

A Review for an Isotropic Landau Model



Maria Galdani and Nicola Zamponi

Abstract We consider the equation

$$u_t = \operatorname{div} (a[u]\nabla u - u\nabla a[u]), \quad -\Delta a = u.$$

This model has attracted some attention in the recent years and several results are available in the literature. We review recent results on existence and smoothness of solutions and explain the open problems.

Keywords Landau equation · Coulomb potential · Isotropic model · Even solutions · Weighted Poincaré and Sobolev inequalities · Regularity estimates

1 Introduction

1.1 The Isotropic Landau Equation

In this manuscript we review recent results on the isotropic Landau equation

$$\begin{aligned} u_t &= \operatorname{div} (a[u]\nabla u - u\nabla a[u]), & -\Delta a &= u & \text{in } \mathbb{R}^3, & t > 0, \\ u(\cdot, 0) &= u_0. \end{aligned} \tag{1}$$

M. Galdani (✉)

Department of Mathematics, George Washington University, Washington, DC, USA

e-mail: galdani@gwu.edu

N. Zamponi

Institute for Analysis and Scientific Computing, Vienna University of Technology, Wien, Austria

e-mail: nicola.zamponi@tuwien.ac.at

This problem has been extensively studied in the recent years. Due to its similarity to the semilinear heat equation, to the Keller-Segel model but mostly to the homogeneous Landau equation

$$\begin{aligned} u_t &= \operatorname{div}(A[u]\nabla u - u\nabla a[u]), \\ A[u] &:= \frac{1}{8\pi} \int_{\mathbb{R}^3} \frac{1}{|y|} \left(Id - \frac{y \otimes y}{|y|^2} \right) u(x-y) dy, \quad a[u] = \operatorname{Tr}(A[u]), \end{aligned} \quad (2)$$

the analysis of existence, uniqueness and regularity of solutions to (1) is a very interesting problem. A modification of (1) was first introduced in [14, 16]; there the authors studied existence and regularity of bounded radially symmetric and monotone decreasing solutions to

$$u_t = a[u]\Delta u + \alpha u^2, \quad \alpha \in \left(0, \frac{74}{75}\right).$$

Existence of global bounded solutions for (1) has been proven in [11] when initial data are radially symmetric and monotone decreasing. Section 2 explains these results more in details. Existence of weak solutions for even initial data has been shown in [13]. See Sect. 3 for more details.

For general initial data the problem of global existence of regular solutions is still open. The main obstacles for the analysis are hidden in the quadratic non-linearity: expanding the divergence term one can formally rewrite (1) as

$$u_t = a[u]\Delta u + u^2.$$

This problem is reminiscent to the semilinear heat equation, which solutions become unbounded after a finite time [9].

Let us mention that the main interest in studying (1) is to gain insights on model (2). It is well known that existence of global smooth solutions for (2), both in the homogeneous and inhomogeneous settings, is still an open problem. For an overview about the problem we refer to [1, 6, 19, 20]. In the very recent years much has been done regarding integrability and regularization for solution to the Landau equation. In that direction we acknowledge the works [2, 10–12, 15, 18] which reflect a renewed increasing interest in this problem by several mathematical communities.

1.2 Conserved Quantities and Entropy Structure

In this section we collect some properties of (1). The isotropic Landau equation shares some of the conservation properties of the classical Landau and Boltzmann equation. We first note that the potential $a[u]$ can be expressed as

$$a[u](x, t) = \int_{\mathbb{R}^3} \frac{u(y, t)}{4\pi|x-y|} dy, \quad x \in \mathbb{R}^3, \quad t > 0,$$

and therefore (1) can also be written as

$$u_t = \operatorname{div} \int_{\mathbb{R}^3} \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{4\pi|x-y|} dy. \tag{3}$$

With this in mind let us define the Maxwell-Boltzmann entropy:

$$H[u] \equiv \int_{\mathbb{R}^3} u \log u \, dx. \tag{4}$$

The function $t \in (0, \infty) \mapsto H[u(t)] \in \mathbb{R}$ is nonincreasing in time: using (1) we can write the entropy production as

$$\begin{aligned} -4\pi \frac{d}{dt} H[u] &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\nabla u(x)}{u(x)} \cdot \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dx dy \\ &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x)u(y)}{|x-y|} \frac{\nabla u(x)}{u(x)} \cdot \left(\frac{\nabla u(x)}{u(x)} - \frac{\nabla u(y)}{u(y)} \right) dx dy \\ &= \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x)u(y)}{|x-y|} \left| \frac{\nabla u(x)}{u(x)} - \frac{\nabla u(y)}{u(y)} \right|^2 dx dy \geq 0. \end{aligned}$$

Clearly $\int_{\mathbb{R}^3} u(x, t) dx = \int_{\mathbb{R}^3} u_0(x) dx, t > 0$. We can say something about the first and second order moments of u . From (1) it follows

$$4\pi \frac{d}{dt} \int_{\mathbb{R}^3} x u(x, t) dx = - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dx dy = 0$$

for obvious symmetry reasons. So the first moment is conserved. As for the second moment

$$\begin{aligned} 4\pi \frac{d}{dt} \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x, t) dx &= - \iint_{\mathbb{R}^3 \times \mathbb{R}^3} x \cdot \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dx dy \\ &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} y \cdot \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dx dy \\ &= -\frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{x-y}{|x-y|} (u(y)\nabla u(x) - u(x)\nabla u(y)) dx dy. \end{aligned}$$

Since

$$\operatorname{div}_x \frac{x-y}{|x-y|} = -\operatorname{div}_y \frac{x-y}{|x-y|} = \left(\operatorname{div}_z \frac{z}{|z|} \right) \Big|_{z=x-y} = \frac{2}{|x-y|},$$

integration by parts yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x, t) dx = \frac{1}{2\pi} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x, t)u(y, t)}{|x-y|} dx dy = 2 \int_{\mathbb{R}^3} u(x, t) a(x, t) dx > 0. \tag{5}$$

This is one of the main differences to the classical Landau equation. The second moment increases with time and a bound is not given a-priori. We will see in Sect. 3 how to find this bound when the initial data are even.

2 Radially Symmetric Solutions

Problem (1) is well understood when initial data are radially symmetric and monotonically decreasing. In [11] the authors prove the following theorem:

Theorem 1 *Let u_0 be a nonnegative function that has finite mass, energy and entropy. Moreover let u_0 be radially symmetric, monotonically decreasing and such that $u_0 \in L^p_{weak}$ for some $p > 6$. Then there exists a function $u(x, t)$ smooth, positive and bounded for all time which solves*

$$u_t = a[u]\Delta u + u^2, \quad u(x, 0) = u_0.$$

We briefly highlight the ideas behind the proof of Theorem 1. The non-local dependence on the coefficients prevents the equation to satisfy comparison principle: in fact given two functions u_1 and u_2 such that $u_1 < u_2$ for $t < t_0$ and $u_1 = u_2$ at (x_0, t_0) we definitely have that $\Delta u_1(x_0, t_0) \leq \Delta u_2(x_0, t_0)$ and $a[u_1](x_0, t_0) \leq a[u_2](x_0, t_0)$. However it is not necessarily true $a[u_1](x_0, t_0)\Delta u_1(x_0, t_0) \leq a[u_2](x_0, t_0)\Delta u_2(x_0, t_0)$. To overcome this shortcoming, the main observation in [11] is that if one proves the existence of a function $g(x) \in L^p$ for some $p > 3/2$ such that $u_0 < g$ and

$$a[u]\Delta g + ug < 0,$$

then comparison principle for the linearized problem implies $u \leq g$ for all $t > 0$. Once higher integrability L^p of u is proved, standard techniques for parabolic equation such as Stampacchia's theorem yield L^∞ bound for $u(x, t)$ and consequent regularity.

3 Even Initial Data

Existence of weak solutions for (1) with general initial data is still an open problem. As already mentioned at the end of Sect. 1.2, the first obstacle that one encounters in the analysis of (1) is the missing bound for the second moment. This bound is essential when one seeks a-priori estimates for the gradient. In [13] the authors overcame this problem when solutions are even. In this section we highlight the basic estimates of [13] that will lead to construction of weak even solutions. For

weak solutions we mean functions $u(x, t)$ such that

$$\begin{aligned} \sqrt{u} &\in L^2\left(0, T; H^1\left(\mathbb{R}^3, \frac{dx}{1+|x|}\right)\right), & u, u \log u &\in L^\infty(0, T; L^1(\mathbb{R}^3)), \\ a &\in L^\infty(0, T; L^3_{loc}(\mathbb{R}^3)), & \nabla a &\in L^\infty(0, T; L^{3/2}_{loc}(\mathbb{R}^3)), \end{aligned}$$

that satisfy the following weak formulation

$$\int_0^T \langle \partial_t u, \phi \rangle dt + \int_0^T \int_{\mathbb{R}^3} (a \nabla u - u \nabla a) \cdot \nabla \phi \, dx dt = 0, \quad \forall \phi \in L^\infty(0, T; W_c^{1,\infty}(\mathbb{R}^3)).$$

All the computations here are formal, meaning we assume that u and all related quantities have enough regularity for the mathematical manipulations to make sense. We refer to [13] for the detailed calculations. Let

$$E(t) := \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x, t) dx, \quad R(t) := 2\sqrt{\frac{E(t)}{\|u_0\|_{L^1}}},$$

and define $B_{R(t)} \equiv \{x \in \mathbb{R}^3 : |x| < R(t)\}$. We point out that, since $\int_{\mathbb{R}^3 \setminus B_{R(t)}} u(x, t) dx \leq \frac{2E(t)}{R(t)^2} = \frac{1}{2} \|u_0\|_{L^1}$, it follows

$$\int_{B_{R(t)}} u(x, t) dx = \|u_0\|_{L^1} - \int_{\mathbb{R}^3 \setminus B_{R(t)}} u(x, t) dx \geq \frac{1}{2} \|u_0\|_{L^1}. \tag{6}$$

A Lower Bound for $a[u]$ From the definition of $a[u]$ it follows

$$4\pi a[u](x, t) = \int_{\mathbb{R}^3} \frac{u(y, t)}{|x-y|} dy \geq \int_{B_{R(t)}} \frac{u(y, t)}{|x-y|} dy \geq \frac{1}{R(t)+|x|} \int_{B_{R(t)}} u(y, t) dy \geq \frac{\|u_0\|_{L^1}}{2(R(t)+|x|)}$$

and therefore

$$a[u](x, t) \geq \frac{1}{16\pi} \frac{\|u_0\|_{L^1}^{3/2}}{E(t)^{1/2} + |x| \|u_0\|_{L^1}^{1/2}}. \tag{7}$$

A Gradient Estimate for Even Solutions We assume here that the solution u of (1) is even w.r.t. each component of x , for $t \geq 0$.

Clearly $|x - y| \leq |x| + |y| \leq (1 + |x|)(1 + |y|)$ for $x, y \in \mathbb{R}^3$. Therefore

$$\begin{aligned} -4\pi \frac{d}{dt} H[u] &\geq \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x, t)u(y, t)}{(1+|x|)(1+|y|)} \left| \frac{\nabla u(x, t)}{u(x, t)} - \frac{\nabla u(y, t)}{u(y, t)} \right|^2 dx dy \\ &= \left(\int_{\mathbb{R}^3} u(x, t) \frac{dx}{1+|x|} \right) \left(\int_{\mathbb{R}^3} \frac{|\nabla u(x, t)|^2}{u(x)} \frac{dx}{1+|x|} \right) - \left| \int_{\mathbb{R}^3} \frac{\nabla u(x, t)}{1+|x|} dx \right|^2. \end{aligned}$$

For the assumption on u it follows that

$$\left| \int_{\mathbb{R}^3} \frac{\nabla u}{1+|x|} dx \right|^2 = \sum_{i=1}^3 \left(\int_{\mathbb{R}^3} \frac{\partial u}{\partial x_i} \frac{dx}{1+|x|} \right)^2 = 0.$$

As a consequence

$$-4\pi \frac{d}{dt} H[u] \geq \left(\int_{\mathbb{R}^3} u(x, t) \frac{dx}{1+|x|} \right) \left(\int_{\mathbb{R}^3} \frac{|\nabla u(x, t)|^2}{u(x)} \frac{dx}{1+|x|} \right).$$

We now wish to show a positive lower bound for $\int_{\mathbb{R}^3} u(x, t) \frac{dx}{1+|x|}$ for $0 \leq t \leq T$. Let $R(t) = 2\sqrt{E(t)/\|u_0\|_{L^1}}$. It holds

$$\int_{\mathbb{R}^3} u(x, t) \frac{dx}{1+|x|} \geq \int_{B_{R(t)}} u(x, t) \frac{dx}{1+|x|} \geq \frac{1}{1+R(t)} \int_{B_{R(t)}} u(x, t) dx.$$

From (6) it follows

$$\frac{1}{\pi} \int_{\mathbb{R}^3} u(x, t) \frac{dx}{1+|x|} \geq \frac{1}{8\pi} \frac{\|u_0\|_{L^1}^{3/2}}{E(t)^{1/2} + \|u_0\|_{L^1}^{1/2}}, \quad t > 0. \quad (8)$$

Since $E(t)$ is increasing, we conclude

$$\frac{1}{\pi} \inf_{t \in [0, T]} \int_{\mathbb{R}^3} u(x, t) \frac{dx}{1+|x|} \geq \kappa(T), \quad (9)$$

with

$$\kappa(t) := \frac{1}{8\pi} \frac{\|u_0\|_{L^1}^{3/2}}{E(t)^{1/2} + \|u_0\|_{L^1}^{1/2}}.$$

Moreover,

$$\frac{dH[u]}{dt} + \kappa(t) \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u(x, t)}|^2}{1+|x|} dx \leq 0, \quad t > 0. \quad (10)$$

Upper Bound for $a[u]$ It holds

$$a[u](x, t) = \int_{|x-y|<1} \frac{u(y, t)}{|x-y|} dy + \int_{|x-y|\geq 1} \frac{u(y, t)}{|x-y|} dy \equiv I_1 + I_2. \quad (11)$$

The integral I_2 can be estimated immediately:

$$I_2 \leq \|u_0\|_{L^1}.$$

For I_1 we first use Hölder: since $\frac{1}{|x|}$ is $L^q_{loc}(\mathbb{R}^3)$ for $q < 3$, we get

$$\begin{aligned} I_1 &= \int_{|x-y|<1} \frac{u(y,t)}{|x-y|} dy \leq \left(\int_{|x-y|<1} u(y,t)^{3/2+\varepsilon} dy \right)^{\frac{1}{3/2+\varepsilon}} \left(\int_{|x-y|<1} |x-y|^{-\frac{3+2\varepsilon}{1+2\varepsilon}} dy \right)^{\frac{1+2\varepsilon}{3+2\varepsilon}} \\ &\leq 4\pi \frac{1+2\varepsilon}{4\varepsilon} \left(\int_{|y|<1+|x|} u(y,t)^{3/2+\varepsilon} dy \right)^{\frac{1}{3/2+\varepsilon}} = \frac{(1+2\varepsilon)\pi}{\varepsilon} \|\sqrt{u(t)}\|_{L^{3+2\varepsilon}(B_{1+|x|})}^2. \end{aligned}$$

The interpolation inequality implies (for $0 < \varepsilon \leq 3/2$):

$$\|\sqrt{u(t)}\|_{L^{3+2\varepsilon}(B_{1+|x|})} \leq \|\sqrt{u(t)}\|_{L^2(B_{1+|x|})}^{1-\theta} \|\sqrt{u(t)}\|_{L^6(B_{1+|x|})}^\theta, \quad \theta = \frac{3}{2} \frac{1+2\varepsilon}{3+2\varepsilon}.$$

Then, the Sobolev embedding $H^1 \hookrightarrow L^6$ implies

$$\|\sqrt{u(t)}\|_{L^{3+2\varepsilon}(B_{1+|x|})} \leq C(|x|) \|u_0\|_{L^1}^{(1-\theta)/2} \|\sqrt{u(t)}\|_{H^1(B_{1+|x|})}^\theta. \quad (12)$$

Notice that the constant C in (12) depends on $|B_{1+|x|}|$ and therefore on $|x|$. However, it is easy to show that such constant (assuming w.l.o.g. that it is optimal) is nonincreasing with respect to $|x|$, thus (12) leads to

$$\|\sqrt{u(t)}\|_{L^{3+2\varepsilon}(B_{1+|x|})} \leq C \|u_0\|_{L^1}^{(1-\theta)/2} \|\sqrt{u(t)}\|_{H^1(B_{1+|x|})}^\theta. \quad (13)$$

From (13) we obtain

$$\begin{aligned} I_1 &\leq \varepsilon^{-1} C \|\sqrt{u(t)}\|_{H^1(B_{1+|x|})}^{2\theta} \leq \varepsilon^{-1} C (1 + \|\nabla \sqrt{u(t)}\|_{L^2(B_{1+|x|})}^2)^\theta \\ &\leq \varepsilon^{-1} C \left(1 + (2 + |x|) \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u(y,t)}|^2}{1 + |y|} dy \right)^\theta \\ &\leq \varepsilon^{-1} C (1 + |x|)^\theta \left(1 + \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u(y,t)}|^2}{1 + |y|} dy \right)^\theta. \end{aligned}$$

The estimates of I_1, I_2 imply

$$a[u](x, t)^{1/\theta} \leq \varepsilon^{-1} C (1 + |x|) \left(1 + \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u(y,t)}|^2}{1 + |y|} dy \right).$$

The entropy estimate obtained earlier

$$\frac{dH[u]}{dt} + \kappa(t) \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u(x,t)}|^2}{1+|x|} dx \leq 0, \quad t > 0,$$

leads to

$$a[u](x,t)^{1/\theta} \leq \varepsilon^{-1} C(1+|x|) \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right), \quad \frac{1}{\theta} = \frac{2(3+2\varepsilon)}{3(1+2\varepsilon)}, \quad 0 < \varepsilon \leq \frac{3}{2}.$$

We can restate the above estimate in a more handy way by defining $p = 1/\theta \in [1, 2)$ and noticing that $\varepsilon^{-1} \leq C(2-p)^{-1}$:

$$a[u](x,t)^p \leq \frac{C}{2-p} (1+|x|) \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right), \quad 1 \leq p < 2, \quad (14)$$

with $\kappa(t)$ given by (8).

Lower Bound for $H[u]$ A lower bound for $H[u(t)]$ is here showed. Being the spatial domain the whole space \mathbb{R}^3 , this lower bound is not straightforward. To prove a lower bound for $H[u]$, we write

$$H[u] = \int_{\mathbb{R}^3} u(x) \log(u(x)) \chi_{\{u < 1\}} dx + \int_{\mathbb{R}^3} u(x) \log(u(x)) \chi_{\{u > 1\}} dx,$$

and apply Hölder's inequality to get

$$\begin{aligned} -H[u] &\leq \int_{\{u < 1\}} u(x) \log \frac{1}{u(x)} dx = \int_{\{u < 1\}} u(x)^{(1-\varepsilon)/2} u(x)^{(1+\varepsilon)/2} \log \frac{1}{u(x)} dx \\ &\leq \left(\int_{\{u < 1\}} u(x)^{1-\varepsilon} dx \right)^{1/2} \left(\int_{\{u < 1\}} u(x)^{1+\varepsilon} \left(\log \frac{1}{u(x)} \right)^2 dx \right)^{1/2}. \end{aligned}$$

Since the function $s \in (0, 1) \mapsto s^{\varepsilon/2} \log(1/s) \in \mathbb{R}$ is bounded, we can estimate the term

$$\int_{\{u < 1\}} u(x)^{1+\varepsilon} \left(\log \frac{1}{u(x)} \right)^2 dx$$

with a constant that only depends on ε and the L^1 norm of the initial data. Therefore

$$-H[u] \leq C_\varepsilon \left(\int_{\{u < 1\}} u(x)^{1-\varepsilon} dx \right)^{1/2} \leq C_\varepsilon \left(\int_{\mathbb{R}^3} u(x)^{1-\varepsilon} dx \right)^{1/2}. \quad (15)$$

Let us now consider the integral

$$\begin{aligned} \int_{\mathbb{R}^3} u(x)^{1-\varepsilon} dx &= \int (1 + |x|^2)^{1-\varepsilon} u(x)^{1-\varepsilon} (1 + |x|^2)^{-(1-\varepsilon)} dx \\ &\leq \left(\int_{\mathbb{R}^3} (1 + |x|^2) u(x) dx \right)^{1-\varepsilon} \left(\int_{\mathbb{R}^3} (1 + |x|^2)^{-(1-\varepsilon)/\varepsilon} dx \right)^\varepsilon. \end{aligned}$$

For $\varepsilon < 2/5$ we obtain

$$\int_{\mathbb{R}^3} u(x)^{1-\varepsilon} dx \leq C_\varepsilon \left(\int_{\mathbb{R}^3} (1 + |x|^2) u(x) dx \right)^{1-\varepsilon}.$$

From the above estimate and (15) we conclude

$$-H[u(t)] \leq C_\varepsilon (1 + E(t))^{(1-\varepsilon)/2}, \quad 0 < \varepsilon < 2/5, \quad t > 0. \quad (16)$$

Estimate for $E(t)$ We recall that $E(t) = \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x, t) dx, t > 0$. From (5), (14) it follows ($p' \equiv p/(p - 1)$):

$$\begin{aligned} \frac{dE(t)}{dt} &\leq 2 \int_{\mathbb{R}^3} a(x, t) u(x, t)^{1/p} u(x, t)^{1/p'} dx \leq 2 \left(\int_{\mathbb{R}^3} a(x, t)^p u(x, t) dx \right)^{1/p} \|u_0\|_{L^1}^{1/p'} \\ &\leq C_p \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right)^{1/p} \left(\int_{\mathbb{R}^3} (1 + |x|) u(x, t) dx \right)^{1/p} \\ &\leq C_p \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right)^{1/p} \left(\int_{\mathbb{R}^3} \left(\frac{3}{2} + \frac{|x|^2}{2} \right) u(x, t) dx \right)^{1/p} \\ &\leq C_p \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right)^{1/p} (1 + E(t))^{1/p}. \end{aligned}$$

The definition (8) of $\kappa(t)$ implies that $\kappa(t)^{-1} \leq C(1 + \sqrt{E(t)}) \leq C\sqrt{1 + E(t)}$, so

$$\frac{dE(t)}{dt} \leq C_p \left(1 - \frac{dH[u(t)]}{dt} \right)^{1/p} (1 + E(t))^{\frac{3}{2p}}.$$

Choosing $p \in (3/2, 2)$, dividing the above inequality times $(1 + E(t))^{3/2p}$ and integrating it in the time interval $[0, t]$ leads to $(E_0 \equiv \int_{\mathbb{R}^3} \frac{|x|^2}{2} u_0(x) dx)$

$$\begin{aligned} (1 + E(t))^{1-3/2p} - (1 + E_0)^{1-3/2p} &\leq C_p \int_0^t \left(1 - \frac{dH[u]}{dt} \right)^{1/p} dt' \\ &\leq C_p t^{1-1/p} \left(\int_0^t \left(1 - \frac{dH[u]}{dt} \right) dt' \right)^{1/p} = C_p t^{1-1/p} (t + H[u_0] - H[u(t)])^{1/p}. \end{aligned}$$

By inserting (16) into the above inequality we get

$$\begin{aligned} (1 + E(t))^{1-3/2p} - (1 + E_0)^{1-3/2p} &\leq C_{p,\varepsilon} t^{1-1/p} (t + H[u_0] + (1 + E(t))^{(1-\varepsilon)/2})^{1/p} \\ &\leq C_{p,\varepsilon} (1 + t)(1 + E(t))^{(1-\varepsilon)/2p}, \quad \frac{3}{2} < p < 2, \quad 0 < \varepsilon < \frac{2}{5}, \quad t > 0. \end{aligned}$$

Let now $9/5 < p < 2$. We want to choose $\varepsilon \in (0, 2/5)$ such that $1 - 3/2p > (1 - \varepsilon)/2p$. This is equivalent to $\varepsilon > 4 - 2p$. Since $p > 9/5$, it follows that $4 - 2p < 2/5$, so this choice of ε is admissible. Therefore Young inequality allows us to estimate the right-hand side of the above inequality as follows

$$(1 + E(t))^{1-3/2p} - (1 + E_0)^{1-3/2p} \leq C_{p,\varepsilon} (1 + t)^\xi + \frac{1}{2} (1 + E(t))^{1-3/2p}, \quad \xi = \frac{2p-3}{2p-4+\varepsilon},$$

and so we conclude

$$E(t) \leq C_{p,\varepsilon} (1 + t^{2p/(2p-4+\varepsilon)}) \quad t > 0, \quad \frac{9}{5} < p < 2, \quad 4 - 2p < \varepsilon < \frac{2}{5}. \quad (17)$$

For example, if $p = (9/5 + 2)/2 = 19/10$ and $\varepsilon = (4 - 2p + 2/5)/2 = 3/10$, then $2p/(2p - 4 + \varepsilon) = 38$.

Bound (17) means that $E \in L_{loc}^\infty(0, \infty)$. A few consequences of this fact are, for example, that for any $T > 0$:

1. the quantity $\kappa(t)$ defined in (8) and appearing e.g. in (14) is uniformly positive for $t \in [0, T]$;
2. the entropy $H[u(t)]$ has a uniform lower bound for $t \in [0, T]$;
3. in Eq. (10) and the mass conservation yield the following estimate:

$$\|\sqrt{u}\|_{L^2(0,T;H^1(\mathbb{R}^3,\gamma(x)dx))} \leq C_T, \quad \gamma(x) \equiv (1 + |x|)^{-1}; \quad (18)$$

4. the lower bound (7) for a is uniform in $t \in [0, T]$.

4 Conditional Smoothness

4.1 Conditional Regularity Estimates

This section concerns results of conditional regularity of solutions to (1). These results are based upon a so-called ε -Poincaré inequality. We say that u satisfies the ε -Poincaré inequality if given $\varepsilon > 0$ as small as one wishes, there exists a constant

C_ε such that the following inequality holds true

$$\int_{\mathbb{R}^d} u \phi^2 dx \leq \varepsilon \int_{\mathbb{R}^d} a[u] |\nabla \phi|^2 dx + C_\varepsilon \int_{\mathbb{R}^d} \phi^2 dx, \tag{19}$$

for any $\phi \in L^1_{loc}(\mathbb{R}^3)$ that makes the right-hand side of (19) convergent.

Theorem 2 (Conditional Regularity) *Let u be a solution to (1). Assume u is such that (19) holds true. Then for any $s_1 > 1$, $s_2 > \frac{1}{3}$, $T > 0$, $R > 0$ there exist constants $C_1 = C_1(T, u_0, s_1, R)$, $C_2 = C_2(T, u_0, s_2)$ such that*

$$\begin{aligned} \|u\|_{L^\infty(B_R \times (t, T))} &\leq C(T, u_0, s_1, R) \left(\frac{1}{t} + 1\right)^{s_1}, & t \in (0, T), \\ \|a[u]\|_{L^\infty(\mathbb{R}^3 \times (t, T))} &\leq C(T, u_0, s_2) \left(\frac{1}{t} + 1\right)^{s_2}, & t \in (0, T), \end{aligned}$$

where $B_R \subset \mathbb{R}^3$ is any ball of radius R .

Weighted Sobolev and Poincaré’s inequalities have been used to obtain informations about eigenvalues for Schrödinger and degenerate elliptic operators [3–5, 7, 8, 17]. Inspired by the similarity of (1) with the degenerate operator $L = -\operatorname{div}(a[u]\nabla) - u$, in [12] the new inequality (19) has been proposed. We refer to [12] for discussions about (19). While (19) is always true provided u solves the Landau equation for soft-potentials [12], the validity of (19) for Coulomb interactions is still an open question, undoubtedly a very interesting and fundamental one. Consequently the results in Theorem 2 should be viewed as conditional.

Very interesting is the rate of decay in the estimate for $\|u\|_{L^\infty(B_R \times (t, T))}$. In fact one would expect a decay with a rate similar to the heat kernel $1/t^{3/2}$. However thanks to a combination of (19) and a non-local Poincaré’s inequality proven in [14] we obtain a decay that can be made arbitrary close to $1/t$.

The proof of Theorem 2 is divided into several lemmas and propositions. We will make use of the following

Lemma 1 (Weighted Sobolev Inequality) *Let u be a solution to (1). Any smooth function ϕ satisfies*

$$\left(\int_I \int \phi^q a[u] dx dt\right)^{2/q} \leq C \left(\int_I \int a[u] |\nabla \phi|^2 dx dt + \sup_I \int \phi^2 dx\right),$$

with

$$q \in \left(1, 2 \left(1 + \frac{2}{3}\right)\right).$$

Proof We refer to [12] for a detailed proof. □

We define $u_k := (u - k)_+$ for a generic constant $k > 0$.

Proposition 1 *The following inequality holds:*

$$\begin{aligned} \partial_t \int \eta^2 u_k^p dx + \frac{4(p-1)}{p} \int a |\nabla(\eta u_k^{p/2})|^2 dx + \frac{p(p-1)\tau}{2} \int \frac{u_k^{p-2}}{u^3} |\nabla u_k|^4 \eta^2 dx \\ \leq \text{(I)} + \text{(II)} + C\tau \int \eta^2 u_k^p dx + C(p)\tau \int \left(1 + \frac{|\nabla \eta|^{4p}}{\eta^{4p}}\right) \eta^2 dx, \end{aligned} \quad (20)$$

where

$$\begin{aligned} \text{(I)} &:= \frac{4(p-2)}{p} \int u_k^{p/2} (a \nabla(\eta u_k^{p/2}), \nabla \eta) dx + \frac{4}{p} \int u_k^p (a \nabla \eta, \nabla \eta) dx, \\ \text{(II)} &:= \int u_k^p (\nabla a, \nabla(\eta^2)) dx + (p-1) \int u \eta^2 u_k^p dx + pk \int u \eta^2 u_k^{p-1} dx. \end{aligned}$$

Proof Consider

$$\psi = p \eta^2 u_k^{p-1}$$

as test function for (1). A direct computation yields,

$$\begin{aligned} p \int \eta^2 u_k^{p-1} \partial_t u_k dx \\ = -p \int (a \nabla u, \nabla(\eta^2 u_k^{p-1})) dx + p \int (u \nabla a, \nabla(\eta^2 u_k^{p-1})) dx \\ = \tilde{\text{(I)}} + \text{(II)}. \end{aligned}$$

Expanding the first integral, we have the expression:

$$\int (a \nabla u, \nabla(\eta^2 u_k^{p-1})) dx = \int (p-1) \eta^2 u_k^{p-2} (a \nabla u_k, \nabla u_k) + 2u_k^{p-1} \eta (a \nabla u_k, \nabla \eta) dx.$$

Let us rewrite this expression in a more convenient form. Note the elementary identity

$$(a \nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) = \frac{p^2}{4} u_k^{p-2} \eta^2 (a \nabla u_k, \nabla u_k) + p \eta u_k^{p-1} (a \nabla u_k, \nabla \eta) + u_k^p (a \nabla \eta, \nabla \eta),$$

and use it to write,

$$\begin{aligned} & (p-1)\eta^2 u_k^{p-2} (a \nabla u_k, \nabla u_k) + 2u_k^{p-1} \eta (a \nabla u_k, \nabla \eta) \\ &= \frac{4(p-1)}{p^2} (a \nabla (\eta u_k^{p/2}), \nabla (\eta u_k^{p/2})) \\ &\quad - \frac{(2p-4)}{p} u_k^{p-1} \eta (a \nabla u_k, \nabla \eta) - \frac{4(p-1)}{p^2} u_k^p (a \nabla \eta, \nabla \eta). \end{aligned}$$

Further, another elementary identity says

$$u_k^{p-1} \eta (a \nabla u_k, \nabla \eta) = \frac{2}{p} u_k^{p/2} (a \nabla (\eta u_k^{p/2}), \nabla \eta) - \frac{2}{p} u_k^p (a \nabla \eta, \nabla \eta).$$

Combining the above, it follows that

$$\begin{aligned} & (p-1)\eta^2 u_k^{p-2} (a \nabla u_k, \nabla u_k) + 2u_k^{p-1} \eta (a \nabla u_k, \nabla \eta) \\ &= \frac{4(p-1)}{p^2} (a \nabla (\eta u_k^{p/2}), \nabla (\eta u_k^{p/2})) \\ &\quad - \frac{4(p-2)}{p^2} u_k^{p/2} (a \nabla (\eta u_k^{p/2}), \nabla \eta) - \frac{4}{p^2} u_k^p (a \nabla \eta, \nabla \eta). \end{aligned}$$

In particular,

$$\begin{aligned} \tilde{\text{I}} &= -\frac{4(p-1)}{p} \int (a \nabla (\eta u_k^{p/2}), \nabla (\eta u_k^{p/2})) dx \\ &\quad + \frac{4(p-2)}{p} \int u_k^{p/2} (a \nabla (\eta u_k^{p/2}), \nabla \eta) dx + \frac{4}{p} \int u_k^p (a \nabla \eta, \nabla \eta) dx. \end{aligned}$$

Thus,

$$\begin{aligned} & \frac{d}{dt} \int \eta^2 u_k^p dx + \frac{4(p-1)}{p} \int (a \nabla (\eta u_k^{p/2}), \nabla (\eta u_k^{p/2})) dx \\ &= \frac{4(p-2)}{p} \int u_k^{p/2} (a \nabla (\eta u_k^{p/2}), \nabla \eta) dx + \frac{4}{p} \int u_k^p (a \nabla \eta, \nabla \eta) dx \\ &\quad + p \int (u \nabla a, \nabla (\eta^2 u_k^{p-1})) dx. \end{aligned}$$

We now analyze (II). Since

$$\begin{aligned} (\nabla a, u \nabla (\eta^2 u_k^{p-1})) &= u u_k^{p-1} (\nabla a, \nabla (\eta^2)) + (p-1) u u_k^{p-2} \eta^2 (\nabla a, \nabla u_k) \\ &= u u_k^{p-1} (\nabla a, \nabla (\eta^2)) + (p-1) (u_k^{p-1} + k u_k^{p-2}) \eta^2 (\nabla a, \nabla u_k) \end{aligned}$$

$$\begin{aligned}
&= (u_k^p + ku_k^{p-1})(\nabla a, \nabla(\eta^2)) \\
&\quad + \eta^2(\nabla a, \nabla(\frac{p-1}{p}u_k^p + ku_k^{p-1})),
\end{aligned}$$

it follows that

$$\begin{aligned}
(\text{II}) &= p \int (u_k^p + ku_k^{p-1})(\nabla a, \nabla(\eta^2)) \, dx \\
&\quad - p \int \left(\frac{p-1}{p}u_k^p + ku_k^{p-1} \right) \operatorname{div}(\eta^2 \nabla a) \, dx.
\end{aligned}$$

From the above inequality and the Poisson equation it follows

$$\begin{aligned}
(\text{II}) &= p \int (u_k^p + ku_k^{p-1})(\nabla a, \nabla(\eta^2)) \, dx - \int ((p-1)u_k^p + pku_k^{p-1})(\nabla a, \nabla(\eta^2)) \, dx \\
&\quad + \int u\eta^2((p-1)u_k^p + pku_k^{p-1}) \, dx \\
&= \int u_k^p(\nabla a, \nabla(\eta^2)) \, dx + \int u\eta^2((p-1)u_k^p + pku_k^{p-1}) \, dx.
\end{aligned}$$

This finishes the proof of the lemma. □

Lemma 2 *Let $p > 1$, then we have the inequality*

$$\begin{aligned}
&\frac{d}{dt} \int \eta^2 u_k^p \, dx + \frac{(p-1)}{p} \int a |\nabla(\eta u_k^{p/2})|^2 \, dx \\
&\leq (p-1) \int \eta^2 u u_k^p \, dx + pk \int \eta^2 u u_k^{p-1} \, dx \\
&\quad + C(p) \int u_k^p (a \nabla \eta, \nabla \eta) \, dx - \int u_k^p \eta \operatorname{Tr}(a D^2 \eta) \, dx,
\end{aligned}$$

where $C(p)$ denotes a constant that is bounded when $p > 1$.

Proof We proceed to bound from above the first term (I) and the first term of (II) resulting from Proposition 1. The aim is to estimate these terms as

$$\begin{aligned}
&\frac{4(p-2)}{p} \int u_k^{p/2} (a \nabla(\eta u_k^{p/2}), \nabla \eta) \, dx + \int u_k^p (\nabla a, \nabla(\eta^2)) \, dx \\
&\leq c_1 \int (a \nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) \, dx + \text{lower order terms},
\end{aligned}$$

where $c_1 < \frac{4(p-1)}{p}$. For the first term we use Cauchy-Schwarz inequality

$$\begin{aligned} & \left| \frac{4(p-2)}{p} (a \nabla(\eta u_k^{p/2}), u_k^{p/2} \nabla \eta) \right| \\ & \leq \frac{2(p-1)}{p} (a \nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) + \frac{2(p-2)^2}{p(p-1)} u_k^p (a \nabla \eta, \nabla \eta). \end{aligned} \quad (21)$$

For the first term in (II) we use the identity

$$\operatorname{div}(a u_k^p \nabla(\eta^2)) = a \operatorname{div}(u_k^p \nabla(\eta^2)) + u_k^p (\nabla a, \nabla(\eta^2)),$$

and conclude that

$$\begin{aligned} \int u_k^p (\nabla a, \nabla(\eta^2)) dx &= - \int a \operatorname{div}(u_k^p \nabla(\eta^2)) dx \\ &= - \int a u_k^p \Delta(\eta^2) dx - \int (a \nabla u_k^p, \nabla \eta^2) dx. \end{aligned}$$

Since

$$\eta \nabla u_k^{p/2} = \nabla(\eta u_k^{p/2}) - u_k^{p/2} \nabla \eta,$$

Young's inequality yields

$$\begin{aligned} - \int (a \nabla u_k^p, \nabla \eta^2) dx &= -4 \int u_k^{p/2} (a \eta \nabla u_k^{p/2}, \nabla \eta) \\ &= -4 \int u_k^{p/2} (a \nabla(\eta u_k^{p/2}), \nabla \eta) dx + 4 \int u_k^p (a \nabla \eta, \nabla \eta) dx \\ &\leq 2\varepsilon \int (a \nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) dx + \left(\frac{2}{\varepsilon} + 4 \right) \int u_k^p (a \nabla \eta, \nabla \eta) dx. \end{aligned}$$

Thus

$$\begin{aligned} \int u_k^p (\nabla a, \nabla(\eta^2)) dx &\leq - \int u_k^p \operatorname{Tr}(a D^2(\eta^2)) dx + 2\varepsilon \int (a \nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) dx \\ &\quad + \left(\frac{2}{\varepsilon} + 4 \right) \int u_k^p (a \nabla \eta, \nabla \eta) dx. \end{aligned} \quad (22)$$

Substituting (22) and (21) into (20) we get by choosing $\varepsilon < \frac{p-1}{2p}$

$$\begin{aligned} & \frac{d}{dt} \int \eta^2 u_k^p dx + \frac{(p-1)}{p} \int (a \nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) dx \\ & \leq C(p) \int u_k^p (a \nabla \eta, \nabla \eta) dx + (p-1) \int \eta^2 u u_k^p dx \\ & \quad + pk \int \eta^2 u u_k^{p-1} dx - \int u_k^p \text{Tr}(a D^2(\eta^2)) dx. \end{aligned}$$

This concludes the proof. \square

Lemma 3 *We have*

$$\begin{aligned} (p-1) \int_t^T \int \eta^2 u u_k^p dx ds & \leq \varepsilon(p-1) \int_t^T \int_{Q_R} a |\nabla(\eta u_k^{p/2})|^2 dx ds + C(R, \varepsilon, p) \int_t^T \int_{Q_R} \eta^2 u_k^p dx ds, \\ pk \int_t^T \int \eta^2 u u_k^{p-1} dx ds & \leq p\varepsilon \int_t^T \int a |\nabla(\eta u_k^{p/2})|^2 dx ds + C(R, \varepsilon, p) \int_t^T \int \eta^2 u_k^p dx ds \\ & \quad + 2pk^2 \int_0^T \int \eta^2 u_k^{p-1} dx ds. \end{aligned}$$

Proof We use here the ε -Poincaré's inequality (19) with

$$\phi = \eta u_k^{p/2}$$

and get

$$\int \eta^2 u u_k^p dx \leq \varepsilon \int a |\nabla(\eta u_k^{p/2})|^2 dx + C(R, \varepsilon) \int \eta^2 u_k^p dx.$$

For the second inequality we get

$$\begin{aligned} pk \int_t^T \int \eta^2 u u_k^{p-1} dx ds & = pk \int_t^T \int \eta^2 [u \chi_{\{u_k \geq k\}} + u \chi_{\{u_k \leq k\}}] u_k^{p-1} dx ds \\ & = pk \int_t^T \int \eta^2 u \chi_{\{u_k \geq k\}} u_k^{p-1} dx ds + pk \int_t^T \int \eta^2 \underbrace{u \chi_{\{u_k \leq k\}}}_{u \leq 2k} u_k^{p-1} dx ds \\ & \leq p \int_t^T \int \eta^2 u u_k^p dx ds + 2pk^2 \int_0^T \int \eta^2 u_k^{p-1} dx ds \\ & \leq p\varepsilon \int_t^T \int a |\nabla(\eta u_k^{p/2})|^2 dx ds + C(R, \varepsilon, p) \int_t^T \int \eta^2 u_k^p dx ds \\ & \quad + 2pk^2 \int_0^T \int \eta^2 u_k^{p-1} dx ds \end{aligned}$$

using (19) once more. \square

Corollary 1 Fix times $0 < T_1 < T_2 < T_3 < T$, $p > 1$ and a cut-off function $\eta(v)$. Then, we have the following inequality

$$\begin{aligned} \sup_{T_2 \leq t \leq T_3} \left\{ \int (\eta u_k^{p/2})^2 dx \right\} &+ \frac{(p-1)}{4p} \int_{T_2}^{T_3} \int a |\nabla(\eta u_k^{p/2})|^2 dx dt \\ &\leq \left(\frac{1}{T_2 - T_1} + C(p, \varepsilon, R) \right) \int_{T_1}^{T_3} \int \eta^2 u_k^p dx dt \\ &\quad + 2pk^2 \int_{T_1}^{T_3} \int \eta^2 u_k^{p-1} dx dt \\ &\quad + C(p) \int_{T_1}^{T_3} \int u_k^p (a \nabla \eta, \nabla \eta) dx dt + \int_{T_1}^{T_3} \int a u_k^p \eta |\Delta \eta| dx dt. \end{aligned}$$

Proof We start with the bound found in Lemma 2

$$\begin{aligned} \frac{d}{dt} \int \eta^2 u_k^p dx + \frac{(p-1)}{p} \int a |\nabla(\eta u_k^{p/2})|^2 dx \\ \leq (p-1) \int \eta^2 u u_k^p dx + pk \int \eta^2 u u_k^{p-1} dx \\ + C(p) \int u_k^p (a \nabla \eta, \nabla \eta) dx - \int a u_k^p \eta \Delta \eta dx. \end{aligned}$$

Integrating this inequality from t_1 to t_2 shows that the term

$$\int \eta^2 u_k^p(t_2) dx - \int \eta^2 u_k^p(t_1) dx + \frac{(p-1)}{p} \int_{t_1}^{t_2} \int a |\nabla(\eta u_k^{p/2})|^2 dx dt$$

is bounded by

$$\begin{aligned} (p-1) \int_{t_1}^{t_2} \int \eta^2 u u_k^p dx dt + pk \int_{t_1}^{t_2} \int \eta^2 u u_k^{p-1} dx dt \\ + C(p) \int_{t_1}^{t_2} \int u_k^p (a \nabla \eta, \nabla \eta) dx dt - \int_{t_1}^{t_2} \int a u_k^p \eta \Delta \eta dx dt. \end{aligned}$$

For a fixed $t_2 \in (T_2, T_3)$, we take the average with respect to $t_1 \in (T_1, T_2)$ in both sides of the inequality. This yields

$$\begin{aligned}
& \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int \eta^2 u_k^p(t_2) \, dx dt_1 + \frac{(p-1)}{p} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int a |\nabla(\eta u_k^{p/2})|^2 \, dx dt dt_1 \\
& \leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int \eta^2 u_k^p(t_1) \, dx dt_1 \\
& \quad + (p-1) \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int \eta^2 u u_k^p \, dx dt dt_1 \\
& \quad + pk \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int \eta^2 u u_k^{p-1} \, dx dt dt_1 \\
& \quad + C(p) \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int u_k^p (a \nabla \eta, \nabla \eta) \, dx dt dt_1 \\
& \quad - \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int a u_k^p \eta \Delta \eta \, dx dt dt_1,
\end{aligned}$$

which implies

$$\begin{aligned}
& \int \eta^2 u_k^p(t_2) \, dx + \frac{(p-1)}{p} \int_{T_2}^{t_2} \int a |\nabla(\eta u_k^{p/2})|^2 \, dx dt \\
& \leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int \eta^2 u_k^p(t) \, dx dt \\
& \quad + (p-1) \int_{T_1}^{t_2} \int \eta^2 u u_k^p \, dx dt + pk \int_{T_1}^{t_2} \int \eta^2 u u_k^{p-1} \, dx dt \\
& \quad + C(p) \int_{T_1}^{t_2} \int u_k^p (a \nabla \eta, \nabla \eta) \, dx dt + \int_{T_1}^{t_2} \int a u_k^p \eta |\Delta \eta| \, dx dt.
\end{aligned}$$

Since this holds for every $t_2 \in (T_2, T_3)$, this implies the inequality

$$\begin{aligned}
& \sup_{T_2 \leq t \leq T_3} \left\{ \int \eta^2 u_k^p(t) \, dx \right\} + \frac{(p-1)}{p} \int_{T_2}^{T_3} \int a |\nabla(\eta u_k^{p/2})|^2 \, dx dt \\
& \leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_3} \int \eta^2 u_k^p(t) \, dx dt \\
& \quad + (p-1) \int_{T_1}^{T_3} \int \eta^2 u u_k^p \, dx dt + pk \int_{T_1}^{T_3} \int \eta^2 u u_k^{p-1} \, dx dt \\
& \quad + C(p) \int_{T_1}^{T_3} \int u_k^p (a \nabla \eta, \nabla \eta) \, dx dt + \int_{T_1}^{T_3} \int a u_k^p \eta |\Delta \eta| \, dx dt.
\end{aligned}$$

As the last step we use Lemma 3 with $\varepsilon < \frac{p-1}{4p^2}$ and get

$$\begin{aligned} \sup_{T_2 \leq t \leq T_3} \left\{ \int \eta^2 u_k^p(t) dx \right\} &+ \frac{(p-1)}{4p} \int_{T_2}^{T_3} \int a |\nabla(\eta u_k^{p/2})|^2 dx dt \\ &\leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_3} \int \eta^2 u_k^p(t) dx dt \\ &+ C(p, \varepsilon, R) \int_{T_1}^{T_3} \int \eta^2 u_k^p dx dt + 2pk^2 \int_{T_1}^{T_3} \int \eta^2 u_k^{p-1} dx dt \\ &+ C(p) \int_{T_1}^{T_3} \int u_k^p (a \nabla \eta, \nabla \eta) dx dt + \int_{T_1}^{T_3} \int a u_k^p \eta |\Delta \eta| dx dt. \end{aligned}$$

□

Corollary 2 *We have*

$$\begin{aligned} \sup_{T_2 \leq t \leq T_3} \left\{ \int u^p(t) dx \right\} &+ \frac{(p-1)}{4p} \int_{T_2}^{T_3} \int a |\nabla(u^{p/2})|^2 dx dt \\ &\leq \left(\frac{1}{T_2 - T_1} + C(p, \varepsilon) \right) \int_{T_1}^{T_3} \int u^p(t) dx dt. \end{aligned}$$

Proof It is a consequence of Corollary 1 if $\eta = 1$ and $k = 0$. □

Lemma 4 (Gain in Integrability) *For each $p > 1$ and integer $n \geq 0$ we have*

$$\sup_{T/4 \leq t \leq T} \left\{ \int u^{p+n}(t) dx \right\} \leq C(p, n) \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t) dx dt.$$

Proof The proof is based on iterating Corollary 2 with a non-local weighted Poincaré’s inequality proven in [14]: for each $p > 0$ any smooth function $u \geq 0$ satisfies

$$\int_{\mathbb{R}^d} u^{p+1} dx \leq \left(\frac{p+1}{p} \right)^2 \int_{\mathbb{R}^d} a[u] |\nabla(u^{p/2})|^2 dx. \tag{23}$$

Consider a sequence of times

$$T_n = \frac{T}{4} \left(1 - \frac{1}{2^{n-1}} \right).$$

We start with Corollary 2 which states that for each $p > 1$

$$\begin{aligned} \sup_{T_2 \leq t \leq T} \left\{ \int u^p(t) dx \right\} + \frac{(p-1)}{4p} \int_{T_2}^T \int a[u] |\nabla(u^{p/2})|^2 dx dt \\ \leq \left(\frac{1}{T_2} + C(p, \varepsilon) \right) \int_0^T \int u^p(t) dx dt. \end{aligned}$$

Inequality (23) implies

$$\frac{p(p-1)}{4(p+1)^2} \int_{T_2}^T \int u^{p+1} dx dt \leq \left(\frac{1}{T_2} + C(p, \varepsilon) \right) \int_0^T \int u^p(t) dx dt.$$

We now apply the energy inequality to u^{p+1} :

$$\begin{aligned} \sup_{T_3 \leq t \leq T} \left\{ \int u^{p+1}(t) dx \right\} + \frac{p}{4(p+1)} \int_{T_3}^T \int a[u] |\nabla(u^{(p+1)/2})|^2 dx dt \\ \leq \left(\frac{1}{T_3 - T_2} + C(p, \varepsilon) \right) \int_{T_2}^T \int u^{p+1}(t) dx dt \\ \leq \frac{4(p+1)^2}{p(p-1)} \left(\frac{1}{T_3 - T_2} + C(p, \varepsilon) \right) \left(\frac{1}{T_2} + C(p, \varepsilon) \right) \int_0^T \int u^p(t) dx dt \\ \leq 2^6 \frac{(p+1)^2}{p(p-1)} \left(\frac{1}{T} + C(p, \varepsilon) \right)^2 \int_0^T \int u^p(t) dx dt. \end{aligned}$$

Iterating the process we get

$$\sup_{T_{n+2} \leq t \leq T} \left\{ \int u^{p+n}(t) dx \right\} \leq 2^{\sum_1^{n+2} k} C(p)^n \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t) dx dt.$$

Since $T_n \leq T/4$ for any $n \geq 0$ we conclude

$$\sup_{T/4 \leq t \leq T} \left\{ \int u^{p+n}(t) dx \right\} \leq 2^{n(n+1)} C(p)^n \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t) dx dt,$$

and the lemma is proven. \square

4.2 Global $L^p L^p$ Estimates

Lemma 5 *There exists a constant that only depends on T and the initial data u_0 such that*

$$\|u\|_{L^1(0, T; L^3(\mathbb{R}^3, \gamma^3 dx))} \leq C(T, u_0).$$

Proof We start with the classical Sobolev inequality in three dimensions:

$$\left(\int_{\mathbb{R}^3} g^6 dx \right)^{\frac{1}{3}} \leq C \int_{\mathbb{R}^3} |\nabla g|^2 dx,$$

and apply it to $g = \frac{\sqrt{u}}{(1+|x|)^{1/2}}$. Since

$$|\nabla g| \leq \frac{|\nabla \sqrt{u}|}{(1+|x|)^{1/2}} + \sqrt{u},$$

Sobolev inequality yields

$$\left(\int_{\mathbb{R}^3} \frac{u^3}{(1+|x|)^3} dx \right)^{\frac{1}{3}} \leq C \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u}|^2}{(1+|x|)} + u dx.$$

Integrating both sides in the time interval $(0, T)$ we get

$$\begin{aligned} \int_0^T \left(\int_{\mathbb{R}^3} \frac{u^3}{(1+|x|)^3} dx \right)^{\frac{1}{3}} dt &\leq C \int_0^T \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u}|^2}{(1+|x|)} dx dt + \int_0^T \int_{\mathbb{R}^3} u dx dt \\ &\leq C(T, u_0), \end{aligned} \tag{24}$$

using mass conservation and estimate (18). □

Lemma 6 *There exists a constant that only depends on T and the initial data u_0 such that*

$$\|u\|_{L^{5/3}(0,T;L^{5/3}(\mathbb{R}^3))} \leq C(T, u_0).$$

Proof Interpolation yields

$$\begin{aligned} \int_{\mathbb{R}^3} u^p dx &= \int_{\mathbb{R}^3} u^{p\theta} u^{p(1-\theta)} (1+|x|)^m (1+|x|)^{-m} dx \\ &\leq \left(\int_{\mathbb{R}^3} u^{pp_1\theta} (1+|x|)^{p_1m} dx \right)^{\frac{1}{p_1}} \left(\int_{\mathbb{R}^3} u^{p(1-\theta)p_2} (1+|x|)^{-mp_2} dx \right)^{\frac{1}{p_2}}, \end{aligned}$$

with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $\theta < 1$. For $m = 1$, $p_1 = 3/2$, $p_2 = 3$, $p = 5/3$ and $\theta = 2/5$ we get

$$\begin{aligned} \int_{\mathbb{R}^3} u^p dx &\leq \left(\int_{\mathbb{R}^3} u(1+|x|)^{3/2} dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} u^3(1+|x|)^{-3} dx \right)^{\frac{1}{3}} \\ &\leq \left(\int_{\mathbb{R}^3} u(1+|x|)^2 dx \right)^{\frac{2}{3}} \left(\int_{\mathbb{R}^3} u^3(1+|x|)^{-3} dx \right)^{\frac{1}{3}}. \end{aligned}$$

Integrating in the time interval $(0, T)$ we get

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} u^p \, dx dt &\leq \int_0^T \left(\int_{\mathbb{R}^3} u(1+|x|)^2 \, dx \right)^{\frac{3}{5}} \left(\int_{\mathbb{R}^3} u^3(1+|x|)^{-3} \, dx \right)^{\frac{1}{3}} dt \\ &\leq C(T, u_0) \int_0^T \left(\int_{\mathbb{R}^3} u^3(1+|x|)^{-3} \, dx \right)^{\frac{1}{3}} dt \leq C(T, u_0), \end{aligned}$$

using conservation of mass and bound of the second momentum for the second inequality and (24) in the last inequality. \square

4.3 Gain in Integrability

The aim of this section is to show that f has enough integrability for $a[u]$ to be uniformly bounded in space and time. A consequence of interpolation and Hölder's inequality is that $a[u](x, t)$, defined as

$$a[u](x, t) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u(y)}{|x-y|} \, dy,$$

is uniformly bounded in space and time if u belongs to $L^\infty(L^p(\mathbb{R}^3))$ with $p > \frac{3}{2}$. This is what we will show next, combining inequality from Lemma 4 with the $L^{5/3}L^{5/3}$ estimate from Lemma 6.

Lemma 7 *For any $0 < t < T$ and any integer n there exists a constant $C(p, T, u_0, n)$ such that for $\alpha = \frac{(n+1)}{(3n+2)}$:*

$$\|a[u]\|_{L^\infty(t, T, \mathbb{R}^3)} \leq C(T, u_0, n) \left(\frac{1}{t} + 1 \right)^\alpha.$$

Proof Let $r > 0$; for $p > 3/2$ we have

$$\begin{aligned} 4\pi a[u](x, t) &= \int_{B_r(x)} \frac{u(y)}{|x-y|} \, dy + \int_{B_r^c(x)} \frac{u(y)}{|x-y|} \, dy \\ &\leq \frac{1}{r} \|u\|_{L^\infty(L^1)} + 4\pi \|u\|_{L^\infty(L^p)} r^{2-3/p}, \end{aligned}$$

applying Hölder inequality. The minimum of the function $F(r) = \frac{c_1}{r} + c_2 r^{2-3/p}$ is reached at the point

$$r_{min} = \left(\frac{c_1}{(2-3/p)c_2} \right)^{p/(3(p-1))},$$

and this implies

$$a[u](x, t) \leq 4 \|u\|_{L^\infty(L^1)}^{\frac{2p-3}{3(p-1)}} \|u\|_{L^\infty(L^p)}^{\frac{p}{3(p-1)}}.$$

From Lemma 4 we know that

$$\sup_{T/4 \leq t \leq T} \left\{ \int u^{p+n}(t) dx \right\} \leq 2^{n(n+1)} C(p)^n \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t) dx dt,$$

and taking $p = 5/3$ and using Lemma 6 we get

$$\|u\|_{L^\infty(T/4, T; L^{5/3+n}(\mathbb{R}^3))} \leq C(n, T, u_0) \left(\frac{1}{T} + 1 \right)^{\frac{n+1}{5/3+n}}. \quad (25)$$

Going back to $a[u]$ this last estimate implies

$$\begin{aligned} \sup_{t \in (T/4, T), x \in \mathbb{R}^3} a[u](x, t) &\leq c(u_0) \|u\|_{L^\infty(T/4, T; L^{5/3+n})}^{\frac{5/3+n}{2+3n}} \\ &\leq C(n, T, u_0) \left(\frac{1}{T} + 1 \right)^{\frac{n+1}{3n+2}}. \end{aligned} \quad (26)$$

□

4.4 De-Giorgi Iteration and L^∞ -Regularization

Proposition 2 *Let $p = \frac{5}{3}$ and q as in Lemma 1. We have*

$$\begin{aligned} \sup_{T_{n+1} \leq t \leq T} \left\{ \int (\eta_n u_n^{p/2})^2 dx \right\} + \frac{(p-1)}{4p} \int_{T_{n+1}}^T \int a |\nabla(\eta_n u_n^{p/2})|^2 dx dt \\ \leq C_0 \int_{T_n}^T \int a (\eta_{n-1} u_{n-1}^{p/2})^q dx dt, \end{aligned}$$

with

$$C_0 := C^{n-1} C(R, p) \left(\frac{1}{T} + 1 \right) \left(\frac{1}{M} \right)^{\frac{p(q-2)}{2} - 1}.$$

Proof Consider the sequence of times and radii

$$T_n = \frac{1}{4} \left(2 - \frac{1}{2^n} \right) T, \quad R_n = \frac{1}{2} \left(1 + \frac{1}{2^n} \right) R,$$

and, for every $n \geq 1$, let B_n denote the ball $B_n := B_{R_n}(0)$.

Let η_n be a C^∞ function supported in B_n , with $0 \leq \eta_n \leq 1$ everywhere, $\eta_n = 1$ in B_{n+1} , $\|\nabla \eta_n\|_\infty \leq C\eta_n 2^{n+1}$ and $\|D^2(\eta_n)\|_\infty \leq C2^{2n+2}$. Corollary 1 says that for $k_n := M \left(1 - \frac{1}{2^n} \right)$, $T_1 = T_n$, $T_2 = T_{n+1}$, $T_3 = T$, $T_{n+1} - T_n = \frac{T}{2^{n+1}}$ and

$$u_n := \left(u - M \left(1 - \frac{1}{2^n} \right) \right)_+$$

we have

$$\begin{aligned} \sup_{T_{n+1} \leq t \leq T} \left\{ \int \eta_n^2 u_n^p(t) dx \right\} &+ \frac{(p-1)}{4p} \int_{T_{n+1}}^T \int a |\nabla(\eta_n u_n^{p/2})|^2 dx dt \\ &\leq \left(\frac{2^{n+2}}{T} + C(\varepsilon, p) \right) \int_{T_n}^T \int \eta_n^2 u_n^p dx dt \\ &+ C(p) \int_{T_n}^T \int u_n^p (a \nabla \eta_n, \nabla \eta_n) dx dt + 2pk_n^2 \int_{T_n}^T \int \eta_n^2 u_n^{p-1} dx dt \\ &+ \int_{T_n}^T \int a u_n^p \eta_n |\Delta \eta_n| dx dt \leq U_n, \end{aligned}$$

with

$$\begin{aligned} U_n &:= \left(\frac{2^{n+2}}{T} + C(\varepsilon, p) \right) \int_{T_n}^T \int \eta_n^2 u_n^p dx dt \\ &+ (C(p) + 1) 2^{2n+2} \int_{T_n}^T \int_{B_n} a \eta_n^2 u_n^p dx dt + 2pk_n^2 \int_{T_n}^T \int \eta_n^2 u_n^{p-1} dx dt. \end{aligned}$$

We start by estimating the last term of U_n : since $\eta_{n-1} = 1$ on B_n and $\chi_{\{u_n \geq 0\}} = \chi_{\{u_{n-1} \geq \frac{M}{2^n}\}}$ we have

$$\begin{aligned} 2pk_n^2 \int_{T_n}^T \int \eta_n^2 u_n^{p-1} dx dt &\leq 2pM^2 \int_{T_n}^T \int_{B_n} u_n^{p-1} dx dt \\ &= 2pM^2 \int_{T_n}^T \int_{B_n} u_n^{p-1} \chi_{\{u_{n-1} \geq \frac{M}{2^n}\}} dx dt \\ &\leq 2pM^2 \int_{T_n}^T \int_{B_n} u_{n-1}^{p-1} \chi_{\{\eta_{n-1}^{2/p} u_{n-1} \geq \frac{M}{2^n}\}} dx dt. \end{aligned}$$

Hölder inequality yields

$$2pk_n^2 \int_{T_n}^T \int \eta_n^2 u_n^{p-1} dx dt \leq 2pM^2 \int_{T_n}^T \left(\int_{B_n} u_{n-1}^{\frac{pq}{2}} dx \right)^{\frac{2(p-1)}{pq}} \cdot \left(\int_{B_n} \chi_{\{\eta_{n-1}^{2/p} u_{n-1} \geq \frac{M}{2^n}\}} dx \right)^{\frac{pq-2(p-1)}{pq}} dt.$$

Using Chebyshev's inequality

$$\int_{B_n} \chi_{\{\eta_{n-1}^{2/p} u_{n-1} \geq \frac{M}{2^n}\}} dx \leq \left(\frac{2^n}{M} \right)^{pq/2} \int (\eta_{n-1}^{2/p} u_{n-1})^{pq/2} dx$$

we get

$$\begin{aligned} 2pk_n^2 \int_{T_n}^T \int \eta_n^2 u_n^{p-1} dx dt &\leq 2pM^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^T \left(\int_{B_n} u_{n-1}^{\frac{pq}{2}} dx \right)^{\frac{2(p-1)}{pq}} \\ &\quad \times \left(\int (\eta_{n-1} u_{n-1}^{p/2})^q dx \right)^{\frac{pq-2(p-1)}{pq}} dt \\ &= 2pM^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^T \left(\int_{B_n} \eta_{n-1}^q u_{n-1}^{\frac{pq}{2}} dx \right)^{\frac{2(p-1)}{pq}} \cdot \\ &\quad \cdot \left(\int (\eta_{n-1} u_{n-1}^{p/2})^q dx \right)^{\frac{pq-2(p-1)}{pq}} dt \\ &= 2pM^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^T \int (\eta_{n-1} u_{n-1}^{p/2})^q dx dt \\ &\leq 2pC(R)M^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^T \int a(\eta_{n-1} u_{n-1}^{p/2})^q dx dt. \end{aligned}$$

We now estimate the first two terms of U_n :

$$\begin{aligned} &\left(\frac{2^{n+2}}{T} + C(\varepsilon, p) \right) \int_{T_n}^T \int \eta_n^2 u_n^p dx dt + (C(p) + 1)2^{2n+2} \int_{T_n}^T \int_{B_n} a\eta_n^2 u_n^p dx dt \\ &\leq 2^{2n+2} \left(\frac{1}{T} + C(p, R) \right) \int_{T_n}^T \int_{B_n} a\eta_n^2 u_n^p dx dt \\ &\leq 2^{2n+2} \left(\frac{1}{T} + C(p, R) \right) \int_{T_n}^T \int_{B_n} a u_{n-1}^p \chi_{\{u_n \geq 0\}} dx dt \\ &\leq 2^{2n+2} \left(\frac{1}{T} + C(p, R) \right) \int_{T_n}^T \int a\eta_{n-1}^2 u_{n-1}^p \chi_{\{u_{n-1} \geq \frac{M}{2^n}\}} dx dt. \end{aligned}$$

Similarly as before, we apply Hölder's and Chebyshev's inequalities and obtain

$$\begin{aligned} \int a \eta_{n-1}^2 u_{n-1}^p \chi_{\{u_{n-1} \geq \frac{M}{2^{n+1}}\}} dx &\leq \left(\int a \eta_{n-1}^q u_{n-1}^{pq/2} dx \right)^{2/q} \left(\int a \chi_{\{\eta_{n-1}^{2/p} u_{n-1} \geq \frac{M}{2^{n+1}}\}} dx \right)^{(q-2)/q} \\ &\leq \left(\int a (\eta_{n-1} u_{n-1}^{p/2})^q dx \right)^{2/q} \left(\left(\frac{2^n}{M} \right)^{pq/2} \int a \eta_{n-1}^q u_{n-1}^{pq/2} dx \right)^{(q-2)/q} \\ &= \left(\frac{2^n}{M} \right)^{p(q-2)/2} \int a (\eta_{n-1} u_{n-1}^{p/2})^q dx, \end{aligned}$$

which implies

$$\begin{aligned} &\left(\frac{2^{n+2}}{T} + C(\varepsilon, p) \right) \int_{T_n}^T \int \eta_n^2 u_n^p dx dt + (C(p) + 1) 2^{2n+2} \int_{T_n}^T \int a \eta_n^2 u_n^p dx dt \\ &\leq 2^{2n+2} \left(\frac{1}{T} + C(p, R) \right) \left(\frac{2^{n+1}}{M} \right)^{p(q-2)/2} \int_{T_n}^T \int a (\eta_{n-1} u_{n-1}^{p/2})^q dx. \end{aligned}$$

Summarizing we obtain:

$$\begin{aligned} U_n &\leq \left(2pC(R)M^2 \left(\frac{2^{n+1}}{M} \right)^{\frac{pq-2(p-1)}{2}} + 2^{2n+2} \left(\frac{1}{T} + C(p, R) \right) \left(\frac{2^n}{M} \right)^{\frac{p(q-2)}{2}} \right) \\ &\quad \times \int_{T_n}^T \int a (\eta_{n-1} u_{n-1}^{p/2})^q dx dt \\ &\leq 4^{n-1} C(R, p) \left(\frac{1}{T} + 1 \right) \left(\frac{1}{M} \right)^{\frac{p(q-2)}{2}-1} \int_{T_n}^T \int a (\eta_{n-1} u_{n-1}^{p/2})^q dx dt. \end{aligned}$$

This completes the proof. \square

Proposition 3 *Let $T > 0$ and $R > 0$. Given any $s > 1$ there exists a constant that only depends on s , R , the mass and second moment of u (hence on T) such that*

$$\sup_{(T/4, T) \times B_{R/2}} u(x, t) \leq c_0(s, R, T) \left(\frac{1}{T} + 1 \right)^s.$$

Proof Lemma 1 for $\phi = \eta_n u_n^{p/2}$ implies

$$\begin{aligned} \left(\int_{T_{n+1}}^T \int a (\eta_n u_n^{p/2})^q dx dt \right)^{2/q} &\leq \sup_{T_{n+1} \leq t \leq T} \left\{ \int (\eta_n u_n^{p/2})^2 dx \right\} \\ &\quad + \frac{(p-1)}{4p} \int_{T_{n+1}}^T \int a |\nabla (\eta_n u_n^{p/2})|^2 dx dt. \end{aligned} \quad (27)$$

Then Proposition 2 says that

$$\begin{aligned}
 \sup_{T_{n+1} \leq t \leq T} \left\{ \int (\eta_n u_n^{p/2})^2 dx \right\} &+ \frac{(p-1)}{4p} \int_{T_{n+1}}^T \int a |\nabla (\eta_n u_n^{p/2})|^2 dx dt \\
 &\leq U_n \leq C_{n,p,T,M} \int_{T_n}^T \int a (\eta_{n-1} u_{n-1}^{p/2})^q dx dt \\
 &\leq C_{n,p,T,M} \left(\sup_{T_n \leq t \leq T} \left\{ \int (\eta_{n-1} u_{n-1}^{p/2})^2 dx \right\} \right. \\
 &\quad \left. + \frac{(p-1)}{4p} \int_{T_n}^T \int a |\nabla (\eta_{n-1} u_{n-1}^{p/2})|^2 dx dt \right)^{\frac{q}{2}} \\
 &\leq C_{n,p,T,M} U_{n-1}^{\frac{q}{2}},
 \end{aligned}$$

with

$$C_{n,p,T,M} := 4^{n-1} \underbrace{C(p, R) \left(\frac{1}{T} + 1 \right) \left(\frac{1}{M} \right)^{\frac{p(q-2)}{2}-1}}_{:=C_{p,R,T,M}}.$$

This leads to a recurrence relation

$$U_n \leq 4^{n-1} C_{p,R,T,M} U_{n-1}^{\frac{q}{2}}.$$

A standard induction argument shows that the above recurrence relation yields

$$\lim_{n \rightarrow +\infty} U_n = 0, \tag{28}$$

provided the initial step

$$U_0 := \left(\frac{1}{T} + C(\varepsilon, p) \right) \int_{T_0}^T \int \eta_0^2 u^p + a \eta_0^2 u^p dx dt, \quad T_0 = T/4,$$

is small enough. For completeness we sketch this last argument: assume for a certain $n \geq 0$

$$4^n U_n^{\frac{q}{2}-1} \leq \frac{1}{C_{p,R,T,M}(8)^{\frac{1}{2}-1}}, \tag{29}$$

we show that the same is true for $n + 1$: using (29) we get

$$\begin{aligned} 4^{n+1} U_{n+1}^{\frac{q}{2}-1} &\leq 4^{n+1} \left(4^n C_{p,R,T,M} U_n^{\frac{q}{2}} \right)^{\frac{q}{2}-1} \leq 4 C_{p,R,T,M}^{\frac{q}{2}-1} \left(C^n U_n^{\frac{q}{2}-1} \right)^{\frac{q}{2}} \\ &\leq 4 C_{p,R,T,M}^{\frac{q}{2}-1} \left(\frac{1}{C_{p,R,T,M} (8)^{\frac{1}{\frac{q}{2}-1}}} \right)^{\frac{q}{2}} \\ &\leq C_{p,R,T,M}^{-1} \frac{4}{(8)^{\frac{q}{2}-1}} \leq \frac{1}{C_{p,R,T,M} (8)^{\frac{1}{\frac{q}{2}-1}}}. \end{aligned}$$

Therefore if (29) holds for U_0 , i.e.

$$U_0^{\frac{q}{2}-1} \leq \frac{1}{C_{p,R,T,M} (8)^{\frac{1}{\frac{q}{2}-1}}}, \quad (30)$$

then

$$\lim_{n \rightarrow +\infty} U_{n+1}^{\frac{q}{2}-1} \leq \lim_{n \rightarrow +\infty} \frac{c}{4^n} = 0,$$

and (28) is proven.

We are left to prove that for M big enough the condition (30) is satisfied. Let $p = 5/3 + n$ with n any positive integer. Inequalities (25) and (26) imply

$$\begin{aligned} U_0 &\leq c(n) \left(\frac{1}{T} + 1 \right) \int_{T/4}^T \int u^{5/3+n} + au^{5/3+n} \, dxdt \\ &\leq c(n) \left(\frac{1}{T} + 1 \right) (\|a\|_{L^\infty((T/4,T) \times \mathbb{R}^3)} + 1) \int_{T/4}^T \int u^{5/3+n} \, dxdt \\ &\leq c(n, u_0, T) \left(\frac{1}{T} + 1 \right)^{1 + \frac{n+1}{3n+2} + n+1} = c(n, u_0, T) \left(\frac{1}{T} + 1 \right)^{\frac{7n+5}{3n+2} + n}. \end{aligned}$$

We chose M big enough so that

$$c(n) \left(\frac{1}{T} + 1 \right)^{\left(\frac{7n+5}{3n+2} + n \right) \left(\frac{q}{2} - 1 \right)} \left(\frac{1}{T} + 1 \right) \left(\frac{1}{M} \right)^{\frac{(5/3+n)(q-2)}{2} - 1} \leq \frac{1}{8^{\frac{1}{\frac{q}{2}-1}}},$$

or equivalently

$$M > c(n) \left(\frac{1}{T} + 1 \right)^{\alpha(n)},$$

with

$$\alpha(n) = \frac{\left(\frac{7n+5}{3n+2} + n\right) \left(\frac{q}{2} - 1\right)}{(5/3 + n)\left(\frac{q}{2} - 1\right) - 1}.$$

Note that $\alpha(n) \geq 0$ for each $n \geq 0$ and $\alpha(n) \rightarrow 0$ as $n \rightarrow +\infty$. Therefore given any $s > 1$ there exists an integer n such that $\alpha(n) < s$ and this concludes the proof. \square

Acknowledgements MPG is supported by NSF DMS-1514761. MPG would like to thank NCTS Mathematics Division Taipei for their kind hospitality. NZ acknowledges support from the Austrian Science Fund (FWF), grants P22108, P24304, W1245.

References

1. Alexandre, R., Villani, C.: On the Landau approximation in plasma physics. *Ann. Inst. Henri Poincaré, C Anal. Non Linéaire* **21**(1), 61–95 (2004)
2. Cameron, S., Silvestre, L., Snelson, S.: Global a priori estimates for the inhomogeneous Landau equation with moderately soft potentials. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **35**(3), 625–642 (2018)
3. Chang, S.-Y.A., Wilson, J.M., Wolff, T.H.: Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helv.* **60**(2), 217–246 (1985)
4. Chanillo, S., Wheeden, R.: L_p -estimates for fractional integrals and Sobolev inequalities with applications to Schrödinger operators. *Commun. Partial Differ. Equ.* **10**(9), 1077–1116 (1985)
5. Chanillo, S., Wheeden, R.: Weighted Poincaré and Sobolev inequalities and estimates for weighted Peano maximal functions. *Am. J. Math.* **107**(5), 1191–1226 (1985)
6. Desvillettes, L.: Entropy dissipation estimates for the Landau equation in the Coulomb case and applications. *J. Funct. Anal.* **269**, 1359–1403 (2015)
7. Fefferman, C.: The uncertainty principle. *Bull. Am. Math. Soc.* **9**, 129–206 (1983)
8. Fefferman, C., Phong, D.H.: On positivity of pseudo-differential operators. *Proc. Natl. Acad. Sci. U. S. A.* **75**(10), 4673–4674 (1978)
9. Giga, Y., Kohn, R.V.: Asymptotically self-similar blow-up of semilinear heat equations. *Commun. Pure Appl. Math.* **38**(3), 297–319 (1985)
10. Golse, F., Imbert, C., Mouhot, C., Vasseur, A.: Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation. *Annali della Scuola Normale Superiore di Pisa* (2017, To appear)
11. Gualdani, M., Guillen, N.: Estimates for radial solutions of the homogeneous Landau equation with Coulomb potential. *Anal. PDE* **9**(8), 1772–1809 (2016)
12. Gualdani, M., Guillen, N.: On A_p weights and the homogeneous Landau equation. *Calc. Var. Partial Differ. Equ.* (2018, to appear)
13. Gualdani, M., Zamponi, N.: Global existence of weak even solutions for an isotropic Landau equation with Coulomb potential. *SIAM J. Math. Anal.* **50**(4), 3676–3714 (2018)
14. Gressman, P., Krieger, J., Strain, R.: A non-local inequality and global existence. *Adv. Math.* **230**(2), 642–648 (2012)
15. Handerson, C., Snelson, S.: C^∞ -smoothing for weak solutions of the inhomogeneous Landau equation (Preprint)
16. Krieger, J., Strain, R.: Global solutions to a non-local diffusion equation with quadratic nonlinearity. *Commun. Partial Differ. Equ.* **37**(4), 647–689 (2012)

17. Sawyer, E., Wheeden, R.: Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Am. J. Math.* **114**(4), 813–874 (1992)
18. Silvestre, L.: Upper bounds for parabolic equations and the Landau equation. *J. Differ. Equ.* **262**(3), 3034–3055 (2017)
19. Villani, C.: On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Ration. Mech. Anal.* **143**(3), 273–307 (1998)
20. Villani, C.: A review of mathematical topics in collisional kinetic theory. In: *Handbook of Mathematical Fluid Dynamics*, vol. 1, pp. 71–74. Elsevier, Amsterdam (2002)