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PDE Models for Multi-Agent Phenomena

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PDE Models for Multi-Agent Phenomena

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Preface

Mathematics plays a crucial role in improving understanding of the behavior of complex multi-scale systems. In particular, models based on partial differential equations for the description of multi-agent phenomena have demonstrated their scientific robustness over the last decade, and their study provided for many advances at both the theoretical and the applied level. A specific characteristic of such systems is given by the active behavior of individuals, who can make decisions based on a set of preferences. This peculiarity is by contrast excluded in the study of physical particle systems, where there is no element of will of the agents.

A paradigmatic example of a mathematical framework for describing rational collective phenomena is the Mean Field Games (MFG) theory, introduced in the literature in the early 2000s by Jean-Michel Lasry and Pierre-Louis Lions, as the limit of noncooperative games in very large populations composed of interacting individuals, each of whom has a small influence on the global behavior of the system.

The partial differential equations describing multi-agent systems may have peculiar structures, which require new techniques as far as both theoretical and numerical aspects are concerned. Some known examples are the forward-backward structure of Nash-MFG equilibria and the coupling between diffusion operators and singular integral operators.

Because of the rapid development of the subject and with the aim of fostering scientific exchange among worldwide experts in the field and young researchers with high potential, the Italian Institute of Higher Mathematics (INdAM) funded the workshop *"PDE models for multi-agent phenomena"*, which took place in Rome (Italy), from November 28th to December 2nd, 2016. This volume covers most of the topics that were addressed and discussed during the workshop.

In particular, two main classes of equations and systems have been considered: kinetic equations and Mean Field Games models.

Regarding kinetic equations, this volumes includes two contributions.

The article by Gualdani and Zamponi focuses on the quadratic, nonlocal, isotropic Landau model of kinetic theory, a model which has gained attention over the last decade and which shares similarities with other equations, such as the Keller-Segel model, as well as with the semilinear heat equation. After reviewing some integral identities, the authors discuss an "ε-Poincaré inequality" which encapsulates spectral information about a linearized equation. They then demonstrate how a De Giorgi-Nash-Moser iteration argument can be used to show that solutions of the isotropic Landau model regularize instantaneously.

The article by Iacobelli gives an overview of the author's results on a dynamical approach to the quantization problem, namely the approximation of a d-dimensional probability density by a convex combination of a finite number $N$of Dirac masses, when $N \to +\infty$.

The rest of the volume contains seven contributions oriented toward the study of Mean Field Games models. Some of these contributions (by Bardi and Cirant, by Cannarsa and Capuani, by Graber and Mouzouni and by Pimentel and Santos) are devoted to theoretical analysis of the system of partial differential equations arising in Mean Field Games. The other contributions (by Cacace and Camilli, by Carlini and Silva and by Festa, Gomes and Velho) investigate the numerical computation of the solution to the Mean Field Games system.

The article by Bardi and Cirant studies the uniqueness of solutions to the MFG system associated with several populations of agents and Neumann boundary conditions: uniqueness holds under a smallness assumption of some data (the length of the time horizon). This complements the existence results for MFG models of segregation phenomena introduced by the authors and Achdou. An application to robust Mean Field Games is also given.

The paper by Cacace and Camilli describes a new class of finite difference methods for the approximation of the stationary MFG system. A large collection of numerical tests in dimensions one and two shows the performance of the proposed method, in terms of both accuracy and computational time.

The article by Cannarsa and Capuani investigates deterministic Mean Field Games with state constraints on the position of each agent. Because of the state constraints, classical techniques, requiring the generic uniqueness of the optimal trajectories, no longer apply. The authors prove the existence of a Mean Field Game equilibrium written in terms of a measure in a space of arcs, by using set-valued fixed point arguments. The uniqueness of such equilibria also holds under the classical monotonicity assumption.

The article by Carlini and Silva proposes a fully discrete scheme for systems of nonlinear Fokker-Planck-Kolmogorov (FPK) equations. The authors consider a system of FPK equations where the dependence of the coefficients is nonlinear and nonlocal in time with respect to the unknowns and extend a numerical scheme previously proposed for a single FPK equation. They analyze the convergence of the scheme and study its applicability in two examples: a population model involving two interacting species and a MFG system with two populations.

In their contribution, Festa, Gomes and Velho introduce a numerical approach for a class of Fokker-Planck (FP) equations. Using the fact that these equations are the adjoint of the linearization of Hamilton-Jacobi (HJ) equations, the authors show how to transfer the properties of schemes for HJ equations to the FP equations and obtain numerical schemes with desirable features such as positivity and mass preservation.

They illustrate this approach in examples that include Mean Field Games and a crowd motion model.

Bertrand and Cournot Mean Field Games models for market competition are explored in the contribution by Graber and Mouzouni. The authors prove the well-posedness of the MFG system and show that it can be written as an optimality condition of a convex minimization problem. They also investigate the vanishing viscosity limit of the system.

The contribution by Pimentel and Santos gives an overview of asymptotic methods recently introduced in regularity theory for fully nonlinear elliptic equations. The presentation focuses mainly on the notion of recession function and details the role of this class of techniques through examples and results.

Paris, France — Pierre Cardaliaguet
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May 2018

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About the Editors

Pierre Cardaliaguet completed his PhD at the U. Paris Dauphine in 1994. He has been a Professor at Brest U. from 2000 to 2010 and is currently Professor at the U. Paris Dauphine.

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Uniqueness of Solutions in Mean Field Games with Several Populations and Neumann Conditions

Martino Bardi and Marco Cirant

Abstract We study the uniqueness of solutions to systems of PDEs arising in Mean Field Games with several populations of agents and Neumann boundary conditions. The main assumption requires the smallness of some data, e.g., the length of the time horizon. This complements the existence results for MFG models of segregation phenomena introduced by the authors and Achdou. An application to robust Mean Field Games is also given.

Keywords Mean field games · Multi-populations · Uniqueness · Neumann boundary conditions · Robust mean field games

1 Introduction

The systems of partial differential equations associated to finite-horizon Mean Field Games (briefly, MFGs) with N populations of agents have the form

$$\begin{cases} -\partial_t v_k - \Delta v_k + H_k(x, Dv_k) = F_k(x, m(t,\cdot)), & in \quad (0,T)\times\Omega, \\ \partial_t m_k - \Delta m_k - \mathrm{div}(D_p H_k(x, Dv_k) m_k) = 0 & in \quad (0,T)\times\Omega, \\ v_k(T,x) = G_k(x, m(T,\cdot)),\ m_k(0,x) = m_{0,k}(x) & in \quad \Omega, \quad k = 1,\dots,N, \end{cases} \tag{1.1}$$

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where the unknown m is a vector of probability densities on Ω, F_k and G_k are function of this vector and represent the running and terminal costs of a representative agent of the k-population, and v_k is the value function of this agent. The first N equations are parabolic of Hamilton-Jacobi-Bellman type and backward in time with a terminal condition, the second N equations are parabolic of Kolmogorov-Fokker-Planck type and forward in time with an initial condition. If the state space $\Omega \subseteq \mathbb{R}^d$ is not all $\mathbb{R}^d$, boundary conditions must also be imposed. In most of the theory of MFGs they are periodic, which are the easiest to handle, here we will consider instead Neumann conditions, i.e.,

$$\partial_n v_k = 0, \quad \partial_n m_k + m_k D_p H_k(x, Dv_k) \cdot n = 0 \quad \text{on } (0,T) \times \partial\Omega. \tag{1.2}$$

There is a large literature on the existence of solutions for these equations, especially in the case of a single population $N = 1$, beginning with the pioneering papers of Lasry and Lions [27–29] and Huang et al. [23–25], see the lecture notes [9, 21, 22], the books [11, 18, 19], the survey [16], and the references therein. Systems with several populations, $N > 1$, were treated with Neumann conditions in [12, 15] for the stationary case and in [1] in the evolutive case, with periodic conditions in [4, 10].

Uniqueness of solutions is a much more delicate issue. For one population Lasry and Lions [27–29] discovered a monotonicity condition on the costs F and G that together with the convexity in p of the Hamiltonian $H(x, p)$ implies the uniqueness of classical solutions. It reads

$$\int_{\mathbb{R}} (F(x,\mu) - F(x,\nu))d(\mu - \nu)(x)) > 0, \quad \text{if } \mu \neq \nu \tag{1.3}$$

and it means that a representative agent prefers the regions of the state space that are less crowded. This is a restrictive condition that is satisfied in some models and not in others. When it fails, non-uniqueness may arise: this was first observed in the stationary case by Lasry and Lions [29] and other counterexamples were shown by Guéant [21], Bardi [3], Bardi and Priuli [6], and Gomes et al. [20]. The need of a condition such as (1.3) for having uniqueness for finite-horizon MFGs was discussed at length in [31], and some explicit examples of non-uniqueness appeared very recently in [8, 14], and in [5] that presents also a probabilistic proof and references on other examples obtained by the probabilistic approach.

For multi-population problems, $N > 1$, there are extensions of the monotonicity condition (1.3) in [5, 12] and they are even more restrictive: they impose not only aversion to crowd within each population, but also that the costs due to this effect dominate the costs due to the interactions with the other populations. This is not the case in the multi-population models of segregation in urban settlements proposed in [1] following the ideas of the Nobel Prize Thomas Schelling [34]. There the interactions between two different populations are the main cause of the dynamics, and in fact examples of multiple solutions were shown in [1] and [15] for the stationary case and in [5] for the evolutive one. Therefore a different criterion giving uniqueness in some cases is particularly desirable when $N > 1$.

A second regime for uniqueness was introduced in a lecture of P.L. Lions on January 9th, 2009 [31]: it occurs if the length T of the time horizon is short enough. To our knowledge Lions' original argument did not appear in print. For finite state MFGs, uniqueness for short time was proved by Gomes et al. [17] as part of their study of the large population limit. For continuous state, an existence and uniqueness result under a "small data" condition was given in [25] for Linear-Quadratic-Gaussian MFGs using a contraction mapping argument to solve the associated system of Riccati differential equations, and similar arguments were used for different classes of linear-quadratic problems in [32, 36]. The well-posedness when $H(x, Dv) - F(x, m)$ is replaced by $\varepsilon\mathcal{H}(x, Dv, m)$ with ε small is studied in [2], and another result for small Hamiltonian is in [35] for nonconvex H.

Very recently the first author and Fischer [5] revived Lions' argument to show that the smoothness of the Hamiltonian is the crucial property to have small-time uniqueness without monotonicity of the costs and convexity of H, and gave an example of non-uniqueness for all $T > 0$ and $H(x, p) = |p|$. The uniqueness theorem for small data in [5] holds for $N = 1$ and $\Omega = \mathbb{R}^d$ with conditions on the behaviour of the solutions at infinity.

In the present paper we focus instead on $N \geq 1$ and Neumann boundary conditions, which is the setting of the MFG models of segregation in [1]. The new difficulties arise from the boundary conditions, that require different methods for some estimates, especially on the L^∞ norm of the densities m_k. Our first uniqueness result assumes a suitable smoothness of the Hamiltonians H_k, but neither convexity nor growth conditions, and that the costs F_k, G_k are Lipschitz in L^2 with respect to the measure m, with no monotonicity. The smallness condition on the data depends on the range of the spacial gradient of the solutions v_k, unless $D_p H_k$ are bounded and globally Lipschitz for all k. Then we complement such result with some a priori gradient estimates on v_k, under an additional quadratic growth condition on H_k and some more regularity of the costs, and get a $\bar{T} > 0$ depending only on the data such that there is uniqueness for all horizons $T \leq \bar{T}$. Finally, we give sufficient conditions ensuring both existence and uniqueness for the system (1.1) with the boundary conditions (1.2), as well as for some robust MFGs considered in [7, 32], which are interesting examples with nonconvex Hamiltonian.

We mention that in the stationary case, uniqueness up to (space) translation may hold without (1.3) in force. A special class of MFG on $\mathbb{R}^d$ enjoying such a feature has been identified in [13].

The paper is organised as follows. Section 2 contains the main result about uniqueness for small data, possibly depending on gradient bounds on the solutions. Section 3 gives further sufficient conditions depending only on the data for uniqueness and existence of solutions. The Appendix recalls a comparison principle for HJB equations with Neumann conditions.

2 The Uniqueness Theorem

Consider the MFG system for N populations

$$\begin{cases} -\partial_t v_k - \Delta v_k + H_k(x, Dv_k) = F_k(x, m(t,\cdot)), & in \quad (0,T)\times\Omega, \\ \partial_t m_k - \Delta m_k - \operatorname{div}(D_p H_k(x, Dv_k) m_k) = 0 & in \quad (0,T)\times\Omega, \\ \partial_n v_k = 0,\ \partial_n m_k + m_k D_p H_k(x, Dv_k)\cdot n = 0 & on \quad (0,T)\times\partial\Omega, \\ v_k(T,x) = G_k(x, m(T,\cdot)),\ m_k(0,x) = m_{0,k}(x)\ in & \Omega \end{cases} \tag{2.1}$$

where $k = 1, \dots, N$, Dv_k denotes the gradient of the k-th component v_k of the unknown v with respect to the space variables, Δ is the Laplacian with respect to the space variables x, $D_p H_k$ is the gradient of the Hamiltonian of the k-th population with respect to the moment variable, $\Omega \subseteq \mathbb{R}^d$ is a bounded open set with boundary $\partial\Omega$ of class $C^{2,\beta}$ for dome $\beta > 0$, and $n(x)$ is its exterior normal at x. The components m_k of the unknown vector m are bounded densities of probability measures on Ω, i.e., m lives in

$$\mathcal{P}_N(\Omega) := \left\{ \mu = (\mu_1, \dots, \mu_N) \in L^\infty(\Omega)^N : \mu_k \geq 0,\ \int_\Omega \mu_k(x)dx = 1 \right\}.$$

F and G represent, respectively, the running and terminal cost of the MFG

$$F : \overline{\Omega} \times \mathcal{P}_N(\Omega) \to \mathbb{R}^N, \quad G : \overline{\Omega} \times \mathcal{P}_N(\Omega) \to \mathbb{R}^N.$$

By classical solutions we will mean functions of (t,x) of class C^1 in t and C^2 in x in $[0,T] \times \overline{\Omega}$.

2.1 The Main Result

Our main assumptions are the smoothness of the Hamiltonians and a Lipschitz continuity of the costs in the norm $\|\cdot\|_2$ of $L^2(\Omega)^N$ that we state next. We consider $H_k : \overline{\Omega} \times \mathbb{R}^d \to \mathbb{R}$ continuous and satisfying

$$D_p H_k(x,p) \text{ is continuous and locally Lipschitz in } p \text{ uniformly in } x \in \overline{\Omega}. \tag{2.2}$$

We will assume F, G satisfy, for all μ, ν,

$$\|F(\cdot,\mu) - F(\cdot,\nu)\|_2^2 \leq L_F \|\mu - \nu\|_2^2, \tag{2.3}$$

$$\|DG(\cdot,\mu) - DG(\cdot,\nu)\|_2^2 \leq L_G \|\mu - \nu\|_2^2, \tag{2.4}$$

Theorem 2.1 *Assume* (2.2)–(2.4)*,* $m_0 \in \mathcal{P}_N(\Omega)$*, and* $(\tilde{v}, \tilde{m}), (\overline{v}, \overline{m})$ *are two classical solutions of* (2.1)*. Denote*

$$\mathcal{C} := co\{D\tilde{v}(t,x), D\overline{v}(t,x) : (t,x) \in (0,T) \times \Omega\},$$

$$C_H := \max_{k=1,\dots,N} \sup_{x\in\Omega,\, p\in\mathcal{C}} |D_p H_k(x,p)|, \tag{2.5}$$

$$\bar{C}_H := \max_{k=1,\dots,N} \sup_{x\in\Omega,\, p,q\in\mathcal{C}} \frac{|D_p H_k(x,p) - D_p H_k(x,q)|}{|p-q|}. \tag{2.6}$$

Then there exists a function Ψ *of* $T, L_F, L_G, C_H, \bar{C}_H, N$ *and* $\max_k \|m_{0,k}\|_\infty$ *(depending also on* Ω*), such that the inequality* $\Psi < 1$ *implies* $\tilde{v}(t,\cdot) = \overline{v}(t,\cdot)$ *and* $\tilde{m}(t,\cdot) = \overline{m}(t,\cdot)$ *for all* $t \in [0,T]$*, and* $\Psi < 1$ *holds if either* T*, or* $\bar{C}_H$*, or the pair* L_F, L_G *is small enough.*

For the proof we need two auxiliary results.

Proposition 2.2 *There are constants* $r > 1$ *and* $C > 0$ *depending only on* d *and* Ω *such that*

$$\|m_k\|_{L^\infty((0,T)\times\Omega)} \le C[1 + \|m_{0,k}\|_\infty + (1+T)\|D_p H_k(\cdot, Dv_k)\|_{L^\infty((0,T)\times\Omega)}]^r, \quad k = 1,\dots,N. \tag{2.7}$$

Proof *Step 1.* We aim at proving that for any $q \in [1, (d+2)/(d+1))$ there exists a constant $\overline{C}$ depending only on d, q and Ω such that any positive classical solution φ of the backward heat equation

$$\begin{cases} -\partial_t \varphi - \Delta\varphi = 0 & \text{on } (0,t) \times \Omega \\ \partial_n \varphi = 0 & \text{on } (0,t) \times \partial\Omega \\ \int_\Omega \varphi(t,x)dx = 1, & \end{cases}$$

satisfies

$$\|\nabla\varphi\|_{L^q((0,t)\times\Omega)} \le \overline{C}(1+t)^{1/q}.$$

We follow the strategy presented in [19, Section 5]. Note first that $\int_\Omega \varphi(s,x)dx = 1$ for all $s \in (0,t)$, by integrating by parts the equation and using the boundary conditions. We proceed in the case $d \ge 3$; if $d = 1$ or $d = 2$, one argues in a similar way (see the discussion below). Let $\alpha \in (0,1)$ to be chosen later; multiplying the equation by $\alpha\varphi^{\alpha-1}$ and integrating by parts yield for all $s \in (0,t)$

$$\int_\Omega |\nabla\varphi^{\alpha/2}(s,x)|^2 dx = \frac{\alpha}{4(\alpha-1)}\partial_t \int_\Omega \varphi^\alpha(s,x)dx.$$

Integrating in time and using the fact that $\int_\Omega \varphi(s,x)dx = 1$ give

$$\int_0^t \int_\Omega |\nabla \varphi^{\alpha/2}|^2 dxds = \frac{\alpha}{4(1-\alpha)} \int_\Omega \varphi^\alpha(0,x)dx - \frac{\alpha}{4(1-\alpha)} \int_\Omega \varphi^\alpha(t,x)dx \le c_1, \tag{2.8}$$

where c_1 depends on d and Ω (the positive constants $c_2, c_3, \dots$ used in the sequel will have the same dependance).

We now exploit the continuous embedding of $W^{1,2}(\Omega)$ into $L^{\frac{2d}{d-2}}(\Omega)$; the adaption of this proof to the cases $d = 1, 2$ is straightforward, as the injection of $W^{1,2}(\Omega)$ is into $L^p(\Omega)$ for all $p \ge 1$. Hence, for all $s \in (0,t)$, by Hölder and Sobolev inequalities

$$\begin{aligned}\int_\Omega \varphi^{\alpha+\frac{2}{d}}(s,x)dx &\le \left(\int_\Omega \varphi(s,x)dx\right)^{\frac{2}{d}} \left(\int_\Omega \varphi^{\frac{\alpha}{2}\frac{2d}{d-2}}(s,x)dx\right)^{\frac{d-2}{d}} \\ &\le c_2\left(\int_\Omega |\nabla \varphi^{\alpha/2}|^2 dx + \int_\Omega \varphi^\alpha dx\right) \le c_2\left(\int_\Omega |\nabla \varphi^{\alpha/2}|^2 dx + 1 + |\Omega|\right),\end{aligned}$$

so

$$\int_0^t \int_\Omega \varphi^{\alpha+\frac{2}{d}} dxds \le c_3\left(\int_0^t \int_\Omega |\nabla \varphi^{\alpha/2}|^2 dxds + t\right). \tag{2.9}$$

Finally, since $q < (d+2)/(d+1)$, we may choose $\alpha \in (0,1)$ such that

$$q\frac{2-\alpha}{2-q} = \alpha + \frac{2}{d},$$

and therefore, by the identity $\nabla \varphi^{\alpha/2} = \frac{\alpha}{2}\varphi^{\frac{\alpha-2}{2}}\nabla\varphi$ and Young's inequality

$$\begin{aligned}\int_0^t \int_\Omega |\nabla \varphi|^q dxds &= \left(\frac{2}{\alpha}\right)^q \int_0^t \int_\Omega |\nabla \varphi^{\alpha/2}|^q \, \varphi^{q\frac{2-\alpha}{2}} dxds \le \\ c_4&\left(\int_0^t \int_\Omega |\nabla \varphi^{\alpha/2}|^2 dxds + \int_0^t \int_\Omega \varphi^{q\frac{2-\alpha}{2-q}} dxds\right) \le c_4(c_1 + c_3(c_1+t))),\end{aligned}$$

in view of (2.8) and (2.9), and the desired estimate follows.

Step 2. Fix $t \in (0,T)$ and $1 < q < (d+2)/(d+1)$. Let φ_0 be any non-negative smooth function on Ω such that $\partial_n \varphi_0 = 0$ on $\partial\Omega$ and $\int_\Omega \varphi_0(x)dx = 1$. Let φ be the solution of the backward heat equation

$$\begin{cases} -\partial_t \varphi - \Delta \varphi = 0 & \text{on } (0,t)\times\Omega \\ \partial_n \varphi = 0 & \text{on } (0,t)\times\partial\Omega \\ \varphi(t,x) = \varphi_0(x) & \text{on } \Omega. \end{cases}$$

Note that φ is positive on $(0, t) \times \Omega$ by the strong maximum principle. Multiply the KFP equation in (2.1), integrate by parts and use the boundary conditions for m_k to get

$$\int_0^t \int_\Omega \partial_t m_k\, \varphi + \nabla m_k \cdot \nabla\varphi + D_p H_k(x, Dv_k) \cdot \nabla\varphi\, m_k\, dxds = 0.$$

Integrating again by parts (in space-time) yields

$$\int_\Omega m_k(t, x)\varphi_0(x) = \int_\Omega m_k(0, x)\varphi(0, x) - \int_0^t \int_\Omega D_p H_k(x, Dv_k) \cdot \nabla\varphi\, m_k\, dxds,$$

using the equation and the boundary condition for φ. Hence,

$$\begin{aligned}\int_\Omega m_k(t, x)\varphi_0(x) &\le \|m_{k,0}\|_\infty + \|D_p H_k(\cdot, Dv_k)\|_{L^\infty((0,t)\times\Omega)} \int_0^t \int_\Omega |\nabla\varphi|\, |m_k|\, dxds, \\ &\le \|m_{k,0}\|_\infty + \overline{C}(1 + t)^{1/q} \|D_p H_k(\cdot, Dv_k)\|_{L^\infty((0,t)\times\Omega)} \|m_k\|_{L^{q'}((0,t)\times\Omega)}\end{aligned}$$

by Step 1. By the arbitrariness of φ_0, one obtains

$$\|m_k(t, \cdot)\|_\infty \le \|m_{k,0}\|_\infty + \overline{C}(1+t)^{1/q} \|D_p H_k(\cdot, Dv_k)\|_{L^\infty((0,t)\times\Omega)} \|m_k\|_{L^{q'}((0,t)\times\Omega)},$$

and since

$$\|m_k\|_{L^{q'}((0,t)\times\Omega)} \le \left(\int_0^t \|m_k(s, \cdot)\|_\infty^{q'-1} \int_\Omega m_k(s, x) dx\, ds\right)^{1/q'} \le \|m_k\|^{1/q}_{L^\infty((0,t)\times\Omega)}\, t^{1/q'},$$

we have

$$\|m_k(t, \cdot)\|_\infty \le \|m_{k,0}\|_\infty + \overline{C}(1 + t) \|D_p H_k(\cdot, Dv_k)\|_{L^\infty((0,t)\times\Omega)} \|m_k\|^{1/q}_{L^\infty((0,t)\times\Omega)}.$$

Passing to the supremum on $t \in (0, T)$, we conclude (r in the statement can be chosen to be q'). □

Lemma 2.3 (A Mean-Value Theorem) *Let $\mathcal{K} \subseteq \mathbb{R}^d$, $f : \overline{\Omega} \times \mathcal{K} \to \mathbb{R}^d$ be continuous and Lipschitz continuous in the second entry with constant L, uniformly in the first. Then there exists a measurable matrix-valued function $M(\cdot, \cdot, \cdot)$ such that*

$$f(x, p) - f(x, q) = M(x, p, q)(p - q), \quad |M(x, p, q)| \le L, \quad \forall x \in \overline{\Omega},\ p, q \in \mathcal{K}. \tag{2.10}$$

Proof Mollify f in the variables p and get a sequence f_n converging to f locally uniformly and with Jacobian matrix satisfying $\|D_p f_n\|_\infty \le L$. Since f_n is C^1 in p the standard mean-value theorem gives

$$f_n(x,p) - f_n(x,q) = \int_0^1 Df_n(x, q + s(p-q))(p-q)\,ds =: M_n(x,p,q)(p-q), \tag{2.11}$$

and $|M_n| \le L$. We define the matrix M componentwise by setting

$$M(x,p,q)_{ij} := \liminf_n M_n(x,p,q)_{ij}, \quad i,j = 1,\dots,d,$$

so that it is measurable in (x,p,q) and satisfies $|M(x,p,q)| \le L$. Now we take the $\liminf_n$ in the i-th component of the identity (2.11) and get the i-th component of the desired identity (2.10). □

Proof of Theorem 2.1 *Step 1*. First observe that, by the regularity of the solutions, $C_H < +\infty$ and $\bar{C}_H < +\infty$. We set

$$v := \tilde{v} - \overline{v}, \quad m := \tilde{m} - \overline{m}, \quad B_k(t,x) := \int_0^1 D_p H_k(x, D\overline{v}(t,x) + s(D\tilde{v} - D\overline{v})(t,x))ds$$

and observe that $|B_k| \le C_H$ for all k and v_k satisfies

$$\begin{cases} -\partial_t v_k + B_k(t,x)\cdot Dv_k = \Delta v_k + F_k(x,\tilde{m}(t)) - F_k(x,\overline{m}(t)) & \text{in } (0,T)\times\Omega \\ \partial_n v_k = 0 \ \text{ on } (0,T)\times\partial\Omega, \quad v_k(T,x) = G_k(x,\tilde{m}(T)) - G_k(x,\overline{m}(T)). \end{cases} \tag{2.12}$$

Step 2. By the divergence theorem and the boundary conditions we compute

$$\begin{aligned} -\int_t^T \int_\Omega \partial_t v_k \Delta v_k\,ds &= \int_t^T \frac{d}{dt}\int_\Omega \frac{|Dv_k|^2}{2} dx ds - \int_t^T \int_{\partial\Omega} \partial_t v_k Dv_k \cdot n\,d\sigma \\ &= \frac{1}{2}\|Dv_k(T,\cdot)\|_2^2 - \frac{1}{2}\|Dv_k(t,\cdot)\|_2^2. \end{aligned}$$

Now we set

$$\bar{F}(t,x) := F(x,\overline{m}) - F(x,\tilde{m}), \quad \bar{G}(t,x) := G(x,\overline{m}) - G(x,\tilde{m}),$$

multiply the PDE in (2.12) by Δv_k, integrate, use the terminal condition in (2.12) and estimate

$$\frac{1}{2}\|Dv_k(t,\cdot)\|_2^2+\int_t^T \|\Delta v_k(s,\cdot)\|_2^2 \le \frac{1}{2}\|D\bar{G}(T,\cdot)\|_2^2+$$

$$\|B_k\|_\infty \int_t^T \left(\frac{1}{2\varepsilon}\|Dv_k(s,\cdot)\|_2^2+\frac{\varepsilon}{2}\|\Delta v_k(s,\cdot)\|_2^2\right)ds+\int_t^T \left(\frac{1}{2\varepsilon}\|\bar{F}(s,\cdot)\|_2^2+\frac{\varepsilon}{2}\|\Delta v_k(s,\cdot)\|_2^2\right)ds.$$

Next we choose ε such that $1=(\|B_k\|_\infty+1)\varepsilon/2$ and use the assumptions (2.4) and (2.3) to get

$$\|Dv_k(t,\cdot)\|_2^2 \le L_G\|m(T,\cdot)\|_2^2+\int_t^T \frac{L_F}{\varepsilon}\|m(s,\cdot)\|_2^2 ds+\frac{\|B_k\|_\infty}{\varepsilon}\int_t^T \|Dv_k(s,\cdot)\|_2^2 ds.$$

Then Gronwall inequality gives, for $c_o:=(\|B_k\|_\infty+1)/2=1/\varepsilon$ and for all $0\le t\le T$,

$$\|Dv_k(t,\cdot)\|_2^2 \le \left(L_G\|m(T,\cdot)\|_2^2+c_oL_F\int_t^T \|m(s,\cdot)\|_2^2 ds\right)e^{c_o\|B_k\|_\infty T}. \tag{2.13}$$

Step 3. In order to write a PDE solved by m we apply Lemma 2.3 to $D_pH_k:\overline{\Omega}\times\mathcal{C}\to\mathbb{R}^d$, which is Lipschitz in p by the assumption in (2.6), and get a matrix M_k such that

$$D_pH_k(x,D\overline{v}_k)-D_pH_k(x,D\tilde{v}_k)=M_k(x,D\overline{v}_k,D\tilde{v}_k)(D\overline{v}_k-D\tilde{v}_k),$$

with $|M_k|\le \bar{C}_H$. Now define

$$\tilde{B}_k(t,x):=D_pH_k(x,D\overline{v}_k),\quad A_k(t,x):=\tilde{m}_kM(x,D\overline{v}_k,D\tilde{v}_k),$$
$$\tilde{F}_k(t,x):=A_k(t,x)(D\overline{v}_k-D\tilde{v}_k).$$

Then m_k satisfies

$$\begin{cases}\partial_t m_k-\operatorname{div}\left(\tilde{B}_km_k\right)=\Delta m_k+\operatorname{div}\tilde{F}_k & \text{in } (0,T)\times\mathbb{R}^d,\\ \partial_n m_k+(m_k\tilde{B}_k+\tilde{F}_k)\cdot n=0 \ \text{ on } (0,T)\times\partial\Omega, & m_k(0,x)=0.\end{cases} \tag{2.14}$$

with $|\tilde{B}_k|\le C_H$ and $|A_k|\le \mathcal{M}\bar{C}_H$ by the assumption (2.6), where

$$\mathcal{M}:=\max_k C[1+\|m_{0,k}\|_\infty+(1+T)\|D_pH_k(\cdot,Dv_k)\|_{L^\infty((0,T)\times\Omega)}]^r$$

is the upper bound on m_k given by Proposition 2.2 (where C depends only on the set Ω).

Step 4. We multiply the PDE in (2.14) by m_k and integrate by parts to get

$$0=\int_0^t \frac{d}{dt}\int_\Omega \frac{m_k^2}{2}\,dxds+\int_0^t\int_\Omega |Dm_k|^2\,dxds-\int_0^t\int_{\partial\Omega} m_k Dm_k\cdot n\,d\sigma ds$$
$$+\int_0^t\int_\Omega m_k\tilde{B}_k\cdot Dm_k\,dxds-\int_0^t\int_{\partial\Omega} m_k^2\tilde{B}_k\cdot n\,d\sigma ds$$
$$+\int_0^t\int_\Omega \tilde{F}_k\cdot Dm_k\,dxds-\int_0^t\int_{\partial\Omega} m_k\tilde{F}_k\cdot n\,d\sigma ds.$$

By the initial and boundary conditions in (2.14) we obtain

$$\frac{1}{2}\|m_k(t,\cdot)\|_2^2+\int_0^t \|Dm_k(s,\cdot)\|_2^2\,ds=-\int_0^t\int_\Omega\left(m_k\tilde{B}_k+\tilde{F}_k\right)\cdot Dm_k\,dxds\le$$
$$\frac{1}{2\varepsilon}\int_0^t\|\tilde{F}_k(s,\cdot)\|_2^2ds+\frac{\|\tilde{B}_k\|_\infty}{2\varepsilon}\int_0^t\|m_k(s,\cdot)\|_2^2ds+\varepsilon\frac{\|\tilde{B}_k\|_\infty+1}{2}\int_0^t\|Dm_k(s,\cdot)\|_2^2\,ds,$$

and with the choice $\varepsilon=2/(\|\tilde{B}\|_\infty+1)=:1/c_1$

$$\|m_k(t,\cdot)\|_2^2\le c_1\int_0^t\|\tilde{F}_k(s,\cdot)\|_2^2ds+c_1\|\tilde{B}_k\|_\infty\int_0^t\|m(s,\cdot)\|_2^2ds.$$

Then Gronwall inequality and the definition of $\tilde{F}_k$ give, for all $0\le t\le T$,

$$\|m_k(t,\cdot)\|_2^2\le c_1e^{c_1\|\tilde{B}_k\|_\infty T}\|A_k\|_\infty^2\int_0^t\|Dv_k(s,\cdot)\|_2^2ds. \tag{2.15}$$

Step 5. Now we set

$$\phi(t):=\|Dv(t,\cdot)\|_2^2=\sum_{k=1}^N\|Dv_k(t,\cdot)\|_2^2$$

and assume w.l.o.g. $C_H\ge 1$, so that $c_o,c_1\le C_H$. By combining (2.13) and (2.15) we get

$$\phi(t)\le Ne^{C_H^2T}\left(L_G\|m(T,\cdot)\|_2^2+C_HL_F\int_t^T\|m(s,\cdot)\|_2^2ds\right)$$
$$\le\bar{C}_H^2C\left(L_G\int_0^T\phi(s)ds+C_HL_F\int_t^T\int_0^\tau\phi(s)ds\,d\tau\right),\quad C:=NC_He^{C_H^4T^2}\mathcal{M}^2.$$

Then $\Phi:=\sup_{0\le t\le T}\phi(t)$ satisfies

$$\Phi\le\Phi\Psi,\quad \Psi:=T\bar{C}_H^2C(L_G+L_FC_HT/2),$$

which implies $\Phi = 0$ if $\Psi < 1$. Therefore under such condition we conclude that $D\tilde{v}_k(t,x) = D\overline{v}_k(t,x)$ for all k, x and $0 \le t \le T$. By the uniqueness of solution for the KFP equation (e.g., Thm. I.2.2, p. 15 of [26]) we deduce $\tilde{m} = \overline{m}$ and then, by the Comparison Principle for the HJB equation in the Appendix, $\tilde{v} = \overline{v}$.

Finally, it is clear that Ψ can be made less than 1 by choosing either T, or $\bar{C}_H$, or both L_G and L_F small enough. □

2.2 Examples and Remarks

Example 2.1 Integral costs. Consider F_k and G_k of the form

$$F_k(x,\mu) = F_o\left(x, \int_\Omega K(x,y)\mu(y)dy\right), \quad G_k(x,\mu) = g_1(x)\int_\Omega \bar{K}(x,y)\cdot\mu(y)dy + g_2(x)$$

with $F_o : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ measurable and Lipschitz in the second variable uniformly in the first, whereas K is an $N \times N$ matrix with components in $L^2(\Omega\times\Omega)$. Then F_k satisfies (2.3). About G_k we assume $g_1, g_2 \in C^1(\Omega)$, Dg_1 bounded, the vector $\bar{K}$ and its Jacobian $D_x\bar{K}$ with components in $L^2(\Omega\times\Omega)$. Then it satisfies (2.4). Of course all the data F_o, K, $\bar{K}$, g_i are allowed to change with the index $k = 1, \dots, N$.

Example 2.2 Local costs. Take $G_k = G_k(x)$ independent of $m(T)$ and F_k of the form $F_k(x,\mu) = F_k^l(x,\mu(x))$ with $F_k^l : \overline{\Omega} \times [0,+\infty)^N \to \mathbb{R}$ measurable and Lipschitz in the second variable uniformly in the first. Then F_k satisfies (2.3).

Example 2.3 Costs depending on the moments. The mean value of the density μ, $M(\mu) = \int_\Omega y\mu(y)dy$, and all its moments $\int_\Omega y^j\mu(y)dy$, $j = 2,3,\dots$, are Lipschitz in L^2 by Example 2.1. Then any F_k (resp., G_k) depending on μ only via these quantities satisfies (2.3) (resp., (2.4)) if it is Lipschitz with respect to them uniformly with respect to x.

Example 2.4 Convex Hamiltonians. The usual Hamiltonians in MFGs are those arising from classical Calculus of Variations, e.g., $H_k(x,p) = b_k(x)(c_k + |p|^2)^{\beta_k/2}$, which satisfies the assumption (2.2) if $b_k \in C(\overline{\Omega})$ and either $c_k > 0$ or $c_k = 0$ and $\beta_k \ge 2$.

A related class of Hamiltonians are those of Bellman type associated to nonlinear systems, affine in the control $\alpha \in \mathbb{R}^d$,

$$H_k(x,p) := \sup_\alpha\{-(f_k(x) + g_k(x)\alpha)\cdot p - L_k(x,\alpha)\} = -f_k(x)\cdot p + L_k^*\left(x, -g_k(x)^T p\right), \tag{2.16}$$

where f_k is a Lipschitz vector field, g_k a Lipschitz square matrix, $L_k(x,\alpha)$ is the running cost of using the control α (adding to $F_k(x,m)$ in the cost functional of a representative player), and $L_k^*(x,\cdot)$ is its convex conjugate with respect to α. In this case one can check the assumption (2.2) on an explicit expression of L_k^*. For instance, if $L_k(x,\alpha) = |\alpha|^\gamma/\gamma$ then

$$H_k(x,p) = -f_k(x)\cdot p + \frac{\gamma-1}{\gamma}|g_k(x)^T p|^{\gamma/(\gamma-1)},$$

which satisfies (2.2) if $\gamma \le 2$.

Example 2.5 Nonconvex Hamiltonians. Two-person 0-sum differential games give rise to the Isaacs Hamiltonians, which are defined in a way similar to (2.16) but as the inf-sup over two sets of controls. A motivation for considering these Hamiltonians in MFGs is proposed in [35]. A relevant example is the case of robust control, or nonlinear H_∞ control, studied in connection with MFGs by Bauso et al. [7] and Moon and Başar [32] (see also the references therein). In this class of problems a deterministic disturbance $\sigma(x)\beta$ affects the control system (σ is a Lipschitz square matrix) and a worst case analysis is performed by assuming that $\beta \in \mathbb{R}^d$ is the control of an adversary who wishes to maximise the cost functional of the representative agent; a term $-\delta|\beta|^2/2$, with $\delta > 0$ is added to the running cost to penalise the energy of the disturbance. The Hamiltonian for robust control then becomes

$$H_k^{(r)}(x,p) := H_k(x,p) + \inf_\beta \left\{-\sigma(x)\beta\cdot p + \delta|\beta|^2/2\right\} = H_k(x,p) - \frac{|\sigma(x)^T p|^2}{2\delta}, \tag{2.17}$$

which is the sum of the convex H_k of the previous example and a concave function of p. Clearly it satisfies the condition (2.2) if and only if H_k does.

Remark 2.1 Continuous dependence on data. Our proof of uniqueness can be adapted to show the Lipschitz dependence of solutions on some data. For instance, in Theorem 2.1 we may assume that $\bar m(0,x) = \bar m_0(x)$ and $\tilde m(0,x) = \tilde m_0(x)$, with $\bar m_0, \tilde m_0 \in \mathcal{P}_N(\Omega)$. Then a simple variant of the proof allows to estimate

$$\|\tilde m(t,\cdot) - \bar m(t,\cdot)\|_2^2 \le \frac{C}{\delta}\|\tilde m_0 - \bar m_0\|_2^2$$

where $0 < \delta \le 1 - \Psi$ and C depends on the same quantities as Ψ. A similar estimate holds for $\|D\tilde v(t,\cdot) - D\bar v(t,\cdot)\|_2^2$. Under some further assumptions on the costs F and G one can also use results on the HJB equation to obtain the continuous dependence of v itself upon the initial data m_0. More precise results on continuous dependence of solutions with respect to data will be given elsewhere.

Remark 2.2 The statement of Theorem 2.1 holds with the same proof for solutions $\mathbb{Z}^d$-periodic in the space variable x in the case that F_k and G_k are $\mathbb{Z}^d$-periodic in x and without Neumann boundary conditions. In such case of periodic boundary conditions a uniqueness result for short T was presented by Lions in [31] for $N = 1$, regularizing running cost F, and for terminal cost G independent of $m(T)$. He used estimates in L^1 norm for m and in L^∞ norm for Dv, instead of the L^2 norms we used here in (2.13) and (2.15). See also [5] for the case of a single population.

Remark 2.3 The constants C_H and $\bar{C}_H$ in the theorem depend only on the data of (2.1) if H_k and $D_p H_k$ are globally Lipschitz in p, uniformly in x, for all k. In this case the smallness condition $\Psi < 1$ does not depend on the solutions $\tilde{v}, \overline{v}$. In the next section we reach the same conclusion for much more general Hamiltonians H_k under some mild additional conditions on the costs F_k, G_k.

Remark 2.4 If the volatility is different among the populations the terms Δv_k, Δm_k in (2.1) are replaced, respectively, by $\nu_k \Delta v_k$ and $\nu_k \Delta m_k$. If the constants ν_k are all positive, the theorem remains true with the function Ψ now depending also on $\nu_1, \ldots, \nu_N$ and minor changes in the proof. The case of volatility depending on x leads to operators of the form trace$(\sigma_k(x)\sigma_k^T(x)D^2 v_k)$ in the HJB equations and their adjoints in the KFP equations. This can also be treated, with some additional work in the proof, if such operators are uniformly elliptic, i.e., the minimal eigenvalue of the matrix $\sigma_k(x)\sigma_k^T(x)$ is bounded away from 0 for $x \in \overline{\Omega}$.

Remark 2.5 The $C^{2,\beta}$ regularity of $\partial\Omega$ can be weakened in Theorem 2.1. Here we used, e.g., Theorem IV.5.3 of [26] to produce a smooth test function φ in the proof of Proposition 2.2. However, we could work instead with a weak solution of the backward heat equation, which exists, for instance, if $\partial\Omega \in C^{1,\beta}$ by Theorem 6.49 of [30], or if it is "piecewise smooth" by Theorem III.5.1 in [26].

3 Special Cases and Applications

The function Ψ of Theorem 2.1 may depend on the solutions $\tilde{v}, \overline{v}$ if the Hamiltonians H_k are not globally Lipschitz or they have unbounded second derivatives, because the constants $C_H, \bar{C}_H$ may depend on the range of $D\tilde{v}$ and $D\overline{v}$. Under some further assumptions we can estimate these quantities and therefore get a uniqueness result where the function Ψ depends only on the data of the problem (2.1). The additional assumptions are

$$|F_k(x,\mu)| \le C_F, \quad |G_k(x,\mu)| \le C_G, \quad \forall x \in \overline{\Omega}, \mu \in \mathcal{P}_N(\Omega), k = 1,\ldots,N, \tag{3.1}$$

$$|H_k(x,p)| \le \alpha(1+|p|^2), \quad |D_p H_k(x,p)|(1+|p|) \le \alpha(1+|p|^2), \quad \forall x,p,k, \tag{3.2}$$

and $x \to G(x,\mu)$ of class C^2, for all $\mu \in \mathcal{P}_N(\Omega)$, with

$$\|DG_k(\cdot,\mu)\|_\infty + \|D^2 G_k(\cdot,\mu)\|_\infty \le C'_G, \forall k. \tag{3.3}$$

Corollary 3.1 *Assume* (2.2)*,* (2.3)*,* (2.4)*,* $m_0 \in \mathcal{P}_N(\Omega)$*,* (3.1)*,* (3.2)*, and* (3.3)*. Then there exists* $\overline{T} > 0$ *such that for all* $T \in (0, \bar{T}]$ *there can be at most one classical solution of* (2.1)*.*

Proof By Assumption (3.1) the functions $\pm(C_G + t(C_F + \alpha))$ are, respectively, a super- and a subsolution of the HJB equation in (2.1) with homogeneous Neumann condition and terminal condition G_k, for any k and m. Then the Comparison Principle in the Appendix gives for any solution of (2.1) the estimate

$$|v_k(t,x)| \le C_G + TC_F, \quad \forall (t,x) \in [0,T] \times \overline{\Omega}, k = 1, \dots, N.$$

Now we can use an estimate of Theorem V.7.2, p. 486 of [26], stating that there is a constant K, depending only on $\max |v_k|$, α, C'_G, and $\partial\Omega$, such that

$$|Dv_k(t,x)| \le K, \quad \forall (t,x) \in [0,T] \times \overline{\Omega}, k = 1, \dots, N.$$

Then the constant C_H in (2.5) is bounded by $C'_H := \alpha(1 + K^2)/(1 + K)$, and $\bar{C}_H$ defined by (2.6) can be estimated by

$$\bar{C}'_H := \max_{k=1,\dots,N} \sup_{x\in\Omega,\, |p|,|q|\le K} \frac{|D_p H_k(x,p) - D_p H_k(x,q)|}{|p-q|}.$$

Now Theorem 2.1 gives the conclusion. □

Remark 3.1 The constant $\bar{T}$ in the Corollary depends only on L_F, L_G, N, α, C_F, C_G, C'_G, $\max_k \|m_{0,k}\|_\infty$, Ω and the constants C'_H, $\bar{C}'_H$ built in the proof. A similar results holds if, instead of T small, we assume L_F and L_G suitably small.

Example 3.1 Costs satisfying the assumptions. The nonlocal costs F_k and G_k of Example 2.1 satisfy Assumption (3.1) if, for instance, K, $\bar{K}$, and g_i are bounded and F_o is continuous.

The Assumption (3.3) is verified if $g_1, g_2 \in C^2(\overline{\Omega})$ and $|D^2_x \bar{K}(x,y)| + |D^2_x \bar{K}(x,y)| \le C$ for all x, y.

For the local cost F_k of Example 2.2, (3.1) holds if F^l_k is bounded.

3.1 Well-Posedness of Segregation Models

Next we combine this uniqueness result with an existence theorem for models of urban settlements and residential choice proposed in [1]. We take for simplicity

$$N = 2, \quad G_k \equiv 0, \quad H_k(x,p) = h_k(x,|p|). \tag{3.4}$$

We endow $\mathcal{P}_2(\Omega)$ with the Kantorovitch-Rubinstein distance and strengthen condition (3.1) to

$$(F_1, F_2) : \overline{\Omega} \times \mathcal{P}_2(\Omega) \to \mathbb{R}^2 \text{ continuous and with bounded range in } C^{1,\beta}(\overline{\Omega}), \tag{3.5}$$

for some $\beta > 0$. We also assume a compatibility condition and further regularity on m_0:

$$\partial_n m_{0,k} = 0 \text{ on } \partial\Omega, \quad m_{0,k} \in C^{2,\beta}(\overline{\Omega}), \quad k = 1, 2. \tag{3.6}$$

Corollary 3.2 *Assume* (2.2), (2.3), (3.2), (3.4), (3.5), (3.6), *and* $H_k \in C^1(\overline{\Omega} \times \mathbb{R}^d)$. *Then there exists* $\overline{T} > 0$ *such that for all* $T \in (0, \bar{T}]$ *there exists a unique classical solution of* (2.1).

Proof The existence of a solution (for any T) follows from Theorem 12 of [1]. Let us only note that, by (3.4), $D_p H_k(x, p) = \partial_{|p|} h_k(x, |p|) p/|p|$, and then the compatibility condition in (3.6) and the Neumann condition for v_k imply also the compatibility condition

$$\partial_n m_{0,k} + m_{0,k} D_p H_k(x, Dv_k(0, x)) \cdot n = 0 \quad \forall x \in \partial\Omega. \tag{3.7}$$

The uniqueness of the solution for small T follows from Corollary 3.1. □

Remark 3.2 Here the constant $\bar{T}$ depends on L_F, α, C_F, $\max_k \|m_{0,k}\|_\infty$, Ω, and the constants $C'_H, \bar{C}'_H$ built in the proof of Corollary 3.1. The solution m and Dv depend in a Lipschitz way from the initial condition m_0, as explained in Remark 2.1.

Example 3.2 Costs of Schelling type. Let $K_k : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ be Lipschitz and such that, for some $U(x)$ neighborhood of x, $K_k(x, y) = 1$ for $y \in U(x)$ and $K_k(x, y) = 0$ for y out of a small neighborhood of $U(x)$. Then

$$N_k(x, \mu_k) := \int_\Omega K_k(x, y)\mu_k(y)dy$$

represents the amount of population k around x. The cost functional for the k-th population introduced in [1] and inspired by the studies on segregation of Schelling [34] is of the form

$$F_k(x, \mu_1, \mu_2) := \left(\frac{N_k(x, \mu_k)}{N_k(x, \mu_k) + N_{3-k}(x, \mu_{3-k}) + \eta} - a_k \right)^-,$$

where $(\ )^-$ denotes the negative part and $\eta > 0$ is very small. It means that if the ratio of the k-th population with respect to the total population in the neighborhood of x is above the threshold a_k, then a representative agent of this population is happy because his cost is 0, whereas below the threshold the agent incurs in a cost and therefore he wants to move from the neighborhood. These costs fall within

Example 3.1 and satisfy (2.3) and (3.1). Moreover $F_k : \Omega \times \mathcal{P}_2(\Omega) \to \mathbb{R}$ is Lipschitz.

To meet the assumptions of Corollary 3.2 we assume the kernel K is of class C^2 in x and we approximate the negative part $(\ )^-$ with a smooth function, e.g.,

$$\varphi_\varepsilon(r) := \frac{\sqrt{r^2+\varepsilon^2}-r}{2},$$

for a small $\varepsilon > 0$. Then the cost functionals

$$F_k^\varepsilon(x, \mu_1, \mu_2) := \varphi_\varepsilon\left(\frac{N_k(x,\mu_k)}{N_k(x,\mu_k)+N_{3-k}(x,\mu_{3-k})+\eta} - a_k\right)$$

satisfy also (3.5).

Example 3.3 Hamiltonians. Typical examples are either $H_k(x, p) = b_k(x)|p|^2$, with $b_k \in C(\overline{\Omega})$, or

$$H_k(x, p) = b_k(x)(1+|p|^2)^{\beta_k/2}, \quad 0 < \beta_k \le 2.$$

They satisfy (2.2) and (3.2), moreover they are in $C^1(\overline{\Omega} \times \mathbb{R}^d)$ if $b_k \in C^1(\overline{\Omega})$.

Remark 3.3 In the last Corollary 3.2 the simplifying assumption $G_k \equiv 0$ can be dropped and replaced with $G_k : \overline{\Omega} \times \mathcal{P}_2(\Omega) \to \mathbb{R}$ continuous, with bounded range in $C^{2,\beta}(\overline{\Omega})$, and satisfying (2.4). Then (3.3) holds and the constant $\bar{T}$ depends also on L_G, C_G, and C'_G. Examples of such terminal costs can be given along the lines of Examples 2.1, 3.1, and 3.2.

3.2 Well-Posedness of Robust Mean Field Games

For simplicity we limit ourselves to a single population of agents, so $N = 1$ and we drop the subscripts k. The representative agent has the dynamics in $\mathbb{R}^d$

$$dX_s = (f(X_s) + g(X_s)\alpha_s + \sigma(X_s)\beta_s)\,ds + dW_s,$$

where f is a C^1 vector field in $\overline{\Omega}$, g and σ are C^1 scalar functions in $\overline{\Omega}$, W_s is a d-dimensional Brownian motion, α_s, β_s take values in $\mathbb{R}^d$ and are, respectively, the control of the agent and a disturbance affecting the system. The cost functional is (for $\delta > 0$)

$$\mathbb{E}\left[\int_0^T \left(F(X_s, m(s,\cdot)) + \frac{|\alpha_s|^2}{2} - \delta\frac{|\beta_s|^2}{2}\right) ds + G(X_T, m(T,\cdot))\right]$$

that the agent wants to minimise whereas the disturbance, modeled as a second player in a 2-person 0-sum game, wants to maximise. This leads to the Hamiltonian

$$H(x,p) = -f(x)\cdot p + g^2(x)\frac{|p|^2}{2} - \sigma^2(x)\frac{|p|^2}{2\delta}. \tag{3.8}$$

Note that here $g(x)$ and $\sigma(x)$ are scalars, different from Examples 2.4 and 2.5. On the costs we assume

$$F,\ G : \overline{\Omega}\times\mathcal{P}_1(\Omega)\to\mathbb{R} \text{ continuous with bounded range, resp., in } C^{1,\beta}(\overline{\Omega}) \text{ and } C^{2,\beta}(\overline{\Omega}) \tag{3.9}$$

for some $\beta > 0$. The compatibility condition and regularity on m_0 now are

$$\partial_n m_0 - m_0 f\cdot n = 0 \text{ on } \partial\Omega, \qquad m_0\in C^{2,\beta}(\overline{\Omega}). \tag{3.10}$$

Corollary 3.3 *Assume $N = 1$ with the Hamiltonian defined by* (3.8)*,* (2.3)*,* (2.4)*,* (3.9)*, and* (3.10)*. Then for all $T > 0$ there is a classical solution of* (2.1)*, and there exists $\overline{T} > 0$ such that for all $T\in(0,\bar{T}]$ such solution is unique.*

Proof The existence of a solution follows from Theorem 12 of [1]. In fact, $H\in C^1(\overline{\Omega}\times\mathbb{R}^d)$ and it has quadratic growth. Moreover

$$D_pH(x,p) = -f(x) + g^2(x)p - \frac{\sigma^2(x)}{\delta}p,$$

and then the compatibility condition in (3.10) and the Neumann condition for v imply again the compatibility condition (3.7).

The uniqueness of the solution for small T follows from Corollary 3.1, since H satisfies also (2.2). □

Remark 3.4 Also here the solution m and Dv depend in a Lipschitz way from the initial condition m_0, as explained in Remark 2.1.

Remark 3.5 Our example of robust MFG is different from the one in [7]. In that paper the state space is $\Omega = \mathbb{R}$, one-dimensional without boundary, the control system is linear in the state X_s, and the volatility is σX_s instead of 1, for some positive constant σ, so the parabolic operators in the HJB and KFP equations of (2.1) are degenerate at the origin. The well-posedness of the MFG system of PDEs in [7] is an open problem.

Appendix: A Comparison Principle

The next result is known but we give its elementary proof for lack of a precise reference.

Proposition 3.1 *Assume $\Omega \subseteq \mathbb{R}^d$ is bounded with C^2 boundary, $H : \overline{\Omega} \times \mathbb{R}^d$ is of class C^1 with respect to p, and $u, v : [0, T] \times \overline{\Omega} \to \mathbb{R}$ are C^1 in t and C^2 in x and satisfy*

$$\begin{cases} -\partial_t u - \Delta u + H(x, Du) \leq -\partial_t v - \Delta v + H(x, Dv), & \textit{in} \quad (0, T) \times \Omega, \\ \partial_n u \leq \partial_n v, & \textit{on} \quad (0, T) \times \partial\Omega, \\ u(T, x) \leq v(T, x) & \textit{in} \quad \Omega. \end{cases}$$

Then $u \leq v$ in $[0, T] \times \overline{\Omega}$.

Proof Let us assume first that

$$-\partial_t(u - v) - \Delta(u - v) + H(x, Du) - H(x, Dv) < 0 \quad \text{in } [0, T) \times \Omega,$$

$\partial_n(u - v) < 0$ on $[0, T) \times \partial\Omega$, and $(u - v)(T, x) \leq \delta$. Then the maximum of $u - v$ can be attained only at $t = T$, which implies $u - v \leq \delta$ in $[0, T] \times \overline{\Omega}$.

Now take $g \in C^2(\overline{\Omega})$ such that $Dg(x) = n(x)$ for all $x \in \partial\Omega$ and define

$$v_\varepsilon(t, x) := v(t, x) + \varepsilon(T - t)C + \varepsilon g(x).$$

Then $\partial_n(u - v_\varepsilon) = \partial_n(u - v) - \varepsilon < 0$ and $(u - v_\varepsilon)(T, x) \leq \varepsilon\|g\|_\infty$. Moreover, by Taylor's formula, for some q with $|q| \leq \|Dg\|_\infty$,

$$-\partial_t(u - v_\varepsilon) - \Delta(u - v_\varepsilon) + H(x, Du) - H(x, Dv_\varepsilon) =$$
$$-\partial_t(u-v)-\Delta(u-v)+H(x, Du)-H(x, Dv)-\varepsilon(C-\Delta g+D_p H(x, q)\cdot Dg) < -\varepsilon$$

if C is chosen large enough. Then

$$u \leq v_\varepsilon + \varepsilon\|g\|_\infty \leq v + \varepsilon(TC + 2\|g\|_\infty)$$

and we conclude by letting $\varepsilon \to 0$. □

Remark 3.1 The result remains true if $\partial\Omega$ is merely C^1 and satisfies an interior sphere condition. This can be proved in a less direct way by linearizing the inequality for $u - v$ and then using the parabolic Strong Maximum Principle and the parabolic version of Hopf's Lemma for linear equations (see, e.g., [33]).

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Finite Difference Methods for Mean Field Games Systems

Simone Cacace and Fabio Camilli

Abstract We discuss convergence results for a class of finite difference schemes approximating Mean Field Games systems either on the torus or a network. We also propose a quasi-Newton method for the computation of discrete solutions, based on a least squares formulation of the problem. Several numerical experiments are carried out including the case with two or more competing populations.

Keywords Mean field games · Networks · Numerical methods · Finite difference · Newton-like methods

1 Introduction

In this paper we describe a class of finite difference methods for the approximation of the stationary Mean Field Games (MFG in short) system

$$\begin{cases} -\nu\Delta u + H(x, Du) + \lambda = V[m] & x \in \mathcal{T}, \\ \nu\Delta m + \operatorname{div}\left(m\,\frac{\partial H}{\partial p}(x, Du)\right) = 0 & x \in \mathcal{T}, \\ \int_{\mathcal{T}} u(x)dx = 0,\ \int_{\mathcal{T}} m(x)dx = 1,\ m \geq 0\,, \end{cases} \tag{1.1}$$

where $\mathcal{T}$ can be either the unit torus $\mathbb{T}^d = [0, 1]^d$ or a network Γ. The system consists in a couple of PDEs, respectively a Hamilton-Jacobi-Bellman equation and a Fokker-Planck equation plus normalization conditions on both u and m. The unknowns are the value function u, the density m and the ergodic constant λ and

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the system also involves the scalar Hamiltonian $H(x, p)$ and the potential V (for a general presentation of the theory of Mean Field Games we refer [12, 15]). The results are based on the papers in [1, 3, 4, 6, 7] and include existence, uniqueness and regularity of the approximate solution, convergence of the scheme and efficient resolution of the discrete problem.

After the introduction of the MFG theory, an important research activity has been pursued for the approximation of the different types of MFG models and several papers have been devoted to this topic. Besides the finite difference method we describe in this chapter, we mention among the others: the semi-Lagrangian scheme proposed in [9]; the optimization algorithm connected with the optimal control interpretation of the MFG system in [14]; the monotone scheme in [13] which exploits the equivalence between the MFG system and a linear system in the case of a quadratic Hamiltonian; the gradient-flow method based on the variational characterization of certain MFG systems in [5].

A numerical method for MFG systems has to face several difficulties: the system is strongly coupled in both the equations, i.e. via the potential term V in the Hamilton-Jacobi-Bellman equation and via the drift term $\frac{\partial H}{\partial p}(x, Du)$ in the Fokker-Planck equation; in the stationary case the system is formally overdetermined, involving three unknowns (u, m, λ) and two equations, while in the evolutive case it has forward-backward structure with respect to the time variable; the approximation of the Hamilton-Jacobi-Bellman equation presents the typical curse of dimensionality issue complicated furthermore by the coupled structure; the constraint $m \geq 0$ may be difficult to impose for algorithms based on Gradient and Newton methods; moreover, in order to obtain convergence and error estimates, a numerical method for MFG systems should reproduce at a discrete level some main properties of the continuous problem: for example, it is well-known that the Fokker-Planck equation in MFG systems is the adjoint equation associated to the linearization of the Hamilton-Jacobi-Bellman equation and, indeed, this relation is usually employed to get several properties of the solution to the problem.

The numerical method introduced in [1] and described in Sect. 2 is designed to reproduce at the discrete level the same adjoint structure of the continuous system. Discretizing the Hamilton-Jacobi-Bellman equation via standard finite differences, then the approximation of the Fokker-Planck equation is obtained by means of the weak formulation of the linearization of the first equation. The adjoint structure of the discrete problem allows one to obtain, as in the continuous case, several properties of the discrete solution, such as existence, uniqueness and regularity. Moreover, since the continuous and the approximate problems have the same adjoint structure, convergence of the scheme and error estimates are obtained by substituting the continuous solution in the discrete problem and estimating the truncation error (see [2, 4]).

In the recent times, there has been an increasing interest in the study of differential models on networks, and in [8] we extend the MFG theory to this framework. In [7], we consider the numerical approximation of the network problem and, following the approach in [1], we find an approximation of the transition conditions at the vertices which preserves, at the discrete level, the adjoint structure of the continuous problem. Also in this case, employing the similarity between the continuous and the approximate problems, we are able to prove the convergence of the scheme. The scheme for the network problem is described in Sect. 3.

Since MFG theory introduces an effective and efficient methodology for handling a wide variety of applications in different fields, it is particularly relevant to design efficient solvers for the discrete problem. Section 4 is devoted to a new method proposed in [6] which allows to compute solutions of (1.1) avoiding costly large-time and ergodic approximations usually employed in this framework. Indeed, once an effective discretizion of (1.1) is introduced, the discrete problem is solved directly, by interpreting the ergodic constant λ as an unknown of the problem and computing the solution of the overdeterminated system by a Newton-like method for inconsistent nonlinear systems. A large collection of numerical tests in dimensions one and two shows the performance of the proposed method, both in terms of accuracy and computational time.

2 A Finite Difference Scheme for Mean Field Games on the Torus

In this section we consider system (1.1) on the torus $\mathbb{T}^d$, i.e. with periodic boundary conditions. The Hamiltonian $H(x, p)$ is assumed to be convex w.r.t. p and regular w.r.t. x and p. The potential term V may be either a local operator, i.e. $V[m(\cdot)](x) = F(m(x))$ for some regular function F; or a non local operator which continuously maps the set of probability measures on $\mathbb{T}^d$ to a bounded subset of the Lipschitz functions on $\mathbb{T}^d$.

To simplify the notations, we assume that the dimension of the state space is $d = 2$, but the scheme can be easily generalized to any dimension. Hence, let $\mathbb{T}_h^2$ be a uniform grid on the two-dimensional torus with step h (assuming that $N_h = 1/h$ is an integer) and denote by $x_{i,j}$ a typical grid node in $\mathbb{T}_h^2$. The values of u and m at $x_{i,j}$ are approximated, respectively by $U_{i,j}$ and $M_{i,j}$. For a grid function U, we consider the finite difference operators

$$(D_1^+U)_{i,j} = \frac{U_{i+1,j} - U_{i,j}}{h}, \qquad (D_2^+U)_{i,j} = \frac{U_{i,j+1} - U_{i,j}}{h},$$

and define

$$[D_hU]_{i,j} = \left((D_1^+U)_{i,j}, (D_1^+U)_{i-1,j}, (D_2^+U)_{i,j}, (D_2^+U)_{i,j-1}\right)^T,$$

$$(\Delta_h U)_{i,j} = -\frac{4U_{i,j} - U_{i+1,j} - U_{i-1,j} - U_{i,j+1} - U_{i,j-1}}{h^2},$$

where T denotes the transposition operator. We approximate $H(\cdot, \nabla u)(x_{i,j})$ by $g(x_{i,j}, [D_h U]_{i,j})$, where the numerical Hamiltonian is a function $g : \mathbb{T}^2 \times \mathbb{R}^4 \to \mathbb{R}$, $(x, q_1, q_2, q_3, q_4) \to g(x, q_1, q_2, q_3, q_4)$ satisfying

(G1) *monotonicity*: g is non increasing w.r.t. q_1, q_3 and nondecreasing w.r.t. to q_2, q_4.
(G2) *consistency:* $g(x, q_1, q_1, q_2, q_2) = H(x, q)\ \forall x \in \mathbb{T}^2, \forall q = (q_1, q_2) \in \mathbb{R}^2$.
(G3) *differentiability*: g is of class C^1.
(G4) *convexity*: for all $x \in \mathbb{T}^2$, $(q_1, q_2, q_3, q_4) \to g(x, q_1, q_2, q_3, q_4)$ is convex.

Numerical Hamiltonians fulfilling these requirements are provided by Lax-Friedrichs or Godunov type schemes, see [16].

The operator $V[m](x_{i,j})$ is approximated by $V_h[M]_{i,j}$. We assume that $V_h[M]$ can be computed in practice. For example, if $V[m]$ is defined as the solution w of the equation $\Delta^2 w + w = m$ in $\mathbb{T}^2$, (Δ^2 being the bi-laplacian), then one can define $V_h[M]$ as the solution W of the discrete problem $\Delta_h^2 W + W = M$ in $\mathbb{T}_h^2$. If V is a local operator, i.e. $V[m](x) = F(m(x))$, then $V_h[M]_{i,j} = F(M_{i,j})$.

For a generic pair of grid functions U, V we define the scalar product $(U, V)_2 = h^2 \sum_{0 \le i,j < N_h} U_{i,j} V_{i,j}$ and we consider the compact and convex set

$$\mathcal{K}_h = \{M = (M_{i,j})_{0 \le i,j < N_h} : (M, \overline{1})_2 = 1; \quad M_{i,j} \ge 0\},$$

where $\overline{1}$ denotes the N_h^2-tuple with all components equal to 1. Note that $\mathcal{K}_h$ can be viewed as the set of the discrete probability measures on $\mathbb{T}_h^2$.

We make the following assumptions on the potential term, V being local or not:

(V1) V_h is continuous.
(V2) V_h is monotone, i.e.

$$\left(V_h[M] - V_h[\tilde{M}], M - \tilde{M}\right)_2 \le 0 \Rightarrow V_h[M] = V_h[\tilde{M}].$$

If $V[m](x) = F(m(x))$, the function F being continuous from $\mathbb{R}^+$ to $\mathbb{R}$, then V_h is continuous on the set of nonnegative grid functions.
If V is a nonlocal operator, we assume that the discrete operator V_h satisfies the following additional properties:
(V3) There exists a constant C independent of h such that, for every grid function $M \in \mathcal{K}_h$, it holds

$$\|V_h[M]\|_\infty \le C, \qquad \left|(V_h[M])_{i,j} - (V_h[M])_{k,\ell}\right| \le C d(x_{i,j}, x_{k,\ell})$$

where $d(x, y)$ is the distance between the two points x and y in the torus $\mathbb{T}^2$.

(V4) There exists a continuous, bounded function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(0) = 0$ and such that, for all $m \in \mathcal{K} := \{m \in L^1(\mathbb{T}^2) : \int_{\mathbb{T}^2} m dx = 1,\ m \geq 0\}$ and for all $M \in \mathcal{K}_h$,

$$\| V[m] - V_h[M] \|_{L^\infty(\mathbb{T}_h^2)} \leq \omega\left(\|m - \mathcal{I}_h M\|_{L^1(\mathbb{T}^2)}\right), \tag{2.1}$$

where $\mathcal{I}_h M$ is the piecewise constant function taking the value $M_{i,j}$ in the square $\left\{|x - x_{i,j}|_\infty \leq h/2\right\}$.

Remark 2.1 If $m \in \mathcal{K}$ and $\mathcal{P}_h m$ is the grid function whose value at $x_{i,j}$ is

$$\int_{|x-x_{i,j}|_\infty \leq h/2} m(x)dx,$$

then $\mathcal{P}_h m \in \mathcal{K}_h$ and (2.1) implies the convergence of the approximation to $V[m]$, i.e.

$$\lim_{h\to 0} \sup_{m\in\mathcal{K}} \| V[m] - V_h[\mathcal{P}_h m] \|_{L^\infty(\mathbb{T}_h^2)} = 0.$$

To approximate the Hamilton-Jacobi-Bellman equation in (1.1), we consider the scheme

$$-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \Lambda = (V_h[M])_{i,j}\,, \tag{2.2}$$

with $\Lambda \in \mathbb{R}$ and subject to the normalization condition $(U, \overline{1})_2 = 0$.

In order to approximate the Fokker-Planck equation in (1.1), we consider the linearization of the Hamilton-Jacobi-Bellman equation at u in the direction w

$$-\nu\Delta w + \frac{\partial H}{\partial p}(x, Du)\, Dw = 0.$$

Note that the weak formulation of previous equation involves the term

$$-\int_{\mathbb{T}^2} \operatorname{div}\left(m\frac{\partial H}{\partial p}(x, \nabla u)\right) w\, dx.$$

which, by periodicity, yields

$$\int_{\mathbb{T}^2} m\frac{\partial H}{\partial p}(x, \nabla u)\cdot \nabla w\, dx$$

for any test function w, and it can be approximated by

$$h^2 \sum_{i,j} M_{i,j} \nabla_q g(x_{i,j}, [D_h U]_{i,j}) \cdot [D_h W]_{i,j}. \tag{2.3}$$

By discrete integration by parts on $\mathbb{T}_h^2$, the sum in (2.3) is readily rewritten as

$$h^2 \sum_{i,j} \mathcal{T}_{i,j}(U, M) W_{i,j},$$

where the operator $\mathcal{T}$ is defined as follows:

$$\begin{aligned}\mathcal{T}_{i,j}(U, M) =& \frac{1}{h}\left[\begin{array}{l} M_{i,j}\frac{\partial g}{\partial q_1}(x_{i,j}, [D_h U]_{i,j}) - M_{i-1,j}\frac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U]_{i-1,j}) \\ +M_{i+1,j}\frac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U]_{i+1,j}) - M_{i,j}\frac{\partial g}{\partial q_2}(x_{i,j}, [D_h U]_{i,j}) \end{array}\right]+ \\ & \frac{1}{h}\left[\begin{array}{l} M_{i,j}\frac{\partial g}{\partial q_3}(x_{i,j}, [D_h U]_{i,j}) - M_{i,j-1}\frac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U]_{i,j-1}) \\ +M_{i,j+1}\frac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U]_{i,j+1}) - M_{i,j}\frac{\partial g}{\partial q_4}(x_{i,j}, [D_h U]_{i,j}) \end{array}\right].\end{aligned}$$

In conclusion the second equation in (1.1) is approximated by

$$\nu(\Delta_h M)_{i,j} + \mathcal{T}_{i,j}(U, M) = 0, \tag{2.4}$$

subject to the normalization conditions $(M, \overline{1})_2 = 1$, $M \geq 0$. As for the continuous problem, the operator $M \mapsto \big(-\nu(\Delta_h M)_{i,j} - \mathcal{T}_{i,j}(U, M)\big)_{i,j}$ is the adjoint of the linearized version of the operator $u \mapsto \big(-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j})\big)_{i,j}$. This is a crucial property in view of the uniqueness and the convergence of the scheme.

Summarizing the finite difference scheme for the system (1.1) is: for all $0 \leq i, j < N_h$

$$\begin{cases} -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) = (V_h[M])_{i,j}), \\ \nu(\Delta_h M)_{i,j} + \mathcal{T}_{i,j}(U, M) = 0, \\ (U, 1)_2 = 0,\ (M, 1)_2 = 1,\ M \geq 0. \end{cases} \tag{2.5}$$

The following theorem is proved in [1].

Theorem 2.1 *If the numerical Hamiltonian g satisfies **(G1)–(G3)** and the potential V satisfies **(V1)**, then (2.5) has a solution (U, M, Λ). Moreover if g also satisfies **(G4)** and V_h also satisfies **(V2)**, then the solution is unique.*

In the previous result ν can also vanish, hence the deterministic case is included. The proof of existence of a solution to (2.5) is based on a fixed point argument and careful estimates of the Lipschitz norm of the solution of (2.2), while uniqueness is proved with a duality argument similar to the one in [15] for the continuous problem. The next proposition gives a regularity result for the solution (2.5) with an estimate of the norm uniform in h.

Proposition 2.1 *Under the same assumptions of Theorem 2.1, assume moreover that* $\nu > 0$ *and*

$$\left|\frac{\partial g}{\partial x}(x,(q_1,q_2,q_3,q_4))\right| \le C(1+|q_1|+|q_2|+|q_3|+|q_4|). \tag{2.6}$$

Then there exists a constant C *independent of* h *such that*

$$\|U\|_\infty + \|D_h U\|_\infty \le C. \tag{2.7}$$

We now focus on the convergence of the scheme (2.5). In the rest of this section we make the following additional assumptions

- $\nu > 0$;
- the Hamiltonian is of the form

$$H(x,p) = \mathcal{H}(x) + |p|^\beta \tag{2.8}$$

  with the function $\mathcal{H} \in C^1(\mathbb{T}^2)$ and $\beta > 1$;
- the system (1.1) admits a unique classical solution (u, m, λ).

To approximate the Hamiltonian in (2.8) we consider a numerical Hamiltonian of the form

$$g(x,q) = \mathcal{H}(x) + G(q_1^-, q_2^+, q_3^-, q_4^+), \tag{2.9}$$

where, for a real number r, $r^+ = \max(r,0)$, $r^- = \max(-r,0)$ and $G : \mathbb{R}^4 \to \mathbb{R}_+$ is given by

$$G(p) = |p|^\beta = (p_1^2 + p_2^2 + p_3^2 + p_4^2)^{\frac{\beta}{2}}.$$

Assumptions **(G1)–(G4)** are satisfied by the numerical Hamiltonian in (2.9), hence Theorem 2.1 guarantees existence and uniqueness of the solution. In the following we denote by u^h (resp. m^h) the piecewise bilinear function in $C(\mathbb{T}^2)$ obtained by interpolating the values $U^h_{i,j}$ (resp $M^h_{i,j}$) of the solution (U^h, M^h, Λ^h) to (2.5) at the nodes of the space grid. For the convergence analysis we distinguish the cases of a nonlocal potential and the case of a local one.

2.1 Convergence for V Nonlocal Operator

We assume that V is monotone, nonlocal and smooth. In this case it is known that there exists a unique classical solution (u, m, λ) of (1.1) such that $m > 0$ [15]. Note that since g in (2.9) verifies condition (2.6), the regularity estimate (2.7) holds and U is uniformly Lipschitz continuous.

Theorem 2.2 *Consider the numerical Hamiltonian given by (2.9) and a discrete potential V_h such that **(V1)–(V4)** hold.*

The case $\beta \geq 2$*: As h goes to 0, the functions u^h converge to u in $W^{1,\beta}(\mathbb{T}^2)$, the functions m^h converge to m in $H^1(\mathbb{T}^2)$, and Λ^h tends to λ.*

The case $\beta \in (1,2)$*: As h goes to 0, the functions u^h converge to u in $W^{1,2}(\mathbb{T}^2)$, the functions m^h converge to m in $L^2(\mathbb{T}^2)$, and Λ^h tends to λ.*

2.2 Convergence for V Local Operator

If V is a local operator, i.e. $V[m](x) = F(m(x))$, existence of a classical solution to (1.1) for any $\beta > 1$ holds, for example, if F is non decreasing and satisfies

$$mF(m) \geq \delta |F(m)|^\gamma - C_1, \qquad \forall m \geq 0 \tag{2.10}$$

for some constant $C_1 > 0$ and $\gamma > 2$ (being 2 the dimension of the space). In the local case, there are no a priori Lipschitz estimates on U such as (2.7). Since these estimates are used several times in the proof of Theorem 2.2, in this case additional difficulties arise and further assumptions are need.

Theorem 2.3 *Consider the numerical Hamiltonian given by (2.9) and a local operator V defined by a continuous function $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying (2.10) and*

$$F'(m) \geq \underline{\delta} \min(m^{\eta_1}, m^{-\eta_2})$$

for $\underline{\delta} > 0$, $\eta_1 > 0$ and $0 < \eta_2 < 1$. As h goes to 0, the functions u^h converge to u in $W^{1,\beta}(\mathbb{T}^2)$, the functions m^h converges to m in $L^{2-\eta_2}(\mathbb{T}^2)$, and Λ^h tends to λ.

The proofs of Theorems 2.2 and 2.3 are rather technical and require several accurate estimates, hence we skip the details here. We only point out that a key ingredient in the proofs is the fundamental identity given in the next lemma (see [4]).

Lemma 2.1 *Let A, B be two grid functions, (U, M, Λ) a solution of (2.5) and $(\tilde{U}, \tilde{M}, \tilde{\Lambda})$ a solution of the perturbed system*

$$\begin{cases} -\nu(\Delta_h \tilde{U})_{i,j} + g(x_{i,j}, [D_h \tilde{U}]_{i,j}) + \tilde{\Lambda} = (V_h[\tilde{M}])_{i,j} + A_{i,j}, \\ \nu(\Delta_h \tilde{M})_{i,j} + \mathcal{T}_{i,j}(\tilde{U}, \tilde{M}) = B_{i,j}, \\ (\tilde{U}, \overline{1})_2 = 0, \ (\tilde{M}, \overline{1})_2 = 1, \ \tilde{M} \geq 0. \end{cases} \tag{2.11}$$

Then the following identity holds

$$\begin{aligned} &\mathcal{G}(M, U, \tilde{U}) + \mathcal{G}(\tilde{M}, \tilde{U}, U) + (V_h[M] - V_h[\tilde{M}], M - \tilde{M})_2 \\ =&(A, M - \tilde{M})_2 + (B, U - \tilde{U})_2, \end{aligned} \tag{2.12}$$

where $\mathcal{G}$ is the nonlinear functional acting on grid functions defined by

$$\mathcal{G}(M, U, \tilde{U}) = \sum_{i,j} M_{i,j}\Big[g(x_{i,j}, [D\tilde{U}]_{i,j}) - g(x_{i,j}, [DU]_{i,j}) \\ - g_q(x_{i,j}, [DU]_{i,j}) \cdot ([D\tilde{U}]_{i,j} - [DU]_{i,j})\Big].$$

The identity (2.12) holds for a general numerical Hamiltonian g and it employs the crucial property that the second equation in (2.5) is the adjoint of the linearized version of the first equation of the system, as already observed. Moreover, if **(G4)** and **(V2)** hold, then the first line of (2.12) is made of three nonnegative terms, hence uniqueness for (2.5) is a straightforward consequence of this identity.

While Theorems 2.2 and 2.3 rely on the existence of a classical solution to (1.1), the convergence analysis has been extended in [2], where the existence of a weak solution of the MFG system is proved via a compactness argument on solutions of the discrete problem.

3 Mean Field Games on Networks

In this section we consider stationary Mean Field Games defined on a network. We first describe a formal derivation of the MFG system in terms of Pareto equilibria for dynamic games defined on a network with a large number of (indistinguishable) players. In this way we deduce the correct transition conditions at the vertices of the network which allow to prove existence and uniqueness of the classical solution to the problem. Hence we propose a finite difference scheme for the MFG system based on the approach of Sect. 2 and a correct approximation of the transition conditions at the vertices.

3.1 Networks and Functional Spaces

We start by describing the constitutive elements of the problem and the main assumptions. A network $\Gamma = (\mathcal{V}, \mathcal{E})$ is a finite collection of points $\mathcal{V} := \{v_i\}_{i\in I}$ in $\mathbb{R}^d$ connected by continuous, non self-intersecting edges $\mathcal{E} := \{e_j\}_{j\in J}$, respectively indexed by two finite sets I and J. Each edge $e_j \in \mathcal{E}$ is parametrized by a smooth function $\pi_j : [0, l_j] \to \mathbb{R}^d$, $l_j > 0$. Given $v_i \in \mathcal{V}$, we denote by $Inc_i := \{j \in J : v_i \in e_j\}$ the set of edges branching out from v_i and by $d_{v_i} := |Inc_i|$ the degree of v_i. A vertex v_i is said a boundary vertex if $d_{v_i} = 1$, otherwise it is said a transition vertex. For simplicity, we assume that the set of boundary vertices is empty. For a function $u : \Gamma \to \mathbb{R}$ we denote by $u_j : [0, l_j] \to \mathbb{R}$ the restriction of u to e_j, i.e. $u(x) = u_j(y)$ for $x \in e_j$, $y = \pi_j^{-1}(x)$, and by $\partial_j u(v_i)$ the oriented derivative of u

at v_i along the arc e_j defined by

$$\partial_j u(v_i) = \begin{cases} \lim_{h\to 0^+}(u_j(h) - u_j(0))/h, & \text{if } v_i = \pi_j(0); \\ \lim_{h\to 0^+}(u_j(l_j - h) - u_j(l_j))/h, & \text{if } v_i = \pi_j(l_j). \end{cases}$$

The integral of a function u on Γ is defined by

$$\int_\Gamma u(x)dx := \sum_{j\in J} \int_0^{l_j} u_j(r)dr.$$

The space $C^k(\Gamma)$, $k \in \mathbb{N}$, consists of all the continuous functions $u : \Gamma \to \mathbb{R}$ such that $u_j \in C^k([0, l_j])$ for $j \in J$ and $\|u\|_{C^k} = \max_{\beta\le k} \|\partial^\beta u\|_{L^\infty} < \infty$. Observe that no continuity condition at the vertices is prescribed for the derivatives of a function $u \in C^k(\Gamma)$.

3.2 A Formal Derivation of the MFG System on a Network

We first show that the transition conditions at the vertices can be deduced in a natural way by the formulation of the differential game associated to the MFG system on the network. Consider a population of agents, distributed at time $t = 0$ according to a probability measure m_0 on Γ; each agent moves on the network Γ and its dynamics inside the edge e_j is governed by the stochastic differential equation

$$dX_s = -\gamma_s\, ds + \sqrt{2\nu_j}\, dW_s,$$

where X_s is the state variable, γ is the control, $\nu_j > 0$ is a diffusion coefficient and W_t is a 1-dimensional Brownian motion. When the agent reaches a vertex $v_i \in \mathcal{V}$, it almost surely spends zero time at v_i and enters in one of the incident edges, say e_j with $j \in Inc_i$, with probability β_{ij} where

$$\beta_{ij} > 0, \quad \sum_{j\in Inc_i} \beta_{ij} = 1.$$

(see [11] for a rigorous definition of stochastic processes on networks). The cost criterion is given by

$$\liminf_{T\to\infty} \mathbb{E}_x \left[\frac{1}{T} \int_0^T \{L(X_t, \gamma_t) + V[m(X_t)]\} dt \right]$$

where m represents the distribution of the overall population of players, L is the Lagrangian and V is an the potential. A formal application of the dynamic programming principle implies that the value function u of the previous control problem satisfies

$$\begin{cases} -\nu_j \partial^2 u_j + H_j(x, \partial u_j) + \lambda = V_j[m], \; x \in e_j, \; j \in J \\ \sum_{j \in Inc_i} \alpha_{ij} \nu_j \partial u_j(v_i) = 0 \qquad v_i \in \mathcal{V}, \\ u_j(v_i) = u_k(v_i), \qquad j, k \in Inc_i, \; v_i \in \mathcal{V}, \end{cases} \tag{3.1}$$

where $\alpha_{ij} := \beta_{ij} \nu_j^{-1}$, λ is the ergodic cost and the Hamiltonian is given on the edge e_j by the Fenchel transformation

$$H_j(x, p) = \sup_{\gamma} \big[-\gamma \cdot p - L_j(x, \gamma) \big].$$

Note that the differential equation inside e_j is defined in terms of the coordinate parametrizing the edge. The second equation in (3.1) is known as the Kirchhoff transition condition and it is consequence of the assumption on the behavior of X_t at the vertices (see [11]). Finally, the third line equation in (3.1) is a constraint prescribing the continuity of u at transition vertices.

In order to derive the equation satisfied by the distribution m of the agents, we follow a standard duality argument. Consider the linearization of Hamilton-Jacobi-Bellman equation at u in the direction w

$$\begin{cases} -\nu_j \partial^2 w_j + \partial_p H_j(x, \partial u_j) \partial w_j = 0, \; x \in e_j, \; j \in J \\ \sum_{j \in Inc_i} \nu_j \alpha_{ij} \partial w_j(v_i) = 0 \qquad v_i \in \mathcal{V} \\ w_j(v_i) = w_k(v_i), \qquad j, k \in Inc_i, \; v_i \in \mathcal{V}. \end{cases} \tag{3.2}$$

Writing the weak formulation of (3.2) for a test function m, integrating by parts along each edge and regrouping the boundary terms corresponding to the same vertex v_i, we get

$$\begin{aligned} 0 &= \sum_{j \in J} \int_{e_j} \big(-\nu_j \partial^2 w_j + \partial_p H_j(x, \partial u_j) \partial w_j \big) m_j \, dx \\ &= \sum_{j \in J} \int_{e_j} \big[-\nu_j \partial^2 m_j - \partial(m_j \partial_p H_j(x, \partial u_j)) \big] w_j \, dx \\ &\quad - \sum_{v_i \in \mathcal{V}} \left[\sum_{j \in Inc_i} \nu_j m_j(v_i) \partial w_j(v_i) - \big(\nu_j \partial m_j(v_i) + \partial_p H(v_i, \partial u_j) m_j(v_i) \big) w_j(v_i) \right]. \end{aligned}$$

By the previous identity we obtain that m satisfies inside each edge e_j the adjoint equation

$$\nu_j \partial^2 m_j + \partial(m_j\, \partial_p H_j(x, \partial u_j)) = 0.$$

Moreover, recalling the Kirchhoff transition condition for w, the first one of the terms computed at the transition vertices vanishes if

$$\frac{m_j(v_i)}{\alpha_{ij}} = \frac{m_k(v_i)}{\alpha_{ik}}, \qquad j, k \in Inc_i,\ v_i \in \mathcal{V}. \tag{3.3}$$

The vanishing of the other term for each $v_i \in \mathcal{V}$, namely

$$\sum_{j \in Inc_i} \nu_j \partial m_j(v_i) + \partial_p H_j(v_i, \partial u_j) m_j(v_i) = 0, \tag{3.4}$$

gives the transition condition for m at the vertices $v_i \in \mathcal{V}$. Note that (3.4) corresponds to the conservation of the total flux of the density m at v_i.

We restrict for simplicity to the case in which all the coefficients in the transition condition for u are equal, i.e. $\alpha_{ij} = \alpha_{ik}\ \forall i \in I,\ j, k \in Inc_i$ and therefore (3.3) reduces to the continuity of m at the vertices. Summarizing we get the following MFG system

$$\begin{cases} -\nu_j \partial^2 u_j + H_j(x, \partial u_j) + \lambda = V_j[m], & x \in e_j,\ j \in J \\ \nu_j \partial^2 m_j + \partial(m_j\, \partial_p H_j(x, \partial u_j)) = 0 & x \in e_j,\ j \in J \\ \sum_{j \in Inc_i} \nu_j \partial u_j(v_i) = 0 & v_i \in \mathcal{V} \\ \sum_{j \in Inc_i} [\nu_j \partial m_j(v_i) + \partial_p H_j(v_i, \partial u_j) m_j(v_i)] = 0 & v_i \in \mathcal{V} \\ u_j(v_i) = u_k(v_i),\ m_j(v_i) = m_k(v_i) & j, k \in Inc_i,\ v_i \in \mathcal{V} \\ \int_\Gamma u(x)dx = 0,\ \int_\Gamma m(x)dx = 1,\ m \geq 0 \end{cases} \tag{3.5}$$

where the ergodic constant $\lambda \in \mathbb{R}$ is also an unknown of the problem. The transition conditions for u and m (continuity and either Kirchhoff condition or conservation of total flux, respectively) give d_{v_i} linear conditions for each function at a vertex $v_i \in \mathcal{V}$, hence they uniquely determine the values $u_j(v_i)$ and $m_j(v_i)$, $j \in Inc_i$.

For the stationary system (3.5) we have the following existence and uniqueness result [8] in the case of a local coupling $V[m](x) = V(m(x))$, $x \in \Gamma$.

Theorem 3.1 *Assume that* $H = \{H_j\}_{j\in J}$*,* $H_j : [0, l_j] \times \mathbb{R} \to \mathbb{R}$*,* $\nu = \{\nu_j\}_{j\in J}$*,* $\nu_j \in \mathbb{R}$*, and* $V = \{V_j\}_{j\in J}$*,* $V_j : \mathbb{R} \to \mathbb{R}$*, satisfy*

$H_j \in C^2([0, l_j] \times \mathbb{R})$;

$H_j(x, \cdot)$ *is convex in* p *for each* $x \in [0, l_j]$;

$\delta|p|^2 - C \le H_j(x, p) \le C|p|^2 + C$ *for* $(x, p) \in [0, l_j] \times \mathbb{R}$ *and some* $\delta, C > 0$,

$\nu_0 := \inf_{j\in J} \nu_j > 0$,

$V_j[m](x) = V_j(m(x))$ *with* $V_j \in C^1([0, +\infty))$ *and bounded.*

Then, there exists a solution $(u, m, \lambda) \in C^2(\Gamma) \times C^2(\Gamma) \times \mathbb{R}$ *to* (3.5)*. Moreover if*

$$\int_\Gamma (V(m_1) - V(m_2))(m_1 - m_2)dx \le 0 \Rightarrow m_1 = m_2,$$

then the solution is unique.

3.3 A Finite Difference Scheme for Mean Field Games on Networks

The differential equations in (3.5) are defined in terms of derivatives with respect to the coordinate $y = \pi_j^{-1}(x) \in [0, l_j]$ parametrizing the arc e_j. Hence the approximation scheme for the MFG system is obtained by discretizing this local coordinate.

Given a discretization step $h = \{h_j\}_{j\in J}$, we consider an uniform partition $y_{j,k} = kh_j$, $k = 0, \dots, N_j^h$, of the interval $[0, l_j]$ parametrizing the edge e_j (we assume that $N_j^h = l_j/h_j$ is an integer). We obtain a spatial grid on Γ by setting

$$\mathcal{G}_h = \{x_{j,k} = \pi_j(y_{j,k}),\ j \in J,\ k = 0, \dots, N_j^h\}.$$

In the notation $x_{j,k}$, the index j refers to the arc e_j, whereas the index k refers to the grid point on e_j. We set

$$|h| = \max_{j\in J}\{h_j\}, \quad N^h = \#(I) + \sum_{j\in J}(N_j^h - 1),$$

i.e. N^h is the total number of the points of $\mathcal{G}_h$ having identified, for each $i \in I$, the $\#(\mathrm{Inc}_i)$ grid points corresponding to the same vertex v_i. We make a partition of Inc_i

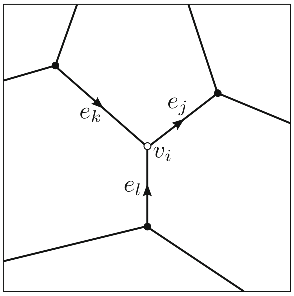


Fig. 1 Incident edges to the vertex v_i: $\mathrm{Inc}_i^+ = \{j\}$, $\mathrm{Inc}_i^- = \{k, l\}$

into the subsets

$$\mathrm{Inc}_i^+ = \{j \in \mathrm{Inc}_i \,:\, v_i = \pi_j(0)\}, \quad \mathrm{Inc}_i^- = \{j \in \mathrm{Inc}_i \,:\, v_i = \pi_j(N_j^h h_j)\},$$

as shown in Fig. 1.

For a grid function $U : \mathcal{G}_h \to \mathbb{R}$ we denote by $U_{j,k}$ its value at the grid point $x_{j,k}$. We say that a grid function $U : \mathcal{G}_h \to \mathbb{R}$ is said to be continuous at v_i if

$$U_{j,\ell} = U_{k,m} := U_i \quad \text{if } v_i = \pi_j(\ell h_j) = \pi_k(m h_k),\ j,k \in \mathrm{Inc}_i,\ \ell \in \{0, N_j^h\},\ \text{and } m \in \{0, N_k^h\},$$

i.e., the value of U at the vertex v_i is independent of the incident edge e_j, $j \in \mathrm{Inc}_i$. A grid function is continuous if it is continuous at v_i, for each $i \in I$.

Given a generic pair of grid functions $U, W : \mathcal{G}_h \to \mathbb{R}$, we define the scalar product

$$(U, W)_2 = \sum_{j \in J} \sum_{k=1}^{N_j^h - 1} h_j U_{j,k} W_{j,k} + \sum_{i \in I} \left(\sum_{j \in \mathrm{Inc}_i^+} \frac{h_j}{2} U_{j,0} W_{j,0} + \sum_{j \in \mathrm{Inc}_i^-} \frac{h_j}{2} U_{j,N_j^h} W_{j,N_j^h} \right).$$

and we introduce the compact and convex set

$$\mathcal{K}_h = \{(M_{j,k})_{j \in J,\, 0 \le k \le N_j^h} : M \text{ is continuous}, M_{j,k} \ge 0,\ (M, 1)_2 = 1\}.$$

We finally define the following finite difference operators

$$(D^+U)_{j,k} = \frac{U_{j,k+1} - U_{j,k}}{h_j},$$

$$[D_h U]_{j,k} = \big((D^+U)_{j,k}, (D^+U)_{j,k-1}\big)^T,$$

$$(D_h^2 U)_{j,k} = \frac{U_{j,k-1} - 2U_{j,k} + U_{j,k+1}}{h_j^2}.$$

In order to approximate the Hamiltonian, $H = \{H_j\}_{j\in J}$, $H_j : [0, l_j] \times \mathbb{R} \to \mathbb{R}$, $j \in J$, we consider a numerical Hamiltonian $g = \{g_j\}_{j\in J}$, $g_j : [0, l_j] \times \mathbb{R}^2 \to \mathbb{R}$, $(x, q_1, q_2) \to g_j(x, q_1, q_2)$ satisfying

(G1) *monotonicity: g_j is non increasing w.r.t. q_1 and nondecreasing w.r.t. q_2;*
(G2) *consistency: $g_j(x, q, q) = H_j(x, q)$ $\forall x \in [0, l_j]$, $\forall q \in \mathbb{R}$;*
(G3) *differentiability: g_j is of class C^1;*
(G4) *convexity: for all $x \in e_j$, $(q_1, q_2) \mapsto g_j(x, q_1, q_2)$ is convex.*

The operator $V[m](x_{j,k})$ is approximated by $V_h[M]_{j,k} = V[\mathcal{I}_h m](x_{j,k})$ where $\mathcal{I}_h m$ is the piecewise constant function taking the value $M_{j,k}$ in the interval $\{|y - y_{j,k}| \le h_j/2\}$, $k = 1, \dots, N_j^h - 1$, $j \in J$ (at the vertices only the half interval contained in $[0, l_j]$ is considered). In particular, if V is a local operator, i.e. $V[m](x) = F(m(x))$, then we set $V_h[M]_{j,k} = F(M_{j,k})$. We assume that

(V1) *V_h is continuous and maps $\mathcal{K}_h$ on a bounded set of grid functions.*
(V2) *V_h is monotone, i.e. $(V_h[M] - V_h[\bar{M}], M - \bar{M})_2 \le 0 \Rightarrow M = \bar{M}$.*

For the discretization of the differential equations in (3.5) inside the edge, we follow the same approach in [1] and we refer to Sect. 2 for motivations and explanations. We just recall that the approximation of the transport operator in the Fokker-Planck equation comes from the discretization of the quantity

$$\int_{e_j} m \frac{\partial H_j}{\partial p}(x, \partial u)\partial w \, dx$$

for a test function w, which is related to the weak formulation of the equation on the network. At the internal grid points we consider the finite difference system

$$\begin{cases} -\nu_j (D_h^2 U)_{j,k} + g_j(x_{j,k}, [D_h U]_{j,k}) + \Lambda = V_h[M]_{j,k}, & k = 1, \dots, N_j^h - 1, \ j \in J \\ \nu_j (D_h^2 M)_{j,k} + \mathcal{B}^h(U, M)_{j,k} = 0, & k = 1, \dots, N_j^h - 1, \ j \in J \\ M \in \mathcal{K}_h, \qquad (U, 1)_2 = 0, & \end{cases} \tag{3.6}$$

where U, M are grid functions and $\Lambda \in \mathbb{R}$. The transport operator $\mathcal{B}^h$ is defined for $j \in J$ and $k = 1$ by

$$\mathcal{B}^h(U, M)_{j,k} = \begin{array}{l} \frac{1}{h_j} \Big[M_{j,k} \frac{\partial g_j}{\partial q_1}(x_{j,k}, [D_h U]_{j,k}) + \\ M_{j,k+1} \frac{\partial g_j}{\partial q_2}(x_{j,k+1}, [D_h U]_{j,k+1}) - M_{j,k} \frac{\partial g_j}{\partial q_2}(x_{j,k}, [D_h U]_{j,k}) \Big]; \end{array}$$

for $k = 2, \dots, N_j^h - 2$ by

$$\mathcal{B}^h(U, M)_{j,k} = \begin{array}{l} \frac{1}{h_j} \Big[M_{j,k} \frac{\partial g_j}{\partial q_1}(x_{j,k}, [D_h U]_{j,k}) - M_{j,k-1} \frac{\partial g_j}{\partial q_1}(x_{j,k-1}, [D_h U]_{j,k-1}) \\ + M_{j,k+1} \frac{\partial g_j}{\partial q_2}(x_{j,k+1}, [D_h U]_{j,k+1}) - M_{j,k} \frac{\partial g_j}{\partial q_2}(x_{j,k}, [D_h U]_{j,k}) \Big]; \end{array}$$

for $k = N_j^h - 1$ by

$$\mathcal{B}^h(U,M)_{j,k} = \begin{array}{l} \frac{1}{h_j}\Big[M_{j,k}\frac{\partial g_j}{\partial q_1}(x_{j,k},[D_hU]_{j,k}) - M_{j,k-1}\frac{\partial g_j}{\partial q_1}(x_{j,k-1},[D_hU]_{j,k-1}) - \\ M_{j,k}\frac{\partial g_j}{\partial q_2}(x_{j,k},[D_hU]_{j,k})\Big]. \end{array}$$

For the approximation of the transition conditions in (3.5), we use a standard first order discretization of the normal derivative of u and m. In particular, we employ forward or backward finite differences depending on whether the vertex is, respectively, the initial or terminal point in the parametrization of the edge. The flux term in the Kirchhoff condition for m is approximated in a upwind fashion taking always into account the orientation of the edge. Moreover we impose the continuity at the vertices of U and M at the vertices so that the full set of discrete transition conditions is given by

$$\begin{cases} \mathcal{S}^h(U, V_h[M]-\Lambda)_i = 0, & i \in I, \\ \mathcal{T}^h(M,U)_i = 0 & i \in I, \\ U, M \text{ continuous at } v_i, & i \in I, \end{cases} \tag{3.7}$$

where,for every triple of grid functions U,V, M, the operators $\mathcal{S}^h : \mathcal{V} \to \mathbb{R}$ and $\mathcal{T}^h : \mathcal{V} \to \mathbb{R}$ are defined by

$$\mathcal{S}^h(U,V)_i = \sum_{j\in \mathrm{Inc}_i^+}\big[\nu_j(D^+U)_{j,0} + \frac{h_j}{2}V_{j,0}\big] - \sum_{j\in \mathrm{Inc}_i^-}\big[\nu_j(D^+U)_{j,N_j^h-1} - \frac{h_j}{2}V_{j,N_j^h}\big],$$

$$\begin{aligned} \mathcal{T}^h(M,U)_i = & \sum_{j\in \mathrm{Inc}_i^+}\big[\nu_j(D^+M)_{j,0} + M_{j,1}\frac{\partial g}{\partial q_2}(x_{j,1},[D_hU]_{j,1})\big] \\ & - \sum_{j\in \mathrm{Inc}_i^-}\big[\nu_j(D^+M)_{j,N_j^h-1} + M_{j,N_j^h-1}\frac{\partial g}{\partial q_1}(x_{j,N_j^h-1},[D_hU]_{j,N_j^h-1})\big] = 0. \end{aligned}$$

We observe that in the approximation of the Kirchhoff condition appears an additional term $\frac{h_j}{2}((V_h[M]) - \Lambda\big)$, vanishing for $h \to 0$. This term is necessary to obtain a fundamental identity analogous to (2.12), suitable for the network problem. Summarizing, the approximation scheme for the stationary problem (3.5) is given by (3.6)–(3.7).

3.4 Existence, Uniqueness and Convergence of the Numerical Scheme

Concerning existence and uniqueness of a solution to (3.6)–(3.7) we have the following result.

Theorem 3.2 *Assume that g satisfies **(G1)–(G3)**, V satisfies **(V1)**. Then the problem* (3.6)–(3.7) *has at least a solution* (U, M, Λ). *If moreover g satisfies **(G4)** and V satisfies **(V2)**, then the solution is unique.*

Existence is proved as in the continuous case by a fixed point argument. For the uniqueness we rely on a fundamental identity similar to (2.12) which is also a crucial tool to prove the convergence of the scheme.

We describe a convergence result for the scheme (3.6)–(3.7) in the reference case

$$H(x, p) = |p|^{\beta} + f(x), \tag{3.8}$$

where $\beta \geq 2$ and $f : \Gamma \to \mathbb{R}$ is a continuous function. We consider a numerical Hamiltonian of the form

$$g(x, p) = G(p_1^-, p_2^+) + f(x) \tag{3.9}$$

where $G(p_1, p_2) = (p_1^2 + p_2^2)^{\beta/2}$ and $p^{\pm}$ denote the positive and negative part of $p \in \mathbb{R}$. We observe that g satisfies assumptions **(G1)–(G4)**.

Theorem 3.3 *Assume* (3.8)*, V is a local* C^1 *potential and g of the form* (3.9)*. Let* (u, m, λ) *be the unique solution of* (3.5) *and let* u^h *(resp.* m^h*) be the piecewise linear function on* Γ *obtained by interpolating the values* $U^h_{j,k}$ *(resp* $M^h_{j,k}$*) of the solution* (U^h, M^h, Λ^h) *to* (3.6) *and* (3.7) *at the nodes of the network grid. Then*

$$\lim_{|h|\to 0} \|u^h - u\|_\infty + \|m^h - m\|_\infty + |\Lambda^h - \lambda| = 0.$$

4 A Quasi-Newton Method for Stationary Mean Field Games

This section is devoted to the actual implementation and test of a numerical solver for stationary MFG systems, both in the Euclidean and Network cases introduced in (1.1) and (3.5) respectively. The main issue from an implementation point of view is that these systems are strongly coupled and, more important, they involve the ergodic constant λ as an additional unknown. A standard way in the literature to overcome this issue is a regularization technique, which is an effective tool both for theoretical and numerical results. The ergodic constant λ is replaced by the zero order term δu_δ (where $\delta > 0$ is a small parameter) or by the time derivative $\frac{\partial u}{\partial t}$ (associated to an initial datum), yielding to well posed problems. Indeed, it can be proved that both $-\delta u_\delta$ and $-\frac{u(\cdot,t)}{t}$ converge uniformly to λ as $\delta \to 0$ and $t \to \infty$ respectively, whereas u_δ and $u(\cdot,t) - \lambda t$ converge to a solution of the original stationary equation. Unfortunately, this procedure introduces an additional approximation in the computation, affecting the accuracy of the corresponding numerical solutions and the computational time to reach convergence.

Here we review a new method that we introduced in [7] for stationary MFG systems on networks and extended in [6] to very general homogenization problems for Hamilton-Jacobi equations. The main novelty is that the discrete stationary MFG system is solved *directly* without any further (small δ or long time t) approximation, treating the ergodic constant λ as it is, an additional unknown.

To avoid cumbersome notations and focus only on the main idea, we keep the discussion at an abstract level. In particular, we no further distinguish between the Euclidean case (1.1) and the Network case (3.5), we only assume that, after the discretization, we end up with a generic lattice of N nodes. We collect all the unknowns in a single vector $X = (U, M, \Lambda)$, whose length turns out to be $2N + 1$. On the other hand, we recast the $2N$ equations in the system (including the transition conditions in the network case) plus the 2 normalization conditions as functions of X with zero right hand sides, obtaining a nonlinear map $F : \mathbb{R}^{2N+1} \to \mathbb{R}^{2N+2}$. The problem is then reduced to

$$\text{Find } X \in \mathbb{R}^{2N+1} \text{ such that } F(X) = 0 \in \mathbb{R}^{2N+2}. \tag{4.1}$$

Note that a zero of F exists and it is unique under the assumptions discussed in the previous sections, but the problem (4.1) is formally *overdetermined*, adopting, with a slight abuse, a terminology usually devoted to linear systems. Hence, the solution to (4.1) should be meant in a *least-squares* sense, namely as a solution of the following optimization problem (where we denote by $\|\cdot\|_2$ the Euclidean norm in $\mathbb{R}^{2N+2}$):

$$\min_{X \in \mathbb{R}^{2N+1}} \frac{1}{2}\|F(X)\|_2^2 .$$

Assuming smoothness of F and defining $\mathcal{F}(X) := \frac{1}{2}\|F(X)\|_2^2$, the classical Newton method for finding critical points of $\mathcal{F}$ is given by

$$\mathcal{H}_{\mathcal{F}}(X^k)(X^{k+1} - X^k) = -\nabla\mathcal{F}(X^k) \qquad k \geq 0 .$$

Computing the gradient $\nabla\mathcal{F}$ and the Hessian $\mathcal{H}_{\mathcal{F}}$ of $\mathcal{F}$ we have

$$\nabla\mathcal{F}(X) = J_F(X)^T F(X) ,$$

$$\mathcal{H}_{\mathcal{F}}(X) = J_F(X)^T J_F(X) + \sum_{i=1}^{2N+2} \frac{\partial^2 F_i}{\partial X^2}(X) F_i(X) ,$$

where the second order term is given by

$$\left(\frac{\partial^2 F_i}{\partial^2 X}(X)\right)_{k,\ell} = \frac{\partial^2 F_i}{\partial X_k \partial X_\ell}(X)\,.$$

Since a solution to the MFG system corresponds to a zero minimum of $\mathcal{F}(X)$, we expect $F(X^k)$ to be small for X^k close enough to a solution. Hence we approximate $\mathcal{H}_{\mathcal{F}}(X^k) \simeq J_F(X^k)^T J_F(X^k)$ and obtain the so called Gauss-Newton method, which requires only Fréchet differentiability of F:

$$J_F(X^k)^T J_F(X^k)\left(X^{k+1} - X^k\right) = -J_F(X^k)^T F(X^k) \qquad k \geq 0\,.$$

From a computational point of view, the presence of the transposed Jacobian restores the square size of the system, but it also squares its condition number, a crucial point for the approximation as N increases.

We proceed in an alternative way, by simply observing that, for $\delta := X^{k+1} - X^k$, the Gauss-Newton step above is just the optimality condition for the following linear least-squares problem:

$$\min_{\delta \in \mathbb{R}^{2N+1}} \frac{1}{2}\|J_F(X^k)\delta + F(X^k)\|_2^2\,, \tag{4.2}$$

which is in turn easily and efficiently solved by means of the *QR factorization* of J_F. Indeed, let $n = 2N + 1$ and suppose that $J_F(X^k) = QR$, where Q is a $(n + 1) \times (n + 1)$ orthogonal matrix (i.e. $Q^{-1} = Q^T$) and R is a $(n + 1) \times n$ matrix of the form $R = \begin{pmatrix} R_1 \\ 0 \end{pmatrix}$, with R_1 of size $n \times n$ and upper triangular. Writing $Q = (Q_1 \quad Q_2)$ with Q_1 of size $(n + 1) \times n$ and Q_2 of size $(n + 1) \times 1$, we get

$$\|J_F(X^k)\delta + F(X^k)\|_2^2 = \|Q^T\left(J_F(X^k)\delta + F(X^k)\right)\|_2^2 = \|Q^T QR\delta + Q^T F(X^k)\|_2^2 =$$

$$= \left\|\begin{pmatrix} R_1\delta \\ 0 \end{pmatrix} + \begin{pmatrix} Q_1^T F(X^k) \\ Q_2^T F(X^k) \end{pmatrix}\right\|_2^2 = \|R_1\delta + Q_1^T F(X^k)\|_2^2 + \|Q_2^T F(X^k)\|_2^2$$

which is finally minimized by getting rid of the first of the two latter terms, i.e. solving the square triangular $n \times n$ linear system $R_1\delta = -Q_1^T F(X^k)$ via back substitution.

Summarizing, we propose the following algorithm:

GIVEN AN INITIAL GUESS X AND A TOLERANCE $\varepsilon > 0$,
REPEAT

1. ASSEMBLE $F(X)$ AND $J_F(X)$
2. SOLVE THE LINEAR SYSTEM $J_F(X)\delta = -F(X)$ IN THE LEAST-SQUARES SENSE, USING THE QR FACTORIZATION OF $J_F(X)$

3. UPDATE $X \leftarrow X + \delta$

UNTIL $\|\delta\|_2^2 < \varepsilon$ AND/OR $\|F(X)\|_2^2 < \varepsilon$

We refer the interested reader to [6] and [7] for implementation details, performance tests of the proposed algorithm and a comparison with existing methods. In the remaining sections we present some simulations in different settings, showing the versatility of the new method to catch interesting features of the corresponding problems.

4.1 MFG in Euclidean Spaces

We consider a MFG system in dimension two, with an eikonal Hamiltonian, a cost function f and a local potential V, namely

$$\begin{cases} -\nu\Delta u + |Du|^2 + f(x) + \lambda = V(m) & x \in \mathbb{T}^2 \\ \nu\Delta m + 2\,\mathrm{div}(m\,Du) = 0 & x \in \mathbb{T}^2 \\ \int_{\mathbb{T}^2} u(x)dx = 0,\ \int_{\mathbb{T}^2} m(x)dx = 1\,, \end{cases}$$

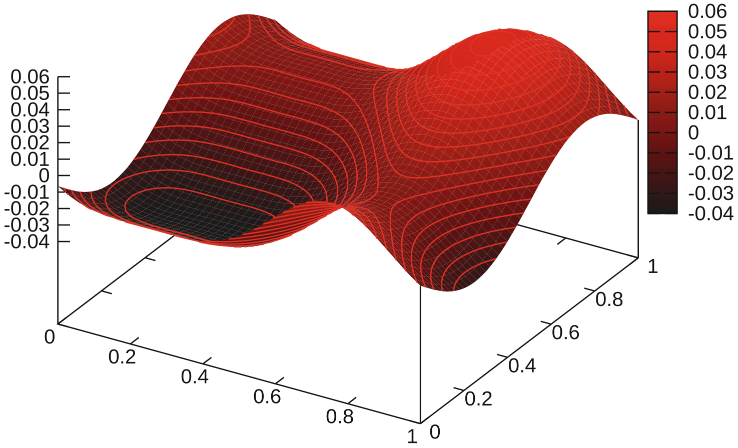
with $f(x) = \sin(2\pi x_1) + \cos(4\pi x_1) + \sin(2\pi x_2)$ and $V(m(x)) = m^2(x)$. We discretize the torus $\mathbb{T}^2$ with $N = 2500$ uniformly distributed nodes, so that the size of the system is 5002×5001, corresponding to 2500 degrees of freedom for U, 2500 for M and 1 for Λ. We choose $U \equiv 0$, $M \equiv 1$ and $\Lambda = 0$ as initial guess for the Newton's method and we set to $\varepsilon = 10^{-6}$ the tolerance for the stopping criterion of the algorithm.

In the first test we set the diffusion coefficient $\nu = 1$. In Fig. 2, we show the surfaces and the level sets of the computed pair of solutions (U, M).

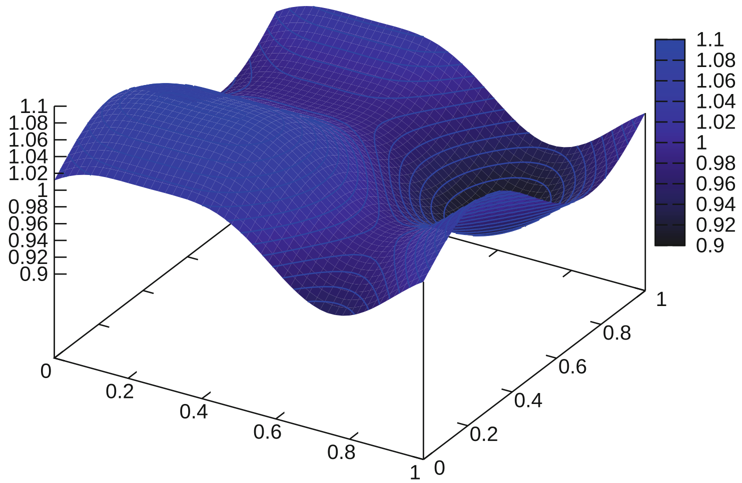
The convergence is fast, just five iterations in 8.06 s and we get $\Lambda = 0.9784$. Moreover, we observe the typical "dual" behavior of the solutions, namely the fact that the local maxima of the mass distribution M correspond to the local minima of the value function U and vice versa. Note that, due to the high diffusion, the mass density is well distributed on the whole domain.

On the contrary, we can push the diffusion close to the deterministic limit, repeating the test with $\nu = 0.01$ to enhance concentration. We reach convergence in 10.72 s with 21 iterations and we get $\Lambda = 1.1878$. In Fig. 3, we show the surfaces and the level sets of the computed pair of solutions (U, M).

We clearly see how the supports of U and M are almost disjoint and that the mass distribution tries to occupy all the region corresponding to the minimum of the value function.



(a)



(b)

Fig. 2 Surfaces and level sets of the solutions U (**a**) and M (**b**)

4.2 Multi-Population MFG in Euclidean Spaces

This is a generalization of (1.1) to the case of P competing populations, each one described by a MFG-system, coupled via a potential term (see [15]). Here we consider the setting recently studied in [10] for problems with Neumann boundary conditions, and we present the case in dimension one and two of an eikonal

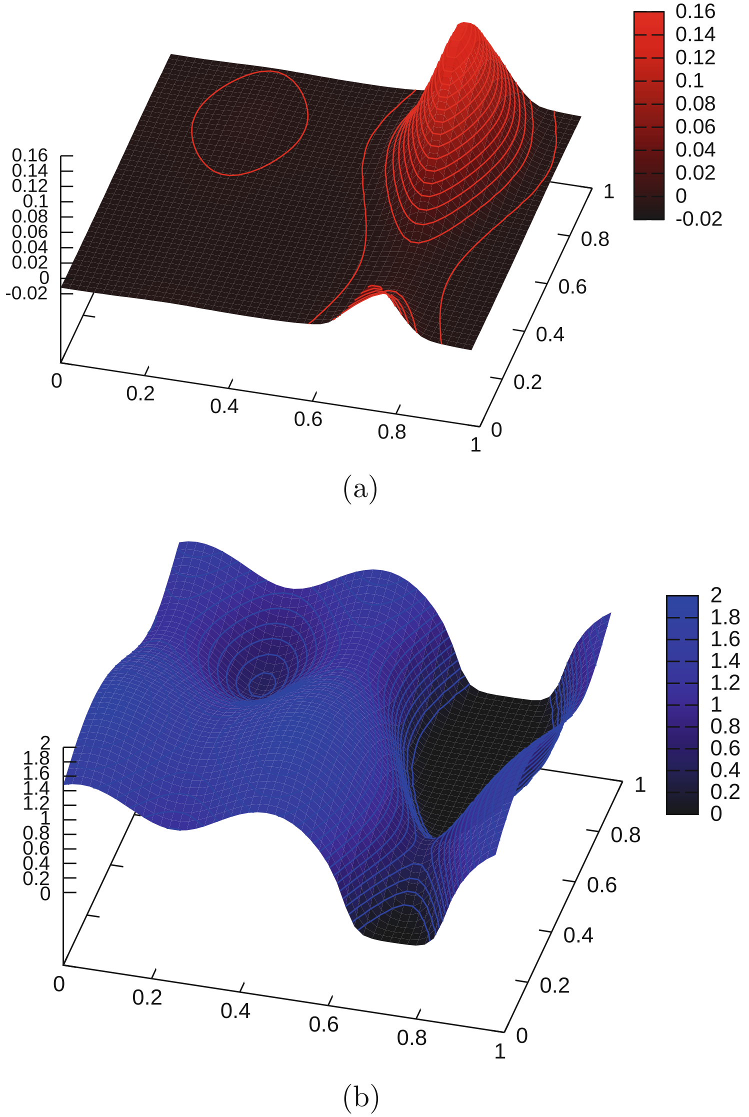


Fig. 3 Surfaces and level sets of the solutions U (**a**) and M (**b**)

Hamiltonian with a linear local potential, namely the problem

$$\begin{cases} -\nu\Delta u_i + |Du_i|^2 + \lambda_i = V_i(m) & \text{in } \Omega\,, \quad i = 1, \dots, P \\ \nu\Delta m_i + 2\mathrm{div}(m_i\, Du_i) = 0 & \text{in } \Omega\,, \quad i = 1, \dots, P \\ \partial_n u_i = 0\,, \quad \partial_n m_i = 0 & \text{on } \partial\Omega\,, \quad i = 1, \dots, P \\ \int_\Omega u_i(x)dx = 0\,, \quad \int_\Omega m_i(x)dx = 1\,,\; i = 1, \dots, P\,, \end{cases}$$

where $\Omega = [0, 1]$ or $\Omega = [0, 1]^2$, the value function $u(x) = (u_1(x), \dots, u_P(x))$ and the mass distribution $m(x) = (m_1(x), \dots, m_P(x))$ are vector functions and $\lambda = (\lambda_1, \dots, \lambda_P) \in \mathbb{R}^P$ is a P-tuple of ergodic constants. Moreover, for $i = 1, \dots, P$, the linear local potential V_i takes the form

$$V_i(m(x)) = \sum_{j=1}^{P} \theta_{ij} m_j(x) ,$$

for some given weights $\theta_{ij} \in \mathbb{R}$, or in matrix notation

$$V = (V_1, \dots, V_P) , \qquad \Theta = (\theta_{ij})_{i,j=1,\dots,P} , \qquad V(m) = \Theta m . \tag{4.3}$$

Existence and uniqueness of a solution (u, m, λ) can be proved under suitable monotonicity assumptions on V (see [10] for details).

Note that this problem is even more *overdetermined* than the previous one. Indeed, discretizing Ω with a uniform grid of N nodes, we end up with $P(2N + 2)$ equations in the $P(2N + 1)$ unknowns (U, M, Λ).

In the special case (4.3) uniqueness is guaranteed assuming that Θ is positive semi-definite and the solution is explicitly given, for $i = 1, \dots, P$, by $u_i \equiv 0$, $m_i \equiv 1$ and $\lambda_i = \sum_{j=1}^{P} \theta_{ij}$. By dropping this condition, the trivial solution is still found, but we expect to observe other more interesting solutions.

We start with some experiments in dimension one, in the case of $P = 2$ populations. We choose the coupling matrix (not positive semi-definite)

$$\Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so that the potential for each population only depends on the other population. Moreover, we discretize the interval $\Omega = [0, 1]$ with $N = 100$ uniformly distributed nodes, we set to $\nu = 0.05$ the diffusion coefficient and to $\varepsilon = 10^{-6}$ the tolerance for the stopping criterion of the algorithm. To avoid the trivial solution, we choose non constant initial guesses, such as piecewise constant pairs with zero mean for U and piecewise constant pairs with mass one for M. Figure 4 shows four computed solutions. In the top panels we show the mass distribution $M = (M_1, M_2)$, while in the bottom panels the corresponding value function $U = (U_1, U_2)$. Segregation of the two populations is expected (see [10]) and clearly visible. This phenomenon can be enhanced by reducing the diffusion coefficient, as shown in Fig. 5, where $\nu = 10^{-4}$, close to the deterministic limit.

We finally consider the more complex and suggestive two dimensional case. We discretize the square $\Omega = [0, 1]^2$ with 25×25 uniformly distributed nodes and we push the diffusion ν up to 10^{-6}, in order to observe segregation among the populations. Moreover, we choose the interaction matrix as before, with all the entries equal to 1 except for the diagonal, which is set to 0.

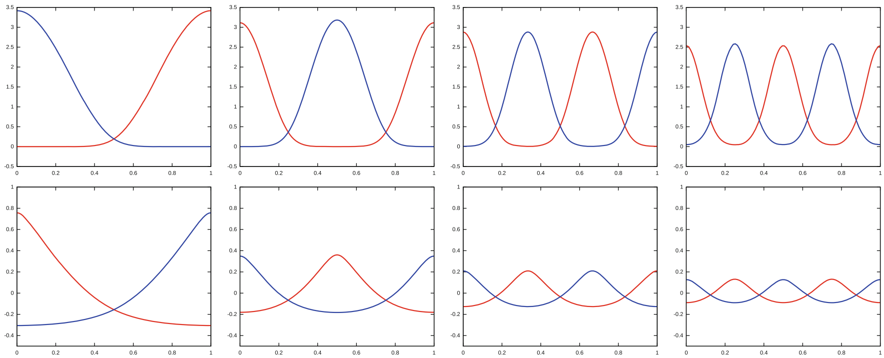


Fig. 4 Two-population MFG solutions ($\nu = 0.05$): mass distribution $M = (M_1, M_2)$ (top panels) and corresponding value function $U = (U_1, U_2)$ (bottom panels)

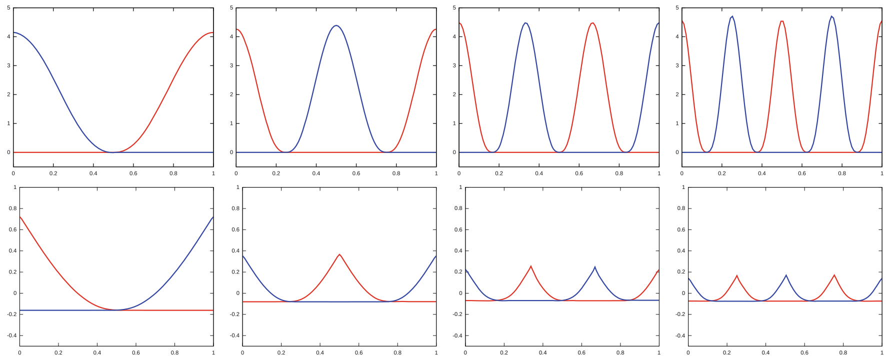


Fig. 5 Two-population MFG solutions ($\nu = 10^{-4}$): mass distribution $M = (M_1, M_2)$ (top panels) and corresponding value function $U = (U_1, U_2)$ (bottom panels)

Figure 6 shows a rich collection of solutions, corresponding to $P = 2$ (top panels), $P = 3$ (middle panels) and $P = 4$ (bottom panels) populations for different initial guesses of the Newton's method. We clearly see how the populations compete to share out all the domain.

4.3 MFG on Networks

Here we show the ability of the proposed method to handle problems on quite complex structures. To this end, we consider a MFG system on a network without boundary, in the special case of an eikonal Hamiltonian, a cost function f and a local potential V, i.e.

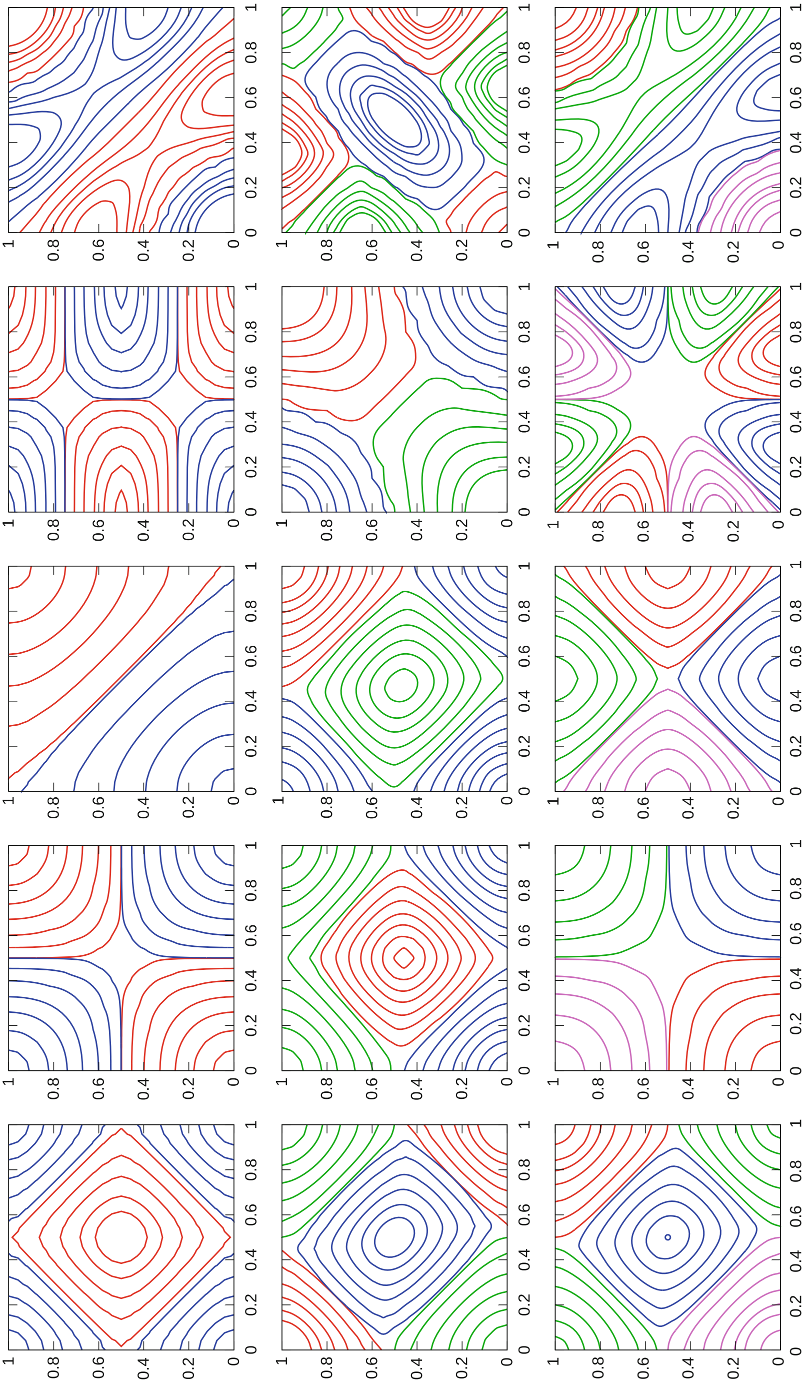


Fig. 6 Multi-population MFG mass distributions ($P = 2, 3, 4$)

$$\begin{cases} -\nu\partial^2 u + |\partial u|^2 + f + \lambda = V[m] & \text{in } \Gamma \\ \nu\partial^2 m + 2\partial(m\,\partial u) = 0 & \text{in } \Gamma \\ \displaystyle\sum_{j\in Inc_i} \nu_j \partial u_j(v_i) = 0 & v_i \in \mathcal{V} \\ \displaystyle\sum_{j\in Inc_i} [\nu_j \partial m_j(v_i) + 2\partial u_j(v_i) m_j(v_i)] = 0 & v_i \in \mathcal{V} \\ u_j(v_i) = u_k(v_i),\ m_j(v_i) = m_k(v_i) & j,k \in Inc_i,\ v_i \in \mathcal{V} \\ \displaystyle\int_\Gamma u(x)dx = 0, \quad \int_\Gamma m(x)dx = 1\,, \end{cases}$$

where $V[m] = m^2$, $\nu = 0.1$, the network Γ is shown in Fig. 7a and the cost f is the restriction to Γ of the function $\min\{|x-(3.5,2.5)|, 1\}$ for $x \in \mathbb{R}^2$, shown in Fig. 7b.

The network consists in 26 vertices and 44 edges, each one uniformly discretized with 50 nodes, yielding a system with 4365 degrees of freedom. We set to $\varepsilon = 10^{-6}$ the tolerance for the stopping criterion of the algorithm. Note that the cost f is maximal ($\equiv 1$) outside of the ball centered in the point $(3.5, 2.5)$ with radius 1 where, as in the Euclidean case, we expect the value function u to attain its minimum and the mass m to be well distributed. This is what is observed in Fig. 8, showing the pair of computed solutions. The algorithm reaches convergence in 14 s after 15 iterations.

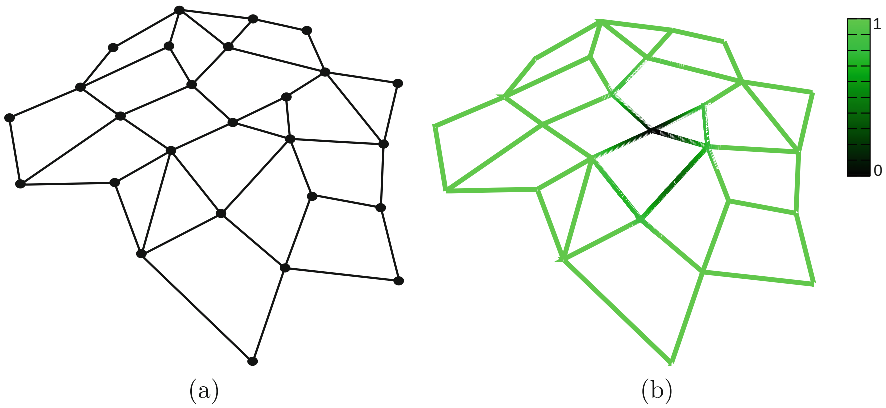


Fig. 7 The network Γ (**a**) and the cost function f (**b**)

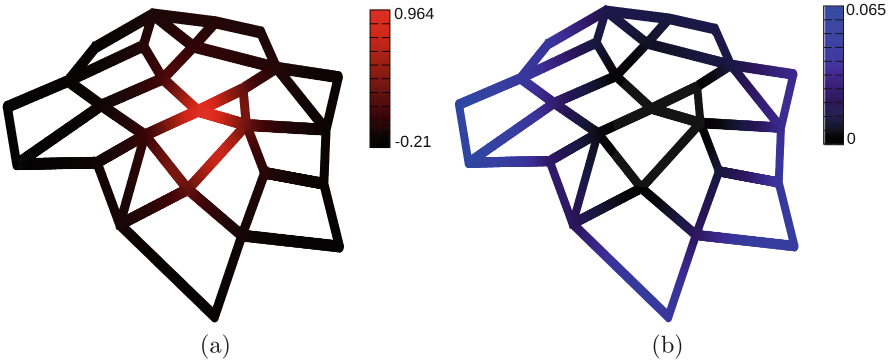


Fig. 8 The value function u (**a**) and the mass distribution m (**b**)

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Existence and Uniqueness for Mean Field Games with State Constraints

Piermarco Cannarsa and Rossana Capuani

Abstract In this paper, we study deterministic mean field games for agents who operate in a bounded domain. In this case, the existence and uniqueness of Nash equilibria cannot be deduced as for unrestricted state space because, for a large set of initial conditions, the uniqueness of the solution to the associated minimization problem is no longer guaranteed. We attack the problem by interpreting equilibria as measures in a space of arcs. In such a relaxed environment the existence of solutions follows by set-valued fixed point arguments. Then, we give a uniqueness result for such equilibria under a classical monotonicity assumption.

Keywords Mean field games · Nash equilibrium · State constraints · Hamilton-Jacobi-Bellman equations

MSC Subject Classifications 49J15, 49J30, 49J53, 49N90

1 Introduction

Mean field games (MFG) theory has been introduced simultaneously by Lasry and Lions [11–13] and by Huang, Malhamé and Caine [8, 9] in order to study large population differential games. The main idea of such a theory is to borrow from statistical physics the general principle of a mean-field approach to describe equilibria in a system of many interacting particles.

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In game theory, for a system with a finite number of players, the natural notion of equilibrium is the one introduced by John Nash. So, the notion of mean-field equilibrium suggested by Lasry-Lions is justified as being the limit, as $N \to \infty$, of the Nash equilibria for N-player games, under the assumption that players are symmetric and rational.

In deterministic settings, the equilibrium found in the mean field limit turns out to be a solution of the forward-backward system of PDEs

$$(MFG) \begin{cases} -\partial_t u + H(x, Du) = F(x, m) & \text{in } [0, T] \times \Omega, \\ \partial_t m - div(m D_p H(x, Du)) = 0 & \text{in } [0, T] \times \Omega, \\ m(0) = m_0 \quad u(x, T) = G(x, m(T)) \end{cases} \tag{1.1}$$

which couples a Hamilton-Jacobi-Bellman equation (for the value function u of the generic player) with a continuity equation (for the density m of players). Here $\Omega \subset \mathbb{R}^n$ represents the domain in the state space in which agents are supposed to operate.

The well-posedness of system (1.1) was developed for special geometries of the domain Ω, namely when Ω equals the flat torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, or the whole space $\mathbb{R}^n$ (see, e.g., [5, 12, 13]). The goal of the present paper is to study the well-posedness of the MFG problem subject to state constraints, that is, when players are confined into a compact domain $\overline{\Omega} \subseteq \mathbb{R}^n$.

In the above references, the solution of (1.1) on $[0, T] \times \mathbb{T}^n$ is obtained by a fixed point argument which uses in an essential way the fact that viscosity solutions of the Hamilton-Jacobi equation

$$-\partial_t u + H(x, Du) = F(x, m) \quad \text{in } [0, T] \times \mathbb{T}^n$$

are smooth on a sufficiently large set to allow the continuity equation

$$\partial_t m - div(m D_p H(x, Du)) = 0 \quad \text{in } [0, T] \times \mathbb{T}^n$$

to be solvable. Specifically, it is known that u is of class $C^{1,1}_{loc}$ outside a closed singular set of zero Lebesgue measure. In this way, the coefficient $D_p H(x, Du)$ in the continuity equation turns out to be locally Lipschitz continuous on a "sufficiently large" open set. Such an "almost smooth" structure is lost in the presence of state constraints [6, Example 1.1]. Therefore, in order to prove the existence of solutions to (1.1) a complete change of paradigm is necessary.

In this paper, following the Lagrangian formulation of the unconstrained MFG problem proposed in [3], we define a "relaxed" notion of constrained MFG equilibria and solutions, for which we give existence and uniqueness results. Such a formulation consists of replacing probability measures on $\overline{\Omega}$ with measures on arcs in $\overline{\Omega}$. More precisely, on the metric space

$$\Gamma = \left\{ \gamma \in AC(0, T; \mathbb{R}^n) : \gamma(t) \in \overline{\Omega}, \quad \forall t \in [0, T] \right\}$$

with the uniform metric, for any $t \in [0, T]$ we consider the evaluation map $e_t : \Gamma \to \overline{\Omega}$ defined by

$$e_t(\gamma) = \gamma(t) \qquad (\gamma \in \Gamma).$$

Given any probability measure m_0 on $\overline{\Omega}$, we denote by $\mathcal{P}_{m_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $e_0 \sharp \eta = m_0$ and we consider, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, the functional

$$J_\eta[\gamma] = \int_0^T \Big[L(\gamma(t), \dot\gamma(t)) + F(\gamma(t), e_t \sharp \eta) \Big] \, dt + G(\gamma(T), e_T \sharp \eta) \qquad (\gamma \in \Gamma).$$

Then, we call a measure $\eta \in \mathcal{P}_{m_0}(\Gamma)$ a *constrained MFG equilibrium* for m_0 if η is supported on the set of all curves $\overline{\gamma} \in \Gamma$ such that

$$J_\eta[\overline{\gamma}] \le J_\eta[\gamma] \ \ \forall \gamma \in \Gamma, \ \gamma(0) = \overline{\gamma}(0).$$

Thus, we obtain the existence of constrained MFG equilibria for m_0 (Theorem 3.1) by applying the Kakutani fixed point theorem [10]. At this point, it is natural to define a *mild solution of the constrained MFG problem* in $\overline{\Omega}$ as a pair $(u, m) \in C([0, T] \times \overline{\Omega}) \times C([0, T]; \mathcal{P}(\overline{\Omega}))$, where m is given by $m(t) = e_t \sharp \eta$ for some constrained MFG equilibrium η for m_0 and

$$u(t, x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t) = x}} \left\{ \int_t^T \Big[L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m(s)) \Big] \, ds + G(\gamma(T), m(T)) \right\}.$$

In this way, the existence of mild solutions of the constrained MFG problem in $\overline{\Omega}$ (Corollary 4.1) becomes an easy corollary of the existence of equilibria for m_0 (Theorem 3.1), whereas the uniqueness issue for such a problem remains a more challenging question. As observed by Lasry and Lions, in absence of state constraints uniqueness can be addressed by imposing suitable monotonicity assumptions on the data. We show that the same general strategy can be adopted even for constrained problems (Theorem 4.1). However, we have to interpret the Lasry-Lions method differently because, as recalled above, solutions are highly nonsmooth in our case.

The results of this paper can be regarded as an initial step of the study of deterministic MFG systems with state constraints. The natural sequel of our analysis would be to show that mild solutions to the constrained MFG problem in $\overline{\Omega}$ satisfy the MFG system in a suitable point-wise sense and, possibly, derive the uniqueness of solutions from such a system.

This paper is organised as follows. In Sect. 2, we introduce the notation and recall preliminary results. In Sect. 3, we define constrained MFG equilibria and we prove their existence. Section 4 is devoted to the study of mild solutions of the constrained MFG problem, in particular to the uniqueness issue.

2 Preliminaries

2.1 Notation

Throughout this paper we denote by $|\cdot|$, $\langle\cdot\rangle$, respectively, the Euclidean norm and scalar product in $\mathbb{R}^n$. For any subset $S \subset \mathbb{R}^n$, $\overline{S}$ stands for its closure, ∂S for its boundary and $S^c = \mathbb{R}^n \setminus S$ for the complement of S. We denote by $\mathbf{1}_S : \mathbb{R}^n \to \{0, 1\}$ the characteristic function of S, i.e.,

$$\mathbf{1}_S(x) = \begin{cases} 1 & x \in S, \\ 0 & x \in S^c. \end{cases}$$

We write $AC(0, T; \mathbb{R}^n)$ for the space of all absolutely continuous $\mathbb{R}^n$-valued functions on $[0, T]$, equipped with the uniform metric. We observe that such a space is not complete.

For any measurable function $f : [0, T] \to \mathbb{R}^n$, we set

$$||f||_2 = \left(\int_0^T |f|^2\, dt\right)^{\frac{1}{2}}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^2$ boundary. The distance function from $\overline{\Omega}$ is the function $d_\Omega : \mathbb{R}^n \to [0, +\infty[$ defined by

$$d_\Omega(x) := \inf_{y \in \overline{\Omega}} |x - y| \quad (x \in \mathbb{R}^n).$$

We define the oriented boundary distance from $\partial\Omega$ by

$$b_\Omega(x) = d_\Omega(x) - d_{\Omega^c}(x) \quad (x \in \mathbb{R}^n). \tag{2.1}$$

We recall that, since the boundary of Ω is of class $\mathcal{C}^2$, there exists $\rho_0 > 0$ such that

$$b_\Omega(\cdot) \in C_b^2 \ \text{ on } \ \partial\Omega + B_{\rho_0} = \Big\{ y \in B(x, \rho_0) : x \in \partial\Omega \Big\}, \tag{2.2}$$

where C_b^2 is the set of all functions with bounded derivates of first and second order. Throughout the paper, we suppose that ρ_0 is fixed so that (2.2) holds.

2.2 Results from Measure Theory

In this section we introduce, without proof, some basic tools needed in the paper (see, e.g., [1]).

Let X be a separable metric space, we denote by $\mathscr{B}(X)$ the family of the Borel subset of X and by $\mathcal{P}(X)$ the family of all Borel probability measures on X. The support of $\mu \in \mathcal{P}(X)$, $supp(\mu)$, is the closed set defined by

$$supp(\mu) := \Big\{x \in X : \mu(V) > 0 \text{ for each neighborhood V of } x\Big\}. \tag{2.3}$$

We say that a sequence $(\mu_n) \subset \mathcal{P}(X)$ is narrowly convergent to $\mu \in \mathcal{P}(X)$ if

$$\lim_{n\to\infty} \int_X f(x)\, d\mu_n(x) = \int_X f(x)\, d\mu(x) \quad \forall f \in C_b^0(X),$$

where $C_b^0(X)$ is the set of all bounded continuous functions on X.

We recall an interesting link between narrow convergence of probability measures and Kuratowski convergence of their supports.

Proposition 2.1 *If $(\mu_n) \subset \mathcal{P}(X)$ is a sequence narrowly converging to $\mu \in \mathcal{P}(X)$ then $supp(\mu) \subset K - \liminf_{n\to\infty} supp(\mu_n)$, i.e.*

$$\forall x \in supp(\mu)\ \exists\, x_n \in supp(\mu_n) : \lim_{n\to\infty} x_n = x.$$

The following theorem is a useful characterization of relatively compact sets with respect to narrow topology.

Theorem 2.1 (Prokhorov's Theorem) *If a set $\mathcal{K} \subset \mathcal{P}(X)$ is tight, i.e.*

$$\forall \epsilon > 0\ \exists\, K_\epsilon \ compact\ in\ X\ such\ that\ \widehat{\eta}(K_\epsilon) \geq 1 - \epsilon \ \ \forall \widehat{\eta} \in \mathcal{K},$$

then $\mathcal{K}$ is relatively compact in $\mathcal{P}(X)$ with respect to narrow topology. Conversely, if X is a separable complete metric space then every relatively compact subset of $\mathcal{P}(X)$ is tight.

Let X be a separable metric space. We recall that X is a Radon space if every Borel probability measure $\mu \in \mathcal{P}(X)$ satisfies

$$\forall B \in \mathscr{B}(X), \forall \epsilon > 0,\ \exists K_\epsilon \text{ compact with } K_\epsilon \Subset B \text{ such that } \mu(B \setminus K_\epsilon) \leq \epsilon.$$

Let us denote by d the distance on X and, for $p \in [1, +\infty)$, by $\mathcal{P}_p(X)$ the set of probability measures m on X such that

$$\int_X d^p(x_0, x)\, dm(x) < +\infty, \quad \forall x_0 \in X.$$

The *Monge-Kantorowich distance* on $\mathcal{P}_p(X)$ is given by

$$d_p(m, m') = \inf_{\lambda \in \Pi(m,m')} \left[\int_{X^2} d(x, y)^p\, d\lambda(x, y)\right]^{1/p}, \tag{2.4}$$

where $\Pi(m, m')$ is the set of Borel probability measures on $X \times X$ such that $\lambda(A \times \mathbb{R}^n) = m(A)$ and $\lambda(\mathbb{R}^n \times A) = m'(A)$ for any Borel set $A \subset X$. In the particular case when $p = 1$, the distance d_p takes the name of Kantorovich-Rubinstein distance and the following formula holds

$$d_1(m, m') = sup\Big\{ \int_X f(x)\, dm(x) - \int_X f(x)\, dm'(x) \mid f : X \to \mathbb{R} \text{ is 1-Lipschitz}\Big\}, \tag{2.5}$$

for all $m, m' \in \mathcal{P}_1(X)$. In the next result, we recall the relationship between the weak-$*$ convergence of measures and convergence with respect to d_p.

Proposition 2.2 *If a sequence of measures $\{\mu_n\}_{n\geq 1} \subset \mathcal{P}_p(X)$ converges to μ for d_p, then $\{\mu_n\}_{n\geq 1}$ weakly converges to μ. "Conversely", if μ_n is concentrated on a fixed compact subset of X for all $n \geq 1$ and $\{\mu_n\}_{n\geq 1}$ weakly converges to μ, then the $\{\mu_n\}_{n\geq 1}$ converges to μ in d_p.*

Given separable metric spaces X_1 and X_2 and a Borel map $f : X_1 \to X_2$, we recall that the push-forward of a measure $\mu \in \mathcal{P}(X_1)$ through f is defined by

$$f\sharp\mu(B) := \mu(f^{-1}(B)) \quad \forall B \in \mathscr{B}(X_2). \tag{2.6}$$

The push-forward is characterized by the fact that

$$\int_{X_1} g(f(x))\, d\mu(x) = \int_{X_2} g(y)\, df\sharp\mu(y) \tag{2.7}$$

for every Borel function $g : X_2 \to \mathbb{R}$.

We conclude this preliminary session by recalling the disintegration theorem.

Theorem 2.2 *Let X, Y be Radon separable metric spaces, $\mu \in \mathcal{P}(X)$, let $\pi : X \to Y$ be a Borel map and let $\eta = \pi\sharp\mu \in \mathcal{P}(Y)$. Then there exists an η-a.e. uniquely determined Borel measurable family*[1] *of probabilities $\{\mu_y\}_{y\in Y} \subset \mathcal{P}(X)$ such that*

$$\mu_y(X \setminus \pi^{-1}(y)) = 0 \ \textit{for } \eta\textit{-a.e. } y \in Y \tag{2.8}$$

and

$$\int_X f(x)\, d\mu(x) = \int_Y \Big(\int_{\pi^{-1}(y)} f(x)\, d\mu_y(x)\Big)\, d\eta(y) \tag{2.9}$$

for every Borel map $f : X \to [0, +\infty]$.

[1]We say that $\{\mu_y\}_{y\in Y}$ is a Borel family (of probability measures) if $y \in Y \longmapsto \mu_y(B) \in \mathbb{R}$ is Borel for any Borel set $B \subset X$.

3 Constrained MFG Equilibria

3.1 Approximation of Constrained Trajectories

Let Ω be a bounded open subset of $\mathbb{R}^n$ with $\mathcal{C}^2$ boundary. Let Γ be the metric subspace of $AC(0, T; \mathbb{R}^n)$ defined by

$$\Gamma = \Big\{\gamma \in AC(0, T; \mathbb{R}^n) : \gamma(t) \in \overline{\Omega}, \quad \forall t \in [0, T]\Big\}.$$

For any $x \in \overline{\Omega}$, we set

$$\Gamma[x] = \{\gamma \in \Gamma : \gamma(0) = x\}. \tag{3.1}$$

Lemma 3.1 *Let $\gamma \in AC(0, T; \mathbb{R}^n)$ and suppose that $d_\Omega(\gamma(t)) < \rho_0$ for all $t \in [0, T]$. Then $d_\Omega \circ \gamma \in AC(0, T)$ and*

$$\frac{d}{dt}(d_\Omega \circ \gamma)(t) = \big\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t)\big\rangle \mathbf{1}_{\Omega^c}(\gamma(t)) \ \ a.e.\ t \in [0, T]. \tag{3.2}$$

Moreover,

$$N_\gamma := \{t \in [0, T] : \gamma(t) \in \partial\Omega,\ \exists\, \dot{\gamma}(t),\ \langle Db_\Omega(\gamma(t)), \dot{\gamma}(t)\rangle \neq 0\} \tag{3.3}$$

is a discrete set.

Proof First we prove that N_γ is a discrete set. Let $t \in N_\gamma$, then there exists $\epsilon > 0$ such that $\gamma(s) \notin \partial\Omega$ for any $s \in (]t - \epsilon, t + \epsilon[\ \setminus \{t\}) \cap [0, T]$. Therefore, N_γ is composed of isolated points and so it is a discrete set.

Let us now set $\phi(t) = (d_\Omega \circ \gamma)(t)$ for all $t \in [0, T]$. We note that $\phi \in AC(0, T)$ because it is the composition of $\gamma \in AC(0, T; \mathbb{R}^n)$ with the Lipschitz continuous function $d_\Omega(\cdot)$. Denote by D the set of $t \in [0, T]$ such that there exists the first order derivative of γ in t, i.e.,

$$D = \{t \in [0, T] :\ \exists\, \dot{\gamma}(t)\ \}.$$

We observe that D has full Lebesgue measure and we decompose D in the following way:

$$D = \underbrace{\{t \in D : \gamma(t) \notin \partial\Omega\}}_{D_0} \cup \underbrace{\{t \in D : \gamma(t) \in \partial\Omega\}}_{D_1}.$$

By Evans and Gariepy [7, Theorem 4, pg 129], for all $t \in D_0$ the first order derivative of ϕ is equal to

$$\dot{\phi}(t) = \begin{cases} 0 & \gamma(t) \in \Omega \\ \big\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t) \big\rangle & \gamma(t) \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Now, consider $t \in D_1 \setminus N_\gamma$. Since $\gamma(t) \in \partial\Omega$, one has that

$$\frac{\phi(t+h) - \phi(t)}{h} = \frac{d_\Omega(\gamma(t+h))}{h},$$

for all $h > 0$. Since $\gamma(t+h) = \gamma(t) + h\dot{\gamma}(t) + o(h)$ and d_Ω is Lipschitz continuous, we obtain

$$0 \le \frac{d_\Omega(\gamma(t+h))}{h} \le \frac{o(h)}{h} + \frac{d_\Omega(\gamma(t) + h\dot{\gamma}(t))}{h}.$$

Hence, one has that

$$0 \le \liminf_{h \to 0} \frac{d_\Omega(\gamma(t+h))}{h} \le \limsup_{h \to 0} \frac{d_\Omega(\gamma(t+h))}{h} \le \limsup_{h \to 0} \frac{d_\Omega(\gamma(t) + h\dot{\gamma}(t))}{h}. \tag{3.4}$$

Moreover, by the regularity of b_Ω, we obtain

$$d_\Omega(\gamma(t) + h\dot{\gamma}(t)) \le |b_\Omega(\gamma(t) + h\dot{\gamma}(t))| \le |h|\, |\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t)\rangle| + o(h). \tag{3.5}$$

Thus, since $t \in D \setminus N_\gamma$, we conclude that

$$\limsup_{h \to 0} \frac{d_\Omega(\gamma(t) + h\dot{\gamma}(t))}{h} \le |Db_\Omega(\gamma(t)), \dot{\gamma}(t)\rangle| = 0. \tag{3.6}$$

So $\dot{\phi}(t) = 0$ and the proof is complete. □

Proposition 3.1 *Let $x_i \in \overline{\Omega}$ be such that $x_i \to x$ and let $\gamma \in \Gamma[x]$. Then there exists $\gamma_i \in \Gamma[x_i]$ such that:*

(i) $\gamma_i \to \gamma$ *uniformly on* $[0, T]$*;*
(ii) $\dot{\gamma}_i \to \dot{\gamma}$ *a.e. on* $[0, T]$*;*
(iii) $|\dot{\gamma}_i(t)| \le C|\dot{\gamma}(t)|$ *for any* $i \ge 1$*, a.e.* $t \in [0, T]$*, and some constant* $C \ge 0$*.*

Proof Let $\widehat{\gamma}_i$ be the trajectory defined by

$$\widehat{\gamma}_i(t) = \gamma(t) + x_i - x. \tag{3.7}$$

We observe that $d_\Omega(\widehat{\gamma}_i(t)) \le \rho_0$ for all $t \in [0, T]$ and all sufficiently large i, say $i \ge i_0$. Indeed,

$$d_\Omega(\widehat{\gamma}_i(t)) \le |\widehat{\gamma}_i(t) - \gamma(t)| = |x_i - x|.$$

Since $x_i \to x$, we have that $d_\Omega(\widehat{\gamma}_i(t)) \le \rho_0$ for all $t \in [0, T]$ and $i \ge i_0$. We denote by γ_i the projection of $\widehat{\gamma}_i$ on $\overline{\Omega}$, i.e.,

$$\gamma_i(t) = \widehat{\gamma}_i(t) - d_\Omega(\widehat{\gamma}_i(t))Db_\Omega(\widehat{\gamma}_i(t)) \quad \forall t \in [0, T]. \tag{3.8}$$

We note that $\gamma_i \in \Gamma[x_i]$. Moreover, γ_i converges uniformly to γ on $[0, T]$. Indeed,

$$|\gamma_i(t) - \gamma(t)| = |x_i - x - d_\Omega(\widehat{\gamma}_i(t))Db_\Omega(\widehat{\gamma}_i(t))| \le 2|x_i - x|, \quad \forall\, t \in [0, T].$$

By Lemma 3.1, $d_\Omega(\widehat{\gamma}_i(\cdot)) \in AC(0, T)$ and $\frac{d}{dt}(d_\Omega(\widehat{\gamma}_i(t))) = \big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\widehat{\gamma}}_i(t)\big\rangle$ $\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))$ a.e. $t \in [0, T]$. Using the regularity of b_Ω, we obtain

$$\dot{\gamma}_i(t) = \dot{\gamma}(t) - \big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\big\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t)) - d_\Omega(\widehat{\gamma}_i(t))D^2 b_\Omega(\widehat{\gamma}_i(t))\dot{\gamma}(t), \text{ a.e. } t \in [0, T].$$

Therefore, there exists a constant $C \ge 0$ such that $|\dot{\gamma}_i(t)| \le C|\dot{\gamma}(t)|$ for any $i \ge i_0$, a.e. $t \in [0, T]$.

Finally, we have to show that $\dot{\gamma}_i \to \dot{\gamma}$ almost everywhere on $[0, T]$. Since $\widehat{\gamma}_i \to \gamma$ and $\gamma \in \Gamma[x]$, one has that

$$d_\Omega(\widehat{\gamma}_i(t))D^2 b_\Omega(\widehat{\gamma}_i(t))\dot{\gamma}(t) \xrightarrow{i\to\infty} 0, \quad \forall t \in [0, T].$$

So, we have to prove that

$$-\big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\big\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t)) \xrightarrow{i\to\infty} 0, \text{ a.e. } t \in [0, T]. \tag{3.9}$$

We note that

$$\Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\big\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))\Big| \le \Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\big\rangle\Big|, \text{ a.e. } t \in [0, T]. \tag{3.10}$$

Fix $t \in [0, T]$ such that (3.10) holds. If $\gamma(t) \in \Omega$ then $\widehat{\gamma}_i(t) \in \Omega$ for i large enough and (3.9) holds. On the other hand, if $\gamma(t) \in \partial\Omega$, then passing to the limit in (3.10), we have that

$$\limsup_{i\to\infty} \Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\big\rangle Db_\Omega(\widehat{\gamma}_i(t))\mathbf{1}_{\Omega^c}(\widehat{\gamma}_i(t))\Big| \le \limsup_{i\to\infty} \Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\big\rangle\Big|.$$

Since $\gamma_i \to \gamma$ uniformly on $[0, T]$, one has that

$$\limsup_{i\to\infty} \Big|\big\langle Db_\Omega(\widehat{\gamma}_i(t)), \dot{\gamma}(t)\big\rangle\Big| = \Big|\big\langle Db_\Omega(\gamma(t)), \dot{\gamma}(t)\big\rangle\Big|. \tag{3.11}$$

By Lemma 3.1, we have that $\langle Db_\Omega(\gamma(t)), \dot\gamma(t)\rangle = 0$ for $t \in [0,T] \setminus N_\gamma$, where N_γ is the discrete set defined in (3.3). So (3.9) holds for a.e. $t \in [0,T]$. Thus, $\dot\gamma_i$ converges almost everywhere to $\dot\gamma$ on $[0,T]$. This completes the proof. $\square$

3.2 Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary. Let $\mathcal{P}(\overline{\Omega})$ be the set of all Borel probability measures on $\overline{\Omega}$ endowed with the Kantorovich-Rubinstein distance d_1 defined in (2.4). We suppose throughout that $F, G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $L : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ are given continuous functions. Moreover, we assume the following conditions.

(L1) $L \in C^1(\overline{\Omega} \times \mathbb{R}^n)$ and for all $(x, v) \in \overline{\Omega} \times \mathbb{R}^n$,

$$|D_x L(x, v)| \le C(1 + |v|^2), \tag{3.12}$$

$$|D_v L(x, v)| \le C(1 + |v|), \tag{3.13}$$

for some constant $C > 0$.

(L2) There exist constants $c_1, c_0 > 0$ such that

$$L(x, v) \ge c_1|v|^2 - c_0, \quad \forall (x, v) \in \overline{\Omega} \times \mathbb{R}^n. \tag{3.14}$$

(L3) $v \longmapsto L(x, v)$ is convex for all $x \in \overline{\Omega}$.

Remark 3.1

- (i) As $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})$ is a compact set, the continuity of F and G implies that they are bounded and uniformly continuous on $\overline{\Omega} \times \mathcal{P}(\overline{\Omega})$.
- (ii) In (L1), L is assumed to be of class $C^1(\overline{\Omega} \times \mathbb{R}^n)$ just for simplicity. All the results of this paper hold true if L is locally Lipschitz—hence, a.e. differentiable—in $\overline{\Omega} \times \mathbb{R}^n$ and satisfies the growth conditions (3.12) and (3.13) a.e. on $\overline{\Omega} \times \mathbb{R}^n$, see Remark 3.3 below.

3.3 Existence of Constrained MFG Equilibria

For any $t \in [0, T]$, we denote by $e_t : \Gamma \to \overline{\Omega}$ the evaluation map defined by

$$e_t(\gamma) = \gamma(t), \quad \forall \gamma \in \Gamma.$$

For any $\eta \in \mathcal{P}(\Gamma)$, we define

$$m^\eta(t) = e_t \sharp \eta, \quad \forall t \in [0, T]. \tag{3.15}$$

Lemma 3.2 *The following holds true.*

(i) $m^\eta \in C([0,T];\mathcal{P}(\overline{\Omega}))$ *for any* $\eta \in \mathcal{P}(\Gamma)$.
(ii) Let $\eta_i,\ \eta \in \mathcal{P}(\Gamma)$, $i \geq 1$, *be such that* η_i *is narrowly convergent to* η. *Then* $m^{\eta_i}(t)$ *is narrowly convergent to* $m^\eta(t)$ *for all* $t \in [0,T]$.

Proof First, we prove point (i). By definition (3.15), it is obvious that $m^\eta(t)$ is a Borel probability measure on $\overline{\Omega}$ for any $t \in [0,T]$. Let $\{t_k\} \subset [0,T]$ be a sequence such that $t_k \to \bar{t}$. We want to show that

$$\lim_{t_k\to\bar{t}} \int_{\overline{\Omega}} f(x) m^\eta(t_k,\, dx) = \int_{\overline{\Omega}} f(x) m^\eta(\bar{t},\, dx), \tag{3.16}$$

for any $f \in C(\overline{\Omega})$. Since $m^\eta(t_k) = e_{t_k}\sharp\eta$ and $e_{t_k}(\gamma) = \gamma(t_k)$, we have that

$$\lim_{t_k\to\bar{t}} \int_{\overline{\Omega}} f(x) m^\eta(t_k,\, dx) = \lim_{t_k\to\bar{t}} \int_{\Gamma} f(e_{t_k}(\gamma))\, d\eta(\gamma) = \lim_{t_k\to\bar{t}} \int_{\Gamma} f(\gamma(t_k))\, d\eta(\gamma).$$

Since $f \in C(\overline{\Omega})$ and $\gamma \in \Gamma$, then $f(\gamma(t_k)) \to f(\gamma(\bar{t}))$ and $|f(\gamma(t_k))| \leq ||f||_\infty$. Therefore, by Lebesgue's dominated convergence theorem, we have that

$$\lim_{t_k\to\bar{t}} \int_{\Gamma} f(\gamma(t_k))\, d\eta(\gamma) = \int_{\Gamma} f(\gamma(\bar{t}))\, d\eta(\gamma). \tag{3.17}$$

Thus, recalling the definition of m^η, we obtain (3.16). Moreover, by Proposition 2.2, we conclude that $d_1(m^\eta(t_k), m^\eta(\bar{t})) \to 0$. This completes the proof of point (i).

In order to prove point (ii), we suppose that η_i is narrowly convergent to η. Then, for all $f \in C(\overline{\Omega})$ we have that

$$\lim_{i\to\infty} \int_{\overline{\Omega}} f(x) m^{\eta_i}(t,\, dx) = \lim_{i\to\infty} \int_{\Gamma} f(\gamma(t))\, d\eta_i(\gamma) = \int_{\Gamma} f(\gamma(t))\, d\eta(\gamma) = \int_{\overline{\Omega}} f(x) m^\eta(t,\, dx).$$

Hence, $m^{\eta_i}(t)$ is narrowly convergent to $m^\eta(t)$ for all $t \in [0,T]$. □

For any fixed $m_0 \in \mathcal{P}(\overline{\Omega})$, we denote by $\mathcal{P}_{m_0}(\Gamma)$ the set of all Borel probability measures η on Γ such that $e_0\sharp\eta = m_0$. For all $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we define

$$J_\eta[\gamma] = \int_0^T \Big[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m^\eta(t))\Big]\, dt + G(\gamma(T), m^\eta(T)), \quad \gamma \in \Gamma. \tag{3.18}$$

Remark 3.2 We note that $\mathcal{P}_{m_0}(\Gamma)$ is nonempty. Indeed, let $j : \overline{\Omega} \to \Gamma$ be the continuous map defined by

$$j(x)(t) = x \quad \forall t \in [0,T].$$

Then,

$$\eta := j\sharp m_0$$

is a Borel probability measure on Γ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$.

For all $x \in \overline{\Omega}$ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$, we define

$$\Gamma^{\eta}[x] = \left\{ \gamma \in \Gamma[x] : J_{\eta}[\gamma] = \min_{\Gamma[x]} J_{\eta} \right\}. \tag{3.19}$$

Definition 3.1 Let $m_0 \in \mathcal{P}(\overline{\Omega})$. We say that $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if

$$supp(\eta) \subseteq \bigcup_{x \in \overline{\Omega}} \Gamma^{\eta}[x]. \tag{3.20}$$

In other words, $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if for η-a.e. $\overline{\gamma} \in \Gamma$ we have that

$$J_{\eta}[\overline{\gamma}] \leq J_{\eta}[\gamma], \qquad \forall \gamma \in \Gamma[\overline{\gamma}(0)].$$

The main result of this section is the existence of constrained MFG equilibria for m_0.

Theorem 3.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L1)-(L3) hold true. Let $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ be continuous. Then, there exists at least one constrained MFG equilibrium for m_0.*

Theorem 3.1 will be proved in Sect. 3.4. Now, we will show some properties of $\Gamma^{\eta}[x]$ that we will use in what follows.

Lemma 3.3 *For all $x \in \overline{\Omega}$ and $\eta \in \mathcal{P}_{m_0}(\Gamma)$ the following holds true.*

(i) $\Gamma^{\eta}[x]$ *is a nonempty set.*
(ii) *All $\gamma \in \Gamma^{\eta}[x]$ satisfy*

$$||\dot{\gamma}||_2 \leq K, \tag{3.21}$$

where

$$K = \frac{1}{\sqrt{c_1}} \left[T \max_{\overline{\Omega}} L(x, 0) + 2T \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |F| + 2 \max_{\overline{\Omega} \times \mathcal{P}(\overline{\Omega})} |G| + T c_0 \right]^{\frac{1}{2}} \tag{3.22}$$

and c_0, c_1 are the constants in (3.14). Consequently, all minimizers $\gamma \in \Gamma^{\eta}[x]$ are $\frac{1}{2}$-Hölder continuous of constant K.

In addition, if $\eta \in \mathcal{P}_{m_0}(\Gamma)$ *is a constrained MFG equilibrium for* m_0*, then* $m^\eta(t) = e_t \sharp \eta$ *is* $\frac{1}{2}$*-Hölder continuous of constant* K*.*

Proof By classical results in the calculus of variation (see, e.g., [4, Theorem 6.1.2]), there exists at least one mimimizer of $J_\eta[\cdot]$ on Γ for any fixed initial point $x \in \overline{\Omega}$. So $\Gamma^\eta[x]$ is a nonempty set.

Let $x \in \overline{\Omega}$ and let $\gamma \in \Gamma^\eta[x]$. By comparing the cost of γ with the cost of the constant trajectory $\gamma(0) \equiv x$, one has that

$$\int_0^T \Big[L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m^\eta(t))\Big] dt + G(\gamma(T), m^\eta(T)) \tag{3.23}$$

$$\leq \int_0^T \Big[L(x, 0) + F(x, m^\eta(t))\Big] dt + G(x, m^\eta(T))$$

$$\leq \Big[T \max_{\overline{\Omega}} L(x, 0) + T \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |F| + \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |G|\Big].$$

Using (3.14) in (3.23), one has that

$$||\dot{\gamma}||_2 \leq \frac{1}{\sqrt{c_1}}\Big[T \max_{\overline{\Omega}} L(x, 0) + 2T \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |F| + 2 \max_{\overline{\Omega}\times\mathcal{P}(\overline{\Omega})} |G| + T c_0\Big]^{\frac{1}{2}} = K, \tag{3.24}$$

where c_0, c_1 are the constants in (3.14). This completes the proof of point (ii) since the Hölder regularity of γ is a direct consequence of the estimate (3.24).

Finally, we claim that, if η is a constrained MFG equilibrium for m_0, then the map $t \to m^\eta(t)$ is $\frac{1}{2}$-Hölder continuous with constant K. Indeed, for any $t_1, t_2 \in [0, T]$, we have that

$$d_1(m^\eta(t_2), m^\eta(t_1)) = \sup_\phi \int_{\bar{\Omega}} \phi(x)\,(m^\eta(t_2, dx) - m^\eta(t_1, dx)), \tag{3.25}$$

where the supremum is taken over the set of all 1-Lipschitz continuous maps $\phi : \overline{\Omega} \to \mathbb{R}$. Since $m^\eta(t) = e_t \sharp \eta$ and the map ϕ is 1-Lipschitz continuous, one has that

$$\int_{\overline{\Omega}} \phi(x)\,(m^\eta(t_2, dx) - m^\eta(t_1, dx)) = \int_\Gamma \Big[\phi(e_{t_2}(\gamma)) - \phi(e_{t_1}(\gamma))\Big] d\eta(\gamma)$$

$$= \int_\Gamma \Big[\phi(\gamma(t_2)) - \phi(\gamma(t_1))\Big] d\eta(\gamma) \leq \int_\Gamma |\gamma(t_2) - \gamma(t_1)|\, d\eta(\gamma).$$

Since η is a constrained MFG equilibrium for m_0, property (ii) yields

$$\int_\Gamma |\gamma(t_2) - \gamma(t_1)|\, d\eta(\gamma) \leq K \int_\Gamma |t_2 - t_1|^{\frac{1}{2}}\, d\eta(\gamma) = K|t_2 - t_1|^{\frac{1}{2}}.$$

Hence, we conclude that

$$d_1(m^\eta(t_2), m^\eta(t_1)) \leq K|t_2 - t_1|^{\frac{1}{2}}, \quad \forall\, t_1, t_2 \in [0, T]$$

and the map $t \longmapsto m^\eta(t)$ is 1/2-Hölder continuous. □

Lemma 3.4 *Let $\eta_i, \eta \in \mathcal{P}_{m_0}(\Gamma)$ be such that η_i narrowly converges to η. Let $x_i \in \overline{\Omega}$ be such that $x_i \to x$ and let $\gamma_i \in \Gamma^{\eta_i}[x_i]$ be such that $\gamma_i \to \overline{\gamma}$. Then $\overline{\gamma} \in \Gamma^\eta[x]$. Consequently, $\Gamma^\eta[\cdot]$ has closed graph.*

Proof We want to prove that

$$J_\eta[\overline{\gamma}] \leq J_\eta[\gamma], \quad \forall \gamma \in \Gamma[x] \text{ such that } \int_0^T |\dot{\gamma}|^2\, dt < \infty. \tag{3.26}$$

We observe that the above request is not restrictive because, by assumption (L2), if $\int_0^T |\dot{\gamma}|^2\, dt = \infty$ then the above inequality is trivial.

Fix $\gamma \in \Gamma[x]$ with $\int_0^T |\dot{\gamma}|^2\, dt < \infty$, by Proposition 3.1, we have that there exists $\widehat{\gamma_i} \in \Gamma[x_i]$ such that $\widehat{\gamma_i} \to \gamma$ uniformly on $[0, T]$, $\dot{\widehat{\gamma}}_i \to \dot{\gamma}$ a.e. on $[0, T]$ and $|\dot{\widehat{\gamma}}_i(t)| \leq C|\dot{\gamma}(t)|$ for any $i \geq 1$, a.e. $t \in [0, T]$, and some constant $C \geq 0$. Since $\gamma_i \in \Gamma^{\eta_i}[x_i]$, one has that

$$J_{\eta_i}[\gamma_i] \leq J_{\eta_i}[\widehat{\gamma_i}], \quad \forall i \geq 1. \tag{3.27}$$

So, in order to prove (3.26), we have to check that

(a) $J_\eta[\overline{\gamma}] \leq \liminf_{i\to\infty} J_{\eta_i}[\gamma_i]$;
(b) $\lim_{i\to+\infty} J_{\eta_i}[\widehat{\gamma_i}] = J_\eta[\gamma]$.

First we show that (a) holds, that is,

$$\begin{aligned} &\liminf_{i\to\infty} \left\{ \int_0^T \Big[L\big(\gamma_i(t), \dot{\gamma}_i(t)\big)\big) + F\big(\gamma_i(t), m^{\eta_i}(t)\big)\Big]\, dt + G(\gamma_i(T), m^{\eta_i}(T)) \right\} \\ &\geq \int_0^T [L\big(\overline{\gamma}(t), \dot{\overline{\gamma}}(t)\big)\big) + F\big(\overline{\gamma}(t), m^\eta(t)\big)\Big]\, dt + G(\overline{\gamma}(T), m^\eta(T)). \end{aligned} \tag{3.28}$$

First of all, we recall that, by Lemma 3.2, $m^{\eta_i}(t)$ narrowly converges to $m^\eta(t)$ for all $t \in [0, T]$. Owing to the convergence of γ_i to $\overline{\gamma}$, the narrow convergence of $m^{\eta_i}(t)$ to $m^\eta(t)$, and our assumption on F and G, we conclude that

$$\int_0^T F(\gamma_i(t), m^{\eta_i}(t))\, dt \xrightarrow{i\to\infty} \int_0^T F(\overline{\gamma}(t), m^\eta(t))\, dt,$$

$$G(\gamma_i(T), m^{\eta_i}(T)) \xrightarrow{i\to\infty} G(\overline{\gamma}(T), m^\eta(T)).$$

Up to taking a subsequence of γ_i, we can assume that $\dot{\gamma}_i \rightharpoonup \dot{\overline{\gamma}}$ in $L^2(0, T; \mathbb{R}^n)$ without loss of generality. By assumption (L3), one has that

$$\int_0^T L(\gamma_i(t), \dot{\gamma}_i(t)) = \int_0^T L(\overline{\gamma}(t), \dot{\gamma}_i(t))\, dt + \int_0^T \Big[L(\gamma_i(t), \dot{\gamma}_i(t)) - L(\overline{\gamma}(t), \dot{\gamma}_i(t))\Big]\, dt$$
$$\geq \int_0^T \Big[L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t)) + \langle D_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t)), \dot{\gamma}_i - \dot{\overline{\gamma}}\rangle\Big]\, dt + \int_0^T \Big[L(\gamma_i(t), \dot{\gamma}_i(t)) - L(\overline{\gamma}(t), \dot{\gamma}_i(t))\Big]\, dt.$$

Since $\gamma_i \in \Gamma^{\eta_i}[x_i]$ and $\gamma_i \to \overline{\gamma}$, by (L1), we obtain

$$\int_0^T \Big[L(\gamma_i(t), \dot{\gamma}_i(t)) - L(\overline{\gamma}(t), \dot{\gamma}_i(t))\Big]\, dt \xrightarrow{i\to\infty} 0.$$

Moreover, since $\dot{\gamma}_i \rightharpoonup \dot{\overline{\gamma}}$ in $L^2(0, T; \mathbb{R}^n)$, one has that

$$\int_0^T \Big\langle D_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t)), \dot{\gamma}_i - \dot{\overline{\gamma}}\Big\rangle\, dt \xrightarrow{i\to\infty} 0.$$

Thus, (3.28) holds.

Finally, we prove (b), i.e.,

$$\lim_{i\to\infty} \left\{\int_0^T L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) + F(\widehat{\gamma}_i(t), m^{\eta_i}(t))\, dt + G(\widehat{\gamma}_i(T), m^{\eta_i}(T))\right\}$$
$$= \int_0^T L(\gamma(t), \dot{\gamma}(t)) + F(\gamma(t), m^{\eta}(t))\, dt + G(\gamma(T), m^{\eta}(T)).$$

Owing to the convergence of $\widehat{\gamma}_i$ to γ, the narrow convergence of $m^{\eta_i}(t)$ to $m^{\eta}(t)$ for all $t \in [0, T]$, and our assumption on F and G, one has

$$\int_0^T F(\widehat{\gamma}_i(t), m^{\eta_i}(t))\, dt \xrightarrow{i\to\infty} \int_0^T F(\gamma(t), m^{\eta}(t))\, dt,$$
$$G(\widehat{\gamma}_i(T), m^{\eta_i}(T)) \xrightarrow{i\to\infty} G(\gamma(T), m^{\eta}(T)).$$

Hence, we only need to prove that

$$\liminf_{i\to\infty} \int_0^T L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t))\, dt = \int_0^T L(\gamma(t), \dot{\gamma}(t))\, dt. \tag{3.29}$$

Owing to (L1), one has that

$$\left|\int_0^T \big[L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\gamma}(t))\big]\, dt\right|$$
$$\leq \int_0^T \Big|L(\widehat{\gamma}_i(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\widehat{\gamma}}_i(t))\Big|\, dt + \int_0^T \Big|L(\gamma(t), \dot{\widehat{\gamma}}_i(t)) - L(\gamma(t), \dot{\gamma}(t))\Big|\, dt$$

$$\leq ||\widehat{\gamma_i} - \gamma||_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2\right) dt + \int_0^T \left| \int_0^1 \left\langle D_v L\big(\gamma(t), \lambda\dot{\widehat{\gamma}}_i + (1-\lambda)\dot{\gamma}\big), \dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t) \right\rangle d\lambda \right| dt$$

$$\leq ||\widehat{\gamma_i} - \gamma||_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2\right) dt + C \int_0^T \int_0^1 \left(1 + |\dot{\widehat{\gamma}}_i| + |\dot{\gamma}|\right) |\dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t)| \, dt.$$

Since $\widehat{\gamma_i} \to \gamma$ uniformly on $[0, T]$ and $|\dot{\widehat{\gamma}}_i(t)| \leq C|\dot{\gamma}(t)|$ for any $i \geq 1$ and for any $t \in [0, T]$, we have that

$$||\widehat{\gamma_i} - \gamma||_\infty \int_0^T \left(1 + |\dot{\widehat{\gamma}}_i|^2\right) dt \xrightarrow{i\to\infty} 0.$$

In addition, since $\dot{\widehat{\gamma}}_i \to \dot{\gamma}$ a.e. on $[0, T]$, by Lebesgue's dominated convergence theorem we obtain

$$C \int_0^T \int_0^1 \left(1 + |\dot{\widehat{\gamma}}_i| + |\dot{\gamma}|\right) \left|\dot{\widehat{\gamma}}_i(t) - \dot{\gamma}(t)\right| \, dt \xrightarrow{i\to\infty} 0. \tag{3.30}$$

This gives (b) and the proof is complete. □

Remark 3.3 The above proof can be adapted to treat the case of a locally Lipschitz Lagrangian L as was mentioned in Remark 3.1. Indeed, it suffices to replace the gradient $D_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t))$ with a measurable selection of the subdifferential $\partial_v L(\overline{\gamma}(t), \dot{\overline{\gamma}}(t))$.

3.4 Proof of Theorem 3.1

In this section we prove Theorem 3.1 using a fixed point argument. First of all, we recall that, by Theorem 2.2, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, there exists a unique Borel measurable family of probabilities $\{\eta_x\}_{x\in\overline{\Omega}}$ on Γ which disintegrates η in the sense that

$$\begin{cases} \eta(d\gamma) = \int_{\overline{\Omega}} \eta_x(d\gamma)\, dm_0(x), \\ supp(\eta_x) \subset \Gamma[x] \;\; m_0 - \text{a.e. } x \in \overline{\Omega}. \end{cases} \tag{3.31}$$

We introduce the set-valued map $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$ by defining, for any $\eta \in \mathcal{P}_{m_0}(\Gamma)$,

$$E(\eta) = \left\{ \widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma) : supp(\widehat{\eta}_x) \subseteq \Gamma^\eta[x]\, m_0 - \text{a.e. } x \in \overline{\Omega} \right\}. \tag{3.32}$$

Then, it is immediate to realize that $\eta \in \mathcal{P}_{m_0}(\Gamma)$ is a constrained MFG equilibrium for m_0 if and only if $\eta \in E(\eta)$. We will therefore show that the set-valued map E has a fixed point. For this purpose, we will apply Kakutani's Theorem [10]. The following lemmas are intended to check that the assumptions of such a theorem are satisfied by E.

Lemma 3.5 *For any $\eta \in \mathcal{P}_{m_0}(\Gamma)$, $E(\eta)$ is a nonempty convex set.*

Proof First, we note that $E(\eta)$ is a nonempty set. Indeed, by (i) of Lemmas 3.3 and Lemma 3.4, and [2, Theorem 8.1.4] we have that $x \longmapsto \Gamma^\eta[x]$ is measurable. Moreover, by [2, Theorem 8.1.3], $x \longmapsto \Gamma^\eta[x]$ has a Borel measurable selection $x \longmapsto \gamma_x^\eta$. Thus, the measure $\widehat{\eta}$, defined by $\widehat{\eta}(\mathcal{B}) = \int_{\overline{\Omega}} \delta_{\{\gamma_x^\eta\}}(\mathcal{B})\, dm_0(x)$ for any $\mathcal{B} \in \mathscr{B}(\Gamma)$, belongs to $E(\eta)$.

Now we want to check that $E(\eta)$ is a convex set. Let $\{\eta_i\}_{i=1,2} \in E(\eta)$ and let $\lambda_1, \lambda_2 \geq 0$ be such that $\lambda_1 + \lambda_2 = 1$. Since η_i are Borel probability measures, $\widehat{\eta} := \lambda_1\eta_1 + \lambda_2\eta_2$ is a Borel probability measure as well. Moreover, for any Borel set $B \in \mathscr{B}(\overline{\Omega})$ we have that

$$e_0\sharp\widehat{\eta}(B) = \widehat{\eta}(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i \eta_i(e_0^{-1}(B)) = \sum_{i=1}^{2} \lambda_i e_0 \sharp \eta_i(B) = \sum_{i=1}^{2} \lambda_i m_0(B) = m_0(B). \tag{3.33}$$

So, $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma)$. Since $\eta_1 \in E(\eta)$, there exists a unique Borel measurable family of probabilities $\{\eta_{1,x}\}_{x\in\overline{\Omega}}$ on Γ which disintegrates η_1 as in (3.31) and there exists $A_1 \subset \overline{\Omega}$, with $m_0(A_1) = 0$, such that

$$supp(\eta_{1,x}) \subset \Gamma^\eta[x], \quad x \in \overline{\Omega} \setminus A_1. \tag{3.34}$$

In the same way, $\eta_2(d\gamma) = \int_{\overline{\Omega}} \eta_{2,x}(d\gamma)\, dm_0(x)$ can be disintegrated and one has that

$$supp(\eta_{2,x}) \subset \Gamma^\eta[x] \quad x \in \overline{\Omega} \setminus A_2, \tag{3.35}$$

where A_2 is such that $m_0(A_2) = 0$. Hence, $\widehat{\eta}$ can be disintegrated in the following way:

$$\begin{cases} \widehat{\eta}(d\gamma) = \int_{\overline{\Omega}} \Big(\lambda_1\eta_{1,x} + \lambda_2\eta_{2,x}\Big)(d\gamma) dm_0(x), \\ supp(\lambda_1\eta_{1,x} + \lambda_2\eta_{2,x}) \subset \Gamma^\eta[x] \quad x \in \overline{\Omega} \setminus (A_1 \cup A_2), \end{cases} \tag{3.36}$$

where $m_0(A_1 \cup A_2) = 0$. This completes the proof. □

Lemma 3.6 *The map $E : \mathcal{P}_{m_0}(\Gamma) \rightrightarrows \mathcal{P}_{m_0}(\Gamma)$ has closed graph.*

Proof Let $\eta_i, \eta \in \mathcal{P}_{m_0}(\Gamma)$ be such that η_i is narrowly convergent to η. Let $\widehat{\eta}_i \in E(\eta_i)$ be such that $\widehat{\eta}_i$ is narrowly convergent to $\widehat{\eta}$. We have to prove that $\widehat{\eta} \in E(\eta)$. Since $\widehat{\eta}_i$ narrowly converges to $\widehat{\eta}$, we have that $\widehat{\eta} \in \mathcal{P}_{m_0}(\Gamma)$ and there exists a unique Borel measurable family of probabilities $\{\widehat{\eta}_x\}_{x\in\overline{\Omega}}$ on Γ such that $\widehat{\eta}(d\gamma) = \int_{\overline{\Omega}} \widehat{\eta}_x(d\gamma)\, dm_0(x)$ and $supp(\widehat{\eta}_x) \subset \Gamma[x]$ for m_0-a.e. $x \in \overline{\Omega}$. Hence, $\widehat{\eta} \in E(\eta)$ if and only if $supp(\widehat{\eta}_x) \subseteq \Gamma^\eta[x]$ for m_0-a.e $x \in \overline{\Omega}$. Let $\Omega_0 \subset \overline{\Omega}$ be an m_0-null set such that

$$supp(\widehat{\eta}_x) \subset \Gamma[x] \quad \forall x \in \overline{\Omega} \setminus \Omega_0.$$

Let $x \in \overline{\Omega} \setminus \Omega_0$ and let $\widehat{\gamma} \in supp(\widehat{\eta}_x)$. Since $\widehat{\eta}_i$ narrowly converges to $\widehat{\eta}$, then, by Proposition 2.1, one has that

$$\exists\, \widehat{\gamma}_i \in \; supp(\widehat{\eta}_i) \; \text{ such that } \lim_{i\to\infty} \widehat{\gamma}_i = \widehat{\gamma}.$$

Let $\widehat{\gamma}_i(0) = x_i$, with $x_i \in \overline{\Omega}$. Since $\widehat{\eta}_i \in E(\eta_i)$ and $\widehat{\gamma}_i \in supp(\widehat{\eta}_i)$, we have that $\widehat{\gamma}_i$ is an optimal trajectory for $J_{\eta_i}[\cdot]$, i.e.,

$$J_{\eta_i}[\widehat{\gamma}_i] \leq J_{\eta_i}[\gamma] \quad \forall \gamma \in \Gamma[x_i]. \tag{3.37}$$

As η_i narrowly converges to η, applying Lemma 3.4, we conclude that $\widehat{\gamma} \in \Gamma^{\eta}[x]$. Since x is any point in $\overline{\Omega} \setminus \Omega_0$, we have shown that $\widehat{\eta} \in E(\eta)$. □

We denote by Γ_K the set of trajectories $\gamma \in \Gamma$ such that γ satisfies (3.21), i.e.,

$$\Gamma_K = \{\gamma \in \Gamma : ||\dot{\gamma}||_2 \leq K\} \tag{3.38}$$

where K is the constant given by (3.22). By the definition of $E(\eta)$ in (3.32) and Lemma 3.3, we deduce that

$$E(\eta) \subseteq \mathcal{P}_{m_0}(\Gamma_K) \quad \forall \eta \in \mathcal{P}_{m_0}(\Gamma). \tag{3.39}$$

Remark 3.4 In general Γ fails to be complete as a metric space. Then, by Theorem 2.1, $\mathcal{P}_{m_0}(\Gamma)$ is not a compact set. On the other hand, if Γ is replaced by Γ_K then $\mathcal{P}_{m_0}(\Gamma_K)$ is a compact convex subset of $\mathcal{P}_{m_0}(\Gamma)$. Indeed, the convexity of $\mathcal{P}_{m_0}(\Gamma_K)$ follows by the same argument used in the proof of Lemma 3.5. As for compactness, let $\{\eta_k\} \subset \mathcal{P}_{m_0}(\Gamma_K)$. Since Γ_K is a compact set, $\{\eta_k\}$ is tight. So, by Theorem 2.1, one finds a subsequence, that we denote again by η_k, which narrowly converges to some probability measure $\eta \in \mathcal{P}_{m_0}(\Gamma_K)$.

We will restrict domain of interest to $\mathcal{P}_{m_0}(\Gamma_K)$, where Γ_K is given by (3.38). Hereafter, we denote by E the restriction $E_{|\mathcal{P}_{m_0}(\Gamma_K)}$.

Conclusion

By Remarks 3.4 and 3.2, $\mathcal{P}_{m_0}(\Gamma_K)$ is a nonempty compact convex set. Moreover, by Lemma 3.5, $E(\eta)$ is nonempty convex set for any $\eta \in \mathcal{P}_{m_0}(\Gamma_K)$ and, by Lemma 3.6, the set-valued map E has closed graph. Then, the assumptions of Kakutani's Theorem are satisfied and so there exists $\overline{\eta} \in \mathcal{P}_{m_0}(\Gamma_K)$ such that $\overline{\eta} \in E(\overline{\eta})$.

4 Mild Solution of the Constrained MFG Problem

In this section we define mild solutions of the constrained MFG problem in $\overline{\Omega}$. Moreover, under the assumptions of Sect. 3.2, we prove the existence of such solutions. Then, we give a uniqueness result under a classical monotonicity assumption on F and G.

Definition 4.1 We say that $(u, m) \in C([0, T] \times \overline{\Omega}) \times C([0, T]; \mathcal{P}(\overline{\Omega}))$ is a mild solution of the constrained MFG problem in $\overline{\Omega}$ if there exists a constrained MFG equilibrium $\eta \in \mathcal{P}_{m_0}(\Gamma)$ such that

(i) $m(t) = e_t \sharp \eta$ for all $t \in [0, T]$;
(ii) u is given by

$$u(t,x) = \inf_{\substack{\gamma \in \Gamma \\ \gamma(t)=x}} \left\{ \int_t^T [L(\gamma(s), \dot\gamma(s)) + F(\gamma(s), m(s))] \, ds + G(\gamma(T), m(T)) \right\}, \tag{4.1}$$

for $(t, x) \in [0, T] \times \overline{\Omega}$.

A direct consequence of Theorem 3.1 is the following result.

Corollary 4.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$ with C^2 boundary and let $m_0 \in \mathcal{P}(\overline{\Omega})$. Suppose that (L1)-(L3) hold true. Let $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ and $G : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ be continuous. There exists at least one mild solution (u, m) of the constrained MFG problem in $\overline{\Omega}$.*

Before proving our uniqueness result, we recall the following definitions.

Definition 4.2 We say that $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is monotone if

$$\int_{\overline{\Omega}} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \geq 0, \tag{4.2}$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$.
We say that F is strictly monotone if

$$\int_{\overline{\Omega}} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) \ \geq \ 0, \tag{4.3}$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$ and $\int_{\overline{\Omega}} (F(x, m_1) - F(x, m_2)) d(m_1 - m_2)(x) = 0$ if and only if $F(x, m_1) = F(x, m_2)$ for all $x \in \overline{\Omega}$.

Example 4.1 Assume that $F : \overline{\Omega} \times \mathcal{P}(\overline{\Omega}) \to \mathbb{R}$ is of the form

$$F(x, m) = \int_{\Omega} f(y, (\phi \star m)(y)) \phi(x - y) \, dy, \tag{4.4}$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth even kernel with compact support and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that $z \to f(x, z)$ is strictly increasing for all $x \in \overline{\Omega}$. Then F satisfies condition (4.3).

Indeed, for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$, we have that

$$\begin{aligned}
&\int_{\overline{\Omega}} [F(x, m_1) - F(x, m_2)]\, d\,(m_1 - m_2)\,(x) \\
&= \int_{\overline{\Omega}} \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))]\, \phi(x - y)\, dy\, d\,(m_1 - m_2)\,(x) \\
&= \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))] \int_{\overline{\Omega}} \phi(x - y)\, d\,(m_1 - m_2)\,(x)\, dy \\
&= \int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))]\, [(\phi \star m_1)(y) - (\phi \star m_2)(y)]\; dy.
\end{aligned}$$

Since $z \to f(x, z)$ is increasing, then one has that

$$\int_{\overline{\Omega}} [f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))]\, [(\phi \star m_1)(y) - (\phi \star m_2)(y)]\; dy \geq 0.$$

Moreover, if the term on the left-side above is equal to zero, then we obtain

$$[f(y, (\phi \star m_1)(y)) - f(y, (\phi \star m_2)(y))]\, [(\phi \star m_1)(y) - (\phi \star m_2(y))] = 0 \quad \text{a.e. } \; y \in \overline{\Omega}.$$

As $z \to f(x, z)$ is strictly increasing, we deduce that $\phi \star m_1(y) = \phi \star m_2(y)$ for any $y \in \overline{\Omega}$ and so $F(\cdot, m_1) = F(\cdot, m_2)$.

Theorem 4.1 *Suppose that F and G satisfy* (4.3)*. Let $\eta_1, \eta_2 \in \mathcal{P}_{m_0}(\Gamma)$ be constrained MFG equilibria and let J_{η_1} and J_{η_2} be the associated functionals. Then J_{η_1} is equal to J_{η_2}. Consequently, if (u_1, m_1), (u_2, m_2) are mild solutions of the constrained MFG problem in $\overline{\Omega}$, then $u_1 = u_2$.*

Proof Let $\eta_1, \eta_2 \in \mathcal{P}_{m_0}(\Gamma)$ be constrained MFG equilibria, such that $m_1(t) = e_t \sharp \eta_1$, $m_2(t) = e_t \sharp \eta_2$ and let u_1, u_2 be the value functions of J_{η_1} and J_{η_2}, respectively. Let $x_0 \in \overline{\Omega}$ and let γ be an optimal trajectory for u_1 at $(0, x_0)$. We get

$$\begin{aligned}
u_1(0, x_0) &= \int_0^T \Big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m_1(s))\Big]\, ds + G(\gamma(T), m_1(T)), \\
u_2(0, x_0) &\leq \int_0^T \Big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m_2(s))\Big]\, ds + G(\gamma(T), m_2(T)).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&G(\gamma(T), m_1(T)) - G(\gamma(T), m_2(T)) \leq u_1(0, x_0) - u_2(0, x_0) \\
&- \int_0^T \Big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m_1(s))\Big]\, ds + \int_0^T \Big[L(\gamma(s), \dot{\gamma}(s)) + F(\gamma(s), m_2(s))\Big]\, ds \\
&= u_1(0, x_0)) - u_2(0, x_0) + \int_0^T F(\gamma(s), m_2(s)) - F(\gamma(s), m_1(s))\, ds.
\end{aligned}
$$

Recalling that η_1 is supported on the set of all curves $\gamma \in \Gamma^{\eta_1}[x]$ for any $x \in \overline{\Omega}$, we integrate on Γ respect to $d\eta_1$ to obtain

$$
\begin{aligned}
&\int_\Gamma \Big[G(\gamma(T), m_1(T)) - G(T, m_2(T))\Big] d\eta_1(\gamma) \leq \\
&\leq \int_\Gamma \Big[u_1(0, \gamma(0)) - u_2(0, \gamma(0))\Big] d\eta_1(\gamma) + \int_\Gamma \int_0^T \Big[F(\gamma(s), m_2(s)) - F(\gamma(s), m_1(s))\Big]\, ds\, d\eta_1(\gamma).
\end{aligned}
$$

Recalling the definition of e_t and using the change of variables $e_t(\gamma) = x$ in the above inequality, we get

$$
\begin{aligned}
\int_\Gamma \Big[G(\overbrace{\gamma(T)}^{e_T(\gamma)}, m_1(T)) - G(\overbrace{\gamma(T)}^{e_T(\gamma)}, m_2(T))\Big] d\eta_1(\gamma) &= \int_{\overline{\Omega}} \Big[G(x, m_1(T)) - G(x, m_2(T))\Big] \\
&\quad \times m_1(T, dx), \\
\int_\Gamma \Big[u_1(0, \overbrace{\gamma(0)}^{e_0(\gamma)}) - u_2(0, \overbrace{\gamma(0)}^{e_0(\gamma)})\Big] d\eta_1(\gamma) &= \int_{\overline{\Omega}} \Big[u_1(0, x) - u_2(0, x)\Big] \\
&\quad \times m_1(0, dx), \\
\int_0^T \int_\Gamma \Big[F(\overbrace{\gamma(s)}^{e_s(\gamma)}, m_2(s)) - F(\overbrace{\gamma(s)}^{e_s(\gamma)}, m_1(s))\Big]\, d\eta_1(\gamma)\, ds &= \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s))\Big] \\
&\quad \times m_1(s, dx)\, ds.
\end{aligned}
$$

So, we deduce that

$$
\begin{aligned}
&\int_{\overline{\Omega}} \Big[G(x, m_1(T)) - G(x, m_2(T))\Big] m_1(T, dx) \qquad (4.5)\\
&\leq \int_{\overline{\Omega}} \Big[u_1(0, x) - u_2(0, x)\Big] m_1(0, dx) + \int_0^T \int_{\overline{\Omega}} \Big[F(x, m_2(s)) - F(x, m_1(s))\, m_1(s, dx)\Big]\, ds.
\end{aligned}
$$

In a similar way, we obtain

$$\int_{\overline{\Omega}} \Big[G(x,m_2(T)) - G(x,m_1(T))\Big] m_2(T,dx) \tag{4.6}$$
$$\leq \int_{\overline{\Omega}} \Big[u_2(0,x) - u_1(0,x)\Big] m_2(0,dx)$$
$$+ \int_0^T \int_{\overline{\Omega}} \Big[F(x,m_1(s)) - F(x,m_2(s))\Big] m_2(s,dx)\,ds.$$

Summing the inequalities (4.5) and (4.6), we deduce that

$$\int_{\overline{\Omega}} [G(x,m_1(T)) - G(x,m_2(T))]\,(m_1(T,dx) - m_2(T,dx))$$
$$\leq \int_{\overline{\Omega}} [u_1(0,x) - u_2(0,x)]\,(m_1(0,dx) - m_2(0,dx))$$
$$+ \int_0^T \int_{\overline{\Omega}} \Big[F(x,m_2(s)) - F(x,m_1(s))\Big]\,(m_1(s,dx) - m_2(s,dx))\,ds$$

Since $m_1(0,dx) = m_2(0,dx) = m_0$, by using the monotonicity assumption on G and F, we obtain that

$$0 \geq \int_0^T \int_{\overline{\Omega}} \Big[F(x,m_2(s)) - F(x,m_1(s))\Big]\,(m_1(s,dx) - m_2(s,dx))\,ds \geq$$
$$\int_{\overline{\Omega}} \Big[G(x,m_1(T)) - G(x,m_2(T))\Big]\,(m_1(T,dx) - m_2(T,dx)) \geq 0.$$

Therefore,

$$\int_{\overline{\Omega}} \Big[F(x,m_2(s)) - F(x,m_1(s))\Big]\,(m_1(s,dx) - m_2(s,dx)) = 0 \quad \text{a.e. } s \in [0,T],$$

and also

$$\int_{\overline{\Omega}} \Big[G(x,m_1(T)) - G(x,m_2(T))\Big]\,(m_1(T,dx) - m_2(T,dx)) = 0.$$

Thus, by the strict monotonicity of F and G, we conclude that $F(x,m_1) = F(x,m_2)$ for all $x \in \overline{\Omega}$ and $G(x,m_1) = G(x,m_2)$ for all $x \in \overline{\Omega}$. Consequently, we have that J_{η_1} is equal to J_{η_2}. □

Remark 4.1 Suppose that G satisfies (4.2) and F satisfies the following condition

$$\int_{\overline{\Omega}} \Big[F(x,m_1) - F(x,m_2)\Big] d(m_1 - m_2)(x) \;>\; 0,$$

for any $m_1, m_2 \in \mathcal{P}(\overline{\Omega})$ with $m_1 \neq m_2$. Then, proceeding as in the proof of Theorem 4.1, one can show that the mild solution (u, m) of the constrained MFG problem in $\overline{\Omega}$ is unique.

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An Adjoint-Based Approach for a Class of Nonlinear Fokker-Planck Equations and Related Systems

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Abstract Here, we introduce a numerical approach for a class of Fokker-Planck (FP) equations. These equations are the adjoint of the linearization of Hamilton-Jacobi (HJ) equations. Using this structure, we show how to transfer properties of schemes for HJ equations to FP equations. Hence, we get numerical schemes with desirable features such as positivity and mass-preservation. We illustrate this approach in examples that include mean-field games and a crowd motion model.

Keywords Numerical methods · Hamilton-Jacobi equations · Fokker-Planck equations · Mean-field games · Hughes model

1 Introduction

Fokker-Plank (FP) equations model the time evolution of a probability density. The general set up is as follows. Given an open subset of $\mathbb{R}^d$, Ω, a terminal time, $T > 0$, and a (*drift*) vector field, $b(x,t) : \Omega \times [0,T] \to \Omega$, we seek to find a time-dependent probability distribution, $\rho : \Omega \times [0,T] \to \mathbb{R}$, solving

$$\begin{cases} \partial_t \rho - \varepsilon \Delta \rho + \operatorname{div}(b(x,t)\rho) = 0 & \text{in } \Omega \times [0,T], \\ \rho(\cdot,0) = \rho_0(\cdot) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Also, we supplement the above problem with boundary conditions on $\partial\Omega \times [0,T]$, where $\partial\Omega$ is the boundary of Ω. We are particularly interested in problems where

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b may depend on ρ either directly or through an unknown function determined by an additional partial differential equation. Two examples discussed in this paper are the forward-forward mean-field game (MFG) and the Hughes model.

The Fokker-Planck equation was introduced in statistical mechanics. This equation has multiple applications in economics [24, 28], crowd motion models [25, 27, 34], and biological models [14, 23]. Due to the complex structure of those equations, the computation of explicit solutions is not possible. Hence, effective numerical methods for approximating solutions of FP equations have a broad interest.

Here, we propose a technique to obtain approximation schemes for FP equations using their representation as the adjoint of the linearization of Hamilton-Jacobi (HJ) equations. In this way, all monotone numerical schemes proposed in the context of HJ equations give rise to consistent schemes for FP equations. In particular, as required by the nature of the problem, these schemes preserve positivity and are conservative, i.e., under suitable boundary conditions mass is preserved.

Previously, the adjoint structure of the FP equation was used by several authors, for example, in [1] and in [2]. In those references, the authors propose a finite-difference scheme which is the adjoint of the linearization of the upwind scheme used to approximate a convex Hamiltonian. In [9–13], the authors propose a semi-Lagrangian numerical method using a slightly different procedure, but based on a similar principle.

The main contribution of the present paper is to show how to use the adjoint structure with a wide class of numerical solvers, and without limitations on the problem dimension. Here, in contrast to the above references, we develop semi-discrete schemes, where the spatial variable is discretized. To construct the semi-discretization, we apply symbolic calculus to assemble the schemes by exact formula manipulation. The evolution in time corresponds to a system of ordinary differential equations (ODE). These can be solved with different methods, depending on the smoothness of the solution and desired accuracy. The semi-discrete schemes obtained symbolically are compiled before the numerical simulation of the resulting system of ODEs, reducing substantially both the computational time and the work of writing the code of such schemes.

Outline of the Paper We end this introduction with an outline of this paper. The adjoint structure is examined in Sect. 2. Next, in Sect. 3, we proof key features of the method: positivity and mass-conservation. In Sect. 4, we describe our numerical framework, its properties, and discuss sample schemes. Finally, in Sect. 5, consider some problems where our framework applies. These included mean-field games and a crowd motion model.

2 Adjoint Structure

The relation between a FP equation and its adjoint equation is well known. In recent works, [5–8, 15, 21, 35], this relation was used to study regularity properties, vanishing viscosity limits, and rates of convergence of numerical methods. Those results are based on the observation that a FP equation is the adjoint of the linearization of a certain HJ equation.

2.1 Linearization and Duality

Here, we discuss the relation between FP and HJ equations. First, we consider the HJ operator

$$HJ(u) := -u_t(x,t) + H(x, Du(x,t)) - \varepsilon\Delta u(x,t), \tag{2.1}$$

with the Hamiltonian $H = H(x,p) : \Omega \times \mathbb{R}^d \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$. Further, we define the nonlinear generator

$$A^{HJ}u := H(x, Du(x,t)) - \varepsilon\Delta u(x,t).$$

Here, we write $Du = D_x u$ for the gradient in the variable $x = (x_1, \cdots, x_d)$. The parameter ε is called the viscosity.

To linearize (2.1) around a generic point u_0, we expand $u = u_0 + \lambda w$, then take the derivative in λ, and, finally, consider the limit $\lambda \to 0$. Now, we compute this linearization. Boundary conditions are discussed in the next subsection.

The expansion $HJ(u_0 + \lambda w)$ gives

$$\begin{aligned} &-\partial_t(u_0 + \lambda w) + H(x, D(u_0 + \lambda w)) - \varepsilon\Delta(u_0 + \lambda w) \\ &\qquad = -(u_0)_t - \lambda w_t + H(x, Du_0 + \lambda Dw) - \varepsilon\Delta u_0 - \lambda\varepsilon\Delta w. \end{aligned}$$

We suppose H has enough regularity so that we can take the derivative of the preceding expression with respect to λ. Next, we let $\lambda \to 0$, obtaining the operator

$$L(w) := -w_t + D_p H(x, Du_0) \cdot Dw - \varepsilon\Delta w, \tag{2.2}$$

the linearization of the HJ operator. The (linear) generator of L is

$$A^L w := D_p H(x, Du_0) \cdot Dw - \varepsilon\Delta w.$$

Finally, we compute the adjoint of L by integration by parts. We fix smooth functions, w and ρ, and derive the identity

$$\iint_{[0,T]\times\Omega} (-w_t + D_pH(x, Du_0)\cdot Dw - \varepsilon\Delta w)\,\rho \tag{2.3}$$
$$= \iint_{[0,T]\times\Omega} \big(\rho_t - \operatorname{div}_x(D_pH(x, Du_0)\,\rho) - \varepsilon\Delta\rho\big)\,w$$
$$+ \iint_{[0,T]\times\partial\Omega} \big(D_pH(x, Du_0)\big)\cdot n\,\rho\,w + \varepsilon\frac{\partial\rho}{\partial n}w - \varepsilon\rho\frac{\partial w}{\partial n}$$
$$- \int_{\Omega} \rho(x,T)\,w(x,T) - \rho(x,0)\,w(x,0),$$

where n is the normal vector to the boundary, $\partial\Omega$. The last calculation shows that the adjoint of L is the following FP operator

$$L^*\rho := \rho_t - \operatorname{div}_x(D_pH(x, Du_0)\,\rho) - \varepsilon\Delta\rho, \tag{2.4}$$

whose generator is $A^{FP}\rho := -\operatorname{div}_x(D_pH(x, Du_0)\,\rho) - \varepsilon\Delta\rho$.

2.2 Boundary Conditions

Now, we address the boundary conditions for (1.1) on $\partial\Omega\times[0,T]$. The discussion of initial conditions is straightforward. Two common boundary conditions for FP equations are Dirichlet data and a prescribed flow via Neumann conditions. In the Dirichlet case, the data vanishes on the boundary. This corresponds to the case where particles exit once they reach the boundary. The prescribed flow case represents a current of particles or agents crossing the boundary. Thus, with a zero flow, the mass is conserved.

Both the Dirichlet condition and the zero flow Neumann condition determine cancellations in the boundary integrals in (2.3). This suggests different conditions for the HJ operator, its linearized version, and its adjoint, the FP operator.

The first case corresponds to a FP equation with Dirichlet boundary conditions:

$$\begin{cases} \rho_t(x,t) - \operatorname{div}(D_pH(x, Du)\,\rho) = \varepsilon\Delta\rho, & \text{in } \Omega\times[0,T],\\ \rho(\cdot,t) = 0, & \text{on } \partial\Omega\times[0,T]. \end{cases}$$

We consider the HJ operator with the boundary conditions

$$\begin{cases} -u_t(x,t) + H(x, Du(x,t)) - \varepsilon\Delta u(x,t), & \text{in } \Omega\times[0,T],\\ u(\cdot,t) = g_1(\cdot,t), \quad \text{for any } g_1, & \text{on } \partial\Omega\times[0,T], \end{cases}$$

and the linearized operator as

$$\begin{cases} -w_t + D_p H(x, Du) \cdot Dw - \varepsilon \Delta w, & \text{in } \Omega \times [0, T], \\ w(\cdot, t) = 0, & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

The second case corresponds to a FP equation with a flux through the boundary

$$\begin{cases} \rho_t(x, t) - \operatorname{div}(D_p H(x, Du)\, \rho) - \varepsilon \Delta \rho(x, t) = 0 & \text{in } \Omega \times [0, T], \\ D_p H(x, Du) \cdot n\rho + \varepsilon \frac{\partial \rho}{\partial n}(x, t) = g_2(x, t), & \text{on } \partial\Omega \times [0, T], \end{cases}$$

where g_2 is the desired in/out-flow through $\partial\Omega$. We can consider diverse boundary conditions for the HJ operator: Dirichlet type, state-constraint, reflection at the boundary, and Neumann type. In the following example, we use Neumann conditions with zero flow. The Hamilton-Jacobi operator is

$$\begin{cases} -u_t(x, t) + H(x, Du(x, t)) - \varepsilon \Delta u(x, t), & \text{in } \Omega \times [0, T], \\ \frac{\partial u}{\partial n}(x, t) = 0, & \text{on } \partial\Omega \times [0, T], \end{cases}$$

with the corresponding linearization

$$\begin{cases} -w_t + D_p H(x, Du) \cdot Dw - \varepsilon \Delta w, & \text{in } \Omega \times [0, T], \\ \frac{\partial w}{\partial n}(\cdot, t) = 0, & \text{on } \partial\Omega \times [0, T]. \end{cases}$$

In the schemes we describe in Sect. 4, we focus on the spatial discretization of the previous operators. The time discretization can be chosen separately, depending on the application. This is the reason we simplified the discussion above and we avoided initial conditions, which are straightforward.

We now discuss the connection between stochastic differential equations with their density formulation, the Fokker-Planck equation, and then the associated Hamiltonian, via the adjoint structure. This pathwise interpretation given by the solution of the SDE provides understanding on how to treat boundary conditions.

A nonlinear FP equation is related to the solution of a stochastic differential equation of McKean-Vlasov type (or mean-field type), see [29–31, 33]. More precisely, we consider the stochastic differential equation (SDE)

$$\begin{cases} dX(t) = b(X(t), \rho(X(t), t), t)\, dt + \sqrt{2\varepsilon}\, dW(t), \\ X(0) = X^0, \end{cases} \tag{2.5}$$

where $b : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}^d$ is a regular vector-valued function, X^0 is a random vector in $\mathbb{R}^d$, independent of the Brownian motion $W(\cdot)$, with density ρ_0, and $\rho(\cdot, t)$ is the density of $X(t)$. It can be shown (see [26]) that under suitable growth and regularity conditions for b (2.5) admits a unique solution and ρ is the unique classical solution of the nonlinear FP equation

$$\partial_t \rho - \varepsilon \Delta \rho + \operatorname{div}(b(x, \rho, t)\rho) = 0.$$

With Dirichlet conditions, those trajectories end at the boundary; for zero-flux conditions, they are reflected, see [3, 19]. Moreover, the Hamilton-Jacobi operator from which (2.2) can be derived is

$$-u_t - \varepsilon \Delta u - b(x, \rho, t) \cdot Du.$$

Remark 2.1 Our methods can be extended to study stationary FP equations. In this case, the associated Hamilton-Jacobi operator is stationary.

3 Properties

In this section, we use a duality argument to obtain properties of FP equations from corresponding properties of the evolution semigroup associated with a HJ equation. The arguments detailed here are valid without any substantial changes for the discretized problems using the semigroups associated with the discretized equation. This discretization can be performed in multiple ways, including finite-differences and semi-Lagrangian schemes.

We denote by $\langle f, g\rangle = \int_\Omega f\; g$ the duality product, and by S_t the semigroup corresponding to the evolution in time given by the linearized operator (2.2). This semigroup preserves order; that is $v \leqslant w$ implies $S_t v \leqslant S_t w$. Moreover, we assume that $S_t 1 = 1$. We note that the property $S_t 1 = 1$ depends on the boundary conditions for (2.2). We define the adjoint S_t^* of S_t by

$$\langle S_t^* u, v\rangle = \langle u, S_t v\rangle.$$

We have then the following results:

Proposition 3.1 (Positivity) *Suppose S_t is monotone. Then, the evolution of the initial density ρ_0 through the adjoint semigroup, S_t^*, preserves positivity. That is, if $\rho_0 \geqslant 0$, we have $S_t^* \rho \geqslant 0$, for all $t \in [0, T]$.*

Proof Denote by w_T the terminal condition for the linearized operator. First, note that $w_T \geqslant 0$ implies $S_t w_T \geqslant 0$. This follows from the maximum principle for HJ equations. Thus, for $w_T \geqslant 0$, we have

$$\langle S_t^* \rho, w_T\rangle = \langle \rho, S_t w_T\rangle \geqslant 0,$$

since $\rho \geqslant 0$, and $S_t w_T \geqslant 0$. Accordingly, $S_t^* \rho \geqslant 0$. ■

Now, we show that if $S_t 1 = 1$ the mass is conserved by S_t^*. For example, under periodic boundary conditions, the evolution of (2.2) preserves constants. This is not the case under Dirichlet boundary conditions, where mass loss through the boundary occurs.

Proposition 3.2 (Conservation of Mass) *Suppose $S_t 1 = 1$. Let ρ_0 be the initial density probability distribution, i.e. $\int_\Omega \rho_0 = 1$. Then, for all $t \in [0, T]$, the evolution of this probability measure through the adjoint semigroup, $S_t^* \rho_0$, is also a probability measure.*

Proof First, observe that $S_t 1 = 1$. Then,

$$\int_\Omega S_t^* \rho_0 = \langle S_t^* \rho_0, 1\rangle = \langle \rho_0, S_t 1\rangle = \langle \rho_0, 1\rangle = \int_\Omega \rho_0 = 1.$$

■

Remark 3.3 The preceding assumptions on S_t are not restrictive for our applications. For discrete problems, the linearized semigroups S_t for HJ equations are usually monotone and, in cases where mass conservation for the FP equation holds, also satisfy $S_t 1 = 1$. Often, it is easier to check monotonicity and that $S_t 1 = 1$ than the corresponding properties for S_t^*.

Remark 3.4 If the viscosity vanishes ($\varepsilon = 0$), our approach is still valid for classical solutions (for first-order equations and systems regularity may fail and our computations need to be justified case by case). A first-order HJ operator gives rise to a continuity equation (CE), i.e. a FP equation without viscosity. This case is considered in Sect. 5, where we extend our numerical scheme to address systems of partial differential equations (PDEs). Those systems arise in multiple applications such as MFGs, population models, traffic flow problems, and modeling in chemotaxis.

4 Numerical Approach

Our numerical approach relies on the relation between the HJ framework and the corresponding adjoint FP equation. Once we choose a semi-discrete (discrete in space) numerical scheme for (2.1), we can reuse it to construct an approximation for (2.4).

Before proceeding, we define additional notation. To simplify, we consider a scheme for the case where the domain Ω is $\mathbb{T}^2$ (2-D torus). Let $\mathbb{T}^2_{\Delta x}$ be an uniform grid on $\mathbb{T}^2$, with constant discretization parameter $\Delta x > 0$. Let $x_{i,j}$ denote a generic point in $\mathbb{T}^2_{\Delta x}$. The space of grid functions defined on $\mathbb{T}^2_{\Delta x}$ is denoted by $\mathcal{G}(\mathbb{T}^2_{\Delta x})$, and the functions $U,\ M \in \mathcal{G}(\mathbb{T}^2_{\Delta x})$ (approximations of respectively u and ρ) are called $U_{i,j}$ and $M_{i,j}$, when evaluated at $x_{i,j}$. We utilize a semi-discrete numerical scheme $N(x, p) : \mathbb{T}^2_{\Delta x} \times \mathbb{R}^d \to \mathbb{R}$ monotone and consistent to approximate the operator

$H(x, p)$ by the discrete operator

$$N(x, \mathcal{D}U), \tag{4.1}$$

where $\mathcal{D}U$ is a discretization of the gradient operator on U. Thanks to the adjoint structure, we use the scheme (4.1) to assemble a discrete operator K

$$K(x, \mathcal{D}U, M) := (D_U N(x, \mathcal{D}U))^T M, \tag{4.2}$$

that discretizes the spatial part of the FP operator (2.4). The discrete approximation M for the solution of the FP equation is then given by the ODE

$$M_t - K(x, \mathcal{D}U, M) - \varepsilon \Delta_d M = 0,$$

or simply,

$$M_t - (D_U N(x, \mathcal{D}U))^T M - \varepsilon \Delta_d M = 0, \tag{4.3}$$

i.e., a discrete equivalent to the adjoint structure seen in (2.4). Here, the nonlinear part of the operator corresponds to the discrete operator $D_U N(x_{i,j}, \mathcal{D}U)$, and $\Delta_d M$ is a discretization of the Laplacian, which is added to the scheme to increase the stability, if necessary.

We note that the operators N and K depend on the monotone approximation scheme used to discretize the HJ equation. The operator K can be computed using a symbolic differentiation as we show in Sect. 5.1. Also, the Hamiltonian $H(x, p)$ must be sufficiently regular in the p variable so that the scheme is properly defined. The properties of positivity and mass conservation are valid at the discrete level as consequence of the semigroup arguments in Sect. 3.

We now prove how the consistency of schemes for HJ equations transfers to schemes for FP equations.

Proposition 4.1 (Consistency) *Suppose u is the solution to a linerized HJ equation and ρ the solution to the associated FP equation. Also, suppose that u and ρ are C^∞, and consider their restriction to the grid points. Denote by S_t^N the linearized semigroup corresponding to a discretization of the HJ equation with $o(1)$ error, i.e.,*

$$S_t^N u = S_t u + o(1), \tag{4.4}$$

with S_t as in the previous sections. Then, the adjoint semigroup $(S_t^N)^*$ operating on the discretization of the FP equation possesses the same order of error, i.e.,

$$(S_t^N)^* \rho = (S_t)^* \rho + o(1). \tag{4.5}$$

Proof By the adjoint structure between the HJ and FP equations, and the hypothesis of consistency in Eq. (4.4), we have

$$\langle (S_t^N)^* \rho, u\rangle = \langle \rho, S_t^N u\rangle = \langle \rho, S_t u\rangle + \langle \rho, o(1)\rangle = \langle S_t^* \rho, u\rangle + o(1),$$

which proves (4.5). ■

Remark 4.2 (Convergence) We stress that the main novelty in the current paper is a systematic approach to build schemes for FP equations from schemes for HJ equations. Naturally, different schemes will have different convergence properties, which must be examined case by case. Furthermore, to address the convergence of a method, we must know the regularity of the vector field in the FP equation and the existence of a solution. In the numerical simulations developed in Sect. 5, this regularity is not always known. Also, for the crowd motion problem we consider, the existence of solutions is not known.

4.1 Finite Differences

Now, we consider an explicit scheme using our method. We describe an upwind discretization for the Hamiltonian, which we assume to be

$$H(x,p) = g(x) + |p|^\alpha, \text{ with } \alpha > 1, \text{ and } p = (p_1, p_2, p_3, p_4). \tag{4.6}$$

We define the standard finite-difference operators as

$$(\mathcal{D}_1^{\pm} u)_{i,j} = \frac{u_{i\pm1,j} - u_{i,j}}{\Delta x}, \quad (\mathcal{D}_2^{\pm} u)_{i,j} = \frac{u_{i,j\pm1} - u_{i,j}}{\Delta x},$$

and

$$\Delta_d u = \frac{1}{\Delta x^2}\left(4u_{i,j} - u_{i+1,j} - u_{i,j+1} - u_{i-1,j} - u_{i,j-1}\right).$$

The approximation of the operator $H(x,p)$ is

$$N(x,p) = g(x) + G(p_1^-, p_2^+, p_3^-, p_4^+),$$

where for a real number r, we define the operators

$$r^+ := \max(0,r), \quad r^- := \max(0,-r), \tag{4.7}$$

and

$$G(p) = G(p_1, p_2, p_3, p_4) := (p_1^2 + p_2^2 + p_3^2 + p_4^2)^{\frac{\alpha}{2}}.$$

The operators r^+ and r^- are chosen to preserve the monotonicity of the scheme for the HJ operator, which is well defined backward in time. Plugging the finite differences operators in $N(x, p)$ we have

$$N(x_{i,j}, [\mathcal{D}U]_{i,j}) = g(x) + G((\mathcal{D}_1^+ u)^-_{i,j}, (\mathcal{D}_1^- u)^+_{i,j}, (\mathcal{D}_2^+ u)^-_{i,j}, (\mathcal{D}_2^- u)^+_{i,j}) =$$
$$g(x) + \left[\left((\mathcal{D}_1^+ u)^-_{i,j}\right)^2 + \left((\mathcal{D}_1^- u)^+_{i,j}\right)^2 + \left((\mathcal{D}_2^+ u)^-_{i,j}\right)^2 + \left((\mathcal{D}_2^- u)^+_{i,j}\right)^2\right]^{\frac{\alpha}{2}}.$$

Now, we compute the operator $K(x, \mathcal{D}U, M)$, and we obtain

$$K(x_{i,j}, [\mathcal{D}U]_{i,j}, M_{i,j}) =$$
$$\frac{1}{\Delta x}\Bigg[M_{i,j}\frac{\partial N}{\partial p_1}(x_{i,j}, [\mathcal{D}U]_{i,j}) - M_{i-1,j}\frac{\partial N}{\partial p_1}(x_{i-1,j}, [\mathcal{D}U]_{i-1,j})$$
$$+ M_{i+1,j}\frac{\partial N}{\partial p_2}(x_{i+1,j}, [\mathcal{D}U]_{i+1,j}) - M_{i,j}\frac{\partial N}{\partial p_2}(x_{i,j}, [\mathcal{D}U]_{i,j})$$
$$+ M_{i,j}\frac{\partial N}{\partial p_3}(x_{i,j}, [\mathcal{D}U]_{i,j}) - M_{i,j-1}\frac{\partial N}{\partial p_3}(x_{i,j-1}, [\mathcal{D}U]_{i,j-1})$$
$$+M_{i,j+1}\frac{\partial N}{\partial p_4}(x_{i,j+1}, [\mathcal{D}U]_{i,j+1}) - M_{i,j}\frac{\partial N}{\partial p_4}(x_{i,j}, [\mathcal{D}U]_{i,j})\Bigg]. \tag{4.8}$$

We then use this expression for the operator in (4.2).

This scheme coincides with the adaptation of the techniques proposed in [1] to our case. The advantage here is the fact that the properties of positivity and mass conservation are automatically obtained from the hypotheses on the discretization of the HJ operator; unlike typical schemes for FP equations, where such properties must be proved a posteriori. Analogously, in case we have chosen the semi-Lagrangian scheme for HJ equations previously presented, we would be creating a "dual semi-Lagrangian" method for FP equations. This is the main point in this work: the possibility of generating "dual methods" with desired properties for FP equations based on originally well-established methods for HJ equations. We explicit this possible method in the following section.

4.2 Semi-Lagrangian Scheme

To describe a semi-Lagrangian scheme appropriate to approximate (4.6), we introduce the operator

$$\mathcal{D}^\gamma U_{i,j} := \max_{\gamma \in B(0,1)} \frac{\mathcal{I}[U](x_{i,j}, \gamma) - U(x_{i,j})}{h}, \tag{4.9}$$

where $B(0, 1)$ is the unitary ball in $\mathbb{R}^2$, h a parameter of the same order of $\sqrt{\Delta x}$, and

$$\mathcal{I}[U](x_{i,j}, \gamma) = \frac{1}{2}\sum_{i=1}^{2}\Big(\mathbb{I}[U](x_{i,j} + \gamma h + e_i\sqrt{2\varepsilon h}) + \mathbb{I}[U](x_{i,j} + \gamma h - e_i\sqrt{2\varepsilon h})\Big). \tag{4.10}$$

Here, $\mathbb{I}[u](x)$ is an interpolation operator on the matrix U, and e_i is the i unitary vector of an orthonormal basis of the space. Details on how to choose the interpolation operator are discussed in [16]. In our case, the discrete operator has the form

$$N(x, \mathcal{D}^\gamma U_{i,j}) := g(x) + (\mathcal{D}^\gamma U_{i,j})^\alpha. \tag{4.11}$$

We take the adjoint of the linearized of N, by using (4.9), and we apply it to (4.2); analogously as performed for the finite-difference scheme. This scheme formally differs from the one proposed in [13], since the parameter γ in (4.10) is computed in the dual formulation of the problem. We note that the operator $N(x, p)$ in (4.11) is monotone by construction, see [16].

5 Applications to Systems of PDEs

One immediate application of our numerical scheme is to solve "measure-potential" systems of PDEs. These systems comprise an equation for the evolution of a measure coupled with a second equation for a potential, or value function. Typically, this potential determines the drift for the convection in the first equation. Many problems have this structure: mean-field games, traffic-flow models, crowd motion, and chemotaxis.

Here, we describe how to use our method in the following examples: two 1-D forward-forward mean-field games (FFMFG) problems and a crowd motion model. All the simulations were performed on a 2.3GHz i7 computer with 16GB of RAM.

5.1 Example: Hughes Model in 1-D

Now, we illustrate the application of our numerical approach to a model for crowd motion model due to Hughes [25]. We discuss this simple 1-D model with the intention of showing the steps connected to the adjoint structure, the discretization of the corresponding operators, and the extension of the methodology to measure-potential systems of PDEs.

The Hughes model comprises a FP equation, describing the evolution of the density of pedestrians/agents, coupled to an Eikonal (EK) equation that gives the optimal movement direction. This system is

$$\begin{cases} \rho_t(x,t) - \operatorname{div}(\rho(1-\rho)^2 Du) = 0, \\ |Du(x)|^2 = \dfrac{1}{(1-\rho)^2}, \end{cases} \tag{5.1}$$

together with an initial condition for the density and Dirichlet/Neumann boundary conditions. The goal is to exit a domain Ω in minimal time, taking into account congestion effects. Due to the stationary character of the EK equation, this system is not of MFG type. The density, ρ, evolves as if at each instant of time, the EK equation sees a frozen density. Then the agents choose the direction that leads to the shortest-time to evacuation, and this process determines the evolution of ρ.

We now describe how the Hughes system can be studied via our framework. Performing the same steps as in Sect. 2, with the HJ operator

$$-u_t + f(\rho)H(x, Du) - \varepsilon\Delta u, \tag{5.2}$$

where $f(\rho)$ is a regular function of the density, we obtain the associated FP equation

$$\rho_t - \operatorname{div}\left(f(\rho)D_pH(x, Du)\rho\right) = \varepsilon\Delta u. \tag{5.3}$$

By setting $f(\rho) = (1-\rho)^2$, and $H(x,p) = \dfrac{|p|^2}{2}$, (5.3) becomes the first equation of (5.1); and (5.2) is the adjoint operator we must study. More explicitly, starting by the spatial part of the HJ equation, the adjoint structure is as follows:

$$(1-\rho)^2\,\frac{|Du|^2}{2} \xrightarrow{\text{Linearization}} (1-\rho)^2\; Du\cdot Dw \xrightarrow{\text{Adjoint}} -\operatorname{div}\left[\rho(1-\rho)^2 Du\right],$$

where u, w, and ρ follow the notation of Sect. 2.

We illustrate the use of finite differences to discretize the generator of the HJ operator. We choose the monotone scheme:

$$N_n(u) \equiv\; (1-\rho_n)^2\left[\frac{\max\{u_n - u_{n-1}, 0\}^2}{2h^2} + \frac{\max\{u_n - u_{n+1}, 0\}^2}{2h^2}\right]. \tag{5.4}$$

Once we discretize the domain Ω (a finite interval of $\mathbb{R}$ in this example), we calculate the previous discrete operators in each grid point, taking also in account the discretized versions of the boundary conditions. We have then a matrix, with the dimensions of the grid, whose entries have the expressions of the form of (5.4). The matrix entries (a vector in the 1-D case) have the u_j's ($j \in \{0, 1, \ldots, P\}$, P being the size of the grid in this 1-D example) as a parameter. The numerical value of this matrix will only be evaluated once we have an approximation for the values of u. This will be done by solving the EK equation with an initial guess. Since the EK equation is a particular case of a HJ equation, we choose to discretize it in space in the same form as the HJ operator associated to the FP equation. We use the monotone scheme

$$\tilde{N}_n(u) \equiv \frac{\max\{u_n - u_{n-1}, 0\}^2}{h^2} + \frac{\max\{u_n - u_{n+1}, 0\}^2}{h^2} - \frac{1}{(1-\rho_n)^2}. \tag{5.5}$$

We are then left to solve, at discrete level, the system

$$\begin{cases} \rho_t + (D_u N_n(u))^T \rho = 0, \\ \tilde{N}_n(u) = 0, \end{cases} \tag{5.6}$$

at each point of the grid. In this way, all the spatial part of the operators was treated. First equation of (5.6) is a time-dependent matrix ODE in ρ, whose spatial part has u_j's as parameters (updated by the solution of the second equation). This ODE is supplemented with a discretization of the initial condition for the density, defined in the original continuous Hughes system (5.1). The second equation, the discretized version of the EK equation, is a difference equation in the u_j variable, with ρ_j as a parameter (updated by the approximation of the solution of the first equation at each time iteration).

Up to now, we have only treated the spatial part of the operators and we have made the choice of discretization with a monotone finite-difference scheme. To solve in time the ODEs in (5.6), we can use standard solvers either explicit or implicit. Since in the practice of the tests presented in this work, the matrix $(D_u N_n(u))^T$ is not stiff, we opted for reasons of simplicity and low computing time for an explicit Euler method. In the case of less regular Hamiltonian and consequently less stable systems we recommend the use of Backward Differentiation Formulas (BDF).

For the second equation in (5.6), which contain a high dimensional non-linear optimization problem, we use a fixed-point iteration as in [32]. More sophisticated techniques can give improved performance for this numerical step, as the policy iteration algorithm [18] or domain decomposition techniques [4, 17].

5.2 *Example: 1-D Forward-Forward Mean-Field Games*

Here, we consider two one-dimensional forward-forward mean-field game problems, see [1, 20, 22]. A special case of such systems is

$$\begin{cases} u_t + H(u_x) = \varepsilon u_{xx} + g(\rho), \\ \rho_t - (H'(u_x)\rho)_x = \varepsilon \rho_{xx}, \end{cases} \tag{5.7}$$

together with the *initial-initial conditions*

$$\begin{cases} u(x,0) = u_0(x), \\ \rho(x,0) = \rho_0(x), \end{cases}$$

and periodic boundary conditions.

Now, we explain how such systems were treated numerically. MFGs have built-in the adjoint structure we have considered so far, i.e., the FP equation of a MFG system is the adjoint of the linearization of the HJ equation present in the MFG system. Hence, we can use the same type of spatial schemes for the discretization of both the FP and the HJ equation. As described in the previous section, each of the discretizations requires solving an ODE in time. Since we must solve the system of FP coupled to a HJ equation, we treat these ODEs as a single system, and we apply a suitable solver for the time discretization. This manner of solving is possible because we are solving a FFMFG, where both equations evolve forward in time. For backward-forward MFG, either a fixed point iteration between the resolution of the two systems or a simultaneous resolution is required. In the next two problems, we use a monotone finite-difference method for the spatial discretization, as in Sect. 4.1, and an explicit Euler method in time.

For the first problem, we set $H(u_x) = \dfrac{u_x^2}{2}$, $g(\rho) = \ln \rho$, and $\varepsilon = 0.01$. We then solve:

$$\begin{cases} u_t + \frac{u_x^2}{2} = 0.01\, u_{xx} + \ln \rho, \\ \rho_t - (u_x \rho)_x = 0.01\, \rho_{xx}, \end{cases} \tag{5.8}$$

with the *initial-initial conditions*:

$$\begin{cases} u_0(x) = 0.3\cos(2\pi x), \\ \rho_0(x) = 1. \end{cases}$$

We depict the solution of this problem in Fig. 1. Here, we see how the oscillations in the evolution of the system reduce illustrating the convergence, for $t \to +\infty$, to a stationary solution. This convergence for systems as (5.8), for $t \to +\infty$ was discussed in [22].

Now, for the second case, we choose $H(u_x, \rho) = \dfrac{(p + u_x)^2}{2\rho^\alpha}$, $g(\rho) = \frac{3}{2}\rho^\alpha$, and $\varepsilon = 0$. This example is a first-order FFMFG with congestion, which is equivalent to a system of conservation laws. Setting $v = p + u_x$, the equivalent system is

$$\begin{cases} v_t + \left(\frac{v^2}{2\rho^\alpha} - \frac{3}{2}\rho^\alpha\right)_x = 0, \\ \rho_t - \left(\rho^{1-\alpha} v\right)_x = 0. \end{cases} \tag{5.9}$$

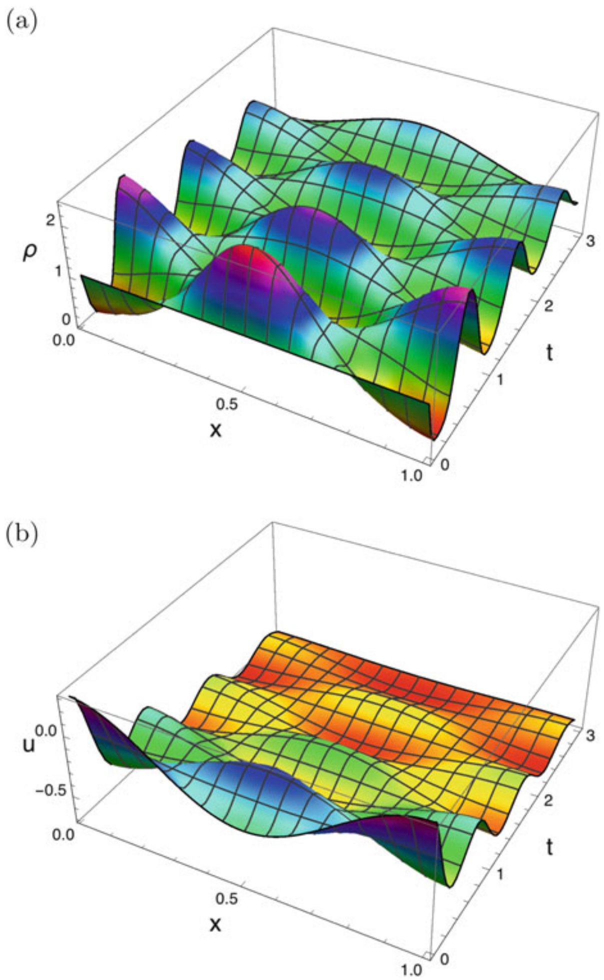

Fig. 1 Solutions for $g(\rho) = \ln \rho$. (**a**) Density. (**b**) Value function

For $\alpha = 1$, and for the *initial-initial conditions*

$$\begin{cases} u_0 = -0.5\,\frac{\cos(2\pi x)}{2\pi}, \\ \rho_0 = 1 + 0.5\sin(2\pi x), \end{cases}$$

the solution for the density in (5.9) is a traveling wave, depicted in Fig. 2. Such failure of convergence is an interesting phenomenon identified in [20] for first-order forward-forward MFGs, and it illustrates an important difference between these two MFG models.

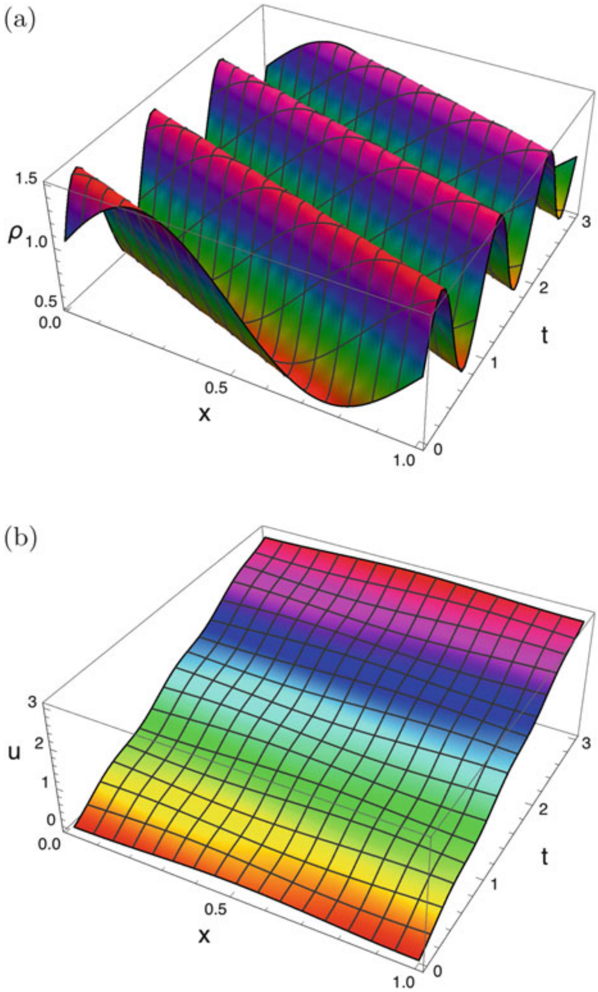

Fig. 2 Solutions for the FFMFG with congestion. (**a**) Density. (**b**) Value function

The simulations corresponding to Figs. 1 and 2 were produced with a spatial grid of 80 points, final time $T = 3$, and 50 points for the time sample. The simulations run in about 1 s.

5.3 *Example: Hughes Model in 2-D*

In this section, we illustrate our approach for 2-D models. We repeat the steps of Sect. 5.1, with the modification of choosing a monotone finite-difference scheme in 2-D. The domain of our example is a rectangle $[0, 3] \times [0, 1]$, modeling a room with walls, and an exit on $[2.25, 3] \times \{1\}$, corresponding to a typical proportion of the size of a door in a room. We set the value of u to $+\infty$ on all the boundary but on its exit, where we fix it equal to zero. The density is set equal zero on the boundary.

We perform the simulations analogously to the Hughes model in 1-D. We depict the initial condition and its evolution in Fig. 3. The spatial grid contains 243 points, and we choose the final time $T = 1.0$. The simulation runs in 12.9 s. We remark that, at each time iteration for the solution of the Fokker-Planck equation one Eikonal equation in 2-D is being solved. In this example, 83 Eikonal equations were solve and they took 78% of the total simulation time.

We end this section by remarking that, in all our simulations both positivity and mass preservation were observed, except when agents are leaving through the boundary; clearly, in the Hughes model, the mass is preserved as long as no agents reach the exit.

6 Conclusions

In this work, we develop an approach for the approximation of nonlinear Fokker-Planck equations via its adjoint Hamilton-Jacobi operator. Our methodology guarantees that the produced schemes preserve mass and positivity. Consistency is also addressed. We then solve systems of PDEs with a Fokker-Planck equation coupled to a Hamilton-Jacobi equation. Our methods apply to a broad range of problems where a measure-potential structure appears, including mean-field games, crowd and traffic models, and chemotaxis.

In future work, we plan to address different schemes developed for HJ equations to study FP equations. Originally, schemes such as the Discontinuous Galerkin or ENO schemes were developed for conservation laws and later gave rise to effective numerical schemes for HJ equations. With our methods we can reverse this process by starting with schemes for Hamilton-Jacobi equations and then deriving schemes for FP equations. Nevertheless, it is clear that, without monotonicity and stability properties, results for the convergence of such schemes are difficult to achieve.

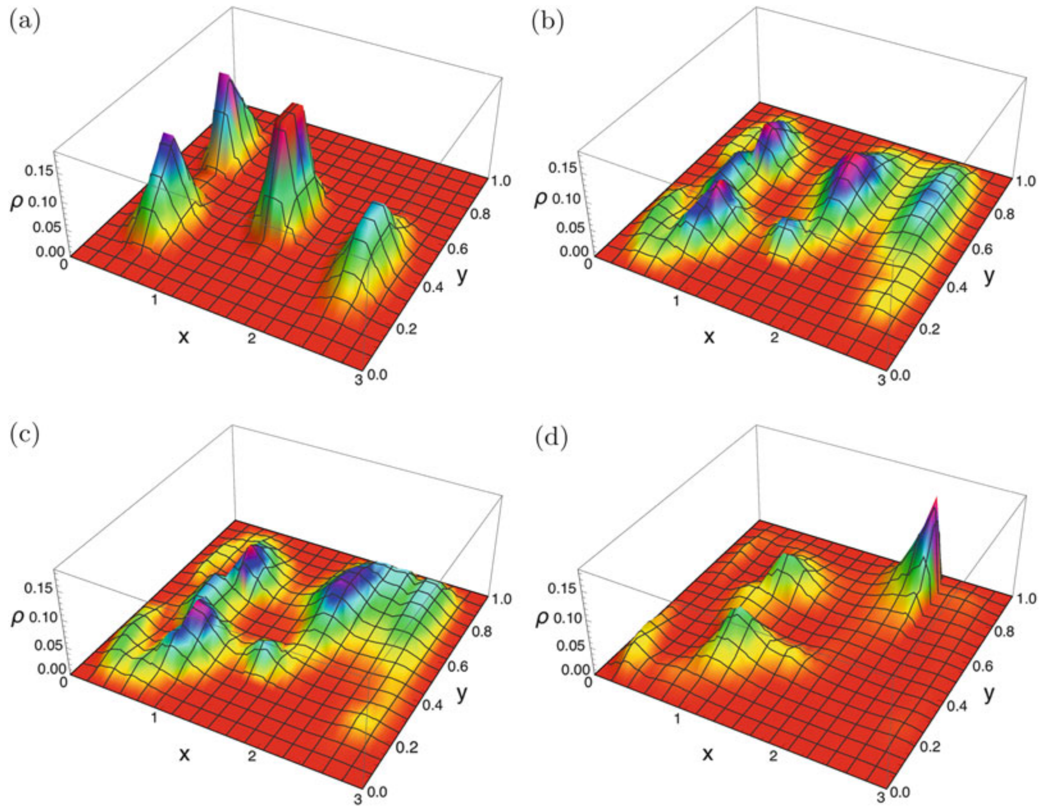

Fig. 3 Evolution of the density for the Hughes model. (**a**) Initial density. (**b**) Density at time 0.33. (**c**) Density at time 0.5. (**d**) Final density at time 1.0

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Variational Mean Field Games for Market Competition

Philip Jameson Graber and Charafeddine Mouzouni

Abstract In this paper, we explore Bertrand and Cournot Mean Field Games models for market competition with reflection boundary conditions. We prove existence, uniqueness and regularity of solutions to the system of equations, and show that this system can be written as an optimality condition of a convex minimization problem. We also provide a short proof of uniqueness to the system addressed in Graber and Bensoussan (Appl Math Optim 77:47–71, 2018), where uniqueness was only proved for small parameters ϵ. Finally, we prove existence and uniqueness of weak solutions to the corresponding first order system at the deterministic limit.

Keywords Cournot competition · Extended mean field games · Optimal control · Forward-backward systems of PDE

1 Introduction

Our purpose is to study the following coupled system of partial differential equations:

$$\begin{cases} (i)\ u_t + \frac{\sigma^2}{2}u_{xx} - ru + G(u_x, m)^2 = 0,\ 0 < t < T,\ 0 < x < L \\ (ii)\ m_t - \frac{\sigma^2}{2}m_{xx} - \{G(u_x, m)m\}_x = 0,\ 0 < t < T,\ 0 < x < L \\ (iii)\ m(0,x) = m_0(x), u(T,x) = u_T(x), \qquad 0 \le x \le L \\ (iv) \qquad u_x(t,0) = u_x(t,L) = 0, \qquad 0 \le t \le T \\ (v)\ \frac{\sigma^2}{2}m_x(t,x) + G(u_x, m)m(t,x) = 0,\ \ 0 \le t \le T, x \in \{0, L\} \end{cases} \tag{1}$$

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where $G(u_x, m) := \frac{1}{2}\left(b + c\int_0^L u_x(t,y)m(t,y)\,\mathrm{d}y - u_x\right)$, σ, b, c, T, L are given positive constants, and $m_0(x)$, $u_T(x)$ are known functions.

System (1) is in the family of models introduced by Guéant et al. [26] as well as by Chan and Sircar in [16, 17] to describe a *mean field game* in which producers compete to sell an exhaustible resource such as oil. The basic notion of mean field games (MFG) was introduced by Lasry and Lions [28–30] and Huang et al. [27]. Here we view the producers as a continuum of rational agents whose is given by the function $m(t, x)$ governed by a Fokker-Planck equation. Each of them must solve an optimal control problem in order to optimize profit, which corresponds to the Hamilton-Jacobi-Bellman equation (1)(i). A solution to the coupled system therefore corresponds (formally) to a Nash equilibrium among infinitely many competitors in the market.

The analysis of this type of PDE system was already addressed in [25] with Dirichlet boundary conditions at $x = 0$. It is a framework where producers have limited stock, and they leave the market as soon as their stock is exhausted. In particular, the density of players is a non-increasing function [25]. By contrast, in studying system (1) we explore a new boundary condition. In terms of the model, we assume that players never leave the game so that the number of producers in the market remains constant. In this particular case, the density of players is a probability density for all the times, which considerably simplifies the analysis of the system of equations. Further details on the interpretation of the problem will be given below in Sect. 1.1.

Applications of mean field games to economics have attracted much recent interest; see [1, 6, 20] for surveys of the topic. Nevertheless, most results from the PDE literature on mean field games are not sufficient to establish well-posedness for models of market behavior such as (1). In particular, many authors have studied existence and uniqueness of solutions to systems of the type

$$\begin{aligned} u_t + \tfrac{1}{2}\sigma^2 u_{xx} - ru + H(t,x,u_x) &= V[m], \\ m_t - \tfrac{1}{2}\sigma^2 m_{xx} - (G(t,x,u_x)m)_x &= 0. \end{aligned} \tag{2}$$

See, for example, [9, 10, 12–14, 18, 22, 23, 31]. In all of these references, the equilibrium condition is determined solely through the distribution of the state variable, rather than that of the control. That is, each player faces a cost determined by the distribution of positions, but not decisions, of other players. For economic production models, by contrast, players must optimize against a cost determined by the distribution of *controls*, since the market price is determined by aggregating all the prices (or quantities) set by individual firms. A mathematical framework which takes this assumption into account has been called both “extended mean field games” [19, 21] and “mean field games of controls” [11]. However, other than the results of [11, 19, 21], there appear to be few existence and uniqueness theorems for PDE models of this type. One of the main difficulties appears to be that the coupling is inherently nonlocal, a feature which is manifest in (1) through the integral term $\int_0^L u_x m\,\mathrm{d}x$.

Inspired by Graber and Bensoussan [25], our goal in this article is to prove the existence and uniqueness of solutions to (1). Because of the change in boundary conditions, many of the arguments becomes considerably simpler and stronger results are possible. Let us now outline our main results. We show in Sect. 2 that there exists a unique classical solution of System (1). Note that, whereas in [25], uniqueness was only proved for small values of $\epsilon := 2c/(1-c)$ (cf. the interpretation in the following subsection), here we improve that result by showing that solutions are unique in general (including in the case of Dirichlet boundary conditions). We show in Sect. 3 that (1) has an interpretation as a system of optimality for a convex minimization problem. Although this feature has been noticed and exploited for mean field games with congestion penalization (see [5] for an overview), here we show that it is also true for certain extended mean field games (cf. [24]). Finally, in Sect. 4 we give an existence result for the first order case where $\sigma = 0$, using a "vanishing viscosity" argument by collecting a priori estimates from Sects. 2 and 3.

1.1 Explanation of the Model

We summarize the interpretation of (1) as follows. Let t be time and x be the producer's capacity. We assume there is a large set of producers and represent it as a continuum.

The first equation in (1) is the Hamilton-Jacobi-Bellman (HJB) equation for the maximization of profit. Each producer's capacity is driven by a stochastic differential equation

$$dX(s) = -q(s)ds + \sigma\, dW(s), \tag{3}$$

where q is determined by the price p through a linear demand schedule

$$q = D(p, \bar{p}) = \frac{1}{1+\epsilon} - p + \frac{\epsilon}{1+\epsilon}\bar{p}, \quad \eta > 0. \tag{4}$$

The presence of noise expresses the short term unpredictable fluctuations of the demand [16]. In (4) $\bar{p}$ represents the market price, that is, the average price offered by all producers; and ϵ is the product substitutability, with $\epsilon = 0$ corresponding to independent goods and $\epsilon = +\infty$ implying perfect substitutability. Thus each producer competes with all the others by responding to the market price.

We define the value function

$$u(t,x) := \sup_{p} \mathbb{E}\left\{\int_t^T e^{-r(s-t)} p(s)q(s)ds + e^{-r(T-t)}u_T(X(T)) \mid X(t) = x\right\} \tag{5}$$

where $q(s)$ is given in terms of $p(s)$ by (4). The optimization problem (5) has the corresponding Hamilton-Jacobi-Bellman equation

$$u_t + \frac{1}{2}\sigma^2 u_{xx} - ru + \max_p \left[\left(\frac{1}{1+\epsilon} - p + \frac{\epsilon}{1+\epsilon}\bar{p}(t)\right)(p - u_x)\right] = 0. \tag{6}$$

The optimal $p^*(t,x)$ satisfies the first order condition

$$p^*(t,x) = \frac{1}{2}\left(\frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon}\bar{p}(t) + u_x(t,x)\right), \tag{7}$$

and we take $q^*(t,x)$ to be the corresponding demand

$$q^*(t,x) = \frac{1}{2}\left(\frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon}\bar{p}(t) - u_x(t,x)\right). \tag{8}$$

Therefore (6) becomes

$$u_t + \frac{1}{2}\sigma^2 u_{xx} - ru + \frac{1}{4}\left(\frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon}\bar{p}(t) - u_x\right)^2 = 0. \tag{9}$$

On the other hand, the density of producers $m(t,x)$ is transported by the optimal control (8); it is governed by the Fokker-Planck equation

$$m_t - (\frac{1}{2}\sigma^2 m)_{xx} - \frac{1}{2}\left(\left(\frac{1}{1+\epsilon} + \frac{\epsilon}{1+\epsilon}\bar{p}(t) - u_x\right)m\right)_x = 0. \tag{10}$$

The coupling takes place through a market clearing condition. With $p^*(t,x)$ the Nash equilibrium price we must have

$$\bar{p}(t) = \int_0^L p^*(t,x)m(t,x)\,\mathrm{d}x, \tag{11}$$

which, thanks to (7), can be rewritten

$$\bar{p}(t) = \frac{1}{2+\epsilon} + \frac{1+\epsilon}{2+\epsilon}\int_0^L u_x(t,x)m(t,x)\,\mathrm{d}x. \tag{12}$$

We recover System (1) by setting

$$b = \frac{2}{2+\epsilon},\ c = \frac{\epsilon}{2+\epsilon}. \tag{13}$$

Boundary Conditions We assume that the maximum capacity of all producers does not exceed $L > 0$. We consider a situation where players are able to renew their stock after exhaustion, so that players stay all the time with a non empty stock. For the sake of simplicity, we do not consider the implications of stock renewal on the cost structure. This situation entails a reflection boundary condition at $x = 0$ instead of an absorbing boundary condition. Therefore, we consider Neumann boundary conditions at $x = 0$ and $x = L$.

1.2 Notation and Assumptions

Throughout this article we define $Q_T := (0, T) \times (0, L)$ to be the domain, $S_T := ([0, T] \times \{0, L\}) \cup (\{T\} \times [0, L])$ to be the parabolic boundary, and at times $\Gamma_T := ([0, T] \times \{0\}) \cup (\{T\} \times [0, L])$ to be the parabolic half-boundary. For any domain X in $\mathbb{R}$ or $\mathbb{R}^2$ we define $L^p(X)$, $p \in [1, +\infty]$ to be the Lebesgue space of p-integrable functions on X; $C^0(X)$ to be the space of all continuous functions on X; $C^\alpha(X)$, $0 < \alpha < 1$ to be the space of all Hölder continuous functions with exponent α on X; and $C^{n+\alpha}(X)$ to be the set of all functions whose n derivatives are all in $C^\alpha(X)$. For a subset $X \subset \overline{Q_T}$ we also define $C^{1,2}(X)$ to be the set of all functions on X which are locally continuously differentiable in t and twice locally continuously differentiable in x. By $C^{\alpha/2,\alpha}(X)$ we denote the set of all functions which are locally Hölder continuous in time with exponent $\alpha/2$ and in space with exponent α.

We will denote by C a *generic* constant, which depends only on the data (namely $u_T, m_0, L, T, \sigma, r$ and ϵ). Its precise value may change from line to line.

Throughout we take the following assumptions on the data:

1. u_T and m_0 are function in $C^{2+\gamma}([0, L])$ for some $\gamma > 0$.
2. u_T and m_0 satisfy compatible boundary conditions : $u_T'(0) = u_T'(L) = 0$ and $m_0(0) = m_0'(0) = m_0(L) = m_0'(L) = 0$.
3. m_0 is probability density.
4. $u_T \geq 0$.

2 Analysis of the System

In this section we give a proof of existence and uniqueness for system (1). Note that most results of this section are an adaptation of those of [25, section 2]. However, unlike the case addressed in [25], we provide uniform bounds on u and u_x which do not depend on σ. We start by providing some a priori bounds on solutions to (1), then we prove existence and uniqueness using the Leray-Schauder fixed point theorem.

Let us start with some basic properties of the solutions.

Proposition 1 *Let* (u, m) *be a pair of smooth solutions to* (1). *Then, for all* $t \in [0, T]$, $m(t)$ *is a probability density, and*

$$u(t, x) \geq 0 \quad \forall t \in [0, T], \forall x \in [0, L]. \tag{14}$$

Moreover, for some constant $C > 0$ *depending on the data, we have*

$$\int_0^T \int_0^L m u_x^2 \leq C. \tag{15}$$

Proof Using (1)(ii) and (1)(v), one easily checks that $m(t)$ is a probability density for all $t \in [0, T]$. Moreover, the arguments used to prove (14) and (15) in [25] hold also for the system (1).

Lemma 1 *Let* (u, m) *be a pair of smooth solution to* (1), *then*

$$\|u\|_\infty + \|u_x\|_\infty \leq C, \tag{16}$$

where the constant $C > 0$ *does not depend on* σ. *In particular we have that*

$$\forall t \in [0, T], \qquad \left| \int_0^L u_x(t, x) m(t, x) \, \mathrm{d}x \right| \leq C, \tag{17}$$

where $C > 0$ *does not depend on* σ.

Proof As in [25, Lemma 2.3, Lemma 2.7], the result is a consequence of using the maximum principle for suitable functions. We give a proof highlighting the fact that C does not depend on σ. Set $f(t) := b + c \int_0^L u_x(t, y) m(t, y) \, \mathrm{d}y$, so that

$$-u_t - \frac{\sigma^2}{2} u_{xx} + ru \leq \frac{1}{2} \left(f^2(t) + u_x^2 \right).$$

Owing to Proposition 1, $f \in L^2(0, T)$. Moreover, if

$$w := \exp\left\{ \sigma^{-2} \left(u + \frac{1}{2} \int_0^t f(s)^2 \, \mathrm{d}s \right) \right\} - 1,$$

then we have

$$-w_t - \frac{\sigma^2}{2} w_{xx} \leq 0.$$

In particular w satisfies the maximum principle, and $w \leq \mu$ everywhere, where

$$\mu = \max_{0 \leq x \leq L} \exp\left\{ \sigma^{-2} \left(u_T + \frac{1}{2} \int_0^T f(s)^2 \, \mathrm{d}s \right) \right\} - 1.$$

Whence, $0 \le u \le \sigma^2 \ln(1+\mu)$, so that

$$\|u\|_\infty \le \|u_T\|_\infty + \frac{1}{2}\int_0^T f(s)^2\,\mathrm{d}s.$$

On the other hand, we have that

$$\max_{\Gamma_T} |u_x| \le \|u'_T\|_\infty, \quad \Gamma_T := ([0,T]\times\{0,L\}) \cup (\{T\}\times[0,L]),$$

so by using the maximum principle for the function $w(t,x) = u_x(t,x)e^{-rt}$, we infer that

$$\|u_x\|_\infty \le e^{rT}\|u'_T\|_\infty.$$

Remark 1 Unlike in [25], where more sophisticated estimates are performed, the estimation of the nonlocal term $\int_0^L u_x(t,x)m(t,x)\,\mathrm{d}x$ follows easily in this case, owing to (16) and the fact that m is a probability density.

Proposition 2 *There exists a constant $C > 0$ depending only on σ and data such that if (u,m) is a smooth solution to (1), then for some $0 < \alpha < 1$,*

$$\|u\|_{C^{1+\alpha/2,2+\alpha}(\overline{Q_T})} + \|m\|_{C^{1+\alpha/2,2+\alpha}(\overline{Q_T})} \le C. \tag{18}$$

Proof See [25, Proposition 2.8].

We now prove the main result of this section.

Theorem 1 *There exists a unique classical solution to (1).*

Proof The proof of existence is the same as in [25, Theorem 3.1] and relies on Leray-Schauder fixed point theorem. Let (u_1,m_1) and (u_2,m_2) be two solutions of (1), and set $u = u_1 - u_2$ and $m = m_1 - m_2$. Define

$$G_i := \frac{1}{2}\left(b + c\int_0^L u_{i,x}(t,y)m_i(t,y)\,\mathrm{d}y - u_{i,x}\right).$$

Note that G_i can be written

$$G_i = \frac{1}{2}\left(\frac{b}{1-c} - \frac{2c}{1-c}\bar{G}_i - u_{i,x}\right), \quad \text{where} \quad \bar{G}_i := \int_0^L G_i(t,y)m_i(t,y)\,\mathrm{d}y.$$

Integration by parts yields

$$\left[e^{-rt}\int_0^L u(t,x)m(t,x)\,\mathrm{d}x\right]_0^T = \int_0^T e^{-rt}\int_0^L (G_2^2 - G_1^2 - G_1u_x)m_1 + (G_1^2 - G_2^2 + G_2u_x)m_2\,\mathrm{d}x\,\mathrm{d}t. \tag{19}$$

The left-hand side of (19) is zero. As for the right-hand side, we check that

$$G_2^2 - G_1^2 - G_1 u_x = (G_2 - G_1)^2 + \frac{2c}{1-c} G_1(\bar{G}_1 - \bar{G}_2)$$

and, similarly,

$$G_1^2 - G_2^2 + G_2 u_x = (G_2 - G_1)^2 - \frac{2c}{1-c} G_2(\bar{G}_1 - \bar{G}_2).$$

Then (19) becomes

$$0 = \int_0^T e^{-rt} \int_0^L (G_1 - G_2)^2 (m_1 + m_2)\, \mathrm{d}x\, \mathrm{d}t + \frac{2c}{1-c} \int_0^T e^{-rt} (\bar{G}_1 - \bar{G}_2)^2\, \mathrm{d}t. \tag{20}$$

It follows that $\bar{G}_1 \equiv \bar{G}_2$. Then by uniqueness for parabolic equations with quadratic Hamiltonians, it follows that $u_1 \equiv u_2$. From uniqueness for the Fokker-Planck equation it follows that $m_1 \equiv m_2$.

2.1 Uniqueness Revisited for the Model of Chan and Sircar

The authors of [16] originally introduced the following model:

$$\begin{cases} (i) & u_t + \frac{1}{2}\sigma^2 u_{xx} - ru + G^2(t, u_x, [mu_x]) = 0, & 0 < t < T,\ 0 < x < L \\ (ii) & m_t - \frac{1}{2}\sigma^2 m_{xx} - (G(t, u_x, [mu_x])m)_x = 0, & 0 < t < T,\ 0 < x < L \\ (iii) & m(0, x) = m_0(x),\ u(T, x) = u_T(x), & 0 \le x \le L \\ (iv) & u(t, 0) = m(t, 0) = 0, \quad u_x(t, L) = 0, & 0 \le t \le T \\ (v) & \frac{1}{2}\sigma^2 m_x(t, L) + G(t, u_x(t, L), [mu_x])m(t, L) = 0, & 0 \le t \le T \end{cases} \tag{21}$$

where

$$G(t, u_x, [mu_x]) = \frac{1}{2}\left(\frac{2}{2+\epsilon\eta(t)} + \frac{\epsilon}{2+\epsilon\eta(t)} \int_0^L u_\xi(t,\xi) m(t,\xi) d\xi - u_x\right), \tag{22}$$

$$\eta(t) := \int_0^L m(t,\xi) d\xi$$

The main difference between (1) and (21) is that in (21) there are Dirichlet boundary conditions on the left-hand side $x = 0$, which also means that m is no longer a density, but might have decreasing mass. In [25], existence and uniqueness of classical solutions for (21) is obtained. However, uniqueness was only proved for small parameters ϵ. Here we improve this result by using the idea of the proof of Theorem 1. (The proof is in fact much simpler than in [25].)

Theorem 2 *There exists a unique classical solution of the system* (21)*.*

Proof Existence was given in [25]. For uniqueness, let (u_1, m_1), (u_2, m_2) be two solutions, and define $u = u_1 - u_2$, $m = m_1 - m_2$, and

$$G_i = \frac{1}{2}\left(\frac{2}{2+\epsilon\eta_i(t)} + \frac{\epsilon}{2+\epsilon\eta_i(t)}\int_0^L u_{i,\xi}(t,\xi)m_i(t,\xi)d\xi - u_{i,x}\right),$$

$$\eta_i(t) := \int_0^L m_i(t,\xi)d\xi.$$

Note that G_i can also be written

$$G_i = \frac{1}{2}(1 - \epsilon\bar{G}_i - u_{i,x}), \qquad \text{where} \qquad \bar{G}_i := \int_0^L G_i(t,y)m_i(t,y)\,\mathrm{d}y.$$

Then integrating by parts as in the proof of Theorem 1, we obtain

$$0 = \int_0^T e^{-rt}\int_0^L (G_1 - G_2)^2(m_1 + m_2)\,\mathrm{d}x\,\mathrm{d}t + \epsilon\int_0^T e^{-rt}(\bar{G}_1 - \bar{G}_2)^2\,\mathrm{d}t. \tag{23}$$

We conclude as before.

3 Optimal Control of Fokker-Planck Equation

The purpose of this section is to prove that (1) is a system of optimality for a convex minimization problem. It was first noticed in the seminal paper by Lasry and Lions [30] that systems of the form (2) have a formal interpretation in terms of optimal control. Since then this property has been made rigorous and exploited to obtain well-posedness in first-order [9, 10, 15] and degenerate cases [14]; see [5] for a nice discussion. However, all of these references consider the case of congestion penalization, which results in an a priori summability estimate on the density. There is no such penalization in (1). Hence, the optimality arguments used in [9], for example, appear insufficient in the present case to prove existence and uniqueness of solutions to the first order system. Furthermore, it is very difficult in the present context to formulate the dual problem, which in the aforementioned works was an essential ingredient in proving existence of an adjoint state. Nevertheless, aside from its intrinsic interest, we will see in Sect. 4 that optimality gives us at least enough to pass to the limit as $\sigma \to 0$.

We make the substitution $\bar{b} = \frac{b}{1-c}, \bar{c} = \frac{c}{1-c}$ (so according to (13) we get $\bar{b} = 1$ and $\bar{c} = \epsilon/2$). Consider the optimization problem of minimizing the objective functional

$$J(m,q) = \int_0^T \int_0^L e^{-rt} \left(q^2(t,x) - \bar{b}q(t,x)\right) m(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

$$+ \bar{c} \int_0^T e^{-rt} \left(\int_0^L q(t,y)m(t,y)\,\mathrm{d}y\right)^2 \mathrm{d}t - \int_0^L e^{-rT} u_T(x)m(T,x)\,\mathrm{d}x \tag{24}$$

for (m,q) in the class $\mathcal{K}$, defined as follows. Let $m \in L^1([0,T] \times [0,L])$ be non-negative, let $q \in L^2([0,T] \times [0,L])$, and assume that m is a weak solution to the Fokker-Planck equation

$$m_t - \frac{\sigma^2}{2} m_{xx} - (qm)_x = 0, \quad m(0) = m_0, \tag{25}$$

equipped with Neumann boundary conditions, where weak solutions are defined as in [31]:

- the integrability condition $mq^2 \in L^1([0,T] \times [0,L])$ holds, and
- (25) holds in the sense of distributions–namely, for all $\phi \in C_c^\infty([0,T) \times [0,L])$ such that $\phi_x(t,0) = \phi_x(t,L) = 0$ for each $t \in (0,T)$, we have

$$\int_0^T \int_0^L (-\phi_t - \frac{\sigma^2}{2}\phi_{xx} + q\phi_x)m \;\,\mathrm{d}x\,\mathrm{d}t = \int_0^L \phi(0)m_0\,\mathrm{d}x.$$

Then we say that $(m,q) \in \mathcal{K}$. We refer the reader to [31] for properties of weak solutions of (25), namely that they are unique and that they coincide with renormalized solutions and for this reason have several useful properties. One property which will be of particular interest to us is the following lemma:

Lemma 2 (Proposition 3.10 in [31]) *Let $(m,q) \in \mathcal{K}$, i.e. let m be a weak solution of the Fokker-Planck equation (25). Then $\|m(t)\|_{L^1([0,L])} = \|m_0\|_{L^1([0,L])}$ for all $t \in [0,T]$. Moreover, if $\log m_0 \in L^1([0,L])$, then for any*

$$\|\log m(t)\|_{L^1([0,L])} \leq C(\|\log m_0\|_{L^1([0,L])} + 1) \;\; \forall t \in [0,T], \tag{26}$$

where C depends on $\|q\|_{L^2}$ and $\|m_0\|_{L^1}$. In particular, if $\log m_0 \in L^1([0,L])$ and (m,q) in $\mathcal{K}$, then $m > 0$ a.e.

Proposition 3 *Let (u,m) be a solution of (1). Set*

$$q = \frac{1}{2}\left(b + c\int_0^L u_x(t,y)m(t,y)\,\mathrm{d}y - u_x\right).$$

Then (m, q) *is a minimizer for problem* (24)*, that is,* $J(m, q) \le J(\tilde{m}, \tilde{q})$ *for all* $(\tilde{m}, \tilde{q})$ *satisfying* (25)*. Moreover, if* $\log m_0 \in L^1([0, L])$ *then the maximizer is unique.*

Proof It is useful to keep in mind that the proof is based on the convexity of J following a change of variables. By abuse of notation we might write

$$J(m, w) = \int_0^T \int_0^L e^{-rt} \left(\frac{w^2(t,x)}{m(t,x)} - \bar{b} w(t,x) \right) \mathrm{d}x\, \mathrm{d}t + \bar{c} \int_0^T e^{-rt} \left(\int_0^L w(t,y)\, \mathrm{d}y \right)^2 \mathrm{d}t - \int_0^L e^{-rT} u_T(x) m(T,x)\, \mathrm{d}x,$$

cf. the change of variables used in [4] and several works which cite that paper. However, in this context we prefer a direct proof.

Using the algebraic identity

$$\tilde{q}^2 \tilde{m} - q^2 m = 2q(\tilde{q}\tilde{m} - qm) - q^2(\tilde{m} - m) + \tilde{m}(\tilde{q} - q)^2,$$

we have

$$\begin{aligned} J(\tilde{m}, \tilde{q}) - J(m, q) &= \bar{c} \int_0^T e^{-rt} \left(\int_0^L \tilde{q}\tilde{m} - qm \;\mathrm{d}y \right)^2 \mathrm{d}t - \int_0^L e^{-rT} u_T(x)(\tilde{m} - m)(T, x)\, \mathrm{d}x \\ &\quad + 2\bar{c} \int_0^T e^{-rt} \left(\int_0^L \tilde{q}\tilde{m} - qm \;\mathrm{d}y \right) \left(\int_0^L qm \;\mathrm{d}y \right) \mathrm{d}t \\ &\quad + \int_0^T \int_0^L e^{-rt} \left(\bar{b}(qm - \tilde{q}\tilde{m}) + 2q(\tilde{q}\tilde{m} - qm) - q^2(\tilde{m} - m) + \tilde{m}(\tilde{q} - q)^2 \right) \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{27}$$

Now using the fact that u is a smooth solution of

$$u_t + \frac{\sigma^2}{2} u_{xx} - ru + q^2 = 0,\; u(T) = 0,\; u_x|_{0,L} = 0 \tag{28}$$

and since

$$(\tilde{m} - m)_t - \frac{\sigma^2}{2}(\tilde{m} - m)_{xx} - (\tilde{q}\tilde{m} - qm)_x = 0, \quad (\tilde{m} - m)(0) = 0$$

in the sense of distributions, it follows that

$$\int_0^T \int_0^L e^{-rt} q^2 (\tilde{m} - m)\, \mathrm{d}x\, \mathrm{d}t + \int_0^L e^{-rT} u_T(x)(\tilde{m} - m)(T, x)\, \mathrm{d}x = -\int_0^T \int_0^L e^{-rt} (\tilde{q}\tilde{m} - qm) u_x \;\mathrm{d}x\, \mathrm{d}t.$$

Putting this into (27) and rearranging, we have

$$J(\tilde{m},\tilde{q}) - J(m,q) = \int_0^T \int_0^L e^{-rt}(qm - \tilde{q}\tilde{m}) \left(\bar{b} - 2q - 2\bar{c}\int_0^L qm \; \mathrm{d}y - u_x \right) \mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T \int_0^L e^{-rt}\tilde{m}(\tilde{q}-q)^2 \,\mathrm{d}x\,\mathrm{d}t + \bar{c}\int_0^T e^{-rt}\left(\int_0^L \tilde{q}\tilde{m} - qm \; \mathrm{d}x\right)^2 \mathrm{d}t. \tag{29}$$

To conclude that $J(\tilde{m},\tilde{q}) \geq J(m,q)$, it suffices to prove that

$$\bar{b} - 2q - 2\bar{c}\int_0^L qm \; \mathrm{d}y - u_x = 0. \tag{30}$$

Recall the definition

$$q = \frac{1}{2}\left(b + c\int_0^L u_x(t,y)m(t,y)\,\mathrm{d}y - u_x\right).$$

Integrate both sides against m and rearrange, using the definition of the constants $\bar{b}, \bar{c}$ to get

$$\int u_x m \; \mathrm{d}y = \bar{b} - 2(\bar{c}+1)\int qm \; \mathrm{d}y.$$

Plugging this into the definition of q proves (30). Thus (m,q) is a minimizer.

On the other hand, suppose $\log m_0 \in L^1([0,L])$ and that $(\tilde{m},\tilde{q})$ is another minimizer. Then (29) implies that

$$\int_0^T \int_0^L e^{-rt}\tilde{m}(\tilde{q}-q)^2 \,\mathrm{d}x\,\mathrm{d}t + \bar{c}\int_0^T e^{-rt}\left(\int_0^L \tilde{q}\tilde{m} - qm \; \mathrm{d}x\right)^2 \mathrm{d}t = 0. \tag{31}$$

Now by Lemma 2, we have $\tilde{m} > 0$ a.e. Therefore (31) implies $\tilde{q} = q$. By uniqueness for the Fokker-Planck equation, we conclude that $\tilde{m} = m$ as well. The proof is complete.

Remark 2 A similar argument shows that System (21), with Dirichlet boundary conditions on the left-hand side, is also a system of optimality for the same minimization problem, except this time with Dirichlet boundary conditions (on the left-hand side) imposed on the Fokker-Planck equation. We omit the details.

4 First-Order Case

In this section we use a vanishing viscosity method to prove that (1) has a solution even when we plug in $\sigma = 0$. We need to collect some estimates which are uniform in σ as $\sigma \to 0$. From now on we will assume $0 < \sigma \leq 1$, and whenever a constant C appears it does not depend on σ.

Lemma 3 $\|u_t\|_2 \le C$.

Proof We first prove that $\sigma^2\|u_{xx}\|_2 \le C$. For this, multiply

$$u_{xt} - ru_x + \frac{\sigma^2}{2}u_{xxx} - Gu_{xx} = 0 \tag{32}$$

by u_x and integrate by parts. We get, after using Young's inequality and (16),

$$\sigma^4 \int_0^T \int_0^L u_{xx}^2 \,\mathrm{d}x\,\mathrm{d}t \le 4\int_0^T \int_0^L (Gu_x)^2 \,\mathrm{d}x\,\mathrm{d}t + 2\sigma^2 \int_0^L u_T'(x)^2 \,\mathrm{d}x \le C,$$

as desired.

Then the claim follows from (1)(i) and Lemma 1.

Lemma 4 $\|u\|_{C^{1/3}} \le C$.

Proof Since $\|u_x\|_\infty \le C$ it is enough to show that u is 1/3-Hölder continuous in time. Let $t_1 < t_2$ in $[0, T]$ be given. Set $\eta > 0$ to be chosen later. We have, by Hölder's inequality,

$$\begin{aligned} |u(t_1, x) - u(t_2, x)| &\le C\eta + \frac{1}{\eta}\int_{x-\eta}^{x+\eta} |u(t_1,\xi) - u(t_2,\xi)| \,\mathrm{d}\xi \\ &\le C\eta + \frac{1}{\eta}\int_{x-\eta}^{x+\eta}\int_{t_1}^{t_2} |u_t(s,\xi)| \,\mathrm{d}s\,\mathrm{d}\xi \\ &\le C\eta + \frac{1}{\eta}\|u_t\|_2\sqrt{2\eta|t_2 - t_1|} \le C\eta + C|t_2 - t_1|^{1/2}\eta^{-1/2}. \end{aligned} \tag{33}$$

Setting $\eta = |t_2 - t_1|^{1/3}$ proves the claim.

To prove compactness estimates for m, we will first use the fact that it is the minimizer for an optimization problem. Let us reintroduce the optimization problem from Sect. 3 with $\sigma \ge 0$ as a variable. We first define the convex functional

$$\mathscr{P}si(m, w) := \begin{cases} \frac{|w|^2}{m} & \text{if } m \ne 0, \\ 0 & \text{if } w = 0, m = 0, \\ +\infty & \text{if } w \ne 0, m = 0. \end{cases} \tag{34}$$

Now we rewrite the functional J, with a slight abuse of notation, as

$$\begin{aligned} J(m, w) = &\int_0^T \int_0^L e^{-rt}\left(\mathscr{P}si(m(t,x), w(t,x)) - \bar{b}w(t,x)\right) \mathrm{d}x\,\mathrm{d}t \\ &+ \bar{c}\int_0^T e^{-rt}\left(\int_0^L w(t,y)\,\mathrm{d}y\right)^2 \mathrm{d}t - \int_0^L e^{-rT}u_T(x)m(T,x)\,\mathrm{d}x, \end{aligned} \tag{35}$$

and consider the problem of minimizing over the class $\mathscr{K}_\sigma$, defined here as the set of all pairs $(m, w) \in L^1((0,T) \times (0,L))_+ \times L^1((0,T) \times (0,L); \mathbb{R}^d)$ such that

$$m_t - \frac{\sigma^2}{2} m_{xx} - w_x = 0, \; m(0) = m_0 \tag{36}$$

in the sense of distributions. By Proposition 3, for every $\sigma > 0$, J has a minimizer in $\mathscr{K}_\sigma$ given by $(m, w) = (m, Gm)$ where (u, m) is the solution of System (1). Since (m, w) is a minimizer, we can derive a priori bounds which imply, in particular, that $m(t)$ is Hölder continuous in the Kantorovich-Rubinstein distance on the space of probability measures, with norm bounded uniformly in σ. We recall that the Kantorovich-Rubinstein metric on $\mathscr{P}(\Omega)$, the space of Borel probability measures on Ω, is defined by

$$\mathbf{d}_1(\mu, \nu) = \inf_{\pi \in \mathscr{P}i(\mu,\nu)} \int_{\Omega \times \Omega} |x - y| \, \mathrm{d}\pi(x, y),$$

where $\mathscr{P}i(\mu, \nu)$ is the set of all probability measures on $\Omega \times \Omega$ whose first marginal is μ and whose second marginal is ν. Here we consider $\Omega = (0, L)$.

Lemma 5 *There exists a constant C independent of σ such that*

$$\| |w|^2/m \|_{L^1((0,T)\times(0,L))} \leq C.$$

As a corollary, m is $1/2$-Hölder continuous from $[0, T]$ into $\mathscr{P}((0, L))$, and there exists a constant (again denoted C) independent of σ such that

$$\mathbf{d}_1(m(t_1), m(t_2)) \leq C|t_1 - t_2|^{1/2}. \tag{37}$$

Proof To see that $\| |w|^2/m \|_{L^1((0,T)\times(0,L))} \leq C$, use $(m_0, 0) \in \mathscr{K}$ as a comparison. By the fact that $J(m, w) \leq J(m_0, 0)$ we have

$$\int_0^T \int_0^L e^{-rt} \frac{|w|^2}{2m} \, \mathrm{d}x \, \mathrm{d}t + \bar{c} \int_0^T e^{-rt} \left(\int_0^L w \, \mathrm{d}x \right)^2 \mathrm{d}t \\ \leq \int_0^L e^{-rT} u_T (m(T) - m_0) \, \mathrm{d}x + \frac{\bar{b}}{2} \int_0^T \int_0^L e^{-rt} m \, \mathrm{d}x \, \mathrm{d}t \leq C.$$

The Hölder estimate (37) follows from [14, Lemma 4.1].

We also have compactness in L^1, which comes from the following lemma.

Lemma 6 *For every $K \geq 0$, we have*

$$\int_{m(t) \geq 2K} m(t) \, \mathrm{d}x \leq 2 \int_0^L (m_0 - K)_+ \, \mathrm{d}x \tag{38}$$

for all $t \in [0, T]$.

Proof Let $K \geq 0$ be given. We define the following auxiliary functions:

$$\phi_{\alpha,\delta}(s) := \begin{cases} 0 & \text{if } s \leq K, \\ \frac{1}{6}(1+\alpha)\alpha\delta^{\alpha-2}(s-K)^3 & \text{if } K \leq s \leq K+\delta, \\ \frac{1}{6}(1+\alpha)\alpha\delta^{\alpha+1} + \frac{1}{2}(1+\alpha)\alpha\delta^{\alpha}(s-K) + (s-K)^{1+\alpha} & \text{if } s \geq K+\delta, \end{cases} \tag{39}$$

where $\alpha, \delta \in (0, 1)$ are parameters going to zero. For reference we note that

$$\phi'_{\alpha,\delta}(s) = \begin{cases} 0 & \text{if } s \leq K, \\ \frac{1}{2}(1+\alpha)\alpha\delta^{\alpha-2}(s-K)^2 & \text{if } K \leq s \leq K+\delta, \\ \frac{1}{2}(1+\alpha)\alpha\delta^{\alpha} + (1+\alpha)(s-K)^{\alpha} & \text{if } s \geq K+\delta, \end{cases} \tag{40}$$

and

$$\phi''_{\alpha,\delta}(s) = \begin{cases} 0 & \text{if } s \leq K, \\ (1+\alpha)\alpha\delta^{\alpha-2}(s-K) & \text{if } K \leq s \leq K+\delta, \\ (1+\alpha)\alpha(s-K)^{\alpha-1} & \text{if } s \geq K+\delta. \end{cases} \tag{41}$$

Observe that $\phi''_{\alpha,\delta}$ is continuous and non-negative. Multiply (1)(ii) by $\phi'_{\alpha,\delta}(m)$ and integrate by parts. After using Young's inequality we have

$$\int_0^L \phi_{\alpha,\delta}(m(t))\,\mathrm{d}x \leq \int_0^L \phi_{\alpha,\delta}(m_0)\,\mathrm{d}x + \frac{\|G\|_\infty^2}{2\sigma^2}\int_0^t\int_0^L \phi''_{\alpha,\delta}(m)m^2\,\mathrm{d}x\,\mathrm{d}t. \tag{42}$$

Since $\phi''_{\alpha,\delta}(s) \leq (1+\alpha)\alpha\delta^{-2}$, after taking $\alpha \to 0$ we have

$$\int_0^L \phi_\delta(m(t))\,\mathrm{d}x \leq \int_0^L \phi_\delta(m_0)\,\mathrm{d}x, \tag{43}$$

where $\phi_\delta(s) = (s-K)\chi_{[K+\delta,\infty)}(s)$. Now letting $\delta \to 0$ we see that

$$\int_0^L (m(t)-K)_+\,\mathrm{d}x \leq \int_0^L (m_0-K)_+\,\mathrm{d}x, \tag{44}$$

where $s_+ := (s+|s|)/2$ denotes the positive part. Whence

$$\int_0^L (m_\sigma(t)-K)_+\,\mathrm{d}x \leq \int_0^L (m_0-K)_+\,\mathrm{d}x, \tag{45}$$

which also implies (38).

We also have a compactness estimate for the function $t \mapsto \int_0^L u_x(t,y)m(t,y)\,\mathrm{d}y$.

Lemma 7 $\sigma^2 \left(\int_0^T \int_0^L \frac{|m_x|^2}{m+1}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \le C.$

Proof Multiply the Fokker-Planck equation by $\log(m+1)$ and integrate by parts. After using Young's inequality, we obtain

$$\frac{\sigma^4}{4}\int_0^T \int_0^L \frac{|m_x|^2}{m+1}\,\mathrm{d}x\,\mathrm{d}t \le \sigma^2 \int_0^L ((m_0+1)\log(m_0+1)-m_0)\,\mathrm{d}x + \|G\|_\infty^2 \int_0^T \int_0^L \frac{m^2}{m+1}$$
$$\le \int_0^L ((m_0+1)\log(m_0+1)-m_0)\,\mathrm{d}x + \|G\|_\infty^2 \int_0^T \int_0^L m\,\mathrm{d}x\,\mathrm{d}t \le C.$$

Lemma 8 *Let* $\zeta \in C_c^\infty((0,L))$. *Then* $t \mapsto \int_0^L u_x(t,x)m(t,x)\zeta(x)\,\mathrm{d}x$ *is* 1/2-*Hölder continuous, and in particular,*

$$\left|\left[\int_0^L u_x(t,x)m(t,x)\zeta(x)\,\mathrm{d}x\right]_{t_1}^{t_2}\right| \le C_\zeta |t_1-t_2|^{1/2} \tag{46}$$

where C_ζ *is a constant that depends on* ζ *but not on* σ.

Proof Integration by parts yields

$$\left[e^{-rt}\int_0^L u_x(t,x)m(t,x)\zeta(x)\,\mathrm{d}x\right]_{t_1}^{t_2}$$
$$= -\sigma^2 \int_{t_1}^{t_2} e^{-rs}\int_0^L u_x(t,x)m_x(t,x)\zeta'(x)\,\mathrm{d}x\,\mathrm{d}s - \frac{\sigma^2}{2}\int_{t_1}^{t_2} e^{-rs}\int_0^L u_x(t,x)m(t,x)\zeta''(x)\,\mathrm{d}x\,\mathrm{d}s$$
$$-\frac{1}{2}\int_{t_1}^{t_2}\left\{\left(b+c\int_0^L u_x(t)m(t)\right)\int_0^L \zeta_x u_x m\,\mathrm{d}x - \int_0^L \zeta_x u_x^2 m\,\mathrm{d}x\right\}\mathrm{d}s. \tag{47}$$

On the one hand,

$$\left|\frac{\sigma^2}{2}\int_{t_1}^{t_2} e^{-rs}\int_0^L u_x(t,x)m(t,x)\zeta''(x)\,\mathrm{d}x\,\mathrm{d}s\right| \le \frac{\|u_x\|_\infty \|\zeta''\|_\infty}{2}|t_1-t_2| \le C\|\zeta''\|_\infty |t_1-t_2|,$$

and

$$\left|\int_{t_1}^{t_2}\left\{\left(b+c\int_0^L u_x(t)m(t)\right)\int_0^L \zeta_x u_x m\,\mathrm{d}x - \int_0^L \zeta_x u_x^2 m\,\mathrm{d}x\right\}\mathrm{d}s\right| \le C\|\zeta'\|_\infty \|u_x\|_\infty^2 |t_1-t_2|.$$

On the other hand, by Hölder's inequality and Lemma 7 we get

$$\left|\sigma^2 \int_{t_1}^{t_2} e^{-rs}\int_0^L u_x(t,x)m_x(t,x)\zeta'(x)\,\mathrm{d}x\,\mathrm{d}s\right|$$
$$\le \|u_x\|_\infty \|\zeta'\|_\infty \sigma^2 \left(\int_{t_1}^{t_2}\int_0^L \frac{|m_x|^2}{m+1}\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\left(\int_{t_1}^{t_2}\int_0^L (m+1)\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}$$
$$\le C\|\zeta'\|_\infty (L+1)^{1/2}|t_1-t_2|^{1/2}.$$

Corollary 1 *The function* $t \mapsto \int_0^L u_x(t,x)m(t,x)\,\mathrm{d}x$ *is uniformly continuous with modulus of continuity independent of* σ.

Proof Let $\delta \in (0, L)$ and fix $\zeta \in C_c^\infty((0, L))$ be such that $0 \le \zeta \le 1$ and $\zeta \equiv 1$ on $[\delta, L-\delta]$. Notice that for any $t_1, t_2 \in [0, T]$

$$\left|\left[\int_0^L u_x(t,x)m(t,x)(1-\zeta(x))\,\mathrm{d}x\right]_{t_1}^{t_2}\right| \le \|u_x\|_\infty \int_{[0,L]\setminus[\delta,L-\delta]} [m(t_1,x)+m(t_2,x)]\,\mathrm{d}x. \tag{48}$$

Now by Lemma 6 we have

$$\begin{aligned}&\int_{[0,L]\setminus[\delta,L-\delta]} m(t,x)\,\mathrm{d}x\\ &\le \int_{\{m(t)<2K\}\cap[0,L]\setminus[\delta,L-\delta]} m(t,x)\,\mathrm{d}x + \int_{\{m(t)\ge 2K\}} m(t,x)\,\mathrm{d}x \le 4K\delta + 2\int_0^L (m_0-K)_+\,\mathrm{d}x\end{aligned} \tag{49}$$

for all $t \in [0, T]$. Combine (48) and (49) with Lemmas 8 and 1 to get

$$\left|\left[\int_0^L u_x(t,x)m(t,x)\,\mathrm{d}x\right]_{t_1}^{t_2}\right| \le C_\zeta|t_1-t_2|^{1/2} + CK\delta + C\int_0^L (m_0-K)_+\,\mathrm{d}x \quad \forall t_1,t_2 \in [0,T]. \tag{50}$$

Let $\eta > 0$ be given. Set K large enough such that $C\int_0^L (m_0-K)_+\,\mathrm{d}x < \eta/3$, then pick δ small enough that $CK\delta < \eta/3$. Finally, fix ζ as described above. Equation (50) implies that if $|t_1-t_2| < \eta^2/(9C_\zeta^2)$, we have $\left|\left[\int_0^L u_x(t,x)m(t,x)\,\mathrm{d}x\right]_{t_1}^{t_2}\right| < \eta$. Thus the function $t \mapsto \int_0^L u_x(t,x)m(t,x)\,\mathrm{d}x$ is uniformly continuous, and since none of the constants here depend on σ, the modulus of continuity is independent of σ.

We are now in a position to prove an existence result for the first-order system.

Theorem 3 *There exists a unique pair* (u, m) *which solves System* (1) *in the following sense:*

1. $u \in W^{1,2}([0,T]\times[0,L]) \cap L^\infty(0,T;W^{1,\infty}(0,L))$ *is a continuous solution of the Hamilton-Jacobi equation*

$$u_t - ru + \frac{1}{4}(f(t)-u_x)^2 = 0,\ \ u(T,x) = u_T(x), \tag{51}$$

equipped with Neumann boundary conditions, in the viscosity sense;

2. *$m \in L^1 \cap C([0,T]; \mathscr{P}([0,L]))$ satisfies the continuity equation*

$$m_t - \frac{1}{2}((f(t) - u_x)m)_x = 0, \;\; m(0) = m_0, \tag{52}$$

equipped with Neumann boundary conditions, in the sense of distributions; and

3. *$f(t) = b + c\int_0^L u_x(t,x)m(t,x)\,\mathrm{d}x$ a.e.*

Proof Existence: Collecting Lemmas 1, 3–6, and Corollary 1, we can construct a sequence $\sigma_n \to 0^+$ such that if (u^n, m^n) is the solution corresponding to $\sigma = \sigma_n$, we have

- $u^n \to u$ uniformly, so that $u \in C([0,T] \times [0,L])$, and also weakly in $W^{1,2}([0,T] \times [0,L])$;
- $u_x^n \rightharpoonup u_x$ weakly* in L^∞;
- $m^n \to m$ in $C([0,T]; \mathscr{P}([0,L]))$, so that $m(t)$ is a well-defined probability measure for every $t \in [0,T]$, $m^n \rightharpoonup m$ weakly in $L^1([0,T] \times [0,L])$, and $m^n(T) \rightharpoonup m(T)$ weakly in $L^1([0,L])$;
- $m^n u_x^n \rightharpoonup w$ weakly in L^1; and
- $f_n(t) := b + c\int_0^L u_x^n(t,x)m^n(t,x)\,\mathrm{d}x \to f(t)$ in $C([0,T])$.

Since $u^n \to u$ and $f_n \to f$ uniformly, by standard arguments, we have that (51) holds in a viscosity sense. Moreover, since $u_x^n \rightharpoonup u_x$ weakly* in L^∞, we also have

$$u_t - ru + \frac{1}{4}(f(t) - u_x)^2 \le 0 \tag{53}$$

in the sense of distributions, i.e. for all $\phi \in C^\infty([0,T] \times [0,L])$ such that $\phi \ge 0$, we have

$$\begin{aligned}&\int_0^L e^{-rT} u_T(x)\phi(T,x)\,\mathrm{d}x - \int_0^L e^{-rT} u(0,x)\phi(0,x)\,\mathrm{d}x \\ &- \int_0^T \int_0^L e^{-rt} u(t,x)\phi_t(t,x)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{4}\int_0^T \int_0^L (f(t) - u_x(t,x))^2 \phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \le 0.\end{aligned} \tag{54}$$

(This follows from the convexity of $u_x \mapsto u_x^2$.)

Since $m^n \rightharpoonup m$ and $m^n u_x^n \rightharpoonup w$ weakly in L^1, it also follows that

$$m_t - \frac{1}{2}(f(t)m - w)_x = 0, \;\; m(0) = m_0 \tag{55}$$

in the sense of distributions. For convenience we define $\upsilon := \frac{1}{2}(f(t)m - w)$. Extend the definition of (m, υ) so that $m(t,x) = m(T,x)$ for $t \ge T$, $m(t,x) = m_0(x)$ for $t \le 0$, and $m(t,x) = 0$ for $x \notin [0,L]$; and so that $\upsilon(t,x) = 0$ for $(t,x) \notin [0,T] \times [0,L]$. Now let $\xi_\delta(t,x)$ be a standard convolution kernel (i.e. a C^∞, positive function whose support is contained in a ball of radius δ and such that $\iint \xi^\delta(t,x)\,\mathrm{d}x\,\mathrm{d}t = 1$). Set $m_\delta = \xi_\delta * m$ and $\upsilon_\delta = \xi_\delta$. Then $m_\delta, \upsilon_\delta$ are smooth functions such that $\partial_t m_\delta = \partial_x \upsilon_\delta$ in $[0,T]\times[0,L]$; moreover m_δ is positive. Using m_δ as a test function in (54) we get

$$\int_0^L e^{-rT} u_T(x) m_\delta(T,x)\,\mathrm{d}x - \int_0^L e^{-rT} u(0,x) m_\delta(0,x)\,\mathrm{d}x \\ + \int_0^T \int_0^L e^{-rt} u_x \upsilon_\delta\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{4}\int_0^T\int_0^L (f(t)-u_x)^2 m_\delta\,\mathrm{d}x\,\mathrm{d}t \le 0.$$

Using the continuity of $m(t)$ in $\mathscr{P}([0,L])$ from Lemma 5, we see that

$$\lim_{\delta\to 0^+} \int_0^L e^{-rT} u_T(x) m_\delta(T,x)\,\mathrm{d}x = \int_0^L e^{-rT} u_T(x) m(T,x)\,\mathrm{d}x,$$

and $\lim_{\delta\to 0^+}\int_0^L e^{-rT} u(0,x) m_\delta(0,x)\,\mathrm{d}x = \int_0^L e^{-rT} u(0,x) m_0(x)\,\mathrm{d}x$. Since $m_\delta \to m$ and $\upsilon_\delta \to \upsilon$ in L^1, we have

$$\int_0^L e^{-rT} u_T(x) m(T,x)\,\mathrm{d}x - \int_0^L e^{-rT} u(0,x) m_0(x)\,\mathrm{d}x \\ + \int_0^T \int_0^L e^{-rt} u_x \upsilon\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{4}\int_0^T\int_0^L (f(t)-u_x)^2 m\,\mathrm{d}x\,\mathrm{d}t \le 0,$$

or

$$\int_0^L e^{-rT} u_T(x) m(T,x)\,\mathrm{d}x - \int_0^L e^{-rT} u(0,x) m_0(x)\,\mathrm{d}x \\ + \int_0^T \int_0^L e^{-rt}\left(\frac{1}{4} m u_x^2 - \frac{1}{2} u_x w\right)\mathrm{d}x\,\mathrm{d}t + \frac{1}{4}\int_0^T\int_0^L f^2(t) m\,\mathrm{d}t \le 0. \tag{56}$$

Recall the definition of $\mathscr{P}si(m,w)$ from (34). From (56) we have

$$\int_0^L e^{-rT} u_T(x) m(T,x)\,\mathrm{d}x - \int_0^L e^{-rT} u(0,x) m_0(x)\,\mathrm{d}x \\ + \frac{1}{4}\int_0^T\int_0^L f^2(t) m\,\mathrm{d}t \le \frac{1}{4}\int_0^T\int_0^L e^{-rt}\mathscr{P}si(m,w)\,\mathrm{d}x\,\mathrm{d}t. \tag{57}$$

On the other hand, for each n we have

$$\int_0^L e^{-rT} u_T(x) m^n(T,x)\,\mathrm{d}x - \int_0^L e^{-rT} u^n(0,x) m_0(x)\,\mathrm{d}x \\ + \frac{1}{4}\int_0^T\int_0^L f_n^2(t) m^n\,\mathrm{d}t = \frac{1}{4}\int_0^T\int_0^L e^{-rt} m^n u_x^2\,\mathrm{d}x\,\mathrm{d}t = \frac{1}{4}\int_0^T\int_0^L e^{-rt}\mathscr{P}si(m^n, m^n u_x^n)\,\mathrm{d}x\,\mathrm{d}t. \tag{58}$$

Since $(m^n, m^n u_x^n) \rightharpoonup (m, w)$ weakly in $L^1 \times L^1$, it follows from weak lower semicontinuity that

$$\int_0^L e^{-rT} u_T(x) m(T,x)\,\mathrm{d}x - \int_0^L e^{-rT} u(0,x) m_0(x)\,\mathrm{d}x \\ + \frac{1}{4}\int_0^T \int_0^L f^2(t) m\,\mathrm{d}t \geq \frac{1}{4}\int_0^T \int_0^L e^{-rt} \mathscr{P}si(m,w)\,\mathrm{d}x\,\mathrm{d}t. \tag{59}$$

From (56), (57), and (59) it follows that

$$\int_0^T \int_0^L e^{-rt}(\mathscr{P}si(m,w) + m u_x^2 - 2u_x w)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

where $\mathscr{P}si(m,w) + m u_x^2 - 2u_x w$ is a non-negative function, hence zero almost everywhere. We deduce that $w = m u_x$ almost everywhere.

Finally, by weak convergence we have

$$f(t) = b + c \lim_{n\to\infty} \int_0^L u_x^n(t,x) m^n(t,x)\,\mathrm{d}x = b + c\int_0^L w(t,x)\,\mathrm{d}x \\ = b + c\int_0^L u_x(t,x) m(t,x)\,\mathrm{d}x \quad a.e.$$

Which entails the existence part of the Theorem.

Uniqueness: The proof of uniqueness is essentially the same as for the second order case, the only difference is the lack of regularity which makes the arguments much more subtle invoking results for transport equations with a non-smooth vector field. Let (u_1, m_1) and (u_2, m_2) be two solutions of system (1) in the sense given above, and let us set $u := u_1 - u_2$ and $m = m_1 - m_2$. We use a regularization process to get the energy estimate (20). Then we get that $u_1 \equiv u_2$ and $\int_0^L u_{1,x} m_1 = \int_0^L u_{2,x} m_2$ in $\{m_1 > 0\} \cup \{m_2 > 0\}$, so that m_1 and m_2 are both solutions to

$$m_t - \frac{1}{2}((f_1(t) - u_{1,x})m)_x = 0, \quad m(0) = m_0,$$

where $f_1(t) := b + c\int_0^L u_{1,x}(t,x) m_1(t,x)\,\mathrm{d}x$ and $u_{1,x} := (u_1)_x$. In order to conclude that $m_1 \equiv m_2$, we invoke the following Lemma:

Lemma 9 *Assume that v is a viscosity solution to*

$$v_t - rv + \frac{1}{4}(f_1(t) - v_x)^2 = 0, \quad v(T,x) = u_T(x),$$

then the transport equation

$$m_t - \frac{1}{2}((f_1(t) - v_x)m)_x = 0, \quad m(0) = m_0$$

possesses at most one weak solution in L^1.

The proof of Lemma 9 (see e.g. [8, Section 4.2]) relies on semi-concavity estimates for the solutions of Hamilton-Jacobi equations [7], and Ambrosio superposition principle [2, 3].

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A Review for an Isotropic Landau Model

Maria Gualdani and Nicola Zamponi

Abstract We consider the equation

$$u_t = \text{div}\,(a[u]\nabla u - u\nabla a[u]), \qquad -\Delta a = u.$$

This model has attracted some attention in the recent years and several results are available in the literature. We review recent results on existence and smoothness of solutions and explain the open problems.

Keywords Landau equation · Coulomb potential · Isotropic model · Even solutions · Weighted Poincaré and Sobolev inequalities · Regularity estimates

1 Introduction

1.1 The Isotropic Landau Equation

In this manuscript we review recent results on the isotropic Landau equation

$$\begin{aligned} &u_t = \text{div}\,(a[u]\nabla u - u\nabla a[u]), \qquad -\Delta a = u \qquad \text{in } \mathbb{R}^3, \ \ t > 0, \\ &u(\cdot, 0) = u_0. \end{aligned} \tag{1}$$

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This problem has been extensively studied in the recent years. Due to its similarity to the semilinear heat equation, to the Keller-Segel model but mostly to the homogeneous Landau equation

$$\begin{aligned} &u_t = \mathrm{div}\,(A[u]\nabla u - u\nabla a[u]),\\ &A[u] := \tfrac{1}{8\pi}\int_{\mathbb{R}^3}\tfrac{1}{|y|}\left(Id - \tfrac{y\otimes y}{|y|^2}\right)u(x-y)\,dy, \qquad a[u] = Tr(A[u]), \end{aligned} \tag{2}$$

the analysis of existence, uniqueness and regularity of solutions to (1) is a very interesting problem. A modification of (1) was first introduced in [14, 16]; there the authors studied existence and regularity of bounded radially symmetric and monotone decreasing solutions to

$$u_t = a[u]\Delta u + \alpha u^2, \quad \alpha \in \left(0, \frac{74}{75}\right).$$

Existence of global bounded solutions for (1) has been proven in [11] when initial data are radially symmetric and monotone decreasing. Section 2 explains these results more in details. Existence of weak solutions for even initial data has been shown in [13]. See Sect. 3 for more details.

For general initial data the problem of global existence of regular solutions is still open. The main obstacles for the analysis are hidden in the quadratic non-linearity: expanding the divergence term one can formally rewrite (1) as

$$u_t = a[u]\Delta u + u^2.$$

This problem is reminiscent to the semilinear heat equation, which solutions become unbounded after a finite time [9].

Let us mention that the main interest in studying (1) is to gain insights on model (2). It is well known that existence of global smooth solutions for (2), both in the homogeneous and inhomogeneous settings, is still an open problem. For an overview about the problem we refer to [1, 6, 19, 20]. In the very recent years much has been done regarding integrability and regularization for solution to the Landau equation. In that direction we acknowledge the works [2, 10–12, 15, 18] which reflect a renewed increasing interest in this problem by several mathematical communities.

1.2 Conserved Quantities and Entropy Structure

In this section we collect some properties of (1). The isotropic Landau equation shares some of the conservation properties of the classical Landau and Boltzmann equation. We first note that the potential $a[u]$ can be expressed as

$$a[u](x,t) = \int_{\mathbb{R}^3} \frac{u(y,t)}{4\pi|x-y|}dy, \qquad x \in \mathbb{R}^3, \;\; t > 0,$$

and therefore (1) can also be written as

$$u_t = \operatorname{div} \int_{\mathbb{R}^3} \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{4\pi|x-y|} dy. \tag{3}$$

With this in mind let us define the Maxwell-Boltzmann entropy:

$$H[u] \equiv \int_{\mathbb{R}^3} u \log u \, dx. \tag{4}$$

The function $t \in (0, \infty) \mapsto H[u(t)] \in \mathbb{R}$ is nonincreasing in time: using (1) we can write the entropy production as

$$\begin{aligned}
-4\pi \frac{d}{dt} H[u] &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\nabla u(x)}{u(x)} \cdot \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dxdy \\
&= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x)u(y)}{|x-y|} \frac{\nabla u(x)}{u(x)} \cdot \left(\frac{\nabla u(x)}{u(x)} - \frac{\nabla u(y)}{u(y)} \right) dxdy \\
&= \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x)u(y)}{|x-y|} \left| \frac{\nabla u(x)}{u(x)} - \frac{\nabla u(y)}{u(y)} \right|^2 dxdy \geq 0.
\end{aligned}$$

Clearly $\int_{\mathbb{R}^3} u(x,t)dx = \int_{\mathbb{R}^3} u_0(x)dx$, $t > 0$. We can say something about the first and second order moments of u. From (1) it follows

$$4\pi \frac{d}{dt} \int_{\mathbb{R}^3} xu(x,t)dx = -\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dxdy = 0$$

for obvious symmetry reasons. So the first moment is conserved. As for the second moment

$$\begin{aligned}
4\pi \frac{d}{dt} \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x,t)dx &= -\iint_{\mathbb{R}^3 \times \mathbb{R}^3} x \cdot \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dxdy \\
&= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} y \cdot \frac{u(y)\nabla u(x) - u(x)\nabla u(y)}{|x-y|} dxdy \\
&= -\frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{x-y}{|x-y|} (u(y)\nabla u(x) - u(x)\nabla u(y)) dxdy.
\end{aligned}$$

Since

$$\operatorname{div}_x \frac{x-y}{|x-y|} = -\operatorname{div}_y \frac{x-y}{|x-y|} = \left(\operatorname{div}_z \frac{z}{|z|} \right)\Big|_{z=x-y} = \frac{2}{|x-y|},$$

integration by parts yields

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x,t)dx = \frac{1}{2\pi} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{u(x,t)u(y,t)}{|x-y|} dxdy = 2 \int_{\mathbb{R}^3} u(x,t)a(x,t)dx > 0. \tag{5}$$

This is one of the main differences to the classical Landau equation. The second moment increases with time and a bound is not given a-priori. We will see in Sect. 3 how to find this bound when the initial data are even.

2 Radially Symmetric Solutions

Problem (1) is well understood when initial data are radially symmetric and monotonically decreasing. In [11] the authors prove the following theorem:

Theorem 1 *Let u_0 be a nonnegative function that has finite mass, energy and entropy. Moreover let u_0 be radially symmetric, monotonically decreasing and such that $u_0 \in L^p_{weak}$ for some $p > 6$. Then there exists a function $u(x,t)$ smooth, positive and bounded for all time which solves*

$$u_t = a[u]\Delta u + u^2, \quad u(x,0) = u_0.$$

We briefly highlight the ideas behind the proof of Theorem 1. The non-local dependence on the coefficients prevents the equation to satisfy comparison principle: in fact given two functions u_1 and u_2 such that $u_1 < u_2$ for $t < t_0$ and $u_1 = u_2$ at (x_0, t_0) we definitely have that $\Delta u_1(x_0,t_0) \leq \Delta u_2(x_0,t_0)$ and $a[u_1](x_0,t_0) \leq a[u_2](x_0,t_0)$. However it is not necessarily true $a[u_1](x_0,t_0)\Delta u_1(x_0,t_0) \leq a[u_2](x_0,t_0)\Delta u_2(x_0,t_0)$. To overcome this shortcoming, the main observation in [11] is that if one proves the existence of a function $g(x) \in L^p$ for some $p > 3/2$ such that $u_0 < g$ and

$$a[u]\Delta g + ug < 0,$$

then comparison principle for the linearized problem implies $u \leq g$ for all $t > 0$. Once higher integrability L^p of u is proved, standard techniques for parabolic equation such as Stampacchia's theorem yield L^∞ bound for $u(x,t)$ and consequent regularity.

3 Even Initial Data

Existence of weak solutions for (1) with general initial data is still an open problem. As already mentioned at the end of Sect. 1.2, the first obstacle that one encounters in the analysis of (1) is the missing bound for the second moment. This bound is essential when one seeks a-priori estimates for the gradient. In [13] the authors overcame this problem when solutions are even. In this section we highlight the basic estimates of [13] that will lead to construction of weak even solutions. For weak solutions we mean functions $u(x,t)$ such that

$$\sqrt{u} \in L^2\left(0, T; H^1\left(\mathbb{R}^3, \frac{dx}{1+|x|}\right)\right), \qquad u, u\log u \in L^\infty(0, T; L^1(\mathbb{R}^3)),$$
$$a \in L^\infty(0, T; L^3_{loc}(\mathbb{R}^3)), \qquad \nabla a \in L^\infty(0, T; L^{3/2}_{loc}(\mathbb{R}^3)),$$

that satisfy the following weak formulation

$$\int_0^T \langle \partial_t u\,, \phi\,\rangle dt + \int_0^T \int_{\mathbb{R}^3} (a\nabla u - u\nabla a)\cdot\nabla\phi\, dxdt = 0, \qquad \forall\phi \in L^\infty(0, T; W^{1,\infty}_c(\mathbb{R}^3)).$$

All the computations here are formal, meaning we assume that u and all related quantities have enough regularity for the mathematical manipulations to make sense. We refer to [13] for the detailed calculations. Let

$$E(t) := \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x,t)dx, \qquad R(t) := 2\sqrt{\frac{E(t)}{\|u_0\|_{L^1}}},$$

and define $B_{R(t)} \equiv \{x \in \mathbb{R}^3 \;:\; |x| < R(t)\}$. We point out that, since $\int_{\mathbb{R}^3\setminus B_{R(t)}} u(x,t)dx \leq \frac{2E(t)}{R(t)^2} = \frac{1}{2}\|u_0\|_{L^1}$, it follows

$$\int_{B_{R(t)}} u(x,t)dx = \|u_0\|_{L^1} - \int_{\mathbb{R}^3\setminus B_{R(t)}} u(x,t)dx \geq \frac{1}{2}\|u_0\|_{L^1}. \tag{6}$$

A Lower Bound for $a[u]$ From the definition of $a[u]$ it follows

$$4\pi a[u](x,t) = \int_{\mathbb{R}^3} \frac{u(y,t)}{|x-y|}dy \geq \int_{B_{R(t)}} \frac{u(y,t)}{|x-y|}dy \geq \frac{1}{R(t)+|x|}\int_{B_{R(t)}} u(y,t)dy \geq \frac{\|u_0\|_{L^1}}{2(R(t)+|x|)}$$

and therefore

$$a[u](x,t) \geq \frac{1}{16\pi}\frac{\|u_0\|_{L^1}^{3/2}}{E(t)^{1/2} + |x|\|u_0\|_{L^1}^{1/2}}. \tag{7}$$

A Gradient Estimate for Even Solutions We assume here that the solution u of (1) is even w.r.t. each component of x, for $t \geq 0$.

Clearly $|x-y| \leq |x| + |y| \leq (1+|x|)(1+|y|)$ for $x, y \in \mathbb{R}^3$. Therefore

$$-4\pi\frac{d}{dt}H[u] \geq \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{u(x,t)u(y,t)}{(1+|x|)(1+|y|)}\left|\frac{\nabla u(x,t)}{u(x,t)} - \frac{\nabla u(y,t)}{u(y,t)}\right|^2 dxdy$$
$$= \left(\int_{\mathbb{R}^3} u(x,t)\frac{dx}{1+|x|}\right)\left(\int_{\mathbb{R}^3}\frac{|\nabla u(x,t)|^2}{u(x)}\frac{dx}{1+|x|}\right) - \left|\int_{\mathbb{R}^3}\frac{\nabla u(x,t)}{1+|x|}dx\right|^2.$$

For the assumption on u it follows that

$$\left|\int_{\mathbb{R}^3} \frac{\nabla u}{1+|x|} dx\right|^2 = \sum_{i=1}^{3} \left(\int_{\mathbb{R}^3} \frac{\partial u}{\partial x_i} \frac{dx}{1+|x|}\right)^2 = 0.$$

As a consequence

$$-4\pi \frac{d}{dt} H[u] \geq \left(\int_{\mathbb{R}^3} u(x,t) \frac{dx}{1+|x|}\right) \left(\int_{\mathbb{R}^3} \frac{|\nabla u(x,t)|^2}{u(x)} \frac{dx}{1+|x|}\right).$$

We now wish to show a positive lower bound for $\int_{\mathbb{R}^3} u(x,t) \frac{dx}{1+|x|}$ for $0 \leq t \leq T$. Let $R(t) = 2\sqrt{E(t)/\|u_0\|_{L^1}}$. It holds

$$\int_{\mathbb{R}^3} u(x,t) \frac{dx}{1+|x|} \geq \int_{B_{R(t)}} u(x,t) \frac{dx}{1+|x|} \geq \frac{1}{1+R(t)} \int_{B_{R(t)}} u(x,t) dx.$$

From (6) it follows

$$\frac{1}{\pi} \int_{\mathbb{R}^3} u(x,t) \frac{dx}{1+|x|} \geq \frac{1}{8\pi} \frac{\|u_0\|_{L^1}^{3/2}}{E(t)^{1/2} + \|u_0\|_{L^1}^{1/2}}, \qquad t > 0. \tag{8}$$

Since $E(t)$ is increasing, we conclude

$$\frac{1}{\pi} \inf_{t \in [0,T]} \int_{\mathbb{R}^3} u(x,t) \frac{dx}{1+|x|} \geq \kappa(T), \tag{9}$$

with

$$\kappa(t) := \frac{1}{8\pi} \frac{\|u_0\|_{L^1}^{3/2}}{E(t)^{1/2} + \|u_0\|_{L^1}^{1/2}}.$$

Moreover,

$$\frac{dH[u]}{dt} + \kappa(t) \int_{\mathbb{R}^3} \frac{|\nabla \sqrt{u(x,t)}|^2}{1+|x|} dx \leq 0, \qquad t > 0. \tag{10}$$

Upper Bound for $a[u]$ It holds

$$a[u](x,t) = \int_{|x-y|<1} \frac{u(y,t)}{|x-y|} dy + \int_{|x-y|\geq 1} \frac{u(y,t)}{|x-y|} dy \equiv I_1 + I_2. \tag{11}$$

The integral I_2 can be estimated immediately:

$$I_2 \leq \|u_0\|_{L^1}.$$

For I_1 we first use Hölder: since $\frac{1}{|x|}$ is $L^q_{loc}(\mathbb{R}^3)$ for $q < 3$, we get

$$I_1 = \int_{|x-y|<1} \frac{u(y,t)}{|x-y|} dy \leq \left(\int_{|x-y|<1} u(y,t)^{3/2+\varepsilon} dy \right)^{\frac{1}{3/2+\varepsilon}} \left(\int_{|x-y|<1} |x-y|^{-\frac{3+2\varepsilon}{1+2\varepsilon}} dy \right)^{\frac{1+2\varepsilon}{3+2\varepsilon}}$$
$$\leq 4\pi \frac{1+2\varepsilon}{4\varepsilon} \left(\int_{|y|<1+|x|} u(y,t)^{3/2+\varepsilon} dy \right)^{\frac{1}{3/2+\varepsilon}} = \frac{(1+2\varepsilon)\pi}{\varepsilon} \|\sqrt{u(t)}\|^2_{L^{3+2\varepsilon}(B_{1+|x|})}.$$

The interpolation inequality implies (for $0 < \varepsilon \leq 3/2$):

$$\|\sqrt{u(t)}\|_{L^{3+2\varepsilon}(B_{1+|x|})} \leq \|\sqrt{u(t)}\|^{1-\theta}_{L^2(B_{1+|x|})} \|\sqrt{u(t)}\|^{\theta}_{L^6(B_{1+|x|})}, \quad \theta = \frac{3}{2}\frac{1+2\varepsilon}{3+2\varepsilon}.$$

Then, the Sobolev embedding $H^1 \hookrightarrow L^6$ implies

$$\|\sqrt{u(t)}\|_{L^{3+2\varepsilon}(B_{1+|x|})} \leq C(|x|)\|u_0\|^{(1-\theta)/2}_{L^1} \|\sqrt{u(t)}\|^{\theta}_{H^1(B_{1+|x|})}. \tag{12}$$

Notice that the constant C in (12) depends on $|B_{1+|x|}|$ and therefore on $|x|$. However, it is easy to show that such constant (assuming w.l.o.g. that it is optimal) is nonincreasing with respect to $|x|$, thus (12) leads to

$$\|\sqrt{u(t)}\|_{L^{3+2\varepsilon}(B_{1+|x|})} \leq C\|u_0\|^{(1-\theta)/2}_{L^1} \|\sqrt{u(t)}\|^{\theta}_{H^1(B_{1+|x|})}. \tag{13}$$

From (13) we obtain

$$I_1 \leq \varepsilon^{-1} C \|\sqrt{u(t)}\|^{2\theta}_{H^1(B_{1+|x|})} \leq \varepsilon^{-1} C (1 + \|\nabla\sqrt{u(t)}\|^2_{L^2(B_{1+|x|})})^{\theta}$$
$$\leq \varepsilon^{-1} C \left(1 + (2+|x|) \int_{\mathbb{R}^3} \frac{|\nabla\sqrt{u(y,t)}|^2}{1+|y|} dy \right)^{\theta}$$
$$\leq \varepsilon^{-1} C (1+|x|)^{\theta} \left(1 + \int_{\mathbb{R}^3} \frac{|\nabla\sqrt{u(y,t)}|^2}{1+|y|} dy \right)^{\theta}.$$

The estimates of I_1, I_2 imply

$$a[u](x,t)^{1/\theta} \leq \varepsilon^{-1} C (1+|x|) \left(1 + \int_{\mathbb{R}^3} \frac{|\nabla\sqrt{u(y,t)}|^2}{1+|y|} dy \right).$$

The entropy estimate obtained earlier

$$\frac{dH[u]}{dt} + \kappa(t)\int_{\mathbb{R}^3} \frac{|\nabla\sqrt{u(x,t)}|^2}{1+|x|}dx \le 0, \qquad t > 0,$$

leads to

$$a[u](x,t)^{1/\theta} \le \varepsilon^{-1}C(1+|x|)\left(1-\frac{1}{\kappa(t)}\frac{dH[u(t)]}{dt}\right), \qquad \frac{1}{\theta} = \frac{2(3+2\varepsilon)}{3(1+2\varepsilon)}, \qquad 0<\varepsilon\le\frac{3}{2}.$$

We can restate the above estimate in a more handy way by defining $p = 1/\theta \in [1,2)$ and noticing that $\varepsilon^{-1} \le C(2-p)^{-1}$:

$$a[u](x,t)^p \le \frac{C}{2-p}(1+|x|)\left(1-\frac{1}{\kappa(t)}\frac{dH[u(t)]}{dt}\right), \qquad 1 \le p < 2, \tag{14}$$

with $\kappa(t)$ given by (8).

Lower Bound for $H[u]$ A lower bound for $H[u(t)]$ is here showed. Being the spatial domain the whole space $\mathbb{R}^3$, this lower bound is not straightforward. To prove a lower bound for $H[u]$, we write

$$H[u] = \int_{\mathbb{R}^3} u(x)\log(u(x))\chi_{\{u<1\}}\,dx + \int_{\mathbb{R}^3} u(x)\log(u(x))\chi_{\{u>1\}}\,dx,$$

and apply Hölder's inequality to get

$$\begin{aligned}-H[u] &\le \int_{\{u<1\}} u(x)\log\frac{1}{u(x)}\,dx = \int_{\{u<1\}} u(x)^{(1-\varepsilon)/2}u(x)^{(1+\varepsilon)/2}\log\frac{1}{u(x)}\,dx \\ &\le \left(\int_{\{u<1\}} u(x)^{1-\varepsilon}dx\right)^{1/2}\left(\int_{\{u<1\}} u(x)^{1+\varepsilon}\left(\log\frac{1}{u(x)}\right)^2 dx\right)^{1/2}.\end{aligned}$$

Since the function $s \in (0,1) \mapsto s^{\varepsilon/2}\log(1/s) \in \mathbb{R}$ is bounded, we can estimate the term

$$\int_{\{u<1\}} u(x)^{1+\varepsilon}\left(\log\frac{1}{u(x)}\right)^2 dx$$

with a constant that only depends on ε and the L^1 norm of the initial data. Therefore

$$-H[u] \le C_\varepsilon\left(\int_{\{u<1\}} u(x)^{1-\varepsilon}dx\right)^{1/2} \le C_\varepsilon\left(\int_{\mathbb{R}^3} u(x)^{1-\varepsilon}dx\right)^{1/2}. \tag{15}$$

Let us now consider the integral

$$\int_{\mathbb{R}^3} u(x)^{1-\varepsilon} dx = \int (1+|x|^2)^{1-\varepsilon} u(x)^{1-\varepsilon} (1+|x|^2)^{-(1-\varepsilon)} dx$$
$$\le \left(\int_{\mathbb{R}^3} (1+|x|^2) u(x) dx \right)^{1-\varepsilon} \left(\int_{\mathbb{R}^3} (1+|x|^2)^{-(1-\varepsilon)/\varepsilon} dx \right)^{\varepsilon}.$$

For $\varepsilon < 2/5$ we obtain

$$\int_{\mathbb{R}^3} u(x)^{1-\varepsilon} dx \le C_\varepsilon \left(\int_{\mathbb{R}^3} (1+|x|^2) u(x) dx \right)^{1-\varepsilon}.$$

From the above estimate and (15) we conclude

$$-H[u(t)] \le C_\varepsilon (1+E(t))^{(1-\varepsilon)/2}, \qquad 0<\varepsilon<2/5, \;\; t>0. \tag{16}$$

Estimate for $E(t)$ We recall that $E(t) = \int_{\mathbb{R}^3} \frac{|x|^2}{2} u(x,t) dx$, $t > 0$. From (5), (14) it follows ($p' \equiv p/(p-1)$):

$$\frac{dE(t)}{dt} \le 2 \int_{\mathbb{R}^3} a(x,t) u(x,t)^{1/p} u(x,t)^{1/p'} dx \le 2 \left(\int_{\mathbb{R}^3} a(x,t)^p u(x,t) dx \right)^{1/p} \|u_0\|_{L^1}^{1/p'}$$
$$\le C_p \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right)^{1/p} \left(\int_{\mathbb{R}^3} (1+|x|) u(x,t) dx \right)^{1/p}$$
$$\le C_p \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right)^{1/p} \left(\int_{\mathbb{R}^3} \left(\frac{3}{2} + \frac{|x|^2}{2} \right) u(x,t) dx \right)^{1/p}$$
$$\le C_p \left(1 - \frac{1}{\kappa(t)} \frac{dH[u(t)]}{dt} \right)^{1/p} (1+E(t))^{1/p}.$$

The definition (8) of $\kappa(t)$ implies that $\kappa(t)^{-1} \le C(1+\sqrt{E(t)}) \le C\sqrt{1+E(t)}$, so

$$\frac{dE(t)}{dt} \le C_p \left(1 - \frac{dH[u(t)]}{dt} \right)^{1/p} (1+E(t))^{\frac{3}{2p}}.$$

Choosing $p \in (3/2, 2)$, dividing the above inequality times $(1+E(t))^{3/2p}$ and integrating it in the time interval $[0,t]$ leads to ($E_0 \equiv \int_{\mathbb{R}^3} \frac{|x|^2}{2} u_0(x) dx$)

$$(1+E(t))^{1-3/2p} - (1+E_0)^{1-3/2p} \le C_p \int_0^t \left(1 - \frac{dH[u]}{dt} \right)^{1/p} dt'$$
$$\le C_p t^{1-1/p} \left(\int_0^t \left(1 - \frac{dH[u]}{dt} \right) dt' \right)^{1/p} = C_p t^{1-1/p} (t + H[u_0] - H[u(t)])^{1/p}.$$

By inserting (16) into the above inequality we get

$$(1+E(t))^{1-3/2p} - (1+E_0)^{1-3/2p} \le C_{p,\varepsilon} t^{1-1/p}(t + H[u_0] + (1+E(t))^{(1-\varepsilon)/2})^{1/p}$$
$$\le C_{p,\varepsilon}(1+t)(1+E(t))^{(1-\varepsilon)/2p}, \qquad \frac{3}{2} < p < 2, \;\; 0 < \varepsilon < \frac{2}{5}, \;\; t > 0.$$

Let now $9/5 < p < 2$. We want to choose $\varepsilon \in (0, 2/5)$ such that $1 - 3/2p > (1-\varepsilon)/2p$. This is equivalent to $\varepsilon > 4 - 2p$. Since $p > 9/5$, it follows that $4 - 2p < 2/5$, so this choice of ε is admissible. Therefore Young inequality allows us to estimate the right-hand side of the above inequality as follows

$$(1+E(t))^{1-3/2p} - (1+E_0)^{1-3/2p} \le C_{p,\varepsilon}(1+t)^{\xi} + \frac{1}{2}(1+E(t))^{1-3/2p}, \quad \xi = \frac{2p-3}{2p-4+\varepsilon},$$

and so we conclude

$$E(t) \le C_{p,\varepsilon}(1 + t^{2p/(2p-4+\varepsilon)}) \qquad t > 0, \quad \frac{9}{5} < p < 2, \quad 4 - 2p < \varepsilon < \frac{2}{5}. \tag{17}$$

For example, if $p = (9/5+2)/2 = 19/10$ and $\varepsilon = (4-2p+2/5)/2 = 3/10$, then $2p/(2p-4+\varepsilon) = 38$.

Bound (17) means that $E \in L^{\infty}_{loc}(0, \infty)$. A few consequences of this fact are, for example, that for any $T > 0$:

1. the quantity $\kappa(t)$ defined in (8) and appearing e.g. in (14) is uniformly positive for $t \in [0, T]$;
2. the entropy $H[u(t)]$ has a uniform lower bound for $t \in [0, T]$;
3. in Eq. (10) and the mass conservation yield the following estimate:

$$\|\sqrt{u}\|_{L^2(0,T;H^1(\mathbb{R}^3,\gamma(x)dx)} \le C_T, \qquad \gamma(x) \equiv (1+|x|)^{-1}; \tag{18}$$

4. the lower bound (7) for a is uniform in $t \in [0, T]$.

4 Conditional Smoothness

4.1 Conditional Regularity Estimates

This section concerns results of conditional regularity of solutions to (1). These results are based upon a so-called *ε-Poincaré inequality*. We say that u satisfies the ε-Poincaré inequality if given $\varepsilon > 0$ as small as one wishes, there exists a constant

C_ε such that the following inequality holds true

$$\int_{\mathbb{R}^d} u\phi^2\,dx \le \varepsilon \int_{\mathbb{R}^d} a[u]|\nabla\phi|^2\,dx + C_\varepsilon \int_{\mathbb{R}^d} \phi^2\,dx, \tag{19}$$

for any $\phi \in L^1_{loc}(\mathbb{R}^3)$ that makes the right-hand side of (19) convergent.

Theorem 2 (Conditional Regularity) *Let u be a solution to* (1). *Assume u is such that (19) holds true. Then for any $s_1 > 1$, $s_2 > \frac{1}{3}$, $T > 0$, $R > 0$ there exist constants $C_1 = C_1(T, u_0, s_1, R)$, $C_2 = C_2(T, u_0, s_2)$ such that*

$$\|u\|_{L^\infty(B_R\times(t,T))} \le C(T, u_0, s_1, R)\left(\frac{1}{t}+1\right)^{s_1}, \qquad t \in (0,T),$$

$$\|a[u]\|_{L^\infty(\mathbb{R}^3\times(t,T))} \le C(T, u_0, s_2)\left(\frac{1}{t}+1\right)^{s_2}, \qquad t \in (0,T),$$

where $B_R \subset \mathbb{R}^3$ is any ball of radius R.

Weighted Sobolev and Poincare's inequalities have been used to obtain informations about eigenvalues for Schrödinger and degenerate elliptic operators [3–5, 7, 8, 17]. Inspired by the similarity of (1) with the degenerate operator $L = -\mathrm{div}(a[u]\nabla) - u$, in [12] the new inequality (19) has been proposed. We refer to [12] for discussions about (19). While (19) is always true provided u solves the Landau equation for soft-potentials [12], the validity of (19) for Coulomb interactions is still an open question, undoubtedly a very interesting and fundamental one. Consequently the results in Theorem 2 should be viewed as conditional.

Very interesting is the rate of decay in the estimate for $\|u\|_{L^\infty(B_R\times(t,T))}$. In fact one would expect a decay with a rate similar to the heat kernel $1/t^{3/2}$. However thanks to a combination of (19) and a non-local Poincare's inequality proven in [14] we obtain a decay that can be made arbitrary close to $1/t$.

The proof of Theorem 2 is divided into several lemmas and propositions. We will make use of the following

Lemma 1 (Weighted Sobolev Inequality) *Let u be a solution to (1). Any smooth function ϕ satisfies*

$$\left(\int_I\int \phi^q a[u]\,dxdt\right)^{2/q} \le C\left(\int_I\int a[u]|\nabla\phi|^2\,dxdt + \sup_I \int \phi^2\,dx\right),$$

with

$$q \in \left(1, 2\left(1+\frac{2}{3}\right)\right).$$

Proof We refer to [12] for a detailed proof. □

We define $u_k := (u-k)_+$ for a generic constant $k > 0$.

Proposition 1 *The following inequality holds:*

$$\partial_t \int \eta^2 u_k^p \, dx + \frac{4(p-1)}{p} \int a|\nabla(\eta u_k^{p/2})|^2 \, dx + \frac{p(p-1)\tau}{2} \int \frac{u_k^{p-2}}{u^3} |\nabla u_k|^4 \eta^2 dx$$
$$\leq (\mathrm{I}) + (\mathrm{II}) + C\tau \int \eta^2 u_k^p \, dx + C(p)\tau \int \left(1 + \frac{|\nabla\eta|^{4p}}{\eta^{4p}}\right) \eta^2 dx, \tag{20}$$

where

$$(\mathrm{I}) := \frac{4(p-2)}{p} \int u_k^{p/2} (a\nabla(\eta u_k^{p/2}), \nabla\eta) \, dx + \frac{4}{p} \int u_k^p (a\nabla\eta, \nabla\eta) \, dx,$$
$$(\mathrm{II}) := \int u_k^p (\nabla a, \nabla(\eta^2)) \, dx + (p-1) \int u\eta^2 u_k^p \, dx + pk \int u\eta^2 u_k^{p-1} \, dx.$$

Proof Consider

$$\psi = p \, \eta^2 \, u_k^{p-1}$$

as test function for (1). A direct computation yields,

$$\begin{aligned} &p \int \eta^2 u_k^{p-1} \partial_t u_k \, dx \\ &= -p \int (a\nabla u, \nabla(\eta^2 u_k^{p-1})) \, dx + p \int (u\nabla a, \nabla(\eta^2 u_k^{p-1})) \, dx \\ &= (\widetilde{\mathrm{I}}) + (\mathrm{II}). \end{aligned}$$

Expanding the first integral, we have the expression:

$$\int (a\nabla u, \nabla(\eta^2 u_k^{p-1})) \, dx = \int (p-1)\eta^2 u_k^{p-2} (a\nabla u_k, \nabla u_k) + 2u_k^{p-1} \eta (a\nabla u_k, \nabla\eta) \, dx.$$

Let us rewrite this expression in a more convenient form. Note the elementary identity

$$(a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) = \frac{p^2}{4} u_k^{p-2} \eta^2 (a\nabla u_k, \nabla u_k) + p\eta u_k^{p-1} (a\nabla u_k, \nabla\eta) + u_k^p (a\nabla\eta, \nabla\eta),$$

and use it to write,

$$
\begin{aligned}
&(p-1)\eta^2 u_k^{p-2}(a\nabla u_k, \nabla u_k) + 2u_k^{p-1}\eta(a\nabla u_k, \nabla\eta) \\
&= \frac{4(p-1)}{p^2}(a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) \\
&\quad - \frac{(2p-4)}{p}u_k^{p-1}\eta(a\nabla u_k, \nabla\eta) - \frac{4(p-1)}{p^2}u_k^p(a\nabla\eta, \nabla\eta).
\end{aligned}
$$

Further, another elementary identity says

$$
u_k^{p-1}\eta(a\nabla u_k, \nabla\eta) = \frac{2}{p}u_k^{p/2}(a\nabla(\eta u_k^{p/2}), \nabla\eta) - \frac{2}{p}u_k^p(a\nabla\eta, \nabla\eta).
$$

Combining the above, it follows that

$$
\begin{aligned}
&(p-1)\eta^2 u_k^{p-2}(a\nabla u_k, \nabla u_k) + 2u_k^{p-1}\eta(a\nabla u_k, \nabla\eta) \\
&= \frac{4(p-1)}{p^2}(a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) \\
&\quad - \frac{4(p-2)}{p^2}u_k^{p/2}(a\nabla(\eta u_k^{p/2}), \nabla\eta) - \frac{4}{p^2}u_k^p(a\nabla\eta, \nabla\eta).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\widetilde{(\mathrm{I})} = &- \frac{4(p-1)}{p}\int (a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2}))\,dx \\
&+ \frac{4(p-2)}{p}\int u_k^{p/2}(a\nabla(\eta u_k^{p/2}), \nabla\eta)\,dx + \frac{4}{p}\int u_k^p(a\nabla\eta, \nabla\eta)\,dx.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{d}{dt}\int \eta^2 u_k^p\,dx + \frac{4(p-1)}{p}\int (a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2}))\,dx \\
&= \frac{4(p-2)}{p}\int u_k^{p/2}(a\nabla(\eta u_k^{p/2}), \nabla\eta)\,dx + \frac{4}{p}\int u_k^p(a\nabla\eta, \nabla\eta)\,dx \\
&\quad + p\int (u\nabla a, \nabla(\eta^2 u_k^{p-1}))\,dx.
\end{aligned}
$$

We now analyze (II). Since

$$
\begin{aligned}
(\nabla a, u\nabla(\eta^2 u_k^{p-1})) &= uu_k^{p-1}(\nabla a, \nabla(\eta^2)) + (p-1)uu_k^{p-2}\eta^2(\nabla a, \nabla u_k) \\
&= uu_k^{p-1}(\nabla a, \nabla(\eta^2)) + (p-1)(u_k^{p-1} + ku_k^{p-2})\eta^2(\nabla a, \nabla u_k)
\end{aligned}
$$

$$= (u_k^p + ku_k^{p-1})(\nabla a, \nabla(\eta^2)) \\ + \eta^2(\nabla a, \nabla(\frac{p-1}{p}u_k^p + ku_k^{p-1})),$$

it follows that

$$\text{(II)} = p\int (u_k^p + ku_k^{p-1})(\nabla a, \nabla(\eta^2))\, dx \\ - p\int \left(\frac{p-1}{p}u_k^p + ku_k^{p-1}\right) \text{div}(\eta^2\nabla a)\, dx.$$

From the above inequality and the Poisson equation it follows

$$\text{(II)} = p\int (u_k^p + ku_k^{p-1})(\nabla a, \nabla(\eta^2))\, dx - \int ((p-1)u_k^p + pku_k^{p-1})(\nabla a, \nabla(\eta^2))\, dx \\ + \int u\eta^2((p-1)u_k^p + pku_k^{p-1})\, dx \\ = \int u_k^p(\nabla a, \nabla(\eta^2))\, dx + \int u\eta^2\left((p-1)u_k^p + pku_k^{p-1}\right)\, dx.$$

This finishes the proof of the lemma. □

Lemma 2 *Let $p > 1$, then we have the inequality*

$$\frac{d}{dt}\int \eta^2 u_k^p\, dx + \frac{(p-1)}{p}\int a|\nabla(\eta u_k^{p/2})|^2\, dx \\ \leq (p-1)\int \eta^2 u u_k^p\, dx + pk\int \eta^2 u u_k^{p-1}\, dx \\ + C(p)\int u_k^p(a\nabla\eta, \nabla\eta)\, dx - \int u_k^p \eta Tr\,(aD^2\eta))\, dx,$$

where $C(p)$ denotes a constant that is bounded when $p > 1$.

Proof We proceed to bound from above the first term (I) and the first term of (II) resulting from Proposition 1. The aim is to estimate these terms as

$$\frac{4(p-2)}{p}\int u_k^{p/2}(a\nabla(\eta u_k^{p/2}, \nabla\eta)\, dx + \int u_k^p(\nabla a, \nabla(\eta^2))\, dx \\ \leq c_1\int (a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2}))\, dx + \textit{lower order terms},$$

where $c_1 < \frac{4(p-1)}{p}$. For the first term we use Cauchy-Schwarz inequality

$$\begin{aligned}&\left|\frac{4(p-2)}{p}(a\nabla(\eta u_k^{p/2}), u_k^{p/2}\nabla\eta)\right| \\ &\leq \frac{2(p-1)}{p}(a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2})) + \frac{2(p-2)^2}{p(p-1)}u_k^p(a\nabla\eta, \nabla\eta). \qquad (21)\end{aligned}$$

For the first term in (II) we use the identity

$$\operatorname{div}(au_k^p\nabla(\eta^2)) = a\operatorname{div}(u_k^p\nabla(\eta^2)) + u_k^p(\nabla a, \nabla(\eta^2)),$$

and conclude that

$$\begin{aligned}\int u_k^p(\nabla a, \nabla(\eta^2))\,dx &= -\int a\operatorname{div}(u_k^p\nabla(\eta^2))\,dx \\ &= -\int au_k^p\Delta(\eta^2)\,dx - \int(a\nabla u_k^p, \nabla\eta^2)\,dx.\end{aligned}$$

Since

$$\eta\nabla u_k^{p/2} = \nabla(\eta u_k^{p/2}) - u_k^{p/2}\nabla\eta,$$

Young's inequality yields

$$\begin{aligned}-\int(a\nabla u_k^p, \nabla\eta^2)\,dx &= -4\int u_k^{p/2}(a\eta\nabla u_k^{p/2}, \nabla\eta) \\ &= -4\int u_k^{p/2}(a\nabla(\eta u_k^{p/2}), \nabla\eta)\,dx + 4\int u_k^p(a\nabla\eta, \nabla\eta)\,dx \\ &\leq 2\varepsilon\int(a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2}))\,dx + \left(\frac{2}{\varepsilon}+4\right)\int u_k^p(a\nabla\eta, \nabla\eta)\,dx.\end{aligned}$$

Thus

$$\begin{aligned}\int u_k^p(\nabla a, \nabla(\eta^2))\,dx \leq &-\int u_k^p\operatorname{Tr}(aD^2(\eta^2))\,dx + 2\varepsilon\int(a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2}))\,dx \\ &+ \left(\frac{2}{\varepsilon}+4\right)\int u_k^p(a\nabla\eta, \nabla\eta)\,dx. \qquad (22)\end{aligned}$$

Substituting (22) and (21) into (20) we get by choosing $\varepsilon < \frac{p-1}{2p}$

$$
\begin{aligned}
\frac{d}{dt}\int \eta^2 u_k^p \, dx + \frac{(p-1)}{p}\int (a\nabla(\eta u_k^{p/2}), \nabla(\eta u_k^{p/2}))\, dx \\
\leq C(p)\int u_k^p (a\nabla\eta, \nabla\eta)\, dx + (p-1)\int \eta^2 u u_k^p \, dx \\
+ pk\int \eta^2 u u_k^{p-1}\, dx - \int u_k^p \mathrm{Tr}(aD^2(\eta^2))\, dx.
\end{aligned}
$$

This concludes the proof. □

Lemma 3 *We have*

$$
(p-1)\int_t^T\int \eta^2 u u_k^p \, dxds \leq \varepsilon(p-1)\int_t^T\int_{Q_R} a|\nabla(\eta u_k^{p/2})|^2\, dxds + C(R,\varepsilon,p)\int_t^T\int_{Q_R}\eta^2 u_k^p\, dxds,
$$

$$
\begin{aligned}
pk\int_t^T\int \eta^2 u u_k^{p-1}\, dxds \leq\ & p\varepsilon\int_t^T\int a|\nabla(\eta u_k^{p/2})|^2\, dxds + C(R,\varepsilon,p)\int_t^T\int \eta^2 u_k^p\, dxds \\
&+ 2pk^2\int_0^T\int \eta^2 u_k^{p-1}\, dxds.
\end{aligned}
$$

Proof We use here the ε-Poincare's inequality (19) with

$$
\phi = \eta u_k^{p/2}
$$

and get

$$
\int \eta^2 u u_k^p \, dx \leq \varepsilon \int a|\nabla(\eta u_k^{p/2})|^2\, dx + C(R,\varepsilon)\int \eta^2 u_k^p\, dx.
$$

For the second inequality we get

$$
\begin{aligned}
pk\int_t^T\int \eta^2 u u_k^{p-1}\, dxds &= pk\int_t^T\int \eta^2[u\chi_{\{u_k\geq k\}} + u\chi_{\{u_k\leq k\}}]u_k^{p-1}\, dxds \\
&= pk\int_t^T\int \eta^2 u\chi_{\{u_k\geq k\}}u_k^{p-1}\, dxds + pk\int_t^T\int \eta^2 \underbrace{u\chi_{\{u_k\leq k\}}}_{u\leq 2k} u_k^{p-1}\, dxds \\
&\leq p\int_t^T\int \eta^2 u u_k^p\, dxds + 2pk^2\int_0^T\int \eta^2 u_k^{p-1}\, dxds \\
&\leq p\varepsilon\int_t^T\int a|\nabla(\eta u_k^{p/2})|^2\, dxds + C(R,\varepsilon,p)\int_t^T\int \eta^2 u_k^p\, dxds \\
&\quad + 2pk^2\int_0^T\int \eta^2 u_k^{p-1}\, dxds
\end{aligned}
$$

using (19) once more. □

Corollary 1 *Fix times* $0 < T_1 < T_2 < T_3 < T$, $p > 1$ *and a cut-off function* $\eta(v)$. *Then, we have the following inequality*

$$\sup_{T_2 \le t \le T_3} \left\{ \int (\eta u_k^{p/2})^2 \, dx \right\} + \frac{(p-1)}{4p} \int_{T_2}^{T_3} \int a |\nabla(\eta u_k^{p/2})|^2 \, dxdt$$
$$\le \left(\frac{1}{T_2 - T_1} + C(p, \varepsilon, R) \right) \int_{T_1}^{T_3} \int \eta^2 u_k^p \, dxdt$$
$$+ 2pk^2 \int_{T_1}^{T_3} \int \eta^2 u_k^{p-1} \, dxdt$$
$$+ C(p) \int_{T_1}^{T_3} \int u_k^p (a\nabla\eta, \nabla\eta) \, dxdt + \int_{T_1}^{T_3} \int a u_k^p \eta |\Delta\eta| \, dxdt.$$

Proof We start with the bound found in Lemma 2

$$\frac{d}{dt} \int \eta^2 u_k^p \, dx + \frac{(p-1)}{p} \int a |\nabla(\eta u_k^{p/2})|^2 \, dx$$
$$\le (p-1) \int \eta^2 u u_k^p \, dx + pk \int \eta^2 u u_k^{p-1} \, dx$$
$$+ C(p) \int u_k^p (a\nabla\eta, \nabla\eta) \, dx - \int a u_k^p \eta \Delta\eta \, dx.$$

Integrating this inequality from t_1 to t_2 shows that the term

$$\int \eta^2 u_k^p(t_2) \, dx - \int \eta^2 u_k^p(t_1) \, dx + \frac{(p-1)}{p} \int_{t_1}^{t_2} \int a |\nabla(\eta u_k^{p/2})|^2 \, dxdt$$

is bounded by

$$(p-1) \int_{t_1}^{t_2} \int \eta^2 u u_k^p \, dxdt + pk \int_{t_1}^{t_2} \int \eta^2 u u_k^{p-1} \, dxdt$$
$$+ C(p) \int_{t_1}^{t_2} \int u_k^p (a\nabla\eta, \nabla\eta) \, dxdt - \int_{t_1}^{t_2} \int a u_k^p \eta \Delta\eta \, dxdt.$$

For a fixed $t_2 \in (T_2, T_3)$, we take the average with respect to $t_1 \in (T_1, T_2)$ in both sides of the inequality. This yields

$$
\begin{aligned}
\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int \eta^2 u_k^p(t_2)\, dx dt_1 + \frac{(p-1)}{p} \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int a |\nabla(\eta u_k^{p/2})|^2\, dx dt dt_1 \\
\leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int \eta^2 u_k^p(t_1)\, dx dt_1 \\
+ (p-1) \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int \eta^2 u u_k^p\, dx dt dt_1 \\
+ pk \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int \eta^2 u u_k^{p-1}\, dx dt dt_1 \\
+ C(p) \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int u_k^p (a \nabla \eta, \nabla \eta)\, dx dt dt_1 \\
- \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int_{t_1}^{t_2} \int a u_k^p \eta \Delta \eta\, dx dt dt_1,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int \eta^2 u_k^p(t_2)\, dx + \frac{(p-1)}{p} \int_{T_2}^{t_2} \int a |\nabla(\eta u_k^{p/2})|^2\, dx dt \\
\leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \int \eta^2 u_k^p(t)\, dx dt \\
+ (p-1) \int_{T_1}^{t_2} \int \eta^2 u u_k^p\, dx dt + pk \int_{T_1}^{t_2} \int \eta^2 u u_k^{p-1}\, dx dt \\
+ C(p) \int_{T_1}^{t_2} \int u_k^p (a \nabla \eta, \nabla \eta)\, dx dt + \int_{T_1}^{t_2} \int a u_k^p \eta |\Delta \eta|\, dx dt.
\end{aligned}
$$

Since this holds for every $t_2 \in (T_2, T_3)$, this implies the inequality

$$
\begin{aligned}
\sup_{T_2 \leq t \leq T_3} \left\{ \int \eta^2 u_k^p(t)\, dx \right\} + \frac{(p-1)}{p} \int_{T_2}^{T_3} \int a |\nabla(\eta u_k^{p/2})|^2\, dx dt \\
\leq \frac{1}{T_2 - T_1} \int_{T_1}^{T_3} \int \eta^2 u_k^p(t)\, dx dt \\
+ (p-1) \int_{T_1}^{T_3} \int \eta^2 u u_k^p\, dx dt + pk \int_{T_1}^{T_3} \int \eta^2 u u_k^{p-1}\, dx dt \\
+ C(p) \int_{T_1}^{T_3} \int u_k^p (a \nabla \eta, \nabla \eta)\, dx dt + \int_{T_1}^{T_3} \int a u_k^p \eta |\Delta \eta|\, dx dt.
\end{aligned}
$$

As the last step we use Lemma 3 with $\varepsilon < \frac{p-1}{4p^2}$ and get

$$\sup_{T_2 \le t \le T_3} \left\{ \int \eta^2 u_k^p(t)\, dx \right\} + \frac{(p-1)}{4p} \int_{T_2}^{T_3} \int a |\nabla(\eta u_k^{p/2})|^2\, dxdt$$

$$\le \frac{1}{T_2 - T_1} \int_{T_1}^{T_3} \int \eta^2 u_k^p(t)\, dxdt$$

$$+ C(p, \varepsilon, R) \int_{T_1}^{T_3} \int \eta^2 u_k^p\, dxdt + 2pk^2 \int_{T_1}^{T_3} \int \eta^2 u_k^{p-1}\, dxdt$$

$$+ C(p) \int_{T_1}^{T_3} \int u_k^p (a\nabla\eta, \nabla\eta)\, dxdt + \int_{T_1}^{T_3} \int a u_k^p \eta |\Delta\eta|\, dxdt.$$

□

Corollary 2 *We have*

$$\sup_{T_2 \le t \le T_3} \left\{ \int u^p(t)\, dx \right\} + \frac{(p-1)}{4p} \int_{T_2}^{T_3} \int a |\nabla(u^{p/2})|^2\, dxdt$$

$$\le \left(\frac{1}{T_2 - T_1} + C(p, \varepsilon) \right) \int_{T_1}^{T_3} \int u^p(t)\, dxdt.$$

Proof It is a consequence of Corollary 1 if $\eta = 1$ and $k = 0$. □

Lemma 4 (Gain in Integrability) *For each* $p > 1$ *and integer* $n \ge 0$ *we have*

$$\sup_{T/4 \le t \le T} \left\{ \int u^{p+n}(t)\, dx \right\} \le C(p, n) \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t)\, dxdt.$$

Proof The proof is based on iterating Corollary 2 with a non-local weighted Poincare's inequality proven in [14]: for each $p > 0$ any smooth function $u \ge 0$ satisfies

$$\int_{\mathbb{R}^d} u^{p+1}\, dx \le \left(\frac{p+1}{p} \right)^2 \int_{\mathbb{R}^d} a[u] |\nabla(u^{p/2})|^2\, dx. \tag{23}$$

Consider a sequence of times

$$T_n = \frac{T}{4} \left(1 - \frac{1}{2^{n-1}} \right).$$

We start with Corollary 2 which states that for each $p > 1$

$$\sup_{T_2 \le t \le T} \left\{ \int u^p(t)\, dx \right\} + \frac{(p-1)}{4p} \int_{T_2}^{T} \int a[u] |\nabla (u^{p/2})|^2\, dxdt \le \left(\frac{1}{T_2} + C(p,\varepsilon) \right) \int_0^T \int u^p(t)\, dxdt.$$

Inequality (23) implies

$$\frac{p(p-1)}{4(p+1)^2} \int_{T_2}^{T} \int u^{p+1}\, dxdt \le \left(\frac{1}{T_2} + C(p,\varepsilon) \right) \int_0^T \int u^p(t)\, dxdt.$$

We now apply the energy inequality to u^{p+1}:

$$\begin{aligned} \sup_{T_3 \le t \le T} \left\{ \int u^{p+1}(t)\, dx \right\} &+ \frac{p}{4(p+1)} \int_{T_3}^{T} \int a[u] |\nabla (u^{(p+1)/2})|^2\, dxdt \\ &\le \left(\frac{1}{T_3 - T_2} + C(p,\varepsilon) \right) \int_{T_2}^{T} \int u^{p+1}(t)\, dxdt \\ &\le \frac{4(p+1)^2}{p(p-1)} \left(\frac{1}{T_3 - T_2} + C(p,\varepsilon) \right) \left(\frac{1}{T_2} + C(p,\varepsilon) \right) \int_0^T \int u^p(t)\, dxdt \\ &\le 2^6 \frac{(p+1)^2}{p(p-1)} \left(\frac{1}{T} + C(p,\varepsilon) \right)^2 \int_0^T \int u^p(t)\, dxdt. \end{aligned}$$

Iterating the process we get

$$\sup_{T_{n+2} \le t \le T} \left\{ \int u^{p+n}(t)\, dx \right\} \le 2^{\sum_1^{n+2} k} C(p)^n \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t)\, dxdt.$$

Since $T_n \le T/4$ for any $n \ge 0$ we conclude

$$\sup_{T/4 \le t \le T} \left\{ \int u^{p+n}(t)\, dx \right\} \le 2^{n(n+1)} C(p)^n \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t)\, dxdt,$$

and the lemma is proven. □

4.2 Global $L^p L^p$ Estimates

Lemma 5 *There exists a constant that only depends on T and the initial data u_0 such that*

$$\|u\|_{L^1(0,T;L^3(\mathbb{R}^3,\gamma^3 dx))} \le C(T, u_0).$$

Proof We start with the classical Sobolev inequality in three dimensions:

$$\left(\int_{\mathbb{R}^3} g^6\,dx\right)^{\frac{1}{3}} \le C\int_{\mathbb{R}^3} |\nabla g|^2\,dx,$$

and apply it to $g = \frac{\sqrt{u}}{(1+|x|)^{1/2}}$. Since

$$|\nabla g| \le \frac{|\nabla\sqrt{u}|}{(1+|x|)^{1/2}} + \sqrt{u},$$

Sobolev inequality yields

$$\left(\int_{\mathbb{R}^3} \frac{u^3}{(1+|x|)^3}\,dx\right)^{\frac{1}{3}} \le C\int_{\mathbb{R}^3} \frac{|\nabla\sqrt{u}|^2}{(1+|x|)} + u\,dx.$$

Integrating both sides in the time interval $(0, T)$ we get

$$\begin{aligned}\int_0^T \left(\int_{\mathbb{R}^3} \frac{u^3}{(1+|x|)^3}\,dx\right)^{\frac{1}{3}} dt &\le C\int_0^T\int_{\mathbb{R}^3} \frac{|\nabla\sqrt{u}|^2}{(1+|x|)}\,dxdt + \int_0^T\int_{\mathbb{R}^3} u\,dxdt\\ &\le C(T, u_0),\end{aligned} \tag{24}$$

using mass conservation and estimate (18). □

Lemma 6 *There exists a constant that only depends on* T *and the initial data* u_0 *such that*

$$\|u\|_{L^{5/3}(0,T;L^{5/3}(\mathbb{R}^3))} \le C(T, u_0).$$

Proof Interpolation yields

$$\begin{aligned}\int_{\mathbb{R}^3} u^p\,dx &= \int_{\mathbb{R}^3} u^{p\theta}u^{p(1-\theta)}(1+|x|)^m(1+|x|)^{-m}\,dx\\ &\le \left(\int_{\mathbb{R}^3} u^{pp_1\theta}(1+|x|)^{p_1 m}\,dx\right)^{\frac{1}{p_1}}\left(\int_{\mathbb{R}^3} u^{p(1-\theta)p_2}(1+|x|)^{-mp_2}\,dx\right)^{\frac{1}{p_2}},\end{aligned}$$

with $\frac{1}{p_1} + \frac{1}{p_2} = 1$ and $\theta < 1$. For $m = 1$, $p_1 = 3/2$, $p_2 = 3$, $p = 5/3$ and $\theta = 2/5$ we get

$$\begin{aligned}\int_{\mathbb{R}^3} u^p\,dx &\le \left(\int_{\mathbb{R}^3} u(1+|x|)^{3/2}\,dx\right)^{\frac{3}{5}}\left(\int_{\mathbb{R}^3} u^3(1+|x|)^{-3}\,dx\right)^{\frac{1}{3}}\\ &\le \left(\int_{\mathbb{R}^3} u(1+|x|)^2\,dx\right)^{\frac{3}{5}}\left(\int_{\mathbb{R}^3} u^3(1+|x|)^{-3}\,dx\right)^{\frac{1}{3}}.\end{aligned}$$

Integrating in the time interval $(0, T)$ we get

$$\int_0^T \int_{\mathbb{R}^3} u^p \, dxdt \le \int_0^T \left(\int_{\mathbb{R}^3} u(1+|x|)^2 \, dx \right)^{\frac{3}{5}} \left(\int_{\mathbb{R}^3} u^3 (1+|x|)^{-3} \, dx \right)^{\frac{1}{3}} \, dt$$
$$\le C(T, u_0) \int_0^T \left(\int_{\mathbb{R}^3} u^3 (1+|x|)^{-3} \, dx \right)^{\frac{1}{3}} \, dt \le C(T, u_0),$$

using conservation of mass and bound of the second momentum for the second inequality and (24) in the last inequality. □

4.3 Gain in Integrability

The aim of this section is to show that f has enough integrability for $a[u]$ to be uniformly bounded in space and time. A consequence of interpolation and Hölder's inequality is that $a[u](x, t)$, defined as

$$a[u](x, t) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u(y)}{|x - y|} \, dy,$$

is uniformly bounded in space and time if u belongs to $L^\infty(L^p(\mathbb{R}^3))$ with $p > \frac{3}{2}$. This is what we will show next, combining inequality from Lemma 4 with the $L^{5/3}L^{5/3}$ estimate from Lemma 6.

Lemma 7 *For any* $0 < t < T$ *and any integer* n *there exists a constant* $C(p, T, u_0, n)$ *such that for* $\alpha = \frac{(n+1)}{(3n+2)}$*:*

$$\|a[u]\|_{L^\infty(t,T,\mathbb{R}^3)} \le C(T, u_0, n) \left(\frac{1}{t} + 1 \right)^\alpha .$$

Proof Let $r > 0$; for $p > 3/2$ we have

$$4\pi a[u](x, t) = \int_{B_r(x)} \frac{u(y)}{|x - y|} \, dy + \int_{B_r^c(x)} \frac{u(y)}{|x - y|} \, dy$$
$$\le \frac{1}{r} \|u\|_{L^\infty(L^1)} + 4\pi \|u\|_{L^\infty(L^p)} r^{2-3/p},$$

applying Hölder inequality. The minimum of the function $F(r) = \frac{c_1}{r} + c_2 r^{2-3/p}$ is reached at the point

$$r_{min} = \left(\frac{c_1}{(2 - 3/p)\, c_2} \right)^{p/(3(p-1))},$$

and this implies

$$a[u](x,t) \leq 4\|u\|_{L^\infty(L^1)}^{\frac{2p-3}{3(p-1)}} \|u\|_{L^\infty(L^p)}^{\frac{p}{3(p-1)}}.$$

From Lemma 4 we know that

$$\sup_{T/4 \leq t \leq T} \left\{ \int u^{p+n}(t)\, dx \right\} \leq 2^{n(n+1)} C(p)^n \left(\frac{1}{T} + 1 \right)^{n+1} \int_0^T \int u^p(t)\, dxdt,$$

and taking $p = 5/3$ and using Lemma 6 we get

$$\|u\|_{L^\infty(T/4,T,L^{5/3+n}(\mathbb{R}^3))} \leq C(n,T,u_0) \left(\frac{1}{T} + 1 \right)^{\frac{n+1}{5/3+n}}. \tag{25}$$

Going back to $a[u]$ this last estimate implies

$$\begin{aligned} \sup_{t \in (T/4,T), x \in \mathbb{R}^3} a[u](x,t) \leq & c(u_0) \|u\|_{L^\infty(T/4,T;L^{5/3+n})}^{\frac{5/3+n}{2+3n}} \\ \leq & C(n,T,u_0) \left(\frac{1}{T} + 1 \right)^{\frac{n+1}{3n+2}}. \end{aligned} \tag{26}$$

□

4.4 De-Giorgi Iteration and L^∞-Regularization

Proposition 2 *Let $p = \frac{5}{3}$ and q as in Lemma 1. We have*

$$\begin{aligned} \sup_{T_{n+1} \leq t \leq T} \left\{ \int (\eta_n u_n^{p/2})^2\, dx \right\} + \frac{(p-1)}{4p} \int_{T_{n+1}}^T \int a |\nabla(\eta_n u_n^{p/2})|^2\, dxdt \\ \leq C_0 \int_{T_n}^T \int a (\eta_{n-1} u_{n-1}^{p/2})^q\, dxdt, \end{aligned}$$

with

$$C_0 := C^{n-1} C(R,p) \left(\frac{1}{T} + 1 \right) \left(\frac{1}{M} \right)^{\frac{p(q-2)}{2} - 1}.$$

Proof Consider the sequence of times and radii

$$T_n = \frac{1}{4}\left(2 - \frac{1}{2^n}\right)T, \quad R_n = \frac{1}{2}\left(1 + \frac{1}{2^n}\right)R,$$

and, for every $n \geq 1$, let B_n denote the ball $B_n := B_{R_n}(0)$.

Let η_n be a C^∞ function supported in B_n, with $0 \leq \eta_n \leq 1$ everywhere, $\eta_n = 1$ in B_{n+1}, $\|\nabla\eta_n\|_\infty \leq C\eta_n 2^{n+1}$ and $\|D^2(\eta_n)\|_\infty \leq C2^{2n+2}$. Corollary 1 says that for $k_n := M\left(1 - \frac{1}{2^n}\right)$, $T_1 = T_n$, $T_2 = T_{n+1}$, $T_3 = T$, $T_{n+1} - T_n = \frac{T}{2^{n+1}}$ and

$$u_n := \left(u - M\left(1 - \frac{1}{2^n}\right)\right)_+$$

we have

$$\begin{aligned}
\sup_{T_{n+1}\leq t\leq T}\left\{\int \eta_n^2 u_n^p(t)\,dx\right\} &+ \frac{(p-1)}{4p}\int_{T_{n+1}}^T\int a|\nabla(\eta_n u_n^{p/2})|^2\,dxdt \\
&\leq \left(\frac{2^{n+2}}{T} + C(\varepsilon, p)\right)\int_{T_n}^T\int \eta_n^2 u_n^p\,dxdt \\
&\quad + C(p)\int_{T_n}^T\int u_n^p(a\nabla\eta_n, \nabla\eta_n)\,dxdt + 2pk_n^2\int_{T_n}^T\int \eta_n^2 u_n^{p-1}\,dxdt \\
&\quad + \int_{T_n}^T\int a u_n^p \eta_n|\Delta\eta_n|\,dxdt \leq U_n,
\end{aligned}$$

with

$$\begin{aligned}
U_n := &\left(\frac{2^{n+2}}{T} + C(\varepsilon, p)\right)\int_{T_n}^T\int \eta_n^2 u_n^p\,dxdt \\
&+ (C(p)+1)2^{2n+2}\int_{T_n}^T\int_{B_n} a\eta_n^2 u_n^p\,dxdt + 2pk_n^2\int_{T_n}^T\int \eta_n^2 u_n^{p-1}\,dxdt.
\end{aligned}$$

We start by estimating the last term of U_n: since $\eta_{n-1} = 1$ on B_n and $\chi_{\{u_n\geq 0\}} = \chi_{\{u_{n-1}\geq\frac{M}{2^n}\}}$ we have

$$\begin{aligned}
2pk_n^2\int_{T_n}^T\int \eta_n^2 u_n^{p-1}\,dxdt &\leq 2pM^2\int_{T_n}^T\int_{B_n} u_n^{p-1}\,dxdt \\
&= 2pM^2\int_{T_n}^T\int_{B_n} u_n^{p-1}\chi_{\{u_{n-1}\geq\frac{M}{2^n}\}}\,dxdt \\
&\leq 2pM^2\int_{T_n}^T\int_{B_n} u_{n-1}^{p-1}\chi_{\{\eta_{n-1}^{2/p}u_{n-1}\geq\frac{M}{2^n}\}}\,dxdt.
\end{aligned}$$

Hölder inequality yields

$$2pk_n^2 \int_{T_n}^{T} \int \eta_n^2 u_n^{p-1}\, dxdt \le 2pM^2 \int_{T_n}^{T} \left(\int_{B_n} u_{n-1}^{\frac{pq}{2}}\, dx \right)^{\frac{2(p-1)}{pq}} \cdot$$

$$\cdot \left(\int_{B_n} \chi_{\{\eta_{n-1}^{2/p} u_{n-1} \ge \frac{M}{2^n}\}}\, dx \right)^{\frac{pq-2(p-1)}{pq}} dt.$$

Using Chebyshev's inequality

$$\int_{B_n} \chi_{\{\eta_{n-1}^{2/p} u_{n-1} \ge \frac{M}{2^n}\}}\, dx \le \left(\frac{2^n}{M} \right)^{pq/2} \int (\eta_{n-1}^{2/p} u_{n-1})^{pq/2}\, dx$$

we get

$$\begin{aligned}
2pk_n^2 \int_{T_n}^{T} \int \eta_n^2 u_n^{p-1}\, dxdt \le\ & 2pM^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^{T} \left(\int_{B_n} u_{n-1}^{\frac{pq}{2}}\, dx \right)^{\frac{2(p-1)}{pq}} \\
& \times \left(\int (\eta_{n-1} u_{n-1}^{p/2})^q\, dx \right)^{\frac{pq-2(p-1)}{pq}} dt \\
=\ & 2pM^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^{T} \left(\int_{B_n} \eta_{n-1}^q u_{n-1}^{\frac{pq}{2}}\, dx \right)^{\frac{2(p-1)}{pq}} \cdot \\
& \cdot \left(\int (\eta_{n-1} u_{n-1}^{p/2})^q\, dx \right)^{\frac{pq-2(p-1)}{pq}} dt \\
=\ & 2pM^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^{T} \int (\eta_{n-1} u_{n-1}^{p/2})^q\, dxdt \\
\le\ & 2pC(R)M^2 \left(\frac{2^n}{M} \right)^{\frac{pq-2(p-1)}{2}} \int_{T_n}^{T} \int a(\eta_{n-1} u_{n-1}^{p/2})^q\, dxdt.
\end{aligned}$$

We now estimate the first two terms of U_n:

$$\begin{aligned}
& \left(\frac{2^{n+2}}{T} + C(\varepsilon, p) \right) \int_{T_n}^{T} \int \eta_n^2 u_n^p\, dxdt + (C(p)+1) 2^{2n+2} \int_{T_n}^{T} \int_{B_n} a\eta_n^2 u_n^p\, dxdt \\
& \le 2^{2n+2} \left(\frac{1}{T} + C(p,R) \right) \int_{T_n}^{T} \int_{B_n} a\eta_n^2 u_n^p\, dxdt \\
& \le 2^{2n+2} \left(\frac{1}{T} + C(p,R) \right) \int_{T_n}^{T} \int_{B_n} a u_{n-1}^p \chi_{\{u_n \ge 0\}}\, dxdt \\
& \le 2^{2n+2} \left(\frac{1}{T} + C(p,R) \right) \int_{T_n}^{T} \int a\eta_{n-1}^2 u_{n-1}^p \chi_{\{u_{n-1} \ge \frac{M}{2^n}\}}\, dxdt.
\end{aligned}$$

Similarly as before, we apply Hölder's and Chebyshev's inequalities and obtain

$$\int a\eta_{n-1}^2 u_{n-1}^p \chi_{\{u_{n-1}\geq \frac{M}{2^n}\}}\,dx \leq \left(\int a\eta_{n-1}^q u_{n-1}^{pq/2}\,dx\right)^{2/q}\left(\int a\chi_{\{\eta_{n-1}^{2/p}u_{n-1}\geq \frac{M}{2^{n+1}}\}}\,dx\right)^{(q-2)/q}$$
$$\leq \left(\int a(\eta_{n-1}u_{n-1}^{p/2})^q\,dx\right)^{2/q}\left(\left(\frac{2^n}{M}\right)^{pq/2}\int a\eta_{n-1}^q u_{n-1}^{pq/2}\,dx\right)^{(q-2)/q}$$
$$= \left(\frac{2^n}{M}\right)^{p(q-2)/2}\int a(\eta_{n-1}u_{n-1}^{p/2})^q\,dx,$$

which implies

$$\left(\frac{2^{n+2}}{T}+C(\varepsilon,p)\right)\int_{T_n}^T\int \eta_n^2 u_n^p\,dxdt + (C(p)+1)2^{2n+2}\int_{T_n}^T\int a\eta_n^2 u_n^p\,dxdt$$
$$\leq\ 2^{2n+2}\left(\frac{1}{T}+C(p,R)\right)\left(\frac{2^{n+1}}{M}\right)^{p(q-2)/2}\int_{T_n}^T\int a(\eta_{n-1}u_{n-1}^{p/2})^q\,dx.$$

Summarizing we obtain:

$$U_n \leq \left(2pC(R)M^2\left(\frac{2^{n+1}}{M}\right)^{\frac{pq-2(p-1)}{2}} + 2^{2n+2}\left(\frac{1}{T}+C(p,R)\right)\left(\frac{2^n}{M}\right)^{\frac{p(q-2)}{2}}\right)$$
$$\times\int_{T_n}^T\int a(\eta_{n-1}u_{n-1}^{p/2})^q\,dxdt$$
$$\leq 4^{n-1}C(R,p)\left(\frac{1}{T}+1\right)\left(\frac{1}{M}\right)^{\frac{p(q-2)}{2}-1}\int_{T_n}^T\int a(\eta_{n-1}u_{n-1}^{p/2})^q\,dxdt.$$

This completes the proof. □

Proposition 3 *Let $T>0$ and $R>0$. Given any $s>1$ there exists a constant that only depends on s, R, the mass and second moment of u (hence on T) such that*

$$\sup_{(T/4,T)\times B_{R/2}} u(x,t) \leq c_0(s,R,T)\left(\frac{1}{T}+1\right)^s.$$

Proof Lemma 1 for $\phi=\eta_n u_n^{p/2}$ implies

$$\left(\int_{T_{n+1}}^T\int a(\eta_n u_n^{p/2})^q\,dxdt\right)^{2/q} \leq \sup_{T_{n+1}\leq t\leq T}\left\{\int(\eta_n u_n^{p/2})^2\,dx\right\} \tag{27}$$
$$+\frac{(p-1)}{4p}\int_{T_{n+1}}^T\int a|\nabla(\eta_n u_n^{p/2})|^2\,dxdt.$$

Then Proposition 2 says that

$$\sup_{T_{n+1}\le t\le T}\left\{\int(\eta_n u_n^{p/2})^2\,dx\right\}+\frac{(p-1)}{4p}\int_{T_{n+1}}^{T}\int a|\nabla(\eta_n u_n^{p/2})|^2\,dxdt$$
$$\le U_n\le C_{n,p,T,M}\int_{T_n}^{T}\int a(\eta_{n-1}u_{n-1}^{p/2})^q\,dxdt$$
$$\le C_{n,p,T,M}\left(\sup_{T_n\le t\le T}\left\{\int(\eta_{n-1}u_{n-1}^{p/2})^2\,dx\right\}\right.$$
$$\left.+\frac{(p-1)}{4p}\int_{T_n}^{T}\int a|\nabla(\eta_{n-1}u_{n-1}^{p/2})|^2\,dxdt\right)^{\frac{q}{2}}$$
$$\le C_{n,p,T,M}\,U_{n-1}^{\frac{q}{2}},$$

with

$$C_{n,p,T,M}:=4^{n-1}\underbrace{C(p,R)\left(\frac{1}{T}+1\right)\left(\frac{1}{M}\right)^{\frac{p(q-2)}{2}-1}}_{:=C_{p,R,T,M}}.$$

This leads to a recurrence relation

$$U_n\le 4^{n-1}C_{p,R,T,M}U_{n-1}^{\frac{q}{2}}.$$

A standard induction argument shows that the above recurrence relation yields

$$\lim_{n\to+\infty}U_n=0, \tag{28}$$

provided the initial step

$$U_0:=\left(\frac{1}{T}+C(\varepsilon,p)\right)\int_{T_0}^{T}\int\eta_0^2u^p+a\eta_0^2u^p\,dxdt,\quad T_0=T/4,$$

is small enough. For completeness we sketch this last argument: assume for a certain $n\ge 0$

$$4^nU_n^{\frac{q}{2}-1}\le\frac{1}{C_{p,R,T,M}(8)^{\frac{1}{\frac{q}{2}-1}}}, \tag{29}$$

we show that the same is true for $n+1$: using (29) we get

$$\begin{aligned}
4^{n+1}U_{n+1}^{\frac{q}{2}-1} &\le 4^{n+1}\left(4^n C_{p,R,T,M} U_n^{\frac{q}{2}}\right)^{\frac{q}{2}-1} \le 4C_{p,R,T,M}^{\frac{q}{2}-1}\left(C^n U_n^{\frac{q}{2}-1}\right)^{\frac{q}{2}}\\
&\le 4C_{p,R,T,M}^{\frac{q}{2}-1}\left(\frac{1}{C_{p,R,T,M}(8)^{\frac{1}{\frac{q}{2}-1}}}\right)^{\frac{q}{2}}\\
&\le C_{p,R,T,M}^{-1}\frac{4}{(8)^{\frac{\frac{q}{2}}{\frac{q}{2}-1}}} \le \frac{1}{C_{p,R,T,M}(8)^{\frac{1}{\frac{q}{2}-1}}}.
\end{aligned}$$

Therefore if (29) holds for U_0, i.e.

$$U_0^{\frac{q}{2}-1} \le \frac{1}{C_{p,R,T,M}(8)^{\frac{1}{\frac{q}{2}-1}}}, \tag{30}$$

then

$$\lim_{n\to+\infty} U_{n+1}^{\frac{q}{2}-1} \le \lim_{n\to+\infty}\frac{c}{4^n} = 0,$$

and (28) is proven.

We are left to prove that for M big enough the condition (30) is satisfied. Let $p = 5/3 + n$ with n any positive integer. Inequalities (25) and (26) imply

$$\begin{aligned}
U_0 &\le c(n)\left(\frac{1}{T}+1\right)\int_{T/4}^{T}\int u^{5/3+n} + au^{5/3+n}\, dxdt\\
&\le c(n)\left(\frac{1}{T}+1\right)\left(\|a\|_{L^\infty((T/4,T)\times\mathbb{R}^3)}+1\right)\int_{T/4}^{T}\int u^{5/3+n}\, dxdt\\
&\le c(n,u_0,T)\left(\frac{1}{T}+1\right)^{1+\frac{n+1}{3n+2}+n+1} = c(n,u_0,T)\left(\frac{1}{T}+1\right)^{\frac{7n+5}{3n+2}+n}.
\end{aligned}$$

We chose M big enough so that

$$c(n)\left(\frac{1}{T}+1\right)^{\left(\frac{7n+5}{3n+2}+n\right)\left(\frac{q}{2}-1\right)}\left(\frac{1}{T}+1\right)\left(\frac{1}{M}\right)^{\frac{(5/3+n)(q-2)}{2}-1} \le \frac{1}{8^{\frac{1}{\frac{q}{2}-1}}},$$

or equivalently

$$M > c(n)\left(\frac{1}{T}+1\right)^{\alpha(n)},$$

with

$$\alpha(n) = \frac{\left(\frac{7n+5}{3n+2} + n\right)\left(\frac{q}{2} - 1\right)}{(5/3 + n)(\frac{q}{2} - 1) - 1}.$$

Note that $\alpha(n) \geq 0$ for each $n \geq 0$ and $\alpha(n) \to 0$ as $n \to +\infty$. Therefore given any $s > 1$ there exists an integer n such that $\alpha(n) < s$ and this concludes the proof. □

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A Gradient Flow Perspective on the Quantization Problem

Mikaela Iacobelli

Abstract In this paper we review recent results by the author on the problem of quantization of measures. More precisely, we propose a dynamical approach, and we investigate it in dimensions 1 and 2. Moreover, we discuss a recent general result on the static problem on arbitrary Riemannian manifolds.

Keywords Quantization of measures · Gradient flows · Riemannian manifolds

1 Introduction

The term *quantization* refers to the process of finding an *optimal* approximation of a d-dimensional probability density by a convex combination of a finite number N of Dirac masses. The quality of such approximation is usually measured in terms of the Monge-Kantorovich or Wasserstein metric.

The need for such approximations first arose in the context of information theory in the early 1950s. The idea was to see the quantized measure as the digitization of an analog signal intended for storage on a data storage medium or transmitted via a channel [6, 12]. Another classical application of the quantization problem concerns numerical integration, where integrals with respect to certain probability measures need to be replaced by integrals with respect to a good discrete approximation of the original measure [22]. Moreover, this problem has applications in cluster analysis, materials science (crystallization and pattern formation [4]), pattern recognition, speech recognition, stochastic processes, and mathematical models in economics [3, 7, 23] (optimal location of service centers). Due to the wide range of applications aforementioned, the quantization problem has been studied with several completely different techniques, and a comprehensive review on the topic goes beyond the purposes of this paper. Nevertheless, it is worth to mention that the problem of

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the quantization of measure has been studied with a Γ-convergence approach in [2, 3, 5, 21]. For a detailed exposition on the quantization problem and a complete list of references see the monograph [16] and [15, Chapter 33].

1.1 A Motivating Example

Question What is the "optimal" way to locate N clinics in a region Ω with population density ρ?

To answer this question we have to choose:

- a suitable notion of "optimality";
- the location of each clinic x^i;
- the capacity of each clinic m_i.

1.2 Setup of the Problem

We now introduce the theoretical setup of the problem. Given $r \geq 1$, consider ρ a probability density on an open set $\Omega \subset \mathbb{R}^d$ with finite r-th moment,

$$\int_\Omega |y|^r \rho(y)dy < \infty.$$

Given N points $x^1, \dots, x^N \in \Omega$, we seek the best approximation of ρ, in the sense of Wasserstein distances,[1] by a convex combination of Dirac masses centered at $x^1, \dots, x^N$:

$$W_r\Big(\rho, \sum_i m_i \delta_{x^i}\Big)^r := \inf_\gamma \bigg\{ \int_{\Omega\times\Omega} |x-y|^r d\gamma(x,y) \; : \; (\pi_1)_\# \gamma = \sum_i m_i \delta_{x^i}, \; (\pi_2)_\# \gamma = \rho(y)dy \bigg\},$$

where γ varies among all probability measures on $\Omega \times \Omega$, and $\pi_i : \Omega \times \Omega \to \Omega$ ($i = 1, 2$) denotes the canonical projection onto the i-th factor (see [1, 23] for more details on the Monge-Kantorovich distance between probability measures).

Remark 1.1 We note the following equivalent definition, which the reader may find more intuitive. Since ρ is absolutely continuous, it follows by the general theory of optimal transport (see for instance [1]) that the Wasserstein distance can also be obtained as an infimum over maps:

$$W_r\Big(\rho, \sum_i m_i \delta_{x^i}\Big)^r := \inf \int_\Omega |y - T(y)|^r \rho(y)\, dy$$

[1] Equivalently known as Monge-Kantorovich distances; we shall use both terms interchangeably.

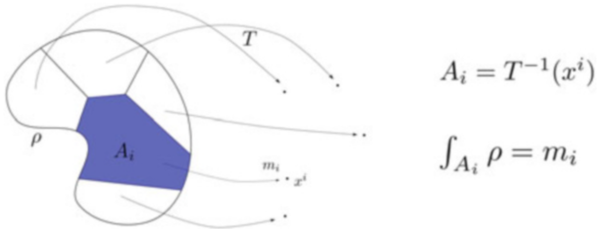

Fig. 1 Transport map

where $T : \Omega \to \Omega$ varies among all maps that transport ρ onto $\sum_i m_i \delta_{x^i}$. In other words, the transport map T partitions a region Ω with population density ρ into N regions, $\{T^{-1}(x_i)\}_{i=1}^N$. Region $T^{-1}(x_i)$ is assigned to the resource (e.g., clinic) located at point x_i of mass m_i. If T is an *optimal* transport map, then it minimizes the L^r distance between the population and the resources (see Fig. 1).

Hence, we minimize

$$\inf \left\{ W_r \left(\sum_i m_i \delta_{x^i}, \rho(y)dy \right)^r \,:\, m_1, \ldots, m_N \geq 0,\ \sum_{i=1}^N m_i = 1 \right\}.$$

As shown in [16], the following facts hold:

1. The best choice of the masses m_i is given by

$$m_i := \int_{W(x^i|\{x^1,\ldots,x^N\})} \rho(y)dy,$$

   where

$$W(x^i|\{x^1,\ldots,x^N\}) := \{y \in \Omega \,:\, |y - x^i| \leq |y - x^j|,\ j \in 1,\ldots,N\}$$

   is the so called *Voronoi cell* of x^i in the set $x^1, \ldots, x^N$ (see Fig. 2).
2. The following identity holds:

$$\inf \left\{ MK_r \left(\sum_i m_i \delta_{x^i}, \rho(y)dy \right) \,:\, m_1, \ldots, m_N \geq 0,\ \sum_{i=1}^N m_i = 1 \right\} = F_{N,r}(x^1, \ldots, x^N),$$

   where

$$F_{N,r}(x^1, \ldots, x^N) := \int_\Omega \min_{1 \leq i \leq N} |x^i - y|^r \rho(y)dy.$$

Fig. 2 20 points and their Voronoi cells.
Image from Wikipedia https://en.wikipedia.org/wiki/Voronoi_diagram

Now that the optimal masses have been found in terms of $x^1, \dots, x^N$, we seek for the optimal location of these points by minimizing $F_{N,r}$. As shown in [16, Chapter 2, Theorem 7.5], if one chooses $x^1, \dots, x^N$ in an optimal way by minimizing the functional $F_{N,r} : (\mathbb{R}^d)^N \to \mathbb{R}^+$, then in the limit as N tends to infinity these points distribute themselves according to a probability density proportional to $\rho^{d/d+r}$. More precisely, under the assumption that

$$\int_{\mathbb{R}^d} |x|^{r+\delta} \rho(x)\, dx < \infty \qquad \text{for some } \delta > 0 \tag{1.1}$$

one has

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x^i} \rightharpoonup \frac{\rho^{d/d+r}}{\int_\Omega \rho^{d/d+r}(y)dy}\, dx \qquad \text{weakly in } \mathcal{P}(\Omega). \tag{1.2}$$

These issues have been extensively studied from the point of view of the calculus of variations [16, Chapter 1, Chapter 2]. In [8], we considered a gradient flow approach to this problem in dimension 1.

Now we will explain the general heuristic of the dynamical approach, and we will later discuss the main difficulties in extending this method to higher dimension.

1.3 A Dynamical Approach to the Quantization Problem

Given N points $x_0^1, \dots, x_0^N$ in $\mathbb{R}^d$, we consider their evolution under the gradient flow generated by $F_{N,r}$, that is, we solve the system of ODEs in $(\mathbb{R}^d)^N$

$$\begin{cases} \big(\dot{x}^1(t), \dots, \dot{x}^N(t)\big) = -\nabla F_{N,r}\big(x^1(t), \dots, x^N(t)\big), \\ \big(x^1(0), \dots, x^N(0)\big) = (x_0^1, \dots, x_0^N). \end{cases} \tag{1.3}$$

As usual in gradient flow theory, as t tends to infinity one expects the points $\big(x^1(t), \dots, x^N(t)\big)$ to converge to a minimizer $(\bar{x}^1, \dots, \bar{x}^N)$ of $F_{N,r}$. Hence, in view of (1.2), the empirical measure

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{\bar{x}^i}$$

is expected to converge to

$$\frac{\rho^{d/d+r}}{\int_\Omega \rho^{d/d+r}(y)dy}dx$$

as $N \to \infty$.

We now want to exchange the limits $t \to \infty$ and $N \to \infty$, and for this we need to take the limit in the ODE above as N goes to infinity. As a way to do this, we take a set of reference points $(\hat{x}^1, \dots, \hat{x}^N)$ and we parameterize a general family of N points x^i as the image of $\hat{x}^i$ via a slowly varying smooth map $X : \mathbb{R}^d \to \mathbb{R}^d$, that is

$$x^i = X(\hat{x}^i).$$

In this way, the functional $F_{N,r}(x^1, \dots, x^N)$ can be rewritten in terms of the map X, that is

$$F_{N,r}(x^1, \dots, x^N) = F_{N,r}\big(X(\hat{x}^1), \dots, X(\hat{x}^N)\big),$$

and (a suitable renormalization of it) should converge to a functional $\mathcal{F}[X]$. Hence, we can expect that the evolution of $x^i(t)$ for N large is well-approximated by the L^2-gradient flow of $\mathcal{F}$. Although this formal argument may look convincing, already the one dimensional case is rather delicate. In the next section, we review the results of [8].

2 The 1D Case

The aim of this section is to describe the GF approach introduced above in the one dimensional case. This case will already show several features of this problem. In particular we will need to study the dynamics of degenerate parabolic equations, and to use several refined estimates on stability of PDEs.

2.1 The Continuous Functional

With no loss of generality let Ω be the open interval $[0, 1]$ and consider ρ a smooth probability density on Ω. In order to obtain a continuous version of the functional

$$F_{N,r}(x^1, \ldots, x^N) = \int_0^1 \min_{1\le i\le N} |x^i - y|^r \rho(y)\, dy,$$

with $0 \le x^1 \le \ldots \le x^N \le 1$, assume that

$$x^i = X\left(\frac{i-1/2}{N}\right), \qquad i = 1, \ldots, N$$

with $X : [0, 1] \to [0, 1]$ a smooth non-decreasing map such that $X(0) = 0$ and $X(1) = 1$. Then the expression for the minimum becomes

$$\min_{1\le j\le N} |y - x^j|^r = \begin{cases} |y - x^i|^r & \text{for } y \in (x^{i-1/2}, x^{i+1/2}), \\ |y|^r & \text{for } y \in (0, x^{1/2}), \\ |y-1|^r & \text{for } y \in (x^{N+1/2}, 1), \end{cases}$$

and $F_{N,r}$ is given by

$$F_{N,r}(x^1, \ldots, x^N) =$$
$$\sum_{i=1}^N \int_{x^{i-1/2}}^{x^{i+1/2}} |y - x^i|^r \rho(y) dy + \int_0^{x^{1/2}} |y|^r \rho(y) dy, + \int_{x^{N+1/2}}^1 |y-1|^r \rho(y) dy.$$

where

$$x^{i+1/2} := \frac{x^i + x^{i+1}}{2} \qquad i = 0, \ldots, N,$$

with the convention $x^0 = 0$ and $x^{N+1} = 1$. Hence, by a Taylor expansion, we get

$$F_{N,r}(x^1, \ldots, x^N) = \frac{C_r}{N^r} \int_0^1 \rho(X(\theta)) |\partial_\theta X(\theta)|^{r+1} d\theta + O\left(\frac{1}{N^{r+1}}\right),$$

where $C_r = \frac{1}{2^r(r+1)}$ and $O\left(\frac{1}{N^{r+1}}\right)$ depends on the smoothness of ρ and X (for instance, $\rho \in C^1$ and $X \in C^2$ is enough). Hence

$$N^r F_{N,r}(x^1, \ldots, x^N) \longrightarrow C_r \int_0^1 \rho(X(\theta)) |\partial_\theta X(\theta)|^{r+1} d\theta := \mathcal{F}[X]$$

as $N \to \infty$.

By a standard computation, we obtain the gradient flow PDE for $\mathcal{F}$ for the L^2-metric,

$$\partial_t X(t,\theta) = C_r\Big((r+1)\partial_\theta\big(\rho(X(t,\theta))|\partial_\theta X(t,\theta)|^{r-1}\partial_\theta X(t,\theta)\big) - \rho'(X(t,\theta))|\partial_\theta X(t,\theta)|^{r+1}\Big), \tag{2.1}$$

coupled with the Dirichlet boundary condition

$$X(t,0) = 0, \qquad X(t,1) = 1. \tag{2.2}$$

Remark In the particular case $\rho \equiv 1$, we get the p-Laplacian equation

$$\partial_t X = C_r\,(r+1)\,\partial_\theta\big(|\partial_\theta X|^{r-1}\partial_\theta X\big) \tag{2.3}$$

with $p - 1 = r$. Hence, in general, the gradient flow PDE for $\mathcal{F}$ is a degenerate parabolic equation. More precisely, the degeneracy comes from the fact that the coefficient $|\partial_\theta X|^{r-1}$ appearing in the equation may vanish or go to infinity. So a natural question becomes:

Degeneracy Issue if $0 < c_0 \le \partial_\theta X_0 \le C_0$, is a similar bound true for all times? Although the answer is easily seen to be positive for the case $\rho \equiv 1$ using that fact that $\partial_\theta X$ solves a "nice" equation, the question becomes much more delicate for a general ρ. In the next section we show how to give a positive answer to the degeneracy issue for a general class of densities ρ.

2.2 An Eulerian Formulation

Define $f \equiv f(t,x)$ by

$$f(t,x)\,dx = X(t,\cdot)_\# d\theta,$$

namely

$$\int_0^1 \varphi(x) f(t,x)\,dx = \int_0^1 \varphi(X(t,\theta))\,d\theta \qquad \text{for all } \varphi \in C^0([0,1]).$$

Performing the change of variable $x = X(t,\theta)$ in the left hand side, the above identity gives (as long as $X(t,\theta) : [0,1] \to [0,1]$ is a diffeomorphism)

$$\int_0^1 \varphi(X(t,\theta)) f(t,X(t,\theta))\partial_\theta X(t,\theta)\,d\theta = \int_0^1 \varphi(X(t,\theta))\,d\theta \qquad \text{for all } \varphi \in C^0([0,1])$$

from which we deduce (by the arbitrariness of φ)

$$f(t, X(t,\theta)) = \frac{1}{\partial_\theta X(t,\theta)}.$$

Then, by a direct computation, we get

$$\begin{cases} \partial_t f = -r\, C_r\, \partial_x \left(f \partial_x \left(\frac{\rho}{f^{r+1}} \right) \right), & x \in \mathbb{R} \\ f(t, x+1) = f(t,x) \end{cases} \tag{2.4}$$

Remark If $\rho \equiv 1$ the Eulerian equation becomes

$$\partial_t f = -C_r\,(r+1)\,\partial_x^2\big(f^{-r}\big)$$

which is an equation of very fast diffusion type.

Let us set $m := \rho^{1/(1+r)}$ and $u := f/m$. Then the Eulerian quantization gradient flow equation becomes

$$\partial_t u = -\frac{(r+1)\,C_r}{m}\partial_x \left(m\,\partial_x \Big(\frac{1}{u^r} \Big) \right). \tag{2.5}$$

For the latter equation we can then prove the following comparison principle [8, Lemma 2.1]:

Lemma 2.1 *If* $u > 0$ *is a solution of* (2.5) *and* $c > 0$*, then*

$$\frac{d}{dt} \int_0^1 (u-c)_+(t,x)\, m(x)\, dx \le 0,$$

$$\frac{d}{dt} \int_0^1 (u-c)_-(t,x)\, m(x)\, dx \le 0.$$

Thanks to this lemma, we deduce that the following implication holds for all constants $0 < c_0 \le C_0$:

$$c_0 \le u(0,x) \le C_0 \qquad \Rightarrow \qquad c_0 \le u(t,x) \le C_0 \qquad \text{for all } t \ge 0.$$

Therefore, we obtain the following comparison principle:

Corollary 2.2 *Assume that* $0 < \lambda \le \rho \le 1/\lambda$ *and* $0 < a_0 \le \partial_\theta X(0) \le A_0$*. Then there exist* $0 < b_0 \le B_0$*, depending only on* λ, a_0, A_0*, such that*

$$0 < b_0 \le \partial_\theta X(t) \le B_0 \qquad \textit{for all } t \ge 0.$$

Remark The Eq. (2.4) is a very fast diffusion equation that has an interest on its own. In the paper [18] we investigated the asymptotic behavior of (2.4) and its natural gradient flows structure in the space of probability measures endowed with the Wasserstein distance. By using this different approach, one can prove convergence results for (2.4) also in situations that are not covered by the results in [8, 9]. Using energy-entropy production techniques, one can prove exponential convergence to equilibrium under minimal assumptions on the data when the functional is not convex in the Wasserstein space. Also, by a detailed analysis of the Hessian of the functional, we can provide sufficient conditions for stability of solutions with respect to the Wasserstein distance.

2.3 Main Result

Our main result in [8] shows that, under the assumptions that $r = 2$, $\|\rho - 1\|_{C^2} \ll 1$, and that the initial datum is smooth and increasing, the discrete and the continuous gradient flows remain *uniformly* close in L^2 for *all* times. In addition, by entropy-dissipation inequalities for the PDE, we show that the continuous gradient flow converges exponentially fast to the stationary state for the PDE, which is seen in Eulerian variables to correspond to the measure $\frac{\rho^{1/3}\,d\theta}{\int \rho^{1/3}}$, as predicted by (1.2). We point out that the assumption $r = 2$ is not essential, and it is imposed just to simplify some computations so as to emphasize the main ideas.

Our main theorem can be informally stated as follows (we refer to [8] for the precise assumptions on the initial data):

Theorem 2.3 *Assume $r = 2$, $\|\rho - 1\|_{C^2} \leq \bar{\varepsilon}$, and let $\big(x^1(t), \ldots, x^N(t)\big)$ be the gradient flow of $F_{N,2}$ starting from $\big(x_0^1, \ldots, x_0^N\big)$. Under some suitable assumptions on the initial data, if $\bar{\varepsilon}$ is small enough, then the continuous and discrete GF remain quantitatively close for all times:*

$$\sup_{t \geq 0} \frac{1}{N} \sum_{i=1}^{N} \Big| x_i(N^3 t) - X(t, \tfrac{i-1/2}{N}) \Big|^2 \leq \frac{C'}{N^4}.$$

In particular

$$W_1\left(\frac{1}{N}\sum_i \delta_{x^i(t)}, \frac{\rho^{1/3}\,d\theta}{\int \rho^{1/3}}\right) \leq \frac{2C'}{N} \qquad \textit{for all } t \geq \frac{N^3 \log N}{c'}.$$

We now give a quick overview of the proof of this result, and we refer the reader to [8] for a detailed proof.

Strategy of the Proof As we shall explain, the proof in the case $\rho \not\equiv 1$ is more involved than the case $\rho \equiv 1$. We begin with the simpler case $\rho \equiv 1$.

- *The case* $\rho \equiv 1$. In this situation the L^2-GF of $\mathcal{F}$ depends on $\partial_\theta X$ and $\partial_{\theta\theta} X$, but not on X itself, see (2.3). By a discrete maximum principle for the incremental quotients, we can show that the discrete monotonicity estimate

$$\frac{c_0}{N} \leq x^{i+1}(t) - x^i(t) \leq \frac{C_0}{N} \qquad \text{for all } i$$

holds for all times, provided it is satisfied at time 0. Thanks to this information, we can perform a Gronwall-type argument on the quantity

$$\frac{1}{N} \sum_{i=1}^{N} \left| x_i(N^3 t) - X(t, \tfrac{i-1/2}{N}) \right|^2,$$

and this allows us to prove that the discrete and the continuous gradient flows remain uniformly close in L^2 for *all* times.
- *The case* $\rho \not\equiv 1$. This case is more delicate because there is no clear way to show the validity of the discrete monotonicity estimate, so the approach for the case $\rho \equiv 1$ completely fails. To circumvent this, we implement a bootstrap argument that combines a finite-time stability in L^∞ with L^2 exponential convergence. This is roughly described in the next 5 steps.

Step 1: We show that

$$\hat{X}(t) := \left(X\left(t, \tfrac{1/2}{N}\right), \ldots, X\left(t, \tfrac{N-1/2}{N}\right) \right)$$

solves the discrete gradient flow equation up to an error of order $1/N^2$.

Step 2: We prove that the discrete and continuous gradient flows stay $1/N^2$-close on a finite interval of time, namely

$$\left| x^i(N^3 t) - X(t, \tfrac{i-1/2}{N}) \right| = O\left(\frac{1+T}{N^2} \right) \qquad \text{for all } i, \text{ for all } t \in [0, T].$$

Step 3: By Step 2, we are able to transfer the discrete monotonicity estimate from $X(t, \frac{i}{N})$ to $x^i(N^3 t)$ on $[0, T]$. More precisely, it follows by Corollary 2.2 that

$$\frac{b_0}{N} \leq X(t, \tfrac{i+1/2}{N}) - X(t, \tfrac{i-1/2}{N}) \leq \frac{B_0}{N} \qquad \text{for all } i, \text{ for all } t \in [0, T],$$

so a triangle inequality yields

$$\frac{b_0}{2N} \leq x^{i+1}(t) - x^i(t) \leq \frac{2B_0}{N} \qquad \text{for all } i, \text{ for all } t \in [0, T],$$

provided T is bounded and N is sufficiently large.

Step 4: Thanks to the monotonicity bound established in Step 3, as in the case $\rho \equiv 1$ we are now able to perform a Gronwall argument in L^2 to deduce that

$$t \mapsto \frac{1}{N} \sum_{i=1}^{N} \left| x^i(N^3 t) - X(t, \tfrac{i-1/2}{N}) \right|^2$$

decreases exponentially in time on $[0, T]$. For this step, the assumption $\|\rho - 1\|_{C^2} \ll 1$ is crucial (see also Sect. 2.4 below).

Step 5: This is the key step: choosing T carefully, for N large enough, the exponential gain from Step 4 allows us to iterate the argument above starting from time T instead of 0, and obtain the previous estimates on $[T, 2T]$. Iterating infinitely many times, this concludes the proof. □

2.4 On the Assumptions $\|\rho - 1\|_{C^2} \ll 1$

As we have seen in the previous section, we have been able to prove the closeness of the discrete and continuous gradient flow, together with an exponential stability estimate, under the assumption $\|\rho - 1\|_{C^2} \ll 1$. The aim now is to show that the hypothesis $\|\rho - 1\|_{C^2} \ll 1$ is *necessary* to ensure the convexity of $\mathcal{F}$ (and therefore to hope to obtain L^2-stability).

It will be convenient to specify the dependence of $\mathcal{F}$ on ρ, so we denote

$$\mathcal{F}_\rho(X) := \int_0^1 \rho(X)\, |\partial_\theta X|^3 \, d\theta.$$

We begin by computing the Hessian of $\mathcal{F}_\rho$

Assume $\lambda \le \rho \le \frac{1}{\lambda}$, and let $X, Y \in L^2([0, 1])$ with $0 \le c \le \partial_\theta X \le C$ and $|\partial_\theta Y| \le C$. Note that, to ensure that $(X + sY)(0) = 0$ and $(X + sY)(1) = 1$ for all s small, we need to assume that

$$X(0) = 0, \qquad X(1) = 1, \qquad Y(0) = 0, \qquad Y(1) = 0.$$

Then

$$\begin{aligned} D^2\mathcal{F}_\rho[X](Y, Y) &= \frac{d^2}{ds^2}|_{s=0}\mathcal{F}_\rho[X + sY] \\ &= 6 \int_0^1 \rho(X)\, \partial_\theta X\, (\partial_\theta Y)^2 \, d\theta \\ &\quad + 6 \int_0^1 \rho'(X)\, (\partial_\theta X)^2\, (\partial_\theta Y)\, Y \, d\theta + \int_0^1 \rho''(X)\, (\partial_\theta X)^3\, Y^2 \, d\theta. \end{aligned}$$

To build a counterexample, we consider $X(t,\theta)=\theta$. By the formula for the Hessian above, we see that for any smooth density $\bar\rho$ and for any smooth function Y,

$$D^2\mathcal{F}_{\bar\rho}(X)[Y,Y]=6\int_0^1\bar\rho\,(\partial_\theta Y)^2\,d\theta+6\int_0^1\bar\rho'\,\partial_\theta Y\,Y\,d\theta+\int_0^1\bar\rho''\,Y^2\,d\theta.$$

Integrating by parts we have

$$\begin{aligned}D^2\mathcal{F}_{\bar\rho}(X)[Y,Y]&=6\int_0^1\bar\rho\,(\partial_\theta Y)^2\,d\theta-6\int_0^1\bar\rho\,(\partial_\theta Y)^2-6\int_0^1\bar\rho\,\partial_\theta^2 Y\,Y\,d\theta\\&\quad+2\int_0^1\bar\rho\Big[(\partial_\theta Y)^2+\partial_\theta^2 Y\,Y\Big]d\theta\\&=2\int_0^1\bar\rho\,(\partial_\theta Y)^2\,d\theta-4\int_0^1\bar\rho\,\partial_\theta^2 Y\,Y\,d\theta.\end{aligned}$$

We now fix $\varepsilon\in(0,1/8)$ to be a small number and define

$$\bar\rho(\theta):=\begin{cases}1 & \text{for } \theta\in\left[\frac12-\varepsilon,\frac12+\varepsilon\right]\\ 0 & \text{for } \theta\in[0,1]\setminus\left[\frac12-\varepsilon,\frac12+\varepsilon\right].\end{cases}$$

Also, let $Y(t,\theta)$ be a Lipschitz function, compactly supported in $(0,1)$, that is smooth on $(0,1/2)\cup(1/2,1)$ and coincides with $|\theta-\frac12|+1$ in $\left[\frac12-\varepsilon,\frac12+\varepsilon\right]$.

Since $\bar\rho$ and Y are not smooth, we first extend both of them by periodicity to the whole real line and define $\rho_\delta:=\bar\rho*\varphi_\delta$ and $Y_\delta:=Y*\varphi_\delta$, with

$$\varphi_\delta(\theta)=\frac{\exp^{-\frac{|\theta|^2}{2\delta}}}{\sqrt{2\pi\delta}}.$$

Then

$$D^2\mathcal{F}_{\rho_\delta}(X)[Y_\delta,Y_\delta]=2\int_0^1\rho_\delta\,(\partial_\theta Y_\delta)^2\,d\theta-4\int_0^1\rho_\delta\,\partial_\theta^2 Y_\delta\,Y_\delta\,d\theta.$$

Noticing that

$$\rho_\delta\to\bar\rho\quad\text{in } L^1,\qquad \rho_\delta\to 1\quad\text{uniformly in } [1/2-\varepsilon/2,1/2+\varepsilon/2],$$

$$Y_\delta\to Y\quad\text{uniformly},\qquad \partial_\theta Y_\delta\to\partial_\theta Y\quad\text{a.e.},\qquad \partial_\theta^2 Y_\delta\rightharpoonup 2\delta_{1/2},$$

we see that

$$D^2\mathcal{F}_{\rho_\delta}(X)[Y_\delta,Y_\delta]\to 2\int_{\frac12-\varepsilon}^{\frac12+\varepsilon}(\partial_\theta Y)^2\,d\theta-8Y\left(\frac12\right)=4\varepsilon-8<0\qquad\text{as }\delta\to 0.$$

In particular, by choosing $\delta > 0$ sufficiently small, we have obtained that the Hessian of $\mathcal{F}_{\rho_\delta}$ in the direction Y_δ is negative when $X(\theta) = \theta$ and $\rho_\delta \in C^\infty([0, 1])$ satisfies $1 \geq \rho_\delta > 0$. Note that ρ_δ does not satisfy the condition $\|\rho_\delta - 1\|_{C^2} \ll 1$.

3 The 2D Case

Our goal now is to extend the result described above to higher dimensions. As a natural first step, we consider the two-dimensional setting. The main advantage in this situation is that optimal configurations are known to be asymptotically triangular lattices[2] [10, 11, 13, 14, 20]. Hence, it looks natural to use the vertices of these lattices as the reference points $\hat{x}^i$ used to parameterize our starting configurations. In this way we obtain a limiting functional $\mathcal{F}$ that involves not only ∇X but also its determinant. Unfortunately, at present there is no general theory for gradient flows of functionals involving the determinant (this is actually a major open problem in the field). Moreover, as we shall see, our functional depends in a singular way on the determinant, so it cannot be a convex functional. For this reason, we shall consider initial configurations that are small perturbations of the hexagonal lattices and perform a detailed analysis of the linearized equation. Combining this with some general ϵ-regularity theorems for parabolic systems, we prove that the nonlinear evolution is governed by the linear dynamics, and in this way we can prove exponential convergence to the hexagonal configurations.

3.1 Setting of the Problem

To state our main result, let us consider a regular hexagonal Voronoi tessellation $\mathscr{L}$ of the Euclidean plan $\mathbf{R}^2$ with sides of length 1. We consider the triangular regular lattice

$$\mathscr{L} := \mathbf{Z}e_1 \oplus \mathbf{Z}e_2\,, \quad e_1 := (1,0)\,, \quad e_2 := (\tfrac{1}{2}; \tfrac{\sqrt{3}}{2}),$$

and we note that the Voronoi cells for the points in $\mathscr{L}$ are regular hexagons. To increase the number of points, we consider its dilations

$$\varepsilon\mathscr{L}, \qquad \varepsilon > 0.$$

Let

$$\Pi := \{ae_1 + be_2 \,:\, |a| \leq 1/2,\ |b| \leq 1/2\},$$

[2] The vertices of the triangular lattice are the centres of a hexagonal tiling.

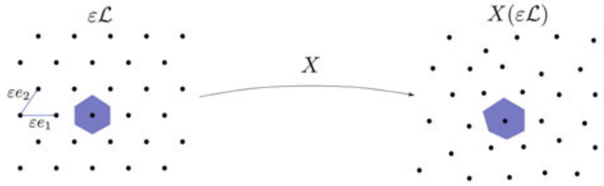

Fig. 3 Π-periodic deformations of $\varepsilon\mathscr{L}$

and observe that the periodicity of Π and $\varepsilon\mathscr{L}$ are compatible for any $\varepsilon = 1/n$.

To modify the regular hexagonal lattice, we look at Π-periodic deformations of $\varepsilon\mathscr{L}$ (see Fig. 3)

$$X(\varepsilon\mathscr{L}), \quad \varepsilon = 1/n, \; n \in \mathbb{N},$$

where $X \in \mathrm{Diff}(\mathbf{R}^2)$ satisfies

$$X \quad \text{is } \Pi\text{-periodic}, \qquad \|X - \mathrm{id}\|_{L^\infty} \ll 1.$$

Note that, up to a translation, we can assume that

$$\int_\Pi X\,dx = \int_\Pi \mathrm{id}\,dx = 0.$$

Our goal is to compute the energy $\mathcal{F}$ of X as $\varepsilon = 1/n \to 0$, and prove that, under the gradient flow of $\mathcal{F}$, the near-hexagonal Voronoi tesselation of $X(\mathscr{L}/n)$ converges to the regular hexagonal tesselation.

3.2 *The Continuous Functional*

Let $(x_1^n, \ldots, x_N^n) = X(\mathscr{L}/n) \cap \Pi$ and consider the functional $F_{N,2}(x_1^n, \ldots, x_N^n)$. By a geometric argument and a delicate computation, we show that[3]

$$F_{N,2}(x_1^n, \ldots, x_N^n) \approx \frac{1}{n^4}\mathcal{F}[X],$$

where

$$\mathcal{F}[X] = \int_\Pi F(\nabla X)\,dx,$$

[3]Note that this corresponds to the quantization of $\rho \equiv 1$ with $d = r = 2$ for $N \approx n^2 \to \infty$.

and, for each $M \in M_2(\mathbf{R})$,

$$F(M) = \tfrac{1}{3} \sum_{\omega \in \{e_1, e_2, e_{12}\}} |M \cdot \omega|^4 \Phi(\omega, M)\Big(3 + \Phi(\omega, M)^2\Big)$$

with

$$\Phi(\omega, M) := \sqrt{\frac{|MR\omega|^2 |MR^T\omega|^2}{\tfrac{3}{4}\det(M)} - 1}$$

for each $\omega \in \mathbf{S}^2$, and

$$R := \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$e_1 = (1, 0), \quad e_2 = Re_1, \quad e_{12} = R^{-1}e_1 = e_1 - e_2.$$

Hence the gradient flow is given by

$$\partial_t X(t, x) = \operatorname{div}\big(\nabla F(\nabla X(t, x))\big)$$

with initial and boundary conditions

$$\begin{cases} X(t) \quad \text{is } \Pi\text{-periodic,} \\ X(0) = X^{in}. \end{cases}$$

Particularly useful for our analysis is the following more manageable formula:

$$\begin{aligned} F(M) := {} & \frac{1}{16\sqrt{3}} \det(M)\,\mathrm{tr}[M^T M(2S - I)] \\ & + \frac{1}{64\sqrt{3}} \frac{[\mathrm{tr}(M^T M)]^2 [\mathrm{tr}(M^T MS)]}{\det(M)} \\ & - \frac{1}{192\sqrt{3}} \frac{[\mathrm{tr}(M^T M)]^3 + 4[\mathrm{tr}(M^T MS)]^3}{\det(M)}, \end{aligned}$$

where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that F depends on $\det(M)$, and blows up as $\det(M) \to 0$. In particular this implies that F cannot be convex.

3.3 The Small Deformation Regime

As mentioned in the introduction, there is no existence theory for gradient flows depending in a singular way on the determinant. For this reason, it makes sense to focus on a perturbative regime. Hence we write $X = \mathrm{id} + \tau Y$ with $|\tau| \ll 1$, and compute

$$\begin{aligned} 3\sqrt{3}\, F(\mathrm{Id} + \tau \nabla Y) &= 10 + 20\,\tau\,\mathrm{div}(Y) \\ &\quad + \tau^2 (14\,\det(\nabla Y) + 10\,\mathrm{div}(Y)^2 + 3\,|\nabla Y|^2) + O(\tau^3). \end{aligned}$$

We note that, by the expansion above, one can see that the function $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is not convex. Luckily the following crucial fact holds as a consequence of the fact that Y is periodic:

$$\int_\Pi \mathrm{div}(Y) = \int_\Pi \det(\nabla Y) = 0.$$

Thus, if we set

$$F_0(A) = F(A) - \frac{20}{3\sqrt{3}}\mathrm{Tr}(A - \mathrm{Id}) - \frac{14}{3\sqrt{3}}\det(A - \mathrm{Id}),$$

then F_0 is uniformly convex if $|A - \mathrm{Id}| \le \eta \ll 1$.

As a consequence of these two facts, we deduce that $\mathcal{F}[X]$ can be rewritten as

$$\mathcal{F}[X] = \int_\Pi F_0(\nabla X)\, dx, \tag{3.1}$$

and $\mathcal{F}$ is uniformly convex on functions that are sufficiently close to the identity in C^1.

3.4 Main Result

Our main theorem shows that the hexagonal lattice is asymptotically optimal and dynamically stable:

Theorem 3.1 *Consider an initial datum such that*

$$\int_\Pi X(0)\, dx = 0, \qquad \|X(0) - \mathrm{id}\|_{W^{\sigma,p}(\Pi)} \le \varepsilon_0,$$

with $p > 2$, and $1 + 2/p < \sigma$. Assume that ε_0 is small enough. Then the gradient flow of $\mathcal{F}$ exists, is unique, and converges exponentially fast to the identity map, that is

$$\|X(t) - \mathrm{id}\|_{L^2} \le \|X(0) - \mathrm{id}\|_{L^2} e^{-\mu t}.$$

for some $\mu > 0$.

Strategy of the Proof We begin by recalling that $\mathcal{F}$ can be rewritten as (3.1), where F_0 is uniformly convex in a neighborhood $B_\eta(\mathrm{Id})$ of the identity matrix.

Step 1: Let $G_0 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a smooth uniformly convex function such that

$$G_0(A) = F_0(A) \qquad \text{for all } A \text{ s.t. } |A - \mathrm{Id}| \le \eta/2,$$

and define

$$\mathcal{G}[X] := \int_\Pi F_0(\nabla X)\, dx.$$

Hence $\mathcal{G}$ is a convex functional that coincides with $\mathcal{F}$ on maps that are C^1-close to the identity.

Step 2: Since G is convex, there exists a unique gradient flow $Y(t)$ for $\mathcal{G}$. Also, again by the standard theory for convex gradient flows, $Y(t)$ converges exponentially fast in L^2 to id.

Step 3: By the Sobolev regularity on the initial datum and propagation of regularity for short times, we show that

$$\|\nabla Y(t) - \mathrm{Id}\|_\infty \le \eta/4 \qquad \text{for all } t \in [0, t_0]$$

for some $t_0 > 0$ small.

Step 4: Since the gradient flow of $\mathcal{G}$ is a system, there is no regularity theory as for classical parabolic equations. Hence, we cannot automatically guarantee that $Y(t)$ is smooth. To circumvent this difficulty, we exploit the L^2 exponential convergence of $Y(t)$ to id with a delicate ϵ-regularity theorem for parabolic systems in order to show that

$$\|\nabla Y(t) - \mathrm{Id}\|_\infty \le \eta/4 \qquad \text{for all } t \ge t_0.$$

Step 5: Combining Steps 3 and 4 we obtain that

$$\|\nabla Y(t) - \mathrm{Id}\|_\infty \le \eta/4 \qquad \text{for all } t \ge 0.$$

Recalling the definition of $\mathcal{G}$ (see Step 1), this implies that $\mathcal{G} = \mathcal{F}$ in a neighborhood of $Y(t)$ for all $t \ge 0$, hence $Y(t)$ *is* the gradient flow for $\mathcal{F}$, and the desired exponential convergence holds. □

Moreover, our numerical simulations confirm the asymptotic optimality of the hexagonal lattice as the number of points tends to infinity (see Figs. 4, 5, and 6). Notice that, in Figs. 4, 5, and 6 the coloured polygons are hexagons. In Fig. 6 it is shown that the minimizers have some small 1-dimensional defects with respect to the hexagonal lattice. This is due to the fact that the boundary conditions in the simulation are not periodic and on the fact that the hexagonal lattice is not the global minimizer for a finite number N of points.

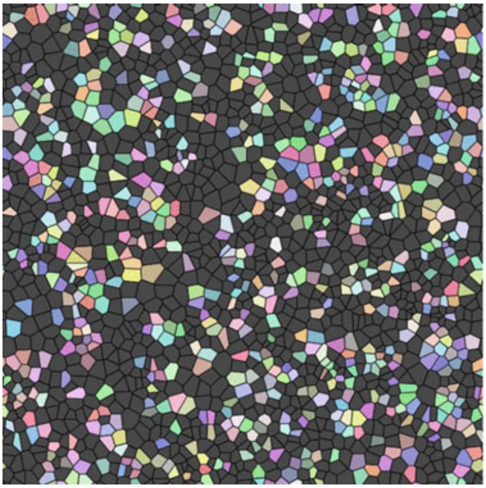

Fig. 4 720 points at time 0

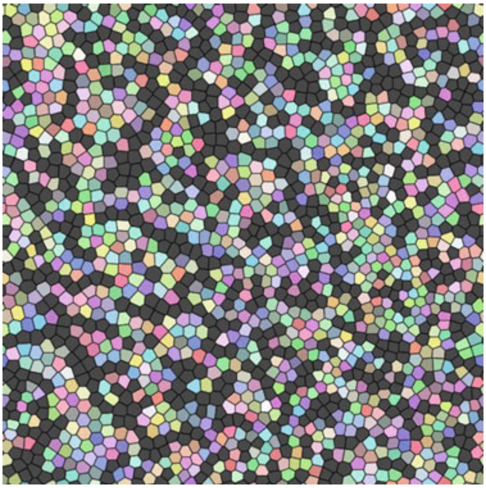

Fig. 5 720 points after 19 iterations

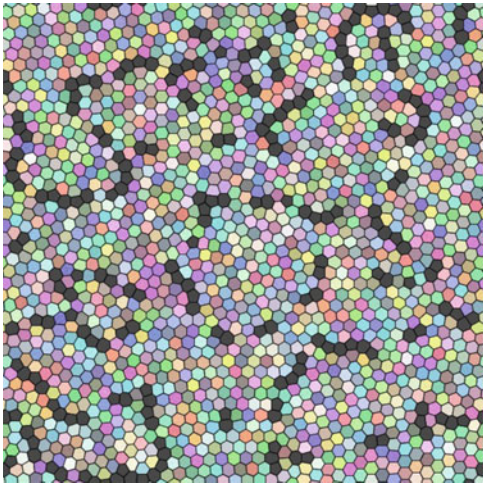

Fig. 6 720 points after 157 iterations

4 What Happens on Riemannian Manifolds?

As described in the introduction, the static version of the quantization problem in $\mathbb{R}^d$ is well understood. The aim of this question is to understand what happens when $\mathbb{R}^d$ is replaced by a Riemannian manifold.

In this section we briefly present the results obtained in [17]. Our results display how geometry can affect the optimal location problem.

4.1 Main Results

While on compact manifolds one can prove (1.2) by using a suitable *localization argument* (see [17, 19]), the situation is very different when the manifold is *non-compact*. Indeed, some global hypotheses on the behavior of the measure at "infinity" have to be imposed. The new growth assumption (4.2) depends on the curvature of the manifold and reduces, in the flat case, to a moment condition. We also build an example showing that our hypothesis is sharp.

To state the result we need to introduce some notation: given a point $x_0 \in \mathcal{M}$, we can consider polar coordinates (R, ϑ) on $T_{x_0}\mathcal{M} \simeq \mathbb{R}^d$ induced by the constant metric g_{x_0}, where ϑ denotes a vector on the unit sphere $\mathbb{S}^{d-1}$. Then, we can define the following quantity that measures the size of the differential of the exponential map when restricted to a sphere $\mathbb{S}_R^{d-1} \subset T_{x_0}\mathcal{M}$:

$$A_{x_0}(R) := R \sup_{v \in \mathbb{S}_R^{d-1},\, w \in T_v \mathbb{S}_R^{d-1},\, |w|_{x_0}=1} \left| d_v \exp_{x_0}(w) \right|_{\exp_{x_0}(v)}. \tag{4.1}$$

The result on non-compact manifolds reads as follows:

Theorem 4.1 *Let* $(\mathcal{M}, g)$ *be a complete Riemannian manifold, and let* $\mu = \rho\, d\mathrm{vol}$ *be a probability measure on* $\mathcal{M}$*. Let* $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ *be the Riemannian distance function.*

Assume that there exist $x_0 \in \mathcal{M}$ *and* $\delta > 0$ *such that*

$$\int_{\mathcal{M}} d(x, x_0)^{r+\delta}\, d\mu(x) + \int_{\mathcal{M}} A_{x_0}\big(d(x, x_0)\big)^r\, d\mu(x) < \infty, \tag{4.2}$$

and let $x^1, \dots, x^N$ *minimize the functional* $F_{N,r} : (\mathcal{M})^N \to \mathbb{R}^+$*. Then* (1.2) *holds.*

Remark 4.2 If $\mathcal{M} = \mathbb{H}^d$ is the hyperbolic space, then $A_{x_0}(R) = \sinh R$ and (4.2) reads as

$$\int_{\mathbb{H}^d} d(x, x_0)^{r+\delta}\, d\mu(x) + \int_{\mathbb{H}^d} \sinh\big(d(x, x_0)\big)^r\, d\mu(x) \approx \int_{\mathbb{H}^d} e^{r\, d(x,x_0)}\, d\mu(x) < \infty.$$

If $\mathcal{M} = \mathbb{R}^d$ then $A_{x_0}(R) = R$ and (4.2) coincides with the finiteness of the $(r + \delta)$-moment of μ, as in (1.1).

We notice that the moment condition (1.1) required on $\mathbb{R}^d$ is not sufficient to ensure the validity of the result on $\mathbb{H}^d$. Indeed, as shown in [17], there exists a measure μ on $\mathbb{H}^2$ such that

$$\int_{\mathbb{H}^2} d(x, x_0)^p\, d\mu < \infty \qquad \text{for all } p > 0, \text{ for all } x_0 \in \mathbb{H}^2,$$

but for which the result fails.

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Asymptotic Methods in Regularity Theory for Nonlinear Elliptic Equations: A Survey

Edgard A. Pimentel and Makson S. Santos

Abstract We survey recent asymptotic methods introduced in regularity theory for fully nonlinear elliptic equations. Our presentation focuses mainly on the recession function. We detail the role of this class of techniques through examples and results. Our applications include regularity in Sobolev and Hölder spaces. In addition, we produce a density result and examine ellipticity-invariant quantities, such as the Escauriaza's exponent.

Keywords Fully nonlinear elliptic equations · Regularity Theory · Asymptotic Methods · Recession Operator

Mathematics Subject Classification (2010) 35J60, 35B65

1 Introduction

In this paper, we examine asymptotic methods in regularity theory for fully nonlinear elliptic equations. We survey recent developments and prove a density result.

At the core of our analysis is the notion of recession operator. Given a (λ, Λ)-elliptic operator $F : \mathcal{S}(d) \to \mathbb{R}$, its recession function is denoted by F^* and defined as follows:

$$F^*(M) := \lim_{\mu \downarrow 0} \mu F(\mu^{-1} M). \tag{1}$$

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We observe that F^* captures the behavior of the operator F at the ends of $\mathcal{S}(d)$. For that reason, we refer to this analysis as *asymptotic* with respect to the space of symmetric matrices.

The notion of recession is imported from the realm of free boundary problems; see for example [1]. In the context of regularity theory for elliptic partial differential equations (PDEs), it appeared in [19]. In that paper, the authors partially reproduce the program developed in [22], replacing the fixed-coefficients operator with the recession function F^*.

In [18], the authors investigate Sobolev regularity for the solutions to

$$F(D^2u) = f \quad \text{in} \quad B_1,$$

with $f \in L^d(B_1)$, through the recession strategy. They produce estimates in $W^{2,p}_{loc}(B_1)$ and $p - BMO_{loc}(B_1)$ by assuming that F^* has $\mathcal{C}^{1,1}$-estimates. See [2, 3]. Regularity theory in Sobolev spaces, for the parabolic problem, is the subject of [4].

Because of its asymptotic character, the recession strategy accesses two additional types of consequences. First, we mention density properties for general (λ, Λ)-elliptic operators. In addition, it enables us to examine ellipticity-invariant quantities (e.g., the Escauriaza's exponent).

The first regularity result for fully nonlinear elliptic equations appeared in the context of the Krylov-Safonov theory, see [10, 11]. This theory accounts for a Harnack's inequality and estimates in $\mathcal{C}^{0,\alpha}$ for the solutions of a *linear* elliptic equation in divergence form. By linearizing the homogeneous problem

$$F(D^2u) = 0 \quad \text{in} \quad B_1, \tag{2}$$

we learn that its solutions and their derivatives satisfy a linear elliptic equation in divergence form. Hence, the Krylov-Safonov theory implies estimates in $\mathcal{C}^{1,\alpha}$ for the solutions to (2).

Under the assumption of convexity of the operator F, Evans and Krylov proved, independently, that solutions are indeed of class $\mathcal{C}^{2,\alpha}$. This is the content of the Evans-Krylov theory.

In [2], Caffarelli introduced a geometric method relating $F(M, x)$ to $F(M, x_0)$, the fixed-coefficients operator. The author supposes that $F(M, x_0)$ is convex with respect to $M \in \mathcal{S}(d)$, for every $x_0 \in B_1$ fixed. In addition, he works under the assumption that the oscillation

$$\beta(x, x_0) := \sup_{M \in \mathcal{S}(d)} \frac{|F(M, x) - F(M, x_0)|}{1 + \|M\|}$$

is small in the L^p-sense; that is

$$\|\beta(\cdot, x_0)\|_{L^p(B_1)} \ll 1,$$

for every $x_0 \in B_1$. Under those conditions, Caffarelli developed a regularity theory covering estimates in Hölder and Sobolev spaces.

This corpus of advances entailed several questions. The most important one regarded the optimal regularity implied by ellipticity alone. In particular, if the Krylov-Safonov estimates were the best regularity level in the absence of further structures of the problem.

This class of questions was set in the negative only recently. In [14–16], Nadirashvili and Vladut produced a number of counterexamples to the theory. For instance, the authors built singular solutions—failing to be of class $\mathcal{C}^{1,1}$—for (λ, Λ)-elliptic operators. Moreover, given a number $\tau \in (0, 1)$, there exists an elliptic operator F_τ, whose solutions fail to be of class $\mathcal{C}^{1,\tau}$.

Those counterexamples reveal important subtleties of the theory. To access more general regularity results, finer methods would be necessary. Of particular interest are techniques capable of accessing general underlying mechanisms governing the regularity of the solutions.

In this context, asymptotic methods have been successful in producing new information with consequences to the general theory of nonlinear PDEs. In the present paper, we detail those methods through examples and applications. Our approach also highlights further classes of information, such as the weak regularity theory (see Sect. 4).

1.1 Outline of the Paper

In Sect. 2 we introduce the recession function associated with F. We discuss properties of this object and address a number of examples; these involve a perturbation of the Monge-Ampère equation. Section 3 discusses two applications of the asymptotic analysis to the theory of nonlinear PDEs; first, we study estimates in Sobolev spaces. Then, we examine the Escauriaza's exponent. Section 4 puts forward a theorem on the density of $\mathcal{C}^{1,Log-Lip}$ in the class of viscosity solutions. We refer to this class of results as *weak regularity theory*.

2 Asymptotic Methods: The Recession Operator

We say that a fully nonlinear operator $F : \mathcal{S}(d) \to \mathbb{R}$ is (λ, Λ)-elliptic if it satisfies

$$\lambda\|N\| \leq F(M + N) - F(M) \leq \Lambda\|N\|,$$

for every $M, N \in \mathcal{S}(d)$, with $N \geq 0$.

Next we introduce the class of viscosity solutions $S(\lambda, \Lambda, f)$. To do so, we present the Pucci's extremal operators:

$$\mathcal{M}^{+}_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

and

$$\mathcal{M}^{-}_{\lambda,\Lambda}(M) := \Lambda \sum_{e_i<0} e_i + \lambda \sum_{e_i>0} e_i$$

where $(e_i)_{i=1}^{d}$ are the eigenvalues of the matrix M. Before we proceed, we present the definition of viscosity solution.

Definition 2.1 (Viscosity Solution) We say that $u \in \mathcal{C}(B_1)$ is a viscosity subsolution [resp. supersolution] to

$$F(D^2u) = f \quad \text{in} \quad B_1$$

if, for every $\phi \in \mathcal{C}^2(B_1)$ such that $u - \phi$ has a local maximum [resp. minimum] at $x_0 \in B_1$, we have

$$F(D^2\phi(x_0) \geq f(x_0)$$

$$[\text{resp.} \qquad F(D^2\phi(x_0) \leq f(x_0) \;].$$

If u is both a viscosity sub and supersolution, we say it is a viscosity solution.

If $u \in \mathcal{C}(B_1)$ is a viscosity solution of

$$\mathcal{M}^{+}_{\lambda,\Lambda}(D^2u) \geq f \quad \text{in} \quad B_1,$$

we say that $u \in \underline{S}(\lambda, \Lambda, f)$. If $u \in \mathcal{C}(B_1)$ is a viscosity solution of

$$\mathcal{M}^{-}_{\lambda,\Lambda}(D^2u) \leq f \quad \text{in} \quad B_1,$$

we say that $u \in \overline{S}(\lambda, \Lambda, f)$. The class of viscosity solutions $S(\lambda, \Lambda, f)$ is defined as

$$S(\lambda, \Lambda, f) := \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f).$$

For any given (λ, Λ)-elliptic operator, we produce the operator F_μ, defined as

$$F_\mu(M) := \mu F(\mu^{-1}M),$$

for $\mu > 0$. Notice that

$$\mu\mu^{-1}\lambda\|N\| \leq F_\mu(M + N) - F_\mu(M) \leq \mu\mu^{-1}\Lambda\|N\|. \tag{3}$$

Therefore, F_μ has the exact same ellipticity constants as the original operator F. To define the recession function associated with F, we consider F_μ and take the limit $\mu \downarrow 0$.

Definition 2.2 (Recession Operator) Let F be a (λ, Λ)-elliptic operator and consider the family $(F_\mu)_{\mu\in(0,1)}$. The recession function F^* associated with F is defined as

$$F^*(M) := \lim_{\mu\downarrow 0} F_\mu(M). \tag{4}$$

When the limit in (4) exists, F^* has the same ellipticity as F. Moreover, the operator F_μ acts as a curve in the space of (λ, Λ)-elliptic operators. For $\mu \equiv 1$, we have $F_1 \equiv F$; however, as μ decreases and approaches 0, the path produced by F_μ approaches the recession operator F^*.

The rationale behind the use of the recession function is the following. Given F, we compute F_μ and produce a path along the space of (λ, Λ)-elliptic operators. For small values of $\mu > 0$, this path approaches a neighborhood of F^*. Suppose this limiting operator has good properties. The idea is to import information from F^* to the original operator along the path parametrized by F_μ. For example, if F^* has $\mathcal{C}^{1,1}$-estimates, we expect to import regularity in $W^{2,p}$ for the original problem. See Fig. 1.

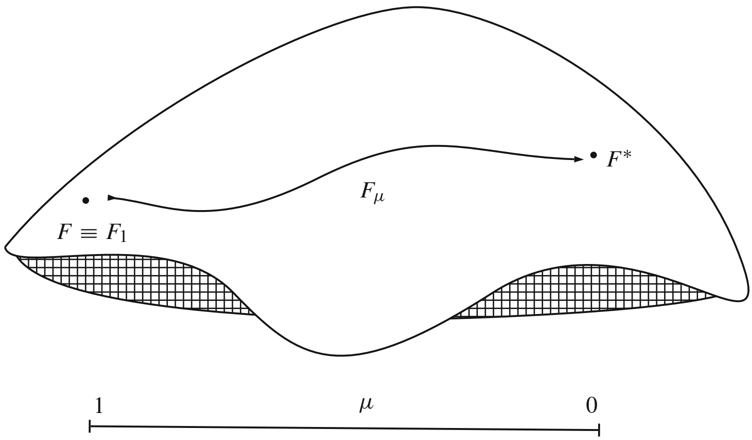

Fig. 1 Recession strategy. The operator F_μ produces a path, parametrized by $\mu \in (0, 1]$, in the space of (λ, Λ)-elliptic operators. Depending on the regularity available for the PDE driven by F^*, we expect to transport information along the path F_μ back to the original operator

We put forward a result relating ellipticity, the recession operator and the behavior of the limit in (4). We begin with a simple lemma on the homogeneity of F^*.

Lemma 2.1 (Positive Homogeneity of Degree 1) *Let $F : \mathcal{S}(d) \to \mathbb{R}$ be a (λ, Λ)-elliptic operator. If the recession function F^* is unique, it is positively homogeneous of degree* 1.

Proof We start by fixing $\rho > 0$. From the definition of recession function we have

$$|F^*(\rho M) - \rho F^*(M)| \leq |F^*(\rho M) - F_\mu(\rho M)| + |F_\mu(\rho M) - \rho F^*(M)|. \tag{5}$$

For every $\delta > 0$, there exists $\varepsilon > 0$ such that $\mu < \varepsilon$ implies

$$|F^*(\rho M) - F_\mu(\rho M)| \leq \delta.$$

In addition, notice that

$$|F_\mu(\rho M) - \rho F^*(M)| = \rho \left| F_{\mu\rho^{-1}}(M) - F^*(M) \right|;$$

the uniqueness of recession function yields

$$\rho \left| F_{\mu\rho^{-1}}(M) - F^*(M) \right| \to 0$$

as $\mu \to 0$. By gathering the former computations, we conclude that

$$|F^*(\rho M) - \rho F^*(M)| \leq \varepsilon^*,$$

for arbitrarily small ε^*. This closes the proof. □

Next, we prove that F_μ converges to F^* uniformly in compact sets of $\mathcal{S}(d)$. For ease of presentation, we suppose the recession function is homogeneous of degree 1. The uniqueness of the recession operator may sound as a too strict condition. However, important applications of this technique involve modifying F outside of a large ball to coincide with F^*. This is at the core of the argument behind density type of results. In this case, the uniqueness of F^* is simple to verify.

Proposition 2.1 (Uniform Convergence) *Let $F : \mathcal{S}(d) \to \mathbb{R}$ be a (λ, Λ)-elliptic operator. Suppose F^* is homogeneous of degree* 1. *Then, F_μ converges locally uniformly to F^*. Moreover, for every $\delta > 0$ there exists $\varepsilon > 0$ so that*

$$\left\| F_\mu(M) - F^*(M) \right\| \leq \varepsilon(1 + \|M\|), \tag{6}$$

provided $\mu \leq \delta$.

Proof Because F_μ is (λ, Λ)-elliptic, it is uniformly Lipschitz continuous in $\mathcal{S}(d)$; see [3, p. 12]. By the Arzelà-Ascoli Theorem, we conclude that F_μ converges locally uniformly, through a subsequence if necessary. The definition of F^* implies that $F_\mu(M)$ converges pointwise to $F^*(M)$, for every $M \in \mathcal{S}(d)$. Therefore, every subsequential limit F_{μ_j} must coincide, as $j \to \infty$. Then, we conclude that F_μ converges uniformly locally to F^*.

As for the estimate in (6), we consider two cases.

Case 1 Suppose that $\|M\| \leq 1$. In this case, (6) is consequential on from the local uniform convergence of F_μ.

Case 2 Let $\|M\| > 1$ and consider

$$\mu_M := \frac{\mu}{\|M\|}.$$

By assumption, F^* is positively homogeneous of degree 1. Then,

$$\frac{1}{\|M\|}|F_\mu(M) - F^*(M)| = \left| F_{\mu_M}\left(\frac{M}{\|M\|}\right) - F^*\left(\frac{M}{\|M\|}\right)\right| \to 0 \tag{7}$$

as $\mu_M \to 0$, where we have used Case 1. It stems from (7) that for $\mu \ll 1$, we have

$$|F_\mu(M) - F^*(M)| \leq \varepsilon\|M\|,$$

which completes the proof. □

Remark 2.1 Instead of supposing that F^* is homogeneous of degree 1, we could have assumed uniqueness of the recession operator. In this case, Lemma 2.1 would produce the homogeneity.

A notable feature of the recession strategy relies on its flexibility. For any (λ, Λ)-elliptic operator F, it is possible to fix a number $L \gg 1$ and propose the following modification:

$$F_L(M) := \begin{cases} F(M) & \text{if} \quad M \in B_L \\ \gamma_L(M) & \text{if} \quad M \in B_{2L} \setminus B_L \\ \mathrm{Tr}(M) & \text{if} \quad M \in B_{2L}^c, \end{cases}$$

with

$$\gamma_L(M) := \frac{2L - \|M\|}{L} F(M) + \frac{\|M\| - L}{L} \mathrm{Tr}(M).$$

In this case, it is clear that F_L^* coincides with the asymptotic profile of the operator; that is, $F_L^*(M) \equiv \mathrm{Tr}(M)$. Hence, the modification in (2) yields the Laplacian operator as the recession profile of F_L (Fig. 2).

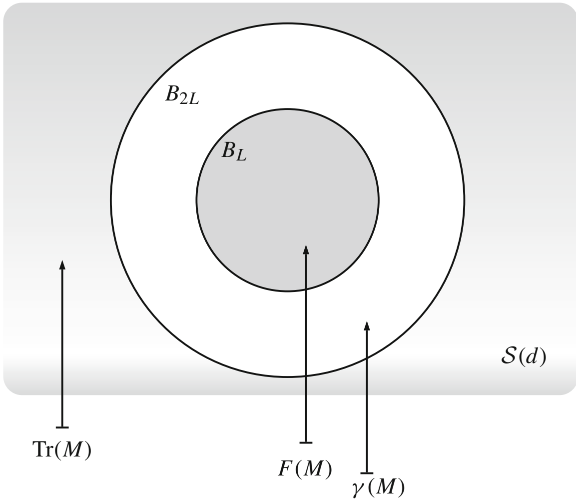

Fig. 2 Modification strategy. The recession operator allows us to modify the original problem outside of a ball of large enough radius $L \gg 1$. In this case, the resulting operator has a recession profile described by the Laplacian

Asymptotic modifications of a given operator are useful in producing density results. We return to this topic in Sect. 4. We close this section with a few examples. We expect to highlight the strength of the recession analysis *as well as its drawbacks and limitations*.

Example 2.1 (Eigenvalue q-Momentum Operator) Let $q \in 2\mathbb{N} + 1$ and consider the operator

$$F_q(M) := \sum_{i=1}^{d} \left(1 + \lambda_i^q\right)^{\frac{1}{q}},$$

where $(\lambda_i)_{i=1}^d$ are the eigenvalues of the matrix M. Notice that

$$\mu F_q(\mu^{-1} M) = \mu^{q/q} \sum_{i=1}^{d} \left(1 + \mu^{-q}\lambda_i^q\right)^{\frac{1}{q}} = \sum_{i=1}^{d} \left(\mu^q + \lambda_i^q\right)^{\frac{1}{q}};$$

therefore,

$$F_q^*(M) = \lim_{\mu \downarrow 0} \sum_{i=1}^{d} \left(\mu^q + \lambda_i^q\right)^{\frac{1}{q}} = \mathrm{Tr}(M).$$

This example shows that the recession operator relates F_q to the Laplacian. Moreover, if we are interested in ellipticity-invariant (or universal) properties of F_q, it suffices to examine the case of the Laplacian operator.

Our next example appears in Differential Geometry. It is called special Lagrangian equation.

Example 2.2 (A Perturbation of the Special Lagrangian Operator) We write the special Lagrangian operator as follows:

$$F(M) := \sum_{i=1}^{d} \arctan(1+\lambda_i) + \alpha_i \lambda_i,$$

where $(\alpha_i)_{i=1}^{d}$ are real numbers. A straightforward computation yields

$$F^*(M) = \sum_{i=1}^{d} \alpha_i \lambda_i;$$

i.e., the operator under analysis relates to a perturbation of the Laplacian.

Example 2.3 (The Log-Monge-Ampère Equation) The log-Monge-Ampère operator is given by

$$F(M) := \ln[\det(M)].$$

If we consider uniformly convex solutions, a scaling argument allows us to suppose the eigenvalues of M are strictly above 1. Consider the following perturbed problem:

$$F_\alpha(M) := \ln[\det(M)] + \sum_{i=1}^{d} \alpha_i \lambda_i,$$

where $\alpha_i \in \mathbb{R}$ are small. Because $\lambda_i > 1$, the sublinearity of the logarithm implies

$$\mu \left[\ln\left[\det\left(\mu^{-1} M\right)\right] + \sum_{i=1}^{d} \alpha_i \mu^{-1} \lambda_i \right] \leq C(d)\sqrt{\mu} + \sum_{i=1}^{d} \alpha_i \lambda_i;$$

therefore,

$$F_\alpha^*(M) = \sum_{i=1}^{d} \alpha_i \lambda_i.$$

We conclude that a small perturbation of the log-Monge-Ampère equation can be related to a linear uniformly elliptic operator. If stability results are available for the strictly convex solutions of the log-Monge-Ampère equation, the recession provides access to information through approximation results.

In the previous examples, the recession strategy related arbitrary operators with simpler ones (e.g., the Laplacian). Since the rationale of our method is to import information from F^* to F, these examples are encouraging. This is because the regularity theory for the Laplacian operator is well-established in most cases and, therefore, more information is available in the limit case.

Though promising, this is not a general outcome. In many important examples, the recession function falls short in producing additional information. Next, we consider the case of the Isaacs equation.

Example 2.4 (The Isaacs Equation) An important example of fully nonlinear elliptic equation is the Isaacs equation

$$F(M) := \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left[-\mathrm{Tr}\left(A_{\alpha,\beta}(x)M\right)\right].$$

The Isaacs equation is homogeneous of degree 1. Therefore,

$$\mu \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left[-\mathrm{Tr}\left(A_{\alpha,\beta}(x)\mu^{-1}M\right)\right] = \sup_{\alpha\in\mathcal{A}} \inf_{\beta\in\mathcal{B}} \left[-\mathrm{Tr}\left(A_{\alpha,\beta}(x)M\right)\right]$$

and

$$F \equiv F_\mu \equiv F^*;$$

i.e., the recession strategy produces no further information in this case.

The Isaacs equation arises in the study of two-player, zero-sum, (stochastic) differential games. See [8, 20]. In [17], an approximation method based on the Bellman equation is introduced to study the regularity of the solutions to the Isaacs operator.

3 Applications to Regularity Theory

In this section we describe two applications of the asymptotic methods. The first one regards regularity theory in Sobolev spaces for fully nonlinear equations, based on the results in [18]. The second application regards an ellipticity-invariant quantity, namely the Escauriaza exponent; see [5].

In what follows, we recur to the concept of *universal constant*. From now on, a universal constant is a real number $C > 0$ depending only on the dimension d and the ellipticity constants λ and Λ.

3.1 Estimates in $W^{2,p}$

In this section we consider the equation

$$F(D^2u) = f \quad \text{in} \quad B_1, \tag{8}$$

where F is a (λ, Λ)-elliptic operator and $f \in L^d(B_1)$. We prove the following theorem:

Theorem 3.1 **($W^{2,p}$-Regularity)** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (8) and suppose that F^* is convex. Then, $u \in W^{2,p}_{loc}(B_1)$ and*

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}\right),$$

where $C > 0$ is a universal constant.

Theorem 3.1 first appeared in [18]. It can be framed as a Calderón-Zygmund estimate. From a geometric viewpoint, Theorem 3.1 regards controlling the curvature of paraboloids touching the graph of the solution u. Because our arguments rely on the measure of sets involving quadratic polynomials, we define these objects in the sequel.

A quadratic polynomial of opening $M > 0$ is a map $P_M : B_1 \to \mathbb{R}$ of the form

$$P_M(x) := \ell(x) + M\frac{|x|^2}{2},$$

where $\ell : B_1 \to \mathbb{R}$ is an affine function.

Next we discuss the main elements of the proof of Theorem 3.1 and highlight the role of the recession operator. We start with a proposition.

Proposition 3.1 **($W^{2,\delta}$-Estimates)** *Let $u \in \mathcal{C}(B_1)$ be a viscosity solutions to (8). There exist $\delta > 0$ and $C > 0$, universal constants, such that $u \in W^{2,\delta}_{loc}(B_1)$ and*

$$\|u\|_{W^{2,\delta}_{loc}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^d(B_1)}\right).$$

This result was proved in the linear case by Lin in [13]. For the fully nonlinear setting, see [3]. The recession strategy builds upon Proposition 3.1 to produce a regime switching of the form $\delta \to p$, for $p > d$. This is based on the decay rate for the measure of a family of sets. We continue with a definition.

Definition 3.1 Let $u \in \mathcal{C}(B_1)$. For $M > 0$ and $H \subset B_1$, we define

$$\underline{G}_M(u, H) := \{x \in H \mid \exists P_M \text{ concave paraboloid touching } u \text{ from below at } x\}$$

and

$$\overline{G}_M(u, H) := \{x \in H \mid \exists P_M \text{ convex paraboloid touching } u \text{ from above at } x\}.$$

We also set

$$\underline{A}_M(u, H) := H \setminus \underline{G}_M(u, H) \qquad \text{and} \qquad \overline{A}_M(u, H) := H \setminus \overline{G}_M(u, H).$$

Finally, we have

$$G_M(u, H) := \underline{G}_M(u, H) \cap \overline{G}_M(u, H)$$

and

$$A_M(u, H) := \underline{A}_M(u, H) \cup \overline{A}_M(u, H).$$

We proceed with a proposition relating the notions of distribution function, maximal operator and norms in Lebesgue spaces.

Proposition 3.2 *Let $g \geq 0$ be a measurable function on B_1 and denote by μ_g its distribution function*

$$\mu_g(t) = |\{x \in B_1 \mid g(x) > t\}|, \qquad t > 0.$$

Fix $\zeta > 0$ and $M > 1$. For $p > 0$, we have

$$g \in L^p(B_1) \qquad \Longleftrightarrow \qquad \sum_{k=1}^{\infty} M^{pk} \mu_g(\zeta M^k) =: S < \infty.$$

Moreover, For some $C = C(\zeta, M, p)$, we have

$$C^{-1} S \leq \|g\|^p_{L^p(B_1)} \leq C(1 + S).$$

For more on Proposition 3.2, we refer the reader to [3, Lemma 7.3]. The following fact is consequential on Proposition 3.2: $D^2 u \in L^p(B_{1/2})$ is equivalent to the summability of

$$\sum_{k=1}^{\infty} M^{pk} \left| A_{M^k}(u, B_{1/2}) \right|,$$

for some M fixed.

Here we use the recession strategy. By assuming that F^* is convex, we infer that solutions to

$$F^*(D^2 u) = 0 \quad \text{in} \quad B_1$$

have estimates in $\mathcal{C}^{2,\alpha}$, for some $\alpha \in (0, 1)$—because of the Evans-Krylov theory; see [6, 9]. These estimates set a competing inequality: when the Hessian of the solutions to (8) starts to grow, the recession profile governs the problem. Because it has $\mathcal{C}^{2,\alpha}$-estimates, the norm of the Hessian decreases and the original operator resumes driving the equation. This process repeats itself. It prevents the Hessian from blowing up in an L^p-sense. To formalize this intuition, we state and prove an Approximation Lemma.

Proposition 3.3 (Approximation Lemma) *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to*

$$F_\mu(D^2u) = f \quad \text{in} \quad B_1,$$

where F is (λ, Λ)-elliptic. Suppose that F^* is convex. For every $\delta > 0$, there exists $\varepsilon > 0$ such that if

$$\mu + \|f\|_{L^d(B_1)} \leq \varepsilon,$$

there exists $h \in \mathcal{C}^{2,\alpha}_{loc}(B_1)$, with

$$\|h\|_{\mathcal{C}^{2,\alpha}(B_{9/10})} \leq C\, \|h\|_{L^\infty(B_1)}\,,$$

satisfying

$$\|u - h\|_{L^\infty(B_{9/10})} \leq \delta,$$

where $C > 0$ and $\alpha \in (0, 1)$ are universal constants. Moreover,

$$u - h \in S\left(\frac{\lambda}{d}, \Lambda, f - F(D^2u)\right).$$

Proof The last assertion of the proposition follows from elementary facts on the class of viscosity solutions; see [3, Proposition 2.13]. As regards the approximation statement, we argue by way of contradiction and use a compactness argument. Suppose the statement of the proposition is false. In this case, there would exist δ_0 such that every function $h \in \mathcal{C}^{2,\alpha}_{loc}(B_{9/10})$ is such that

$$\|u - h\|_{L^\infty(B_{9/10})} \geq \delta_0.$$

Consider a sequence of real numbers $(\mu_n)_{n\in\mathbb{N}}$ and sequences of functions $(u_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ such that

$$\mu_n \to 0 \quad \text{and} \quad \|f_n\|_{L^d(B_1)} \to 0$$

and

$$F_{\mu_n}(D^2 u_n) = f_n \quad \text{in} \quad B_1.$$

By the Krylov-Safonov theory, the sequence $(u_n)_{n\in\mathbb{N}}$ is equibounded in $\mathcal{C}^{1,\alpha}_{loc}(B_1)$. Therefore, through a subsequence if necessary, $u_n \to u_\infty$ in the $\mathcal{C}^{1,\alpha}$-topology. Standard stability results in the theory of viscosity solutions imply that

$$F^*(D^2 u_\infty) = 0 \quad \text{in} \quad B_{9/10}.$$

Because of the Evans-Krylov theory, $u_\infty \in \mathcal{C}^{2,\alpha}_{loc}(B_{9/10})$ and

$$\|u - u_\infty\|_{L^\infty(B_{9/10})} \to 0,$$

as $n \to \infty$. By taking $h \equiv u_\infty$, we get a contradiction and complete the proof. □

Next, we combine Proposition 3.3 with Proposition 3.1 to control the measure of $G_M(u, B_1) \cap Q_1$. We notice that, throughout the paper, Q_ℓ stands for the d-dimensional cube of side length ℓ.

Lemma 3.1 *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (8) and suppose*

$$-|x|^2 \le u(x) \le |x|^2 \quad \textit{in} \quad B_1 \setminus B_{3/4}.$$

Under the assumptions of Proposition 3.3, there exists $M > 0$, depending only on the dimension, and $\rho \in (0, 1)$ such that

$$|G_M(u, B_1) \cap Q_1| \ge 1 - \rho.$$

Proof Take h, the function from Proposition 3.3 and consider its restriction to $B_{1/2}$. Extend h outside $\overline{B}_{1/2}$ continuously in such a way that

$$h \equiv u \quad \text{in} \quad B_1 \setminus B_{3/4}$$

and

$$\|u - h\|_{L^\infty(B_1)} = \|u - h\|_{L^\infty(B_{3/4})}.$$

These choices imply that

$$-2 - |x|^2 \le h(x) \le 2 + |x|^2 \quad \text{in} \quad B_1 \setminus B_{1/2}.$$

It is easy to verify the existence of a number $N > 0$ so that

$$Q_1 \subset G_N(h, B_1).$$

For a constant ρ_0 to be determined later, we set

$$\vartheta := \rho_0 (u - h).$$

We gather Propositions 3.3 and 3.1 to conclude that $\vartheta \in W^{2,\delta}_{loc}(B_1)$. Therefore,

$$|A_t(\vartheta, B_1) \cap Q_1| \leq Ct^{-\delta},$$

which follows from the definition of A_t. Because A_N and G_N are complement to each other, we conclude that

$$|G_N(u - h, B_1) \cap Q_1| \geq 1 - \rho_0,$$

for some $N > 1$. Finally,

$$|G_{2N}(u, B_1) \cap Q_1| \geq 1 - \rho_0,$$

which completes the proof. □

An application of Lemma 3.1 yields the following result:

Lemma 3.2 *Let $u \in \mathcal{C}(B_1)$ be a viscosity solutions to* (8). *Under the assumptions of Proposition 3.3, we have*

$$G_1(u, B_1) \cap Q_3 \neq \emptyset \quad \Longrightarrow \quad |G_M(u, B_1) \cap Q_1| \geq 1 - \rho,$$

where $M > 0$ and $\rho > 0$ are as in Lemma 3.1.

Proof For a proof of this result, we refer the reader to [3]; see also [18, Lemma 5.2]. □

The maximal function associated with $f \in L^1_{loc}(\mathbb{R}^d)$ is denoted by $m(f)$ and given by

$$m(f)(x) := \sup_{\ell > 0} \frac{1}{|Q_\ell(x)|} \int_{Q_\ell(x)} |f(y)| dy.$$

Lemma 3.3 *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to*

$$F_\mu(D^2 u) = f \quad in \quad B_1.$$

Suppose

$$\mu + \|f\|_{L^d(B_1)} \ll 1.$$

Suppose further the assumptions of Proposition 3.3 are in force. Extend f outside of B_1 by zero. Define

$$A := A_{M^{k+1}}(u, B_1) \cap Q_1$$

and

$$B := \left(A_{M^k}(u, B_1) \cap Q_1\right) \cup \left\{x \in Q_1 \,|\, m(f^d)(x) \geq \left(cM^k\right)^d\right\}.$$

Then, there exists $\varepsilon \in (0, 1)$ such that

$$|A| \leq \varepsilon |B|.$$

Proof As before, for the proof of this result we refer the reader to [3] and [18, Lemma 5.3]. □

Finally, we consider the distribution function of $\Theta(x)$, defined as

$$\Theta(x) := \inf\left\{M \,|\, x \in G_M(u, B_{1/2})\right\}.$$

The integrability of D^2u is closely related to the integrability of Θ, in the sense that

$$\|\Theta\|_{L^p(B_1)} \sim \left\|D^2 u\right\|_{L^p(B_1)}.$$

See, for instance, [12].

Once the former lemmas are available, we present the proof of Theorem 3.1. It relies on the properties of the maximal function associated with $f \in L^d(B_1)$.

Proof of Theorem 3.1 We take $M > 0$ from Lemma 3.3 and define ρ as follows:

$$\rho := \frac{1}{2M^p}.$$

In addition, set

$$\alpha_k := \left|A_{M^k}(u, B_1) \cap Q_1\right|$$

and

$$\beta_k := \left|\left\{x \in Q_1 \,|\, m(f^d)(x) \geq (CM^k)^d\right\}\right|.$$

Because of Lemma 3.3,

$$\alpha_k \leq \rho^k + \sum_{i=0}^{k-1} \rho^{k-i} \beta_i.$$

Moreover, $m(f^d) \in L^{\frac{p}{d}}(\mathbb{R}^d)$ and

$$\left\| m(f^d) \right\|_{L^{\frac{p}{d}}(\mathbb{R}^d)} \leq c \, \| f \|^d_{L^p(B_1)} \leq C.$$

Therefore, Proposition 3.2 implies

$$\sum_{k=0}^{\infty} M^{pk} \beta_k \leq C.$$

On the other hand we have

$$\mu_\Theta(t) \leq \left| A_t(u, B_{1/2}) \right| \leq \left| A_t(u, B_{1/2}) \cap Q_1 \right|.$$

Because of Proposition 3.2, the proof is complete if we verify that

$$\sum_{k=1}^{\infty} M^{pk} \alpha_k \leq C.$$

However,

$$\begin{aligned} \sum_{k=1}^{\infty} M^{pk} \alpha_k &\leq \sum_{k=1}^{\infty} \left(\rho M^p \right)^k + \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \rho^{k-i} M^{p(k-i)} M^{pi} \beta_i \\ &\leq \sum_{k=1}^{\infty} 2^{-k} + \left(\sum_{i=0}^{\infty} M^{pi} \beta_i \right) \left(\sum_{j=1}^{\infty} 2^{-j} \right) \leq C. \end{aligned}$$

□

We close this section with a number of remarks on the consequences and applications of Theorem 3.1.

Remark 3.1 Theorem 3.1 implies that, for every $p > d$, $D^2 u \in p - BMO_{loc}(B_1)$, where

$$u \in p - BMO(B_1) \qquad \Leftrightarrow \qquad \sup_{\ell > 0} \int_{B_\ell} \left| u(x) - \langle u \rangle_\ell \right|^p dx < \infty,$$

and

$$\langle u \rangle_\ell := \frac{1}{|B_\ell|} \int_{B_\ell} u(x) dx.$$

In fact, ellipticity builds upon Sobolev regularity to produce an integrability level for the Hessian above L^p, for every $p > 1$, and strictly below L^∞. See [18].

Remark 3.2 Theorem 3.1 extends to operators of the form

$$F : \mathcal{S}(d) \times \mathbb{R}^d \times \mathbb{R} \times B_1 \to \mathbb{R},$$

provided the dependence of $F(M, p, u, x)$ with respect to p, u and x is properly controlled. In case F is globally Lipschitz with respect to p, has a modulus of continuity with respect to u and small oscillation with respect to x, Theorem 3.1 extends to equations of the form

$$F(D^2u, Du, u, x) = f \quad \text{in} \quad B_1.$$

See [21] for details. Similar arguments produce global estimates, as in [23], under asymptotic conditions on the problem.

Remark 3.3 We work under the assumption that F^* is convex. However, the result holds even if F^* has only $W^{2,q}$ estimates; see [12]. In this case, estimates in $W^{2,p}$ would be available for $d < p < q$.

3.2 The Escauriaza's Exponent

Among the assumptions of Theorem 3.1 is the restriction $p > d$. See [2, 18]; also, [3, Chapter 7]. In [5], Escauriaza extended Caffarelli's estimates under the condition $p > d - \varepsilon$, for some constant $\varepsilon = \varepsilon(\lambda, \Lambda, d)$.

Proposition 3.4 (Escauriaza's Exponent) *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (8) and suppose that F^* is convex. Then, $u \in W^{2,p}_{loc}(B_1)$ and*

$$\|u\|_{W^{2,p}(B_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^p(B_1)} \right),$$

for $p > d - \varepsilon$, where $C > 0$ is a universal constant and $\varepsilon = \varepsilon(\lambda, \Lambda, d)$ is the Escauriaza's exponent.

Proposition 3.4 requires lower integrability of the source term to ensure estimates in Sobolev spaces. This weaker requirement is quantified by ε. Although a function of λ, Λ and the dimension, a precise formula for this quantity remains unknown. Next, we use the recession strategy to examine some examples of operators and produce asymptotic information on ε.

The key to the lower integrability of the source term is related to F. In fact, it comes from the integrability of the Green's function associated with F through its linearized operator L. The following proposition accounts for the integrability of the Green's function of a linear (λ, Λ)-elliptic operator. It is due to Fabes and Stroock; see [7].

Proposition 3.5 *Let L be a (λ, Λ)-elliptic operator with measurable coefficients. Let $G(x, y)$ be its Green's function in B_1. Then,*

1. *There exists $C > 0$ and $\varepsilon > 0$ such that if $p > d - \varepsilon$,*

$$\int_{B_1} G(x, y)^{p'} dy \leq C,$$

for all $x \in B_1$, where

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

2. *There exists $\beta > 0$ such that if $E \subset B_r \subset B_{1/2}$, we have*

$$\left(\frac{|E|}{|B_r|}\right)^{\beta} \int_{B_r} G(x, y)dy \leq C \int_E G(x, y)dy.$$

For the proof of Proposition 3.5 we refer to [7]. Consequential on this result in the following Harnack's inequality.

Proposition 3.6 (Harnack's Inequality) *Let $u \in \mathcal{C}(B_1)$ be a nonnegative viscosity solutions to*

$$F(D^2u) = f \quad \text{in} \quad B_1,$$

where F is a (λ, Λ)-elliptic operator and $f \in L^{d-\varepsilon}(B_1)$. Then, there exists $C > 0$, a universal constant, such that

$$\sup_{B_{r/2}} u \leq C \left(\inf_{B_{r/2}} u + r^{2-\frac{d}{d-\varepsilon}} \|f\|_{L^{d-\varepsilon}(B_1)} \right).$$

The proof of Proposition 3.6 is in [5]. This result has many consequences to the general theory of elliptic PDEs. We mention the universal modulus of continuity produced in [22]. Indeed, solutions to (8) satisfy

$$\|u\|_{\mathcal{C}^{0,\frac{d-2\varepsilon}{d-\varepsilon}}(B_{1/2})} \leq C \left(\|u\|_{L^\infty(B_1)} + \|f\|_{L^{d-\varepsilon}(B_1)} \right).$$

Notice that Escauriaza's exponent depends only on the integrability of the Green's function associated with F and the dimension. Hence, ε is invariant with respect to the ellipticity. Therefore, for a fixed dimension d, two operators with the same ellipticity must have the same exponent ε. Here the recession strategy plays a role.

When the limit

$$F^*(M) = \lim_{\mu \downarrow 0} F_\mu(M)$$

exists, the recession operator F^* has the same ellipticity as F. If the Green's function associated with F^* is known, or we infer its integrability, it would be possible to compute the Escauriaza's exponent for F^*, say ε_{F^*}. By knowing this quantity, we recover ε_F. In what follows, we examine an example and explicitly compute the Escauriaza's exponent.

Example 3.1 (Eigenvalue q-Momentum Operator) We revisit Example 2.1, where the operator F_q is defined:

$$F_q(M) := \sum_{i=1}^{d} \left(1 + \lambda_i^q\right)^{\frac{1}{q}}.$$

To linearize this operator and evaluate the integrability of the associated Green's function in a ball might be not even possible. However, we learned that $F_q^*(D^2u) = \Delta u$. In addition, the Escauriaza's exponent for the Laplacian, ε_Δ, is known to be $d/2$. Therefore,

$$\varepsilon_{F_q} = \varepsilon_\Delta = \frac{d}{2}.$$

Moreover, we conclude that Theorem 3.1 is available for F_q provided the source term satisfies $f \in L^{\frac{d}{2}}(B_1)$.

In the prior example, $\varepsilon_{F_q} = d/2$. Every fully nonlinear operator whose recession profile coincides with the Laplacian has the same exponent ε_Δ.

4 Weak Regularity in $\mathcal{C}^{1,Log-Lip}$

In this section we prove a weak regularity result. We understand *weak regularity result* as the density of regular enough solutions in the class of viscosity solutions. As indicated in the works of Nadirashvili and Vladut, the optimal level of regularity implied by ellipticity is $\mathcal{C}^{1,\alpha}$. This is due to the Krylov-Safonov theory.

However, for many applications, it is enough that solutions to $F(D^2u) = f$ are approximated by regular functions. For example, in [18] the authors proved that $W^{2,p}_{loc}(B_1) \cap S(\lambda^-, \Lambda^+, f)$ is dense in $S(\lambda, \Lambda, f)$. Therefore, when studying properties closed under uniform limits, the starting point of the theory shifts to $W^{2,p}$-estimates. We refer to a result in this spirit as a weak regularity result.

The main result of this section regards $\mathcal{C}^{1,Log-Lip}$-weak regularity. We say that a function $u \in \mathcal{C}^{1,Log-Lip}(B_1)$ if and only if there exists a constant $C > 0$ satisfying

$$\sup_{x \in B_r} |u(x) - [u(0) + Du(0) \cdot x]| \leq -Cr^2 \ln r.$$

In what follows, we consider operators with explicit dependence on the space variable $x \in B_1$. It leads to the following problem:

$$F(x, D^2u) = f \quad \text{in} \quad B_1. \tag{9}$$

Theorem 4.1 (Weak Estimates in $\mathcal{C}^{1,Log-Lip}$) *Let $u \in \mathcal{C}(B_1)$ be a continuous viscosity solution to (9). Suppose F is a (λ, Λ)-elliptic operator and $f \in L^\infty(B_1)$. Then, there exists a sequence of functions $\{u_j\}_{j \geq 1} \subset \mathcal{C}^{1,Log-Lip}_{loc}(B_1) \cap S(\lambda^-, \Lambda^+, f)$ that converges locally uniformly to u.*

The proof of Theorem 4.1 relies on three main structures. The first one is the Approximation Lemma (Proposition 3.3). It ensures the existence of a quadratic polynomial that approximates the solution u. The second main ingredient in the proof is a further application of Proposition 3.3; in this case, it produces estimates in $\mathcal{C}^{1,Log-Lip}$ for operators whose recession is convex. Finally, an asymptotic modification of F completes the argument. We start with a lemma.

Lemma 4.1 *Let $u \in \mathcal{C}(B_1)$ be a viscosity solution to (9). Under the assumptions of Proposition 3.3, there exist a second order polynomial P such that $||P|| \leq C$ and*

$$||u - P||_{L^\infty(B_r)} \leq r^2,$$

where $C > 0$ and $0 < r \ll 1$ are universal constants.

Proof Let h be the function from Proposition 3.3. Let P denote the second order Taylor's expansion of h at the origin. Thus

$$||u - P||_{L^\infty}(B_r) \leq ||u - h||_{L^\infty(B_r)} + ||h - P||_{L^\infty(B_r)} \leq \delta + Cr^{2+\alpha}.$$

We choose r small enough so that $Cr^\alpha < \dfrac{1}{2}$ and $\delta = \dfrac{r^2}{2}$ and we obtain

$$||u - P||_{L^\infty}(B_r) \leq r^2.$$

□

Remark 4.1 We notice that the choice of r in Lemma 4.1 determines $\delta > 0$ in Proposition 3.3 and, therefore, sets the smallness regime involving F_μ and the norms of the source term.

The next result regards the regularity of the solutions to (9) in $\mathcal{C}^{1,Log-Lip}$. It appeared for the first time in [19]. Compare with [22, Theorem 3].

Theorem 4.2 (Regularity) *Leu $u \in \mathcal{C}(B_1)$ be a viscosity solutions to (9). Suppose F^* is convex and $f \in L^\infty(B_1)$. Suppose further that*

$$\lim_{\mu \to 0} \mu F(x, \mu^{-1} M) = F^*(M)$$

is uniform in M. Then, $u \in C^{1,Log-Lip}_{loc}(B_1)$ and there exists $C > 0$, universal, such that

$$\sup_{B_r} |u(x) - u(x_0) - Du(x_0) \cdot x| \leq Cr^2 \ln r^{-1},$$

for every $x_0 \in B_{1/2}$.

Proof We split the proof in several steps

Step 1 We prove the result for $x_0 = 0$. For all $M \in \mathcal{S}(d)$, we can find $\varepsilon > 0$ such that for all $\mu < \varepsilon$ we have $||F_\mu(M) - F^*(M)|| \leq \delta$, where $\delta > 0$ is the number from Lemma 4.1. We choose $r_0 \sim \sqrt{\varepsilon}$ and define

$$u_0(x) = \varepsilon \max\{1, ||u||_{L^\infty}, ||f||_{L^\infty}\}^{-1} u(r_0 x).$$

It is clear that $||u_0||_{L^\infty} \leq 1$ and

$$D^2 u(r_0 x) = \frac{1}{\varepsilon r_0^2} \max\{1, ||u||_{L^\infty}, ||f||_{L^\infty}\} D^2 u_0(x);$$

thus, u_0 satisfies

$$\tau F\left(\tau^{-1} D^2 u_0(x)\right) = \tau f(r_0 x),$$

where

$$\tau = \frac{\varepsilon r_0^2}{\max\{1, ||u||_{L^\infty}, ||f||_{L^\infty}\}}.$$

Note that $\tilde{f} = \tau f(r_0 x)$ satisfies $||\tilde{f}||_{L^\infty} \leq \varepsilon$.

Step 2 Let $0 < r < r_0$. Next, we show the existence of a sequence of quadratic polynomials $(P_k)_{k \in \mathbb{N}}$,

$$P_k(x) := a_k + b_k \cdot X + \frac{1}{2} x^t M_k x,$$

such that

$$F^*(M_k) = 0 \tag{10}$$

$$\sup_{B_{r^k}} |u_0 - P_k| \le r^{2k} \tag{11}$$

$$|a_k - a_{k-1}| + r^{k-1}|b_k - b_{k-1}| + r^{2(k-1)}|M_k - M_{k-1}| \le Cr^{(2(k-1))}. \tag{12}$$

The constant r in (11) and (12) is the one from Lemma 4.1. We shall verify (10)–(12) by induction. We set $P_0 = P_{-1} = 0$, and the first step $k = 0$ is immediately satisfied, since $F^*(0) = 0$ and $||u_0||_{L^\infty} \le 1$. Suppose we have verified the thesis of induction for $k = 0, 1, \ldots, i$. Define the function

$$v(x) = \frac{u_0(r^i x) - P_i(r^i x)}{r^{2i}}.$$

From (11), we have $|v| \le 1$, and furthermore

$$D^2 v(x) = D^2 u_0(r^i x) - M_i;$$

thus v satisfies

$$\mu F(\mu^{-1}(D^2 v + M_i)) = \tilde{f}(r^i x).$$

If we define $F_i(M) = F(M + M_i)$ and $F_i^*(M) = F^*(M + M_i)$, it follows that $||F_{\mu,i}(M) - F_i^*(M)|| \le \delta$. Furthermore, since $F^*(M_i) = 0$, the equation $F^*(D^2\zeta) = 0$ has the same estimates as F^*. Now, since $F_{\mu,i}(D^2 v) = 0$, from Lemma 4.1 there exists a quadratic polynomial $\tilde{P}$ such that $||v - \tilde{P}||_{L^\infty(B_r)} \le r^2$. Then

$$\frac{|u_0(r^i x) - P_i(r^i x) - r^{2i}\tilde{P}(x)|}{r^{2i}} \le r^2$$

and

$$|u_0(x) - (P_i(x) + r^{2i}\tilde{P}(r^{-1}x))| \le r^{2(i+1)};$$

taking

$$P_{i+1}(x) := P_i(x) + r^{2i}\tilde{P}(r^{-i}x),$$

we verify (11).

Step 3 We define $P_{i+1}(x) = P_i(x) + r^{2i}\tilde{P}(r^{-i}x)$ and since $P_0 = 0$ we obtain

$$P_k(x) = \sum_{j=1}^{k} r^{2(j-1)}h(0) + \sum_{j=1}^{k} r^{(j-1)}Dh(0)x + k\frac{x^t D^2h(0)x}{2}.$$

Indeed, we shall verify this by induction. For $k = 1$ we have

$$P_1 = h(0) + Dh(0)x + \frac{x^t D^2h(0)x}{2} = \tilde{P}(x).$$

Now, suppose we have verified for $k = 1, 2, \ldots, i$. Since $P_{i+1}(x) = P_i(x) + r^{2i}\tilde{P}(r^{-i}x)$, we obtain

$$\begin{aligned} P_{i+1}(x) &= \sum_{j=1}^{i} r^{2(j-1)}h(0) + \sum_{j=1}^{i} r^{(j-1)}Dh(0)x + i\frac{x^t D^2h(0)x}{2} + r^{2i}h(0) \\ &\quad + r^i Dh(0)x + \frac{x^t D^2h(0)x}{2} \\ &= \sum_{j=1}^{i+1} r^{2(j-1)}h(0) + \sum_{j=1}^{i+1} r^{(j-1)}Dh(0)x + (i+1)\frac{x^t D^2h(0)x}{2}, \end{aligned}$$

thus we conclude the induction.

Step 4 In addition,

$$|a_{k+1} - a_k| + r^k|b_{k+1} - b_k| + r^{2k}|M_{k+1} - M_k| \le Cr^{2k},$$

since

$$|a_{k+1} - a_k| = \left|\sum_{j=1}^{k+1} r^{2(j-1)} - \sum_{j=1}^{k} r^{2(j-1)}\right| |h(0)| = r^{2k}|h(0)| \le Cr^{2k},$$

$$|b_{k+1} - b_k| = \left|\sum_{j=1}^{k+1} r^{(j-1)} - \sum_{j=1}^{k} r^{(j-1)}\right| |Dh(0)| = r^k|Dh(0)| \le Cr^k,$$

and

$$|M_{k+1} - M_k| = |k + 1 - k| = 1.$$

This proves (12).

From (11) we have $|u_0 - a_k| < r^{2k}$. Futhermore $|Du_0(0) - b_k| \le Cr^k$ and $|M_k| = |kD^2h(0)| \le Ck$.

Finally, for any $0 < \rho < \dfrac{1}{4}$, let k such that $r^{k+1} < \rho \leq r^k$. From estimates above, we obtain

$$
\begin{aligned}
\sup_{B_{r^k}} |u_0(x) - (u_0(0) + Du_0(0)\cdot x)| &= \sup_{B_{r^k}} \Big|(u_0 - P_k) + a_k - u_0(0) \\
&\quad + b_k \cdot x - Du_0(0)\cdot x + \frac{x^t M_k x}{2}\Big| \\
&\leq r^{2k} + Cr^{2k} + Cr^{2k} + \frac{C}{2} k r^{2k} \\
&\leq C(r^{2k} + kr^{2k}) \\
&= \frac{C}{r^2}(r^{2(k+1)} + r^2 k r^{2k}) \\
&\leq \frac{C}{r^2}(\rho^2 + k\rho^2) \\
&= C\rho^2(1+k).
\end{aligned}
$$

Since $\rho < r^k$ we obtain $k < \dfrac{\ln \rho}{\ln r}$ and

$$
\begin{aligned}
\sup_{B_{r^k}} |u_0(x) - (u_0(0) + Du_0(0)\cdot x)| &\leq C\rho^2 \left(1 + \frac{\ln \rho}{\ln r}\right) \\
&= c\rho^2(1 + \ln \rho - \ln r) \\
&\leq c\rho^2(-\ln r),
\end{aligned}
$$

provided $\rho < \frac{1}{4}$.

Since $\rho \leq r^k$ we have $-\frac{1}{k}\ln \rho \geq -\ln r$, and thus

$$
\sup_{B_{r^k}} |u_0(x) - (u_0(0) + Du_0(0)\cdot x)| \leq -c\frac{1}{k}\rho^2 \ln \rho = -C\rho^2 \ln \rho.
$$

This finishes the poof. □

Proof of Theorem 4.1 We construct a sequence of operators F_j as follows: given $\delta > 0$, define

$$
L_\delta(M) := (\Lambda + \delta)\sum_{e_i > 0} e_i + (\lambda + \delta)\sum_{e_i < 0} e_i,
$$

where e_i are the eigenvalues of $M \in \mathcal{S}(d)$. Now, define

$$
F^j(x, M) := \max\{F(x, M), L_\delta(M) - C_j\},
$$

where C_j is a sequence of positive numbers to be determined. From the (λ, Λ)-ellipticity, we obtain

$$\begin{aligned}
F(x, M) &\geq \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i \\
&\geq \lambda \sum_{e_i>0} e_i - \Lambda||M|| \\
&= L_\delta(M) - L_\delta(M) + \lambda \sum_{e_i>0} e_i - \Lambda||M|| \\
&= L_\delta(M) - (\Lambda + \delta - \lambda) \sum_{e_i>0} e_i - (\lambda - \delta) \sum_{e_i<0} e_i - \Lambda||M|| \\
&= L_\delta(M) - (\Lambda + \delta - \lambda) \left[\sum_{e_i>0} e_i - \sum_{e_i<0} e_i \right] - \Lambda \sum_{e_i<0} e_i - \Lambda||M|| \\
&= L_\delta(M) - (2\Lambda + \delta - \lambda)||M|| - \Lambda \sum_{e_i<0} e_i \\
&\geq L_\delta(M) - (2\Lambda + \delta - \lambda)||M|| \\
&\geq L_\delta(M) - C_j
\end{aligned}$$

provided we set $C_j := j(2\Lambda - \lambda + \delta)$ and $||M|| \leq j$. Here, we use $||M|| := \sum_{i=1}^{d} |e_i|$.

This shows that

$$F^j = F \text{ in } B_j \subset \mathcal{S}(d).$$

To compute the recession function of F^j, we find

$$F^j_\mu(x, M) = \mu F(x, \mu^{-1}M) = \max\{F_\mu(x, M), L_\delta(M) - \mu C_j\}.$$

Now, since F_μ is (λ, Λ)-elliptic, we have

$$\begin{aligned}
F_\mu(x, M) &\leq \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i \\
&= L_\delta(M) - L_\delta(M) + \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i \\
&= L_\delta(M) - \delta \sum_{e_i>0} e_i + \delta \sum_{e_i<0} e_i \\
&= L_\delta(M) - \delta||M|| \\
&\leq L_\delta(M) - \mu C_j,
\end{aligned}$$

provided $||M|| \geq \dfrac{\mu C_j}{\delta}$.

Then, we have $F^j_\mu = L_\delta(M) - \mu C_j$ outside the ball of radius C_j and

$$(F^j)^* = \lim_{\mu\to 0} F^j_\mu = \lim_{\mu\to 0}(L_\delta(M) - \mu C_j) = L_\delta(M).$$

Thus, from Theorem 4.2 for each j fixed, the operator F^j have a priori estimates in $C^{1,Log-Lip}(\Omega)$.

Finally, we constructed u_j to be the viscosity solution of the Dirichlet problem

$$\begin{cases} F^j(x, D^2u_j) = f(x) \text{ in } B_1 \\ u_j = u \text{ on } \partial B_1. \end{cases}$$

Thus, each u_j is locally in $C^{1,Log-Lip}$, and since $F^j = F$ in B_j, we have that up to a subsequence, $u_j \to u$ locally in the $C^{0,\alpha}$-topology. The convergence is ensured by stability results in the theory of viscosity solutions. □

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A Fully-Discrete Scheme for Systems of Nonlinear Fokker-Planck-Kolmogorov Equations

Elisabetta Carlini and Francisco J. Silva

Abstract We consider a system of Fokker-Planck-Kolmogorov (FPK) equations, where the dependence of the coefficients is nonlinear and nonlocal in time with respect to the unknowns. We extend the numerical scheme proposed and studied in Carlini and Silva (SIAM J. Numer. Anal., 2018, To appear) for a single FPK equation of this type. We analyse the convergence of the scheme and we study its applicability in two examples. The first one concerns a population model involving two interacting species and the second one concerns two populations Mean Field Games.

Keywords Systems of nonlinear Fokker-Planck-Kolmogorov equations · Numerical Analysis · Semi-Lagrangian schemes · Markov chain approximation · Mean Field Games

AMS-Subject Classification 35Q84, 65N12, 65C40

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1 Introduction

In this note we consider the following system of nonlinear Fokker-Planck-Kolmogorov (FPK) equations

$$\begin{aligned} \partial_t m^\ell - \tfrac{1}{2}\sum_{1\le i,j\le d} \partial^2_{x_i,x_j}\left(a^\ell_{i,j}(m,x,t)m^\ell\right) + \textstyle\sum_{i=1}^{d_\ell}\partial_{x_i}\left(b^\ell(m,x,t)m^\ell\right) = 0, &\quad \text{in } \mathbb{R}^{d_\ell}\times(0,T),\\ m^\ell(0) = \bar{m}^\ell_0 &\quad \text{in } \mathbb{R}^{d_\ell}, \end{aligned} \tag{FPK}$$

where $\ell = 1,\dots,M$ and $d_\ell \in \mathbb{N}\backslash\{0\}$. In the system above, we look for M unknowns $m = (m^1,\dots,m^M)$ such that for each $\ell = 1,\dots,M$, m^ℓ belongs to the space $C([0,T];\mathcal{P}_1(\mathbb{R}^{d_\ell}))$, where $\mathcal{P}_1(\mathbb{R}^{d_\ell})$ is the set of probability measures on $\mathbb{R}^{d_\ell}$ with finite first order moment. This set is endowed with the standard Monge-Kantorovic distance (see Sect. 2 below). The coefficients in (FPK) are given by functions

$$b^\ell : \prod_{\ell'=1}^{M} C([0,T];\mathcal{P}_1(\mathbb{R}^{d_{\ell'}}))\times\mathbb{R}^{d_\ell}\times[0,T]\to\mathbb{R}^{d_\ell}, \qquad a^\ell_{i,j}=\sum_{p=1}^{r_\ell}\sigma^\ell_{i,p}\cdot\sigma^\ell_{j,p} \quad \forall\, i,\ j=1,\dots d,$$

where $r_\ell \in \mathbb{N}\setminus\{0\}$ and for all $p = 1,\dots,r_\ell$

$$\sigma^\ell_{i,p} : \prod_{\ell'=1}^{M} C([0,T];\mathcal{P}_1(\mathbb{R}^{d_{\ell'}}))\times\mathbb{R}^{d_\ell}\times[0,T]\to\mathbb{R}.$$

Finally, the prescribed initial distributions $\bar{m}_0 := (\bar{m}^1_0,\dots,\bar{m}^N_0)$ are assumed to be probability measures with finite second order moments, i.e. $\int_{\mathbb{R}^{d_\ell}}|x|^2\mathrm{d}\bar{m}^\ell_0 < \infty$ for all $\ell = 1,\dots,M$. Note that system (FPK) is highly nonlinear because the dependence on m of the coefficients b^ℓ and $a^\ell_{i,j}$ can be nonlocal in time. A priori these coefficients depend of the entire trajectory $t \in [0,T] \to m(t) \in \prod_{\ell=1}^{M}\mathcal{P}_1(\mathbb{R}^{d_\ell})$.

When $M = 1$, and the coefficients b^1 and σ^1 do not depend on m, the resulting equation is the classical FPK equation that describes the law of a diffusion process whose drift and volatility coefficients are given by b^1 and σ^1, respectively. We refer the reader to the monograph [4] for a rather complete account of analytical results related to this equation and to the references in introduction of [9] for the numerical approximation of its solutions.

Let us now comment on the probabilistic interpretation of (FPK) when $M > 1$. Formally speaking, provided that for each $\ell = 1,\dots,M$, the equation

$$\mathrm{d}X_\ell(t) = b^\ell(m,X_\ell(t),t)\mathrm{d}t + \sum_{p=1}^{r_\ell}\sigma^\ell_{\cdot,p}(m,X_\ell(t),t)\mathrm{d}W^\ell_p(t) \quad t\in[0,T], \quad X_\ell(0) = X^\ell_0, \tag{1.1}$$

is well-posed (let us say in a weak sense), system (FPK) describes the time evolution of the laws of $[0, T] \ni t \mapsto X_\ell(t) \in \mathbb{R}^{d_\ell}$. In (1.1), the Brownian motions $\{W_p^\ell \; ; \; p = 1, \ldots, M, \quad m = 0, \ldots, r_\ell\}$ are mutually independent and independent of $(X_0^\ell)_{\ell=1}^M$, where, for each ℓ, the distribution of X_0^ℓ is given by m_0^ℓ. In addition, the map $m : [0, T] \to \Pi_{\ell=1}^M \mathcal{P}_1(\mathbb{R}^{d_\ell})$ is given by $m(t) = (\text{Law}(X_1(t)), \ldots, \text{Law}(X_M(t)))$.

Our aim in this paper is to use this probabilistic interpretation in order to provide a convergent fully discrete scheme for (FPK). The analysis of the proposed approximation, that we will present in Sect. 3, is a rather straightforward extension of the study done in [9], where $M = 1$. On the other hand, as we will show in the next section, it is easy to see that solutions of (FPK) can be found as the marginal laws of a single FPK equation whose solution takes values in $\mathcal{P}_1(\prod_{\ell=1}^M \mathbb{R}^{d_\ell})$ at each time. Therefore, the scheme in [9] could, in principle, be used to approximate (FPK). However, from the practical point of view, this roadmap has serious difficulties because the numerical efficiency of the scheme in [9] depends heavily on the dimension of the state space. In this sense, the study of a scheme that can be directly applied to system (FPK) is interesting in its own right.

We implement the scheme in two examples. In the first one we consider a diffusive version, introduced in [6], of a system of FPK equations proposed in [12] modelling the evolution of two interacting species under attraction and repulsion effects. Since in [6] some of the drift terms depend on the densities of the species distributions, we need to regularize these terms in order to obtain a convergent approximation in our framework. Our discretization produces rather similar numerical results to those in [3, Section 5.1]. In the second example, we consider a particular instance of a two population Mean Field Game (MFG) (see e.g. [10]). The system we consider, introduced in [1, Section 6.2.1], is symmetric with respect to both populations and aims to model xenophobia effects on urban settlements. In [1] it is shown that even if at the microscopic level the xenophobic effect is small, segregation occurs at the macroscopic level, indicating that Schelling's principle (see [17]) is also valid in the context of MFGs. In the tests that we have implemented, we recover the numerical results in [1] for the viscosity parameters the authors consider, but we are also able to deal with very small, or null, viscosity parameters, capturing, for these cases, different segregated configurations than those in [1]. We believe that the possibility of dealing with small or null viscosity parameters, as well as large time steps, is an important feature of the scheme that we propose.

The article is organized as follows. In the next section we introduce some standard notations and our main assumptions. In Sect. 3 we introduce the scheme that we propose, which is a straightforward extension of the one in [9], and we study its main properties, including the convergence analysis. Finally, in Sect. 4, we present our numerical results for the two examples described in the previous paragraph.

2 Preliminaries and Main Assumptions

Let us first set some standard notations and assumptions that we will use in the rest of the paper. For the sake of notational convenience we will assume that $M = 2$, but our results admit straightforward generalizations for arbitrary $M \in \mathbb{N}$. The set $\mathcal{P}_i(\mathbb{R}^d)$ ($d, i \in \mathbb{N} \setminus \{0\}$) denotes the set of Borel probability measures over $\mathbb{R}^d$ with finite i-th order moment. We endow $\mathcal{P}_i(\mathbb{R}^d)$ with the standard Monge-Kantorovic metric

$$d_i(\mu_1, \mu_2) := \inf \left\{ \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^i \mathrm{d}\gamma(x, y) \right)^{\frac{1}{i}} \mid \Pi_x \sharp \gamma = \mu_1, \quad \Pi_y \sharp \gamma = \mu_2 \right\},$$

where $\Pi_x(x, y) := x$, $\Pi_y(x, y) := y$ for all $x, y \in \mathbb{R}^d$ and given a Borel map $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ and a Borel measure μ on $\mathcal{B}(\mathbb{R}^m)$, the *push-forward* measure $\Phi \sharp \mu$ is defined as $\Phi \sharp \mu(A) := \mu(\Phi^{-1}(A))$. Let $\mathcal{K} \subseteq \mathcal{P}_i(\mathbb{R}^d)$ be given. A useful compactness result in $\mathcal{P}_i(\mathbb{R}^d)$ states that if for a given $\mathcal{K} \subseteq \mathcal{P}_i(\mathbb{R}^d)$ there exists $C > 0$ such that

$$\int_{\mathbb{R}^d} |x|^{i+\delta} \mathrm{d}\mu(x) \leq C \qquad \text{for some } \delta > 0 \text{ and all } \mu \in \mathcal{K}, \tag{2.1}$$

then $\mathcal{K}$ is relatively compact (see e.g. [2, Proposition 7.1.5]).

Define $\mathcal{M} := C([0, T]; \mathcal{P}_1(\mathbb{R}^{d_1})) \times C([0, T]; \mathcal{P}_1(\mathbb{R}^{d_2}))$. We say that $m = (m^1, m^2) \in \mathcal{M}$ is a weak solution of (FPK) if for all $\ell = 1, 2$, $t \in [0, T]$ and $\phi \in C_0^\infty(\mathbb{R}^{d_\ell})$ (the space of C^∞ real-valued functions defined on $\mathbb{R}^{d_\ell}$ and with compact support) we have that

$$\begin{aligned} \int_{\mathbb{R}^{d_\ell}} \varphi(x) \mathrm{d}m^\ell(t)(x) = \int_{\mathbb{R}^{d_\ell}} \varphi(x) \mathrm{d}\bar{m}_0^\ell(x) + \int_0^t \int_{\mathbb{R}^{d_\ell}} \left[b^\ell(m, x, s) \cdot \nabla \varphi(x) \right] \mathrm{d}m^\ell(s)(x) \mathrm{d}s \\ + \int_0^t \int_{\mathbb{R}^{d_\ell}} \left[\tfrac{1}{2} \sum_{i,j} a_{i,j}^\ell(m, x, s) \partial^2_{x_i, x_j} \varphi(x) \right] \mathrm{d}m^\ell(s)(x) \mathrm{d}s, \end{aligned} \tag{2.2}$$

provided that the second and third terms in the right hand side are meaningful.

The main assumptions in this paper are continuity and uniform linear growths of b^ℓ and σ^ℓ, respectively, with respect to the space variables. More precisely, we will suppose that

(H) For $\ell = 1, 2$

(i) $\bar{m}_0^\ell \in \mathcal{P}_2(\mathbb{R}^d)$.
(ii) The maps b^ℓ and σ^ℓ are continuous.
(iii) There exists $C > 0$ such that

$$|b^\ell(m, x, t)| + |\sigma^\ell(m, x, t)| \leq C(1 + |x|) \;\; \forall\, m \in \mathcal{M}, \quad x \in \mathbb{R}^{d_\ell}, \;\; t \in [0, T]. \tag{2.3}$$

Note that system (FPK) can be analysed with the help of a single FPK equation. Indeed, let $\bar{m}_0 \in \mathcal{P}_2(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ be such that its marginal in $\mathbb{R}^{d_\ell}$ ($\ell = 1, 2$) is given by $\bar{m}_0^\ell$. Given $\mu \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}))$ denote by $\hat{\mu} := (\mu^1, \mu^2) \in \mathcal{M}$ the marginals in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ of $t \in [0,T] \to \mu(t) \in \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Writing $x = (x^1, x^2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ define the coefficients

$$
\begin{aligned}
b &: C\left([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\right) \times (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \times [0,T] \to \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, \\
\sigma &: C\left([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\right) \times (\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}) \times [0,T] \to \mathbb{R}^{d_1 \times r_1} \times \mathbb{R}^{d_2 \times r_2}.
\end{aligned}
$$

as

$$
b(\mu, x, t) := \left(b^1(\hat{\mu}, x^1, t), b^2(\hat{\mu}, x^2, t)\right), \quad \sigma(\mu, x, t) := \left(\sigma^1(\hat{\mu}, x^1, t), \sigma^2(\hat{\mu}, x^2, t)\right), \tag{2.4}
$$

for all $\mu \in C\left([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\right)$, $x \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $t \in [0,T]$. Finally, for all $\ell_1, \ell_2 = 1, 2$ we set

$$
a_{i,j}^{\ell_1,\ell_2}(\mu, x, t) := \begin{cases} \sum_{p=1}^{d_{\ell_1}} \sigma_{i,p}^{\ell_1}(\hat{\mu}, x^{\ell_1}, t)\sigma_{j,p}^{\ell_1}(\hat{\mu}, x^{\ell_1}, t) & \text{if } \ell_1 = \ell_2, \\ 0 & \text{if } \ell_1 \neq \ell_2. \end{cases}
$$

Consider the problem of finding $m \in C\left([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\right)$ such that

$$
\begin{aligned}
\partial_t m - \frac{1}{2} \sum_{\substack{1 \le \ell_1, \ell_2 \le 2 \\ 1 \le i,j \le d_\ell}} \partial^2_{x_i^{\ell_1}, x_j^{\ell_2}} \left(a_{i,j}^{\ell_1,\ell_2}(m, x, t) m\right) + \operatorname{div}\left(b(m, x, t) m\right) &= 0 \quad \text{in } \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times [0,T], \\
m(0) &= \bar{m}_0 \quad \text{in } \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}.
\end{aligned} \tag{FPK'}
$$

If **(H)** holds, then the coefficients b and σ, defined in (2.4), also satisfy **(H)** in the corresponding spaces. More precisely, b and σ are continuous and there exists $C > 0$ such that

$$
|b(m,x,t)| + |\sigma(m,x,t)| \le C(1 + |x|) \quad \forall\, m \in C\left([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\right), \; x \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}, \; t \in [0,T]. \tag{2.5}
$$

Thus, by the results in [15, 16] (see also [9, Theorem 4.2]) we have that (FPK') admits at least one solution $m \in C\left([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})\right)$. Moreover, from the results in [9] we have the existence of $C > 0$ such that

$$
\sup_{t \in [0,T]} \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |x|^2 \mathrm{d}m(t)(x) \le C. \tag{2.6}
$$

Now, for $R > 0$ and $x' \in \mathbb{R}^{d_2}$ we set $\xi_R(x') := \xi(x'/R)$, where $\xi \in C_0^\infty(\mathbb{R}^{d_2})$ is such that $0 \leq \xi \leq 1$, $\xi(x') = 1$ if $|x'| \leq 1/2$ and $\xi(x') = 0$ if $|x'| \geq 1$. The function ξ_R belongs to $C_0^\infty(\mathbb{R}^{d_2})$ and, as $R \uparrow \infty$, approximate the constant function equal to 1 in $\mathbb{R}^{d_2}$. Given $\varphi \in C_0^\infty(\mathbb{R}^{d_1})$, let us define $\varphi_R^1 : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ as $\varphi_R^1(x) := \varphi(x^1)\xi_R(x^2)$, which belongs to $C_0^\infty(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. By considering this test function in (FPK'), using (2.5) and (2.6) and letting $R \uparrow \infty$ we obtain that $m^1 \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1}))$ (defined for all $t \in [0,T]$ as the marginal of $m(t)$ with respect to $\mathbb{R}^{d_1}$) satisfies (2.2) with $\ell = 1$. A similar construction shows that $m^2 \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_2}))$ (defined for all $t \in [0,T]$ as the marginal of $m(t)$ with respect to $\mathbb{R}^{d_2}$) satisfies (2.2) with $\ell = 2$. As a result (m^1, m^2) solves (FPK).

From the analytical point of view, the argument above is useful in order to obtain existence and properties of solutions to (FPK). On the other hand, as we comment in Remark 3.2 in the next section, this simplification is useless from the numerical point of view.

3 The Fully Discrete Scheme

We consider a time step $h = T/N^T$ ($N^T \in \mathbb{N}$) and space steps $\rho^1, \rho^2 > 0$. We define $t_k = kh$ ($k = 0, \ldots, N_T$), the time grid $\{0, t_1, \ldots, t_{N^T-1}, T\}$ and the space grids $\mathcal{G}_{\rho^\ell} := \{x_i^\ell = \rho^\ell i \mid i \in \mathbb{Z}^{d_\ell}\}$ ($\ell = 1, 2$). We consider two regular lattices $\mathcal{T}_{\rho^1}$ and $\mathcal{T}_{\rho^2}$ of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, where the vertices of the square elements belong to $\mathcal{G}_{\rho^1}$ and $\mathcal{G}_{\rho^2}$, respectively. Associated to these lattices and their vertices, we consider two $\mathbb{Q}_1$ bases $\{\beta_i^\ell\ ; i \in \mathbb{Z}^{d_\ell}\}$ ($\ell = 1, 2$) . By definition, for $\ell = 1, 2$ and $i \in \mathbb{Z}^{d_\ell}$, the functions $\beta_i^\ell : \mathbb{R}^{d_\ell} \to \mathbb{R}_+$ (where $\mathbb{R}_+$ denotes the set of non negative real numbers) are polynomials of degree less than or equal to 1 with respect to each variable $(x_1, \ldots, x_{d_\ell})$ on each square $Q \in \mathcal{T}_{\rho^\ell}$, have compact support and satisfy that $\beta_i^\ell(x_j^\ell) = \delta_{i,j}$ (where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$, otherwise) and $\sum_{i \in \mathbb{Z}^{d_\ell}} \beta_i^\ell(x) = 1$ for all $x \in \mathbb{R}^{d_\ell}$. In order to define a discretization of the initial condition $\bar{m}_0^\ell$ we define the sets

$$E_i^\ell := \left\{ x \in \mathbb{R}^{d_\ell}\ ;\ |x - x_i|_\infty \leq \frac{\rho^\ell}{2} \right\}.$$

Since we will let ρ^ℓ tend to 0 later, without loss of generality we can assume that $\bar{m}_0^\ell(\partial E_i^\ell) = 0$ for all $i \in \mathbb{Z}^{d_\ell}$. We then set

$$m_{i,0}^\ell = \bar{m}_0^\ell(E_i^\ell) \qquad \forall\, i \in \mathbb{Z}^{d_\ell}.$$

Since $\bar{m}_0^\ell(\mathbb{R}^{d_\ell}) = 1$, we have that $\left\{m_{i,0}^\ell \mid i \in \mathbb{Z}^{d_\ell}\right\}$ belongs to the *simplex*

$$\mathcal{S}^{\rho_\ell} := \left\{\mu \in [0,1]^{\mathbb{Z}^{d_\ell}} \mid \sum_{i\in\mathbb{Z}^{d_\ell}} \mu_i = 1\right\}.$$

Given $\mu = \left\{\mu_{i,k} \mid i \in \mathbb{Z}^{d_\ell},\ k = 0,\dots,N_T\right\} \in (\mathcal{S}^{\rho_\ell})^{N_T+1}$, we identify μ with an element in $C([0,T];\mathcal{P}_1(\mathbb{R}^{d_\ell}))$ via a linear interpolation

$$\mu(t) := \left(\frac{t-t_k}{h}\right)\sum_{i\in\mathbb{Z}^{d_\ell}} \mu_{i,k+1}\delta_{x_i^\ell} + \left(\frac{t_{k+1}-t}{h}\right)\sum_{i\in\mathbb{Z}^{d_\ell}} \mu_{i,k}\delta_{x_i^\ell} \quad \text{if } t \in [t_k, t_{k+1}[. \tag{3.1}$$

Now, we have all the elements to introduce the discretization of (FPK) we consider. For the sake of clarity, we first recall the fully-discrete scheme introduced in [9] when $M = 1$. In this case the (FPK) system is given by

$$\begin{aligned} \partial_t m - \tfrac{1}{2}\sum_{1\le i,j\le d} \partial^2_{x_i,x_j}\left(a_{i,j}(m,x,t)m\right) + \textstyle\sum_{i=1}^d \partial_{x_i}\left(b(m,x,t)m\right) = 0, \qquad &\text{in } \mathbb{R}^d\times(0,T),\\ m(0) = \bar{m}_0 \qquad &\text{in } \mathbb{R}^d, \end{aligned} \tag{3.2}$$

where we have omitted the superfluous index $\ell = 1$. The fully discrete scheme for (3.2) reads: Find $m \in (\mathcal{S}^\rho)^{N_T+1}$ such that

$$\begin{aligned} m_{i,0} &= \bar{m}_0(E_i) \quad \forall\, i \in \mathbb{Z}^d,\\ m_{i,k+1} &= \tfrac{1}{2r}\sum_{p=1}^{r}\sum_{j\in\mathbb{Z}^d}\left[\beta_i(\Phi^{p,+}_{j,k}[m]) + \beta_i(\Phi^{p,-}_{j,k}[m])\right] m_{j,k} \quad \forall\, i\in\mathbb{Z}^d,\ \ k=0,\dots,N_T-1, \end{aligned} \tag{3.3}$$

where the *one-step* discrete characteristics starting from x_j at time t_k are defined as

$$\Phi^{p,+}_{j,k}[m] := x_j + hb(m,x_j,t_k) + \sqrt{rh}\,\sigma_p(m,x_j,t_k), \quad \Phi^{p,-}_{j,k}[m] := x_j + hb(m,x_j,t_k) - \sqrt{rh}\,\sigma_p(m,x_j,t_k),$$

with b and σ_p being defined, as a function of m, through the extension (3.1).

Existence of at least one solution $m^{\rho,h}$ to (3.3) has been proved in [9, Proposition 3.1]. Moreover, under an additional local Lipschitz assumption on b and σ, as ρ and h tend to 0 and $\rho^2 = o(h)$, the sequence $m^{\rho,h}$ in $C([0,T];\mathcal{P}_1(\mathbb{R}^d))$, defined again through the extensions (3.1), has at least one limit point $m \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$, and every such limit point solves (FPK) (see [9, Theorem 4.1]).

Remark 3.1 By regularizing the coefficients b and σ using standard mollifiers, and modifying the scheme accordingly, this convergence result is also shown to hold under assumption **(H)** only (see [9, Theorem 4.2]).

In order to grasp the probabilistic interpretation of (3.3), it is useful to think this problem as the one of finding a fixed point of a suitable mapping. Indeed, given $\mu \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ and a solution $m[\mu] \in (\mathcal{S}^\rho)^{N_T+1}$ to

$$\begin{aligned} m_{i,0} &= \bar{m}_0(E_i) \quad \forall\, i \in \mathbb{Z}^d, \\ m_{i,k+1} &= \tfrac{1}{2r}\sum_{p=1}^{r}\sum_{j\in\mathbb{Z}^d}\Big[\beta_i(\Phi_{j,k}^{p,+}[\mu]) + \beta_i(\Phi_{j,k}^{p,-}[\mu])\Big] m_{j,k} \quad \forall\, i \in \mathbb{Z}^d,\ \ k = 0,\dots,N_T-1, \end{aligned} \tag{3.4}$$

we can construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and Markov chain $\{X_k[\mu] \mid k = 0,\dots,N_T\}$, defined on it, taking values in $\mathcal{G}_\rho$ and whose marginal laws and transition probabilities are given, respectively, by $m[\mu]_{(\cdot),k} \in \mathcal{S}^\rho$ and

$$\mathbb{P}\left(X_{k+1}[\mu]=x_i \mid X_k[\mu]=x_j\right)=\frac{1}{2r}\sum_{p=1}^{r}\Big[\beta_i(\Phi_{j,k}^{p,+}[\mu]) + \beta_i(\Phi_{j,k}^{p,-}[\mu])\Big] \quad \forall\, i,j \in \mathbb{Z}^d,\ \ k = 0,\dots,N_T-1. \tag{3.5}$$

In [9] the Markov chain defined above is shown to satisfy the *consistency conditions* introduced by Kushner (see e.g. [14]). Hence, we can expect that its marginal laws will approximate the law of a weak solution $X[\mu]$ to

$$\mathrm{d}X(t) = b(\mu, X(t), t)\mathrm{d}t + \sum_{p=1}^{r}\sigma_{\cdot,p}(\mu, X(t), t)\mathrm{d}W_p(t) \ \ t \in [0,T], \quad X(0) = X_0, \tag{3.6}$$

where the distribution of X_0 is given by $\bar{m}_0$. As explained in [9], a solution to (FPK), when $M = 1$, corresponds to a fixed point $m \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$ of the application $C([0,T];\mathcal{P}_1(\mathbb{R}^d)) \ni \mu \to m[\mu](\cdot) \in C([0,T];\mathcal{P}_1(\mathbb{R}^d))$, where, for every $t \in [0,T]$, the measure $m[\mu](t)$ is defined as the law of $X[\mu](t)$. Based on this interpretation, scheme (3.3) can be interpreted as the analogous fixed point problem for the approximating Markov chain $\{X_k[\mu] \mid k = 0,\dots,N_T\}$.

Having the previous observations in mind, the extension of scheme (3.3) to the case $M = 2$ is straightforward. We consider the problem of finding $m = (m^1, m^2) \in (\mathcal{S}^{\rho^1})^{N_T+1} \times (\mathcal{S}^{\rho^2})^{N_T+1}$ such that, for $\ell = 1, 2$, we have

$$\begin{aligned} m_{i,0}^{\ell} &= \bar{m}_0^{\ell}(E_i^{\ell}) \quad \forall\, i \in \mathbb{Z}^{d_\ell}, \\ m_{i,k+1}^{\ell} &= \tfrac{1}{2r_\ell}\sum_{p=1}^{r_\ell}\sum_{j\in\mathbb{Z}^{d_\ell}}\Big[\beta_i^{\ell}(\Phi_{j,k}^{\ell,p,+}[m]) + \beta_i^{\ell}(\Phi_{j,k}^{\ell,p,-}[m])\Big] m_{j,k}^{\ell} \quad \forall\, i \in \mathbb{Z}^{d_\ell},\ \ k=0,\dots,N_T-1, \end{aligned} \tag{3.7}$$

where

$$\Phi_{j,k}^{\ell,p,+}[m] := x_j^\ell + hb^\ell(m, x_j^\ell, t_k) + \sqrt{r_\ell h}\sigma_p(m, x_j^\ell, t_k),$$
$$\Phi_{j,k}^{\ell,p,-}[m] := x_j^\ell + hb^\ell(m, x_j^\ell, t_k) - \sqrt{r_\ell h}\sigma_p(m, x_j^\ell, t_k).$$

Arguing exactly as in the proof of Proposition 3.1 in [9], the existence of at least one solution $m_{\rho,h}$ is a consequence of **(H)** and Schauder fixed-point theorem. We also point out that the scheme is conservative. Indeed, for $\ell = 1, 2$ and $k = 0, \dots, N_T$ we have

$$\sum_{i\in\mathbb{Z}^{d_\ell}} m_{i,k+1}^\ell = \sum_{j\in\mathbb{Z}^{d_\ell}} m_{j,k}^\ell \frac{1}{2r_\ell} \sum_{p=1}^{r_\ell} \sum_{i\in\mathbb{Z}^{d_\ell}} \left[\beta_i^\ell(\Phi_{j,k}^{\ell,p,+}[m]) + \beta_i^\ell(\Phi_{j,k}^{\ell,p,-}[m])\right] = \sum_{j\in\mathbb{Z}^{d_\ell}} m_{j,k}^\ell = 1,$$

where the last equality follows from $\sum_{j\in\mathbb{Z}^{d_\ell}} m_{j,0}^\ell = 1$.

Remark 3.2

(i) As we discussed at the end of the previous section, we could approximate a solution to (FPK) by first approximating a solution of (FPK') and then taking its marginals with respect to $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$. The problem of this approach is that if we use scheme (3.3) in order to approximate (FPK'), then we should consider a discretization of $\mathbb{R}^{d_1+d_2}$ instead of discretizing $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ separately (as we do with scheme (3.7)), which affects enormously the computational time. Of course, in our numerical experiments we must consider bounded space grids (see the next section), but the same difficulty arises.

(ii) Note that if for each $(x, t) \in \mathbb{R}^{d_\ell} \times [0, T]$ ($\ell = 1, 2$) the functions

$$C([0, T]; \mathcal{P}_1(\mathbb{R}^{d_\ell}))^2 \ni (m^1, m^2) \mapsto b^\ell(m^1, m^2, x, t) \in \mathbb{R}^{d_\ell}$$
$$\text{and } C([0, T]; \mathcal{P}_1(\mathbb{R}^{d_\ell}))^2 \ni (m^1, m^2) \mapsto \sigma^\ell(m^1, m^2, x, t) \in \mathbb{R}^{d_\ell\times r_\ell},$$

depend on $\{(m_1(s), m_2(s)) \mid 0 \le s \le t\}$, then the scheme (3.7) is explicit and, as a consequence, it admits a unique solution. On the other hand, if $b^\ell(m^1, m^2, x, t)$, or $\sigma^\ell(m^1, m^2, x, t)$, depends on values $(m^1(s), m^2(s))$, for some $s \in [t, T]$, then the scheme is implicit and ad-hoc techniques should be used in order to compute a solution numerically.

3.1 Convergence

In this section we analyse the limit behaviour of solutions (m_n^1, m_n^2) to (3.7) with steps ρ_n^1, ρ_n^2 and $h_n := 1/N_T^n$ tending to zero as $n \to \infty$. We work with the extensions, defined through (3.1), of m_n^1 and m_n^2 to $C([0, T]; \mathcal{P}_1(\mathbb{R}^{d_1}))$ and $C([0, T]; \mathcal{P}_1(\mathbb{R}^{d_2}))$, respectively.

The first important remark is that, as the next result shows, the sequence (m_n^1, m_n^2) is equicontinuous in $C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1})) \times C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_2}))$ (see (3.8)) and, for each $t \in [0,T]$, we have that $(m_n^1(t), m_n^2(t))$ belongs to a fixed relatively compact subset of $\mathcal{P}_1(\mathbb{R}^{d_1}) \times \mathcal{P}_1(\mathbb{R}^{d_2})$ (see (3.9) and (2.1)).

Proposition 3.1 *Suppose that* ***(H)*** *holds true and that, as* $n \to \infty$, $\rho_1^n + \rho_2^n = O(h_n^2)$. *Then, there exists a constant* $C > 0$ *(independent of* n*) such that*

$$d_1(m_n^1(t), m_n^1(s)) + d_1(m_n^2(t), m_n^2(s)) \leq C\sqrt{|t-s|} \qquad \forall\, t,\ s \in [0,T], \tag{3.8}$$

$$\int_{\mathbb{R}^{d_1}} |x|^2 \mathrm{d}m_n^1(t)(x) + \int_{\mathbb{R}^{d_2}} |x|^2 \mathrm{d}m_n^2(t)(x) \leq C \qquad \forall\, t \in [0,T]. \tag{3.9}$$

The proofs of (3.8) and (3.9) are analogous to the proofs of [9, Proposition 4.1] and [9, Proposition 4.2], respectively, and will therefore be omitted. As a consequence of the previous result and the Arzelà-Ascoli theorem, there exists at least one limit point $(m^1, m^2) \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1})) \times C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_2}))$ of (m_n^1, m_n^2). In order to prove that any limit point of (m_n^1, m_n^2) solves (FPK), we will assume in addition

(Lip) For $\ell = 1, 2$, $\mu \in \mathcal{M}$ and compact set $K_\ell \subseteq \mathbb{R}^{d_\ell}$, there exists $C_\ell = C(\mu, K_\ell) > 0$ such that

$$|b^\ell(\mu,y,t) - b^\ell(\mu,x,t)| + |\sigma^\ell(\mu,y,t) - \sigma^\ell(\mu,x,t)| \leq C_\ell |y-x| \quad \forall\, x,\ y \in K_\ell,\quad t \in [0,T].$$

Theorem 3.1 *Suppose that* **(H)**-**(Lip)** *hold true and that, as* $n \to \infty$, $\rho_1^n + \rho_2^n = o(h_n^2)$. *Then, every limit point* (m^1, m^2) *of* (m_n^1, m_n^2) *(there exists at least one) solves* (FPK).

Proof The proof is analogous to the proof of [9, Theorem 4.1] and so we only sketch the main steps. Let $(m^1, m^2) \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1})) \times C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_2}))$ be a limit point of (m_n^1, m_n^2) and consider a subsequence, still labelled by n, such that $(m_n^1, m_n^2) \to (m^1, m^2)$ as $n \to \infty$. Then, for any $t \in [0,T]$ and $\varphi \in C_0^\infty(\mathbb{R}^{d_\ell})$ $(\ell = 1, 2)$ we have

$$\int_{\mathbb{R}^{d_\ell}} \varphi(x) \mathrm{d}m_n^\ell(t_{n'})(x) = \int_{\mathbb{R}^{d_\ell}} \varphi(x) \mathrm{d}m_n^\ell(0)(x) + \sum_{k=0}^{n'-1} \int_{\mathbb{R}^{d_\ell}} \varphi(x) \mathrm{d}\left[m_n^\ell(t_{k+1}) - m_n^\ell(t_k)\right](x), \tag{3.10}$$

where $n' \in \{0, \ldots, N_T^n\}$ is such that $t_{n'} = n'h_n \to t$. Using (3.7), we obtain that

$$\begin{aligned} \textstyle\int_{\mathbb{R}^{d_\ell}} \varphi(x) \mathrm{d}m_n^\ell(t_{k+1})(x) &= \textstyle\sum_{i \in \mathbb{Z}^{d_\ell}} \varphi(x_i) m_{k+1,i}^\ell \\ &= \textstyle\sum_{i \in \mathbb{Z}^{d_\ell}} \varphi(x_i) \frac{1}{2r_\ell} \sum_{p=1}^{r_\ell} \sum_{j \in \mathbb{Z}^{d_\ell}} \left[\beta_i^\ell(\Phi_{j,k}^{\ell,p,+}[m_n]) + \beta_i^\ell(\Phi_{j,k}^{\ell,p,-}[m_n])\right] m_{j,k}^\ell \end{aligned}$$

$$= \sum_{j\in\mathbb{Z}^{d_\ell}} \frac{m^\ell_{j,k}}{2r_\ell} \sum_{p=1}^{r_\ell} \Big[I[\varphi](\Phi^{\ell,p,+}_{j,k}[m_n]) + I[\varphi](\Phi^{\ell,p,-}_{j,k}[m_n]) \Big]$$
$$= \sum_{j\in\mathbb{Z}^{d_\ell}} \frac{m^\ell_{j,k}}{2r_\ell} \sum_{p=1}^{r_\ell} \Big[\varphi\left(\Phi^{\ell,p,+}_{j,k}[m_n]\right) + \varphi\left(\Phi^{\ell,p,-}_{j,k}[m_n]\right) \Big] + O((\rho^\ell_n)^2),$$

where in the last equality we have used that $\sup_{x\in\mathbb{R}^{d_\ell}} |I[\varphi](x) - \varphi(x)| = O((\rho^\ell_n)^2)$. By a Taylor expansion, we obtain

$$\varphi\left(\Phi^{\ell,p,+}_{j,k}[m_n]\right) + \varphi\left(\Phi^{\ell,p,-}_{j,k}[m_n]\right) = 2\phi(x_j) + 2h_n \nabla\varphi(x_j)\cdot b^\ell(m^1_n, m^2_n, x_j, t_k)$$
$$+ r_\ell h_n \sum_{1\le i',j'\le d_\ell} \partial_{x_{i'},x_{i'}}\varphi(x_j)\sigma^\ell_{i',p}\sigma^\ell_{j',p}$$
$$+ O(h_n^2),$$

where we have omitted the dependence of $\sigma^\ell_{i',p}$ and $\sigma^\ell_{j',p}$ on (m^1_n, m^2_n, x_j, t_k). This implies that

$$\frac{1}{2r_\ell}\sum_{p=1}^{r_\ell} \Big[\varphi\left(\Phi^{\ell,p,+}_{j,k}[m_n]\right) + \varphi\left(\Phi^{\ell,p,-}_{j,k}[m_n]\right) \Big] = \phi(x_j) + h_n \nabla\varphi(x_j)\cdot b^\ell(m^1_n, m^2_n, x_j, t_k)$$
$$+ \frac{h_n}{2} \sum_{1\le i',j'\le d_\ell} \partial_{x_{i'},x_{i'}}\varphi(x_j) a^\ell_{i',j'}(m^1_n, m^2_n, x_j, t_k)$$
$$+ O(h_n^2).$$

Thus, using (3.10), we obtain

$$\int_{\mathbb{R}^{d_\ell}} \varphi(x)\mathrm{d}m^\ell_n(t_{n'})(x) = \int_{\mathbb{R}^{d_\ell}} \varphi(x)\mathrm{d}m^\ell_n(0)(x)$$
$$+ h_n \sum_{k=0}^{n'-1} \int_{\mathbb{R}^{d_\ell}} \left[\nabla\varphi(x)\cdot b^\ell(m^1_n, m^2_n, x, t_k) + \frac{h_n}{2} \sum_{1\le i,j\le d_\ell} \partial_{x_i,x_i}\varphi(x) a^\ell_{i,j}(m^1_n, m^2_n, x, t_k) \right] \mathrm{d}m^\ell_n(t_k)$$
$$+ O\left(h_n + \frac{(\rho^\ell_n)^2}{h_n}\right).$$

Finally, using that $m^\ell_n \to m^\ell \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_\ell}))$ by **(H)** we have that $b^\ell(m^1_n, m^2_n, \cdot,\cdot) \to b^\ell(m^1, m^2, \cdot,\cdot)$ and $a^\ell_{i,j}(m^1_n, m^2_n, \cdot,\cdot) \to a^\ell_{i,j}(m^1, m^2, \cdot,\cdot)$ uniformly in $\mathrm{supp}(\varphi)\times[0,T]$ (where $\mathrm{supp}(\varphi)$ denotes the support of φ, which is a compact set). Using this fact and assumption **(Lip)**, we can argue in the same manner than in [9, Theorem 4.1] and pass to the limit in the expression above to obtain that m^ℓ satisfies (2.2). The result follows.

Remark 3.3 As in [9, Theorem 4.2], we can get rid of assumption **(Lip)** at the price of regularizing by convolution the coefficients b^ℓ and σ^ℓ and considering the associated scheme with the regularized coefficients.

In practice we have not always access to the coefficients b^ℓ and $a^\ell_{i,j}$ and they have to be approximated. As we will see in the next section, this is the case of multi-population MFGs systems. Consider a sequence of space steps ρ_n^1, ρ_n^2 and a sequence of time steps h_n satisfying the assumptions of the previous result. Assume that for each n we have

$$b_n^\ell : C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_\ell})) \times \mathbb{R}^{d_\ell} \times [0,T] \to \mathbb{R}^{d_\ell},$$

$$\sigma_n^\ell : C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_\ell})) \times \mathbb{R}^{d_\ell} \times [0,T] \to \mathbb{R}^{d_\ell \times r_\ell},$$

such that:

(H')

(i) for each fixed $t \in [0,T]$, the mappings $b_n^\ell(\cdot,\cdot,t)$ and $\sigma_n^\ell(\cdot,\cdot,t)$ are continuous.
(ii) the growth condition (2.5) holds for a constant $C > 0$ independent of n.
(iii) for any sequence $\mu_n \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_\ell}))$ and $\mu \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_\ell}))$ satisfying that $\mu_n \to \mu$ we have

$$b_n^\ell(\mu_n,\cdot,\cdot) \to b^\ell(\mu,\cdot,\cdot), \qquad \sigma_n^\ell(\mu_n,\cdot,\cdot) \to \sigma^\ell(\mu,\cdot,\cdot)$$

uniformly on compact subsets of $\mathbb{R}^{d_\ell} \times [0,T]$.

Consider the scheme (3.7) constructed with discrete characteristics

$$(\Phi_{j,k}^{\ell,p,+})_n[m] := x_j + h b_n^\ell(m, x_j, t_k) + \sqrt{rh}(\sigma_n^\ell)_p(m, x_j, t_k),$$

$$(\Phi_{j,k}^{\ell,p,-})_n[m] := x_j + h b_n^\ell(m, x_j, t_k) - \sqrt{rh}(\sigma_n^\ell)_p(m, x_j, t_k),$$

which, by similar arguments to those in the case of coefficients independent of n, admits at least one solution (m_n^1, m_n^2). Then, we have the following result, whose proof is analogous to the proof of Theorem 3.1.

Theorem 3.2 *Under* **(H)-(Lip)** *and the previous assumptions, the sequence* (m_n^1, m_n^2) *admits at least one limit point* $(m^1, m^2) \in C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1})) \times C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_2}))$. *Moreover, every such limit point solves* (FPK).

4 Simulations

We show the performance of our scheme by applying it to approximate the solution of two instances of (FPK) with $M = 2$. In the first example we consider a variation of a PDE system treated analytically in [6] and numerically in [3], which models the evolution of two interacting species. In our framework, the drifts b^1 and b^2 have non local cross interaction terms and also a term that will approximate a nonlinear diffusion term present in [3, 6]. In the second example, we consider a particular instance of a two population MFG system modelling segregation (see e.g. [1, 10]). As discussed in [9], standard MFGs can be seen as a particular (FPK) equation with $M = 1$, where the drift term b^1 satisfies that for each $(x,t) \in \mathbb{R}^{d_1} \times [0,T]$

the function $C([0,T]; \mathcal{P}_1(\mathbb{R}^{d_1})) \ni m \mapsto b^1(m,x,t) \in \mathbb{R}^{d_1}$ depends on the values $\{m(s) \mid s \in (t,T]\}$. When $M \neq 1$, the situation is similar and hence, as explained in Remark 3.2(ii), the scheme is implicit.

Since the scheme (3.7) is defined on the unbounded space grid $\mathcal{G}_\rho$, in our numerical examples we need to change this grid to a bounded one. In order to maintain the total mass constant, we impose homogeneous Neumann boundary conditions and near the boundary we approximate the discrete flow by using a projected Euler scheme, as proposed in [11]. The proof of convergence of the modified scheme is postponed to a future work.

In all tests that we chose the discretization parameters (ρ, h) satisfying $h = O(\rho^{3/2})$, which is less restrictive than the classical parabolic CFL condition for explicit finite difference schemes. Larger time step would produce loss of accuracy close to the boundary. The question on how to modify the scheme at the boundary maintaining large time steps will also be addressed in a future work.

In the examples that we present below, at each time $t \in [0,T]$ the solution (m^1, m^2) is shown to admit a density with respect to the Lebesgue measure. For each $\ell = 1, 2$ we approximate the density of m^ℓ by defining $\mathbf{m}^\ell_{\rho,h}(x,t) := m^\ell_{i,k}/\rho^{d_\ell}$ if $(x,t) \in E^\ell_i \times [t_k, t_{k+1})$. For fixed t, $\mathbf{m}^\ell_{\rho,h}$ is a density which is uniform on each E^ℓ_i.

4.1 Interacting Species

We consider a system of two interacting species proposed first in the first order case in [12] and then extended in [6] to the case where a nonlinear diffusion term is also added to the system. The densities m^1 and m^2 of the two species are coupled through the drift by non local terms. The system studied in [6] reads

$$\begin{cases} \partial_t m^1 - \operatorname{div}\left(m^1\left(\nabla E'(m^1) + \nabla U_1(m^1, m^2, x, t)\right)\right) = 0, \\ \partial_t m^2 - \operatorname{div}\left(m^2\left(\nabla E'(m^2) + \nabla U_2(m^1, m^2, x, t)\right)\right) = 0, \\ m^1(\cdot, 0) = m^1_0(\cdot), \qquad m^2(\cdot, 0) = m^2_0(\cdot). \end{cases} \tag{4.1}$$

In (4.1), m^ℓ_0 $(\ell = 1, 2)$ represent two absolutely continuous probability measures whose densities are still denoted by m^ℓ_0. The term $E(m) := \frac{1}{2}m^3$ corresponds to an internal energy which introduces the nonlinear diffusion term $-\operatorname{div}(m^\ell(\nabla E'(m^\ell)) = -\Delta(m^\ell)^3$ in (4.1). It is assumed that $\int_{\mathbb{R}^d}(m^\ell_0(x))^3 \mathrm{d}x < +\infty$ for $\ell = 1, 2$. The potentials $U_1, U_2 : C([0,T]; \mathcal{P}_1(\mathbb{R}^d))^2 \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ are cross interactions terms and they are given by convolution with smooth functions

$$U_1(m^1, m^2, x, t) = W_{11} * \left[m^1(t)\right](x) + W_{21} * [m^2(t)](x),$$
$$U_2(m^1, m^2, x, t) = W_{12} * [m^1(t)](x) + W_{22} * [m^2(t)](x),$$

where $*$ denotes the space convolution and $W_{11}(x) = W_{21}(x) = W_{22}(x) := \frac{|x|^2}{2}$, $W_{12}(x) := \frac{-|x|^2}{2}$. With these choices, the drift terms

$$-\nabla\left(W_{11} * m^1(t)\right)(x) = \int_{\mathbb{R}^2} (y-x)\mathrm{d}m^1(t)(y), \quad -\nabla\left(W_{22} * m^2(t)\right)(x) = \int_{\mathbb{R}^2} (y-x)\mathrm{d}m^2(t)(y) \tag{4.2}$$

model self-interactions for the first and second species, respectively, whereas the terms

$$-\nabla\left(W_{21} * m^2(t)\right)(x) = \int_{\mathbb{R}^2} (y-x)\mathrm{d}m^2(t)(y), \quad -\nabla\left(W_{12} * m^1(t)\right)(x) = -\int_{\mathbb{R}^2} (y-x)\mathrm{d}m^1(t)(y), \tag{4.3}$$

model the facts that the first species is attracted by the second one and that the latter is repelled by first one, respectively. Note that the drift terms in (4.2)–(4.3) do not satisfy **(H)** because the linear growth is not uniform w.r.t. m^ℓ. This can be easily fixed by considering suitable compactly supported C^∞ approximations of the function $y - x$. In our simulations, we work on a bounded domain and so we work directly with the coefficients (4.2)–(4.3). It is easy to see that these drift terms satisfy **(Lip)**.

Existence and uniqueness results of weak solutions to (4.1) has been proved in [12] when $E_1 = E_2 = 0$. In the diffusive case, existence of at least one weak solution, which is absolutely continuous w.r.t. the Lebesgue measure, has been proved in [6]. We refer the reader to [3] for the numerical resolution of (4.1) by the so-called JKO scheme combined with the augmented Lagrangian method.

Since under **(H)** the coefficients should be continuous with respect to the weak convergence of probability measures, we need to regularize the local term $E'(m) = \frac{3}{2}m^2$. We do this by convolution. More precisely, given a regularization parameter $\delta > 0$ we define $E'_\delta : C([0,T]; \mathcal{P}_1(\mathbb{R}^d))) \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ as

$$E'_\delta(m, x, t) := \frac{3}{2}(m(t) * \phi_\delta(x))^2,$$

where $\phi_\delta(x) = \sqrt{2\pi}\delta \exp\left(-|x|^2/(2\delta^2)\right)$. We then consider the following variation of (4.1):

$$\begin{cases} \partial_t m^1 - \mathrm{div}(m^1(\nabla E'_\delta(m^1) + \nabla U_1(m^1, m^2))) = 0, \\ \partial_t m^2 - \mathrm{div}(m^2(\nabla E'_\delta(m^2) + \nabla U_2(m^1, m^2))) = 0, \\ m^1(\cdot, 0) = \bar{m}_0^1(\cdot), \qquad m^2(\cdot, 0) = \bar{m}_0^2(\cdot), \end{cases} \tag{4.4}$$

which satisfies **(H)**, with the suitable modifications of (4.2)–(4.3).

4.1.1 Numerical Test

We numerically solve system (4.4) with $d_1 = d_2 = 2$ on a domain $\Omega \times [0, T] = [-1, 1] \times [-1, 1] \times [0, 5]$, with homogeneous Neumann boundary conditions, $\delta = 0.02$ and initial conditions

$$m^1(x, 0) = \frac{\nu_1(x)}{\bar{\nu}_1} \qquad \text{and} \qquad m^2(x, 0) = \frac{\nu_2(x)}{\bar{\nu}_2},$$

where

$$\nu_1(x_1, x_2) := \left[0.2 - (x_1 - 0.5)^2 - \tfrac{(x_2+0.5)^2}{2}\right]_+^2,$$
$$\nu_2(x_1, x_2) := \left[0.2 - (x_1 + 0.5)^2 - \tfrac{(x_2-0.5)^2}{2}\right]_+^2,$$

and, for $a \in \mathbb{R}$, $a^+ := \max\{0, a\}$, and $\bar{\nu}_1$, $\bar{\nu}_2$ are two positive constants such that

$$\int_\Omega m^1(x, 0)\mathrm{d}x = \int_\Omega m^2(x, 0)\mathrm{d}x = 1.$$

In Fig. 1 we display the evolution of the two densities at the times $t = 0, 1, 2, 3, 4, 5$ computed with $\rho = 2e^{-2}$ and $h = \frac{1}{3}\rho^{3/2}$. The first plot on the top left shows

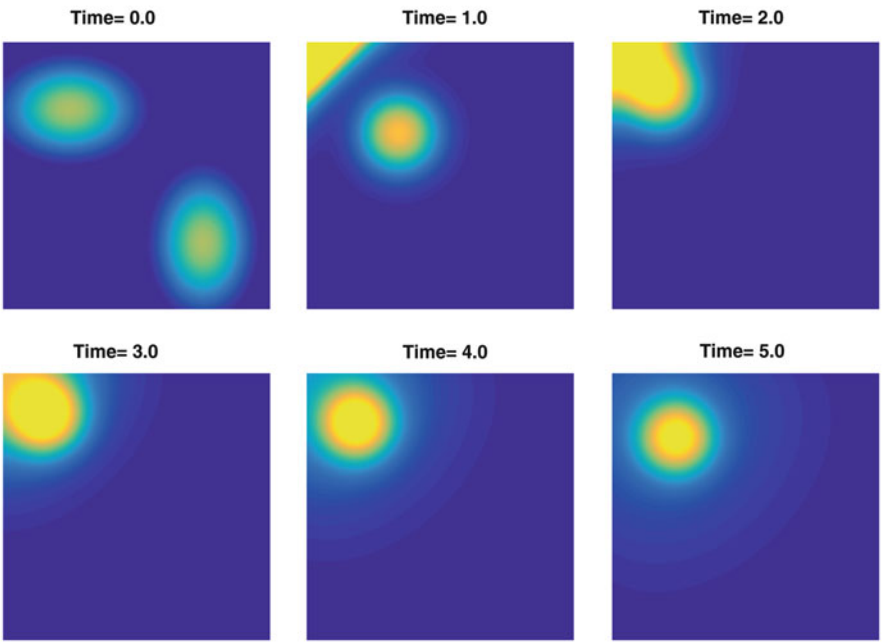

Fig. 1 Evolution of the two densities $\mathbf{m}^1_{\rho,h}$ and $\mathbf{m}^2_{\rho,h}$ at the times $t = 0, 1, 2, 3, 4, 5$

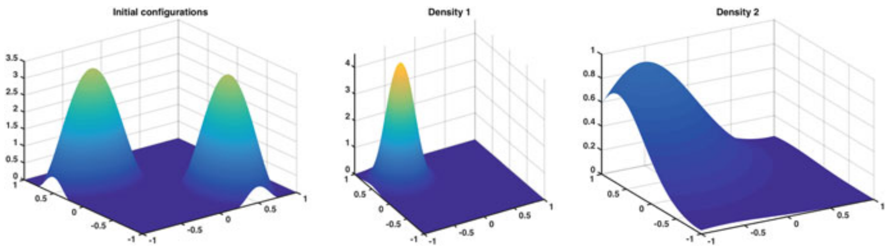

Fig. 2 3D view of the initial configuration (left), of the final configuration of $\mathbf{m}^1_{\rho,h}$ (center) and $\mathbf{m}^2_{\rho,h}$ (right)

the initial configurations: $\mathbf{m}^1_{\rho,h}$ is represented by the density located on the bottom right and $\mathbf{m}^2_{\rho,h}$ by the density located on the top left of the numerical domain. As time evolves, we observe the density $\mathbf{m}^1_{\rho,h}$ moving towards the density $\mathbf{m}^2_{\rho,h}$, which is instead repelled by $\mathbf{m}^1_{\rho,h}$. Due to the presence of Neumann boundary conditions, $\mathbf{m}^2_{\rho,h}$ get finally captured in the upper left corner of the domain. We can also observe the effect of the regularization of the nonlinear diffusion terms along with the effect of the attraction potential W_{11}: the numerical support of the density $\mathbf{m}^1_{\rho,h}$ takes a circular shape. In Fig. 2, we show a 3D view of the initial configuration (left) and the final configurations of $\mathbf{m}^1_{\rho,h}$(center) and $\mathbf{m}^2_{\rho,h}$(right).

4.2 Two Populations Mean Field Games

In this section, we consider the following MFG system

$$\begin{cases} -\partial_t v^1 - \nu\Delta v^1 + \frac{1}{2}|\nabla v^1|^2 = V(m^1, m^2), \\ -\partial_t v^2 - \nu\Delta v^2 + \frac{1}{2}|\nabla v^2|^2 = V(m^2, m^1), \\ v^1(\cdot, T) = 0, \;\; v^2(\cdot, T) = 0, \\ \partial_t m^1 - \nu\Delta m^1 - \operatorname{div}(\nabla v^1 m^1) = 0, \\ \partial_t m^2 - \nu\Delta m^2 - \operatorname{div}(\nabla v^2 m^2) = 0, \\ m^1(\cdot, 0) = \bar{m}^1_0(\cdot), \;\; m^2(\cdot, 0) = \bar{m}^2_0(\cdot). \end{cases} \tag{MFG}$$

In the system above, $\nu \geq 0$, $\bar{m}^1_0$, $\bar{m}^2_0 \in L^\infty(\mathbb{R}^d)$ ($d \in \mathbb{N} \setminus \{0\}$) are densities with compact support and the local coupling term $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$V(m^1, m^2) = \left(\frac{m^1}{m^1 + m^2} - 0.7\right)^- + (m^1 + m^2 - 8)^+, \tag{4.5}$$

where, for $a \in \mathbb{R}$, we set $a^- := a^+ - a$. This system has been proposed in [1] and models interactions between two populations with xenophobia and aversion to overcrowded regions effects. As in the previous example, we need to regularize the local coupling term V in order to obtain a function that is continuous with respect to the weak convergence of probability measures. We proceed as in [1, Section 6.2.1]. Given $\eta, \delta > 0$, we define $V_{\eta,\delta} : C([0,T]; \mathcal{P}_1(\mathbb{R}^d))^2 \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ as

$$
\begin{aligned}
V_{\eta,\delta}(m^1, m^2, x, t) = \Psi_{-,\eta}&\left(\tfrac{m^1(t)*\phi_\delta(x)}{m^1(t)*\phi_\delta(x)+m^2(t)*\phi_\delta(x)+\eta} - 0.7\right)\\
&+\Psi_{+,\eta}\left(m^1(t)*\phi_\delta(x) + m^2(t)*\phi_\delta(x) - 8\right),
\end{aligned}
$$

where

$$
\Psi_{-,\eta}(y) := \begin{cases} -y + \frac{\eta}{2}(e^{\frac{y}{\eta}} - 1) & y \le 0,\\ \frac{\eta}{2}(e^{-\frac{y}{\eta}} - 1) & y > 0, \end{cases} \qquad \Psi_{+,\eta}(y) := \begin{cases} \frac{\eta}{2}(e^{\frac{y}{\eta}} - 1) & y \le 0,\\ y + \frac{\eta}{2}(e^{-\frac{y}{\eta}} - 1) & y > 0, \end{cases}
$$

are smooth approximations of $(\cdot)^-$ and $(\cdot)^+$, respectively, and $m^1(t)*\phi_\delta(\cdot)$, $m^2(t)*\phi_\delta(\cdot)$ are defined as the convolutions of $m^1(t)$ and $m^2(t)$ with $\mathbb{R}^d \ni x \mapsto \phi_\delta(x) = \sqrt{2\pi}\delta \exp(-|x|^2/(2\delta^2)) \in \mathbb{R}$.

When $\nu > 0$ and $\bar{m}_0^\ell$ ($\ell = 1, 2$) are sufficiently regular, the existence of classical solutions to (MFG) can be proved by standard methods (see [1, Theorem 12], where the proof is provided when the space domain in (MFG) is bounded and Neumann boundary conditions are imposed on its boundary).

In order to write (MFG) as (FPK), note that by standard arguments in stochastic control theory (see e.g. [13]) the first and second equations in (MFG) are equivalent to

$$
\begin{aligned}
v^1(x,t) &= \inf_{\alpha_1} \ \mathbb{E}\left(\int_t^T \left[\tfrac{1}{2}|\alpha_1(s)|^2 + V_{\eta,\delta}\left(m^1, m^2, X_1^{x,t,\alpha_1}(s), s\right)\right] \mathrm{d}s\right),\\
v^2(x,t) &= \inf_{\alpha_2} \ \mathbb{E}\left(\int_t^T \left[\tfrac{1}{2}|\alpha_2(s)|^2 + V_{\eta,\delta}\left(m^2, m^1, X_2^{x,t,\alpha_2}(s), s\right)\right] \mathrm{d}s\right),
\end{aligned} \tag{4.6}
$$

where the expectation $\mathbb{E}$ is taken in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which two independent d-dimensional Brownian motion W^1 and W^2 are defined, the $\mathbb{R}^d$-valued processes α_1 and α_2 are adapted to the natural filtration generated by W^1 and W^2, respectively, and they satisfy $\mathbb{E}\left(\int_0^T |\alpha^\ell(t)|^2 \mathrm{d}t\right) < \infty$ ($\ell = 1, 2$). Finally, the processes X_ℓ^{x,t,α_ℓ} ($\ell = 1, 2$) are defined as the unique solutions of

$$
\mathrm{d}X_\ell(s) = \alpha_\ell(s)\mathrm{d}s + \sqrt{2\nu}\mathrm{d}W^\ell(s) \;\; s \in (t,T), \qquad X_\ell(t) = x. \tag{4.7}
$$

By a verification argument (see e.g. [13, Chapter III, Section 8]), the optimal dynamics for the problems defining v^ℓ ($\ell = 1, 2$) are given by the solutions of

$$
\mathrm{d}X_\ell(s) = -\nabla v^\ell(X_\ell(s), s)\mathrm{d}s + \sqrt{2\nu}\mathrm{d}W^\ell(s) \;\; s \in (t,T), \qquad X_\ell(t) = x.
$$

Therefore, redefining $v^\ell : C([0,T]; \mathcal{P}_1(\mathbb{R}^d))^2 \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ as

$$\begin{aligned} v^1(\mu^1,\mu^2,x,t) &= \inf_{\alpha_1} \ \mathbb{E}\left(\int_t^T \left[\tfrac{1}{2}|\alpha_1(s)|^2 + V_{\eta,\delta}\left(\mu^1,\mu^2,X_1^{x,t,\alpha_1}(s),s\right)\right] \mathrm{d}s\right), \\ v^2(\mu^1,\mu^2,x,t) &= \inf_{\alpha_2} \ \mathbb{E}\left(\int_t^T \left[\tfrac{1}{2}|\alpha_2(s)|^2 + V_{\eta,\delta}\left(\mu^2,\mu^1,X_2^{x,t,\alpha_2}(s),s\right)\right] \mathrm{d}s\right), \end{aligned} \tag{4.8}$$

we have that (MFG), with $V_{\eta,\delta}$ instead of V on the right hand side of the first and second equations, is equivalent to (FPK) with $d_1 = r_1 = d_2 = r_2 = d$ and

$$b^\ell(\mu^1,\mu^2,x,t) = -\nabla v^\ell(\mu^1,\mu^2,x,t) \ \text{ and } \ \sigma^\ell(\mu^1,\mu^2,x,t) = \sqrt{2\nu} I_{d\times d}, \tag{4.9}$$

where $I_{d\times d}$ is the $d \times d$ identity matrix. Arguing as in [9] for the one population case, if $\nu > 0$, it is easy to prove that for these drift terms, assumptions **(H)** and **(Lip)** are satisfied.

As (4.6) shows, at the equilibrium (MFG) a typical player of population ℓ minimizes a cost that penalizes its speed, modelled by the quadratic penalization on α^ℓ, as well as a cost depending of its position, and the distribution of his and the other populations. Recalling that $V_{\eta,\delta}$ is an approximation of V, defined in (4.5), the cost $V_{\eta,\delta}$ models a xenophobia effect (the regularization of the first term in V) and penalizes overcrowded regions taking into account the sum of both populations (the regularization of the second term in V).

Note that the coefficients b^ℓ in (4.9) depend on the value functions v^ℓ, which do not admit an explicit expression. Moreover, as (4.8) shows, $b^\ell(\mu^1,\mu^2,x,t)$ depends on the values $(\mu^1(s),\mu^2(s))$ with $s \in (t,T)$, and so the scheme (3.7) is implicit (see Remark 3.2(ii)). In order to obtain an implementable scheme, we approximate b by computable vector fields. More precisely, we use a Semi-Lagrangian scheme to approximate v^1 and v^2, as described in [8] and in Section 5.3 of [9] for the case of a single population. We then call $v^{\ell,\rho,h} : C([0,T]; \mathcal{P}_1(\mathbb{R}^d))^2 \times \mathbb{R}^d \times [0,T] \to \mathbb{R}$ ($\ell = 1,2$) the resulting interpolated discrete value functions and we regularize them by using space convolution

$$v^{\ell,\rho,h,\varepsilon}[\mu^1,\mu^2](\cdot,t) := \phi_\varepsilon * v^{\ell,\rho,h}[\mu](\cdot,t) \quad \forall\, t \in [0,T],$$

where $\phi_\varepsilon(x) = \sqrt{2\varepsilon}\delta \exp(-|x|^2/(2\varepsilon^2))$. Next, we approximate the drifts in (4.9) by

$$b^\ell[\mu^1,\mu^2](x,t) := -\nabla_x v^{\ell,\rho,h,\varepsilon}[\mu^1,\mu^2](x,t).$$

Consider sequences ρ_n, h_n and ε_n converging to 0 as $n \to \infty$ and define (m_n^1, m_n^2) as the sequence in $C([0,T]; \mathcal{P}_1(\mathbb{R}^d))^2$ constructed with the scheme (3.7) and the extension (3.1), by considering discrete characteristics computed with the drifts $b_n^\ell[\mu^1,\mu^2](x,t) := -\nabla_x v^{\ell,\rho_n,h_n,\varepsilon_n}[\mu^1,\mu^2](x,t)$. Then, arguing exactly as in [9, Section 5.3], we can prove that if $\nu > 0$ and $\rho_n^2 = o(h_n)$ and $\rho_n = o(\varepsilon_n)$

and $(\mu_n^1, \mu_n^2) \to (\mu^1, \mu^2)$ in $C([0,T]; \mathcal{P}_1(\mathbb{R}^d))^2$ we have that **(H')** is satisfied. Therefore, we can apply Theorem 3.2 to deduce that (m_n^1, m_n^2) admits at least one limit point (m^1, m^2) and every such limit point solves (FPK). When $\nu = 0$, the situation is more delicate because we need to construct approximations which are absolutely continuous with respect to the Lebesgue measure (see [9, Remark 4.2 and Remark 5.1(ii)]). The resulting scheme is the natural extension of the one proposed in [7] to the multipopulation case. Arguing as in the proof of Theorem 3.12 in [7], we can obtain a convergence result under the additional assumptions that $d = 1$ and $h_n = o(\varepsilon_n)$.

4.2.1 Numerical Tests

As in [1, Section 6.2.1], we solve system (MFG), with V replaced by $V_{\eta,\delta}$, on the one dimensional space domain $\Omega = [-0.5, 0.5]$. We set the final time $T = 4$ and we consider homogeneous Neumann boundary conditions. The initial densities are given by

$$m^1(x,0) = 3/4 + 1/2\mathbb{I}_{[-1/2,-1/4]\cup[0,1/4]}(x) \quad \text{and} \quad m^2(x,0) = 3/4 + 1/2\mathbb{I}_{[-1/4,0]\cup[1/4,1/2]}(x),$$

where for $A \subseteq \mathbb{R}$, $\mathbb{I}_A(x) = 1$ if $x \in A$ and $\mathbb{I}_A(x) = 0$, otherwise. We choose $\rho = 0.02$ and $h = \rho^{\frac{3}{2}}$. The regularizing parameters are set to $\delta = \varepsilon = 0.025$ and $\eta = 10^{-5}$.

In order to compute the solution of the fully discrete system, we have used the learning procedure proposed in [5] in the continuous framework. We point out that a rigorous study of the convergence of this method for the resolution of discretizations of MFG systems has not been established yet and remains as an interesting research subject. We stop the procedure when the difference between two successive discrete densities, measured in the maximum discrete norm, is smaller than 5×10^{-3}.

Due to the symmetry of the initial conditions and to the form of the coupling terms, the evolutions of the two populations are symmetric to each other. This symmetry can be observed in all the simulations. We also observe that the evolutions present a *turnpike* property since most of the time after and before the $t = 0$ and $t = T = 4$, respectively, the distribution is near a stationary configuration.

In Fig. 3, computed with $\nu = 0.05$, we show the evolution of the two densities at the times $t = 0, 0.1, 0.5, 2, 3, 4$. We can observe that the two densities separate from each other, with only a small overlap region at the end. We also observe that the configurations at times $t = 2$ and $t = 3$ have the same shape, which is near a stationary configuration (see [1, Section 6.1]). For this viscosity parameter, our results are almost identical to those in [1, Section 6.2.1, Figure 8].

In Fig. 4, computed with $\nu = 0.001$, we show the configuration at the times $t = 0, 0.1, 0.2, 1, 2, 4$. The two densities separate faster than the previous case, reaching a nearly steady-state solution already at time $t = 1$. We can observe that the resulting segregated configurations differ considerably from the previous case, computed with $\nu = 0.05$.

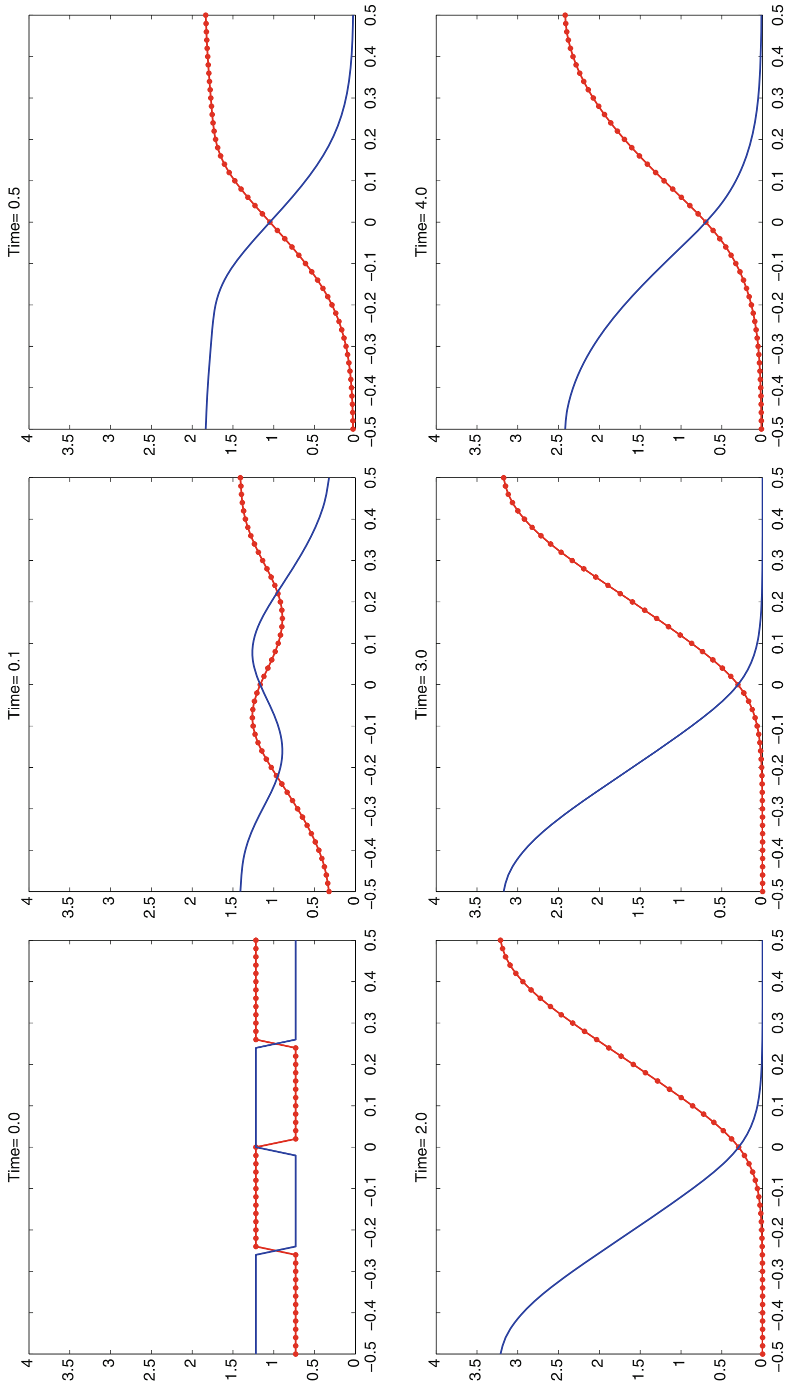

Fig. 3 Evolution of $\mathbf{m}^1_{\rho,h}$ (*blue line*) and $\mathbf{m}^2_{\rho,h}$ (*red line with asterisk*) at the time $t = 0, 0.1, 0.5, 2, 3, 4$ computed with $\nu = 0.05$

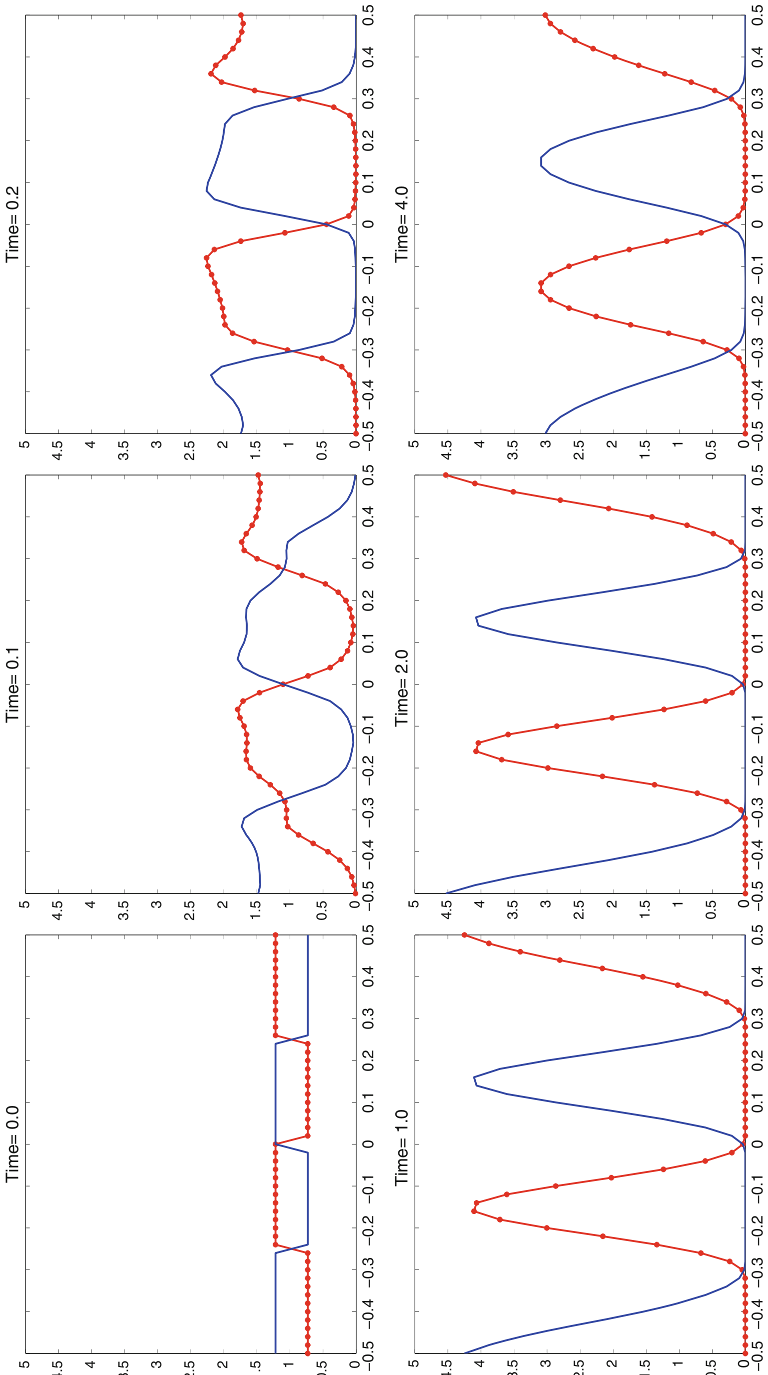

Fig. 4 Evolution of $\mathbf{m}^1_{\rho,h}$ (*blue line*) and $\mathbf{m}^2_{\rho,h}$ (*red line with asterisk*) at the times $t = 0, 0.1, 0.2, 1, 2, 4$ with $\nu = 0.001$

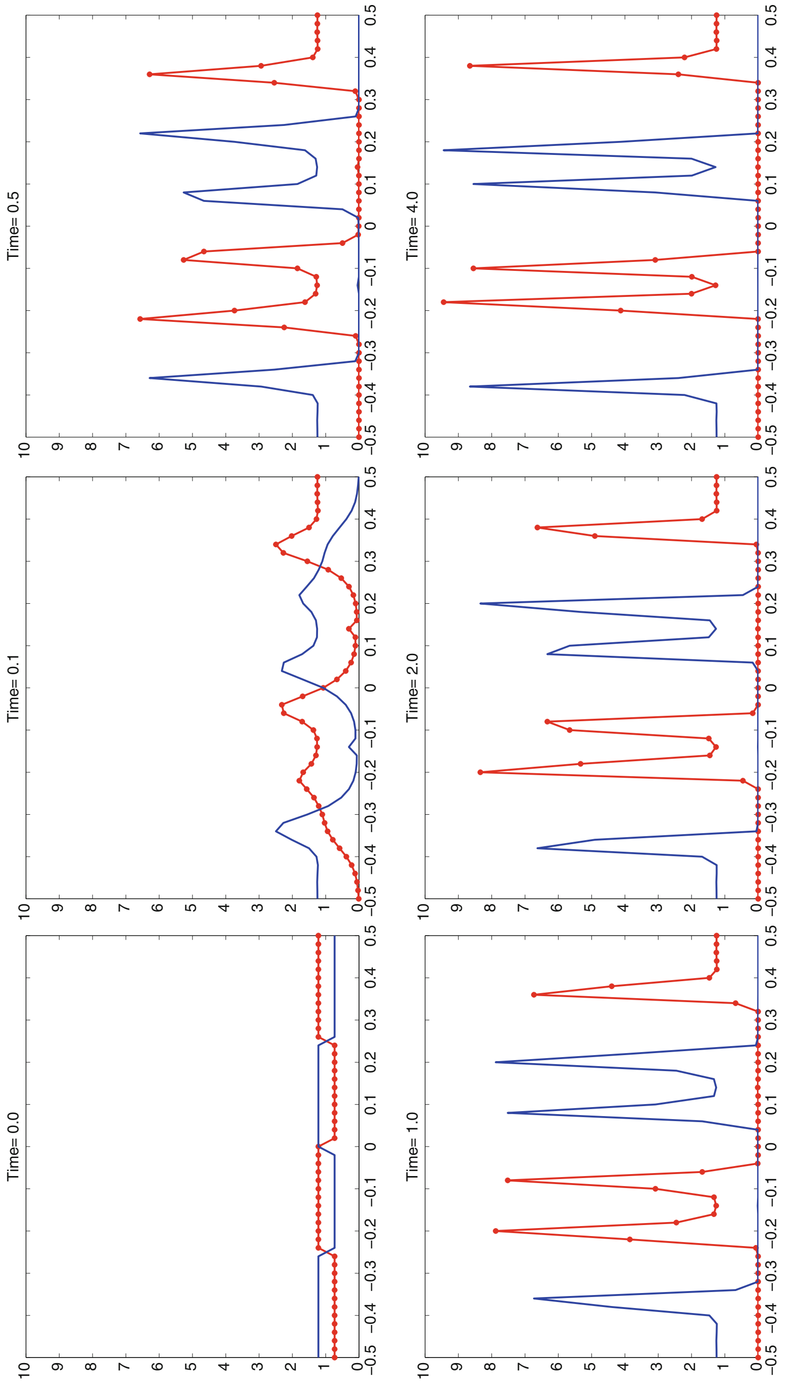

Fig. 5 Evolution of $m^1_{\rho,h}$ (*blue line*) and $m^2_{\rho,h}$ (*red line with asterisk*) at the time $t = 0, 0.1, 0.5, 1, 2, 4$ computed with $\nu = 0$

In Fig. 5, computed with $\nu = 0$, we show the configuration of the two measures at the times $t = 0, 0.1, 0.5, 1, 2, 4$. As expected in the deterministic case, the evolution is much less smooth. Compared to the diffusive cases, at the final time T, the supports of the densities $\mathbf{m}^1_{\rho,h}$ and $\mathbf{m}^2_{\rho,h}$ are disjoint and separated by much larger sets. We insist that, for the previous and the current tests, the solutions captured by the scheme differ importantly from the ones computed with larger viscosity parameters (see Fig. 3 and [1, Section 6.2.1]).

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