Chapter 9
Constructive Martingale Representation in Functional Itô Calculus: A Local Martingale Extension

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Abstract The constructive martingale representation theorem of functional Itô calculus is extended, from the space of square integrable martingales, to the space of local martingales. The setting is that of an augmented filtration generated by a Wiener process.

Keywords Functional Itô calculus · Martingale representation

9.1 Introduction

Consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which lives an n-dimensional Wiener process W. Let $\underline{\mathcal{F}} = (\mathcal{F}_t)_{0 \le t \le T}$ denote the augmentation under $\mathbb{P}$ of the filtration generated by W until the constant terminal time $T < \infty$. One of the main results of Itô calculus is the martingale representation theorem which in the present setting is as follows: *Let M be a RCLL local martingale relative to* $(\mathbb{P}, \underline{\mathcal{F}})$, *then there exists a progressively measurable n-dimensional process φ such that*

$$M(t) = M(0) + \int_0^t \varphi(s)' dW(s), 0 \le t \le T, \text{ and } \int_0^T |\varphi(t)|^2 dt < \infty \text{ a.s.}$$

In particular, M has continuous sample paths a.s.

Considerable effort has in the literature been made in order to find explicit formulas for the integrand φ, i.e. in order to find constructive representations of martingales, mainly using Malliavin calculus, see e.g. [8, 15, 16, 20] and the references therein. The recently developed *functional Itô calculus* includes a new type of constructive representation of square integrable martingales due to Cont and Fournié see e.g. [1, 3–5]. The main result of the present paper is an extension of this result to local martingales.

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The organization of the paper is as follows. Section 9.2 is based on [1] and contains a brief and heuristic account of the relevant parts of functional Itô calculus including the constructive martingale representation theorem for square integrable martingales. Section 9.3 contains the local martingale extension of this theorem and a simple example.

9.2 Constructive Representation of Square Integrable Martingales

Denote an n-dimensional sample path by ω. Denote a sample path stopped at t by ω_t, i.e. let $\omega_t(s) = \omega(t \wedge s), 0 \leq s \leq T$. Consider a real-valued functional of sample paths $F(t, \omega)$ which is *non-anticipative* (essentially meaning that $F(t, \omega) = F(t, \omega_t)$). The *horizontal derivative* at (t, ω) is defined by

$$\mathcal{D}F(t, \omega) = \lim_{h \searrow 0} \frac{F(t + h, \omega_t) - F(t, \omega_t)}{h}.$$

The *vertical derivative* at (t, ω) is defined by

$$\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega), i = 1, ..., n)',$$

where

$$\partial_i F(t, \omega) = \lim_{h \to 0} \frac{F(t, \omega_t + h e_i I_{[t,T]}) - F(t, \omega_t)}{h}.$$

Higher order vertical derivatives are obtained by vertically differentiating vertical derivatives.

One of the main results of functional Itô calculus is the functional Itô formula, which is just the standard Itô formula with the usual time and space derivatives replaced by the horizontal and vertical derivatives. If the functional F is sufficiently regular (regarding e.g. continuity and boundedness of its derivatives), which we write as $F \in \mathbb{C}_b^{1,2}$, then the functional Itô formula holds, see [1, ch. 5,6]. We remark that [12] contains another version of this result.

Using the functional Itô formula it easy to see that if Z is a martingale satisfying

$$Z(t) = F(t, W_t) \;\; dt \times d\mathbb{P}\text{-a.e., with } F \in \mathbb{C}_b^{1,2}, \tag{9.1}$$

then, for every $t \in [0, T]$,

$$Z(t) = Z(0) + \int_0^t \nabla_\omega F(s, W_s)' dW(s) \;\; a.s.$$

We may therefore define the vertical derivative with respect to the process W of a martingale Z satisfying (9.1) as the $dt \times d\mathbb{P}$-a.e. unique process $\nabla_W Z$ given by

$$\nabla_W Z(t) = \nabla_\omega F(t, W_t), 0 \leq t \leq T. \tag{9.2}$$

Let $\mathcal{C}_b^{1,2}(W)$ be the space of processes Z which allow the representation in (9.1). Let $\mathcal{L}^2(W)$ be the space of progressively measurable processes φ satisfying the condition $E[\int_0^T \varphi(s)'\varphi(s)ds] < \infty$. Let $\mathcal{M}^2(W)$ be the space of square integrable martingales with initial value 0. Let $D(W) = \mathcal{C}_b^{1,2}(W) \cap \mathcal{M}^2(W)$.

It can be shown that $\{\nabla_W Z : Z \in D(W)\}$ is dense in $\mathcal{L}^2(W)$ and that $D(W)$ is dense in $\mathcal{M}^2(W)$ [1, ch. 7]. Using this it is possible to show that the vertical derivative operator $\nabla_W(\cdot)$ admits a unique extension to $\mathcal{M}^2(W)$, in the following sense: *For $Y \in \mathcal{M}^2(W)$ the (weak) vertical derivative $\nabla_W Y$ is the unique element in $\mathcal{L}^2(W)$ satisfying*

$$E[Y(T)Z(T)] = E\left[\int_0^T \nabla_W Y(t)'\nabla_W Z(t)dt\right] \tag{9.3}$$

for every $Z \in D(W)$, where $\nabla_W Z$ is defined in (9.2). The constructive martingale representation theorem ([1, ch. 7]) follows:

Theorem 9.1 (Cont and Fournié) *For any square integrable martingale Y relative to* $(\mathbb{P}, \underline{\mathcal{F}})$ *and every $t \in [0, T]$,*

$$Y(t) = Y(0) + \int_0^t \nabla_W Y(s)'dW(s) \;\; a.s.$$

9.3 Constructive Representation of Local Martingales

This section contains an extension of the vertical derivative $\nabla_W(\cdot)$ and the constructive martingale representation in Theorem 9.1 to local martingales. Let $\mathcal{M}^{\text{loc}}(W)$ denote the space of local martingales relative to $(\mathbb{P}, \underline{\mathcal{F}})$ with initial value zero and RCLL sample paths. In Theorem 9.2 we extend the vertical derivative to $\mathcal{M}^{\text{loc}}(W)$. Using this extension we can formulate the constructive martingale representation theorem also for local martingales, see Theorem 9.3.

Before extending the definition of the vertical derivative to $\mathcal{M}^{\text{loc}}(W)$ we recall the definition of a local martingale.

Definition 9.1 M is said to be a local martingale if there exists a sequence of non-decreasing stopping times $\{\theta_n\}$ with $\lim_{n\to\infty} \theta_n = \infty$ a.s. such that the stopped local martingale $M(\cdot \wedge \theta_n)$ is a martingale for each $n \geq 1$.

Theorem 9.2 (Definition of $\nabla_W(\cdot)$ on $\mathcal{M}^{loc}(W)$)

- *There exists a progressively measurable* $dt \times d\mathbb{P}$*-a.e. unique extension of the vertical derivative* $\nabla_W(\cdot)$ *from* $\mathcal{M}^2(W)$ *to* $\mathcal{M}^{loc}(W)$*, such that, for* $M \in \mathcal{M}^{loc}(W)$*,*

$$\begin{aligned} M(t) = \textstyle\int_0^t \nabla_W M(s)' dW(s), 0 \leq t \leq T, \textit{and} \\ \textstyle\int_0^T |\nabla_W M(t)|^2 dt < \infty \textit{a.s.} \end{aligned} \tag{9.4}$$

- *Specifically, for* $M \in \mathcal{M}^{loc}(W)$ *the vertical derivative* $\nabla_W M$ *is defined as the progressively measurable* $dt \times d\mathbb{P}$*-a.e. unique process satisfying*

$$\nabla_W M(t) = \lim_{n \to \infty} \nabla_W M_n(t) \quad dt \times d\mathbb{P}\text{-}a.e. \tag{9.5}$$

where $\nabla_W M_n$ *is the vertical derivative of* $M_n := M(\cdot \wedge \tau_n) \in \mathcal{M}^2(W)$ *and* τ_n *is given by*

$$\tau_n = \theta_n \wedge \inf\{s \in [0, T] : |M(s)| \geq n\} \wedge T \tag{9.6}$$

where $\{\theta_n\}$ *is an arbitrary sequence of stopping times of the kind described in Definition 9.1.*

Remark 9.1 Note that if M in Theorem 9.2 satisfies

$$M(t) = \int_0^t \gamma(s)' dW(s), 0 \leq t \leq T \text{ a.s.}$$

for some process γ, then $\gamma = \nabla_W M \; dt \times d\mathbb{P}$-a.e. It follows that the extended vertical derivative $\nabla_W M$ defined in Theorem 9.2 does not depend (modulo possibly on a null set $dt \times d\mathbb{P}$) on the particulars of the chosen stopping times $\{\theta_n\}$.

Proof The martingale representation theorem implies that, for $M \in \mathcal{M}^{\text{loc}}(W)$, there exists a progressively measurable process φ satisfying

$$M(t) = \int_0^t \varphi(s)' dW(s), 0 \leq t \leq T, \text{ and } \int_0^T |\varphi(t)|^2 dt < \infty \text{ a.s.} \tag{9.7}$$

Therefore, if we can prove that

$$\lim_{n \to \infty} \nabla_W M_n(t) = \varphi(t) \quad dt \times d\mathbb{P}\text{-a.e.}, \tag{9.8}$$

then it follows that there exists a progressively measurable process, denote it by $\nabla_W M$, which is $dt \times d\mathbb{P}$-a.e. uniquely defined by (9.5) and satisfies

$$\nabla_W M(t) = \varphi(t) \quad dt \times d\mathbb{P}\text{-a.e.},$$

which in turn implies that the integrals of $\nabla_W M$ and φ coincide in the way that (9.7) implies (9.4). All we have to do is therefore to prove that (9.8) holds.

Let us recall some results about stopping times and martingales. The stopped local martingale $M(\cdot \wedge \theta_n)$ is a martingale for each n, by Definition 9.1. Stopped RCLL martingales are martingales. The minimum of two stopping times is a stopping time and the hitting time

$$\inf\{s \in [0, T] : |M(s)| \geq n\}$$

is, for each n, in the present setting, a stopping time. Using these results we obtain that $M(\cdot \wedge \theta_n \wedge \inf\{s \in [0, T] : |M(s)| \geq n\} \wedge T) = M(\cdot \wedge \tau_n)$ is a martingale, for each n. Moreover, M is by the standard martingale representation result a.s. continuous. Hence, we may define a sequence of, a.s. continuous, martingales $\{M_n\}$ by

$$M_n = M(\cdot \wedge \tau_n) = \int_0^{\cdot \wedge \tau_n} \varphi(s)' dW(s) \text{ a.s.} \tag{9.9}$$

where the last equality follows from (9.7). Now, use the definition of τ_n in (9.6) to see that

$$|M_n(t)| = \left| \int_0^{t \wedge \tau_n} \varphi(s)' dW(s) \right| \leq n \text{ a.s.}$$

for any t and n, and that in particular M_n is, for each n, a square integrable martingale. Moreover, (9.9) implies that M_n satisfies

$$M_n(t) = \int_0^t I_{\{s \leq \tau_n\}} \varphi(s)' dW(s), 0 \leq t \leq T \text{ a.s.} \tag{9.10}$$

Since each M_n is a square integrable martingale we may use Theorem 9.1 on M_n, which together with (9.10) implies that

$$\begin{aligned} M_n(t) &= \int_0^t \nabla_W M_n(s)' dW(s) \\ &= \int_0^t I_{\{s \leq \tau_n\}} \varphi(s)' dW(s), 0 \leq t \leq T \text{ a.s.} \end{aligned} \tag{9.11}$$

where $\nabla_W M_n$ is the vertical derivative of M_n with respect to W (defined in (9.3)) and where we also used the continuity of the Itô integrals. The equality of the two Itô integrals in (9.11) implies that

$$\nabla_W M_n(t) = I_{\{t \leq \tau_n\}} \varphi(t) \quad dt \times d\mathbb{P}\text{-a.e.} \tag{9.12}$$

The local martingale property of M implies that $\lim_{n \to \infty} \theta_n = \infty \; a.s.$ Using this and the definition of τ_n in (9.6) we conclude that for almost every $\omega \in \Omega$ and each

$t \in [0, T]$ there exists an $N(\omega, t)$ such that

$$n \geq N(\omega, t) \Rightarrow \sup_{0 \leq s \leq t} |M(\omega, s)| \leq n \text{ and } t \leq \theta_n(\omega) \Rightarrow t \leq \tau_n(\omega). \tag{9.13}$$

It follows from (9.12) and (9.13) that there exists an $N(\omega, t)$ such that

$$n \geq N(\omega, t) \Rightarrow \nabla_W M_n(\omega, t) = \varphi(\omega, t) \quad dt \times d\mathbb{P}\text{-a.e.}$$

which means that (9.8) holds. □

If M is a RCLL local martingale then $M - M(0) \in \mathcal{M}^{\text{loc}}(W)$, which implies that $\nabla_W(M - M(0))$ is defined in Theorem 9.2. This observation allows us to extend the definition of the vertical derivative to RCLL local martingales not necessarily starting at zero in the following obvious way.

Definition 9.2 The vertical derivative of a local martingale M relative to $(\mathbb{P}, \underline{\mathcal{F}})$ with RCLL sample paths is defined as the progressively measurable $dt \times d\mathbb{P}$-a.e. unique process $\nabla_W M$ satisfying

$$\nabla_W M(t) = \nabla_W(M - M(0))(t), 0 \leq t \leq T, \tag{9.14}$$

where $\nabla_W(M - M(0))(t)$ is defined in Theorem 9.2.

The following result is an immediate consequence of Theorem 9.2 and Definition 9.2.

Theorem 9.3 *If M is a local martingale relative to $(\mathbb{P}, \underline{\mathcal{F}})$ with RCLL sample paths, then*

$$M(t) = M(0) + \int_0^t \nabla_W M(s)' dW(s), 0 \leq t \leq T, \textit{ and}$$
$$\int_0^T |\nabla_W M(t)|^2 dt < \infty \textit{ a.s.},$$

where $\nabla_W M(s)$ is defined in Definition 9.2.

Let us try to clarify the theory by studying a simple example. It is straightforward to extend the results above to the case when the Wiener process W is replaced by an adapted process X given by

$$X(t) = X(0) + \int_0^t \sigma(s) dW(s), \tag{9.15}$$

where σ is a matrix-valued adapted process satisfying suitable assumptions, mainly invertibility, see also [1, 4]. Thus, a local martingale M can be represented as

$$M(t) - M(0) = \int_0^t \nabla_W M(s)' dW(s) = \int_0^t \nabla_X M(s)' dX(s),$$

and the relationship between the vertical derivatives with respect to W and X is $\nabla_W M(t)' = (\nabla_X M(t)')\sigma(t)$, cf. (9.15). As example consider the one-dimensional case and let X with $X(0) = 0$ be given by (9.15) under the assumption that $\sigma(s)$ is a deterministic function of time and let M be given by $M(t) = F(t, X_t)$ where F is the non-anticipative functional $F(t, \omega) = \omega^3(t) - 3\int_0^t \omega(s)\sigma^2(s)ds$, i.e. let M be the local martingale defined by

$$M(t) = X^3(t) - 3\int_0^t X(s)\sigma^2(s)ds.$$

In this case the vertical derivative simplifies to the standard derivative, that is, $\nabla F_\omega(t, \omega) = 3\omega^2(t)$, see also [1, 4] (we remark that the horizontal derivative is $\mathcal{D}F(t, \omega) = -3\omega(t)\sigma^2(t)$). In this case, $\nabla_X M(t) = 3X^2(t)$ and

$$M(t) = \int_0^t 3X^2(s)dX(s) = \int_0^t 3X^2(s)\sigma(s)dW(s),$$

which we remark is easily found using the standard Itô formula. Note that this also means that $\nabla_W M(t) = 3X^2(t)\sigma(t) = \nabla_X M(t)\sigma(t)$.

Concluding Remarks

Many of the applications that rely on martingale representation are within mathematical finance. A particular application that may benefit from the local martingale extension of the present paper is optimal investment theory, in which the discounted (using the state price density) optimal wealth process is a (not necessarily square integrable) martingale, see e.g. [9, ch. 3], see also [13]. In particular, using functional Itô calculus it is possible to derive an explicit formula for the optimal portfolio in terms of the vertical derivative of the discounted optimal wealth process, see also [14]. Similar explicit formulas for optimal portfolios based on the Malliavin calculus approach to constructive martingale representation have, under restrictive assumptions, been studied extensively, see e.g. [2, 6, 7, 10, 11, 17–19]. The general connection between Malliavin calculus and functional Itô calculus is studied in e.g. [1, 4].

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