Chapter 5
The Impact of Teaching Mathematics Upon the Development of Mathematical Practices

Gert Schubring

Abstract This chapter discusses interfaces between the development of mathematics and the teaching of mathematics. Contrary to traditional convictions of teaching as being restricted to a receptive and passive role, productive interactions between the two poles are analysed here. Four cases even for an impact of teaching upon mathematical practices will be presented and discussed, featuring the issue of elements and elementarisation, the institutional impact of teacher education on research in pure mathematics, and the dissemination of set theory and of non-Euclidean geometry by German school textbooks in the second half of the nineteenth century.

Keywords D'Alembert · Destutt de Tracy · Elements · Elementarisation · Friedrich Meyer, Hermann Wagner · Interfaces · Non-Euclidean geometry · Richard Dedekind · Set theory · Teacher education

1 Introduction: Issues of Methodology

Traditionally, in mathematics and historiography, the teaching of mathematics has been seen as having no influence on mathematical practices and their development. The contents of teaching are seen as a certain kind of projection of academic mathematics, as a certain sedimentation. Therefore, the relation between the development of mathematical practices and the teaching of mathematics is often conceived of as unilateral, without an impact of teaching upon research. This chapter undertakes it to show that there are productive interactions between the two poles. Four cases for an impact of teaching upon mathematical practices will be presented and discussed. While the first one will discuss the importance of the notion of element and elementarisation in the interface between mathematical development and teaching, and the

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second will discuss the impact of teacher education for research profiles, the third and the fourth will discuss conceptual developments of nineteenth century mathematics.

A paradigmatic case for the traditional position is the French mathematics educator Yves Chevallard who has made it the foundation of a theoretical generalisation, elaborated as *transposition didactique* and widely influential in mathematics education. The conception of the didactic transposition proposes to examine how academic knowledge of mathematics ("savoir savant") becomes school mathematical knowledge. For this, Chevallard distinguished three types of knowledge:

– "Objet de savoir"—object of knowledge, i.e. the knowledge achieved by mathematics.
– "Objet d'enseigner"—subject to be taught: The academic knowledge becomes teachable knowledge by the efforts of mathematics educators (their community being called "noosphère").
– "Objet d'enseignement"—teaching subject: The subject to be taught as adapted by teachers (Chevallard 1985, 39).

Analysing this conception, it becomes clear that the transposition notion offered conceives of a unilateral process: it has as its starting point, a pole designed as advanced, the academic or university knowledge and as its final point another pole inferior to it, occurring in schools and involving the teacher in the classroom.

Willem Kuyk—the author of "Complementarity in mathematics" (1977)—, however, had denounced this traditional view in 1978, in stating: "Mathematics is not a stalactite hanging over a stalagmite"; Kuyk thus denied the view that mathematics education grows only by receiving some drops from above, from the supreme instance (Schubring 2001, p. 297).

A historiographical endeavour where one might expect a reflection about the interfaces between mathematical research and the teaching of mathematics is the monumental work *Writing the History of Mathematics*, edited by Joseph Dauben and Christoph Scriba in 2002, where the historiography of mathematics is analysed in a most comprehensive chronological and geographical manner. Yet, given the fact that historiography of mathematics had largely been written by mathematicians, historiography followed essentially the preoccupations of mathematicians "with respect to chronology and where questions about priorities and the actual sequence of internal mathematical developments are concerned", given their primary interest "in the history of concepts and methods" (Dauben and Christoph 2002, p. xxiv).

Thus, although the editors asked, "are there any general historiographic principles that emerge from these studies, ones that seem to transcend time and national boundaries?" (ibid., p. xxiii), the study does not go beyond what the respective analysed authors had elaborated from their traditional viewpoint. Being descriptive, the volume documents that historiography was practised until very recently by mathematicians—with the notable exception of France, where philosophers and *épistémologues* took the lead in the twentieth century—and that their focus was on the internal history of ideas. The professionalisation of historiography dates from recent times. While the section "History of Mathematics and Mathematics Education" in the *Postscriptum* might have addressed new functions of mathematics education, it remains restricted, however, to the use of history in teaching mathematics. The following section "History of Mathematics: Recent Trends" does not address interfaces between research and teaching (ibid., pp. 335 ff.).

While this volume documents that mathematics historiography is still strongly marked by the opposition between "internal" and "external" approaches, a new German *Handbuch Wissenschaftsgeschichte* of 2017 declares this dispute as overcome and is open to much broader conceptual approaches, understanding science as just one form of knowledge—history of science being hence a part of *Wissensgeschichte*, the history of knowledge (Sommer et al. 2017, p. 3). In fact, this handbook realised an ambitious endeavour to reflect the methodology of history of science research; it presents, in particular, systematic chapters on recent research approaches. Pertinent for research on our issue of interfaces is the chapter on cultural sciences and science history (Brandt 2017). Likewise, the series of chapters on places of knowledge production is novel.

2 Examples for Introducing the Interface Approach

To give the first piece of evidence for the productive role of teaching: as is well known, Dedekind emphasised in the preface of his book *Stetigkeit und irrationale Zahlen* (1872), which became decisive for establishing a rigorous concept of real numbers, that it was his experience in teaching the infinitesimal calculus for the first time at the *Eidgenössische Technische Hochschule* (ETH) Zürich, in 1858, that made him conscious of the missing fundaments for the number concept (see Fig. 5.1):

> The reflections which form the subject of this little work date from the autumn of 1858. At that time, as a professor at the Swiss Polytechnic in Zurich, I was in a position to lecture the elements of differential calculus for the first time, and felt more sensitive than ever before to the lack of a truly scientific justification of arithmetic. Regarding the concept of a variable quantity approaching a fixed limit, and especially in the proof of the proposition that every quantity which grows steadily, but not beyond all limits, must certainly approach a limit, I resorted to geometrical evidence (Dedekind 1872, p. 1).[1]

To add a second piece of evidence: Belhoste recalled that the project which initiated the collective work of the Bourbaki group in the 1930s was to elaborate an analysis textbook: it was intended to be more modern in particular than the textbook by Édouard Goursat, *Cours d'analyse mathématique*, first published in 1902 and dominant in France since the early twentieth century (Belhoste 1998, p. 300). In beginning this initially restricted task, the group was lead to search for the

[1] Die Betrachtungen, welche den Gegenstand dieser kleinen Schrift bilden, stammen aus dem Herbst des Jahres 1858. Ich befand mich damals als Professor am eidgenössischen Polytechnikum zu Zürich zum erste Male in der Lage, die Elemente der Differentialrechnung vortragen zu müssen, und fühlte dabei empfindlicher als jemals früher den Mangel einer wirklich wissenschaftlichen Begründung der Arithmetik. Bei dem Begriffe der Annäherung einer veränderlichen Größe an einen festen Grenzwerth und namentlich bei dem Beweise des Satzes, daß jede Größe, welche beständig, aber nicht über alle Grenzen wächst, sich gewiß einem Grenzwerth nähern muß, nahm ich meine Zuflucht zu geometrischen Evidenzen.

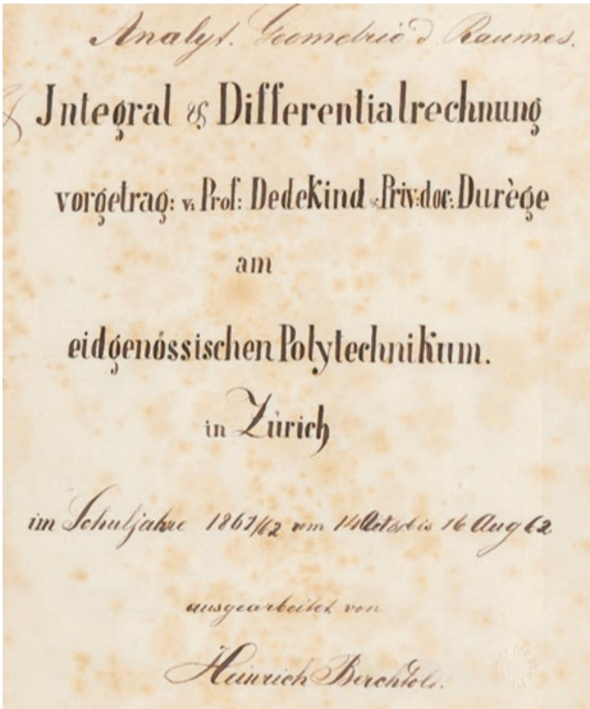

Fig. 5.1 Lecture notes of Dedekind's differential calculus lectures, taken by the student Heinrich Berchtold in the winter term 1861/62 (the next term, due to Dedekind's move to Braunschweig, Heinrich Durège continued the course). ETH-Bibliothek

foundations of analysis—so that their focus became to elaborate the textbook on set theory as the first *fascicule*. This turned out to be so complicated and challenging that this volume, *Théorie des ensembles*, took a long time to be ready for publishing. Bourbaki's search for a rigorous presentation of analysis had thus even more profound and comprehensive outcomes than Dedekind's search—as evidenced by the common title of Bourbaki's work: *Éléments de mathématique*. Both historically significant examples reveal us the key function of the notion of *element*.

3 The Notion of Element and of Elementarisation

In fact, the notion of element connects the development of mathematics and the modes of teaching mathematics in a fundamental way. Since Euclid's geometry textbook, the term "elements" expresses the intention to give a systematic presentation of a mathematical theory, constructed from its basic components (see Trouvé 2008, pp. 21 ff.). While thus fixing the state of knowledge of mathematics or of one of its branches for a certain time and period, a textbook represents a stage in the development of mathematics. At the same time, such a textbook provides the material for teaching mathematics.

Remarkably, it was in France that the notion of elements and its role for textbooks and in particular for elementarising knowledge was reflected most explicitly. The reflections began practically right with Modern Times with criticism of how Euclid elementarised mathematics: Pierre de la Ramée or Petrus Ramus (1515–1572) refuted Euclid's *Elements* as the model of a rigorous and methodical presentation of mathematics. Ramus did not just criticise particular propositions or the exactitude or rigour of certain statements, but much more fundamentally their methodology. In Ramus's view, the Elements were not, as traditionally judged, the primordial model for rigorous reasoning and for logical deduction, but rather revealed a lack of a natural, methodical order. Ramus, on the one hand, developed rules for methodical thought, and on the other hand proposed an entirely different order and architecture for mathematics: that it should begin with the general—the general being, in Ramus's view, not geometry, but arithmetic. In addition, arithmetic and geometry should be treated first separately and then combined (Ramus 1569).

Ramus's approach was continued and perfected by a new type of textbook that realised his methodological conceptions: by Antoine Arnauld (1612–1694), a Jansenist philosopher and theologian. Arnauld dared, for the first time, to challenge Euclid's model by claiming to be able to realise an alternative and better model, the title of his textbook is already emblematic and programmatic: ***Nouveaux** élémens de géométrie* (1667), with the subtitle "contenant Outre un ordre tout nouveau, & de nouvelles demonstrations des propositions les plus communes" expressing this ambition.

The reflections on elementarisation were taken up and deepened by d'Alembert in the Encyclopédie, as an essential part of the Enlightenment programme to make knowledge generally accessible. There is an extensive entry in the Encyclopédie, "élémens des sciences", where d'Alembert published these reflections. He started from a first, rough distinction, calling "elements" the first and original components of a whole:

> On appelle en général *élémens d'un tout*, les parties primitives & originaires dont on peut supposer que ce tout est formé. (d'Alembert 1755, col. 491, l)

According to him, it would be easy to identify these original parts, which serve as basis:

> il est facile de distinguer les propositions ou vérités générales qui servent de base aux autres, & dans lesquelles celles-ci sont implicitement renfermées. (d'Alembert 1755, col. 491r)

That the other, more developed concepts would be implicitly enclosed in the basic ones reveals d'Alembert's specific conception of elementarisation, since he continued:

> Ces propositions réunies en un corps, formeront, à proprement parler, les élémens de la science, puisque ces *élémens* seront comme un germe qu'il suffiroit de développer pour connoitre les objets de la science fort en détail.[2] (ibid.)

The key term here is "germ". And this biological analogy means that the element is a kind of nucleus, which already contains all possibilities of unfolding, of development. Its unfolding will hence result in a coherent structure. It is in this sense that Bourbaki used to speak of the architecture of mathematics (see Bourbaki 1948). "Elementarising", therefore, means to expose a mathematical theory as structured and built from its elements, understood in this way.

D'Alembert expressed the characteristic optimistic vision of the Enlightenment that, by this elementarisation, knowledge can be universally disseminated and understood:

> Tout ce qui est vrai, surtout dans les sciences de pur raisonnement, a toûjours des principes clairs & sensibles, & par conséquent peut être mis à la portée de tout le monde sans aucune obscurité (d'Alembert 1755, col. 492r).[3]

D'Alembert thus launched the conception of *livres élémentaires*, intended to be the primary preoccupation for education in the first stages of the French Revolution. D'Alembert had called on scientists to elaborate these textbooks, criticising that they so far preferred to strive for their personal fame:

> Uniquement occupés de faire de nouveaux progrès dans l'art, pour s'élever, s'il leur est possible, au-dessus de leurs prédécesseurs ou de leurs contemporains, & plus jaloux de l'admiration que de la reconnoissance publique, ils ne pensent qu'à découvrir & à jouir, & préfèrent la gloire d'augmenter l'édifice au soin d'en éclairer l'entrée (d'Alembert 1755, col. 496r).[4]

This conception of *livres élémentaires* became adopted during the first stages of the French Revolution; one of the first plans for a system of public education, by Talleyrand, postulated:

> Il faut […] que des livres élémentaires […] rendent universellement familières toutes les verités (quoted from Schubring 1988, p. 160).[5]

The first projects for a new educational system were in fact based on elaborating livres élémentaires. In 1792, L. F. A. Arbogast proposed a *concours* for composing such textbooks for the disciplines of the primary schools to be created. He emphasised the urgency in order to have the new books before opening the schools:

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[2] These propositions united in a body will, properly speaking, form the elements of science, since these elements will be like a germ from which it would be sufficient to develop knowledge of the objects of science in great detail.

[3] All that is true, especially in the sciences of pure reasoning, always has clear and sensible principles, and consequently can be made accessible to everyone without any obscurity.

[4] Only occupied with making new progress in their science, in order to rise, if possible, above their predecessors or their contemporaries, and more jealous of admiration than of public recognition, they intend only to discover and enjoy, and prefer the glory of increasing the building of science rather than take care to light its entrance.

[5] It is necessary [...] that *livres élémentaires* [...] turn all truths universally familiar.

> le moyen le plus efficace pour la régénération de l'enseignement, c'est la composition des livres élémentaires. Il étoit de la plus grande urgence [...] de hâter la composition de ces ouvrages (Arbogast 1792, p. 2).

Yet, due to political problems, it took until 18 January of 1794 for the *concours* to be decided by Parliament; within five months, manuscripts for those *livres* were to be submitted for the ten projected teaching subjects in the primary schools. However, the process of composing proved to be much longer. The final evaluation occurred only after one and a half year. And the results were disappointing: for all the ten disciplines, only seven manuscripts were judged to be qualified. Already in October 1794, Joseph Lakanal gave an intermediary evaluation of the *concours* process. In criticising the conception of many submitted manuscripts, he confirmed and elaborated d'Alembert's conception of elementarisation:

> qui avaient confondus généralement deux objets très différents, des *élémentaires* avec des *abregés*. Resserrer, coarcter un long ouvrage, c'est l'abréger; présenter les *premiers germes* et en quelque sorte la *matrice* d'une science, c'est l'élémenter: ainsi, l'abrégé, c'est précisement l'opposé de l'élémentaire (quoted from Schubring 1988, p. 161).[6]

Noteworthy in particular is the opposition between an abridged handbook and a truly elementarised textbook, characterised here not only by the term "germ", but also by "matrix".

An even more revealing result of the experience with this first *concours* for textbooks for public schools proved to be a deepened understanding of the inter-relation between research and teaching. It was the French philosopher Destutt de Tracy (1754-1836), one of the leading *idéologues*—the then influential French group of philosophers—who evaluated in 1801 the project to elaborate the "livres élémentaires", meant to be the basis for this profoundly new type of teaching. Among the various reasons for the few results of this effort, Destutt de Tracy had outlined that composing a textbook frequently leads the author to tasks of research:

> Often, in exposing a fact, one remarks that this requires new observations, and, when examined more thoroughly, it presents itself in a completely different light: on other occasions it is the principles themselves which have to be revised, or, to connect them with each other, there are many gaps to be filled; in a word, it is not only a question of exposing the truth, but of discovering it (Destutt de Tracy 1801, pp. 4–5; my transl., G.S.).[7]

This assessment of the historical experience reveals a decisive pattern for the interface between the development of mathematics and its teaching: upon preparing teaching—either as oral lecture or as written textbook—one will become aware of missing connections in a logical deduction or remark on problems in the

[6] who had generally confounded two very different objects, the elementary with abbreviated ones. To constrict, to coarct a long work, is to shorten it; to present the first germs and, in a way, the matrix of a science is to elementarise it: thus, the abridged is exactly the opposite of the elementary.

[7] Souvent, en rendant compte d'un fait, on s'aperçoit qu'il exige de nouvelles observations, et, mieux examiné, il se présente sous un tout autre aspect: d'autres fois, ce sont les principes eux-mêmes qui sont à refaire, ou, pour les lier entre eux, il y a beaucoup de lacunes à remplir; en un mot, il ne s'agit pas seulement d'exposer la vérité, mais de la découvrir.

foundations of the theory so that one is incited to research for providing the needed conceptions.

It is likewise characteristic that, in this context of reflection about the elementarisation of science, the role of the textbook author also became investigated and even credited. While the share of textbook composition in establishing the elements of science was valued, the textbook author was also assessed in his productive contribution to science. A first such crediting was published in 1796, in a review of the second edition of J.A.J. Cousin's calculus textbook: *Leçons de Calcul Différentiel et de Calcul Intégral* (1796). The review was published in *La Décade*, the journal of the *idéologues*. Its anonymous author assumed the novel stance of attributing to a textbook author the rank of "inventor"—a notion in the discourse on science, that designated an innovative scientist since Clairaut and d'Alembert:

> The author of an elementary book attains the rank of an inventor if he can present the elements, first, in the best order, in the most simple and the most clear manner: if he removes from the science all its technical wrapping and if he illustrates after each step the space traversed in such a manner that the student always knows well where he is (quoted from Schubring 1987, p. 43).

And Sylvestre-François Lacroix (1765–1843), the prolific and successful textbook author since the first periods of the French Revolution, was distinguished even by the *Institut*—the new form of the Academy of Sciences since the Revolution—in being attributed a rank equal to an inventor. The distinction had been given in the *Institut's* report on the project presented by Lacroix to publish a treatise on the differential and integral calculus. In fact, he published this treatise as a three volumes textbook from 1797 to 1799. The report explained:

> To present difficult theories with clarity, to connect them with other known theories, to dismantle some of the systematic or erroneous parts which might have obscured them at the time of their emergence, to spread an equal degree of enlightenment and precision over the whole; or, put shortly: to produce a book which is at the same time elementary and up to the mark in science. This is the objective which Citizen Lacroix has taken to himself and which he could not have attained without engaging himself in profound research and by progressing often at the same level as the inventors (quoted from: ibid.).

This assessment, made still in an Enlightenment period, expresses in a paradigmatic manner the programme of elementarisation and the interface between research and teaching.

4 The Impact of Teacher Education

Recent research upon the social history of mathematics confirms the decisive role of teaching for the progress of research practices. In fact, it was the establishing of study courses for mathematics teacher training in higher education which proved to constitute the predominant structural pattern, initiating for the first time within universities the enabling of research activities for the professors performing the lectures and supervisions for this teacher education study course.

Before the French Revolution, mathematics basically could not be studied for obtaining a degree in mathematics. Lectures by mathematics professors in the Arts Faculty had either a propaedeutic character, preparing for professional studies (and degrees) in one of the three higher faculties, or were encyclopaedic, for a broad, non-specialised use. The first time that proper study courses were established for mathematics occurred as a part of Marquis de Pombal's profound university reforms of Coimbra University in Portugal from 1772: not only was the first Mathematics Faculty created then, but likewise a study course, leading to degrees which should assure its graduates the best teaching positions in the country (Silva 1991).

Admittedly, there were only few graduates of these study courses, but one of the first graduates was Francisco de Borja Garçao Stockler (1759–1829) who published important research about the fundaments of analysis from 1794.

The profound reforms of the educational system in Prussia had a much more far-reaching effect from 1806 onward: the Philosophy Faculty became upgraded, providing for the first time proper degrees for professions—namely and noteworthy for the teaching profession. The mathematics teaching profession was included, since mathematics became one of the three major teaching disciplines at the likewise reformed secondary schools, the *Gymnasien*. Within two decades, the profile of the mathematics professors at the Prussian universities changed completely: the formerly encyclopaedic lectures became replaced by specialised high-level lectures and the professors themselves became specialised researchers—in marked contrast to the other German states where the traditional patterns were continued until up to the middle of the nineteenth century (Schubring 1991a).

The *facultas* degree for teaching at secondary schools remained the only degree throughout the entire nineteenth century, which could be obtained by studying mathematics—the same period, which is renowned and famous for the establishment of the new era of rigour by German mathematics! A second degree—the diploma for applying mathematics in other professions and for higher education careers—became established only in 1942. The doctorate as a degree independent of the teaching profession and leading to university careers had been sought for and achieved only by a few students since about the late 1860s (Schubring 1990).

This key role of teacher training for professionalising mathematical research and constituting mathematical communities is even confirmed by a more recent example from Brazil. Upon the establishment of higher education in Brazil in 1810, mathematics could not be studied as a proper study course, but the mathematics lectures functioned as service courses for engineering professions at military academies and polytechnic schools. No universities were founded throughout the nineteenth century, due to the model function of the French higher education structure of *écoles spéciales* (Schubring 1991b). Universities were founded in Brazil only from the 1930s on, due to changed social and cultural conditions. And then, in the first two universities—the *Universidade de São Paulo* (USP) and *Universidade do Distrito Federal* (UDF), resp. the *Universidade do Brasil* (in Rio de Janeiro) —it was the study course for the *magistério*, the teaching profession, within the equivalent of a Philosophy Faculty, which enabled a "take-off" of practising mathematical research (Pereira 2017).

The first university to be founded was the USP, in 1934. Its distinctive new feature was a Faculty, which basically resembled the German Faculties of Philosophy: the FFCL—*Faculdade de Filosofia, Ciências e Letras*—which constituted in fact the kernel of disciplinary development. The founding decree of the USP, of 25 January 1934, in art. 5, § 1, stipulated the introduction of the teaching licence for those trained to become teachers at secondary schools as the "*licença para o magistério secundário*". The degree afforded studies of a scientific discipline at the FFCL and accompanying pedagogical studies at the Institute of Education, attached to the Faculty. It is even more revealing that the statutes projected doctoral studies; for such studies, only students having the *licenciado* diploma were mentioned to be admitted for an additional two years of studies (§ 12 of the decree).[8] Hence, a direct continuation was established: studying for a teaching licence, and possible continuation for a doctorate.

At the UDF, founded in 1935, here, too, there was a new Faculty besides the integration of various former professional schools, like the polytechnic schools, which was at first called *Escola de Ciências*. It had as its principal function the formation of teachers for secondary schools. The § 25 of the founding statutes, of 5 April 1935, attributed the function of providing study courses for the "candidato ao professorado secundário das ciências" in four different courses: for teachers of mathematics, physics, chemistry, and natural sciences. Doctoral studies were not yet instituted (Fávero and de Castro Lopes 2009, pp. 193–194).

5 An Early Teaching of Set Theory in Germany

In 1885, when Georg Cantor was still perfecting his set theory providing new fundaments for mathematics, Friedrich Meyer (1842–1898) —friend of Cantor and mathematics teacher at the Gymnasium in Halle—elaborated a schoolbook on arithmetic and algebra, as reorganised from this basis in set theory.

The fact that the transposition of new knowledge into school knowledge does not necessarily take a path through the scientific community is shown by set theory, which is regarded as the key example of imposing scientific concepts into school teaching: to my own surprise, in my research on the development of school mathematics in the nineteenth century, I encountered a textbook which was not only the first implementation of Cantor's set theory, but which also propagated a radical reconstruction of arithmetic and algebra for schools on the basis of the concepts of set theory, seventy years before the corresponding effect of Bourbaki on school mathematics. It is the book by Friedrich Meyer: *Elemente der Arithmetik und Algebra*, of 1885.

[8] Source: https://www.al.sp.gov.br/repositorio/legislacao/decreto/1934/decreto-6283-25.01.1934.html. I am grateful to Prof. Rogério Monteiro de Siqueira (USP, Sao Paulo) for communicating me these sources.

Meyer was born near Kulm in East-Prussia in 1842, and he completed the Gymnasium in Kulm and studied in Breslau and Halle, but above all at the University of Berlin, mainly with the number theorist Ernst Eduard Kummer. In 1868 he became a teacher of mathematics at the Gymnasium in Halle, where he worked as a highly respected and highly renowned scholar and educator until his early death in 1898. His extensive scholarship was praised in particular, even beyond mathematics and the natural sciences (Hoffmann 1899). In 1894, he received an honorary doctorate from the University of Halle, especially because of his set theory textbook.[9] Cantor himself greatly appreciated this book and recommended it especially to mathematics teachers (Hoffmann 1899).[10] Wilhelm Lorey, known both as a historian of mathematics and of mathematics teaching, he emphasised the importance of this book in his address for the celebration of Cantor's 70th birthday:

> [Meyer] was one of the first to recognize the far-reaching significance of your ideas. In a time when the scientific world was opposed to you, also in our own country, he had already presented the basic concepts of set theory in his textbook destined for schools, and in the foreword he recommended the study of your writings intensely to the teachers of mathematics (Lorey 1915, p. 273).

Meyer's achievement is all the more significant as Cantor's ideas of set theory were not yet completely elaborated in 1885; in their most elaborate form, they were only published in 1895/96 (important parts were accessible since 1883). It turns out that Cantor's ideas were for Meyer, in effect, only a trigger for developing fundamental concepts that had already been developed by mathematics teachers for a long time. In fact, Meyer did not present the concept of a set as something new, but as belonging to a tradition going back to the ancient Greeks; he referred in particular to Nikomachus (about 100 CE).

I have also shown earlier that Johann Schultz (1739–1805) used the set concept for his infinity concept in 1788, and especially for his attempts to prove the 11th postulate on parallel lines. The notion of set ("Menge") was for him well known and used abundantly for conceiving of infinite sets of numbers, and Schultz developed the number concept based on the set concept (Schubring, 1982). Moreover, in the 1810s and 1820s in Germany, when the programme of algebraisation was still in practice and not yet substituted by the return to valuing synthetic geometry, school textbooks existed that constructed arithmetic and the number concept from the basis of the set notion. Two such examples are the arithmetic textbooks by Mathias Metternich (1818), in Hesse, and by Carl Seebold (1821), in Hesse (see Schubring 1991a, b, p. 190).

Meyers's book is also particularly interesting as a dissemination of Cantor's concepts. Walter Purkert was able to show that Cantor was not surprised by the antinomies of set theory because they had been known to him for a long time and because he had assumed that he had excluded inconsistent multiplicities through his

[9] Information from the archives of the University of Halle-Wittenberg.

[10] Yet, in a letter to the Swedish mathematician Ivar Bendixson with whom he was cooperating on set theory, Cantor expressed some doubts regarding the rigour of Meyer's proofs in this schoolbook (Purkert & Ilgauds 1987, p. 132).

definitions of concepts. In letters to Hilbert, Cantor explained that his 1895 definition of a set (summary of certain well-defined objects […] to a whole) served the function of excluding inconsistent sets, and already in his formulation of 1883:

> "Every Multiplicity which one is able to think of as a One" (Purkert 1986, pp. 18ff.; also Purkert & IIgauds 1987).

Remarkably, Meyer also adopted this 1883 definition of a set in his textbook, saying: "Im Begriffe der Menge wird vieles zu einem verbunden" (Meyer 1885, p. 1).[11]

The 1885 edition is given as the second edition, but it was not possible to find a printed first edition; probably, it had circulated only as a manuscript among Meyer's colleagues. In fact, Meyer presented his book as serving for cooperation between the mathematics teachers of his Gymnasium and the school's students for repetition of the teaching in the classroom (ibid., p. iii). From various indications, it becomes clear that Meyer had used it in the upper grades (e.g. Meyer 1891, p. 29).

The preface begins with a rather epistemological discussion. The basis of introducing set theory is the notion of "Anzahl",[12] which has according to Meyer the status of a category—in the Kantian sense, like space and time, thus as a given dimension of thinking. By contrast, all other numbers of elementary arithmetic not being an "Anzahl", i.e. not positive integer numbers, are qualified as "inventions", due to the capacity of equations to generate new number types. At the end of the preface, Meyer strongly recommended the study of Cantor's publications, referring in particular to those collected in Volume II (December 1883) of the Swedish journal *Acta Mathematica*.

The first chapter on *Anzahl, Zählen. Zahlzeichen* introduces set as first notion, being presented as likewise primary, like time and space. As its first characteristic, the notion of *Mächtigkeit*—potency—is presented, discussing sets of equal and unequal potency and finite and infinite sets (Meyer 1885, p. 1). Though a finite set admits an ordering, not only an ordered set is defined, but also an "wohlgeordnet", a well-ordered set (ibid., p. 2). The notion of denumerability follows immediately, by determining two well-ordered sets as denumerably related when each element of the one can be related to one of the other. After this, the definition of "Anzahl" is introduced as a general concept or category, comprising well-ordered mutually denumerable sets (ibid., p. 3). Propositions on well-ordered denumerable sets follow; in particular, complete mathematical induction is presented. The number concept is then derived from the concept of *Anzahl* and *Mächtigkeit*. At first, the signs of the first nine "Anzahlen" are explained and, then, how to count the elements of a

[11] In the concept of set a Multiplicity is connected to a One.

[12] Actually, the English language has no translation for "Anzahl" that would distinguish it from "number" for "Zahl". Dictionaries only give "number". Joseph W. Dauben, in his publications about the history of Cantor's set theory, uses "numbering". He draw my attention to a paper by W. W. Tait which relates controversies about an adapted English translation of *Anzahl*: "counting number" versus "enumeral" (Tait 2000, p. 275). Another translation of Cantor's works uses to put just "*number (Anzahl)*", or simply "number"—in italics—, as in the translation of Cantor's treatise on *Grundlagen der Mannigfaltigkeitslehre* of 1883, by Uwe Papart, in *The Campaigner*, vol. 9, no.s 1 & 2. I will use here the German term, in italics.

finite set. The introduction of ordinal and cardinal numbers follows (ibid., p. 6). To operate with the numbers, the signs for "greater than" and "less than" are introduced—and, to assure generality not for concrete numbers but only for general signs of numbers, namely for "letter numbers" *a, b, c, … x, y, z*. The textbook shows to be an axiomatically structured textbook, which is already quite modern. Thus, in operating with numbers, one finds them presented via the axioms of identity, commutativity, associativity, and distributivity (ibid., p. 8 & 22).

The next chapter is devoted to the operations of the first kind, adding and subtracting. Here the domain of "Anzahl", of positive integers, is extended to relative numbers, i.e. to positive and negative integers and to zero, by extending the operations of addition and subtraction already introduced (ibid., pp. 16 ff.). Another chapter introduces multiplication and division as operations of the second kind, for this extended domain. The product is defined here in set theory terms:

> § 27. Sind *a* und *b* Anzahlen, ist ferner $\mathfrak{A}'$, $\mathfrak{A}''$, $\mathfrak{A}'''$, … eine wohlgeordnete Menge von der Anzahl *b*, während jedes der Elemente *A* selbst eine wohlgeordnete Menge von der Anzahl *a* ist, so entsteht durch Auflösung eines jeden $\mathfrak{A}$ in seine Elemente wiederum eine wohlgeordnete Menge, deren Anzahl das Produkt aus *a* und *b* genannt und durch *ab* […] bezeichnet wird (ibid., p. 20).[13]

Then, the extensions to rational and to irrational numbers are described before decimal fractions. In another part, series of numbers are introduced to be continuous in order to use them to introduce real numbers. A third part deals with the operations of the third kind: to exponentiate, square root and logarithmise. Further chapters are: imaginary and complex numbers, theory of equations, theory of permutations and combinations, progressions, theorems from number theory, and finally continued fractions—actually quite demanding subjects.

An examination of the impact of Meyer's textbook also produces remarkable results. At first the book seems to have left his colleagues speechless, for in the three major journals—the *Archiv für Mathematics und Physik*, the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht* and the *Zeitschrift für Mathematik und Physik*—which constantly published reviews of a large number of mathematics textbooks, no review of Meyer's book appeared. Eventually, however, the book had not only found the recognition of leading specialists, for instance by Max Simon, but also provided ready access to Cantor's set theory for mathematics teachers (Lietzmann 1909, p. 61). Lietzmann, the leading mathematics educator in Germany since the 1910s, was sceptical as to whether set theory could be taught as the fundament for arithmetic teaching, as is clear from his remark:

> Unfortunately, it is impossible to infer from the book the manner in which the author used it for his teaching (ibid.).

[13] § 27. If *a* and *b* are *Anzahlen*, and A', A'', A''',... is a well-ordered set of the *Anzahl b*, while each of the elements *A* itself is a well-ordered set of the *Anzahl a*, then by dissolution of each *A* into its elements arises, in turn, a well-ordered set, the *Anzahl* of which is called the product of *a* and *b* and denoted by *ab* [...].

One must be aware that Felix Klein had emphatically polemicised against Meyer's approach in his *Elementarmathematik*. Set theory was a case for Klein where this theoretical development was too fresh, not yet accomplished, and even further from having matured to the point of having induced an intra-disciplinary process of integration and restructuration. Thus it was not yet elementarised: the concepts of set theory had not (yet) provide new elements for mathematics, hence Klein's polemic against Friedrich Meyer's schoolbook of 1885 (see Klein 2016, p. 289, note 181). Klein sharply criticised this schoolbook in his first edition of 1908 but softened his critique in subsequent editions (see ibid.).

6 Non-Euclidean Geometry in German Gymnasien

The last case concerns non-Euclidean geometry: in 1874, shortly after the first establishments of mathematical practices with the new geometries, which still met with strong resistance from many mathematicians and philosophers, a mathematics teacher at a Hamburg Gymnasium published a geometry textbook according to Bolyai's notion of absolute geometry.

In historiography, the decisive cause for accepting non-Euclidean geometry by a larger part of the mathematical community is attributed to the 1868 work of the Italian mathematician Eugenio Beltrami (1835–1900). In this work, the author presented Lobachevski's ideas in a geometric construction, establishing a description of the points inside a disc (Gray 1994, p. 881). The textbook presented here, hitherto unknown and not considered in historiography, reveals, however, another access to the new developments of geometry. The author, Hermann Wagner, having obtained a doctorate in mathematics and being a teacher of mathematics at a secondary school in Hamburg refers in fact not to Beltrami, but mainly to Bolyai and Riemann, mentioning Lobachevski briefly. The work he cites as a central reference for his approach is Riemann's famous masterpiece, published posthumously in 1867: *Über die Hypothesen, welche der Geometrie zu Grunde liegen*.

Wagner represents, similarly to Friedrich Meyer, the Prussian neo-humanist teacher, a profile socially recognised as that of a scholar, and with a professional performance that assumes to structure teaching in harmony with the coherence and rigour of its science. His book addressed two distinct audiences: the preface was written for his "Fachgenossen", the colleagues of his discipline, and the text itself was for school students of both the classic and modern streams. As at that time the teaching of geometry itself began in the "Quarta", corresponding to the third grade of the secondary school, and as this textbook was intended for the normal teaching of geometry, one can assume that it was written for students with an age of 12 years and over.

The entire book, and in particular its preface, shows that the author has no problem accepting the existence of different geometries; in fact, the intent is to make this recent breakthrough in mathematics accessible to students. Wagner explains at the beginning of the book that, on the one hand, geometry is one of the few sciences that has achieved a high degree of perfection, while, on the other, it is precisely its first foundations that still lack clarity and sufficient certainty. He highlights the absence of a definition of the straight line and the lack of a proof, which has been sought for centuries, of the axiom of parallels (equivalent to the theorem of the sum of angles in the triangle). The author notes with satisfaction that it has at last been proved in this century that all attempts at demonstrations of such an axiom should fail, attributing to Riemann the main merit of this result. For Wagner, this new approach in geometry represented a great epistemological significance: contrary to Kant's conceptions of geometry as an abstract science, of "reiner Anschauung", Riemann would have shown perfectly that the foundations of planimetry are grounded in experience and that, therefore, geometry presents itself as an "Erfahrungswissenschaft", an empirical science. This epistemological concept was of great importance for the author because it legitimates the introduction of the first concepts of geometry empirically—contrary to the dominant practice of his time (Wagner 1874, p. iii).

In fact, the existence of another quality of our "Raumform", the form of space, constitutes for Wagner a characteristic of the empirical. However, Wagner asserted that Bolyai had demonstrated the possibility of such a geometry in a "widerspruchsfrei" way, without contradictions, referring then to an axiomatic method.

The specific approach of his book stems from a book published in 1872 by Johannes Frischauf (1837–1924), professor of mathematics at the Austrian university of Graz: *Absolute Geometrie gemäß Johann Bol*yai. Although Frischauf incorrectly attributed the results of Wolfgang Bolyai to his father, who published his son's work as an appendix in a book, Wagner attributes to Frischauf's book the merit of having made "the genuine being of geometry" accessible to the general public. The author then proposed, as his task, to make the new concepts accessible to beginners, the students in secondary school. Since the traditional teaching of geometry was to present a logical and strictly related and deductible system of knowledge, the lack of clarity in the first foundations always presented a dilemma for teachers of mathematics (Wagner 1874, p. iv). As the task of his textbook, Wagner set out to begin with the simplest preconditions in planimetry, to demonstrate in the book only that which can be truly demonstrated, explaining that what cannot be demonstrated must be legitimised by experience.

In the rest of the preface, the author explained to his colleagues how he understands and teaches the basic concepts, point, line, plane and space, making one of the traditional choices for such an introduction: starting from the point as an infinitely small element of space and the line as movement of a point, etc. Of importance was his conceptualisation of the notion of direction, for the investigation of parallels in particular, leading to the concept of curvature in the case of spheres and the degree of curvature. Wagner hastened to assert to his colleagues that he did not intend to teach the concept of curvature to students of that age, but that he only explained it in the preface so as not to be accused of lack of understanding. Wagner also mentioned that the notion of angle presents problems still unresolved in its definition.

6.1 About the Contents of Wagner's Schoolbook

The structure of the book is very interesting—I have yet to encounter an analogue. The book has two parts:

- Absolute geometry, with three sections
- Euclidean geometry, with 5 sections

The absolute geometry sections deal with the straight line, the triangle composed of straight lines, and the sum of the angles in the triangle and the parallel lines. The "Euclidean" sections deal with the quadrilateral, the circle, the polygons (inscribed and circumscribed), proportionality, and the calculation of the content of the plane figures. Somehow, this structure corresponds even to the Elements of Euclid, since Euclid does not use the axiom of parallels at the beginning of the first book; the axiom happens to be used only from proposition 29 of book I onward.

Already in the introductory part of the book one remarks on an application of the empirical approach: given the key role of the concepts of congruence and equality, Wagner introduces them as "Erfahrungssatz"—a proposition legitimised by experience:

> All spatial quantities are independent of the place where they are (ibid., p.2).

Here Wagner also develops a definition of the straight line that proves to be sufficient for the definition of direction, angle, and finally of parallels in absolute geometry and in Euclidean geometry (ibid., pp. 3-4 and passim).

The third section is clearly the one of greatest interest here. In it, Wagner exposes and demonstrates all the results obtained since the eighteenth century on the sum of the angles in a triangle, including in particular the results of Legendre. He also adopts the use of Legendre's infinitely small, by "flattening" a triangle—namely by degenerating a triangle into a straight line (see Fig. 5.2).

More important are the theorems that state that the sum of the angles of a triangle cannot exceed two right angles; that from the sum of two right angles in *one* triangle, it follows that the sum would be the same in *each* triangle; and that the sum of each triangle is equal to two right angles or to a value smaller than this (see Fig. 5.3).

Following this is a well-argued discussion on the characteristics of parallel lines and the different cases of existence of only one parallel to a point in a line vertical

* § 73.

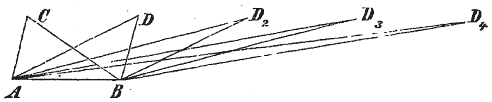
Lehrsatz: Jedes Dreieck läßt sich allmälich in ein anderes von gleicher Winkelsumme verwandeln, in welchem die Summe zweier Winkel beliebig klein ist.

Beweis: Um dies einzusehen, verwandle man (Fig. 35) $\triangle ACB$ zunächst in $\triangle ADB$, in welchem nach § 72:

$$\angle CAB = \angle DAB + \angle ADB.$$

Das so erhaltene Dreieck ADB behandle man wieder wie vorhin das Dreieck ACB. Man erhält alsdann ein Dreieck D_2AB (Fig. 36), in welchem $\angle D_2AB + \angle AD_2B$ nur noch so groß ist, wie $\angle DAB$.

Fig. 36.



Mit dem Dreiecke D_2AB kann man wieder ebenso verfahren und erhält, wenn man dies Verfahren beliebig oft wiederholt, eine Reihe von Dreiecken, in denen die Summe zweier Winkel beständig kleiner und kleiner wird. Nichts hindert, dies Verfahren bis ins Unbegrenzte fortgesetzt zu denken. So gelangt man schließlich zu Dreiecken, in welchen die Summe zweier Winkel kleiner ist, als jede noch so kleine Größe, so daß der dritte Winkel mit beliebiger Annäherung die ganze Winkelsumme des Dreiecks darstellt.

Fig. 5.2 Transforming a triangle into another with the same sum of angles, but in which the sum of two angles becomes "as small as one likes"

* § 75.

Zusatz: Die Summe der drei Winkel eines Dreiecks ist entweder gleich zwei Rechten oder kleiner als zwei Rechte.

Fig. 5.3 Resuming the propositions about the sum of angles in a triangle

to a straight line or of several parallels. The definitions, explanations and theorems are well explained and discussed, considering the level of the students.

7 Conclusion

Sufficient evidence for a productive, forward-looking function of mathematics teaching has been presented here. On the other hand, it is not possible to close our eyes to the fact that dogmatic, formalising impulses for the development of science have also emerged from the school in a fundamentalist exaggeration of the search for firm foundations.

Likewise, even institutionalised teaching of mathematics does not need to instigate production practices in mathematics. A striking example for such patterns deviating from the patterns presented in the second case is the medieval universities in Europe. While the quadrivium used to be part of teaching in the Arts Faculty, this teaching was not only relatively marginal, with regard to the trivium—the teachers of the quadrivium were freshly graduated *baccalaurei* who continued their studies in the higher faculties, without a specific qualification for these lecturers. Hence, institutionalisation, as a means for constituting a community developing their practices, has to be complemented by some professionalisation.

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