Q}_s(q)^{-1}\, dq_j q,
\end{aligned}$$

which equals

$$\begin{aligned}
4(2\pi)^2 \breve{P}^2 &= \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\big[F_R(s,T)q - \bar{s}F_R(s,T)\big]\, \mathcal{Q}_s(q)^{-1}\, dq_j\, q \\
&\quad - \int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\big[F_L(q,T)q - \bar{s}F_L(q,T)\big]\, \mathcal{Q}_s(q)^{-1}\, dq_j\, q.
\end{aligned}$$

Since $\overline{G_1} \subset G_2$, for every $s \in \partial(G_2 \cap \mathbb{C}_j)$ the functions

$$q \mapsto q\mathcal{Q}_s(q)^{-1} = q(q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$$

and

$$q \mapsto \mathcal{Q}_s(q)^{-1} = (q^2 - 2\mathrm{Re}(s)q + |s|^2)^{-1}$$

are intrinsic slice hyperholomorphic on $\overline{G_1}$. By the Cauchy integral theorem, we therefore have

$$\int_{\partial(G_1\cap\mathbb{C}_j)} q\mathcal{Q}_s(q)^{-1}\,dq_j\,q = 0 \quad \text{and} \quad \int_{\partial(G_1\cap\mathbb{C}_j)} \mathcal{Q}_s(q)^{-1}\,dq_j\,q = 0,$$

and it follows that

$$\int_{\partial(G_2\cap\mathbb{C}_j)} ds_j \int_{\partial(G_1\cap\mathbb{C}_j)} s\big[F_R(s,T)q - \bar{s}F_R(s,T)\big]\,\mathcal{Q}_s(q)^{-1}\,dq_j\,q = 0.$$

Thus, we obtain

$$\begin{aligned}\breve{P}^2 = -\frac{1}{4(2\pi)^2}&\int_{\partial(G_2\cap\mathbb{C}_j)} s\,ds_j\\ &\times\int_{\partial(G_1\cap\mathbb{C}_j)} \big[\,(F_L(q,T)q - \bar{s}F_L(q,T))\,\big]\,\mathcal{Q}_s(q)^{-1}\,dq_j q,\end{aligned}$$

and by exchanging the order of integration and applying Lemma 4.1.2, we finally obtain

$$\breve{P}^2 = -\frac{1}{4(2\pi)}\int_{\partial(G_1\cap\mathbb{C}_j)} F_L(q,T)\,dq_j\,q^2 = \breve{P}.$$

We furthermore deduce from (7.21) that

$$\begin{aligned}\breve{P}T &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^2\,ds_j\,F_R(s,T)T\\ &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^3\,ds_j\,F_R(s,T) - \frac{16}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^2\,ds_j\,\mathcal{Q}_s(T)^{-2}.\end{aligned}$$

Since $s^3\chi_{G_1}(s)$ is intrinsic slice hyperholomorphic, this equals

$$\begin{aligned}\breve{P} &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} s^2\,ds_j\,F_R(s,T)T\\ &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,s^3 - \frac{16}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} \mathcal{Q}_s(T)^{-2}\,ds_j\,s^2\\ &= -\frac{4}{2\pi}\int_{\partial(G_1\cap\mathbb{C}_j)} TF_L(s,T)\,ds_j\,s^2 = T\breve{P},\end{aligned}$$

where we applied (7.20) in the third identity. □

7.5 The Cauchy–Fueter Functional Calculus

We recall the Cauchy formula for Cauchy–Fueter regular functions (or Fueter regular functions), and we use it to define the Cauchy–Fueter functional calculus.

We will not give all the details but just the main definitions. The function $\mathcal{G}(q)$ defined by

$$\mathcal{G}(q) = \frac{q^{-1}}{|q|^2} = \frac{\bar{q}}{|q|^4} \tag{7.24}$$

is called the *Cauchy–Fueter kernel*, and it is left and right Fueter regular on $\mathbb{H}\setminus\{0\}$.

Theorem 7.5.1 (Cauchy–Fueter formula). *Let f be a left Fueter regular function on an open set that contains $\overline{U}$. If U is a four-dimensional compact, oriented manifold with smooth boundary ∂U, then*

$$f(q) = \frac{1}{2\pi^2} \int_{\partial U} \mathcal{G}(p-q)\, Dp\, f(p), \quad q \in U, \tag{7.25}$$

the differential form Dp is given by $Dp = \eta(p)dS(p)$, where $\eta(p)$ is the outer unit normal to ∂U at the point p, and $dS(p)$ is the scalar element of surface area on ∂U. If f is a right Fueter regular function on U, then

$$f(q) = \frac{1}{2\pi^2} \int_{\partial U} f(p)\, Dp\, \mathcal{G}(p-q), \quad q \in U. \tag{7.26}$$

Fueter regular functions do not admit power series expansions, but there exist series expansions in terms of suitable homogeneous functions. For every triple $\nu = (n_1, n_2, n_3)$ with $|\nu| := n_1 + n_2 + n_3 = n$, we define

$$\partial_\nu = \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} \quad \text{and} \quad \mathcal{G}_\nu(q) = \partial_\nu \mathcal{G}(q),$$

where $\mathcal{G}(q)$ is the Cauchy–Fueter kernel. Furthermore, we define the set $\Gamma(\nu)$ as the set of all n-tuples $(\lambda_1, \dots, \lambda_n)$ with exactly n_1 entries that equal 1, exactly n_2 entries that equal 2, and exactly n_3 entries that equal 3. In other words, if we set $\lambda_1, \dots, \lambda_{n_1} = 1$ and $\lambda_{n_1+1}, \dots, \lambda_{n_1+n_2} = 2$ and $\lambda_{n_1+n_2+1}, \dots \lambda_n = 3$, then

$$\Gamma(\nu) = \{(\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) : \quad \sigma \in \text{perm}(n)\},$$

where perm(n) denotes the group of permutations of n elements. Furthermore, let us denote by σ_n the set of all triples $\nu = (n_1, n_2, n_3)$ with $|v| = n_1 + n_2 + n_3 = n$. For every $n > 0$, the set σ_n contains $\frac{1}{2}(n+1)(n+2)$ triples. If $n = 0$, we set $\nu = \emptyset$ and $P_\nu \equiv 1$. For every $\nu \in \sigma_n$ and for $q = q_0 + \sum_{\ell=1}^{3} q_\ell e_\ell$, we define

$$P_\nu(q) = \frac{1}{n!} \sum_{(\lambda_1,\dots,\lambda_n)\in\Gamma(n)} (q_0 e_{\lambda_1} - q_{\lambda_1}) \cdots (q_0 i_{\lambda_n} - q_{\lambda_n}).$$

The polynomials $P_\nu(q)$ play the role of the powers z^n in the Taylor expansion of a function $\sum_{n=0}^{+\infty} a_n z^n$ holomorphic at the origin.

Let $\mathcal{U}_n$ be the quaternionic right vector space of functions $f : \mathbb{H} \to \mathbb{H}$ that are left Fueter regular and homogeneous of degree $n \geq 0$ over $\mathbb{R}$, i.e., such that $f(\alpha q) = \alpha^n f(q)$ for every $\alpha \in \mathbb{R}$. We have the following result.

Theorem 7.5.2. *The polynomials P_ν, $\nu \in \sigma_n$, are left Fueter regular and form a basis for $\mathcal{U}_n$. Moreover, if $f \in \mathcal{U}_n$, then*

$$f(q) = \sum_{\nu \in \sigma_n} (-1)^n P_\nu(q) \partial_\nu f(0). \tag{7.27}$$

If f is right Fueter regular, then the polynomials P_ν are right Fueter regular and

$$f(q) = \sum_{\nu \in \sigma_n} (-1)^n \partial_\nu f(0) P_\nu(q).$$

The introduction of the polynomials P_ν and the derivatives $\mathcal{G}_\nu$ allows one to prove two results that generalize the Taylor and the Laurent expansion series.

Theorem 7.5.3. *Let $f : U \subseteq \mathbb{H} \to \mathbb{H}$ be left Fueter regular, $p \in U$. Then there exists a ball $|q-p| < \delta$ with radius $\delta < \operatorname{dist}(p, \partial U)$ in which f can be represented by a uniformly convergent series of the form*

$$f(q) = \sum_{n=0}^{+\infty} \sum_{\nu \in \sigma_n} P_\nu(q-p) a_\nu,$$

where

$$a_\nu = (-1)^n \partial_\nu f(p) = \frac{1}{2\pi^2} \int_{|q-p|=\delta} \mathcal{G}_\nu(q-p)\,, Dq\, f(q).$$

If $f : U \to \mathbb{H}$ is right Fueter regular, then

$$f(q) = \sum_{n=0}^{+\infty} \sum_{\nu \in \sigma_n} a_\nu P_\nu(q-p),$$

where

$$a_\nu = (-1)^n \partial_\nu f(p) = \frac{1}{2\pi^2} \int_{|q-p|=\delta} f(q)\, Dq\, \mathcal{G}_\nu(q-p).$$

Let T be a quaternionic bounded linear operator with commuting components on a two-sided quaternionic Banach space X. Recall that such a set is denoted by $\mathcal{BC}(X)$. In this case, we consider the function $\mathcal{G}(q,p) := G(p-q)$ written in series expansion, and we replace p by T. We get

$$\mathcal{G}(q,T) = \sum_{n \geq 0} \sum_{\nu \in \sigma_n} P_\nu(T) \mathcal{G}_\nu(q) = \sum_{n \geq 0} \sum_{\nu \in \sigma_n} \mathcal{G}_\nu(q) P_\nu(T). \tag{7.28}$$

The expansions hold for $\|T\| < |q|$ and define a bounded operator. It is natural to give the following definition:

Definition 7.5.4. The maximal open set $\rho(T)$ in $\mathbb{H}$ on which the series (7.28) converges in the operator norm to a bounded operator is called the *resolvent set of T*. The *spectral set $\sigma(T)$ of T* is defined as the complement set in $\mathbb{H}$ of the resolvent set.

Definition 7.5.5. A function f is said to be *locally right Cauchy–Fueter regular* on the spectral set $\sigma(T)$ of an operator $T \in \mathcal{BC}(X)$ if there exists an open set $U \subset \mathbb{H}$ containing $\sigma(T)$ whose boundary ∂U is a rectifiable 3-cell and such that f is regular in every connected component of U. We denote by $\mathcal{CF}_L(\sigma(T))$ the set of locally left Cauchy–Fueter regular functions on $\sigma(T)$. We denote by $\mathcal{CF}_R(\sigma(T))$ the set of locally right Cauchy–Fueter regular functions on $\sigma(T)$.

Definition 7.5.6 (The Cauchy–Fueter functional calculus)**.** Let $f \in \mathcal{CF}_L(\sigma(T))$ and $T \in \mathcal{BC}(V)$ be such that $T = T_1 e_1 + T_2 e_2 + T_3 e_3$, and assume that the operators T_ℓ, $\ell = 1, 2, 3$, have real spectrum. We define

$$f(T) := \frac{1}{2\pi^2} \int_{\partial U} \mathcal{G}(q, T) Dq f(q).$$

Let $f \in \mathcal{CF}_R(\sigma(T))$ and $T \in \mathcal{BC}(V)$. We define

$$f(T) := \frac{1}{2\pi^2} \int_{\partial U} f(q) Dq \mathcal{G}(q, T),$$

where U is an open set in $\mathbb{H}$ containing $\sigma(T)$ as in Definition 7.5.5.

The definition is well posed, since the integrals that define the Cauchy–Fueter functional calculus do not depend on the open set U. This is a consequence of the Cauchy–Fueter regularity of the operator-valued function $\mathcal{G}(q, T)$. We point out that the series expansion of the Cauchy-Fueter resolvent operator in (7.28) has a closed form if T has commuting components, namely

$$\mathcal{G}(q, T) = (q\mathcal{I} - T)^{-2}(\overline{q\mathcal{I} - T})^{-1}.$$

This operator is then associated with the left spectrum of T. A closed form of the sum $\mathcal{G}(q, T)$ in the general case, without the assumption that the components of T commute, would naturally lead to a notion of spectrum of the operator T for the case of Fueter regularity. But if we want to replace operators with noncommuting components, then it is not clear what is the closed formula for the Cauchy–Fueter resolvent. Observe that for the slice hyperholomorphic case, a closed form of the series $\sum_{n\geq 0} T^n s^{-1-n}$ can be found. It is

$$\sum_{n\geq 0} T^n s^{-1-n} = -(T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I})^{-1}(T - \overline{s}\mathcal{I})$$

for $\|T\| < |s|$, and this identity does not depend on the commutativity of the components of T. This is one of its great advantages.

7.6 Comments and Remarks

Comments on the references. The F-functional calculus has been developed in the papers [20, 54, 78, 81, 86]. It is based on the Fueter mapping theorem in integral form, and it is a monogenic functional calculus in the spirit of McIntosh and collaborators, see [159–161, 164, 166], but it is associated with slice hyperholomorphicity. The W-functional calculus is also a monogenic functional calculus, and it was introduced in the paper [70].

7.6.1 The F-Functional Calculus for n-Tuples of Operators

The F-functional calculus can be extended to the case of n-tuples of commuting operators. Because of the structure of the Fueter–Sce mapping theorem in integral form, the F-functional calculus depends on the dimension of the Clifford algebra. The Fueter–Sce–Qian mapping theorem, one should say, was proved by Michele Sce [187] for n odd and by Tao Qian [176] for the case in which n is even. Later on, Fueter's theorem was generalized to the case in which a slice hyperholomorphic function f is multiplied by a monogenic homogeneous polynomial of degree k, see [162] [172] [173], and to the case in which the function f is defined on an open set U not necessarily chosen in the upper complex plane; see [175–177]. We need to recall the definition of monogenic functions.

Definition 7.6.1 (Monogenic functions)**.** Let U be an open set in $\mathbb{R}^{n+1}$. A real differentiable function $f : U \to \mathbb{R}_n$ is *left monogenic* if

$$\frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} f(x) = 0.$$

It is *right monogenic* if

$$\frac{\partial}{\partial x_0} f(x) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x) e_i = 0.$$

We recall the theorem of Sce to produce monogenic functions from complex-valued functions (the case of odd dimension of $\mathbb{R}_n$):

We consider a holomorphic function $f(z)$ that depends on a complex variable $z = u + \iota v$ in an open set of the upper complex half-plane. We write

$$f(z) = f_0(u, v) + \iota f_1(u, v),$$

where f_0 and f_1 are $\mathbb{R}$-valued functions that satisfy the Cauchy–Riemann system. For every paravector x such that $u + \iota v$ belongs to the domain of f, we replace the complex imaginary unit ι in $f(z) = f_0(u, v) + \iota f_1(u, v)$ by the Clifford imaginary unit $j = \underline{x}/|\underline{x}|$ and we set $u = x_0$ and $v = |\underline{x}|$. We then define

$$f(x) = f_0(x_0, |\underline{x}|) + j f_1(x_0, |\underline{x}|).$$

This function in slice hyperholomorphic with values in the Clifford algebra $\mathbb{R}_n$ (or slice monogenic). Then we apply the $(n-1)/2$th power of the Laplace operator $\Delta^{(n-1)/2}$ in dimension $n+1$ to f. The function

$$\breve{f}(x_0, |\underline{x}|) := \Delta^{(n-1)/2}(f_0(x_0, |\underline{x}|) + j f_1(x_0, |\underline{x}|))$$

is then left monogenic, i.e., it is in the kernel of the Dirac operator. If we replace $f_0(x_0, |\underline{x}|) + jf_1(x_0, |\underline{x}|)$ by $f_0(x_0, |\underline{x}|) + f_1(x_0, |\underline{x}|)j$ in the above procedure, we obtain a right monogenic function.

Proposition 7.6.2. *Let n be an odd number and let $x, s \in \mathbb{R}^{n+1}$ be such that $x \notin [s]$. Let $S_L^{-1}(s,x)$ and $S_R^{-1}(s,x)$ be the slice hyperholomorphic Cauchy kernels in form II. Then:*

- *The function $\Delta^{\frac{n-1}{2}} S_L^{-1}(s,x)$ is a left monogenic function in the variable x and right slice hyperholomorphic in s.*
- *The function $\Delta^{\frac{n-1}{2}} S_R^{-1}(s,x)$ is a right monogenic function in the variable x and left slice hyperholomorphic in s.*

Based on the explicit computations of functions

$$(s,x) \mapsto \Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) \quad \text{and} \quad (s,x) \mapsto \Delta^{\frac{n-1}{2}} S_R^{-1}(s,x),$$

for $s \notin [x]$, we define the F_n-kernels.

Definition 7.6.3 (The F_n-kernels)**.** Let n be an odd number and let $x, s \in \mathbb{R}^{n+1}$. We define, for $s \notin [x]$, the F_n^L*-kernel* as

$$F_n^L(s,x) := \Delta^{\frac{n-1}{2}} S_L^{-1}(s,x) = \gamma_n (s-\bar{x})(s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}},$$

and the F_n^R*-kernel* as

$$F_n^R(s,x) := \Delta^{\frac{n-1}{2}} S_R^{-1}(s,x) = \gamma_n (s^2 - 2\mathrm{Re}(x)s + |x|^2)^{-\frac{n+1}{2}}(s-\bar{x}),$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} \Big[\Big(\frac{n-1}{2}\Big)!\Big]^2. \tag{7.29}$$

Theorem 7.6.4 (The Fueter–Sce mapping theorem in integral form)**.** *Let $U \subset \mathbb{R}^{n+1}$ be a slice Cauchy domain and choose $j \in \mathbb{S}$. Let n be an odd number.*

(a) *If $f \in \mathcal{SM}_L(O)$ for some set O with $\overline{U} \subset O$, then the left monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} F_n^L(s,x)\, ds_j\, f(s). \tag{7.30}$$

(b) *If $f \in \mathcal{SM}_R(O)$ for some set O with $\overline{U} \subset O$, then the right monogenic function $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ admits the integral representation*

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, F_n^R(s,x). \tag{7.31}$$

The integrals depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

We refer to the section on comments and remarks at the end of chapter on the properties of the S-functional calculus for bounded operators for the definition of the Clifford algebra $\mathbb{R}_n$ and for the functional setting on paravector operators $T = T_0 + T_1 e_1 + \cdots + T_n e_n$. In the sequel, we will consider bounded paravector operators T, with commuting components $T_\ell \in \mathcal{B}(X)$ for $\ell = 0, 1, \ldots, n$. Such subsets of $\mathcal{B}(X_n)$ will be denoted by $\mathcal{BC}^{0,1}(X_n)$. The F-functional calculus is based on the commutative version of the S-spectrum (often called F-spectrum in the literature). So we define the F-resolvent operators.

Definition 7.6.5 (F-resolvent operators)**.** Let n be an odd number and let $T \in \mathcal{BC}^{0,1}(X_n)$. For $s \in \rho_S(T)$ we define the *left F-resolvent operator* by

$$F_n^L(s,T) := \gamma_n (s\mathcal{I} - \overline{T}) \mathcal{Q}_{c,s}(T)^{-\frac{n+1}{2}}, \tag{7.32}$$

and the *right F-resolvent operator* by

$$F_n^R(s,T) := \gamma_n \mathcal{Q}_{c,s}(T)^{-\frac{n+1}{2}} (s\mathcal{I} - \overline{T}), \tag{7.33}$$

where the constants γ_n are given by (7.29).

Let $T \in \mathcal{BC}^{0,1}(X_n)$. We denote by $\mathcal{SM}_L(\sigma_S(T))$, $\mathcal{SM}_R(\sigma_S(T))$ the set of all left (or right) slice hyperholomorphic functions f with $\sigma_S(T) \subset \mathcal{D}(f)$.

Definition 7.6.6 (The F-functional calculus for bounded operators)**.** Let n be an odd number, let $T \in \mathcal{BC}^{0,1}(X_n)$ be such that $T = T_1 e_1 + T_2 e_2 + T_3 e_3$, and assume that the operators T_ℓ, $\ell = 1, \ldots, n$, have real spectrum. Set $ds_j = ds/j$. For every function $f \in \mathcal{SM}_L(\sigma_S(T))$, we define

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} F_n^L(s,T)\, ds_j\, f(s). \tag{7.34}$$

For every $f \in \mathcal{SM}_R(\sigma_S(T))$, we define

$$\breve{f}(T) := \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_j)} f(s)\, ds_j\, F_n^R(s,T), \tag{7.35}$$

where $j \in \mathbb{S}$ and U is a slice Cauchy domain U.

The definition of the F-functional calculus is well posed, since the integrals in (7.9) and (7.35) depend neither on U nor on the imaginary unit $j \in \mathbb{S}$.

7.6.2 The Inverse Fueter–Sce Mapping Theorem

In recent years, new problems related to the inversion of the Fueter–Sce mapping theorem have been solved. For the sake of simplicity here we mention the inversion problem of axially monogenic functions. The results can be found in [83], while more general cases are treated in the papers [84, 85, 87, 103].

Definition 7.6.7 (Axially monogenic function)**.** Let U be an axially symmetric open set in $\mathbb{R}^{n+1}$, and let $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in U$, for $\underline{\omega} \in \mathbb{S}$. Assume that $\breve{f} : U \to \mathbb{R}_n$ is a monogenic function, i.e., it is in the kernel of the Dirac operator. We say that $\breve{f}$ is an axially monogenic function if there exist two functions $A = A(x_0, r)$ and $B = B(x_0, r)$, independent of $\underline{\omega} \in \mathbb{S}^{n-1}$ and with values in $\mathbb{R}_n$, such that

$$\breve{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r).$$

We denote by $\mathcal{AM}(U)$ the set of left axially monogenic functions on the open set U.

The problem is as follows: suppose that $\breve{f}$ is an axially monogenic function and f is a slice monogenic function such that $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. Determine a slice monogenic function f in terms of the components $A(x_0, r)$ and $B(x_0, r)$ of the axially monogenic function $\breve{f}(x) = A(x_0, r) + \underline{\omega} B(x_0, r)$. It is important to give the definition of a Fueter–Sce primitive.

Definition 7.6.8 (Fueter–Sce primitive)**.** Let n be an odd number and let $U \subseteq \mathbb{R}^{n+1}$ be an axially symmetric domain. Suppose that $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is a left slice monogenic function. We say that f is a *Fueter–Sce primitive* of $\breve{f} \in \mathcal{M}(U)$ if $\Delta^{\frac{n-1}{2}} f(x) = \breve{f}(x)$ on U.

The definition of a Fueter–Sce primitive of $\breve{f}$ is well posed, since slice monogenic functions are infinitely differentiable. The monogenic Cauchy kernel $\mathcal{G}(x)$ is defined for $x \in \mathbb{R}^{n+1} \setminus \{0\}$ as

$$\mathcal{G}(x) = \frac{1}{A_{n+1}} \, \frac{\overline{x}}{|x|^{n+1}}, \tag{7.36}$$

where $A_{n+1} = \frac{2\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})}$ is the area of the unit sphere in $\mathbb{R}^{n+1}$. As we will see, $\mathcal{G}(x)$ plays a crucial role in the inversion formula of monogenic functions.

Definition 7.6.9 (The kernels $\mathcal{N}_n^+(x)$ and $\mathcal{N}_n^-(x)$)**.** Let $\mathcal{G}(x)$ be the monogenic Cauchy kernel defined in (7.36) with $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$, and for $\underline{y} = r\underline{\omega} \in \mathbb{R}^n$ we assume $r = 1$ and $\underline{\omega} \in \mathbb{S}^{n-1}$. We define the kernels

$$\mathcal{N}_n^+(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x - \underline{\omega})\, dS(\underline{\omega}), \quad \mathcal{N}_n^-(x) = \int_{\mathbb{S}^{n-1}} \mathcal{G}(x - \underline{\omega})\, \underline{\omega}\, dS(\underline{\omega}), \tag{7.37}$$

where $dS(\underline{\omega})$ is the scalar element of surface area of $\mathbb{S}^{n-1}$.

Theorem 7.6.10 (The structure of the Fueter–Sce primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$)**.** *Let n be an odd number and denote by $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ the Fueter–Sce primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$, respectively. Consider the functions*

$$\mathcal{W}_n^+(x_0) := \frac{\mathcal{C}_n}{\mathcal{K}_n} \, D^{-(n-1)} \frac{x_0}{(x_0^2 + 1)^{(n+1)/2}},$$
$$\mathcal{W}_n^-(x_0) := -\frac{\mathcal{C}_n}{\mathcal{K}_n} \, D^{-(n-1)} \frac{1}{(x_0^2 + 1)^{(n+1)/2}},$$

where $\frac{\mathcal{C}_n}{\mathcal{K}_n}$ is a given constant and the symbol $D^{-(n-1)}$ stands for $(n-1)$ integrations with respect to x_0. Then, by replacing x_0 by x in $\mathcal{W}_n^+(x_0)$ and in $\mathcal{W}_n^-(x_0)$, we get $\mathcal{W}_n^+(x)$ and $\mathcal{W}_n^-(x)$, respectively. Moreover, the functions $\mathcal{W}_n^+(x)$ and $\mathcal{W}_n^-(x)$ are extendable to slice monogenic functions defined for all $x \in \{x_0 + r\underline{\omega} : (x_0, r) \neq (0,1)\}$.

The Fueter–Sce primitives of $\mathcal{N}_n^+$ and $\mathcal{N}_n^-$ can be explicitly computed. For example, when $n = 3$ they are given by

$$\mathcal{W}_3^+(x) = \frac{1}{2\pi} \arctan x, \quad \mathcal{W}_3^-(x) = -\frac{1}{2\pi} x \arctan x.$$

Theorem 7.6.11 (The inverse Fueter–Sce mapping theorem)**.** *Let $\breve{f}(x) = A(x_0,\rho) + \underline{\omega}B(x_0,\rho)$ be an axially monogenic function defined on an axially symmetric domain $U \subseteq \mathbb{R}^{n+1}$. Let Γ be the boundary of an open bounded subset $\mathcal{V}$ of the half-plane $\mathbb{R} + \underline{\omega}\mathbb{R}^+$ and let*

$$V = \{x = u + \underline{\omega}v,\ (u,v) \in \mathcal{V},\ \underline{\omega} \in \mathbb{S}^{n-1}\} \subset U.$$

Moreover, suppose that Γ is a regular curve whose parametric equations $y_0 = y_0(s)$, $\rho = \rho(s)$ are expressed in terms of the arc length $s \in [0,L]$, $L > 0$. Then the function

$$\begin{aligned} f(x) = &\int_\Gamma \mathcal{W}_n^-\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0\, A(y_0,\rho) - d\rho\, B(y_0,\rho)) \\ &- \int_\Gamma \mathcal{W}_n^+\Big(\frac{1}{\rho}(x - y_0)\Big)\rho^{n-2}(dy_0 B(y_0,\rho) - d\rho A(y_0,\rho)). \end{aligned} \tag{7.38}$$

is a Fueter–Sce primitive of $\breve{f}(x)$ on V, where $\mathcal{W}_n^+$ and $\mathcal{W}_n^-$ are as in Theorem 7.6.10.

This theorem has several generalizations, and this topic is still under investigation.

Chapter 8

The F-Functional Calculus for Unbounded Operators

Similar to the S-functional calculus, we can also extend the F-functional calculus to unbounded operators by suitably transforming the operator and the function and then applying the theory for bounded operators. Let us first specify the type of operator for which this is possible.

Let $X = X_{\mathbb{R}} \otimes \mathbb{H}$ be a quaternionic two-sided Banach space and let $T_\ell : \mathcal{D}(T_\ell) \subset X_{\mathbb{R}} \to X_{\mathbb{R}}$ be linear closed operators for $\ell = 0, \ldots, 3$ such that $T_\ell T_\kappa = T_\kappa T_\kappa$ on $\mathcal{D}(T_\ell T_\kappa) \cap \mathcal{D}(T_\kappa T_\ell)$ for $\ell, \kappa = 0, \ldots, 3$. Then

$$\mathcal{D}(T) = \bigcap_{\ell=0}^{3} \mathcal{D}(T_\ell)$$

is the domain of the quaternionic right linear operators

$$T = T_0 + \sum_{\ell=1}^{3} e_\ell T_\ell : \quad \mathcal{D}(T) \subset X \to X$$

and

$$\overline{T} = T_0 - \sum_{\ell=1}^{3} e_\ell T_\ell : \quad \mathcal{D}(T) \subset X \to X.$$

Definition 8.0.1. We denote the set of closed right linear operators with commuting components as discussed above by $\mathcal{KC}(X)$.

For operators in $\mathcal{KC}(X)$, we can characterize their *S-resolvent set* and *S-spectrum* just as in Theorem 4.5.6 as

$$\rho_S(T) = \left\{ s \in \mathbb{H} : \quad \mathcal{Q}_{c,s}(T)^{-1} \in \mathcal{B}(X) \right\}$$

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_8

with

$$\mathcal{Q}_{c,s}(T) = (s^2\mathcal{I} - 2sT_0 + T\overline{T})^{-1}.$$

Definition 8.0.2 (The F-resolvent operators for the unbounded operators)**.** Let $T \in \mathcal{KC}(X)$. For $s \in \rho_S(T)$, we define the *(left) F-resolvent operator* as

$$F_L(s,T) := -4(s\mathcal{I} - \overline{T})\mathcal{Q}_{c,s}(T)^{-2}.$$

8.1 Relations Between F-Resolvent Operators

The following results are important since they will lead us to the definition of F-functional calculus for unbounded operators.

Proposition 8.1.1. *Let $T \in \mathcal{KC}(X)$ and assume that there exists a point $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and set $A := (T - \alpha\mathcal{I})^{-1}$ as in Theorem 5.2.3. For $p = (s-\alpha)^{-1}$, we have*

$$\mathcal{Q}_{c,p}(A)^{-1} = \left(A\overline{A}\right)^{-1}\mathcal{Q}_{c,s}(T)^{-1}p^{-2} = \mathcal{Q}_{c,\alpha}(T)\mathcal{Q}_{c,s}(T)^{-1}p^{-2}$$

and

$$\mathcal{Q}_{c,p}(A)^{-2} = \left(A\overline{A}\right)^{-2}\mathcal{Q}_{c,s}(T)^{-2}p^{-4} = \mathcal{Q}_{c,\alpha}(T)^2\mathcal{Q}_{c,s}(T)^{-2}p^{-4}.$$

Proof. Observe that

$$\begin{aligned}
&p^2\mathcal{I} - p\left(A + \overline{A}\right) + A\overline{A} \\
&= \left(p^2\left(A\overline{A}\right)^{-1} - p\left(A + \overline{A}\right)\left(A\overline{A}\right)^{-1} + \mathcal{I}\right)\left(A\overline{A}\right) \\
&= \left(p^2\left(A\overline{A}\right)^{-1} - p\left(A^{-1} + \overline{A}^{-1}\right) + \mathcal{I}\right)\left(A\overline{A}\right),
\end{aligned}$$

where we have used the fact that $\left(A\overline{A}\right)^{-1} = A^{-1}\overline{A}^{-1} = \overline{A}^{-1}A^{-1}$. Recalling that $A := (T - \alpha\mathcal{I})^{-1}$ and $\overline{A} := \left(\overline{T} - \alpha\mathcal{I}\right)^{-1}$, we obtain

$$\overline{A}^{-1}A^{-1} = \alpha^2\mathcal{I} - \alpha\left(T + \overline{T}\right) + T\overline{T} = \mathcal{Q}_{c,\alpha}(T)$$

and

$$A^{-1} + \overline{A}^{-1} = T + \overline{T} - 2\alpha\mathcal{I},$$

so that we obtain

$$\begin{aligned}
\mathcal{Q}_{c,p}(A)^{-1} &= \left(p^2\mathcal{I} - p\left(A + \overline{A}\right) + A\overline{A}\right)^{-1} \\
&= \left(A\overline{A}\right)^{-1}\left(p^2\left(A\overline{A}\right)^{-1} - p\left(A^{-1} + \overline{A}^{-1}\right) + \mathcal{I}\right)^{-1} \\
&= \mathcal{Q}_{c,\alpha}(T)\left(p^2\ \mathcal{Q}_{c,\alpha}(T) - p\left(T + \overline{T} - 2\alpha\mathcal{I}\right) + \mathcal{I}\right)^{-1}.
\end{aligned}$$

Observe now that $T+\overline{T}=2T_0$ and $T\overline{T}=\sum_{\ell=0}^{3}T_\ell^2$ are scalar operators and thus commute with p, so we have

$$\mathcal{Q}_{c,p}(A)^{-1}=\mathcal{Q}_{c,\alpha}(T)\left(\mathcal{Q}_{c,\alpha}(T)-p^{-1}\left(T+\overline{T}-2\alpha\mathcal{I}\right)+p^{-2}\mathcal{I}\right)^{-1}p^{-2}.$$

Finally, we also get

$$\begin{aligned}&\mathcal{Q}_{c,\alpha}(T)-p^{-1}\left(T+\overline{T}-2\alpha\mathcal{I}\right)+p^{-2}\mathcal{I}\\&=\alpha^2\mathcal{I}-\alpha\left(T+\overline{T}\right)+T\overline{T}-p^{-1}\left(T+\overline{T}\right)+2\alpha p^{-1}\mathcal{I}+p^{-2}\mathcal{I}\\&=T\overline{T}-\left(p^{-1}+\alpha\right)\left(T+\overline{T}\right)+\left(p^{-2}+\alpha^2+2\alpha p^{-1}\right)\mathcal{I}\\&=T\overline{T}-s\left(T+\overline{T}\right)+s^2\mathcal{I}=\mathcal{Q}_{c,s}(T),\end{aligned}$$

and so

$$\mathcal{Q}_{c,p}\left(A\right)^{-1}=\mathcal{Q}_{c,\alpha}(T)\mathcal{Q}_{c,s}\left(T\right)^{-1}p^{-2}.$$

Since $\alpha\in\mathbb{R}$, we have

$$sp=s\left(s-\alpha\right)^{-1}=ps,$$

and so $\mathcal{Q}_{c,\alpha}(T)$, $\mathcal{Q}_s\left(T\right)$, and p^{-2} commute mutually. Therefore, we also obtain

$$\mathcal{Q}_{c,p}\left(A\right)^{-2}=\mathcal{Q}_{\alpha}(A)^2\mathcal{Q}_s\left(T\right)^{-2}p^{-4}.$$ □

From Proposition 8.1.1, we deduce now two important relations between the F-resolvents of T and A.

Theorem 8.1.2. *Let $T\in\mathcal{KC}(X)$, let $\alpha\in\rho_S(T)\cap\mathbb{R}\neq\emptyset$, and define $A=(T-\alpha\mathcal{I})^{-1}$. For $s\in\rho_S(T)$ with $s\notin\sigma_L(T)$ and $p=(s-\alpha)^{-1}$, we have*

$$F_L(s,T)=-\overline{A}A^2F_L(p,A)p^3. \tag{8.1}$$

Proof. We recall that $F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\mathcal{Q}_{c,p}(A)^{-2}$. Due to Proposition 8.1.1, we have

$$F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\overline{A}^{-2}A^{-2}\mathcal{Q}_{c,s}(T)^{-2}p^{-4}.$$

Since $s=p^{-1}+\alpha$ commutes with p, we have

$$F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\overline{A}^{-2}A^{-2}p^{-1}\mathcal{Q}_{c,s}(T)^{-2}p^{-3},$$

and so

$$F_L(p,A)=-4\left(p\mathcal{I}-\overline{A}\right)\overline{A}^{-2}A^{-2}p^{-1}\left(s\mathcal{I}-\overline{T}\right)^{-1}\left(s\mathcal{I}-\overline{T}\right)\mathcal{Q}_{c,s}(T)^2p^{-3}.$$

Observe that $\overline{A}^{-2}A^{-2}=\mathcal{Q}_{\alpha,s}(T)^2$ is a scalar operator since $\alpha\in\mathbb{R}$ and hence commutes with p and so also with $\left(p\mathcal{I}-\overline{A}\right)$. Since $F_L(s,T)=-4\left(s\mathcal{I}-\overline{T}\right)\mathcal{Q}_{c,s}(T)^{-2}$, we obtain

$$F_L(p,A)=\overline{A}^{-2}A^{-2}\left(p\mathcal{I}-\overline{A}\right)p^{-1}\left(s\mathcal{I}-\overline{T}\right)^{-1}F_L(s,T)p^{-3}.$$

Replacing $s = p^{-1} + \alpha$, we get

$$\begin{aligned} F_L(p,A) &= \overline{A}^{-2}A^{-2}\left(p\mathcal{I} - \overline{A}\right)p^{-1}\left(p^{-1}\mathcal{I} + \alpha\mathcal{I} - \overline{T}\right)^{-1}F_L(s,T)p^{-3} \\ &= \overline{A}^{-2}A^{-2}\left(p\mathcal{I} - \overline{A}\right)p^{-1}\left(p^{-1}\mathcal{I} - \overline{A}^{-1}\right)^{-1}F_L(s,T)p^{-3} \end{aligned}$$

and hence

$$F_L(p,A) = -\overline{A}^{-2}A^{-2}\left(p\mathcal{I} - \overline{A}\right)p^{-1}\left(p\left(p\mathcal{I} - \overline{A}\right)^{-1}\overline{A}\right)F_L(s,T)p^{-3}.$$

We thus get the statement because

$$\begin{aligned} F_L(p,A) &= -\left(\overline{A}\right)^{-2}A^{-2}\overline{A}F_L(s,T)p^{-3} \\ &= -\overline{A}\left(\overline{A}\right)^{-2}A^{-2}F_L(s,T)p^{-3} = -\left(\overline{A}\right)^{-1}A^{-2}F_L(s,T)p^{-3}. \end{aligned}$$ □

Theorem 8.1.3. *Let $T \in \mathcal{KC}(X)$, let $\alpha \in \rho_S(T)\cap\mathbb{R} \neq \emptyset$, and define $A = (T - \alpha\mathcal{I})^{-1}$. For $s \in \rho_S(T)$ and $p = (s-\alpha)^{-1}$, we have*

$$(A\overline{A})^{-1}F_L(p,A)p^4 = -4p\mathcal{Q}_{c,s}(T)^{-1} - F_L(s,T). \tag{8.2}$$

Proof. We recall that

$$A\overline{A} = (\alpha^2\mathcal{I} - \alpha(T + \overline{T}) + T\overline{T})^{-1} : \ \mathcal{D}(T\overline{T}) \to V$$

and that

$$A + \overline{A} = (T + \overline{T} - 2\alpha\mathcal{I})A\overline{A} \quad : \ \mathcal{D}(T\overline{T}) \to \mathcal{D}(T).$$

Using the relation $s = p^{-1} + \alpha$, we get

$$p^2\mathcal{I} - p(A + \overline{A}) + A\overline{A} = p^2\left(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}\right)(T - \alpha\mathcal{I})^{-1}(\overline{T} - \alpha\mathcal{I})^{-1}, \tag{8.3}$$

where the right-hand side of (8.3) is the composition of the maps

$$(T - \alpha\mathcal{I})^{-1}(\overline{T} - \alpha\mathcal{I})^{-1} : \ V \to \mathcal{D}(T\overline{T})$$

and

$$\left(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}\right) : \quad \mathcal{D}(T\overline{T}) \to V.$$

We write $F_L(p,A)$ now in terms of the above positions and get

$$\begin{aligned} F_L(p,A) = -4&[(p\mathcal{I} - (\overline{T} - \alpha\mathcal{I})^{-1})(\overline{T} - \alpha\mathcal{I})(T - \alpha\mathcal{I})] \\ &\times (\overline{T} - \alpha\mathcal{I})(T - \alpha\mathcal{I})\left(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}\right)^{-2}p^{-8}. \end{aligned}$$

Due to $s = p^{-1} + \alpha$, we have

$$\begin{aligned} &[(p\mathcal{I} - (\overline{T} - \alpha\mathcal{I})^{-1})(\overline{T} - \alpha\mathcal{I})(T - \alpha\mathcal{I})] \\ &= [p(\alpha^2\mathcal{I} - \alpha(T + \overline{T}) + T\overline{T}) + \alpha\mathcal{I} - T] \\ &= p\mathcal{Q}_{c,s}(T) - (s\mathcal{I} - \overline{T}), \end{aligned}$$

from which we conclude

$$F_L(p,A) = -4(\overline{T}-\alpha\mathcal{I})(T-\alpha\mathcal{I})[p\mathcal{Q}_{c,s}(T)-(s\mathcal{I}-\overline{T})] \\ \times \left(s^2\mathcal{I}-s(T+\overline{T})+T\overline{T}\right)^{-2}p^{-4},$$

which gives

$$(A\overline{A})^{-1}F_L(p,A)p^4 = -4p\mathcal{Q}_{c,s}(T)^{-1}-F_L(s,T).$$

□

8.2 The F-Functional Calculus for Unbounded Operators

Let $T\in\mathcal{KC}(X)$ with $\rho_S(T)\cap\mathbb{R}\neq\emptyset$. For $\alpha\in\rho_S\cap\mathbb{R}$, we define $\Phi_\alpha:\overline{\mathbb{H}}\to\overline{\mathbb{H}}$ as

$$\Phi_\alpha(s)=(s-\alpha)^{-1},\quad \Phi_\alpha(\alpha)=\infty,\quad \Phi_\alpha(\infty)=0, \tag{8.4}$$

and set $A:=(T-\alpha\mathcal{I})^{-1}$. We recall that by Theorem 5.2.3, we have $\Phi(\overline{\sigma}_S(T))=\sigma_S(A)$ and that

$$\mathcal{SH}_L(\sigma_S(A))=\left\{f\circ\Phi_\alpha^{-1}:\quad f\in\mathcal{SH}_L(\overline{\sigma}_S(T))\right\}.$$

Definition 8.2.1 (F-functional calculus for unbounded operators)**.** Let $T\in\mathcal{KC}(X)$ with $\rho_S(T)\cap\mathbb{R}\neq\emptyset$, let $\alpha\in\rho_S(T)\cap\mathbb{R}$, and define Φ_α and A as in (8.4). For $f\in\mathcal{SH}_L(\overline{\sigma}_S(T))$ with $f(\alpha)=0$, we consider the functions

$$\phi(q):=(f\circ\Phi_\alpha^{-1})(q),$$
$$\breve{\psi}(q):=\Delta(q^2\phi(q)),$$

and define the operator $\breve{f}(T)$ for $\breve{f}=\Delta f$ as

$$\breve{f}(T):=(A\overline{A})^{-1}\breve{\psi}(A), \tag{8.5}$$

where $\breve{\psi}(A)$ is intended in the sense of Definition 7.1.11.

Remark 8.2.2. Observe that the condition $f(\alpha)=0$ is not a restriction in the above definition. Indeed, if $f(\alpha)\neq 0$, then we can consider the function $\tilde{f}(q)=f(q)-f(\alpha)$ and we find that also $\tilde{f}\in\mathcal{SH}_L(\overline{\sigma}_S(T))$ with $\breve{f}=\Delta\tilde{f}$, but now $\tilde{f}=0$. We will take this fact into account in the next result.

Theorem 8.2.3. *Let $T\in\mathcal{KC}(X)$ with $\rho_S(T)\cap\mathbb{R}\neq\emptyset$, let $\alpha\in\rho_S(T)\cap\mathbb{R}$, and define Φ_α and A as in* (8.4)*. For $\breve{f}=\Delta f$ with $f\in\mathcal{SH}_L(\overline{\sigma}_S(T))$ with $f(\alpha)=0$, the operator $\breve{f}(T)$ defined in* (8.5) *satisfies*

$$\breve{f}(T)=\int_{\partial(U\cap\mathbb{C}_j)}F_L(s,T)\,ds_j\,f(s), \tag{8.6}$$

where U is any unbounded slice Cauchy domain with $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$ and j is any imaginary unit in $\mathbb{S}$.

In particular, $\breve{f}(T)$ is independent of α. If $f_* = f - c$ with $c \in \mathbb{H}$ such that $f_*(\beta) = 0$ with $\beta \in \rho_S(T) \cap \mathbb{R}$, we can define $\breve{f}_*(T)$ using β instead of α. Then $\breve{f} = \breve{f}_*$ and $\breve{f}(T) = \breve{f}_*(T)$.

Proof. Let $j \in \mathbb{S}$ and let U be a slice Cauchy domain as above. Furthermore, we assume that $\alpha \notin \overline{U}$. If this is not the case, we can replace U by the axially symmetric slice Cauchy domain $U \setminus \overline{B_\varepsilon(0)}$ with sufficiently small $\varepsilon > 0$ without altering the value of the integral in (8.6) by the Cauchy integral theorem.

The set $V = \Phi_\alpha(U)$ is a bounded slice Cauchy domain with $\sigma_S(T) \subset V$ and $\overline{V} \subset \mathcal{D}(f \circ \Phi_\alpha^{-1}) = \Phi(\mathcal{D}(f))$.

Using the second relation between $F_L(p, A)$ and $F_L(s, T)$, see formula (8.2), we have

$$\begin{aligned}&\int_{\partial(U\cap\mathbb{C}_j)} \left(-4p\mathcal{Q}_{c,s}(T)^{-1} - F_L(s,T)\right)\, ds_j\, f(s)\\&= (A\overline{A})^{-1}\int_{\partial(V\cap\mathbb{C}_j)} F_L(p,A)\, dp_j\, p^2\phi(p).\end{aligned} \tag{8.7}$$

Now we work on the left-hand side:

$$\begin{aligned}&\int_{\partial(U\cap\mathbb{C}_j)} \left(-4p\mathcal{Q}_{c,s}(T)^{-1} - F_L(s,T)\right)\, ds_j\, f(s)\\&= \int_{\partial(U\cap\mathbb{C}_j)} -4p\mathcal{Q}_{c,s}(T)^{-1} ds_j f(s) - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T) ds_j f(s)\\&= -4\int_{\partial(U\cap\mathbb{C}_j)} (s-\alpha)^{-1}\, ds_j \mathcal{Q}_{c,s}(T)^{-1} f(s)\\&\quad - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T) ds_j f(s)\\&= -4(2\pi)\mathcal{Q}_\alpha(T)^{-1} f(\alpha) - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\, ds_j\, f(s).\end{aligned}$$

The last identity follows because $\mathcal{Q}_{c,s}(T)^{-1}\, ds_j = ds_j\, \mathcal{Q}_{c,s}(T)^{-1}$, since $T + \overline{T}$ and $T\overline{T}$ are scalar operators, so that

$$\begin{aligned}&-4\int_{\partial(U\cap\mathbb{C}_j)} (s-\alpha)^{-1}\, ds_j\, \mathcal{Q}_{c,s}(T)^{-1} f(s)\\&= -4\int_{\partial(U\cap\mathbb{C}_j)} S_L^{-1}(s,\alpha)\, ds_j\, \mathcal{Q}_{c,s}(T)^{-1} f(s) = -4(2\pi)\mathcal{Q}_{c,\alpha}(T)^{-1} f(\alpha)\end{aligned}$$

by Cauchy's integral formula because $s \mapsto \mathcal{Q}_{c,s}(T)^{-1} f(s)$ is left slice hyperholomorphic. Indeed, for $s = u + jv$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right)\mathcal{Q}_{c,s}(T)^{-1}f(s) = \left(\frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right)\mathcal{Q}_{c,s}(T)^{-1}\right)f(s) + \mathcal{Q}_{c,s}(T)^{-1}\left(\frac{1}{2}\left(\frac{\partial}{\partial u} + j\frac{\partial}{\partial v}\right)f(s)\right) = 0$$

because $\mathcal{Q}_s(T)^{-1}$ commutes with j, since $T + \overline{T}$ and $T\overline{T}$ are scalar operators.

The identity (8.7) therefore turns into

$$\begin{aligned} &- 4(2\pi)\mathcal{Q}_\alpha(T)^{-1}f(\alpha) - \int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f(s) \\ &= (A\overline{A})^{-1}\int_{\partial(V\cap\mathbb{C}_j)} F_L(p,A)\,dp_j\,p^2\phi(p). \end{aligned}$$

Since by assumption $f(\alpha) = 0$, we conclude that

$$\begin{aligned} \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f(s) &= \frac{1}{2\pi}(A\overline{A})^{-1}\int_{\partial(\mathcal{V}\cap\mathbb{C}_j)} F_L(p,A)\,dp_j\,p^2\phi(p) \\ &= (A\overline{A})^{-1}\breve{\psi}(A) = \breve{f}(T). \end{aligned}$$

Finally, if $f_* = f + c$ with $f_*(\beta) = 0$ for some $\beta \in \rho_S(T) \cap \mathbb{R}$, then we find that

$$\begin{aligned} \breve{f}_*(T) &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f_*(s) \\ &= \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,f(s) \\ &\quad + \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,c = \breve{f}(T), \end{aligned}$$

since

$$\frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_j)} F_L(s,T)\,ds_j\,c = 0$$

by Cauchy's integral theorem. □

8.3 Comments and Remarks

The definition of the F-functional calculus can be extended to the case of n-tuples of unbounded operators. As is well known in the case of unbounded operators, the notion of commutativity is more delicate, and one has to pay attention to the domains of the operators. The situation is simpler when just one of the operators $T_j : \mathcal{D}(T_j) \subset X \to X$, $j = 0, 1, \dots, n$, is unbounded; see [78].

8.3.1 F-Functional Calculus for n-Tuples of Unbounded Operators

The definition of the F-functional calculus for unbounded operators is less intuitive than the S-functional calculus for unbounded operators. The reason is that the S-functional calculus is defined by a Cauchy formula, while the F-functional calculus is defined by an integral transform that maps slice monogenic functions to monogenic functions.

Definition 8.3.1 (Admissible operators)**.** Let X be a real Banach space and $X_n = X \otimes \mathbb{R}_n$. Let $T_j : \mathcal{D}(T_j) \subset X \to X$ be linear closed operators for $j = 0, 1, \dots, n$, such that $T_j T_i x = T_i T_j x$, for all $x \in \mathcal{D}(T_j T_i) \cap \mathcal{D}(T_i T_j)$ for $i, j = 0, 1, \dots, n$. Let $\mathcal{D}(T) = \bigcap_{j=0}^{n} \mathcal{D}(T_j)$ be the domain of the operator $T = T_0 + \sum_{j=1}^{n} e_j T_j \ : \mathcal{D}(T) \subset X_n \to X_n$. We say that T is an *admissible operator* if

1) $\bigcap_{j=0}^{n} \mathcal{D}(T_j)$ is dense in X_n,
2) $s\mathcal{I} - \overline{T}$ is densely defined in X_n for $s \in \mathbb{R}^{n+1}$,
3) $\mathcal{D}(T\overline{T}) \subset \mathcal{D}(T)$ is dense in X_n.

We need the following definitions:

- Let $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and let n be an odd number and let $p = (s - \alpha)^{-1}$. Set $A := (T - \alpha\mathcal{I})^{-1}$.
- Let $\alpha \in \mathbb{R}$ and define the homeomorphism $\Phi : \overline{\mathbb{R}}^{n+1} \to \overline{\mathbb{R}}^{n+1}$,

$$p = \Phi(s) = (s - \alpha)^{-1}, \quad \Phi(\infty) = 0, \quad \Phi(\alpha) = \infty. \tag{8.8}$$

Definition 8.3.2 (The F-functional calculus for n-tuples of unbounded operators)**.** Let n be an odd number and let $T : \mathcal{D}(T) \to X_n$ be an admissible operator with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{SM}_L(\overline{\sigma}_S(T))$. Let us consider the functions

$$\phi(p) := f(\Phi^{-1}(p)),$$

$$\breve{\psi}(p) := \Delta_p^{\frac{n-1}{2}} (p^{n-1} \phi(p)),$$

where Δ_p is the Laplace operator in dimension $n + 1$, and recall that

$$A := (T - \alpha\mathcal{I})^{-1}, \quad \text{for some } \alpha \in \rho_S(T) \cap \mathbb{R}.$$

With the notation above, we define

$$\widetilde{f}(T) := (A\overline{A})^{-\frac{n-1}{2}} \breve{\psi}(A) \tag{8.9}$$

for functions f such that $f(\alpha) = 0$.

The definition seems unnatural, but it is suggested by the two relations between the resolvents $F_n(p, A)$ and $F_n(s, T)$.

Theorem 8.3.3 (First relation between the F-resolvents)**.** *Let T be admissible, let $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$, let n be an odd number, and let $p = (s - \alpha)^{-1}$. Let us put $A := (T - \alpha\mathcal{I})^{-1}$ and suppose that $p \in \rho_S(A)$ and $p \neq 0$. Then we have*

$$F_n(s,T) = -(\overline{A})^{\frac{n-1}{2}} A^{\frac{n+1}{2}} F_n(p,A)p^n. \tag{8.10}$$

Theorem 8.3.4 (Second relation between $F_n(p,A)$ and $F_n(s,T)$)**.** *Let $\alpha \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and let n be an odd number and let $p = (s-\alpha)^{-1}$. Recall that $A := (T-\alpha\mathcal{I})^{-1}$ for T admissible. Let $s \in \rho_S(T)$ and $p \neq 0$. Then we have*

$$(A\overline{A})^{-\frac{n-1}{2}} F_n(p,A)p^{n+1} = \gamma_n p(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-\frac{n-1}{2}} - F_n(s,T), \tag{8.11}$$

where γ_n are defined in (7.29), i.e., $\gamma_n := (-1)^{(n-1)/2}2^{(n-1)/2}\left[\left(\frac{n-1}{2}\right)!\right]^2$.

Thanks to Theorem 8.3.3 and (8.11), we can prove that for n an odd number, if $k \in \rho_S(T) \cap \mathbb{R} \neq \emptyset$ and Φ, ϕ are as above, then $\Phi(\overline{\sigma}_S(T)) = \sigma_S(A)$, and the relation $\phi(p) := f(\Phi^{-1}(p))$ determines a one-to-one correspondence between $f \in \mathcal{SM}^L(\overline{\sigma}_S(T))$ and $\phi \in \mathcal{SM}(\overline{\sigma}_S(A))$, and so the integral representation theorem of the F-functional calculus is what we expect:

Theorem 8.3.5. *Let n be an odd number and let T be admissible with $\rho_S(T) \cap \mathbb{R} \neq \emptyset$ and suppose that $f \in \mathcal{SM}_L(\overline{\sigma}_S(T))$ and set $ds_j = -dsj$ for $j \in \mathbb{S}$. If $f(k) = 0$, then the operator $\widetilde{f}(T) := (A\overline{A})^{-\frac{n-1}{2}}\breve{\psi}(A)$, defined in (8.9), does not depend on $k \in \rho_S(T) \cap \mathbb{R}$. Moreover, we have the integral formula*

$$\widetilde{f}(T) = \int_{\partial(W\cap\mathbb{C}_j)} F_n^L(s,T)ds_j f(s), \tag{8.12}$$

where W is a suitable Cauchy domain.

The reason we have defined the F-functional calculus as in (8.9) is essentially due to the relation in Theorems 8.3.3 and 8.3.4. Thanks to this relation, we can prove that $\widetilde{f}(T)$ is independent of k and admits the integral representation (8.12). A similar definition can be found for $f \in \mathcal{SM}_R(\overline{\sigma}_{\mathcal{S}}(T))$.

Chapter 9

Quaternionic Operators on a Hilbert Space

9.1 Preliminary Results

In this section we recall some preliminary definitions and results on quaternionic Hilbert spaces and on quaternionic linear operators. The proofs of the results that are too similar to the case of complex Hilbert spaces will be omitted. We also recall some definitions that we have already stated for quaternionic Banach spaces in the previous chapters for better clarity.

Definition 9.1.1. (i) A *right* $\mathbb{H}$*-module* is an abelian group with a right scalar multiplication that satisfies the distributive properties

$$(x+y)q = xq + yq, \quad x(p+q) = xp + xq, \quad \text{for all } x, y \in \mathcal{H},\ p, q \in \mathbb{H},$$

and the associative property

$$x(pq) = (xp)q, \quad \text{for all } x, y \in \mathcal{H},\ p, q \in \mathbb{H}.$$

(ii) A *Hermitian quaternionic scalar product* is a map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{H}$, $(x, y) \to \langle x, y \rangle$ that satisfies the following properties:

$$\begin{aligned} \langle x\alpha + y\beta, z \rangle &= \langle x, z \rangle \alpha + \langle y, z \rangle \beta, \\ \langle x, y \rangle &= \overline{\langle y, x \rangle}, \\ \langle x, x \rangle &\geq 0 \quad \text{and} \quad \langle x, x \rangle = 0 \Longleftrightarrow x = 0, \end{aligned}$$

for every $\alpha, \beta \in \mathbb{H}$, and $x, y, z \in \mathcal{H}$,

From the above relations it follows that

$$\langle x, y\alpha + z\beta \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle, \quad \text{for every } \alpha, \beta \in \mathbb{H}, \text{ and } x, y, z \in \mathcal{H}.$$

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_9

Definition 9.1.2. A *quaternionic pre-Hilbert space* $\mathcal{H}$ is a right $\mathbb{H}$-module such that there exists a Hermitian quaternionic scalar product.

The Hermitian scalar product satisfies the Cauchy–Schwarz inequality

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|.$$

As in the complex case, on $\mathcal{H}$ we can define the natural norm

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad x \in \mathcal{H}.$$

Definition 9.1.3. A quaternionic pre-Hilbert space is called *quaternionic Hilbert space* if it is complete with respect to the natural distance defined as

$$\operatorname{dist}(x, y) := \|x - y\|.$$

When $\langle x, y\rangle = 0$ for $x, y \in \mathcal{H}$, we say that x is orthogonal to y and we write $x \perp y$. When $\mathcal{E} \subset \mathcal{H}$ and $\mathcal{F} \subset \mathcal{H}$, the notation $\mathcal{E} \perp \mathcal{F}$ means that $x \perp y$ for all $x \in \mathcal{E}$ and $y \in \mathcal{F}$. We denote by $\mathcal{E}^{\perp}$ the points $y \in \mathcal{H}$ that are orthogonal to every $x \in \mathcal{E}$.

Theorem 9.1.4. *If $\mathcal{E}$ is a closed subspace of $\mathcal{H}$, then*

$$\mathcal{H} = \mathcal{E} \oplus \mathcal{E}^{\perp}.$$

This means that $\mathcal{E}$ and $\mathcal{E}^{\perp}$ are closed subspaces of $\mathcal{H}$ whose intersection is $\{0\}$ and whose sum is $\mathcal{H}$. The subspace $\mathcal{E}^{\perp}$ is called the orthogonal complement of $\mathcal{E}$. It follows that if $\mathcal{E}$ is a closed subspace of $\mathcal{H}$, then

$$(\mathcal{E}^{\perp})^{\perp} = \mathcal{E}.$$

Definition 9.1.5. We will call a subset $\mathcal{N} \subseteq \mathcal{H}$ an *orthonormal basis* if

$$\langle x, y\rangle = 0 \quad \text{for } x, y \in \mathcal{H} \text{ such that } x \neq y, \tag{9.1}$$

$$\langle x, x\rangle = 1 \quad \text{for } x \in \mathcal{H}, \tag{9.2}$$

$$\{x \in \mathcal{H} : \ \langle x, y\rangle \text{ for all } y \in \mathcal{N}\} = \{0\}. \tag{9.3}$$

It can be checked in a similar manner to the classical complex Hilbert space case that every vector $x \in \mathcal{H}$ can be written as

$$x = \sum_{y \in \mathcal{N}} y \langle x, y\rangle. \tag{9.4}$$

The proofs of following propositions are analogous to those of the complex case.

Theorem 9.1.6. *Let $\mathcal{N}$ be an orthonormal basis of a quaternionic Hilbert space $\mathcal{H}$. Then every $x \in \mathcal{H}$ can be decomposed uniquely via*

$$x = \sum_{z \in \mathcal{N}} z \langle x, z\rangle, \tag{9.5}$$

where

$$\sum_{z\in\mathcal{N}} z\langle x,z\rangle := \sup\{\sum_{z\in\mathcal{N}_f} z\langle x,z\rangle :\ \ \mathcal{N}_f \text{ is a nonempty finite subset of } \mathcal{N}\}.$$

Theorem 9.1.7. *Let $\{x_n\}$ be a sequence of pairwise orthogonal vectors in $\mathcal{H}$. Then each of the following conditions implies the other two:*

(i) *the series $\sum_{n=1}^{\infty} x_n$ converges in the norm topology of $\mathcal{H}$,*

(ii) $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$,

(iii) *the series $\sum_{n=1}^{\infty} \langle x_n, y\rangle$ converges for every $y \in \mathcal{H}$.*

Let $\mathcal{D}(T)$ denote the domain of a linear operator T. In the following we will consider right linear operators $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{H}$, that is, those operators such that

$$T(x\alpha + y\beta) = (Tx)\alpha + (Ty)\beta,$$

for all $x, y \in \mathcal{D}(T)$ and $\alpha, \beta \in \mathbb{H}$. The set of right linear operators on $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$. Given $T \in \mathcal{L}(\mathcal{H})$, the range and kernel of T will be given by

$$\text{ran}(T) = \{y \in \mathcal{H} :\ \ Tx = y \text{ for } x \in \mathcal{D}(T)\}$$

and

$$\ker(T) = \{x \in \mathcal{D}(T) :\ \ Tx = 0\},$$

respectively. We will denote by $\mathcal{B}(\mathcal{H})$ the right Banach space of all bounded right linear operators $T : \mathcal{H} \to \mathcal{H}$ endowed with the natural norm, i.e.,

$$\|T\| = \sup_{\|x\|\leq 1} \|Tx\|.$$

Lemma 9.1.8. *Fix a right linear quaternionic Hilbert space $\mathcal{H}$. A right linear subspace $\mathcal{K}$ of $\mathcal{H} \oplus \mathcal{H}$ satisfies*

$$\mathcal{K} = \{(x, Tx) :\ \ x \in \mathcal{D}(T)\} \tag{9.6}$$

for some $T \in \mathcal{L}(\mathcal{H})$ if and only if

$$(0, y) \in \mathcal{K} \Longrightarrow y = 0. \tag{9.7}$$

Proof. If $\mathcal{K}$ is as in (9.6), then (9.7) follows directly from $T0 = 0$. Conversely, if (9.7) holds, then (x, y) and (x, z) belonging to $\mathcal{K}$ implies that $y = z$, i.e., there exists a function $T : \mathcal{D}(T) \to \mathcal{H}$. The fact that $T \in \mathcal{L}(\mathcal{H})$ follows easily from the right linearity of $\mathcal{K}$. Thus, (9.6) holds. □

Definition 9.1.9. • An operator $T \in \mathcal{L}(\mathcal{H})$ is called *closed* if the set $\{(x, Tx) : x \in \mathcal{D}(T)\}$ is a closed subset of $\mathcal{H} \times \mathcal{H}$.

- Let S and T both belong to $\mathcal{L}(\mathcal{H})$. We write $S = T$ if $\mathcal{D}(S) = \mathcal{D}(T)$ and $Sx = Tx$ for all $x \in \mathcal{D}(S) = \mathcal{D}(T)$.
- We write $S \subseteq T$ if $\mathcal{D}(S) \subseteq \mathcal{D}(T)$ and $Sx = Tx$ for all $x \in \mathcal{D}(S)$.
- Clearly, $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *closable* if there exists a closed operator $U \in \mathcal{L}(\mathcal{H})$ such that $T \subseteq U$.

Theorem 9.1.10. *Let $T \in \mathcal{L}(\mathcal{H})$. Then T is closable if and only if*

$$\overline{\{(x, Tx) : \ x \in \mathcal{D}(T)\}} = \{(x, Ux) : \ \text{for some operator } U \in \mathcal{L}(\mathcal{H})\}. \tag{9.8}$$

Proof. If $S \in \mathcal{L}(\mathcal{H})$ is any closed operator such that $T \subseteq S$, then

$$\{(x, Tx) : \ x \in \mathcal{D}(T)\} \subseteq \{(x, Sx) : \ x \in \mathcal{D}(S)\}.$$

Hence, since S is closed,

$$\overline{\{(x, Tx) : \ x \in \mathcal{D}(T)\}} \subseteq \{(x, Sx) : \ x \in \mathcal{D}(S)\}.$$

Therefore, in view of Lemma 9.1.8, (9.8) holds. Conversely, if (9.8) holds, then $T \subseteq U$, and hence U is closed, since

$$\{(x, Ux) : \ x \in \mathcal{D}(U)\}$$

is closed. Thus, T is closable. □

Definition 9.1.11. Let $T \in \mathcal{L}(\mathcal{H})$ be closable. We let

$$\overline{T}x := \lim_{n \to +\infty} T(x_n)$$

denote the operator in $\mathcal{L}(\mathcal{H})$ with domain

$$\mathcal{D}(\overline{T}) = \Big\{x \in \mathcal{H} : \ x = \lim_{n \to +\infty} x_n \text{ for } \{x_n\}_{n=0}^{+\infty} \subseteq \mathcal{D}(T) \text{ and } \{T(x_n)\}_{n=0}^{+\infty} \text{ converges in } \mathcal{H}\Big\}.$$

Remark 9.1.12. In view of Theorem 9.1.10, the definition of $\overline{T}$ is independent of the choice of sequence $\{x_n\}_{n=0}^{+\infty}$. Note that for every closed operator $U \in \mathcal{L}(\mathcal{H})$ such that $T \subseteq U$,

$$\overline{T} \subseteq U.$$

Definition 9.1.13 (Adjoint operator)**.** Given $T \in \mathcal{L}(\mathcal{H})$ that is densely defined, we let $T^* \in \mathcal{L}(\mathcal{H})$ denote the unique operator (called the *adjoint*) such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad x \in \mathcal{D}(T).$$

The domain of T^* is given by

$$\mathcal{D}(T^*) = \{y \in \mathcal{H} : \text{ there exists } z \in \mathcal{H} \text{ with } \langle Tx, y\rangle = \langle x, z\rangle \text{ for every } x \in \mathcal{D}(T)\}.$$

Theorem 9.1.14. *If $T \in \mathcal{L}(\mathcal{H})$ is densely defined and $W \in \mathcal{L}(\mathcal{H})$, then:*

(i) *$T^* \in \mathcal{L}(\mathcal{H})$ is closed.*

(ii) $\operatorname{ran}(T)^\perp = \ker(T^*)$.

(iii) *If $T \subseteq W$, then $W^* \subseteq T^*$.*

(iv) $\ker(T) \subseteq \operatorname{ran}(T^*)^\perp$.

(v) *When T is closed and $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$, then*

$$\ker(T) = \operatorname{ran}(T^*)^\perp.$$

Proof. The proofs can completed in much the same way as in the case in which $\mathcal{H}$ is a complex Hilbert space (see, e.g., Proposition 1.6 in [191]). □

Theorem 9.1.15. *If $T \in \mathcal{L}(\mathcal{H})$ is densely defined, then:*

(i) *T is closable if and only if $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$.*

(ii) *If T is closable, then $\overline{T} = T^{**}$.*

(iii) *T is closed if and only if $T = T^{**}$.*

(iv) *If T is closable and* $\ker(T) = \{0\}$, *then T^{-1} is closable if and only if* $\ker(\overline{T}) = \{0\}$. *Moreover,*

$$(\overline{T})^{-1} = \overline{T^{-1}}.$$

Proof. The proofs can completed in much the same way as in the case in which $\mathcal{H}$ is a complex Hilbert space (see, e.g., Theorem 1.8 in [191]). □

Definition 9.1.16. Let $T \in \mathcal{L}(\mathcal{H})$. We call T *normal* if T is densely defined, T is closed, $\mathcal{D}(T) = \mathcal{D}(T^*)$, and $TT^* = T^*T$.

Lemma 9.1.17. *Let $T \in \mathcal{L}(\mathcal{H})$ be normal. If $S \in \mathcal{L}(\mathcal{H})$ so that $T \subseteq S$ and $\mathcal{D}(S) \subseteq D(S^*)$, then $S = T$.*

Proof. If $T \subseteq S$, then $S^* \subseteq T^*$ and hence

$$\mathcal{D}(T) \subseteq \mathcal{D}(S) \subseteq \mathcal{D}(S^*) \subseteq \mathcal{D}(T^*) = \mathcal{D}(T),$$

i.e., $\mathcal{D}(S) = \mathcal{D}(T)$. Therefore, $S = T$. □

Definition 9.1.18. Let $T \in \mathcal{L}(\mathcal{H})$. We call T

- *self-adjoint* if $T = T^*$,
- *anti-self-adjoint* if $T = -T^*$
- *unitary* if $TT^* = T^*T = \mathcal{I}$.

9.2 The S-Spectrum of Some Classes of Operators

As in the complex case there are different ways of splitting the spectrum of a closed linear operator. For the spectral theorem, the splitting of the spectrum in terms of the point spectrum, continuous spectrum, and residual spectrum is very natural.

Definition 9.2.1. Let $T \in \mathcal{L}(\mathcal{H})$ be densely defined and let $\mathcal{Q}_s(T) : \mathcal{D}(T^2) \to \mathcal{H}$ be given by

$$\mathcal{Q}_s(T)x = (T^2 - 2\text{Re}(s)T + |s|^2\mathcal{I})x, \quad x \in \mathcal{D}(T^2).$$

The *S-resolvent set of T* is defined as follows:

$$\rho_S(T) = \{s \in \mathbb{H} : \ \ker(\mathcal{Q}_s(T)) = \{0\}, \text{ran}(\mathcal{Q}_s(T)) \text{ is dense in } \mathcal{H} \text{ and } \mathcal{Q}_s(T)^{-1} \in \mathcal{B}(\mathcal{H})\}.$$

The *S-spectrum* is defined as

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$

We recall Theorem 3.1.13 for the particular case of Hilbert spaces:

Theorem 9.2.2. *Let $T \in \mathcal{B}(\mathcal{H})$. Then the S-spectrum is a compact nonempty subset of $\mathbb{H}$ and*

$$\sigma_S(T) \subseteq \{p \in \mathbb{H} : \ 0 \leq |p| \leq \|T\|\}. \tag{9.9}$$

Moreover, we recall that the axially symmetric structure of the S-spectrum will be crucial in the followng.

Theorem 9.2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be densely defined. If $p = p_0 + ip_1 \in \sigma_S(T)$ for $i \in \mathbb{S}$ and $p_0, p_1 \in \mathbb{R}$, then $p_0 + jp_1 \in \sigma_S(T)$ for all $j \in \mathbb{S}$.*

We will use the following splitting of the S-spectrum:

Definition 9.2.4. Let $T : D(T) \to \mathcal{H}$. We split the S-spectrum into the three disjoint sets:

(P) The *point S-spectrum* of T:

$$\sigma_{PS}(T) = \{s \in \mathbb{H} : \ \ker(\mathcal{Q}_s(T)) \neq \{0\}\}.$$

(R) The *residual S-spectrum* of T:

$$\sigma_{RS}(T) = \left\{s \in \mathbb{H} : \ \ker(\mathcal{Q}_s(T)) = \{0\}, \ \overline{\text{ran}(\mathcal{Q}_s(T))} \neq \mathcal{H}\right\}.$$

(C) The *continuous S-spectrum* of T:

$$\sigma_{CS}(T) = \left\{s \in \mathbb{H} : \ \ker(\mathcal{Q}_s(T)) = \{0\}, \ \overline{\text{ran}(\mathcal{Q}_s(T))} = \mathcal{H}, \ \mathcal{Q}_s(T)^{-1} \notin \mathcal{B}(\mathcal{H})\right\}.$$

We observe that from the definitions of S-spectrum and S-resolvent set we have the following results, which were proved in [142].

Theorem 9.2.5. *Let $T : \mathcal{D}(T) \to \mathcal{H}$ be a right linear quaternionic operator with dense domain.*

(i) *If T is self-adjoint, then $\sigma_S(T)$ is real and $\sigma_{RS}(T)$ is empty.*

(ii) *If T is anti-self-adjoint, then $\sigma_S(T)$ is purely imaginary and $\sigma_{RS}(T)$ is empty.*

Proof. Let us prove (i). To prove that the S-spectrum is real, we show that the S-resolvent set consists of quaternions $s = s_0 + \underline{s}$ such that $\underline{s} \neq 0$. Observe that

$$\mathcal{Q}_s(T) = T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} = (T - \mathcal{I}s_0)^2 + \mathcal{I}|\underline{s}|^2.$$

Since T is self-adjoint, by standard arguments it follows that also $(T - \mathcal{I}s_0)$, its squared $(T - \mathcal{I}s_0)^2$, and $\mathcal{Q}_s(T)$ are self-adjoint operators. Take $x \in \mathcal{D}(T^2)$, so that

$$\langle x, (T - \mathcal{I}s_0)^2 x\rangle = \langle (T - \mathcal{I}s_0)x, (T - \mathcal{I}s_0)x\rangle \geq 0.$$

We observe that

$$\begin{aligned}\|\mathcal{Q}_s(T)x\|^2 &= \langle ((T - \mathcal{I}s_0)^2 + \mathcal{I}|\underline{s}^2)x, ((T - \mathcal{I}s_0)^2 + \mathcal{I}|\underline{s}|^2)x\rangle \\ &= \|(T - \mathcal{I}s_0)^2 x\| + 2|\underline{s}|^2\langle x, (T - \mathcal{I}s_0)^2 x\rangle + |\underline{s}|^4\|x\|^2 \\ &\geq |\underline{s}|^4\|x\|^2.\end{aligned}$$

So from the estimate

$$\|\mathcal{Q}_s(T)x\| \geq |\underline{s}|^2\|x\|, \quad x \in \mathcal{D}(T^2),$$

we have that $\ker(\mathcal{Q}_s(T)) = \{0\}$ and $\mathcal{Q}_s(T)^{-1} : \mathrm{ran}(\mathcal{Q}_s(T)) \to \mathcal{D}(T^2)$ is a bounded operator. Now observe by Theorem 9.1.14 that

$$\begin{aligned}\mathrm{ran}(\mathcal{Q}_s(T)) &= (\mathrm{ran}(\mathcal{Q}_s(T)))^{\perp})^{\perp} \\ &= \ker(\mathcal{Q}_s(T)^*))^{\perp} \\ &= \ker(\mathcal{Q}_s(T)))^{\perp} \\ &= \{0\}^{\perp} = \mathcal{H}.\end{aligned}$$

This proves that s is in the S-resolvent set, and so the S-spectrum is real. Now suppose that the residual spectrum is nonempty. We get the following contradiction:

$$\{0\} = \ker(\mathcal{Q}_s(T)) = \ker(\mathcal{Q}_s(T)^*) = \mathrm{ran}(\mathcal{Q}_s(T)))^{\perp} \neq \{0\},$$

so $\sigma_{RS}(T)$ is the empty set.

Let us prove (ii). To prove that the S-spectrum is purely imaginary, we show that the S-resolvent set consists of quaternions $s = s_0 + \underline{s}$ such that $s_0 \neq 0$. In analogy with the previous statement we want to show that $\ker(\mathcal{Q}_s(T)) = \{0\}$ and

$\mathcal{Q}_s(T)^{-1} : \operatorname{ran}(\mathcal{Q}_s(T)) \to \mathcal{D}(T^2)$ is a bounded operator. This follows from the inequality

$$\|\mathcal{Q}_s(T)x\| \geq s_0^2\|x\|, \quad x \in \mathcal{D}(T^2). \tag{9.10}$$

Precisely, since T is anti-self-adjoint, the relations

$$\langle T^2x, Tx\rangle + \langle Tx, T^2x\rangle = 0, \ \ \langle Tx, x\rangle + \langle x, Tx\rangle = 0, \ \ \langle T^2x, x\rangle + \langle x, T^2x\rangle = -2\|x\|^2,$$

imply

$$\|\mathcal{Q}_s(T)x\|^2 = \|T^2x\|^2 + |s|^4\|x\|^2 + 2(s_0^2 - |\underline{s}|^2)\|Tx\|^2.$$

When $s_0^2 - |\underline{s}|^2 \geq 0$, the estimate $\|\mathcal{Q}_s(T)x\| \geq s_0^2\|x\|$ holds. When $s_0^2 - |\underline{s}|^2 \leq 0$, it still holds, since from the estimate

$$2(s_0^2 - |\underline{s}|^2)\|Tx\|^2 \geq 2(s_0^2 - |\underline{s}|^2)\|x\|\|T^2x\|,$$

we get

$$\begin{aligned}
\|\mathcal{Q}_s(T)x\|^2 &\geq \|T^2x\|^2 + |s|^4\|x\|^2 + 2(s_0^2 - |\underline{s}|^2)\|x\|\|T^2x\| \\
&= (\|T^2x\| - |\underline{s}|\|x\|)^2 + 2s_0^2\|x\|\|T^2x\| + (s_0^4 + 2s_0^2|\underline{s}|^2)\|x\|^2 \\
&\geq s_0^4\|x\|^2.
\end{aligned}$$

We now recall that as in the complex case, if T is a closed linear quaternionic operator, then $\mathcal{D}(TT^*)$ is dense in $\mathcal{H}$ and T^*T is self-adjoint. We will use the above fact to show that $\mathcal{D}(TT^*)$ is dense in $\mathcal{H}$ and that $\mathcal{Q}_s(T)^* = \mathcal{Q}_{-s_0+\underline{s}}(T)$. Indeed, the operator $T^2 + s_0^2\mathcal{I}$ is self-adjoint. Since for the property of adjoint operators

$$T_1^* + T_2^* \subset (T_1 + T_2)^*$$

when $\mathcal{D}(T_1 + T_2)$ is dense in $\mathcal{H}$ we get

$$\begin{aligned}
T^2 + s_0^2\mathcal{I} &= (T - s_0\mathcal{I})^2 + 2s_0T)^* \\
&\supset ((T - s_0\mathcal{I})^2)^* + 2s_0T^* \\
&= ((T - s_0\mathcal{I})^2)^* - 2s_0T,
\end{aligned}$$

it follows that

$$(T + s_0\mathcal{I})^2 = (T - s_0\mathcal{I})^2 + 4s_0T \supset ((T - s_0\mathcal{I})^2)^*.$$

Since $T_1^*T_2^* \subset (T_2T_1)^*$ if $\mathcal{D}((T_2T_1))$ is dense in $\mathcal{H}$, we have

$$((T - s_0\mathcal{I})^2)^* \supset (T^* - s_0\mathcal{I})^2 = (T + s_0\mathcal{I})^2,$$

so we get

$$((T - s_0\mathcal{I})^2)^* = (T + s_0\mathcal{I})^2,$$

and so

$$\mathcal{Q}_s(T)^* = ((T - s_0\mathcal{I})^2 + |\underline{s}|^2\mathcal{I})^* = (T + s_0\mathcal{I})^2 + |\underline{s}|^2\mathcal{I} = \mathcal{Q}_{-s_0+\underline{s}}(T).$$

We now apply (9.10) to get

$$\|\mathcal{Q}_{-s_0+\underline{s}}(T)\| \geq s_0^2\|x\|, \quad x \in \mathcal{D}(T^2).$$

In particular, this implies that

$$\ker(\mathcal{Q}_{-s_0+\underline{s}}(T)) = \{0\},$$

and moreover, with similar considerations as in point (i), we have

$$\operatorname{ran}(\mathcal{Q}_{-s_0+\underline{s}}(T)) = \mathcal{H}.$$

This means that $s \in \rho_S(T)$, and so the S-spectrum is purely imaginary. The fact that the residual S-spectrum is empty follows by contradiction as in case (i). □

Theorem 9.2.6. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then we have*

$$\sigma_{PS}(T) = \sigma_{PS}(T^*), \quad \sigma_{RS}(T) = \sigma_{RS}(T^*) = 0, \quad \sigma_{CS}(T) = \sigma_{CS}(T^*).$$

Proof. Since T is normal and $\mathcal{Q}_s(T)^* = \mathcal{Q}_s(T^*)$, it is clear that $\mathcal{Q}_s(T)^*$ is normal. For bounded linear operators, the kernel T and the kernel of its adjoint are equal, so

$$\ker(\mathcal{Q}_s(T)) = \ker(\mathcal{Q}_s(T^*)).$$

So by the definition of point S-spectrum, we have

$$\sigma_{PS}(T) = \sigma_{PS}(T^*).$$

The fact that $\sigma_{RS}(T) = \sigma_{RS}(T^*) = 0$ follows by contradiction. In fact, if $0 \neq s \in \sigma_{RS}(T)$, we get

$$\{0\} = \ker(\mathcal{Q}_s(T)) = \ker(\mathcal{Q}_s(T^*)) = (\operatorname{ran}(\mathcal{Q}_s(T))^{\perp} \neq \{0\}.$$

In the same way we can prove that $\sigma_{RS}(T^*) = 0$. Since T and T^* have the same S-spectrum and the three components of the S-spectrum, by definition, are pairwise disjoint, it follows that $\sigma_{CS}(T) = \sigma_{CS}(T^*)$. □

Theorem 9.2.7. *Let $T \in \mathcal{B}(\mathcal{H})$. Then we have the following:*

(i) *If T is unitary, then*

$$\sigma_S(T) \subset \{s \in \mathbb{H} : \ |s| = 1\}.$$

(ii) *If T is anti-self-adjoint and unitary, then*

$$\sigma_S(T) = \mathbb{S}.$$

Proof. We study the invertibility of $\mathcal{Q}_s(T)$ for $|s| \geq 0$. If $s = 0$, then $\mathcal{Q}_0(T) = T^2$ has a bounded inverse. Since T is unitary, $\|T\| = 1$, and for $|s| > \|T\|$ we know that $\mathcal{Q}_s(T)$ has a bounded inverse. When $0 < |s| < 1$, from the identity

$$\mathcal{Q}_s(T) = |s|^2 \mathcal{Q}_{s^{-1}}(T^*)T^2,$$

the operator T^* is unitary and $|s^{-1}| > 1$, so $\mathcal{Q}_{s^{-1}}(T^*)$ is bijective and has a bounded inverse. From the above identity, also $\mathcal{Q}_s(T)$ is bijective and has a bounded inverse, so we conclude that $s \in \rho_S(T)$ if $|s| \neq 1$. Finally, the fact that if T is anti-self-adjoint and unitary, then $\sigma_S(T) = \mathbb{S}$ follows for the previous point and from the fact that the S-spectrum of an anti-self-adjoint operator is purely imaginary. □

We recall that the splitting of the spectrum is defined according to where an operator is not invertible. A quaternionic bounded linear operator A that satisfies the two conditions

(i) there exists $K > 0$ such that $\|Av\| \geq K\|v\|$ for $v \in D(A)$ (bounded from below),

(ii) the range of A is dense,

is invertible. In the paper [49], the authors studied the invariant subspaces of quaternionic normal operators, and the more natural splitting of the spectrum is based on the previous theorem. So in analogy to the classical case for the S-spectrum, we have the following definition:

Definition 9.2.8. Let T be a quaternionic bounded linear operator. The *approximate point S-spectrum* of T, denoted by $\Pi_S(T)$, is defined as

$$\Pi_S(T) = \{s \in \mathbb{H} : \ T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} \text{ is not bounded from below}\}.$$

The *compression S-spectrum*of T, denoted by $\Gamma_S(T)$, is defined as

$$\Gamma_S(T) = \{s \in \mathbb{H} : \ \text{the range of } T^2 - 2\mathrm{Re}(s)T + |s|^2\mathcal{I} \text{ is not dense}\}.$$

The set $\Pi_S(T)$ contains the S-eigenvalues.

9.3 The Splitting of a Normal Operator and Consequences

This section starts with two classical results on the square root of a positive definite quaternionic linear operator and the polar decomposition of a bounded quaternionic linear operator. Even though the proofs are the same as in the complex case, they are of crucial importance for the Teichmüller decomposition of a quaternionic normal operator.

Let us define an order relation on bounded self-adjoint operators on a quaternionic Hilbert space $\mathcal{H}$ denoted by

$$A \succeq B \quad \text{or} \quad B \preceq A$$

whenever

$$\langle Ax, x\rangle \geq \langle Bx, x\rangle \quad \text{for all } x \in \mathcal{H}. \tag{9.11}$$

Clearly, (9.11) forces

$$\|A\| \leq \|B\| \quad \text{whenever } A \preceq B,$$

and

$$A \succeq 0 \Longleftrightarrow A \text{ is positive semidefinite.}$$

Fix a positive semidefinite operator $A \in \mathcal{B}(\mathcal{H})$, i.e., $A \succeq 0$. We will make use of the so called *generalized Cauchy–Schwarz inequality* for A, namely,

$$|\langle Ax, y\rangle|^2 \leq \langle Ax, x\rangle\langle Ay, y\rangle, \quad \text{for } x, y \in \mathcal{H}. \tag{9.12}$$

In order to justify (9.12), note that if

$$z_t = x + y(\langle Ax, y\rangle t), \quad t \in \mathbb{R},$$

then

$$0 \leq \langle Az_t, z_t\rangle = \langle Ax, x\rangle + 2t|\langle Ax, y\rangle|^2 + t^2|\langle Ax, y\rangle|^2\langle Ay, y\rangle.$$

Since a nonnegative quadratic polynomial of a real variable with real coefficients cannot have two distinct zeros, we obtain

$$4|\langle Ax, y\rangle|^4 - 4|\langle Ax, y\rangle|^2\langle Ax, x\rangle\langle Ay, y\rangle \leq 0,$$

i.e., (9.12) holds.

Lemma 9.3.1. *Every bounded monotonic sequence of self-adjoint operators* $(A_n)_{n=1}^\infty$ *converges strongly to a bounded self-adjoint operator* $A \in \mathcal{B}(\mathcal{H})$.

Proof. It suffices to consider the case

$$0 \preceq A_1 \preceq \cdots \preceq I.$$

Let $A_{mn} := A_n - A_m \succeq 0$ for $n > m$. It follows from (9.12) that

$$\|A_{mn}x\|^4 = \langle A_{mn}x, A_{mn}x\rangle^2 \leq \langle A_{mn}x, x\rangle\langle A_{mn}^2 x, A_{mn}x\rangle.$$

Since $\|A_{mn}\| \leq 1$, we obtain

$$\|A_n x - A_m x\|^4 \leq (\langle A_n x, x\rangle - \langle A_m x, x\rangle)\|x\|^2.$$

Finally, since $(\langle A_n x, x\rangle)_{n=1}^\infty$ is bounded and monotonically increasing, the above inequality shows that $(A_{mn}x)_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{H}$. □

The following proofs of the existence of a square root of a bounded positive operator and the polar decomposition of a bounded linear operator are exactly the same as in the case of complex linear operators [193].

Theorem 9.3.2. *Every positive semidefinite operator $A \in \mathcal{B}(\mathcal{H})$ has a unique positive square root $A^{1/2}$ that satisfies $(A^{1/2})^2) = A$. Moreover, every operator $B \in \mathcal{B}(\mathcal{H})$ that commutes with A also commutes with $A^{1/2}$.*

Proof. The goal is to solve the equation $A = X^2$ for $X \succeq 0$. Without loss of generality, suppose that

$$A \preceq I.$$

If we let $C = I - A \succeq 0$ and $Y = I - X$, then

$$A = X^2 \Longleftrightarrow Y = \frac{1}{2}(C + Y^2).$$

Consider the recurrent sequence of positive semidefinite mutually commuting operators in $\mathcal{B}(\mathcal{H})$ given by $Y_0 = 0$ and

$$Y_{n+1} = \frac{1}{2}(C + Y_n^2), \quad n = 1, 2, \ldots.$$

Note that $Y_n Y_m = Y_m Y_n$ for all $m, n = 1, 2, \ldots$, since Y_n is a polynomial in C. We claim that

$$\|Y_n\| \leq 1$$

and $Y_{n+1} - Y_n \succeq 0$ for $n = 1, 2, \ldots$. Both claims can be established by induction. Indeed, since $\|Y_0\| = 0 < 1$ and

$$\|Y_{n+1}\| \leq \frac{1}{2}(\|C\| + \|Y_n\|^2) \leq \frac{1}{2}(1 + \|Y_n\|^2) \leq 1$$

when $\|Y_n\| \leq 1$, we have the first claim. For the second claim, we obviously have $Y_1 = \frac{1}{2}C \succeq Y_0$, since $C \succeq 0$ and $Y_0 = 0$. Using $Y_m Y_n = Y_n Y_m$ and $Y_n - Y_{n-1} \succeq 0$ as well as $Y_n + Y_{n-1} \succeq 0$, we have

$$Y_{n+1} - Y_n = \frac{1}{2}(Y_n + Y_{n-1})(Y_n - Y_{n-1}) \succeq 0 \quad \text{for } n = 1, 2, \ldots.$$

Thus, we may invoke Lemma 9.3.1 to obtain a limit $Y \succeq 0$ with $\|Y\| \leq 1$ for $(Y_n)_{n=1}^{\infty}$, and hence if we put $X = I - Y \succeq 0$, we have a desired solution $A^{1/2} := X$ to $A = X^2$.

If $B \in \mathcal{B}(\mathcal{H})$ commutes with A, then it also commutes with each of the operators Y_n, since they are real polynomials in A. Consequently, B also commutes with the limit $Y = \lim_{n \to +\infty} Y_n$ and in turn also with $A^{1/2} = X = I - X$.

Finally, we will now show that X is unique. Suppose that there are two operators $X \succeq 0$ and $\widetilde{X} \succeq 0$ such that

$$A = X^2 \quad \text{and} \quad A = \widetilde{X}^2.$$

Let $y = (X - \widetilde{X})x$ for every $x \in \mathcal{H}$. We may use the above construction to obtain bounded operators $Z \succeq 0$ and $\widetilde{Z} \succeq 0$ such that

$$X = Z^2 \quad \text{and} \quad \widetilde{X} = \widetilde{Z}^2,$$

respectively. But then

$$\begin{aligned} \|Zy\|^2 + \|\widetilde{Z}y\|^2 &= \langle Z^2 y, y\rangle + \langle \widetilde{Z}^2 y, y\rangle \\ &= \langle Xy, y\rangle + \langle \widetilde{X}y, y\rangle \\ &= \langle X(X - \widetilde{X})x, y\rangle + \langle \widetilde{X}(X - \widetilde{X})x, y\rangle \\ &= \langle (X + \widetilde{X})(X - \widetilde{X})x, y\rangle \\ &= \langle (A - A)x, y\rangle = 0. \end{aligned}$$

Thus, $Zy = \widetilde{Z}y = 0$ and hence also $Xy = \widetilde{X}y = 0$. Consequently,

$$\|(X - \widetilde{X})x\|^2 = \langle (X - \widetilde{X})^2 x, x\rangle = \langle (X - \widetilde{X})y, x\rangle = 0,$$

and thus $X = \widetilde{X}$. □

The next theorem motivates the following definition.

Definition 9.3.3. For every operator $T \in \mathcal{B}(\mathcal{H})$, we define $|T| := (T^*T)^{1/2}$.

Theorem 9.3.4 (Polar decomposition in $\mathcal{B}(\mathcal{H})$)**.** *Every operator $T \in \mathcal{B}(\mathcal{H})$ admits a unique factorization*

$$T = UP \tag{9.13}$$

into the product of a positive operator P and a partial isometry U on $\overline{\operatorname{ran} P}$ (that is, $\|Ux\| = \|x\|$ for every $x \in \overline{\operatorname{ran} P}$ and $Ux = 0$ for every $x \in \overline{(\operatorname{ran} P)^{\perp}}$). The operator P is furthermore given by $P := (T^*T)^{1/2}$, and $\operatorname{ran}(U) = \overline{\operatorname{ran}(T)}$.

If T is normal, then P and U commute mutually and with every operator in $\mathcal{B}(\mathcal{H})$ that commutes with both T and T^* and U is (anti)-self-adjoint if and only if T is. Furthermore, $\overline{\operatorname{ran}(U)} = \overline{\operatorname{ran}(P)} = \overline{\operatorname{ran}(T)}$, and so U defines in this case a unitary operator on $\overline{\operatorname{ran}(T)}$.

Proof. We will first prove the existence of (9.13). In view of Theorem 9.3.2, the positive operator $T^*T \in \mathcal{B}(\mathcal{H})$ has a unique positive square root $P := (T^*T)^{1/2}$ in $\mathcal{B}(\mathcal{H})$. Consequently,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle P^2 x, x\rangle = \langle Px, Px\rangle = \|Px\|^2 \tag{9.14}$$

for all $x \in \mathcal{H}$ and hence

$$P(x - y) = 0 \Longleftrightarrow T(x - y) = 0,$$

so that $P(x) = P(y)$ if and only if $T(x) = T(y)$ for all $x, y \in \mathcal{H}$. Thus, we may define

$$U : \begin{cases} \operatorname{ran} P & \to \operatorname{ran} T, \\ Px & \mapsto Tx. \end{cases} \tag{9.15}$$

Because of (9.14) this operator is isometric and hence extends to an isometry defined on $\overline{\operatorname{ran} P}$. We can extend U to all of $\mathcal{H}$ by setting

$$Ux = \begin{cases} Ux & \text{for } x \in \overline{\operatorname{ran} P}, \\ 0 & \text{for } x \in (\operatorname{ran} P)^{\perp}. \end{cases} \tag{9.16}$$

If $x, y \in \overline{\operatorname{ran}\ P}$, then

$$\langle y, x \rangle = \langle Uy, Ux \rangle = \langle y, U^* U x \rangle$$

and hence

$$U^* U x - x \in (\operatorname{ran} P)^{\perp}.$$

Since, on the other hand, $U^* : H \to \overline{\operatorname{ran} P}$, we obtain $U^*Ux - x \in (\operatorname{ran} P)^{\perp} \cap \overline{\operatorname{ran} P}$, and so

$$U^* U x - x = 0, \quad x \in \overline{\operatorname{ran} P}.$$

Thus $U^*Ux = x$ for every $x \in \overline{\operatorname{ran} P}$, and so U is actually a partial isometry on $\overline{\operatorname{ran} P}$.

Let now $T = UP$ be an arbitrary decomposition of the form (9.14). Since U is a partial isometry on $\overline{\operatorname{ran} P}$, the operator U^*U is the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran} P}$ and so $T^*T = P^*U^*UP = P^*P = P^2$. Since the positive square root of a positive operator is unique, we find that $P = (T^*T)^{1/2}$ and in turn that U must be the operator defined in (9.15) and (9.16).

Suppose now T is normal and consider the factorization $T = UP$ with $P^2 = |T|^2 = T^*T$ in (9.13). Since $T^*T = TT^*$, we then have

$$T^*T = TT^* = UPPU^* = UP^2U^* = UT^*TU^*.$$

Since U^*U is the orthogonal projection of $\mathcal{H}$ onto $\overline{\operatorname{ran} T} \supset \operatorname{ran}(TT^*) = \operatorname{ran}(T^*T)$, applying U^* to the above equation yields $U^*(T^*T) = (TT^*)U^*$, and taking the adjoint, we finally obtain $U(T^*T) = (T^*T)U$. Since $P = |T| = (T^*T)^{1/2}$ commutes with every operator that commutes with T^*T, the operator U therefore also commutes with P.

Since U is a partial isometry on $\overline{\operatorname{ran} P}$, its restriction $U : \overline{\operatorname{ran} P} \to \operatorname{ran} U$ is a unitary operator. Due to (9.14), we however have $\ker T = \ker P$, and since $(\ker T)^{\perp} = \overline{\operatorname{ran} T}$ because T is normal, this implies

$$\overline{\operatorname{ran} P} = (\ker P)^{\perp} = (\ker T)^{\perp} = \operatorname{ran} T = \overline{\operatorname{ran} U},$$

so that U defines a unitary operator on this space.

Finally, an arbitrary operator $A \in \mathcal{B}(\mathcal{H})$ that commutes with T and T^* also commutes with T^*T and hence also with $P = (T^*T)^{1/2}$. Furthermore, it is easy to show that each component of the orthogonal decomposition $\mathcal{H} = \ker(T) \oplus \overline{\operatorname{ran}(T)}$ is left invariant by A. Hence, we find for $x \in \ker(T) = \ker(U)$ that $AUx = 0 = UAx$. On the other hand, by the previous arguments, we have

$$UAP = UPA = TA = AT = AUP,$$

and so $UAx = AUx$ also for every $x \in \overline{\operatorname{ran} P} = \overline{\operatorname{ran} T}$. Altogether, we obtain $UA = AU$.

Finally, for self-adjoint T, we have $0 = T - T^* = UP - PU^* = (U - U*)P$ and hence $U = U^*$ on $\overline{\operatorname{ran}(P)}$. Since U vanishes on U^*, we conclude that also U^* vanishes on $\ker P$, and hence $U = U^*$ on $\mathcal{H} = \ker(T) \oplus \overline{\operatorname{ran}(T)}$. The anti-self-adjointness of U for anti-self-adjoint T follows by similar arguments. □

The following result is due to Teichmüller [195].

Theorem 9.3.5. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then there exists a triple (A, J, B) of mutually commuting operators in $\mathcal{B}(\mathcal{H})$ all of which commute with T such that*

$$T = A + JB, \tag{9.17}$$

where $A = A^*$, $B \succeq 0$, and J is anti-self-adjoint and a partial isometry on $\ker(T - T^*)^\perp$. The operators A and B are given by

$$A = \frac{1}{2}(T + T^*), \quad B = \frac{1}{2}|T - T^*|,$$

and J is the partial symmetry that appears in the polar decomposition of the operator $\frac{1}{2}|T - T^*|$. Finally, the adjoint of T equals $T^* = A - JB$, and every operator in $\mathcal{B}(\mathcal{H})$ commutes with T and T^* if and only if it commutes with A, B, and J.

Proof. We obviously have

$$T = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*).$$

If $A = \frac{1}{2}(T + T^*)$, then $A = A^*$. If we apply Theorem 9.3.4 to

$$C := \frac{1}{2}(T - T^*) \in \mathcal{B}(\mathcal{H}),$$

we obtain a positive operator $B := \frac{1}{2}|T - T^*|$ and a partial symmetry $J \in \mathcal{B}(\mathcal{H})$ on $\operatorname{ran}(T - T^*)$ such that $C = JB$. Since C is anti-self-adjoint, the operators B and J commute, and the operator J is anti-self-adjoint too.

Obviously T and A commute. Moreover, since these operators both commute with C and C is normal, they also commute with the factors B and J in the polar decomposition of C.

Finally, we have

$$T^* = (A + JB)^* = A^* + B^* J^* = A - BJ = A - JB.$$

Every operator that commutes with A, B, and J therefore obviously also commutes with T and T^*. If, on the other hand, $N \in \mathcal{B}(\mathcal{H})$ commutes with T and T^*, then it also commutes with $A = \frac{1}{2}(T + T^*)$ and $C = \frac{1}{2}(T - T^*)$. Since J and B are the factors of the polar decomposition of C, they commute with N. □

The operator J in the above decomposition is of fundamental importance for developing the spectral theory of the normal operator $T = A + JB$. As we will see later, it determines how to multiply the sphere $\mathbb{S}$ of imaginary units onto vectors in $\mathcal{H}$ in order to be in accordance with the operator T when one performs spectral integration. We therefore call operators of this type imaginary operators.

Definition 9.3.6. An anti-self-adjoint operator $J \in \mathcal{B}(\mathcal{H})$ is called *imaginary* if the restriction of J to $\operatorname{ran} J = (\ker J)^{\perp}$ is a unitary operator on $\operatorname{ran} J$. An imaginary operator is called *fully imaginary* if $\ker J = \{0\}$, that is, if J is a unitary anti-self-adjoint operator on $\mathcal{H}$.

Remark 9.3.7. The operator J in the decomposition $T = A + JB$ can be extended to a fully imaginary operator that commutes with T and T^*. Since $\mathcal{H}_0 := \ker J = \ker(T - T^*)$, we find that $T|_{\mathcal{H}_0} = T^*|_{\mathcal{H}_0} = T|_{\mathcal{H}_0}^*$, and so

$$A_0 := A|_{\mathcal{H}_0} = \frac{1}{2}(T + T^*)|_{\mathcal{H}_0} = T|_{\mathcal{H}_0}.$$

The operator A_0 is bounded and self-adjoint, and hence the spectral theorem for bounded self-adjoint operators on a quaternionic Hilbert space based on the S-spectrum (which can be proven in much the same way the complex Hilbert space case, see, e.g., Section 31.3 in [163], with the aid of the spectral mapping theorem given in Theorem 4.2.1) implies the existence of

(i) a measure space $(\Omega, \mathcal{A}, \mu)$ with $\mu \geq 0$,

(ii) a unitary operator $U : \mathcal{H}_0 \to L^2(\Omega, \mathbb{H}, \mu)$ from $\mathcal{H}_0$ to the space of quaternion-valued functions on Ω that are square-integrable with respect to μ, and

(iii) an essentially bounded measurable function $\varphi : \Omega \to \mathbb{R}$

such that

$$A_0 = U^* M_\varphi U,$$

where M_φ denotes the multiplication operator $(M_\varphi f)(\xi) = \varphi(\xi) f(\xi)$ on $L^2(\Omega, \mathbb{H}, \mu)$. We can choose then an arbitrary imaginary unit $j \in \mathbb{S}$ and set $J_0 = U^* M_j U$, where M_j is the multiplication operator $(M_j f)(\xi) = j f(\xi)$ on $L^2(\Omega, \mathbb{H}, \mu)$. The operator M_j is unitary and anti-self-adjoint on $L^2(\Omega, \mathbb{H}, \mu)$, since

$$(M_j)^* = M_{\overline{j}} = M_{-j} = -M_j = (M_j)^{-1}.$$

Since U is unitary, also $J_0 = U^* M_j U$ is unitary and anti-self-adjoint. Since the function φ is real-valued, it commutes with j, and hence

$$\begin{aligned} A_0 J_0 &= U^* M_\varphi U U^* M_j U = U^* M_\varphi M_j U \\ &= U^* M_j M_\varphi U = U^* M_j U^* U M_\varphi U = J_0 A_0. \end{aligned}$$

If E_0 is the orthogonal projection of $\mathcal{H}_0$ and E_1 is the projection of $\mathcal{H}$ onto $\mathcal{H}_0^{\perp} = \operatorname{ran} J$, then we find due to the above arguments that $\tilde{J} = J_0 E_0 + J E_1$ is a fully imaginary operator that, by construction, commutes with T and T^*. The details of this construction were explained in [142]. The unitary operator U and the imaginary unit $j \in \mathbb{S}$ are, however, not determined by T and in particular are not unique. Also the operator J_0 on $\mathcal{H}_0$ and the extension of J to $\tilde{J}$ are in turn not determined by T and in particular are not unique unless $\ker J = \{0\}$.

Corollary 9.3.8. *If J is an imaginary operator, then $-J^2$ is the orthogonal projection onto* $\operatorname{ran} J$*.*

Proof. Since $J_0 = J|_{\operatorname{ran} J}$ is a unitary operator on $\operatorname{ran} J$ and $(J_0)^* = J^*|_{\operatorname{ran} J} = -J_0$, we obtain $J_0^{-1} = J_0^* = -J_0$, and so $-J^2 x = -J_0^2 x = x$ for every $x \in \operatorname{ran} J$. Since obviously $-J^2 x \in \operatorname{ran} J$ for every $x \in \mathcal{H}$, we conclude that $(-J^2)^2 x = -J^2 x$ for every $x \in \mathcal{H}$, and hence $-J_0^2$ is a projection. Since J is anti-self-adjoint, the projection $-J^2$ is self-adjoint, since $(-J^2)^* = -(J^*)^2 = -J^2$, and hence it is the orthogonal projection on $\operatorname{ran}(-J^2) = \operatorname{ran} J$. □

Every imaginary operator allows one to split the space $\mathcal{H}$ into three complex linear subspaces, on which the J is the multiplication with only one quaternion.

Lemma 9.3.9. *If J is an imaginary operator and $j \in \mathbb{S}$, then*

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j \tag{9.18}$$

with

$$\mathcal{H}_0 := \{x \in \mathcal{H} : \ Jx = 0\} \quad \text{and} \quad \mathcal{H}_\pm^j := \{x \in \mathcal{H} : \ Jx = x(\pm j)\}.$$

The spaces $\mathcal{H}_\pm^j$ are nontrivial if $J \neq 0$, they are complex Hilbert spaces over $\mathbb{C}_j$ with the structure they inherit from $\mathcal{H}$, and the orthogonality in (9.18) is intended in the sense of the $\mathbb{C}_j$-Hilbert space structure on $\mathcal{H}$.

Proof. We obviously have $\mathcal{H} = \ker J \oplus \operatorname{ran} J = \mathcal{H}_0 \oplus \operatorname{ran} J$. We hence have to show that $\operatorname{ran} J = \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ for $j \in \mathbb{S}$. Let therefore $x \in \operatorname{ran} J$. Then

$$x = \frac{1}{2}(x - Jxj) + \frac{1}{2}(x + Jxj).$$

Setting $x_+ := \frac{1}{2}(x - Jxj)$ and $x_- := \frac{1}{2}(x + Jxj)$, we obtain $x = x_+ + x_-$ with

$$Jx_+ = \frac{1}{2}(Jx - J^2 xj) = \frac{1}{2}(-Jxj + x)j = x_+ j$$

and

$$Jx_- = \frac{1}{2}(Jx + J^2 xj) = \frac{1}{2}(-Jxj - x)j = x_+(-j)$$

due to Corollary 9.3.8. Hence, (9.18) holds.

Obviously $\mathcal{H}_+^j$ and $\mathcal{H}_-^j$ are $\mathbb{C}_j$-linear vector spaces that are closed in the topology of $\mathcal{H}$. Moreover, if $x, y \in \mathcal{H}_+^j$, then

$$j\langle x, y\rangle = \langle x(-j), y\rangle = \langle -Jx, y\rangle = \langle x, Jy\rangle = \langle x, y\rangle j.$$

Hence, $\langle x, y\rangle$ belongs to $\mathbb{C}_j$, and so $\mathcal{H}_+^j$ is actually a Hilbert space over $\mathbb{C}_j$. Similarly, we can also show that $\mathcal{H}_j^-$ is a Hilbert space over $\mathbb{C}_j$.

Finally, the spaces $\mathcal{H}_0$, $\mathcal{H}_+^j$, and $\mathcal{H}_-^j$ are orthogonal if we consider $\mathcal{H}$ a Hilbert space over $\mathbb{C}_j$ with the scalar product

$$\langle x, y\rangle_j := \xi_0 + \xi_1 j \quad \text{if } \langle x, y\rangle = \xi_0 + \xi_1 j + \xi_2 i + \xi_3 ji$$

with $i \in \mathbb{S}$ so that $i \perp j$. Since $\mathcal{H}_0 \perp \operatorname{ran} J$, we obviously have $\mathcal{H}_0 \perp \mathcal{H}_\pm^j$. For $x \in \mathcal{H}_+^j$ and $y \in \mathcal{H}_-^j$, on the other hand, we have

$$j\langle x, y\rangle = \langle x(-j), y\rangle = \langle -Jx, y\rangle = \langle x, Jy\rangle = \langle x, y\rangle(-j).$$

Since $\langle x, y\rangle$ anti-commutes with j, it is of the form $\langle x, y\rangle = \xi_2 i + \xi_3 ji$, and hence $\langle x, y\rangle_j = 0$. □

Definition 9.3.10. Let J be an imaginary operator. We define, according to the direct sum decomposition in (9.18), the $\mathbb{C}_j$-linear projections

$$\Pi_0 : \mathcal{H} \to \mathcal{H}_0, \quad \Pi_+^j : \mathcal{H} \to \mathcal{H}_+^j, \quad \Pi_-^j : \mathcal{H} \to \mathcal{H}_-^j.$$

These projections are orthogonal in the $\mathbb{C}_j$-Hilbert space structure of $\mathcal{H}$.

9.4 The Continuous Functional Calculus

In this section we introduce the continuous functional calculus of a normal operator on a quaternionic Hilbert space. This functional calculus applies to continuous intrinsic slice function, and therefore we start by investigating this class of functions in more detail.

Definition 9.4.1. Let $\Omega \subset \mathbb{H}$ be an axially symmetric open set. We denote the sets of left, right, and intrinsic slice functions on Ω that are continuous respectively by $\mathcal{SC}_L(\Omega)$, $\mathcal{SC}_R(\Omega)$, and $\mathcal{SC}(\Omega)$.

Remark 9.4.2. The set $\mathcal{C}(\Omega, \mathbb{H})$ of all continuous quaternion-valued functions on a compact axially symmetric set $\Omega \subset \mathbb{H}$ is a two-sided quaternionic Banach space with the pointwise multiplications $(af)(q) = af(q)$ and $(fa)(q) = f(q)a$ and with the supremum norm $\|f\|_\infty := \sup_{q\in\Omega} |f(q)|$. If follows from the structure formula in Theorem 2.1.9 that the uniform limit of a sequence of continuous left, right, or intrinsic slice functions is again a continuous left, right, or intrinsic slice function on Ω. Hence, the set $\mathcal{SC}_L(\Omega)$ is a closed quaternionic right linear subspace of $\mathcal{H}$ and so a quaternionic right Banach space, and the set $\mathcal{SC}_R(\Omega)$ is a closed quaternionic left linear subspace of $\mathcal{H}$ and so a quaternionic left Banach space.

The set of intrinsic slice functions is, however, not invariant under multiplication by quaternionic scalars, neither from the left nor from the right, but only under multiplication by real scalars. Hence $\mathcal{SC}(\Omega)$ is only a closed $\mathbb{R}$-linear subspace of $\mathcal{C}(\Omega, \mathbb{H})$, and so it is only a real Banach space. Since the pointwise product of two intrinsic slice functions is again an intrinsic slice function and since the pointwise product of two intrinsic slice functions is commutative, as one can verify easily, $\mathcal{SC}(\Omega)$ is even a commutative real Banach algebra. The sets $\mathcal{SC}_L(\Omega)$ and $\mathcal{SC}_R(\Omega)$, on the other hand, are by Theorem 2.1.4 invariant only under multiplication by intrinsic slice functions, and hence they do not form an algebra with the pointwise product.

Lemma 9.4.3. *Let $\Omega \subset \mathbb{H}$ be axially symmetric and let*

$$\Omega^+ := \{(u, v) \in \mathbb{R} \times [0, +\infty) : \ u + \mathbb{S}v \subset \Omega\}.$$

A function $f : \Omega \to \mathbb{H}$ is a left slice function if and only if there exist functions $F_0, F_1 : \Omega^+ \to \mathbb{H}$, where $F_1(u, v) = 0$ if $v = 0$, such that

$$f(q) = F_0(u, v) + jF_1(u, v)$$

for all $q = u + jv \in \Omega$ with $v \geq 0$, and it is a right slice function if and only if there exist functions $F_0, F_1 : \Omega^+ \to \mathbb{H}$, where $F_1(u, v) = 0$ if $v = 0$, such that

$$f(q) = F_0(u, v) + F_1(u, v)j$$

for all $q = u + jv \in \Omega$ with $v \geq 0$. In this case, the function f is intrinsic if and only if F_0 and F_1 take values in $\mathbb{R}$, and it is continuous if and only if F_0 and F_1 are continuous.

Proof. If f is a left slice function, then $f(q) = f_0(u, v) + jf_1(u, v)$ for every $q = u + jv \in \Omega$ with arbitrary $j \in \mathbb{S}$, with functions $f_\ell : \Omega \to \mathbb{H}$ that satisfy the compatibility condition

$$f_0(u, v) = f_0(u, -v) \quad \text{and} \quad f_1(u, -v) = -f_1(u, v)$$

in (2.4), where

$$\tilde{\Omega} := \{(u, v) \in \mathbb{R} \times \mathbb{R} : \ u + \mathbb{S}v \subset \Omega\}.$$

We can obviously set $F_\ell(u, v) = f_\ell(u, v)$ for $(u, v) \in \Omega^+$ and $\ell = 1, 2$, and we find that $f(q) = F_0(u, v) + jF_1(u, v)$ if $q = u + jv$ with $v \geq 0$ belongs to Ω. Since $f_1(u, -v) = -f_1(u, v)$, we moreover obtain

$$F_1(u, 0) = f_1(u, 0) = -f_1(u, 0) = -F_1(u, 0),$$

and hence $F_1(u, 0) = 0$ for every $u + j0 \in \Omega$.

Conversely, if $f(q) = F_0(u,v) + jF_1(u,v)$ for every $q = u + jv \in \Omega$ with $v \geq 0$, then we can simply define, for $(u,v) \in \tilde{\Omega}$,

$$f_0(u,v) := \begin{cases} F_0(u,v), & v \geq 0, \\ F_0(u,-v), & v < 0, \end{cases} \quad f_1(u,v) := \begin{cases} F_1(u,v), & v \geq 0, \\ -F_1(u,-v), & v < 0. \end{cases}$$

Then $f(q) = f_0(u,v) + jf_1(u,v)$ for all $q = u + jv \in \Omega$, and f_0 and f_1 satisfy the compatibility condition (2.4) because $F_1(u,0) = 0$ for all $(u,0) \in \Omega^+$. Hence f is a left slice function.

Obviously f is intrinsic if and only if F_0 and F_1 are real-valued. Moreover, we have

$$F_0(u,v) = \frac{1}{2}\left(f(u+jv) + f(u-jv)\right)$$

and

$$F_1(u,v) = \frac{1}{2}j(f(u-jv) - f(u+jv))$$

for every $(u,v) \in \Omega^+$ and any $j \in \mathbb{S}$. Hence F_0 and F_1 are continuous if f is continuous. Conversely, assume that F_0 and F_1 are continuous. Then $f(q) = F_0(u,v) + jF_1(u,v)$ is continuous on $\Omega \setminus \mathbb{R}$ because u, v, and j depend continuously on q on this set. On the real line, the terms $F_0(u,v)$ and $F_1(u,v)$ depend continuously on $q = u + jv$, but the imaginary unit j does not. However, since $q \mapsto F_1(u,v)$ tends to zero as q approaches the real line, the function f is also continuous at points in $\mathbb{R}$ and hence continuous on all of Ω. For right slice functions, we can argue similarly. □

Remark 9.4.4. From the above result it is clear that the functions f_0 and f_1 are simply straightforward extensions of F_0 and F_1 to all of $\tilde{\Omega}$ that we obtain by imposing the compatibility condition (2.4). In Remark 2.1.3, we argued that the compatibility condition is necessary in order to ensure that $f(q)$ is well defined and independent of the choice of the imaginary unit j that we use to represent $q = u + jv$. Indeed, if $v < 0$, then we can write $q = u + (-j)v$ with $-j \in \mathbb{S}$ and find that

$$f(q) = f_0(u,v) + jf_1(u,v) = f_0(u,-v) + (-j)f_1(u,-v)$$

due to (2.4). If we always choose the imaginary unit $j_q := \underline{q}/|\underline{q}|$ in the representation of q, then always $v \geq 0$, and hence F_0 and F_1 are sufficient to describe f. However, in order to define slice hyperholomorphicity, the extended functions f_0 and f_1 are necessary. The set Ω^+ does not contain neighborhoods of real points. In order to consider the partial derivatives of the component functions, which for slice hyperholomorphic functions must satisfy the Cauchy–Riemann equations, one hence needs to work with the extended component functions f_0 and f_1 in order to avoid technical problems when differentiating on the real line.

The following theorem is a classical readaptation of the classical Stone–Weierstrass theorem; cf. also [142].

Theorem 9.4.5. *Every polynomial P in q of the form*

$$P(q) = \sum_{0 \leq |\ell| \leq n} a_\ell q^{\ell_1} \overline{q}^{\ell_2} \tag{9.19}$$

with coefficients $a_\ell \in \mathbb{R}$ for every multi-index $\ell = (\ell_1, \ell_2)$ is a continuous intrinsic slice function on $\mathbb{H}$. For every compact axially symmetric set $\Omega \subset \mathbb{H}$ the set of polynomials of the form (9.19) is furthermore dense in $\mathcal{SC}(\Omega)$.

Proof. The functions $q \mapsto q$ and $q \mapsto \overline{q}$ are obviously continuous intrinsic slice functions. Since the set of continuous intrinsic slice functions is closed under pointwise multiplication and pointwise multiplication by a real number, we conclude that every polynomial of the form (9.19) is also a continuous intrinsic slice function.

Let now $\Omega \subset \mathbb{H}$ be axially symmetric and compact and let us consider a function $f \in \mathcal{SC}(\Omega)$. The function $f_j = f|_{\Omega_j^+}$ with $\Omega_j^+ := \Omega \cap \mathbb{C}_j^+$ is then a continuous $\mathbb{C}_j$-valued function on the compact set $\Omega \cap \mathbb{C}_j^+$. The Stone–Weierstrass approximation theorem implies the existence of a sequence of polynomials $Q_n(z) = \sum_{0 \leq |\ell| \leq n} b_{n,\ell} z^{\ell_1} \overline{z}^{\ell_2}$ with $b_{n,\ell} \in \mathbb{C}_j$ that converges uniformly to f_j on Ω_j. We set $P_n(z) := \frac{1}{2}\left(Q_n(z) + \overline{Q_n(\overline{z})}\right)$ and we denote the coefficients of P_n by $a_{n,\ell}$, so that $P_n(z) = \sum_{0 \leq |\ell| \leq n} a_{n,\ell} z^{\ell_1} \overline{z}^{\ell_2}$. Obviously, $\overline{P_n(z)} = P_n(\overline{z})$, and so we find for arbitrary $t \in \mathbb{R}$ that

$$\sum_{0 \leq |\ell| \leq n} \overline{a_\ell} t^{\ell_1 + \ell_2} = \overline{P_n(t)} = P_n(t) = \sum_{0 \leq |\ell| \leq n} a_\ell t^{\ell_1 + \ell_2}.$$

Hence P_n has real coefficients. Its natural extension $P_n(q) = \sum_{0 \leq |\ell| \leq n} a_\ell q t^{\ell_1} \overline{q}^{\ell_2}$ to $\mathbb{H}$ is therefore of the form (9.19). Furthermore, it tends uniformly to f on Ω, since for every $s = u + iv \in U$, we can set $z = u + jv \in \Omega_j^+$ and find due to the structure formula (2.9) that

$$\begin{aligned} |f(s) - P_n(s)| &\leq |f(z) - P_n(z)| + |f(\overline{z}) - P_n(\overline{z})| \\ &= |f(z) - P_n(z)| + \left|\overline{f(z) - P_n(z)}\right| \leq 2 \sup_{z \in \Omega_j} |f(\overline{z}) - P_n(\overline{z})| \overset{n \to +\infty}{\longrightarrow} 0. \square \end{aligned}$$

Theorem 9.4.5 allows us now to define the continuous functional calculus. This functional calculus relies on the $T = A + JB$ decomposition introduced by Teichmüller, cf. Theorem 9.3.5. Following the usual strategy, we can first define $P(T)$ for a normal operator T and any polynomial of the form (9.19) in the natural way. Due to the density of these polynomials, we can then extend this functional calculus to arbitrary continuous intrinsic slice functions.

Definition 9.4.6. Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. For every polynomial

$$P(q) = \sum_{0 \leq |\ell| \leq n} a_\ell q^{\ell_1} \overline{q}^{\ell_2} \tag{9.20}$$

with real coefficients as in (9.19), we define the operator

$$P(T) := \sum_{0\leq|\ell|\leq n} a_\ell T^{\ell_1}(T^*)^{\ell_2}. \tag{9.21}$$

Theorem 9.4.7. *Let $T \in \mathcal{B}(\mathcal{H})$ be a normal operator. For every polynomial $P(q) = \sum_{0\leq|\ell|\leq n} a_\ell q^{\ell_1}\overline{q}^{\ell_2}$ with real coefficients, the operator $P(T)$ is a normal operator that commutes with T and T^*, and*

$$\sigma_S(P(T)) = P(\sigma_S(T)).$$

In particular, this implies $\|P(T)\| = \max_{s\in\sigma_S(T)} |P(s)|$.

Proof. The adjoint of $P(T)$ is $P(T)^* = \sum_{0\leq|\ell|\leq n} a_\ell (T^*)^{\ell_1} T^{\ell_2}$. Since T and T^* commute, since T is normal and the coefficients a_ℓ are real, this operator obviously commutes with T and T^* and in turn also with $P(T)$. Hence $P(T)$ is normal.

Let $T = A+JB$ be the decomposition (9.17) of T and recall that the operators $A = \frac{1}{2}(T+T^*)$, $B = \frac{1}{2}|T-T^*|$ and the imaginary operator J commute mutually. The Hilbert space $\mathcal{H}$ can then decomposed into the orthogonal sum

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

with $\mathcal{H}_0 := \ker J = \ker B$ and $\mathcal{H}_1 := \operatorname{ran} J = \overline{\operatorname{ran} B}$. Since T and T^* leave $\mathcal{H}_0$ and $\mathcal{H}_1$ invariant, also the operator $P(T)$ leaves $\mathcal{H}_0$ and $\mathcal{H}_1$ invariant.

If A is an arbitrary bounded operator on $\mathcal{H}$ that leaves $\mathcal{H}_0$ and $\mathcal{H}_1$ invariant, then $A_\ell := A|_{\mathcal{H}_\ell}$ belongs to $\mathcal{B}(\mathcal{H}_\ell)$ for $\ell = 0, 1$ and we obtain

$$\sigma_S(A) = \sigma_S(A_0) \cup \sigma_S(A_1). \tag{9.22}$$

Indeed, if $s \in \rho_S(A)$, then $\mathcal{Q}_s(A_\ell)^{-1} = \mathcal{Q}_s(A)^{-1}|_{\mathcal{H}_\ell} \in \mathcal{B}(\mathcal{H}_\ell)$ for $\ell = 0, 1$, and hence $s \in \rho_S(A_0) \cap \rho_S(A_1)$. Conversely, if $s \in \rho_S(A_0) \cap \rho_S(A_1)$, then the inverse of $\mathcal{Q}_s(A)$ is the operator $\mathcal{Q}_s(A)^{-1} = \mathcal{Q}_s(A_0)^{-1}E_0 + \mathcal{Q}_s(A_1)^{-1}E_1$, where E_ℓ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\ell$. This operator is obviously bounded, and so $s \in \rho_S(T)$. We conclude that $\rho_S(A) = \rho_S(A_0) \cap \rho_S(A_1)$, and by taking the complement, we arrive at (9.22).

If we set $T_\ell = T|_{\mathcal{H}_\ell}$, then $(T_\ell)^* = T^*|_{\mathcal{H}_\ell}$, and so $P(T_\ell) = P(T)|_{\mathcal{H}_\ell}$. Since $\mathcal{H}_0 = \ker|T-T^*| = \ker(T-T^*)$, we find that $T^* = T$ on $\mathcal{H}_0$, and so $T_0^* = T_0$. Hence $P(T_0) = \sum_{0\leq|\ell|\leq n} a_\ell T_0^{\ell_1+\ell_2}$, and we conclude from Theorem 4.2.1 that $\sigma_S(P(T_0)) = P(\sigma_S(T_0))$.

Before we consider the operator T_1, let us recall that for $j \in \mathbb{S}$ we can split $\mathcal{H}_1$ into the direct sum

$$\mathcal{H}_1 = \operatorname{ran} J = \mathcal{H}_+^j \oplus \mathcal{H}_-^j,$$

where $\mathcal{H}_\pm^j = \{x \in \mathcal{H} : Jx = x(\pm j)\}$ are complex Hilbert spaces over $\mathbb{C}_j$ by Lemma 9.3.9. If C is a bounded quaternionic right linear operator on $\mathcal{H}_1$ that commutes with J, then $J(Tx) = T(Jx) = T(xj) = (Tx)j$, and so $Tx \in \mathcal{H}_1$. Hence

$C_j := C|_{\mathcal{H}_j^+}$ defines a bounded $\mathbb{C}_j$-linear operator on $\mathcal{H}_j^+$. Conversely, if C_j is a bounded $\mathbb{C}_j$-linear operator on $\mathcal{H}_+^j$, then it extends naturally to a quaternionic linear operator on $\mathcal{H}_1$. Indeed, if $i \in \mathbb{S}$, with $i \perp j$, then $J(xi) = (Jx)i = xji = (xi)(-j)$ because i and j anti-commute. Hence $x \mapsto xi$ maps $\mathcal{H}_+^j$ to $\mathcal{H}_-^j$. Since its inverse is given by $x \mapsto x(-i)$, this function is even a bijection. We can therefore write every vector $x = x_+ + x_- \in \mathcal{H}_1 = \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ as $x = x_1 + x_2 i$ with two components $x_1 = x_+$ and $x_2 = x_-(-i)$ in $\mathcal{H}_j^+$. We moreover obtain

$$\begin{aligned} \|x\|^2 = \langle x, x\rangle_{\mathcal{H}} &= \langle x_1, x_1\rangle + \langle x_2, x_1\rangle i - i\langle x_1, x_2\rangle - i\langle x_2, x_2\rangle i \\ &= \langle x_1, x_1\rangle + \langle x_2, x_2\rangle = \|x_1\|^2 + \|x_2\|^2. \end{aligned} \tag{9.23}$$

The natural quaternionic linear extension of C_j is then simply

$$Cx := C_j x_1 + (C_j x_2) i.$$

This operator is $\mathbb{C}_j$-linear because C_j is $\mathbb{C}_j$-linear, and it satisfies

$$\begin{aligned} C(xi) &= C(x_1 i - x_2) = C_j(x_1) i - C_j(x_2) \\ &= (C_j(x_1) + (C_j x_2) i) i = (Cx) i. \end{aligned}$$

If we write $a \in \mathbb{H}$ as $a = a_1 + a_2 i$ with $a_1, a_2 \in \mathbb{C}_j$, we therefore obtain

$$\begin{aligned} C(xa) &= C(xa_1) + C(xa_2 i) = C(xa_1) + C(xa_2) i \\ &= C(x)a_1 + C(x)a_2 i = C(x)a, \end{aligned}$$

and hence C is actually $\mathbb{H}$-linear. Moreover, we have on the one hand

$$\begin{aligned} \|C_j\| &= \sup\{\|C_j x\| : x \in \mathcal{H}_+^j, \|x\| = 1\} \\ &= \sup\{\|Cx\| : x \in \mathcal{H}_+^j, \|x\| = 1\} \\ &\leq \sup\{\|Cx\| : x \in \mathcal{H}_1, \|x\| = 1\} = \|C\|, \end{aligned}$$

and due to (9.23), on the other hand

$$\begin{aligned} \|Cx\|^2 &= \|C_j x_1 + (C_j x_2) i\|^2 = \|C_j x_1\|^2 + \|C_j x_2\|^2 \\ &\leq \|C_j\|^2 \left(\|x_1\|^2 + \|x_2\|^2\right) = \|C_j\|^2 \|x\|^2, \end{aligned}$$

which implies $\|C\| \leq \|C_j\|$. Altogether, $\|C\| = \|C_j\|$.

The S-spectrum of C and the spectrum of C_j satisfy the relation

$$\sigma_S(C) \cap \mathbb{C}_j = \sigma(C_j) \cup \overline{\sigma(C_j)}, \tag{9.24}$$

which will be essential for us. For every $x \in \mathcal{H}_+^j$ and $z = z_0 + z_1 j \in \mathbb{C}_j$, we namely have

$$\left(z\mathcal{I}_{\mathcal{H}_+^j}\right) x = xz = xz_0 + xz_1 j = (z_0 \mathcal{I}_{\mathcal{H}_+}^j + z_1 J)x,$$

where we recall that the multiplication of vectors in $\mathcal{H}_+^j$ by scalars in $\mathbb{C}_j$ is simply the multiplication of vectors by quaternionic scalars from the right in $\mathcal{H}$ that is restricted to $\mathbb{C}_j$. For every $x \in \mathcal{H}_+^j$, we have $-J^2 x = -xj^2 = x$, and so

$$\begin{aligned}\mathcal{Q}_z(C)x &= \mathcal{Q}_z(C_j)x = (C_j^2 - 2z_0 C_j + |z|^2\mathcal{I})x \\ &= (z_0\mathcal{I} - z_1 J - C_j)(z_0\mathcal{I} + z_1 J - C_j)x = (z\mathcal{I}_{\mathcal{H}_+^j} - C_j)(\overline{z}\mathcal{I}_{\mathcal{H}_+^j} - C_j)x.\end{aligned} \tag{9.25}$$

Hence if $z \in \rho_S(C) \cap \mathbb{C}_j$, then the resolvent $R_z(C_j) = (z\mathcal{I}_{\mathcal{H}_+^j} - C_j)^{-1}$ of C_j at z is $R_z(C_j) = (\overline{z}\mathcal{I}_{\mathcal{H}_+^j} - C_j)\mathcal{Q}_z(C)^{-1}$, and so $z \in \rho(C_j)$. We conclude that $\rho_S(C)\cap\mathbb{C}_j \subset \rho(C_j)$. Due to the axial symmetry of $\rho_S(C)$, we also have $\rho_S(C)\cap\mathbb{C}_j = \overline{\rho_S(C) \cap \mathbb{C}_j} \subset \overline{\rho(C_j)}$, and so

$$\rho_S(C) \cap \mathbb{C}_j \subset \rho(C_j) \cap \overline{\rho(C_j)}.$$

Conversely, if $z \in \rho(C_j) \cap \overline{\rho(C_j)}$, then z and $\overline{z}$ both belong to $\rho(C_j)$, and we conclude from (9.25) that the pseudo-resolvent $\mathcal{Q}_z(C)^{-1}$ of C at z is the quaternionic linear extension of the operator $R_{\overline{z}}(C)R_z(C)$. Hence $z \in \rho_S(C)$, and we conclude that

$$\rho_S(C) \cap \mathbb{C}_j \supset \rho(C_j) \cap \overline{\rho(C_j)}$$

and in turn

$$\rho_S(C) \cap \mathbb{C}_j = \rho(C_j) \cap \overline{\rho(C_j)}.$$

Taking the complement of this set in $\mathbb{C}_j$, we arrive at (9.24).

Let us now return to the operator $T_1 = T|_{\mathcal{H}_1}$. Since this operator commutes with $J_1 = J|_{\mathcal{H}_1}$, it is by the above arguments the quaternionic linear extension of the complex linear operator $T_j = T_1|_{\mathcal{H}_+^j} = T|_{\mathcal{H}_+^j} \in \mathcal{B}(\mathcal{H}_+^j)$, and we have $\sigma_S(T_1) \cap \mathbb{C}_j = \sigma(T_j) \cup \overline{\sigma(T_j)}$. Moreover, $T_j^* = T^*|_{\mathcal{H}_+^j}$, as one can check easily. Hence T_j is normal and $P_j(T_j) = \sum_{0\leq|\ell|\leq n} a_\ell T_j^{\ell_1}(T_j^*)^{\ell_2} = P(T_1)|_{\mathcal{H}_+^j}$, where $P_j = P|_{\mathbb{C}_j}$. The spectral mapping property of the continuous functional calculus for normal operators on a complex Hilbert space (cf. for instance [105, 183]) implies

$$\sigma(P_j(T_j)) = P_j(\sigma(T_j)).$$

Since P is an intrinsic function, we have $P_j\left(\overline{\sigma(T_j)}\right) = \overline{P_j(\sigma(T_j))}$, and so

$$\begin{aligned}\sigma_S(P(T_1)) \cap \mathbb{C}_j &= \sigma(P_j(T_j)) \cup \overline{\sigma(P_j(T_j))} \\ &= P_j(\sigma(T_j)) \cup \overline{P_j(\sigma(T_j))} = P_j\left(\sigma(T_j) \cup \overline{\sigma(T_j)}\right).\end{aligned}$$

As an intrinsic slice function, P is compatible with the axially symmetric hull, that is, $[P(\Delta)] = P([\Delta])$ for every $\Delta \subset \mathbb{C}_j$. Hence, we finally obtain

$$\begin{aligned}\sigma_S(P(T_1)) &= [\sigma_S(P(T_1)) \cap \mathbb{C}_j] = \left[P_j\left(\sigma(T_j) \cup \overline{\sigma(T_j)}\right)\right] \\ &= \left[P\left(\sigma(T_j) \cup \overline{\sigma(T_j)}\right)\right] = P\left(\left[\sigma(T_j) \cup \overline{\sigma(T_j)}\right]\right) = P(\sigma_S(T_1)).\end{aligned}$$

If we turn our attention back to the operator T that is defined on the entire space $\mathcal{H}$, we find due to (9.22) and the identities $P(T_0) = P(T)|_{\mathcal{H}_0}$ and $P(T_1) = P(T)|_{\mathcal{H}_1}$ that

$$\begin{aligned}\sigma_S(P(T)) &= \sigma_S(P(T_0)) \cup \sigma(P(T_1)) \\ &= P(\sigma_S(T_0)) \cup P(\sigma_S(T_1)) = P(\sigma_S(T_0) \cup \sigma_S(T_1)) = P(\sigma_S(T)).\end{aligned}$$

Finally, since the norm of a normal operator coincides with its S-spectral radius, which is as in the complex case an easy consequence of Gelfand's formula for the S-spectral radius in Theorem 4.2.3, we obtain

$$\|P(T)\| = \max_{s\in\sigma_S(P(T))} |s| = \max_{s\in P(\sigma_S(T))} |s| = \max_{s\in\sigma_S(T)} |P(s)|. \qquad \square$$

Remark 9.4.8. For the operator $T_j = T_{\mathcal{H}_+^j}$, the identity $\sigma_S(T_1)\cap\mathbb{C}_j = \sigma(T_j)\cup\overline{\sigma(T_j)}$ in (9.24) can be further specified. Since $T = A + JB$, we have with $A_j := A|_{\mathcal{H}_+^j}$ and $B_j := B|_{\mathcal{H}_+^j}$ that $T_j = A_j + jB_j$ because $T = A + JB$ and $J|_{\mathcal{H}_+^j} = j\mathcal{I}_{\mathcal{H}_+^j}$. For every $z = z_0 + jz_1 \in \sigma_S(T)$, we have $z_1 = \frac{j}{2}(-z + \overline{z})$. By the spectral mapping theorem of the continuous functional calculus for operators on complex Hilbert spaces (cf. [105, 183]), we therefore have

$$\begin{aligned}\{z_1 : z_0 + jz_1 \in \sigma(T_j)\} &= \sigma\left(\frac{j}{2}(-T_j + T_j^*)\right) \\ &= \sigma\left(\frac{j}{2}(-A_j - jB_j + A_j - jB_j)\right) = \sigma(B_j) \subset [0, +\infty)\end{aligned}$$

because B, and hence also B_j, is positive. Therefore, every $z \in \sigma(T_j)$ belongs to the upper half-plane $\mathbb{C}_j^+ := \{z_0 + z_1 j \in \mathbb{C}_j : z_1 \geq 0\}$, and we conclude from (9.24) that

$$\sigma(T_j) = \sigma_S(T) \cap \mathbb{C}_j^+.$$

Theorem 9.4.9 (Continuous functional calculus of a normal quaternionic operator)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. There exists a unique continuous homomorphism of real unital $*$-algebras*

$$\Psi_T : \begin{cases} \mathcal{SC}(\sigma_S(T)) & \to \quad \mathcal{B}(\mathcal{H}), \\ \quad f & \mapsto \quad \Psi_T(T) := f(T), \end{cases}$$

such that $s(T) = T$, where s denotes the identity function $s \mapsto s$. The homomorphism has furthermore the following properties:

(i) *The homomorphism Ψ_T is isometric, since $\|f(T)\| = \max_{s\in\sigma_S(T)} |f(s)|$.*

(ii) *Every operator $f(T)$ is normal, and it commutes with T and T^* (or equivalently with A, B, and J where $T = A + JB$ is the decomposition (9.17) of T).*

(iii) *The spectral mapping property* $\sigma_S(f(T)) = f(\sigma_S(T))$ *holds, and for every function* $g \in \mathcal{SC}(\sigma_S(f(T)))$, *we have* $g(f(T)) = (g \circ f)(T)$.

Proof. Let us first show the existence of the homomorphism Ψ_T. For every function $f \in \mathcal{SC}(\sigma_S(T))$, there exists due to Theorem 9.4.5 a sequence $P_n(q) = \sum_{0\leq|\ell|\leq n} a_{n,\ell}q^{\ell_1}\overline{q}^{\ell_2}$ of polynomials of the form (9.19) that converges uniformly to f on $\sigma_S(T)$. In particular, P_n is a Cauchy sequence in $\mathcal{SC}(\sigma_S(T))$, and hence the sequence $P_n(T) := \sum_{0\leq|\ell|\leq n} a_{n,\ell}T^{\ell_1}(T^*)^{\ell_2}$ is a Cauchy sequence in $\mathcal{B}(\mathcal{H})$, since

$$\|P_n(T) - P_m(T)\| = \|(P_n - P_m)(T)\| = \max_{s\in\sigma_S(T)} |P_n(s) - P_m(T)| \tag{9.26}$$

by Theorem 9.4.7. Hence $P_n(T)$ converges in $\mathcal{B}(\mathcal{H})$, and we can define

$$f(T) := \lim_{n\to+\infty} P_n(T).$$

The operator $f(T)$ does not depend on the choice of the polynomials P_n and is hence well defined. If $\tilde{P}_n$ is a different sequence of polynomials that tends uniformly to f on $\sigma_S(T)$, then $P_n - \tilde{P}_n$ tends uniformly to zero on $\sigma_S(T)$, and we conclude from

$$\left\|P_n(T) - \tilde{P}_n(T)\right\| = \max_{s\in\sigma_S(T)} \left|P_n(s) - \tilde{P}_n(s)\right| \overset{n\to+\infty}{\longrightarrow} 0$$

that $\lim_{n\to+\infty} P_n(T) = \lim_{n\to+\infty} \tilde{P}_n(T)$. The mapping $P \mapsto P(T)$ defined for polynomials of the form (9.19) is obviously a homomorphism of real unital $*$-algebras, and hence also the above defined continuous extension $f \mapsto f(T)$ is a homomorphism of real unital $*$-algebras.

The homomorphism Ψ_T is obviously uniquely determined by the property $s(T) = T$. Due to the homomorphism property, this determines $P(T)$ for every polynomial of the form (9.19), and polynomials of this type are dense in $\mathcal{SC}(\sigma_S(T))$ by Theorem 9.4.5. Hence by continuity, the requirement $s(T) = T$ determines the entire homomorphism Ψ_T.

Since each of the approximating operators $P_n(T)$ is normal by Theorem 9.4.7, also the limit $f(T)$ is normal and commutes with T and T^*. By Theorem 9.3.5 these operators also commute with the operators A, B, and J in the decomposition (9.17) of the form $T = A + JB$. Since Φ_T is isometric on the set of polynomials of the form (9.19) and since this set is dense in $\mathcal{SC}(\sigma_S(T))$, we obtain $\|f(T)\| = \max_{s\in\sigma_S(T)} |f(s)|$ for every $f \in \mathcal{SC}(\sigma_S(T))$.

Let us finally prove the spectral mapping property and the composition rule. Since f is continuous and $\sigma_S(T)$ is compact, we first of all observe that $f(\sigma_S(T))$ is a compact subset of $\mathbb{H}$, too. Let now $\varepsilon > 0$ and let P be a polynomial of the form (9.19) such that

$$\|P(T) - f(T)\| = \max_{s\in\sigma_S(T)} |P(s) - f(s)| < \varepsilon. \tag{9.27}$$

We then have

$$f(\sigma_S(T)) \subset B_\varepsilon(P(\sigma_S(T))) := \{s \in \mathbb{H} :\ \operatorname{dist}(s, P(\sigma_S(T))\} < \varepsilon.$$

Since P and f commute, also $P(T)$ and $f(T)$ commute. Hence $f(T) = P(T) + \Theta$ with $\Theta = f(T) - P(T)$. The operator Θ commutes with $P(T)$, and it satisfies $\|\Theta\| \leq \varepsilon$, so that Theorem 4.4.12 implies

$$\sigma_S(f(T)) \subset B_\varepsilon(\sigma_S(P(T))) = B_\varepsilon(P(\sigma_S(T))) \subset B_{2\varepsilon}(f(\sigma_S(T))).$$

Since $\varepsilon > 0$ was arbitrary, we obtain

$$\sigma_S(f(T)) \subset \overline{f(\sigma_S(T))} = f(\sigma_S(T)).$$

On the other hand, (9.27) also implies $P(\sigma_S(T)) \subset B_\varepsilon(f(\sigma_S(T))$. Writing $P(T) = f(T) + (-\Theta)$ and applying again Theorem 4.4.12, we obtain

$$f(\sigma_S(T)) \subset B_\varepsilon(P(\sigma_S(T))) = B_\varepsilon(\sigma_S(P(T))) = B_{2\varepsilon}(\sigma_S(f(T))).$$

Since $\varepsilon > 0$ was arbitrary, we conclude that also

$$f(\sigma_S(T)) \subset \sigma_S(f(T)).$$

Altogether, $f(\sigma_S(T)) = \sigma_S(f(T))$.

Finally, due to the homomorphism property, the composition rule holds if g is a polynomial of the form (9.19). If g is an arbitrary function in $\mathcal{SC}(f(\sigma_S(T)))$, then we can choose a sequence of polynomials P_n of the form (9.19) that converges uniformly to g on $f(\sigma_S(f(T))$. Then $P_n \circ f$ converges uniformly to $g \circ f$ on $\sigma_S(T)$, and due to the continuity of Ψ_T, we obtain

$$g(f(T)) = \lim_{n\to+\infty} P_n(f(T)) = \lim_{n\to+\infty} (P_n \circ f)(T) = (g \circ f)(T). \qquad \square$$

Let $f \in \mathcal{SC}(\sigma_S(T))$ and let F_0 and F_1 be the component functions determined in Lemma 9.4.3, so that $f(q) = F_0(u,v) + jF_1(u,v)$ if $q = u + jv$ with $v \geq 0$. Since $u = \frac{1}{2}(q + \overline{q})$ and $v = \frac{1}{2}|q - \overline{q}|$, we can consider F_0 and F_1 functions of q and $\overline{q}$ and then apply the functional calculus defined above. As the next theorem shows, the functional calculus is compatible with these component functions too.

Before we discuss this, let us, however, first show how the approximation in terms of polynomials translates into an approximation of the component functions.

Lemma 9.4.10. *Let $K \subset \mathbb{H}$ be a compact axially symmetric set, let $f = F_0 + jF_1$ in $\mathcal{SC}(K)$, and let $P_n(q) = \sum_{0 \leq |\ell| \leq n} a_{n,\ell} q^{\ell_1} \overline{q}^{\ell_2}$ be a sequence of polynomials of the form (9.19) that converges uniformly to f on K. Then P_n is of the form*

$$P_n(q) = Q_n(u,v) + jvR_n(u,v) \quad \text{for } q = u + jv,$$

where Q_n and R_n are real polynomials such that $Q_n(u,v) \to F_0(u,v)$ and such that $vR_n(u,v) \to F_1$ uniformly on K as n tends to infinity. Furthermore, $Q_n(u,v)$ and $R_n(u,v)$ contain only even powers of v, so that after the identification $u = \frac{1}{2}(q+\overline{q})$ and $v = (-j)\frac{1}{2}(q - \overline{q})$ they are polynomials with real coefficients in q and $\overline{q}$ and hence of the form (9.19).

Proof. For $q = u + jv$, we have

$$P_n(q) = \sum_{0\leq|\ell|\leq n} a_{n,\ell} q^{\ell_1}\overline{q}^{\ell_2} = \sum_{0\leq|\ell|\leq n} a_{n,\ell}(u+jv)^{\ell_1}(u-jv)^{\ell_2}. \tag{9.28}$$

The terms $(u+jv)^{\ell_1}$ and $(u-jv)^{\ell_2}$ are polynomials in u and jv, namely

$$\begin{aligned}(u+jv)^{\ell_1} &= \sum_{\kappa=0}^{\ell_1}\binom{\ell_1}{\kappa}u^{\kappa}(jv)^{\ell_1-\kappa},\\ (u-jv)^{\ell_2} &= \sum_{\kappa=0}^{\ell_2}(-1)^{\ell_2-\kappa}\binom{\ell_2}{\kappa}u^{\kappa}(jv)^{\ell_2-\kappa}.\end{aligned} \tag{9.29}$$

If we apply the identities (9.29) in (9.28) and rearrange the terms, then due to $j^2 = -1$, we are left with an expression of the form

$$\begin{aligned}P_n(q) &= \sum_{\substack{0\leq|\ell|\leq n\\ \ell_2 \text{ even}}} b_{n,\ell}u^{\ell_1}v^{\ell_2} + j\sum_{\substack{0\leq|\ell|\leq n\\ \ell_2 \text{ odd}}} c_{n,\ell}u^{\ell_1}v^{\ell_2}\\ &= \sum_{\substack{0\leq|\ell|\leq n\\ \ell_2 \text{ even}}} b_{n,\ell}u^{\ell_1}v^{\ell_2} + jv\sum_{\substack{0\leq|\ell|\leq n\\ \ell_2 \text{ odd}}} c_{n,\ell}u^{\ell_1}v^{\ell_2-1},\end{aligned}$$

with real coefficients $b_{n,\ell}$ and $c_{n,\ell}$. If we set

$$Q_n(u,v) = \sum_{\substack{0\leq|\ell|\leq n\\ \ell_2 \text{ even}}} b_{n,\ell}u^{\ell_1}v^{\ell_2}, \quad R_n(u,v) := \sum_{\substack{0\leq|\ell|\leq n\\ \ell_2 \text{ odd}}} c_{n,\ell}u^{\ell_1}v^{\ell_2-1},$$

then we find that P_n is of the desired form $P_n(q) = Q_n(u,v) + jvR_n(u,v)$. Finally, since $P_n \to f$ uniformly on K, we find that $\mathrm{Re}(P_n(u,v)) = Q_n(u,v)$ tends uniformly on K to $\mathrm{Re}(f(u,v)) = F_0(u,v)$ and that $\mathrm{Im}(P_n(u,v)) = jvR_n(u,v)$ tends uniformly to $\mathrm{Im}(f(u,v)) = jF_1(u,v)$, which implies that $(-j)\mathrm{Im}(P_n(u,v)) = vR_n(u,v)$ tends uniformly to $(-j)\mathrm{Im}(f(u,v)) = F_1(u,v)$. (Note that again, we do not have any problems on the real line, where $q = u + jv \mapsto j$ is not well defined, because $vR_n(u,v)$ and $F_1(u,v)$ equal 0 if $v = 0$.) □

Theorem 9.4.11. *Let $T = A + JB \in \mathcal{B}(\mathcal{H})$ be a normal operator and let $f = F_0 + jF_1 \in \mathcal{SC}(\sigma_S(T))$. Then*

$$f(T) = F_0(T) + JF_1(T).$$

Moreover, the operators $F_0(T)$ and $F_1(T)$ can be expressed as functions of the operators A and B in terms of the continuous functional calculus for n-tuples of commuting self-adjoint operators as

$$F_0(T) = F_0(A,B) \quad \textit{and} \quad F_1(T) = F_0(A,B).$$

In particular, they hence do not depend on the imaginary operator J.

Proof. Let $P_n(q)$ be a sequence of polynomials of the form (9.19) that converges uniformly to $F_0(T)$ on $\sigma_S(T)$. By Lemma 9.4.10, we have $P_n(q) = Q_n(u,v) + jvP_n(u,v)$ with real polynomials Q_n and P_n such that $Q_n(u,v) \to F_0(u,v)$ and $vR_n(u,v) \to F_1(u,v)$ uniformly for $q = u + jv \in \sigma_S(T)$. Since Q_n and R_n are real polynomials in q and $\overline{q}$ after the identification $u = \frac{1}{2}(q+\overline{q})$ and $v = \frac{1}{2}(q-\overline{q})$, the functions $P_n(T)$ and $Q_n(T)$ can be explicitly computed by (9.21).

We obviously have

$$f(T) = \frac{1}{2}(f(T) + f(T)^*) + \frac{1}{2}(f(T) - f(T)^*).$$

Since $f \mapsto f(T)$ is a $*$-homeomorphism and $F_0(q) = \frac{1}{2}\left(f(q) + \overline{f(q)}\right)$, we conclude that

$$\frac{1}{2}(f(T) + f(T)^*) = \frac{1}{2}\left(f + \overline{f}\right)(T) = F_0(T),$$

and since $(jF_1((q) = \frac{1}{2}\left(F(q) - \overline{F(q)}\right)$, we have

$$\frac{1}{2}(f(T) - f(T)^*) = \frac{1}{2}\left(f - \overline{f}\right)(T) = (jF_1)(T).$$

We can, however, not apply the $*$-homeomorphism property in order to show that $(jF_1)(T) = j(T)F_1(T) = JF_1(T)$. The mapping $j : q = u + jv \mapsto j$ is not continuous on the real line, and hence it does not in general belong to $\mathcal{SC}(\sigma_S(T))$. The function $jv = \frac{1}{2}(q-\overline{q})$, on the other hand, belongs to $\mathcal{SC}(\sigma_S(T))$ and $(jv)(T) = \frac{1}{2}(T-T^*) = JB$. Hence we can use the approximating sequence $jvR_n(u,v)$ in order to see that

$$\begin{aligned}(jF_1)(T) &= \lim_{n\to+\infty}(jvR_n)(T) = \lim_{n\to+\infty}(jv)(T)R_n(T)\\ &= \lim_{n\to+\infty} JBR_n(T) = J\lim_{n\to+\infty}(vR_n)(T) = JF_1(T),\end{aligned}$$

where we used that $v(q) = |q - q^*|$, and so $v(T) = |T - T^*| = B$.

Finally, we observe that $u(T) = A$ and $v(T) = B$ for the functions $u(q) = u$ and $v(q) = v$ for $q = u + jv$, and so

$$F_0(T) = \lim_{n\to+\infty} Q_n(T) = \sum_{n\to+\infty} Q_n(A,B) = F_0(A,B)$$

and

$$F_1(T) = \lim_{n\to+\infty}(vR_n)(T) = BR_n(A,B) = F_1(A,B),$$

where $F_0(A,B)$ and $F_1(A,B)$ are intended in the sense of the continuous functional calculus for n-tuples of commuting self-adjoint operators. (One constructs functions of an n-tuple $(T_1,\ldots,T_n)$ of commuting self-adjoint operators similar to the above procedure by approximating a function $f(x_1,\ldots,x_n)$ in n real variables uniformly by a sequence $P_n(x_1,\dots,x_n)$ of polynomials in n real variables. One formally replaces the real variables $(x_1,\dots,x_n)$ by the operators $T_1,\dots,T_n$ and defines $f(T_1,\dots,T_n) := \lim_{n\to+\infty} P_n(T_1,\dots,T_n)$. See, for instance, Theorem 5.6.5 in the book [193].) □

Remark 9.4.12. The continuous functional calculus for normal operators on a quaternionic Hilbert space was first introduced in [142]. In that paper, the authors extend the operator J in the decomposition $T = A + JB$ to a fully imaginary operator J_E on $\mathcal{H}$ that commutes with T. Recall that according to Lemma 9.3.9, the imaginary operator J allows one to decompose the Hilbert space $\mathcal{H}$ into the subspaces

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j,$$

where

$$\mathcal{H}_0 = \{x \in \mathcal{H} : \ \ Jx = x0\} \quad \text{and} \quad \mathcal{H}_\pm^j = \{x \in \mathcal{H} : \ \ Jx = x(\pm j)\}.$$

The extension of J to a fully imaginary operator J_E artificially introduces a splitting of the space $\mathcal{H}_0$ of the form

$$\mathcal{H}_0 = \mathcal{H}_{0,+} \oplus \mathcal{H}_{0,-}$$

into two $\mathbb{C}_j$-linear subspaces. We then have

$$\mathcal{H} = \mathcal{E}_+^j \oplus \mathcal{E}_-^j$$

with

$$\mathcal{E}_\pm^j := \{x \in \mathcal{H} : \ \ J_E x = x(\pm j)\} = \mathcal{H}_{0,\pm} \oplus \mathcal{H}_\pm^j.$$

The operator T is then simply the quaternionic linear extension of the $\mathbb{C}_j$-linear normal operator $T_j := T|_{\mathcal{E}_j^+}$ on $\mathcal{E}_j^+$ with $\sigma(T_j) = \sigma_S(T) \cap \mathbb{C}_j^+$. In principle, even though this is not done explicitly in the paper mentioned above, the continuous functional calculus can then be constructed by defining $f(T)$ as the quaternionic linear extension of $f_j(T_j)$, where $f_j(T_j)$ is the operator obtained by applying the continuous functional calculus for complex linear normal operators to $f_j = f|_{\mathbb{C}_j^+}$. This approach seems quite straightforward, but several technical steps have to be added in this case in order to show that $f(T)$ is independent of the choice of both the imaginary unit j and the extension J_E of J to a fully imaginary operator.

In our approach, we consider only the operator $T_1 := T|_{\mathcal{H}_1}$ with $\mathcal{H}_1 = \operatorname{ran} J$ as the quaternionic linear extension of a complex linear operator; cf. the proof of Theorem 9.4.7. This is the subspace on which the operator T naturally induces a complex structure. We then have to treat $T_0 = T|_{\mathcal{H}_0}$ separately. The advantage, however, is that we do not have to introduce any undetermined structure, namely an extension J_E of J, in order to split the space $\mathcal{H}_0$ into two complex linear subspaces. Since this needs to be done in accordance with T in order to guarantee that J_E and T commute, it requires a lot of avoidable technical work; cf. Remark 9.3.7.

9.5 Comments and Remarks

As we will see in the next chapters, the main ingredients to proving the spectral theorem for bounded normal operators are:

- The Riesz representation theorem for the dual of $\mathcal{C}(X, \mathbb{R})$, where X is a compact Hausdorff space.
- The Riesz representation theorem for quaternionic Hilbert spaces.
- The Teichmüller decomposition of a normal bounded operator $T = A + JB$.
- The continuous functional calculus based on the S-spectrum.

To prove the spectral theorem for unbounded normal operators, we have to introduce the notion of spectral integrals that depend on the imaginary operator J. Precisely, the main ingredients can be summarized in the following points:

- The spectral theorem for bounded normal operators.
- The spectral integrals in the quaternionic setting depending on the imaginary operator J.
- Suitable transformations (in the spirit of von Neumann) that reduce the case of unbounded operators to the case of bounded operators.

Finally, using the spectral integrals, we define a functional calculus for unbounded normal operators.

Chapter 10

Spectral Integrals

In this chapter, we define spectral integrals in the quaternionic setting. The aim is to define them for a suitably large class of functions that allows us to prove the spectral theorem for unbounded operators in Section 12. To this end, we adapt part of Chapter 4 of the book [191] to the quaternionic setting. Most of the proofs of the properties of spectral integrals are easily adapted from the classical case presented in [191], i.e., when $\mathcal{H}$ is a complex Hilbert space. However, some facts require additional arguments, which we will highlight.

Definition 10.0.1. Let $\Omega \subset \mathbb{H}$ be axially symmetric. We denote the *σ-algebra of axially symmetric Borel sets in* Ω by $\mathfrak{B}_S(\Omega)$, and for each $j \in \mathbb{S}$, the *σ-algebra of Borel sets of* $\Omega_j^+ := \Omega \cap \mathbb{C}_j^+$ with $\mathbb{C}_j^+ := \{u + jv : u \in \mathbb{R}, v \geq 0\}$ by $\mathfrak{B}(\Omega_j^+)$.

Remark 10.0.2. Every point $q \in \mathbb{C}_j^+$ corresponds to a sphere in $[q] \subset \mathbb{H}$. Similarly, every set $\Delta_j \in \mathfrak{B}(\mathbb{C}_j^+)$ corresponds to an axially symmetric set $\Delta \subset \mathbb{H}$, and the two sets are related via

$$\Delta_j = \Delta \cap \mathbb{C}_j^+ \quad \text{and} \quad \Delta = [\Delta_j].$$

Spectral integrals in the quaternionic setting can be defined with respect to spectral measures that are defined either on $\mathfrak{B}_S(\mathbb{H})$ or on $\mathfrak{B}(\mathbb{C}_j^+)$ for some $j \in \mathbb{S}$. Both approaches are equivalent for intrinsic slice functions, but they follow different intuitions. We work in this chapter with spectral measures defined on $\mathfrak{B}(\mathbb{C}_j^+)$ and present the second approach in Chapter 14, where we also discuss the equivalence of the two methods.

Definition 10.0.3. Let $\mathcal{H}$ be a quaternionic Hilbert space and let $j \in \mathbb{S}$. A *spectral measure over* $\mathbb{C}_j^+$ is a map $E : \mathfrak{B}(\mathbb{H}) \to \mathcal{B}(\mathcal{H})$, whose values are orthogonal projections, such that

(a) $E(\mathbb{C}_j^+) = \mathcal{I}$,

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_10

(b) E is countably additive:

$$E\left(\bigcup_{n=1}^{\infty} \sigma_n\right) = \sum_{n=1}^{\infty} E(\sigma_n)$$

for every sequence $(\sigma_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathfrak{B}(\mathbb{C}_j^+)$.

If $\mathcal{O}_E$ denotes the set of open sets in $\mathbb{C}_j^+$ with $E(O) = 0$, then the support of E is the set

$$\Omega_j^+ := \mathbb{C}_j^+ \setminus \bigcup_{O\in\mathcal{O}_E} O.$$

As in the complex setting, the spectral measure E has the following additional properties; cf. Section 4.2 in [191].

Lemma 10.0.4. *Let E be a spectral measure on $\mathbb{C}_j^+$. For all Borel sets σ and τ in $\mathfrak{B}(\mathbb{C}_j^+)$, we have*

(i) $E(\sigma) = E(\sigma)^*$.

(ii) $\|E(\sigma)\| \leq 1$.

(iii) $E(\emptyset) = 0$.

(iv) $E(\sigma\cap\tau) = E(\sigma)E(\tau)$.

(v) $E(\sigma)^2 = E(\sigma)$.

(vi) $E(\sigma)$ *and* $E(\tau)$ *commute.*

10.1 Spectral Integrals for Bounded Measurable Functions

Throughout this chapter, we fix an imaginary unit $j \in \mathbb{S}$, a spectral measure E over $\mathbb{C}_j$, and an imaginary operator $J \in \mathcal{B}(\mathcal{H})$ that commutes with E such that $\ker J = E(\mathbb{R})$ and $\operatorname{ran} J = \operatorname{ran} E(\mathbb{C}_j^+ \setminus \mathbb{R})$. This is by Corollary 9.3.8 equivalent to

$$E(\mathbb{C}_j^+ \setminus \mathbb{R}) = -J^2. \tag{10.1}$$

Before we continue, let us first discuss the intuition of the above definition. Spectral integrals are defined via approximation of a bounded measureable function f by a sequence of simple functions

$$f_n(q) = \sum_{\ell=1}^{n} a_{\ell,n}\chi_{\Delta_{n,\ell}}(q),$$

where χ_{Δ_n} denotes the characteristic function of the set $\Delta_n \in \mathbb{C}_j^+$. One then sets

$$\int_{\mathbb{C}_j^+} f_n(s)\, dE(s) := \sum_{\ell=1}^{n} a_{n,\ell} E(\Delta_{n,\ell}) \tag{10.2}$$

and defines

$$\int_{\mathbb{C}_j^+} f(s)\, dE(s) := \lim_{n\to+\infty} \int_{\mathbb{C}_j^+} f_n(s)\, dE(s).$$

A quaternionic Hilbert space is a priori, however, only a right vector space, and so $\mathcal{B}(\mathcal{H})$ is not a quaternionic Banach space, but only a real Banach space. Indeed, a quaternionic scalar multiplication on $\mathcal{B}(\mathcal{H})$ is supposed to act as $(Ta)(x) = T(ax)$ and $(aT)(x) = a(Tx)$ for $T \in \mathcal{B}(\mathcal{H})$, $a \in \mathbb{H}$, and $x \in \mathcal{H}$, which is meaningful only if a left multiplication is defined on $\mathcal{H}$. Using the right multiplication on $\mathcal{H}$ to define a multiplication of operators with scalars as $(Ta) = T(xa)$ yields $(Ta)(xb) = T(xba) = T(x)ba$, but $((Ta)x) = T(xa)b = T(x)ab$, and these two expressions are equal for every $b \in \mathbb{H}$ if and only if $a \in \mathbb{R}$. Hence, (Ta) is quaternionic linear only if a belongs to $\mathbb{R}$ and $\mathcal{B}(\mathcal{H})$ is in turn only a real Banach space.

As a consequence, the expression (10.2) is meaningful only if the coefficients a_n are real. This is sufficient for self-adjoint operators, but in order to develop the spectral theory of normal operators that are not self-adjoint, one has to be able to define spectral values for functions that are not real-valued.

Since the continuous functional calculus is based on the class of intrinsic slice functions, one has at least to be able to define spectral integrals of intrinsic slice functions, which are of the form $f(q) = f_0(u,v) + jf_1(u,v)$ for $q = u + jv$. The imaginary operator J tells us how to multiply the imaginary unit j. Since intrinsic slice functions take real values on the real line, this multiplication does, however, not need to be defined on the subspace that is associated with the real line. This is expressed in the condition (10.1)

One might wonder why one works with this minimal structure and does not simply define a complete left multiplication on the space $\mathcal{H}$. This would even allow the integration of more general functions than only intrinsic slice functions. However, it turns out that spectral integration of functions other than intrinsic slice functions is not meaningful, since such techniques cannot follow the usual intuition of spectral integration. Moreover, as we will see in the next chapter, a normal operator T defines only a spectral measure E and an imaginary operator J satisfying (10.1), but it does not define a left multiplication on the entire space $\mathcal{H}$ that could be used for studying this operator; cf. also Remark 9.4.12. For more details, we refer to the discussion in Chapter 14.

Definition 10.1.1. Let $\Omega \subset \mathbb{H}$ be axially symmetric. We denote the *set of all bounded* $\mathfrak{B}_S(\Omega)$-$\mathfrak{B}(\mathbb{H})$*-measurable intrinsic slice functions* by $\mathcal{SM}^\infty(\Omega)$, and with the notation $\Omega_j^+ := \Omega \cap \mathbb{C}_j^+$, we define

$$\mathcal{SM}_j^\infty(\Omega_j^+) := \{f|_{\Omega_j^+} : f \in \mathcal{SM}^\infty(\Omega)\}.$$

Remark 10.1.2. The spaces $\mathcal{SM}^\infty(\Omega)$ and $\mathcal{SM}_j^\infty(\Omega_j^+)$ are real Banach spaces with the supremum norm

$$\|f\|_\infty = \sup_{q\in\Omega}|f(q)|, \quad \text{resp.} \quad \|f|_{\Omega_j}\|_\infty = \sup_{q\in\Omega_j^+}|f(q)|.$$

Furthermore, if $f = f_0 + jf_1 \in \mathcal{SM}^\infty(\Omega)$, then we have for every $q = u + iv \in \Omega$ that

$$|f(q)|^2 = |f_0(u,v) + jf_1(u,v)|^2 = |f_0(u,v)|^2 + |f_1(u,v)|^2.$$

The modulus $|f(q)|$ therefore does not depend on the imaginary unit $i \in \mathbb{S}$ and therefore it is constant on each sphere $[q]$. We conclude that $\|f\|_\infty = \|f|_{\Omega_j^+}\|_\infty$ and that the restriction $f \mapsto f|_{\Omega_j}$ is an isometric bijection between $\mathcal{SM}^\infty(\Omega)$ and $\mathcal{SM}_j^\infty(\Omega_j)$.

As an immediate consequence of Lemma 9.4.3, we obtain also the following lemma.

Lemma 10.1.3. *Let $\Omega \subset \mathbb{H}$ be axially symmetric and set $\Omega_j^+ := \Omega \cap \mathbb{C}_j^+$. A function f_j belongs to $\mathcal{SM}_j^\infty(\Omega_j^+)$ if and only if it is of the form*

$$f_j(q) = f_0(u,v) + jf_1(u,v), \quad q = u + jv \in \Omega_j^+, \tag{10.3}$$

where the component functions $f_\ell : \Omega_j^+ \to \mathbb{R}$ for $\ell = 0,1$ are measurable real-valued functions and $f_1(u,v) = 0$ for every $q = u + jv \in \Omega_j^+ \cap \mathbb{R}$ whenever $v = 0$.

The above discussion shows that we do not have to distinguish between globally defined intrinsic slice functions and functions of the form (10.3) that are defined only on one slice $\mathbb{C}_j^+$. We can jump back and forth by extending, resp. restricting, the respective functions.

Definition 10.1.4. Let $\Omega_j^+ \in \mathfrak{B}(\mathbb{C}_j)$. We denote by $\mathcal{EF}(\Omega_j^+)$ the *subset of simple functions in $\mathcal{SM}_j^\infty(\Omega_j^+)$*, all $\mathbb{C}_j$-valued functions of the form

$$f(q) = \sum_{m=1}^{n} c_m \chi_{\sigma_m}(q),$$

where $\sigma_1, \ldots \sigma_n$ are pairwise disjoint sets in $\mathfrak{B}(\Omega_j^+)$, where $c_1, \ldots, c_n \in \mathbb{C}_j$ and $c_m \in \mathbb{R}$ if $\sigma_m \cap \mathbb{R} \neq \emptyset$ and where

$$\chi_\sigma(q) = \begin{cases} 1 & \text{if} \quad q \in \sigma, \\ 0 & \text{if} \quad q \notin \sigma. \end{cases}$$

For $f \in \mathcal{EF}(\Omega_j^+)$, we define

$$\mathbb{I}(f) = \int_{\Omega_j^+} f(p)\, dE(p) := \sum_{m=1}^{n} (\mathrm{Re}(c_m)\mathcal{I} + \mathrm{Im}(c_m)J\} E(\sigma_m). \tag{10.4}$$

Remark 10.1.5. If $f(q) = \sum_{m=1}^{n} c_m \chi_{\sigma_m}(q)$ belongs to $\mathcal{SM}_j^\infty(\Omega_j^+)$, then the property that $c_m \in \mathbb{R}$ whenever $\sigma_m \cap \mathbb{R} \neq \emptyset$ follows from the fact that f is the restriction of an intrinsic slice function. Such functions map the real line into itself, and hence every coefficient that determines the value of f at a real point must be real.

Lemma 10.1.6. *If $f \in \mathcal{EF}(\Omega_j^+)$, then*

$$\|\mathbb{I}(f)\| \leq \|f\|_\infty. \tag{10.5}$$

Proof. If $f = \sum_{m=1}^{n} c_m \chi_{\sigma_m}$ in $\mathcal{EF}(\Omega_j^+)$, then, using properties (ii), (iii), (iv), and (a) in Lemma 10.0.4 and the fact that $\|J\| = 1$, we have

$$\begin{aligned}
\|\mathbb{I}(f)x\|^2 &= \left\| \sum_{m=1}^{n} (\mathrm{Re}(c_m) + \mathrm{Im}(c_m)J)E(\sigma_m)x \right\|^2 \\
&= \sum_{m=1}^{n} \|(\mathrm{Re}(c_m) + \mathrm{Im}(c_m)J)E(\sigma_m)x\|^2 \\
&\leq \sum_{m=1}^{n} |c_m|^2 \|E(\sigma_m)x\|^2 \leq \|f\|_\infty^2 \|x\|^2.
\end{aligned}$$

Thus (10.5) holds. ☐

Fix $f \in \mathcal{SM}^\infty(\Omega_j^+)$. Since $\mathcal{EF}(\Omega_j)$ is a dense subset of $\mathcal{SM}_j^\infty(\Omega_j^+)$, there exists a sequence of functions $(f_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{EF}(\Omega_j^+)$ such that

$$\lim_{n\to+\infty} \|f_n - f\|_\infty = 0.$$

In view of (10.5), $(\mathbb{I}(f_n)x)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$. Let $\mathbb{I}(f)$ be given by

$$\mathbb{I}(f)x = \lim_{n\to+\infty} \mathbb{I}(f_n)x, \quad x \in \mathcal{H}.$$

Note that f does not depend on the choice of the sequence $(f_n)_{n\in\mathbb{N}}$, and consequently, neither does $\mathbb{I}(f)$.

Lemma 10.1.7. *If $f = f_0 + f_1 j$ and $g = g_0 + g_1 j$ belong to $\mathcal{SM}_j^\infty(\Omega_j^+)$, where f_0, f_1, g_0, and g_1 are real-valued, $\alpha, \beta \in \mathbb{C}_j$, and $x, y \in \mathcal{H}$, then:*

(i) $\mathbb{I}(f)^* = \mathbb{I}(\bar{f})$, $\quad \mathbb{I}(\alpha f + \beta g) = \alpha\, \mathbb{I}(f) + \beta\, \mathbb{I}(g)$.

(ii) $\langle \mathbb{I}(f)x, y\rangle = \int_\Omega f_0(p) d\langle E(p)x, y\rangle + \int_\Omega f_1(p) d\langle JE(p)x, y\rangle$.

(iii) $\mathbb{I}(fg) = \mathbb{I}(f)\mathbb{I}(g)$.

(iv) $\|\mathbb{I}(f)x\|^2 = \int_\Omega |f(p)|^2 d\langle E(p)x, x\rangle$.

(v) $\|\mathbb{I}(f)\| \leq \|f\|_\infty$.

Proof. In view of the density of $\mathcal{EF}(\Omega_j^+)$ in $\mathcal{SM}^\infty(\Omega_j^+)$ and (10.5), it suffices to check (i)–(v) when $f, g \in \mathcal{EF}(\Omega_j^+)$. The assumption that $E(\sigma)$ and J commute for all $\sigma \in \mathfrak{B}(\Omega_j^+)$ will be a useful tool for checking (i)–(v) and will be used without further mention. If $f = \sum_{m=1}^n c_m \chi_{\sigma_m}$ and $g = \sum_{m=1}^n d_m \chi_{\tau_m}$ belong to $\mathcal{EF}(\Omega_j^+)$, then

$$\begin{aligned}\langle \mathbb{I}(f)^* y, x\rangle &= \sum_{m=1}^n \langle y, \{\mathrm{Re}(c_m)\mathcal{I} + \mathrm{Im}(c_m)J\}E(\sigma_m)x\rangle \\ &= \sum_{m=1}^n \langle \{\mathrm{Re}(c_m)\mathcal{I} - \mathrm{Im}(c_m)J\}E(\sigma_m)y, x\rangle \\ &= \langle \mathbb{I}(\bar{f})y, x\rangle, \quad x, y \in \mathcal{H}.\end{aligned}$$

Thus, the first assertion in (i) holds. The second assertion in (i) is easily checked.

We will now check that (ii) holds. Since

$$\begin{aligned}\langle \mathbb{I}(f)x, y\rangle &= \sum_{m=1}^n \{\mathrm{Re}(c_m)\langle E(\sigma_m)x, y\rangle + \mathrm{Im}(c_m)\langle JE(\sigma_m)x, y\rangle\} \\ &= \int_\Omega f_0(p)d\langle E(p)x, y\rangle + \int_\Omega f_1(p)d\langle JE(p)x, y\rangle,\end{aligned}$$

where f_0 and f_1 are real-valued functions that satisfy $f = f_0 + f_1 j$, (ii) holds.

We will now check that (iii) holds. Since

$$\begin{aligned}\mathbb{I}(fg) &= \sum_{\ell,m=1}^n \{\mathrm{Re}(c_\ell)\mathrm{Re}(d_m) - \mathrm{Im}(c_\ell)\mathrm{Im}(d_m)\}E(\sigma_\ell \cap \tau_m) \\ &\quad + \sum_{\ell,m=1}^n \{\mathrm{Re}(c_\ell)\mathrm{Im}(d_m) + \mathrm{Im}(c_\ell)\mathrm{Re}(d_m)\}JE(\sigma_\ell \cap \tau_m) \\ &= \sum_{\ell,m=1}^n \{\mathrm{Re}(c_\ell)\mathcal{I} + \mathrm{Im}(c_\ell)J\}\{\mathrm{Re}(d_m)\mathcal{I} + \mathrm{Im}(d_m)J\}E(\sigma_m)E(\tau_n) \\ &= \left\{\sum_{\ell=1}^n \{\mathrm{Re}(c_\ell)\mathcal{I} + \mathrm{Im}(c_\ell)J\}E(\sigma_\ell)\right\}\left\{\sum_{m=1}^n \{\mathrm{Re}(d_m)\mathcal{I} + \mathrm{Im}(d_m)J\}E(\tau_m)\right\} \\ &= \mathbb{I}(f)\mathbb{I}(g),\end{aligned}$$

(iii) holds.

Assertion (iv) is a direct consequence of (i), (ii) and (iii). Indeed,

$$\begin{aligned}\|\mathbb{I}(f)x\|^2 &= \langle \mathbb{I}(f)x, \mathbb{I}(f)x\rangle \\ &= \langle \mathbb{I}(|f|^2)x, x\rangle \\ &= \int_{\Omega_j^+} |f(p)|^2 d\langle E(p)x, x\rangle, \quad x \in \mathcal{H}.\end{aligned}$$

Finally, assertion (v) is a direct consequence of assertion (iv). □

Remark 10.1.8. If $T \in \mathcal{B}(\mathcal{H})$ is normal and E_j is the spectral measure with support $\sigma_S(T) \cap \mathbb{C}_j^+$ for $j \in \mathbb{S}$ appearing in Theorem 11.2.1 in the next chapter, then item (ii) of Lemma 10.1.7 ensures that

$$f(T) = \mathbb{I}(f), \quad f \in \mathcal{C}(\sigma_S(T) \cap \mathbb{C}_j^+, \mathbb{C}_j). \tag{10.6}$$

Finally, let us show that the choice of the imaginary unit $j \in \mathbb{S}$ is irrelevant.

Lemma 10.1.9. *Let E be the spectral measure over $\mathbb{C}_j^+$ and let $i \in \mathbb{S}$. If we define for $\Delta \in \mathfrak{B}(\mathbb{C}_i^+)$ the set*

$$\Delta_j := \{u + jv : \ u + iv \in \Delta\}$$

and set

$$\tilde{E}(\Delta) = E(\Delta_j),$$

then $\tilde{E}$ is a spectral measure over $\mathbb{C}_i^+$, and for every $f \in \mathcal{SM}^\infty(\Omega)$, we have after setting $f_j := f|_{\Omega_j^+}$ and $f_i := f|_{\Omega_i^+}$ that

$$\int_{\Omega_j^+} f_j(p)\, dE(p) = \int_{\Omega_i^+} f_i(p)\, d\tilde{E}(p).$$

Proof. It is immediate that $\tilde{E}$ is a spectral measure over $\mathbb{C}_i^+$. If furthermore $f \in \mathcal{SM}^\infty(\Omega_j^+)$ is such that $f_i = \sum_{m=0}^m c_m \chi_{\sigma_m} \in \mathcal{EF}(\Omega_i^+)$ with $c_m = \mathrm{Re}(c_m) + j\mathrm{Im}(c_m)$, then $f_j = \sum_{m=0}^n \{\mathrm{Re}(c_m) + j\mathrm{Im}(c_m)\}\chi_{\sigma_{m,j}}$ and we obtain

$$\begin{aligned}\int_{\Omega_j^+} f_j(p)\, dE(p) &= \sum_{m=0}^n \{\mathrm{Re}(c_m)\mathcal{I} + \mathrm{Im}(c_m)J\} E(\sigma_{m,j}) \\ &= \sum_{m=0}^n \{\mathrm{Re}(c_m)\mathcal{I} + \mathrm{Im}(c_m)J\} \tilde{E}(\sigma_m) = \int_{\Omega_j^+} f_i(p)\, d\tilde{E}(p).\end{aligned}$$ □

10.2 Spectral Integrals for Unbounded Measurable Functions

We will now define spectral integrals for a more general class of functions than $\mathcal{SM}^\infty(\Omega_j^+)$. This will be useful in proving the spectral theorem for unbounded normal operators. Again we fix an imaginary unit $j \in \mathbb{S}$, a spectral measure E over $\mathbb{C}_j$, and an imaginary operator $J \in \mathcal{B}(\mathcal{H})$ that commutes with E such that $E(\mathbb{C}_j^+ \setminus \mathbb{R}) = -J^2$.

Definition 10.2.1. Let $\mathcal{SM}_E^{\#}(\Omega_j^+)$ denote the space of all $\mathfrak{B}(\Omega_j^+)$-measurable functions $f : \Omega_j^+ \to \mathbb{C}_j \cup \{\infty\}$ that are the restriction of an intrinsic slice function and E-a.e. finite, i.e., such that

$$E(\{p \in \Omega : \ f(p) = \infty\}) = 0.$$

Furthermore, we let $\mathcal{SM}_E^{\infty}(\Omega_j^+)$ denote the set of all $\mathfrak{B}(\Omega_j^+)$-measurable functions $f : \Omega_j^+ \to \mathbb{C}_j \cup \{\infty\}$ that are the restriction of an intrinsic slice function and essentially bounded, i.e., such that

$$\|f\|_{E,\infty} := \inf\{a \in \mathbb{R} : \ E(\Delta_a(f)) = 0\},$$

where $\Delta_a(f) = \{q \in \Omega_j^+ : |f(q)| \geq a\}$.

Definition 10.2.2. A sequence of sets $(\sigma_n)_{n\in\mathbb{N}}$, where $\sigma_n \in \mathfrak{B}(\Omega_j^+)$ for $n \in \mathbb{N}$, is called a *bounding sequence* for a subset of functions $\mathfrak{F} \subseteq \mathcal{SM}_E^{\#}(\Omega_j^+)$ if

(i) $f \in \mathfrak{F}$ is bounded on σ_n for $n = 0, 1, \ldots$,

(ii) $\sigma_n \subseteq \sigma_{n+1}$ for $n = 0, 1, \ldots$,

(iii) $E(\bigcup_{n=0}^{+\infty} \sigma_n) = I_{\mathcal{H}}$,

where E is a spectral measure.

Remark 10.2.3. If $(\sigma_n)_{n\in\mathbb{N}}$ is a bounding sequence, then the following assertions follow from the definition of a spectral measure:

(i) $E(\sigma_n) \preceq E(\sigma_{n+1})$.

(ii) $E(\sigma_n)x \to x$ as $n \to +\infty$ for any $x \in \mathcal{H}$.

(iii) The set $\bigcup_{n=0}^{+\infty} E(\sigma_n)\mathcal{H}$ is dense in $\mathcal{H}$.

We will now give meaning to $\mathbb{I}(f)$ for $f \in \mathcal{SM}_E^{\#}(\Omega_j)$.

Definition 10.2.4. Let $f \in \mathcal{SM}_E^{\#}(\Omega_j^+)$ and let $(\sigma_n)_{n\in\mathbb{N}}$ be a bounding sequence for f. We define the operator

$$\mathbb{I}(f)x = \lim_{n\to+\infty} \mathbb{I}(\chi_{\sigma_n} f)x \tag{10.7}$$

with domain

$$\mathcal{D}(\mathbb{I}(f)) = \left\{ x \in \mathcal{H} : \ \int_{\Omega_j^+} |f(q)|^2 d\langle E(q)x, x\rangle < +\infty \right\}. \tag{10.8}$$

Given a quaternionic measure μ on $\Omega_j^+ \subseteq \mathbb{C}_j$, we will let $L_2(\Omega_j^+, \mu)$ consist of all measurable functions such that

$$\|f\|_{L_2(\Omega_j^+,\mu)} := \left(\int_{\Omega_j^+} |f(q)|^2 d|\mu|(q) \right)^{1/2} < +\infty,$$

where $|\mu|$ denotes the *total variation* of μ defined by

$$|\mu|(\sigma) := \sup\left\{\sum_{m\in\mathbb{N}} |\mu(\sigma_m)| : \quad \sigma = \bigsqcup_{m=1}^{+\infty} \sigma_m\right\} \qquad \forall \sigma \in \mathfrak{B}(\Omega_j^+).$$

Here $\bigsqcup$ denotes a disjoint union of sets in $\mathfrak{B}(\Omega_j^+)$.

Lemma 10.2.5. *For $x, y \in \mathcal{H}$, the quaternionic measure $\mu_{x,y}(\sigma) = \langle E(\sigma)x, y\rangle$ and $f, g \in \mathcal{SM}_E^{\#}(\Omega_j^+)$, we have:*

(i) $|\mu_{x,y}(\sigma)| \leq \mu_x(\sigma)^{1/2}\mu_y(\sigma)^{1/2}$ *for* $\sigma \in \mathfrak{B}(\Omega_j^+)$.

(ii) *If $f \in L_2(\Omega_j^+, \mu_x)$ and $g \in L_2(\Omega_j^+, \mu_y)$, then*

$$\begin{aligned}\left|\int_{\Omega_j^+} \mathrm{Re}\{(fg)(p)\} d\mu_{x,y}(p) + \int_{\Omega} \mathrm{Im}\{(fg)(p)\} d\mu_{x,-Jy}(p)\right| \\ \leq 2\|f\|_{L_2(\Omega_j^+,\mu_x)}\|g\|_{L_2(\Omega_j^+,\mu_{Jy})}\end{aligned} \tag{10.9}$$

Proof. The proof of Lemma 4.8 (i) in [191] can easily be adapted to obtain item (i) in our present setting. Since

$$\begin{aligned}\left|\int_{\Omega_j^+} \mathrm{Re}\{(f\bar{g})(p)\} d\mu_{x,y}(p)\right| &\leq \int_{\Omega_j^+} |\mathrm{Re}\{(f\bar{g})(p)\}|\, d|\mu_{x,y}|(p) \\ &\leq \int_{\Omega_j^+} |(f\bar{g})(p)|\, d|\mu_{x,y}|(p),\end{aligned}$$

one may proceed as in the proof of Lemma 4.8(ii) in [191] to obtain

$$\int_{\Omega_j^+} |(f\bar{g})(p)|\, d|\mu_{x,y}|(p) \leq \|f\|_{L_2(\Omega_j^+,\mu_{x,x})}\|g\|_{L_2(\Omega_j^+,\mu_{y,y})}$$

and hence

$$\left|\int_{\Omega_j^+} \mathrm{Re}\{(f\bar{g})(p)\} d\mu_{x,y}(p)\right| \leq \|f\|_{L_2(\Omega_j^+,\mu_{x,x})}\|g\|_{L_2(\Omega_j^+,\mu_{y,y})}.$$

Similarly, one can show that

$$\left|\int_{\Omega_j^+} \mathrm{Im}\{(f\bar{g})(p)\} d\mu_{x,y}(p)\right| \leq \|f\|_{L_2(\Omega_j^+,\mu_{x,x})}\|g\|_{L_2(\Omega_j^+,\mu_{-Jy,-Jy})}.$$

But since

$$\mu_{-Jy,-Jy}(\sigma) = \langle E(\sigma)(-Jy), -Jy\rangle = \langle E(\sigma)y, y\rangle = \mu_{y,y}(\sigma),$$

we have the advertised upper bound. □

Lemma 10.2.6. *If $f \in \mathcal{SM}_E^{\#}(\Omega_j^+)$ and $(\sigma_n)_{n\in\mathbb{N}}$ is a bounding sequence for f, then*

(i) *A vector $x \in \mathcal{H}$ belongs to $\mathcal{D}(\mathbb{I}(f))$ if and only if the sequence $(\mathbb{I}(\chi_{\sigma_n} f)x)_{n\in\mathbb{N}}$ converges in $\mathcal{H}$, or equivalently,*

$$\sup_{n\in\mathbb{N}} \|\mathbb{I}(f\chi_{\sigma_n})x\| < +\infty.$$

(ii) *$\mathbb{I}(f)$ does not depend on the choice of the bounding sequence for f.*

(iii) *The set $\bigcup_{n=0}^{+\infty} E(\sigma_n)\mathcal{H}$ is a dense subset of $\mathcal{D}(\mathbb{I}(f))$ with respect to the norm*

$$\|x\|_{\mathbb{I}(f)} = \|x\| + \|\mathbb{I}(f)x\|, \quad x \in \mathcal{D}(\mathbb{I}(f)).$$

Moreover,

$$E(\sigma_n)\mathbb{I}(f) \subseteq \mathbb{I}(f)E(\sigma_n) = \mathbb{I}(f\chi_{\sigma_n}), \quad n = 0, 1, \dots. \tag{10.10}$$

Proof. Since $E(\sigma)^2 = E(\sigma)$ and $E(\sigma)^* = E(\sigma)$, the operator $E(\sigma)$ is a positive operator for every $\sigma \in \mathfrak{B}(\Omega_j^+)$. Thus, μ_x is a positive measure on Ω_j^+, where $\mu_x(\sigma) = \langle E(\sigma)x, x\rangle$. Consequently, the proof of items (i)–(iii) can be completed in much the same way as items (i)–(iii) of Theorem 4.13 in [191]. □

In the following theorem, $\overline{W}$ denotes the closure of an operator $W \in \mathcal{L}(\mathcal{H})$, while $\bar{f}$ denotes the usual complex conjugation of the function f.

Theorem 10.2.7. *If $f, g \in \mathcal{SM}_E^{\#}(\Omega_j^+)$ and $\alpha, \beta \in \mathbb{R}$, then:*

(i) $\mathbb{I}(\bar{f}) = \mathbb{I}(f)^*$.

(ii) $\mathbb{I}(\alpha f + \beta g) = \overline{\alpha\mathbb{I}(f) + \beta\mathbb{I}(g)}$.

(iii) $\mathbb{I}(fg) = \overline{\mathbb{I}(f)\mathbb{I}(g)}$.

(iv) *$\mathbb{I}(f)$ is a closed normal operator on $\mathcal{H}$ and*

$$\mathbb{I}(f)^*\mathbb{I}(f) = \mathbb{I}(f\bar{f}) = \mathbb{I}(\bar{f}f).$$

(v) $\mathcal{D}(\mathbb{I}(f)\mathbb{I}(g)) = \mathcal{D}(\mathbb{I}(g)) \cap \mathcal{D}(\mathbb{I}(fg))$.

(vi) *If $x \in \mathcal{D}(\mathbb{I}(f))$ and $y \in \mathcal{D}(\mathbb{I}(g))$, then*

$$\langle \mathbb{I}(f)x, \mathbb{I}(g)y\rangle = \int_{\Omega_j^+} \operatorname{Re}(f(p)\overline{g(p)})\, d\langle E(p)x, y\rangle + \int_{\Omega_j^+} \operatorname{Im}(f(p)\overline{g(p)})\, d\langle JE(p)x, y\rangle.$$

(vii) *If $x \in \mathcal{D}(\mathbb{I}(f))$, then*

$$\|\mathbb{I}(f)x\|^2 = \int_{\Omega_j^+} |f(p)|^2 d\langle E(p)x, x\rangle.$$

Proof. The proof of items (i)–(iv) when $\mathcal{H}$ is a complex Hilbert space (see items (i)–(v) of Theorem 4.16 in [191]) can easily be adapted to the case in which $\mathcal{H}$ is a quaternionic Hilbert space. Item (vii) follows directly from item (vi) when $g = f$ and $x = y$. What remains is to show (vi). To this end, we will adapt the argument for the proof of Proposition 4.15 in [191].

In view of items (i) and (ii) of Lemma 10.2.6,

$$\begin{aligned}\int_{\Omega_j^+} \operatorname{Re}\{(f\bar{g}\chi_m)(p)\}d\langle E(p)x,y\rangle + \int_{\Omega_j^+} \operatorname{Im}\{(f\bar{g}\chi_m)(p)\}d\langle JE(p)x,y\rangle \\ = \langle \mathbb{I}(f\bar{g}\chi_{\sigma_m})x,y\rangle = \langle \mathbb{I}(f\chi_{\sigma_m})x, \mathbb{I}(g\chi_{\sigma_m})y\rangle.\end{aligned} \tag{10.11}$$

Since $x \in \mathcal{D}(\mathbb{I}(f))$ and $y \in \mathcal{D}(\mathbb{I}(g))$, we have $f \in L_2(\Omega, \mu_{x,x})$ and $g \in L_2(\Omega, \mu_{y,y})$, where $\mu_{x,y}(\sigma) = \langle E(\sigma)x, y\rangle$, $\sigma \in \mathfrak{B}(\Omega)$. Therefore, we may use Lemma 10.2.5 to get that the integrals given in

$$\kappa_m := \int_{\Omega_j^+} \operatorname{Re}\{(f\bar{g}\chi_m)(p)\}d\langle E(p)x,y\rangle + \int_{\Omega_j^+} \operatorname{Im}\{(f\bar{g}\chi_m)(p)\}d\langle JE(p)x,y\rangle$$

exist and hence

$$\begin{aligned}\Bigg|\int_{\Omega_j^+} &\operatorname{Re}\{(f\bar{g}\chi_{\sigma_m})(p)\}d\langle E(p)x,y\rangle \\ &+ \int_{\Omega_j^+} \operatorname{Im}\{(f\bar{g}\chi_{\sigma_m})(p)\}d\langle JE(p)x,y\rangle - \kappa_m\Bigg| \to 0\end{aligned}$$

as $m \to +\infty$. But then the formula advertised in (vi) follows from letting m tend to $+\infty$ in (10.11). □

Lemma 10.2.8. *The operator $\mathbb{I}(f)$ is bounded if and only if $f \in \mathcal{SM}_E^\infty(\Omega_j^+)$. In this case $\|\mathbb{I}(f)\| = \|f\|_{E,\infty}$.*

Proof. The proof when $\mathcal{H}$ is a complex Hilbert space (see Proposition 4.18 in [191]) can easily be adapted to the case in which $\mathcal{H}$ is a quaternionic Hilbert space. □

Theorem 10.2.9. *If $f \in \mathcal{SM}_E^\#(\Omega_j^+)$, then $\mathbb{I}(f)$ is invertible if and only if f does not vanish E-a.e. on Ω_j^+. In this case,*

$$\mathbb{I}(f)^{-1} = \mathbb{I}(1/f), \tag{10.12}$$

where we use the convention that $1/0 = \infty$ and $1/\infty = 0$.

Proof. Also here, the proof when $\mathcal{H}$ is a complex Hilbert space (see Proposition 4.19 in [191]) can easily be adapted to the case in which $\mathcal{H}$ is a quaternionic Hilbert space. □

Lemma 10.2.10. *If $f \in \mathcal{SM}_E^{\#}(\Omega_j^+)$, then*

$$\sigma_S(\mathbb{I}(f)) \cap \mathbb{C}_j^+ = \left\{\lambda \in \mathbb{C}_j^+ : \quad E(\sigma_\varepsilon(\lambda))) \neq 0 \ \forall \varepsilon > 0\right\}, \tag{10.13}$$

where

$$\sigma_\varepsilon(\lambda) := \{z \in \Omega_j^+ : \quad |f(z)^2 - 2\mathrm{Re}(\lambda)f(z) + |\lambda|^2| < \varepsilon\}.$$

Proof. We have $q \in \rho_S(\mathbb{I}(f)) \cap \mathbb{C}_j^+$ if and only if

$$\mathbb{I}(f)^2 - 2\mathrm{Re}(q)\mathbb{I}(f) + |q|^2\mathcal{I}$$

has a bounded inverse. This is the case if and only if

$$g_q(t) = f(t)^2 - 2\mathrm{Re}(q)f(t) + |q|^2 \neq 0 \quad E\text{-a.e. on } \Omega_j^+$$

and the function $g_q^{-1}(t)$ is essentially bounded and hence belongs to $\mathcal{SM}_E^{\infty}(\Omega_j^+)$. In other words, there exists a constant $c > 0$ such that

$$E(\{z \in \Omega_j^+ : \quad |g_q(z)| \geq c\}) = 0.$$

Thus, $\lambda \in \sigma_S(\mathbb{I}(f)) \cap \mathbb{C}_j^+$ if and only if

$$E(\{z \in \Omega_j^+ : \quad |g_\lambda(z)| < \varepsilon\}) \neq 0 \quad \forall \varepsilon > 0,$$

and we have (10.13). □

As a direct consequence of Lemma 10.1.9, we also find that spectral integrals of functions in $\mathcal{SM}_E^{\#}(\Omega_j)$ are also not dependent on the imaginary unit $j \in \mathbb{S}$.

Lemma 10.2.11. *Let E be the spectral measure over $\mathbb{C}_j^+$ and let $i \in \mathbb{S}$. If we define for $\Delta \in \mathfrak{B}(\mathbb{C}_i^+)$ the set*

$$\Delta_j := \{u + jv : \quad u + iv \in \Delta\}$$

and set

$$\tilde{E}(\Delta) = E(\Delta_j),$$

then $\tilde{E}$ is a spectral measure over $\mathbb{C}_i^+$, and for every $f \in \mathcal{SM}_E^{\#}(\Omega)$, we have after setting $f_j := f|_{\Omega_j^+}$ and $f_i := f|_{\Omega_i^+}$ that

$$\int_{\Omega_j^+} f_j(q)\, dE(q) = \mathbb{I}_E(f_j) = \mathbb{I}_{\tilde{E}}(f_i) = \int_{\Omega_i^+} f_i(q)\, d\tilde{E}(q).$$

Furthermore, we have

$$\int_{\Omega_j^+} f_j(p)\, dE(p) = \int_{\Omega_i^+} f_i(p)\, d\tilde{E}(p).$$

Proof. Since $\mathbb{I}(f)$ is defined as the limit $\mathbb{I}(f)x := \lim_{n\to+\infty} \mathbb{I}(\chi_{\sigma_n} f)x$ of spectral integrals of functions in $\mathcal{SM}^{\infty}(\Omega_j^+)$ and the statement holds for such functions by Lemma 10.1.9, it also holds for $f \in \mathcal{SM}_E^{\#}(\Omega_j^+)$. □

10.3 Comments and remarks

The reader is encouraged to see the book of Schmüdgen for a very good and detailed write-up of spectral integrals in the complex Hilbert space case. The main difference with respect to the complex case is that the quaternionic spectral integrals depend on the imaginary operator J, which is considered to be multiplication by the imaginary unit j from the left.

Chapter 11

The Spectral Theorem for Bounded Normal Operators

In this chapter we prove the spectral theorem for bounded normal operators T in $\mathcal{B}(\mathcal{H})$. Our approach has analogies with the well-known approach for complex bounded normal operators on a complex Hilbert space, see for example [163], but it has to take into account the axially symmetric structure of the S-spectrum of T and the (A, J, B)-decomposition $T = A + JB$ of the quaternionic bounded normal operators. As we will see, the spectral measures E are constructed using just the two self-adjoint operators A and B, and only later, we take into account the imaginary operator J for the spectral representation of T. We present the original proof from [13] using the Teichmüller decomposition $T = A + JB$. The following representation theorems will be used in the sequel.

Theorem 11.0.1 (Riesz representation theorem for real-valued functions)**.** *Let X be a compact Hausdorff space and let $\mathcal{C}(X, \mathbb{R})$ denote the normed space of real-valued continuous functions on X together with the supremum norm $\|\cdot\|_\infty$. Corresponding to every bounded linear functional $\psi : \mathcal{C}(X, \mathbb{R}) \to \mathbb{R}$ there exists a signed Borel measure μ on X such that*

$$\psi(f) = \int_X f(t)d\mu(t) \quad \textit{for all } f \in \mathcal{C}(X, \mathbb{R}). \tag{11.1}$$

If, in addition, ψ is a positive linear functional, then μ is a positive Borel measure on X. In both cases μ is unique.

For a proof of Theorem 11.0.1 we refer to Theorem D in Section 56 of [157] for the case in which ψ is a positive bounded linear functional on X and, e.g., Chapter 21 in [182] for the more general case.

Theorem 11.0.2 (Riesz representation theorem for quaternionic Hilbert spaces)**.** *Let $\mathcal{H}$ be a quaternionic right Hilbert space with quaternionic inner product $\langle \cdot, \cdot \rangle$,*

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*, Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_11

and let φ be a continuous right linear functional on $\mathcal{H}$. Then there exists a unique $y_\varphi \in \mathcal{H}$ such that

$$\varphi(x) = \langle x, y_\varphi \rangle, \quad \text{for all } x \in \mathcal{H}.$$

Theorem 11.0.2 can be found in [33]. We also want to mention Proposition 1.10 in [167] for a statement and proof in a more general Clifford algebra setting.

11.1 Construction of the Spectral Measure

We can now construct the spectral measures. We will use the Riesz representation theorem for continuous real-valued functions and the Riesz representation theorem for quaternionic Hilbert spaces.

In this chapter we consider a normal operator $T \in \mathcal{B}(\mathcal{H})$ and fixed imaginary unit $j \in \mathbb{S}$ and define $\Omega = \sigma_S(T)$ and

$$\Omega_j^+ := \Omega \cap \mathbb{C}_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+.$$

We recall that $\mathcal{C}(\Omega_j^+, \mathbb{R})$ denotes the space of continuous real-valued functions on Ω_j^+. By Lemma 9.4.3, every function $f_j \in \mathcal{C}(\Omega_j^+, \mathbb{R})$ is the restriction $f_j = f|_{\Omega_j^+}$ of a real-valued continuous slice function f on $\Omega = \sigma_S(T)$. We denote the set of continuous real-valued slice functions on Ω by $\mathcal{SC}(\Omega, \mathbb{R})$, and in the following, we do not distinguish between the function f_j and the function f unless that could cause confusion.

We consider for $x \in \mathcal{H}$ the mapping

$$\ell_x(g) = \langle g(T)x, x \rangle, \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}) \cong \mathcal{SC}(\Omega, \mathbb{R}),$$

where $g(T)$ is the operator obtained by the continuous function calculus introduced in Theorem 9.4.11, where $g(T)$ stands for $F_0(T)$ and $F_1(T)$. Since T is a bounded operator, its S-spectrum $\sigma_S(T)$ is a compact and nonempty set. It is readily checked that ℓ_x is a real-valued bounded linear functional on $\mathcal{C}(\Omega_j^+, \mathbb{R})$. Moreover, ℓ_x is a positive functional. Indeed, if h is a continuous nonnegative function on Ω_j^+, then we can consider the function $g(u,v) = \sqrt{h(u,v)}$ and find $g \in \mathcal{C}(\Omega_j^+, \mathbb{R})$ with $g(T) = g(T)^*$. Thus

$$\ell_x(h) = \langle h(T)x, x \rangle = \langle g(T)x, g(T)x \rangle = \|g(T)x\|^2 \geq 0.$$

Theorem 11.0.1 yields the existence of a uniquely determined positive-valued measure μ_x on the Borel sets $\mathfrak{B}(\Omega_j^+)$, so that

$$\ell_x(g) = \int_{\Omega_j^+} g(p)\, d\mu_x(p), \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}). \tag{11.2}$$

In view of (11.2), we may use the formula

$$\begin{aligned}4\langle g(T)x, y\rangle = &\langle g(T)(x+y), x+y\rangle - \langle g(T)(x-y), x-y\rangle \\ &+ e_1\langle g(T)(x+ye_1), x+ye_1\rangle - e_1\langle g(T)(x-ye_1), x-ye_1\rangle \\ &+ e_1\langle g(T)(x-ye_2), x-ye_2\rangle e_3 - e_1\langle g(T)(x+ye_2), x+ye_2\rangle e_3 \\ &+ \langle g(T)(x+ye_3), x+ye_3\rangle e_3 - \langle g(T)(x-ye_3), x-ye_3\rangle e_3, \quad (11.3)\end{aligned}$$

where $\{1, e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{H}$, to obtain for every $x, y \in \mathcal{H}$ a uniquely determined $\mathbb{H}$-valued measure $\mu_{x,y}$ such that

$$\langle g(T)x, y\rangle = \int_{\Omega_j^+} g(p)d\mu_{x,y}(p), \quad g \in \mathcal{C}(\Omega_j^+, \mathbb{R}), \tag{11.4}$$

where

$$\begin{aligned}4\mu_{x,y} = &\mu_{x+y} - \mu_{x-y} + e_1\mu_{x+ye_1} - e_1\mu_{x-ye_1} \\ &+ e_1\mu_{x-ye_2}e_3 - e_1\mu_{x+ye_2}e_3 + \mu_{x+ye_3}e_3 - \mu_{x-ye_3}e_3.\end{aligned} \tag{11.5}$$

Lemma 11.1.1. *Let $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{H}$. The $\mathbb{H}$-valued measures $\mu_{x,y}$ given in (11.5) enjoy the following properties*

(i) $\mu_{x\alpha+y\beta,z} = \mu_{x,z}\alpha + \mu_{y,z}\beta$,

(ii) $\mu_{x,y\alpha+z\beta} = \bar{\alpha}\mu_{x,y} + \bar{\beta}\mu_{x,z}$,

(iii) $|\mu_{x,y}(\Omega_j^+)| \leq \|x\|\|y\|$,

(iv) $\bar{\mu}_{x,y} = \mu_{y,x}$.

Proof. Properties (i)–(iii) are easily obtained from (11.4) using the uniqueness of $\mu_{x,y}$ and the properties of $\langle\cdot,\cdot\rangle$. Property (iv) follows from properties (i) and (ii). □

It follows from properties (i) and (iii) in Lemma 11.1.1 that for every fixed $y \in \mathcal{H}$ and every fixed $\sigma \in \mathfrak{B}(\Omega_j^+)$, the mapping

$$\Phi_y(x) = \mu_{x,y}(\sigma)$$

is a continuous right linear functional on $\mathcal{H}$. Moreover, it follows from property (ii) in Lemma 11.1.1 that

$$\Phi_{y\alpha}(x) = \overline{\alpha}\,\Phi_y(x), \quad \alpha \in \mathbb{H}.$$

It follows from the Riesz representation theorem for quaternionic Hilbert spaces, see Theorem 11.0.2, that corresponding to every $x \in \mathcal{H}$, there exists a unique vector $w \in \mathcal{H}$ such that

$$\Phi_y(x) = \langle x, w\rangle, \tag{11.6}$$

i.e., $\mu_{x,y}(\sigma) = \langle x, w\rangle$. Since the left-hand side of (11.6) depends linearly on x and anti-linearly on y and the right-hand side depends linearly on x, it follows that $\Phi_y(x)$ depends linearly on x and anti-linearly on y, so

$$E(\sigma)y = w,$$

for some operator

$$E(\sigma) \in \mathcal{B}(\mathcal{H}).$$

Thus,

$$\mu_{x,y}(\sigma) = \langle x, E(\sigma)y\rangle, \quad \sigma \in \mathfrak{B}(\Omega_j^+),$$

and in view of property (iv) in Lemma 11.1.1,

$$E(\sigma) = E(\sigma)^*, \quad \sigma \in \mathfrak{B}(\Omega_j^+), \tag{11.7}$$

and hence

$$\mu_{x,y}(\sigma) = \langle E(\sigma)x, y\rangle, \quad \sigma \in \mathfrak{B}(\Omega_j^+). \tag{11.8}$$

Since μ_x is countably additive, $\mu_{x,y}$ is also countably additive. Consequently, the $\mathcal{B}(\mathcal{H})$-valued measure E is also countably additive, i.e.,

$$E\left(\bigcup_{n=0}^{+\infty} \sigma_n\right) = \sum_{n=0}^{+\infty} E(\sigma_n) \tag{11.9}$$

for every sequence of pairwise disjoint sets $(\sigma_n)_{n\in\mathbb{N}}$ in $\mathfrak{B}(\Omega_j^+)$. The limit in (11.9) is intended with respect to the strong operator topology.

We recall that $\mathcal{SC}(\Omega)$ denotes the space of all continuous intrinsic slice functions on Ω, and we denote by

$$\mathcal{SC}_j(\Omega_j^+) := \{f_j := f|_{\Omega_j^+} : \ f \in \mathcal{SC}(\Omega)\}$$

the set of all restrictions of functions in $\mathcal{SC}(\Omega)$. Again we do not distinguish between a function f and its restriction f_j unless this could cause confusion.

Lemma 11.1.2. *Let J be the imaginary component in the $T = A+JB$ decomposition (9.17) of the normal operator $T \in \mathcal{B}(\mathcal{H})$ and let E be the spectral measure on $\mathfrak{B}(\Omega_j^+)$ with $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ defined above. The following statements hold:*

(i) *If $g \in \mathcal{C}(\Omega_j^+, \mathbb{R}) \cong \mathcal{SC}(\Omega, \mathbb{R})$, then for all $x, y \in \mathcal{H}$*

$$\langle g(T)x, y\rangle = \int_{\Omega_j^+} g(p)\, d\langle E(p)x, y\rangle. \tag{11.10}$$

(ii) *If $f = f_0 + jf_1 \in \mathcal{SC}_j(\Omega_j^+) \cong \mathcal{SC}(\Omega)$, then we have for all $x, y \in \mathcal{H}$,*

$$\langle f(T)x, y\rangle = \int_{\Omega_j^+} f_0(p)\, d\langle E(p)x, y\rangle + \int_{\Omega_j^+} f_1(p)\, d\langle JE(p)x, y\rangle. \tag{11.11}$$

(iii) *$E(\sigma)$ and J commute for all $\sigma \in \mathfrak{B}(\Omega_j^+)$ and $-J^2 = E(\Omega_j^+ \setminus \mathbb{R})$.*

Proof. Assertion (i) follows directly from (11.4) and (11.8). We will now prove assertion (11.11). In view of (11.10) and Theorems 9.4.9 and 9.4.11, we have

$$\begin{aligned}\langle f(T)x, y\rangle &= \langle \{f_0(T) + f_1(T)J\}x, y\rangle \\ &= \langle f_0(T)x, y\rangle + \langle f_1(T)Jx, y\rangle \\ &= \int_{\Omega_j^+} f_0(p) d\langle E(p)x, y\rangle + \int_{\Omega_j^+} f_1(p) d\langle E(p)Jx, y\rangle, \quad x, y \in \mathcal{H}.\end{aligned}$$

Thus, the proof of (11.11) will be complete on showing that

$$d\langle E(p)Jx, y\rangle = d\langle JE(p)x, y\rangle, \quad x, y \in \mathcal{H}.$$

To see this, let $g \in \mathcal{C}(\Omega_j^+, \mathbb{R})$ and use (11.10) and the fact that $g(T)$ and J commute to obtain

$$\int_{\Omega_j^+} g(p) d\langle E(p)Jx, y\rangle = \langle g(T)Jx, y\rangle = \langle Jg(T)x, y\rangle = \int_{\Omega_j^+} g(p) d\langle JE(p)x, y\rangle.$$

If we write $\nu = \langle E(p)Jx, y\rangle$ and $\widetilde{\nu} = \langle JE(p)x, y\rangle$ and then

$$\nu = \nu_0 \mathrm{e}_0 + \nu_1 \mathrm{e}_1 + \nu_2 \mathrm{e}_2 + \nu_3 \mathrm{e}_3$$

and

$$\widetilde{\nu} = \widetilde{\nu}_0 \mathrm{e}_0 + \widetilde{\nu}_1 \mathrm{e}_1 + \widetilde{\nu}_2 \mathrm{e}_2 + \widetilde{\nu}_3 \mathrm{e}_3,$$

where ν_ℓ and $\widetilde{\nu}_\ell$, $\ell = 0, \ldots, 3$, are real signed measures and $e_\ell, = 0, \ldots, 3$ is the standard basis for $\mathbb{H}$, then it follows from Theorem 11.0.1 that $\nu_\ell = \widetilde{\nu}_\ell$ for $\ell = 0, \ldots, 3$. Therefore, items (iii) and (ii) hold.

Finally, we have due to (i) and due to Lemma 10.1.7(iii) that

$$BE(\mathbb{R}) = \int_{\Omega_j^+} |\mathrm{Im}(q)| \, dE(q)E(\mathbb{R}) = \int_{\Omega_j^+} |\mathrm{Im}(q)| \chi_{\mathbb{R}} \, dE(q) = 0,$$

so that

$$\operatorname{ran} E(\mathbb{R}) \subset \ker B = \ker J,$$

where B is the positive operator in the decomposition $T = A + JB$. If, on the other hand, $x \in \ker J = \ker B$, then

$$0 = \langle Bx, x\rangle = \int_{\Omega_j^+} |\mathrm{Im} q|^2 d\mu_{x,x}(q).$$

Since the measure $\mu_{x,x}(\sigma) = \langle E(\sigma)x, x\rangle$ and the function $\varphi(q) := |\mathrm{Im}(q)|^2$ are nonnegative, this implies

$$0 = \mu_{x,x}(\varphi^{-1}(\mathbb{R} \setminus \{0\}) = \mu_{x,x}(\Omega_j^+ \setminus \mathbb{R}) = \langle x, E(\Omega_j^+ \setminus \mathbb{R})x\rangle = \|E(\Omega_j^+ \setminus \mathbb{R})\|^2.$$

Hence $E(\Omega_j^+ \setminus \mathbb{R})x = 0$, and in turn, $x \in \operatorname{ran} E(\mathbb{R})$. Therefore,

$$\operatorname{ran} E(\mathbb{R}) \supset \ker B = \ker J,$$

and in turn,

$$\operatorname{ran} E(\mathbb{R}) = \ker J.$$

Since $-J^2$ is the orthogonal projection onto $(\ker J)^\perp = \operatorname{ran} J$ by Corollary 9.3.8 and $E(\Omega_j^+ \setminus \mathbb{R})$ is the orthogonal projection onto $(\operatorname{ran} E(\mathbb{R}))^\perp$, we conclude that $-J^2 = E(\Omega_j^+ \setminus \mathbb{R})$. □

The properties of the spectral measure can be checked directly as in the following result.

Theorem 11.1.3. *The $\mathcal{B}(\mathcal{H})$-valued countably additive measure E, given by (11.8), for all $\sigma, \tau \in \mathfrak{B}(\Omega_j^+)$, enjoys the following properties:*

(i) $E(\sigma) = E(\sigma)^*$.

(ii) $\|E(\sigma)\| \le 1$.

(iii) $E(\emptyset) = 0$ *and* $E(\sigma_S(T) \cap \mathbb{C}_j^+) = \mathcal{I}$.

(iv) $E(\sigma \cap \tau) = E(\sigma)E(\tau)$.

(v) $E(\sigma)^2 = E(\sigma)$.

(vi) $E(\sigma)$ *commutes with* $f(T)$ *for all* $f \in \mathcal{SC}_j(\Omega_j^+) \cong \mathcal{SC}(\Omega)$.

(vii) $E(\sigma)$ *and* $E(\tau)$ *commute.*

Proof. The proof is broken into steps.

Step 1: *Show* (i) and (ii).

Property (i) has already been noted in (11.7). Property (ii) follows directly from property (iii) in Lemma 11.1.1. Indeed, if $x = y$ in property (iii) in Lemma 11.1.1, then

$$\mu_{x,x}(\sigma) \le \mu_{x,x}(\Omega_j^+) \le \|x\|^2$$

and hence

$$\langle E(\sigma)x, x\rangle \le \|x\|^2 \quad \text{for } x \in \mathcal{H},$$

i.e., $\mathcal{I} - E(\sigma)$ is a positive operator for all $\sigma \in \mathfrak{B}(\Omega_j^+)$. Therefore, property (ii) holds.

Step 2: *Show* (iii).

Since $\mu_{x,y}(\emptyset) = 0$, we may use (11.4) to deduce $E(\emptyset) = 0$. Similarly, putting $g(p) = 1$ in (11.4) yields $g(T) = \mathcal{I}$ for all $x, y \in \mathcal{H}$ and thus

$$\langle x, y\rangle = \int_{\Omega_j^+} d\mu_{x,y} = \langle E(\Omega_j^+)x, y\rangle,$$

i.e., $E(\Omega_j^+) = \mathcal{I}$.

Step 3: *Show* (iv).

Recall that for all real-valued polynomials ϕ and ψ on Ω_j^+, we have set $\phi(T) := \phi(A,B)$ and $\psi(T) := \psi(A,B)$. Clearly we have $(\phi\psi)(T) = \phi(T)\psi(T)$, $\phi(T) = \phi(T)^*$, and $\psi(T) = \psi(T)^*$. Thus,

$$\int_{\sigma_S(T)\cap\mathbb{C}_j^+} \phi(p)\, d\mu_{\psi(T)x,x}(p) = \langle \phi(T)\psi(T)x, x\rangle$$
$$= \langle (\phi\psi)(T)x, x\rangle = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \phi(p)\psi(p) d\mu_{x,x}(p). \tag{11.12}$$

Since $E(\sigma) = E(\sigma)^*$, (11.8) implies that

$$\mu_{x,x}(\sigma) \in \mathbb{R} \text{ for all } \sigma \in \mathfrak{B}(\Omega_j^+).$$

Similarly, since $\langle \psi(T)x, x\rangle$ is real, (11.8) implies that

$$\mu_{\psi(T)x,x}(\sigma) \in \mathbb{R} \text{ for all } \sigma \in \mathfrak{B}(\sigma_S(T)\cap\mathbb{C}_j^+).$$

In view of the density of real-valued polynomials in the space $\mathcal{C}(\Omega_j^+, \mathbb{R})$ and the Riesz representation theorem given in Theorem 11.0.1, (11.12) implies that

$$d\mu_{\psi(T)x,x}(p) = \psi(p) d\mu_{x,x}(p).$$

But then we may use the identity (11.5) and the fact that $\psi(p)$ is real-valued to obtain

$$d\mu_{\psi(T)x,y}(p) = \psi(p) d\mu_{x,y}(p).$$

Thus, in view of (11.8),

$$\langle E(\sigma)\psi(T)x, y\rangle = \int_\sigma \psi(p)\, d\mu_{x,y}(p) \quad \text{for } \sigma \in \mathfrak{B}(\Omega_j^+).$$

Since $E(\sigma) = E(\sigma)^*$ for $\sigma \in \mathfrak{B}(\Omega_j^+)$,

$$\int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\, d\mu_{x,E(\sigma)y} = \langle \psi(T)x, E(\sigma)y\rangle$$
$$= \langle E(\sigma)\psi(T)x, y\rangle = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\,\chi_\sigma\, d\mu_{x,y},$$

where

$$\chi_\sigma(p) = \begin{cases} 1 & \text{if } \; p \in \sigma, \\ 0 & \text{if } \; p \notin \sigma. \end{cases}$$

Since ψ is real-valued, we also have

$$\int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\, d\mu^{(m)}_{x,E(\sigma)y} = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \psi\,\chi_\sigma\, d\mu^{(m)}_{x,y} \quad \text{for } m = 0,\dots,3, \tag{11.13}$$

where $\mu^{(m)}_{x,y}$ and $\mu^{(m)}_{x,E(\sigma)y}$ are real-valued signed measures given by

$$\mu_{x,y} = \sum_{m=0}^{3} \mu^{(m)}_{x,y} e_m$$

and

$$\mu^{(m)}_{x,E(\sigma)y} = \sum_{m=0}^{3} \mu^{(m)}_{x,E(\sigma)y} e_m.$$

Recall that $(e_m)_{\ell=0,\dots,3}$ is the standard basis for $\mathbb{H}$.

In view of the density of real-valued polynomials in the space $\mathcal{C}(\Omega_j^+,\mathbb{R})$ and the Riesz representation theorem given in Theorem 11.0.1, the identity (11.13) implies that

$$d\mu^{(m)}_{x,E(\sigma)y} = \chi_\sigma\, d\mu^{(m)}_{x,y} \quad \text{for } m = 0,\dots,3,$$

and hence

$$d\mu_{x,E(\sigma)y} = \chi_\sigma\, d\mu_{x,y}.$$

Therefore,

$$\mu_{x,E(\sigma)y}(\tau) = \int_{\Omega_j^+\cap\tau} \chi_\sigma d\mu_{x,y} = \mu_{x,y}(\sigma\cap\tau)$$

for $\sigma,\tau\in\mathfrak{B}(\Omega_j^+)$. Since

$$\mu(\sigma) = \langle E(\sigma)x, y\rangle \quad \text{for } \sigma\in\mathfrak{B}(\Omega_j^+),$$

we obtain $E(\sigma)E(\tau) = E(\sigma\cap\tau)$ for $\sigma,\tau\in\mathfrak{B}(\Omega_j^+)$.

Step 4: *Show* (v).

Property (v) can be obtained from Property (iv) when $\sigma=\tau$.

Step 5: *Show* (vi).

Let A, B, and J be as in Theorem 9.3.5. We have already observed in item (iii) of Lemma 11.1.2 that $E(\sigma)$ and J commute. One can show in a similar fashion that A and $E(\sigma)$ commute and B and $E(\sigma)$ commute. Thus, in view of the construction of $f(T)$, we have that $f(T)$ and $E(\sigma)$ commute.

Step 6: *Show* (vii).

Property (vii) follows from Property (iv) on interchanging τ and σ. □

Remark 11.1.4. The spectral measure E was constructed using only operators $g(T)$ that were generated by functions $g \in \mathcal{C}(\Omega_j^+, \mathbb{R})$, that is, by real-valued functions. By Theorem 9.4.11, for such functions, the operator $g(T)$, however, does not depend on all the information we have about T, but only on the factors A and B in the $T = A + JB$ decomposition of T. Hence E is actually a joint spectral measure of the self-adjoint operators A and B. This in particular implies that $T = A + JB$ and $T^* = A - JB$ have the same spectral measure E.

In the quaternionic setting, invariant subspaces are not associated with individual eigenvalues, but with spheres $[s]$ of equivalent eigenvalues, because the eigenvalue equation $T(x) - xs = 0$ associated with a single (nonreal) eigenvalue is not linear. The correct interpretation of the above observation is therefore that the spectral measure E associates invariant subspaces of T to sets of spectral spheres, while the imaginary operator J orients the spheres. It determines how the different spectral values in these spheres need to be multiplied onto the vectors in the associated subspaces in order to fit the operator T. A more detailed discussion of this idea will be given in Chapter 14.

11.2 The Spectral Theorem and Some Consequences

We conclude this chapter with the main result, the spectral theorem for bounded operators.

Theorem 11.2.1 (The spectral theorem for bounded normal operators)**.** *Let $T \in \mathcal{B}(\mathcal{H})$ be normal, let $J \in \mathcal{B}(\mathcal{H})$ be the imaginary operator in the Teichmüller decomposition $T = A + JB$ of Theorem 9.3.5, and fix $j \in \mathbb{S}$. Let $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ and let Π_0 and $\Pi_\pm^j$ denote the orthogonal $\mathbb{C}_j$-linear projections defined in Definition 9.3.10 corresponding to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ given in Lemma 9.3.9.*

Then there exists a unique spectral measure E_j on Ω_j^+ such that for all $x, y \in \mathcal{H}$,

$$\langle Tx, y\rangle = \int_{\Omega_j^+} \operatorname{Re}(q)\, d\langle E(q)x, y\rangle + \int_{\Omega_j^+} \operatorname{Im}(q)\, d\langle E(q)Jx, y\rangle. \tag{11.14}$$

For every function $f = f_0 + jf_1 \in \mathcal{SC}_j(\Omega_j^+)$ and $x, y \in \mathcal{H}$, we moreover have

$$\langle f(T)x, y\rangle = \int_{\Omega_j^+} f_0(p)\, d\langle E_j(p)x, y\rangle + \int_{\Omega_j^+} f_1(p)\, d\langle JE_j(p)x, y\rangle, \tag{11.15}$$

or, equivalently,

$$\begin{aligned}\langle f(T)x,y\rangle = \int_{\Omega_j^+} d\langle \Pi_0 E_j(p)x,y\rangle\, f_0(p) \\ + \int_{\Omega_j^+} d\langle \Pi_+^j E_j(p)x,y\rangle\, f(p) \\ + \int_{\Omega_j^+} d\langle \Pi_-^j E_j(p)x,y\rangle \overline{f(p)}.\end{aligned} \tag{11.16}$$

Moreover, on identifying the complex plane $\mathbb{C}_k$ *with* $\mathbb{C}_j$ *in the natural way by the mapping* $\varphi_{kj} : u+kv \mapsto u+jv$, *we have* $E_j(\varphi_{kj}(\sigma)) = E_k(\sigma)$ *for all* $\sigma \in \mathfrak{B}(\Omega_k^+)$ *for all* $j,k \in \mathbb{S}$.

Proof. Formula (11.15) was established in item (ii) of Lemma 11.1.2. Formula (11.16) follows from (11.15). Indeed, if we write $y = y_0 + y_+ + y_- \in \mathcal{H}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ as in Lemma 9.3.9, then

$$\begin{aligned}\langle f(T)x,y\rangle &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)x,y\rangle + \int_{\Omega_j^+} f_1(p) d\langle JE_j(p)x,y)\rangle \\ &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)x,y\rangle + \int_{\Omega_j^+} f_1(p) d\langle E_j(p)x_+ j,y\rangle \\ &\quad + \int_{\Omega_j^+} f_1(p) d\langle E_j(p)x_-(-j),y\rangle \\ &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)x_0,y\rangle + \int_{\Omega_j^+} d\langle E_j(p)x_+,y\rangle (f_0(q)+jf_1(q)) \\ &\quad + \int_{\Omega_j^+} d\langle E_j(p)x_-,y\rangle (f_0(q)-jf_1(q)) \\ &= \int_{\Omega_j^+} f_0(p) d\langle E_j(p)\Pi_0 x,y\rangle + \int_{\Omega_j^+} d\langle E_j(p)\Pi_+^j x,y\rangle f(q) \\ &\quad + \int_{\Omega_j^+} d\langle E_j(p)\Pi_-^j x,y\rangle \overline{f(q)}.\end{aligned}$$

The fact that there is only one spectral measure E_j on $\sigma_S(T) \cap \mathbb{C}_j^+$ such that (11.15) holds follows directly from the uniqueness of the measure $\mu_{x,y}(\sigma) = \langle E(\sigma)x,y\rangle$ on Ω_j^+ (see (11.5)). The claimed invariance $E_j(\varphi_{jk}(\sigma)) = E_k(\sigma)$ relative to $j,k \in \mathbb{S}$ drops out easily from the aforementioned uniqueness of E_j and Theorem 9.2.3. □

Corollary 11.2.2. *In the setting of Theorem 11.2.1, the following statements hold:*

(i) *If* $T \in \mathcal{B}(\mathcal{H})$ *is a positive operator, then there exists a unique positive operator* $T^{1/2} := W \in \mathcal{B}(\mathcal{H})$ *such that* $W^2 = T$.

(ii) *$T \in \mathcal{B}(\mathcal{H})$ is self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{[-\|T\|,\|T\|]} t\, d\langle E_j(t)x, y\rangle, \quad x, y \in \mathcal{H}. \tag{11.17}$$

(iii) *$T \in \mathcal{B}(\mathcal{H})$ is anti-self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{[0,\|T\|]} t\, d\langle JE_j(t)x, y\rangle, \quad x, y \in \mathcal{H}. \tag{11.18}$$

(iv) *$T \in \mathcal{B}(\mathcal{H})$ is unitary if and only if*

$$\langle Tx, y\rangle = \int_{[0,\pi]} \cos(t) d\langle E_j(t)x, y\rangle + \int_{[0,\pi]} \sin(t) d\langle JE_j(t)x, y\rangle. \tag{11.19}$$

Proof. If $T \in \mathcal{B}(\mathcal{H})$ is a positive operator, then $\sigma_S(T) \subseteq [0, \|T\|]$. Thus, using Theorem 11.2.1, we have the existence of a uniquely determined spectral measure E_j such that

$$\langle Tx, y\rangle = \int_{[0,\|T\|]} t\, d\langle E_j(t)x, y\rangle. \tag{11.20}$$

Let $g(t) = t^{1/2}$ for $t \in \mathbb{R}$. Since $g \in \mathcal{C}(\sigma_S(T), \mathbb{R})$, it follows from Theorem 11.2.1 that

$$\langle Wx, y\rangle := \langle g(T)x, y\rangle = \int_{[0,\|T\|]} t^{1/2}\, d\langle E_j(t)x, y\rangle$$

satisfies $W^2 = T$. Thus, we have established the existence of a positive operator $W \in \mathcal{B}(\mathcal{H})$ such that $W^2 = T$. The proof that W is unique follows from the uniqueness of the spectral measure E_j, just as in the case that $\mathcal{H}$ is a complex Hilbert space.

The proofs of (ii)–(iv) follow readily from Theorem 11.2.1 and (9.9). □

11.3 Comments and Remarks

The spectral theorem based on the S-spectrum was proved in the following papers: the general case for bounded and unbounded normal operators was shown in [13]. A different proof for unitary operators was given in [14], and the simple case of compact normal operators was shown in [143].

Results related to the quaternionic spectral theorem can furthermore be found in [57, 74]. For quaternionic matrices, the spectral theorem based on the right spectrum was proved in [108]. The right spectrum is in the finite-dimensional case, however, equal to the S-spectrum.

The main application of the quaternionic spectral theorem is in quaternionic quantum mechanics. In the list of references there are also papers related to quaternionic quantum mechanics [107], [109], [158] in which the notion of right spectrum was used.

Chapter 12

The Spectral Theorem for Unbounded Normal Operators

In this section we will consider normal operators T that are unbounded. The strategy will be to transform T into a normal operator $Z_T \in \mathcal{B}(\mathcal{H})$ and use Theorem 11.2.1 and a change of variable argument to obtain a spectral theorem for T based on the S-spectrum. Obtaining a spectral theorem for unbounded operators in the aforementioned way has been done in the classical case, i.e., when $\mathcal{H}$ is a complex Hilbert space; see, e.g., the book of Schmüdgen [191].

12.1 Some Transformations of Operators

Given $T \in \mathcal{L}(\mathcal{H})$, we let

$$Z_T = TC_T^{1/2}, \tag{12.1}$$

where $C_T = (\mathcal{I} + T^*T)^{-1} \in \mathcal{B}(\mathcal{H})$ (the proof that C_T is bounded and positive can be carried out in a similar manner to the classical complex Hilbert case; see, e.g., Proposition 3.18(i) in [191]).

Theorem 12.1.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a densely defined closed operator on $\mathcal{H}$. The operator Z_T has the following properties:*

(i) *$Z_T \in \mathcal{B}(\mathcal{H})$, $\|Z_T\| \leq 1$, and*

$$C_T = (\mathcal{I} + T^*T)^{-1} = \mathcal{I} - Z_T^* Z_T. \tag{12.2}$$

(ii) $(Z_T)^* = Z_{T^*}$.

(iii) *If T is normal, then Z_T is normal.*

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*, Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_12

Proof. The proof is based on the proof of Lemma 5.8 in [191] and is broken into three steps.

Step 1: *Prove* (i).

First note that

$$\{C_T x: \ x \in \mathcal{H}\} = \mathcal{D}(\mathcal{I} + T^*T) = \mathcal{D}(T^*T). \tag{12.3}$$

Consequently, if $x \in \mathcal{H}$, then

$$\begin{aligned}\|TC_T^{1/2}C_T^{1/2}x\|^2 &= \langle T^*TC_Tx, C_Tx\rangle \\ &\leq \langle (\mathcal{I} + T^*T)C_Tx, C_Tx\rangle \\ &= \langle C_T^{-1}C_Tx, C_Tx\rangle \\ &= \langle x, C_Tx\rangle \\ &= \|C_T^{1/2}x\|^2.\end{aligned}$$

Thus if $y \in \{C_T^{1/2}x\colon x \in \mathcal{H}\}$, then

$$\|Z_Ty\| = \|TC_T^{1/2}y\| \leq \|y\|. \tag{12.4}$$

Since $\ker(C_T) = \{0\}$, we have that $\ker(C_T^{1/2}) = \{0\}$, and thus $\{C_T^{1/2}x\colon x \in \mathcal{H}\}$ is a dense subset of $\mathcal{H}$. Since T is a closed operator by assumption and $C_T^{1/2} \in \mathcal{B}(\mathcal{H})$, we get that Z_T is closed as well. Thus, we have $\{C_T^{1/2}x\colon x \in \mathcal{H}\} \subseteq \mathcal{D}(T)$, $\mathcal{D}(Z_T) = \mathcal{H}$, and in view of (12.4), $\|Z_T\| \leq 1$.

Next, it follows from (12.4) and $C^{1/2}T^* \subseteq Z_T^*$ that

$$\begin{aligned}(\mathcal{I} - C_T)C_T^{1/2} &= C_T^{1/2}(\mathcal{I} + T^*T)C_T - C_T^{1/2}C_T \\ &= C_T^{1/2}T^*TC_T^{1/2}C_T^{1/2} \\ &\subseteq Z_T^*Z_TC_T^{1/2}.\end{aligned}$$

Thus, $Z_T^*Z_TC_T^{1/2} = (\mathcal{I} - C_T)C_T^{1/2}$, and since $\{C_T^{1/2}x\colon x \in \mathcal{H}\}$ is a dense subset of $\mathcal{H}$, we get (12.2).

Step 2: *Prove* (ii).

Using (12.2) we get that $C_{T^*} = (\mathcal{I} + TT^*)^{-1}$. If $x \in \mathcal{D}(T^*)$, then let $y = C_{T^*}x$. Therefore,

$$x = (\mathcal{I} + TT^*)y$$

and

$$T^*x = T^*(\mathcal{I} + TT^*)y = (\mathcal{I} + T^*T)T^*y.$$

Thus, $C_{T^*}x \in \mathcal{D}(T^*)$ and hence

$$C_TT^*x = T^*y = T^*C_{T^*}x. \tag{12.5}$$

It follows easily from (12.5) and (12.2) that $p(C_{T^*})x \in \mathcal{D}(T^*)$ and

$$p(C_T)T^*x = T^*p(C_{T^*})x$$

for every real polynomial p of a real variable. By the Weierstrass approximation theorem, there exists a sequence of real polynomials $\{\phi_n\}_{n=0}^{+\infty}$ that converges uniformly to the function $t \mapsto t^{1/2}$ on $[0,1]$. Since the continuous functional calculus is norm-preserving, we find that

$$\lim_{n\to+\infty} \|\phi_n(C_T) - C_T^{1/2}\| = \lim_{n\to+\infty} \|\phi_n(C_{T^*}) - C_{T^*}^{1/2}\| = 0.$$

Since T is a closed operator, T^* is also a closed operator. Thus, we have

$$\begin{aligned} C_T^{1/2}T^*x &= \lim_{n\to+\infty} \phi_n(C_T)T^*x = \lim_{n\to+\infty} T^*\phi_n(C_{T^*})x \\ &= T^*(C_{T^*})^{1/2}x \quad \text{for} \quad x \in \mathcal{D}(T^*). \end{aligned}$$

Since $C_T^{1/2}T^* \subseteq (TC_T^{1/2})^* = Z_{T^*}$, we get that

$$Z_{T^*}x = C_T^{1/2}T^*x = T^*(C_{T^*})^{1/2}x = (Z_T)^*x$$

for $x \in \mathcal{D}(T^*)$. Finally, since $\mathcal{D}(T^*)$ is dense in $\mathcal{H}$, we have that $Z_{T^*}x = (Z_T)^*x$ for all $x \in \mathcal{H}$, i.e., $Z_{T^*} = (Z_T)^*$.

Step 3: *Prove* (iii).

Using (12.2) on T and T^* and the fact that $TT^* = T^*T$, we have

$$\mathcal{I} - Z_T^*Z_T = (\mathcal{I} + T^*T)^{-1} = (\mathcal{I} + TT^*)^{-1} = \mathcal{I} - Z_{T^*}^*Z_{T^*}.$$

Making use of Property (ii), we have that

$$\mathcal{I} - Z_T^*Z_T = \mathcal{I} - Z_TZ_T^*,$$

i.e., Z_T is normal. □

12.2 The Spectral Theorem for Unbounded Normal Operators

We are now ready to state and prove a spectral theorem for unbounded normal operators on a quaternionic Hilbert space.

Theorem 12.2.1. *Let T be an unbounded right linear normal operator on $\mathcal{H}$ and $j \in \mathbb{S}$. There exists a uniquely determined spectral measure E_j on $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ such that for $x \in \mathcal{D}(T)$ and $y \in \mathcal{H}$,*

$$\langle Tx, y\rangle = \int_{\Omega_j^+} \mathrm{Re}(p)d\langle E_j(p)x, y\rangle + \int_{\Omega_j^+} \mathrm{Im}(p)d\langle JE_j(p)x, y\rangle, \tag{12.6}$$

or equivalently,

$$\begin{aligned}\langle Tx, y\rangle = &\int_{\Omega_j^+} \mathrm{Re}(p)\, d\langle \Pi_0 E_j(p)x, y\rangle \\ &+ \int_{\Omega_j^+} d\langle \Pi_+^j E_j(p)x, y\rangle p \\ &+ \int_{\Omega_j^+} d\langle \Pi_-^j E_j(p)x, y\rangle \overline{p}.\end{aligned} \tag{12.7}$$

The operator J in the above equation is the imaginary operator appearing in the Teichmüller decomposition $Z_T = A + JB$ of Z_T defined in Theorem 9.3.5 and Π_0 and $\Pi_\pm^j$ are the associated projections defined in Definition 9.3.10. The operator J commutes with E and satisfies $-J^2 = E(\mathbb{H} \setminus \mathbb{R})$.

Moreover, on identifying the complex plane $\mathbb{C}_k$ with $\mathbb{C}_j$ in the natural way by the mapping φ_{kj}, we have $E_j(\varphi_{kj}(\sigma)) = E_k(\sigma), \sigma \in \mathfrak{B}(\Omega_k^+)$, for all $j, k \in \mathbb{S}$.

Proof. The proof is broken into two steps.

Step 1: *Show that a spectral measure E_j exists such that (12.6) holds.*

Let $\mathbb{B} = \{p \in \mathbb{H} : |p| < 1\}$, $\partial\mathbb{B} = \{p \in \mathbb{H} : |p| = 1\}$, and $\overline{\mathbb{B}} = \mathbb{B} \cup \partial\mathbb{B}$. If T is normal, then using Properties (i) and (iii) in Theorem 12.1.1, we get that $\|Z_T\| \leq 1$ and Z_T is normal, respectively. Thus, we may use Theorem 11.2.1 to obtain a uniquely determined spectral measure F on $\sigma_S(Z_T) \cap \mathbb{C}_j^+$ such that

$$f(Z_T) = \mathbb{I}(f) = \int_{\sigma_S(Z_T)\cap\mathbb{C}_j^+} f(p)\, dF(p) \tag{12.8}$$

for $f \in \mathcal{SC}_j(\sigma_S(Z_T) \cap \mathbb{C}_j^+)$. In addition, it follows from Theorem 3.1.13 that

$$\sigma_S(Z_T) \subseteq \{p \in \mathbb{H} : \ |p| \leq \|Z_T\|\}$$

and hence

$$\sigma_S(Z_T) \cap \mathbb{C}_j^+ \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_j^+.$$

If $x \in \mathcal{H}$ and $\sigma \in \mathfrak{B}(\sigma_S(Z_T) \cap \mathbb{C}_j^+)$, then in view of item (v) in Lemma 10.1.7 and (12.8), we have

$$\langle (\mathcal{I} - Z_T^* Z_T)F(\sigma)x, F(\sigma)x\rangle = \int_\sigma (1 - |p|^2) d\langle F(p)x, x\rangle. \tag{12.9}$$

Recall that $\mathcal{I} - Z_T^* Z_T = (\mathcal{I} + T^*T)^{-1}$, and so $\ker(\mathcal{I} - Z_T^* Z_T) = \{0\}$. Thus, using (12.9) with

$$\sigma = \mathbb{B} \cap \mathbb{C}_j^+,$$

we get that $\operatorname{supp} F \subseteq \overline{\mathbb{B}} \cap \mathbb{C}_j^+$ and $F(\partial\mathbb{B} \cap \mathbb{C}_j^+) = 0$. Therefore,

$$F(\mathbb{B} \cap \mathbb{C}_j^+) = F[(\overline{\mathbb{B}} \cap \mathbb{C}_j^+) \setminus \partial(\mathbb{B} \cap \mathbb{C}_j^+)] = \mathcal{I}.$$

If $\varphi(p) = p(1-|p|^2)^{-1/2}$, then $\varphi \in \mathcal{SM}_F^{\#}(\sigma_S(Z_T) \cap \mathbb{C}_j^+)$. In view of item (iii) and (v) of Theorem 10.2.7, we have

$$\mathbb{I}(\varphi) = \mathbb{I}(f)\mathbb{I}(g),$$

where

$$f(p) = p \quad \text{and} \quad g(p) = \frac{1}{\sqrt{1-|p|^2}},$$

and $\mathcal{D}(\mathbb{I}(\varphi)) = \mathcal{D}(\mathbb{I}(g))$. Using Theorem 10.2.9, we have

$$\mathbb{I}(g) = \mathbb{I}(1/g)^{-1}.$$

Consequently, we may use item (i) in Corollary 11.2.2 to obtain

$$\mathbb{I}(g) = \{(\mathbb{I}(h))^{1/2}\}^{-1},$$

where

$$h(p) = 1 - |p|^2 \in \mathcal{SM}_j^{\infty}(\sigma_S(Z_T) \cap \mathbb{C}_j^+).$$

Putting these observations together, we obtain

$$\mathbb{I}(\varphi) = Z_T(C_T^{1/2})^{-1}. \tag{12.10}$$

Since $Z_T = TC_T^{1/2}$, we obtain $\varphi(Z_T) \subseteq T$. Using $C_T = (\mathcal{I} - Z_T^* Z_T)^{1/2}$, we get that $\mathbb{I}(\varphi) \subseteq T$. Thus, using Lemma 9.1.17, we get that

$$\mathbb{I}(\varphi) = T.$$

Let $E_j(\sigma) = F(\varphi^{-1}(\sigma))$, where

$$\varphi^{-1}(\sigma) = \{p \in \mathbb{H} : \ \varphi(p) \in \sigma\} \quad \text{for } \sigma \in \mathfrak{B}(\sigma_S(T) \cap \mathbb{C}_j^+).$$

It is readily checked that $E_j = F(\varphi^{-1})$ defines a spectral measure on $\mathbb{C}_j^+$, and thus using Lemma 10.2.11, we get (12.6). The equivalent assertion (12.7) is established in much the same way as the analogous assertion in Theorem 11.2.1.

Since the imaginary operator J in the Teichmüller decomposition of Z_T commutes with the spectral measure F, it also commutes with $E_j = F(\varphi^{-1})$. Furthermore, since φ maps $\mathbb{R}$ into itself and $\mathbb{C}_j^+ \setminus \mathbb{R}$ into itself, we obtain

$$E_j(\mathbb{C}_j^+ \setminus \mathbb{R}) = F(\varphi^{-1}(\mathbb{C}_j^+ \setminus \mathbb{R})) = F(\mathbb{C}_j^+ \setminus \mathbb{R}) = -J^2.$$

Step 2: *Show that E_j from Step 1 is unique.*

If E_j and $\widetilde{E}_j$ are spectral measures on $\sigma_S(T) \cap \mathbb{C}_j^+$ that satisfy (12.6), then $F = E_j(\varphi)$ and $\widetilde{F} = \widetilde{E}_j(\varphi)$ are both spectral measures such that for $x, y \in \mathcal{H}$,

$$\begin{aligned}\langle Z_T\, x, y\rangle &= \int_{\mathbb{B}\cap\mathbb{C}_j^+} \operatorname{Re}(p)\, d\langle F(p)x, y\rangle + \int_{\mathbb{B}\cap\mathbb{C}_j^+} \operatorname{Im}(p)\, d\langle JF(p)x, y\rangle \\ &= \int_{\mathbb{B}\cap\mathbb{C}_j^+} \operatorname{Re}(p)\, d\langle \widetilde{F}(p)x, y\rangle + \int_{\mathbb{B}\cap\mathbb{C}_j^+} \operatorname{Im}(p)\, d\langle J\widetilde{F}(p)x, y\rangle. \end{aligned} \tag{12.11}$$

Consider now a polynomial $\Phi(p) = \sum_{0 \le |\ell| \le n} a_\ell p^{\ell_1} \overline{p}^{\ell_2}$ with real coefficients as in (9.19). In view of Lemma 10.1.7 and Remark 10.1.8, the identity (12.11) implies

$$\begin{aligned}\langle \psi(Z_T)x, y\rangle &= \int_{\sigma_S(Z_T)\cap \mathbb{C}_j^+} \psi(p) d\langle F(p)x, y\rangle \\ &= \int_{\sigma_S(Z_T)\cap \mathbb{C}_j^+} \psi(p) d\langle \widetilde{F}(p)x, y\rangle.\end{aligned}$$

Since the set of polynomials of this type is by Theorem 9.4.5 dense in $\mathcal{SC}_j(\sigma_S(Z_T)\cap \mathbb{C}_j^+)$, we have that

$$\int_{\sigma_S(Z_T)\cap \mathbb{C}_j^+} \phi(p) d\langle F(p)x, x\rangle = \int_{\sigma_S(Z_T)\cap \mathbb{C}_j^+} \phi(p) d\langle \widetilde{F}(p)x, x\rangle$$

for all $\phi \in \mathcal{SC}_j(\sigma_S(Z_T)\cap \mathbb{C}_j^+)$. Hence in view of construction of the spectral measure given in Section 11, $F = \widetilde{F}$. Therefore, $E_j = \widetilde{E}_j$. The final assertion concerning E_j and E_k is proved in a similar manner to an analogous assertion in Theorem 11.2.1. □

12.3 Some Consequences of the Spectral Theorem

We conclude this chapter with some consequences of the spectral theorem for unbounded normal operators, Moreover, in the last corollary we state the functional calculus for unbounded normal operators, which is a direct consequence of the definition and the properties of the spectral integrals, which depend of the operator J.

Corollary 12.3.1. *In the setting of Theorem 12.2.1, the following statements hold:*

(i) *If $T \in \mathcal{L}(\mathcal{H})$ is a positive operator, then there exists a unique positive operator $W \in \mathcal{L}(\mathcal{H})$ such that $W^2 = T$.*

(ii) *$T \in \mathcal{L}(\mathcal{H})$ is self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{\mathbb{R}} t\, d\langle E(t)x, y\rangle, \quad x \in \mathcal{D}(T),\ \ y \in \mathcal{H}. \tag{12.12}$$

(iii) *$T \in \mathcal{L}(\mathcal{H})$ is anti-self-adjoint if and only if*

$$\langle Tx, y\rangle = \int_{[0,\infty)} t\, d\langle JE(t)x, y\rangle, \quad x \in \mathcal{D}(T),\ \ y \in \mathcal{H}. \tag{12.13}$$

Proof. Using Theorem 12.2.1, the proof is completed as in Corollary 11.2.2. □

Remark 12.3.2. We remind the reader that the functional calculus mentioned in Section 10 is applicable to unbounded normal operators $T \in \mathcal{L}(\mathcal{H})$. We conclude this section by stating, in the following corollary, such a functional calculus.

Corollary 12.3.3. *Let T, E_j, and J be as in Theorem 12.2.1. If $f, g \in \mathcal{SM}_E^{\#}(\Omega_j^+)$ with $\Omega_j^+ = \sigma_S(T) \cap \mathbb{C}_j^+$ and $\alpha, \beta \in \mathbb{R}$, then:*

(i) $\mathbb{I}(\bar{f}) = \mathbb{I}(f)^*$.

(ii) $\mathbb{I}(\alpha f + \beta g) = \overline{\alpha \mathbb{I}(f) + \beta \mathbb{I}(g)}$.

(iii) $\mathbb{I}(fg) = \overline{\mathbb{I}(f)\mathbb{I}(g)}$.

(iv) $\mathbb{I}(f)$ *is a closed normal operator on* $\mathcal{H}$ *and*

$$\mathbb{I}(f)^*\mathbb{I}(f) = \mathbb{I}(f\bar{f}) = \mathbb{I}(\bar{f}f).$$

(v) $\mathcal{D}(\mathbb{I}(f)\mathbb{I}(g)) = \mathcal{D}(\mathbb{I}(g)) \cap \mathcal{D}(\mathbb{I}(fg))$.

(vi) *If $x \in \mathcal{D}(\mathbb{I}(f))$ and $y \in \mathcal{D}(\mathbb{I}(g))$, then*

$$\langle \mathbb{I}(f)x, \mathbb{I}(g)y\rangle = \int_{\Omega_j^+} \mathrm{Re}(f(p)\overline{g(p)})d\langle E(p)x, y\rangle + \int_{\Omega_j^+} \mathrm{Im}(f(p)\overline{g(p)})d\langle JE(p)x, y\rangle.$$

(vii) *If $x \in \mathcal{D}(\mathbb{I}(f))$, then*

$$\|\mathbb{I}(f)x\|^2 = \int_{\Omega_j^+} |f(p)|^2 d\langle E(p)x, x\rangle.$$

Theorem 12.3.4. *Let T be as in Theorem 12.2.1 and let J be the imaginary operator in the Teichmüller decomposition of Z_T. Then there exist strongly commuting operators A and B that commute with J, where $A \in \mathcal{L}(\mathcal{H})$ is self-adjoint and $B \in \mathcal{L}(\mathcal{H})$ is positive with* $\ker B = \ker J$ *such that*

$$T = A + JB. \tag{12.14}$$

Proof. To verify assertion (iv), let E be the spectral measure of T and define

$$A\,x = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \mathrm{Re}(p)\, dE(p)x, \quad x \in \mathcal{D}(T),$$

$$B\,x = \int_{\sigma_S(T)\cap\mathbb{C}_j^+} \mathrm{Im}(p)\, dE(p)x, \quad x \in \mathcal{D}(T).$$

If we set $E_0(\sigma) =: E(\{z \in \mathbb{C}_j^+ : \mathrm{Re}(p) \in \sigma\})$ and $E_1(\sigma) := E(\{z \in \mathbb{C}_j^+ : \mathrm{Im}(p) \in \sigma\})$ for $\sigma \in \mathfrak{B}(\mathbb{R})$, then the change of measure principle implies

$$A\,x = \int_{\mathbb{R}} t\, dE_0(t)x, \quad x \in \mathcal{D}(T),$$

$$B\,x = \int_0^{+\infty} t\, dE_1(t)x, \quad x \in \mathcal{D}(T).$$

Hence A and B are self-adjoint, and their spectral measures are E_0 and E_1. Since all projections $E(\sigma)$ with $\sigma \in \mathfrak{B}(\mathbb{C}_j^+)$ commute mutually and with J, we find that also E_0 and E_1 commute mutually and with J. Hence, A and B commute strongly, and they commute with J. Finally, we have

$$\ker B = \operatorname{ran} E_1(\{0\}) = \operatorname{ran} E(\{z \in \mathbb{C}_j^+ : \ \operatorname{Im}(z) = 0\}) = \operatorname{ran} E(\mathbb{R}) = \ker J. \qquad \square$$

Theorem 12.3.5 (Spectral mapping theorem)**.** *Let T be as in Theorem 12.2.1 and let $f \in \mathcal{SC}(\sigma_S(T))$. Then*

$$\sigma_S(f(T)) = \overline{f(\sigma_S(T))}. \tag{12.15}$$

Proof. First of all, observe that $\overline{f(\sigma_S(T))}$ is an axially symmetric set because $\sigma_S(T)$ is axially symmetric and f maps axially symmetric sets to axially symmetric sets since it is intrinsic. Let $\lambda \in \overline{f(\sigma_S(T)) \cap \mathbb{C}_j^+}$, let $\varepsilon > 0$, and choose $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon}(\tilde{\varepsilon} + 2|\operatorname{Im}(\lambda)|) < \frac{\varepsilon}{2}.$$

We can then find $z_\varepsilon \in \sigma_S(T)$ such that

$$|\lambda - f(z_\varepsilon)| < \tilde{\varepsilon},$$

and since $\lambda \in \mathbb{C}_j^+$ and f maps each complex plane $\mathbb{C}_i$ into itself, we even find that $z_\varepsilon \in \sigma_S(T) \cap \mathbb{C}_j$. (The function f, however, does not necessarily map each half-plane $\mathbb{C}_i^+$ into itself, and hence z_ε might belong to $\mathbb{C}_j^-$. In this case, $\overline{z_\varepsilon} \in \mathbb{C}_j^+$.) Then

$$\begin{aligned} &\left|f(z_\varepsilon)^2 - 2\operatorname{Re}(\lambda) f(z_\varepsilon) + |\lambda|^2\right| = |f(z_\varepsilon) - \lambda| \left|f(z_\varepsilon) - \overline{\lambda}\right| \\ &\le |f(z_\varepsilon) - \lambda| \, |f(z_\varepsilon) - \lambda| \left|\lambda - \overline{\lambda}\right| < \tilde{\varepsilon}(\tilde{\varepsilon} + 2|\operatorname{Im}(\lambda)|) < \frac{\varepsilon}{2}. \end{aligned}$$

The map $z \mapsto \mathcal{Q}_\lambda(f(z)) := f(z)^2 - 2\operatorname{Re}(\lambda) f(z) + |\lambda|^2$ is continuous, and hence there exists $\delta > 0$ such that for $z \in \mathbb{C}_j$ with $|z - z_\varepsilon| < \delta$, we have

$$|\mathcal{Q}_\lambda(f(z)) - \mathcal{Q}_\lambda(f(z_\varepsilon))| < \frac{\varepsilon}{2}$$

and in turn

$$|\mathcal{Q}_\lambda(f(z))| \le |\mathcal{Q}_\lambda(f(z)) - \mathcal{Q}_\lambda(f(z_\varepsilon))| + |\mathcal{Q}_\lambda(f(z_\varepsilon))| < \varepsilon.$$

Moreover,

$$|\mathcal{Q}_\lambda\left(f\left(\overline{z}\right)\right)| = \left|\mathcal{Q}_\lambda\left(\overline{f\left(z\right)}\right)\right| = \left|\overline{\mathcal{Q}_\lambda\left(f\left(z\right)\right)}\right| = |\mathcal{Q}_\lambda\left(f\left(z\right)\right)| < \varepsilon.$$

If $z_{\varepsilon,+} := [z] \cap \mathbb{C}_j^+$, that is, $z_{\varepsilon,+} = z_\varepsilon$ if $z_\varepsilon \in \mathbb{C}_j^+$ and $z_{\varepsilon,+} = \overline{z_\varepsilon}$ if $z_\varepsilon \in \mathbb{C}_j^-$, it follows that

$$\begin{aligned} U_\delta &:= \{z \in \sigma_S(T) \cap \mathbb{C}_j^+ : |z - z_{\varepsilon,+}| < \delta\} \\ &\subset \sigma_\varepsilon(\lambda) := \{z \in \sigma_S(T) \cap \mathbb{C}_j^+ : |\mathcal{Q}_z(f(z))| < \varepsilon\}. \end{aligned}$$

Since U_δ is an open set in $\sigma_S(T) \cap \mathbb{C}_j^+$, which is exactly the support of E, we find that $E(U_\delta) \neq 0$ and hence also $E(\sigma_\varepsilon) \neq 0$. We conclude from Lemma 10.2.10 that $\lambda \in \sigma_S(f(T))$, and so

$$\overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+ \subset \sigma_S(f(T)) \cap \mathbb{C}_j^+.$$

On the other hand, if $\lambda \notin \overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+$, then

$$\begin{aligned}\sigma_\varepsilon(\lambda) &= \left\{z \in \sigma_S(T) \cap \mathbb{C}_j^+ : \ |\mathcal{Q}_\lambda(f(z))| < \varepsilon\right\} \\ &\subset \{z \in \sigma_S(T) \cap \mathbb{C}_j : \ |\mathcal{Q}_\lambda(f(z))| < \varepsilon\}\end{aligned}$$

is empty for $\varepsilon > 0$ sufficiently small. Thus, Lemma 10.2.10 yields that $\lambda_0 \notin \sigma_S(f(T)) \cap \mathbb{C}_j^+$. We conclude that

$$\overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+ \supset \sigma_S(f(T)) \cap \mathbb{C}_j^+,$$

and in turn,

$$\overline{f(\sigma_S(T))} \cap \mathbb{C}_j^+ = \sigma_S(f(T)) \cap \mathbb{C}_j^+.$$

Taking the axially symmetric hull, we arrive at (12.15). □

12.4 Comments and Remarks

Several papers have appeared in the literature that claimed to introduce a spectral theorem for normal operators on a quaternionic Hilbert space (see [107, 109, 195, 197]). However, in all of the aforementioned papers, a precise notion of spectrum is not made clear. We will now enter into a discussion concerning the papers of Teichmüller [195] and Viswanath [197].

Teichmüller's paper [195] was the first to claim a spectral theorem for normal operators; it appeared in 1936. Despite not making the notion of spectrum clear, [195] does have a number of valid and important observations (even though some details for the precise proofs may be missing) such as the decomposition $T = A + JB$ (see Theorem 9.3.5) and also the fact that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+^j \oplus \mathcal{H}_-^j$ (see (9.18)). Finally, the spectral resolution in [195] takes the form

$$N = \int_{-\infty}^{\infty} \int_0^{\infty} (\lambda' + T_0 \lambda'') dQ_{\lambda''} dP_{\lambda'}, \tag{12.16}$$

where N is a normal operator, T_0 is an "Imaginäroperator" on $\overline{\operatorname{ran} B}$, i.e., $T_0 T_0^* = \mathcal{I}_{\operatorname{ran} B}$ and $T_0^* = -T_0$ (thus T_0 is playing the role of the operator J in Theorem 12.2.1), and Q and P are projection-valued measures. This bears some resemblance to (11.15).

In 1971 the paper [197] of Viswanath also claimed to have a spectral theorem for normal operators on a quaternionic Hilbert space. It is worth noting that [195] is not cited in Viswanath's paper [197]. The approach of [197] is very different from [195] in so far as the symplectic image of a normal operator is used and the spectral theorem is allegedly deduced from the classical spectral theorem and some kind of lifting argument. Viswanath's spectral resolution takes the form

$$T = \int_{\mathbb{C}_+} \lambda dE, \tag{12.17}$$

where T is a normal operator, E is a projection-valued measure E. Viswanath claims to deduce an antecedent to the decomposition in Theorem 9.3.5 from (12.17). However, the details are not given.

Beyond the spectral theorem there is the theory of the characteristic operator function, which was initiated in [28].

On the equivalent formulations of complex and quaternionic quantum mechanics see [126]. For recent applications of the spectral theory on the S-spectrum to quantum mechanics see [170, 171] and also [168, 196]. For coherent state transforms and the Weyl equation in Clifford analysis, see [169].

Chapter 13

Spectral Theorem for Unitary Operators

The spectral theorem for unitary operators is a particular case of the spectral theorem for bounded normal operators proved in Chapter 11. However, as in the complex case, the spectral theorem for unitary operators can be deduced from the quaternionic version of Herglotz's theorem proved in [16]. The spectral theorem for unitary operators based on Herglotz's theorem was proved in [14].

13.1 Herglotz's Theorem in the Quaternionic Setting

We recall some classical results and also their quaternionic analogues, which will be useful in proving a spectral theorem for quaternionic unitary operators. We need to recall some classical results in order to prove the quaternionic version of Herglotz's theorem.

Theorem 13.1.1 (Herglotz's theorem)**.** *The function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{C}^{s\times s}$ is positive definite if and only if there exists a unique $\mathbb{C}^{s\times s}$-valued measure μ on $[0,2\pi]$ such that*

$$r(n) = \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}. \tag{13.1}$$

Theorem 13.1.2. *Let μ and ν be $\mathbb{C}^{s\times s}$-valued measures on $[0,2\pi]$. If*

$$\int_0^{2\pi} e^{int} d\mu(t) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z},$$

then $\mu = \nu$.

In the above theorems we used the imaginary unit i for the complex plane. Given $P \in \mathbb{H}^{s\times s}$, there exist unique $P_1, P_2 \in \mathbb{C}^{s\times s}$ such that $P = P_1 + P_2 j$. Recall

F. Colombo et al., *Spectral Theory on the S-Spectrum for Quaternionic Operators*,
Operator Theory: Advances and Applications 270, https://doi.org/10.1007/978-3-030-03074-2_13

the bijective homomorphism $\chi : \mathbb{H}^{s\times s} \to \mathbb{C}^{2s\times 2s}$ given by

$$\chi P = \begin{pmatrix} P_1 & P_2 \\ -\overline{P_2} & \overline{P_1} \end{pmatrix}, \quad \text{where } P = P_1 + P_2 j. \tag{13.2}$$

Definition 13.1.3. Given an $\mathbb{H}$-valued measure ν, we may always write $\nu = \nu_1 + \nu_2 j$, where ν_1 and ν_2 are uniquely determined $\mathbb{C}$-valued measures. We call a measure $d\nu$ on $[0, 2\pi]$ *q-positive* if the $\mathbb{C}^{2\times 2}$-valued measure

$$\mu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3 \end{pmatrix}, \quad \text{where } \nu_3(t) = \nu_1(2\pi - t),\ t \in [0, 2\pi], \tag{13.3}$$

is positive and in addition,

$$\nu_2(t) = -\nu_2(2\pi - t), \quad t \in [0, 2\pi].$$

Remark 13.1.4. If ν is q-positive, then $\nu = \nu_1 + \nu_2 j$, where ν_1 is a uniquely determined positive measure and ν_2 is a uniquely determined $\mathbb{C}$-valued measure.

Remark 13.1.5. If $r = (r(n))_{n\in\mathbb{Z}}$ is an $\mathbb{H}$-valued sequence on $\mathbb{Z}$ such that

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t),$$

where $d\nu$ is a q-positive measure, then r is Hermitian, i.e., $\overline{r(-n)} = r(n)$.

The following result is a particular case of [16, Theorem 5.5] ($\mathbb{H}^{s\times s}$-valued positive sequences for $s > 1$ were also considered in [16]).

Theorem 13.1.6 (Herglotz's theorem for the quaternions)**.** *The function $n \mapsto r(n)$ from $\mathbb{Z}$ into $\mathbb{H}$ is positive definite if and only if there exists a unique q-positive measure ν on $[0, 2\pi]$ such that*

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}. \tag{13.4}$$

Proof. We give the proof for the general case. Let $(r(n))_{n\in\mathbb{Z}}$ be a positive definite sequence and write $r(n) = r_1(n) + r_2(n)j$, where $r_1(n), r_2(n) \in \mathbb{C}^{s\times s}$, $n \in \mathbb{Z}$. Put $R(n) = \chi r(n)$, $n \in \mathbb{Z}$. It is easily seen that $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence if and only if $(r(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{H}^{s\times s}$-valued sequence. Thus by Theorem 13.1.1, there exists a unique positive $\mathbb{C}^{2s\times 2s}$-valued measure μ on $[0, 2\pi]$ such that

$$R(n) = \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}. \tag{13.5}$$

Write

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{12}^* & \mu_{22} \end{pmatrix} : \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix} \to \begin{matrix} \mathbb{C}^s \\ \oplus \\ \mathbb{C}^s \end{matrix}.$$

It follows from

$$R(n) = \begin{pmatrix} r_1(n) & r_2(n) \\ -\overline{r_2(n)} & \overline{r_1(n)} \end{pmatrix}, \quad n \in \mathbb{Z},$$

and (13.5) that

$$r_1(n) = \int_0^{2\pi} e^{int} d\mu_{11}(t) = \int_0^{2\pi} e^{-int} d\bar{\mu}_{22}(t), \quad n \in \mathbb{Z},$$

and hence

$$\int_0^{2\pi} e^{int} d\mu_{11}(t) = \int_0^{2\pi} e^{int} d\bar{\mu}_{22}(2\pi - t), \quad n \in \mathbb{Z}.$$

Thus, Theorem 13.1.2 yields that $d\mu_{11}(t) = d\bar{\mu}_{22}(2\pi - t)$ for $t \in [0, 2\pi)$. Similarly,

$$r_2(n) = \int_0^{2\pi} e^{int} d\mu_{12}(t) = -\int_0^{2\pi} e^{-int} d\mu_{12}(t)^T, \quad n \in \mathbb{Z},$$

and hence

$$\int_0^{2\pi} e^{int} d\mu_{12}(t) = \int_0^{2\pi} e^{int} (-d\mu_{12}(2\pi - t)^T), \quad n \in \mathbb{Z}.$$

Thus, Theorem 13.1.2 yields that $d\mu_{12}(t) = -d\mu_{12}(2\pi - t)^T$ for $t \in [0, 2\pi)$.

It is easy to show that

$$\begin{pmatrix} I_s & -jI_s \end{pmatrix} R(n) \begin{pmatrix} I_s \\ jI_s \end{pmatrix} = 2r(n),$$

and hence (13.5) yields

$$\begin{aligned} 2r(n) &= \int_0^{2\pi} \begin{pmatrix} e^{int} & -je^{int} \end{pmatrix} \begin{pmatrix} d\mu_{11}(t) + d\mu_{12}(t)j \\ d\mu_{12}(t)^* + d\mu_{22}(t)j \end{pmatrix} \\ &= \int_0^{2\pi} e^{int} d\mu_{11}(t) + \int_0^{2\pi} e^{int} d\mu_{12}(t)j - \int_0^{2\pi} e^{-int} d\mu_{12}(t)^T j \\ &\quad + \int_0^{2\pi} e^{-int} d\bar{\mu}_{22}(t) \\ &= \int_0^{2\pi} e^{int} d\mu_{11}(t) + \int_0^{2\pi} e^{int} d\mu_{12}(t)j - \int_0^{2\pi} e^{int} d\mu_{12}(2\pi - t)^T j \\ &\quad + \int_0^{2\pi} e^{int} d\bar{\mu}_{22}(2\pi - t) \\ &= 2\int_0^{2\pi} e^{int} d\mu_{11}(t) + 2\int_0^{2\pi} e^{int} d\mu_{12}(t)j, \quad n \in \mathbb{Z}, \end{aligned}$$

where the last line follows from $d\mu_{11}(t) = d\bar{\mu}_{22}(2\pi - t)$ and $d\mu_{12}(t) = -d\mu_{12}(2\pi - t)^T$. If we put $\nu = \mu_{11} + \mu_{12}j$, then ν is a q-positive measure that satisfies (13.4).

Conversely, suppose $\nu = \nu_1 + \nu_2 j$ is a q-positive measure on $[0, 2\pi]$ and put

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}.$$

Since ν is q-positive,

$$\mu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3 \end{pmatrix}, \quad \text{where } d\nu_3(t) = d\bar{\nu}_1(2\pi - t),\ t \in [0, 2\pi),$$

is a positive $\mathbb{C}^{2s\times 2s}$-valued measure on $[0, 2\pi]$ and

$$d\nu_2(t) = -d\nu_2(2\pi - t)^T, \quad t \in [0, 2\pi).$$

Since μ is a positive $\mathbb{C}^{2s\times 2s}$-valued measure, $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence, where

$$R(n) := \int_0^{2\pi} e^{int} d\mu(t), \quad n \in \mathbb{Z}.$$

Moreover, $R(n)$ can be written in the form

$$R(n) = \begin{pmatrix} r_1(n) & r_2(n) \\ -\overline{r_2(n)} & \overline{r_1(n)} \end{pmatrix}, \quad n \in \mathbb{Z},$$

where

$$r_1(n) = \int_0^{2\pi} e^{int} d\nu_1(t), \quad n \in \mathbb{Z};$$
$$r_2(n) = \int_0^{2\pi} e^{int} d\nu_2(t), \quad n \in \mathbb{Z}.$$

Thus, $R(n) = \chi\, r(n)$, where

$$r(n) = r_1(n) + r_2(n) j = \int_0^{2\pi} e^{int} d\nu(t).$$

Since $(R(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{C}^{2s\times 2s}$-valued sequence, we get that $(r(n))_{n\in\mathbb{Z}}$ is a positive definite $\mathbb{H}^{s\times s}$-valued sequence.

Finally, suppose that the q-positive measure ν were not unique, i.e., that there existed $\tilde{\nu}$ such that $\tilde{\nu} \neq \nu$ and

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t) = \int_0^{2\pi} e^{int} d\tilde{\nu}(t), \quad n \in \mathbb{Z}.$$

Write $\nu = \nu_1 + \nu_2 j$ and $\tilde{\nu} = \tilde{\nu}_1 + \tilde{\nu}_2 j$ as in Remark 13.1.4. If we consider $R(n) = \chi\, r(n), n \in \mathbb{Z}$, then it follows from Theorem 13.1.1 that $\nu_1 = \tilde{\nu}_1$ and $\nu_2 = \tilde{\nu}_2$ and hence that $\nu = \tilde{\nu}$, a contradiction. □

Remark 13.1.7. For every $i \in \mathbb{S}$, there exists $j \in \mathbb{S}$ such that $ij = -ji$. Thus, $\mathbb{H} = \mathbb{C}_i \oplus \mathbb{C}_i j$, and we may rewrite (13.4) as

$$r(n) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}, \tag{13.6}$$

where $\nu = \nu_1 + \nu_2 j$ is a q-positive measure (in the sense that

$$\mu = \begin{pmatrix} \nu_1 & \nu_2 \\ \nu_2^* & \nu_3 \end{pmatrix}$$

is positive). Here $\nu_3(t) = \nu_1(2\pi - t)$.

For our purpose the scalar case will be important.

13.2 Preliminaries for the Spectral Resolution

We start with a preliminary result.

Lemma 13.2.1. *Let U be a unitary operator on $\mathcal{H}$ and let $r_x(n) = \langle U^n x, x\rangle$ for $x \in \mathcal{H}$. Then $r_x = (r_x(n))_{n\in\mathbb{Z}}$ is an $\mathbb{H}$-valued positive definite sequence.*

Proof. If $\{p_0, \ldots, p_N\} \subset \mathbb{H}$, then

$$\begin{aligned}
\sum_{m,n=0}^{N} \bar{p}_m r_x(n-m) p_n &= \sum_{m,n=0}^{N} \bar{p}_m \langle U^{n-m} x, x\rangle p_n \\
&= \sum_{m,n=0}^{N} \langle U^{n-m} x p_n, x p_m\rangle \\
&= \sum_{m,n=0}^{N} \langle U^n x p_n, U^m x p_m\rangle \\
&= \langle \sum_{n=0}^{N} U^n x p_n, \sum_{m=0}^{N} U^m x p_m\rangle \\
&= \left\| \sum_{n=0}^{N} U^n x p_n \right\|^2 \geq 0.
\end{aligned}$$

Thus, r_x is a positive definite $\mathbb{H}$-valued sequence. □

Let r_x be as in Lemma 13.2.1. It follows from Theorem 13.1.6 that there exists a unique q-positive measure $d\nu_x$ such that

$$r_x(n) = \langle U^n x, x\rangle = \int_0^{2\pi} e^{int} d\nu_x(t), \quad n \in \mathbb{Z}. \tag{13.7}$$

One can check that

$$\begin{aligned} 4\langle U^n x, y\rangle = {} & \langle U^n(x+y), x+y\rangle - \langle U^n(x-y), x-y\rangle \qquad (13.8)\\ & + i\langle U^n(x+yi), x+yi\rangle \\ & - i\langle U^n(x-yi), x-yi\rangle + i\langle U^n(x-yj), x-yj\rangle k \qquad (13.9)\\ & - i\langle U^n(x+yj), x+yj\rangle k \\ & + \langle U^n(x+yk), x+yk\rangle k - \langle U^n(x-yk), x-yk\rangle k, \qquad (13.10) \end{aligned}$$

and hence letting

$$\begin{aligned} 4\nu_{x,y} := {} & \nu_{x+y} - \nu_{x-y} + i\nu_{x+yi} - i\nu_{x-yi} + i\nu_{x-yj}k - i\nu_{x+yj}k \\ & + \nu_{x+yk}k - \nu_{x-yk}k, \qquad (13.11) \end{aligned}$$

then

$$\langle U^n x, y\rangle = \int_0^{2\pi} e^{int} d\nu_{x,y}(t), \quad x, y \in \mathcal{H} \text{ and } n \in \mathbb{Z}. \qquad (13.12)$$

Theorem 13.2.2. *The $\mathbb{H}$-valued measures $\nu_{x,y}$ defined on $\mathbf{B}([0,2\pi])$ enjoy the following properties:*

(i) $\nu_{x\alpha+y\beta,z} = \nu_{x,z}\alpha + \nu_{y,z}\beta, \quad \alpha, \beta \in \mathbb{H}$;

(ii) $\nu_{x,y\alpha+z\beta} = \bar{\alpha}\nu_{x,y} + \bar{\beta}\nu_{x,z}, \quad \alpha, \beta \in \mathbb{C}_i$;

(iii) $\nu_{x,y}([0,2\pi]) \leq \|x\|\|y\|$;

where $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{H}$.

Proof. Formula (13.12) yields

$$\begin{aligned} \int_0^{2\pi} e^{int} d\nu_{x\alpha+y\beta,z}(t) &= \langle U^n x, z\rangle\alpha + \langle U^n y, z\rangle\beta \\ &= \int_0^{2\pi} e^{int}(d\nu_{x,z}(t)\alpha + d\nu_{y,z}(t)\beta), \quad n \in \mathbb{Z}. \end{aligned}$$

The uniqueness of the q-positive measure proved in Theorem 13.1.6 allows us to conclude that

$$\nu_{x\alpha+y\beta,z}(t) = \nu_{x,z}(t)\alpha + \nu_{y,z}(t)\beta,$$

and hence we have proved (i). Property (ii) is proved in a similar fashion, observing that $\bar{\alpha}$, $\bar{\beta}$ commute with e^{int}.

If $n = 0$ in (13.12), then

$$\langle x, y\rangle = \int_0^{2\pi} d\nu_{x,y}(t) = \nu_{x,y}([0,2\pi]),$$

and thus we can use an analogue of the Cauchy–Schwarz inequality (see Lemma 5.6 in [33]) to obtain

$$\nu_{x,y}([0,2\pi]) \leq \|x\|\|y\|,$$

and hence we have proved (iii). □

Remark 13.2.3. In contrast to the classical complex Hilbert space setting, $\nu_{x,y}$ need not equal $\bar{\nu}_{y,x}$.

It follows from statements (i), (ii), and (iii) in Theorem 13.2.2 that $\phi(x) = \nu_{x,y}(\sigma)$, where $y \in \mathcal{H}$ and $\sigma \in \mathbf{B}([0, 2\pi])$ are fixed, is a continuous right linear functional. Moreover, an analogue of the Riesz representation theorem (see Theorem 6.1 in [33] or Theorem 7.6 in [47]) gives that corresponding to every $x \in \mathcal{H}$, there exists a uniquely determined vector $w \in \mathcal{H}$ such that

$$\phi(x) = \langle x, w \rangle,$$

i.e., $\nu_{x,y}(\sigma) = \langle x, w \rangle$. Use (i) and (ii) in Theorem 13.2.2 to deduce that $w = E(\sigma)^* y$. The uniqueness of E follows readily from the construction. Thus, we have

$$\nu_{x,y}(\sigma) = \langle E(\sigma)x, y \rangle, \quad x, y \in \mathcal{H} \text{ and } \sigma \in \mathbf{B}([0, 2\pi]), \tag{13.13}$$

whence

$$\langle U^n x, y \rangle = \int_0^{2\pi} e^{int} \langle dE(t)x, y \rangle. \tag{13.14}$$

To prove the main properties of the operator E we need a uniqueness result on quaternionic measures that is a corollary of the following:

Theorem 13.2.4. *Let μ and ν be $\mathbb{C}$-valued measures on $[0, 2\pi]$. If*

$$r(n) = \int_0^{2\pi} e^{int} d\mu(t) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}, \tag{13.15}$$

then $\mu = \nu$.

Proof. See, e.g., Theorem 1.9.5 in [186]. □

Theorem 13.2.5. *Let μ and ν be $\mathbb{H}$-valued measures on $[0, 2\pi]$. If*

$$r(n) = \int_0^{2\pi} e^{int} d\mu(t) = \int_0^{2\pi} e^{int} d\nu(t), \quad n \in \mathbb{Z}, \tag{13.16}$$

then $\mu = \nu$.

Proof. Write $r(n) = r_1(n) + r_2(n)j$, $\mu = \mu_1 + \mu_2 j$, and $\nu = \nu_1 + \nu_2 j$, where $r_1(n), r_2(n) \in \mathbb{C}$ and $\mu_1, \mu_2, \nu_1, \nu_2$ are $\mathbb{C}$-valued measures on $[0, 2\pi]$. It follows from (13.16) that

$$r_1(n) = \int_0^{2\pi} e^{int} d\mu_1(t) = \int_0^{2\pi} e^{int} d\nu_1(t), \quad n \in \mathbb{Z},$$

and

$$r_2(n) = \int_0^{2\pi} e^{int} d\mu_2(t) = \int_0^{2\pi} e^{int} d\nu_2(t), \quad n \in \mathbb{Z}.$$

Use Theorem 13.2.4 to conclude that $\mu_1 = \nu_1$, $\mu_2 = \nu_2$ and hence that $\mu = \nu$. □

Theorem 13.2.6. *The operator E given in (13.13) enjoys the following properties:*

(i) $\|E(\sigma)\| \le 1$;

(ii) $E(\emptyset) = 0$ *and* $E([0, 2\pi]) = \mathcal{I}$;

(iii) *If* $\sigma \cap \tau = \emptyset$, *then* $E(\sigma \cup \tau) = E(\sigma) + E(\tau)$;

(iv) $E(\sigma \cap \tau) = E(\sigma)E(\tau)$;

(v) $E(\sigma)^2 = E(\sigma)$;

(vi) $E(\sigma)$ *commutes with* U *for all* $\sigma \in \mathbf{B}([0, 2\pi])$.

Proof. Use (13.13) with $y = E(\sigma)x$ and (iii) in Theorem (13.2.2) to obtain

$$\|E(\sigma)x\|^2 \le \|x\|\|E(\sigma)x\|,$$

whence we have shown (i). The first part of property (ii) follows directly from the fact that $\nu_{x,y}(\emptyset) = 0$. The last part follows from (13.14) when $n = 0$. Statement (iii) follows easily from the additivity of the measure $\nu_{x,y}$.

We will now prove property (iv). It follows from (13.14) that

$$\begin{aligned}\langle U^{n+m}x, y\rangle &= \int_0^{2\pi} e^{int}e^{imt}\langle dE(t)x, y\rangle \\ &= \langle U^n(U^m x), y\rangle \\ &= \int_0^{2\pi} e^{int} d\langle E(t)U^m x, y\rangle.\end{aligned}$$

Using the uniqueness in Theorem 13.2.5 we obtain

$$e^{imt} d\langle E(t)x, y\rangle = \langle dE(t)U^m x, y\rangle,$$

and hence denoting by $\mathbf{1}_\sigma$ the characteristic function of the set σ, we have

$$\int_0^{2\pi} \mathbf{1}_\sigma(t)e^{imt}\langle dE(t)x, y\rangle = \langle E(\sigma)U^m x, y\rangle.$$

But

$$\int_0^{2\pi} \mathbf{1}_\sigma(t)e^{imt}\langle dE(t)x, y\rangle = \langle U^k x, E(\sigma)^* y\rangle = \int_0^{2\pi} e^{imt} d\langle E(t)x, E(\sigma)^* y\rangle.$$

Using the uniqueness in Theorem 13.2.5 once more, we get

$$\mathbf{1}_\sigma(t) d\langle E(t)x, y\rangle = \langle dE(t)x, E(\sigma)^* y\rangle$$

and hence

$$\int_0^{2\pi} \mathbf{1}_\tau(t)\mathbf{1}_\sigma(t)\langle dE(t)x, y\rangle = \langle E(t)x, E(\sigma)^* y\rangle$$

and thus

$$\langle E(\sigma \cap \tau)x, y\rangle = \langle E(\sigma)E(\tau)x, y\rangle.$$

Property (v) is obtained from (iv) by letting $\sigma = \tau$.

Finally, since U is unitary, one can check that

$$\langle U(x \pm U^*y), x \pm U^*y\rangle = \langle U(Ux \pm y), Ux \pm y\rangle,$$

and hence from (13.12) and the uniqueness in Theorem 13.2.5 we obtain $\nu_{x \pm U^*y} = \nu_{Ux \pm y}$. Similarly,

$$\nu_{x \pm U^*yi} = \nu_{Ux \pm yi},$$
$$\nu_{x \pm U^*yj} = \nu_{Ux \pm yj},$$

and

$$\nu_{x \pm U^*yk} = \nu_{Ux \pm yk}.$$

It follows from (13.11) that

$$\nu_{x, U^*y} = \nu_{Ux, y}.$$

Now use (13.13) to obtain

$$\langle E(\sigma)x, U^*y\rangle = \langle E(\sigma)Ux, y\rangle,$$

i.e.,

$$\langle UE(\sigma)x, y\rangle = \langle E(\sigma)Ux, y\rangle, \quad x, y \in \mathcal{H}. \qquad \square$$

Given any quaternionic Hilbert space $\mathcal{H}$, there exists a subspace $\mathcal{M} \subset \mathcal{H}$ on $\mathbb{C}$ such that for every $x \in \mathcal{H}$ we have

$$x = x_1 + x_2 j, \quad x_1, x_2 \in \mathcal{M}.$$

Theorem 13.2.7. *Let U be a unitary operator on a quaternionic Hilbert space $\mathcal{H}$ and let E be the corresponding operator given by (13.13). E is self-adjoint if and only if $U : \mathcal{M} \to \mathcal{M}$, where $\mathcal{M}$ is as above.*

Proof. If $E = E^*$, then it follows from (13.13) that $\nu_{x,y} = \bar{\nu}_{y,x}$ for all $x, y \in \mathcal{H}$. In particular, we get $\nu_{x,x} = \bar{\nu}_{x,x}$, i.e.,

$$\nu_x = \bar{\nu}_x, \quad x \in \mathcal{H}. \tag{13.17}$$

Since ν_x is a q-positive measure, we may write $\nu_x = \alpha_x + \beta_x j$, where α_x is a positive Borel measure on $[0, 2\pi]$ and β_x is a complex Borel measure on $[0, 2\pi]$. It follows from (13.17) that

$$\beta_x = -\beta_x,$$

i.e., $\beta_x = 0$. Thus, we may make use of the spectral theorem for unitary operators on a complex Hilbert space (see, e.g., Section 31.7 in [163]) to deduce that $U : \mathcal{M} \to \mathcal{M}$. Conversely, if $U : \mathcal{M} \to \mathcal{M}$, then the spectral theorem for unitary operators on a complex Hilbert space yields that $E = E^*$. $\square$

If $U : \mathbb{H}^n \to \mathbb{H}^n$ is unitary, then (13.14) and Theorem 13.2.6 assert that

$$U = \sum_{a=1}^{n} e^{i\theta_a} P_a, \tag{13.18}$$

where $\theta_1, \dots, \theta_n \in [0, 2\pi]$ and $P_1, \dots, P_n$ are oblique projections (i.e., $(P_a)^2 = P_a$ but $(P_a)^*$ need not equal P_a). Corollary 6.2 in Zhang [199] asserts, in particular, the existence of $V : \mathbb{H}^n \to \mathbb{H}^n$ that is unitary and $\theta_1, \dots, \theta_n \in [0, 2\pi]$ such that

$$U = V^* \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) V. \tag{13.19}$$

In the following remark we will explain how (13.18) and (13.19) are consistent.

Remark 13.2.8. Let $U : \mathbb{H}^n \to \mathbb{H}^n$ be unitary. Let V and $\theta_1, \dots, \theta_n$ be as above. If we let $e_a = (0, \dots, 0, 1, 0, \dots, 0)^T \in \mathbb{H}^n$, where the 1 is the ath position, then we can rewrite (13.19) as

$$U = \sum_{a=1}^{n} V^* e^{i\theta_a} e_a e_a^* V.$$

Note that $V^* e^{i\theta_a} e_a e_a^* V = e^{i\theta_a} V^* e_a e_a^* V$ if and only if $V : \mathbb{C}^n \to \mathbb{C}^n$. In this case $U : \mathbb{C}^n \to \mathbb{C}^n$ and

$$U = \sum_{a=1}^{n} e^{i\theta_a} P_a,$$

where P_a denotes the orthogonal projection given by $V^* e^{i\theta_a} e_a e_a^* V$.

Remark 13.2.9. Observe that in the proof of the spectral theorem for U^n we have taken the imaginary units i, j, k for the quaternions and we have determined spectral measures $\langle dE(t)x, y\rangle$ that are supported on the unit circle in $\mathbb{C}_i$. If one uses other orthogonal units i', j', and $k' \in \mathbb{S}$ to represent quaternions, then the spectral measures are supported on the unit circle in $\mathbb{C}_{i'}$.

Observe that (13.14) provides a vehicle to define a functional calculus for unitary operators on a quaternionic Hilbert space. For a continuous $\mathbb{H}$-valued function f on the unit circle, which will be approximated by the polynomials $\sum_k e^{ikt} a_k$. We will consider a subclass of continuous quaternionic-valued functions defined as follows, see [142]: It is important to note that every polynomial of the form $P(u + jv) = \sum_{k=0}^{n} (u + jv)^n a_n$, $a_n \in \mathbb{H}$ is a slice continuous function in the whole of $\mathbb{H}$. A trigonometric polynomial of the form $P(e^{jt}) = \sum_{m=-n}^{n} e^{jmt} a_m$ is a slice continuous function on $\partial\mathbb{B}$, where $\mathbb{B}$ denotes the unit ball of quaternions.

Let us now denote by $\mathcal{PS}(\sigma_S(T))$ the set of slice continuous functions $f(u + iv) = \alpha(u, v) + i\beta(u, v)$, where α, β are polynomials in the variables u, v.

In the sequel we will work in the complex plane $\mathbb{C}_i$ and we denote by $\mathbb{T}_i$ the boundary of $\mathbb{B} \cap \mathbb{C}_i$. Any other choice of an imaginary unit in the unit sphere $\mathbb{S}$ will provide an analogous result.

Remark 13.2.10. For every $i \in \mathbb{S}$, there exists $j \in \mathbb{S}$ such that $ij = -ji$. Bearing in mind Remark 13.1.7, we can construct $\nu_{x,y}^{(j)}$ such that (13.12) can also be written as

$$\langle U^n x, y\rangle = \int_0^{2\pi} e^{int} d\nu_{x,y}^{(j)}(t), \quad x, y \in \mathcal{H} \text{ and } n \in \mathbb{Z}. \tag{13.20}$$

Consequently, (13.14) can be written as

$$\langle U^n x, y\rangle = \int_0^{2\pi} e^{int} \langle E_j(t)x, y\rangle, \tag{13.21}$$

where E_j is given by

$$\nu_{x,y}^{(j)}(\sigma) = \langle E_j(\sigma)x, y\rangle, \quad x, y \in \mathcal{H} \text{ and } \sigma \in \mathrm{B}(\mathbb{T}_i).$$

Moreover, the E_j satisfy properties (i)–(v) listed in Theorem 13.2.6.

13.3 Further Properties of Quaternionic Riesz Projectors

An axially symmetric set $\sigma \subseteq \sigma_S(T)$ that is both open and closed in $\sigma_S(T)$ in its relative topology, is called an S-spectral set. Denote by Ω_σ an axially symmetric domain that contains the spectral set σ but not any other points of the S-spectrum. We recall the Riesz projectors

$$\mathcal{P}(\sigma) = \frac{1}{2\pi} \int_{\partial(\Omega_\sigma \cap \mathbb{C}_j)} S_L^{-1}(s, T) ds_j$$

and the fact that $\mathcal{P}(\sigma)$ can be given using the right S-resolvent operator $S_R^{-1}(s, T)$, that is,

$$\mathcal{P}(\sigma) = \frac{1}{2\pi} \int_{\partial(\Omega_\sigma \cap \mathbb{C}_j)} ds_j S_R^{-1}(s, T).$$

We have the following properties.

Theorem 13.3.1. *Let T be a quaternionic linear operator. Then the family of operators $\mathcal{P}(\sigma)$ has the following properties:*

(i) $(\mathcal{P}(\sigma))^2 = \mathcal{P}(\sigma)$;

(ii) $T\mathcal{P}(\sigma) = \mathcal{P}(\sigma)T$;

(iii) $\mathcal{P}(\sigma_S(T)) = \mathcal{I}$;

(iv) $\mathcal{P}(\emptyset) = 0$;

(v) $\mathcal{P}(\sigma \cup \delta) = \mathcal{P}(\sigma) + \mathcal{P}(\delta); \quad \sigma \cap \delta = \emptyset$;

(vi) $\mathcal{P}(\sigma \cap \delta) = \mathcal{P}(\sigma)\mathcal{P}(\delta)$.

Proof. Properties (i) and (ii) are proved in Theorem 4.1.5. Property (iii) follows from the quaternionic functional calculus, since

$$T^m = \frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j\ s^m, \quad m\in\mathbb{N}_0,$$

for $\sigma_S(T)\subset\Omega$, which for $m=0$ gives

$$\mathcal{I} = \frac{1}{2\pi}\int_{\partial(\Omega\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j.$$

Property (iv) is a consequence of the functional calculus as well.

Property (v) follows from

$$\begin{aligned}\mathcal{P}(\sigma\cup\delta) &= \frac{1}{2\pi}\int_{\partial(\Omega_{\sigma\cup\delta}\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j\\ &= \frac{1}{2\pi}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j + \frac{1}{2\pi}\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j\\ &= \mathcal{P}(\sigma)+\mathcal{P}(\delta).\end{aligned}$$

To prove (vi), assume that $\sigma\cap\delta\neq\emptyset$, and for simplicity set

$$\mathcal{Q}_s(p)^{-1} := (p^2-2\mathrm{Re}(s)p+|s|^2)^{-1}, \quad p\notin[s],$$

and consider

$$\begin{aligned}\mathcal{P}(\sigma)\mathcal{P}(\delta) &= \frac{1}{(2\pi)^2}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j S_R^{-1}(s,T)\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} S_L^{-1}(p,T)dp_j\\ &= \frac{1}{(2\pi)^2}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [S_R^{-1}(s,T)-S_L^{-1}(p,T)]p\mathcal{Q}_s(p)^{-1}dp_j\\ &\quad - \frac{1}{(2\pi)^2}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} \overline{s}[S_R^{-1}(s,T)-S_L^{-1}(p,T)]\mathcal{Q}_s(p)^{-1}dp_j,\end{aligned}$$

where we have used the S-resolvent equation (see Theorem 3.1.15). We rewrite the above relation as

$$\begin{aligned}\mathcal{P}(\sigma)\mathcal{P}(\delta) &= -\frac{1}{(2\pi)^2}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s}S_R^{-1}(s,T)-S_R^{-1}(s,T)p]\mathcal{Q}_s(p)^{-1}dp_j\\ &\quad + \frac{1}{(2\pi)^2}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s}S_L^{-1}(p,T)-S_L^{-1}(p,T)p]\mathcal{Q}_s(p)^{-1}dp_j\\ &:= \mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\mathcal{J}_1 &= -\frac{1}{(2\pi)^2}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j \int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s}S_R^{-1}(s,T) - S_R^{-1}(s,T)p]\mathcal{Q}_s(p)^{-1}dp_j \\ &= \frac{1}{2\pi}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} ds_j S_R^{-1}(s,T), \quad \text{for } s \in \Omega_\delta \cap \mathbb{C}_j \\ &= \frac{1}{2\pi}\int_{\partial(\Omega_\sigma\cap\mathbb{C}_j)} S_L^{-1}(s,T)ds_j, \quad \text{for } s \in \Omega_\delta \cap \mathbb{C}_j,\end{aligned}$$

while $\mathcal{J}_1 = 0$ when $s \notin \Omega_\delta \cap \mathbb{C}_j$, since

$$\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} [\overline{s}S_R^{-1}(s,T) - S_R^{-1}(s,T)p]\mathcal{Q}_s(p)^{-1}dp_j = 0.$$

Similarly, one can show that

$$\mathcal{J}_2 = \frac{1}{2\pi}\int_{\partial(\Omega_\delta\cap\mathbb{C}_j)} S_L^{-1}(p,T)dp_j, \quad \text{for } p \in \Omega_\sigma \cap \mathbb{C}_j,$$

while $\mathcal{J}_2 = 0$ when $p \notin \Omega_\sigma \cap \mathbb{C}_j$. The integrals $\mathcal{J}_1$, $\mathcal{J}_2$ are either both zero or both nonzero, so with a change of variable we get

$$\mathcal{J}_1 + \mathcal{J}_2 = \frac{1}{2\pi}\int_{\partial(\Omega_{\sigma\cap\delta}\cap\mathbb{C}_j)} S_L^{-1}(r,T)dr_j = \mathcal{P}(\sigma\cap\delta). \qquad \square$$

We recall that if U is a unitary operator on $\mathcal{H}$, then the S-spectrum of U belongs to the unit sphere of the quaternions; see Theorem 9.2.7. We denote the Borel sets in $[0, 2\pi]$ by $\mathbf{B}([0, 2\pi])$.

Lemma 13.3.2. *Let $x, y \in \mathcal{H}$ and let $\mathcal{P}(\sigma)$ be the projector associated with the unitary operator U. We define*

$$m_{x,y}(\sigma) := \langle \mathcal{P}(\sigma)x, y\rangle, \quad x,\, y \in \mathcal{H},\ \sigma \in \mathbf{B}([0, 2\pi]).$$

Then the $\mathbb{H}$-valued measures $m_{x,y}$ defined on $\mathbf{B}([0, 2\pi])$ enjoy the following properties:

(i) $m_{x\alpha+y\beta,z} = m_{x,z}\alpha + m_{y,z}\beta$;

(ii) $m_{x,y\alpha+z\beta} = \overline{\alpha}m_{x,y} + \overline{\beta}m_{x,z}$;

(iii) $m_{x,y}([0, 2\pi]) \leq \|x\|\|y\|$;

where $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{H}$.

Proof. Properties (i) and (ii) follow from the properties of the quaternionic scalar product, while (iii) follows from Property (iii) in Theorem 13.3.1 and the Cauchy–Schwarz inequality. $\square$

13.4 The Spectral Resolution

We are now in a position to prove the spectral theorem for quaternionic unitary operators.

Theorem 13.4.1 (The spectral theorem for quaternionic unitary operators)**.** *Let U be a unitary operator on a right linear quaternionic Hilbert space $\mathcal{H}$. Let $i, j \in \mathbb{S}$, i orthogonal to j. Then there exists a unique spectral measure E_j defined on the Borel sets of $\mathbb{T}_i$ such that for every slice continuous function $f \in \mathcal{S}(\sigma_S(U))$, we have*

$$f(U) = \int_0^{2\pi} f(e^{it}) dE_j(t).$$

Proof. Let us consider a polynomial $P(t) = \sum_{m=-n}^{n} e^{imt} a_m$ defined on $\mathbb{T}_i$. Then using (13.21), we have

$$\langle U^m x, y \rangle = \int_0^{2\pi} e^{imt} \langle dE_j(t)x, y \rangle, \quad x, y, \in \mathcal{H}.$$

By linearity, we can define

$$\langle P(U)x, y \rangle = \int_0^{2\pi} P(e^{it}) \langle dE_j(t)x, y \rangle, \quad x, y, \in \mathcal{H}.$$

The map $\Psi : \mathcal{PS}(\sigma_S(U)) \to \mathcal{H}$ defined by $\psi_U(P) = P(U)$ is $\mathbb{R}$-linear. By fixing a basis for $\mathbb{H}$, e.g., the basis $1, i', j', k'$, each slice continuous function f can be decomposed using intrinsic functions, i.e., $f = f_0 + f_1 i' + f_2 j' + f_3 k'$ with $f_\ell \in \mathcal{S}_{\mathbb{R}}(\sigma_S(U))$, $\ell = 0, \dots, 3$. For these functions the spectral mapping theorem holds; thus $f_\ell(\sigma_S(U)) = \sigma_S(f_\ell(U))$, and so $\|f_\ell(U)\| = \|f_\ell\|_\infty$. The map ψ is continuous, and so there exists $C > 0$, which does not depend on ℓ, such that

$$\|P(U)\|_{\mathcal{H}} \leq C \max_{t \in \sigma_S(U)} |P(t)|.$$

A slice continuous function $f \in \mathcal{S}(\sigma_S(U))$ is defined on an axially symmetric subset $K \subseteq \mathbb{T}$, and thus it can be written as a function $f(e^{jt}) = \alpha(\cos t, \sin t) + j\beta(\cos t, \sin t)$. By fixing a basis of $\mathbb{H}$, e.g., $1, i', j', k'$, f can be decomposed into four slice continuous intrinsic functions $f_\ell(\cos t, \sin t) = \alpha_\ell(\cos t, \sin t) + j\beta_\ell(\cos t, \sin t)$, $\ell = 0, \dots, 3$, such that $f = f_0 + f_1 i' + f_2 j' + f_3 k'$.

By the Weierstrass approximation theorem for trigonometric polynomials, see, e.g., Theorem 8.15 in [183], each function f_ℓ can be approximated by a sequence of polynomials

$$\tilde{R}_{\ell n} = \tilde{a}_{\ell n}(\cos t, \sin t) + j\tilde{b}_{\ell n}(\cos t, \sin t),$$

$\ell = 0, \dots, 3$, which tend uniformly to f_ℓ. These polynomials do not necessarily belong to the class of the continuous slice functions, since $\tilde{a}_{\ell n}, \tilde{b}_{\ell n}$ do not satisfy, in general, the even and odd conditions of slice continuous functions. However, by setting

$$a_{\ell n}(u,v) = \frac{1}{2}(\tilde{a}_{\ell n}(u,v) + \tilde{a}_{\ell n}(u,-v)),$$
$$b_{\ell n}(u,v) = \frac{1}{2}(\tilde{b}_{\ell n}(u,-v) - \tilde{b}_{\ell n}(u,v)),$$

we obtain that the sequence of polynomials $a_{\ell n} + j' b_{\ell n}$ still converges to f_ℓ, $\ell = 0,\dots,3$. By putting $R_{\ell n} = a_{\ell n}(\cos t, \sin t) + j b_{\ell n}(\cos t, \sin t)$, $\ell = 0,\dots,3$, and $R_n = R_{0n} + R_{1n} i' + R_{2n} j' + R_{3n} k'$ we have a sequence of slice continuous polynomials $R_n(e^{jt})$ converging to $f(e^{jt})$ uniformly on $\mathbb{R}$.

By the previous discussion, $\{R_n(U)\}$ is a Cauchy sequence in the space of bounded linear operators, since

$$\|R_n(U) - R_m(U)\| \le C \max_{t\in\sigma_S(U)} |R_n(t) - R_m(t)|;$$

so $R_n(U)$ has a limit, which we denote by $f(U)$. □

Remark 13.4.2. Fix $j \in \mathbb{S}$. It is worth pointing out that $f(u + jv) = (u + jv)^{-1}$ is an intrinsic function on $\mathbb{C}_j \cap \partial\mathbb{B}$, where $\partial\mathbb{B} = \{q \in \mathbb{H} : |q| = 1\}$, since

$$f(u + jv) = \frac{u}{u^2 + v^2} + \left(\frac{-v}{u^2 + v^2}\right) j.$$

Thus, using Theorem 13.4.1, we may write

$$U^{-1} = \int_0^{2\pi} e^{-it} dE_j(t) \tag{13.1}$$

and

$$U = \int_0^{2\pi} e^{it} dE_j(t). \tag{13.2}$$

We are now ready to prove the following fundamental result, which shows the relation between the spectral measures and the S-spectrum.

Theorem 13.4.3. *Fix $i, j \in \mathbb{S}$, with i orthogonal to j. Let U be a unitary operator on a right linear quaternionic Hilbert space $\mathcal{H}$ and let $E(t) = E_j(t)$ be its spectral measure. Assume that $\sigma_S^0(U) \cap \mathbb{C}_i$ is contained in the arc of the unit circle in $\mathbb{C}_i$ with endpoints t_0 and t_1. Then*

$$\mathcal{P}(\sigma_S^0(U)) = E(t_1) - E(t_0).$$

Proof. The spectral theorem implies that the operator $S_R^{-1}(s, U)$ can be written as

$$S_R^{-1}(s, U) = \int_0^{2\pi} S_R^{-1}(e^{it}, s) dE(t).$$

The Riesz projector is given by

$$\mathcal{P}(\sigma_S^0(U)) = \frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i S_R^{-1}(s,U),$$

where Ω_0 is an open set containing $\sigma_S^0(U)$ such that $\partial(\Omega_0\cap\mathbb{C}_i)$ is a smooth closed curve in $\mathbb{C}_i$. Write

$$\mathcal{P}(\sigma_S^0(U)) = \frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i \Big(\int_0^{2\pi} S_R^{-1}(e^{it},s)dE(t)\Big)$$

and use Fubini's theorem to obtain

$$\mathcal{P}(\sigma_S^0(U)) = \int_0^{2\pi} \Big(\frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i S_R^{-1}(e^{it},s)\Big)dE(t).$$

It follows from the Cauchy formula that

$$\frac{1}{2\pi}\int_{\partial(\Omega_0\cap\mathbb{C}_i)} ds_i S_R^{-1}(e^{it},s) = \mathbf{1}_{[t_0,t_1]},$$

where $\mathbf{1}_{[t_0,t_1]}$ is the characteristic function of the set $[t_0,t_1]$, and thus the statement follows, since

$$\mathcal{P}(\sigma_S^0(U)) = \int_0^{2\pi} \mathbf{1}_{[t_0,t_1]} dE(t) = E(t_1) - E(t_2). \qquad \square$$

We will close by establishing a connection between the spectral resolutions for a unitary operator presented in Theorem 11.2.1 and Theorem 13.4.1. Let $U \in \mathcal{B}(\mathcal{H})$ be unitary. Since $U \in \mathcal{B}(\mathcal{H})$ is normal, we may write

$$U = A + JB,$$

where A, J, and B are as in Theorem 9.3.5. Thus, Theorem 11.2.1 asserts the existence of a spectral measure E (in the usual sense) on $\Omega := [0,\pi]\cap\sigma_S(U)$ such that if $n\in\mathbb{Z}$, then

$$\langle U^n x, y\rangle = \int_\Omega \cos(n\theta) d\langle E(\theta)x,y\rangle + \int_\Omega \sin(n\theta) d\langle JE(\theta)x,y\rangle, \quad x,y\in\mathcal{H}. \tag{13.3}$$

On the other hand, Theorem 13.4.1 asserts the existence of a $\mathcal{B}(\mathcal{H})$-valued measure F that satisfies most of the properties of a spectral measure (see Theorem 13.2.6) such that if $n\in\mathbb{Z}$, then

$$\langle U^n x, y\rangle = \int_0^{2\pi} e^{in\theta} d\langle F(\theta)x,y\rangle, \quad x,y\in\mathcal{H}. \tag{13.4}$$

Consequently, if we let $d\nu_x(\theta) := d\langle E(\theta)x, x\rangle$ and $d\mu_x(\theta) := d\langle F(\theta)x, x\rangle$, then $d\nu_x$ is a positive measure and $d\mu_x := d\mu_x^{(0)} + d\mu_x^{(1)} j$ is a q-positive measure (and hence $d\mu_x^{(0)}$ is a positive measure). Now (13.3) implies that

$$\frac{1}{2}\langle (U^n + U^{*n})x, x\rangle = \int_0^{\pi} \cos(n\theta) d\nu_x(\theta),$$

while (13.4) implies that

$$\frac{1}{2}\langle (U^n + U^{*n})x, x\rangle = \int_0^{2\pi} \cos(n\theta) d\mu_x^{(0)}(\theta).$$

Since $d\mu_x^{(0)}$ and $d\nu_x$ are positive measures, the uniqueness assertion in Theorem 13.1.1 forces $d\mu_x^{(0)} = d\nu_x$ and hence $d\langle E(\theta)x, x\rangle = \mathrm{Re}\langle F(\theta)x, x\rangle$.

13.5 Comments and Remarks

Theorem 13.1.6 is taken from [16], and it helped give rise to a spectral theorem for unitary operators based on the S-spectrum in [14]. In addition, Theorem 13.1.6 can be used to generate a quaternionic analogue of the Herglotz representation on a slice (see Theorem 8.1 in [16]). More precisely, if $f : \mathbb{B} \to \mathbb{H}$ is slice hyperholomorphic with $\mathrm{Re}(f(p)) \geq 0$ for all $p \in \mathbb{B} := \{p \in \mathbb{H} : |p| < 1\}$ and $i, j \in \mathbb{S}$ with i and j orthogonal, then there exists a $\mathbb{C}_j$-valued measure $d\mu_j(t) = d\mu_1(t) + d\mu_2(t)j$ of finite total variation with μ_1 positive and μ_2 signed such that the restriction $f_j(z) = f|_{\mathbb{C}_j} = F(z) + G(z)j$ admits the representation

$$f_j(z) = i[\mathrm{Im}F(0) + \mathrm{Im}G(0)j] + \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu_j(t). \tag{13.5}$$

A half-space analogue of (13.5) was treated in [9] (albeit with stronger conditions on f and the corresponding measure).

Chapter 14

Spectral Integration in the Quaternionic Setting

Before we begin the study of quaternionic spectral operators, we discuss in this chapter spectral integration in the quaternionic setting. There have existed several different approaches to this topic in the literature, but these approaches required the introduction of a left multiplication on the Hilbert space (even though this multiplication was sometimes assumed to be defined only for quaternions in one complex plane and not for all $q \in \mathbb{H}$). This left multiplication was in general only partially determined by the a priori given mathematical structures; cf. also Remarks 9.3.7 and 9.4.12. It had to be extended randomly, and the necessary procedure does not generalize to the Banach space setting, in which we want to develop the theory of quaternionic spectral operators.

In this chapter we therefore develop an approach to spectral integration of intrinsic slice functions on a quaternionic right Banach space. This integration is done with respect to a spectral system instead of a spectral measure, a concept that makes specific ideas of [197]. It has a clear and intuitive interpretation in terms of the right linear structure of the space, and it is compatible with the complex theory. The prototype of a spectral system is a pair (E, J) on a Hilbert space that consists of a spectral measure E and an imaginary operator J with $E(\mathbb{H} \setminus \mathbb{R}) = -J^2$. This is exactly the structure that we used to define spectral integration on Hilbert spaces in Chapter 10. In this chapter we consider, however, spectral measures that are defined on axially symmetric subsets of $\mathbb{H}$ instead of subsets of a complex half-plane $\mathbb{C}_j^+$. Both approaches are equivalent: we can identify any axially symmetric set with its intersection with one complex half-plane $\mathbb{C}_j^+$ in order to obtain a bijective relation between these two types of sets. The two notations stress, however, two different philosophies. While the imaginary operator J was in Chapter 10 considered a multiplication by the imaginary unit j from the left, we stress in this chapter that J can also be considered a right linear multiplication

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by the entire set of imaginary units $\mathbb{S}$ form the right. This allows us to give a clear interpretation of spectral integration in terms of the right linear structure on the space.

The results in this chapter are taken from [125] and [128]. We want to point out that in this chapter and in the next one it is very important to distinguish between left and right Banach spaces. So to avoid confusion with the previous chapters we will denote the left Banach spaces by V_L, right Banach spaces by V_R and the two-sided ones by V.

14.1 Spectral Integrals of Real-Valued Slice Functions

The basic idea of spectral integration is well known: it generates a multiplication operator in a way that generalizes the multiplication by eigenvalues in the discrete case. If, for instance, $\lambda \in \sigma(A)$ of some normal operator $A : \mathbb{C}^n \to \mathbb{C}^n$, then we can define $E(\{\lambda\})$ to be the orthogonal projection of $\mathbb{C}^n$ onto the eigenspace associated with λ and we obtain $A = \sum_{\lambda\in\sigma(A)} \lambda\, E(\{\lambda\})$. Setting $E(\Delta) = \sum_{\lambda\in\Delta} E(\{\lambda\})$, one obtains a discrete measure on $\mathbb{C}$, the values of which are orthogonal projections on $\mathbb{C}^n$, and A is the integral of the identity function with respect to this measure. Changing the notation accordingly, we have

$$A = \sum_{\lambda\in\sigma(A)} \lambda\, E(\{\lambda\}) \quad \Longrightarrow \quad A = \int_{\sigma(A)} \lambda\, dE(\lambda). \tag{14.1}$$

Via functional calculus it is possible to define functions of an operator. The fundamental intuition of a functional calculus is that $f(A)$ should be defined by the action of f on the spectral values (resp. the eigenvalues) of A. For our normal operator A on $\mathbb{C}^n$ the operator $f(A)$ is the operator with the following property: if $y \in \mathbb{C}^n$ is an eigenvector of A with respect to λ, then y is an eigenvector of $f(A)$ with respect to $f(\lambda)$, just as happens, for instance, naturally for powers and polynomials of A. Using the above notation, we thus have

$$f(A) = \sum_{\lambda\in\sigma(A)} f(\lambda)\, E(\{\lambda\}) \quad \Longrightarrow \quad f(A) = \int_{\sigma(A)} f(\lambda)\, dE(\lambda). \tag{14.2}$$

In infinite-dimensional Hilbert spaces, the spectrum of a normal operator might be not discrete, so that the expressions on the left-hand side of (14.1) and (14.2) do not make sense. If E, however, is a suitable projection-valued measure, then it is possible to give the expression (14.2) a meaning by following the usual way of defining integrals via the approximation of f by simple functions. The spectral theorem then shows that for every normal operator T there exists a spectral measure such that (14.1) holds.

If we want to introduce similar concepts in the quaternionic setting, we are—even in the discrete case—faced with several unexpected phenomena.

(P1) The space of bounded linear operators on a quaternionic Banach space V_R is only a real Banach space and not a quaternionic one. Hence the expressions in (14.1) and (14.2) are defined a priori only if λ and $f(\lambda)$, respectively, are real. Otherwise, one needs to give meaning to the multiplication of the operator $E(\{\lambda\})$ by nonreal scalars.

(P2) The multiplication by a (nonreal) scalar on the right is not linear, so that $aE(\{\lambda\})$ for $a \in \mathbb{H}$ cannot be defined as $(aE(\{\lambda\}))(ya) = (E(\{\lambda\})y)a$. Moreover, the set of eigenvectors associated with a specific eigenvalue does not constitute a linear subspace of V_R: if, for instance, $Ty = ys$ with $s = s_0 + j_s s_1$ and $i \in \mathbb{S}$ with $j_s \perp i$, then $T(yi) = (Ty)i = (ys)i = (yi)\overline{s} \neq (yi)s$.

(P3) Finally, the set of eigenvalues is in general not discrete: if $s \in \mathbb{H}$ is an eigenvalue of T with $Ty = ys$ for some $y \neq 0$ and $s_j = s_0 + js_1 \in [s]$, then there exists $h \in \mathbb{H} \setminus \{0\}$ such that $s_j = h^{-1}sh$, and so

$$T(yh) = T(y)h = ysh = (yh)h^{-1}sh = (yh)s_j. \tag{14.3}$$

Thus, every $s_j \in [s]$ is also an eigenvalue of T.

As a first consequence of items (P2) and (P3), the notions of eigenvalue and eigenspace have to be adapted: linear subspaces are in the quaternionic setting not associated with individual eigenvalues s but with spheres $[s]$ of equivalent eigenvalues.

Definition 14.1.1. Let $T \in \mathcal{L}(V_R)$ and let $s \in \mathbb{H}\backslash\mathbb{R}$. We say that $[s]$ is an eigensphere of T if there exists a vector $y \in V_R \setminus \{0\}$ such that

$$(T^2 - 2s_0T + |s|^2\mathcal{I})y = \mathcal{Q}_s(T)y = 0. \tag{14.4}$$

The eigenspace of T associated with $[s]$ consists of all those vectors that satisfy (14.4).

Remark 14.1.2. For real values, things remain as we know them from the classical case: a quaternion $s \in \mathbb{R}$ is an eigenvalue of T if $Ty - ys = 0$ for some $y \neq 0$. The quaternionic right linear subspace $\ker(T - s\mathcal{I})$ is then called the eigenspace of T associated with s.

Every eigenvector y that satisfies $T(y) = ys_j$ with $s_j = s_0 + js_1$ for some $j \in \mathbb{S}$ belongs to the eigenspace associated with the eigensphere $[s]$. Note, however, that the eigenspace associated with an eigensphere $[s]$ does not consist only of eigenvectors. It contains also linear combinations of eigenvectors associated with different eigenvalues in $[s]$, as the next lemma shows.

Lemma 14.1.3. *Let $T \in \mathcal{L}(V_R)$, let $[s]$ be an eigensphere of T, and let $j \in \mathbb{S}$. A vector y belongs to the eigenspace associated with $[s]$ if and only if $y = y_1 + y_2$ such that $Ty_1 = y_1 s_j$ and $Ty_2 = y_2\overline{s_j}$, where $s_j = s_0 + js_1$.*

Proof. Observe that

$$\mathcal{Q}_s(T)y = T^2y - Ty2s_0 + y|s|^2 = T(Ty - y\overline{s_j}) - (Ty - y\overline{s_j})s_j \tag{14.5}$$

and

$$\mathcal{Q}_s(T)y = T^2y - Ty2s_0 + y|s|^2 = T(Ty - ys_j) - (Ty - ys_j)\overline{s_j}. \tag{14.6}$$

Hence $\mathcal{Q}_s(T)y = 0$ for every eigenvector associated with s_j or $\overline{s_j}$ and in turn also for every y that is the sum of two such vectors.

If, conversely, y satisfies (14.4), then we deduce from (14.5) that $Ty - y\overline{s_j}$ is a right eigenvector associated with s_j and that $Ty - ys_j$ is a right eigenvalue of T associated with $\overline{s_j}$. Since s_j and j commute, the vectors $y_1 = (Ty - y\overline{s_j})\frac{-j}{2s_1}$ and $y_2 = (Ty - ys_j)\frac{j}{2s_1}$ are right eigenvectors associated with s resp. $\overline{s_j}$, too. Hence we have obtained the desired decomposition as

$$y_1 + y_2 = (Ty - y\overline{s_j})\frac{-j}{2s_1} + (Ty - ys_j)\frac{j}{2s_1} = y(\overline{s_j} - s_j)\frac{j}{2s_1} = y. \qquad \square$$

Remark 14.1.4. If $i \in \mathbb{S}$ with $i \perp j$, then $\tilde{y}_2 := y_2(-i)$ is an eigenvector of T associated with s. Hence we can write y also as $y = y_1 + \tilde{y}_2 i$, where $y_1, \tilde{y}_2$ are both eigenvectors associated with s_j.

Since the eigenspaces of quaternionic linear operators are not associated with individual eigenvalues but instead with eigenspheres, quaternionic spectral measures must not be defined on arbitrary subsets of the quaternions. Instead, their natural domains of definition consist of axially symmetric subsets of $\mathbb{H}$, so that they associate subspaces of V_R not to sets of spectral values but to sets of spectral spheres. This is also consistent with the fact that the S-spectrum of an operator is axially symmetric.

Definition 14.1.5. We denote the σ-algebra of axially symmetric Borel sets on $\mathbb{H}$ by $\mathfrak{B}_S(\mathbb{H})$. Furthermore, we denote the set of all real-valued $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}(\mathbb{R})$-measurable functions defined on $\mathbb{H}$ by $\mathcal{M}_S(\mathbb{H}, \mathbb{R})$ and the set of all such functions that are bounded by $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$.

Remark 14.1.6. The restrictions of functions in $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ to a complex half-plane $\mathbb{C}_j^+$ are exactly the functions that were used to construct the spectral measure of a quaternionic normal operator in the previous chapters.

Definition 14.1.7. A quaternionic spectral measure on a quaternionic right Banach space V_R is a function $E : \mathfrak{B}_S(\mathbb{H}) \to \mathcal{B}(V_R)$ that satisfies

(i) $E(\Delta)$ is a continuous projection and $\|E(\Delta)\| \leq K$ for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$ with a constant $K > 0$ independent of Δ,

(ii) $E(\emptyset) = 0$ and $E(\mathbb{H}) = \mathcal{I}$,

(iii) $E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$ for every $\Delta_1, \Delta_2 \in \mathfrak{B}_S(\mathbb{H})$, and

(iv) for every sequence $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathfrak{B}_S(\mathbb{H})$ we have

$$E\left(\bigcup_{n\in\mathbb{N}}\Delta_n\right)y = \sum_{n=1}^{+\infty} E(\Delta_n)y \quad \text{for all } y \in V_R.$$

Corollary 14.1.8. *Let E be a spectral measure on V_R and let V_R^* be its dual space, the left Banach space consisting of all continuous right linear mappings from V_R to $\mathbb{H}$. For every $y \in V_R$ and $y^* \in V_R^*$, the mapping $\Delta \mapsto \langle y^*, E(\Delta)y\rangle$ is a quaternion-valued measure on $\mathfrak{B}_S(\mathbb{H})$.*

Remark 14.1.9. In the literature, authors have considered quaternionic spectral measures defined on the Borel sets $\mathfrak{B}\big(\mathbb{C}_j^+\big)$ of one of the closed complex half-planes $\mathbb{C}_j^+ = \{s_0 + js_1 : s_0 \in \mathbb{R}, s_1 \geq 0\}$, and we also did this in Chapter 10. This is equivalent to E being defined on $\mathfrak{B}_S(\mathbb{H})$. Indeed, if $\tilde{E}$ is defined on $\mathfrak{B}\big(\mathbb{C}_j^+\big)$, then setting

$$E(\Delta) := \tilde{E}\big(\Delta \cap \mathbb{C}_j^+\big) \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H})$$

yields a spectral measure in the sense of Definition 14.1.7 that is defined on $\mathfrak{B}_S(\mathbb{H})$. If, on the other hand, we start with a spectral measure E defined on $\mathfrak{B}_S(\mathbb{H})$, then setting

$$\tilde{E}(\Delta) := E([\Delta]) \quad \forall \Delta \in \mathfrak{B}\big(\mathbb{C}_j^+\big)$$

yields the respective measure on $\mathfrak{B}\big(\mathbb{C}_j^+\big)$. Although both definitions are equivalent, in this chapter we prefer $\mathfrak{B}_S(\mathbb{H})$ as the domain of E because it does not suggest a dependence on the imaginary unit j.

For a function $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$, we can now define the spectral integral with respect to a spectral measure E as in the classical case. If f is a simple function, i.e., $f(s) = \sum_{k=1}^n \alpha_k \chi_{\Delta_k}(s)$ with pairwise disjoint sets $\Delta_k \in \mathfrak{B}_S(\mathbb{H})$, where χ_{Δ_k} denotes the characteristic function of Δ_k, then we set

$$\int_{\mathbb{H}} f(s)\, dE(s) := \sum_{k=1}^n \alpha_k E(\Delta_k). \tag{14.7}$$

There exists a constant $C_E > 0$ that depends only on E such that

$$\left\| \int_{\mathbb{H}} f(s)\, dE(s) \right\| \leq C_E \|f\|_\infty, \tag{14.8}$$

where $\|.\|_\infty$ denotes the supremum norm. If $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ is arbitrary, then we can find a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $\|f - f_n\|_\infty \to 0$ as $n \to +\infty$. In this case we can set

$$\int_{\mathbb{H}} f(s)\, dE(s) := \lim_{n\to+\infty} \int_{\mathbb{H}} f_n(s)\, dE(s), \tag{14.9}$$

where this sequence converges in the operator norm because of (14.8).

Lemma 14.1.10. *Let E be a quaternionic spectral measure on V_R. The mapping $f \mapsto \int_{\mathbb{H}} f(s)\,dE(s)$ is a continuous homomorphism from $\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ to $\mathcal{B}(V_R)$. Moreover, if T commutes with E, i.e., it satisfies $TE(\Delta) = E(\Delta)T$ for all sets $\Delta \in \mathfrak{B}_S(\mathbb{H})$, then T commutes with $\int_{\mathbb{H}} f(s)\,dE(s)$ for every $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$.*

Corollary 14.1.11. *Let E be a quaternionic spectral measure on V_R and let $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$. For every $y \in V_R$ and $y^* \in V_R^*$, we have*

$$\left\langle y^*, \left[\int_{\mathbb{H}} f\,dE\right] y\right\rangle = \int_{\mathbb{H}} f(s)\,d\langle y^*, E(s)y\rangle.$$

Proof. Let $f_n = \sum_{k=1}^{N_n} \alpha_{n,k}\chi_{\Delta_{n,k}} \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ be such that $\|f - f_n\| \to 0$ as $n \to +\infty$. Since all coefficients $\alpha_{n,k}$ are real, we have

$$\begin{aligned}\left\langle y^*, \left[\int_{\mathbb{H}} f\,dE\right] y\right\rangle &= \lim_{n\to\infty} \left\langle y^*, \left[\sum_{k=1}^{N_n} \alpha_{n,k} E(\Delta_{n,k})\right] y\right\rangle \\ &= \lim_{n\to\infty} \sum_{k=1}^{N_n} \alpha_{n,k}\,\langle y^*, E(\Delta_{n,k})y\rangle = \int_{\mathbb{H}} f(s)\,d\langle y^*, E(s)y\rangle. \quad \square\end{aligned}$$

Remark 14.1.12. The above definitions are well posed and the properties given in Lemma 14.1.10 can be shown as in the classical case, so we omit their proofs. One can also deduce them directly from the classical theory: if we consider V_R a real Banach space and E a spectral measure with values in the space $\mathcal{B}_{\mathbb{R}}(V_R)$ of bounded $\mathbb{R}$-linear operators on V_R, which obviously contains $\mathcal{B}(V_R)$, then $\int_{\mathbb{H}} f(s)\,dE(s)$ defined in (14.7), resp. (14.9), is nothing but the spectral integral of f with respect to E in the classical sense. Since every α_k in (14.7) is real and since each $E(\Delta)$ is a quaternionic right linear projection, the integral of every simple function f with respect to E is a quaternionic right linear operator and hence belongs to $\mathcal{B}(V_R)$. The space $\mathcal{B}(V_R)$ is closed in $\mathcal{B}_{\mathbb{R}}(V_R)$, and hence the property of being quaternionic linear survives the approximation by simple functions such that $\int_{\mathbb{H}} f(s)\,dE(s) \in \mathcal{B}(V_R)$ for every $f \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ even if we consider it the integral with respect to a (real) spectral measure on the real Banach space V_R.

14.2 Imaginary Operators

The techniques introduced so far allow us to integrate real-valued functions with respect to a spectral measure. This is obviously insufficient, even for formulating the statement corresponding to (14.1) in the quaternionic setting unless $\sigma_S(T)$ is real. In order to define spectral integrals for functions that are not real-valued, we need additional information.

This fits another observation: in contrast to the complex case, even for the simple case of a normal operator on a finite-dimensional quaternionic Hilbert space, a decomposition of the space V_R into the eigenspaces of T is not sufficient to recover the entire operator T. Let $j, i \in \mathbb{S}$ with $j \neq i$ and consider, for instance, the operators T_1, T_2, and T_3 on $\mathbb{H}^2$, which are given by their matrix representations

$$T_1 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \quad T_2 = \begin{pmatrix} j & 0 \\ 0 & i \end{pmatrix}, \quad T_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}. \tag{14.10}$$

For each $\ell = 1, 2, 3$, we have $\sigma_S(T_\ell) = \mathbb{S}$ and that the only eigenspace of T_ℓ is the entire space $\mathbb{H}^2$. The spectral measure E that is associated with T_ℓ is hence given by $E(\Delta) = 0$ if $\mathbb{S} \not\subset \Delta$ and $E(\Delta) = \mathcal{I}$ if $\mathbb{S} \subset \Delta$. Obviously, the spectral measures associated with these operators agree, although these operators do not coincide.

Since the eigenspace of an operator T that is associated with some eigensphere $[s]$ contains eigenvectors associated with different eigenvalues, we need some additional information to understand how to multiply the eigensphere onto the associated eigenspace, i.e., to understand which vector in the eigenspace must be multiplied by which eigenvalue in the corresponding eigensphere $[s]$. This information will be provided by a suitable imaginary operator. Such operators generalize the properties of the anti-self-adjoint partially unitary operator J in the Teichmüller decomposition

$$T = A + JB = \frac{1}{2}(T + T^*) + \frac{1}{2}J|T - T^*| \tag{14.11}$$

(where J is an anti-self-adjoint partial isometry with $\ker J = \ker(T - T^*)$ that is determined by the polar decomposition of $\frac{!}{2}(T - T^*)$) of a normal operator on a Hilbert space to the Banach space setting.

Definition 14.2.1. An operator $J \in \mathcal{B}(V_R)$ is called *imaginary* if $-J^2$ is the projection of V_R onto $\operatorname{ran} J$ along $\ker J$. We call J *fully imaginary* if $-J^2 = \mathcal{I}$, i.e., if in addition, $\ker J = \{0\}$.

Corollary 14.2.2. *An operator $J \in \mathcal{B}(V_R)$ is an imaginary operator if and only if*

(i) $-J^2$ *is a projection and*

(ii) $\ker J = \ker J^2$.

Proof. If J is an imaginary operator, then obviously item (i) and item (ii) hold. Assume, on the other hand, that item (i) and item (ii) hold. Obviously $\operatorname{ran}(-J^2) \subset \operatorname{ran} J$. For every $x \in V_R$, we have $(-J^2)x - x \in \ker(-J^2) = \ker J$ because

$$(-J^2)\left((-J^2)x - x\right) = (-J^2)^2 x - (-J)^2 x = (-J^2)x - (-J)^2 x = 0,$$

since $(-J^2)$ is a projection. Therefore

$$0 = J(-J^2 x - x) = (-J^2)Jx - Jx,$$

and hence $y = (-J^2)y$ for every $y = Jx \in \operatorname{ran} J$. Consequently, $\operatorname{ran}(-J^2) \supset \operatorname{ran} J$, and in turn $\operatorname{ran} J = \operatorname{ran}(-J^2)$. Since $\ker J = \ker(-J^2)$, we find that $-J^2$ is the projection of V_R onto $\operatorname{ran} J$ along $\ker J$, i.e., that J is an imaginary operator. $\square$

Remark 14.2.3. The above implies that every anti-self-adjoint partially unitary operator J on a quaternionic Hilbert space $\mathcal{H}$ is an imaginary operator. Indeed, for every $y \in \ker J$, we obviously have $-J^2 y = 0$. Since the restriction of J to $\mathcal{H}_0 := \operatorname{ran} J = \ker J^{\perp}$ is unitary and J is anti-self-adjoint, we furthermore have for $y \in \mathcal{H}_0$ that $-J^2 y = J^* J y = (J|_{\mathcal{H}_0})^* (J|_{\mathcal{H}_0}) y = (J|_{\mathcal{H}_0})^{-1} (J|_{\mathcal{H}_0}) y = y$. Hence $-J^2$ is the orthogonal projection onto $\mathcal{H}_0 = \operatorname{ran} J$, and so J is an imaginary operator. In particular, every unitary anti-self-adjoint operator is fully imaginary. Cf. also Corollary 9.3.8.

Lemma 14.2.4. *If $J \in \mathcal{B}(V_R)$ is an imaginary operator, then $\sigma_S(T) \subset \{0\} \cup \{\mathbb{S}\}$.*

Proof. Since the operator $-J^2$ is a projection, its S-spectrum $\sigma_S(-J^2)$ is a subset of $\{0,1\}$. Indeed, for every projection $P \in \mathcal{B}(V)$, a simple calculation shows that the pseudo-resolvent of P at every $s \in \mathbb{H} \setminus \{0,1\}$ is given by

$$\mathcal{Q}_s(P)^{-1} = -\frac{1}{|s|^2}\left(\frac{1-2\operatorname{Re}(s)}{1-2\operatorname{Re}(s)+|s|^2} P - \mathcal{I}\right)$$

such that $s \in \rho_S(P)$. As a consequence of the spectral mapping theorem, we find that

$$-\sigma_S(J)^2 = \{-s^2 : \ s \in \sigma_S(J)\} = \sigma_S(-J^2) \subset \{0,1\}.$$

But if $-s^2 \in \{0,1\}$, then $s \in \{0\} \cup \mathbb{S}$ and hence $\sigma_S(J) \subset \{0\} \cup \mathbb{S}$. □

Remark 14.2.5. If $J = 0$, then J is an imaginary operator with $\sigma_S(T) = \{0\}$. If, on the other hand, $\ker J = \{0\}$ (i.e., if J is fully imaginary), then $\sigma_S(T) = \mathbb{S}$. In every other case we obviously have $\sigma_S(T) = \{0\} \cup \mathbb{S}$.

Our next goal is to arrive at Theorem 14.2.10, which gives a complete characterization of imaginary operators on V_R. It is the analogue of Lemma 9.3.9. Before we prove this result, however, we prove a crucial relation between the concepts of quaternionic spectral theory and the concepts of classical complex operator theory.

Every quaternionic right Banach space V_R can in a natural way be considered a complex Banach space over any of the complex planes $\mathbb{C}_j$ by restricting the multiplication by quaternionic scalars from the right to $\mathbb{C}_j$. In order to deal with the different structures on V_R, we introduce the following notation.

Definition 14.2.6. Let V_R be a quaternionic right Banach space. For $j \in \mathbb{S}$, we denote the space V_R considered as a complex Banach space over the complex field $\mathbb{C}_j$ by $V_{R,j}$. If T is a quaternionic right linear operator on V_R, then $\rho_{\mathbb{C}_j}(T)$ and $\sigma_{\mathbb{C}_j}(T)$ shall denote its *resolvent set* and *spectrum* as a $\mathbb{C}_j$-complex linear operator on $V_{R,j}$. If A is a $\mathbb{C}_j$-complex linear, but not quaternionic linear, operator on $V_{R,j}$, then we denote its *spectrum* as usual by $\sigma(A)$.

If we want to distinguish between the identity operator on V_R and the identity operator on $V_{R,j}$, we denote them by $\mathcal{I}_{V_R}$ and $\mathcal{I}_{V_{R,j}}$. We point out that the operator $\lambda \mathcal{I}_{V_{R,j}}$ for $\lambda \in \mathbb{C}_j$ acts as $\lambda \mathcal{I}_{V_{R,j}} y = y\lambda$ because the multiplication by scalars on $V_{R,j}$ is defined as the quaternionic right scalar multiplication on V_R restricted to $\mathbb{C}_j$.

Theorem 14.2.7. *Let $T \in \mathcal{L}(V_R)$ and choose $j \in \mathbb{S}$. The spectrum $\sigma_{\mathbb{C}_j}(T)$ of T considered as a closed complex linear operator on $V_{R,j}$ equals $\sigma_S(T) \cap \mathbb{C}_j$, i.e.,*

$$\sigma_{\mathbb{C}_j}(T) = \sigma_S(T) \cap \mathbb{C}_j. \tag{14.12}$$

For every λ in the resolvent set $\rho_{\mathbb{C}_j}(T)$ of T as a complex linear operator on $V_{R,j}$, the $\mathbb{C}_j$-linear resolvent of T is given by $R_\lambda(T) = \left(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T\right)\mathcal{Q}_\lambda(T)^{-1}$, i.e.,

$$R_\lambda(T)y := \mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y. \tag{14.13}$$

For every $i \in \mathbb{S}$ with $j \perp i$, we moreover have

$$R_{\overline{\lambda}}(T)y = -[R_\lambda(T)(yi)]i. \tag{14.14}$$

Proof. Let $\lambda \in \rho_S(T) \cap \mathbb{C}_j$. The resolvent $(\lambda\mathcal{I}_{V_{R,j}} - T)^{-1}$ of T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$ is then given by (14.13). Indeed, since T and $Q_\lambda(T)^{-1}$ commute, we have for $y \in \mathcal{D}(T)$ that

$$\begin{aligned}
&(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T)\mathcal{Q}_\lambda(T)^{-1}(y\lambda - Ty)\\
&= (\overline{\lambda}\mathcal{I}_{V_{R,j}} - T)\left(\mathcal{Q}_\lambda(T)^{-1}y\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\right)\\
&= \mathcal{Q}_\lambda(T)^{-1}y\lambda\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\lambda + T^2\mathcal{Q}_\lambda(T)^{-1}y\\
&= (|\lambda|^2\mathcal{I}_{V_{R,j}} - 2\lambda_0 T + T^2)\mathcal{Q}_\lambda(T)^{-1}y = y.
\end{aligned}$$

Similarly, for $y \in V_{R,j} = V_R$, we have

$$\begin{aligned}
&(\lambda\mathcal{I}_{V_{R,j}} - T)R_\lambda(T)y\\
&= (\lambda\mathcal{I}_{V_{R,j}} - T)\left(\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} - T\mathcal{Q}_\lambda(T)^{-1}y\right)\\
&= \mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda}\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\lambda - T\mathcal{Q}_\lambda(T)^{-1}y\overline{\lambda} + T^2\mathcal{Q}_\lambda(T)^{-1}y\\
&= (|\lambda|^2\mathcal{I}_{V_{R,j}} - 2\lambda_0 T + T^2)\mathcal{Q}_\lambda(T)^{-1}y = y.
\end{aligned}$$

Since $\mathcal{Q}_\lambda(T)^{-1}$ maps $V_{R,j}$ to $\mathcal{D}(T^2) \subset \mathcal{D}(T)$, we find that the operator $R_\lambda(T) = (\lambda\mathcal{I}_{V_{R,j}} - T)\mathcal{Q}_\lambda(T)^{-1}$ is bounded, and so λ belongs to the resolvent set$\rho_{\mathbb{C}_j}(T)$ of T considered as a $\mathbb{C}_j$-linear operator on $V_{R,j}$. Hence, $\rho_S(T) \cap \mathbb{C}_j \subset \rho_{\mathbb{C}_j}(T)$, and in turn $\sigma_{\mathbb{C}_j}(T) \subset \sigma_S(T) \cap \mathbb{C}_j$. Together with the axial symmetry of the S-spectrum, this further implies

$$\sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)} \subset (\sigma_S(T) \cap \mathbb{C}_j) \cup (\overline{\sigma_S(T) \cap \mathbb{C}_j}) = \sigma_S(T) \cap \mathbb{C}_j, \tag{14.15}$$

where $\overline{A} = \{\overline{z} : z \in A\}$.

If λ and $\overline{\lambda}$ both belong to $\rho_{\mathbb{C}_j}(T)$, then $[\lambda] \subset \rho_S(T)$ because

$$\begin{aligned}
&(\lambda\mathcal{I}_{V_{R,j}} - T)(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T)y\\
&= (y\overline{\lambda})\lambda - (Ty)\lambda - T(y\overline{\lambda}) + T^2 y\\
&= (T^2 - 2\lambda_0 T + |\lambda|^2)y
\end{aligned}$$

and hence $\mathcal{Q}_\lambda(T)^{-1} = R_\lambda(T)R_{\overline{\lambda}}(T) \in \mathcal{B}(V_R)$. Thus $\rho_S(T) \cap \mathbb{C}_j \supset \rho_{\mathbb{C}_j}(T) \cap \overline{\rho_{\mathbb{C}_j}(T)}$, and in turn

$$\sigma_S(T) \cap \mathbb{C}_j \subset \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)}. \tag{14.16}$$

The two relations (14.15) and (14.16) together yield

$$\sigma_S(T) \cap \mathbb{C}_j = \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)}. \tag{14.17}$$

What remains to show is that $\rho_{\mathbb{C}_j}(T)$ and $\sigma_{\mathbb{C}_j}(T)$ are symmetric with respect to the real axis, which then implies

$$\sigma_S(T) \cap \mathbb{C}_j = \sigma_{\mathbb{C}_j}(T) \cup \overline{\sigma_{\mathbb{C}_j}(T)} = \sigma_{\mathbb{C}_j}(T). \tag{14.18}$$

Let $\lambda \in \rho_{\mathbb{C}_j}(T)$ and choose $i \in \mathbb{S}$ with $j \perp i$. We show that $R_{\overline{\lambda}}(T)$ equals the mapping $Ay := -\left[R_\lambda(T)(yi)\right] i$. Since $\lambda i = i\overline{\lambda}$ and $i\lambda = \overline{\lambda} i$, we have for $y \in \mathcal{D}(T)$ that

$$\begin{aligned} A\left(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T\right) y &= A\left(y\overline{\lambda} - Ty\right) \\ &= -\left[R_\lambda(T)\left((y\overline{\lambda})i - (Ty)i\right)\right] i \\ &= -\left[R_\lambda(T)((yi)\lambda - T(yi))\right] i \\ &= -\left[R_\lambda(T)(\lambda\mathcal{I}_{V_{R,j}} - T)(yi)\right] i = -yii = y. \end{aligned}$$

Similarly, for arbitrary $y \in V_{R,j} = V_R$, we have

$$\begin{aligned} \left(\overline{\lambda}\mathcal{I}_{V_{R,j}} - T\right) Ay &= (Ay)\,\overline{\lambda} - T\,(Ay) \\ &= -\left[R_\lambda(T)(yi)\right] i\overline{\lambda} + T\left(\left[R_\lambda(T)(yi)\right] i\right) \\ &= -\left[R_\lambda(T)(yi)\lambda - T(R_\lambda(T)(yi))\right] i \\ &= -\left[(\lambda\mathcal{I}_{V_{R,j}} - T)R_\lambda(T)(yi)\right] i = -yii = y. \end{aligned}$$

Hence if $\lambda \in \rho_{\mathbb{C}_j}(T)$, then $R_{\overline{\lambda}}(T) = -\left[R_\lambda(T)(yi)\right] i$ so that in particular, $\overline{\lambda} \in \rho_{\mathbb{C}_j}(T)$. Consequently, $\rho_{\mathbb{C}_j}(T)$ and in turn also $\sigma_{\mathbb{C}_j}(T)$, are symmetric with respect to the real axis, so that (14.18) holds. □

Definition 14.2.8. Let $T \in \mathcal{L}(V_R)$. We define the V_R-valued function

$$\mathcal{R}_s(T; y) = \mathcal{Q}_s(T)^{-1} y\overline{s} - T\mathcal{Q}_s(T)^{-1} y \quad \forall y \in V_R,\ s \in \rho_S(T).$$

Remark 14.2.9. By Theorem 14.2.7, the mapping $y \mapsto \mathcal{R}_s(T; y)$ coincides with the resolvent of T at s applied to y if T is considered a $\mathbb{C}_{j_s}$-linear operator on V_{R,j_s}.

Let us now turn back to characterizing imaginary operators on Banach spaces. Just as with imaginary operators on a Hilbert space, we can find three subspaces of V_R on which such an operator is simply multiplication by 0, j, or $-j$.

Theorem 14.2.10. *Let $J \in \mathcal{B}(V_R)$ be an imaginary operator. For every $j \in \mathbb{S}$, the Banach space V_R admits a direct sum decomposition as*

$$V_R = V_{J,0} \oplus V_{J,j}^{+} \oplus V_{J,j}^{-} \tag{14.19}$$

with

$$\begin{aligned} V_{J,0} &= \ker(J), \\ V_{J,j}^{+} &= \{y \in V : Jy = yj\}, \\ V_{J,j}^{-} &= \{y \in V : Jy = y(-j)\}. \end{aligned} \tag{14.20}$$

The spaces $V_{J,j}^{+}$ and $V_{J,j}^{-}$ are complex Banach spaces over $\mathbb{C}_j$ with the natural structure inherited from V_R, and for each $i \in \mathbb{S}$ with $j \perp i$, the map $y \mapsto yi$ is a $\mathbb{C}_j$-antilinear and isometric bijection between $V_{J,j}^{+}$ and $V_{J,j}^{-}$.

Conversely, let $j, i \in \mathbb{S}$ with $j \perp i$ and assume that V_R is the direct sum $V_R = V_0 \oplus V_+ \oplus V_-$ of a closed ($\mathbb{H}$-linear) subspace V_0 and two closed $\mathbb{C}_j$-linear subspaces V_+ and V_- of V_R such that $\Psi : y \mapsto yi$ is a bijection between V_+ and V_-. Let E_+ and E_- be the $\mathbb{C}_j$-linear projections onto V_+ and V_- along $V_0 \oplus V_-$, resp. $V_0 \oplus V_+$. The operator $Jy := E_+ yj + E_- y(-j)$ for $y \in V_R$ is an imaginary operator on V_R.

Proof. We first assume that J is an imaginary operator and show the existence of the corresponding decomposition of V_R. Let $j \in \mathbb{S}$ and let $V_{R,j}$ denote the space V_R considered as a complex Banach over $\mathbb{C}_j$. Furthermore, let us assume that $J \neq 0$, since the statement is obviously true in this case. Then J is a bounded $\mathbb{C}_j$-linear operator on $V_{R,j}$, and by Theorem 14.2.7 and Lemma 14.2.4, the spectrum of J as an element of $\mathcal{B}(V_{R,j})$ is $\sigma_{\mathbb{C}_j}(J) = \sigma_S(J) \cap \mathbb{C}_j \subset \{0, j, -j\}$. We define now for $\tau \in \{0, j, -j\}$ the projection E_τ as the spectral projection associated with $\{\tau\}$ obtained from the Riesz–Dunford functional calculus. If we choose $0 < \varepsilon < \frac{1}{2}$, then the relation $R_z(J) = (\overline{z}\mathcal{I}_{V_{R,j}} - J)\mathcal{Q}_z(J)^{-1}$ in Theorem 14.2.7 implies

$$E_\tau y = \int_{\partial U_\varepsilon(\tau;\mathbb{C}_j)} R_z(J) y \, dz \frac{1}{2\pi i} = \int_{\partial U_\varepsilon(\tau;\mathbb{C}_j)} \mathcal{Q}_z(J)^{-1}(y\overline{z} - Jy) \, dz \frac{1}{2\pi i},$$

where $U_\varepsilon(\tau;\mathbb{C}_j)$ denotes the ball of radius ε in $\mathbb{C}_j$ that is centered at τ. (Since we assumed $\ker J \neq V$, the projections E_j and E_{-j} are not trivial. It might, however, happen that $E_0 = 0$, but this is not a problem in the following argumentation.)

We set

$$V_{J,0} = E_0 V_{R,j}, \quad V_{J,j}^{+} = E_j V_{R,j}, \quad \text{and} \quad V_{J,j}^{-} = E_{-j} V_{R,j}.$$

Obviously these are closed $\mathbb{C}_j$-linear subspaces of $V_{R,j}$, resp. V_R, and (14.19) holds.

Let us now show that the relation (14.20) holds. We first consider the subspace $V_{J,j}^{+}$. Since it is the range of the Riesz projector E_j associated with the spectral set $\{j\}$, this is a $\mathbb{C}_j$-linear subspace of $V_{R,j}$ that is invariant under J, and the restriction $J_+ := J|_{V_{J,j}^+}$ has spectrum $\sigma(J_+) = \{j\}$. Now observe that $-J_+^2 = -J^2|_{V_{J,j}^+}$ is the restriction of a projection onto an invariant subspace and hence a projection itself. Since $0 \notin \sigma(-J_+^2) = -\sigma(J_+)^2 = \{1\}$, we find that $\ker -J_+^2 = \{0\}$ and in turn $\mathcal{I}_+ := \mathcal{I}_{V_{J,j}^+} = -J_+^2$. For $y \in V_{J,j}^+$ we therefore have

$$\begin{aligned}-y = J_+^2 y &= (J_+ - j\mathcal{I}_+ + j\mathcal{I}_+)^2 y = (J_+ - j\mathcal{I}_+ + j\mathcal{I}_+)((J_+ - j\mathcal{I}_+)y + yj)\\ &= (J_+ - j\mathcal{I}_+)^2 y + (J_+ - j\mathcal{I}_+)yj + (J_+ - j\mathcal{I}_+)yj + yj^2.\end{aligned}$$

Since $j^2 = -1$, this is equivalent to

$$(J_+ - j\mathcal{I}_+)^2 y = (J_+ - j\mathcal{I}_+)y(-2j).$$

Hence $(J_+ - j\mathcal{I}_+)y$ is either 0 or an eigenvector of $J_+ - j\mathcal{I}_+$ associated with the eigenvalue $-2j$. By the spectral mapping theorem, $\sigma(J_+ - j\mathcal{I}_+) = \sigma(J_+) - j = \{0\}$. Hence $J_+ - j\mathcal{I}_+$ cannot have an eigenvector with respect to the eigenvalue $-2j$, and so $(J_+ - j\mathcal{I}_+)y = 0$. Therefore, $J_+ = \mathcal{I}_+ i$ and $Jy = J_+ y = yj$ for all $y \in V_{J,j}^+$.

With similar arguments, one shows that $Jy = y(-j)$ for every $y \in V_{J,j}^-$. Finally, $\sigma(-J_0^2) = -\sigma(J_0)^2 = \{0\}$ for $J_0 := J|_{V_{J,0}}$. Since $-J_0^2 = -J^2|_{V_{J,0}}$ is the restriction of a projection to an invariant subspace and thus a projection itself, we find that $-J_0^2$ is the zero operator, and hence $V_{J,0} = \ker(-J_0)^2 \subset \ker(J^2) = \ker J$. On the other hand, $\ker J \subset V_{J,0}$, since $V_{J,0}$ is the invariant subspace associated with the spectral value 0 of J. Thus $V_{J,0} = \ker J$, and so (14.20) is true.

Finally, if $i \in \mathbb{S}$ with $j \perp i$ and $y \in V_+$ then $(Jyi) = J(y)i = yji = (yi)(-j)$. Hence $\Psi : y \to yi$ maps $V_{J,j}^+$ to $V_{J,j}^-$. It is obviously $\mathbb{C}_j$-antilinear, isometric, and a bijection, since $y = -(yi)i$, so that the proof of the first statement is finished.

Now let $j, i \in \mathbb{S}$ with $j \perp i$ and assume that $V_R = V_0 \oplus V_+ \oplus V_-$ with subspaces V_0, V_+, and V_- as in the assumptions. We define $Jy := E_+ yj + E_- y(-j)$. Obviously, J is a continuous $\mathbb{C}_j$-linear operator on $V_{R,j}$. The mapping $\Psi : y \mapsto yi$ maps V_+ bijectively to V_-, but since $\Psi^{-1} = -\Psi$, it also maps V_- bijectively to V_+. Moreover, as an $\mathbb{H}$-linear subspace, V_0 is invariant under Ψ. For $y = y_0 + y_+ + y_- \in V_0 \oplus V_+ \oplus V_- = V_R$, we therefore obtain

$$\begin{aligned}J(yi) &= E_+(yi)j + E_-(yi)(-j) = y_- ij + y_+ i(-j)\\ &= y_-(-j)i + y_+ ji = (E_- y(-j))\, i + (E_+ yj)\, i = (Jy)i.\end{aligned}$$

If now $a \in \mathbb{H}$, then we can write $a = a_1 + a_2 i$ with $a_1, a_2 \in \mathbb{C}_j$ and find due to the $\mathbb{C}_j$-linearity of J that

$$J(ya) = J(ya_1) + J(ya_2 i) = J(y)a_1 + J(y)a_2 i = J(y)(a_1 + a_2 i) = J(y)a.$$

Hence J is quaternionic linear and therefore belongs to $\mathcal{B}(V_R)$.

Since $E_+E_- = E_-E_+ = 0$, we furthermore observe that

$$\begin{aligned}-J^2 y &= -J(E_+ yj + E_- y(-j))\\ &= -\left(E_+^2 yj^2 + E_+E_- y(-j^2) + E_-E_+ y(-j^2) + E_-^2 y(-j)^2\right) = (E_+ + E_-)y.\end{aligned}$$

Hence $-J^2$ is the projection onto $V_+ \oplus V_- = \operatorname{ran}(J)$ along $\ker J = V_0$, so that J is actually an imaginary operator. □

14.3 Spectral Systems and Spectral Integrals of Intrinsic Slice Functions

As pointed out above, invariant subspaces of an operator are in the quaternionic setting not associated with spectral values but with entire spectral spheres. Hence quaternionic spectral measures associate subspaces of V_R with sets of entire spectral spheres and not with arbitrary sets of spectral values. If we want to integrate a function f that takes nonreal values with respect to a spectral measure E, then we need some additional information. We need to know how to multiply the different values that f takes on a spectral sphere onto the vectors associated with the different spectral values in this sphere. This information is given by a suitable imaginary operator. Similar to [197], we hence introduce now the notion of a spectral system.

Definition 14.3.1. A spectral system on V_R is a pair (E, J) consisting of a spectral measure and an imaginary operator J such that

(i) E and J commute, i.e., $E(\Delta)J = JE(\Delta)$ for all $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and

(ii) $E(\mathbb{H} \setminus \mathbb{R}) = -J^2$, that is, $E(\mathbb{R})$ is the projection onto $\ker J$ along $\operatorname{ran} J$, and $E(\mathbb{H} \setminus \mathbb{R})$ is the projection onto $\operatorname{ran} J$ along $\ker J$.

Definition 14.3.2. We denote by $\mathcal{SM}^\infty(\mathbb{H})$ the set of all bounded intrinsic slice functions on $\mathbb{H}$ that are measurable with respect to the usual Borel sets $\mathfrak{B}(\mathbb{H})$ on $\mathbb{H}$.

Lemma 14.3.3. *A function $f : \mathbb{H} \to \mathbb{H}$ belongs to $\mathcal{SM}^\infty(\mathbb{H})$ if and only if it is of the form $f(s) = f_0(s) + j_s f_1(s)$ with $f_0, f_1 \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ and $f_1(s) = 0$ for $s \in \mathbb{R}$.*

Proof. If $f(s) = f_0(s) + j_s f_1(s)$ with $f_0, f_1 \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ and $f_1(s) = 0$ for $s \in \mathbb{R}$, then we can set $f_0(s_0, s_1) := f_0(s_0 + js_1)$ and $f_1(s_0, s_1) = f_1(s + js_1)$ and $f_1(s_0, -s_1) := -f_1(s_0 + js_1)$ with $j \in \mathbb{S}$ arbitrary. Since $f_0(s)$ and $f_1(s)$ are $\mathfrak{B}_S(\mathbb{H})$-measurable, they are constant on each sphere $[s]$, and so this definition is independent of the chosen imaginary unit j. Since $f_1(s) = 0$ for real s, $f_1(s_0, s_1)$ is moreover well defined for $s_1 = 0$. We find that $f(s) = f_0(s) + j_s f_1(s) = f_0(s_0, s_1) + j_s f_1(s_0, s_1)$ with $f_0(s_0, s_1)$ and $f_1(s_0, s_1)$ taking real values and satisfying (2.4), so that f is actually an intrinsic slice function. Moreover, the functions $f_0(s)$ and $f_1(s)$ and the function $\varphi(s) := j_s$ if $s \notin \mathbb{R}$ and $\varphi(s) := 0$ if $s \in \mathbb{R}$ are $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable. Since $f_1(s) = 0$ if $s \in \mathbb{R}$, we have $f(s) = f_0(s) + j_s f_1(s) = f_0(s) + \varphi(s) f_1(s)$, and hence the function f is $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable too.

If, on the other hand, $f \in \mathcal{SM}^\infty(\mathbb{H})$ with $f(s) = f_0(s_0, s_1) + j_s f_1(s_0, s_1)$, then also $f_0(s) := \frac{1}{2}\left(f(s) + f(\overline{s})\right) = f_0(s_0, s_1)$ and $f_1(s) := \frac{1}{2}\varphi(s)\left(f(\overline{s}) - f(s)\right) = f_1(s_0, s_1)$ with $\varphi(s)$ as above are $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable. Moreover $f_1(s) = 0$ if

$s_1 = 0$. Since f is intrinsic, these functions take values in $\mathbb{R}$, and hence they are $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{R})$-measurable. They are, moreover, constant on each sphere $[s]$, so that the preimages $f_0^{-1}(A)$ and $f_1^{-1}(A)$ of each set $A \in \mathfrak{B}(\mathbb{R})$ are axially symmetric Borel sets in $\mathbb{H}$. Consequently, f_0 and f_1 are $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}(\mathbb{R})$-measurable. Finally, $|f|^2 = |f_0|^2 + |f_1|^2$, so that f is bounded if and only if f_0 and f_1 are bounded. □

Corollary 14.3.4. *Every function $f \in \mathcal{SM}^\infty(\mathbb{H})$ is $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}_S(\mathbb{H})$-measurable.*

Proof. Let $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Its inverse image $f^{-1}(\Delta)$ is a Borel set in $\mathbb{H}$ because f is $\mathfrak{B}(\mathbb{H})$-$\mathfrak{B}(\mathbb{H})$-measurable. If $s \in f^{-1}(\Delta)$, then $f(s) = f_0(s_0, s_1) + j_s f_1(s_0, s_1) \in \Delta$. The axial symmetry of Δ implies then that for every $s_j = s_0 + js_1 \in [s]$ with $j \in \mathbb{S}$ also $f(s_j) = f_0(s_0, s_1) + j_s f_1(s_0, s_1) \in \Delta$ and hence $s_j \in f^{-1}(\Delta)$. Thus $s \in f^{-1}(\Delta)$ implies $[s] \subset f^{-1}(\Delta)$ and so $f^{-1}(\Delta) \in \mathfrak{B}_S(\mathbb{H})$. □

We observe that Lemma 14.3.3 implies that the spectral integrals of the component functions f_0 and f_1 of every $f = f_0 + j_s f_1 \in \mathcal{SM}^\infty(\mathbb{H})$ are defined by Definition 14.1.7.

Definition 14.3.5. Let (E, J) be a spectral system on V_R. For $f \in \mathcal{SM}^\infty(\mathbb{H})$ with $f(s) = f_0(s) + j_s f_1(s)$ we define the *spectral integral of f with respect to* (E, J) as

$$\int_{\mathbb{H}} f(s)\, dE_J(s) := \int_{\mathbb{H}} f_0(s)\, dE(s) + J \int_{\mathbb{H}} f_1(s)\, dE(s). \tag{14.21}$$

The estimate (14.8) generalizes to

$$\left\| \int_{\mathbb{H}} f(s)\, dE(s) \right\| \leq C_E \|f_0\|_\infty + C_E \|J\| \|f_1\|_\infty \leq C_{E,J} \|f\|_\infty \tag{14.22}$$

with

$$C_{E,J} := C_E(1 + \|J\|).$$

As a consequence of Lemma 14.1.10 and the fact that J and E commute, we immediately obtain the following result.

Lemma 14.3.6. *Let (E, J) be a spectral system on V_R. The mapping*

$$f \mapsto \int_{\mathbb{H}} f(s)\, dE_J(s)$$

is a continuous homomorphism from $(\mathcal{SM}^\infty(\mathbb{H}), \|.\|_\infty)$ to $\mathcal{B}(V_R)$. Moreover, if $T \in \mathcal{B}(V_R)$ commutes with E and J, then it commutes with $\int_{\mathbb{H}} f(s)\, dE_J(s)$ for every $f \in \mathcal{SM}^\infty(\mathbb{H})$.

From Corollary 14.1.11 we furthermore immediately obtain the following lemma, which is an analogue of Lemma 5.3 in [13]. See also the chapter on spectral integrals.

Corollary 14.3.7. *Let (E, J) be a spectral system on V_R and let $f = f_0 + jf_1 \in \mathcal{SM}^\infty(\mathbb{H})$. For every $y \in V_R$ and $y^* \in V_R^*$, we have*

$$\left\langle y^*, \left[\int_{\mathbb{H}} f(s)\, dE_J(s)\right] y \right\rangle = \int_{\mathbb{H}} f_0(s)\, d\langle y^*, E(s)y\rangle + \int_{\mathbb{H}} f_1(s)\, d\langle y^*, E(s)Jy\rangle\,.$$

Similar to the what happens for the S-functional calculus, there exists a deep relation between quaternionic and complex spectral integrals on V_R.

Lemma 14.3.8. *Let (E, J) be a spectral system on V_R, let $j \in \mathbb{S}$, let E_+ be the projection of V_R onto $V_{J,j}^+$ along $V_{J,0} \oplus V_{J,j}^-$, and let E_- be the projection of V_R onto $V_{J,j}^-$ along $V_{J,0} \oplus V_{J,j}^+$; cf. Theorem 14.2.10. For $\Delta \in \mathfrak{B}(\mathbb{C}_j)$, we set*

$$E_j(\Delta) := \begin{cases} E_+E([\Delta]) & \text{if } \Delta \subset \mathbb{C}_j^+, \\ E(\Delta) & \text{if } \Delta \subset \mathbb{R}, \\ E_-E(\Delta) & \text{if } \Delta \subset \mathbb{C}_j^-, \\ E_j(\Delta \cap \mathbb{C}_j^+) + E_j(\Delta \cap \mathbb{R}) + E_j(\Delta \cap \mathbb{C}_j^-) & \text{otherwise}, \end{cases} \tag{14.23}$$

where $\mathbb{C}_j^+$ and $\mathbb{C}_j^-$ are the open upper and lower half-plane in $\mathbb{C}_j$. Then E_j is a spectral measure on $V_{R,j}$. For every $f \in \mathcal{SM}^\infty(\mathbb{H})$, we have with $f_j := f|_{\mathbb{C}_j}$ that

$$\int_{\mathbb{H}} f(s)\, dE_J(s) = \int_{\mathbb{C}_j} f_j(z)\, dE_j(s). \tag{14.24}$$

Proof. Recall that E and J commute. For $y_+ \in V_{J,j}^+$, we thus have $JE(\Delta)y_+ = E(\Delta)Jy_+ = E(\Delta)y_+j$, so that $E(\Delta)y_+ \in V_{J,j}^+$ and in turn $E_+E(\Delta)y_+ = E(\Delta)y_+$. Similarly, we see that $E(\Delta)y_\sim \in V_{J,0} \oplus V_{J,1}^-$ for $y_\sim \in V_{J,0} \oplus V_{J,j}^-$, so that $E_+E(\Delta)y_\sim = 0$. Hence if we decompose $y \in V_R$ as $y = y_+ + y_\sim$ with $y_+ \in V_{J,j}^+$ and $y_\sim \in V_{J,0} \oplus V_{J,j}^-$ according to Theorem 14.2.10, then $E_+E(\Delta)y = E_+E(\Delta)y_+ + E_+E(\Delta)y_\sim = E(\Delta)y_+$ and $E(\Delta)E_+y = E(\Delta)y_+$, so that altogether, $E(\Delta)E_+y = E_+E(\Delta)y$. Analogous arguments show that $E_-E(\Delta) = E(\Delta)E_-$ and hence E_+, E_-, and $E(\Delta)$, $\Delta \in \mathfrak{B}_S(\mathbb{H})$, commute mutually.

Let us now show that E_j is actually a $\mathbb{C}_j$-complex linear spectral measure on $V_{R,j}$. For each $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ set $\Delta_+ := \Delta \cap \mathbb{C}_j^+$, $\Delta_- := \Delta \cap \mathbb{C}_j^-$, and $\Delta_{\mathbb{R}} := \Delta \cap \mathbb{R}$ for neatness and recall that $[\,\cdot\,]$ denotes the axially symmetric hull of a set. For every $\Delta, \sigma \in \mathfrak{B}_S(\mathbb{H})$, we have then

$$\begin{aligned} E([\Delta_+])E(\sigma_{\mathbb{R}}) = E(\Delta_{\mathbb{R}})E([\sigma_+]) = 0, \\ E([\Delta_-])E(\sigma_{\mathbb{R}}) = E(\Delta_{\mathbb{R}})E([\sigma_-]) = 0, \end{aligned} \tag{14.25}$$

because of item (iii) in Definition 14.1.7. Moreover, E_+ and E_- as well as $E([\Delta+])$, $E([\Delta_-])$, and $E(\Delta_{\mathbb{R}})$ are projections that commute mutually, as we just showed.

Since in addition, $E_+E_- = E_-E_+ = 0$, we have

$$\begin{aligned} E_j(\Delta)^2 &= (E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-]))^2 \\ &= E_+^2E([\Delta_+])^2 + E_+E([\Delta_+])E(\Delta_{\mathbb{R}}) + E_+E_-E([\Delta_+])E([\Delta_-]) \\ &\quad + E_+E(\Delta_{\mathbb{R}})E([\Delta_+]) + E(\Delta_{\mathbb{R}})^2 + E_-E(\Delta_{\mathbb{R}})E([\Delta_-]) \\ &\quad + E_-E_+E([\Delta_-])E([\Delta_+]) + E_-E([\Delta_-])E(\Delta_{\mathbb{R}}) + E_-^2E([\Delta_-])^2 \\ &= E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-]) = E_j(\Delta). \end{aligned} \tag{14.26}$$

Hence $E_j(\Delta)$ is a projection that is moreover continuous, since $\|E_j(\Delta)\| \leq K(1 + \|E_+\| + \|E_-\|)$, where $K > 0$ is the constant in Definition 14.1.7. Altogether, we find that E takes values that are uniformly bounded projections in $\mathcal{B}(V_{R,j})$.

We obviously have $E_j(\emptyset) = 0$. Since $E_+ + E_- = E(\mathbb{H} \setminus \mathbb{R})$ because of item (ii) in Definition 14.3.1, also

$$\begin{aligned} E_j(\mathbb{C}_j) &= E_+E([\mathbb{C}_j^+]) + E(\mathbb{R}) + E_-E([\mathbb{C}_j^-]) \\ &= (E_+ + E_-)E(\mathbb{H} \setminus \mathbb{R}) + E(\mathbb{R}) = E(\mathbb{H}) = \mathcal{I}. \end{aligned}$$

Using the same properties of E_+, E_-, and $E(\Delta)$ as in (14.26), we find that for $\Delta, \sigma \in \mathfrak{B}(\mathbb{C}_j)$,

$$\begin{aligned} &E_j(\Delta)E(\sigma) \\ &= \big(E_+E([\Delta_+]) + E(\Delta_{\mathbb{R}}) + E_-E([\Delta_-])\big)\big(E_+E([\sigma_+]) + E(\sigma_{\mathbb{R}}) + E_-E([\sigma_-])\big) \\ &= E_+^2E([\Delta_+])E([\sigma_+]) + E_+E([\Delta_+])E(\sigma_{\mathbb{R}}) + E_+E_-E([\Delta_+])E([\sigma_-]) \\ &\quad + E_+E(\Delta_{\mathbb{R}})E([\sigma_+]) + E(\Delta_{\mathbb{R}})E(\sigma_{\mathbb{R}}) + E_-E(\Delta_{\mathbb{R}})E([\sigma_-]) \\ &\quad + E_-E_+E([\Delta_-])E([\sigma_+]) + E_-E([\Delta_-])E(\sigma_{\mathbb{R}}) + E_-^2E([\Delta_-])E([\sigma_-]) \\ &= E_+E([\Delta_+] \cap [\sigma_+]) + E(\Delta_{\mathbb{R}} \cap \sigma_{\mathbb{R}}) + E_-E([\Delta_-] \cap [\sigma_-]). \end{aligned}$$

In general it not true that $[A] \cap [B] = [A \cap B]$ for $A, B \subset \mathbb{C}_j$. (Just think, for instance, about $A = \{j\}$ and $B = \{-j\}$ with $[A] \cap [B] = \mathbb{S} \cap \mathbb{S} = \mathbb{S}$ and $[A \cap B] = [\emptyset] = \emptyset$.) For every axially symmetric set C we have, however,

$$C = \left[C \cap \mathbb{C}_i^+\right] \quad \forall i \in \mathbb{S}.$$

If A and B belong to the same complex half-plane $\mathbb{C}_i^+$, then

$$\begin{aligned} [A] \cap [B] &= \left[([A] \cap [B]) \cap \mathbb{C}_i^+\right] \\ &= \left[\left([A] \cap \mathbb{C}_i^+\right) \cap \left([B] \cap \mathbb{C}_i^+\right)\right] = [A \cap B]. \end{aligned} \tag{14.27}$$

Hence $[\Delta_+] \cap [\sigma_+] = [(\Delta \cap \sigma)_+]$ and $[\Delta_-] \cap [\sigma_-] = [(\Delta \cap \sigma)_-]$, so that altogether

$$\begin{aligned} E_j(\Delta)E_j(\sigma) &= E_+E([(\Delta \cap \sigma)_+]) + E(\Delta_{\mathbb{R}} \cap \sigma_{\mathbb{R}}) + E_-E([(\Delta \cap \sigma)_-]) \\ &= E_j(\Delta \cap \sigma). \end{aligned}$$

Finally, we find for $y \in V_{R,j} = V_R$ and every countable family $(\Delta_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets that

$$\begin{aligned}
&E_j\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) y \\
&= E_+ E\left(\left[\bigcup_{n\in\mathbb{N}} \Delta_{n,+}\right]\right) y + E\left(\bigcup_{n\in\mathbb{N}} \Delta_{n,\mathbb{R}}\right) y + E_- E\left(\left[\bigcup_{n\in\mathbb{N}} \Delta_{n,-}\right]\right) y \\
&= E_+ E\left(\bigcup_{n\in\mathbb{N}} [\Delta_{n,+}]\right) y + E\left(\bigcup_{n\in\mathbb{N}} \Delta_{n,\mathbb{R}}\right) y + E_- E\left(\bigcup_{n\in\mathbb{N}} [\Delta_{n,-}]\right) y.
\end{aligned}$$

Since the sets $\Delta_{n,+}$, $n \in \mathbb{N}$, are pairwise disjoint sets in the upper half-plane $\mathbb{C}_j^+$, their axially symmetric hulls also are disjoint because of (14.27). Similarly, the axially symmetric hulls of the sets $\Delta_{n,-}$, $n \in \mathbb{N}$ are also pairwise disjoint, so that

$$\begin{aligned}
&E_j\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) y \\
&= \sum_{n\in\mathbb{N}} E_+ E_j\left([\Delta_{n,+}]\right) y + \sum_{n\in\mathbb{N}} E\left(\Delta_{n,\mathbb{R}}\right) y + \sum_{n\in\mathbb{N}} E_- E\left([\Delta_{n,-}]\right) y \\
&= \sum_{n\in\mathbb{N}} E_j(\Delta_n) y.
\end{aligned}$$

Altogether, we see that E_j is actually a $\mathbb{C}_j$-linear spectral measure on $V_{\mathbb{R},j}$.

Now let us consider spectral integrals. We start with the simplest real-valued function possible: $f = \alpha\chi_\Delta$ with $\alpha \in \mathbb{R}$ and $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Since $f_j = \alpha\chi_{\Delta\cap\mathbb{C}_j}$ and $E(\Delta) = E_j(\Delta_j \cap \mathbb{C}_j)$, we have for such a function

$$\int_{\mathbb{H}} f(s)\, dE(s) = \alpha E(\Delta) = \alpha E_j(\Delta \cap \mathbb{C}_j) = \int_{\mathbb{C}_j} f_j(z)\, dE(z).$$

By linearity we find that (14.24) holds for every simple function

$$f(s) = \sum_{\ell=1}^{n} \alpha_k \chi_{\Delta(s)}$$

in $\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$. Since these functions are dense in $\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$, it even holds for every function in $\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$. Now consider the function $\varphi(s) = j_s$ if $s \in \mathbb{H}\setminus\mathbb{R}$ and $\varphi(s) = 0$ if $s \in \mathbb{R}$. Since $\varphi_j(z) = j\chi_{\mathbb{C}_j^+} + (-j)\chi_{\mathbb{C}_j^-}$ and $E_j(\mathbb{C}_j^+) = E_+$ and $E_j^- = E_-$, the integral of φ_j with respect to E_j is

$$\begin{aligned}
\int_{\mathbb{C}_j} \varphi(z)\, dE_j(z) y &= \left(jE_j(\mathbb{C}_j)\right) y + \left((-j)E_j(\mathbb{C}_j^-)\right) y \\
&= E_+ yj + E_- y(-j) = Jy
\end{aligned}$$

for all $y \in V_{R,j} = V_R$. If f is now an arbitrary function in $\mathcal{SM}^\infty(\mathbb{H})$, then $f(s) = f_0(s) + \varphi(s)f_1(s)$ with $f_0, f_1 \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ and $f_1(s) = 0$ if $s \in \mathbb{R}$ by Lemma 14.3.3. By what we have shown so far and the homomorphism properties of both quaternionic and the complex spectral integrals, we thus obtain

$$\begin{aligned}
&\int_{\mathbb{H}} f(s)\, dE_J(s)\\
&= \int_{\mathbb{H}} f_0(s)\, dE(s) + J \int_{\mathbb{H}} f_1(s)\, dE(s)\\
&= \int_{\mathbb{C}_j} f_{0,j}(z)\, dE_j(z) + \left(\int_{\mathbb{C}_j} \varphi_j(z)\, dE_j(z)\right)\left(\int_{\mathbb{C}_j} f_{1,j}(z)\, dE_j(z)\right)\\
&= \int_{\mathbb{C}_j} f_{0,j}(z) + \varphi_j(z) f_{1,j}(z)\, dE_j(z) = \int_{\mathbb{C}_j} f_j(z)\, dE_j(z).
\end{aligned}$$ □

Working on a quaternionic Hilbert space, one might consider only spectral measures whose values are orthogonal projections. If J is an anti-self-adjoint partially unitary operator, as happens, for instance, in the spectral theorem for normal operators in [13], then E_j has values that are orthogonal projections.

Corollary 14.3.9. *Let $\mathcal{H}$ be a quaternionic Hilbert space, let (E, J) be a spectral system on $\mathcal{H}$, let $j \in \mathbb{S}$, and let E_j be the spectral measure defined in (14.23). If $E(\Delta)$ is for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$ an orthogonal projection on $\mathcal{H}$ and J is an anti-self-adjoint partially unitary operator, then $E_j(\Delta_j)$ is for every $\Delta_j \in \mathfrak{B}(\mathbb{C}_j)$ an orthogonal projection on $(\mathcal{H}, \langle\cdot,\cdot\rangle_j)$, where $\langle x, y\rangle_j := \{\langle x, y\rangle\}_j$ is the $\mathbb{C}_j$-part of $\langle x, y\rangle$ defined as $\{a\}_j = a_1$ if $a = a_1 + a_2 i$ with $a_1, a_2 \in \mathbb{C}_j$ and $i \in \mathbb{S}$ with $j \perp i$.*

Proof. If $x, y \in \mathcal{H}^+_{J,j}$, then

$$\langle x, y\rangle = \langle x, -J^2 y\rangle = \langle Jx, Jy\rangle = \langle xj, yj\rangle = (-j)\langle x, y\rangle j,$$

so that $j\langle x, y\rangle = \langle x, y\rangle j$. Since a quaternion commutes with $j \in \mathbb{S}$ if and only if it belongs to $\mathbb{C}_j$, we have $\langle x, y\rangle \in \mathbb{C}_j$. Hence if we choose $i \in \mathbb{S}$ with $j \perp i$, then $\langle x, yi\rangle = \langle x, y\rangle i \in \mathbb{C}_j i$, so that in turn, $\langle x, yi\rangle_j = \{\langle x, y\rangle\}_j = 0$ for $x, y \in \mathcal{H}^+_{J,j}$. Since $\mathcal{H}^-_{J,j} = \{yi : y \in \mathcal{H}^+_{J,j}\}$ by Theorem 14.2.10, we obtain $\mathcal{H}^-_{J,j} \perp_j \mathcal{H}^+_{J,j}$, where $\perp_j$ denotes orthogonality in $\mathcal{H}_j$. Furthermore, we have for $x \in \mathcal{H}_0 = \ker J$ and $y \in \mathcal{H}^+_{J,j}$ that

$$\langle x, y\rangle = \langle x, Jy\rangle(-j) = \langle Jx, y\rangle j = \langle 0, y\rangle j = 0,$$

and so $\langle x, y\rangle_j = \{\langle x, y\rangle\}_j = 0$ and in turn $\mathcal{H}^+_{J,j} \perp \mathcal{H}_0$. Similarly, we see that also $\mathcal{H}^-_{J,j} \perp_j \mathcal{H}_0$. Hence the direct sum decomposition $\mathcal{H}_j = \mathcal{H}_{J,0} \oplus \mathcal{H}^+_{J,j} \oplus \mathcal{H}^-_{J,j}$ in (14.19) is actually a decomposition into orthogonal subspaces of $\mathcal{H}_j$. The projection E_+ of $\mathcal{H}$ onto $\mathcal{H}^+_{J,j}$ along $\mathcal{H}_{J,0} \oplus \mathcal{H}^-_{J,j}$ and the projection E_- of $\mathcal{H}$ onto $\mathcal{H}^-_{J,j}$ along $\mathcal{H}_{J,0} \oplus \mathcal{H}^+_{J,j}$ are hence orthogonal projections on $\mathcal{H}_j$.

Since the operator $E(\Delta)$ is for $\Delta \in \mathfrak{B}_S(\mathbb{H})$ an orthogonal projection on $\mathcal{H}$, it is an orthogonal projection on $\mathcal{H}_j$. A projection is orthogonal if and only if it is self-adjoint. Since E_+, E_-, and E commute mutually, we find for every $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ and $x, y \in \mathcal{H}_j = \mathcal{H}$ that

$$\begin{aligned}
&\langle x, E_j(\Delta)y\rangle_j \\
&= \langle x, E_+E([\Delta \cap \mathbb{C}_j^+])y\rangle_j + \langle x, E(\Delta \cap \mathbb{R})y\rangle_j + \langle x, E_-E([\Delta \cap \mathbb{C}_j^-])y\rangle_j \\
&= \langle E_+E([\Delta \cap \mathbb{C}_j^+])x, y\rangle_j + \langle E(\Delta \cap \mathbb{R})x, y\rangle_j + \langle E_-E([\Delta \cap \mathbb{C}_j^-])x, y\rangle_j \\
&= \langle E_j(\Delta)x, y\rangle_j.
\end{aligned}$$

Hence $E_j(\Delta)$ is an orthogonal projection on $\mathcal{H}_j$. □

We present two easy examples of spectral systems that illustrate the intuition behind the concept of a spectral system.

Example 14.3.10. We consider a compact normal operator T on a quaternionic Hilbert space $\mathcal{H}$. The spectral theorem for compact normal operators in [143] implies that the S-spectrum consists of a (possibly finite) sequence $[s_n] = s_{n,0} + \mathbb{S}s_{n,1}, n \in \Upsilon \subset \mathbb{N}$, of spectral spheres that are (apart from possibly the sphere $[0]$) isolated in $\mathbb{H}$. Moreover, it implies the existence of an orthonormal basis of eigenvectors $(b_\ell)_{\ell\in\Lambda}$ associated with eigenvalues $s_\ell = s_{\ell,0} + j_{s_\ell}s_{\ell,1}$ with $j_{s_\ell} = 0$ if $s_\ell \in \mathbb{R}$ such that

$$Ty = \sum_{\ell\in\Lambda} b_\ell s_\ell \langle b_\ell, y\rangle. \tag{14.28}$$

Each eigenvalue s_ℓ obviously belongs to one spectral sphere, namely to $[s_{n(\ell)}]$ with $s_{n(\ell),0} = s_{\ell,0}$ and $s_{n(\ell),1} = s_{\ell,1}$, and for $[s_n] \neq \{0\}$ only finitely many eigenvalues belong to the spectral sphere $[s_n]$. We can hence rewrite (14.28) as

$$Ty = \sum_{[s_n]\in\sigma_S(T)} \sum_{s_\ell\in[s_n]} b_\ell s_\ell\langle b_\ell, y\rangle = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell s_\ell\langle b_\ell, y\rangle.$$

The spectral measure E of T is then given by

$$E(\Delta)y = \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\Delta}} \sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}).$$

If $f \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$, then obviously

$$\int_{\mathbb{H}} f(s)\, dE(s)y = \sum_{n\in\Upsilon} E([s_n])yf(s_n) = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle f(s_n). \tag{14.29}$$

In particular,

$$\int_{\mathbb{H}} s_0\, dE(s)y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle s_{\ell,0}$$

and

$$\int_{\mathbb{H}} s_1\, dE(s)y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle s_{\ell,1}.$$

If we define

$$Jy := \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell j_{s_\ell}\langle b_\ell, y\rangle,$$

then J is an anti-self-adjoint partially unitary operator and (E, J) is a spectral system. One can check easily that E and J commute, and since $j_{s_\ell} = 0$ for $s_\ell \in \mathbb{R}$ and $j_{s_\ell} \in \mathbb{S}$ with $j_{s_\ell}^2 = -1$ otherwise, one has

$$-J^2 y = -\sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell j_{s_\ell}^2\langle b_\ell, y\rangle = \sum_{n\in\Upsilon:[s_n]\subset\mathbb{H}\setminus\mathbb{R}}\ \sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle = E(\mathbb{H}\setminus\mathbb{R})y.$$

In particular, $\ker J = \overline{\operatorname{span}_{\mathbb{H}}\{b_\ell : s_\ell \in \mathbb{R}\}} = E(\mathbb{R})$. Note, moreover, that J is completely determined by T.

For every function $f = f_0 + jf_1 \in \mathcal{SM}^\infty(\mathbb{H})$, we have because of (14.29) and $\langle b_\ell, b_\kappa\rangle = \delta_{\ell,\kappa}$ that

$$\begin{aligned}
\int_{\mathbb{H}} f(s)\, dE_J(s)y &= \int_{\mathbb{H}} f_0(s)\, dE(s)y + J\int_{\mathbb{H}} f(s)\, dE(s)y \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle f_0(s_{n,0}, s_{n,1}) \\
&\quad + \sum_{m,n\in\Upsilon}\ \sum_{\substack{n(\ell)=n\\ n(\kappa)=m}} b_\ell j_{s_\ell}\langle b_\ell, b_\kappa\rangle\langle b_\kappa, y\rangle f_1(s_{m,0}, s_{m,1}) \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell f_0(s_{\ell,0}, s_{\ell,1})\langle b_\ell, y\rangle \\
&\quad + \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell j_{s_\ell} f_1(s_{\ell,0}, s_{\ell,1})\langle b_\ell, y\rangle \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\big(f_0(s_{\ell,0}, s_{\ell,1}) + j_{s_\ell} f_1(s_{\ell,0}, s_{\ell,1})\big)\langle b_\ell, y\rangle,
\end{aligned}$$

and so

$$\int_{\mathbb{H}} f(s)\, dE_J(s)y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell f(s_\ell)\langle b_\ell, y\rangle. \tag{14.30}$$

In particular,

$$\int_{\mathbb{H}} s\, dE_J(s) = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell s_\ell\langle b_\ell, y\rangle = Ty.$$

We have in particular $T = A + JB$ with $A = \int_{\mathbb{H}} s_0 \, dE(s)$ self-adjoint, $B = \int_{\mathbb{H}} s_1 \, dE(s)$ positive and J anti-self-adjoint and partially unitary as in (14.11). Moreover, E corresponds via Remark 14.1.9 to the spectral measure obtained from the spectral theorem for bounded normal operators.

We choose now $j, i \in \mathbb{S}$ with $j \perp i$, and for each $\ell \in \Lambda$ with $s_\ell \notin \mathbb{R}$ we choose $h_\ell \in \mathbb{H}$ with $|h_\ell| = 1$ such that $h_\ell^{-1} j_{s_\ell} h_\ell = j$ and in turn

$$h_\ell^{-1} s_\ell h_\ell = s_{\ell,0} + h_\ell^{-1} j_{s_\ell} h_\ell s_1 = s_{\ell,0} + j s_{\ell,1} =: s_{\ell,j}.$$

In order to simplify the notation we also set $h_\ell = 1$ and $j_{s_\ell} = 0$ if $s_\ell \in \mathbb{R}$. Then $\tilde{b}_\ell := b_\ell h_\ell, \ell \in \Lambda$ is another orthonormal basis consisting of eigenvectors of T, and since $h_\ell^{-1} = \overline{h_\ell}/|h_\ell|^2 = \overline{h_\ell}$, we have

$$\begin{aligned} Ty &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} b_\ell (h_\ell h_\ell^{-1}) s_\ell (h_\ell h_\ell^{-1}) \langle b_\ell, y \rangle \\ &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} (b_\ell h_\ell)(h_\ell^{-1} s_\ell h_\ell) \langle b_\ell h_\ell, y \rangle = \sum_{n \in \Upsilon} \sum_{n(\ell)=n} \tilde{b}_\ell s_{\ell,j} \langle \tilde{b}_\ell, y \rangle \end{aligned} \tag{14.31}$$

and similarly

$$\begin{aligned} Jy &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} b_\ell (h_\ell h_\ell^{-1}) j_\ell (h_\ell h_\ell^{-1}) \langle b_\ell, y \rangle \\ &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} (b_\ell h_\ell)(h_\ell^{-1} j_\ell h_\ell) \langle b_\ell h_\ell, y \rangle = \sum_{n \in \Upsilon} \sum_{n(\ell)=n} \tilde{b}_\ell j \langle \tilde{b}_\ell, y \rangle. \end{aligned}$$

Recall that $j\lambda = \lambda j$ for every $\lambda \in \mathbb{C}_j$ and $ji = -ij$. The splitting of $\mathcal{H}$ obtained from Theorem 14.2.10 is therefore given by

$$\mathcal{H}_{J,0} = \ker J = \overline{\operatorname{span}_{\mathbb{H}} \{\tilde{b}_\ell : \ s_\ell \in \mathbb{R}\}}, \quad \mathcal{H}_{J,j}^{+} := \overline{\operatorname{span}_{\mathbb{C}_j} \{\tilde{b}_\ell : \ s_\ell \notin \mathbb{R}\}},$$

and

$$\mathcal{H}_{J,j}^{-} = \overline{\operatorname{span}_{\mathbb{C}_j} \{\tilde{b}_\ell i : \ s_\ell \notin \mathbb{R}\}} = \mathcal{H}_{J,j}^{+} i.$$

If $\langle b_\ell, y \rangle = a_\ell = a_{\ell,1} + a_{\ell,2} i$ with $a_{\ell,1}, a_{\ell,2} \in \mathbb{C}_j$, then (14.31) implies

$$\begin{aligned} Ty &= \sum_{n \in \Upsilon} \sum_{n(\ell)=n} \tilde{b}_\ell s_{\ell,j} a_\ell \\ &= \sum_{\substack{n \in \Upsilon \\ [s_n] \subset \mathbb{R}}} \sum_{n(\ell)=n} \tilde{b}_\ell a_\ell s_\ell + \sum_{\substack{n \in \Upsilon \\ [s_n] \subset \mathbb{H} \setminus \mathbb{R}}} \sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,1} s_{\ell,j} + \sum_{\substack{n \in \Upsilon \\ [s_n] \subset \mathbb{H} \setminus \mathbb{R}}} \sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,2} i \overline{s_{\ell,j}}. \end{aligned} \tag{14.32}$$

If $f \in \mathcal{SM}^{\infty}(\mathbb{H})$, then the representation (14.30) of $\int_{\mathbb{H}} f(s) \, dE_J(s)$ in the basis

$\tilde{b}_\ell, \ell \in \Lambda$ implies

$$\begin{aligned}\int f(s)\,dE_J(s)y &= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell f(s_{\ell,j})a_\ell \\ &= \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_\ell f(s_\ell) \\ &\quad + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,1} f(s_{\ell,j}) + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,2} i\overline{f(s_{\ell,j})} \\ &= \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_\ell f(s_\ell) \\ &\quad + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,1} f(s_{\ell,j}) + \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\mathbb{H}\setminus\mathbb{R}}}\sum_{n(\ell)=n} \tilde{b}_\ell a_{\ell,2} i f(\overline{s_{\ell,j}}), \end{aligned} \tag{14.33}$$

since $f(s_\ell)\in\mathbb{R}$ for $s_\ell\in\mathbb{R}$ and $\overline{f(s_{\ell,j})} = f(\overline{s_{\ell,j}})$ because f is intrinsic. Note that the representations (14.32) and (14.33) show clearly that $f(T)$ is defined by letting f act on the right eigenvalues of T.

Example 14.3.11. Let us consider the space $L^2(\mathbb{R},\mathbb{H})$ of all quaternion-valued functions on $\mathbb{R}$ that are square-integrable with respect to the Lebesgue measure λ. Endowed with the pointwise multiplication $(fa)(t) = f(t)a$ for $f\in L^2(\mathbb{R},\mathbb{H})$ and $a\in\mathbb{H}$ and with the scalar product

$$\langle g, f\rangle = \int_{\mathbb{R}} \overline{g(t)}f(t)\,d\lambda(t) \quad \forall f,g\in L^2(\mathbb{R},\mathbb{H}), \tag{14.34}$$

this space is a quaternionic Hilbert space. Let us now consider a bounded measurable function $\varphi:\mathbb{R}\to\mathbb{H}$ and the multiplication operator $(M_\varphi f)(s) := \varphi(s)f(s)$. This operator is normal with $(M_\varphi)^* = M_{\overline{\varphi}}$, and its S-spectrum is the set $\overline{\varphi(\mathbb{R})}$. Indeed, writing $\varphi(t) = \varphi_0(t) + j_{\varphi(t)}\varphi_1(t)$ with $\varphi_0(t)\in\mathbb{R}$, $\varphi_1(t)>0$, and $j_{\varphi(t)}\in\mathbb{S}$ for $\varphi(t)\in\mathbb{H}\setminus\mathbb{R}$ and $j_{\varphi(t)}=0$ for $\varphi(t)\in\mathbb{R}$, we find that

$$\begin{aligned}\mathcal{Q}_s(M_\varphi)f(t) &= M_\varphi^2 f(t) - 2s_0 M_\varphi f(t) + |s|^2 f(t)\\ &= (\varphi^2(t) - 2s_0\varphi(t) + |s|^2)f(t)\\ &= (\varphi(t) - s_{j_{\varphi(t)}})(\varphi(t) - \overline{s_{j_{\varphi(t)}}})f(t)\end{aligned}$$

with $s_{j_{\varphi(t)}} = s_0 + j_{\varphi(t)}s_1$, and hence

$$\mathcal{Q}_s(M_\varphi)^{-1}f(t) = (\varphi(t) - s_{j_{\varphi(t)}})^{-1}(\varphi(t) - \overline{s_{j_{\varphi(t)}}})^{-1}f(t)$$

is a bounded operator if $s\notin\overline{\varphi(\mathbb{R})}$. If we define $E(\Delta) = M_{\chi_{\varphi^{-1}(\Delta)}}$ for all $\Delta\in\mathfrak{B}_S(\mathbb{H})$, then we obtain a spectral measure on $\mathfrak{B}_S(\mathbb{H})$, namely

$$E(\Delta)f(t) = \chi_{\varphi^{-1}(\Delta)}(t)f(t).$$

If we set

$$J := M_{j_\varphi} \quad \text{i.e.,} \quad (Jf)(t) = j_{\varphi(t)} f(t),$$

then we find that (E, J) is a spectral system. Obviously J is anti-self-adjoint and partially unitary and hence an imaginary operator that commutes with E. Since $j_{\varphi(t)} = 0$ if $\varphi(t) \in \mathbb{R}$ and $j_{\varphi(t)} \in \mathbb{S}$ otherwise, we have, moreover,

$$(-J^2 f)(t) = -j_{\varphi(t)}^2 f(t) = \chi_{\varphi^{-1}(\mathbb{H}\setminus\mathbb{R})} f(t) = (E(\mathbb{H} \setminus \mathbb{R})f)(t).$$

If $g \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$, then let $g_n(s) = \sum_{\ell=1}^{N_n} a_{n,\ell}\chi_{\Delta_{n,\ell}}(s) \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ be a sequence of simple functions that converges uniformly to g. Then

$$\int_{\mathbb{H}} g(s)\, dE(s) f(t) = \lim_{n\to\infty} \sum_{\ell=1}^{N_n} a_{n,\ell} E(\Delta_{n,\ell}) f(t) = \lim_{n\to\infty} \sum_{\ell=1}^{N_n} a_{n,\ell} \chi_{\varphi^{-1}(\Delta)}(t) f(t)$$

$$= \lim_{n\to\infty} \sum_{\ell=1}^{N_n} a_{n,\ell} \chi_{\Delta}(\varphi(t)) f(t) = \lim_{n\to\infty} (g_n \circ \varphi)(t) f(t) = (g \circ \varphi)(t) f(t).$$

Hence if $g(s) =_0 (s) + j_s f_1(s) \in \mathcal{SM}^\infty(\mathbb{H})$, then

$$\begin{aligned}
\int_{\mathbb{H}} g(s)\, dE_J(s) f(t) &= \int_{\mathbb{H}} g(s)\, dE(s) f(t) \\
&= \int_{\mathbb{H}} g_0(s)\, dE(s) f(t) + J \int_{\mathbb{H}} f_1(s)\, dE(s) f(t) \\
&= g_0(\varphi(t)) f(t) + j_{\varphi(t)} f_1(\varphi(t)) f(t) \\
&= (g_0(\varphi(t)) + j_{\varphi(t)} f_1(\varphi(t)) f(t) = (g \circ \varphi)(t) f(t),
\end{aligned}$$

and so

$$\int_{\mathbb{H}} g(s)\, dE_J(s) = M_{g\circ\varphi}.$$

Choosing $g(s) = s$, we find in particular that $T = A + JB$ with $A = \int_{\mathbb{H}} s_0\, dE(s)$ self-adjoint, $B = \int_{\mathbb{H}} s_1\, dE(s)$ positive, and J anti-self-adjoint and partially unitary as in the Teichmüller decomposition. The spectral measure E corresponds via Remark 14.1.9 to the spectral measure obtained in Theorem 11.2.1.

14.4 On the Different Approaches to Spectral Integration

The approach to spectral integration presented in this chapter specifies some ideas in [197]. We now compare this approach with the approaches in [13] and [144]. In [13], the authors consider a spectral measure E over $\mathbb{C}_j^+$ and a unitary and anti-self-adjoint operator J (i.e., a fully imaginary operator J in the terminology of this book) that commutes with E. They define a left multiplication on $\mathcal{H}$ by the imaginary unit J as $jy := Jy$ for $y \in \mathcal{H}$. (If one tries to develop the spectral theory of a normal operator T, then J is simply the extension of the imaginary operator in the Teichmüller decomposition of T to a fully imaginary operator; cf. Remark 9.3.7.) One can then define the multiplication of an operator A in $\mathcal{B}(\mathcal{H})$ by the imaginary unit j as $jA = JA$ and $Aj := AJ$, and this makes the integration of $\mathbb{C}_j$-valued functions on $f : \mathbb{C}_j^+ \to \mathbb{C}_j$ possible. The procedure

$$\int_{\mathbb{C}_j^+} f(s)\, dE(s) := \lim_{n\to+\infty} \int_{\mathbb{C}_j^+} f_n(s)\, dE(s) := \lim_{n\to+\infty} \sum_{k=1}^{N_n} \alpha_{n,k} E(\Delta_{n,k}), \tag{14.35}$$

where $f_n := \sum_{k=1}^{N_n} \alpha_{n,k}\chi_{\Delta_{n,k}}$ with $\Delta_{n,k} \in \mathfrak{B}(\mathbb{C}_j^+)$ is a sequence of simple functions that uniformly converges to f, is in this case also well defined if the coefficients $\alpha_{n,k}$ belong to $\mathbb{C}_j$, and not only if they belong to $\mathbb{R}$.

The authors of [144] go one step further: they define a second unitary and anti-self-adjoint operator K that commutes with E and anti-commutes with J, and they define a full left multiplication on $\mathcal{H}$. They choose $i \in \mathbb{S}$ with $j \perp i$ and define $L_j := J$ and $L_i := K$ and the left multiplication

$$\mathcal{L} : \begin{cases} \mathbb{H} & \to & \mathcal{B}(\mathcal{H}), \\ a = a_0 + a_1 j + a_2 i + a_3 ji & \mapsto & L_a := a_0\mathcal{I} + a_1 j + a_2 i + a_3 ji, \end{cases}$$

so that

$$ay := L_a y = ya_0 + L_j y a_1 + L_i y a_2 + L_j L_i y a_3 \quad \forall y \in \mathcal{H}.$$

They call a pair $\mathcal{E} := (E, \mathcal{L})$ consisting of a spectral measure over $\mathbb{C}_j^+$ and a left multiplication that commutes with E an intertwining quaternionic projection-valued measure (iqPVM for short). Such iqPVMs allow one to define spectral integrals for functions $f : \mathbb{C}_j^+ \to \mathbb{H}$ with arbitrary values in $\mathbb{H}$, since the coefficients $a_{n,k}$ in (14.35) are in this case meaningful for arbitrary values $a_{n,k} \in \mathbb{H}$. The authors arrive then at the following version of the spectral theorem [144, Theorem 4.1].

Theorem 14.4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be normal and let $j \in \mathbb{S}$. There exists an iqPVM $\mathcal{E} = (E, \mathcal{L})$ over $\mathbb{C}_j^+$ on $\mathcal{H}$ such that*

$$T = \int_{\mathbb{C}_j^+} z\, d\mathcal{E}(z). \tag{14.36}$$

The spectral measure E is uniquely determined by T, and the left multiplication is uniquely determined for $a \in \mathbb{C}_j$ on $\ker(T - T^*)^\perp$. Precisely, we have for every other left multiplication $\mathcal{L}'$ such that $\mathcal{E}' = (E, \mathcal{L}')$ is an iqPVM satisfying (14.36) that $L_a y = L'_a y$ for every $a \in \mathbb{C}_j$ and $y \in \ker(T - T^*)^\perp$. (Even more specifically, we have $jy = Jy$ for every $y \in \ker(T - T^*)^\perp = \operatorname{ran} J$, where J is the imaginary operator in the Teichmüller decomposition of T.)

All three approaches are consistent if things are interpreted correctly. Let us first consider a spectral measure E over $\mathbb{C}_j^+$ for some $j \in \mathbb{S}$, the values of which are orthogonal projections on a quaternionic Hilbert space $\mathcal{H}$. Furthermore, let J be a unitary anti-self-adjoint operator on $\mathcal{H}$ that commutes with E and let us interpret the application of J as multiplication by j from the left as in [13]. By Remark 14.1.9, we obtain a quaternionic spectral measure on $\mathfrak{B}_S(\mathbb{H})$ if we set $\tilde{E}(\Delta) := E\big(\Delta \cap \mathbb{C}_j^+\big)$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, and obviously we have

$$\int_{\mathbb{H}} f(s)\, d\tilde{E}(s) = \int_{\mathbb{C}_j^+} f_j(z)\, dE(z) \quad \forall f \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R}),$$

where $f_j = f|_{\mathbb{C}_j^+}$. If we set $J := \mathsf{J}\tilde{E}(\mathbb{H} \setminus \mathbb{R}) = \mathsf{J}E(\mathbb{C}_j^+ \setminus \mathbb{R})$, then J is an imaginary operator and we find that $(\tilde{E}, J)$ is a spectral system on $\mathcal{H}$. Now let $f(s) = f_0(s) + jf_1(s) \in \mathcal{SM}^\infty(\mathbb{H})$ and let again $f_j = f|_{\mathbb{C}_j^+}$, $f_{0,j} = \alpha|_{\mathbb{C}_j^+}$ and $f_{1,j} = f_1|_{\mathbb{C}_j^+}$. Since $f_1(s) = 0$ if $s \in \mathbb{R}$, we have $f_1(s) = \chi_{\mathbb{H}\setminus\mathbb{R}}(s) f_1(s)$ and in turn

$$\begin{aligned}
\int_{\mathbb{C}_j^+} f_j(z)\, dE(z) &= \int_{\mathbb{C}_j^+} f_{0,j}(z)\, dE(z) + \mathsf{J} \int_{\mathbb{C}_j^+} f_{1,j}(z)\, dE(z) \\
&= \int_{\mathbb{H}} f_0(s)\, d\tilde{E}(s) + \mathsf{J} \int_{\mathbb{H}} \chi_{\mathbb{H}\setminus\mathbb{R}}(s) f_1(s)\, d\tilde{E}(s) \\
&= \int_{\mathbb{H}} f_0(s)\, d\tilde{E}(s) + \mathsf{J}E(\mathbb{H} \setminus \mathbb{R}) \int_{\mathbb{H}} f_1(s)\, d\tilde{E}(s) \\
&= \int_{\mathbb{H}} f_0(s)\, d\tilde{E}(s) + J \int_{\mathbb{H}} f_1(s)\, d\tilde{E}(s) = \int_{\mathbb{H}} f(s)\, d\tilde{E}_J(s).
\end{aligned} \tag{14.37}$$

Hence for every measurable intrinsic slice function f, the spectral integral of f with respect to the spectral system $(\tilde{E}, J)$ coincides with the spectral integral of $f|_{\mathbb{C}_j^+}$ with respect to E, where we interpret the application of J as multiplication by j from the left. Since the mapping $f \mapsto f|_{\mathbb{C}_j^+}$ is a bijection between the set of all measurable intrinsic slice functions on $\mathbb{H}$ and the set of all measurable $\mathbb{C}_j$-valued functions on $\mathbb{C}_j^+$ that map the real line into itself, both approaches are equivalent for real slice functions if we identify $\tilde{E}$ with E and f with f_j. The same identifications show that the approach in [144] is equivalent to our approach, as long as we consider only intrinsic slice functions. Indeed, if $\mathcal{E} = (E, \mathcal{L})$ is an iqPVM over $\mathbb{C}_j^+$ on $\mathcal{H}$, then $\mathsf{J}y := L_j y = jy$ is a unitary and anti-self-adjoint operator on $\mathcal{H}$. As before, we can set $\tilde{E}(\Delta) = E(\Delta \cap \mathbb{C}_j^+)$ and $J := \mathsf{J}\tilde{E}(\mathbb{H} \setminus \mathbb{R}) = L_j E(\mathbb{C}_j^+ \setminus \mathbb{R})$. We then find as in (14.37) that

$$\int_{\mathbb{C}_j^+} f_j(z)\, d\mathcal{E}(z) = \int_{\mathbb{H}} f(s)\, d\tilde{E}_J(s) \quad \forall f \in \mathcal{SM}^\infty(\mathbb{H}). \tag{14.38}$$

For intrinsic slice functions, all three approaches are hence consistent.

Let us continue our discussion of how our approach to spectral integration fits into the existing theory. We recall that every normal operator T on $\mathcal{H}$ can be decomposed as

$$T = A + JB,$$

with the self-adjoint operator $A = \frac{1}{2}(T + T^*)$, the positive operator $B = \frac{1}{2}|T - T^*|$, and the imaginary operator J with $\ker J = \ker(T - T^*)$ and $\operatorname{ran} J = \ker(T - T^*)^{\perp}$. Let $\mathcal{E} = (E, \mathcal{L})$ be an iqPVM of T obtained from Theorem 14.4.1. From [144, Theorem 3.13], we know that $\left(\int_{\mathbb{C}_j^+} \varphi(z)\, d\mathcal{E}(z)\right)^* = \int_{\mathbb{C}_j^+} \overline{\varphi(z)}\, d\mathcal{E}(z)$ and $\ker \int_{\mathbb{C}_j^+} \varphi(z)\, d\mathcal{E}(z) = \operatorname{ran} E(\varphi^{-1}(0))$. Hence

$$T - T^* = \int_{\mathbb{C}_j^+} z\, d\mathcal{E}(z) - \int_{\mathbb{C}_j^+} \overline{z}\, d\mathcal{E}(z) = \int_{\mathbb{C}_j^+} 2jz_1\, d\mathcal{E}(z).$$

Since $z_1 = 0$ if and only if $z \in \mathbb{R}$, we find that $\ker J = \ker(T - T^*) = \operatorname{ran} E(\mathbb{R})$ and in turn $\operatorname{ran} J = \ker(T - T^*)^{\perp} = \operatorname{ran} E\big(\mathbb{C}_j^+ \setminus \mathbb{R}\big)$.

If we identify E with the spectral measure $\tilde{E}$ on $\mathfrak{B}_S(\mathbb{H})$ that is given by $\tilde{E}(\Delta) = E\big(\Delta \cap \mathbb{C}_j^+\big)$, then $\mathbf{J} = L_j E(\mathbb{C}_j^+ \setminus \mathbb{R})$ is an imaginary operator such that $(\tilde{E}, \mathbf{J})$ is a spectral system, as we showed above. The spectral integral of every measurable intrinsic slice function f with respect to $(\tilde{E}, \mathbf{J})$ coincides with the spectral integral of $f|_{\mathbb{C}_j^+}$ with respect to $\mathcal{E}$. Since $\operatorname{ran} E(\mathbb{C}_j^+ \setminus \mathbb{R}) = \ker(T - T^*)^{\perp} = \operatorname{ran} J$ and $L_j y = \mathbf{J} y$ for all $y \in \ker(T - T^*)^{\perp}$ (this follows from the construction of $\mathcal{L}$ and in particular L_j in [144]), we moreover find that $J = \mathbf{J}$. Therefore $(\tilde{E}, J)$ is the spectral system that for integration of intrinsic slice functions is equivalent to $\mathcal{E}$. We can hence rewrite the spectral theorem in the terminology of spectral systems as follows.

Theorem 14.4.2. *Let $T = A + JB \in \mathcal{B}(\mathcal{H})$ be a normal operator. There exists a unique quaternionic spectral measure E on $\mathfrak{B}_S(\mathbb{H})$ with $E(\mathbb{H} \setminus \sigma_S(T)) = 0$, the values of which are orthogonal projections on $\mathcal{H}$, such that (E, J) is a spectral system and such that*

$$T = \int_{\mathbb{H}} s\, dE_J(s).$$

We want to point out that the spectral system (E, J) is completely determined by T—unlike the unitary anti-self-adjoint operator $\mathbf{J}$ that extends J used in [13] and unlike the iqPVM used in [144]. We also want to stress that the proof of the spectral theorem presented in Chapter 11 translates directly into the language of spectral systems: one can pass to the language of spectral systems by the identification described above without any problems.

Example 14.4.3. In order to discuss the relations described above, let us return to Example 14.3.10, in which we considered normal compact operators on a quaternionic Hilbert space given by

$$Ty = \sum_{n \in \Upsilon} \sum_{n(\ell) = n} b_\ell s_\ell \langle b_\ell, y \rangle,$$

whose spectral system (E, J) was

$$E(\Delta)y = \sum_{\substack{n\in\Upsilon\\ [s_n]\subset\Delta}} \sum_{n(\ell)=n} b_\ell \langle b_\ell, y\rangle \quad \text{and} \quad Jy = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell j_{s_\ell} \langle b_\ell, y\rangle.$$

The integral of a function $f \in \mathcal{SM}^\infty(\mathbb{H})$ with respect to (E, J) is then given by (14.30) as

$$\int_{\mathbb{H}} f(s)\, dE_J(s)y = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell f(s_\ell) \langle b_\ell, y\rangle. \tag{14.39}$$

Let $j \in \mathbb{S}$. If we set $\tilde{E}(\Delta) = E([\Delta])$ for all $\Delta \in \mathfrak{B}\big(\mathbb{C}_j^+\big)$, then $\tilde{E}$ is a quaternionic spectral measure over $\mathbb{C}_j^+$. In [13] the authors extend J to an anti-self-adjoint and unitary operator J that commutes with T and interpret applying this operator as multiplication by j from the left in order to define spectral integrals. One possibility to do this is to define $\iota(\ell) = j_{s_\ell}$ if $s_\ell \notin \mathbb{R}$ and $\iota(\ell) \in \mathbb{S}$ arbitrary if $s_\ell \in \mathbb{R}$ and to set

$$\mathsf{J}y = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell \iota(\ell) \langle b_\ell, y\rangle$$

and $iy = \mathsf{J}y$.

In [144] the authors go even one step further and extend this multiplication by scalars from the left to a full left multiplication $\mathcal{L} = (L_a)_{a\in\mathbb{H}}$ that commutes with E in order obtain an iqPVM $\mathcal{E} = (E, \mathcal{L})$. We can do this by choosing for each $\ell \in \Lambda$ an imaginary unit $\jmath(\ell) \in \mathbb{S}$ with $\jmath(\ell) \perp \iota(\ell)$ and by defining

$$\mathsf{K}y = \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell \jmath(\ell) \langle b_\ell, y\rangle.$$

If we choose $i \in \mathbb{S}$ and define for $a = a_0 + a_1 j + a_2 i + a_3 ji \in \mathbb{H}$,

$$\begin{aligned} ay = L_a y &:= y a_0 + \mathsf{i} y a_1 + \mathsf{K} y a_2 + \mathsf{JK} y a_3 \\ &= \sum_{n\in\Upsilon} \sum_{n(\ell)=n} b_\ell (a_0 + a_1 \iota(\ell) + a_2 \jmath(\ell) + a_3 \iota(\ell) \jmath(\ell) \langle b_\ell, y\rangle, \end{aligned}$$

then $\mathcal{L} = (L_a)_{a\in\mathbb{H}}$ is obviously a left multiplication that commutes with E, and hence $\mathcal{E} = (\tilde{E}, \mathcal{L})$ is an iqPVM over $\mathbb{C}_j^+$.

Set $s_{n,j} = [s_n] \cap \mathbb{C}_j^+$. For $f_j : \mathbb{C}_j^+ \to \mathbb{H}$, the integral of f_j with respect to $\mathcal{E}$ is

$$\begin{aligned}
&\int_{\mathbb{C}_j^+} f_j(z)\, d\mathcal{E}(z) \\
&= \sum_{n\in\Upsilon} f_j(s_{n,j})\tilde{E}(\{s_{n,j}\})y = \sum_{n\in\Upsilon} f_j(s_{n,j})E([s_n])y \\
&= \sum_{n\in\Upsilon} \big(F_0(s_{n,j}) + F_1(s_{n,j})\mathsf{J} + F_2(s_{n,j})\mathsf{K} + F_3(s_{n,j})\mathsf{JK}\big) \sum_{n(\ell)=n} b_\ell\langle b_\ell, y\rangle \\
&= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} b_\ell\big(F_0(s_{n,j}) + F_1(s_{n,j})\imath(\ell) + F_2(s_{n,j})\jmath(\ell) + F_3(s_{n,j})\imath(\ell)\jmath(\ell)\big)\langle b_\ell, y\rangle,
\end{aligned} \tag{14.40}$$

where $F_0, \ldots, F_3$ are the real-valued component functions such that

$$f_j(z) = F_0(z) + F_1(z)j + F_2(z)i + F_3(z)ji.$$

If now f_j is the restriction of an intrinsic slice function $f(s) = f_0(s) + j_s f_1(s)$, then $F_0(s_{n(\ell),j}) = f_0(s_{\ell,j}) = f_0(s_\ell)$ and $F_1(s_{n(\ell),j}) = f_1(s_{\ell,j}) = f_1(s_\ell)$ and $F_2(z) = F_3(z) = 0$. Since moreover $F_1(s_{n(\ell),j}) = f_1(s_\ell) = 0$ if $s_\ell \in \mathbb{R}$ and $\imath(\ell) = j_{s_\ell}$ if $s_\ell \notin \mathbb{R}$, we find that (14.40) actually equals (14.39) in this case. Note, however, that for every other function f_j, the integral (14.40) depends on the random choice of the functions $\imath(\ell)$ and $\jmath(\ell)$, which are not fully determined by T.

Let us now investigate the relation between (14.40) and the right linear structure of T. Let us therefore change to the eigenbasis $\tilde{b}_\ell, \ell \in \Lambda$, with $T\tilde{b}_\ell = \tilde{b}_\ell s_{\ell,j}$ defined in Example 14.3.10. For convenience let us furthermore choose $\imath(\ell)$ and $\jmath(\ell)$ such that

$$\mathsf{J}y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell j\langle\tilde{b}_\ell, y\rangle \quad \text{and} \quad \mathsf{K}y = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell i\langle\tilde{b}_\ell, y\rangle.$$

The left multiplication $\mathcal{L}$ is hence exactly the left multiplication induced by the basis $\tilde{b}_\ell, \ell \in \Lambda$, and multiplication of y by $a \in \mathbb{H}$ from the left exactly corresponds to multiplying the coordinates $\langle\tilde{b}_\ell, y\rangle$ by a from the left, i.e., $ay = \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell a\langle\tilde{b}_\ell, y\rangle$. (Note, however, that unlike multiplication by scalars from the right, multiplication by scalars from the left corresponds to multiplication of the coordinates only in this basis. This relation is lost if we change the basis.)

Let us define $\langle\tilde{b}_\ell, y\rangle = a_\ell$ with $a_\ell = a_{\ell,1} + a_{\ell,2}i$ with $a_{\ell,1}, a_{\ell,2} \in \mathbb{C}_j$ and let $f_j : \mathbb{C}_j^+ \to \mathbb{H}$. If we write $f_j(z) = f_1(z) + f_2(z)i$, this time with $\mathbb{C}_j$-valued components $f_1, f_2 : \mathbb{C}_j^+ \to \mathbb{C}_j$, then (14.40) yields

$$\begin{aligned}\int_{\mathbb{C}_j^+} f_j(z)\,d\mathcal{E}(z) &= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell\big(f_1(s_{n,j}) + f_2(s_{n,j})i\big)(a_1 + a_2 i)\\ &= \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell\big(a_1 f_1(s_{n,j}) + \overline{a_1} f_2(s_{n,j})i\big)\\ &\quad + \sum_{n\in\Upsilon}\sum_{n(\ell)=n} \tilde{b}_\ell\big(a_2 i\overline{f_2(s_{n,j})} - \overline{a_2} f_2(s_{n,j})\big).\end{aligned} \tag{14.41}$$

If we compare this with (14.32), then we find that $\int_{\mathbb{C}_j^+} f_j(z)\,d\mathcal{E}(z)$ corresponds to an application of f_j to the right eigenvalues of T only if $f_2 \equiv 0$ and f_1 can be extended to a function on all of $\mathbb{C}_j$ such that $f_1(\overline{s_{\ell,j}}) = \overline{f_1(s_{\ell,j})}$. This is, however, the case if and only if $f_j = f_1$ is the restriction of an intrinsic slice function to $\mathbb{C}_j^+$.

As pointed out above, spectral integrals of intrinsic slice functions defined in the sense of [13] or [144] can be considered spectral integrals with respect to a suitably chosen spectral system. The other two approaches—in particular the approach using iqPVMs in [144]—allow, however, the integration of a larger class of functions.

The authors of [144] argue in the introduction that the approach of spectral integration in [13] is complex in nature, since it allows one to integrate only $\mathbb{C}_j$-valued functions defined on $\mathbb{C}_j^+$ for some $j \in \mathbb{S}$. They argue that their approach using iqPVMs, on the other hand, is quaternionic in nature, since it allows one to integrate functions that are defined on a complex half-plane and take arbitrary values in the quaternions. It is rather the other way around. It is the approach to spectral integration using spectral systems that is quaternionic in nature, although they allow one to integrate only intrinsic slice functions, and we have three main arguments in favor of this point of view:

(i) **Spectral integration with respect to a spectral system does not require the random introduction of any undetermined structure.**

If we consider a normal operator $T = A + JB$ on a quaternionic Hilbert space, then only its spectral system J is uniquely defined. The extension of J to a unitary anti-self-adjoint operator $\mathbf{J}$ that can be interpreted as multiplication $L_j = \mathbf{J}$ by some $j \in \mathbb{S}$ from the left is not determined by T. Also, multiplication L_i by some $i \in \mathbb{S}$ with $i \perp j$ that extends L_j to the left multiplication $\mathcal{L}$ in an iqPVM $\mathcal{E} = (E, \mathcal{L})$ associated with T is not determined by T. The construction in [142] and [144] is based on the spectral theorems for quaternionic self-adjoint operators and for complex linear normal operators.

As we shall see in Chapter 15, the spectral orientation J of a spectral operator T—that is, the imaginary operator in the spectral system (E, J) associated with T—on a right Banach space can be constructed once the spectral measure E associated with T is known. Since the spectral theorems for self-adjoint operators and for complex linear operators are not available on Banach spaces, it is not clear how to extend J to a fully imaginary operator or even further to something that generalizes an iqPVM and whether this is possible at all.

(ii) **Spectral integration with respect to a spectral system has a clear interpretation in terms of the right linear structure on the space.**

The natural domain of a right linear operator is a right Banach space. If a left multiplication is defined on the Banach space, then the operator's spectral properties should be independent of this left multiplication. Integration with respect to a spectral system (E, J) has a clear and intuitive interpretation with respect to the right linear structure of the space: the spectral measure E associates (right) linear subspaces to spectral spheres, and the spectral orientation determines how to multiply the spectral values in the corresponding spectral spheres (from the right) onto the vectors in these subspaces.

The role of the left multiplication in an iqPVM in terms of the right linear structure is less clear. Indeed, we doubt that there exists a similarly clear and intuitive interpretation in view of the fact that no relation between left and right eigenvalues has been discovered up to now.

(iii) **Extending the class of integrable functions toward non-intrinsic slice functions does not seem to bring any benefit and might not even be meaningful.**

Extending the class of admissible functions for spectral integration beyond the class of measurable intrinsic slice functions seems to add little value to the theory. As pointed out above, the proof of the spectral theorem in [13] translates directly into the language of spectral systems, and hence spectral systems offer a framework that is sufficient to prove the most powerful result of spectral theory.

Even more, spectral integrals of functions that are not intrinsic slice functions cannot follow the basic intuition of spectral integration. In particular, if we define a measurable functional calculus via spectral integration, then this functional calculus only follows the fundamental intuition of a functional calculus, namely that $f(T)$ should be defined by the action of f on the spectral values of T if the underlying class of functions consists of intrinsic slice functions.

Chapter 15

Bounded Quaternionic Spectral Operators

We turn our attention now to the study of quaternionic linear spectral operators, in which we generalize the complex linear theory in [106]. The results presented in this chapter can be found in [125] and in [128].

15.1 The Spectral Decomposition of a Spectral Operator

A complex spectral operator is a bounded operator A that has a spectral resolution, i.e., there exists a spectral measure E defined on the Borel sets $\mathfrak{B}(\mathbb{C})$ on $\mathbb{C}$ such that $\sigma_S(A|_\Delta) \subset \overline{\Delta}$ with $A_\Delta = A|_{\operatorname{ran} E(\Delta)}$ for all $\Delta \in \mathfrak{B}(\mathbb{C})$. Chapter 14 showed that spectral systems take over the role of spectral measures in the quaternionic setting. If E is a spectral measure that reduces an operator $T \in \mathcal{B}(V_R)$, then there will in general exist infinitely many imaginary operators J such that (E, J) is a spectral system. We thus have to find a criterion for identifying the one among them that fits the operator T and that can hence serve as its spectral orientation. A first and quite obvious requirement is that T and J commute. This is, however, not sufficient. Indeed, if J and T commute, then also $-J$ and T commute. More generally, every operator that is of the form $\tilde{J} := -E(\Delta)J + E(\mathbb{H} \setminus \Delta)J$ with $\Delta \in \mathfrak{B}_S(\mathbb{H})$ is an imaginary operator such that $(E, \tilde{J})$ is a spectral system that commutes with T.

We develop a second criterion by analogy with the finite-dimensional case. Let $T \in \mathcal{B}(\mathbb{H}^n)$ be the operator on $\mathbb{H}^n$ that is given by the diagonal matrix $T = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ and let us assume $\lambda_\ell \notin \mathbb{R}$ for $\ell = 1, \ldots, n$. We intuitively identify the operator $J = \operatorname{diag}(j_{\lambda_1}, \ldots, j_{\lambda_n})$ as the spectral orientation for T, cf. also Example 14.3.10. Obviously J commutes with T. Moreover, if $s_0 \in \mathbb{R}$ and

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$s_1 > 0$ are arbitrary, then the operator $(s_0\mathcal{I} - s_1 J) - T$ is invertible. Indeed, one has

$$(s_0\mathcal{I} - s_1 J) - T = \operatorname{diag}(\overline{s_{j_{\lambda_1}}} - \lambda_1, \dots, \overline{s_{j_{\lambda_n}}} - \lambda_n),$$

where $s_{j_{\lambda_\ell}} = s_0 + j_{\lambda_\ell} s_1$. Since $\overline{s_{j_{\lambda_\ell}}} - \lambda_\ell = (s_0 - \lambda_{\ell,0}) + j_{\lambda_\ell}(-s_1 - \lambda_{\ell,1})$ and both $s_1 > 0$ and $\lambda_{\ell,1} > 0$ for all $\ell = 1, \dots, n$, each of the diagonal elements has an inverse, and so

$$((s_0\mathcal{I} - s_1 J) - T)^{-1} = \operatorname{diag}\left((\overline{s_{j_{\lambda_1}}} - \lambda_1)^{-1}, \dots, (\overline{s_{j_{\lambda_n}}} - \lambda_n)^{-1}\right).$$

This invertibility is the criterion that uniquely identifies J.

Definition 15.1.1. An operator $T \in \mathcal{B}(V_R)$ is called a *spectral operator* if there exists a spectral decomposition for T, i.e., a spectral system (E, J) on V_R such that the following three conditions hold:

(i) The spectral system (E, J) commutes with T, i.e., $E(\Delta)T = TE(\Delta)$ for all $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and $TJ = JT$.

(ii) If we set $T_\Delta := T|_{V_\Delta}$ with $V_\Delta = E(\Delta)V_R$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, then

$$\sigma_S(T_\Delta) \subset \overline{\Delta} \quad \text{for all } \Delta \in \mathfrak{B}_S(\mathbb{H}).$$

(iii) For all $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, the operator $((s_0\mathcal{I} - s_1 J) - T)|_{V_1}$ has a bounded inverse on $V_1 := E(\mathbb{H} \setminus \mathbb{R})V_R = \operatorname{ran} J$.

The spectral measure E is called a *spectral resolution* for T, and the imaginary operator J is called a *spectral orientation* of T.

A first easy result, which we shall use frequently, is that the restriction of a spectral operator to an invariant subspace $E(\Delta)V_R$ is again a spectral operator.

Lemma 15.1.2. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator on V_R and let (E, J) be a spectral decomposition for T. For every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, the operator $T_\Delta = T|_{V_\Delta}$ with $V_\Delta = \operatorname{ran} E(\Delta)$ is a spectral operator on V_Δ. A spectral decomposition for T_Δ is (E_Δ, J_Δ) with $E_\Delta(\sigma) = E(\sigma)|_{V_\Delta}$ and $J_\Delta = J|_{V_\Delta}$.*

Proof. Since $E(\Delta)$ commutes with $E(\sigma)$ for $\sigma \in \mathfrak{B}_S(\mathbb{H})$ and J, the restrictions $E_\Delta(\sigma) = E(\sigma)|_{V_\Delta}$ and $J_\Delta = J|_{V_\Delta}$ are right linear operators on V_Δ. It is immediate that E_Δ is a spectral measure on V_Δ. Moreover,

$$\ker J_\Delta = \ker J \cap V_\Delta = \operatorname{ran} E(\mathbb{R}) \cap V_\Delta = \operatorname{ran} E_\Delta(\mathbb{R})$$

and

$$\operatorname{ran} J_\Delta = \operatorname{ran} J \cap V_\Delta = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) \cap V_\Delta = \operatorname{ran} E_\Delta(\mathbb{H} \setminus \mathbb{R}).$$

Since

$$-J_\Delta^2 = -J^2|_{V_\Delta} = E(\mathbb{H} \setminus \mathbb{R})|_{V_\Delta} = E_\Delta(\mathbb{H} \setminus \mathbb{R}),$$

the operator $-J_\Delta^2$ is the projection of V_Δ onto $\operatorname{ran} J_\Delta$ along $\ker J_\Delta$. Hence J_Δ is an imaginary operator on V_Δ. Moreover, (E_Δ, J_Δ) is a spectral system. Since

$$E_\Delta(\sigma)T_\Delta E(\Delta) = E(\sigma)TE(\Delta) = TE(\sigma)E(\Delta) = T_\Delta E_\Delta(\sigma)E(\Delta),$$

and similarly

$$J_\Delta T_\Delta E(\Delta) = JTE(\Delta) = TJE(\Delta) = T_\Delta J_\Delta E(\Delta),$$

this spectral system commutes with T_Δ.

If $\sigma \in \mathfrak{B}_S(\mathbb{H})$ and we set $V_{\Delta,\sigma} = \operatorname{ran} E_\Delta(\sigma)$, then

$$V_{\Delta,\sigma} = \operatorname{ran} E(\sigma)|_{V_\Delta} = \operatorname{ran} E(\sigma)E(\Delta) = \operatorname{ran} E(\sigma \cap \Delta) = V_{\Delta\cap\sigma}.$$

Thus $T_\Delta|_{V_{\Delta,\sigma}} = T|_{V_{\sigma\cap\Delta}}$ and so $\sigma_S(T_{\Delta,\sigma}) = \sigma_S(T_{\Delta\cap\sigma}) \subset \Delta \cup \sigma \subset \sigma$. Hence E_Δ is a spectral resolution for T_Δ. Finally, for $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, the operator $s_0\mathcal{I} - s_1 J - T$ leaves the subspace $V_{\Delta,1} := \operatorname{ran} E_\Delta(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} E(\Delta \cap (\mathbb{H} \setminus \mathbb{R}))$ invariant because it commutes with E. Hence the restriction of $(s_0\mathcal{I} - s_1 J - T)|_{V_1}^{-1}$ to $V_{\Delta,1} \subset V_1 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ is a bounded linear operator on $V_{\Delta,1}$. It obviously is the inverse of $(s_0\mathcal{I} - s_1 J_\Delta - T_\Delta)|_{V_{\Delta,1}}$. Therefore (E_Δ, J_Δ) is actually a spectral decomposition for T_Δ, which hence is in turn a spectral operator. □

The remainder of this section considers the questions of uniqueness and existence of the spectral decomposition (E, J) of T. We recall the V_R-valued right slice hyperholomorphic function $\mathcal{R}_s(T; y) := \mathcal{Q}_s(T)^{-1}y\overline{s} - T\mathcal{Q}_s(T)^{-1}y$ on $\rho_S(T)$ for $T \in \mathcal{L}(V_R)$ and $y \in V_R$, which was defined in Definition 14.2.8. If T is bounded, then $\mathcal{Q}_s(T)^{-1}$ and T commute, and we have

$$\mathcal{R}_s(T; y) := \mathcal{Q}_s(T)^{-1}(y\overline{s} - Ty).$$

Definition 15.1.3. Let $T \in \mathcal{B}(V_R)$ and let $y \in V_R$. A V_R-valued right slice hyperholomorphic function f defined on an axially symmetric open set $\mathcal{D}(f) \subset \mathbb{H}$ with $\rho_S(T) \subset \mathcal{D}(f)$ is called a *slice hyperholomorphic extension* of $\mathcal{R}_s(T; y)$ if

$$(T^2 - 2s_0T + |s|^2\mathcal{I})f(s) = y\overline{s} - Ty \quad \forall s \in \mathcal{D}(f). \tag{15.1}$$

Obviously such an extension satisfies

$$f(s) = \mathcal{R}_s(T; y) \quad \text{for } s \in \rho_S(T).$$

Definition 15.1.4. Let $T \in \mathcal{B}(V_R)$ and let $y \in V_R$. The function $\mathcal{R}_s(T; y)$ is said to have the *single-valued extension property* if every two slice hyperholomorphic extensions f and g of $\mathcal{R}_s(T; y)$ satisfy $f(s) = g(s)$ for $s \in \mathcal{D}(f) \cap \mathcal{D}(g)$. In this case,

$$\rho_S(y) := \bigcup \{\mathcal{D}(f) : \ f \text{ is a slice hyperholomorphic extension of } \mathcal{R}_s(T; y)\}$$

is called the *S-resolvent set* of y, and $\sigma_S(y) = \mathbb{H} \setminus \rho_S(y)$ is called the *S-spectrum* of y.

Since it is the union of axially symmetric sets, $\rho_S(y)$ is axially symmetric. Moreover, there exists a unique maximal extension of $\mathcal{R}_s(T;y)$ to $\rho_S(y)$. We shall denote this extension by $y(s)$.

We shall see soon that the single-valued extension property holds for $\mathcal{R}_s(T;y)$ for every $y \in V_R$ if T is a spectral operator. This is, however, not true for an arbitrary operator $T \in \mathcal{B}(V_R)$. A counterexample can be constructed analogously to [106, p. 1932].

Lemma 15.1.5. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution for T. Let $s \in \mathbb{H}$ and let $\Delta \subset \mathbb{H}$ be a closed axially symmetric set such that $s \notin \Delta$. If $y \in V_R$ satisfies $(T^2 - 2s_0T + |s|^2\mathcal{I})y = 0$, then*

$$E(\Delta)y = 0 \quad \textit{and} \quad E([s])y = y.$$

Proof. Assume that $y \in V_R$ satisfies $(T^2 - 2s_0T + |s|^2\mathcal{I})y = 0$ and let T_Δ be the restriction of T to the subspace $V_\Delta = E(\Delta)V$. Since $s \notin \Delta$, we have $s \in \rho_S(T_\Delta)$, and so $\mathcal{Q}_s(T_\Delta)$ is invertible. Since $\mathcal{Q}_s(T_\Delta)^{-1} = \mathcal{Q}_s(T)^{-1}|_{V_\Delta}$, we have

$$\mathcal{Q}_s(T_\Delta)^{-1}(T^2 - 2s_0T + |s|^2\mathcal{I})E(\Delta) = E(\Delta),$$

from which we deduce

$$\begin{aligned} E(\Delta)y &= \mathcal{Q}_s(T_\Delta)^{-1}(T^2 - 2s_0T + |s|^2\mathcal{I})E(\Delta)y \\ &= \mathcal{Q}_s(T_\Delta)^{-1}E(\Delta)(T^2 - 2s_0T + |s|^2\mathcal{I})y = 0. \end{aligned}$$

Now define for $n \in \mathbb{N}$ the closed axially symmetric set

$$\Delta_n = \left\{ p \in \mathbb{H} : \ \operatorname{dist}(p, [s]) \geq \frac{1}{n} \right\}.$$

By the above, we have $E(\Delta_n)y = 0$ and in turn

$$(\mathcal{I} - E([s]))y = \lim_{n\to\infty} E(\Delta_n)y = 0,$$

so that $y = E([s])y$. □

Lemma 15.1.6. *If $T \in \mathcal{B}(V_R)$ is a spectral operator, then for every $y \in V_R$, the function $\mathcal{R}_s(T;y)$ has the single-valued extension property.*

Proof. Let $y \in V_R$ and let f and g be two slice hyperholomorphic extensions of $\mathcal{R}_s(T;y)$. We set $h(s) = f(s) - g(s)$ for $s \in \mathcal{D}(h) = \mathcal{D}(f) \cap \mathcal{D}(g)$.

If $s \in \mathcal{D}(h)$, then there exists an axially symmetric neighborhood $U \subset \mathcal{D}(h)$ of s, and for every $p \in U$ we have

$$\begin{aligned} (T^2 - 2p_0T + |p|^2\mathcal{I})h(p) &= (T^2 - 2p_0T + |p|^2\mathcal{I})f(p) - (T^2 - 2p_0T + |p|^2)g(p) \\ &= (y\overline{p} - Ty) - (y\overline{p} - Ty) = 0. \end{aligned}$$

If E is a spectral resolution of T, then we can conclude from the above and Lemma 15.1.5 that $E([p])h(p) = h(p)$ for $p \in U$. We consider now a sequence $s_n \in U$ with $s_n \neq s$ for $n \in U$ such that $s_n \to s$ as $n \to \infty$ and obtain

$$0 = E([s])E([s_n])h(s_n) = E([s])h(s_n) \to E([s])h(s) = h(s).$$

Hence $f(s) = g(s)$, and $\mathcal{R}_s(T, y)$ has the single-valued extension property. □

Corollary 15.1.7. *If $T \in \mathcal{B}(V_R)$ is a spectral operator, then for every $y \in V_R$, the function $\mathcal{R}_s(T; y)$ has a unique maximal slice hyperholomorphic extension to $\rho_S(y)$. We denote this maximal slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$ by $y(\cdot)$.*

Corollary 15.1.8. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let $y \in V_R$. Then $\sigma_S(y) = \emptyset$ if and only if $y = 0$.*

Proof. If $y = 0$, then $y(s) = 0$ is the maximal slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$. It is defined on all of $\mathbb{H}$, and hence $\sigma_S(y) = \emptyset$.

Now assume that $\sigma_S(y) = \emptyset$ for some $y \in V_R$ such that the maximal slice hyperholomorphic extension $y(\cdot)$ of $\mathcal{R}_s(T; y)$ is defined on all of $\mathbb{H}$. For every $w^* \in V_R^*$, the function $s \to \langle w^*, y(s)\rangle$ is an entire right slice hyperholomorphic function. From the fact that $\mathcal{R}_s(T; y)$ equals the resolvent of T as a bounded operator on V_{R,j_s}, we deduce $\lim_{s\to\infty} \mathcal{R}_s(T; y) = 0$ and then

$$\lim_{s\to\infty} \langle w^*, y(s)\rangle = \lim_{s\to\infty} \langle w^*, \mathcal{R}_s(T; y)\rangle = 0.$$

Liouville's theorem for slice hyperholomorphic functions therefore implies that $\langle w^*, y(s)\rangle = 0$ for all $s \in \mathbb{H}$. Since w^* was arbitrary, we obtain $y(s) = 0$ for all $s \in \mathbb{H}$.

Finally, we can choose $s \in \rho_S(T)$ such that the operator $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2\mathcal{I}$ is invertible, and we find because of (15.1) that

$$\begin{aligned} 0 = y(s)s - Ty(s) &= \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)y(s)s - T\mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)y(s) \\ &= \mathcal{Q}_s(T)^{-1}\left(\mathcal{Q}_s(T)y(s)s - T\mathcal{Q}_s(T)y(s)\right) \\ &= \mathcal{Q}_s(T)^{-1}((y\overline{s} - Ty)s - T(y\overline{s} - Ty)) \\ &= \mathcal{Q}_s(T)^{-1}\left(T^2y - Ty2s_0 + y|s|^2\right) = \mathcal{Q}_s(T)^{-1}\mathcal{Q}_s(T)y = y. \end{aligned}$$ □

Theorem 15.1.9. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution for T. If $\Delta \in \mathfrak{B}_S(\mathbb{H})$ is closed, then*

$$E(\Delta)V_R = \{y \in V_R : \ \sigma_S(y) \subset \Delta\}.$$

Proof. Let $V_\Delta = E(\Delta)V_R$ and let T_Δ be the restriction of T to V_Δ. Since Δ is closed, Definition 15.1.1 implies $\sigma_S(T_\Delta) \subset \Delta$. Moreover $\mathcal{Q}_s(T_\Delta) = \mathcal{Q}_s(T)|_{V_\Delta}$ for $s \in \mathbb{H}$. If $y \in V_\Delta$, then

$$\mathcal{Q}_s(T)\mathcal{R}_s(T; y) = \mathcal{Q}_s(T_\Delta)\mathcal{Q}_s(T_\Delta)^{-1}(y\overline{s} - T_\Delta y) = y\overline{s} - Ty$$

for $s \in \rho_S(T_\Delta)$, and hence $\mathcal{R}_s(T_\Delta; y)$ is a slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$ to $\rho_S(T_\Delta) \supset \mathbb{H} \setminus \Delta$. Thus $\sigma_S(y) \subset \Delta$. Since $y \in V_R$ was arbitrary, we obtain $E(\Delta)V_R \subset \{y \in V_R : \sigma_S(y) \subset \Delta\}$.

In order to show the converse relation, we assume that $\sigma_S(y) \subset \Delta$. We consider a closed subset $\sigma \in \mathfrak{B}_S(\mathbb{H})$ of the complement of Δ and set $T_\sigma = T|_{V_\sigma}$ with $V_\sigma = E(\sigma)V_R$. As above, $\mathcal{R}_s(T_\sigma; E(\sigma)y)$ is then a slice hyperholomorphic extension of $\mathcal{R}_s(T; E(\sigma)y)$ to $\mathbb{H} \setminus \sigma$. If, on the other hand, $y(s)$ is the unique maximal slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$, then

$$\begin{aligned} \mathcal{Q}_s(T)E(\sigma)y(s) &= E(\sigma)\mathcal{Q}_s(T)y(s) \\ &= E(\sigma)(y\overline{s} - Ty) = (E(\sigma)y)\overline{s} - T(E(\sigma)y) \end{aligned}$$

for $s \in \mathbb{H} \setminus \Delta$, and so $E(\sigma)y(s)$ is a slice hyperholomorphic extension of $\mathcal{R}_s(T; E(\sigma)y)$ to $\mathbb{H} \setminus \Delta$. Combining these two extensions, we find that $\mathcal{R}_s(T; E(\sigma)y)$ has a slice hyperholomorphic extension to all of $\mathbb{H}$. Hence $\sigma_S(E(\sigma)y) = \emptyset$, so that $E(\Delta)y = 0$ by Corollary 15.1.8.

Let us now choose an increasing sequence of closed subsets $\sigma_n \in \mathfrak{B}_S(\mathbb{H})$ of $\mathbb{H} \setminus \Delta$ such that $\bigcup_{n\in\mathbb{N}} \sigma_n = \mathbb{H} \setminus \Delta$. By the above arguments, $E(\sigma_n)y = 0$ for every $n \in \mathbb{N}$. Hence

$$E(\mathbb{H} \setminus \Delta)y = \lim_{n\to\infty} E(\Delta_n)y = 0,$$

so that in turn $E(\Delta)y = y$. We thus obtain $E(\Delta)V_R \supset \{y \in V_R : \sigma_S(y) \subset \Delta\}$. □

The following corollaries are immediate consequences of Theorem 15.1.9.

Corollary 15.1.10. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let E be a spectral resolution of T. Then $E(\sigma_S(T)) = \mathcal{I}$.*

Corollary 15.1.11. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator and let $\Delta \in \mathfrak{B}_S(\mathbb{H})$ be closed. The set of all $y \in V_R$ with $\sigma_S(y) \subset \Delta$ is a closed right subspace of V_R.*

Lemma 15.1.12. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator. If $A \in \mathcal{B}(V_R)$ commutes with T, then A commutes with every spectral resolution E for T. Moreover, $\sigma_S(Ay) \subset \sigma_S(y)$ for all $y \in V_R$.*

Proof. For $y \in V_R$ we have

$$\begin{aligned} (T^2 - 2s_0T + |s|^2\mathcal{I})Ay(s) &= A(T^2 - 2s_0T + |s|^2\mathcal{I})y(s) \\ &= A(y\overline{s} - Ty) = (Ay)\overline{s} - T(Ay). \end{aligned}$$

The function $Ay(s)$ is therefore a slice hyperholomorphic extension of $\mathcal{R}_s(T; Ay)$ to $\rho_S(y)$, and so $\sigma_S(Ay) \subset \sigma_S(y)$. From Theorem 15.1.9 we deduce that

$$AE(\Delta)V \subset E(\Delta)V$$

for every closed axially symmetric subset Δ of $\mathbb{H}$.

If σ and Δ are two disjoint closed axially symmetric sets, we therefore have

$$E(\Delta)AE(\Delta) = AE(\Delta) \quad \text{and} \quad E(\Delta)AE(\sigma) = E(\Delta)E(\sigma)AE(\sigma) = 0.$$

If we choose again an increasing sequence of closed sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$ with $\mathbb{H}\setminus\Delta = \bigcup_{n\in\mathbb{N}} \Delta_n$, we therefore have

$$E(\Delta)AE(\mathbb{H}\setminus\Delta)y = \lim_{n\to\infty} E(\Delta)AE(\Delta_n)y = 0 \quad \forall y \in V_R$$

and hence

$$E(\Delta)A = E(\Delta)A[E(\Delta) + E(\mathbb{H}\setminus\Delta)] = E(\Delta)AE(\Delta) = AE(\Delta). \tag{15.2}$$

Since Δ was an arbitrary closed set in $\mathfrak{B}_S(\mathbb{H})$ and since the sigma-algebra $\mathfrak{B}_S(\mathbb{H})$ is generated by sets of this type, we finally conclude that (15.2) holds for every set $\sigma \in \mathfrak{B}_S(\mathbb{H})$. □

Lemma 15.1.13. *The spectral resolution E of a spectral operator $T \in \mathcal{B}(V_R)$ is uniquely determined.*

Proof. Let E and $\tilde{E}$ be two spectral resolutions of T. For every closed set $\Delta \in \mathfrak{B}_S(\mathbb{H})$, Theorem 15.1.9 implies

$$\tilde{E}(\Delta)E(\Delta) = E(\Delta) \quad \text{and} \quad E(\Delta)\tilde{E}(\Delta) = \tilde{E}(\Delta),$$

and we deduce from Lemma 15.1.12 that $E(\Delta) = \tilde{E}(\Delta)$. Since the sigma algebra $\mathfrak{B}_S(\mathbb{H})$ is generated by the closed sets in $\mathfrak{B}_S(\mathbb{H})$, we obtain $E = \tilde{E}$, and hence the spectral resolution of T is uniquely determined. □

Before we consider the uniqueness of the spectral orientation, we observe that for certain operators, the existence of a spectral resolution already implies the existence of a spectral orientation and is hence sufficient for them to be a spectral operator.

Proposition 15.1.14. *Let $T \in \mathcal{B}(V_R)$ and assume that there exists a spectral resolution E for T. If $\sigma_S(T)\cap\mathbb{R} = \emptyset$, then there exists an imaginary operator $J \in \mathcal{B}(V_R)$ that is a spectral orientation for T such that T is a spectral operator with spectral resolution (E, J). Moreover, this spectral orientation is unique.*

Proof. Since $\sigma_S(T)$ is closed with $\sigma_S(T)\cap\mathbb{R} = \emptyset$, we have $\operatorname{dist}(\sigma_S(T),\mathbb{R}) > 0$. We choose $j \in \mathbb{S}$ and consider T a complex linear operator on $V_{R,j}$. Because of Theorem 14.2.7, the spectrum of T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$ is $\sigma_{\mathbb{C}_j}(T) = \sigma_S(T)\cap\mathbb{C}_j$. Since $\operatorname{dist}(\sigma_S(T),\mathbb{R}) > 0$, the sets

$$\sigma_+ = \sigma_{\mathbb{C}_j}(T)\cap\mathbb{C}_j^+ \quad \text{and} \quad \sigma_- = \sigma_{\mathbb{C}_j}(T)\cap\mathbb{C}_j^-$$

are open and closed subsets of $\sigma_{\mathbb{C}_j}(T)$ such that $\sigma_+\cup\sigma_- = \sigma_{\mathbb{C}_j}(T)$. Via the Riesz–Dunford functional calculus we can hence associate spectral projections E_+ and

E_- onto closed invariant $\mathbb{C}_j$-linear subspaces of $V_{R,j}$ to σ_+ and σ_-. The resolvent of T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$ at $z \in \rho_{\mathbb{C}_j}(T)$ is $R_z(T)y := \mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)$, and hence these projections are given by

$$\begin{aligned} E_+ y &:= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\, dz \frac{1}{2\pi j}, \\ E_- y &:= \int_{\Gamma_-} \mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\, dz \frac{1}{2\pi j}, \end{aligned} \tag{15.3}$$

where Γ_+ is a positively oriented Jordan curve that surrounds σ_+ in $\mathbb{C}_j^+$ and Γ_- is a positively oriented Jordan curve that surrounds σ_- in $\mathbb{C}_j^-$. We set

$$Jy := E_- y(-j) + E_+ yj.$$

From Theorem 14.2.10 we deduce that J is an imaginary operator on V_R if $\Psi : y \mapsto yi$ is a bijection between $V_+ := E_+ V_R$ and $V_- := E_- V_R$ for $i \in \mathbb{S}$ with $i \perp j$. This is indeed the case: due to the symmetry of $\sigma_{\mathbb{C}_j}(T) = \sigma_S(T) \cap \mathbb{C}_j$ with respect to the real axis, we obtain $\sigma_+ = \overline{\sigma_-}$, so that we can choose $\Gamma_-(t) = \overline{\Gamma_+(1-t)}$ for $t \in [0,1]$ in (15.3). Because of the relation (14.14) established in Theorem 14.2.7, the resolvent $R_z(T)$ of T as an operator on $V_{R,j}$ satisfies $R_{\overline{z}}(T)y = -\left[R_z(T)(yi)\right] i$, and so

$$\begin{aligned} E_- y &= \int_{\Gamma_-} R_z(T)y\, dz \frac{1}{2\pi j} = -\int_{\Gamma_+} R_{\overline{z}}(T)y\, d\overline{z} \frac{1}{2\pi j} \\ &= \int_{\Gamma_+} \left[R_z(T)(yi)\right] i\, d\overline{z} \frac{1}{2\pi j} = \int_{\Gamma_+} \left[R_z(T)(yi)\right]\, dz \frac{1}{2\pi j}(-i) = \left[E_+(yi)\right](-i). \end{aligned}$$

Hence we have

$$(E_- y)\, i = E_+(yi) \quad \forall y \in V_R. \tag{15.4}$$

If $y \in V_-$, then $yi = (E_- y)i = E_+(yi)$, and so $yi \in V_+$. Replacing y by yi in (15.4), we find that also $(E_- yi)\, i = -E_+(y)$ and in turn $E_-(yi) = E_+(y)i$. For $y \in V_+$ we thus find that $yi = E_+(y)i = E_-(yi)$, and so $yi \in V_-$. Hence Ψ maps V_+ to V_- and V_- to V_+, and since $\Psi^{-1} = -\Psi$, it is even bijective. We conclude that J is actually an imaginary operator.

Let us now show that (i) in Definition 15.1.1 holds. For every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, the operator $\mathcal{Q}_z(T)^{-1}$ commutes with $E(\Delta)$. Hence

$$\begin{aligned} E(\Delta)E_+ y &= \int_{\Gamma_+} E(\Delta)\mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\, dz \frac{1}{2\pi j} \\ &= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(E(\Delta)y\overline{z} - TE(\Delta)y)\, dz \frac{1}{2\pi j} = E_+ E(\Delta)y \end{aligned} \tag{15.5}$$

for every $y \in V_{R,j} = V_R$, and so $E_+ E(\Delta) = E(\Delta)E_+$. Similarly, one can show that also $E(\Delta)E_- = E_- E(\Delta)$. By construction, the operator J hence commutes with T and with $E(\Delta)$ for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, since

$$TJy = TE_-y(-j) + TE_+yj = E_-Ty(-j) + E_+Tyj = JTy$$

and

$$\begin{aligned} E(\Delta)Jy &= E(\Delta)E_-y(-j) + E(\Delta)E_+yj \\ &= E_-E(\Delta)y(-j) + E_+E(\Delta)yj = JE(\Delta)y. \end{aligned}$$

Moreover, since $\sigma_S(T) \cap \mathbb{R} = \emptyset$, Corollary 15.1.10 implies $\operatorname{ran} E(\mathbb{R}) = \{0\} = \ker J$ and $\operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = V_R = \operatorname{ran} J$. Hence (E, J) is actually a spectral system that moreover commutes with T.

Let us now show condition (iii) of Definition 15.1.1. If $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, then set $s_j := s_0 + js_1$. Since $E_+ + E_- = \mathcal{I}$, we then have

$$\begin{aligned} &((s_0\mathcal{I} - s_1J) - T)y \\ &= (E_+ + E_-)ys_0 - (E_+y)js_1 - (E_-y)(-j)s_1 - T(E_+ + E_-)y \\ &= (E_+y)(s_0 - s_1j) - T(E_+y) + (E_-y))(s_0 + s_1j) - T(E_-y) \\ &= (E_+y)\overline{s_j} - T(E_+y) + (E_-y)s_j - T(E_-y) \\ &= (\overline{s_j}\mathcal{I}_{V_{R,j}} - T)E_+y + (s_j\mathcal{I}_{V_{R,j}} - T)E_-y. \end{aligned}$$

Since E_+ and E_- are the Riesz projectors associated to σ_+ and σ_-, the spectrum $\sigma(T_+)$ of $T_+ := T|_{V_+}$ is $\sigma_+ \subset \mathbb{C}_j^+$ and the spectrum $\sigma(T_-)$ of $T_- := T|_{V_-}$ is $\sigma_- \subset \mathbb{C}_j^-$. Since s_j has positive imaginary part, we find that $\overline{s_j} \in \mathbb{C}_j^- \subset \rho(T_+)$ and $s_j \in \mathbb{C}_j^+ \subset \rho(T_-)$, so that $R_{\overline{s}_j}(T_+) := \left(\overline{s_j}\mathcal{I}_{V_+} - T_+\right)^{-1} \in \mathcal{B}(V_+)$ and $R_{s_j}(T)^{-1} := \left(s_j\mathcal{I}_{V_-} - T_-\right)^{-1} \in \mathcal{B}(V_-)$ exist. Since $E_+|_{V_+} = \mathcal{I}_{V_+}$ and $E_-|_{V_+} = 0$, they satisfy the relations

$$E_+R_{\overline{s_j}}(T_+)E_+ = R_{\overline{s_j}}(T_+)E_+ \quad \text{and} \quad E_-R_{\overline{s_j}}(T_+)E_+ = 0 \tag{15.6}$$

and similarly also

$$E_-R_{s_j}(T_-)E_- = R_{s_j}(T_-)E_- \quad \text{and} \quad E_+R_{s_j}(T_-)E_- = 0. \tag{15.7}$$

Setting $R(s_0, s_1) := R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-$, we obtain a bounded $\mathbb{C}_j$-linear operator that is defined on the entire space $V_{R,j} = V_R$. Because E_+ and E_- commute with T and satisfy $E_+E_- = E_-E_+ = 0$ and because (15.6) and (15.7) hold, we obtain for every $y \in V_R$,

$$\begin{aligned} &R(s_0, s_1)((s_0\mathcal{I} - s_1J) - T)y \\ &= \left[R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-\right]\left[(\overline{s_j}\mathcal{I}_{V_{R,j}} - T)E_+y + (s_j\mathcal{I}_{V_{R,j}} - T)E_-y\right] \\ &= R_{\overline{s_j}}(T_+)(\overline{s_j}\mathcal{I}_{V_{R,j}} - T_+)E_+y + R_{s_j}(T_-)E_-(s_j\mathcal{I}_{V_{R,j}} - T_-)E_-y \\ &= E_+y + E_-y = y \end{aligned}$$

and

$$\begin{aligned}&((s_0\mathcal{I} - s_1 J) - T)R(s_0, s_1)y\\&= \left[(\overline{s_j}\mathcal{I}_{V_{R,j}} - T)E_+ + (s_j\mathcal{I}_{V_{R,j}} - T)E_-\right]\left[R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-\right] y\\&= (\overline{s_j}\mathcal{I}_{V_+} - T_+)R_{\overline{s_j}}(T_+)E_+y + (s_j\mathcal{I}_{V_-} - T_-)R_{s_j}(T_-)E_-y\\&= E_+y + E_-y = y.\end{aligned}$$

Hence $R(s_0, s_1) \in \mathcal{B}(V_{R,j})$ is the $\mathbb{C}_j$-linear bounded inverse of $(s_0\mathcal{I} - s_1 J) - T$. Since $(s_0\mathcal{I} - s_1 J) - T$ is quaternionic right linear, its inverse is quaternionic right linear too, so that even $((s_0\mathcal{I} - s_1 J) - T)^{-1} \in \mathcal{B}(V_R)$. Therefore, J is actually a spectral orientation for T, and T is in turn a spectral operator with spectral decomposition (E, J).

Finally, we show the uniqueness of the spectral orientation J. Assume that $\widetilde{J}$ is an arbitrary spectral orientation for T. We show that $\widetilde{V_+} := V^+_{\widetilde{J},j}$ equals $V_+ = V^+_{J,j}$. Theorem 14.2.10 implies then $J = \widetilde{J}$ because $\ker J = \ker \widetilde{J} = \operatorname{ran} E(\mathbb{R}) = \{0\}$ and $V^-_{J,j} = V_+ i = \widetilde{V_+} i = V^-_{\widetilde{J},j}$.

Since $\widetilde{J}$ commutes with T, we have $\widetilde{J}E_+ = E_+\widetilde{J}$, since

$$\begin{aligned}\widetilde{J}E_+y &= \int_{\Gamma_+} \widetilde{J}\mathcal{Q}_z(T)^{-1}(y\overline{z} - Ty)\, dz\frac{1}{2\pi j}\\&= \int_{\Gamma_+} \mathcal{Q}_z(T)^{-1}(\widetilde{J}y\overline{z} - T\widetilde{J}y)\, dz\frac{1}{2\pi j} = E_+\widetilde{J}y.\end{aligned} \tag{15.8}$$

The projection E_+ therefore leaves $\widetilde{V_+}$ invariant because

$$\widetilde{J}(E_+y) = E_+\big(\widetilde{J}y\big) = (E_+y)j \in \widetilde{V_+}$$

for every $y \in \widetilde{V_+}$. Hence $E_+|_{\widetilde{V_+}}$ is a projection on $\widetilde{V_+}$.

We show now that $\ker E_+|_{\widetilde{V_+}} = \{0\}$, so that $E_+|_{\widetilde{V}_+} = \mathcal{I}_{V_+}$ and hence $\widetilde{V_+} \subset \operatorname{ran} E_+ = V_+$. We do this by constructing a slice hyperholomorphic extension of $\mathcal{R}_s(T; y)$ that is defined on all of $\mathbb{H}$ and applying Corollary 15.1.8 tor any $y \in \ker E_+|_{\widetilde{V_+}}$.

Let $y \in \ker E_+|_{\widetilde{V_+}}$. Since $\ker E_+|_{\widetilde{V_+}} \subset \ker E_+ = \operatorname{ran} E_- = V_-$, we obtain $y \in V_-$. For $z = z_0 + z_1 j \in \mathbb{C}_j$, we define the function

$$f_j(z; y) := \begin{cases} R_z(T_-)y, & z_1 \geq 0,\\ \Big(z_0\mathcal{I} + z_1\widetilde{J} - T)\Big)^{-1} y, & z_1 < 0.\end{cases}$$

This function is (right) holomorphic on $\mathbb{C}_j$. On $\mathbb{C}_j^+$ this is obvious because the resolvent of T_- is a holomorphic function. For $z_1 < 0$, we have

$$\begin{aligned}
&\frac{1}{2}\left(\frac{\partial}{\partial z_0} f_j(z;y) + \frac{\partial}{\partial z_1} f_j(z;y)j\right)\\
&= \frac{1}{2}\left(-\left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2} y - \left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2}\widetilde{J}yj\right)\\
&= \frac{1}{2}\left(-\left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2} y - \left(z_0\mathcal{I} + z_1\widetilde{J} - T)\right)^{-2} yj^2\right) = 0,
\end{aligned}$$

since $\widetilde{J}y = yj$ because $y \in \widetilde{V_+} = V^+_{\widetilde{J},j}$. The slice extension $f(s;y)$ of $f_j(s;y)$ obtained from Lemma 2.1.11 is a slice hyperholomorphic extension of $\mathcal{R}_s(T;y)$ to all of $\mathbb{H}$ in the sense of Definition 15.1.3. Indeed, since

$$\mathcal{Q}_z(T)|_{V_-} = \mathcal{Q}_z(T_-) = (\mathcal{I}_{V_-}\overline{z} - T_-)(\mathcal{I}_{V_-}z - T_-),$$

we find for $s \in \mathbb{C}_j^+$ that

$$\begin{aligned}
&\mathcal{Q}_s(T)f(s;y) = \mathcal{Q}_s(T_-)f_j(s;y)\\
&= (\overline{s}\mathcal{I}_{V_-} - T_-)(s\mathcal{I}_{V_-} - T_-)R_s(T_-)y\\
&= (\overline{s}\mathcal{I}_{V_-} - T_-)y = ys - T_-y = ys - Ty.
\end{aligned}$$

On the other hand, the facts that T and $\widetilde{J}$ commute and that $-\widetilde{J}^2 = \mathcal{I}$ because $\widetilde{J}$ is an imaginary operator with $\operatorname{ran}\widetilde{J} = V_R$ imply

$$\begin{aligned}
&\left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)\\
&= s_0^2\mathcal{I} - s_0s_1\widetilde{J} - s_0T + s_0s_1\widetilde{J} - s_1^2\widetilde{J}^2 - s_1\widetilde{J}T - s_0T + s_1T\widetilde{J} + T^2\\
&= |s|^2\mathcal{I} - 2s_0T + T^2 = \mathcal{Q}_s(T).
\end{aligned}$$

For $s = s_1 + (-j)s_1 \in \mathbb{C}_j^-$, we find thus because of $y \in \widetilde{V_+} = V^+_{\widetilde{J},j}$ that

$$\begin{aligned}
\mathcal{Q}_s(T)f(s;y) &= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)f_j(s;y)\\
&= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)\left(s_0\mathcal{I} - s_1\widetilde{J} - T\right)^{-1} y\\
&= \left(s_0\mathcal{I} + s_1\widetilde{J} - T\right)y = ys_0 + yjs_1 - T = y\overline{s} - Ty.
\end{aligned}$$

Finally, for $s \notin \mathbb{C}_j$, the representation formula yields

$$\begin{aligned}
\mathcal{Q}_s(T)f(s;y) &= \mathcal{Q}_s(T)f_j(s_j;y)(1 - jj_s)\frac{1}{2} + \mathcal{Q}_s(T)f_j\left(\overline{s_j};y\right)(1 + jj_s)\frac{1}{2}\\
&= \left(y\overline{s_j} - Ty\right)(1 - jj_s)\frac{1}{2} + (ys_j - Ty)(1 + jj_s)\frac{1}{2}\\
&= y\left(\overline{s_j}(1 - jj_s) + s(1 + jj_s)\right)\frac{1}{2} - Ty\left((1 - jj_s) + (1 + jj_s)\right)\frac{1}{2}\\
&= y(s_j + \overline{s_j} + (s_j - \overline{s_j})jj_s)\frac{1}{2} - Ty = y(s_0 - s_1j_s) - Ty = y\overline{s} - Ty.
\end{aligned}$$

From Corollary 15.1.8, we hence deduce that $y = 0$, and so $\ker E_+|_{\widetilde{V_+}} = \{0\}$. Since $E_+|_{\widetilde{V_+}}$ is a projection on $\widetilde{V_+}$, we have $\widetilde{V_+} = \ker E_+|_{\widetilde{V_+}} \oplus \operatorname{ran} E_+|_{\widetilde{V_+}} = \{0\} \oplus \operatorname{ran} E_+|_{\widetilde{V_+}}$. We conclude that $\widetilde{V_+} = \operatorname{ran} E_+|_{\widetilde{V_+}} \subset \operatorname{ran} E_+ = V_+$. We therefore have

$$V_R = \widetilde{V_+} \oplus \widetilde{V_+} i \subset V_+ \oplus V_+ i = V_R.$$

This implies $V_+ = \widetilde{V_+}$ and in turn $J = \widetilde{J}$. □

Corollary 15.1.15. *Let $T \in \mathcal{B}(V_R)$ and assume that there exists a spectral resolution for T as in Proposition 15.1.14. If $\sigma_S(T) = \Delta_1 \cup \Delta_2$ with closed sets $\Delta_1, \Delta_2 \in \mathfrak{B}_S(\mathbb{H})$ such that $\Delta_1 \subset \mathbb{R}$ and $\Delta_2 \cap \mathbb{R} = \emptyset$, then there exists a unique imaginary operator $J \in \mathcal{B}(V_R)$ that is a spectral orientation for T such that T is a spectral operator with spectral decomposition (E, J).*

Proof. Let $T_2 = T_2|_{V_2}$, where $V_2 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} E(\Delta_2)$. Then the spectral measure $E_2(\Delta) := E(\Delta)|_{V_2}$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$ is by Lemma 15.1.2 a spectral resolution for T_2. Since $\sigma_S(T_2) \subset \Delta_2$ and $\Delta_2 \cap \mathbb{R} = \emptyset$, Proposition 15.1.14 implies the existence of a unique spectral orientation J_2 for T_2.

The fact that (E_2, J_2) is a spectral system implies $\operatorname{ran} J_2 = \operatorname{ran} E_2(\mathbb{H}\setminus\mathbb{R})V_2 = V_2$ because $E_2(\mathbb{H} \setminus \mathbb{R}) = E(\mathbb{H} \setminus \mathbb{R})|_{V_2} = \mathcal{I}_{V_2}$. If we set $J = J_2 E(\mathbb{H} \setminus \mathbb{R})$, we find that $\ker J = \operatorname{ran} E(\mathbb{R})$ and $\operatorname{ran} J = V_2 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. We also have

$$\begin{aligned} E(\Delta)J &= E(\Delta \cap \mathbb{R})J_2 E(\mathbb{H} \setminus \mathbb{R}) + E(\Delta \setminus \mathbb{R})J_2 E(\mathbb{H} \setminus \mathbb{R}) \\ &= E_2(\Delta \setminus \mathbb{R})J_2 E(\mathbb{H} \setminus \mathbb{R}) = J_2 E_2(\Delta \setminus \mathbb{R})E(\mathbb{H} \setminus \mathbb{R}) \\ &= J_2 E(\Delta \setminus \mathbb{R})E(\mathbb{H} \setminus \mathbb{R}) = J_2 E(\mathbb{H} \setminus \mathbb{R})E(\Delta \setminus \mathbb{R}) = JE(\Delta), \end{aligned}$$

where the last identity used that $E(\mathbb{H} \setminus \mathbb{R})E(\Delta \cap \mathbb{R}) = 0$. Moreover, we have

$$-J^2 = -J_2 E(\mathbb{H} \setminus \mathbb{R})J_2 E(\mathbb{H} \setminus \mathbb{R}) = -J_2^2 E(\mathbb{H} \setminus \mathbb{R}) = E(\mathbb{H} \setminus \mathbb{R}),$$

so that $-J^2$ is a projection onto $\operatorname{ran} J = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ along $\ker J = \operatorname{ran} E(\mathbb{R})$. Hence, J is an imaginary operator and (E, J) is a spectral system on V_R. Finally, for every $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we have

$$\left((s_0\mathcal{I} - s_1 J - T)|_{V_2}\right)^{-1} = (s_0\mathcal{I}_{V_2} - s_1 J_2 - T_2)^{-1} \in \mathcal{B}(V_2),$$

and hence (E, J) is actually a spectral decomposition for T.

In order to show the uniqueness of J we consider an arbitrary spectral orientation $\widetilde{J}$ for T. Then

$$\ker \widetilde{J} = E(\mathbb{R})V_R = \ker J \quad \text{and} \quad \operatorname{ran} \widetilde{J} = E(\mathbb{H} \setminus \mathbb{R})V_R = \operatorname{ran} J. \tag{15.9}$$

By Lemma 15.1.2, the operator $\widetilde{J}|_{V_2}$ is a spectral orientation for T_2. The spectral orientation of T_2 is, however, unique by Proposition 15.1.14, and hence $\widetilde{J}|_{V_2} = J_2 = J|_{V_2}$. We conclude that $\widetilde{J} = J$. □

Finally, we can now show the uniqueness of the spectral orientation of an arbitrary spectral operator.

Theorem 15.1.16. *The spectral decomposition* (E, J) *of a spectral operator* $T \in \mathcal{B}(V_R)$ *is uniquely determined.*

Proof. The uniqueness of the spectral resolution E has already been shown in Lemma 15.1.13. Let J and $\widetilde{J}$ be two spectral orientations for T. Since (15.9) holds also in this case, we can reduce the problem to showing that $J|_{V_1} = \widetilde{J}|_{V_1}$ with $V_1 := \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. The operator $T_1 := T|_{V_1}$ is a spectral operator on V_1. By Lemma 15.1.2, (E_1, J_1) and $(E_1, \widetilde{J}_1)$ with $E_1(\Delta) = E(\Delta)|_{V_1}$ and $J_1 = J|_{V_1}$ and $\widetilde{J}_1 := \widetilde{J}|_{V_1}$ are spectral decompositions of T_1. Since $E_0(\mathbb{R}) = 0$, it is hence sufficient to show the uniqueness of the spectral orientation of a spectral operator under the assumption $E(\mathbb{R}) = 0$.

Therefore, let T be a spectral operator with spectral decomposition (E, J) such that $E(\mathbb{R}) = 0$. If $\operatorname{dist}(\sigma_S(T), \mathbb{R}) > 0$, then we already know that the statement holds. We have shown this in Proposition 15.1.14. Otherwise, we choose a sequence of pairwise disjoint sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$ with $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ that cover $\sigma_S(T) \setminus \mathbb{R}$. We can choose, for instance,

$$\Delta_n := \left\{ s \in \mathbb{H} : \ -\|T\| \le s_0 \le \|T\|,\ \frac{\|T\|}{n+1} < s_1 \le \frac{\|T\|}{n} \right\}.$$

By Corollary 15.1.10 and since $E(\mathbb{R}) = 0$, we have

$$E(\sigma_S(T) \setminus \mathbb{R}) = E(\sigma_S(T) \setminus \mathbb{R}) + E(\sigma_S(T) \cap \mathbb{R}) = E(\sigma_S(T)) = \mathcal{I}.$$

We therefore obtain $\sum_{n=0}^{+\infty} E(\Delta_n) y = E\left(\bigcup_{n\in\mathbb{N}} \Delta_n\right) y = y$ for every $y \in V_R$ because we have $\sigma_S(T) \setminus \mathbb{R} \subset \bigcup_{n\in\mathbb{N}} \Delta_n$.

Since $E(\Delta_n)$ and J commute, the operator J leaves $V_{\Delta_n} := \operatorname{ran} E(\Delta_n)$ invariant. Hence $J_{\Delta_n} = J|_{V_{\Delta_n}}$ is a bounded operator on V_{Δ_n}, and we have

$$Jy = J \sum_{n=0}^{+\infty} E(\Delta_n) y = \sum_{n=1}^{+\infty} J E(\Delta_n) y = \sum_{n=1}^{+\infty} J_{\Delta_n} E(\Delta_n) y.$$

Similarly, we see that also $\widetilde{J_{\Delta_n}} := \widetilde{J}|_{V_{\Delta_n}}$ is a bounded operator on V_{Δ_n} and that $\widetilde{J} y = \sum_{n=1}^{+\infty} \widetilde{J}_{\Delta_n} E(\Delta_n) y$.

Now observe that T_{Δ_n} is a spectral operator. Its spectral resolution is given by $E_n(\Delta) := E(\Delta)|_{V_{\Delta_n}}$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, as one can check easily. Its spectral orientation is given by J_{Δ_n}: for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$, we have

$$E_n(\Delta) J_{\Delta_n} E(\Delta_n) = E(\Delta) J E(\Delta_n) = J E(\Delta) E(\Delta_n) = J_{\Delta_n} E_n(\Delta) E(\Delta_n)$$

and hence $E_n(\Delta) J_{\Delta_n} = J_{\Delta_n} E(\Delta_n)$ on V_{Δ_n}. Since $\ker J_{\Delta_n} = \{0\} = E_n(\mathbb{R})$ and $\operatorname{ran} J_{\Delta_n} = V_{\Delta_n} = E_n(\mathbb{H} \setminus \mathbb{R})$, the pair (E, J_{Δ_n}) is actually a spectral system. Furthermore, the operators T_{Δ_n} and J_{Δ_n} commute, since

$$T_{\Delta_n} J_{\Delta_n} E(\Delta_n) = T J E(\Delta_n) = J T E(\Delta_n) = J_{\Delta_n} T_{\Delta_n} E(\Delta_n).$$

Finally, for all $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we obtain

$$(s_0\mathcal{I}_{V_{\Delta_n}} - s_1 J_{\Delta_n} - T_{\Delta_n})^{-1} = (s_0\mathcal{I} - s_1 J - T)^{-1}|_{V_{\Delta_n}},$$

so that (E_n, J_{Δ_n}) is actually a spectral decomposition for T_{Δ_n}. However, the same arguments show that also $\left(E_n, \widetilde{J}_{\Delta_n}\right)$ is a spectral decomposition for T_{Δ_n}. Since, however, $\sigma_S(T_{\Delta_n}) \subset \Delta_n$ and $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$, Proposition 15.1.14 implies that the spectral orientation of T_{Δ_n} is unique such that $J_{\Delta_n} = \widetilde{J}_{\Delta_n}$. We thus obtain

$$Jy = \sum_{n=1}^{+\infty} J_{\Delta_n} E(\Delta_n)y = \sum_{n=1}^{+\infty} \widetilde{J}_{\Delta_n} E(\Delta_n)y = \widetilde{J}y. \qquad \square$$

Remark 15.1.17. In Proposition 15.1.14 and Corollary 15.1.15 we showed that under certain assumptions the existence of a spectral resolution E for T already implies the existence of a spectral orientation and is hence a sufficient condition for T to be a spectral operator. One may wonder whether this is true in general. An intuitive approach for showing this follows the idea of the proof of Theorem 15.1.16. We can cover $\sigma_S(T)\setminus\mathbb{R}$ by pairwise disjoint sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$ with $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ for each $n \in \mathbb{N}$. On each of the subspaces $V_n := \operatorname{ran} E(\Delta_n)$, the operator T induces the operator $T_n := T|_{V_n}$ with $\sigma_S(T_n) \subset \overline{\Delta_n}$. Since $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$, we can then define $\Delta_{n,+} := \Delta_n \cap \mathbb{C}_j^+$ and $\Delta_{n,-} := \Delta_n \cap \mathbb{C}_j^-$ for an arbitrary imaginary unit $j \in \mathbb{S}$ and consider the Riesz projectors $E_{n,+} := \chi_{\Delta_{n,+}}(T_n)$ and $E_{n,-} := \chi_{\Delta_{n,-}}(T_n)$ of T_n on $V_{n,j}$ associated with $\Delta_{n,+}$ and $\Delta_{n,-}$. Just as we did in the proof of Proposition 15.1.14, we can then construct a spectral orientation for T_n by setting $J_n y = E_{n,+}yj + E_{n,-}y(-j)$ for $y \in V_n$. The spectral orientation of J must then be

$$Jy = \sum_{n=1}^{+\infty} J_n E(\Delta_n)y = \sum_{n=1}^{+\infty} E_{n,+}E(\Delta_n)yj + E_{n,-}E(\Delta_n)y(-j). \tag{15.10}$$

If T is a spectral operator, then $E_{n,+} = E_+|_{V_n}$ and $E_{n,-} = E_-|_{V_n}$, where E_+ and E_- are as usual the projections of V_R onto $V_{J,j}^+$ and $V_{J,j}^-$ along $V_0 \oplus V_{J,j}^-$ resp. $V_0 \oplus V_{J,j}^+$. Hence the Riesz projectors $E_{n,+}$ and $E_{n,-}$ are uniformly bounded in $n \in \mathbb{N}$, and the above series converges. The spectral orientation of T can therefore be constructed as described above if T is a spectral operator.

This procedure, however, fails if the Riesz projectors $E_{n,+}$ and $E_{n,-}$ are not uniformly bounded, because the convergence of the above series is in this case not guaranteed. The next example presents an operator for which the above series does actually not converge for this reason although the operator has a quaternionic spectral resolution. Hence the existence of a spectral resolution does not in general imply the existence of a spectral orientation.

Example 15.1.18. Let $\ell^2(\mathbb{H})$ be the space of all square-summable sequences with quaternionic entries and choose $j, i \in \mathbb{S}$ with $j \perp i$. We define an operator T on

$\ell^2(\mathbb{H})$ by the following rule: if $(b_n)_{n\in\mathbb{N}} = T(a_n)_{n\in\mathbb{N}}$, then

$$\begin{pmatrix} b_{2m-1} \\ b_{2m} \end{pmatrix} = \frac{1}{m^2}\begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix}\begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix}. \tag{15.11}$$

For neatness, let us denote the matrix in the above equation by J_m and let us set $T_m := \frac{1}{m^2}J_m$, that is,

$$J_m := \begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix} \quad \text{and} \quad T_m := \frac{1}{m^2}\begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix}.$$

Since all matrix norms are equivalent, there exists a constant $C > 0$ such that

$$\|M\| \leq C \max_{\ell,\kappa\in\{1,2\}} |m_{\ell,\kappa}| \quad \forall M = \begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{pmatrix} \in \mathbb{H}^{2\times 2}, \tag{15.12}$$

such that $\|J_m\| \leq 2Cm$. We thus find for (15.11) that

$$\|(b_{2m-1}, b_{2m})^T\|_2 \leq \frac{2C}{m}\|(a_{2m-1}, a_{2m})^T\|_2 \leq 2C\|(a_{2m-1}, a_{2m})^T\|_2,$$

and in turn

$$\begin{aligned} \|T(a_n)_{n\in\mathbb{N}}\|^2_{\ell^2(\mathbb{H})} &= \sum_{m=1}^{+\infty} |b_{2m-1}|^2 + |b_{2m}|^2 \\ &\leq \sum_{m=1}^{+\infty} 4C^2\left(|a_{2m-1}|^2 + |a_{2m}|^2\right) = 4C^2\,\|(a_n)_{n\in\mathbb{N}}\|^2_{\ell^2(\mathbb{H})}. \end{aligned} \tag{15.13}$$

Hence T is a bounded right-linear operator on $\ell^2(\mathbb{H})$.

We show now that the S-spectrum of T is the set $\Lambda = \{0\} \cup \cup_{n\in\mathbb{N}}\frac{1}{n^2}\mathbb{S}$. For $s \in \mathbb{H}$, the operator $\mathcal{Q}_s(T) = T^2 - 2s_0T + |s|^2$ is given by the following relation: if $(c_n)_{n\in\mathbb{N}} = \mathcal{Q}_s(T)(a_n)_{n\in\mathbb{N}}$, then

$$\begin{pmatrix} c_{2m-1} \\ c_{2m} \end{pmatrix} = \begin{pmatrix} -\frac{1}{m^2} - 2j\frac{s_0}{m^2} + |s|^2 & -4j\frac{s_0}{m} \\ 0 & -\frac{1}{m^2} - 2j\frac{s_0}{m^2} + |s|^2 \end{pmatrix}\begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix}. \tag{15.14}$$

The inverse of the above matrix is

$$\begin{aligned} \mathcal{Q}_s(T_m)^{-1} &= \begin{pmatrix} \frac{m^4}{|s|^2m^4 - 2is_0m^2 - 1} & \frac{4im^7s_0}{|s|^4m^8 + 2\left(s_0^2 - s_1^2\right)m^4 + 1} \\ 0 & \frac{m^4}{|s|^2m^4 + 2is_0m^2 - 1} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\left(s_j - \frac{j}{m^2}\right)\left(\overline{s_j} - \frac{j}{m^2}\right)} & \frac{4is_0}{m\left(s_j + \frac{j}{m^2}\right)\left(s_j - \frac{j}{m^2}\right)\left(\overline{s_j} + \frac{j}{m^2}\right)\left(\overline{s_j} - \frac{j}{m^2}\right)} \\ 0 & \frac{1}{\left(s_j + \frac{j}{m^2}\right)\left(\overline{s_j} + \frac{j}{m^2}\right)} \end{pmatrix} \end{aligned}$$

with $s_j = s_0 + js_1$. Hence $\mathcal{Q}_s(T_m)^{-1}$ exists for $s_j \neq \frac{1}{m^2}j$. We have

$$\left|s_j - \frac{j}{m^2}\right|\left|\overline{s_j} - \frac{j}{m^2}\right| = \left|s_j + \frac{j}{m^2}\right|\left|\overline{s_j} + \frac{j}{m^2}\right| \geq 2\left|s_j - \frac{j}{m^2}\right| = 2\mathrm{dist}\left(s, \left[\frac{j}{m}\right]\right),$$

and so

$$\left\|\mathcal{Q}_s(T_m)^{-1}\right\| \leq C \max\left\{\frac{1}{2\mathrm{dist}\left(s,\left[\frac{j}{m^2}\right]\right)}, \frac{|s_0|}{m\left(\mathrm{dist}\left(s,\left[\frac{j}{m^2}\right]\right)\right)^2}\right\}, \tag{15.15}$$

where C is the constant in (15.12). If $s \notin \Lambda$, then $0 < \mathrm{dist}(s, \Lambda) \leq \mathrm{dist}\left(s, \left[\frac{j}{m^2}\right]\right)$ and hence the matrices $\mathcal{Q}_s(T_m)^{-1}$ are for $m \in \mathbb{N}$ uniformly bounded by

$$\left\|\mathcal{Q}_s(T_m)^{-1}\right\| \leq C \max\left\{\frac{1}{2\mathrm{dist}\,(s,\Lambda)}, \frac{|s_0|}{\mathrm{dist}\,(s,\Lambda)^2}\right\}.$$

The operator $\mathcal{Q}_s(T)^{-1}$ is then given by the relation

$$\begin{pmatrix} a_{2m-1} \\ a_{2m} \end{pmatrix} = \mathcal{Q}_s(T_m)^{-1} \begin{pmatrix} c_{2m-1} \\ c_{2m} \end{pmatrix}, \tag{15.16}$$

for $(a_n)_{n\in\mathbb{N}} = Q_s(T)^{-1}(c_n)_{n\in\mathbb{N}}$. A computation similar to the one in (15.13) shows that this operator is bounded on $\ell^2(\mathbb{H})$. Thus $s \in \rho_S(T)$ if $s \notin \Lambda$ and in turn $\sigma_S(T) \subset \Lambda$.

For every $m \in \mathbb{N}$, we set $s_m = \frac{1}{m^2}j$. The sphere $[s_m] = \frac{1}{m^2}\mathbb{S}$ is an eigensphere of T and the associated eigenspace V_m is the right-linear span of $\mathbf{e}_{2m-1}$ and $\mathbf{e}_{2m}$, where $\mathbf{e}_n = (\delta_{n,\ell})_{\ell\in\mathbb{N}}$, as one can see easily from (15.14). A straightforward computation, moreover, shows that the vectors $y_{2m-1} := \mathbf{e}_{2m-1}$ and $y_{2m} := -\mathbf{e}_{2m-1}i + \frac{1}{m}\mathbf{e}_{2m}i$ are eigenvectors of T with respect to the eigenvalue s_m. Hence $[s_m] \subset \sigma_S(T)$. Since $\sigma_S(T)$ is closed, we finally obtain $\Lambda = \overline{\bigcup_{m\in\mathbb{N}}[s_m]} \subset \sigma_S(T)$ and in turn $\sigma_S(T) = \Lambda$.

Let E_m for $m \in \mathbb{N}$ be the orthogonal projection of $\ell^2(\mathbb{H})$ onto the subspace $V_m := \mathrm{span}_{\mathbb{H}}\{\mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}$, that is, $E_m(a_n)_{n\in\mathbb{N}} = \mathbf{e}_{2m-1}a_{2m-1} + \mathbf{e}_{2m}a_{2m}$. We define for every set $\Delta \in \mathfrak{B}_S(\mathbb{H})$ the operator

$$E(\Delta) = \sum_{m\in I_\Delta} E_m \quad \text{with} \quad I_\Delta := \left\{m \in \mathbb{N} : \frac{1}{m^2}\mathbb{S} \subset \Delta\right\}.$$

It is immediate that E is a spectral measure on $\ell^2(\mathbb{H})$, that $\|E(\Delta)\| \leq 1$ for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and that $E(\Delta)$ commutes with T for every $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Moreover, if $s \notin \overline{\Delta}$, then the pseudo-resolvent $\mathcal{Q}_s(T_\Delta)^{-1}$ of $T_\Delta = T|_{V_\Delta}$ with $V_\Delta = \mathrm{ran}\,E(\Delta)$ is given by

$$\mathcal{Q}_s(T_\Delta)^{-1} = \left.\left(\sum_{m\in I_\Delta} \mathcal{Q}_s(T_m)^{-1}E_m\right)\right|_{\mathrm{ran}\,E(\Delta)}.$$

Since $0 < \mathrm{dist}\left(s, \bigcup_{m\in I_\Delta}\left[\frac{j}{m^2}\right]\right) = \inf_{m\in I_\Delta} \mathrm{dist}\left(s, \left[\frac{j}{m^2}\right]\right)$, the operators $\mathcal{Q}_s(T_m)^{-1}$ are uniformly bounded for $m \in I_\Delta$. Computations similar to (15.13) show that $\mathcal{Q}_s(T_\Delta)^{-1}$ is a bounded operator on V_Δ. Hence $s \in \rho_S(T_\Delta)$ and in turn $\sigma_S(T_\Delta) \subset \overline{\Delta}$. Altogether we obtain that E is a spectral resolution for T.

In order to construct a spectral orientation for T, we first observe that J_m is a spectral orientation for T_m. For $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, we have

$$s_0\mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m = \begin{pmatrix} s_0 - \left(s_1 + \frac{1}{m^2}\right) j & -\left(s_1 + \frac{1}{m^2}\right) 2mj \\ 0 & s_0 + \left(s_1 + \frac{1}{m^2}\right) j \end{pmatrix},$$

the inverse of which is given by the matrix

$$(s_0\mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m)^{-1} = \begin{pmatrix} \frac{1}{s_0 - \left(s_1 + \frac{1}{m^2}\right)j} & \frac{2jm\left(\frac{1}{m^2}+s_1\right)}{s_0^2 + \left(\frac{1}{m^2}+s_1\right)^2} \\ 0 & \frac{1}{s_0 + \left(\frac{1}{m^2}+s_1\right)j} \end{pmatrix}.$$

Since $s_1 > 0$, each entry has nonzero denominator, and hence we have that the operator $(s_0\mathcal{I}_{\mathbb{H}^2} - s_1 J_m - T_m)^{-1}$ belongs to $\mathcal{B}(\mathbb{H}^2)$.

If $J \in \mathcal{B}(\ell^2(\mathbb{H}))$ is a spectral orientation for T, then the restriction $J|_{V_m}$ of J to $V_m = \operatorname{span}_{\mathbb{H}}\{\mathbf{e}_{2m-1}, \mathbf{e}_{2m}\}$ is also a spectral orientation for T_m. The uniqueness of the spectral orientation implies $J|_{V_m} = J_m$ and hence

$$J = \sum_{m=1}^{+\infty} J|_{V_m} E\left(\frac{1}{m^2}\mathbb{S}\right) = \sum_{m=1}^{+\infty} J_m E_m.$$

This series does not, however, converge, because the operators J_{V_m} are not uniformly bounded. Hence, it does not define a bounded operator on $\ell^2(\mathbb{H})$. Indeed, the sequence $a_{2m-1} = 0$, $a_{2m} = m^{-\frac{3}{2}}$, for instance, belongs to $\ell^2(\mathbb{H})$, but

$$\begin{aligned}\left\| \sum_{m=1}^{+\infty} J_m E_m (a_n)_{n\in\mathbb{N}} \right\|_{\ell^2(\mathbb{H})}^2 &= \sum_{m=1}^{+\infty} \left\| \begin{pmatrix} j & 2mj \\ 0 & -j \end{pmatrix} \begin{pmatrix} 0 \\ m^{-\frac{3}{2}} \end{pmatrix} \right\|_2^2 \\ &= 2\sum_{m=1}^{+\infty} 4\frac{1}{m} + \frac{1}{m^3} = +\infty.\end{aligned}$$

Hence there cannot exist a spectral orientation for T, and in turn T is not a spectral operator on $\ell^2(\mathbb{H})$.

We conclude this example with a remark on its geometric intuition. Let us identify $\mathbb{H}^2 \cong \mathbb{C}_j^4$, which is for every $i \in \mathbb{S}$ with $i \perp j$ spanned by the basis vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} i \\ 0 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_4 = \begin{pmatrix} 0 \\ i \end{pmatrix}.$$

The vectors $y_{m,1} = b_1$ and $y_{m,2} = -b_2 + \frac{1}{m}b_4$ are eigenvectors of J_m with respect to j, and the vectors $y_1 i = b_2$ and $y_{m,2} = b_1 - \frac{1}{m}b_3$ are eigenvectors of J_m with respect to $-j$. We thus obtain $V^+_{J_m,j} = \operatorname{span}_{\mathbb{C}_j}\{b_1, -b_2 + \frac{1}{m}b_4\}$ and $V^-_{J_m,j} = V^+_{J_m,j}i$. However, as m tends to infinity, the vector y_2 tends to $y_1 i$ and $y_2 i$ tends to y_1. Hence intuitively, in the limit $V^-_{J_m,j} = V^+_{J_m,j}i = V^+_{J_m,j}$, and consequently the projections of $\mathbb{H}^2 = \mathbb{C}_j^4$ onto $V^+_{J_m,j}$ along $V^-_{J_m,j}$ become unbounded.

Finally, the notion of quaternionic spectral operator is backward compatible with the complex theory on $V_{R,j}$.

Theorem 15.1.19. *An operator $T \in \mathcal{B}(V_R)$ is a quaternionic spectral operator if and only if it is a spectral operator on $V_{R,j}$ for some (and hence every) $j \in \mathbb{S}$. (See [106] for the complex theory.) If furthermore (E, J) is the quaternionic spectral decomposition of T and E_j is the spectral resolution of T as a complex $\mathbb{C}_j$-linear operator on $V_{R,j}$, then*

$$\begin{aligned} E(\Delta) &= E_j(\Delta \cap \mathbb{C}_j) \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}), \\ Jy &= E_j(\mathbb{C}_j^{+,\circ}\backslash)yj + E_j(\mathbb{C}_j^{-,\circ})y(-j) \quad \forall y \in V_R \end{aligned} \tag{15.17}$$

with

$$\mathbb{C}_j^{\pm,\circ} := \mathbb{C}_j^{\pm} \setminus \mathbb{R} = \{z_0 + jz_1 : \ z_0 \in \mathbb{R}, z_1 > 0\}.$$

Conversely, E_j is the spectral measure on V_R determined by (E, J) that was constructed in Lemma 14.3.8.

Proof. Let us first assume that $T \in \mathcal{B}(V_R)$ is a quaternionic spectral operator with spectral decomposition (E, J) in the sense of Definition 15.1.1 and let $j \in \mathbb{S}$. Let E_+ be the projection of $\operatorname{ran} J = V_{J,j}^+ \oplus V_{J,j}^-$ onto $V_{J,j}^+$ along $V_{J,j}^-$ and let E_- be the projection of $\operatorname{ran} J$ onto $V_{J,j}^-$ along $V_{J,j}^+$; cf. Theorem 14.2.10. Since T and $E(\Delta)$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$ commute with J, they leave the spaces $V_{J,j}^+$ and $V_{J,j}^-$ invariant, and hence they commute with E_+ and E_-. By Lemma 14.3.8, the set function E_j on $\mathbb{C}_j$ defined in (14.23), which is given by

$$E_j(\Delta) = E_+ E\left(\left[\Delta \cap \mathbb{C}_j^{+,\circ}\right]\right) + E(\Delta \cap \mathbb{R}) + E_- E\left(\left[\Delta \cap \mathbb{C}_j^{-,\circ}\right]\right), \tag{15.18}$$

for $\Delta \in \mathfrak{B}(\mathbb{C}_j)$, is a spectral measure on $V_{R,j}$. Since the spectral measure E and the projections E_+ and E_- commute with T, the spectral measure E_j commutes with T too.

If $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ is a subset of $\mathbb{C}_j^{+,\circ}$, then $Jy = yj$ for $y \in V_{j,\Delta} := \operatorname{ran} E_j(\Delta)$, since $\operatorname{ran} E_j(\Delta) = \operatorname{ran}(E_+ E([\Delta])) \subset V_{J,j}^+$. For $z = z_0 + jz_1 \in \mathbb{C}_j$ and $y \in V_{j,\Delta}$, we thus have

$$\begin{aligned} (z\mathcal{I}_{V_{j,\Delta}} - T)y &= yz_0 + yjz_1 - Ty \\ &= yz_0 + Jyz_1 - Ty = (z_0\mathcal{I}_{V_{j,\Delta}} + z_1 J - T)y. \end{aligned}$$

If $z \in \mathbb{C}_j^{-,\circ}$, then the inverse of $(z_0\mathcal{I}_{V_{R,j}} + z_1 J - T)|_{\operatorname{ran} J}$ exists because J is the spectral orientation of T. We thus have $R_z(T_\Delta) = (z_0\mathcal{I}_{V_{R,j}} + z_1 J - T)^{-1}|_{V_{j,\Delta}}$, and so $\mathbb{C}_j^{-,\circ} \subset \rho(T_\Delta)$. If, on the other hand, $z \in \mathbb{C}_j^+ \setminus \overline{\Delta}$, then $z \in \rho_S(T_{[\Delta]})$, where $T_{[\Delta]} = T|_{V_{[\Delta]}}$ with $V_{[\Delta]} = \operatorname{ran} E([\Delta])$. Hence $\mathcal{Q}_z(T_{[\Delta]})$ has a bounded inverse on $V_{[\Delta]}$. By the construction of E_j we have $V_{j,\Delta} = E_+ V_{[\Delta]}$, and since $T_{[\Delta]}$ and E_+ commute, $\mathcal{Q}_z(T_{[\Delta]})^{-1}$ leaves $V_{j,\Delta}$ invariant, so that $\mathcal{Q}_z(T_{[\Delta]})^{-1}\big|_{V_{j,\Delta}}$ defines a bounded $\mathbb{C}_j$-linear operator on $V_{j,\Delta}$. Because of Theorem 14.2.7, the resolvent of T_Δ at z is therefore given by

$$R_z(T)y = \mathcal{Q}_s(T_{[\Delta]})^{-1}(y\overline{z} - T_\Delta y) \quad \forall y \in V_{j,\Delta}.$$

Altogether, we conclude that $\rho(T_\Delta) \supset \mathbb{C}_j^{-,\circ} \cup \left(\mathbb{C}_j^{+} \setminus \overline{\Delta}\right) = \mathbb{C}_j \setminus \overline{\Delta}$ and in turn $\sigma(T_\Delta) \subset \overline{\Delta}$. Similarly, we see that $\sigma(T_\Delta) \subset \overline{\Delta}$ if $\Delta \subset \mathbb{C}_j^{-,\circ}$. If, on the other hand, $\Delta \subset \mathbb{R}$, then $E_j(\Delta) = E(\Delta)$, so that T_Δ is a quaternionic linear operator with $\sigma_S(T_\Delta) \subset \overline{\Delta}$. By Theorem 14.2.7, we have $\sigma(T_\Delta) = \sigma_{\mathbb{C}_j}(T_\Delta) = \sigma_S(T) \subset \overline{\Delta}$. Finally, if $\Delta \in \mathfrak{B}(\mathbb{C}_j)$ is arbitrary and $z \notin \overline{\Delta}$, we can set $\Delta_+ := \Delta \cap \mathbb{C}_j^{+,\circ}$, $\Delta_- := \Delta \cap \mathbb{C}_j^{-,\circ}$, and $\Delta_{\mathbb{R}} := \Delta \cap \mathbb{R}$. Then z belongs to the resolvent sets of each of the operators T_{Δ_+}, T_{Δ_-}, and $T_{\Delta_{\mathbb{R}}}$, and we obtain

$$R_z(T) = R_z(T_{\Delta_+})E_j(\Delta_+) + R_z(T_{\Delta_{\mathbb{R}}})E(\Delta_{\mathbb{R}}) + R_z(T_{\Delta_-})E_j(\Delta_-).$$

We thus have $\sigma(T_\Delta) \subset \overline{\Delta}$. Hence T is a spectral operator on $V_{R,j}$, and E_j is its ($\mathbb{C}_j$-complex) spectral resolution on $V_{R,j}$.

Now assume that T is a bounded quaternionic linear operator on V_R and that for some $j \in \mathbb{S}$ there exists a $\mathbb{C}_j$-linear spectral resolution E_j for T as a $\mathbb{C}_j$-linear operator on $V_{R,j}$. Following Definition 6 of [104, Chapter XV.2], an analytic extension of $R_z(T)y$ with $y \in V_{R,j} = V_R$ is a holomorphic function f defined on a set $\mathcal{D}(f)$ such that $(z\mathcal{I}_{V_{R,j}} - T)f(z) = y$ for $z \in \mathcal{D}(f)$. The resolvent $\rho(y)$ is the domain of the unique maximal analytic extension of $R_z(T)y$, and the spectrum $\sigma(y)$ is the complement of $\rho(y)$ in $\mathbb{C}_j$. (We defined the quaternionic counterparts of these concepts in Definition 15.1.3 and Definition 15.1.4.) Analogously to Theorem 15.1.9, we have

$$E_j(\Delta)V_{R,j} = \{y \in V_{R,j} = V_R : \ \sigma(y) \subset \Delta\}, \quad \forall \Delta \in \mathfrak{B}(\mathbb{C}_j). \tag{15.19}$$

Let $y \in V_{R,j}$, let $i \in \mathbb{S}$ with $j \perp i$, and let f be the unique maximal analytic extension of $R_z(T)y$ defined on $\rho(y)$. The mapping $z \mapsto f\left(\overline{z}\right)i$ is then holomorphic on $\overline{\rho(y)}$: for every $z \in \overline{\rho(y)}$, we have $\overline{z} \in \rho(y)$ and in turn

$$\lim_{h\to 0} \left(f\left(\overline{z+h}\right)i - f\left(\overline{z}\right)i\right)h^{-1} = \lim_{h\to 0}\left(f(\overline{z} + \overline{h}) - f(\overline{z})\right)\overline{h}^{-1} i = f'\left(\overline{z}\right)i.$$

Since T is quaternionic linear, we moreover have for $z \in \overline{\rho(y)}$ that

$$\left(z\mathcal{I}_{V_{R,j}} - T\right)\left(f\left(\overline{z}\right)i\right) = f\left(\overline{z}\right)iz - T\left(f\left(\overline{z}\right)i\right) = \left(f\left(\overline{z}\right)\overline{z} - T\left(f\left(\overline{z}\right)\right)\right)i = yi.$$

Hence $z \mapsto f\left(\overline{z}\right)i$ is an analytic extension of $R_z(T)(yi)$ that is defined on $\overline{\rho(y)}$. Consequently $\rho(yi) \supset \overline{\rho(y)}$, and in turn $\sigma(yi) \subset \overline{\sigma(y)}$. If $\tilde{f}$ is the maximal analytic extension of $R_z(T)(yi)$, then similar arguments show that $z \mapsto \tilde{f}\left(\overline{z}\right)(-i)$ is an analytic extension of $R_z(T)y$. Since this function is defined on $\overline{\rho(yi)}$, we obtain

$\rho(y) \supset \overline{\rho(yi)}$ and in turn $\sigma(y) \subset \overline{\sigma(yi)}$. Altogether, we obtain $\sigma(y) = \overline{\sigma(yi)}$ and $\tilde{f}(z) = f(\overline{z})\, i$. From (15.19) we deduce

$$\begin{aligned} \operatorname{ran} E_j\left(\overline{\Delta}\right) &= \left\{y \in V_{R,j} = V_R : \ \ \sigma(y) \subset \overline{\Delta}\right\} \\ &= \{yi \in V_{R,j} = V_R : \ \ \sigma(y) \subset \Delta\} = (\operatorname{ran} E_j(\Delta))\, i. \end{aligned} \tag{15.20}$$

In order to construct the quaternionic spectral resolution of T, we define now

$$E(\Delta) := E_j(\Delta \cap \mathbb{C}_j), \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}).$$

Obviously this operator is a bounded $\mathbb{C}_j$-linear projection on $V_R = V_{R,j}$. We show now that it is also quaternionic linear. Due to the axial symmetry of Δ, the identity (15.20) implies

$$\begin{aligned} (\operatorname{ran} E(\Delta))i = (\operatorname{ran} E_j(\Delta \cap \mathbb{C}_j))\, i &= \operatorname{ran} E_j\left(\overline{\Delta \cap \mathbb{C}_j}\right) \\ &= \operatorname{ran} E_j\left(\Delta \cap \mathbb{C}_j\right) = \operatorname{ran} E(\Delta). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} (\ker E(\Delta))i &= (\ker E_j(\Delta \cap \mathbb{C}_j))i = (\operatorname{ran} E_j(\mathbb{C}_j \setminus \Delta))i \\ &= \operatorname{ran} E_j\left(\overline{\mathbb{C}_j \setminus \Delta}\right) = \operatorname{ran} E_j\left(\mathbb{C}_j \setminus \Delta\right) \\ &= \ker E_j(\Delta \cap \mathbb{C}_j) = \ker E(\Delta). \end{aligned}$$

If we write $y \in V_R$ as $y = y_0 + y_1$ with $y_0 \in \ker E(\Delta)$ and $y_1 \in \operatorname{ran} E(\Delta)$, we thus have

$$E(\Delta)(yi) = E(\Delta)(y_0 i) + E(\Delta)(y_1 i) = y_1 i = (E(\Delta)y)i.$$

Writing $a \in \mathbb{H}$ as $a = a_1 + ia_2$ with $a_1, a_2 \in \mathbb{C}_j$, we find due to the $\mathbb{C}_j$-linearity of $E(\Delta)$ that even

$$E(\Delta)(ya) = (E(\Delta)y)a_1 + (E(\Delta)yi)a_2 = (E(\Delta)y)a_1 + (E(\Delta)y)ia_2 = (E(\Delta)y)a.$$

Hence the set function $\Delta \mapsto E(\Delta)$ defined in (15.18) takes values that are bounded quaternionic linear projections on V_R. It is immediate that it moreover satisfies items (i) to (iv) in Definition 14.1.7 because E_j is a spectral measure on $V_{R,j}$ and hence has the respective properties. Consequently, E is a quaternionic spectral measure. Since E_j commutes with T, also E commutes with T. From Theorem 14.2.7 and the fact that $\sigma(T|_{\operatorname{ran} E_j(\Delta_j)}) \subset \overline{\Delta_j}$ for $\Delta_j \in \mathfrak{B}(\mathbb{C}_j)$, we deduce for $T_\Delta = T|_{\operatorname{ran} E(\Delta)} = T|_{\operatorname{ran} E_j(\Delta \cap \mathbb{C}_j)}$ that

$$\sigma_S(T_\Delta) = \left[\sigma_{\mathbb{C}_j}(T_\Delta)\right] \subset \left[\overline{\Delta \cap \mathbb{C}_j}\right] = \overline{[\Delta \cap \mathbb{C}_j]} = \overline{\Delta}.$$

Therefore E is a spectral resolution for T.

Let us now set $V_0 = \operatorname{ran} E_j(\mathbb{R})$ as well as $V_+ := \operatorname{ran} E_j\left(\mathbb{C}_j^{+,\circ}\right)$ and $V_- := \operatorname{ran} E_j\left(\mathbb{C}_j^{-,\circ}\right)$. Then $V_{R,j} = V_0 \oplus V_+ \oplus V_-$ is a decomposition of V_R into closed

$\mathbb{C}_j$-linear subspaces. The space $V_0 = \operatorname{ran} E_j(\mathbb{R}) = \operatorname{ran} E(\mathbb{R})$ is even a quaternionic right linear subspace of V_R because $E(\mathbb{R})$ is a quaternionic right linear operator. Moreover, (15.20) shows that $y \mapsto yi$ is a bijection from V_+ to V_-. By Theorem 14.2.10, the operator

$$Jy = E_j\left(\mathbb{C}_j^{+,\circ}\right) yj + E_j\left(\mathbb{C}_j^{-,\circ}\right) y(-j)$$

is an imaginary operator on V_R. Since E_j commutes with T and $E(\Delta)$ for $\Delta \in \mathfrak{B}_S(\mathbb{H})$, also J commutes with T and $E(\Delta)$. Moreover, $\ker J = V_0 = \operatorname{ran} E(\mathbb{R})$ and $\operatorname{ran} J = \operatorname{ran} E_j(\mathbb{C}_j^{+,\circ}) \oplus \operatorname{ran} E_j(\mathbb{C}_j^{-,\circ}) = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$, and hence (E, J) is a spectral system that commutes with T. Finally, we have $\sigma(T_+) \subset \mathbb{C}_j^+$ for $T_+ = T|_{V_+} = T|_{\operatorname{ran} E_j(\mathbb{C}_j^{+,\circ})}$, and hence the resolvent of $R_z(T_+)$ exists for every $z \in \mathbb{C}_j^{-,\circ}$. Similarly, the resolvent $R_z(T_-)$ with $T_- = T|_{V_-} = T|_{\operatorname{ran} E_j(\mathbb{C}_j^{-,\circ})}$ exists for every $z \in \mathbb{C}_j^{+,\circ}$. For $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$ we can hence set $s_j = s_0 + js_1$ and define by

$$R(s_0, s_1) := \left(R_{\overline{s_j}}(T_+)E_+ + R_{s_j}(T_-)E_-\right)\big|_{V_+ \oplus V_-}$$

with $E_+ = E_j(\mathbb{C}_j^{+,\circ})$ and $E_- = E_j(\mathbb{C}_j^{-,\circ})$ a bounded operator on $V_+ \oplus V_- = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$. Since T leaves V_+ and V_- invariant, we then have for $y = y_+ + y_- \in V_+ \oplus V_-$ that

$$\begin{aligned}
&R(s_0, s_1)(s_0\mathcal{I} - s_1 J - T)y \\
&= R(s_0, s_1)\left(y_+ s_0 - Jy_+ s_1 - Ty_+ + y_- s_0 - Jy_- s_1 - Ty_-\right) \\
&= R(s_0, s_1)\left(y_+ \overline{s_j} - Ty_+\right) + R(s_0, s_1)\left(y_- s_j - Ty_-\right) \\
&= R_{\overline{s_j}}(T_+)\left(y_+ \overline{s_j} - T_+ y_+\right) + R_{s_j}(T_-)\left(y_- s_j - T_- y_-\right) = y_+ + y_- = y.
\end{aligned}$$

Similarly we find that

$$\begin{aligned}
&(s_0\mathcal{I} - s_1 J - T)R(s_0, s_1)y \\
&= (s_0\mathcal{I} - s_1 J - T)R_{\overline{s_j}}(T_+)y_+ + (s_0\mathcal{I} - s_1 J - T)R_{s_j}(T_-)y_- \\
&= R_{\overline{s_j}}(T_+)y_+ s_0 - J(R_{\overline{s_j}}(T_+)y_+)s_1 - TR_{\overline{s_j}}(T_+)y_+ \\
&\quad + R_{s_j}(T_-)y_- s_0 - J(R_{s_j}(T_-)y_-)s_1 - TR_{s_j}(T_-)y_- \\
&= R_{\overline{s_j}}(T_+)y_+(s_0 - js_1) - R_{\overline{s_j}}(T_+)T_+ y_+ \\
&\quad + R_{s_j}(T_-)y_-(s_0 + js_1) - R_{s_j}(T_-)T_- y_- \\
&= R_{\overline{s_j}}(T_+)\left(y_+ \overline{s} - T_+ y_+\right) + R_{s_j}(T_-)\left(y_- s - T_- y_-\right) = y_+ + y_- = y.
\end{aligned}$$

Hence $R(s_0, s_1)$ is the bounded inverse of $(s_0\mathcal{I} - s_1 J - T)|_{\operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})}$, and so J is actually a spectral orientation for T. Consequently, T is a quaternionic spectral operator, and the relation (15.17) holds.

□

Remark 15.1.20. We want to stress that Theorem 15.1.19 showed a one-to-one relation between quaternionic spectral operators on V_R and $\mathbb{C}_j$-complex spectral operators on $V_{R,j}$ that are furthermore compatible with the quaternionic scalar multiplication. It did not show a one-to-one relation between quaternionic spectral operators on V_R and $\mathbb{C}_j$-complex spectral operators on $V_{R,j}$. There exist $\mathbb{C}_j$-complex spectral operators on $V_{R,j}$ that are not quaternionic linear and hence cannot be quaternionic spectral operators.

15.2 Canonical Reduction and Intrinsic S-Functional Calculus for Quaternionic Spectral Operators

As in the complex case, every bounded quaternionic spectral operator T can be decomposed into the sum $T = S + N$ of a scalar operator S and a quasi-nilpotent operator N. The intrinsic S-functional calculus for a spectral operator can then be expressed as a Taylor series similar to the one that involves functions of S obtained via spectral integration and powers of N. Analogously to the complex case in [106], the operator $f(T)$ is therefore already determined by the values of f on $\sigma_S(T)$ and not only by its values on a neighborhood of $\sigma_S(T)$.

Definition 15.2.1. An operator $S \in \mathcal{B}(V_R)$ is said to be *of scalar type* if it is a spectral operator and satisfies the identity

$$S = \int s\, dE_J(s), \tag{15.21}$$

where (E, J) is the spectral decomposition of S.

Remark 15.2.2. If we start from a spectral system (E, J) and S is the operator defined by (15.21), then S is an operator of scalar type and (E, J) is its spectral decomposition. This can easily be checked by direct calculations or indirectly via the following argument: by Lemma 14.3.8, we can choose $j \in \mathbb{S}$ and obtain

$$S = \int_{\mathbb{H}} s\, dE_J(s) = \int_{\mathbb{C}_j} z\, dE_j(z),$$

where E_j is the spectral measure constructed in (14.23). From the complex theory in [106], we deduce that S is a spectral operator on $V_{R,j}$ with spectral decomposition E_j that is furthermore quaternionic linear. By Theorem 15.1.19, this is equivalent to S being a quaternionic spectral operator on V_R with spectral decomposition (E, J).

Lemma 15.2.3. *Let S be an operator of scalar type with spectral decomposition (E, J). An operator $A \in \mathcal{B}(V_R)$ commutes with S if and only if it commutes with the spectral system (E, J).*

Proof. If $A \in \mathcal{B}(V_R)$ commutes with (E, J), then it commutes with $S = \int_{\mathbb{H}} s\, dE_J(s)$ because of Lemma 14.3.6. If, on the other hand, A commutes with S, then it also commutes with E by Lemma 15.1.12. By Lemma 14.1.10, it commutes in turn with the operator $f(T) = \int_{\mathbb{H}} f(s)\, dE(s)$ for every $f \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$. If we define

$$S_0 := \int_{\mathbb{H}} \operatorname{Re}(s)\, dE(s) \quad \text{and} \quad S_1 := \int_{\mathbb{H}} \underline{s}\, dE_J(S) = J \int_{\mathbb{H}} |\underline{s}|\, dE(s),$$

where $\underline{s} = j_s s_1$ denotes the imaginary part of a quaternion s, then $AS = SA$ and $AS_0 = S_0 A$ and in turn

$$AS_1 = A(S - S_0) = AS - AS_0 = SA - S_0 A = (S - S_0)A = S_1 A.$$

We can now choose pairwise disjoint sets $\Delta_n \in \mathfrak{B}_S(\mathbb{H})$, $n \in \mathbb{N}$, such that $\sigma_S(T) \setminus \mathbb{R} = \bigcup_{n\in\mathbb{N}} \Delta_n$ and such that $\operatorname{dist}(\Delta_n, \mathbb{R}) > 0$ for every $n \in \mathbb{N}$. Then $s \mapsto |\underline{s}|^{-1} \chi_{\Delta_n}(s)$ belongs to $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ for every $n \in \mathbb{N}$, and in turn

$$\begin{aligned}
AJE(\Delta_n) &= AJ \left(\int_{\mathbb{H}} |\underline{s}||\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) \\
&= AJ \left(\int_{\mathbb{H}} |\underline{s}|\, dE(s) \right) \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) \\
&= AS_1 \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) \\
&= S_1 \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) A \\
&= J \left(\int_{\mathbb{H}} |\underline{s}|\, dE(s) \right) \left(\int_{\mathbb{H}} |\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) A \\
&= J \left(\int_{\mathbb{H}} |\underline{s}||\underline{s}|^{-1} \chi_{\Delta_n}(s)\, dE(s) \right) E(\Delta_n) A = JE(\Delta_n) A.
\end{aligned}$$

Since $\sigma_S(S) \setminus \mathbb{R} \subset \bigcup_{n\in\mathbb{N}} \Delta_n$, we have $\sum_{n=0}^{+\infty} E(\Delta_n) y = E(\sigma_S(T) \setminus \mathbb{R}) y = E(\mathbb{H} \setminus \mathbb{R}) y$ for all $y \in V_R$ by Corollary 15.1.10. Since $J = JE(\mathbb{H} \setminus \mathbb{R})$, we hence obtain

$$\begin{aligned}
AJy = AJE(\mathbb{H} \setminus \mathbb{R}) y &= \sum_{n=1}^{+\infty} AJE(\Delta_n) y \\
&= \sum_{n=1}^{+\infty} JE(\Delta_n) A y = JE(\mathbb{H} \setminus \mathbb{R}) A y = JAy,
\end{aligned}$$

which finishes the proof. □

Definition 15.2.4. An operator $N \in \mathcal{B}(V_R)$ is called *quasi-nilpotent* if

$$\lim_{n\to\infty} \|N^n\|^{\frac{1}{n}} = 0. \tag{15.22}$$

The following corollaries are immediate consequences of Gelfand's formula

$$r(T) = \lim_{n\to+\infty} \|T^n\|^{\frac{1}{n}},$$

for the spectral radius $r(T) = \max_{s\in\sigma_S(T)} |s|$ of T.

Corollary 15.2.5. *An operator $N \in \mathcal{B}(V_R)$ is quasi-nilpotent if and only if $\sigma_S(T) = \{0\}$.*

Corollary 15.2.6. *Let $S, N \in \mathcal{B}(V_R)$ be commuting operators and let N be quasi-nilpotent. Then $\sigma_S(S+N) = \sigma_S(S)$.*

We are now ready to show the main result of this section: the canonical reduction of a spectral operator, the quaternionic analogue of Theorem 5 in [106, Chapter XV.4.3].

Theorem 15.2.7. *An operator $T \in \mathcal{B}(V_R)$ is a spectral operator if and only if it is the sum $T = S + N$ of a bounded operator S of scalar type and a quasi-nilpotent operator N that commutes with S. Furthermore, this decomposition is unique, and T and S have the same S-spectrum and the same spectral decomposition (E, J).*

Proof. Let us first show that every operator $T \in \mathcal{B}(V_R)$ that is the sum $T = S+N$ of an operator S of scalar type and a quasi-nilpotent operator N commuting with S is a spectral operator. If (E, J) is the spectral decomposition of S, then Lemma 15.2.3 implies $E(\Delta)N = NE(\Delta)$ for all $\Delta \in \mathfrak{B}_S(\mathbb{H})$ and $JN = NJ$. Since $T = S + N$, we find that also T commutes with (E, J).

Let now $\Delta \in \mathfrak{B}_S(\mathbb{H})$. Then $T_\Delta = S_\Delta + N_\Delta$, where as usual the subscript Δ denotes the restriction of an operator to $V_\Delta = E(\Delta)V_R$. Since N_Δ inherits the property of being quasi-nilpotent from N and commutes with S_Δ, we deduce from Corollary 15.2.6,that

$$\sigma_S(T_\Delta) = \sigma_S(S_\Delta + N_\Delta) = \sigma_S(S_\Delta) \subset \overline{\Delta}.$$

Thus (E, J) satisfies items (i) and (ii) of Definition 15.1.1. It remains to show that also item (iii) holds true. Therefore, let $V_0 = \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R})$ and set $T_0 = T|_{V_0}$, $S_0 = S|_{V_0}$, $N_0 = N|_{V_0}$, and $J_0 = J|_{V_0}$ and choose $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$. Since (E, J) is the spectral resolution of S, the operator $s_0\mathcal{I}_{V_0} - s_1J_0 - S_0$ has a bounded inverse $R(s_0, s_1) = (s_0\mathcal{I}_{V_0} - s_1J_0 - S_0)^{-1} \in \mathcal{B}(V_0)$. The operator N_0 is quasi-nilpotent because N is quasi-nilpotent, and hence it satisfies (15.22). The root test thus shows the convergence of the series $\sum_{n=0}^{+\infty} N_0^n R(s_0, s_1)^{n+1}$ in $\mathcal{B}(V_0)$.

Since T_0, N_0, S_0, and J_0 commute mutually, we have

$$\begin{aligned}
&(s_0\mathcal{I}_{V_0} - s_1 J_0 - T_0)\sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1}\\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1}(s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0 - N_0)\\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1}(s_0\mathcal{I}_{V_0} - s_1 J_0 - S_0) - \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n+1} N_0\\
&= \sum_{n=0}^{+\infty} N_0^n R(s_0,s_1)^{n} - \sum_{n=0}^{+\infty} N_0^{n+1} R(s_0,s_1)^{n+1} = \mathcal{I}_{V_0}.
\end{aligned}$$

We find that $s_0\mathcal{I}_0 - s_1J_0 - T_0$ has a bounded inverse for $s_0, s_1 \in \mathbb{R}$ with $s_1 > 0$, so that J is a spectral orientation for T. Hence, T is a spectral operator and T and S have the same spectral decomposition (E, J).

Since the spectral decomposition of T is uniquely determined, $S = \int_{\mathbb{H}} s\, dE_J(s)$ and in turn also $N = T - S$ are uniquely determined. Moreover, Corollary 15.2.6 implies that $\sigma_S(T) = \sigma_S(S)$.

Now assume that T is a spectral operator and let (E, J) be its spectral decomposition. We set

$$S := \int_{\mathbb{H}} s\, dE_J(s) \quad \text{and} \quad N := T - S.$$

By Remark 15.2.2, the operator S is of scalar type, and its spectral decomposition is (E, J). Since T commutes with (E, J), it commutes with S by Lemma 15.2.3. Consequently, $N = T - S$ also commutes with S and with T. What remains to show is that N is quasi-nilpotent. In view of Corollary 15.2.5, it is sufficient to show that $\sigma_S(N)$ is for every $\varepsilon > 0$ contained in the open ball $B_\varepsilon(0)$ of radius ε centered at 0 .

For arbitrary $\varepsilon > 0$, we choose $\alpha > 0$ such that $0 < (1 + C_{E,J})\alpha < \varepsilon$, where $C_{E,J} > 0$ is the constant in (14.22). We decompose $\sigma_S(T)$ into the union of disjoint axially symmetric Borel sets $\Delta_1, \ldots, \Delta_n \in \mathfrak{B}_S(\mathbb{H})$ such that for each $\ell \in \{1, \ldots, n\}$, the set Δ_ℓ is contained in a closed axially symmetric set whose intersection with every complex half-plane is a half-disk of diameter α. More precisely, we assume that there exist points $s_1, \ldots, s_n \in \mathbb{H}$ such that for all $\ell = 1, \ldots, n$,

$$\Delta_\ell \subset B_\alpha^+([s_\ell]) = \{p \in \mathbb{H} : \ \operatorname{dist}(p, [s_\ell]) \leq \alpha \text{ and } p_1 \geq s_{\ell,1}\}.$$

Observe that we have either $s_\ell \in \mathbb{R}$ or $B_\alpha^+([s_\ell]) \cap \mathbb{R} = \emptyset$.

We set $V_{\Delta_\ell} = E(\Delta_\ell)V_R$. Since T and S commute with $E(\Delta_\ell)$, also $N = T - S$ does, and so $NV_{\Delta_\ell} \subset V_{\Delta_\ell}$. Hence $N_{\Delta_\ell} = N|_{V_{\Delta_\ell}} \in \mathcal{B}(V_{\Delta_\ell})$. If s belongs to $\rho_S(N_{\Delta_\ell})$

for all $\ell \in \{1, \dots, n\}$, we can set

$$\mathcal{Q}(s)^{-1} := \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell),$$

where

$$\mathcal{Q}_s(N_{\Delta_\ell})^{-1} = \left(N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}\right)^{-1} \in \mathcal{B}(V_{\Delta_\ell})$$

is the pseudo-resolvent of N_{Δ_ℓ} as s. The operator $\mathcal{Q}(s)^{-1}$ commutes with $E(\Delta_\ell)$ for every $\ell \in \{1, \dots, n\}$, so that

$$\begin{aligned}
&(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) \mathcal{Q}(s)^{-1} \\
&= \sum_{\ell=1}^{n} (N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}) \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell) = \sum_{\ell=1}^{n} E(\Delta_\ell) = \mathcal{I}_{V_R}
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{Q}(s)^{-1}(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) \\
&= \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} E(\Delta_\ell)(N^2 - 2s_0 N + |s|^2 \mathcal{I}_{V_R}) \\
&= \sum_{\ell=1}^{n} \mathcal{Q}_s(N_{\Delta_\ell})^{-1} (N_{\Delta_\ell}^2 - 2s_0 N_{\Delta_\ell} + |s|^2 \mathcal{I}_{V_{\Delta_\ell}}) E(\Delta_\ell) \\
&= \sum_{\ell=1}^{n} E(\Delta_\ell) = \mathcal{I}_{V_R}.
\end{aligned}$$

Therefore, we find $s \in \rho_S(N)$ such that $\bigcap_{\ell=1}^{n} \rho_S(N_{\Delta_\ell}) \subset \rho_S(N)$ and in turn $\sigma_S(N) \subset \bigcup_{\ell=1}^{n} \sigma_S(N_{\Delta_\ell})$. It is hence sufficient to show that $\sigma_S(N_{\Delta_\ell}) \subset B_\varepsilon(0)$ for all $\ell = 1, \dots, n$.

We distinguish two cases: if $s_\ell \in \mathbb{R}$, then we write

$$N_{\Delta_\ell} = (T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}).$$

Since $s_\ell \in \mathbb{R}$, we have for $p \in \mathbb{H}$ that

$$\begin{aligned}
&\mathcal{Q}_p(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) \\
&= (T_{\Delta_\ell}^2 - 2s_\ell T_{\Delta_\ell} + s_\ell^2 \mathcal{I}_{V_{\Delta_\ell}} - 2p_0(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (p_0^2 + p_1^2)\mathcal{I}_{V_{\Delta_\ell}} \\
&= T_{\Delta_\ell}^2 - 2(p_0 - s_\ell) T_{\Delta_\ell} + \left((p_0 - s_\ell)^2 + p_1^2\right) \mathcal{I}_{V_{\Delta_\ell}} = \mathcal{Q}_{p - s_\ell}(T_{\Delta_\ell})
\end{aligned}$$

and thus

$$\begin{aligned}
\sigma_S(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_\ell}) &= \{p - s_\ell \in \mathbb{H} : p \in \sigma_S(T_{\Delta_\ell})\} \\
&\subset \{p - s_\ell \in \mathbb{H} : p \in B_\alpha^+(s_\ell)\} = B_\alpha(0).
\end{aligned} \tag{15.23}$$

Moreover, the function $f(s) = (s_\ell - s)\chi_{\Delta_\ell}(s)$ is an intrinsic slice function because $s_\ell \in \mathbb{R}$. Since it is bounded, its integral with respect to (E, J) is defined and

$$s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell} = \left(\int_{\mathbb{H}} (s_\ell - s)\chi_{\Delta_\ell}(s)\, dE_J(s) \right)\Big|_{V_{\Delta_\ell}}.$$

We thus have

$$\|s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}\| \le C_{E,J} \|(s_\ell - s)\chi_{\Delta_\ell}(s)\|_\infty \le C_{E,J}\alpha \tag{15.24}$$

because $\Delta_\ell \subset B_\alpha([s_\ell])^+ = \overline{B_\alpha(s_\ell)}$. Since the operator $T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}$ and the operator $s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}$ commute, we conclude from Theorem 4.4.12 together with (15.23) and (15.24) that

$$\begin{aligned}
&\sigma_S(T_{\Delta\ell}) = \sigma_S\left((T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} - S_{\Delta_\ell}) \right) \\
&\subset \left\{ s \in \mathbb{H} : \operatorname{dist}\left(s, \sigma_S\big(T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}}\big) \right) \le C_{E,J}\alpha \right\} \subset B_{\alpha(1+C_{E,J})}(0) \subset B_\varepsilon(0).
\end{aligned}$$

If $s_\ell \notin \mathbb{R}$, then let us write

$$N_{\Delta_\ell} = (T_{\Delta_\ell} - s_\ell \mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1} J_{\Delta_\ell}) + (s_\ell \mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1} J_{\Delta_\ell} - S_{\Delta_\ell}) \tag{15.25}$$

with $J_{\Delta_\ell} = J|_{V_{\Delta_\ell}}$. Since $E(\Delta_\ell)$ and J commute, J_{Δ_ℓ} is an imaginary operator on V_{Δ_ℓ} and it moreover commutes with T_{Δ_ℓ}. Since $-J_{\Delta_\ell}^2 = -J^2|_{V_\Delta} = E(\mathbb{H}\setminus\mathbb{R})|_{V_{\Delta_\ell}} = \mathcal{I}_{V_{\Delta_\ell}}$ because $\Delta_\ell \subset \mathbb{H} \setminus \mathbb{R}$, we find for $s = s_0 + j_s s_1 \in \mathbb{H}$ with $s_1 \ge 0$ that

$$\begin{aligned}
&\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big)\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big) \\
&= s_0^2 - s_1^2 J_{\Delta_\ell}^2 - 2s_0 T_{\Delta_\ell} + T_{\Delta_\ell}^2 = \mathcal{Q}_s(T_{\Delta_\ell}).
\end{aligned} \tag{15.26}$$

Because of condition (iii) in Definition 15.1.1, the operator $(s_0\mathcal{I} - s_1 J - T)|_{\operatorname{ran} E(\mathbb{H}\setminus\mathbb{R})}$ is invertible if $s_1 > 0$. Since this operator commutes with $E(\Delta_\ell)$, the restriction of its inverse to V_{Δ_ℓ} is the inverse of $(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell})$ in $\mathcal{B}(V_{\Delta_\ell})$. Hence if $s_1 > 0$, then $(s_0 \mathcal{I}_{V_{\Delta_\ell}} - s_1 J_{\Delta_\ell} - T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell})$, and we conclude from (15.26) that

$$\big(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}\big)^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \iff \mathcal{Q}_s(T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell}). \tag{15.27}$$

If, on the other hand, $s_1 = 0$, then both factors on the left-hand side of (15.26) agree, and so (15.27) holds also in this case. Hence $s \in \rho_S(T_{\Delta_\ell})$ if and only if the operator $(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T)$ has an inverse in $\mathcal{B}(V_{\Delta_\ell})$. Since

$$\sigma_S(T_{\Delta_\ell}) \subset \overline{\Delta_\ell} \subset B_\alpha^+([s_\ell]) \subset \{ s = s_0 + j_s s_1 \in \mathbb{H} : \ \ s_1 \ge s_{\ell,1} \},$$

the operator $s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell}$ is in particular invertible for every quaternion $s \in \mathbb{H}$ with $0 \le s_1 < s_{\ell,1}$. Since J_{Δ_ℓ} is a spectral orientation for T_{Δ_ℓ}, this operator is also invertible if $s_1 < 0$, and hence we even obtain

$$(s_0 \mathcal{I}_{V_{\Delta_\ell}} + s_1 J_{\Delta_\ell} - T_{\Delta_\ell})^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \quad \forall s_0, s_1 \in \mathbb{R} : \ \ s_1 < s_{\ell,1}. \tag{15.28}$$

We can use these observations to deduce a spectral mapping property: a straightforward computation using the facts that T_{Δ_ℓ} and J_{Δ_ℓ} commute and that $J_{\Delta_\ell}^2 = -\mathcal{I}_{V_{\Delta_\ell}}$ shows that

$$\begin{aligned}&\mathcal{Q}_s(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})\\ &= \Big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_1 + s_{\ell,1})J_{\Delta_\ell} - T_{\Delta_\ell}\Big)\\ &\quad \cdot \Big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_{\ell,1} - s_1)J_{\Delta_\ell} - T_{\Delta_\ell}\Big).\end{aligned} \tag{15.29}$$

If $s_1 > 0$, then the second factor is invertible because of (15.28). Hence we have $s \in \rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})$ if and only if the first factor in (15.29) is also invertible, i.e., if and only if

$$\big((s_0 + s_{\ell,0})\mathcal{I}_{V_{\Delta_\ell}} + (s_1 + s_{\ell,1})J_{\Delta_\ell} - T_{\Delta_\ell}\big)^{-1} \in \mathcal{B}(V_{\Delta_\ell}) \tag{15.30}$$

exists. If, on the other hand, $s_1 = 0$, then both factors in (15.29) agree. Hence also in this case, s belongs to $\rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})$ if and only if the operator in (15.30) exists. By (15.27), the existence of (15.30) is, however, equivalent to

$$s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})\mathbb{S} \subset \rho_S(T_\Delta),$$

so that

$$\rho_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}) = \{s \in \mathbb{H} : \ \ s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})j_s \in \rho_S(T_{\Delta_\ell})\}$$

and in turn

$$\begin{aligned}&\sigma_S(T_{\Delta_\ell} - s_{\ell,0}\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell})\\ &= \{s \in \mathbb{H} : \ \ s_0 + s_{\ell,1} + (s_1 + s_{\ell,1})j_s \in \sigma_S(T_{\Delta_\ell})\}\\ &\subset \{s \in \mathbb{H} : \ \ s_0 + s_{\ell,0} + (s_1 + s_{\ell,1})j_s \in B_\alpha^+(s_\ell)\} = B_\alpha(0).\end{aligned}$$

For the second operator in (15.25), we have again

$$s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell} = \left(\int_{\mathbb{H}} (s_{\ell,0} + i_s s_{\ell,1} - s)\chi_{\Delta_\ell}(s)\, dE_J(s)\right)\Bigg|_{V_{\Delta_\ell}},$$

and so

$$\|s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell}\| \leq C_{E,J}\|(s_{\ell,0} + i_s s_{\ell,1} - s)\chi_{\Delta_\ell}(s)\|_\infty \leq C_{E,J}\alpha.$$

Since the operators $T_{\Delta_\ell} - s_\ell\mathcal{I}_{V_{\Delta_\ell}} - s_{\ell,1}J_{\Delta_\ell}$ and $s_\ell\mathcal{I}_{V_{\Delta_\ell}} + s_{\ell,1}J_{\Delta_\ell} - S_{\Delta_\ell}$ commute, we conclude as before from Theorem 4.4.12 that $\sigma_S(T_{\Delta_\ell}) \subset B_{\alpha(1+C_{E,J})}(0) = B_\varepsilon(0)$.

Altogether, we obtain that N is quasi-nilpotent, which concludes the proof. □

Remark 15.2.8. Twice we applied Theorem 4.4.12 in the above proof, even though we are working on a right Banach space and the theory in Chapter 4 was developed on a two-sided Banach space. Using Theorem 14.2.7, one can, however, define the S-functional calculus also on right-sided Banach spaces, so that this result is actually applicable. For details, we refer to [125].

Definition 15.2.9. Let $T \in \mathcal{B}(V_R)$ be a spectral operator and decompose $T = S+N$ as in Theorem 15.2.7. The scalar operator S is called the *scalar part of* T, and the quasi-nilpotent operator N is called the *radical part of* T.

Remark 15.2.10. Let $T \in \mathcal{B}(V_R)$ be a spectral operator. The canonical decomposition of T into its scalar part and its radical part obviously coincides for every $j \in \mathbb{S}$ with the canonical decomposition of T as a $\mathbb{C}_j$-linear spectral operator on V_j.

The remainder of this section discusses the S-functional calculus for spectral operators. Similar to the complex case, one can express $f(T)$ for every intrinsic function f as a formal Taylor series in the radical part N of T. The Taylor coefficients are spectral integrals of f with respect to the spectral decomposition of T. Hence these coefficients, and in turn also $f(T)$, depend only on the values of f on the S-spectrum $\sigma_S(T)$ of T and not on the values of f on an entire neighborhood of $\sigma_S(T)$. The operator $f(T)$ is again a spectral operator, and its spectral decomposition can easily be constructed from the spectral decomposition of T.

In the following we consider an operator that is again defined on a two-sided Banach space V.

Proposition 15.2.11. *Let $S \in \mathcal{B}(V)$ be an operator of scalar type on a two-sided quaternionic Banach space V. If $f \in \mathcal{N}(\sigma_S(S))$, then*

$$f(S) = \int_{\mathbb{H}} f(s)\, dE_J(s), \tag{15.31}$$

where $f(S)$ is intended in the sense of the S-functional calculus.

Proof. Since $1(T) = \mathcal{I} = \int_{\mathbb{H}} 1\, dE_J(s)$ and $s(S) = S = \int_{\mathbb{H}} s\, dE_J(s)$, the product rule and the $\mathbb{R}$-linearity of both the S-functional calculus and the spectral integration imply that (15.31) holds for every intrinsic polynomial. It in turn also holds for every intrinsic rational function in $\mathcal{N}(\sigma_S(S))$, i.e., for every function r of the form $r(s) = p(s)q(s)^{-1}$ with intrinsic polynomials p and q such that $q(s) \neq 0$ for every $s \in \sigma_S(S)$.

Let now $f \in \mathcal{N}(\sigma_S(S))$ be arbitrary and let U be a bounded axially symmetric open set such that $\sigma_S(T) \subset U$ and $\overline{U} \subset \mathcal{D}(f)$. Runge's theorem for slice hyperholomorphic functions implies the existence of a sequence of intrinsic rational functions $r_n \in \mathcal{N}(\overline{U})$ such that $r_n \to f$ uniformly on $\overline{U}$. Because of Lemma 14.3.6, we thus have

$$\int_{\mathbb{H}} f(s)\, dE_J(s) = \lim_{n\to+\infty} \int_{\mathbb{H}} r_n(s)\, dE_J(s) = \lim_{n\to+\infty} r_n(S) = f(S). \qquad \square$$

Theorem 15.2.12. *Let $T \in \mathcal{B}(V)$ be a spectral operator on a two-sided quaternionic Banach space V with spectral decomposition (E, J) and let $T = S + N$ be the decomposition of T into scalar and radical parts. If $f \in \mathcal{N}(\sigma_S(T))$, then*

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \int_{\mathbb{H}} (\partial_S^n f)(s)\, dE_J(s), \tag{15.32}$$

where $f(T)$ is intended in the sense of the S-functional calculus and the series converges in the operator norm.

Proof. Since $T = S + N$ with $SN = NS$ and $\sigma_S(N) = \{0\}$, it follows from Theorem 4.4.14 that

$$f(T) = \sum_{n=0}^{+\infty} N^n \frac{1}{n!} \left(\partial_S^n f\right)(S).$$

What remains to show is that

$$(\partial_S^n f)(S) = \int_{\mathbb{H}} (\partial_S^n f)(s)\, dE_J(s), \tag{15.33}$$

but this follows immediately from Proposition 15.2.11. □

The operator $f(T)$ is again a spectral operator, and its radical part can be easily obtained from the above series expansion.

Definition 15.2.13. A spectral operator $T \in \mathcal{B}(V)$ on a two-sided quaternionic Banach space V is said to be of *type* $m \in \mathbb{N}$ if its radical part satisfies $N^{m+1} = 0$.

Lemma 15.2.14. *A spectral operator $T \in \mathcal{B}(V)$ on a two-sided quaternionic Banach space V with spectral resolution (E, J) and radical part N is of type m if and only if*

$$f(T) = \sum_{n=0}^{m} N^n \frac{1}{n!} \int_{\mathbb{H}} (\partial_S^n f)(s)\, dE_J(s) \quad \forall f \in \mathcal{N}(\sigma_S(T)). \tag{15.34}$$

In particular, T is a scalar operator if and only if it is of type 0.

Proof. If T is of type m, then the above formula follows immediately from Theorem 15.2.12 and $N^{m+1} = 0$. If, on the other hand, (15.34) holds, then we choose $f(s) = \frac{1}{m!} s^m$ in (15.32) and (15.34) and subtract these two expressions. We obtain

$$0 = N^{m+1} \int_{\mathbb{H}} dE_J(s) = N^{m+1}.$$

□

Theorem 15.2.15. *Let $T \in \mathcal{B}(V_R)$ be a spectral operator with spectral decomposition (E, J). If $f \in \mathcal{N}(\sigma_S(T))$, then $f(T)$ is a spectral operator, and the spectral decomposition $(\tilde{E}, \tilde{J})$ of $f(T)$ is given by*

$$\tilde{E}(\Delta) = E\big(f^{-1}(\Delta)\big) \quad \forall \Delta \in \mathfrak{B}_S(\mathbb{H}) \quad \text{and} \quad \tilde{J} = \int_{\mathbb{H}} j_{f(s)}\, dE_J(s),$$

where $j_{f(s)} = 0$ *if* $f(s) \in \mathbb{R}$ *and* $j_{f(s)} = \underline{f(s)}/|\underline{f(s)}|$ *if* $f(s) \in \mathbb{H} \setminus \mathbb{R}$*. For every* $g \in \mathcal{SM}^\infty(\mathbb{H})$ *we have*

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}_{\tilde{J}}(s) = \int_{\mathbb{H}} (g \circ f)(s)\, dE_J(s), \tag{15.35}$$

and if S *is the scalar part of* T*, then* $f(S)$ *is the scalar part of* $f(T)$*.*

Proof. We first show that $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$. By Corollary 14.3.4, the function f is $\mathfrak{B}_S(\mathbb{H})$-$\mathfrak{B}_S(\mathbb{H})$-measurable, so that $\tilde{E}$ is a well-defined spectral measure on $\mathfrak{B}_S(\mathbb{H})$.

The operator $\tilde{J}$ obviously commutes with E. Moreover, writing $f(s) = f_0(s) + j_s f_1(s)$ as in Lemma 14.3.3, we have $j_{f(s)} = j_s \operatorname{sgn}(f_1(s))$. If we set $\Delta_+ = \{s \in \mathbb{H} : f_1(s) > 0\}$, $\Delta_- = \{s \in \mathbb{H} : f_1(s) < 0\}$, and $\Delta_0 = \{s \in \mathbb{H} : f_1(s) = 0\}$, we therefore have

$$\tilde{J} = JE(\Delta_+) - JE(\Delta_-).$$

Since $f_1(s) = 0$ for every $s \in \mathbb{R}$, we have $\mathbb{R} \subset \Delta_0$ and hence $V_+ = \operatorname{ran} E(\Delta_+) \subset \operatorname{ran} E(\mathbb{H} \setminus \mathbb{R}) = \operatorname{ran} J$ and similarly also $V_- = \operatorname{ran} E(\Delta_-) \subset \operatorname{ran} J$. Since J and E commute, V_+ and V_- are invariant subspaces of J contained in $\operatorname{ran} J$, so that J_+ and J_- define bounded surjective operators on V_+, resp. V_-. Moreover, $\ker J = \operatorname{ran} E(\mathbb{R})$, and hence $\ker J|_{V_+} = V_+ \cap \ker J = \{0\}$ and $\ker J|_{V_-} = V_- \cap \ker J = \{0\}$, so that $\ker \tilde{J} = \operatorname{ran} E(\Delta_0)$ and $\operatorname{ran} \tilde{J} = \operatorname{ran} E(\Delta_+) \oplus \operatorname{ran} E(\Delta_-) = \operatorname{ran} E(\Delta_+ \cup \Delta_-)$.

Now observe that $f(s) \in \mathbb{R}$ if and only if $f_1(s) = 0$. Hence $f^{-1}(\mathbb{R}) = \Delta_0$ and $f^{-1}(\mathbb{H} \setminus \mathbb{R}) = \Delta_+ \cup \Delta_-$, and we obtain

$$\operatorname{ran} \tilde{J} = \operatorname{ran} E(\Delta_+ \cup \Delta_-) = \operatorname{ran} E\left(f^{-1}(\mathbb{H} \setminus \mathbb{R})\right) = \operatorname{ran} \tilde{E}(\mathbb{H} \setminus \mathbb{R})$$

and

$$\ker \tilde{J} = \operatorname{ran} E(\Delta_0) = \operatorname{ran} E\left(f^{-1}(\mathbb{R})\right) = \operatorname{ran} \tilde{E}(\mathbb{R}).$$

Moreover, since $E(\Delta_+)E(\Delta_-) = E(\Delta_-)E(\Delta_+) = 0$ and $-J^2 = E(\mathbb{H} \setminus \mathbb{R})$, we have

$$\begin{aligned} -\tilde{J}^2 &= -J^2 E(\Delta_+)^2 - (-J^2) E(\Delta_-)^2 \\ &= E(\mathbb{H} \setminus \mathbb{R}) E(\Delta_+) + E(\mathbb{H} \setminus \mathbb{R}) E(\Delta_-) \\ &= E(\Delta_+ \cup \Delta_-) = \tilde{E}(\mathbb{H} \setminus \mathbb{R}), \end{aligned}$$

where we used that $\Delta_+ \subset \mathbb{H} \setminus \mathbb{R}$ and $\Delta_- \subset \mathbb{H} \setminus \mathbb{R}$ as $\mathbb{R} \subset \Delta_0$. Hence $-\tilde{J}^2$ is the projection onto $\operatorname{ran} \tilde{J}$ along $\ker \tilde{J}$, and so $\tilde{J}$ is actually an imaginary operator, and $(\tilde{E}, \tilde{J})$ in turn is a spectral system.

Let $g = \sum_{\ell=0}^n a_\ell \chi_{\Delta_\ell} \in \mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ be a simple function. Then $(g \circ f)(s) = \sum_{\ell=0}^n a_\ell \chi_{f^{-1}(\Delta_\ell)}(s)$ is also a simple function in $\mathcal{M}_S^\infty(\mathbb{H}, \mathbb{R})$ and

$$\int_{\mathbb{H}} g(s)\, d\tilde{E}(s) = \sum_{\ell=0}^n a_\ell \tilde{E}(\Delta_\ell) = \sum_{\ell=0}^n a_\ell E\big(f^{-1}(\Delta_\ell)\big) = \int_{\mathbb{H}} (g \circ f)(s)\, dE(s).$$

Due to the density of simple functions in $(\mathcal{M}_S^\infty(\mathbb{H},\mathbb{R}), \|.\|_\infty)$, we hence obtain

$$\int_{\mathbb{H}} g(s)\,d\tilde{E}(s) = \int_{\mathbb{H}} (g\circ f)(s)\,dE(s), \quad \forall g \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R}).$$

If $g \in \mathcal{SM}^\infty(\mathbb{H})$, then we deduce from Lemma 14.3.3 that $g(s) = \gamma(s) + j_s\delta(s)$ with $\gamma,\delta \in \mathcal{M}_S^\infty(\mathbb{H},\mathbb{R})$ and $j_s = \underline{s}/|\underline{s}|$ if $s \notin \mathbb{R}$ and $j_s = \delta(s) = 0$ if $s \in \mathbb{R}$. We then have $(g\circ f)(s) = \gamma(f(s)) + j_{f(s)}\delta(f(s))$, and we obtain

$$\begin{aligned}
\int_{\mathbb{H}} g(s)\,d\tilde{E}_{\tilde{J}}(s) &= \int_{\mathbb{H}} \gamma(s)\,d\tilde{E}(s) + \tilde{J}\int_{\mathbb{H}} \delta(s)\,d\tilde{E}(s)\\
&= \int_{\mathbb{H}} (\gamma\circ f)(s)\,dE(s) + \tilde{J}\int_{\mathbb{H}} (\delta\circ f)(s)\,dE(s)\\
&= \int_{\mathbb{H}} (\gamma\circ f)(s)\,dE_J(s) + \int_{\mathbb{H}} j_{f(s)}\,dE_J(s)\int_{\mathbb{H}} (\delta\circ f)(s)\,dE(s)\\
&= \int_{\mathbb{H}} (\gamma\circ f)(s) + j_{f(s)}(\delta\circ f)(s)\,dE_J(s) = \int_{\mathbb{H}} (g\circ f)(s)\,dE_J(s),
\end{aligned}$$

and hence (15.35) holds. Choosing in particular $g(s) = s$, we deduce from Proposition 15.2.11 that

$$f(S) = \int_{\mathbb{H}} f(s)\,dE_J(s) = \int_{\mathbb{H}} s\,d\tilde{E}_{\tilde{J}}(s).$$

By Remark 15.2.2, $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E},\tilde{J})$.

Theorem 15.2.12 implies $f(T) = f(S) + \Theta$ with

$$\Theta := \sum_{n=1}^{+\infty} N^n \frac{1}{n!}(\partial_S^n f)(S).$$

If we can show that Θ is a quasi-nilpotent operator, then the statement of the theorem follows from Theorem 15.2.7. We first observe that each term in the sum is a quasi-nilpotent operator because N^n and $(\partial_S^n f)(S)$ commute due to Lemmas 14.3.6 and 15.2.3, so that

$$0 \le \lim_{k\to\infty}\left\|\left(N^n\frac{1}{n!}(\partial_S^n f)(S)\right)^k\right\|^{\frac{1}{k}} \le \left\|\frac{1}{n!}(\partial_S^n f)(S)\right\|\left(\lim_{k\to\infty}\left\|N^{nk}\right\|^{\frac{1}{nk}}\right)^n = 0.$$

Corollary 15.2.5 thus implies $\sigma_S\left(N^n\frac{1}{n!}(\partial_S^n f(S))\right) = \{0\}$.

By induction we conclude from Taylor's formula and Corollary 15.2.5 that for each $m \in \mathbb{N}m$, the finite sum $\Theta_1(m) := \sum_{n=1}^m N^n\frac{1}{n!}(\partial_S^n f)(S)$ is quasi-nilpotent and satisfies $\sigma_S(\Theta(m)) = \{0\}$.

Since the series Θ converges in the operator norm, for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that $\Theta_2(m_\varepsilon) := \sum_{n=m_\varepsilon+1}^{+\infty} N^n\frac{1}{n!}(\partial_S^n f)(S)$ satisfies $\|\Theta_2(m_\varepsilon)\| < \varepsilon$.

Hence $\sigma_S(\Theta_2(m_\varepsilon)) \subset B_\varepsilon(0)$, and since $\Theta = \Theta_1(m_\varepsilon) + \Theta_2(m_\varepsilon)$ and $\Theta_1(m_\varepsilon)$ and $\Theta_2(m_\varepsilon)$ commute, we conclude from Theorem 4.4.12 that $\sigma_S(\Theta) \subset B_\varepsilon(0)$. Since $\varepsilon > 0$ was arbitrary, we obtain $\sigma_S(\Theta) = \{0\}$. By Corollary 15.2.5, Θ is quasi-nilpotent.

We have shown that $f(T) = f(S) + \Theta$, that $f(S)$ is a scalar operator with spectral decomposition $(\tilde{E}, \tilde{J})$, and that Θ is quasi-nilpotent. From Theorem 15.2.7 we therefore deduce that $f(T)$ is a spectral operator with spectral decomposition $(\tilde{E}, \tilde{J})$, that $f(S)$ is its scalar part, and that Θ is its radical part. This concludes the proof. □

Corollary 15.2.16. *Let $T \in \mathcal{B}(\mathcal{H})$ be a spectral operator and let $f \in \mathcal{N}(\sigma_S(T))$. If T is of type $m \in \mathbb{N}$, then $f(T)$ is of type m too.*

Proof. If $T = S + N$ is the decomposition of T into its scalar and radical parts and T is of type m such that $N^{m+1} = 0$, then the radical part Θ of $f(T)$ is given, due to Lemma 15.2.14 and Theorem 15.2.15, by

$$\Theta = f(T) - f(S) = \sum_{n=1}^{+\infty} N^n \frac{1}{n!}(\partial_S^n f)(S) = \sum_{n=1}^{m} N^n \frac{1}{n!}(\partial_S^n f)(S).$$

Obviously also $\Theta^{m+1} = 0$. □

Contents of the Monograph: Quaternionic Closed Operators, Fractional Powers and Fractional Diffusion Processes

The natural continuation of this book is the monograph [56]: *Quaternionic closed operators, fractional powers and fractional diffusion processes*. In [56] the study of quaternionic operator theory has been continued and it has been considered a new class of fractional diffusion problems that are naturally defined using this theory. The book has 12 chapters whose contents are as follows.

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