Parabolic Anderson Model with Rough Dependence in Space

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Abstract This paper studies the one-dimensional parabolic Anderson model driven by a Gaussian noise which is white in time and has the covariance of a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{4}, \frac{1}{2})$ in the space variable. We derive the Wiener chaos expansion of the solution and a Feynman-Kac formula for the moments of the solution. These results allow us to establish sharp lower and upper asymptotic bounds for the nth moment of the solution.

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1 Introduction

A recent paper [9] studies the stochastic heat equation for $(t, x) \in (0, \infty) \times \mathbb{R}$

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2} \frac{\partial^2 u}{\partial x^2} + \sigma(u)\, \dot{W}\,, \tag{1}$$

where $\dot{W}$ is a centered Gaussian noise which is white in time and behaves as fractional Brownian motion with Hurst parameter $1/4 < H < 1/2$ in space, and σ may be a nonlinear function with some smoothness.

However, the specific case $\sigma(u) = u$, i.e.

$$\frac{\partial u}{\partial t} = \frac{\kappa}{2} \frac{\partial^2 u}{\partial x^2} + u\, \dot{W} \tag{2}$$

deserves some specific treatment due to its simplicity. Indeed, this linear equation turns out to be a continuous version of the parabolic Anderson model, and is related to challenging systems in random environment like KPZ equation [3, 6] or polymers [1, 4]. The localization and intermittency properties of (2) have thus been thoroughly studied for equations driven by a space-time white noise (see [13] for a nice survey), while a recent trend consists in extending this kind of result to equations driven by very general Gaussian noises [5, 8, 10, 11]. However, the rough noise $\dot{W}$ presented in this work is not covered by the aforementioned references.

To fill this gap, we first tackle the existence and uniqueness problem. Although the existence and uniqueness of the solution in the general nonlinear case (1) has been established in [9], in this linear case (2), one can implement a rather simple procedure involving Fourier transforms. Since this point of view is interesting in its own right and is short enough, we develop it in Sect. 3.1. In Sect. 3.2, we study the random field solution using chaos expansion. Following the approach introduced in [8, 10], we obtain an explicit formula for the kernels of the Wiener chaos expansion and we show its convergence, and thus obtain the existence and uniqueness of the solution. It is worth noting these methods treat different classes of initial data which are more general than in [9] and different from [2].

We then move to a Feynman-Kac type representation for the moments of the solution. In fact, we cannot expect a Feynman-Kac formula for the solution, because the covariance is rougher than the space-time white noise case, and this type of formula requires smoother covariance structures (see, for instance, [11]). However, by means of Fourier analysis techniques as in [8, 10], we are able to obtain a Feynman-Kac formula for the moments that involves a fractional derivative of the Brownian local time.

Finally, the previous considerations allow us to handle, in the last section of the paper, the intermittency properties of the solution. More precisely, we show sharp lower bounds for the moments of the solution of the form $\mathbf{E}[u(t,x)^n] \geq \exp(Cn^{1+\frac{1}{H}} t)$, for all $t \geq 0$, $x \in \mathbb{R}$ and $n \geq 2$, where C is independent of $t \geq 0$,

$x \in \mathbb{R}$ and n. These bounds entail the intermittency phenomenon and match the corresponding estimates for the case $H > \frac{1}{2}$ obtained in [10]. After the completion of this work, three of the authors have studied the parabolic Anderson model in more details in [12]. Existence and uniqueness results are extended to wider class of initial data. In particular, exact long term asymptotics for the moments of the solution of the form $\limsup \frac{1}{t} \sup_{|x|>\alpha t} \log \mathbb{E}(|u(t,x)|^p)$ are obtained.

2 Preliminaries

Let us start by introducing our basic notation on Fourier transforms of functions. The space of Schwartz functions is denoted by $\mathcal{S}$. Its dual, the space of tempered distributions, is $\mathcal{S}'$. The Fourier transform of a function $u \in \mathcal{S}$ is defined with the normalization

$$\mathcal{F}u(\xi) = \int_{\mathbb{R}} e^{-i\xi x} u(x) dx,$$

so that the inverse Fourier transform is given by $\mathcal{F}^{-1}u(\xi) = (2\pi)^{-1}\mathcal{F}u(-\xi)$. The Fourier transform of a tempered distribution can also be defined (see [18]).

Let $\mathcal{D}((0,\infty) \times \mathbb{R})$ denote the space of real-valued infinitely differentiable functions with compact support on $(0,\infty) \times \mathbb{R}$. Taking into account the spectral representation of the covariance function of the fractional Brownian motion in the case $H < \frac{1}{2}$ proved in [17, Theorem 3.1], we represent our noise W by a zero-mean Gaussian family $\{W(\varphi), \varphi \in \mathcal{D}((0,\infty) \times \mathbb{R})\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, whose covariance structure is given by

$$\mathbf{E}\left[W(\varphi)\, W(\psi)\right] = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\varphi(s,\xi)\, \overline{\mathcal{F}\psi(s,\xi)}\, |\xi|^{1-2H}\, ds d\xi, \tag{3}$$

where the Fourier transforms $\mathcal{F}\varphi$, $\mathcal{F}\psi$ are understood as Fourier transforms in space only and

$$c_{1,H} = \frac{1}{2\pi} \Gamma(2H+1) \sin(\pi H)\,. \tag{4}$$

We denote by $\mathfrak{H}$ the Hilbert space obtained by completion of $\mathcal{D}((0,\infty) \times \mathbb{R})$ with respect to the inner product

$$\langle \varphi, \psi \rangle_{\mathfrak{H}} = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\varphi(s,\xi) \overline{\mathcal{F}\psi(s,\xi)} |\xi|^{1-2H} d\xi ds\,. \tag{5}$$

The next proposition is from Theorem 3.1 and Proposition 3.4 in [17].

Proposition 2.1 *If $\mathfrak{H}_0$ denotes the class of functions $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R})$ such that*

$$\int_{\mathbb{R}_+\times\mathbb{R}} |\mathcal{F}\varphi(s,\xi)|^2 |\xi|^{1-2H} d\xi ds < \infty ,$$

then $\mathfrak{H}_0$ is not complete and the inclusion $\mathfrak{H}_0 \subset \mathfrak{H}$ is strict.

We recall that the Gaussian family W can be extended to $\mathfrak{H}$ and this produces an isonormal Gaussian process, for which Malliavin calculus can be applied. We refer to [16] and [7] for a detailed account of the Malliavin calculus with respect to a Gaussian process. On our Gaussian space, the smooth and cylindrical random variables F are of the form

$$F = f(W(\phi_1), \ldots, W(\phi_n)) ,$$

with $\phi_i \in \mathfrak{H}$, $f \in C_p^\infty(\mathbb{R}^n)$ (namely f and all its partial derivatives have polynomial growth). For this kind of random variable, the derivative operator D in the sense of Malliavin calculus is the $\mathfrak{H}$-valued random variable defined by

$$DF = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(W(\phi_1), \ldots, W(\phi_n))\phi_j .$$

The operator D is closable from $L^2(\Omega)$ into $L^2(\Omega; \mathfrak{H})$ and we define the Sobolev space $\mathbb{D}^{1,2}$ as the closure of the space of smooth and cylindrical random variables under the norm

$$\|DF\|_{1,2} = \sqrt{\mathbf{E}[F^2] + \mathbf{E}[\|DF\|^2_{\mathfrak{H}}]} .$$

We denote by δ the adjoint of the derivative operator (called divergence operator) given by the duality formula

$$\mathbf{E}\left[\delta(u)F\right] = \mathbf{E}\left[\langle DF, u\rangle_{\mathfrak{H}}\right], \tag{6}$$

for any $F \in \mathbb{D}^{1,2}$ and any element $u \in L^2(\Omega; \mathfrak{H})$ in the domain of δ.

For any integer $n \geq 0$ we denote by $\mathbf{H}_n$ the nth Wiener chaos of W. We recall that $\mathbf{H}_0$ is simply $\mathbb{R}$ and for $n \geq 1$, $\mathbf{H}_n$ is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_n(W(\phi)), \phi \in \mathfrak{H}, \|\phi\|_{\mathfrak{H}} = 1\}$, where H_n is the nth Hermite polynomial. For any $n \geq 1$, we denote by $\mathfrak{H}^{\otimes n}$ (resp. $\mathfrak{H}^{\odot n}$) the nth tensor product (resp. the nth symmetric tensor product) of $\mathfrak{H}$. Then, the mapping $I_n(\phi^{\otimes n}) = H_n(W(\phi))$ can be extended to a linear isometry between $\mathfrak{H}^{\odot n}$ (equipped with the modified norm $\sqrt{n!}\|\cdot\|_{\mathfrak{H}^{\otimes n}}$) and $\mathbf{H}_n$.

Consider now a random variable $F \in L^2(\Omega)$ which is measurable with respect to the σ-field $\mathcal{F}$ generated by W. This random variable can be expressed as

$$F = \mathbf{E}\left[F\right] + \sum_{n=1}^{\infty} I_n(f_n), \tag{7}$$

where the series converges in $L^2(\Omega)$, and the elements $f_n \in \mathfrak{H}^{\odot n}$, $n \geq 1$, are determined by F. This identity is called the Wiener chaos expansion of F.

The Skorohod integral (or divergence) of a random field u can be computed by using the Wiener chaos expansion. More precisely, suppose that $u = \{u(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is a random field such that for each (t,x), $u(t,x)$ is an $\mathcal{F}$-measurable and square-integrable random variable. Then, for each (t,x) we have a Wiener chaos expansion of the form

$$u(t,x) = \mathbf{E}\left[u(t,x)\right] + \sum_{n=1}^{\infty} I_n(f_n(\cdot, t, x)). \tag{8}$$

Suppose that $\mathbf{E}[\|u\|_{\mathfrak{H}}^2]$ is finite. Then, we can interpret u as a square-integrable random function with values in $\mathfrak{H}$ and the kernels f_n in the expansion (8) are functions in $\mathfrak{H}^{\otimes(n+1)}$ which are symmetric in the first n variables. In this situation, u belongs to the domain of the divergence operator (that is, u is Skorohod integrable with respect to W) if and only if the following series converges in $L^2(\Omega)$

$$\delta(u) = \int_0^{\infty} \int_{\mathbb{R}^d} u(t,x)\, \delta W(t,x) = W(\mathbf{E}[u]) + \sum_{n=1}^{\infty} I_{n+1}(\widetilde{f_n}), \tag{9}$$

where $\widetilde{f_n}$ denotes the symmetrization of f_n in all its $n+1$ variables.

For each $t \geq 0$, let $\mathcal{F}_t$ be the σ-field generated by W up to time t. Define the predictable σ-field as the σ-field of subsets of $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ generated by the collection of sets $\{A \times (s,t] \times B$, where $0 \leq s < t$, $A \in \mathcal{F}_s$ and B is a Borel set in $\mathbb{R}$. Denote by Λ_H the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathfrak{H}$ and $\mathbf{E}[\|g\|_{\mathfrak{H}}^2] < \infty$. Then, if $g \in \Lambda_H$, the Skorohod integral of g with respect to W coincides with the Itô integral defined in [9] and we have the isometry

$$\mathbf{E}\left[\left(\int_{\mathbb{R}_+}\int_{\mathbb{R}} g(s,x) W(ds,dx)\right)^2\right] = \mathbb{E}\|g\|_{\mathfrak{H}}^2\,. \tag{10}$$

Now we are ready to state the definition of the solution to Eq. (2). Denote by $p_t(x)$ the heat kernel on the real line related to $\frac{\kappa}{2}\Delta$. We denote by $*$ the convolution operation.

Definition 2.2 Let $u = \{u(t,x), 0 \le t \le T, x \in \mathbb{R}\}$ be a real-valued predictable stochastic process such that for all $t \in [0,T]$ and $x \in \mathbb{R}$ the process $\{p_{t-s}(x-y)u(s,y)\,\mathbf{1}_{[0,t]}(s), s \ge 0, y \in \mathbb{R}\}$ belongs to Λ_H. We say that u is a mild solution of (2) if for all $t \in [0,T]$ and $x \in \mathbb{R}$ we have

$$u(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)u(s,y)W(ds,dy) \quad a.s., \tag{11}$$

where and in what follows the stochastic integral is always understood in the sense of Itô and coincides with the Skorohod integral defined by (6).

3 Existence and Uniqueness

In this section we prove the existence and uniqueness result for the solution to Eq. (2) by means of two different methods: one is via Fourier transform and the other is via chaos expansion.

3.1 Existence and Uniqueness via Fourier Transform

In this subsection we discuss the existence and uniqueness of Eq. (2) using techniques of Fourier analysis.

Let $\dot{H}_0^{\frac{1}{2}-H}$ be the set of functions $f \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} |\mathcal{F}f(\xi)|^2 |\xi|^{1-2H} d\xi < \infty$. This space is the time independent analogue to the space $\mathfrak{H}_0$ introduced in Proposition 2.1. We know that $\dot{H}_0^{\frac{1}{2}-H}$ is not complete with the seminorm $\left[\int_{\mathbb{R}} |\mathcal{F}f(\xi)|^2 |\xi|^{1-2H} d\xi\right]^{\frac{1}{2}}$ (see [17]). However, it is not difficult to check that the space $\dot{H}_0^{\frac{1}{2}-H}$ is complete for the seminorm $\|f\|^2_{\mathcal{V}(H)} := \int_{\mathbb{R}} |\mathcal{F}f(\xi)|^2 (1 + |\xi|^{1-2H}) d\xi$.

In the next theorem we show the existence and uniqueness result assuming that the initial condition belongs to $\dot{H}_0^{\frac{1}{2}-H}$ and using estimates based on the Fourier transform in the space variable. To this purpose, we introduce the space $\mathcal{V}_T(H)$ as the completion of the set of elementary $\dot{H}_0^{\frac{1}{2}-H}$-valued stochastic processes

$$u(t) = \sum_{i=0}^{n-1} \mathbf{1}_{(t_i,t_{i+1}]}(t) u_i\,, \quad t \in [0,T]\,,$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0, T]$ and $u_i \in \dot{H}_0^{\frac{1}{2}-H}$, with respect to the seminorm

$$\|u\|^2_{\mathcal{V}_T(H)} := \sup_{t\in[0,T]} \mathbf{E}\|u(t,\cdot)\|^2_{\mathcal{V}(H)}. \tag{12}$$

We now state a convolution lemma.

Proposition 3.1 *Consider a function $u_0 \in \dot{H}_0^{\frac{1}{2}-H}$ and $\frac{1}{4} < H < \frac{1}{2}$. For any $v \in \mathcal{V}_T(H)$ we set $\Gamma(v) = V$ in the following way:*

$$\Gamma(v) := V(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)v(s,y)W(ds,dy), \quad t \in [0,T],\ x \in \mathbb{R}.$$

Then Γ is well-defined as a map from $\mathcal{V}_T(H)$ to $\mathcal{V}_T(H)$. Furthermore, there exist two positive constants c_1, c_2 such that the following estimate holds true on $[0, T]$:

$$\|V(t,\cdot)\|^2_{\mathcal{V}(H)} \le c_1 \|u_0\|^2_{\mathcal{V}(H)} + c_2 \int_0^t (t-s)^{2H-3/2}\|v(s,\cdot)\|^2_{\mathcal{V}(H)}\, ds\,. \tag{13}$$

Proof Let v be a process in $\mathcal{V}_T(H)$ and set $V = \Gamma(v)$. The stochastic integral appearing in the definition of $\Gamma(v)$ exists as an Itô (or Skorohod) integral, because the process $\{p_{t-s}(x-y)v(s,y), \mathbf{1}_{[0,t]}(s), s \ge 0, y \in \mathbb{R}\}$ is predictable and square integrable. We focus on the bound (13) for V.

Notice that the Fourier transform of V can be computed easily. Indeed, setting $v_0(t,x) = p_t * u_0(x)$ and invoking a stochastic version of Fubini's theorem, which can be easily proved in our framework, we get

$$\mathcal{F}V(t,\xi) = \mathcal{F}v_0(t,\xi) + \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{ix\xi}\, p_{t-s}(x-y)\, dx\right) v(s,y)W(ds,dy)\,.$$

According to the expression of $\mathcal{F}p_t$, we obtain

$$\mathcal{F}V(t,\xi) = \mathcal{F}v_0(t,\xi) + \int_0^t \int_{\mathbb{R}} e^{-i\xi y} e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,y)W(ds,dy)\,.$$

We now evaluate the quantity $\mathbf{E}[\int_{\mathbb{R}} |\mathcal{F}V(t,\xi)|^2|\xi|^{1-2H}d\xi]$ in the definition of $\|V\|_{\mathcal{V}_T(H)}$ given by (12). We thus write

$$\mathbf{E}\left[\int_{\mathbb{R}} |\mathcal{F}V(t,\xi)|^2|\xi|^{1-2H}d\xi\right] \le 2\int_{\mathbb{R}} |\mathcal{F}v_0(t,\xi)|^2|\xi|^{1-2H}d\xi$$
$$+2\int_{\mathbb{R}} \mathbf{E}\left[\left|\int_0^t \int_{\mathbb{R}} e^{-i\xi y}e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,y)W(ds,dy)\right|^2\right] |\xi|^{1-2H}d\xi := 2\,(I_1 + I_2)\ ,$$

and we handle the terms I_1 and I_2 separately.

The term I_1 can be easily bounded by using that $u_0 \in \dot{H}_0^{\frac{1}{2}-H}$ and recalling $v_0 = p_t * u_0$. That is,

$$I_1 = \int_{\mathbb{R}} |\mathcal{F}u_0(\xi)|^2 e^{-\kappa t|\xi|^2} |\xi|^{1-2H} d\xi \le C \, \|u_0\|^2_{\mathcal{V}(H)}.$$

We thus focus on the estimation of I_2, and we set $f_\xi(s,\eta) = e^{-i\xi\eta} e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,\eta)$. Applying the isometry property (10) we have:

$$\mathbf{E}\left[\left| \int_0^t \int_{\mathbb{R}} e^{-i\xi y} e^{-\frac{\kappa}{2}(t-s)\xi^2} v(s,y) W(ds,dy) \right|^2 \right] = c_{1,H} \int_0^t \int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}_\eta f_\xi(s,\eta)|^2 \right] |\eta|^{1-2H} \, ds d\eta,$$

where $\mathcal{F}_\eta$ is the Fourier transform with respect to η. It is obvious that the Fourier transform of $e^{-i\xi y} V(y)$ is $\mathcal{F}V(\eta+\xi)$. Thus we have

$$\begin{aligned} I_2 &= C \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2} \, \mathbf{E}\left[|\mathcal{F}v(s,\eta+\xi)|^2 \right] |\eta|^{1-2H} |\xi|^{1-2H} \, d\eta d\xi ds \\ &= C \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2} \, \mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2 \right] |\eta-\xi|^{1-2H} |\xi|^{1-2H} \, d\eta d\xi ds\,. \end{aligned}$$

We now bound $|\eta-\xi|^{1-2H}$ by $|\eta|^{1-2H} + |\xi|^{1-2H}$, which yields $I_2 \le I_{21} + I_{22}$ with:

$$\begin{aligned} I_{21} &= C \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2} \, \mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2 \right] |\eta|^{1-2H} |\xi|^{1-2H} \, d\eta d\xi ds \\ I_{22} &= C \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\kappa(t-s)\xi^2} \, \mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2 \right] |\xi|^{2-4H} \, d\eta d\xi ds\,. \end{aligned}$$

Performing the change of variable $\xi \to (t-s)^{-1/2}\xi$ and then trivially bounding the integrals of the form $\int_{\mathbb{R}} |\xi|^\beta e^{-\kappa\xi^2} d\xi$ by constants, we end up with

$$\begin{aligned} I_{21} &\le C \int_0^t (t-s)^{H-1} \int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2 \right] |\eta|^{1-2H} \, d\eta \, ds \\ I_{22} &\le C \int_0^t (t-s)^{2H-3/2} \int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\eta)|^2 \right] d\eta \, ds. \end{aligned}$$

Observe that for $H \in (\frac{1}{4}, \frac{1}{2})$ the term $(t-s)^{2H-3/2}$ is more singular than $(t-s)^{H-1}$, but we still have $2H - \frac{3}{2} > -1$ (this is where we need to impose $H > 1/4$).

Summarizing our consideration up to now, we have thus obtained

$$\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}V(t,\xi)|^2\right]|\xi|^{1-2H}d\xi$$
$$\leq C_{1,T}\,\|u_0\|^2_{\mathcal{V}(H)} + C_{2,T}\int_0^t (t-s)^{2H-3/2}\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\xi)|^2\right](1+|\xi|^{1-2H})\,d\xi\,ds, \tag{14}$$

for two strictly positive constants $C_{1,T}, C_{2,T}$.

The term $\mathbf{E}[\int_{\mathbb{R}}|\mathcal{F}V(t,\xi)|^2 d\xi]$ in the definition of $\|V\|_{\mathcal{V}_T(H)}$ can be bounded with the same computations as above, and we find

$$\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}V(t,\xi)|^2\right]d\xi$$
$$\leq C_{1,T}\,\|u_0\|^2_{\mathcal{V}(H)} + C_{2,T}\int_0^t (t-s)^{H-1}\int_{\mathbb{R}} \mathbf{E}\left[|\mathcal{F}v(s,\xi)|^2\right](1+|\xi|^{1-2H})\,d\eta\,ds\,. \tag{15}$$

Hence, gathering our estimates (14) and (15), our bound (13) is easily obtained, which finishes the proof. □

As in the forthcoming general case, Proposition 3.1 is the key to the existence and uniqueness result for Eq. (2).

Theorem 3.2 *Suppose that u_0 is an element of $\dot{H}_0^{\frac{1}{2}-H}$ and $\frac{1}{4} < H < \frac{1}{2}$. Fix $T > 0$. Then there is a unique process u in the space $\mathcal{V}_T(H)$ such that for all $t \in [0,T]$,*

$$u(t,\cdot) = p_t * u_0 + \int_0^t \int_{\mathbb{R}} p_{t-s}(\cdot - y)u(s,y)W(ds,dy). \tag{16}$$

Proof The proof follows from the standard Picard iteration scheme, where we just set $u_{n+1} = \Gamma(u_n)$. Details are left to the reader for the sake of conciseness. □

3.2 Existence and Uniqueness via Chaos Expansions

Next, we provide another way to prove the existence and uniqueness of the solution to Eq. (2), by means of chaos expansions. This will enable us to obtain moment estimates. Before stating our main theorem in this direction, let us label an elementary lemma borrowed from [10] for further use.

Lemma 3.3 *For $m \geq 1$ let $\alpha \in (-1+\varepsilon, 1)^m$ with $\varepsilon > 0$ and set $|\alpha| = \sum_{i=1}^m \alpha_i$. For $t \in [0,T]$, the m-th dimensional simplex over $[0,t]$ is denoted by $T_m(t) =$*

$\{(r_1, r_2, \ldots, r_m) \in \mathbb{R}^m : 0 < r_1 < \cdots < r_m < t\}$. *Then there is a constant* $c > 0$ *such that*

$$J_m(t,\alpha) := \int_{T_m(t)} \prod_{i=1}^{m} (r_i - r_{i-1})^{\alpha_i} dr \le \frac{c^m t^{|\alpha|+m}}{\Gamma(|\alpha| + m + 1)},$$

where by convention, $r_0 = 0$.

Let us now state a new existence and uniqueness theorem for our equation of interest (2).

Theorem 3.4 *Suppose that* $\frac{1}{4} < H < \frac{1}{2}$ *and that the initial condition* u_0 *satisfies*

$$\int_{\mathbb{R}} (1 + |\xi|^{\frac{1}{2}-H}) |\mathcal{F}u_0(\xi)| d\xi < \infty \,. \tag{17}$$

Then there exists a unique solution to Eq. (2)*, that is, there is a unique process u such that the Itô (or Skorohod) integral of the process* $\{p_{t-s}(x-y)u(s,y)\boldsymbol{1}_{[0,t]}(s), s \ge 0, y \in \mathbb{R}\}$ *exists for any* $(t,x) \in [0,T] \times \mathbb{R}$ *and relation* (11) *holds true.*

Remark 3.5

(i) The formulation of Theorem 3.4 yields the definition of our solution u for all $(t,x) \in [0,T] \times \mathbb{R}$. This is in contrast with Theorem 3.2 which gives a solution sitting in $\dot{H}_0^{\frac{1}{2}-H}$ for every value of t, and thus defined a.e. in x only.
(ii) Obviously a constant can be considered as a tempered distribution. Condition (17) is satisfied by constant functions.

Remark 3.6 In the later paper [12], the existence and uniqueness in Theorem 3.4 is obtained under a more general initial condition. Since the proof of Theorem 3.4 for condition (17) is easier and shorter, we present the proof as follows.

Proof of Theorem 3.4 Suppose that $u = \{u(t,x), t \ge 0, x \in \mathbb{R}^d\}$ is a solution to Eq. (11) in Λ_H. Then according to (7), for any fixed (t,x) the random variable $u(t,x)$ admits the following Wiener chaos expansion

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot,t,x)) \,, \tag{18}$$

where for each (t,x), $f_n(\cdot,t,x)$ is a symmetric element in $\mathfrak{H}^{\otimes n}$. Hence, thanks to (9) and using an iteration procedure, one can find an explicit formula for the kernels f_n for $n \ge 1$. Indeed, we have:

$$\begin{aligned} & f_n(s_1, x_1, \ldots, s_n, x_n, t, x) \\ & = \frac{1}{n!} p_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) p_{s_{\sigma(1)}} u_0(x_{\sigma(1)}) \,, \end{aligned} \tag{19}$$

where σ denotes the permutation of $\{1, 2, \dots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ (see, for instance, formula (4.4) in [8] or formula (3.3) in [10]). Then, to show the existence and uniqueness of the solution it suffices to prove that for all (t, x) we have

$$\sum_{n=0}^{\infty} n! \| f_n(\cdot, t, x) \|^2_{\mathfrak{H}^{\otimes n}} < \infty \,. \tag{20}$$

The remainder of the proof is devoted to prove relation (20).

Starting from relation (19), some elementary Fourier computations show that

$$\mathcal{F} f_n(s_1, \xi_1, \dots, s_n, \xi_n, t, x) = \frac{c_H^n}{n!} \int_{\mathbb{R}} \prod_{i=1}^{n} e^{-\frac{\kappa}{2}(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_{\sigma(i)} + \cdots + \xi_{\sigma(1)} - \zeta|^2} \times e^{-ix(\xi_{\sigma(n)} + \cdots + \xi_{\sigma(1)} - \zeta)} \mathcal{F} u_0(\zeta) e^{-\frac{\kappa s_{\sigma(1)} |\zeta|^2}{2}} d\zeta,$$

where we have set $s_{\sigma(n+1)} = t$. Hence, owing to formula (5) for the norm in $\mathfrak{H}$ (in its Fourier mode version), we have

$$n! \| f_n(\cdot, t, x) \|^2_{\mathfrak{H}^{\otimes n}} = \frac{c_H^{2n}}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}} \prod_{i=1}^{n} e^{-\frac{\kappa}{2}(s_{\sigma(i+1)} - s_{\sigma(i)})|\xi_i + \cdots + \xi_1 - \zeta|^2} e^{-ix(\xi_{\sigma(n)} + \cdots + \xi_{\sigma(1)} - \zeta)} \mathcal{F} u_0(\zeta) e^{-\frac{\kappa s_{\sigma(1)} |\zeta|^2}{2}} d\zeta \right|^2 \times \prod_{i=1}^{n} |\xi_i|^{1-2H} d\xi ds \,, \tag{21}$$

where $d\xi$ denotes $d\xi_1 \cdots d\xi_n$ and similarly for ds. Then using the change of variable $\xi_i + \cdots + \xi_1 = \eta_i$, for all $i = 1, 2, \dots, n$ and a linearization of the above expression, we obtain

$$n! \| f_n(\cdot, t, x) \|^2_{\mathfrak{H}^{\otimes n}} = \frac{c_H^{2n}}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^2} \prod_{i=1}^{n} e^{-\frac{\kappa}{2}(s_{\sigma(i+1)} - s_{\sigma(i)})(|\eta_i - \zeta|^2 + |\eta_i - \zeta'|^2)} \mathcal{F} u_0(\zeta) \overline{\mathcal{F} u_0(\zeta')} \times e^{ix(\zeta - \zeta')} e^{-\frac{\kappa s_{\sigma(1)}(|\zeta|^2 + |\zeta'|^2)}{2}} \prod_{i=1}^{n} |\eta_i - \eta_{i-1}|^{1-2H} d\zeta d\zeta' d\eta ds \,,$$

where we have set $\eta_0 = 0$. Then we use Cauchy-Schwarz inequality and bound the term $\exp(-\kappa s_{\sigma(1)}(|\zeta|^2 + |\zeta'|^2)/2)$ by 1 to get

$$n! \| f_n(\cdot, t, x) \|^2_{\mathfrak{H}^{\otimes n}} \le \frac{c_H^{2n}}{n!} \int_{\mathbb{R}^2} \left(\int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i - \zeta|^2} \prod_{i=1}^{n} |\eta_i - \eta_{i-1}|^{1-2H} d\eta ds \right)^{\frac{1}{2}} \times \left(\int_{[0,t]^n} \int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-\kappa(s_{\sigma(i+1)} - s_{\sigma(i)})|\eta_i - \zeta'|^2} \prod_{i=1}^{n} |\eta_i - \eta_{i-1}|^{1-2H} d\eta ds \right)^{\frac{1}{2}} \left| \mathcal{F} u_0(\zeta) \right| \left| \mathcal{F} u_0(\zeta') \right| d\zeta d\zeta' .$$

Arranging the integrals again, performing the change of variables $\eta_i := \eta_i - \zeta$ and invoking the trivial bound $|\eta_i - \eta_{i-1}|^{1-2H} \le |\eta_{i-1}|^{1-2H} + |\eta_i|^{1-2H}$, this yields

$$n!\|f_n(\cdot,t,x)\|^2_{\mathfrak{H}^{\otimes n}} \le \frac{c_H^{2n}}{n!}\left(\int_{\mathbb{R}} L^{\frac{1}{2}}_{n,t}(\zeta)\left|\mathcal{F}u_0(\zeta)\right| d\zeta\right)^2, \qquad (22)$$

where $L_{n,t}(\zeta)$ is

$$\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i|^2}(|\zeta|^{1-2H}+|\eta_1|^{1-2H})\times\prod_{i=2}^n(|\eta_i|^{1-2H}+|\eta_{i-1}|^{1-2H})d\eta ds.$$

Let us expand the product $\prod_{i=2}^n(|\eta_i|^{1-2H} + |\eta_{i-1}|^{1-2H})$ in the integral defining $L_{n,t}(\zeta)$. We obtain an expression of the form $\sum_{\alpha\in D_n}\prod_{i=1}^n|\eta_i|^{\alpha_i}$, where D_n is a subset of multi-indices of length $n-1$. The complete description of D_n is omitted for the sake of conciseness, and we will just use the following facts: $\mathrm{Card}(D_n) = 2^{n-1}$ and for any $\alpha \in D_n$ we have

$$|\alpha| \equiv \sum_{i=1}^n \alpha_i = (n-1)(1-2H), \quad \text{and} \quad \alpha_i \in \{0, 1-2H, 2(1-2H)\}, \quad i = 1,\dots,n.$$

This simple expansion yields the following bound

$$\begin{aligned} L_{n,t}(\zeta) \le |\zeta|^{1-2H}&\sum_{\alpha\in D_n}\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i|^2}\prod_{i=1}^n|\eta_i|^{\alpha_i}d\eta ds\\ &+\sum_{\alpha\in D_n}\int_{[0,t]^n}\int_{\mathbb{R}^n}\prod_{i=1}^n e^{-\kappa(s_{\sigma(i+1)}-s_{\sigma(i)})|\eta_i|^2}|\eta_1|^{1-2H}\prod_{i=1}^n|\eta_i|^{\alpha_i}d\eta ds\,. \end{aligned}$$

Perform the change of variable $\xi_i = (\kappa(s_{\sigma(i+1)} - s_{\sigma(i)}))^{1/2}\eta_i$ in the above integral, and notice that $\int_{\mathbb{R}} e^{-\xi^2}|\xi|^{\alpha_i}d\xi$ is bounded by a constant for $\alpha_i > -1$. Changing the integral over $[0,t]^n$ into an integral over the simplex, we get

$$\begin{aligned} L_{n,t}(\zeta) \le C|\zeta|^{1-2H}n!c_H^n&\sum_{\alpha\in D_n}\int_{T_n(t)}\prod_{i=1}^n(\kappa(s_{i+1}-s_i))^{-\frac{1}{2}(1+\alpha_i)}ds.\\ &+Cn!c_H^n\sum_{\alpha\in D_n}\int_{T_n(t)}(\kappa(s_2-s_1))^{-\frac{2-2H+\alpha_1}{2}}\prod_{i=2}^n(\kappa(s_{i+1}-s_i))^{-\frac{1}{2}(1+\alpha_i)}ds. \end{aligned}$$

We observe that whenever $\frac{1}{4} < H < \frac{1}{2}$, we have $\frac{1}{2}(1+\alpha_i) < 1$ for all $i = 2,\dots n$, and it is easy to see that α_1 is at most $1-2H$ so $\frac{1}{2}(2-2H+\alpha_1) < 1$. Condition $H > 1/4$ comes from the requirement that when $\alpha_1 = 1-2H$, we need $\frac{1}{2}(2-2H+\alpha_1)=\frac{1}{2}(3-4H)<1$. Thanks to Lemma 3.3 and recalling that $\sum_{i=1}^n \alpha_i=(n-1)(1-2H)$ for all $\alpha \in D_n$, we thus conclude that

$$L_{n,t}(\zeta) \le \frac{C(1+t^{\frac{1}{2}-H}\kappa^{\frac{1}{2}-H}|\zeta|^{1-2H})n!c^n c_H^n t^{nH}\kappa^{nH-n}}{\Gamma(nH+1)}\,.$$

Plugging this expression into (22), we end up with

$$n!\|f_n(\cdot,t,x)\|^2_{\mathfrak{H}^{\otimes n}} \le \frac{Cc_H^n c^n t^{nH}\kappa^{nH-n}}{\Gamma(nH+1)}\left(\int_{\mathbb{R}}(1+t^{\frac{1}{2}-H}\kappa^{\frac{1}{2}-H}|\zeta|^{\frac{1}{2}-H})\left|\mathcal{F}u_0(\zeta)\right|d\zeta\right)^2. \tag{23}$$

The proof of (20) is now easily completed thanks to the asymptotic behavior of the Gamma function and our assumption of u_0. This finishes the existence and uniqueness proof. □

4 Moment Formula and Bounds

In this section we derive the Feynman-Kac formula for the moments of the solution to Eq. (2) and the upper and lower bounds for the moments of the solution to Eq. (2) which allow us to conclude on the intermittency of the solution. We proceed by first getting an approximation result for u, and then deriving the upper and lower bounds for the approximation.

4.1 Approximation of the Solution

The approximation of the solution we consider is based on the following approximation of the noise W. For each $\varepsilon > 0$ and $\varphi \in \mathfrak{H}$, we define

$$W_\varepsilon(\varphi)=\int_0^\infty\int_{\mathbb{R}}[\rho_\varepsilon * \varphi](s,x)W(ds,dy)=\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(s,x)\rho_\varepsilon(x-y)W(ds,dy)dx\,, \tag{24}$$

where $\rho_\varepsilon(x)=(2\pi\varepsilon)^{-\frac{1}{2}}e^{-x^2/(2\varepsilon)}$. Notice that the covariance of W_ε can be read (either in Fourier or direct coordinates) as:

$$\mathbf{E}\left[W_\varepsilon(\varphi)W_\varepsilon(\psi)\right]=c_{1,H}\int_0^\infty\int_{\mathbb{R}}\mathcal{F}\varphi(s,\xi)\,\overline{\mathcal{F}\psi(s,\xi)}\,e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi ds \tag{25}$$

$$=c_{1,H}\int_0^\infty\int_{\mathbb{R}}\int_{\mathbb{R}}\varphi(s,x)f_\varepsilon(x-y)\psi(s,y)\,dxdyds,$$

for every φ, ψ in $\mathfrak{H}$, where f_ε is given by $f_\varepsilon(x) = \mathcal{F}^{-1}(e^{-\varepsilon|\xi|^2}|\xi|^{1-2H})$. In other words, W_ε is still a white noise in time but its space covariance is now given by f_ε. Note that f_ε is a real positive-definite function, but is not necessarily positive. The noise W_ε induces an approximation to the mild formulation of Eq. (2), namely

$$u_\varepsilon(t,x) = p_t * u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) u_\varepsilon(s,y)\, W_\varepsilon(ds,dy), \tag{26}$$

where the integral is understood (as in Sect. 3.1) in the Itô sense. We will start by a formula for the moments of u_ε.

Proposition 4.1 *Let W_ε be the noise defined by (24), and assume $\frac{1}{4} < H < \frac{1}{2}$. Assume u_0 is such that $\int_{\mathbb{R}}(1+|\xi|^{\frac{1}{2}-H})|\mathcal{F}u_0(\xi)|d\xi < \infty$. Then*

(i) *Equation (26) admits a unique solution.*
(ii) *For any integer $n \geq 2$ and $(t,x) \in [0,T] \times \mathbb{R}$, we have*

$$\boldsymbol{E}\left[u_\varepsilon^n(t,x)\right] = \boldsymbol{E}_B\left[\prod_{j=1}^n u_0(x + B_{\kappa t}^j) \exp\left(c_{1,H} \sum_{1\leq j\neq k\leq n} V_{t,x}^{\varepsilon,j,k}\right)\right], \tag{27}$$

with

$$V_{t,x}^{\varepsilon,j,k} = \int_0^t f_\varepsilon(B_{\kappa r}^j - B_{\kappa r}^k) dr = \int_0^t \int_{\mathbb{R}} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H} e^{i\xi(B_{\kappa r}^j - B_{\kappa r}^k)}\, d\xi dr. \tag{28}$$

In formula (28), $\{B^j;\, j = 1,\ldots,n\}$ is a family of n independent standard Brownian motions which are also independent of W and $\boldsymbol{E}_B$ denotes the expected value with respect to the randomness in B only.

(iii) *The quantity $\boldsymbol{E}[u_\varepsilon^n(t,x)]$ is uniformly bounded in ε. More generally, for any $a > 0$ we have*

$$\sup_{\varepsilon>0} \boldsymbol{E}_B\left[\exp\left(a \sum_{1\leq j\neq k\leq n} V_{t,x}^{\varepsilon,j,k}\right)\right] \equiv c_a < \infty.$$

Proof The proof of item (i) is almost identical to the proof of Theorem 3.4, and is omitted for the sake of conciseness. Moreover, in the proof of (ii) and (iii), we may take $u_0(x) \equiv 1$ for simplicity.

In order to check item (ii), set

$$A_{t,x}^\varepsilon(r,y) = \rho_\varepsilon(B_{\kappa(t-r)}^x - y), \quad \text{and} \quad \alpha_{t,x}^\varepsilon = \|A_{t,x}^\varepsilon\|_{\mathfrak{H}}^2. \tag{29}$$

Then one can prove, similarly to Proposition 5.2 in [8], that u_ε admits a Feynman-Kac representation of the form

$$u_\varepsilon(t,x) = \mathbf{E}_B\left[\exp\left(W(A^\varepsilon_{t,x}) - \frac{1}{2}\alpha^\varepsilon_{t,x}\right)\right]. \tag{30}$$

Now fix an integer $n \geq 2$. According to (30) we have

$$\mathbf{E}\left[u^n_\varepsilon(t,x)\right] = \mathbf{E}_W\left[\prod_{j=1}^n \mathbf{E}_B\left[\exp\left(W(A^{\varepsilon,B^j}_{t,x}) - \frac{1}{2}\alpha^{\varepsilon,B^j}_{t,x}\right)\right]\right],$$

where for any $j = 1,\ldots,n$, $A^{\varepsilon,B^j}_{t,x}$ and $\alpha^{\varepsilon,B^j}_{t,x}$ are evaluations of (29) using the Brownian motion B^j. Therefore, since $W(A^{\varepsilon,B^j}_{t,x})$ is a Gaussian random variable conditionally on B, we obtain

$$\begin{aligned}\mathbf{E}\left[u^n_\varepsilon(t,x)\right] &= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^n A^{\varepsilon,B^j}_{t,x}\|^2_{\mathfrak{H}} - \frac{1}{2}\sum_{j=1}^n \alpha^{\varepsilon,B^j}_{t,x}\right)\right]\\ &= \mathbf{E}_B\left[\exp\left(\frac{1}{2}\|\sum_{j=1}^n A^{\varepsilon,B^j}_{t,x}\|^2_{\mathfrak{H}} - \frac{1}{2}\sum_{j=1}^n \|A^{\varepsilon,B^j}_{t,x}\|^2_{\mathfrak{H}}\right)\right]\\ &= \mathbf{E}_B\left[\exp\left(\sum_{1\leq i<j\leq n}\langle A^{\varepsilon,B^i}_{t,x}, A^{\varepsilon,B^j}_{t,x}\rangle_{\mathfrak{H}}\right)\right].\end{aligned}$$

The evaluation of $\langle A^{\varepsilon,B^i}_{t,x}, A^{\varepsilon,B^j}_{t,x}\rangle_{\mathfrak{H}}$ easily yields our claim (27), the last details being left to the patient reader.

Let us now prove item (iii), namely

$$\sup_{\varepsilon>0}\sup_{t\in[0,T],x\in\mathbb{R}} \mathbf{E}\left[u^n_\varepsilon(t,x)\right] < \infty. \tag{31}$$

To this aim, notice first from the expression (27) that $\mathbf{E}\left[u^n_\varepsilon(t,x)\right]$ does not depend on $x \in \mathbb{R}$ (since $u_0(x) \equiv 1$) so that the $\sup_{t\in[0,T],x\in\mathbb{R}}$ in (31) can be reduced to a sup in t only. Next, still resorting to formula (27), it is readily seen that it suffices to show that for two independent Brownian motions B and $\tilde{B}$, we have

$$\sup_{\varepsilon>0,t\in[0,T]} \mathbf{E}_B\left[\exp\left(c\,F^\varepsilon_t\right)\right] < \infty, \quad \text{with} \quad F^\varepsilon_t \equiv \int_0^t\int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B_{\kappa r}-\tilde{B}_{\kappa r})}d\xi dr, \tag{32}$$

for any positive constant c. In order to prove (32), we expand the exponential and write:

$$\mathbf{E}_B\left[\exp(c\,F_t^{\varepsilon})\right] = \sum_{l=0}^{\infty} \frac{\mathbf{E}_B\left[(c\,F_t^{\varepsilon})^l\right]}{l!}\,. \tag{33}$$

Next, we have

$$\begin{aligned}\mathbf{E}_B\left[\left(F_t^{\varepsilon}\right)^l\right] &= \mathbf{E}_B\left[\int_{[0,t]^l}\int_{\mathbb{R}^l}\prod_{j=1}^{l} e^{-i\xi_j(B_{\kappa r_j}-\tilde{B}_{\kappa r_j})-\varepsilon|\xi_j|^2}|\xi_j|^{1-2H}d\xi dr\right]\\ &\le \int_{[0,t]^l}\int_{\mathbb{R}^l}\prod_{j=1}^{l} e^{-\kappa(t-r_{\sigma(j)})|\xi_j+\cdots+\xi_1|^2}\,|\xi_j|^{1-2H}\,d\xi dr\,,\end{aligned}$$

where σ is the permutation on $\{1,2,\ldots,l\}$ such that $t \ge r_{\sigma(l)} \ge \cdots \ge r_{\sigma(1)}$. We have thus gone back to an expression which is very similar to (21). We now proceed as in the proof of Theorem 3.4 to show that (31) holds true from Eq. (33). □

Starting from Proposition 4.1, let us take limits in order to get the moment formula for the solution u to Eq. (2).

Theorem 4.2 *Assume* $\frac{1}{4} < H < \frac{1}{2}$ *and consider* $n \ge 1$, $j,k \in \{1,\ldots,n\}$ *with* $j \ne k$. *For* $(t,x) \in [0,T]\times\mathbb{R}$, *denote by* $V_{t,x}^{j,k}$ *the limit in* $L^2(\Omega)$ *as* $\varepsilon \to 0$ *of*

$$V_{t,x}^{\varepsilon,j,k} = \int_0^t\int_{\mathbb{R}} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B_{\kappa r}^j-B_{\kappa r}^k)}d\xi dr.$$

Then $\boldsymbol{E}\left[u_{\varepsilon}^n(t,x)\right]$ *converges as* $\varepsilon \to 0$ *to* $\boldsymbol{E}[u^n(t,x)]$, *which is given by*

$$\boldsymbol{E}[u^n(t,x)] = \boldsymbol{E}_B\left[\prod_{j=1}^{n} u_0(B_{\kappa t}^j + x)\exp\left(c_{1,H}\sum_{1\le j\ne k\le n} V_{t,x}^{j,k}\right)\right]. \tag{34}$$

We note that in a recent paper [12], the moment formula for general covariance function is obtained. However we present the proof here for the sake of completeness.

Proof As in Proposition 4.1, we will prove the theorem for $u_0 \equiv 1$ for simplicity. For any $p \ge 1$ and $1 \le j < k \le n$, we can easily prove that $V_{t,x}^{\varepsilon,j,k}$ converges in $L^p(\Omega)$ to $V_{t,x}^{j,k}$ defined by

$$V_{t,x}^{j,k} = \int_0^t\int_{\mathbb{R}} |\xi|^{1-2H}e^{i\xi(B_{\kappa r}^j-B_{\kappa r}^k)}d\xi dr. \tag{35}$$

Indeed, this is due to the fact that $e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}e^{i\xi(B^j_{\kappa r}-B^k_{\kappa r})}$ converges to $|\xi|^{1-2H}e^{i\xi(B^j_{\kappa r}-B^k_{\kappa r})}$ in the $d\xi\otimes dr\otimes d\mathbf{P}$ sense, plus standard uniform integrability arguments. Now, taking into account relation (27), Proposition 4.1 (iii), the fact that $V^{\varepsilon,j,k}_{t,x}$ converges to $V^{j,k}_{t,x}$ in $L^2(\Omega)$ as $\varepsilon\to 0$, and the inequality $|e^x-e^y|\le(e^x+e^y)|x-y|$, we obtain

$$\begin{aligned}
&\mathbf{E}_B\left|\exp\left(c_{1,H}\sum_{1\le j\ne k\le n}V^{\epsilon,j,k}_{t,x}\right)-\exp\left(c_{1,H}\sum_{1\le j\ne k\le n}V^{j,k}_{t,x}\right)\right|\\
&\le\sup_{\epsilon>0}2\left(\mathbf{E}_B\left|\exp\left(2c_{1,H}\sum_{1\le j\ne k\le n}V^{\epsilon,j,k}_{t,x}\right)+\exp\left(2c_{1,H}\sum_{1\le j\ne k\le n}V^{j,k}_{t,x}\right)\right|^2\right)^{\frac12}\\
&\quad\times\left(\mathbf{E}_B\left|c_{1,H}\sum_{1\le j\ne k\le n}V^{\epsilon,j,k}_{t,x}-c_{1,H}\sum_{1\le j\ne k\le n}V^{j,k}_{t,x}\right|^2\right)^{\frac12},
\end{aligned}$$

which implies

$$\begin{aligned}
\lim_{\varepsilon\to0}\mathbf{E}\left[u^n_\varepsilon(t,x)\right]&=\lim_{\varepsilon\to0}\mathbf{E}_B\left[\exp\left(c_{1,H}\sum_{1\le j\ne k\le n}V^{\varepsilon,j,k}_{t,x}\right)\right]\\
&=\mathbf{E}_B\left[\exp\left(c_{1,H}\sum_{1\le j\ne k\le n}V^{j,k}_{t,x}\right)\right].
\end{aligned}\tag{36}$$

To end the proof, let us now identify the right hand side of (36) with $\mathbf{E}[u^n(t,x)]$, where u is the solution to Eq. (2). For $\varepsilon,\varepsilon'>0$ we write

$$\mathbf{E}\left[u_\varepsilon(t,x)\,u_{\varepsilon'}(t,x)\right]=\mathbf{E}_B\left[\exp\left(\langle A^{\varepsilon,B^1}_{t,x},A^{\varepsilon',B^2}_{t,x}\rangle_{\mathfrak{H}}\right)\right],$$

where we recall that $A^{\varepsilon,B}_{t,x}$ is defined by relation (29). As for (36) we can show that the above $\mathbf{E}\left[u_\varepsilon(t,x)\,u_{\varepsilon'}(t,x)\right]$ converges as ε,ε' tend to zero. So, $u_\varepsilon(t,x)$ converges in L^2 to some limit $v(t,x)$. For any positive integer k notice the identity

$$\mathbf{E}|u_\varepsilon(t,x)-u_{\varepsilon'}(t,x)|^{2k}=\sum_{j=0}^{2k}\frac{(-1)^j(2k)!}{j!(2k-j)!}\mathbb{E}\left[u_\varepsilon(t,x)^{2k-j}u_{\varepsilon'}(t,x)^j\right].\tag{37}$$

We can find the limit of $\mathbb{E}\left[u_\varepsilon(t,x)^{2k-j}u_{\varepsilon'}(t,x)^j\right]$ and then show that (37) converges to 0 as ε and ε' tend to 0. It is now clear that $u_\varepsilon(t,x)$ converges to $v(t,x)$ in L^p for all $p \geq 1$. Moreover, $\mathbf{E}[v^n(t,x)]$ is equal to the right hand side of (36). Finally, for any smooth random variable F which is a linear combination of $W(\mathbf{1}_{[a,b]}(s)\varphi(x))$, where φ is a C^∞ function with compact support, using the duality relation (6), we have

$$\mathbf{E}\left[Fu_\varepsilon(t,x)\right] = \mathbf{E}\left[F\right] + \mathbf{E}\left[\langle Y^\varepsilon, DF\rangle_{\mathfrak{H}}\right], \tag{38}$$

where

$$Y^{t,x}(s,z) = \left(\int_{\mathbb{R}} p_{t-s}(x-y)\, p_\varepsilon(y-z)u_\varepsilon(s,y)\,dy\right)\mathbf{1}_{[0,t]}(s).$$

Letting ε tend to zero in Eq. (38), after some easy calculation we get

$$\mathbf{E}[Fv_{t,x}] = \mathbf{E}[F] + \mathbf{E}\left[\langle DF, vp_{t-\cdot}(x-\cdot)\rangle_{\mathfrak{H}}\right].$$

This equation is valid for any $F \in \mathbb{D}^{1,2}$ by approximation. So the above equation implies that the process v is the solution of Eq. (2), and by the uniqueness of the solution we have $v = u$. □

4.2 Intermittency Estimates

In this subsection we prove some upper and lower bounds on the moments of the solution which entail the intermittency phenomenon.

Theorem 4.3 *Let $\frac{1}{4} < H < \frac{1}{2}$, and consider the solution u to Eq. (2). For simplicity we assume that the initial condition is $u_0(x) \equiv 1$. Let $n \geq 2$ be an integer, $x \in \mathbb{R}$ and $t \geq 0$. Then there exist some positive constants c_1, c_2, c_3 independent of n, t and κ with $0 < c_1 < c_2 < \infty$ satisfying*

$$\exp(c_1 n^{1+\frac{1}{H}}\kappa^{1-\frac{1}{H}}t) \leq \boldsymbol{E}\left[u^n(t,x)\right] \leq c_3 \exp\left(c_2 n^{1+\frac{1}{H}}\kappa^{1-\frac{1}{H}}t\right). \tag{39}$$

Remark 4.4 It is interesting to point out from the above inequalities that when $\kappa \downarrow 0$, the moments of u go to infinity. This is because the equation $\frac{\partial u}{\partial t} = u\dot{W}$ has no classical solution due to the singularity of the noise $\dot{W}$ in spatial variable x.

Proof of Theorem 4.3 We divide this proof into upper and lower bound estimates.

Step 1: Upper bound. Recall from Eq. (18) that for $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, $u(t,x)$ can be written as: $u(t,x)=\sum_{m=0}^{\infty} I_m(f_m(\cdot,t,x))$. Moreover, as a consequence of the hypercontractivity property on a fixed chaos we have (see [16, p. 62])

$$\|I_m(f_m(\cdot,t,x))\|_{L^n(\Omega)}\le (n-1)^{\frac{m}{2}}\,\|I_m(f_m(\cdot,t,x))\|_{L^2(\Omega)},$$

and substituting the above right hand side by the bound (23), we end up with

$$\|I_m(f_m(\cdot,t,x))\|_{L^n(\Omega)}\le n^{\frac{m}{2}}\,\|I_m(f_m(\cdot,t,x))\|_{L^2(\Omega)}\le \frac{c^{\frac{n}{2}}n^{\frac{m}{2}}t^{\frac{mH}{2}}\kappa^{\frac{Hm-m}{2}}}{\Gamma(mH/2+1)}.$$

Therefore from by the asymptotic bound of Mittag-Leffler function $\sum_{n\ge 0} x^n/\Gamma(\alpha n+1)\le c_1\exp(c_2x^{1/a})$ (see [14], Formula (1.8.10)), we get:

$$\|u(t,x)\|_{L^n(\Omega)}\le\sum_{m=0}^{\infty}\|J_m(t,x)\|_{L^n(\Omega)}\le\sum_{m=0}^{\infty}\frac{c^{\frac{m}{2}}n^{\frac{m}{2}}t^{\frac{mH}{2}}\kappa^{\frac{Hm-m}{2}}}{\left(\Gamma(mH+1)\right)^{\frac{1}{2}}}\le c_1\exp\left(c_2tn^{\frac{1}{H}}\kappa^{\frac{H-1}{H}}\right),$$

from which the upper bound in our theorem is easily deduced.
Step 2: Lower bound for u_ε. For the lower bound, we start from the moment formula (27) for the approximate solution, and write

$$\begin{aligned}&\mathbf{E}\left[u_\varepsilon^n(t,x)\right]\\&=\mathbf{E}_B\left[\exp\left(c_{1,H}\left[\int_0^t\int_{\mathbb{R}}e^{-\varepsilon|\xi|^2}\left|\sum_{j=1}^n e^{-iB_{\kappa r}^j\xi}\right|^2|\xi|^{1-2H}d\xi dr-nt\int_{\mathbb{R}}e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi\right]\right)\right].\end{aligned}$$

In order to estimate the expression above, notice first that the obvious change of variable $\lambda=\varepsilon^{1/2}\xi$ yields $\int_{\mathbb{R}}e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}d\xi=C\varepsilon^{-(1-H)}$ for some constant C. Now for an additional arbitrary parameter $\eta>0$, consider the set

$$A_\eta=\left\{\omega;\ \sup_{1\le j\le n}\sup_{0\le r\le t}|B_{\kappa r}^j(\omega)|\le\frac{\pi}{3\eta}\right\}.$$

Observe that classical small balls inequalities for a Brownian motion (see (1.3) in [15]) yield $\mathbf{P}(A_\eta)\ge c_1e^{-c_2\eta^2n\kappa t}$ for a large enough η. In addition, if we assume that A_η is realized and $|\xi|\le\eta$, some elementary trigonometric identities show that the following deterministic bound hold true: $|\sum_{j=1}^{n} e^{-iB^{j}_{\kappa r}\xi}| \geq \frac{n}{2}$. Gathering those considerations, we thus get

$$\mathbf{E}\left[u_{\varepsilon}^{n}(t,x)\right] \geq \exp\left(c_1 n^2 \int_0^t \int_0^{\eta} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H} d\xi dr - c_2 n t \varepsilon^{H-1}\right) \mathbf{P}\left(A_{\eta}\right)$$

$$\geq C \exp\left(c_1 n^2 t \varepsilon^{-(1-H)} \int_0^{\varepsilon^{1/2}\eta} e^{-|\xi|^2}|\xi|^{1-2H} d\xi - c_2 n t \varepsilon^{-(1-H)} - c_3 n \kappa t \eta^2\right).$$

We now choose the parameter η such that $\kappa\eta^2 = \varepsilon^{-(1-H)}$, which means in particular that $\eta \to \infty$ as $\varepsilon \to 0$. It is then easily seen that $\int_0^{\varepsilon^{1/2}\eta} e^{-|\xi|^2}|\xi|^{1-2H} d\xi$ is of order $\varepsilon^{H(1-H)}$ in this regime, and some elementary algebraic manipulations entail

$$\mathbf{E}\left[u_{\varepsilon}^{n}(t,x)\right] \geq C \exp\left(c_1 n^2 t \kappa^{H-1} \varepsilon^{-(1-H)^2} - c_2 n t \varepsilon^{-(1-H)}\right) \geq C \exp\left(c_3 t \kappa^{1-\frac{1}{H}} n^{1+\frac{1}{H}}\right),$$

where the last inequality is obtained by choosing $\varepsilon^{-(1-H)} = c\,\kappa^{\frac{H-1}{H}} n^{\frac{1}{H}}$ in order to optimize the second expression. We have thus reached the desired lower bound in (39) for the approximation u^{ε} in the regime $\varepsilon = c\,\kappa^{\frac{1}{H}} n^{-\frac{1}{H(1-H)}}$.

Step 3: Lower bound for u. To complete the proof, we need to show that for all sufficiently small ε, $\mathbf{E}\left[u_{\varepsilon}^{n}(t,x)\right] \leq \mathbf{E}[u^{n}(t,x)]$. We thus start from Eq. (27) and use the series expansion of the exponential function as in (33). We get

$$\mathbf{E}\left[u_{\varepsilon}^{n}(t,x)\right] = \sum_{m=0}^{\infty} \frac{c_{1,H}^{m}}{m!} \mathbf{E}_B\left[\left(\sum_{1\leq j\neq k\leq n} V_{t,x}^{\varepsilon,j,k}\right)^{m}\right], \qquad (40)$$

where we recall that $V_{t,x}^{\varepsilon,j,k}$ is defined by (28). Furthermore, expanding the mth power above, we have

$$\mathbf{E}_B\left[\left(\sum_{1\leq j\neq k\leq n} V_{t,x}^{\varepsilon,j,k}\right)^{m}\right] = \sum_{\alpha\in K_{n,m}} \int_{[0,t]^m} \int_{\mathbb{R}^m} e^{-\varepsilon\sum_{l=1}^{m}|\xi_l|^2} \mathbf{E}_B\left[e^{iB^{\alpha}(\xi)}\right] \prod_{l=1}^{m} |\xi_l|^{1-2H} d\xi dr,$$

where $K_{n,m}$ is a set of multi-indices defined by

$$K_{n,m} = \left\{\alpha = (j_1,\ldots,j_m,k_1,\ldots,k_m) \in \{1,\ldots,n\}^{2m};\ j_l < k_l \text{ for all } l = 1,\ldots,n\right\},$$

and $B^\alpha(\xi)$ is a shorthand for the linear combination $\sum_{l=1}^m \xi_l(B^{j_l}_{\kappa r_l} - B^{k_l}_{\kappa r_l})$. The important point here is that $E_B e^{iB^\alpha(\xi)}$ is positive for any $\alpha \in K_{n,m}$. We thus get the following inequality, valid for all $m \geq 1$

$$\mathbf{E}_B\left[\left(\sum_{1\leq j\neq k\leq n} V^{\varepsilon,j,k}_{t,x}\right)^m\right] \leq \sum_{\alpha\in K_{n,m}} \int_{[0,t]^m}\int_{\mathbb{R}^m} \mathbf{E}_B\left[e^{iB^\alpha(\xi)}\right]\prod_{l=1}^m |\xi_l|^{1-2H}\, d\xi dr$$
$$= \mathbf{E}_B\left[\left(\sum_{1\leq j\neq k\leq n} V^{j,k}_{t,x}\right)^m\right],$$

where $V^{j,k}_{t,x}$ is defined by (35). Plugging this inequality back into (40) and recalling expression (34) for $\mathbf{E}[u^n(t,x)]$, we easily deduce that $\mathbf{E}[u^n_\varepsilon(t,x)] \leq \mathbf{E}[u^n(t,x)]$, which finishes the proof. □

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