Heavy Tailed Random Matrices: How They Differ from the GOE, and Open Problems

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Abstract Since the pioneering works of Wishart and Wigner on random matrices, matrices with independent entries with finite moments have been intensively studied. Not only it was shown that their spectral measure converges to the semi-circle law, but fluctuations both global and local were analyzed in fine details. More recently, the domain of universality of these results was investigated, in particular by Erdos-Yau et al and Tao-Vu et al. This survey article takes the opposite point of view by considering matrices which are not in the domain of universality of Wigner matrices: they have independent entries but with heavy tails. We discuss the properties of these matrices. They are very different from Wigner matrices: the limit law of the spectral measure is not the semi-circle distribution anymore, the global fluctuations are stronger and the local fluctuations may undergo a transition and remain rather mysterious.

1 Introduction

Random matrices were introduced by Wishart [36] in the twenties to study large arrays of data and then in the 1950s by Wigner [35] to model Hamiltonians of quantum systems. In both cases, it appeared natural to assume the dimension of the matrices to be large. Moreover, it is natural to take the entries as independent as possible within the known constraints of the model. A typical model for such a matrix is to take a symmetric matrix filled with independent equidistributed Gaussian random variables: the so-called Gaussian orthogonal ensemble (GOE). To fix the ideas, the matrix will be $N \times N$ with independent centered Gaussian entries with variance $1/N$ (and variance $2/N$ on the diagonal). The properties of

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the spectrum and the eigenvectors of such matrices were studied in details, thanks to the fact that the law of such matrices is invariant under multiplication by orthogonal matrices and that the law of the eigenvalues has a simple expression as particles in Coulomb-gas interaction. Understanding how the details of the model could influence the spectral properties of the random matrices then became a central question. Assuming the entries to be still independent, it was shown that if the entries have sufficiently light tails, the fluctuations of the extreme eigenvalues are similar to that of the GOE [30] and in the bulk [21]. A series of remarkable works then focussed on obtaining optimal assumptions on the tails, which are a finite fourth (respectively the second) moment to observe the same fluctuations [18, 20, 25, 33]. However, there are matrices of interest which do not belong to this domain of universality. Typically, these matrices will have most entries which are very tiny, but a finite number of entries per row or column will be of order one. This is in contrast with light tails matrices where all entries are of order $1/\sqrt{N}$. An example is given by the adjacency matrix of an Erdös-Rényi graph whose entries are independent Bernoulli variables which are equal to one with probability c/N for some finite constant c. Such matrices are much less known. We shall in this note outline the main results and open problems related to such matrices. Roughly, the convergence of the empirical measure of the eigenvalues can be derived under rather general assumptions, but the limiting measure is not anymore the semi-circle law and is not compactly supported [6, 7, 37]. Under slightly more demanding hypotheses, the central limit theorem around this convergence can be derived: fluctuations occur in larger scale than for light tail matrices, in fact the usual central limit rescaling by a square root of the dimension is needed as soon as the moment of order two of the entries is infinite [9]. Local law could be derived only for α-stable entries [12, 13]. It shows a transition in the regime where $\alpha < 1$: for small eigenvalues the eigenvectors are delocalized whereas for large eigenvalues they are localized, again a phenomenon which does not occur for light tail matrices. Even words in independent heavy tail matrices behave differently: they are not asymptotically free in general and one need a new non-commutative framework, namely traffic distributions, to analyze them.

    In the sequel, a Wigner matrix will be a symmetric matrix with centered independent equidistributed entries. The case of Hermitian matrices with complex entries is similar but will not be treated here for simplicity. We will denote **X** a Wigner matrix with light tails and **A** a Wigner matrix with heavy tails.

2 Macroscopic Limit

Going back to Wigner [35], it was shown that the spectral measure of random matrices with light tail entries converges towards the same asymptotic law: the semi-circle law. In this section, we discuss the convergence of the spectral measure of random matrices with heavy tails matrices and show that it converges towards

different limiting measures. Let us be more precise. Let $\mathbf{X}^N$ be a symmetric matrix so that $(X_{ij}^N)_{i \leq j}$ are independent and such that

$$\mathbb{E}[\mathbf{X}_{ij}^N] = 0, \quad \lim_{N \to \infty} \max_{1 \leq i,j \leq N} |N\mathbb{E}[|\mathbf{X}_{ij}^N|^2] - 1| = 0. \tag{1}$$

Assume moreover that for all $k \in \mathbb{N}$ we have

$$B_k := \sup_{N \in \mathbb{N}} \sup_{(i,j) \in \{1,\cdots,N\}^2} \mathbb{E}[|\sqrt{N}\mathbf{X}_{ij}^N|^k] < \infty. \tag{2}$$

Then, Wigner proved the almost sure convergence

$$\lim_{N \to \infty} \frac{1}{N} \operatorname{Tr}\left((\mathbf{X}^N)^k\right) = \begin{cases} 0 & \text{if k is odd,} \\ C_{\frac{k}{2}} & \text{otherwise,} \end{cases} \tag{3}$$

where $C_{k/2} = \dfrac{\binom{k}{\frac{k}{2}}}{\frac{k}{2}+1}$ are the Catalan numbers. The proof is based on an expansion of the trace of moments of matrices in terms of the entries, together with the observation that the indices which will contribute to the first order of this expansion can be described by rooted trees. Based on the fact that the Catalan numbers are the moments of the semi-circle distribution

$$\sigma(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}\, 1_{|x| \leq 2} dx. \tag{4}$$

one can use density arguments (see e.g. [4]) to show that as soon as B_3 (in fact "$B_{2+\varepsilon}$") is finite, the eigenvalues $(\lambda_1, \cdots, \lambda_N)$ of $\mathbf{X}^N$ satisfy the almost sure convergence

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f(\lambda_i) = \int f(x) d\sigma(x) \tag{5}$$

where f is a bounded continuous function.

In contrast, the limit my be different as soon as $B_{2+\varepsilon}$ is infinite. The new hypothesis is that all moments are of order $1/N$: Assume that $\mathbb{E}[\mathbf{A}_{ij}^N] = 0$ and

$$\lim_{N \to \infty} N\mathbb{E}[(\mathbf{A}_{ij}^N)^{2k}] = M_k, \qquad \forall k \in \mathbb{N}. \tag{6}$$

Note that this includes the case of the adjacency matrix of a Erdös-Rényi graph with $M_k = c$ for all k. Then, Zakharevich [37] showed that $N^{-1} \operatorname{Tr}((\mathbf{A}^N)^p)$ goes to zero if p is odd and

$$\lim_{N \to \infty} \mathbb{E} \left[\frac{1}{N} \operatorname{Tr}((\mathbf{A}^N)^{2k}) \right] = \sum_{G=(V,E) \in \mathcal{T}_k} \sum_{P \in P_k(G)} \prod_{e \in E} M_{m(P,e)/2} \,. \tag{7}$$

where $\mathcal{T}_k$ is the set of rooted trees with at most k edges, $P_k(G)$ the set of closed paths on G with $2k$ steps, going through all edges of G, starting from the root and $m(P, e)$ the (even) number of times that the path goes through the edge e. The probability measure with the above moments is very different from the semi-circle law: in general it has unbounded support. One can generalize this result to the case where the entries have no moments at all by using convergence of the Stieltjes transform $G_\mu(z) = \int (z - x)^{-1} d\mu(x)$. Assume that the law μ_N of A_{ij}^N satisfies

$$\lim_{N \to \infty} N \left(\int (e^{-iux^2} - 1) d\mu_N(x) \right) = \Phi(u) \tag{8}$$

with Φ such that there exists g on $\mathbb{R}^+$, with $g(y)$ bounded by Cy^κ for some $\kappa > -1$, such that for $u \in \mathbb{C}^-$,

$$\Phi(u) = \int_0^\infty g(y) e^{\frac{iy}{u}} dy. \tag{9}$$

An example is given by α stable laws with $\Phi(u) = c(iu)^{\alpha/2}$ and $g(y) = Cy^{\alpha/2-1}$ for some constants c, C. Another example is provided by the adjacency matrix of Erdös-Rényi graph with $\Phi(u) = c(e^{iu} - 1)$ and g a Bessel function [9]. Then, it was shown in [7, 9] that $G_N(z) = \frac{1}{N} \operatorname{Tr}(z - \mathbf{A}^N)^{-1}$ converges almost surely towards G given by, for $z \in \mathbb{C}^+$

$$G(z) = i \int e^{itz} e^{\rho_z(t)} dt \tag{10}$$

where $\rho_z : \mathbb{R}^+ \to \{x + iy; x \leq 0\}$ is the unique solution analytic in $z \in \mathbb{C}^+$ of the equation

$$\rho_z(t) = \int_0^\infty g(y) e^{\frac{iy}{t} z + \rho_z(\frac{y}{t})} dy \tag{11}$$

This entails the convergence of the spectral measure of $\mathbf{A}^N$ as in (5), with σ replaced by a probability measure with Stieltjes transform given by (10). To give some heuristics of the proof of such convergence, let us take $z \in \mathbb{C} \backslash \mathbb{R}$. Then, Schur complement formula reads

$$(z - \mathbf{A})_{ii}^{-1} = \frac{1}{z - A_{ii} - \langle A_i, (z - \mathbf{A}^{(i)})^{-1} A_i \rangle}$$

where $A_i = (A_{ij})_{j\neq i}$ and $\mathbf{A}^{(i)}$ is the $(N-1) \times (N-1)$ matrix obtained from $\mathbf{A}$ by removing the ith row and column. A_{ii} goes to zero with N, but

$$\langle A_i, (z - \mathbf{A}^{(i)})^{-1} A_i \rangle \simeq \sum_{j\neq i} A_{ij}^2 (z - \mathbf{A}^{(i)})_{jj}^{-1}$$

is a non trivial random variable in the heavy tail case. We can compute its Fourier transform thanks to our hypothesis (8), and deduce fractional moments of the resolvent as follows. Observe that for $\beta > 0$, there exists a constant C_β such that for all $z = a + ib, b < 0$

$$\frac{1}{z^\beta} = C_\beta \int_0^\infty dx x^{\beta-1} e^{-ixz} .$$

As a consequence, we can guess thanks to (8) that

$$\mathbb{E}[((z - \mathbf{A})_{ii}^{-1})^\beta] \simeq C_\beta \int_0^\infty dx x^{\beta-1} e^{-ixz} \mathbb{E}[e^{-ix \sum_{j\neq i} A_{ij}^2 (z-\mathbf{A}^{(i)})_{jj}^{-1}}]$$

$$\simeq C_\beta \int_0^\infty dx x^{\beta-1} e^{-ixz} e^{\frac{1}{N} \sum \Phi(x(z-\mathbf{A}^{(i)})_{jj}^{-1})} \simeq C_\beta \int_0^\infty dx x^{\beta-1} e^{-ixz} e^{\rho_z^N(x)}$$

with the order parameter $\rho_z^N(x) := \mathbb{E}[\frac{1}{N} \sum \Phi(x(z - \mathbf{A}^{(i)})_{jj}^{-1})]$. Here, we used self-averaging of the order parameter. We next can get an equation for the order parameter thanks to hypothesis (9) which implies, again by Schur complement formula, that

$$\rho_z^N(x) \simeq \int_0^\infty g(y) e^{\frac{iy}{t}z + \rho_z^N(\frac{y}{t})} dy .$$

Hence, if the above heuristics are true, we get convergence of fractional moments of the resolvent as soon as the equation for ρ_z as a unique solution, to which the order parameter ρ_z^N converges. In particular, the Stieltjes transform

$$G_N(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{z - \lambda_i} = \frac{1}{N} \sum_{i=1}^N (z - \mathbf{A})_{ii}^{-1}$$

converges towards $C_1 \int_0^\infty dx x^{\beta-1} e^{-ixz} e^{\rho_z(x)}$. The above arguments were made rigorous in [7–9].

It is quite difficult to study the limiting probability measure whose Stieltjes transform is given by the intricate fixed point Eq. (10). It is known that it has unbounded support. The case of α-stable laws is easier: they have a smooth density except possibly at finitely many points [6], their density at the origin can be computed and this law can be interpreted as the spectral measure of the adjacency matrix of the PWIT [11]. But in general, simple properties such as the existence of

an absolutely continuous part are difficult, see e.g. [17] in the case of the Erdös-Rényi graph.

3   Central Limit Theorem for Linear Statistics

One can push the previous arguments to study the fluctuations of the linear statistics. Let us first consider light tails matrices, in fact matrices satisfying (2) and the fluctuations of

$$M_k^N := \sum_{i=1}^N \left(\lambda_i^k - \mathbb{E}[\lambda_i^k] \right) \tag{12}$$

Then, it was shown in [5], see also [24] for the Gaussian case, that M_k^N converges in distribution towards a Gaussian variable whose covariance depends on the fourth moment of the entries. Again, such convergence can be generalized to heavier tails by replacing taking as test functions smooth enough bounded test functions instead of moment: it was indeed shown, see [27, 31], that $\sum_{i=1}^N f(\lambda_i) - \mathbb{E}[\sum_{i=1}^N f(\lambda_i)]$ converges in distribution to a Gaussian variable provided f is $C^{1/2+\varepsilon}$, $\varepsilon > 0$. One can also recenter with respect to $N \int f(x) d\sigma(x)$, see e.g. [9], inducing in general a non trivial mean to the limiting Gaussian variable.

Hence, we see that the spectral measure of light tails matrices fluctuates much less than the empirical measure of independent variables which is of order $1/\sqrt{N}$ and not $1/N$. The situation changes drastically when one considers heavy tails matrices, and in fact as soon as the fourth moment of the entries is infinite. If one considers α stable laws with $\alpha \in (2, 4)$, the fluctuations are of order $N^{-1+\alpha/4}$ [10]. For heavy tails matrices satisfying (6) or (8), the fluctuations are of size $1/\sqrt{N}$ [9], as for independent variables. Test functions are assumed to be smooth enough in these cases, and centering in general holds with respect to expectation (additional hypothesis concerning the errors in (6) or (8) are required otherwise). To give some heuristics of the proof of such central limit theorem, let us take $f = (z - .)^{-1}$ for $z \in \mathbb{C} \backslash \mathbb{R}$. Then, recall that Schur complement formula shows that

$$(z - \mathbf{A})_{ii}^{-1} \simeq \frac{1}{z - Y_i^N(z)} \text{ with } Y_i^N(z) = \sum_{j \neq i} A_{ij}^2 (z - \mathbf{A}^{(i)})_{jj}^{-1} \, .$$

The $(Y_i^N(z))_{1 \leq i \leq N}$ converge (jointly for finite marginals) towards independent $\alpha/2$-stable laws with parameter ρ_z given by (11). Hence, the diagonal elements of the resolvent behave like independent equidistributed random variables, so that their sum, once renormalized by $\sqrt{N}$, converges towards a Gaussian variable.

4 Local Law

In order to get more local information, one would like to be able to take less smooth functions in the previous result, in fact functions which are supported in intervals going to zero as N goes to infinity. This idea was first developed for light tails matrices by Erdös-Schlein-Yau [22]. It reads as follows. Assume that μ, the law of the entries, have stretched exponential decay, i.e. there exists $\alpha > 0$ and $C < \infty$ such that for all $t \geq 0$

$$\mu(|x| \geq \sqrt{N}^{-1}t) \leq Ce^{-t^\alpha}. \tag{13}$$

Let for $I \subset]-2, 2[$, N_I be the number of eigenvalues in I. Then for all $\kappa \in (0, 2)$, all $\eta > \frac{(\log N)^4}{N}$ sufficiently small, there exists $c > 0$ such that we have for all $\delta \leq c\kappa$,

$$\mathbb{P}\left(\sup_{|E|\leq 2-\kappa}\left|\frac{N_{[E-\eta,E+\eta]}}{2N\eta} - \rho_{sc}(E)\right| > \frac{(\log N)^c}{\sqrt{\eta}N}|I|\right) \leq N^{-\log\log N}. \tag{14}$$

Such estimates were shown to hold under much weaker hypothesis afterwards and it was extended to the neighborhood of $\{-2\}$ and $\{2\}$, the boundary of the support of the semi-circle, see e.g. [19, 20, 26] or [32]. This allowed to prove that the eigenvalues are rigid, that is do not fluctuate much around their deterministic limit. Indeed, if we now order the eigenvalues $\lambda_1 \leq \lambda_2 \cdots \leq \lambda_N$ and let γ_i^N be the ith quantile given by $\sigma([-2, \gamma_i^N]) = i/N$, then, with probability greater than $1 - N^{-N}$, for all i

$$|\lambda_i - \gamma_i^N| \leq (\log N)^2 N^{-2/3}\min\{(N-i)^{-1/3}, i^{-1/3}\}.$$

Of course, one can not expect the eigenvalues to be as rigid in the heavy tails case since this would contradict the central limit theorems of the previous section. However, one could still expect the local law to be true inside the bulk: in [9], corresponding to entries decaying like $x^{-\alpha}$ for some $\alpha \in (2, 4)$, it was shown that global fluctuations hold in the scale $N^{-\alpha/4}$ whereas local law inside the bulk was derived in [1]. Hence, in this case, large eigenvalues should be less rigid, creating large fluctuations. For heavier tails, local laws have not yet been established except for the case of α-stable entries [12]. The following result was proved if the A_{ij} are α-stable variables: for all $t \in \mathbb{R}$,

$$\mathbb{E}[\exp(itA_{11})] = \exp(-\frac{1}{N}w_\alpha|t|^\alpha), \tag{15}$$

for some $0 < \alpha < 2$ and $w_\alpha = \pi/(\sin(\pi\alpha/2)\Gamma(\alpha))$. We let μ_α be the equilibrium measure and put

$$\rho = \begin{cases} \frac{1}{2} & \text{if } \frac{8}{5} \leq \alpha < 2 \\ \frac{\alpha}{8-3\alpha} & \text{if } 1 < \alpha < \frac{8}{5} \\ \frac{\alpha}{2+3\alpha} & \text{if } 0 < \alpha \leq 1. \end{cases} \tag{16}$$

Then, there exists a finite set $\mathcal{E}_\alpha \subset \mathbb{R}$ such that if $K \subset \mathbb{R}\backslash\mathcal{E}_\alpha$ is a compact set and $\delta > 0$, the following holds. There are constants $c_0, c_1 > 0$ such that for all integers $N \geq 1$, if $I \subset K$ is an interval of length $|I| \geq c_1 N^{-\rho}(\log N)^2$, then

$$\left|N_I - N\mu_\alpha(I)\right| \leq \delta N|I|, \tag{17}$$

with probability at least $1 - 2\exp\left(-c_0 N\delta^2|I|^2\right)$.

In both light and heavy tails, the main point is to estimate the Stieltjes transform $G_N(z) = \frac{1}{N}\sum_{i=1}^{N}(z-\lambda_i)^{-1}$ for z going to the real axis: $z = E + i\eta$ with η of order nearly as good as N^{-1} for light tails, $N^{-\rho}$ for heavy tails. This is done by showing that G_N is characterized approximately by a closed set of equations. In the case of lights tails, one has simply a quadratic equation for G_N and needs to show that the error terms remain small as z approaches the real line. In the heavy tails case, the equations are much more complicated, see (10), and therefore more difficult to handle. Even in the α-stable case it is not clear what should be the optimal local law. We believe ρ should be at least equal to $1/2$ for all $\alpha \in (1, 2)$. Similar questions are completely open for other heavy tails matrices.

5   Localization and Delocalization of the Eigenvectors

Based on the local law, it was shown that the eigenvectors of Wigner matrices with light entries (for instance with sub exponential tail) are strongly delocalized [22, 23], for any $p \in (2, \infty]$ and $\epsilon > 0$, with high probability,

$$\max_{1\leq k\leq N} \|u_k\|_p = O(N^{1/p-1/2+\epsilon}), \tag{18}$$

where for $u \in \mathbb{R}^n$, $\|u\|_p = \left(\sum_{i=1}^{n}|u_i|^p\right)^{1/p}$ and $\|u\|_\infty = \max|u_i|$. This phenomenon seems to be quite robust and continues to hold even if a fraction of the entries vanish. For instance, if the entries vanish outside a band around the diagonal of width W, it is conjectured that the eigenvectors remain delocalized as long as $W \gg \sqrt{N}$, but start being localized when $W \ll \sqrt{N}$. Universality was shown for $W \simeq N$, see [15].

It was shown in [12] that the eigenvectors of matrices with α-stable entries are also delocalized if $\alpha \in (1, 2)$: there is a finite set $\mathcal{E}_\alpha$ such if $K \subset \mathbb{R} \backslash \mathcal{E}_\alpha$ is a compact set, for any $\varepsilon > 0$, with high probability,

$$\max \left\{ \|u_k\|_\infty : 1 \leq k \leq N, \lambda_k \in K \right\} = O(N^{-\delta+\epsilon}), \tag{19}$$

where $\delta = (\alpha - 1)/((2\alpha) \vee (8 - 3\alpha))$. Since $\|u\|_p \leq \|u\|_\infty^{1-2/p} \|u\|_2^{2/p}$, it implies that the L^p-norm of the eigenvectors is $O(N^{2\delta/p-\delta+o(1)})$. Notice that when $\alpha \to 2$, then $\delta \to 1/4$ and it does not match with (18): we expect that this result could be improved.

However, for $\alpha \in (0, 1)$, we observe a new phenomenon, closer to what can be observed for random Schrodinger operators, see e.g. [3]: eigenvectors are delocalized if they correspond to eigenvalues which are small, but are localized if they correspond to large eigenvalues. In [14], Bouchaud and Cizeau conjectured the existence of a mobility edge, $E_\alpha > 0$ where this transition occurs (a value for E_α is predicted in [34]). However, the sense of localization/delocalization has to be precised. In [12, 13], we considered

$$P_I(k) = \frac{1}{|\Lambda_I|} \sum_{u \in \Lambda_I} \langle u, e_k \rangle^2, \, Q_I = N \sum_{k=1}^{N} P_I(k)^2 \, \in [1, N].$$

and showed that for $\alpha \in (0, 2/3)$, for $I = [E - \eta, E + \eta]$ with η going to zero with N, Q_I goes to infinity if E is large enough, whereas it is bounded for small enough E. This localization/delocalization of the eigenvectors should be related with the local fluctuations of the spectrum. Bouchaud and Cizeau conjectured that the small eigenvalues should behave like the eigenvalues of the Gaussian ensemble when $\alpha \in (1, 2)$. Also, for $\alpha \in (0, 1)$ and large eigenvalues, one expects a Poisson distribution. However, for the two remaining regimes, they predict something between Poisson and Sine-kernel. In [34], the authors predict a phase transition at a mobility edge between the localized and delocalized regimes. While this article was under print, it was shown in [2] that in the regime of delocalization, the local statistics are given by the GOE statistics and that, then, the eigenvectors are completely delocalized in the sense that (19) holds with the optimal rate $\delta = 1/2$.

6   Heavy-Tailed Operators in Free Probability

Another important feature of random matrices is their role in free probability, as a toy example of matrices whose large dimension limit are free. Free probability is a theory of non-commutative variables equipped with a notion of freeness. Let us consider self-adjoint non-commutative variables $X_1, \ldots, X_d$. We equip the set of

polynomials in these non-commutative variables with the involution

$$(z X_{i_1} X_{i_2} \cdots X_{i_k})^* = \bar{z} X_{i_k} \cdots X_{i_1}.$$

Distributions of d self adjoint variables are simply linear functions τ on this set of polynomials in non-commutative variables such that

$$\tau(PP^*) \geq 0, \quad \tau(1) = 1, \qquad \tau(PQ) = \tau(QP),$$

for all choices of polynomials P, Q. Freeness is a condition on the joint distribution of non-commutative variables. For instance, we say that $X_1, \ldots, X_d$ are free under τ iff

$$\tau(P_1(X_{i_1}) \cdots P_\ell(X_{i_\ell})) = 0 \tag{20}$$

as soon as $\tau(P_j(X_j)) = 0$ for all j and $i_j \neq i_{j+1}$, $1 \leq j \leq \ell - 1$. The latter property was introduced by Voiculescu and named freeness, as it is related with the usual notion of free generators of a group. Taking d independent Wigner matrices $\mathbf{X}_1^N, \ldots, \mathbf{X}_d^N$ with light tails, one finds that for all choices of $i_1, \ldots, i_k \in \{1, \ldots, d\}^k$,

$$\lim_{N \to \infty} \frac{1}{N} \operatorname{Tr}(\mathbf{X}_{i_1}^N \cdots \mathbf{X}_{i_k}^N) = \sigma^d(X_{i_1} \cdots X_{i_k}) \qquad a.s$$

where σ^d is uniquely described by saying that the moments of a single X_i are given by the Catalan numbers, and their joint moments satisfy (20). Voiculescu also showed that matrices $\mathbf{Y}_j = U_j D_j U_j^*$ with deterministic matrices D_j and independent Haar distributed orthogonal matrices satisfy at the large N limit the freeness property (20). Hence, matrices become asymptotically free if the position of their eigenvectors are "sufficiently" independent. One could then wonder whether Wigner matrices with heavy tails are also asymptotically free. All these matrices share the invariance by multiplication by permutation matrices. It is clear that matrices conjugated by independent permutation matrices are not asymptotically free. Indeed, for instance if one takes two diagonal matrices with given spectral measure, it will have a different joint law if it is conjugated by unitary matrices than if it is conjugated by permutations (which does not change the law) since then they will commute. Similarly, it is not enough to know the spectral measure of a heavy tail matrix to derive the joint law of several of them. In the one matrix case, this could already be guessed in view of the additional parameter ρ_z. In fact, this parameter appears naturally as the large N limit of $\frac{1}{N} \sum_{k=1}^N \Phi(t(z - \mathbf{A})_{ii}^{-1})$ which is not a function of the spectral measure. To remedy this point, another non-commutative framework was introduced by C. Male: the distribution of traffics and their free product [28]. Traffics distributions are now linear maps of a set of functionals that generalize the non-commutative polynomials, called graph polynomials. Namely, if we are given d $N \times N$ self-adjoint random matrices

$(A_1^N, \ldots, A_d^N)$, a finite connected graph $G = (V, E)$ and γ a map from E into $\{1, \ldots, d\}$, we define the observables given by

$$\Phi_{A^N}(G, \gamma) = \mathbb{E}\left[\frac{1}{N} \sum_{\phi:V \mapsto \{1,\ldots,N\}} \prod_{e=(v,w)\in E} A_{\gamma(e)}(\phi(w), \phi(v))\right],$$

where the sum is taken over all maps and $N \geq |V|$. For instance if G is a cycle $V = \{v_1, \ldots, v_k\}$, $E = \{e_\ell = (v_\ell, v_{\ell+1})\}_{1\leq\ell\leq k}$ with $v_{k+1} = v_1$, we get the trace of the word $A_{\gamma(e_1)} \cdots A_{\gamma(e_k)}$. If V is as before, but $E = \{e_\ell = (v_\ell, v_{\ell+1})\}_{1\leq\ell\leq k} \cup \{e_{\ell+k} = (v_\ell, v_{\ell+1})\}_{1\leq\ell\leq k}$ while $\gamma(e_\ell) = 1$ for $\ell \leq k$ and 2 for $\ell \geq k+1$, we get the trace of the kth moment of the Hadamard product $A_1^N \circ A_2^N$. More generally, we can obtain all the the normalized trace of Hadamard products of polynomials in the matrices $A_1^N, \ldots, A_d^N$

$$\mathbb{E}\left[\frac{1}{N} \text{Tr}(P_1(A_1^N, \ldots, A_d^N) \circ \cdots \circ P_k(A_1^N, \ldots, A_d^N))\right].$$

The collection of all $\Phi_{A^N}(G, \gamma)$ defines the distribution of the traffics $(A_1^N, \ldots, A_d^N)$. A sequence $(A_1^N, \ldots, A_d^N)$ of matrices converges in traffics iff $\Phi_{A^N}(G, \gamma)$ converges for all finite connected graphs G and all map γ. The model of heavy Wigner matrices was the initial motivation to introduce it: matrices satisfying (6) can be seen to converge in traffic. Traffic distribution comes together with the notion of traffic independence, which is more complicated than freeness in the sense that it involves non algebraic (combinatorial) formulas (see [28, Definition 3.10]). However, it prescribes uniquely the traffic distribution of two families **A** and **B** from the traffic distributions of **A** and **B**. One can see that traffic independence does not imply free independence. Let us consider two asymptotically traffic independent families of matrices $\mathbf{A}_N$ and $\mathbf{B}_N$ (that is with traffic distribution which converges towards a distribution of two traffic independent families). If

$$\kappa(A_N, P) = \frac{1}{N} \text{Tr}\left[P(\mathbf{A}_N) \circ P^*(\mathbf{A}_N)\right] - |\frac{1}{N} \text{Tr} P(\mathbf{A}_N)|^2$$

does not go to zero for some polynomial P and the same hold for $\mathbf{B}_N$, then $\mathbf{A}_N$ and $\mathbf{B}_N$ are not asymptotically freely independent [28, Section 3.3]. This criterion applies for heavy Wigner matrices, which shows in particular that heavy Wigner matrices are not asymptotically freely independent, and not asymptotically freely independent with diagonal matrices. On the contrary, if $\kappa(A_N, P)$ and $\kappa(B_N, P)$ tend to zero for all polynomial P, then A_N and B_N are asymptotically free independent [16, Section 3.2]. This is the case of adjacency matrices of Erdos-Renyi matrices with parameter $\frac{c_N}{N}$ when c_N goes to infinity [29]. Traffic independence is difficult to manipulate, still we can deduce from it a system of equations which characterizes the limiting distribution of independent heavy Wignerand deterministic diagonal matrices. It involves again limits of normalized trace of

Hadamard products of polynomials in matrices. It implies another characterization of the spectrum of a single heavy Wigner matrix in term of the maps $G(\lambda)^k = \frac{1}{N} \sum_i \left[(\lambda - X)^{-1}_{ii} \right]^k$ [29].

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