Chapter 4
"Je n'ai point ambitionnée d'être neuf": Modern Geometry in Early Nineteenth-Century French Textbooks

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Abstract This article aims to show how early nineteenth-century French geometry textbooks incorporated concepts from modern geometry. As will be shown, textbook authors in this time period rarely incorporated new developments from research mathematics into their teaching material. Modern geometry could only enter textbooks when authors had opportunities to learn new research and were willing to challenge the increasingly prescribed state geometry curriculum. Finally, and most importantly, the types of modern geometry that entered textbooks had to have perceived value for a student audience. A systematic study will illustrate how pedagogical values shaped the presentation and integration of modern geometry in ways that persisted through twentieth-century iterations.

Keywords Modern geometry · Practical geometry · Projective geometry · Nineteenth-century French mathematics

1 Introduction

The early nineteenth century was a fertile time for geometry research in France. New institutions like the École Polytechnique and the *Annales des mathématiques pures et appliquées* encouraged and published findings. In the mid-1820s, these advances were chronicled by Auguste Cournot in the *Bulletin des sciences mathématiques, astronomiques, physiques et chimiques*.

> It is not off-topic to call the attention of our readers to the progress that geometric studies have made in recent times. Long neglected for mathematical research of another kind, pure geometry, this elder sister of all the sciences, is newly in favor; new and keen studies bear fruit. Whereas, in what one calls analysis, behind an apparent richness of procedures and methods, good minds have discovered real poverty (so that often the importance of applications can only compensate for the dryness of the work), the elegant and varied results, which the science of extension enriches each day, show what an inexhaustible mine of

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> research this simple notion opens for the human mind. Distinguished geometers, among whom one must cite MM. Gergonne, Poncelet, Steiner and several others, have understood that, in order to advance science, one must leave at once both the methods of Greek geometers and this geometry, called analytic, which has only truly embraced a very limited part of the theory of extension. (Cournot 1827, p. 298)[1]

Many of these cited geometers who advanced pure geometry also expanded analytic geometry, often translating the objects of pure geometry into coordinate equations.[2] At the same time, new journals of mathematics emerged enabling authors to more quickly put their results into print and before multiple audiences. While books continued to appear, authors increasingly relied on the faster and more convenient article format to publicize and propagate their new findings and methods.

The beginning of modern geometry, as Poncelet called it (Poncelet 1817)—or projective geometry, as it would come to be called[3]—is well documented in the historical literature. Beginning in 1837 with Michel Chasles' *Aperçu Historique*, through mid-century necrologies and biographies, to the geometry articles of Felix Klein's *Encyklopädie der mathematischen Wissenschaften*, nineteenth-century geometers were eager to trace the historical development of their research (Chasles 1837; Fano 1907; Schoenflies 1909; Loria 1887).

In the mid-twentieth century, René Taton situated the emergence of "modern geometry" in the early nineteenth century. Taton recognized that certain defining aspects of modern geometry (under different terminology) dated back to ancient times, including the study of conjugate diameters with Apollonius and anharmonic ratios with Pappus. Further, modern geometry owed its origins to the foundational work of Girard Desargues and Gaspard Monge, particularly with infinite and imaginary points. Yet, Taton considered these geometers, along with Lazare Carnot, as constituting the prehistory of modern geometry:

> We will here limit this study of the prehistory of modern geometry because the study of the work of the disciples of Monge and Carnot already belongs to the history of this branch of

[1] Il n'est pas hors de propos d'appeler l'attention de nos lecteurs sur les progrès qu'ont faits dans ces derniers temps les études géométriques. Long-temps délaissée pour des recherches mathématiques d'un autre ordre, la géométrie pure, cette soeur aînée de toutes les sciences, reprend une faveur nouvelle; des aperçus nouveaux et piquants viennent la féconder. Tandis que, dans ce qu'on appelle analyse, derrière une apparente richesse de procédés et de méthodes, de bons esprits ont découvert une pauvreté réelle (en sorte que souvent l'importance des applications peut seule compenser l'aridité du travail), les résultats élégants et variés, dont s'enrichit chaque jour la science de l'étendue montrent assez quelle mine inépuisable de recherches cette notion si simple ouvre à l'esprit humain. Des géomètres distingués, parmi lesquels il faudrait citer MM. Gergonne, Poncelet, Steiner et plusieurs autres, ont compris que, pour faire avancer la science, il fallait sortir à la fois et des méthodes des géomètres grecs, et de cette géométrie, dite analytique, qui n'embrasse vraiment qu'un côté fort restreint de la théorie de l'étendue.

[2] The analytic geometers best known from this time period include Charles Dupin, Joseph-Diez Gergonne, August Möbius, and Julius Plücker. Carl Boyer's chapter on the "Golden Age of Geometry" in Boyer (1956) provides a helpful overview of this period. For more detailed information, see Clebsch (1872), Otero (1997), and Gérini (2010).

[3] "Projective geometry" was first coined by Olry Terquem in 1859 as one of eight geometries that exist today "distinguées les unes des autres par des différences logiques" (Terquem 1859).

> geometry whose developments were so rapid and fruitful over the course of the 19th century. (Taton 1949, p. 212)[4]

Subsequent historians have continued to study the "disciples of Monge and Carnot" and developments in the study of projective properties (Nabonnand 2011, 2015; Lombard 2011; Friedelmeyer 2011), ideal and imaginary objects (Rowe 1997; Nabonnand 2016), and the principle of duality (Chemla 1989; Chemla and Pahaut 1988).

While these studies focus on the history of geometrical research, by the mid-1860s entire textbooks on modern geometry were used for teaching in higher education courses throughout Western Europe and the United States into the twentieth century (Housel 1865; Reye 1866; Cremona 1873; Mulcahy 1862). Unlike descriptive geometry, which was first disseminated in the classroom, the process of establishing the so-called modern geometry as a standard course of study was gradual, multifaceted, and led to numerous iterations (projective geometry, geometry of position, etc.).[5] Nevertheless, both descriptive and modern geometries remained closely linked with similar modes of evolution. As Evelyn Barbin has documented in her study of how descriptive geometry changed over the nineteenth century, "journals are a good vehicle to move ideas between teachers of *Classes Préparatoires*, and to propagate new methods among secondary schools teachers" (Barbin 2015, p. 59). Similarly, we will see how concepts introduced in journals motivated changes in pedagogical content with respect to teaching modern geometry.

However, as Bruno Belhoste warns in "Pour une réévaluation du rôle de l'enseignement dans l'histoire des mathématiques," teaching modern geometry was not simply a process of reproduction from the research context:

> This is because most still consider the communication, transmission, and popularization of mathematical knowledge as secondary and peripheral activities. Under this indifference hides in fact the false idea that mathematical production can be separated a priori by the historian from the conditions of its reproduction. (Belhoste 1998, p. 289)[6]

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[4] Nous limiterons ici cette étude de la préhistoire de la géométrie moderne, car l'étude de l'oeuvre des disciples de Monge et Carnot appartient déjà à l'histoire de cette branche de la géométrie dont les développe-ments furent, au cours du XIXe siècle si rapides et si fructueux.

[5] The difficulty of determining what this subject should be called is exemplified in Luigi Cremona's introduction:

> Various names have been given to this subject of which we are about to develop the fundamental principles. I prefer not to adopt that of Higher Geometry (Géométrie supérieure, höhere Geometrie), because that to which the title "higher" at one time seemed appropriate, may today have become very elementary; nor that of Modern Geometry (neuere Geometrie), which in like manner expresses a merely relative idea; and is moreover open to the objection that although the methods may be regarded as modern, yet the matter is to a great extent old. Nor does the title Geometry of position (Geometrie der Lage) as used by STAUDT seem to me a suitable one, since it excludes the consideration of the metrical properties of figures. I have chosen the name of Projective Geometry, as expressing the true nature of the methods, which are based essentially on central projection or perspective. And one reason which has determined this choice is that the great PONCELET, the chief creator of the modern methods, gave to his immortal book the title of *Traité des propriétés projectives des figures* (1822). (Cremona 1885)

[6] C'est que la plupart considèrent encore la communication, la transmission, et la vulgarisation du

Indeed, the content and practices of teaching modern geometry by the late nineteenth century reflected an evolution and active restructuring of the subject that only resembled cited foundational texts, like Poncelet's *Traité des propriétés projectives*. Rather than a one-way transmission, teaching practices were shaped by decades of feedback among overlapping audiences and contributors. Looking back to the beginning of the century, early research in modern geometry was never far from teaching. Journals like the *Correspondance sur l'École Polytechnique*, the *Annales*, and the *Journal der reine und angewandte Mathematik* included posed problems to encourage students to apply new methods and engage in accessible research.[7] Some of the most famous modern geometers—Poncelet, Gergonne, Plücker, Steiner — often cited their teaching experiences in their publications.

In this chapter, I will focus on the adaptation of modern geometry within French textbooks during the first third of the nineteenth century. This study comprises a small fraction of the process of developing autonomous modern geometry courses, which spanned diverse temporal and local variations. Even so, documenting modern geometry in French textbooks will illustrate how pedagogical values shaped the presentation and integration of modern geometry in ways that persisted through twentieth-century iterations.

2 Historiography of Mathematics Textbooks

The historical study of textbooks in nineteenth-century France has expanded greatly over the past three decades. In Jean Dhombres' 1985 statistical study of "French mathematical textbooks from Bézout to Cauchy," he shows that, as compared with other contemporary disciplines, mathematical textbook writing constituted a significantly larger proportion of mathematical writing than research articles (Dhombres 1985). Dhombres explains the proliferation of textbooks with respect to three of the four roles of mathematics in the period from 1775 to 1825:

> First, mathematics was a favored field of education since the Revolution, appreciated both by students and by educators. We have already given statistical figures for students. The aims of teachers might have been different, but to discuss it requires the analysis of the content of the textbooks, which we postpone to another paper. Second, mathematics was a means of selecting candidates to higher positions through an elitist process, which nonetheless satisfied the egalitarian ethos of the Revolution. This process tended to be more and more organized in various fields with the model being the *École Polytechnique*. This elitist process, via a selection based on mathematics, has already begun with the military schools organized after 1770, but it obtained its peak when only an exam on mathematics was required to enter the *École Polytechnique*. Third, mathematics appeared as the necessary

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savoir mathématique comme des activités secondaires et périphériques. Sous cette indifférence se cache en fait l'idée fausse que la production mathématique peut être séparée a priori par l'historien des conditions de sa reproduction.

[7] Through the early nineteenth century, posed problems attracted a diverse range of respondents. See Despeaux (2008), Gérini and Verdier (2007), Rollet and Nabonnand (2013), Delcourt (2011).

> language for developing all other sciences (Condillac used the expression "la langue des calculs", which became the title of one of his posthumous books published at the end of the Revolution). (Dhombres 1985, p. 116)

Indeed, all three roles (the fourth is mathematics for its own sake) motivated the inclusion of modern geometry within the textbook literature.

Further, Dhombres points out that "mathematical books reached a far larger audience than mathematicians" (136). The fact that textbooks are emblematic of the wider public facing side of mathematics presents another tool toward understanding which parts of modern geometry were assimilated into textbook literature. Textbook authors could determine which parts of mathematical research seemed most valuable and appropriate for more general consumption. For instance, geometry textbooks reveal an absence of imaginary or ideal objects.

While Dhombres overviews the diversity of textbooks in a given location and time period, Gert Schubring focuses his 1987 study on "Lacroix as Textbook Author" in order to outline a methodology of textbook analysis. He proposes a three-dimensional historical scheme beginning with comparing a single textbook across multiple editions, then examining corresponding changes in contemporary textbooks in light of "changes in the syllabus, administerial decrees, didactical debates, evolution of mathematics, changes in epistemology, etc." (Schubring 1987, p. 45).

The institutional factors behind textbook production are further examined by Belhoste and Renaud d'Enfert, respectively. In *Les Sciences dans l'enseignement secondaire français: textes officiels*, Belhoste explores how government policies shaped the teaching of science between 1789 and 1914. During the first third of the nineteenth century, mathematics education remained fairly static as a result of officially sanctioned texts and institutional entrance exams. The standards for mathematics education were set in Paris, and up until 1840 many decisions with respect to instruction were determined by the mathematician Siméon-Denis Poisson, who served as a member of the *Conseil Royal de l'Instruction Publique* from 1820 to 1840.

> True «patron» of mathematics in France, he is at once the exit examiner for the *École Polytechnique*, which allows him to keep an eye on the preparatory course, and the president of the jury of the science agrégation, which assures him the control of recruiting mathematics and physics teachers. (Belhoste 1995, p. 30)[8]

Thus, it is no surprise that most textbooks were written by Parisian authors, often associated with the École Polytechnique, and for purposes of exam preparation (see the Appendix for publication data).

By contrast, d'Enfert portrays regional variation in his study of mathematics education for workers. Although the movement to provide regular evening courses

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[8] Véritable «patron» des mathématiques en France, il est. à la fois examinateur de sortie à l'École Polytechnique, ce qui lui permet d'avoir un oeil sur la filière préparatoire, et président du jury d'agrégation des sciences, ce qui lui assure le contrôle du recrutement des professeurs de mathématiques et de sciences physiques.

on geometry and mechanics for French workers initiated with Charles Dupin in Paris, by 1830 records show enrollment of 4000 to 5000 students in 109 towns:

> A variety of local situations respond to this general movement. This variety also affects the nature of teaching dispensed by the teachers themselves. All the courses instituted in the second half of the decade 1820 were not exactly modeled on Dupin. (D'Enfert and Fonteneau 2011, p. 89)[9]

While these instructors had more liberty to personalize their courses than their counterparts in formal education, d'Enfert observes that including "more theoretical new mathematical knowledge" in this practical context was perceived as controversial (99). Textbooks in this genre could thus only incorporate modern geometry insofar as it could be useful to the intended audience.

Most recently, Guillaume Moussard has investigated the circulation of problems and methods within elementary and analytic geometry textbooks in France between 1794 and 1891 (Moussard 2015). Of particular interest here is his chapter "L'essor de la géométrie rationnelle: nouvelles notions et méthodes" on how new geometrical notions and methods informed two textbooks on teaching geometry to workers during the 1830s. Moussard concludes that during this period modern geometry (what he calls "géométrie rationelle") did not enter secondary teaching in the strictly regulated lycées or colleges.

> Finally, we will research the presence of this rational geometry in secondary teaching. We will see that we find it less in the classical teaching of Lycées and Collèges than in industrial teaching, where the geometry teaching texts of Claude Lucien Bergery in 1826 and Étienne Bobillier in 1832 incorporated numerous elements. (Moussard 2015, p. 68)[10]

This article will be similar in that it also examines the presence of new "notions" in geometry textbooks. However, while Moussard compares methods for teaching geometry from the late eighteenth to early twentieth centuries, here modern geometry will be compared synchronously across a range of textbooks. In complement to Moussard's findings, I will examine multiple motivations for how and why different early nineteenth century authors introduced, situated, and changed certain objects from modern geometry.

All mathematics education remained fairly conservative due to strict centralized content regulations and unchanging standards of admission through the first half of the nineteenth century. In a summary of mathematics education in France from 1800 to 1980, Hélène Gispert discusses the initially bifurcated French education system, where secondary schools taught theoretical mathematics and primary schools taught practical mathematics. In both these situations, mathematics above

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[9] À ce mouvement d'ensemble répond la variété des situations locales. Cette variété concerne aussi bien la nature de l'enseignement dispensé que les professeurs eux-mêmes. Tous les cours institués dans la seconde moitié de la décennie 1820 ne sont pas exactement calqués sur le modèle de Dupin.

[10] Ensuite, nous recherchons la présence de cette géométrie rationnelle dans l'enseignement secondaire. Nous verrons que nous la trouvons moins dans l'enseignement classique des Lycées et Collèges que dans l'enseignement industriel, où les ouvrages d'enseignement de la géométrie de Claude Lucien Bergery en 1826 et d'Étienne Bobillier en 1832 en intègrent de nombreux éléments.

the elementary level was considered accessory to other subjects. Mathematics only "occupied an important place in the specialized courses that were offered, often in private institutions […] and that prepared for the écoles spéciales of the government, of which the École Polytechnique held the highest rank" (Gispert 2014, p. 230). As Caroline Ehrhardt observes in her study of algebra education,

> In spite of successive reforms about the general scientific training in high schools between 1808 and 1830, the program of the mathematical courses for students who wanted to make a scientific career remained mostly unchanged from the first years of the century to the 1830s. (Ehrhardt 2010, p. 93)

Studies of early nineteenth-century textbooks show that the most prolific geometry textbooks were those prescribed by the government and authored by Silvestre-François Lacroix and Adrien-Marie Legendre. Between 1799 and 1832, Lacroix's *Élémens de géométrie* and Legendre's *Éléments de géométrie* each ran fourteen editions with little change in content.[11] Consequently, many mathematics teachers of the 1830s essentially taught from the same textbooks that they had learned from as students.

Charles Dupin claimed "Les progrès de la science ne sont vraiment fructueux, que quand ils amènent aussi le progrès des Traités élémentaires" (Dupin 1813). This sentiment was far from universal. In fact, as will be shown, textbook authors during the early nineteenth century rarely incorporated new developments from research mathematics into their teaching material. Modern geometry could only enter textbooks when authors had opportunities to learn of new research and were willing to challenge the increasingly state prescribed geometry curriculum. Finally, and most importantly, the types of modern geometry that entered textbooks had to have perceived value for a student audience.

3 Finding Textbooks

To identify the presence of modern geometry in textbooks, this article will focus on the presence of new research objects in geometry. Admittedly, this is a rather conservative marker and may miss certain textbooks with subtler forms of modern geometry, such as the theory of transversals following Carnot. However, as will be shown, almost every textbook that emphasized new content in the introduction also included some of the new objects from research publications in the body of the text.[12] Further, this criterion coincides with the observed pattern in contemporary research articles, in which geometers praised and adopted new vocabulary in advance of new methods or theories.

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[11] The perceived values of these two textbooks and the relationship between their authors are described in Schubring (1987).

[12] The exception here is Charles Dupin, who introduced his own new objects within his textbooks that later became part of differential geometry.

> Analysts perceiving that certain quite complicated functions are reproduced frequently in their calculations, have called them exponentials, logarithms, sines, tangents, factorial derivatives, etc.; they have created abbreviated signs to designate them, and their formulas have acquired greater clarity and conciseness. And thus for certain points, certain lines and certain circles whose consideration is frequently represented in geometric speculations, it is natural to do the same with respect to them, and to call them, following their properties, similitude centers, radical centers, polars, similitude axes, radical axes, circles of common power, etc. This attention must inevitably introduce analogous simplifications in the statement of theorems and in the solution of problems, which belong to the science of magnitude. (Anonymous 1827a, p. 279)[13]

This quote from an anonymous *Bulletin* review of Steiner, provides a list of new objects that emerged in the *Journal de l'École Polytechnique* (radical axes (Gaultier 1813)), *Correspondance sur l'École Polytechnique* (similitude centers (Hachette and Monge 1813)), *Annales des mathématiques pures et appliquées* (polars (Servois 1810)), and *Journal für die reine und angewandte Mathematik* (circles of common power (Steiner 1826)). These objects propagated through research articles, often independently from the methodological context in which they first emerged.

Significantly, most of these objects persisted into the textbooks of the twentieth century. Thus, though not capturing all of the ways in which modern pure geometry might transition from research to teaching, the paths of new objects tell significant and enduring accounts in the story. The use of poles, polars, similitude, and radicals signaled a foray into the modern geometry of the early nineteenth century that would later be characterized as projective geometry.[14]

To obtain an appropriate corpus of contemporary geometry textbooks, I first queried the *Bibliothèque nationale de France* library catalog for all texts that included the keyword "Géométrie" and had been published between 1800 and 1833 (www.bnf.fr). This search returned 113 available texts, some of which were multiple editions of the same title.[15]

Certainly, this form of search did not gather every single book on geometry published in French between 1800 and 1833.[16] Nevertheless, this search appears to be

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[13] Les analystes s'étant aperçu que certaines fonctions assez compliquées se reproduisaient fréquemment dans leurs calculs, les ont appelées exponentiels, logarithmes, sinus, tangentes, dérivées factorielles, etc.; ils ont créé des signes abréviatifs pour les désigner, et leurs formules en ont acquis beaucoup de clarté et de concision. Puis donc qu'il est. certains points, certaines droites et certains cercles dont la considération se représente fréquemment dans les spéculations de la géométrie, il est. naturel d'en user de même à leur égard, et de les appeler, suivant leurs propriétés, centers de similitude, centers radicaux, polaires, axes de similitude, axes radicaux, cercles de commune puissance, etc. Cette attention doit introduire inévitablement des simplifications analogues dans l'énoncé des théorèmes et dans la solution des problèmes qui appartiennent à la science de l'étendue.

[14] For instance, in David Eugene Smith's very brief *History of Modern Mathematics* he points to "the theory of the radical axis" as one of several contributions that affected elementary geometry during the nineteenth century (Smith 1906).

[15] Several texts were listed in the BnF catalog, but reported "hors usage," and thus could not be accessed.

[16] For example, Poncelet's 1822 *Traité des propriétés projectives* was not found in this search because this first edition did not receive any classification and the word "géométrie" is cut-off from

representative, which I confirmed by conducting the same search through the Library of Congress online catalog (http://catalog.loc.gov/). The Library of Congress search added one additional text, the 1812 edition of Étienne Bézout's *Cours de mathématiques* originally published in 1772 (Bézout and Reynaud 1812). The same search through the Catalogue collectif de France (http://ccfr.bnf.fr/portailccfr/jsp/index.jsp), not including the Bibliothèque nationale, returned 18 new texts, of which I was able to consult 12.

In this chapter, textbooks will be defined as books that explicitly advertised to an audience of teachers or students through the title, subtitle, dedication, preface, or introduction. Acknowledging that other books might still have been used in classrooms or for self-study, this criterion applied to 79 of the 113 texts. Thus, a direct reference to the intended audience was a fairly common practice. For instance, the 1803 edition of Lacroix's analytic geometry textbooks contained a page listing all his textbooks included in the "Cours de Mathématiques pures, à l'usage de l'École centrale des Quatre-Nations" (Lacroix 1803b). In contrast to this formality, Alexandre Vincent dedicated his 1826 *Cours de géométrie élémentaire* to "students."

> This work belongs to you in more than one way: it is for you, it is with you that I wrote it: receive its dedication. May it nourish in you, as you recall the hours of our meetings, that love of study that will soon set you as well (I hope) to pay the tribute you owe to public utility. (Vincent 1826, p. i)[17]

In general, textbooks so-defined were written for teachers to use with their students, or, less frequently, for immediate student consumption.

The cost of production may help to explain why so few books appeared that weren't textbooks, and why only textbooks were reprinted in quick succession. Many of the well-known and widely republished names in turn of the century geometry—Monge, Lacroix, Legendre—wrote books almost exclusively for a student audience. Textbooks catered to an existent market, while research books were expensive and risked not being sold.[18]

Most textbook titles indicate their subject as elementary geometry, elementary analytic geometry, descriptive geometry, or practical geometry. This is in marked contrast to research articles. For instance, there were no courses corresponding to the popular *Annales* subject headings: *Géométrie de la règle*, *Géométrie de situation*, *Géométrie transcendante*, *Géométrie pure*, *or Géométrie des courbes et surfaces*.

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the full title within the library catalog, it reads "Traité des propriétés projectives des figures..." The 1865 editions were classified as Géométrie descriptive and the full title is printed, thus these do show up if there is no date restriction.

[17] Cet ouvrage vous appartient à plus d'un titre: c'est pour vous, c'est avec vous que je l'ai composé: recevez-en la dédicace. Puisse-t-il, en vous rappelant les heures de nos entretiens, alimenter en vous cet amour de l'étude qui vous mettra bientôt à même (je l'espère) de payer le tribut que vous devez à l'utilité publique.

[18] The cost of production has been studied by Norbert Verdier in his thesis on Liouville's Journal (Verdier (2009). Jean and Nicole Dhombres addressed these issues from the perspective of books, and particularly textbooks in Dhombres (1985) and Dhombres and Dhombres (1989).

In this corpus, only the textbooks of Olry Terquem, who emphasized his different approach, proposed introducing geometry alongside algebra (Terquem 1829). Otherwise, analytic geometry was the next most advanced geometry, to be learned by those who mastered both elementary geometry and algebra, and continued to pursue mathematics. Descriptive geometry appeared after elementary geometry, either before or after analytic geometry. Finally, practical geometry was for a different group of students, often industrial workers in public courses, and might serve as their only mathematics training beyond basic arithmetic.

For each of the 79 texts, I consulted the title page, table of contents, any prefatory remarks, and the sheets of figures (nearly always located at the very end of the volume).[19] When the table of contents or introduction included any of the new objects cited above, referenced recently published articles, or broadly mentioned new geometric content then I included the text as part of my corpus of textbooks containing modern geometry. This turned out to be a very small corpus of only seven titles, several in multiple editions.

To understand why modern geometry entered textbooks, I will first consider how textbooks justified their existence and attracted readers through claims of novelty. For the majority of textbooks, novelty was framed in terms of pedagogical values. By contrast, in the seven textbooks that did contain modern geometry, authors also emphasized the novelty of the content. I will then take a closer look as to how authors developed specific aspects of modern geometry within a teaching context, simultaneously extending the tools for learning geometry while remaining within the bounds of constructive practices.

4 Claims for Novelty By Textbook Authors

4.1 The Majority View

Most claims for novelty in textbooks concerned best teaching practices. Authors debated whether theorems should appear before or after their proofs, whether problems should be embedded in the text or collected in an appendix (Develey 1812; Legendre 1800; Vincent 1826); the appropriate use of proof by contradiction (Lacroix 1803a; Schwab 1813; Olivier 1835), and how much rigour could be obtained without sacrificing the more important quality of simplicity (Lacroix 1799; Vincent 1826; Develey 1812; Clairaut 1830; Mutel 1831; Terquem 1829, etc.). Distinct forms of teaching could be subtle but were still advertised, such as the decision by Louis-Etienne Develey, Auguste Mutel, and Vincent to state propositions without reference to the lettered figure, in order that the wording might more easily be committed to memory, which all three highlighted as important decisions in their

[19] Unfortunately, when consulting scanned texts, the figure pages were often poorly copied. While disappointing, this feature was in general not a detriment toward understanding the book's content nor the author's textual use of figures.

introductions. As a further example, the Abbé de la Caille allowed his text to be more or less advanced through restricting "less useful or less easy" material to small font that the reader could include or ignore depending on preference (de LaCaille and Labey 1811, 1741, p. iv).

Within their introductions, authors both acknowledged and criticized the work of contemporary textbook writers, such as when Develey described the ongoing dialog on the best form of presenting the elements:

> A lot has been written on the best form to give to the Elements of Geometry; I do not wish to repeat what others have said and very well for I could not do it. But with these excellent directions, do we achieve perfect Elements? I do not think so; and I am far from believing that mine are thus. Several authors have taken great steps toward this perfection as we see everything in perspective; I have also attempted some efforts; perhaps one day someone luckier, but above all abler than I, will achieve the desired goal. (Develey 1812, p. v)[20]

Authors often described their work as supplementing rather than replacing previous treatments. Antoine Charles Poullet-Delisle assured the reader that his publication should not be perceived as a criticism of contemporaries, and only intended to be useful. He professed: "I have no ambition to be new: in a work of this kind that would be undoubtably a ridiculous pretension" (Poullet-Delisle 1809, p. v).[21]

When evaluating who had succeeded in writing geometry, Legendre was portrayed as the standard. Legendre himself began each new edition by thanking the various geometers who had recently offered new and relevant material including over the years Lhuilier, Cauchy, and Querret (Legendre 1800, 1812, 1832). Although feedback from other mathematicians could be useful, the ultimate test of a text's success, as Biot observed, was by experiment, "test it on the minds of the students, and verify by this proof the goodness of the chosen methods" (Biot 1810, p. vi).

The expression "modern" possessed a more traditional connotation in most textbooks, particularly those with editions dating back to the eighteenth century. As elementary geometry was considered the "method of the ancients," so analytic geometry was considered "modern." Bossut, whose text originally appeared in 1772, described analytic geometry as producing a "revolution" in "the empire of mathematics" (Bossut 1800, p. xii). Late eighteenth and early nineteenth-century geometers credited the origin of this modern geometry to Viète and Descartes, admired the work of Newton, and were inspired by both the form and content of Euler's trigonometric and analytic texts. For instance, citing Viète, Descartes, Newton, Euler, and Cramer, Lacroix provided a brief history of analytic geometry, which he prefaced in praising the "moderns."

[20] On a beaucoup écrit sur la meilleure forme à donner aux Élémens de Géométrie; je ne voudrais pas répéter ce que d'autres ont dit, et bien mieux que je ne pourrais le faire. Mais avec ces excellentes directions, sommes-nous parvenus à avoir des Élémens parfaits? Je ne le pense pas; et je suis bien loin de croire que les miens le soient. Quelques auteurs ont fait de grands pas vers cette perfection que nous voyons tous en perspective; j'ai voulu hasarder aussi quelques efforts; peut-être un jour quelqu'un plus heureux, mais surtout plus habile que moi, atteindra-t-il le but désiré.

[21] Je n'ai point ambitionnée d'être neuf: dans un ouvrage de cette espèce, ce serait sans doute une prétention ridicule.

> Then came the application of algebra to geometry; this branch, due entirely to the moderns, and whose discovery soon gave them a huge advantage over the ancients, had to change form in measure as it was extended and perfected. (Lacroix 1803b, p. vi)[22]

Yet citations back to the seventeenth century suggest that claims to modernity in analytic geometry did not necessarily imply recent development nor attention to new research. Algebraic solutions that indicated imaginary, infinite, and to some extent negative points or curves were usually dismissed as impossible or absurd.[23] Solutions that could not be represented on paper were non-existent. In fact, as will be shown, new research was just as infrequently adapted to analytic geometry as to any other geometry textbook.

4.2 Textbooks with Modern Geometry

The presence of modern geometry from contemporary research coincided with markedly different claims for novelty among textbook writers. A chronological introduction to the authors, titles, and circumstances of publication will provide a background against which such claims can be better evaluated.

Dupin

Charles Dupin (1784–1873) is both the epitome and the exception among the other authors in this study. His commitment to developing pure and analytic methods within research and teaching provided him with a remarkable professional status among his contemporaries exhibited by citations and dedications. Beginning in 1813, Dupin's call for teaching new geometry to researchers, students, and workers modeled later efforts to bring modern geometry into the textbook literature. His contributions more closely aligned with what would become differential geometry than projective geometry, but since this distinction did not yet exist, it would be artificial to remove Dupin from a study of modern geometry. Nevertheless, in the interest of space, I will leave aside a more technical discussion of his texts.[24]

In the introduction to his *Développements de Géométrie, avec des Applications à la stabilité des Vaisseaux, aux Déblais et Remblais, au Défilement, à l'Optique, etc.*, Dupin called for new concepts in elementary treatises. He intended his elemen-

[22] Vient ensuite l'application de l'algèbre à la géométrie; cette branche, due entièrement aux modernes, et dont la découverte leur a bientôt donné une immense supériorité sur les anciens, devait nécessairement changer de forme à mesure qu'elle s'étendoit et se perfectionnoit.

[23] As the history of complex numbers in the nineteenth century indicates, imaginary numbers held an ambiguous status within mathematics, and geometry in particular, through the 1820s (Flament 1997; Schubring 2005).

[24] For additional historical analyses of Dupin's contributions, see Christen-Lécuyer and Vatin (2009), and Bradley (2012).

tary treatise to serve as a sequel to the descriptive and analytic geometry introduced by Monge, most famously in *Géométrie descriptive* and *Application de l'Analyse à la Géométrie à l'usage de l'École Impériale Polytechnique* (Monge 1798, 1807, 1795). To accomplish this, *Développements de Géométrie* appeared in two parts, "Théorie" and "Applications" published, respectively, in 1813 and 1822 (Dupin 1813, 1822). He had studied descriptive geometry at the École Polytechnique with Monge, to whom he dedicated his text, and by the time the first part appeared, Dupin was already an acclaimed engineer and mathematician. Dupin described his work as written for "les élèves de l'École Polytechnique, ou des corps du Génie" (Dupin 1813, p. viii). Yet, while he declared his work a textbook, at the same time he promised to introduce new research.

> The progress of science is not truly fruitful, except when it also leads to the progress of elementary Treatises; it is through these writings that new concepts, reserved first for a small number of superior minds, finally becomes general knowledge, and extends its benefits into all parts that wait only for an intelligent application. (ibid, p. vii)[25]

In particular, Dupin promised to include results derived between 1805 and 1807, some of which had been previously published in the *Correspondance sur l'École Polytechnique*. Dupin further signaled his awareness of recent developments in geometry by summarizing the contributions contemporary geometers, and in particular former polytechniciens. Most of all, Dupin credited Monge and Carnot, who in turn provided a positive review of the book. Their recommendation, written along with Poisson on behalf of the *Académie des sciences*, was printed as a further introduction.

Dupin distinguished this book from his earlier articles, in that the treatment here would be simpler. The reviewers echoed this sentiment, acknowledging that Dupin contributed to both research and public works and pointed to "remarkably simple" new discoveries.

> The research that we are going to present proves that in the midst of the work with which he has been charged, M. Dupin has not lost sight of the objects of his first studies. It makes us wish that an engineer who reunites such extensive knowledge in geometry and analysis, would soon publish the work in which he proposes to apply them to questions of practice and public utility. (Dupin 1813, p. xx)[26]

The reviewers saw this enterprise as reflecting the founding goals of the École Polytechnique. Indeed, the entire "Théorie" text reflects the balance between writing for beginning students and experienced researchers. On the one hand, Dupin

[25] Les progrès de la science ne sont vraiment fructueux, que quand ils amènent aussi le progrès des Traités élémentaires; c'est par ces écrits que les conceptions nouvelles, réservées d'abord au petit nombre des esprits supérieurs, deviennent enfin des connaissances générales, et ramifient leurs bienfaits dans toutes les parties qui n'attendent qu'une application intelligente.

[26] Les recherches que nous venons d'exposer prouvent qu'au milieu des travaux dont il a été chargé, M. Dupin n'a pas perdu de vue les objets de ses premières études. Elles font desirer qu'un Ingénieur qui réunit des connaissances si étendues en géométrie et en analyse, publie bientôt l'ouvrage dans lequel il se propose de les appliquer à des questions de pratique et d'utilité publique.

occasionally apologized for providing too many details in a very elementary treatment.

> Perhaps, despite this, people well-versed in considerations of Geometry, will find still that I entered into too many details; but if these developments make that which seems too elementary easier, they will certainly not be superfluous for all the readers. (ibid, p. 25)[27]

On the other hand, Dupin at times chose his methods in order to maintain the practicality desired by engineers.

> If we only wrote for Geometers, we would have freed this latter part from all infinitesimal considerations; but in following the beautiful methods of the author of Fonctions Analytiques, it would have been less easy; and that ease is above all what we would like to be able to make possible, in order to generalize the study of theories truly useful to Engineers. (ibid, p. 68)[28]

These sentiments suggest a growing separation between professions in France, despite the goals of the École Polytechnique and Dupin's own contributions to both engineering and geometry. This distance seemed even more apparent by 1822 when his *Applications de Géométrie et de Méchanique, à la marine, aux ponts et chaussées, etc., pour faire suite aux Développements de Géométrie* appeared. Despite the many concrete applications within *Développements de géométrie* Dupin explained in his introduction to *Applications* that the first text had presented "abstract truths" that were "without practical utility" (Dupin 1822, p. xx). This sequel, which presumably could be read independently of the prefatory theory, would not be subject to "the same judgment." Though Dupin's endeavor to write at once for researchers and students was not emulated, he was joined in his commitment to introducing new geometry at the elementary level.

Garnier

Jean Guillaume Garnier (1766–1840) published the first edition of *Elémens de géométrie analytique* in 1808 as a "Traité que j'offre aux élèves" (Garnier 1808, p. iv). Garnier identified himself on the title page as an "Ancien Professeur à l'École Polytechnique, et Instituteur, à Paris"—indeed, he had been an assistant to Lagrange's courses between 1798 and 1802. By 1808, he was a teacher of transcendental mathematics at a lycée in Rouen. Moreover, as noted on the back cover, Garnier had published other textbooks on arithmetic, algebra, elementary geometry, statics, and differential and integral calculus. In the introduction to his first edition,

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[27] Peut-être, malgré cela, les personnes très-versées dans les considérations de la Géométrie, trouveront-elles encore que je suis entré dans trop de détails; mais si ces développements rendent plus facile ce qui leur semblera trop élémentaire, ils ne seront certainement pas superflus pour tous les lecteurs.

[28] Si nous n'écrivions que pour des Géomètres, nous aurions pu dégager cette dernière partie de toute considération infinitésimale; mais en le faisant d'après les belles méthodes de l'auteur des Fonctions Analy-tiques, nous aurions été moins faciles; et c'est surtout ce que nous voudrions pouvoir être le plus possible, afin de généraliser l'étude des théories vraiment utiles à des Ingénieurs.

he credited the work of many other textbook authors associated with his former, prestigious, institution including Lacroix, Prony, Biot, Lefrançois, Boucharlat, Dinet, Puissant, Monge, Hachette, and Poisson.

In 1813, Garnier published a second edition, under a slightly different title, *Géométrie analytique ou application de l'algèbre à la géométrie*. He explained the need for this new edition by harshly criticizing his first edition.

> The first Edition of this Work lacks method, and consequently that which forms the principal merit of an elementary book: it desired several formulas which, without being exclusively preferable to others, advantageously replace them in the solution of a great number of questions; several solutions are not complete or thorough enough, others are diffcult; the problems of space are mixed with problems of two dimensions; finally the notation is often defective. (Garnier 1813, p. v)[29]

Garnier described this new treatise as "plus méthodique, plus soigné et plus complet" and credited particularly "les précieux matériaux" from Gergonne's Annales as well as the geometry research of L'Huilier and Puissant. As will be seen in the following section, Garnier included objects from modern geometry among this "precious material." Garnier was certainly familiar with Gergonne's *Annales* as he had submitted a brief article to the journal, which was published in 1813. Though most of Garnier's writings around 1813 were for textbooks, he would later contribute many brief articles to his own journal, *Correspondance mathématique et physique* (1825—1839), in almost every domain of pure and applied mathematics.

Biot

Garnier's inclusion of modern mathematics in 1813 demonstrates an exceptionally early adoption of certain recently published research. While Jean Baptiste Biot (1774–1862) thanked Garnier in the preface of his 1813 *Essai de Géométrie Analytique, appliquée aux courbes et aux surfaces du second ordre* (Biot 1813), not until the sixth edition, ten years later, did he also begin to include some of this same new content. The first edition of Biot's textbook appeared in 1802, written for prospective École Polytechnique students.

> This work is principally destined for the young people who are studying to enter the École Polytechnique. It results from lessons that I gave at the École Centrale de l'Oise. (Biot 1802, p. i)[30]

Part of Biot's qualifications included his own experience as a student at the École Polytechnique, where he enrolled in 1794. By 1803, he was a member of the *Institut*

[29] La première Édition de cet Ouvrage manque de méthode, et conséquemment de ce qui fait le principal mérite d'un livre élémentaire: elle laisse à désirer plusieurs formules qui, sans être exclusivement préférables à d'autres, les remplacent avantageusement dans la résolution d'un grand nombre de questions; quelques solutions ne sont pas complètes ou assez approfondies, d'autres sont pénibles; les problèmes de l'espace sont mêlés avec les problèmes à deux dimensions; enfin la notation est souvent défectueuse.

[30] Cet ouvrage est principalement destiné aux jeunes gens qui étudient pour entrer à l'École Polytechnique. Il est résultat des leçons que j'ai données à l'École Centrale de l'Oise.

de France, a professor of mathematics and physics at the *Collége du France*, and a professor of Astronomy at the *Faculté des Sciences* de Paris. The textbook appears to have been popular, as the next four editions quickly followed over the next ten years without many changes from the original volume. In the preface to his 1823 edition Biot apologized for his long hiatus, explaining understandably that he was prevented by other "occupations plus obligées, ou plus attrayantes" (Biot 1823, p. vii).[31] Even more than Garnier, Biot is connected to Parisian mathematics. Nevertheless, like Garnier, he credited the *Annales*, published in Nîmes and not formally connected to Parisian mathematical activity, for the new geometry he included in this edition.[32]

> I also believed I must no longer pass over in silence the properties of poles and polar lines first considered by Monge, and to which authors of the *Annales de Mathématiques* have given such elegant analytic developments. (Biot 1823, p. vii)[33]

Biot also cited Lagrange, Lacroix, and his brother-in-law Brisson for other modifications to his treatment of curves in this volume. In these numerous citations, Biot established a broad base of support for his new contents.

Vincent

The market for preparing future École Polytechnique students also included teaching elementary geometry. Alexandre Vincent (1797–1868) had been a student at the *École Normale* between 1816 and 1820, and first wrote an elementary geometry textbook dedicated to his students at the Collége royal de Reims in 1826. As he noted in the subtitle to the first edition, the *Cours de Géométrie Élémentaire* was "à l'usage des élèves qui se destinent à l'école Polytechnique ou aux écoles militaires." Vincent highlighted the pedagogical improvements to his approach, including distinct placement of practice problems and the statement of propositions without reference to lettered figures. He also announced additional material for strong students:

> For the rest, the things which are not indispensable are printed in small type, one could leave them aside, or reserve them as exercises for the strongest students. (Vincent 1826, p. iv)[34]

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[31] Biot utilizes the exact same preface for his subsequent 1826 edition, which is identical to the fifth edition except for minor typographical corrections.

[32] In Otero (1997), Mario Otero statistically analyzes the distribution of content in Gergonne's *Annales* and finds that geometry was overrepresented as compared to other contemporary research publications.

[33] J'ai cru aussi devoir ne plus passer sous silence les propriétés des pôles et des lignes polaires considérées d'abord par Monge, et auxquelles les auteurs des Annales de Mathématiques ont donné des développemens analytiques si élégans.

[34] Au reste, les choses qui ne sont pas indispensables étant imprimées en petit caractère, on pourra les laisser de côté, ou les réserver comme exercices pour les élèves les plus forts.

Fig. 4.1 Vincent's notation in Vincent (1832)

The use of small font enabled those who wanted to focus on only the entrance exam material to skip these sections. With respect to content, Vincent credited Lacroix, Francoeur, Legendre, Dupin, Develey, and Gergonne "dont j'ai plus d'une fois consulté les intéressantes annales" (ibid, p. v).

Vincent published a second edition in 1832 based on feedback from other teachers, a review by Augustin Cournot in Lycée, and "un rapport très étendu adressé par M. Ampère au Conseil royal de l'instruction publique" (Vincent 1832, p. v). Many of these changes were organizational, such as better in-text references to corresponding problems. Vincent eliminated the use of small font, due to complaints about legibility. However, rather than deleting the challenging content, he added more, highlighting the elementary principal properties of transversals, radical axes, and poles and polars. In this edition, Vincent denoted the extracurricular status of this material with using the symbol of a left and right facing sideways M (Fig. 4.1).

> In addition, I have noted these theories, like several others, as well as a great number of propositions which one does not require students to prepare for exams, by an ostensible sign that advertises to the reader in a hurry to arrive at the goal, and lacking necessary time or volition to explore in detail the numerous avenues of the science of extension, that he can pass over without being subsequently required to retrace his steps. (ibid, p. xi)[35]

Since geometry exams did not contain new content, any modern geometry could only be included as supplementary in this genre of textbook. As in 1826, Vincent acknowledged a large number of his contemporaries, here also adding Bergery, Terquem, and finally "des Annales de Mathématiques, dont le savant rédacteur a eu l'obligeance de m'adresser en outre diverses spécialement appropriées à mon ouvrage" (ibid, p. xv). In fact, Vincent wrote three articles for Gergonne's *Annales* between 1825 and 1826 though none of these were on subjects of elementary geometry.

Didiez

While Garnier, Biot, and Vincent composed geometry textbooks connected to Paris, and more particularly the École Polytechnique, outside this mathematical center authors also took opportunities to write textbooks with modern content.

Even so, N. J. Didiez is the only one of the authors in this corpus apparently without ties to the École Polytechnique. Little is known today of Didiez beyond his

[35] Au surplus, j'ai noté ces théories, comme plusieurs autres, ainsi qu'un grand nombre de propositions que l'on n'exige point des élèves qui se présentent aux examens, d'une signe ostensible qui avertira le lecteur pressé d'arriver au but, et manquant du temps ou de la volonté nécessaire pour explorer en détail les avenues nombreuses de la science de l'étendue, qu'il peut passer outre sans se trouver exposé par la suite à revenir sur ses pas.

published books and their reviews. He published the first part of his *Cours Complet du Géométrie* on planar elementary geometry in 1828, advertising this text as the first in a four-part series that would progress through three-dimensional elementary geometry, planar analytic geometry, and finally three-dimensional analytic geometry (Didiez 1828). These books represented an ongoing private mathematics course that Didiez had been teaching for the past eight years. The course and associated texts are described in an *Annales* review of Didiez's volume on arithmetic, published in 1825.

> M. Didiez has been giving public mathematics courses in Paris for several years. Preferring to surrender to his own ideas than to subject himself to follow those of another, but wanting to avoid the loss of time which the dictation of lesson entails, he proposes to publish a simple summary of his lessons; and it is the summary of those of arithmetic that he presents today. (Gergonne 1826a)[36]

Following these geometry texts, Didiez promised a subsequent series on applications to the "arts d'imitation et de construction" (Didiez 1828, p. i). However, if any of the other anticipated volumes ever appeared, there are no longer any publicly available extant copies. Though the circumstances of publication might indicate a less established author, Didiez's geometry textbook was published by Bachelier with drawings engraved by Adam—both well-respected individuals in the textbook medium.

Didiez dedicated his book to Dupin in a very elaborate full-page spread that listed the latter's many accomplishments: Dupin's membership at the Institut de France, various public honors, and position at the Conservatoire des Arts et Métiers. By 1828, following an initiative by Dupin in Paris, after-hours courses for workers were widely available in many French metropolitan areas. Likewise, Didiez may have decided to offer his own courses in the evening in order to attract a wide range of students and employed persons.

Bergery

Claude Lucien Bergery (1787–1863) more directly emulated Dupin by spearheading the public education efforts in Metz, which resulted in his books *Cours de sciences industrielles. Géométrie appliquée à l'industrie* (Bergery 1825), 1826).[37]

Bergery categorized his subject as "géométrie pratique." Practical geometry, as defined by François Joseph Servois in *Solutions peu connues de différens problèmes de géométrie-pratiques* concerned the study of executing "diverse geometric operations on the terrain" (Servois 1803, p. 1). Bergery devoted his introduction to addressing "les ouvriers et artistes" from Metz and explained that he had wanted to

[36] M. Didiez fait à Paris, depuis plusieurs années des cours publics de mathématiques. Aimant mieux s'abandonner à ses propres idées que de s'astreindre à suivre celles d'autrui, mais voulant éviter la perte de temps qu'entraîne la dictée des cahiers, il se propose de publier un simple résumé de ses leçons; et c'est le résumé de celles d'arithmétique qu'il présente aujourd'hui.

[37] See Vatin (2007) for a scientific biography of Bergery.

teach such a course since 1821 but only with the movement toward public education initiated to Dupin had such intentions been realized.[38] Bergery elaborated the practical potential of an education in geometry.

> The one Geometry has three distinct branches: the geometry of the straight line and circle, whose use is daily; that of curves, which explains many wonders; descriptive geometry, which one can call the language of constructions, and which applies to architecture properly speaking, to stone cutting, to carpentry, to painting, to sculpture and to a great number of other arts. (Bergery 1825, p. xix)[39]

The first volume was intended for students with only a basic knowledge of arithmetic. Bergery followed with a "second part" on the geometry of curves applied to industry, published in 1826 (Bergery 1826).[40]

Bergery enrolled at the École Polytechnique in 1806 and had taught geometry and engineering at the École royale de l'artillerie in Metz since 1817. There he worked with Poncelet, whom he mentioned as another instructor in the first edition of 1825. By Bergery's 1828 second edition, Poncelet appears as a primary influence (Bergery 1828a). Bergery framed this new edition as providing the necessary prerequisites for a young geometer to study "sans peine, les Propriétés projectives des figures dans le bel ouvrage de M. Poncelet, et de s'élever à des connaissances qui, jusqu'à présent, ont été rangées dans la Géométrie transcendante" (Bergery 1828a, p. vii).

Nevertheless, Bergery departed from his colleague on certain issues of simplicity, generality, and vocabulary, which will be explored in the following section. Bergery promised the most clear, methodical, and complete volume of practical geometry. Achieving this required a balance between accessible and comprehensive scope. Bergery refrained from too much technical language:

> I have abstained from several scientific expressions which, in the end, teach nothing, and each time that I have been obliged to employ them, I have taken care to explain them by equivalent expressions taken from common language. (ibid, p. xx)[41]

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38 Dupin's efforts toward public education within the context of engineering are discussed in Grattan-Guinness (1984), particularly Sect. 8.

39 La seule Géométrie a trois branches distinctes: la géométrie de la ligne droite et du cercle, dont l'usage est journalier; celles des courbes, qui explique tant de merveilles; la géométrie descriptive, qu'on peut appeler la langue des constructions, et qui s'applique à l'architecture proprement dite, à la coupe des pierres, à la charpenterie, à la peinture, à la sculpture et à un grand nombre d'autres arts.

40 This brief second volume, *Cours de Sciences Industrielles. Seconde Partie. Géométrie des courbes appliquée à l'industrie* covers the properties and construction of conic sections, lemniscates, spirals, cycloids, and a wide variety of other curves and analogous surfaces. Bergery directed the reader interested in "demonstrations de ceux de principes que nous avons seulement énoncés" to the *Annales*, Poncelet's *Traité* Brianchon's *Mémoire sur les lignes du second ordre*, among other contemporary texts. However, none of the modern geometry contained in these suggested readings is in this volume except in the form of succinctly stated results, where any modern techniques were obscured.

41 Je me suis abstenu de plusieurs expressions scientifiques qui, dans le fond, n'apprennent rien, et chaque fois que j'ai été obligé d'en employer, j'ai en soin de les expliquer par des équivalens pris dans le langage vulgaire.

By contrast, Bergery argued that it was necessary to introduce the recently discovered objects and expressions from contemporary geometry research into elementary geometry books.

> For some time I have regretted not finding in elementary books any notion of Transversals which make the practice of geometry so simple, Poles and Polars, conjugate Points, radical Axes, similitude Centers, Centers of gravity, and traces, rather frequently used, many of which result from recently discovered principles. Why, in effect, not try to place these new riches from science at the door of practitioners who can use them daily? (ibid, pp. vi—vii)[42]

Thus, this second edition marked a radical departure from his earlier texts in terms of new content and contemporary references. While Bergery wrote for a local audience, his text achieved wide distribution throughout France as well as at bookstores as far as Liége and London. A third edition, which retained the modern geometry from the second, appeared in 1835.

Terquem

Each of the aforementioned authors demonstrated considerably more reliance on contemporary research articles—particularly those in the *Annales*—than their fellow textbook writers. Nevertheless, Olry Terquem (1782—1862) eclipsed them all in his efforts to connect elementary geometry to modern geometry. Terquem published his *Manuel de géométrie* in 1829 for "l'usage des personnes privées des secours d'un maître" (Terquem 1829, p. i). The book opened with a two-column page of authors cited alphabetically from Anonyme to Vincent. These names ranged in time and fame from Archimedes to Durrande (a young geometer, who had published several articles in elementary geometry in Gergonne's *Annales* before his death in his early 20s). Gergonne is the most widely represented, with four listed citations.

Terquem was a teacher of mathematics and the librarian at the Dépôt Central de l'Artillerie Paris, but he is perhaps better known for co-founding the *Nouvelles Annales de Mathématiques* with Gerono in 1842. Like his manuals from a decade before, in this journal Terquem would strive to engage young geometers in new research and by many accounts succeeded. As observed by Chasles in an obituary from 1863,

> These *Nouvelles Annales*, in the modest format of 1 in octavo and a moderate price, were destined especially for teachers and numerous candidates to the Écoles of Government: Écoles Normale, Polytechnique, Militaire, de Marine, etc. M. Terquem, in exciting young geometers about research on posed questions, and welcoming their attempts, in making

[42] Depuis quelque temps on regrettait de ne trouver dans les livres élémentaires aucune notion sur les Transversales qui rendent si simple la pratique de la Géométrie, sur les Pôles et les Polaires, sur les Points conjugués, sur les Axes radicaux, sur les Centres de similitude, sur les Centres de gravité, et sur des tracés, d'un usage assez fréquent, dont plusieurs résultent de principes récemment découverts. Pourquoi, en effet, ne pay essayer de mettre ces nouvelles richesses de la science à la portée des praticiens qui peuvent s'en servir tous les jours?

> them aware of new facts of science, either by this publications or by his individual communications, rendered a great service to mathematical studies. (Chasles 1863, p. 245)[43]

Terquem's *Manuel de géométrie* aimed for a similar audience, but in the format of a compact textbook. In this book, Terquem criticized the standard curriculum in which students progressed slowly from elementary geometry, to planar and spherical trigonometry, to conic sections, to second-degree surfaces, and finally to projective procedures and descriptive geometry. Many students dropped out along the way, even though for physical science and industrial arts "les propriétés des sections coniques, les moyens graphiques sont au moins aussi importans à connaître que la mesure des distances, des aires, des volumes, but ordinaire de la géométrie élémentaire" (Terquem 1829, p. iv).

To correct this omission, Terquem proposed a one-year geometry course that would blend elementary, analytic, and descriptive geometry into a single subject accessible to any student with a previous course in algebra.[44] Along with condensing several years of geometry, Terquem intended to blend the writings of ancient texts with contemporary geometers:

> We have applied ourselves to editing this Manual following the ideas just given, remaining in the limits prescribed to this nature of work; we have given all that is essential in the ancient treatises and in the writings of contemporary geometers. (ibid, p. vi)[45]

This was not an idle promise. Of all the texts in our corpus, Terquem's is the most closely correlated with the methods and directions in recent research publications. Rather than a result of centralized administration, individual innovation drove the use of modern geometry in these seven titles. This personal initiative marked other forms of nineteenth-century education. In *Espaces de l'enseignement scientifique et technique*, historians d'Enfert and Virginie Fonteneau describe the potential for the individual in the evolution of teaching.

> One such approach leads equally to consider the relations between the individuals and the institutions within which they evolve, between individual actions and collective enterprises. The questions then concern constraints of the environment where these individuals exert their action as well as the margins of movement or the possible options available to them. For a number of actors evoked in this work, the realization of their projects or those, which

[43] Ces Nouvelles Annales, dans le modeste format de 1 in octavo et d'un prix modéré, étaient destinés surtout aux professeurs et aux nombreux candidats aux Écoles du Gouvernement: Écoles Normale, Poly-technique, Militaire, de Marine, etc. M. Terquem, en excitant les jeunes géomètres à des recherches sur des questions proposées, en accueillant leurs essais, en les tenant au courant des faits nouveaux de la science, tant par cette publication que par ses communications individuelles, rendait un grand service aux études mathématiques.

[44] Of historiographical interest, Terquem noted that he would not be straying too far from the "method of the ancients" since Euclid was essentially using algebra "sans signes, mais en phrases" in "five of his fifteen books" (iv). Terquem's reference to fifteen books of Euclid indicates that he was working from a different manuscript tradition than his contemporaries. For instance, François Peyrard divided Euclid's *Elements* into twelve books in his French translation (Peyrard 1804).

[45] On s'est appliqué à rédiger ce Manuel d'après les idées qu'on vient d'émettre, se tenant dans les limites prescrites à cette nature d'ouvrages; on a donné tout ce qu'il y a d'essentiel dans les anciens traités et dans les écrits des géomètres contemporains.

> they had been assigned is not exempt from personal interests in terms of career, status and social recognition. (D'Enfert and Fonteneau 2011, p. 11)[46]

For Garnier, Biot, and Vincent, the content of their textbooks was prescribed by the course of study at the École Polytechnique, and the inclusion of modern geometry required circumventing these prescriptions. In this aspect, elementary and analytic geometry appear equally conservative. Bergery and Terquem were less institutionally bound, but nevertheless utilized their introductions to justify the inclusion of newer concepts as providing practical shortcuts for students. Whether in introductions or citations, each of these authors indicated their knowledge of contemporary geometry research, often through specific articles published in the *Annales*. However, the majority viewpoint as expressed by great names like Lacroix and Legendre indicates that knowing about modern geometry was necessary, but not sufficient, for including modern geometry in a textbook. Authors also had to believe that there were educational advantages to these innovations.

The following section will examine what objects from modern geometry were perceived as worth importing and the contexts in which they were employed. These decisions reveal how geometers attempted to resolve tensions between theory and application and to strike a delicate balance between stating general principles and practicing specific constructions.

5 New Objects

New research in pure geometry coincided with new, specialized vocabulary. This trend is especially apparent in the *Annales des mathématiques pures et appliquées*, in which the terms pole and polar were first introduced and radical and ideal objects quickly proliferated (Servois 1810). Within these articles, the adoption of these terms signaled an awareness of contemporary results as well as a willingness to employ new results in further research. However, the use of an author's vocabulary did not necessarily coincide with support of his underlying method. Similarly in books, the new vocabulary of modern geometry could be adapted to more conservative contexts. Citations suggest that textbook authors were also aware of diverse contemporary approaches, and deliberately chose definitions that could benefit a pedagogical setting.

In particular, this section will focus on the concepts of pole and polar and then radical and similitude through the introduction of these objects between authors and

[46] Une telle approche conduit également à considérer les relations entre les individus et les institutions au sein desquelles ils évoluent, entre les actions individuelles et les entreprises collectives. Les interrogations portent alors sur les contraintes du milieu où ces individus exercent leur action ainsi que sur les marges de manoeuvre ou les possibilités de choix dont ils disposent. Pour nombre d'acteurs évoqués dans cet ouvrage, la réalisation de leurs projets ou de ceux qui leur ont été assignés n'est d'ailleurs pas exempte d'intérêts personnels en terme de carrière, de statut et de reconnaissance sociale.

editions. Finally, the treatment of imaginary points in the work of some authors reflected a willingness to extend teaching to the forefront of research.

Together these authors demonstrate common strategies for incorporating new material into textbooks. The following cases will demonstrate that geometric objects entered the textbook literature when they could be adapted to multiple contexts and represented through simple constructive language and figures.

5.1 Poles and Polars

The pole of a line was first defined by Servois in his 1810 solution to a posed problem published in the first volume of the *Annales*. His definition comprises the opening paragraph to his article:

> A line and a second degree curve being given, I call pole of the line, a point in the plane of this line and the curve around which turn all the chords of contact points of pairs of tangents to the curve from different points on the line: (Servois 1810, p. 338)[47]

The fact that such a point uniquely existed was often attributed to Monge, who had not assigned any special name to this property. In 1810, Servois simply stated and then applied the definition to finding a triangle circumscribing a given curve. In the third volume of the *Annales*, Gergonne introduced the corresponding polar of a point and proved the existence of the pole and polar in terms of coordinate equations (Gergonne 1813). Following Poncelet's publications on polar reciprocity in 1822 and Gergonne's use of duality in 1826, pole and polar were often associated with dual relationships between definitions and theorems (Poncelet 1822; Gergonne 1826b). Textbook authors' introductions to pole and polar can be classified under three strategies, which can be summarized as copy and paste, constructive innovations, and new applications and properties.

Copy and Paste

In an 1812 article on finding the distance between the centers of circles inscribed and circumscribed to a given triangle, Garnier promised a new edition of his *Application de l'algèbre à la géométrie*, which appeared the following year (Garnier 1812, p. 347). Both the article and his new textbook demonstrated a confident literacy in recent research published in the *Annales*, and, indeed, the addition of poles and polars in Garnier's second edition marked the influence of the new journal.

Garnier included poles and polars in a section on problems concerning tangents of second-degree curves. He began with a secant and a second-degree curve of the

[47] Une droite et une ligne du second ordre étant assignées, j'appelle pôle de la droite, le point du plan de cette droite et de la courbe autour duquel tournent toutes les cordes des points de contact des paires de tangentes à la courbe issues des différens points de la droite:

form, $ay^2 + cx^2dy + ex = 0$. Each secant passed through the curve in two points and tangent lines drawn from these two points would intersect in a point on the plane. Garnier then showed that when the secant turned around a given point G with coordinates $(g; h)$, then the locus of tangent intersection points was given by the equation

$$(2cg + e)x + (2ah + d)y + eg + dh = 0$$

a straight line. He added that "inversely" if the tangent intersection points lay on a straight line, then the corresponding secants all passed through the same point.

Following this theorem, Garnier provided a definition.

> Because of the relation, which exists between the point G and the line which is the locus of vertices of circumscribed angles, this point has been called the pole of this line, and one calls the line the polar of the point G. (Garnier 1813, p. 165)[48]

Though Garnier later used the result relating secants and tangents, this definition was the only mention of pole and polar in his text. In fact, Garnier's exposition was nearly identical to that of Gergonne from the *Annales* in 1813 down to the use of coefficients and the concluding paragraph. For comparison,

> Because of the relation which exists between the point (P) and the line (Q), this point has been called the Pole of this line; and one can, inversely, call the line (Q) the Polar of the point (P). (Gergonne 1813, p. 297)[49]

Undoubtedly, Garnier took his descriptions of poles and polars from this article, or perhaps a previous unpublished version, though Gergonne was not directly cited. While this might seem like plagiarism today, the complete appropriation of Gergonne's proof aligns with the acknowledged lack of originality and infrequent citations in textbook writing of the early nineteenth century. More surprisingly, Garnier here exhibited a remarkably quick publication process with the ability to integrate the previous year's newest results and vocabulary. The novelty of pole and polar at this time may also explain why they did not appear elsewhere in Garnier's text, despite their obvious abbreviating power.

Constructive Innovations

Garnier offered little innovation, except possibly introducing pole and polar to a new audience. By contrast, ten years later Biot used his analytic geometry textbook to provide a more constructive, graphical treatment of poles and polars. First, Biot defined pole and polar with respect to a circle centered at a point C as shown in his figure 43 (see Fig. 4.2). He proposed that the definition could be extended by analogy to all second-order curves.

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[48] A cause de la relation qui existe entre le point G et la droite qui est le lieu des sommets des angles circonscrits, ce point a été appelé le pôle de cette droite, et on peut appeler la droite la polaire du point G.

[49] A cause de la relation qui existe entre le point (P) et la droite (Q), ce point a été appelé le Pôle de cette droite; et on peut, à l'inverse, appeler la droite (Q) la Polaire du point (P).

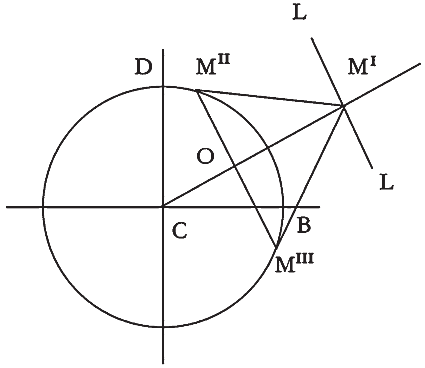

Fig. 4.2 Figure 43 in (Biot 1823)

> As analogous properties are found in all second order curves, one employs abbreviated denominations to express them. The point where the chords meet is called the pole of the line LL, from where the tangents are drawn, and reciprocally; this line is called the polar line of the point O. (Biot 1823, pp. 197–198)[50]

This quote might seem to suggest that Biot would simply rely on his circle construction and analogy, but as the text progressed he systematically provided constructions for the pole and polar of an ellipse (figure 58) and a parabola (figure 76), and only applied analogy to the case of the hyperbola (see Fig. 4.3).

> In continuing to follow, in calculations, the analogy between two curves, we will arrive at a similar construction, to determine the line which contains the vertices of the pairs of tangents, when one knows the intersection of the chords, and reciprocally. The similitude is so perfect that there is no need to explain here the application of this method, and it will suffice, to realize it, to cast ones' eyes on fig. 91. (ibid, p. 307)[51]

Earlier articles in pure geometry included constructions of poles and polars for the circle, but Biot's specific constructions for the ellipse and parabola appear unique. His constructions emphasized a different property of poles and polars than that stated in the definition as can be seen in the case of finding the polar of a point O with respect to a parabola.

Beginning with a parabola and a coplanar point O, draw a line OM parallel to the axis of the parabola and meeting the curve at the point M. Through the point M draw a tangent TMT′ to the curve. Then draw the chord M″M‴ through the point O and parallel to TMT′, where M‴ is the point where the chord meets the parabola. From

[50] Comme des propriétés analogues se retrouvent dans toutes les lignes du second ordre, on a employé des dénominations abrégées pour les exprimer. Le point où l cordes concourent, s'appelle le pôle de la droite LL, d'où les tangentes sont menées, et réciproquement; cette droite se nomme la ligne polaire du point O.

[51] En continuant de suivre, dans les calculs, l'analogie des deux courbes, on arrivera à une construction pareille, pour déterminer la droite qui contient les sommets des couples de tangentes, quand on connaîtra le point de concours des cordes, et réciproquement. La similitude est si parfaite, qu'il n'est pas besoin d'expliquer ici l'application de cette méthode, et qu'il suffira, pour s'en rendre compte, de jeter les yeux sur la fig. 91.

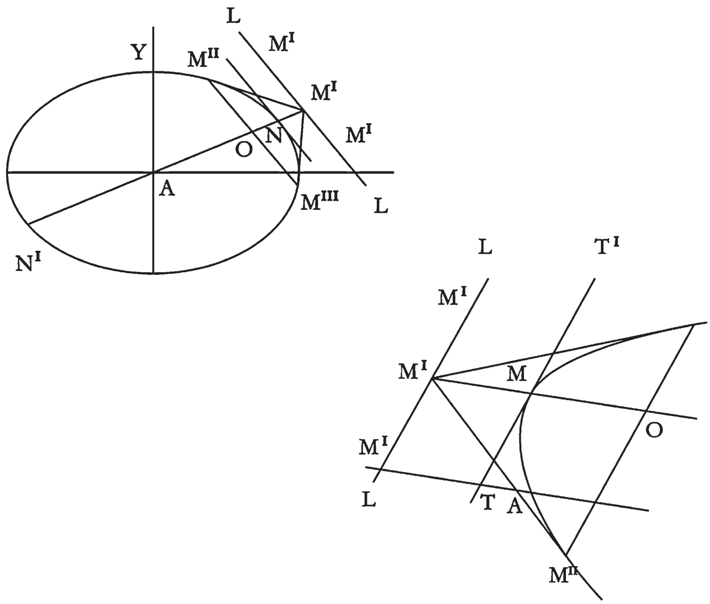

Fig. 4.3 Figures 58 and 76 in (Biot 1823)

the point M″ draw a new tangent that will meet the line OM in the point M′. Then the polar is the line through M′ and parallel to TT′, which is drawn as LM′L.

This construction is fairly simple to execute as it only involved drawing two tangents. Since Biot was writing a textbook in analytic geometry, his commitment to demonstrating particular cases is striking. Indeed, he proved each construction analytically using the specific equation of the given second-order curve rather than a general second-order curve. Biot also made notice of exceptional cases, such as if the point O lays on the axis of the parabola.

New Applications and Properties

Without the use of coordinate equations, Bergery and Vincent limited their exposition to the case of the circle. Nevertheless, these authors reveal the wide-ranging potential of objects from modern geometry to play various roles within the elementary geometry context.[52]

[52] Within his chapter on "Des systèmes qu'on peut former sur un plan avec trois lignes droites, ou circulaires" Didiez showed that:

130. When two lines are tangent to the same circumference, if one imagines that the intersection point of these tangents moves along a straight line drawn arbitrarily through this point, the tangent lines and points will change position; this will be the same for the chord of contact, but in

Bergery used the vocabulary of pole and polar, stating a defining property. He explained that the name pole "signified pivot of rotation" and then provided the construction of a polar using two chords and their associated tangents, which he had just shown how to construct in an earlier section (Bergery 1828a). To find the pole, one could either follow a similar construction of tangents or use the fact that the pole of a line lies on the perpendicular drawn from the center of the circle to the given line.

Bergery followed his constructions with practical applications. First, he described how to use poles and polars in a pivoting physical model that could produce a circular movement from a rectilinear movement without the use of gears (ibid., pp. 137–138). Bergery's applications show the potential benefit of poles and polars beyond theoretical geometry. Further, his construction demonstrated that these new objects were no more difficult to find than tangent lines to circles.

An alternative construction can be found in the second edition of Vincent's textbook (Vincent 1832). Vincent introduced pole and polar in an optional section on the propreties of transversal lines. For a given line OA and coplanar point P not on the line, one could construct lines through the point that meet the given line at points A; B; C; D; .... Then from the point O, any transversal to these new lines would meet them at points, respectively, denoted a; b; c; d; .... Each of the pairs of diagonals Ab and aB, Bc and bC, Cd and cD, ... will meet at a point, p; q; r; ... and the geometric locus of these points was a line through the point O. This line Op was the polar of the point P with respect to the angle AOa and reciprocally, P was the pole of the line Op.

Though two intersecting lines form a degenerate case of a conic section, Vincent's construction of pole and polar without an obvious curve was unusual. He then considered a circle centered at O with radius OA. If on the line OA, one took two points P; Q on the same side of the circle's center such that the product of their distances to the circle's center equaled the distance OA^2, then these points would be conjugate to each other with respect to the given circle.

Finally, if one drew a perpendicular through P or Q to the line PQ, this perpendicular would be the polar of the other conjugate point. That is, perpendicular PM was the polar of Q and perpendicular QN is the polar of P with respect to the circle

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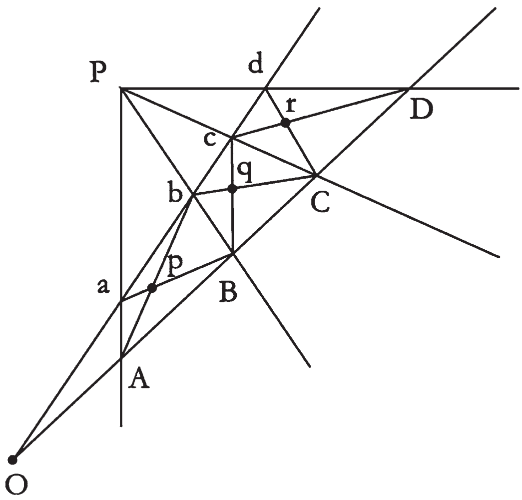
all the positions that the latter can take, it will not stop passing through the same point situated on a line drawn from the center of the circumference, perpendicular to the direction according to which one moves the intersection points of the tangents. (Didiez (1828), 124)

130. Lorsque deux droites sont tangentes à une même circonférence, si l'on conçoit que le point de concours de ces tangentes se meuve le long d'une droite menée comme on voudra par ce point, les tangentes et les points de contact changeront de position; il en sera de même de la corde de contact, mais dans toutes les positions que cette dernière pourra prendre, elle ne cessera de passer par un même point situé sur une droite menée du centre de la circonférence, perpendiculairement à la direction suivant laquelle on fait mouvoir le point de concours des tangentes.

Didiez's proof relied on the fact that the radius of the circle would be the mean proportional between the distance from the center to the chord of contact and the distance from the center to the tangents' intersection point. Though Didiez seemed comfortable with new vocabulary, he did not use the terms pole and polar here and this result concluded the section on "The system formed by two straight lines and a circle" without further applications.

Fig. 4.4 Illustration of Vincent's pole and polar construction



OA. In parentheses, Vincent noted that the names were based on the fact that these points and lines have analogous properties with respect to the circle as the above defined poles and polars have with respect to the given angle (see Fig. 4.4).

Having shown the relationship between transversal lines, conjugate points, and poles and polars, Vincent proceeded to prove several theorems about properties of poles and polars. Many of these theorems appeared in reciprocal pairs, such as the pole of all lines through a given point is on the polar of this point and reciprocally the polar of all points on a given line passes through the pole of this line. These proofs mainly utilized the fixed product definition of conjugate points discussed above. The theorems culminated in a pair of theorems (now known as Pascal's and Brianchon's theorems)—that for all hexagons inscribed to a circle, the points of intersection of opposite sides taken two by two are collinear, and reciprocally for circumscribed hexagons. Vincent followed these by several corollaries on quadrilaterals and triangles. He concluded with a parenthetical reference.

> See the *Annales de Mathématiques* in various places, and notably volume XIV, page 39 and following.—See also the *Correspondance sur l'École Polytechnique*. (Vincent 1832, p. 216)[53]

Indeed, Vincent's theorems summarized many recent results from these two publications including articles by Brianchon, Gergonne, and Poncelet.

As writers of analytic geometry textbooks, both Garnier and Biot would have expected their readers to be familiar with elementary geometry from a previous course, while Didiez, Bergery, and Vincent only anticipated their students knew elementary arithmetic. In his geometry textbook, Terquem attempted to create a volume that introduced beginning students to elementary and analytic geometry at

[53] Voyez les *Annales de Mathématiques* en divers endroits, et notamment Tome XIV, page 39 et suiv.– Voyez aussi la *Correspondance sur l'École Polytechnique*.

the same time. This meant a lengthier text with a combination of purely constructive and coordinate equation definitions and examples. Terquem's definition of poles and polars is computationally similar to Bergery's, emphasizing the turning property of the chords. Terquem then proved the validity of this property by computing with the harmonic ratios created by the intercepted segments.

Terquem stated that the intersection of two given polar lines is the pole of the line containing their poles with respect to the same circle. He claimed this property was "easy to prove" and proceeded by showing how it could be applied in problem solving. For instance, he employed poles and polars in showing how to find a third point such that its distance to two given points is in a given ratio (Terquem 1829, p. 147).

Though Terquem initially defined poles and polars with respect to circles, he later concluded that "the polar properties of circles […] belong to second degree curves," which he argued by considering the poles and polars as "the angular projections of analogous lines and points situated in a circle." This projective relationship furnished "an easy means to draw a tangent to a second degree curve through a point not on the curve" (ibid, p. 350). Terquem explained the consequences of this relationship in particular cases, such as when the pole of a diameter is at infinity and that the directrix of a hyperbola is the polar of its closest focus.

Finally, in a note to a discussion of the volume of ellipsoids and elliptic paraboloids, Terquem generalized polar reciprocity with the use of coordinate equations.

> If from a point A lying on a curve of degree p, one draws tangents to a curve of order m, the tangent points are situated on a line of order m–1 (13.); each position of the point A responds to another curve of tangent points; all are tangent to a curve of degree
> $(m + p - 2)^2$. In the particular case where $p = 1$, this latter curve reduces to $(m-1)^2$ points through the tangent curves constantly pass. This proposition with its reciprocal contains the general theory of polar curves. (ibid, p. 444)[54]

Terquem located this result at the very end of his text, which served to show his familiarity with the general theory of polar lines without alienating his intended audience of beginners. He was exceptionally generous with citations to contemporaries in the majority of his book, but did not provide any references for poles and polars. This may also reflect his knowledge and involvement with research mathematics, where the concepts were simply part of the standard lexicon by the end of the 1820s.

The use of poles and polars in textbooks reveal a range of methods for incorporating objects from modern geometry. Garnier represents one end of the spectrum, excerpting the treatment of poles and polars directly from Gergonne's article without any significant modification, commentary, or applications. In contrast, both Biot and Bergery provided more practically oriented texts by showing the visual and

[54] Si d'un point A situé sur une ligne du degré p, on mène des tangentes à une ligne de l'ordre m, les points de contact sont situés sur une ligne de l'ordre m–1 (13.); chaque position du point A répond à une autre courbe de points de contact; toutes sont tangentes à une courbe du degré $(m + p - 2)^2$. Dans le cas particulier où $p = 1$, cette dernière ligne se réduit à $(m - 1)^2$ points par lesquels passent constamment les courbes de contact. Cette proposition avec sa réciproque renferme la théorie générale des lignes polaires.

concrete properties. Biot presented precise constructions for almost all cases of conics, rather than simply giving a generic or circle-based construction. Bergery followed his definition of poles and polars with many examples of potential practical applications, oriented toward physical tools of measurement and design. Finally, Vincent and Terquem summarized recently proved theorems and solved problems that exhibited the significant role played by pole and polar in the past two decades. Since Vincent professed to be writing this section for advanced students, he could also introduce them to the latest research and even suggest ways in which they might contribute.

Though Garnier showed that textbooks could simply copy and paste from research articles, the latter presentations suggest ways in which textbook literature instead provided more nuanced understanding of poles and polars, in drawing connections to practical considerations and in synthesizing accumulated results to show the current state of knowledge.

5.2 Centers and Axes

Like poles and polars, the concepts of similitude and radical defined relationships between points and lines with respect to other coplanar figures.

Similitude Without Radicals

In Friedelmeyer's history of transformations in the nineteenth century, he explains how similitude was generalized from polygons to general curves at the end of the eighteenth century "soit par une traduction analytique, soit par une mise en relation d'éléments homologues" (Friedelmeyer 2016, p. 22). Euler introduced the "similitude center" in *De centro similitudinis*, but most early nineteenth century French authors attributed similitude centers and axes to Monge (pp. 24–27). For instance, Monge provided a brief account of similitude in an article published in the *Correspondance sur l'École Polytechnique* in 1814 (Monge 1814). In this two-page article, he showed how to calculate the coordinates of a similitude center for any two second-degree curves of the form

$$Ax^2 + By^2 + Cxy + 2Dx + 3Ey - 1 = 0$$

He concluded by referencing the *Traité des surfaces du second degré* by Monge and Hachette. In this text, the authors proved that when second-degree surfaces are cut by parallel planes, any two sections are similar and similarly placed curves, and so can be considered as parallel sections of a conic surface (Hachette and Monge 1813). While similarity is an important subject of this book, the expression similitude does not appear. The lack of systematic vocabulary makes the concept of similitude somewhat difficult to trace.

For instance, in his chapter on "Systems that one can form on a plane with three straight or circular lines," Didiez provided an extensive discussion of similarity, similitude, and homology for triangles. Following French convention, he attributed the similitude center of two similar and similarly placed triangles to Monge.

> This intersection point of the three lines drawn through the homologous vertices of two similar and similarly placed triangles has been named by MONGE, the similitude center of two triangles. It is the direct similitude center or the inverse similitude center, according to whether the two triangles are directly or inversely similar. (Didiez 1828, p. 88)[55]

However, he surprisingly referenced a specific volume and page number in the *Annales* for his definition of the similitude axis (p. 89). Though Didiez did not mention the author, the article is Gergonne's 1827 interpretation of Steiner's 1826 article on circle tangency first published in Crelle's *Journal für die reine und angewandte mathematik* (Steiner and Gergonne 1827). While this article contains much new vocabulary including radical axes, Gergonne made no claim to originality in the use of similitude axes. Further, Didiez restricted this initial definition to triangles, and Gergonne, in the cited text, defined similitude centers and axes for general polygons and circles. The citation thus appears merely as a jumping off point for Didiez, who modified the scope and order to suit the prominent role of triangles in his text.

Only in subsequent chapters did Didiez define similitude centers of axes for arcs of circles (Didiez 1828, p. 132) and then similar polygons (p. 179), each with references back to his initial triangle definition.[56]

In his study of circles, Didiez explained that the point of contact between two tangent circles would "evidently" be a direct or inverse similitude center depending on the kind of tangency. He then used this property to solve the problem of describing a circle passing through a given point A and tangent to two given circles on a plane. This question has four solutions, which utilized properties of similitude centers, as can be seen in the case where the tangent circles are exterior.

> Suppose the question is solved, and AED is a circumference passing through the point A and exteriorly tangent at D and E to the given circumferences. The tangent points D and E will be the inverse similitude centers of the circumferences to which they belong (no. 142). (Didiez 1828, p. 204)[57]

[55] Ce point de concours des trois droites menées par les sommets homologues, de deux triangles semblables et semblablement situés, a été nommé par MONGE, le centre de similitude des deux triangles. Il est centre de similitude directe ou centre de similitude inverse, suivant que les deux triangles sont directement ou inversement semblables.

[56] A nearly identical definition of similitude can be found in the second edition of Étienne Bobillier's Cours de Géométrie from 1834. Based on the table of contents from the 1832 edition (only available at Archives Départamentales de la Marne, and thus not included in our corpus), little changed between the first and second edition. In both of these editions, Bobillier's text is very brief (less than 100 pages in the first edition) and similitude is the only concept from modern geometry adopted in these first two editions. For more on Bobillier and this text, see dos Santos (2015).

[57] Supposons la question résolue, et soit AED une circonférence passant par le point A et touchant extérieurement en D et E les circonférences données. Les points de contact D et E seront les centres de similitude inverse des circonférences auxquelles ils appartiennent (no. 142).

In turn, these solutions form the basis of how Didiez solved the Apollonius problem, one of the most famous geometry problems of nineteenth-century geometry. Didiez had introduced similitude with triangles, but he found the largest scope of application in considering similitude between circles.

Uniting Similitude with Radicals

The history of radical axes and centers is less ambiguous. Louis Gaultier first defined radical axes and centers for two and three given circles in the *Journal de l'École Polytechnique* in 1813 (Gaultier 1813). Like similitude, these radical objects proliferated through geometry research articles by the 1820s (for instance, Steiner and Gergonne 1827; Plücker 1826; Bobillier 1827). Although these centers and axes were introduced roughly contemporaneously with poles and polars and appeared in the same journals, they were even less frequently used in textbooks—neither appears in the books of Didiez, Garnier, or Biot. The earliest instance that I found of these objects is in textbooks from the mid-1820s.

Vincent included similitude centers for polygons in the optional content of both his 1826 and 1832 editions. His definition was essentially the same as the one given by Didiez (except using "internal" for "inverse" and "external" for "direct"), which he extended to tetrahedra and then general polyhedra in a later chapter. In the second edition, Vincent significantly expanded and updated this material.

First, Vincent incorporated recent publications. In his discussion of polygons, he mentioned the "série de propositions sur les figures semblables, nouvellement démontrées par M. Chasles" though without an exact citation (Vincent 1832, p. 179). Further, he extended the concept of similitude from his first edition. By considering circles as regular polygons, Vincent determined there would be two similitude centers for any pair of circles. This definition could be applied to all possible cases of circle position (internal, external, tangent, concentric) as well as the degenerate cases where one of the given circles was a straight line or a point. In a later section, Vincent showed that the three centers of similitude of three similar and parallel polygons would lie on a straight line, the similitude axis. He then considered the case of three circles, which would have three internal and three external similitude centers, which determine four similitude axes. Here, too, Vincent demonstrated his knowledge of contemporary articles:

> These axes are the only common homologous lines that the three polygons or three circles can create. (See the *Annales de Mathématiques* of M. Gergonne, Volume XIII, page 197.) (ibid, 206)[58]

This is an article on tangent circles written anonymously as a letter to the editor of the journal, but subsequently attributed to Gergonne (1823).

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[58] Ces axes sont les seules droites homologues communes que puisent avoir les trois polygones ou les trois cercles. (Voy. les *Annales de Mathématiques* de M. Gergonne, Tome XIII, page 197.)

Vincent adopted the concept of radical in his second edition, defining a radical axis as the locus of points from which one can draw tangent lines of equal length to two given circles. In examining particular cases, Vincent concluded that two concentric circles would not have a radical axis since there are no points from which tangents of equal length can be drawn. He defined the radical center of three given circles as the point of intersection of their three radical axes.

The use of sideways M symbols to set off each of these results gives the impression of disjoint results. Thus when Vincent followed this definition with a theorem on similitude centers, he appeared to be changing topics. He proved that when one drew two secants through the similitude center of two given circles, O; O′, the eight resulting points of intersection could be taken four by four to define four new circumferences. Vincent called these four new circles the reciprocal circles to O, O′ relative to their similitude center. With this new concept, he then returned to radical axes in a corollary, revealing that each similitude center of two circles was the radical center of all their reciprocal circles relative to this similitude center. Thus, the new objects emerged as interdependent, consequently strengthening the relative importance of similitude in this second edition.

All of the problems are located at the end of Vincent's text, where he applied these new objects from modern geometry in several constructions, such as finding a circle tangent to three given circles. For this problem, Vincent provided three solutions. The first did not invoke any modern geometry. The second employed both similitude centers and radical axes, drawing on their common properties through reciprocal circles. Vincent explained that this second solution was superior to the first.

> Apart from the exceptional case that we have just signaled, the second construction has the great advantage of being applicable, when it is conveniently modified, to problems that were solved following number 276 inclusive. (Vincent 1832, p. 327)[59]

These earlier problems that Vincent referenced were versions of finding a circle subject to three conditions including passing through a given point or being tangent to a given line. Finally, Vincent attributed his third solution, which employed similitude axes and poles, to Gergonne in volume XVII of the *Annales*. This was the same article cited earlier by Didiez, Gergonne's interpretation of Steiner (Steiner and Gergonne 1827). For more on this problem, Vincent recommended the recent text of Bergery.

> We also encourage students to consult the Géométrie of M. Bergery. They will find there the discussion of different cases that can lead to the second construction, which, moreover, is due to M. Poncelet. (Vincent 1832, p. 328)[60]

[59] En mettant à part le cas d'exception que nous venons de signaler, la deuxième construction a le grand avantage de pouvoir s'appliquer, lorsqu'elle est convenablement modifiée, aux problèmes qui ont été résolus depuis le numéro 276 inclusivement.

[60] Nous engageons aussi les élèves à consulter la Géométrie de M. Bergery. Ils y trouveront la discussion des divers cas que peut présenter la deuxième construction que l'on doit d'ailleurs à M. Poncelet.

This citation suggests that textbook innovation could be contagious, or at least that these authors were aware of the novelties in each other's work. Rather than starting with triangles, Bergery first defined similitude for circles. He introduced similitude in a section on drawing secants, using proportions between segments to show that "the common secants to two circles, determined by parallel radii, have two points of intersection A; A' which are always conjugated to each other, in such a manner that the ratio of the two parts formed by each of these points, on the line of the centers, is equal to the ratio of the radii" (Bergery 1828a, p. 129). After explaining the designations "direct" and "inverse," Bergery promised that "we will see the basis of these denominations, when we will study similar polygons," which formed the topic of a later section (p. 130).

In an investigation of intersecting circles, Bergery applied similitude to the study of radical circles, axes, and centers. For a given circle and a coplanar point A, if one drew a secant intersecting the circle at D and C and passing through its center, then the circle centered at A with radius equal to the mean proportional between AC and AD would be radical to the first (p. 150). Bergery explicitly limited his study to radical circles where the point A lay outside of the given circle. Then the locus of centers of all circles radical to two given circles would be their radical axis, and analogously three given circles would define the radical center.

Returning to similitude, Bergery found that one could always "describe a circle that cuts two others A; B into four points C; D; E; F where they are met by two secants through one of their centers of similitude, as long as the four points are not on parallel radii" (p. 151). This new circle, Bergery named a reciprocal circle on account of the relationship between a similitude center I and the four points, namely ID: IC:: IF: IE. Combining all of these new terms, he concluded this section in showing that "one or the other of the similitude centers of two circles A; B, is the radical center of all reciprocal circles relative to this similitude center" (p. 152). This result, using slightly different vocabulary, had only just been published in the research articles of Steiner and Plücker (Steiner 1826, Plücker 1827). Bergery did not reference these geometers, but later credited Poncelet's solution to finding a circle tangent in the same way to three given circles. The "elegant construction due to M. Poncelet" utilized similitude centers and axes, radical centers and axes, and poles (Bergery 1828a, p. 162).

Having demonstrated how to construct tangent circles, Bergery emphasized their practical applications.

> Drawing tangent circles is frequently used in the construction of machines; gears that fit with other gears, or pinions, or lantern gears rest on these drawings. […] Tangent circles also form the curves that workers call ovals when they are completed or closed, and anses de panier when they are only halves. (ibid., p. 174)[61]

[61] Le tracé des cercles qui se touchent, est d'un usage fréquent dans la construction des machines: c'est sur ce tracé que repose celui des roues dentées qui engrènent soit avec d'autres roues dentées, soit avec des pignons, soit avec des lanternes. [...] Ce sont aussi des cercles tangens les uns aux autres qui forment ces courbes que les ouvriers nomment ovales quand elles sont complètes ou fermées, et anses de panier lorsqu'une moitié manque.

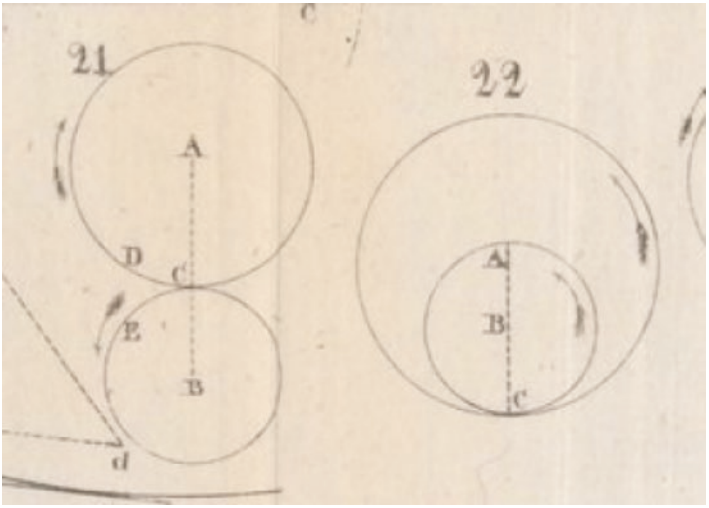

Fig. 4.5 Figures 21 and 22 in (Bergery 1828a), planche VI

Bergery illustrated these applications in more detail with several helpful figures. In figures 21 and 22 (Fig. 4.5), he used arrows to portray how the rotation of externally tangent circles will result in opposite motion, while internal tangency will create rotations in the same direction. In figure 24, Bergery drew *anses de panier* using the arcs of tangent circles. Bergery developed numerous methods for constructing several different kinds and sizes of arcs and ovals using tangent circles. These figures could achieve both aesthetic appeal and architectural utility. Bergery was thus able to connect some of the most recent research in planar geometry to the anticipated practices of his working students.

Like Didiez, Vincent, and Bergery, Terquem also attended to new findings from contemporary sources. However, he created some of his own new vocabulary (which did not catch on) to replace the term radical axis with dishomologous lines in order to emphasize its homologous relationship to the two given circles (Terquem 1829).

Terquem first used similitude in the context of similar polygons, defining the similitude center as the intersection of lines that pass through the homologous vertices between two similar polygons (p. 168). After showing the use of similitude in constructing similar polygons, he extended the concept to similar curves since "the polygons ABCDE, abcde are similar, and have the same similitude center as the similar curves in which they are inscribed" (p. 173). Analogously, he defined similitude centers in his study of similar polyhedra (p. 271).

Terquem applied these centers and axes toward finding similar and tangent curves. For instance, he demonstrated that the diagonals of a hexagon circumscribed to a conic section intersected in the same point, a property he attributed to Brianchon, and, as noted above, Vincent also included (p. 227). Terquem limited his proof to the case of the circle, which he attributed to another *Annales* writer, M. Durrande. He constructed a series of circles centered around the vertices of the circumscribed hexagon to determine similitude centers as well as poles and polars that could be applied to verify concurrence. While Terquem slightly modified Gaultier's vocabulary, he otherwise consistently credited properties and proofs as originating in the works of other authors.

The frequent, precise citations in these textbooks demonstrate a significant overlap between research and teaching mediums. While the flow of referenced information usually progressed from articles to textbooks, these authors also extended the applications and connections between similitude and radical centers and axes beyond what had appeared in research publications. As with poles and polars, Bergery displayed concrete examples of how these new objects could be applied to industry. Further, several authors exploited the interrelations between similitude and radicals to create a system of objects that interacted within solutions and proofs. While this systematicity may be found in some contemporary research articles — such as Steiner (1826) and Steiner and Gergonne (1827)—the longer format of textbooks enabled authors to present numerous and detailed examples of how similitude, radical, and polar relations operated in tandem.

The adaptability and abbreviating power of these objects to different formats may explain why they were attractive to a diversity of textbook writers. For instance, as similitude could be introduced with respect to polygons or circles, these concepts could be adjusted to suit the order of a textbook that introduced circles early in a Euclidean style or defined them later as the limit of regular polygons. Since many of these textbooks were not first editions, this meant that authors did not have to dramatically reformat their texts in order to incorporate aspects of modern geometry.

The selection, organization, and presentation of these objects from modern geometry exemplify a "process of elementarization" described by Schubring as "the transposition of knowledge into teachable knowledge and a related method" (Schubring 1987, p. 47). Knowing the elements of a subject could serve to develop deeper and more advanced research. However, not all of modern geometry was easily "elementarized" in this time period.

5.3 Imaginaries

Research publications in modern pure geometry—-particularly those of Poncelet, Brianchon, and Chasles—-did not shy from interpreting the imaginary points and lines that emerged from the application of algebra to coordinate equations (Poncelet 1822; Brianchon 1817; Chasles 1828).[62] Yet in the first third of the nineteenth century even textbooks on analytic geometry only gingerly treated imaginary objects.

[62] The question of representing imaginary points in geometry was a different one than that solved by the complex plane and also debated in Gergonne's *Annales* in this same time period.

Briefly overviewing the presence and absence of imaginaries in the above texts underlines a deep ambivalence in large part driven by concerns for constructive practicality.

A footnote by Dupin on the language of geometry serves to situate the inherent limitations of imaginary numbers in geometry textbooks.

> Often, in transcendental geometry, when one considers extension in all degrees of generality, one must speak at times of points, curves, surfaces, volumes. In order to avoid this long enumeration, we thought it necessary to designate all the magnitudes by the general expression graphic magnitudes, that is to say, capable of being drawn. (Dupin 1813, p. 15)[63]

The understanding that geometric quantities must be graphic and therefore drawable created a firm distinction between the real and the imaginary. In early nineteenth-century geometry, imaginary points, lines, surfaces, etc. could not be figured. Further, there were no attempts within these textbooks to assign an ontological status to imaginary objects. Nevertheless, the expression "imaginary" could play one of several roles within textbooks.

Imaginary values entered geometry textbooks through square roots. Once introduced, they could be discarded or studied. To take a common example, in using coordinate equations to represent second-degree curves, the geometers Dupin, Garnier, Didiez, and Terquem derived one real and one imaginary diameter or axis of a hyperbola. Unlike the real diameter, the imaginary diameter did not intersect the hyperbola. Garnier showed that if the half-diameter of a hyperbola, A', was real then the conjugate half-diameter B' would be of the form $B'\sqrt{-1}$. While the $\sqrt{-1}$ indicated that the points of intersection would be imaginary, Garnier employed the real B′ coefficient to show that the difference of squares of conjugate demi-diameters was equal to the difference of squares of the demi-axes (Garnier 1813, p. 148). Thus, the quantitative value of the imaginary diameter continued to display information about the curve. Including imaginary conjugate diameters also reinforced a general treatment of conic sections without exceptional cases.

The use of real coefficients allowed imaginary diameters to serve a function in better understanding properties of conic sections. Imaginary diameters thus functioned as an extension of imaginary points of intersection, when two curves shared an imaginary point and no real points, geometers concluded that the curves did not intersect. Yet calculations that resulted in imaginary values were also read as indicating a lack of existence (Garnier 1813, p. 75) or an impossible situation (Terquem 1829, p. 439). To give a sense of the language, consider how Biot introduced the square roots of negative numbers as impossible roots.

> Finally, if *B* extends past *A*, the circle described by the point *C* as center, with *A* as radius, will never cut the indefinite line *AB*. The points *X*; *X'*, thus cannot be found in this circum-

[63] Souvent, dans la géométrie transcendante, où l'on considère l'étendue dans tous les degrés de généralité, on doit parler à la fois de points, de lignes, de surfaces, de volumes. C'est pour éviter cette longue énumération, que nous avons cru devoir désigner toutes ces grandeurs par l'expression générale de grandeurs graphiques, c'est-à-dire, susceptibles d'être figurées.

> stance, and so the solution of the proposed question will be impossible. This is also what the equation between the numerical values shows; because, if b extends past a, the radical part $\sqrt{a^2 - b^2}$, which is common to the two roots, becomes imaginary, and consequently, the two roots are impossible. (Biot 1823, p. 19)[64]

This is also illustrated in Garnier's "Example II" where he showed that the parabola $x^2 + yx = 0$ was imaginary when $y > 1/4$. This part of the parabola was invisible.

Though less common, imaginary solutions also emerged in elementary geometry. For instance, in finding a circle of given radius that is tangent to a given line and passes through a given point, Vincent noted the possible cases: "There will be two solutions which can reduce to one only or become imaginary" (Vincent 1832, p. 312).[65] Vincent did not explicitly define what an imaginary solution was, and only used the term in the context of finding points—in this case the center of the desired circle. He more generally described geometric problems that do not lead to a real constructive solution as "impossible" such as in one of the cases of constructing a circle tangent to a given circle and a given line and passing through a given point: "Finally, the problem is impossible when the point is interior and the line is exterior to the given circle" (ibid, 321).[66] By contrast, Vincent also explained for exactly which configurations a construction would be possible, thus implying other cases were not.

To varying degrees the expressions: does not intersect, no longer exists, and becomes impossible, served to convey the non-constructive status of imaginary values. Given the practical constructive aims of these textbooks, it is not surprising that Poncelet's ideal chords comprised of imaginary points were not adapted for beginning students in the same way as radical and similitude axes. Even Poncelet's admitted admirers, like Bergery and Terquem, or those who used ideals in their own research articles, like Bobillier, continued to dismiss imaginary solutions (Bobillier 1828).

6 Reception

Though these texts cracked the traditional textbook mold, the majority was not received as particularly groundbreaking. Some acknowledged reception could be found in reviews published in the *Bulletin*. The review of Vincent comments on his decision to place problems at the end of the text, rather than interspersed with the

[64] Enfin, si B surpassait A, le cercle décrit du point C comme centre, avec A pour rayon, ne couperait pas du tout la droite indéfinie AB. Les points X; X', ne pourraient donc pas s'obtenir dans cette circonstance, et ainsi la solution de la question proposée serait impossible. C'est aussi ce que l'équation entre les valeurs numériques montre; car, si b surpasse a, la partie radicale $\sqrt{a^2 - b^2}$, qui est commune aux deux racines, devient imaginaire, et conséquemment, les deux racines sont impossibles.

[65] Il y aura deux solutions qui pourront se réduire à une seule ou devenir imaginaires.

[66] Enfin, le problème est impossible quand le point est. intérieur et le droite extérieure au cercle donné.

definitions and theorems (Anonymous 1827b, p. 82). Didiez was praised for his care and zeal and recommended to "teachers and students" (Anonymous 1828, p. 321). Though both reviewers summarized the contents of the respective textbooks, the authors' uses of modern geometry went unmentioned. Terquem's reviewer (signed D.—which from the list of contributors suggests either Duhamel or Dupin) noted the author's ambitious plan to combine elementary geometry, rectilinear and spherical trigonometry, conic sections, second-degree surfaces, and descriptive geometry into "un petit volume" (D. 1829, p. 1). However, the review focused primarily on Terquem's citations to contemporary geometers, even offering a correction to attribute a proof to Lacroix rather than Querret. Certainly, attention to geometers' new theorems and proofs is one of the unusual features of Terquem's textbook, but it did not reveal much of the book's contents. Similarly, in reviewing the first edition of Vincent's *Cours de géométrie élémentaire* for *Le Lycée, Journal de l'instruction rédigé par une société de professeurs, d'anciens élèves de l'Ecole normale*, Cournot mentioned that the author "distinguishes, by a very small typeface, less essential theories" without elaborating what these theories were (Cournot 2010, p. 551).

By contrast, Dupin's textbooks appeared to be quite influential through the 1820s, particularly for his unified approach to theory and practice. In an article based wholly on Dupin's *Developpements*, entitled "Démonstration des principaux Théorèmes de M. Dupin sur la courbure des surfaces," Gergonne proposed to introduce Dupin's work to a wider audience:

> We initially dreamed to give a simple analysis of the work of M. Dupin; but, this task having already been accomplished by several journals, we thought to do something more convenient and more useful simultaneously, in presenting here the principal points of the doctrine of the author rather briefly in order to enable its introduction in elementary treatises, where its importance must henceforth be found. (Gergonne 1814, p. 368)[67]

Thus, Dupin's success in obtaining a research article for his textbook may be attributed to his multiplatform publication approach, which combined articles, presentations to the Institut des sciences, and reviews in diverse publications (a strategy Poncelet also utilized to gain readers for his *Traité*).

Dupin's commitment to education also inspired the subsequent pedagogical literature of both Didiez and Bergery.[68] Indeed, in a self-review and defense written for the *Société des lettres, sciences, arts et agriculture*, Bergery framed his textbook as following Dupin's commitment to bring new results to elementary geometry and a more inclusive audience.

> M. Ch. DUPIN, in creating the Courses of industrial science, has opened the methodical and logical path; I dared to enlarge it and push it further; but authors who have preceded and followed us, remain in the narrow paths of the workshop routine; believing that workers are

[67] Nous avions d'abord songé à donner une simple analise de l'ouvrage de M. Dupin; mais, cette tâche ayant déjà été remplie par plusieurs journaux, nous avons pensé faire une chose plus convenable et plus utile à la fois, en présentant ici les principaux points de la doctrine de l'auteur dans un cadre assez resserré pour qu'il soit permis de l'introduire dans les traités élémentaires, où son importance doit désormais lui faire trouver place.

[68] Though neither geometer included the indicatrix or conjugate tangents in their own textbooks.

> ignorant of the simplest facts, they are dedicated to describe them more or less well, without the least attempt to explain them. March thus and you will arrive, they have said; as for why, you do not need it; we know it for you, that is enough. (Bergery 1828b, p. 20)[69]

Bergery claimed to include the new theory behind these practices, but faced criticism in a review by fellow textbook author Francoeur that appeared in the *Revue Encyclopédique*. Though overall impressed with Bergery's text, Francoeur doubted that the readers would benefit from Bergery's more advanced treatment:

> But we do not see that this geometry is more appropriate for teaching this class of men than that for all types of students; and, except for the choice of examples, which are in effect appropriate for industry, the work could also be well placed in the hands of all genres of readers. It appears to me that a geometry for artisans must be a simple collection of propositions, clarified by easy demonstrations, when that is possible, and by numerous applications to the arts. [...] We do not know how it is more useful to the student to teach him the succession of truths which compose an elementary treatise than to clearly conceive the details, and how these details themselves, when they are too numerous, are detrimental to the general instruction, that we want to give. (Francoeur 1828b, pp. 753–754)[70]

In his defense, Bergery countered that the theory enabled viewing the connection between various results and actually served to attract the interest of workers toward science:

These are not the geometric laws of nature that we would like to see disappear; these eternal applications which reveal a supreme intelligence, greatly excite the interest of workers and are very well suited to create love of science, so their suppression is not an evil. (ibid., p. 18).[71]

Bergery described his elementary treatise as the most extended and fruitful yet written. Though Francoeur had not singled out poles, polars, radicals, or similitude in his review, these aspects of modern geometry might similarly be criticized as outside the domain of useful pedagogical instruction. Francoeur's own textbook,

[69] M. Ch. DUPIN, en créant les Cours de sciences industrielles, a ouvert la voie méthodique et logique; j'ai osé l'élargir et la pousser plus avant; mais des auteurs qui nous ont précédés ou suivis, se sont plus à rester dans les étroits sentiers de la routine des ateliers; croyant les faits les plus simples ignorés des ouvriers, ils se sont attachés à les décrire plus ou moins bien, sans chercher le moins de monde à les expliquer. Marchez ainsi et vous arriverez, ont-ils dit; quant au pourquoi, vous n'en avez pas besoin; nous le savons pour vous, cela suffit.

[70] Mais on ne voit pas que cette géométrie soit plus propre à l'enseignement de cette classe d'hommes qu'à celui de toute espèce d'étudians; et, sauf le choix des exemples, qui sont en effet appropriés à l'industrie, l'ouvrage pourrait tout aussi bien être mis entre les mains de tous les genres de lecteurs. Il me paraît qu'une géométrie pour les artisans devrait être un simple recueil de propositions, éclairées par des démonstrations faciles, lorsque cela se peut, et par de nombreuses applications aux arts. [...] On ne sait pas assez combien il est plus utile à l'étudiant de lui faire saisir l'enchaînement des vérités qui composent un traité élémentaire, que d'en concevoir nettement les détails, et combien ces détails eux-mêmes, lorsqu'ils sont trop multipliés, nuisent à l'instruction générale, qu'on veut donner.

[71] Ce ne sont pas non plus les lois géométriques de la nature qu'on voudrait voir disparaître; ces applications éternelles qui révèlent une suprême intelligence, excitent trop l'intérêt des ouvriers et sont bien trop propres à faire aimer la science, pour que leur suppression ne soit pas un mal.

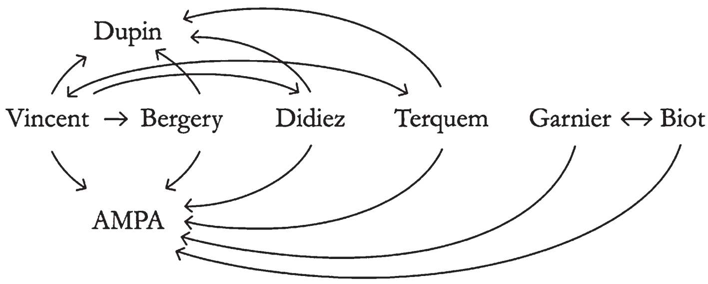

Fig. 4.6 Diagram of citations between textbooks

Cours complet de mathématiques pures, did not include any of these new objects (Francoeur 1828a).

Nevertheless, textbook authors could and did lead others to include modern geometry. As noted above, many of the first textbooks to promote modern geometry were second or subsequent editions, often revised following suggestions from colleagues and students. There was a substantial amount of citation among the authors in this study illustrated in the diagram in Fig. 4.6.

While Gergonne's *Annales* appears as a clear source of modern geometry, it is otherwise difficult to ascertain who borrowed from whom. Yet, looking slightly beyond the first third of the nineteenth century, there is some evidence of a ripple effect.

In the first two editions of his *Cours de Géométrie*, Étienne Bobillier (1798–1840) included the similitude center and axes with respect to similar polygons and polyhedra (Bobillier 1832, 1834). Though Bobillier was a frequent contributor to the *Annales* and utilized poles, polars, and ideal chords in this research, none of these objects were included in this course taught in Angers between 1831 and 1832 and published by the École Royale des arts et métiers of Châlons (Bobillier 1827, 1828).

However, the third edition of this text appeared in 1837 with an added dedication to A. Vincent (Bobillier 1837). Here, the content is much expanded and includes poles, polars, radicals, and the Apollonius problem. Though the treatment is not identical to Vincent's 1832 *Cours de géométrie élémentaire*, the overlap in scope is remarkable. It appears that Vincent's use of modern geometry inspired Bobillier in rewriting this edition.

But textbooks could also lose modern content. While Bobillier died in 1840, his text was subsequently adopted by the minister of agriculture and commerce for the Écoles nationales d'Arts et Métiers. Curiously, by the tenth edition (printed in 1850), most traces of the modern geometry, except for similitude, had disappeared and the text is much closer to the second than the third edition (Bobillier 1850). Likewise, in the third edition of Vincent's text, published in 1834, the author explained his decision to suppress the modern geometry that he had previously included in small font or offset by unique notation.

> The extension that I have given to several theories, having brought the volume of the preceding edition beyond the limits between that which one is accustomed to see included in

> elementary Geometry, I thought it necessary to suppress here the chapter on Transversals and Polars, the majority of the problems on Tangents, a chapter where I had very briefly shown the principles of the theory of Projections, and finally, a portion of the Numerical Problems, which are found multiplied beyond measure. One can consult, for the theory of Transversals, and that of Tangents, special works, notably those of Carnot, of MM. Brianchon, Poncelet, Gaultier de Tours, the *Annales de Mathématiques* of M. Gergonne, and finally, the Treatises of Geometry of MM. Bergery and Didiez, whose plan, less restricted than mine, admits developments which, for me, were nothing other than inconveniences. (Vincent 1834, p. v)[72]

The citations to Bergery and Didiez further emphasize the scarcity of textbooks containing modern geometry at this time. Vincent's reference to restraints implies that the text's role in preparing students for entrance exams may have curtailed the inclusion of new objects. The case of Bobillier's tenth edition suggests a similar institutional oversight and limitation.

7 Conclusion

In 1810, when Gergonne began publishing his *Annales*, he discussed the many functions and advantages of a journal devoted to mathematics.

> […] a periodical that allows Geometers to establish a commerce among themselves or, to put it better, a kind of community of views and ideas; a periodical that spares them from vainly engaging in research already undertaken by others; a periodical which guarantees to each the priority of the new results that they come across; a periodical finally, which assures everyone's work publicity, not less honourable for them than useful to the progress of science. (Gergonne 1810, pp. i—ii)[73]

This public exchange of new ideas aimed toward scientific progress provides a contrast to the slow repetition characteristic of most geometry textbooks, even though pedagogical goals extended to journal publications as well. Gergonne introduced

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[72] L'extension que j'avais donnée à plusieurs théories, ayant porté le volume de l'édition précédente au-delà des limites entre les quelles on est accoutumé à voir renfermer la Géométrie élémentaire, j'ai cru devoir supprimer dans celui-ci, le chapitre des Transversales et des Polaires, la plus grande partie des problèmes sur les Contacts, un chapitre où j'avais très brièvement exposé les principes de la théorie des Projections, et enfin, une portion des Problèmes Numériques, qui se trouvaient multipliés outre mesure. On pourra consulter, pour la théorie des Transversales et celle des Contacts, les ouvrages spéciaux, notamment ceux de Carnot, de MM. Brianchon, Poncelet, Gaultier de Tours, les *Annales de Mathématiques* de M. Gergonne, et enfin, les Traités de Géométrie de MM. Bergery et Didiez, dont le plan, moins restreint que le mien, admettait des développemens qui, pour moi, n'étaient pas sans inconvéniens.

[73] [...] un recueil qui permette aux Géomètres d'établir entre eux un commerce ou, pour mieux dire, une sorte de communauté de vues et d'idées; un recueil qui leur épargne les recherches dans lesquelles ils ne s'engagent que trop souvent en pure perte, faute de savoir que déjà elles ont été entreprises; un recueil qui garantisse à chacun la priorité des résultats nouveaux auxquels il parvient; un recueil enfin qui assure aux travaux de tous une publicité non moins honorable pour eux qu'utile au progrès de la science."

the *Annales* as above all consecrated to "recherches qui auront pour objet d'en perfectionner et d'en simplifier l'enseignement" (p. ii). However, while most articles in the *Annales* and similar journals could certainly be read by students or used by their instructors in creating teaching material, they were not explicitly presented by their authors as such. Instead, the research was framed as an end in itself, or to be used by other participants in the shaping "a community of views and ideas."

Without the community of readers and writers afforded by journals, books were self-sufficient by default. Book authors noted in their prefaces whether any arithmetic, algebra, or additional geometry might be required in advance, and if so, occasionally cited a few sources that might serve as preliminaries. Articles contained none of these explicit prerequisites, instead adopting intext references to cite particular concepts or results. Most books contained very few such references. Further, despite the prevalent redundancy, there is no evidence in the texts of opposition among textbook writers with respect to priority or potential plagiarism. Authors primarily restricted their particular criticisms to pedagogy and order of exposition.

Authors who participated in research and read articles seem to have been more likely to integrate new objects into their teachings, but due to institutional restrictions and limitations on student mathematical background, these instances remained rare and tentative. Textbook introductions demonstrated awareness of this novelty, by often highlighting material beyond the common curriculum. Even so, authors who included objects from modern geometry in some textbooks did not always continue to do so in later editions or other titles.

When authors adapted modern geometry to textbooks, they demonstrated careful consideration. First, the objects appeared as practical and constructive. Students received explicit instructions on finding and applying poles, polars, similitude, and radicals to solving problems from geometry, design, and engineering. The potential audience of mathematics textbooks mostly consisted of students who would not become mathematicians. Nevertheless, these students could find utility in certain concepts from recent research when presented in concrete terms.

Secondly, the length and summary nature of textbooks provided the possibility of bringing together systems of objects and displaying several methods for deriving results. While these objects and results were not original, textbook authors could curate an informative selection. Consequently, students could better observe the multiple potentials of these objects as tools through examples. No textbook claimed to be exhaustive in this respect, but even bringing together material from more than one source added value for the student interested in future research. The efficacy of these contributions is only speculative, however. It would be interesting, for example, to see whether Vincent's more advanced students actually pursued research in modern geometry at a greater rate than the average candidate for the École Polytechnique.

By the mid-twentieth century, many considered projective geometry as a teaching subject, with little research potential (Coolidge 1934, Bourbaki 1960). This article shows one aspect of this evolution—how textbooks began to represent modern geometry. This process began as research was still in flux, but notably poles, polars, similitude, and radicals featured in projective geometry textbooks a century later. Rather than signaling the ossification of modern geometry, these early manifestations for a student audience served to expand potential applications and contexts in an emergent discipline.

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Appendix Chronological table of geometry textbooks consulted

Date	Author	Title	Publisher	Place
1798	Gaspard Monge	Géométrie descriptive	Baudouin	Paris
1799	Silvestre-François Lacroix	Élémens de géométrie	Duprat	Paris
1800 (6th edition)	Charles Bossut	Cours de mathématiques	Firmin Didot	Paris
1800 (3rd edition)	Adrien-Marie Legendre	Éléments de géométrie	Firmin Didot	Paris
1802 (2nd edition)	Silvestre-François Lacroix	Essais de géométrie sur les plans et les surfaces courbes: Élémens de Géométrie descriptive	Duprat	Paris
1803 (3rd edition)	Silvestre-François Lacroix	Élémens de géométrie	Courcier	Paris
1803 (3rd edition)	Sylvestre-François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie	Courcier	Paris
1804	François Servois	Solutions peu connues de différens problèmes de géométrie-pratique	Bachelier	Paris
1806	Christian Kramp	Élémens de géométrie	Hansen	Cologne
1807 (4th edition)	Silvestre-François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie.	Courcier	Paris
1809 (4th edition)	Gaspard Monge	Application de l'Analyse à la Géométrie à l'usage de l'École Impériale Polytechnique	Vve Bernard	Paris
1809	Antoine Charles Marcelin Poullet-Deslile	Application de l'algèbre à la géométrie	Courcier	Paris
1810	Jean-Guillaume Garnier	Réciproques de la géométrie, suivies d'un recueil de théorèmes et de problèmes	Courcier	Paris
1810 (2nd edition)	Jean-Louis Boucharlat	Théorie des courbes et des surfaces du second ordre, précédée des principes fondamentaux de la géométrie analytique	Vve Courcier	Paris

(continued)

Appendix (continued)

Date	Author	Title	Publisher	Place
1810 (4th edition)	Jean-Baptiste Biot	Essai de géométrie analytique, appliqué aux courbes et aux surfaces du second ordre	J. Klostermann fils	Paris
1811 (5th edition)	Nicolas-Louis de LaCaille	Leçons élémentaires de mathématiques	Courcier	Paris
1811 (9th edition)	Silvestre-François Lacroix	Élémens de géométrie	Vve Courcier	Paris
1811	Claude-Jacques Toussaint	Traité de géométrie et d'architecture théorique et pratique, simplifié	Hocquet et Compe	Paris
1812 (2nd edition)	Louis Bertrand	Élémens de géométrie	J. J. Paschoud	Paris
1812	Emanuel Develey	Élémens de géométrie	Vve Courcier	Paris
1812 (4th edition)	Silvestre-François Lacroix	Essais de géométrie sur les plans et les surfaces courbes: Élémens de Géométrie descriptive	Vve Courcier	Paris
1812 (9th edition)	Adrien-Marie Legendre	Éléments de géométrie	Firmin Didot	Paris
1813	Charles Dupin	Développements de géométrie	Vve Courcier	Paris
1813 (6th edition)	Silvestre-François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie	Vve Courcier	Paris
1813	Jacques Schwab	Élémens de géométrie	Hissette	Nancy
1813 (5th edition)	Jean-Baptiste Biot	Essai de géométrie analytique, appliqué aux courbes et aux surfaces du second ordre	J. Klostermann fils	Paris
1813	Jean-Guillaume Garnier	Géométrie analytique, ou Application de l'algèbre à la géométrie	Vve Courcier	Paris
1815	J. de Stainville	Mélanges d'analyse algébrique et de géométrie	Vve Courcier	Paris
1816 (2nd edition)	Emanuel Develey	Élémens de géométrie	Vve Courcier	Paris
1817	Jean-Nicholas-Pierre Hachette	Éléments de géométrie à trois dimensions. Partie synthétique et partie algébrique	Vve Courcier	Paris
1817	Charles Michel Potier	Traité de géométrie descriptive	Firmin Didot	Paris

(continued)

Appendix (continued)

Date	Author	Title	Publisher	Place
1818	Gabriel Lamé	Examen des différentes méthodes employées pour résoudre les problèmes de géométrie	Vve Courcier	Paris
1819 (11th edition)	Silvestre-François Lacroix	Élémens de géométrie	Vve Courcier	Paris
1819	Antoine-André-Louis Reynaud	Traité d'application de l'algèbre à la géométrie, et de trigonométrie	Vve Courcier	Paris
1819	Paul-Marie-Gabriel Treuil	Essais de mathématiques, contenant quelques détails sur l'arithmétique, l'algèbre, la géométrie et la statique	Vve Courcier	Paris
1819	Louis-Léger Vallée	Traité de la géométrie descriptive	Vve Courcier	Paris
1821	L. J. George	Essai de géométrie pratique, destiné aux instituteurs primaires aux élèves des collèges	Beaucolin	Neufchateau
1821	Luis-Léger Vallée	Traité de la science du dessin, contenant la théorie générale des ombres, la perspective linéaire, la théorie générale des images d'optique et la perspective aérienne appliquée au lavis, pour faire suite à la Géométrie descriptive	Mme Vve Courcier	Paris
1822 (5th edition)	Silvestre-François Lacroix	Essais de géométrie sur les plans et les surfaces courbes: Élémens de Géométrie descriptive	Bachelier	Paris
1822	Jean-Nicolas Noël	Mélanges de mathématiques, ou Application de l'algèbre à la géométrie élémentaire	C. Lamort	Metz
1822	Charles Dupin	Applications de géométrie et de mécanique à la marine, aux ponts-et-chaussées, etc.	Bachelier	Paris
1823	Alexandre Denuelle	Traité simple et concis de géométrie pratique (2nd edition)	C. L. F. Panckoucke	Paris
1823	Joseph Adhémar	Cours de géométrie descriptive	Chaignieau fils ainé	Paris
1824	A. Person de Teyssèdre	Notions élémentaires d'arithmétique, de géométrie, de mécanique, de physique, de dessin linéaire, perspective et architecture	Fain	Paris
1825	Pierre Louis Marie Bourdon	Application de l'algèbre à la géométrie	Bachelier	Paris
1825	Claude-Lucien Bergery	Géométrie appliquée à l'industrie	Lamort	Metz

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Appendix (continued)

Date	Author	Title	Publisher	Place
1825	P. A. B. Dupont	Éléments de géométrie théorique et pratique	Boucher	Paris
1825	Charles Dupin	Géométrie et mécanique des arts et métiers et des beaux-arts	Bachelier	Paris
1826 (7th edition)	Jean-Baptiste Biot	Essai de géométrie analytique, appliqué aux courbes et aux surfaces du second ordre	Bachelier	Paris
1826	Claude-Lucien Bergery	Géométrie des sciences industrielles Seconde partie. Géométrie des courbes appliquée à l'industrie	Lamort	Metz
1826	Alexandre Vincent	Cours de géométrie élémentaire	Bachelier	Paris
1826	Nicolas Fourneau	Essais pratiques de géométrie	Firmin Didot	Paris
1826	Pierre Desnanot	Pratique du toisé géométrique, ou Géométrie pratique	Thibaud-Landriot	Clermont-Ferrand
1827	Lancelot	Dessin linéaire et géométrie pratique	Boniez-Lambert	Châlons
1827	A. Person de Teyssèdre	Géométrie des artistes et ouvriers	Decourchant	Paris
1827	Guillaume Henri Dufour	Géométrie perspective	Bachelier	Paris
1827	Louis Gaultier	Notions de géométrie pratique (2nd edition)	L. Colas	Paris
1827 (8th edition)	Sylvestre-François Lacroix	Traité élémentaire de trigonométrie rectiligne et sphérique, et d'application de l'algèbre à la géométrie.	Bachelier	Paris
1827	Louis-Etienne Lefébure de Fourcy	Leçons de géométrie analytique	Bachelier	Paris
1827	A. Lefevre	Applications de la géométrie à la mesure des lignes inaccessibles et des surfaces planes	Bachelier	Paris
1827 (5th edition)	Gaspard Monge (Barnabé Brisson)	Géométrie descriptive	Bachelier	Paris
1828	N. J. Didiez	Cours complet de géométrie	Bachelier	Paris
1828	Charles Dupin	Géométrie et mécanique des arts et métiers et des beaux-arts (2nd edition)	Bachelier	Paris
1828	E. Duchesne	Éléments de géométrie descriptive, à l'usage des élèves qui se destinent à l'École Polytechnique, à l'École militaire, à l'École de marine	H. Balzac	Paris
1828	Gabriel Gascheau	Géométrie descriptive	Bachelier	Paris

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Date	Author	Title	Publisher	Place
1828 (2nd edition)	Lorenzo Mascheroni (trans. A. M. Carette)	Géométrie du compas	Bachelier	Paris
1828	Claude-Lucien Bergery	Géométrie appliquée à l'industrie (2nd edition)	Lamort	Metz
1828	L. J. George	Géométrie pratique à l'usage des artistes et des ouvriers	C. J. Hissette	Nancy
1828	Émile Martin	Géométrie de l'ouvrier, ou Application de la règle, de l'équerre et du compas à la solution des problèmes de la géométrie	Audot	Paris
1829	Charles Mareschal-Duplessis	La Géométrie des gens due monde	Eberhart	Paris
1829 (2nd edition)	E. Duchesne	Éléments de géométrie descriptive, à l'usage des élèves qui se destinent à l'École Polytechnique, à l'École militaire, à l'École de marine	H. Balzac	Paris
1829 (3rd edition)	Enrico Giamboni (trans. D. Roux)	Éléments d'algèbre, d'arithmétique et de géométrie, où l'arithmétique et la géométrie se déduisent des premières notions de l'algèbre	Bachelier	Paris
1829	Amand-Denis Vergnaud	Manuel de perspective du dessinateur et du peintre (third edition)	Roret	Paris
1829	Olry Terquem	Manuel de géométrie, ou exposition élémentaire des principes de cette science	Roret	Paris
1829	Enrico Giamboni	Éléments d'algèbre, d'arithmétique et de géométrie, où l'arithmétique et la géométrie se déduisent des premières notions de l'algèbre (translated from edition)	Bachelier	Paris
1830 (1741)	Alexis-Claude Clairaut	Élémens de géométrie	Bachelier	Paris
1830	Louis Gaultier	Notions de géométrie pratique (2nd edition)	J. Renouard	Paris
1830 (14th edition)	Silvestre François Lacroix	Élémens de géométrie	Bachelier	Paris
1830	H. Vernier	Géométrie élémentaire à l'usage des classes d'humanités et des écoles primaires	L. Hachette	Paris
1830	Hippolyte Véron Vernier	Géométrie élémentaire à l'usage des classes d'humanités et des écoles primaires	A. Felin	Paris

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Date	Author	Title	Publisher	Place
1831	Auguste Mutel	Cours de géométrie et de trigonométrie	Vve Bernard	Paris
1831	Claude-Lucien Bergery	Géométrie des écoles primaires	P. Wittersheim	Metz
1831	Mathieu Bransiet	Abrégé de géométrie pratique appliquée au dessin linéaire	Moronval	Paris
1832	A. Delhorbe	Nouveau Traité de géométrie pratique	Guyot-Roblet	Rheims
1832	François-Cheri Duhousset	Application de la géométrie à la topographie	Migneret	Paris
1832 (14th edition)	Adrien-Marie Legendre	Éléments de géométrie	H. Remy	Brussels
1832	Alexandre Meissas	Cours de géométrie	A. Pihan Delaforest	Paris
1833	Antoine-André-Louis Reynaud	Théorèmes et problèmes de géométrie	Bachelier	Paris
1833	Alphonse-Louise-Bernard Boubée Lespin	Traité de géométrie et d'arpentage	Lecointe et Pougin	Paris
1835 (3rd edition)	G. F. Olivier	Géométrie usuelle	Maire-Nyon	Paris