# Asymptotic Methods in Regularity Theory for Nonlinear Elliptic Equations: A Survey

**Edgard A. Pimentel and Makson S. Santos**

**Abstract** We survey recent asymptotic methods introduced in regularity theory for fully nonlinear elliptic equations. Our presentation focuses mainly on the recession function. We detail the role of this class of techniques through examples and results. Our applications include regularity in Sobolev and Hölder spaces. In addition, we produce a density result and examine ellipticity-invariant quantities, such as the Escauriaza's exponent.

**Keywords** Fully nonlinear elliptic equations · Regularity Theory · Asymptotic Methods · Recession Operator

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## 1 Introduction

In this paper, we examine asymptotic methods in regularity theory for fully nonlinear elliptic equations. We survey recent developments and prove a density result.

At the core of our analysis is the notion of recession operator. Given a $(\lambda, \Lambda)$-elliptic operator $F : \mathcal{S}(d) \to \mathbb{R}$, its recession function is denoted by $F^*$ and defined as follows:

$$F^*(M) := \lim_{\mu \downarrow 0} \mu F(\mu^{-1} M). \tag{1}$$

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We observe that $F^*$ captures the behavior of the operator $F$ at the ends of $\mathcal{S}(d)$. For that reason, we refer to this analysis as *asymptotic* with respect to the space of symmetric matrices.

The notion of recession is imported from the realm of free boundary problems; see for example [1]. In the context of regularity theory for elliptic partial differential equations (PDEs), it appeared in [19]. In that paper, the authors partially reproduce the program developed in [22], replacing the fixed-coefficients operator with the recession function $F^*$.

In [18], the authors investigate Sobolev regularity for the solutions to

$$F(D^2u) = f \quad \text{in} \quad B_1,$$

with $f \in L^d(B_1)$, through the recession strategy. They produce estimates in $W^{2,p}_{loc}(B_1)$ and $p-BMO_{loc}(B_1)$ by assuming that $F^*$ has $\mathcal{C}^{1,1}$-estimates. See [2, 3]. Regularity theory in Sobolev spaces, for the parabolic problem, is the subject of [4].

Because of its asymptotic character, the recession strategy accesses two additional types of consequences. First, we mention density properties for general $(\lambda, \Lambda)$-elliptic operators. In addition, it enables us to examine ellipticity-invariant quantities (e.g., the Escauriaza's exponent).

The first regularity result for fully nonlinear elliptic equations appeared in the context of the Krylov-Safonov theory, see [10, 11]. This theory accounts for a Harnack's inequality and estimates in $\mathcal{C}^{0,\alpha}$ for the solutions of a *linear* elliptic equation in divergence form. By linearizing the homogeneous problem

$$F(D^2u) = 0 \quad \text{in} \quad B_1, \tag{2}$$

we learn that its solutions and their derivatives satisfy a linear elliptic equation in divergence form. Hence, the Krylov-Safonov theory implies estimates in $\mathcal{C}^{1,\alpha}$ for the solutions to (2).

Under the assumption of convexity of the operator $F$, Evans and Krylov proved, independently, that solutions are indeed of class $\mathcal{C}^{2,\alpha}$. This is the content of the Evans-Krylov theory.

In [2], Caffarelli introduced a geometric method relating $F(M, x)$ to $F(M, x_0)$, the fixed-coefficients operator. The author supposes that $F(M, x_0)$ is convex with respect to $M \in \mathcal{S}(d)$, for every $x_0 \in B_1$ fixed. In addition, he works under the assumption that the oscillation

$$\beta(x, x_0) := \sup_{M \in \mathcal{S}(d)} \frac{|F(M, x) - F(M, x_0)|}{1 + \|M\|}$$

is small in the $L^p$-sense; that is

$$\|\beta(\cdot, x_0)\|_{L^p(B_1)} \ll 1,$$

for every $x_0 \in B_1$. Under those conditions, Caffarelli developed a regularity theory covering estimates in Hölder and Sobolev spaces.

This corpus of advances entailed several questions. The most important one regarded the optimal regularity implied by ellipticity alone. In particular, if the Krylov-Safonov estimates were the best regularity level in the absence of further structures of the problem.

This class of questions was set in the negative only recently. In [14–16], Nadirashvili and Vladut produced a number of counterexamples to the theory. For instance, the authors built singular solutions—failing to be of class $\mathcal{C}^{1,1}$—for $(\lambda, \Lambda)$-elliptic operators. Moreover, given a number $\tau \in (0, 1)$, there exists an elliptic operator $F_\tau$, whose solutions fail to be of class $\mathcal{C}^{1,\tau}$.

Those counterexamples reveal important subtleties of the theory. To access more general regularity results, finer methods would be necessary. Of particular interest are techniques capable of accessing general underlying mechanisms governing the regularity of the solutions.

In this context, asymptotic methods have been successful in producing new information with consequences to the general theory of nonlinear PDEs. In the present paper, we detail those methods through examples and applications. Our approach also highlights further classes of information, such as the weak regularity theory (see Sect. 4).

### *1.1 Outline of the Paper*

In Sect. 2 we introduce the recession function associated with $F$. We discuss properties of this object and address a number of examples; these involve a perturbation of the Monge-Ampère equation. Section 3 discusses two applications of the asymptotic analysis to the theory of nonlinear PDEs; first, we study estimates in Sobolev spaces. Then, we examine the Escauriaza's exponent. Section 4 puts forward a theorem on the density of $\mathcal{C}^{1,Log-Lip}$ in the class of viscosity solutions. We refer to this class of results as *weak regularity theory*.

## 2 Asymptotic Methods: The Recession Operator

We say that a fully nonlinear operator $F : \mathcal{S}(d) \to \mathbb{R}$ is $(\lambda, \Lambda)$-elliptic if it satisfies

$$\lambda\|N\| \leq F(M + N) - F(M) \leq \Lambda\|N\|,$$

for every $M, N \in \mathcal{S}(d)$, with $N \geq 0$.

Next we introduce the class of viscosity solutions $S(\lambda, \Lambda, f)$. To do so, we present the Pucci's extremal operators:

$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i<0} e_i + \lambda \sum_{e_i>0} e_i$$

where $(e_i)_{i=1}^d$ are the eigenvalues of the matrix $M$. Before we proceed, we present the definition of viscosity solution.

**Definition 2.1 (Viscosity Solution)** We say that $u \in \mathcal{C}(B_1)$ is a viscosity subsolution [resp. supersolution] to

$$F(D^2u) = f \quad \text{in} \quad B_1$$

if, for every $\phi \in \mathcal{C}^2(B_1)$ such that $u - \phi$ has a local maximum [resp. minimum] at $x_0 \in B_1$, we have

$$F(D^2\phi(x_0) \geq f(x_0)$$

$$[\text{resp.} \qquad F(D^2\phi(x_0) \leq f(x_0) \ ].$$

If $u$ is both a viscosity sub and supersolution, we say it is a viscosity solution.

If $u \in \mathcal{C}(B_1)$ is a viscosity solution of

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) \geq f \quad \text{in} \quad B_1,$$

we say that $u \in \underline{S}(\lambda, \Lambda, f)$. If $u \in \mathcal{C}(B_1)$ is a viscosity solution of

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leq f \quad \text{in} \quad B_1,$$

we say that $u \in \overline{S}(\lambda, \Lambda, f)$. The class of viscosity solutions $S(\lambda, \Lambda, f)$ is defined as

$$S(\lambda, \Lambda, f) := \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f).$$

For any given $(\lambda, \Lambda)$-elliptic operator, we produce the operator $F_\mu$, defined as

$$F_\mu(M) := \mu F(\mu^{-1}M),$$

for $\mu > 0$. Notice that

$$\mu\mu^{-1}\lambda\|N\| \le F_\mu(M+N) - F_\mu(M) \le \mu\mu^{-1}\Lambda\|N\|. \tag{3}$$

Therefore, $F_\mu$ has the exact same ellipticity constants as the original operator $F$. To define the recession function associated with $F$, we consider $F_\mu$ and take the limit $\mu \downarrow 0$.

**Definition 2.2 (Recession Operator)** Let $F$ be a $(\lambda, \Lambda)$-elliptic operator and consider the family $(F_\mu)_{\mu\in(0,1)}$. The recession function $F^*$ associated with $F$ is defined as

$$F^*(M) := \lim_{\mu\downarrow 0} F_\mu(M). \tag{4}$$

When the limit in (4) exists, $F^*$ has the same ellipticity as $F$. Moreover, the operator $F_\mu$ acts as a curve in the space of $(\lambda, \Lambda)$-elliptic operators. For $\mu \equiv 1$, we have $F_1 \equiv F$; however, as $\mu$ decreases and approaches 0, the path produced by $F_\mu$ approaches the recession operator $F^*$.

The rationale behind the use of the recession function is the following. Given $F$, we compute $F_\mu$ and produce a path along the space of $(\lambda, \Lambda)$-elliptic operators. For small values of $\mu > 0$, this path approaches a neighborhood of $F^*$. Suppose this limiting operator has good properties. The idea is to import information from $F^*$ to the original operator along the path parametrized by $F_\mu$. For example, if $F^*$ has $\mathcal{C}^{1,1}$-estimates, we expect to import regularity in $W^{2,p}$ for the original problem. See Fig. 1.

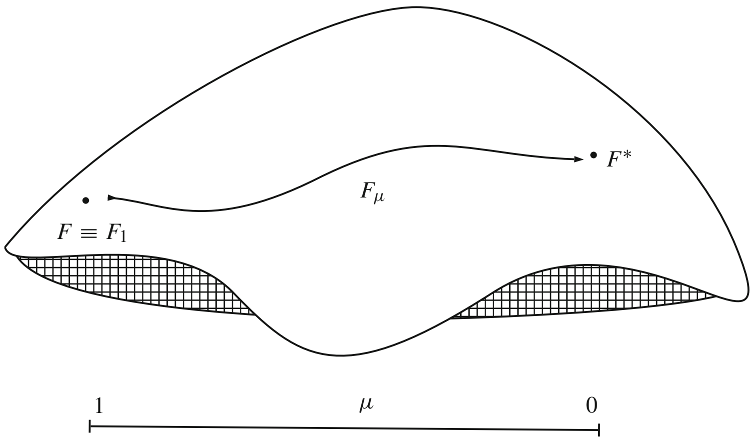


**Fig. 1** Recession strategy. The operator $F_\mu$ produces a path, parametrized by $\mu \in (0, 1]$, in the space of $(\lambda, \Lambda)$-elliptic operators. Depending on the regularity available for the PDE driven by $F^*$, we expect to transport information along the path $F_\mu$ back to the original operator

We put forward a result relating ellipticity, the recession operator and the behavior of the limit in (4). We begin with a simple lemma on the homogeneity of $F^*$.

**Lemma 2.1 (Positive Homogeneity of Degree 1)** *Let $F : \mathcal{S}(d) \to \mathbb{R}$ be a $(\lambda, \Lambda)$-elliptic operator. If the recession function $F^*$ is unique, it is positively homogeneous of degree* 1.

*Proof* We start by fixing $\rho > 0$. From the definition of recession function we have

$$|F^*(\rho M) - \rho F^*(M)| \leq |F^*(\rho M) - F_\mu(\rho M)| + |F_\mu(\rho M) - \rho F^*(M)|. \tag{5}$$

For every $\delta > 0$, there exists $\varepsilon > 0$ such that $\mu < \varepsilon$ implies

$$|F^*(\rho M) - F_\mu(\rho M)| \leq \delta.$$

In addition, notice that

$$|F_\mu(\rho M) - \rho F^*(M)| = \rho \left| F_{\mu\rho^{-1}}(M) - F^*(M) \right|;$$

the uniqueness of recession function yields

$$\rho \left| F_{\mu\rho^{-1}}(M) - F^*(M) \right| \to 0$$

as $\mu \to 0$. By gathering the former computations, we conclude that

$$|F^*(\rho M) - \rho F^*(M)| \leq \varepsilon^*,$$

for arbitrarily small $\varepsilon^*$. This closes the proof. □

Next, we prove that $F_\mu$ converges to $F^*$ uniformly in compact sets of $\mathcal{S}(d)$. For ease of presentation, we suppose the recession function is homogeneous of degree 1. The uniqueness of the recession operator may sound as a too strict condition. However, important applications of this technique involve modifying $F$ outside of a large ball to coincide with $F^*$. This is at the core of the argument behind density type of results. In this case, the uniqueness of $F^*$ is simple to verify.

**Proposition 2.1 (Uniform Convergence)** *Let $F : \mathcal{S}(d) \to \mathbb{R}$ be a $(\lambda, \Lambda)$-elliptic operator. Suppose $F^*$ is homogeneous of degree* 1. *Then, $F_\mu$ converges locally uniformly to $F^*$. Moreover, for every $\delta > 0$ there exists $\varepsilon > 0$ so that*

$$\left\| F_\mu(M) - F^*(M) \right\| \leq \varepsilon(1 + \|M\|), \tag{6}$$

*provided $\mu \leq \delta$.*

*Proof* Because $F_\mu$ is $(\lambda, \Lambda)$-elliptic, it is uniformly Lipschitz continuous in $\mathcal{S}(d)$; see [3, p. 12]. By the Arzelà-Ascoli Theorem, we conclude that $F_\mu$ converges locally uniformly, through a subsequence if necessary. The definition of $F^*$ implies that $F_\mu(M)$ converges pointwise to $F^*(M)$, for every $M \in \mathcal{S}(d)$. Therefore, every subsequential limit $F_{\mu_j}$ must coincide, as $j \to \infty$. Then, we conclude that $F_\mu$ converges uniformly locally to $F^*$.

As for the estimate in (6), we consider two cases.

*Case 1* Suppose that $\|M\| \leq 1$. In this case, (6) is consequential on from the local uniform convergence of $F_\mu$.

*Case 2* Let $\|M\| > 1$ and consider

$$\mu_M := \frac{\mu}{\|M\|}.$$

By assumption, $F^*$ is positively homogeneous of degree 1. Then,

$$\frac{1}{\|M\|}|F_\mu(M) - F^*(M)| = \left| F_{\mu_M}\left(\frac{M}{\|M\|}\right) - F^*\left(\frac{M}{\|M\|}\right)\right| \to 0 \tag{7}$$

as $\mu_M \to 0$, where we have used Case 1. It stems from (7) that for $\mu \ll 1$, we have

$$|F_\mu(M) - F^*(M)| \leq \varepsilon\|M\|,$$

which completes the proof. □

*Remark 2.1* Instead of supposing that $F^*$ is homogeneous of degree 1, we could have assumed uniqueness of the recession operator. In this case, Lemma 2.1 would produce the homogeneity.

A notable feature of the recession strategy relies on its flexibility. For any $(\lambda, \Lambda)$-elliptic operator $F$, it is possible to fix a number $L \gg 1$ and propose the following modification:

$$F_L(M) := \begin{cases} F(M) & \text{if} \quad M \in B_L \\ \gamma_L(M) & \text{if} \quad M \in B_{2L} \setminus B_L \\ \mathrm{Tr}(M) & \text{if} \quad M \in B_{2L}^c, \end{cases}$$

with

$$\gamma_L(M) := \frac{2L - \|M\|}{L}F(M) + \frac{\|M\| - L}{L}\mathrm{Tr}(M).$$

In this case, it is clear that $F_L^*$ coincides with the asymptotic profile of the operator; that is, $F_L^*(M) \equiv \mathrm{Tr}(M)$. Hence, the modification in (2) yields the Laplacian operator as the recession profile of $F_L$ (Fig. 2).

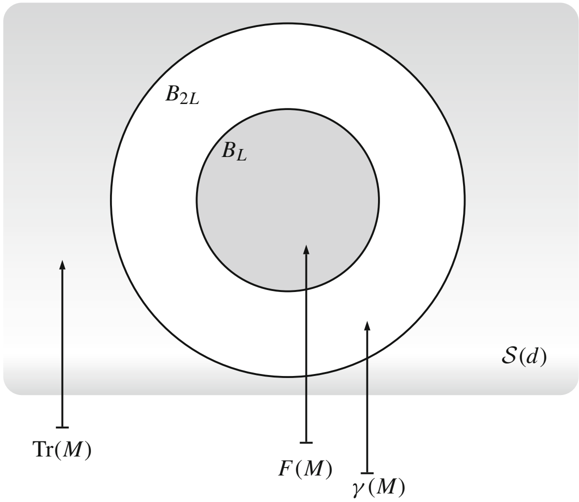


**Fig. 2** Modification strategy. The recession operator allows us to modify the original problem outside of a ball of large enough radius $L \gg 1$. In this case, the resulting operator has a recession profile described by the Laplacian

Asymptotic modifications of a given operator are useful in producing density results. We return to this topic in Sect. 4. We close this section with a few examples. We expect to highlight the strength of the recession analysis *as well as its drawbacks and limitations*.

*Example 2.1 (Eigenvalue q-Momentum Operator)* Let $q \in 2\mathbb{N}+1$ and consider the operator

$$F_q(M) := \sum_{i=1}^{d} \left(1 + \lambda_i^q\right)^{\frac{1}{q}},$$

where $(\lambda_i)_{i=1}^d$ are the eigenvalues of the matrix $M$. Notice that

$$\mu F_q(\mu^{-1}M) = \mu^{q/q} \sum_{i=1}^{d} \left(1 + \mu^{-q}\lambda_i^q\right)^{\frac{1}{q}} = \sum_{i=1}^{d} \left(\mu^q + \lambda_i^q\right)^{\frac{1}{q}};$$

therefore,

$$F_q^*(M) = \lim_{\mu \downarrow 0} \sum_{i=1}^{d} \left(\mu^q + \lambda_i^q\right)^{\frac{1}{q}} = \operatorname{Tr}(M).$$

This example shows that the recession operator relates $F_q$ to the Laplacian. Moreover, if we are interested in ellipticity-invariant (or universal) properties of $F_q$, it suffices to examine the case of the Laplacian operator.

Our next example appears in Differential Geometry. It is called special Lagrangian equation.

*Example 2.2 (A Perturbation of the Special Lagrangian Operator)* We write the special Lagrangian operator as follows:

$$F(M) := \sum_{i=1}^{d} \arctan(1+\lambda_i) + \alpha_i \lambda_i ,$$

where $(\alpha_i)_{i=1}^d$ are real numbers. A straightforward computation yields

$$F^*(M) = \sum_{i=1}^{d} \alpha_i \lambda_i ;$$

i.e., the operator under analysis relates to a perturbation of the Laplacian.

*Example 2.3 (The Log-Monge-Ampère Equation)* The log-Monge-Ampère operator is given by

$$F(M) := \ln[\det(M)] .$$

If we consider uniformly convex solutions, a scaling argument allows us to suppose the eigenvalues of $M$ are strictly above 1. Consider the following perturbed problem:

$$F_\alpha(M) := \ln[\det(M)] + \sum_{i=1}^{d} \alpha_i \lambda_i ,$$

where $\alpha_i \in \mathbb{R}$ are small. Because $\lambda_i > 1$, the sublinearity of the logarithm implies

$$\mu \left[ \ln\left[\det\left(\mu^{-1}M\right)\right] + \sum_{i=1}^{d} \alpha_i \mu^{-1} \lambda_i \right] \leq C(d)\sqrt{\mu} + \sum_{i=1}^{d} \alpha_i \lambda_i ;$$

therefore,

$$F_\alpha^*(M) = \sum_{i=1}^{d} \alpha_i \lambda_i .$$

We conclude that a small perturbation of the log-Monge-Ampère equation can be related to a linear uniformly elliptic operator. If stability results are available for the strictly convex solutions of the log-Monge-Ampère equation, the recession provides access to information through approximation results.

In the previous examples, the recession strategy related arbitrary operators with simpler ones (e.g., the Laplacian). Since the rationale of our method is to import information from $F^*$ to $F$, these examples are encouraging. This is because the regularity theory for the Laplacian operator is well-established in most cases and, therefore, more information is available in the limit case.

Though promising, this is not a general outcome. In many important examples, the recession function falls short in producing additional information. Next, we consider the case of the Isaacs equation.

*Example 2.4 (The Isaacs Equation)* An important example of fully nonlinear elliptic equation is the Isaacs equation

$$F(M) := \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left[-\mathrm{Tr}\left(A_{\alpha,\beta}(x)M\right)\right].$$

The Isaacs equation is homogeneous of degree 1. Therefore,

$$\mu \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left[-\mathrm{Tr}\left(A_{\alpha,\beta}(x)\mu^{-1}M\right)\right] = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left[-\mathrm{Tr}\left(A_{\alpha,\beta}(x)M\right)\right]$$

and

$$F \equiv F_\mu \equiv F^*;$$

i.e., the recession strategy produces no further information in this case.

The Isaacs equation arises in the study of two-player, zero-sum, (stochastic) differential games. See [8, 20]. In [17], an approximation method based on the Bellman equation is introduced to study the regularity of the solutions to the Isaacs operator.

## 3 Applications to Regularity Theory

In this section we describe two applications of the asymptotic methods. The first one regards regularity theory in Sobolev spaces for fully nonlinear equations, based on the results in [18]. The second application regards an ellipticity-invariant quantity, namely the Escauriaza exponent; see [5].

In what follows, we recur to the concept of *universal constant*. From now on, a universal constant is a real number $C > 0$ depending only on the dimension $d$ and the ellipticity constants $\lambda$ and $\Lambda$.

## 3.1 Estimates in $W^{2,p}$

In this section we consider the equation

$$F(D^2u) = f \quad \text{in} \quad B_1, \tag{8}$$

where $F$ is a $(\lambda, \Lambda)$-elliptic operator and $f \in L^d(B_1)$. We prove the following theorem:

**Theorem 3.1** **($W^{2,p}$-Regularity)** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (8) and suppose that $F^*$ is convex. Then, $u \in W^{2,p}_{loc}(B_1)$ and*

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}\right),$$

*where $C > 0$ is a universal constant.*

Theorem 3.1 first appeared in [18]. It can be framed as a Calderón-Zygmund estimate. From a geometric viewpoint, Theorem 3.1 regards controlling the curvature of paraboloids touching the graph of the solution $u$. Because our arguments rely on the measure of sets involving quadratic polynomials, we define these objects in the sequel.

A quadratic polynomial of opening $M > 0$ is a map $P_M : B_1 \to \mathbb{R}$ of the form

$$P_M(x) := \ell(x) + M\frac{|x|^2}{2},$$

where $\ell : B_1 \to \mathbb{R}$ is an affine function.

Next we discuss the main elements of the proof of Theorem 3.1 and highlight the role of the recession operator. We start with a proposition.

**Proposition 3.1** **($W^{2,\delta}$-Estimates)** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solutions to (8). There exist $\delta > 0$ and $C > 0$, universal constants, such that $u \in W^{2,\delta}_{loc}(B_1)$ and*

$$\|u\|_{W^{2,\delta}_{loc}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}\right).$$

This result was proved in the linear case by Lin in [13]. For the fully nonlinear setting, see [3]. The recession strategy builds upon Proposition 3.1 to produce a regime switching of the form $\delta \to p$, for $p > d$. This is based on the decay rate for the measure of a family of sets. We continue with a definition.

**Definition 3.1** Let $u \in \mathcal{C}(B_1)$. For $M > 0$ and $H \subset B_1$, we define

$$\underline{G}_M(u, H) := \{x \in H \mid \exists P_M \text{ concave paraboloid touching } u \text{ from below at } x\}$$

and

$$\overline{G}_M(u, H) := \{x \in H \mid \exists P_M \text{ convex paraboloid touching } u \text{ from above at } x\}.$$

We also set

$$\underline{A}_M(u, H) := H \setminus \underline{G}_M(u, H) \qquad \text{and} \qquad \overline{A}_M(u, H) := H \setminus \overline{G}_M(u, H).$$

Finally, we have

$$G_M(u, H) := \underline{G}_M(u, H) \cap \overline{G}_M(u, H)$$

and

$$A_M(u, H) := \underline{A}_M(u, H) \cup \overline{A}_M(u, H).$$

We proceed with a proposition relating the notions of distribution function, maximal operator and norms in Lebesgue spaces.

**Proposition 3.2** *Let $g \geq 0$ be a measurable function on $B_1$ and denote by $\mu_g$ its distribution function*

$$\mu_g(t) = |\{x \in B_1 \mid g(x) > t\}|, \qquad t > 0.$$

*Fix $\zeta > 0$ and $M > 1$. For $p > 0$, we have*

$$g \in L^p(B_1) \qquad \Longleftrightarrow \qquad \sum_{k=1}^{\infty} M^{pk} \mu_g(\zeta M^k) =: S < \infty.$$

*Moreover, For some $C = C(\zeta, M, p)$, we have*

$$C^{-1} S \leq \|g\|^p_{L^p(B_1)} \leq C(1 + S).$$

For more on Proposition 3.2, we refer the reader to [3, Lemma 7.3]. The following fact is consequential on Proposition 3.2: $D^2 u \in L^p(B_{1/2})$ is equivalent to the summability of

$$\sum_{k=1}^{\infty} M^{pk} \left| A_{M^k}(u, B_{1/2}) \right|,$$

for some $M$ fixed.

Here we use the recession strategy. By assuming that $F^*$ is convex, we infer that solutions to

$$F^*(D^2 u) = 0 \quad \text{in} \quad B_1$$

have estimates in $\mathcal{C}^{2,\alpha}$, for some $\alpha \in (0, 1)$—because of the Evans-Krylov theory; see [6, 9]. These estimates set a competing inequality: when the Hessian of the solutions to (8) starts to grow, the recession profile governs the problem. Because it has $\mathcal{C}^{2,\alpha}$-estimates, the norm of the Hessian decreases and the original operator resumes driving the equation. This process repeats itself. It prevents the Hessian from blowing up in an $L^p$-sense. To formalize this intuition, we state and prove an Approximation Lemma.

**Proposition 3.3 (Approximation Lemma)** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to*

$$F_\mu(D^2u) = f \quad in \quad B_1,$$

*where $F$ is $(\lambda, \Lambda)$-elliptic. Suppose that $F^*$ is convex. For every $\delta > 0$, there exists $\varepsilon > 0$ such that if*

$$\mu + \|f\|_{L^d(B_1)} \le \varepsilon,$$

*there exists $h \in \mathcal{C}^{2,\alpha}_{loc}(B_1)$, with*

$$\|h\|_{\mathcal{C}^{2,\alpha}(B_{9/10})} \le C\,\|h\|_{L^\infty(B_1)},$$

*satisfying*

$$\|u - h\|_{L^\infty(B_{9/10})} \le \delta,$$

*where $C > 0$ and $\alpha \in (0, 1)$ are universal constants. Moreover,*

$$u - h \in S\left(\frac{\lambda}{d}, \Lambda, f - F(D^2u)\right).$$

*Proof* The last assertion of the proposition follows from elementary facts on the class of viscosity solutions; see [3, Proposition 2.13]. As regards the approximation statement, we argue by way of contradiction and use a compactness argument. Suppose the statement of the proposition is false. In this case, there would exist $\delta_0$ such that every function $h \in \mathcal{C}^{2,\alpha}_{loc}(B_{9/10})$ is such that

$$\|u - h\|_{L^\infty(B_{9/10})} \ge \delta_0.$$

Consider a sequence of real numbers $(\mu_n)_{n\in\mathbb{N}}$ and sequences of functions $(u_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ such that

$$\mu_n \to 0 \quad \text{and} \quad \|f_n\|_{L^d(B_1)} \to 0$$

and

$$F_{\mu_n}(D^2 u_n) = f_n \quad \text{in} \quad B_1.$$

By the Krylov-Safonov theory, the sequence $(u_n)_{n\in\mathbb{N}}$ is equibounded in $\mathcal{C}^{1,\alpha}_{loc}(B_1)$. Therefore, through a subsequence if necessary, $u_n \to u_\infty$ in the $\mathcal{C}^{1,\alpha}$-topology. Standard stability results in the theory of viscosity solutions imply that

$$F^*(D^2 u_\infty) = 0 \quad \text{in} \quad B_{9/10}.$$

Because of the Evans-Krylov theory, $u_\infty \in \mathcal{C}^{2,\alpha}_{loc}(B_{9/10})$ and

$$\|u - u_\infty\|_{L^\infty(B_{9/10})} \to 0,$$

as $n \to \infty$. By taking $h \equiv u_\infty$, we get a contradiction and complete the proof. □

Next, we combine Proposition 3.3 with Proposition 3.1 to control the measure of $G_M(u, B_1) \cap Q_1$. We notice that, throughout the paper, $Q_\ell$ stands for the $d$-dimensional cube of side length $\ell$.

**Lemma 3.1** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (8) and suppose*

$$-|x|^2 \leq u(x) \leq |x|^2 \quad \textit{in} \quad B_1 \setminus B_{3/4}.$$

*Under the assumptions of Proposition 3.3, there exists $M > 0$, depending only on the dimension, and $\rho \in (0, 1)$ such that*

$$|G_M(u, B_1) \cap Q_1| \geq 1 - \rho.$$

*Proof* Take $h$, the function from Proposition 3.3 and consider its restriction to $B_{1/2}$. Extend $h$ outside $\overline{B}_{1/2}$ continuously in such a way that

$$h \equiv u \quad \text{in} \quad B_1 \setminus B_{3/4}$$

and

$$\|u - h\|_{L^\infty(B_1)} = \|u - h\|_{L^\infty(B_{3/4})}.$$

These choices imply that

$$-2 - |x|^2 \leq h(x) \leq 2 + |x|^2 \quad \text{in} \quad B_1 \setminus B_{1/2}.$$

It is easy to verify the existence of a number $N > 0$ so that

$$Q_1 \subset G_N(h, B_1).$$

For a constant $\rho_0$ to be determined later, we set

$$\vartheta := \rho_0 (u - h).$$

We gather Propositions 3.3 and 3.1 to conclude that $\vartheta \in W^{2,\delta}_{loc}(B_1)$. Therefore,

$$|A_t(\vartheta, B_1) \cap Q_1| \leq Ct^{-\delta},$$

which follows from the definition of $A_t$. Because $A_N$ and $G_N$ are complement to each other, we conclude that

$$|G_N(u - h, B_1) \cap Q_1| \geq 1 - \rho_0,$$

for some $N > 1$. Finally,

$$|G_{2N}(u, B_1) \cap Q_1| \geq 1 - \rho_0,$$

which completes the proof. □

An application of Lemma 3.1 yields the following result:

**Lemma 3.2** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solutions to* (8). *Under the assumptions of Proposition 3.3, we have*

$$G_1(u, B_1) \cap Q_3 \neq \emptyset \quad \Longrightarrow \quad |G_M(u, B_1) \cap Q_1| \geq 1 - \rho,$$

*where $M > 0$ and $\rho > 0$ are as in Lemma 3.1.*

*Proof* For a proof of this result, we refer the reader to [3]; see also [18, Lemma 5.2]. □

The maximal function associated with $f \in L^1_{loc}(\mathbb{R}^d)$ is denoted by $m(f)$ and given by

$$m(f)(x) := \sup_{\ell > 0} \frac{1}{|Q_\ell(x)|} \int_{Q_\ell(x)} |f(y)| dy.$$

**Lemma 3.3** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to*

$$F_\mu(D^2 u) = f \quad in \quad B_1.$$

*Suppose*

$$\mu + \|f\|_{L^d(B_1)} \ll 1.$$

*Suppose further the assumptions of Proposition 3.3 are in force. Extend $f$ outside of $B_1$ by zero. Define*

$$A := A_{M^{k+1}}(u, B_1) \cap Q_1$$

*and*

$$B := \left(A_{M^k}(u, B_1) \cap Q_1\right) \cup \left\{x \in Q_1 \,|\, m(f^d)(x) \geq \left(cM^k\right)^d\right\}.$$

*Then, there exists $\varepsilon \in (0,1)$ such that*

$$|A| \leq \varepsilon |B|.$$

*Proof* As before, for the proof of this result we refer the reader to [3] and [18, Lemma 5.3]. □

Finally, we consider the distribution function of $\Theta(x)$, defined as

$$\Theta(x) := \inf\left\{M \,|\, x \in G_M(u, B_{1/2})\right\}.$$

The integrability of $D^2u$ is closely related to the integrability of $\Theta$, in the sense that

$$\|\Theta\|_{L^p(B_1)} \sim \left\|D^2 u\right\|_{L^p(B_1)}.$$

See, for instance, [12].

Once the former lemmas are available, we present the proof of Theorem 3.1. It relies on the properties of the maximal function associated with $f \in L^d(B_1)$.

*Proof of Theorem 3.1* We take $M > 0$ from Lemma 3.3 and define $\rho$ as follows:

$$\rho := \frac{1}{2M^p}.$$

In addition, set

$$\alpha_k := \left|A_{M^k}(u, B_1) \cap Q_1\right|$$

and

$$\beta_k := \left|\left\{x \in Q_1 \,|\, m(f^d)(x) \geq (CM^k)^d\right\}\right|.$$

Because of Lemma 3.3,

$$\alpha_k \leq \rho^k + \sum_{i=0}^{k-1} \rho^{k-i} \beta_i.$$

Moreover, $m(f^d) \in L^{\frac{p}{d}}(\mathbb{R}^d)$ and

$$\left\| m(f^d) \right\|_{L^{\frac{p}{d}}(\mathbb{R}^d)} \le c \, \| f \|^d_{L^p(B_1)} \le C.$$

Therefore, Proposition 3.2 implies

$$\sum_{k=0}^{\infty} M^{pk} \beta_k \le C.$$

On the other hand we have

$$\mu_\Theta(t) \le \left| A_t(u, B_{1/2}) \right| \le \left| A_t(u, B_{1/2}) \cap Q_1 \right|.$$

Because of Proposition 3.2, the proof is complete if we verify that

$$\sum_{k=1}^{\infty} M^{pk} \alpha_k \le C.$$

However,

$$\begin{aligned} \sum_{k=1}^{\infty} M^{pk} \alpha_k &\le \sum_{k=1}^{\infty} \left(\rho M^p\right)^k + \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \rho^{k-i} M^{p(k-i)} M^{pi} \beta_i \\ &\le \sum_{k=1}^{\infty} 2^{-k} + \left( \sum_{i=0}^{\infty} M^{pi} \beta_i \right) \left( \sum_{j=1}^{\infty} 2^{-j} \right) \le C. \end{aligned}$$

□

We close this section with a number of remarks on the consequences and applications of Theorem 3.1.

*Remark 3.1* Theorem 3.1 implies that, for every $p > d$, $D^2 u \in p - BMO_{loc}(B_1)$, where

$$u \in p - BMO(B_1) \qquad \Leftrightarrow \qquad \sup_{\ell > 0} \int_{B_\ell} \left| u(x) - \langle u \rangle_\ell \right|^p dx < \infty,$$

and

$$\langle u \rangle_\ell := \frac{1}{|B_\ell|} \int_{B_\ell} u(x) dx.$$

In fact, ellipticity builds upon Sobolev regularity to produce an integrability level for the Hessian above $L^p$, for every $p > 1$, and strictly below $L^\infty$. See [18].

*Remark 3.2* Theorem 3.1 extends to operators of the form

$$F : \mathcal{S}(d) \times \mathbb{R}^d \times \mathbb{R} \times B_1 \to \mathbb{R},$$

provided the dependence of $F(M, p, u, x)$ with respect to $p$, $u$ and $x$ is properly controlled. In case $F$ is globally Lipschitz with respect to $p$, has a modulus of continuity with respect to $u$ and small oscillation with respect to $x$, Theorem 3.1 extends to equations of the form

$$F(D^2u, Du, u, x) = f \quad \text{in} \quad B_1.$$

See [21] for details. Similar arguments produce global estimates, as in [23], under asymptotic conditions on the problem.

*Remark 3.3* We work under the assumption that $F^*$ is convex. However, the result holds even if $F^*$ has only $W^{2,q}$ estimates; see [12]. In this case, estimates in $W^{2,p}$ would be available for $d < p < q$.

## 3.2 The Escauriaza's Exponent

Among the assumptions of Theorem 3.1 is the restriction $p > d$. See [2, 18]; also, [3, Chapter 7]. In [5], Escauriaza extended Caffarelli's estimates under the condition $p > d - \varepsilon$, for some constant $\varepsilon = \varepsilon(\lambda, \Lambda, d)$.

**Proposition 3.4 (Escauriaza's Exponent)** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (8) and suppose that $F^*$ is convex. Then, $u \in W^{2,p}_{loc}(B_1)$ and*

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)}\right),$$

*for $p > d - \varepsilon$, where $C > 0$ is a universal constant and $\varepsilon = \varepsilon(\lambda, \Lambda, d)$ is the Escauriaza's exponent.*

Proposition 3.4 requires lower integrability of the source term to ensure estimates in Sobolev spaces. This weaker requirement is quantified by $\varepsilon$. Although a function of $\lambda$, $\Lambda$ and the dimension, a precise formula for this quantity remains unknown. Next, we use the recession strategy to examine some examples of operators and produce asymptotic information on $\varepsilon$.

The key to the lower integrability of the source term is related to $F$. In fact, it comes from the integrability of the Green's function associated with $F$ through its linearized operator $L$. The following proposition accounts for the integrability of the Green's function of a linear $(\lambda, \Lambda)$-elliptic operator. It is due to Fabes and Stroock; see [7].

**Proposition 3.5** *Let $L$ be a $(\lambda, \Lambda)$-elliptic operator with measurable coefficients. Let $G(x, y)$ be its Green's function in $B_1$. Then,*

1. *There exists $C > 0$ and $\varepsilon > 0$ such that if $p > d - \varepsilon$,*

$$\int_{B_1} G(x, y)^{p'} dy \leq C,$$

*for all $x \in B_1$, where*

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

2. *There exists $\beta > 0$ such that if $E \subset B_r \subset B_{1/2}$, we have*

$$\left(\frac{|E|}{|B_r|}\right)^{\beta} \int_{B_r} G(x, y) dy \leq C \int_E G(x, y) dy.$$

For the proof of Proposition 3.5 we refer to [7]. Consequential on this result in the following Harnack's inequality.

**Proposition 3.6 (Harnack's Inequality)** *Let $u \in \mathcal{C}(B_1)$ be a nonnegative viscosity solutions to*

$$F(D^2 u) = f \quad \text{in} \quad B_1,$$

*where $F$ is a $(\lambda, \Lambda)$-elliptic operator and $f \in L^{d-\varepsilon}(B_1)$. Then, there exists $C > 0$, a universal constant, such that*

$$\sup_{B_{r/2}} u \leq C \left( \inf_{B_{r/2}} u + r^{2-\frac{d}{d-\varepsilon}} \|f\|_{L^{d-\varepsilon}(B_1)} \right).$$

The proof of Proposition 3.6 is in [5]. This result has many consequences to the general theory of elliptic PDEs. We mention the universal modulus of continuity produced in [22]. Indeed, solutions to (8) satisfy

$$\|u\|_{\mathcal{C}^{0,\frac{d-2\varepsilon}{d-\varepsilon}}(B_{1/2})} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^{d-\varepsilon}(B_1)} \right).$$

Notice that Escauriaza's exponent depends only on the integrability of the Green's function associated with $F$ and the dimension. Hence, $\varepsilon$ is invariant with respect to the ellipticity. Therefore, for a fixed dimension $d$, two operators with the same ellipticity must have the same exponent $\varepsilon$. Here the recession strategy plays a role.

When the limit

$$F^*(M) = \lim_{\mu \downarrow 0} F_\mu(M)$$

exists, the recession operator $F^*$ has the same ellipticity as $F$. If the Green's function associated with $F^*$ is known, or we infer its integrability, it would be possible to compute the Escauriaza's exponent for $F^*$, say $\varepsilon_{F^*}$. By knowing this quantity, we recover $\varepsilon_F$. In what follows, we examine an example and explicitly compute the Escauriaza's exponent.

*Example 3.1 (Eigenvalue $q$-Momentum Operator)* We revisit Example 2.1, where the operator $F_q$ is defined:

$$F_q(M) := \sum_{i=1}^{d} \left(1 + \lambda_i^q\right)^{\frac{1}{q}}.$$

To linearize this operator and evaluate the integrability of the associated Green's function in a ball might be not even possible. However, we learned that $F_q^*(D^2u) = \Delta u$. In addition, the Escauriaza's exponent for the Laplacian, $\varepsilon_\Delta$, is known to be $d/2$. Therefore,

$$\varepsilon_{F_q} = \varepsilon_\Delta = \frac{d}{2}.$$

Moreover, we conclude that Theorem 3.1 is available for $F_q$ provided the source term satisfies $f \in L^{\frac{d}{2}}(B_1)$.

In the prior example, $\varepsilon_{F_q} = d/2$. Every fully nonlinear operator whose recession profile coincides with the Laplacian has the same exponent $\varepsilon_\Delta$.

## 4 Weak Regularity in $\mathcal{C}^{1,Log-Lip}$

In this section we prove a weak regularity result. We understand *weak regularity result* as the density of regular enough solutions in the class of viscosity solutions. As indicated in the works of Nadirashvili and Vladut, the optimal level of regularity implied by ellipticity is $\mathcal{C}^{1,\alpha}$. This is due to the Krylov-Safonov theory.

However, for many applications, it is enough that solutions to $F(D^2u) = f$ are approximated by regular functions. For example, in [18] the authors proved that $W^{2,p}_{loc}(B_1) \cap S(\lambda^-, \Lambda^+, f)$ is dense in $S(\lambda, \Lambda, f)$. Therefore, when studying properties closed under uniform limits, the starting point of the theory shifts to $W^{2,p}$-estimates. We refer to a result in this spirit as a weak regularity result.

The main result of this section regards $\mathcal{C}^{1,Log-Lip}$-weak regularity. We say that a function $u \in \mathcal{C}^{1,Log-Lip}(B_1)$ if and only if there exists a constant $C > 0$ satisfying

$$\sup_{x \in B_r} |u(x) - [u(0) + Du(0) \cdot x]| \leq -Cr^2 \ln r.$$

In what follows, we consider operators with explicit dependence on the space variable $x \in B_1$. It leads to the following problem:

$$F(x, D^2u) = f \quad \text{in} \quad B_1. \tag{9}$$

**Theorem 4.1 (Weak Estimates in $\mathcal{C}^{1,Log-Lip}$)** *Let $u \in \mathcal{C}(B_1)$ be a continuous viscosity solution to (9). Suppose $F$ is a $(\lambda, \Lambda)$-elliptic operator and $f \in L^\infty(B_1)$. Then, there exists a sequence of functions $\{u_j\}_{j \geq 1} \subset \mathcal{C}^{1,Log-Lip}_{loc}(B_1) \cap S(\lambda^-, \Lambda^+, f)$ that converges locally uniformly to $u$.*

The proof of Theorem 4.1 relies on three main structures. The first one is the Approximation Lemma (Proposition 3.3). It ensures the existence of a quadratic polynomial that approximates the solution $u$. The second main ingredient in the proof is a further application of Proposition 3.3; in this case, it produces estimates in $\mathcal{C}^{1,Log-Lip}$ for operators whose recession is convex. Finally, an asymptotic modification of $F$ completes the argument. We start with a lemma.

**Lemma 4.1** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (9). Under the assumptions of Proposition 3.3, there exist a second order polynomial $P$ such that $||P|| \leq C$ and*

$$||u - P||_{L^\infty(B_r)} \leq r^2,$$

*where $C > 0$ and $0 < r \ll 1$ are universal constants.*

*Proof* Let $h$ be the function from Proposition 3.3. Let $P$ denote the second order Taylor's expansion of $h$ at the origin. Thus

$$||u - P||_{L^\infty}(B_r) \leq ||u - h||_{L^\infty(B_r)} + ||h - P||_{L^\infty(B_r)} \leq \delta + Cr^{2+\alpha}.$$

We choose $r$ small enough so that $Cr^\alpha < \dfrac{1}{2}$ and $\delta = \dfrac{r^2}{2}$ and we obtain

$$||u - P||_{L^\infty}(B_r) \leq r^2.$$

□

*Remark 4.1* We notice that the choice of $r$ in Lemma 4.1 determines $\delta > 0$ in Proposition 3.3 and, therefore, sets the smallness regime involving $F_\mu$ and the norms of the source term.

The next result regards the regularity of the solutions to (9) in $\mathcal{C}^{1,Log-Lip}$. It appeared for the first time in [19]. Compare with [22, Theorem 3].

**Theorem 4.2 (Regularity)** *Leu $u \in \mathcal{C}(B_1)$ be a viscosity solutions to (9). Suppose $F^*$ is convex and $f \in L^\infty(B_1)$. Suppose further that*

$$\lim_{\mu \to 0} \mu F(x, \mu^{-1} M) \ = \ F^*(M)$$

*is uniform in M. Then, $u \in C^{1,Log-Lip}_{loc}(B_1)$ and there exists $C > 0$, universal, such that*

$$\sup_{B_r} |u(x) \ - \ u(x_0) \ - \ Du(x_0) \cdot x| \ \leq \ Cr^2 \ln r^{-1},$$

*for every $x_0 \in B_{1/2}$.*

*Proof* We split the proof in several steps

**Step 1** We prove the result for $x_0 = 0$. For all $M \in \mathcal{S}(d)$, we can find $\varepsilon > 0$ such that for all $\mu < \varepsilon$ we have $||F_\mu(M) - F^*(M)|| \leq \delta$, where $\delta > 0$ is the number from Lemma 4.1. We choose $r_0 \sim \sqrt{\varepsilon}$ and define

$$u_0(x) = \varepsilon \max\{1, ||u||_{L^\infty}, ||f||_{L^\infty}\}^{-1} u(r_0 x).$$

It is clear that $||u_0||_{L^\infty} \leq 1$ and

$$D^2 u(r_0 x) = \frac{1}{\varepsilon r_0^2} \max\{1, ||u||_{L^\infty}, ||f||_{L^\infty}\} D^2 u_0(x);$$

thus, $u_0$ satisfies

$$\tau F\left(\tau^{-1} D^2 u_0(x)\right) = \tau f(r_0 x),$$

where

$$\tau = \frac{\varepsilon r_0^2}{\max\{1, ||u||_{L^\infty}, ||f||_{L^\infty}\}}.$$

Note that $\tilde{f} = \tau f(r_0 x)$ satisfies $||\tilde{f}||_{L^\infty} \leq \varepsilon$.

**Step 2** Let $0 < r < r_0$. Next, we show the existence of a sequence of quadratic polynomials $(P_k)_{k \in \mathbb{N}}$,

$$P_k(x) := a_k + b_k \cdot X + \frac{1}{2} x^t M_k x,$$

such that

$$F^*(M_k) = 0 \tag{10}$$

$$\sup_{B_{r^k}} |u_0 - P_k| \leq r^{2k} \tag{11}$$

$$|a_k - a_{k-1}| + r^{k-1}|b_k - b_{k-1}| + r^{2(k-1)}|M_k - M_{k-1}| \leq Cr^{(2(k-1))}. \tag{12}$$

The constant $r$ in (11) and (12) is the one from Lemma 4.1. We shall verify (10)–(12) by induction. We set $P_0 = P_{-1} = 0$, and the first step $k = 0$ is immediately satisfied, since $F^*(0) = 0$ and $||u_0||_{L^\infty} \leq 1$. Suppose we have verified the thesis of induction for $k = 0, 1, \ldots, i$. Define the function

$$v(x) = \frac{u_0(r^i x) - P_i(r^i x)}{r^{2i}}.$$

From (11), we have $|v| \leq 1$, and furthermore

$$D^2 v(x) = D^2 u_0(r^i x) - M_i;$$

thus $v$ satisfies

$$\mu F(\mu^{-1}(D^2 v + M_i)) = \tilde{f}(r^i x).$$

If we define $F_i(M) = F(M + M_i)$ and $F_i^*(M) = F^*(M + M_i)$, it follows that $||F_{\mu,i}(M) - F_i^*(M)|| \leq \delta$. Furthermore, since $F^*(M_i) = 0$, the equation $F^*(D^2\zeta) = 0$ has the same estimates as $F^*$. Now, since $F_{\mu,i}(D^2 v) = 0$, from Lemma 4.1 there exists a quadratic polynomial $\tilde{P}$ such that $||v - \tilde{P}||_{L^\infty(B_r)} \leq r^2$. Then

$$\frac{|u_0(r^i x) - P_i(r^i x) - r^{2i}\tilde{P}(x)|}{r^{2i}} \leq r^2$$

and

$$|u_0(x) - (P_i(x) + r^{2i}\tilde{P}(r^{-1}x))| \leq r^{2(i+1)};$$

taking

$$P_{i+1}(x) := P_i(x) + r^{2i}\tilde{P}(r^{-i}x),$$

we verify (11).

**Step 3** We define $P_{i+1}(x) = P_i(x) + r^{2i}\tilde{P}(r^{-i}x)$ and since $P_0 = 0$ we obtain

$$P_k(x) = \sum_{j=1}^{k} r^{2(j-1)}h(0) + \sum_{j=1}^{k} r^{(j-1)}Dh(0)x + k\frac{x^t D^2h(0)x}{2}.$$

Indeed, we shall verify this by induction. For $k = 1$ we have

$$P_1 = h(0) + Dh(0)x + \frac{x^t D^2h(0)x}{2} = \tilde{P}(x).$$

Now, suppose we have verified for $k = 1, 2, \ldots, i$. Since $P_{i+1}(x) = P_i(x) + r^{2i}\tilde{P}(r^{-i}x)$, we obtain

$$\begin{aligned} P_{i+1}(x) &= \sum_{j=1}^{i} r^{2(j-1)}h(0) + \sum_{j=1}^{i} r^{(j-1)}Dh(0)x + i\frac{x^t D^2h(0)x}{2} + r^{2i}h(0) \\ &\quad + r^i Dh(0)x + \frac{x^t D^2h(0)x}{2} \\ &= \sum_{j=1}^{i+1} r^{2(j-1)}h(0) + \sum_{j=1}^{i+1} r^{(j-1)}Dh(0)x + (i+1)\frac{x^t D^2h(0)x}{2}, \end{aligned}$$

thus we conclude the induction.

**Step 4** In addition,

$$|a_{k+1} - a_k| + r^k|b_{k+1} - b_k| + r^{2k}|M_{k+1} - M_k| \leq Cr^{2k},$$

since

$$|a_{k+1} - a_k| = \left|\sum_{j=1}^{k+1} r^{2(j-1)} - \sum_{j=1}^{k} r^{2(j-1)}\right| |h(0)| = r^{2k}|h(0)| \leq Cr^{2k},$$

$$|b_{k+1} - b_k| = \left|\sum_{j=1}^{k+1} r^{(j-1)} - \sum_{j=1}^{k} r^{(j-1)}\right| |Dh(0)| = r^k|Dh(0)| \leq Cr^k,$$

and

$$|M_{k+1} - M_k| = |k + 1 - k| = 1.$$

This proves (12).

From (11) we have $|u_0 - a_k| < r^{2k}$. Futhermore $|Du_0(0) - b_k| \leq Cr^k$ and $|M_k| = |kD^2h(0)| \leq Ck$.

Finally, for any $0 < \rho < \dfrac{1}{4}$, let $k$ such that $r^{k+1} < \rho \le r^k$. From estimates above, we obtain

$$\begin{aligned}
\sup_{B_{r^k}} |u_0(x) - (u_0(0) + Du_0(0)\cdot x)| &= \sup_{B_{r^k}} \Big|(u_0 - P_k) + a_k - u_0(0) \\
&\quad + b_k \cdot x - Du_0(0)\cdot x + \frac{x^t M_k x}{2}\Big| \\
&\le r^{2k} + Cr^{2k} + Cr^{2k} + \frac{C}{2} k r^{2k} \\
&\le C(r^{2k} + kr^{2k}) \\
&= \frac{C}{r^2}(r^{2(k+1)} + r^2 k r^{2k}) \\
&\le \frac{C}{r^2}(\rho^2 + k\rho^2) \\
&= C\rho^2(1+k).
\end{aligned}$$

Since $\rho < r^k$ we obtain $k < \dfrac{\ln \rho}{\ln r}$ and

$$\begin{aligned}
\sup_{B_{r^k}} |u_0(x) - (u_0(0) + Du_0(0)\cdot x)| &\le C\rho^2\left(1 + \frac{\ln\rho}{\ln r}\right) \\
&= c\rho^2(1 + \ln\rho - \ln r) \\
&\le c\rho^2(-\ln r),
\end{aligned}$$

provided $\rho < \frac{1}{4}$.

Since $\rho \le r^k$ we have $-\frac{1}{k}\ln\rho \ge -\ln r$, and thus

$$\sup_{B_{r^k}} |u_0(x) - (u_0(0) + Du_0(0)\cdot x)| \le -c\frac{1}{k}\rho^2 \ln\rho = -C\rho^2\ln\rho.$$

This finishes the poof. □

*Proof of Theorem 4.1* We construct a sequence of operators $F_j$ as follows: given $\delta > 0$, define

$$L_\delta(M) := (\Lambda + \delta)\sum_{e_i > 0} e_i + (\lambda + \delta)\sum_{e_i < 0} e_i,$$

where $e_i$ are the eigenvalues of $M \in \mathcal{S}(d)$. Now, define

$$F^j(x, M) := \max\{F(x, M), L_\delta(M) - C_j\},$$

where $C_j$ is a sequence of positive numbers to be determined. From the $(\lambda, \Lambda)$-ellipticity, we obtain

$$\begin{aligned}
F(x, M) &\geq \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i \\
&\geq \lambda \sum_{e_i>0} e_i - \Lambda ||M|| \\
&= L_\delta(M) - L_\delta(M) + \lambda \sum_{e_i>0} e_i - \Lambda ||M|| \\
&= L_\delta(M) - (\Lambda + \delta - \lambda) \sum_{e_i>0} e_i - (\lambda - \delta) \sum_{e_i<0} e_i - \Lambda ||M|| \\
&= L_\delta(M) - (\Lambda + \delta - \lambda) \left[ \sum_{e_i>0} e_i - \sum_{e_i<0} e_i \right] - \Lambda \sum_{e_i<0} e_i - \Lambda ||M|| \\
&= L_\delta(M) - (2\Lambda + \delta - \lambda) ||M|| - \Lambda \sum_{e_i<0} e_i \\
&\geq L_\delta(M) - (2\Lambda + \delta - \lambda) ||M|| \\
&\geq L_\delta(M) - C_j
\end{aligned}$$

provided we set $C_j := j(2\Lambda - \lambda + \delta)$ and $||M|| \leq j$. Here, we use $||M|| := \sum_{i=1}^{d} |e_i|$.

This shows that

$$F^j = F \text{ in } B_j \subset \mathcal{S}(d).$$

To compute the recession function of $F^j$, we find

$$F^j_\mu(x, M) = \mu F(x, \mu^{-1} M) = \max\{F_\mu(x, M), L_\delta(M) - \mu C_j\}.$$

Now, since $F_\mu$ is $(\lambda, \Lambda)$-elliptic, we have

$$\begin{aligned}
F_\mu(x, M) &\leq \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i \\
&= L_\delta(M) - L_\delta(M) + \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i \\
&= L_\delta(M) - \delta \sum_{e_i>0} e_i + \delta \sum_{e_i<0} e_i \\
&= L_\delta(M) - \delta ||M|| \\
&\leq L_\delta(M) - \mu C_j,
\end{aligned}$$

provided $||M|| \geq \dfrac{\mu C_j}{\delta}$.

Then, we have $F^j_\mu = L_\delta(M) - \mu C_j$ outside the ball of radius $C_j$ and

$$(F^j)^* = \lim_{\mu\to 0} F^j_\mu = \lim_{\mu\to 0}(L_\delta(M) - \mu C_j) = L_\delta(M).$$

Thus, from Theorem 4.2 for each j fixed, the operator $F^j$ have a priori estimates in $C^{1,Log-Lip}(\Omega)$.

Finally, we constructed $u_j$ to be the viscosity solution of the Dirichlet problem

$$\begin{cases} F^j(x, D^2u_j) = f(x) \text{ in } B_1 \\ u_j = u \text{ on } \partial B_1. \end{cases}$$

Thus, each $u_j$ is locally in $C^{1,Log-Lip}$, and since $F^j = F$ in $B_j$, we have that up to a subsequence, $u_j \to u$ locally in the $C^{0,\alpha}$-topology. The convergence is ensured by stability results in the theory of viscosity solutions. □

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