Propagation of Analytic Singularities for Short and Long Range Perturbations of the Free Schrödinger Equation

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Abstract We study the propagation of the analytic wave front set for solutions to the Schrödinger equation associated with perturbations of the free Laplacian.

1 Introduction

We are interested in the analytic singularities of the distributions $u = u(t, x)$ that are solutions in $\mathbb{R} \times \mathbb{R}^n$ to the Schrödinger equation,

$$(\text{Sch}) : \qquad \begin{cases} i\frac{\partial u}{\partial t} = Pu; \\ u_{|t=0} = u_0, \end{cases}$$

where $P = P(x, D_x)$ is a second-order symmetric differential operator on $\mathbb{R}^n$ with analytic coefficients (typically a perturbation of the Laplace operator $P_0 := -\frac{1}{2}\Delta$), and u_0 is in $L^2(\mathbb{R}^n)$ or, more generally, in some Sobolev space.

For such a problem, it is quite natural to wonder if the analyticity of u_0 implies that of $u(t)$ at time $t \neq 0$. But actually this is not true, as it can be seen from the example where $P = P_0$ and $u_0 = (-2i\pi)^{-\frac{n}{2}} e^{-i|x|^2/2}$. In this case, using that the distributional kernel of e^{-itP_0} is $(2i\pi t)^{-\frac{n}{2}} e^{i|x-y|^2/2t}$, one can see that $u(t)$ just coincides with $v(t-1)$, where v solves the same Schrödinger equation with initial date $v(0) = \delta$ (the Dirac measure at $x = 0$). In particular, $u(1) = \delta$ is singular, while $u(0)$ is analytic. Such a phenomenon is called "infinite propagation speed of singularities", and a question one may ask is: Is there any way to read the singularities of $u(t)$ easily on u_0?

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As we shall see, the answer is essentially yes, in the sense that (under some non-trapping conditions) the analytic wave front set of $e^{itP_0}u(t)$ propagates in a very precise way (while that of $u(t)$ does not at all!).

As an example, in the particular case $P = P_0 + V$ where $V = V(x)$ is an analytic function tending to 0 at infinity (and thus, in that case, $u(t) = e^{-itP}u_0$), we will prove that, for all $t \in \mathbb{R}$, one has,

$$WF_a(e^{itP_0}u(t)) = WF_a(u_0)$$

or, equivalently,

$$WF_a(u(t)) = WF_a(e^{-itP_0}u_0).$$

Here, WF_a stands for the analytic wave front set, and the details of the proofs of the results we present here can be found in [7, 8] (see also [6] for related results).

2 Assumptions and Results

Let

$$P = \frac{1}{2}\sum_{j,k=1}^{n} D_j a_{j,k}(x) D_k + \frac{1}{2}\sum_{j=1}^{n}(a_j(x)D_j + D_j a_j(x)) + a_0(x)$$

on $\mathcal{H} = L^2(\mathbb{R}^n)$, where $D_j = -i\partial_{x_j}$, and assume that the coefficients $\{a_\alpha(x)\}$ satisfy to the following hypothesis. For $\nu > 0$ we denote

$$\Gamma_\nu = \left\{ z \in \mathbb{C}^n \;\middle|\; |\mathrm{Im}\, z| < \nu\langle \mathrm{Re}\, z\rangle \right\}.$$

Assumption A For each α, $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$ is real-valued and can be extended to a holomorphic function on Γ_ν with some $\nu > 0$. Moreover, for $x \in \mathbb{R}^n$, the matrix $(a_{j,k}(x))_{1\le j,k\le n}$ is symmetric and positive definite, and there exists $\sigma > 0$ such that,

$$\begin{aligned} &\left|a_{j,k}(x) - \delta_{j,k}\right| \le C_0\langle x\rangle^{-\sigma}, \quad j,k = 1,\dots,n,\\ &\left|a_j(x)\right| \le C_0\langle x\rangle^{1-\sigma}, \qquad j = 1,\dots,n,\\ &\left|a_0(x)\right| \le C_0\langle x\rangle^{2-\sigma}, \end{aligned}$$

for $x \in \Gamma_\nu$ and with some constant $C_0 > 0$.

The case $\sigma > 1$ will be referred to as the *short range case*, while the case $\sigma \in (0, 1]$ as the *long range case*.

We denote by $p(x,\xi) := \frac{1}{2}\sum_{j,k=1}^{n} a_{j,k}(x)\xi_j\xi_k$ the principal symbol of P, and by $P_0 := -\frac{1}{2}\Delta$ the free Laplace operator. For any $(x,\xi) \in \mathbb{R}^{2n}$, we also denote by $(y(t;x,\xi), \eta(t;x,\xi)) = \exp tH_p(x,\xi)$ the solution to the Hamilton system,

$$\frac{dy}{dt} = \frac{\partial p}{\partial \xi}(y, \eta), \ \frac{d\eta}{dt} = -\frac{\partial p}{\partial x}(y, \eta), \tag{2.1}$$

with initial condition $(y(0), \eta(0)) = (x, \xi)$.

We say that a point $(x, \xi) \in T^*\mathbb{R}^n \backslash 0$ is forward non-trapping (respectively backward non-trapping) when $|y(t, x, \xi)| \to \infty$ as $t \to +\infty$ (resp. as $t \to -\infty$).

In that case, one can prove the existence of $\eta_+(x, \xi) \in \mathbb{R}^n$ (resp. $\eta_-(x, \xi)$) such that $\eta(t, x, \xi) \to \eta_+(x, \xi)$ as $t \to +\infty$ (resp. $\eta(t, x, \xi) \to \eta_-(x, \xi)$ as $t \to -\infty$).

If in addition $\sigma > 1$ (short range case), then one can also prove the existence of $y_\pm(x, \xi) \in \mathbb{R}^n$ such that,

$$|y_+(x, \xi) + t\eta_+(x, \xi) - y(t, x, \xi)| \to 0 \text{ as } t \to +\infty,$$

(resp. $|y_-(x, \xi) + t\xi_-(x, \xi) - y(t, x, \xi)| \to 0$ as $t \to -\infty$).

A proof of these two facts can be found, e.g., in [1], Lemma 2.2 (indeed, though only the short range case is treated, the proof given for the existence of $\eta_\pm(x, \xi)$ still works in the long range case).

Denoting by NT^+ (resp. NT^-) the set of forward (resp. backward) non-trapping points, we define the applications,

$$S_\pm : NT^\pm \to \mathbb{R}^{2n}$$

by

$$S_\pm(x, \xi) := (y_\pm(x, \xi), \eta_\pm(x, \xi)).$$

They respectively correspond to the forward and backward classical wave maps. For any distribution $u \in \mathcal{D}'(\mathbb{R}^n)$, we denote by $WF_a(u)$ the analytic wave front set of u (see, e.g., [13]), that can be described by introducing the FBI transform T defined by,

$$Tu(z, h) = \int e^{-(z-y)^2/2h} u(y) dy,$$

where $z \in \mathbb{C}^n$ and $h > 0$ is a small extra-parameter. Then, Tv belongs to the Sjöstrand space $H^{loc}_{\Phi_0}$ with $\Phi_0(z) := |\text{Im } z|^2/2$ (see [13]), and a point (x, ξ) is not in $WF_a(u)$ if and only if there exists some $\delta > 0$ such that $Tu = \mathcal{O}(e^{(\Phi_0(z)-\delta)/h})$ uniformly for z close enough to $x - i\xi$ and $h > 0$ small enough (in this case, we also use the notation: $Tu \sim 0$ in $H_{\Phi_0, x-i\xi}$). By Cauchy-formula, this is also equivalent to the existence of some $\delta' > 0$ such that $\|e^{-\Phi_0/h} Tu\|_{L^2(\Omega)} = \mathcal{O}(e^{-\delta'/h})$ for some complex neighborhood Ω of $x - i\xi$.

In the **short range case**, our main result is,

Theorem 2.1 *Suppose Assumption A with $\sigma > 1$, and let $u_0 \in L^2(\mathbb{R}^n)$. Then,*

(i) For any $t < 0$, one has,

$$WF_a(e^{-itP}u_0) \cap NT^+ = S_+^{-1}(WF_a(e^{-itP_0}u_0)); \tag{2.2}$$

(ii) For any $t > 0$, one has,

$$WF_a(e^{-itP}u_0) \cap NT^- = S_-^{-1}(WF_a(e^{-itP_0}u_0)). \tag{2.3}$$

Remark 2.2 In the particular case where the metric is globally non-trapping, this result gives a complete characterization of the analytic wave front set of $u(t)$ in terms of that of $e^{-itP_0}u_0$.

Remark 2.3 By substituting $e^{itP}u_0$ to u_0, and $-t$ to t, this result implies that one has,

$$\forall t > 0, \quad WF_a(e^{itP_0}u(t)) = S_+(WF_a(u_0) \cap NT^+);$$
$$\forall t < 0, \quad WF_a(e^{itP_0}u(t)) = S_-(WF_a(u_0) \cap NT^-).$$

In particular, this set does not depend on $t > 0$ (resp. $t < 0$).

In the important case where $a_{j,k} = \delta_{j,k}$, then one has $NT^\pm = \mathbb{R}^{2n}\backslash 0$ and $S_\pm = Id$, and we obtain the following immediate corollary:

Corollary 2.4 *Suppose Assumption A with $\sigma > 1$ and $a_{j,k} = \delta_{j,k}$ for all pair (j,k). Then, for all $t \in \mathbb{R}$ and all $u_0 \in L^2(\mathbb{R}^n)$, one has,*

$$WF_a(e^{-itP}u_0) = WF_a(e^{-itP_0}u_0).$$

Remark 2.5 In the C^∞ setting, analogous results have been obtained Hassell and Wunsch in [2]. They involve a notion of "scattering wave front set" in a more general context of manifolds. In the case of $\mathbb{R}^n$, this notion mainly coincides with that of $WF(e^{itP_0}u)$ (see also [3, 4, 9–12, 14] for related questions).

Remark 2.6 Using the FBI transform (see, e.g., [5, 13]) and the expression of the distributional kernel of e^{-itP_0}, one can see that a point $(x_0, \xi_0) \in \mathbb{R}^{2n}\backslash 0$ is not in $WF_a(e^{-itP_0}u_0)$ if and only if there exists some $\delta > 0$ such that the quantity,

$$\mathcal{T}u_0(x, \xi : h) := \int e^{i(x-hy)\xi/h-(x-hy)^2/2h} e^{iy^2/2t} u_0(y)dy,$$

is $\mathcal{O}(e^{-\delta/h})$, uniformly for $h > 0$ small enough and (x, ξ) in a neighborhood of $(-\frac{1}{t}\xi_0, \frac{1}{t}x_0)$.

In the **long range case** ($0 < \sigma \leq 1$), the maps $S_\pm$ are not defined anymore, and one need to modify the free evolution near infinity in order to be able to define similar maps.

For $h > 0$ sufficiently small and $(x, \xi) \in \mathbb{R}^{2n}$, we denote by $\widetilde{p}(x, \xi; h)$ the quantity,

$$\widetilde{p}(x, \xi) := \frac{1}{2}\sum_{j,k} a_{j,k}(x)\xi_j\xi_k + h\sum_j a_j(x)\xi_j + h^2 a_0(x),$$

and by $(\widetilde{y}(t,x,\xi;h),\widetilde{\eta}(t,x,\xi;h)) := \exp tH_{\widetilde{p}}(x,\xi)$ the corresponding Hamilton flow. Then, we have the preliminary result,

Lemma 2.7 *For any $\delta_0 > 0$, there exist two h-dependent smooth functions,*

$$W_\pm : \mathbb{R}_\pm \times \{\xi \in \mathbb{R}^n ; |\xi| > \delta_0\} \to \mathbb{R},$$

that are solutions to,

$$\frac{\partial W_\pm}{\partial t}(t,\xi) = \widetilde{p}(\nabla_\xi W_\pm(t,\xi),\xi;h), \tag{2.4}$$

and such that, for any $\pm t > 0$ and $(x,\xi) \in NT^\pm$, the quantity,

$$\widetilde{y}(t/h,x,\xi) - \nabla_\xi W_\pm(t/h,\widetilde{\eta}(t/h,x,\xi)) + \nabla_\xi W_\pm(0,\eta_\pm(x,\xi)) \tag{2.5}$$

admits a limit $\widetilde{y}_\pm(x,\xi) \in \mathbb{R}^n$ independent of t as $h \to 0_+$.

Remark 2.8 Actually, Eq. (2.4) must be satisfied up to short range terms only, in order to have (2.5). For instance, in the previous short range case, one can take $W_\pm(t,\xi) = t\xi^2/2$, that gives $\widetilde{y}_\pm(x,\xi) = y_\pm(x,\xi)$.

Using the notations of the previous lemma, we set,

$$\begin{aligned} \widetilde{S}_\pm(x,\xi) &:= (\widetilde{y}_\pm(x,\xi),\eta_\pm(x,\xi)), \quad ((x,\xi) \in NT^\pm);\\ z_\pm(x,\xi) &:= \widetilde{y}_\pm(x,\xi) - i\eta_\pm(x,\xi);\\ \widetilde{W}_\pm(t,\xi) &:= W_\pm(t,\xi) - W_\pm(0,\xi). \end{aligned} \tag{2.6}$$

Then, the result for the long range case is,

Theorem 2.9 *Suppose Assumption A with $0 < \sigma \le 1$, and let $u_0 \in L^2(\mathbb{R}^n)$. Then, with the notations (2.6), one has,*

(i) For any $t < 0$ and $(x,\xi) \in NT^+$, one has the equivalence,

$$(x,\xi) \notin WF_a(e^{-itP}u_0) \iff e^{i\widetilde{W}_+(-t/h,hD_z)/h}Tu_0 \sim 0 \textit{ in } H_{\Phi_0,z_+(x,\xi)};$$

(ii) For any $t > 0$ and $(x,\xi) \in NT^-$, one has the equivalence,

$$(x,\xi) \notin WF_a(e^{-itP}u_0) \iff e^{i\widetilde{W}_-(-t/h,hD_z)/h}Tu_0 \sim 0 \textit{ in } H_{\Phi_0,z_-(x,\xi)};$$

Remark 2.10 Here, the operator $e^{i\widetilde{W}_\pm(-t/h,hD_z)/h}$ appearing in the statement is not defined by the Spectral Theorem, but rather as a Fourier integral operator acting on Sjöstrand's spaces (see [8]).

Remark 2.11 Actually, $W_\pm$ can be constructed in such a way that the quantity $W_1^\pm(t,\xi) := \widetilde{W}_\pm(-t/h, hD_x)/h$ does not depend on h, and in principle, the fact that $e^{i\widetilde{W}_\pm(-t/h,hD_z)/h}Tu_0 \sim 0$ in $H_{\Phi_0,z_\pm(x,\xi)}$ essentially means that $\widetilde{S}_\pm(x,\xi) \notin WF_a(e^{iW_1^\pm(-t,D_x)}u_0)$ (and in this sense, the result is very similar to that of the C^∞ setting appearing in [11]). However, in order to define $e^{iW_1^\pm(-t,D_x)}$ properly one needs to extend $\widetilde{W}_\pm$ to all values of $\xi \in \mathbb{R}^n$, and this requires the use of cut-off functions. In the analytic setting, this introduces technical difficulties that can probably be overcome by the use of analytic pseudodifferential operators on the real domain (see [13]).

3 Sketch of Proof

We explain the proof for the forward non-trapping case only (the backward non-trapping case being similar), and we start by considering the short range case with a flat metric (that is, $a_{j,k} = \delta_{j,k}$ for all j, k, and thus $S_\pm(x,\xi) = (x,\xi)$).

Replacing u_0 by $e^{itP}u_0$, and then changing t to $-t$, we see that we have to prove that for any $t > 0$, one has

$$WF_a(u_0) = WF_a(e^{itP_0}e^{-itP}u_0).$$

Following [10], we set $v(t) := e^{itP_0}e^{-itP}u_0$, that solves the system,

$$i\frac{\partial v}{\partial t} = L(t)v \quad ; \quad v(0) = u_0. \tag{3.1}$$

Here,

$$L(t) = e^{itP_0}(P - P_0)e^{-itP_0} = L_2(t) + L_1(t) + L_0(t), \tag{3.2}$$

with,

$$\begin{aligned}
L_2(t) &:= \frac{1}{2}\sum_{j,k=1}^{n} D_j(a_{j,k}^W(x+tD_x) - \delta_{j,k})D_k \\
L_1(t) &:= \frac{1}{2}\sum_{\ell=1}^{n}(a_\ell^W(x+tD_x)D_\ell + D_\ell a_\ell^W(x+tD_x)) \\
L_0(t) &:= a_0^W(x+tD_x),
\end{aligned}$$

where we have denoted by $a^W(x,D_x)$ the usual Weyl-quantization of a symbol $a(x,\xi)$, defined by,

$$a^W(x,D_x)u(x) = \frac{1}{(2\pi)^n}\int e^{i(x-y)\xi}a((x+y)/2,\xi)u(y)dyd\xi.$$

Observe that, in the flat case, one has $L_2(t) = 0$. The expressions for $L_j(t)$, $0 \le j \le 2$ can be proved directly (using the fact that $e^{\pm iP_0}$ is just the multiplication by $e^{\pm i\xi^2/2}$ in the Fourier variables), but they also result from the standard Egorov theorem (that becomes exact in this case).

Since the FBI transform T is a convolution operator, we immediately observe that $TD_{x_j} = D_{z_j}T$. However, in order to study the action of $L(t)$ after transformation by T, we need the following key-lemma that will allow us to enter the framework of Sjöstrand's microlocal analytic theory. Mainly, this lemma tells us that, if f is holomorphic near Γ_ν, then, the operator $\tilde{T} := T \circ f^W(x + thD_x)$ is a FBI transform with the same phase as T, but with some symbol $\tilde{f}(t, z, x; h)$.

Lemma 3.1 ([7], Lemma 3.1) *Let f be a holomorphic function on Γ_ν, verifying $f(x) = \mathcal{O}(\langle x\rangle^\rho)$ for some $\rho \in \mathbb{R}$, uniformly on Γ_ν. Let also K_1 and K_2 be two compact subsets of $\mathbb{R}^n$, with $0 \notin K_2$. Then, there exists a function $\tilde{f}(t, z, x; h)$ of the form,*

$$\tilde{f}(t, z, x; h) = \sum_{k=0}^{1/Ch} h^k f_k(t, z, x), \tag{3.3}$$

where f_k is defined, smooth with respect to t and holomorphic with respect to (z, x) near $\Sigma := \mathbb{R}_t \times \{(z, x)\,;\ \mathrm{Re}\, z \in K_1,\ |\mathrm{Re}\,(z - x)| + |\mathrm{Im}\, x| \le \delta_0,\ \mathrm{Im}\, z \in K_2\}$ with $\delta_0 > 0$ small enough, and such that, for any $u \in L^2(\mathbb{R}^n)$, one has,

$$Tf^W(x + thD_x)u(z, h) = \int_{|x-\mathrm{Re}\, z|<\delta_0} e^{-(z-x)^2/2h}\tilde{f}(t, z, x, h)u(x)dx + \mathcal{O}(\langle t\rangle^{\rho_+} e^{(\Phi_0(z)-\varepsilon)/h}),$$

for some $\varepsilon = \varepsilon(u) > 0$ and uniformly with respect to $h > 0$ small enough, z in a small enough neighborhood of $K := K_1 + iK_2$, and $t \in \mathbb{R}$. (Here, we have set $\rho_+ = \max(\rho, 0)$.)

Moreover, the f_k's verify,

$$f_0(t, z, x) = f(x + it(z - x))\,;$$
$$|\partial_{z,x}^\alpha f_k(t, z, x)| \le C^{k+|\alpha|+1}(k + |\alpha|)!\langle t\rangle^\rho,$$

for some constant $C > 0$, and uniformly with respect to $k \in \mathbb{Z}_+$, $\alpha \in \mathbb{Z}_+^{2n}$, and $(t, z, x) \in \Sigma$.

Thanks to this lemma, and using again Sjöstrand's theory of microlocal analytic singularities [13], we deduce the existence of an analytic second-order (that is, with a symbol $\mathcal{O}(h^{-2})$) pseudodifferential operator $Q(t, h)$ on $H^{loc}_{\Phi_0}(\mathbb{C}^n\backslash\{\mathrm{Im}\, z = 0\})$, such that,

$$TL(t) = Q(t, h)T.$$

Moreover, in the flat case, $Q(t, h)$ becomes of the first order, and its symbol is mainly given by,

$$q(t,h;z,\zeta)\sim h^{-1}\sum_{\ell=1}^{n}a_\ell(z+i\zeta+th^{-1}\zeta)\zeta_\ell+a_0(z+i\zeta+th^{-1}\zeta).$$

Actually, using Lemma 3.1, an exact formula can be obtained for the symbol of $Q(t,h)$, that coincides with the previous expression up to $\mathcal{O}(1)$-terms as $h\to 0_+$. We refer to [7], Sect. 4, for more details.

Then, applying T to (3.1), multiplying it by h^2, and changing the time-scale by setting $s:=t/h$, we obtain the new evolution equation,

$$ih\frac{\partial Tv}{\partial s}=B(s,h)Tv \quad ; \quad Tv(0)=Tu_0, \tag{3.4}$$

where $B(s,h)$ is an analytic pseudodifferential operator of order -1 (still in the sense of [13]), acting on $H^{loc}_{\Phi_0}(\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\})$, with symbol $b(s,h)$ verifying,

$$b(s,h)\sim\sum_{k\geq 1}h^k b_k(s)$$

(in the sense of analytic symbols), with

$$\begin{aligned}&b_1(s;z,\zeta)=\mathcal{O}(\langle s\rangle^{1-\sigma});\\ &b_k(s;z,\zeta)=\mathcal{O}(\langle s\rangle^{2-\sigma})\text{ for } k\geq 2,\end{aligned} \tag{3.5}$$

uniformly with respect to $s>0$, and locally uniformly with respect to $z\in\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\}$ and ζ close enough to $-\mathrm{Im}\, z$ (note that, in particular, for $k\geq 2$ and $s=\mathcal{O}(h^{-1})$, one also has: $hb_k=\mathcal{O}(\langle s\rangle^{1-\sigma})$.)

Let us recall from [13] that the quantization of such a symbol $b(s,h;z,\zeta)$ on $H^{loc}_{\Phi_0}$ is given by,

$$B(s,h)w(z;h)=\frac{1}{(2\pi h)^n}\int_{\gamma(z)}e^{i(z-y)\zeta/h}b(s,h;z,\zeta)w(y)dyd\zeta,$$

where $\gamma(z)$ is a complex contour of the form,

$$\gamma(z)\,:\,\zeta=-\mathrm{Im}\, z+iR(\overline{z-y})\ ;\ |y-z|<r,$$

with $R>0$ is fixed large enough, and $r>0$ can be taken arbitrarily small. In particular, we deduce from (3.5) that $B(s,h)$ can be written as,

$$B(s,h)=hB_1(s,h),$$

where $B_1(s,h)$ admit a symbol uniformly $\mathcal{O}(\langle s\rangle^{1-\sigma}+h\langle s\rangle^{2-\sigma})$, for $s>0$, z in a compact subset of $\mathbb{C}^n\backslash\{\mathrm{Im}\, z=0\}$, and $(y,\zeta)\in\gamma(z)$.

Then, for $z_0 \in \mathbb{C}^n\backslash\{\text{Im } z = 0\}$ and $\varepsilon_0 > 0$, if we set,

$$L^2_{\Phi_0}(z_0, \varepsilon_0) := L^2(\{|z - z_0| < \varepsilon_0\}; e^{-2\Phi_0/h} d\text{Re } z\, d\text{Im } z) \cap H_{\Phi_0}(|z - z_0| < \varepsilon_0),$$

we see that $B_1(s, h)$ is a bounded operator from $L^2_{\Phi_0}(z_0, \varepsilon_0)$ to $L^2_{\tilde{\Phi}_0}(z_0, \varepsilon_0/2)$, and its norm can be easily estimated in terms of the supremum of its symbol. Thus, here we obtain,

$$\|B_1(s)\|_{\mathcal{L}\left(L^2_{\Phi_0}(z_0,\varepsilon_0); L^2_{\Phi_0}(z_0,\varepsilon_0/2)\right)} = \mathcal{O}(\langle s\rangle^{1-\sigma} + h\langle s\rangle^{2-\sigma}) = \mathcal{O}(\langle s\rangle^{1-\sigma}), \tag{3.6}$$

uniformly with respect to $h > 0$ small enough and $|s| \leq T_0/h$ ($T_0 > 0$ fixed arbitrarily).

Now, let us denote by $\tilde{\Phi}_0 = \tilde{\Phi}_0(z, \overline{z})$ a smooth real-valued function defined near $z = z_0$, such that $|\tilde{\Phi}_0 - \Phi_0|$ and $|\nabla_{(z,\overline{z})}(\tilde{\Phi}_0 - \Phi_0)|$ are small enough, and verifying,

$$\tilde{\Phi}_0 \geq \Phi_0 \text{ in } \{|z - z_0| \leq \varepsilon_0\}; \tag{3.7}$$

$$\tilde{\Phi}_0 = \Phi_0 \text{ in } \{|z - z_0| \leq \varepsilon_0/4\}; \tag{3.8}$$

$$\tilde{\Phi}_0 > \Phi_0 + \varepsilon_1 \text{ in } \{|z - z_0| \geq \varepsilon_0/2\}, \tag{3.9}$$

for some $\varepsilon_1 > 0$. By modifying the contour defining $B_1(s)$ (see [13], Remarque 4.4), we know that $B_1(s)$ is also bounded from $L^2_{\tilde{\Phi}_0}(z_0, \varepsilon_0)$ to $L^2_{\tilde{\Phi}_0}(z_0, \varepsilon_0/2)$, and its norm on these space verifies the same estimate (3.6) as on $L^2_{\Phi_0}$.

Setting $w = Tv$, Eq. (3.4) gives,

$$i\partial_s w(s) = B_1(s, h)w(s) \text{ in } H_{\Phi_0}(|z - z_0| < \varepsilon_0), \tag{3.10}$$

with $\varepsilon_0 > 0$ fixed small enough, and thus,

$$\partial_s \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} = 2\text{Im } \langle B_1(s)w(s), w(s)\rangle_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)}.$$

Using Cauchy–Schwarz inequality and (3.6), we obtain,

$$\left|\partial_s \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)}\right| = \mathcal{O}(\langle s\rangle^{1-\sigma})\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0)}. \tag{3.11}$$

On the other hand, using (3.9) and the fact that $\|v(t)\|_{L^2} = \|u_0\|_{L^2}$ does not depend on t, we also have the estimate,

$$\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0)} = \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + \mathcal{O}(e^{-\varepsilon_1/h}),$$

that, inserted into (3.11), gives,

$$\left|\partial_s \|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)}\right| \leq C\langle s\rangle^{1-\sigma}\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + Ce^{-\varepsilon_1/h},$$

with some constant $C > 0$. Setting $g(s) := C\int_0^s \langle s'\rangle^{1-\sigma} ds'$, and using Gronwall's lemma, we finally obtain,

$$\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} \leq e^{g(s)}\|w(0)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + C\int_0^s e^{g(s)-g(s')-\varepsilon_1/h} ds';$$
$$\|w(0)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} \leq e^{g(s)}\|w(s)\|^2_{L^2_{\tilde{\Phi}_0}(z_0,\varepsilon_0/2)} + C\int_0^s e^{g(s')-\varepsilon_1/h} ds'.$$

Then, replacing s by t/h and observing that $g(s) = \mathcal{O}(\langle s\rangle^{2-\sigma}) = \mathcal{O}(h^{\sigma-2}) = o(h^{-1})$, the equivalence $(x_0,\xi_0) \notin WF_a(u_0) \Longleftrightarrow (x_0,\xi_0) \notin WF_a(u(t))$ follows immediately, and the result is proved in this case.

Now, let us still consider the case where the perturbation is short range, but the metric is not necessarily flat anymore. Then, the result we have to prove is the following: for any $t > 0$ and $(x_0,\xi_0) \in NT^+$, one has the equivalence,

$$(x_0,\xi_0) \in WF_a(u_0) \Longleftrightarrow S_+(x_0,\xi_0) \in WF_a(e^{itP_0}e^{-itP}u_0).$$

Proceeding as in the flat case, we arrive again at Eq. (3.4), but this time $B(s,h)$ is of order 0, and can be written as,

$$B(s,h) = B_0(s,h) + hB_1(s,h),$$

where B_1 is as before, and the symbol of B_0 is,

$$b_0(s;z,\zeta) = \frac{1}{2}\sum_{j,k=1}^{n}(a_{j,k}(z+i\zeta+s\zeta) - \delta_{j,k})\zeta_j\zeta_k.$$

Then, in order to get rid of $B_0(s)$, we construct a Fourier integral operator $F(s,h)$ on H_{Φ_0,z_0}, verifying,

$$\begin{cases} ih\partial_s F(s,h) - B_0(s,h)F(s,h) \sim \mathcal{O}(h); \\ F\,|_{s=0} = I. \end{cases}$$

More precisely, we look for $F(s,h)$ of the form,

$$F(s)v(z) = \frac{1}{(2\pi h)^n}\int_{\gamma_s(z)} e^{i(\psi(s,z,\eta)-y\eta)/h}v(y)dyd\eta, \tag{3.12}$$

where $\gamma_s(z)$ is a convenient contour and ψ is a holomorphic function that must solve the system (eikonal equation),

$$\begin{cases} \partial_s \psi + b_0(s, z, \nabla_z \psi) = 0; \\ \psi \,|_{s=0} = z.\eta. \end{cases} \tag{3.13}$$

The construction of $\psi(s)$ for small s just follows from standard Hamilton-Jacobi theory, and the extension to larger values of s can be made by using the classical flow R_s of $b_0(s)$, that is related to the Hamilton flow of p through the formula,

$$R_s = \kappa \circ \exp(-sH_{p_0}) \circ \exp sH_p \circ \kappa^{-1}, \tag{3.14}$$

where $\kappa(x, \xi) = (x - i\xi, \xi)$ is the complex canonical transformation associated with T. We refer to [7], Sect. 6, for the detailed construction.

In that way, we find a solution $\psi(s, \zeta, \eta)$ of (3.13), defined for $s \in \mathbb{R}$, z close to $z_0 := x_0 - i\xi_0$ (where $(x_0, \xi_0) \in NT^+$ is fixed arbitrarily), and η close to ξ_0. One also has the relation,

$$(z, \nabla_z \psi(s, z, \eta)) = R_s(\nabla_\eta \psi(s, z, \eta), \eta), \tag{3.15}$$

which means that ψ is a generating function of the complex canonical transformation R_s. In other words, the operator $F(s, h)$ defined by (3.12) quantizes the canonical relation R_s, and, setting $z_s := \pi_z R_s(z_0, \xi_0)$ (where $\pi_z : (z, \zeta) \mapsto z$), one can show that for any $\varepsilon_0 > 0$ small, $F(s, h)$ acts as,

$$F(s) \,:\, H_{\Phi_0}(|z - z_0| < \varepsilon_0) \to H_{\Phi_0}(|z - z_s| < \varepsilon_1), \tag{3.16}$$

for some $\varepsilon_1 = \varepsilon_1(\varepsilon_0) > 0$. A priori, ε_1 also depends on s, but as a matter of fact, since R_s tends to $R_\infty := \kappa \circ S_+ \circ \kappa^{-1}$ on a neighborhood of (z_0, ξ_0) as $s \to +\infty$, one can prove that $F(s; h)$ admits a limit $F_\infty(h)$ that is a FIO quantizing R_∞. Then, the action (3.16) remains valid for $0 \le s \le +\infty$ (with $z_\infty := \pi_z R_\infty(z_0, \xi_0)$), ε_1 can be taken independent of s, and the norm of $F(s)$ is uniformly bounded both with respect to h and $s \ge 0$.

Now, by construction, for $s \in \mathbb{R}$, $F(s)$ verifies,

$$ih\partial_s F(s) - B_0(s)F(s) = hF_1(s),$$

where $F_1(s) \,:\, H_{\Phi_0}(|z - z_0| < \varepsilon_0) \to H_{\Phi_0}(|z - z_s| < \varepsilon_1)$ is of the form,

$$F_1(s)v(z) = \frac{1}{(2\pi h)^n} \int_{\gamma_s(z)} e^{i(\psi(s,z,\eta) - y\eta)/h} f_1(s, z, \eta; h) v(y) dy d\eta,$$

with f_1 is an analytic symbol that is $\mathcal{O}(\langle s \rangle^{-1-\sigma})$ as $s \to \infty$.

In the same way, for any y close enough to z_0, we can define a Fourier integral operator $\tilde{F}(s)$ of the form,

$$\tilde{F}(s)v(y) := \frac{1}{(2\pi h)^n} \int_{\tilde{\gamma}_s(y)} e^{i(y\eta - \psi(s,z,\eta))/h} v(z) dz d\eta,$$

(where $\tilde{\gamma}_s(y)$ is again a convenient contour), such that $\tilde{F}(s)$ maps $H_{\Phi_0}(|z - z_s| < \varepsilon_0)$ into $H_{\Phi_0}(|z - z_0| < \varepsilon_1)$, and verifies,

$$ih\partial_s \tilde{F}(s) + \tilde{F}(s)B = h\tilde{F}_1(s), \tag{3.17}$$

where $\tilde{F}_1(s) : H_{\Phi_0}(|z - z_s| < \varepsilon_0) \to H_{\Phi_0}(|z - z_0| < \varepsilon_1)$ is a FIO with same phase as $\tilde{F}(s)$ and symbol $\tilde{f}_1 = \mathcal{O}(\langle s\rangle^{-1-\sigma})$.

Now, setting,

$$\tilde{w}(s) = \tilde{F}(s)Tu(hs) \in H_{\Phi_0}(|z - z_0| < \varepsilon_1),$$

by (3.4) and (3.17), we see that $\tilde{w}$ verifies,

$$i\partial_s \tilde{w}(s) = \left[\tilde{F}(s)B_1(s) + \tilde{F}_1(s)\right] Tu(hs).$$

Moreover, since $A(s) := F(s)\tilde{F}(s)$ is an elliptic pseudodifferential operator on H_{Φ_0,z_s}, by taking a parametrix $\tilde{A}(s)$, we have,

$$Tu(hs) = \tilde{A}(s)F(s)w(s) \ \text{ in } H_{\Phi_0}(|z - z_s| < \varepsilon), \tag{3.18}$$

(for some $\varepsilon > 0$ independent of s), and thus, we obtain,

$$i\partial_s \tilde{w}(s) = \tilde{B}_1(s)\tilde{w}(s). \tag{3.19}$$

in $H_{\Phi_0}(|z - z_0| < \varepsilon')$, where $\tilde{B}_1(s) := \left[\tilde{F}(s)B_1(s) + \tilde{F}_1(s)\right]\tilde{A}(s)F(s)$ is a pseudodifferential operator on $H_{\Phi_0}(|z - z_0| < \varepsilon')$ with the same properties as $B_1(s)$ when $s \to +\infty$.

Thus, we are reduced to a situation completely similar to that of the flat case, and, if for instance $(x_0, \xi_0) \notin WF_a(u_0)$, the same arguments show that,

$$\|w(s)\|_{L^2_{\Phi_0}(z_0,\delta)} \le Ce^{-\delta/h},$$

for some positive constant δ independent of $h > 0$ small enough and $s \in [0, T/h]$. As a consequence, using (3.18) and the fact that $\tilde{A}(s)F(s)$ is uniformly bounded from $L^2_{\Phi_0}(z_0, \delta)$ to $L^2_{\Phi_0}(z_s, \delta')$ for some $\delta' > 0$, we obtain (with some new constant $C > 0$),

$$\|Tu(hs)\|_{L^2_{\Phi_0}(z_s,\delta')} \le Ce^{-\delta/h}.$$

Replacing s by t/h with $t > 0$ fixed, and observing that $z_{t/h}$ tends to $\kappa \circ S_+(x_0, \xi_0)$ as $h \to 0_+$, we conclude that $S_+(x_0, \xi_0) \notin WF_a(u(t))$. The converse can be seen in the same way, and thus Theorem 2.1 is proved.

In the **long range** case, the construction of $W_\pm$ results from standard Hamilton-Jacobi theory, and the proof is very similar, except that we now have to handle expressions like

$$e^{i\widetilde{W}_\pm(s,hD_z)/h}v(z;h) := \int_{\gamma(s,z)} e^{i(z-y)\zeta/h+i\widetilde{W}_\pm(s,\zeta)/h}v(y)dydz,$$

where $\gamma(s,z)$ is a good contour in the sense of [13], with some uniformity as $s \to \infty$. Then, one can show that $e^{i\widetilde{W}_\pm(s,hD_z)/h}$ is a Fourier integral operator acting of Sjöstrand's spaces H_{Φ_0}, in the sense that one has,

$$e^{i\widetilde{W}_\pm(s,hD_z)/h} : H_{\Phi_0}(\Omega_s(z_0,\varepsilon_1)) \to H_{\Phi_0}(\Omega_s(Z_s(z_0),\varepsilon_2)),$$

with $\varepsilon_1, \varepsilon_2 > 0$ small enough, and where we have set,

$$\Omega_s(Z,\varepsilon) := \{z \in \mathbb{C}^n ; \langle s\rangle^{-1}|\text{Re } (z-Z)| + |\text{Im } (z-Z)| < \varepsilon\}.$$

We refer to [8] for more details.

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